Derivation of the Maxwell-Schrödinger Equations from the Pauli-Fierz Hamiltonian

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Abstract

We consider the spinless Pauli-Fierz Hamiltonian which describes a quantum system of non-relativistic identical particles coupled to the quantized electromagnetic field. We study the time evolution in a mean-field limit where the number N of charged particles gets large while the coupling to the radiation field is rescaled by $1/\sqrt{N}$. At time zero we assume that almost all charged particles are in the same one-body state (a Bose-Einstein condensate) and we assume also the photons to be close to a coherent state. We show that at later times and in the limit $N \to \infty$ the charged particles as well as the photons exhibit condensation, with the time evolution approximately described by the Maxwell-Schrödinger system, which models the coupling of a non-relativistic particle to the classical electromagnetic field. Our result is obtained by an extension of the „method of counting", introduced in [14], to condensates of charged particles in interaction with their radiation field.

I Introduction and Main Result

I.1 Setting of the problem

The existence of light quanta, later named photons, was first postulated by Albert Einstein in his renowned paper „On a heuristic point of view about the creation and conversion of light"[3]. This led to the invention of Quantum Electrodynamics and supplemented the nature of light, which was formerly described as a wave in classical electromagnetism, with a particle interpretation. During the last decades the predictions of Quantum Electrodynamics has been tested up to highest accuracy. Nevertheless, in a lot of situations the corpuscular character of light is subordinate and the second-quantized electromagnetic field can be approximated by a classical field satisfying Maxwell's equations. In this paper, the validity of such an approximation is justified in the mean-field regime. More explicitly, we derive the Maxwell-Schrödinger

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equations from the spinless Pauli-Fierz Hamiltonian. Such a derivation is of great interest to fundamental physics. Moreover, since the applied mean-field approximation reduces the degrees of freedom of the original system tremendously explicit error bounds might also be of interest for numerical simulations.
In order to derive the Maxwell-Schrödinger equations from Many-body Quantum Dynamics, we consider a system of N identical charged particles interacting with a photon field, described by a wave function $\Psi_N \in \mathcal{H}^{(N)}$. Here,

$$\mathcal{H}^{(N)} := L^2\left(\mathbb{R}^{3N}\right) \otimes \mathcal{F}_p, \tag{1}$$

where the photon field is represented by elements of the Fock space

$$\mathcal{F}_p := \bigoplus_{n\geq 0} \left[L^2(\mathbb{R}^3) \otimes \mathbb{C}^2\right]^{\otimes_s^n}. \tag{2}$$

The subscript s indicates symmetry under interchange of variables. The Hilbert space $\mathfrak{h} := L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$ consists of wave functions $f(k,\lambda)$, with wave number $k \in \mathbb{R}^3$ and helicity $\lambda = 1, 2$. It is equipped with the inner product

$$\langle f, g\rangle_{\mathfrak{h}} := \sum_{\lambda=1,2} \int d^3k f^*(k,\lambda) g(k,\lambda). \tag{3}$$

The time evolution of Ψ_N is governed by the Schrödinger equation

$$i\partial_t \Psi_N(t) = H_m^N \Psi_N(t), \quad \Psi_N(0) = \Psi_{N0}, \tag{4}$$

where

$$H_m^N = \sum_{j=1}^N \left(-i\nabla_j - \frac{\hat{\boldsymbol{A}}_\kappa(x_j)}{\sqrt{N}}\right)^2 + \frac{1}{N}\sum_{1\leq j<k\leq N} v(x_j - x_k) + H_f \tag{5}$$

denotes the Pauli-Fierz Hamiltonian and

$$\hat{\boldsymbol{A}}_\kappa(x) = \sum_{\lambda=1,2} \int d^3k\ \tilde{\kappa}(k) \frac{1}{\sqrt{2|k|}} \boldsymbol{\epsilon}_\lambda(k) \left(e^{ikx} a(k,\lambda) + e^{-ikx} a^*(k,\lambda)\right) \tag{6}$$

the quantized transverse vector potential. The function

$$\tilde{\kappa}(k) = (2\pi)^{-3/2}\ \mathbb{1}_{|k|\leq\Lambda}(k), \quad \text{with } \mathbb{1}_{|k|\leq\Lambda}(k) = \begin{cases} 1 & \text{if } |k| \leq \Lambda, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

cuts off the high frequency modes of the radiation field. There are two real polarization vectors $\boldsymbol{\epsilon}_1(k)$ and $\boldsymbol{\epsilon}_2(k)$ with

$$|\boldsymbol{\epsilon}_1(k)| = |\boldsymbol{\epsilon}_2(k)| = 1,\ \boldsymbol{\epsilon}_1(k) \cdot k = \boldsymbol{\epsilon}_2(k) \cdot k = \boldsymbol{\epsilon}_1(k) \cdot \boldsymbol{\epsilon}_2(k) = 0. \tag{8}$$

The operator valued distributions $a(k,\lambda)$ and $a^*(k,\lambda)$ ($k \in \mathbb{R}^3, \lambda \in \{1,2\}$) are the usual pointwise annihilation and creation operators in $\mathcal{F}_p$, satisfying

$$[a(k,\lambda), a^*(l,\mu)] = \delta_{\lambda,\mu}\delta(k-l), \quad [a(k,\lambda), a(l,\mu)] = [a^*(k,\lambda), a^*(l,\mu)] = 0. \tag{9}$$

The energy of the photon field is given by

$$H_f = \sum_{\lambda=1,2} \int d^3k |k| a^*(k,\lambda) a(k,\lambda) \tag{10}$$

and the potential v describes a direct interaction between the charged particles. We assume:

(A1) The (repulsive) interaction potential v is a positive, real, and even function satisfying

$$||v||_{L^2+L^\infty} = \inf_{v=v_1+v_2} \{||v_1||_{L^2(\mathbb{R}^3)} + ||v_2||_{L^\infty(\mathbb{R}^3)}\} < \infty \tag{11}$$

such that the Pauli-Fierz Hamiltonian H_m^N is self-adjoint on the domain $\mathcal{D}(H_m^N) := \mathcal{D}(\sum_{i=1}^N (-\Delta_i) + H_f)$, [8] and [20, p.164].

The mean-field scaling 1/N in front of the interaction potential and the scaling $1/\sqrt{N}$ in front of the vector potential ensure that the kinetic and potential energy of H_m^N are of the same order. At first, we are interested in the dynamics generated by H_m^N for initial conditions of the product form

$$\varphi_0^{\otimes N} \otimes W(\sqrt{N}\alpha_0)\Omega. \tag{12}$$

Here, Ω denotes the vacuum in $\mathcal{F}_p$ and $W(f)$ is the Weyl operator

$$W(f) := \exp\left(\sum_{\lambda=1,2} \int d^3k f(k,\lambda) a^*(k,\lambda) - f^*(k,\lambda) a(k,\lambda) \right). \tag{13}$$

This choice of initial data corresponds to situations in which both the charged particles and the photons exhibit condensation. (Later, we also formulate a theorem for condensates with weak correlations.) Due to different types of interactions, correlations take place and the time evolved state will no longer have an exact product structure. However, for large N and times of order one it can be approximated, in a sense specified below, by a state of the product form $\varphi_t^{\otimes N} \otimes W(\sqrt{N}\alpha_t)\Omega$, where

$$|k|^{1/2} \alpha_t(k,\lambda) = \frac{1}{\sqrt{2}} \boldsymbol{\epsilon}_\lambda(k) \cdot \left(|k| \mathcal{FT}[\boldsymbol{A}](k,t) - i \mathcal{FT}[\boldsymbol{E}](k,t) \right) \tag{14}$$

and $(\varphi_t, \boldsymbol{A}(t), \boldsymbol{E}(t))$ solve the Maxwell-Schrödinger system*

$$\begin{cases} i\partial_t \varphi_t(x) &= \left((-i\nabla - (\kappa \star \boldsymbol{A})(x,t))^2 + (v \star |\varphi_t|^2)(x) \right) \varphi_t(x), \\ \nabla \cdot \boldsymbol{A}(x,t) &= 0, \\ \partial_t \boldsymbol{A}(x,t) &= -\boldsymbol{E}(x,t), \\ \partial_t \boldsymbol{E}(x,t) &= (-\Delta \boldsymbol{A})(x,t) - (1 - \nabla \mathrm{div} \Delta^{-1})(\kappa \star \boldsymbol{j}_t)(x), \\ \boldsymbol{j}_t(x) &= 2 \left(\mathrm{Im}(\varphi_t^* \nabla \varphi_t)(x) - |\varphi_t|^2(x) (\kappa \star \boldsymbol{A})(x,t) \right) \end{cases} \tag{15}$$

* Hereby, $(\kappa \star \boldsymbol{A})(x,t) = \int d^3k e^{ikx} \tilde{\kappa}(k) \boldsymbol{A}(k,t)$.

with initial datum

$$\begin{cases} \varphi_0, \\ \boldsymbol{A}(x,0) = (2\pi)^{-3/2} \sum_{\lambda=1,2} \int d^3k \frac{1}{\sqrt{2|k|}} \boldsymbol{\epsilon}_\lambda(k) \left(e^{ikx}\alpha_0(k,\lambda) + e^{-ikx}\alpha_0^\star(k,\lambda) \right), \\ \boldsymbol{E}(x,0) = (2\pi)^{-3/2} \sum_{\lambda=1,2} d^3k \sqrt{\frac{|k|}{2}} \boldsymbol{\epsilon}_\lambda(k) i \left(e^{ikx}\alpha_0(k,\lambda) - e^{-ikx}\alpha_0^\star(k,\lambda) \right). \end{cases} \tag{16}$$

These equations determine the time evolution of a single quantum particle interacting with the classical electromagnetic field it generates. The solution theory of this system is well studied, see [13] and references therein.

I.2 Main result

The physical situation we are interested in is the dynamical description of a Bose-Einstein condensate of charged particles. We start with an initial wave function of product form (12) (a condition that will be relaxed later) and show that the condensate is stable over time, i.e. correlations are small at later times. Let $\Psi_N(t) \in \left(L_s^2\left(\mathbb{R}^{3N}\right) \otimes \mathcal{F}_p\right) \cap \mathcal{H}^{(N)}$ with $||\Psi_N(t)|| = 1$. On the Hilbert space $L^2(\mathbb{R}^3)$, define the „reduced one particle density matrix of the charged particles“ by

$$\gamma_N^{(1,0)}(t) := \mathrm{Tr}_{2,\dots,N} \otimes \mathrm{Tr}_{\mathcal{F}} |\Psi_N(t)\rangle\langle\Psi_N(t)|, \tag{17}$$

where $\mathrm{Tr}_{2,\dots,N}$ denotes the partial trace over the coordinates $x_2, \dots, x_N$ and $\mathrm{Tr}_{\mathcal{F}}$ the trace over Fock space. Then the charged particles of the Many-body state Ψ_N are said to exhibit complete asymptotic Bose-Einstein condensation at time t, if there exists $\varphi_t \in L^2(\mathbb{R}^3)$ with $||\varphi_t|| = 1$, such that

$$\mathrm{Tr}_{L^2(\mathbb{R}^3)} |\gamma_N^{(1,0)}(t) - |\varphi_t\rangle\langle\varphi_t|| \to 0, \tag{18}$$

as $N \to \infty$. Such φ_t is called the condensate wave function. For other indicators of condensation and their relation we refer to [12]. Given $\Psi_N(t) \in \mathcal{D}(H_f)$ with $||\Psi_N(t)|| = 1$, we introduce the „reduced one particle energy matrix of the photons “ with kernel

$$\gamma_N^{(0,1)}(t,k,\lambda;k',\lambda') := N^{-1}|k|^{1/2}|k'|^{1/2} \langle \Psi_N(t), a^*(k',\lambda')a(k,\lambda)\Psi_N(t)\rangle_{\mathcal{H}^{(N)}}. \tag{19}$$

$\gamma_N^{(0,1)}$ is a positive trace class operator on $\mathfrak{h}$ with $\mathrm{Tr}_{\mathfrak{h}}(\gamma_N^{(0,1)}) = N^{-1}\langle\Psi_N(t), H_f\Psi_N(t)\rangle_{\mathcal{H}^{(N)}}$. It is important to note, that (19) differs from the usual definition (e.g. [18, p.8]) by the weight factor $|k||k'|\langle\Psi_N, \mathcal{N}\Psi_N\rangle_{\mathcal{H}^{(N)}}/N$ with $\mathcal{N}$ being the number of photons operator. Our choice ensures that we neglect photons with small energies and measure only deviations from the photon field that are at least of order N. This is reasonable because due to the scaled coupling many photon states with a mean particle number smaller than of order N only have a subleading effect on the dynamics of the charged particles. We say the photons exhibit asymptotic Bose-Einstein condensation, if there exists a state $u_t \in \mathfrak{h}$, such that

$$\mathrm{Tr}_{\mathfrak{h}} |\gamma_N^{(0,1)}(t) - |u_t\rangle\langle u_t|| \to 0, \tag{20}$$

as $N \to \infty$.

In the absence of a cutoff function and v being the Coulomb potential, the Maxwell-Schrödinger system is globally well-posed in the space† $C(\mathbb{R}_t, H^3(\mathbb{R}^3) \oplus H^3(\mathbb{R}^3) \oplus H^2(\mathbb{R}^3))$ [13]. We assume that this also holds in presence of the ultraviolet cutoff $\tilde{\kappa}$ and for potentials of the form (A1). More specific, we choose $\varphi_0 \in H^3(\mathbb{R}^3)$ and $\alpha_0 \in \mathfrak{h}$ such that $(\boldsymbol{A}(0), \boldsymbol{E}(0))$, defined by (16), is in $(H^3(\mathbb{R}^3) \oplus H^2(\mathbb{R}^3))$. Then, we assume

$$\sup_{t\in[0,T]} \{||\varphi_t||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{A}(t)||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{E}(t)||_{H^2(\mathbb{R}^3)}\} < \infty \tag{21}$$

for any $T \in \mathbb{R}^+$. As a result (26) one has that u_t, defined by

$$u_t(k,\lambda) := |k|^{1/2}\alpha(k,\lambda) = \frac{1}{\sqrt{2}}\boldsymbol{\epsilon}_\lambda(k) \cdot \left(|k|\mathcal{FT}[\boldsymbol{A}](k,t) - i\mathcal{FT}[\boldsymbol{E}](k,t)\right), \tag{22}$$

is an element of the Hilbert space $\mathfrak{h}$.

Theorem I.1. *Let v satisfy (A1), $\varphi_0 \in L^2(\mathbb{R}^3)$ with $||\varphi_0|| = 1$, $\alpha_0 \in \mathfrak{h}$ such that $(\boldsymbol{A}(0), \boldsymbol{E}(0)) \in (H^3(\mathbb{R}^3) \oplus H^2(\mathbb{R}^3))$, $\Psi_{N0} = \varphi_0^{\otimes N} \otimes W(\sqrt{N}\alpha_0)\Omega$. Let Ψ_{Nt} the unique solution of (4). Let $(\varphi_t, \boldsymbol{A}(t), \boldsymbol{E}(t))$ the unique solution of (15), u_t defined by (22) and assume $\sup_{t\in[0,T]}\{||\varphi_t||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{A}(t)||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{E}(t)||_{H^2(\mathbb{R}^3)}\} < \infty$ for any $T \in \mathbb{R}^+$. Then, for any $t > 0$ there exist two constants C_1, C_2 (C_2 is time-dependent) such that*

$$Tr_{L^2(\mathbb{R}^3)}|\gamma_N^{(1,0)}(t) - |\varphi_t\rangle\langle\varphi_t|| \leq N^{-1/2}\Lambda^2 C_1 e^{\Lambda^4 C_2(t)}(1 + \Lambda^2 C_2^{1/2}(t)), \tag{23}$$

$$Tr_{\mathfrak{h}}|\gamma_N^{(0,1)}(t) - |u_t\rangle\langle u_t|| \leq N^{-1/2}\Lambda^2 C_1 e^{\Lambda^4 C_2(t)}(1 + \Lambda^2 C_2^{1/2}(t)). \tag{24}$$

Remarks:

(R1) Theorem I.1 holds for a larger class of initial conditions, see Theorem IV.1. To state this precisely we first introduce an alternative indicator of condensation in Section III.

(R2) Assumption (A1) allows to consider the Coulomb potential $v(x) = |x|^{-1}$. The requirements on the interaction potential can easily be relaxed because our estimates only rely on the finiteness of $||v \star |\varphi|^2||_\infty$ and $||v^2 \star |\varphi|^2||_\infty$. This is captured by (A1) and $\varphi \in H^3(\mathbb{R}^3)$ but also by other means.

(R3) For simplicity we apply the mean-field scaling $1/N$ in front of the direct interaction. Using techniques from [15] and [16] it seems possible to treat the direct interaction also in the NLS or Gross-Pitaevskii regime.

(R4) The ultraviolet cutoff is essential in our derivation but can be chosen N-dependent.

I.3 Comparison with the literature

Derivations of classical field equations from Many-body Quantum Dynamics has been established in a series of works: In [7], Ginibre, Nironi and Velo derived the Schrödinger-Klein-Gordon system of equations from the Nelson model with cutoff. They considered

†The direct sum of the Sobolev spaces refers to $(\varphi_t, \boldsymbol{A}(t), \boldsymbol{E}(t))$.

a mean field limit where a finite number of charged particles interacts with a coherent state of gauge bosons whose particle number goes to infinity. Falconi [4] derived the Schrödinger-Klein-Gordon system of equations in a mean-field limit where both the number of the charged particles and the gauge bosons go to infinity. The result was obtained by means of Hepp's method but can also be derived with the techniques from the present paper [10]. Making use of a Wigner measure approach Ammari and Falconi [1] were able to establish the classical limit of the renormalized Nelson model without cutoff. The replacement of quantized radiation fields by classical interactions has also been justified in other limits. Teufel [21] considered the adiabatic limit of the Nelson model and showed that the interaction mediated by the quantized radiation field is well approximated by a direct Coulomb interaction. In [5] and [6], Frank, Gang and Schlein showed that in the strong coupling limit the dynamics of a polaron is described by an effective equation, in which the phonon field is treated as a classical field. In [19] it is shown that the semiclassical set of coupled Maxwell-Schrödinger equations is obtained by neglecting certain terms of the Pauli-Fierz Hamiltonian. To our best knowledge, this is the first rigorous result concerning a mean-field limit of the Pauli-Fierz Hamiltonian. This work continues the master thesis [22].

I.4 Acknowledgements

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I.5 Notations:

We set Planck's constant $\hbar$, the speed of light c, the charge e, and twice the mass of the particles $2m$ equal to one. Except in definitions, results, and where confusion might be possible, we refrain from indicating the explicit dependence of a quantity on the time t. We use the notations $\varphi(t)$ and φ_t interchangeably to denote a quantity φ at time t. The symbol C is used as a generic positive constant that depend on some fixed parameters, like $||\nabla\varphi||_{L^2(\mathbb{R}^3)}$. If not stated otherwise, C will be independent of the cutoff Λ. Both $\tilde{f}$ and $\mathcal{FT}[f]$ stand for the Fourier transform of f. With a slight abuse of notation $\boldsymbol{A}$ and $\boldsymbol{E}$ denote the vector potential and the electric field, but also their respective Fourier transforms. If we write $\boldsymbol{A}(t)$ or $\boldsymbol{E}(t)$, we always refer to the coordinate representation of the electromagnetic fields. Furthermore, we use the short-hand notation $\boldsymbol{A}_\kappa(x,t) := (\kappa \star \boldsymbol{A})(x,t)$.
We abbreviate $||\cdot||_{L^2(\mathbb{R}^3)} = ||\cdot||$ and $||\cdot||_{\mathcal{H}} = ||\cdot||$. $H^s(\mathbb{R}^3)$ $(s \in \mathbb{R})$ stands for the Sobolev space with norm $||f||_{H^s(\mathbb{R}^3)} = \left|\left|(1+|k|^2)^{s/2}\tilde{f}\right|\right|_{L^2(\mathbb{R}^3)}$ and $||A||_{HS} = \sqrt{TrA^*A}$ is used for the Hilbert-Schmidt norm. The symbol $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ denotes the scalar product on $\mathcal{H}^{(N)}$. Furthermore, we use the notation $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{;y} = \int d^3y \langle\!\langle\cdot,\cdot\rangle\!\rangle$ and $||\cdot||_{;y} = \sqrt{\int d^3y \langle\!\langle\cdot,\cdot\rangle\!\rangle}$. The scalar product on the Fock space $\mathcal{F}_p$ is called $\langle\cdot,\cdot\rangle_{\mathcal{F}}$ and $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{L^2(\mathbb{R}^3)}$.

II Organization of the proof

Our result is obtained by generalizing the „method of counting“, introduced in [14], to condensates of charged particles in interaction with their radiation field [10]. The key idea is not to prove condensation in terms of reduced density matrices but to consider a different indicator of condensation which is easy to work with. More specific, we introduce a functional $\beta(t) : \mathcal{H}^{(N)} \times L^2(\mathbb{R}^3) \times (L^2(\mathbb{R}^3) \otimes \mathbb{C}^2) \to \mathbb{R}_0^+$ such that

(i) $\beta(0) := \beta[\Psi_0, \varphi_0, \alpha_0] \leq C_1 N^{-\delta}$ for some constants $C_1 \geq 0$ and $\delta > 0$.

(ii) $d_t\beta(t)$ can be estimated by $\beta(t) + 1/N$. Using a Grönwall type estimate it follows that $\beta(t) \leq N^{-\min\{1,\delta\}} C_1 e^{C_2(t)} (1 + C_2(t))$ with time dependent constant $C_2(t)$.

(iii) $\beta(t) \to 0$ as $N \to \infty$ implies (18) and (20).

The proof is organized as follows:

(i) In Section III we define the counting functional. Afterwards, we show that convergence of the functional to zero in the limit $N \to \infty$ implies condensation in terms of reduced density matrices.

(ii) In Section IV we prove Theorem I.1 for a larger class of initial conditions, Theorem IV.1: First, we provide preliminary estimates and then control β during the time evolution.

(iii) Section V shows that states of product type (12) satisfy the initial conditions of Theorem IV.1. This concludes the proof of Theorem I.1.

In our estimates, we need the regularity conditions

$$||\varphi_t||_\infty < \infty, \quad ||\nabla\varphi_t||_\infty < \infty, \quad ||\nabla\varphi_t|| < \infty, \quad ||\Delta\varphi_t|| < \infty, \tag{25}$$

$$||\boldsymbol{A}_\kappa(t)||_\infty < \infty, \quad \mathcal{E}_f(t) := \sum_{\lambda=1,2} \int d^3k |k| |\alpha_t(k,\lambda)|^2 < \infty, \tag{26}$$

$$\mathcal{E}_{f^2}(t) := \sum_{\lambda=1,2} \int d^3k |k|^2 |\alpha_t(k,\lambda)|^2 < \infty. \tag{27}$$

Assuming $\sup_{t\in[0,T]}\{||\varphi_t||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{A}(t)||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{E}(t)||_{H^2(\mathbb{R}^3)}\} < \infty$ for any $T \in \mathbb{R}^+$ the first line follows from Sobolev inequalities. To continue, we define the functions

$$\tilde{\kappa}_<(k) := (2\pi)^{-3/2} \mathbb{1}_{|k|\leq 1}(k), \qquad \tilde{\kappa}_>(k) := (2\pi)^{-3/2} |k|^{-2} \mathbb{1}_{1\leq|k|\leq\Lambda}(k) \tag{28}$$

with

$$||\kappa_<||_2^2 = \langle\tilde{\kappa}_<, \tilde{\kappa}_<\rangle = (2\pi)^{-3} \int_{|k|\leq 1} d^3k = \frac{1}{6\pi^2}, \tag{29}$$

$$||\kappa_>||_2^2 = \langle\tilde{\kappa}_>, \tilde{\kappa}_>\rangle = (2\pi)^{-3} \int_{1\leq|k|\leq\Lambda} d^3k |k|^{-4} = \frac{1}{4\pi^2}(1 - 1/\Lambda) \leq \frac{1}{4\pi^2}. \tag{30}$$

This gives

$$\boldsymbol{A}_\kappa(x,t) =(2\pi)^3\int d^3k e^{ikx}\mathbb{1}_{|k|\leq\Lambda}(k)\boldsymbol{A}(k,t) = (2\pi)^3\int d^3k e^{ikx}\mathbb{1}_{|k|\leq 1}(k)\boldsymbol{A}(k,t) \tag{31}$$

$$+(2\pi)^3\int d^3k e^{ikx}|k|^{-2}\mathbb{1}_{1\leq|k|\leq\Lambda}(k)|k|^2\boldsymbol{A}(k,t) \tag{32}$$

$$= (\kappa_< \star \boldsymbol{A})(x,t) - (\kappa_> \star \Delta\boldsymbol{A})(x,t). \tag{33}$$

and

$$||\boldsymbol{A}_\kappa(t)||_\infty \leq ||(\kappa_< \star \boldsymbol{A})(t)||_\infty + ||(\kappa_> \star \Delta\boldsymbol{A})(t)||_\infty \tag{34}$$

$$\leq ||\kappa_<||\,||\boldsymbol{A}|| + ||\kappa_>||\,||\Delta\boldsymbol{A}|| < \infty \tag{35}$$

where we made use of Young's inequality. By means of

$$\sum_{\lambda=1,2}\boldsymbol{\epsilon}^i_\lambda(k)\boldsymbol{\epsilon}^j_\lambda(k) = \delta_{ij} - \frac{k_ik_j}{|k|^2} \tag{36}$$

one easily shows

$$\sum_{\lambda=1,2}\int d^3k|k||\alpha(k,\lambda)|^2 =1/2\int d^3k\left(|k|^2\boldsymbol{A}^2(k,t) + \boldsymbol{E}^2(k,t)\right) \tag{37}$$

$$\leq ||\boldsymbol{A}(t)||_{H^1(\mathbb{R}^3)} + ||\boldsymbol{E}(t)||\,, \tag{38}$$

$$\sum_{\lambda=1,2}\int d^3k|k|^2|\alpha(k,\lambda)|^2 =1/2\int d^3k\left(|k|^3\boldsymbol{A}^2(k,t) + |k|\boldsymbol{E}^2(k,t)\right) \tag{39}$$

$$\leq ||\boldsymbol{A}(t)||_{H^1(\mathbb{R}^3)}\,||\boldsymbol{A}(t)||_{H^2(\mathbb{R}^3)} + ||\boldsymbol{E}(t)||\,||\boldsymbol{E}(t)||_{H^1(\mathbb{R}^3)}\,. \tag{40}$$

III The counting functional

In this section, we introduce a new indicator of condensation referred to as the „counting functional“. Our system under consideration (4) describes the interaction of charged particles with a radiation field. Initially, we assume the charges and photons to exhibit condensation and we would like to show that both condensates are stable over time. In case of the charges, this is done by means of a functional, denoted by β^a, which counts for each time t the relative number of charges which are not in the state φ_t. Under suitable conditions on the photon field it is then possible to show that the rate of particles which leave the condensate is small, if initially almost all particles are in the state φ_0. The situation is different for the radiation field because the number of photons is not a conserved quantity. On that account not only existing photons gets correlated but also new photons are created or destroyed. One should note that the high frequency modes of the radiation field do not interact with the charges due to the ultraviolet cutoff (7) and evolve according to the free evolution. This is why neither the number of photons changes nor the photon state shows correlations for wave-numbers $|k| \geq \Lambda$. However in the long wave-length sector of $\mathcal{F}_b$ correlations take place and the number of photons varies. To show that the photon field remains

coherent we introduce the functional β^b measuring for each time t the flu ctuations of the photon field around the classical mode function. The main difficulties in our derivation arise from the minimal coupling term in the Pauli-Fierz Hamiltonian which causes a quadratic interaction between the charged particles and the radiation field. On that account we have to control expectation values of certain unbounded operators, see Subsection IV.1. This is established by β^c which restricts our consideration to a subspace of Many-body states whose energy per particle only fluctuates little around the energy functional of the effective system.
In order to define the counting functional we introduce the projectors p_j and q_j.

Definition III.1. Let $\varphi_t \in L^2(\mathbb{R}^3)$ with $||\varphi_t|| = 1$ and $f \in L^2(\mathbb{R}^{3N})$. For any $1 \leq j \leq N$ the projectors $p_j^{\varphi_t} : L^2(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N})$ and $q_j^{\varphi_t} : L^2(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N})$ are given by

$$p_j^{\varphi_t} f := \varphi_t(x_j) \int d^3\tilde{x}_j \varphi_t^*(\tilde{x}_j) f(x_1, \ldots, \tilde{x}_j, \ldots, x_N) \tag{41}$$

and $q_j^{\varphi_t} := 1 - p_j^{\varphi_t}$. For ease of notation, we omit the superscript φ_t in the following.

Moreover, we define the energy functional of the Maxwell-Schrödinger system:

$$\mathcal{E}_M[\varphi_t, \alpha_t] := ||(-i\nabla - \boldsymbol{A}_\kappa(t))\, \varphi_t||^2 + 1/2 \langle \varphi_t, \left(v \star |\varphi_t|^2\right) \varphi_t \rangle + \sum_{\lambda=1,2} \int d^3k |k| |\alpha_t(k,\lambda)|^2. \tag{42}$$

Note that $\mathcal{E}_M[\varphi_t, \alpha_t]$ is finite under (A1) and $\sup_{t\in[0,T]}\{||\varphi_t||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{A}(t)||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{E}(t)||_{H^2(\mathbb{R}^3)}\} < \infty$ for any $T \in \mathbb{R}^+$. The counting functional is defined by

Definition III.2. Let $\Psi_{Nt} \in \mathcal{D}(H_m^N)$, $\varphi_t \in L^2(\mathbb{R}^3)$ with $||\varphi_t|| = 1$ and assume $\sup_{t\in[0,T]}\{||\varphi_t||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{A}(t)||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{E}(t)||_{H^2(\mathbb{R}^3)}\} < \infty$ for any $T \in \mathbb{R}^+$. We define the functional $\beta : \mathcal{H}^{(N)} \times L^2(\mathbb{R}^3) \times (L^2(\mathbb{R}^3) \otimes \mathbb{C}^2) \to \mathbb{R}_0^+$ by

$$\beta[\Psi_{Nt}, \varphi_t, \alpha_t] := \beta^a[\Psi_{Nt}, \varphi_t] + \beta^b[\Psi_{Nt}, \alpha_t] + \beta^c[\Psi_{Nt}, \varphi_t, \alpha_t] \tag{43}$$

with

$$\beta^a := \langle\!\langle \Psi_{Nt}, q_1 \otimes \mathbb{1}_{\mathcal{F}_p} \Psi_{Nt} \rangle\!\rangle, \tag{44}$$

$$\beta^b := \sum_{\lambda=1,2} \int d^3k |k| \langle\!\langle \Psi_{Nt}, \left(\frac{a^*(k,\lambda)}{\sqrt{N}} - \alpha_t^*(k,\lambda) \right) \left(\frac{a(k,\lambda)}{\sqrt{N}} - \alpha_t(k,\lambda) \right) \Psi_{Nt} \rangle\!\rangle, \tag{45}$$

$$\beta^c := \langle\!\langle \left(\frac{H_m^N}{N} - \mathcal{E}_M[\varphi_t, \alpha_t] \right) \Psi_{Nt}, \left(\frac{H_m^N}{N} - \mathcal{E}_M[\varphi_t, \alpha_t] \right) \Psi_{Nt} \rangle\!\rangle. \tag{46}$$

The functional β^a was already used in [2], [9], [11], [14], [15],[16], [17] and others to derive the Hartree and Gross-Pitaevskii equation, while β^b and β^c are introduced to control the interaction with the radiation field.

III.1 Relation to reduced density matrices

Next, we show that condensation indicated by the counting functional, $\beta \to 0$ as $N \to \infty$, implies condensation in terms of reduced density matrices.

Lemma III.3. *For the quantities defined in* (17), (19) *and Definition III.2, one has*

$$\beta^a[\Psi_{Nt}, \varphi_t] \le Tr_{L^2(\mathbb{R}^3)}|\gamma_N^{(1,0)}(t) - |\varphi_t\rangle\langle\varphi_t|| \le \sqrt{8\beta^a[\Psi_{Nt}, \varphi_t]}, \tag{47}$$

$$Tr_{\mathfrak{h}}|\gamma_N^{(0,1)}(t) - |u_t\rangle\langle u_t|| \le 3\beta^b[\Psi_{Nt}, \alpha_t] + 6\,||u_t||_{\mathfrak{h}}\sqrt{\beta^b[\Psi_{Nt}, \alpha_t]}. \tag{48}$$

Proof. The first inequality follows from

$$\beta^a = 1 - \langle\!\langle \Psi_N, p_1 \Psi_N\rangle\!\rangle = 1 - \langle \varphi, \gamma_N^{(1,0)}\varphi\rangle = \mathrm{Tr}_{L^2(\mathbb{R}^3)}(|\varphi\rangle\langle\varphi| - |\varphi\rangle\langle\varphi|\gamma_N^{(1,0)}) \tag{49}$$

$$\le ||p_1||_{\mathrm{op}}\, \mathrm{Tr}_{L^2(\mathbb{R}^3)}|\gamma_N^{(1,0)} - |\varphi\rangle\langle\varphi|| = \mathrm{Tr}_{L^2(\mathbb{R}^3)}|\gamma_N^{(1,0)} - |\varphi\rangle\langle\varphi||. \tag{50}$$

In order to proof the remaining inequalities we use

$$\mathrm{Tr}|\gamma - p| \le 2\,||\gamma - p||_{HS} + \mathrm{Tr}(\gamma - p), \tag{51}$$

valid for any one-dimensional projector p and non-negative density matrix γ. The original argument of the proof was first observed by Robert Seiringer, see [18]. We present a version that is found in [2]: Let $(\lambda_n)_{n\in\mathbb{N}}$ be the sequence of eigenvalues of the trace class operator $A := \gamma - p$. Since p is a rank one projection, A has at most one negative eigenvalue. If there is no negative eigenvalue, $\mathrm{Tr}|A| = \mathrm{Tr}(A)$ and (51) holds. If there is one negative eigenvalue λ_1, we have $\mathrm{Tr}|A| = |\lambda_1| + \sum_{n\ge 2}\lambda_n = 2|\lambda_1| + \mathrm{Tr}(A)$. Because of $|\lambda_1| \le ||A||_{\mathrm{op}} \le ||A||_{HS}$, inequality (51) follows.

For the upper bound of (47) we notice that $\mathrm{Tr}_{L^2(\mathbb{R}^3)}(\gamma_N^{(1,0)} - |\varphi\rangle\langle\varphi|) = 0$. Then, (51) reduces to

$$\mathrm{Tr}_{L^2(\mathbb{R}^3)}|\gamma_N^{(1,0)} - |\varphi\rangle\langle\varphi|| \le 2\left|\left|\gamma_N^{(1,0)} - |\varphi\rangle\langle\varphi|\right|\right|_{HS} \tag{52}$$

and (47) follows from

$$\mathrm{Tr}_{L^2(\mathbb{R}^3)}(\gamma_N^{(1,0)} - |\varphi\rangle\langle\varphi|)^2 = 1 - 2\mathrm{Tr}_{L^2(\mathbb{R}^3)}(|\varphi\rangle\langle\varphi|\gamma_N^{(1,0)}) + \mathrm{Tr}_{L^2(\mathbb{R}^3)}((\gamma_N^{(1,0)})^2) \tag{53}$$

$$\le 2(1 - \mathrm{Tr}_{L^2(\mathbb{R}^3)}(|\varphi\rangle\langle\varphi|\gamma_N^{(1,0)})) = 2\beta^a. \tag{54}$$

To proof inequality (48) it is useful to write the kernel of $\gamma_N^{(0,1)} - |u\rangle\langle u|$ as

$$(\gamma_N^{(0,1)} - |u\rangle\langle u|)(k,\lambda,l,\mu) \tag{55}$$

$$= |k|^{1/2}|l|^{1/2}\left(\langle\!\langle \Psi_N, \frac{a^*(l,\mu)a(k,\lambda)}{N}\Psi_N\rangle\!\rangle - \alpha^*(l,\mu)\alpha(k,\lambda)\right) \tag{56}$$

$$= |k|^{1/2}|l|^{1/2}\langle\!\langle \left(\frac{a(l,\mu)}{\sqrt{N}} - \alpha(l,\mu)\right)\Psi_N, \left(\frac{a(k,\lambda)}{\sqrt{N}} - \alpha(k,\lambda)\right)\Psi_N\rangle\!\rangle \tag{57}$$

$$+ |k|^{1/2}|l|^{1/2}\alpha(k,\lambda)\langle\!\langle \left(\frac{a(l,\mu)}{\sqrt{N}} - \alpha(l,\mu)\right)\Psi_N, \Psi_N\rangle\!\rangle \tag{58}$$

$$+ |k|^{1/2}|l|^{1/2}\alpha^*(l,\mu)\langle\!\langle \Psi_N, \left(\frac{a(k,\lambda)}{\sqrt{N}} - \alpha(k,\lambda)\right)\Psi_N\rangle\!\rangle. \tag{59}$$

Cauchy-Schwarz inequality gives

$$|\gamma_N^{(0,1)} - |u\rangle\langle u||^2(k,\lambda,l,\mu) \tag{60}$$

$$\leq |k||l| \langle\!\langle \left(\frac{a(k,\lambda)}{\sqrt{N}} - \alpha(k,\lambda)\right)\Psi_N, \left(\frac{a(k,\lambda)}{\sqrt{N}} - \alpha(k,\lambda)\right)\Psi_N\rangle\!\rangle \tag{61}$$

$$\times \langle\!\langle \left(\frac{a(l,\mu)}{\sqrt{N}} - \alpha(l,\mu)\right)\Psi_N, \left(\frac{a(l,\mu)}{\sqrt{N}} - \alpha(l,\mu)\right)\Psi_N\rangle\!\rangle \tag{62}$$

$$+|k||l||\alpha(l,\mu)|^2 \langle\!\langle \left(\frac{a(k,\lambda)}{\sqrt{N}} - \alpha(k,\lambda)\right)\Psi_N, \left(\frac{a(k,\lambda)}{\sqrt{N}} - \alpha(k,\lambda)\right)\Psi_N\rangle\!\rangle \tag{63}$$

$$+|k||l||\alpha(k,\lambda)|^2 \langle\!\langle \left(\frac{a(l,\mu)}{\sqrt{N}} - \alpha(l,\mu)\right)\Psi_N, \left(\frac{a(l,\mu)}{\sqrt{N}} - \alpha(l,\mu)\right)\Psi_N\rangle\!\rangle \tag{64}$$

and

$$\left|\left|\gamma_N^{(0,1)} - |u\rangle\langle u|\right|\right|_{HS}^2 = \sum_{\lambda,\mu\in\{1,2\}^2} \int\int d^3k d^3l |\gamma_N^{(0,1)} - |u\rangle\langle u||^2(k,\lambda,l,\mu) \tag{65}$$

$$\leq (\beta^b)^2 + 2\,||u||_{\mathfrak{h}}^2\,\beta^b \tag{66}$$

follows. Similarly,

$$\mathrm{Tr}_{\mathfrak{h}}(\gamma_N^{(0,1)} - |u\rangle\langle u|) \leq \sum_{\lambda=1,2} \int d^3k |\gamma_N^{(0,1)} - |u\rangle\langle u||(k,\lambda,k,\lambda) \tag{67}$$

$$\leq \sum_{\lambda=1,2} \int d^3k |k| \langle\!\langle \left(\frac{a(k,\lambda)}{\sqrt{N}} - \alpha(k,\lambda)\right)\Psi_N, \left(\frac{a(k,\lambda)}{\sqrt{N}} - \alpha(k,\lambda)\right)\Psi_N\rangle\!\rangle \tag{68}$$

$$+2\sum_{\lambda=1,2} \int d^3k |u(k,\lambda)||k|^{1/2} \left|\left|\left(\frac{a(k,\lambda)}{\sqrt{N}} - \alpha(k,\lambda)\right)\Psi_N\right|\right|_{\mathcal{H}^{(N)}}. \tag{69}$$

Applying Cauchy-Schwarz inequality with respect to the scalar product of $\mathfrak{h}$ yields

$$\mathrm{Tr}_{\mathfrak{h}}(\gamma_N^{(0,1)} - |\alpha\rangle\langle\alpha|) \leq \beta^b \tag{70}$$

$$+2\,||u||_{\mathfrak{h}} \left(\sum_{\lambda=1,2} \int d^3k |k| \langle\!\langle \left(\frac{a(k,\lambda)}{\sqrt{N}} - \alpha(k,\lambda)\right)\Psi_N, \left(\frac{a(k,\lambda)}{\sqrt{N}} - \alpha(k,\lambda)\right)\Psi_N\rangle\!\rangle\right)^{1/2} \tag{71}$$

$$\leq \beta^b + 2\,||u||_{\mathfrak{h}}\sqrt{\beta^b}. \tag{72}$$

Monotonicity of the square root and (51) give rise to (48).

IV Condensation in terms of the counting functional

The functional β allows to formulate Theorem I.1 for a larger class of initial states.

Theorem IV.1. *Let v satisfy (A1), $\varphi_0 \in L^2(\mathbb{R}^3)$ with $||\varphi_0|| = 1$, $\alpha_0 \in \mathfrak{h}$ such that $(\boldsymbol{A}(0), \boldsymbol{E}(0)) \in (H^3(\mathbb{R}^3) \oplus H^2(\mathbb{R}^3))$, $\Psi_{N0} \in \left(L_s^2\left(\mathbb{R}^{3N}\right) \otimes \mathcal{F}_p\right) \cap \mathcal{D}(H_m^N)$ such that $\beta[\Psi_{N0}, \varphi_0, \alpha_0] \leq C_1 N^{-\delta}$. Let Ψ_{Nt} the unique solution of (4). Let $(\varphi_t, \boldsymbol{A}(t), \boldsymbol{E}(t))$ the* *unique solution of* (15) *and assume* $\sup_{t\in[0,T]}\{||\varphi_t||_{H^3(\mathbb{R}^3)}+||\boldsymbol{A}(t)||_{H^3(\mathbb{R}^3)}+||\boldsymbol{E}(t)||_{H^2(\mathbb{R}^3)}\} < \infty$ *for any* $T \in \mathbb{R}^+$. *For any* $t > 0$ *there exist two constants* C_1, C_2 *(*C_2 *is time-dependent) such that*

$$\beta(t) \leq N^{-\min\{1,\delta\}} C_1 e^{\Lambda^4 C_2(t)} \left(1 + \Lambda^4 C_2(t)\right). \tag{73}$$

Here, the constant C_1 might depend on Λ. Using Lemma III.3 this gives

Corollary IV.2. *Let* v *satisfy (A1),* $\varphi_0 \in L^2(\mathbb{R}^3)$ *with* $||\varphi_0|| = 1$, $\alpha_0 \in \mathfrak{h}$ *such that* $(\boldsymbol{A}(0), \boldsymbol{E}(0)) \in (H^3(\mathbb{R}^3) \oplus H^2(\mathbb{R}^3))$, $\Psi_{N0} \in \left(L_s^2\left(\mathbb{R}^{3N}\right) \otimes \mathcal{F}_p\right) \cap \mathcal{D}(H_m^N)$ *such that* $\beta[\Psi_{N0}, \varphi_0, \alpha_0] \leq C_1 N^{-\delta}$. *Let* Ψ_{Nt} *the unique solution of* (4). *Let* $(\varphi_t, \boldsymbol{A}(t), \boldsymbol{E}(t))$ *the unique solution of* (15), u_t *defined by* (22) *and assume* $\sup_{t\in[0,T]}\{||\varphi_t||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{A}(t)||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{E}(t)||_{H^2(\mathbb{R}^3)}\} < \infty$ *for any* $T \in \mathbb{R}^+$. *For any* $t > 0$ *there exist two constants* C_3, C_4 *(*C_4 *is time-dependent) such that*

$$Tr_{L^2(\mathbb{R}^3)}|\gamma_N^{(1,0)}(t) - |\varphi_t\rangle\langle\varphi_t|| \leq N^{-\min\{1/2,\delta/2\}} C_3 e^{\Lambda^4 C_4(t)} \left(1 + \Lambda^2 C_4^{1/2}(t)\right), \tag{74}$$

$$Tr_{\mathfrak{h}}|\gamma_N^{(0,1)}(t) - |u_t\rangle\langle u_t|| \leq N^{-\min\{1/2,\delta/2\}} C_3 e^{\Lambda^4 C_4(t)} \left(1 + \Lambda^2 C_4^{1/2}(t)\right). \tag{75}$$

The rest of this section is devoted to prove Theorem IV.1. For the proof, it is convenient to introduce the positive and negative frequency parts of the quantum mechanical and classical vector potential:

$$\hat{\boldsymbol{A}}_\kappa^+(x) := \sum_{\lambda=1,2} \int d^3k\ \tilde{\kappa}(k) \frac{1}{\sqrt{2|k|}} \boldsymbol{\epsilon}_\lambda(k) e^{ikx} a(k,\lambda), \tag{76}$$

$$\hat{\boldsymbol{A}}_\kappa^-(x) := \sum_{\lambda=1,2} \int d^3k\ \tilde{\kappa}(k) \frac{1}{\sqrt{2|k|}} \boldsymbol{\epsilon}_\lambda(k) e^{-ikx} a^*(k,\lambda), \tag{77}$$

$$\boldsymbol{A}_\kappa^+(x,t) := \sum_{\lambda=1,2} \int d^3k\ \tilde{\kappa}(k) \frac{1}{\sqrt{2|k|}} \boldsymbol{\epsilon}_\lambda(k) e^{ikx} \alpha_t(k,\lambda), \tag{78}$$

$$\boldsymbol{A}_\kappa^-(x,t) := \sum_{\lambda=1,2} \int d^3k\ \tilde{\kappa}(k) \frac{1}{\sqrt{2|k|}} \boldsymbol{\epsilon}_\lambda(k) e^{-ikx} \alpha_t^*(k,\lambda). \tag{79}$$

Moreover, it is helpful to define the positive and negative frequency parts of the quantum mechanical and classical electric field.

$$\hat{\boldsymbol{E}}_\kappa^+(x) := \sum_{\lambda=1,2} \int d^3k\ \tilde{\kappa}(k) \sqrt{\frac{|k|}{2}} \boldsymbol{\epsilon}_\lambda(k) i e^{ikx} a(k,\lambda), \tag{80}$$

$$\hat{\boldsymbol{E}}_\kappa^-(x) := \sum_{\lambda=1,2} \int d^3k\ \tilde{\kappa}(k) \sqrt{\frac{|k|}{2}} \boldsymbol{\epsilon}_\lambda(k) (-i) e^{-ikx} a^*(k,\lambda), \tag{81}$$

$$\boldsymbol{E}_\kappa^+(x,t) := \sum_{\lambda=1,2} \int d^3k\ \tilde{\kappa}(k) \sqrt{\frac{|k|}{2}} \boldsymbol{\epsilon}_\lambda(k) i e^{ikx} \alpha_t(k,\lambda), \tag{82}$$

$$\boldsymbol{E}_\kappa^-(x,t) := \sum_{\lambda=1,2} \int d^3k\ \tilde{\kappa}(k) \sqrt{\frac{|k|}{2}} \boldsymbol{\epsilon}_\lambda(k) (-i) e^{-ikx} \alpha_t^*(k,\lambda). \tag{83}$$

For $\sharp \in \{\ ,+,-\}$, we introduce the shorthand notations

$$\boldsymbol{\mathcal{E}}^{\sharp}(x,t) := \frac{\hat{\boldsymbol{E}}_\kappa^{\sharp}(x)}{\sqrt{N}} - \boldsymbol{E}_\kappa^{\sharp}(x,t), \quad \boldsymbol{\mathcal{A}}^{\sharp}(x,t) := \frac{\hat{\boldsymbol{A}}_\kappa^{\sharp}(x)}{\sqrt{N}} - \boldsymbol{A}_\kappa^{\sharp}(x,t). \tag{84}$$

By means of the cutoff function

$$\tilde{\eta}(k) = |k|^{-1}\tilde{\kappa}(k) = (2\pi)^{-\frac{3}{2}}|k|^{-1}\mathbb{1}_{|k|\leq\Lambda}(k) \tag{85}$$

we are able to express the vector potential in terms of the electric field.

Lemma IV.3. *Let η be the Fourier transform of* (85)*, then*

$$\hat{\boldsymbol{A}}_\kappa^{+}(x) = -\,i\left(\eta \star \hat{\boldsymbol{E}}_\kappa^{+}\right)(x), \qquad \boldsymbol{A}_\kappa^{+}(x,t) = -i\left(\eta \star \boldsymbol{E}_\kappa^{+}\right)(x,t), \tag{86}$$

$$\hat{\boldsymbol{A}}_\kappa^{-}(x) = i\left(\eta \star \hat{\boldsymbol{E}}_\kappa^{-}\right)(x), \qquad \hat{\boldsymbol{A}}_\kappa^{-}(x,t) = i\left(\eta \star \boldsymbol{E}_\kappa^{-}\right)(x,t). \tag{87}$$

Proof. The proof is a simple application of the convolution theorem. □

At various points in our estimates, we replace the vector potential by the electric field and make use of (see Lemma VI.1)

$$\int d^3y \langle\!\langle \Psi_N, \left(\frac{\hat{\boldsymbol{E}}_\kappa^{-}(y)}{\sqrt{N}} - \hat{\boldsymbol{E}}_\kappa^{-}(y,t)\right)\left(\frac{\hat{\boldsymbol{E}}_\kappa^{+}(y)}{\sqrt{N}} - \hat{\boldsymbol{E}}_\kappa^{+}(y,t)\right)\Psi_N\rangle\!\rangle \leq \beta^b. \tag{88}$$

To obtain proper bounds it is crucial that the L^2-norm of the cutoff functions

$$||\kappa||_2^2 = \frac{\Lambda^3}{6\pi^2} \quad \text{and} \quad ||\eta||_2^2 = \frac{\Lambda}{2\pi^2} \tag{89}$$

is finite.

IV.1 Preliminary estimates

We have to ensure that quantities like $\langle\!\langle \Psi_N, q_1 \frac{\hat{\boldsymbol{A}}_\kappa^2(x_1)}{N} q_1 \Psi_N\rangle\!\rangle$ are not only finite but bounded by β. This holds for every bounded operator B because

$$\langle\!\langle \Psi_N, q_1 B q_1 \Psi_N\rangle\!\rangle \leq C\,||q_1\Psi_N||^2 \leq C\beta^a, \tag{90}$$

but is not true in general. In case of unbounded operators smallness can sometimes be shown on a subclass of states which have sufficient decay in the occupation of eigenstates. For a self-adjoint operator O with $[O, q_1] \approx 0$ and $c \in \mathbb{R}$ one has

$$\langle\!\langle \Psi_N, q_1 O q_1 \Psi_N\rangle\!\rangle \approx \langle\!\langle \Psi_N, q_1 O \Psi_N\rangle\!\rangle = \langle\!\langle \Psi_N, q_1 (O-c)\Psi_N\rangle\!\rangle + c\langle\!\langle \Psi_N, q_1\Psi_N\rangle\!\rangle \tag{91}$$

$$\leq (c+1)\langle\!\langle \Psi_N, q_1\Psi_N\rangle\!\rangle + \langle\!\langle \Psi_N, (O-c)^2\Psi_N\rangle\!\rangle. \tag{92}$$

Thus, $\langle\!\langle \Psi_N, q_1 O q_1 \Psi_N\rangle\!\rangle$ is small if Ψ_N occupies eigenstates of O with eigenvalues $\lambda \neq c$ only with small probability. This is in the spirit of Chebyshev's inequality from probability theory. Requiring $\langle\!\langle \Psi_{N0}, (O-c)^2\Psi_{N0}\rangle\!\rangle \approx 0$ initially does not imply smallness at later times. But it turns out that this quantity is conserved during the time evolution, if O is a constant of motion, see β^c. This is why we relate the vector potential and the Laplace-operator to the energy per particle H_m^N/N of the Many-body system.

Lemma IV.4. *Let $y \in \mathbb{R}^3$ or $y \in \{x_1, \dots, x_N\}$ and $\chi_N \in \mathcal{D}(H_m^N)$. Then*

$$\left|\left|\frac{\hat{\boldsymbol{A}}_\kappa^+(y)}{\sqrt{N}}\chi_N\right|\right|^2 \leq \frac{\Lambda}{2\pi^2}\langle\!\langle \chi_N, \frac{H_f}{N}\chi_N\rangle\!\rangle, \tag{93}$$

$$\left|\left|\frac{\hat{\boldsymbol{A}}_\kappa^-(y)}{\sqrt{N}}\chi_N\right|\right|^2 \leq \frac{\Lambda}{2\pi^2}\langle\!\langle \chi_N, \frac{H_f}{N}\chi_N\rangle\!\rangle + \frac{\Lambda^2}{4\pi^2 N}\left|\left|\chi_N\right|\right|^2, \tag{94}$$

$$\left|\left|\frac{\hat{\boldsymbol{A}}_\kappa(y)}{\sqrt{N}}\chi_N\right|\right|^2 \leq \frac{2\Lambda}{\pi^2}\langle\!\langle \chi_N, \frac{H_f}{N}\chi_N\rangle\!\rangle + \frac{\Lambda^2}{2\pi^2 N}\left|\left|\chi_N\right|\right|^2. \tag{95}$$

Proof. To ease notation, we define the vector-valued function $\boldsymbol{f}(k,\lambda) := \frac{\tilde{\kappa}(k)}{\sqrt{2|k|}}\boldsymbol{\epsilon}_\lambda(k)$. The first estimate follows from Cauchy-Schwarz inequality

$$\left|\left|\sum_{\lambda=1,2}\int d^3k \; \boldsymbol{f}(k,\lambda)e^{\pm iky}a(k,\lambda)\chi_N\right|\right|_{\mathcal{H}}^2 \tag{96}$$

$$\leq \left(\sum_{\lambda=1,2}\int d^3k|\boldsymbol{f}(k,\lambda)||k|^{-1/2}\left|\left||k|^{1/2}a(k,\lambda)\chi_N\right|\right|_{\mathcal{H}}^2\right)^2 \tag{97}$$

$$\leq \left(\sum_{\lambda=1,2}\int d^3k|\boldsymbol{f}(k,\lambda)|^2|k|^{-1}\right)\left(\sum_{\lambda=1,2}\int d^3k|k|\left|\left|a(k,\lambda)\chi_N\right|\right|_{\mathcal{H}}^2\right) \tag{98}$$

$$=\frac{\Lambda}{2\pi^2}\langle\!\langle \chi_N, H_f\chi_N\rangle\!\rangle. \tag{99}$$

By use of the canonical commutation relations (9), the second bound is obtained via

$$\left|\left|\sum_{\lambda=1,2}\int d^3k \; \boldsymbol{f}(k,\lambda)e^{\pm iky}a^*(k,\lambda)\chi_N\right|\right|_{\mathcal{H}}^2 \tag{100}$$

$$=\left|\left|\boldsymbol{f}\right|\right|_{\mathfrak{h}}^2\left|\left|\chi_N\right|\right|_{\mathcal{H}}^2 + \left|\left|\sum_{\lambda=1,2}\int d^3k \; \overline{\boldsymbol{f}(k,\lambda)}e^{\mp iky}a(k,\lambda)\chi_N\right|\right|_{\mathcal{H}}^2 \tag{101}$$

$$\leq\frac{\Lambda^2}{4\pi^2}\left|\left|\chi_N\right|\right|_{\mathcal{H}}^2 + \frac{\Lambda}{2\pi^2}\langle\!\langle \chi_N, H_f\chi_N\rangle\!\rangle. \tag{102}$$

The last estimate follows by triangular inequality. Lemma IV.4 leads to

Corollary IV.5. *For $\Psi_N \in \mathcal{D}(H_m^N)$ we have*

$$\left|\left|\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}q_1\Psi_N\right|\right|^2 \leq\frac{2\Lambda}{\pi^2}\langle\!\langle \Psi_N, q_1\frac{H_f}{N}q_1\Psi_N\rangle\!\rangle + \frac{\Lambda^2}{2\pi^2 N}\beta^a, \tag{103}$$

$$\left|\left|\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}p_1q_2\Psi_N\right|\right|^2 \leq\frac{2\Lambda}{\pi^2}\langle\!\langle \Psi_N, q_1\frac{H_f}{N}q_1\Psi_N\rangle\!\rangle + \frac{\Lambda^2}{2\pi^2 N}\beta^a. \tag{104}$$

Lemma IV.6. *Let v satisfy (A1), $\varphi_t \in L^2(\mathbb{R}^3)$ with $||\varphi_t|| = 1$, $\Psi_{Nt} \in \left(L_s^2\left(\mathbb{R}^{3N}\right) \otimes \mathcal{F}_p\right) \cap \mathcal{D}(H_m^N)$ and assume $\sup_{t\in[0,T]}\{||\varphi_t||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{A}(t)||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{E}(t)||_{H^2(\mathbb{R}^3)}\} < \infty$ for any $T \in \mathbb{R}^+$. Then*

$$\langle\!\langle \Psi_N, q_1 \frac{H_m^N}{N} q_1 \Psi_N \rangle\!\rangle \leq C\left(\beta + \frac{\Lambda}{N}\right). \tag{105}$$

Proof of Lemma IV.6. First, we split the Pauli Fierz Hamiltonian in its different parts

$$\langle\!\langle \Psi_N, q_1 \frac{H_m^N}{N} q_1 \Psi_N \rangle\!\rangle = \langle\!\langle \Psi_N, q_1 \frac{1}{N} \sum_{j=1}^{N} \left(-i\nabla_j - \frac{\hat{\boldsymbol{A}}_\kappa(x_j)}{\sqrt{N}}\right)^2 q_1 \Psi_N \rangle\!\rangle \tag{106}$$

$$+ \langle\!\langle \Psi_N, q_1 \frac{1}{N^2} \sum_{1\leq j<k\leq N} v(x_j - x_k) q_1 \Psi_N \rangle\!\rangle \tag{107}$$

$$+ \langle\!\langle \Psi_N, q_1 \frac{H_f}{N} q_1 \Psi_N \rangle\!\rangle. \tag{108}$$

The first line can be written as

$$(106) = \langle\!\langle \Psi_N, q_1 \frac{1}{N} \sum_{j=1}^{N} \left(-i\nabla_j - \frac{\hat{\boldsymbol{A}}_\kappa(x_j)}{\sqrt{N}}\right)^2 \Psi_N \rangle\!\rangle \tag{109}$$

$$+ \frac{1}{N} \sum_{j=1}^{N} \langle\!\langle \Psi_N, q_1 \left[\left(-i\nabla_j - \frac{\hat{\boldsymbol{A}}_\kappa(x_j)}{\sqrt{N}}\right)^2, q_1\right] \Psi_N \rangle\!\rangle \tag{110}$$

$$= \langle\!\langle \Psi_N, q_1 \frac{1}{N} \sum_{j=1}^{N} \left(-i\nabla_j - \frac{\hat{\boldsymbol{A}}_\kappa(x_j)}{\sqrt{N}}\right)^2 \Psi_N \rangle\!\rangle \tag{111}$$

$$+ \frac{1}{N} \langle\!\langle \Psi_N, q_1 \left[\left(-i\nabla_1 - \frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)^2, q_1\right] \Psi_N \rangle\!\rangle. \tag{112}$$

Moreover,

$$(107) = \langle\!\langle \Psi_N, q_1 \frac{1}{N^2} \sum_{1\leq j<k\leq N} v(x_j - x_k) \Psi_N \rangle\!\rangle \tag{113}$$

$$+ \frac{1}{N^2} \sum_{1\leq j<k\leq N} \langle\!\langle \Psi_N, q_1 \left[v(x_j - x_k), q_1\right] \Psi_N \rangle\!\rangle \tag{114}$$

$$= \langle\!\langle \Psi_N, q_1 \frac{1}{N^2} \sum_{1\leq j<k\leq N} v(x_j - x_k) \Psi_N \rangle\!\rangle \tag{115}$$

$$+ \frac{N-1}{N^2} \langle\!\langle \Psi_N, q_1 \left[v(x_1 - x_2), q_1\right] \Psi_N \rangle\!\rangle. \tag{116}$$

Since H_f commutes with operators acting only on the charged particles, we obtain

$$\begin{aligned}
\langle\langle\Psi_N, q_1\frac{H_m^N}{N}q_1\Psi_N\rangle\rangle =&\langle\langle\Psi_N, q_1\frac{H_m^N}{N}\Psi_N\rangle\rangle && (117)\\
&+\frac{1}{N}\langle\langle\Psi_N, q_1\left[\left(-i\nabla_1-\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)^2, q_1\right]\Psi_N\rangle\rangle && (118)\\
&+\frac{N-1}{N^2}\langle\langle\Psi_N, q_1\left[v(x_1-x_2), q_1\right]\Psi_N\rangle\rangle. && (119)
\end{aligned}$$

We estimate

$$\begin{aligned}
|(118)| \leq&\frac{1}{N}|\langle\langle\Psi_N, q_1\left[\left(-i\nabla_1-\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)^2, p_1\right]\Psi_N\rangle\rangle| && (120)\\
=&\frac{1}{N}|\langle\langle\Psi_N, q_1\left(-i\nabla_1-\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)^2 p_1\Psi_N\rangle\rangle| \leq C\frac{\Lambda}{N} && (121)
\end{aligned}$$

and

$$\begin{aligned}
|(119)| \leq&\frac{1}{N}|\langle\langle\Psi_N, q_1\left[v(x_1-x_2), p_1\right]\Psi_N\rangle\rangle| && (122)\\
=&\frac{1}{N}|\langle\langle\Psi_N, q_1 v(x_1-x_2)p_1\Psi_N\rangle\rangle| && (123)\\
\leq&\frac{1}{2}\left|\left|q_1\Psi_N\right|\right|^2+\frac{1}{2N^2}\left|\left|v(x_1-x_1)p_1\Psi_N\right|\right|^2 && (124)\\
=&\frac{1}{2}\beta^a+\frac{1}{2N^2}\langle\langle\Psi_N, p_1 v^2(x_2-x_1)p_1\Psi_N\rangle\rangle && (125)\\
=&\frac{1}{2}\beta^a+\frac{1}{2N^2}\langle\langle\Psi_N, p_1\left(v^2\star|\varphi|^2\right)(x_2)\Psi_N\rangle\rangle && (126)\\
\leq&\frac{1}{2}\beta^a+\frac{1}{2N^2}\left|\left|v^2\star|\varphi|^2\right|\right|_\infty. && (127)
\end{aligned}$$

Thus,

$$|(118)+(119)| \leq C\left(\beta+\frac{\Lambda}{N}\right). \qquad (128)$$

and

$$\begin{aligned}
\langle\langle\Psi_N, q_1\frac{H_m^N}{N}q_1\Psi_N\rangle\rangle \leq&|\langle\langle\Psi_N, q_1\frac{H_m^N}{N}\Psi_N\rangle\rangle|+|(118)+(119)| && (129)\\
\leq&|\langle\langle\Psi_N, q_1\left(\frac{H_m^N}{N}-\mathcal{E}_M\right)\Psi_N\rangle\rangle+\mathcal{E}_M\beta^a|+C\left(\beta+\frac{\Lambda}{N}\right) && (130)\\
\leq&|\langle\langle\Psi_N, q_1\left(\frac{H_m^N}{N}-\mathcal{E}_M\right)\Psi_N\rangle\rangle|+C\left(\beta+\frac{\Lambda}{N}\right) && (131)\\
\leq&\frac{1}{2}\langle\langle\Psi_N, \left(\frac{H_m^N}{N}-\mathcal{E}_M\right)^2\Psi_N\rangle\rangle+\frac{1}{2}\beta^a+C\left(\beta+\frac{\Lambda}{N}\right) && (132)\\
\leq&C\left(\beta+\frac{\Lambda}{N}\right). && (133)
\end{aligned}$$

Lemma IV.7. *Let v satisfy (A1), $\varphi_t \in L^2(\mathbb{R}^3)$ with $||\varphi_t|| = 1$, $\Psi_{Nt} \in \left(L^2_s\left(\mathbb{R}^{3N}\right) \otimes \mathcal{F}_p\right) \cap \mathcal{D}(H^N_m)$ and assume $\sup_{t\in[0,T]}\{||\varphi_t||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{A}(t)||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{E}(t)||_{H^2(\mathbb{R}^3)}\} < \infty$ for any $T \in \mathbb{R}^+$. Then, one has*

$$\left|\left|\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} q_1 \Psi_N\right|\right|^2 \leq C\Lambda\left(\beta + \frac{\Lambda}{N}\right), \tag{134}$$

$$\left|\left|\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} q_2 \Psi_N\right|\right|^2 \leq C\Lambda\left(\beta + \frac{\Lambda}{N}\right), \tag{135}$$

$$\left|\left|\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} p_1 q_2 \Psi_N\right|\right|^2 \leq C\Lambda\left(\beta + \frac{\Lambda}{N}\right). \tag{136}$$

Proof. Because v is positive, we have

$$\langle\!\langle \Psi_N, q_1 \frac{H_f}{N} q_1 \Psi_N \rangle\!\rangle \leq \langle\!\langle \Psi_N, q_1 \frac{H^N_m}{N} q_1 \Psi_N \rangle\!\rangle. \tag{137}$$

Lemma IV.6 and Corollary IV.5 then give

$$\langle\!\langle \Psi_N, q_1 \frac{H_f}{N} q_1 \Psi_N \rangle\!\rangle \leq C\left(\beta + \frac{\Lambda}{N}\right) \tag{138}$$

and

$$\left|\left|\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} q_1 \Psi_N\right|\right|^2 \leq C\Lambda\left(\beta + \frac{\Lambda}{N}\right) + C\frac{\Lambda^2}{N}\beta^a \leq C\Lambda\left(\beta + \frac{\Lambda}{N}\right). \tag{139}$$

The other inequalities are shown analogously.

Lemma IV.8. *Let v satisfy (A1), $\varphi_t \in L^2(\mathbb{R}^3)$ with $||\varphi_t|| = 1$, $\Psi_{Nt} \in \left(L^2_s\left(\mathbb{R}^{3N}\right) \otimes \mathcal{F}_p\right) \cap \mathcal{D}(H^N_m)$, $\sup_{t\in[0,T]}\{||\varphi_t||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{A}(t)||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{E}(t)||_{H^2(\mathbb{R}^3)}\} < \infty$ for any $T \in \mathbb{R}^+$ and $\hat{n} := \left(N^{-1}\sum_{j=1}^N q_j\right)^{1/2}$. Then*

$$\left|\left|\hat{n}\kappa(x_1 - y)\left(-i\nabla_1 - \frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)\Psi_N\right|\right|^2_{;y} \leq ||\kappa||^2_2\, C\left(\beta + \frac{\Lambda}{N}\right) \leq C\Lambda^3\left(\beta + \frac{\Lambda}{N}\right).$$

Proof.

$$\left|\left|\hat{n}\kappa(x_1-y)\left(-i\nabla_1-\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)\Psi_N\right|\right|^2_{;y} \tag{140}$$

$$=\langle\!\langle\kappa(x_1-y)\left(-i\nabla_1-\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)\Psi_N,\hat{n}^2\kappa(x_1-y)\left(-i\nabla_1-\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)\Psi_N\rangle\!\rangle_{;y} \tag{141}$$

$$=\frac{1}{N}\langle\!\langle\kappa(x_1-y)\left(-i\nabla_1-\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)\Psi_N,q_1\kappa(x_1-y)\left(-i\nabla_1-\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)\Psi_N\rangle\!\rangle_{;y} \tag{142}$$

$$+\frac{N-1}{N}\langle\!\langle\kappa(x_1-y)\left(-i\nabla_1-\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)\Psi_N,q_2\kappa(x_1-y)\left(-i\nabla_1-\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)\Psi_N\rangle\!\rangle_{;y} \tag{143}$$

$$\leq\frac{1}{N}\langle\!\langle\left(-i\nabla_1-\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)\Psi_N,\int d^3y|\kappa(x_1-y)|^2\left(-i\nabla_1-\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)\Psi_N\rangle\!\rangle \tag{144}$$

$$+\frac{N-1}{N}\langle\!\langle\left(-i\nabla_1-\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)q_2\Psi_N,\int d^3y|\kappa(x_1-y)|^2\left(-i\nabla_1-\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)q_2\Psi_N\rangle\!\rangle \tag{145}$$

$$\leq\frac{1}{N}\,||\kappa||_2^2\,\langle\!\langle\Psi_N,\left(-i\nabla_1-\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)^2\Psi_N\rangle\!\rangle \tag{146}$$

$$+\frac{N-1}{N}\,||\kappa||_2^2\,\langle\!\langle\Psi_N,q_2\left(-i\nabla_1-\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)^2q_2\Psi_N\rangle\!\rangle. \tag{147}$$

Inserting the identity $1=p_1+q_1$ and making use of symmetry we obtain

$$\left|\left|\hat{n}\kappa(x_1-y)\left(-i\nabla_1-\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)\Psi_N\right|\right|^2_{;y} \tag{148}$$

$$\leq\frac{1}{N}\,||\kappa||_2^2\,\langle\!\langle\Psi_N,q_1\left(-i\nabla_1-\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)^2q_1\Psi_N\rangle\!\rangle \tag{149}$$

$$+\frac{2}{N}\,||\kappa||_2^2\,|\langle\!\langle\Psi_N,q_1\left(-i\nabla_1-\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)^2p_1\Psi_N\rangle\!\rangle| \tag{150}$$

$$+\frac{1}{N}\,||\kappa||_2^2\,|\langle\!\langle\Psi_N,p_1\left(-i\nabla_1-\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)^2p_1\Psi_N\rangle\!\rangle| \tag{151}$$

$$+\frac{1}{N}\,||\kappa||_2^2\sum_{j=2}^N\langle\!\langle\Psi_N,q_1\left(-i\nabla_j-\frac{\hat{\boldsymbol{A}}_\kappa(x_j)}{\sqrt{N}}\right)^2q_1\Psi_N\rangle\!\rangle \tag{152}$$

$$=\frac{1}{N}\,||\kappa||_2^2\sum_{j=1}^N\langle\!\langle\Psi_N,q_1\left(-i\nabla_j-\frac{\hat{\boldsymbol{A}}_\kappa(x_j)}{\sqrt{N}}\right)^2q_1\Psi_N\rangle\!\rangle \tag{153}$$

$$+\frac{2}{N}\,||\kappa||_2^2\,|\langle\!\langle\Psi_N, q_1\left(-i\nabla_1-\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)^2 p_1\Psi_N\rangle\!\rangle| \tag{154}$$

$$+\frac{1}{N}\,||\kappa||_2^2\,|\langle\!\langle\Psi_N, p_1\left(-i\nabla_1-\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)^2 p_1\Psi_N\rangle\!\rangle| \tag{155}$$

$$\leq\frac{1}{N}\,||\kappa||_2^2\sum_{j=1}^{N}\langle\!\langle\Psi_N, q_1\left(-i\nabla_j-\frac{\hat{\boldsymbol{A}}_\kappa(x_j)}{\sqrt{N}}\right)^2 q_1\Psi_N\rangle\!\rangle+C\,||\kappa||_2^2\,\frac{\Lambda}{N}. \tag{156}$$

Because H_f and v are positive operators, this is bounded by

$$||\kappa||_2^2\,\langle\!\langle\Psi_N, q_1\frac{H_m^N}{N}q_1\Psi_N\rangle\!\rangle+C\,||\kappa||_2^2\,\frac{\Lambda}{N} \tag{157}$$

and Lemma IV.6 gives

$$\left|\left|\hat{n}\kappa(x_1-y)\left(-i\nabla_1-\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)\Psi_N\right|\right|_{;y}^2\leq C\,||\kappa||_2^2\left(\beta+\frac{\Lambda}{N}\right)\leq C\Lambda^3\left(\beta+\frac{\Lambda}{N}\right). \tag{158}$$

IV.2 Bound on $d_t\beta^a$:

Lemma IV.9. *Let v satisfy (A1), $\varphi_t\in L^2(\mathbb{R}^3)$ with $||\varphi_t||=1$, $\alpha_0\in\mathfrak{h}$ such that $(\boldsymbol{A}(0),\boldsymbol{E}(0))\in(H^3(\mathbb{R}^3)\oplus H^2(\mathbb{R}^3))$, $\Psi_{N0}\in\left(L_s^2\left(\mathbb{R}^{3N}\right)\otimes\mathcal{F}_p\right)\cap\mathcal{D}(H_m^N)$. Let Ψ_{Nt} the unique solution of (4), $(\varphi_t,\boldsymbol{A}(t),\boldsymbol{E}(t))$ the unique solution of (15) and assume $\sup_{t\in[0,T]}\{||\varphi_t||_{H^3(\mathbb{R}^3)}+||\boldsymbol{A}(t)||_{H^3(\mathbb{R}^3)}+||\boldsymbol{E}(t)||_{H^2(\mathbb{R}^3)}\}<\infty$ for any $T\in\mathbb{R}^+$. Then*

$$|d_t\beta^a|\leq C\Lambda^2\left(\beta+\frac{\Lambda}{N}\right). \tag{159}$$

Proof. With the help of $d_tq_1=-i\left[H_1^{HM},q_1\right]$ we calculate

$$d_t\beta^a=d_t\langle\!\langle\Psi_N,q_1\Psi_N\rangle\!\rangle=i\langle\!\langle\Psi_N,\left[\left(H_m^N-H_1^{HM}\right),q_1\right]\Psi_N\rangle\!\rangle \tag{160}$$

$$=-2\langle\!\langle\Psi_N,\left[\left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}-\boldsymbol{A}_\kappa(x_1,t)\right)\cdot\nabla_1,q_1\right]\Psi_N\rangle\!\rangle \tag{161}$$

$$+i\langle\!\langle\Psi_N,\left[\left(\frac{\hat{\boldsymbol{A}}_\kappa^2(x_1)}{N}-\boldsymbol{A}_\kappa(x_1,t)\right),q_1\right]\Psi_N\rangle\!\rangle \tag{162}$$

$$+i\langle\!\langle\Psi_N,\left[\left(\frac{1}{N}\sum_{1\leq j<k\leq N}v(x_j-x_k)-\left(v\star|\varphi|^2\right)(x_1)\right),q_1\right]\Psi_N\rangle\!\rangle \tag{163}$$

$$=-4\mathrm{Re}\langle\!\langle\Psi_N,\left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}-\boldsymbol{A}_\kappa(x_1,t)\right)\cdot\nabla_1q_1\Psi_N\rangle\!\rangle \tag{164}$$

$$-2\mathrm{Im}\langle\!\langle\Psi_N,\left(\frac{\hat{\boldsymbol{A}}_\kappa^2(x_1)}{\sqrt{N}}-\boldsymbol{A}_\kappa^2(x_1,t)\right)q_1\Psi_N\rangle\!\rangle \tag{165}$$

$$-2\mathrm{Im}\langle\!\langle\Psi_N,\left(\frac{N-1}{N}v(x_1-x_2)-\left(v\star|\varphi|^2\right)(x_1)\right)q_1\Psi_N\rangle\!\rangle. \tag{166}$$

Inserting $\mathbb{1} = p_1 + q_1$ and using the relations

$$\mathrm{Re}\langle\!\langle \Psi_N, q_1 \left(\tfrac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} - \boldsymbol{A}_\kappa(x_1,t)\right) \cdot \nabla_1 q_1 \Psi_N \rangle\!\rangle = 0,$$

$$\mathrm{Im}\langle\!\langle \Psi_N, q_1 \left(\tfrac{\hat{\boldsymbol{A}}^2_\kappa(x_1)}{N} - \boldsymbol{A}^2_\kappa(x_1,t)\right) q_1 \Psi_N \rangle\!\rangle = 0,$$

$$\mathrm{Im}\langle\!\langle \Psi_N, q_1 \left(\tfrac{N-1}{N} v(x_1 - x_2) - \left(v \star |\varphi|^2\right)(x_1)\right) q_1 \Psi_N \rangle\!\rangle = 0,$$

leads to

$$\begin{aligned} d_t \beta^a =& - 4\mathrm{Re}\langle\!\langle \Psi_N, p_1 \left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} - \boldsymbol{A}_\kappa(x_1,t)\right) \cdot \nabla_1 q_1 \Psi_N \rangle\!\rangle && (167)\\ & - 2\mathrm{Im}\langle\!\langle \Psi_N, p_1 \left(\frac{\hat{\boldsymbol{A}}^2_\kappa(x_1)}{\sqrt{N}} - \boldsymbol{A}^2_\kappa(x_1,t)\right) q_1 \Psi_N \rangle\!\rangle && (168)\\ & - 2\mathrm{Im}\langle\!\langle \Psi_N, p_1 \left(\frac{N-1}{N} v(x_1 - x_2) - \left(v \star |\varphi|^2\right)(x_1,t)\right) q_1 \Psi_N \rangle\!\rangle. && (169) \end{aligned}$$

In the following, each line is estimated separately.

IV.2.1 Bound on (167):

To simplify notation, we use (84). Integration by parts and triangular inequality let us estimate

$$\begin{aligned} |(167)| &\leq 4|\langle\!\langle \Psi_N, p_1 \left(\boldsymbol{\mathcal{A}}^+(x_1,t) + \boldsymbol{\mathcal{A}}^-(x_1,t)\right) \cdot \nabla_1 q_1 \Psi_N \rangle\!\rangle| && (170)\\ &\leq 4|\langle\!\langle \nabla_1 p_1 \Psi_N, \boldsymbol{\mathcal{A}}^-(x_1,t) q_1 \Psi_N \rangle\!\rangle| && (171)\\ &+ 4|\langle\!\langle \nabla_1 p_1 \Psi_N, \boldsymbol{\mathcal{A}}^+(x_1,t) q_1 \Psi_N \rangle\!\rangle|. && (172) \end{aligned}$$

By means of Lemma IV.3, we bound the first line by

$$\begin{aligned} (171) =& 4|\langle\!\langle \nabla_1 p_1 \Psi_N, \boldsymbol{\mathcal{A}}^-(x_1,t) q_1 \Psi_N \rangle\!\rangle| = 4|\langle\!\langle \nabla_1 p_1 \Psi_N, \left(\eta \star \boldsymbol{\mathcal{E}}^-\right)(x_1,t) q_1 \Psi_N \rangle\!\rangle| && (173)\\ =& 4|\langle\!\langle \boldsymbol{\mathcal{E}}^+(y,t) \nabla_1 p_1 \Psi_N, \eta(x_1 - y) q_1 \Psi_N \rangle\!\rangle_{;y}| && (174)\\ \leq& 4 \left|\left|\boldsymbol{\mathcal{E}}^+(y,t) \cdot \nabla_1 p_1 \Psi_N\right|\right|_{;y} \left|\left|\eta(y - x_1) q_1 \Psi_N\right|\right|_{;y} && (175)\\ \leq& 2 \left|\left|\boldsymbol{\mathcal{E}}^+(y,t) \cdot \nabla_1 p_1 \Psi_N\right|\right|^2_{;y} + 2 \left|\left|\eta\right|\right|_2^2 \left|\left|q_1 \Psi_N\right|\right|^2 \leq \Lambda \pi^{-2} \beta^a + C\beta^b \leq C\Lambda\beta, && (176) \end{aligned}$$

where we made use of Lemma VI.1 and (89).

We introduce the operator $\hat{n} = \left(N^{-1} \sum_{j=1}^N q_j\right)^{1/2}$ and its inverse $\hat{n}^{-1}$ (see e.g. [14]) and continue with

$$\begin{aligned} (172) =& 4|\langle\!\langle \nabla_1 p_1 \Psi_N, \int d^3y\, \eta(x_1 - y) \boldsymbol{\mathcal{E}}^+(y,t) q_1 \Psi_N \rangle\!\rangle| && (177)\\ =& 4|\langle\!\langle \nabla_1 p_1 \Psi_N, \eta(x_1 - y) \boldsymbol{\mathcal{E}}^+(y,t) q_1 \Psi_N \rangle\!\rangle_{;y}| && (178)\\ =& 4|\langle\!\langle \hat{n}\eta(x_1 - y) \nabla_1 p_1 \Psi_N, \hat{n}^{-1} \boldsymbol{\mathcal{E}}^+(y,t) q_1 \Psi_N \rangle\!\rangle_{;y}| && (179)\\ \leq& 2 \left|\left|\hat{n}\eta(x_1 - y) \nabla_1 p_1 \Psi_N\right|\right|^2_{;y} + 2 \left|\left|\hat{n}^{-1} \boldsymbol{\mathcal{E}}^+(y,t) q_1 \Psi_N\right|\right|^2_{;y}. && (180) \end{aligned}$$

Making use of Lemma VI.1, this gives

$$(172) \leq 2\beta^b + 2\,||\hat{n}\eta(x_1-y)\nabla_1 p_1\Psi_N||^2_{;y} \tag{181}$$
$$= 2\beta^b + 2\langle\!\langle \eta(x_1-y)\nabla_1 p_1\Psi_N, \hat{n}^2\eta(x_1-y)\nabla_1 p_1\Psi_N\rangle\!\rangle_{;y} \tag{182}$$
$$= 2\beta^b + 2\langle\!\langle \eta(x_1-y)\nabla_1 p_1\Psi_N, (N^{-1}\sum_{j=1}^{N} q_j)\eta(x_1-y)\nabla_1 p_1\Psi_N\rangle\!\rangle_{;y} \tag{183}$$
$$= 2\beta^b + 2N^{-1}\langle\!\langle \eta(x_1-y)\nabla_1 p_1\Psi_N, q_1\eta(x_1-y)\nabla_1 p_1\Psi_N\rangle\!\rangle_{;y} \tag{184}$$
$$+ 2(N-1)N^{-1}\langle\!\langle \eta(x_1-y)\nabla_1 p_1\Psi_N, q_2\eta(x_1-y)\nabla_1 p_1\Psi_N\rangle\!\rangle_{;y} \tag{185}$$
$$\leq 2\beta^b + 2N^{-1}\langle\!\langle \eta(x_1-y)\nabla_1 p_1\Psi_N, \eta(x_1-y)\nabla_1 p_1\Psi_N\rangle\!\rangle_{;y} \tag{186}$$
$$+ 2(N-1)N^{-1}\langle\!\langle \eta(x_1-y)\nabla_1 p_1 q_2\Psi_N, \eta(x_1-y)\nabla_1 p_1 q_2\Psi_N\rangle\!\rangle_{;y}. \tag{187}$$

Interchanging the order of integration we have

$$(172) \leq 2\beta^b + 2N^{-1}\langle\!\langle \nabla_1 p_1\Psi_N, \left(\int d^3y|\eta(x_1-y)|^2\right)\nabla_1 p_1\Psi_N\rangle\!\rangle \tag{188}$$
$$+ 2(N-1)N^{-1}\langle\!\langle \nabla_1 p_1 q_2\Psi_N, \left(\int d^3y|\eta(x_1-y)|^2\right)\nabla_1 p_1 q_2\Psi_N\rangle\!\rangle \tag{189}$$
$$= 2\beta^b + 2N^{-1}\,||\eta||_2^2\,\langle\!\langle \Psi_N, p_1\left(-\Delta_1\right)p_1\Psi_N\rangle\!\rangle \tag{190}$$
$$+ 2(N-1)N^{-1}\,||\eta||_2^2\,\langle\!\langle \Psi_N, q_2 p_1\left(-\Delta_1\right)p_1 q_2\Psi_N\rangle\!\rangle. \tag{191}$$

By virtue of $p_1\left(-\Delta\right)p_1 = p_1\,||\nabla\varphi||_2^2$, this becomes

$$(172) \leq 2\beta^b + 2N^{-1}\,||\eta||_2^2\,||\nabla\varphi||_2^2\,\langle\!\langle \Psi_N, p_1\Psi_N\rangle\!\rangle \tag{192}$$
$$+2\,||\eta||_2^2\,||\nabla\varphi||_2^2\,\langle\!\langle \Psi_N, q_2 p_1 q_2\Psi_N\rangle\!\rangle \tag{193}$$
$$\leq C\,||\eta||_2^2\left(\beta^a + \beta^b + N^{-1}\right) \leq C\Lambda\left(\beta + N^{-1}\right) \tag{194}$$

ans we obtain

$$|(167)| \leq C\Lambda\left(\beta + N^{-1}\right). \tag{195}$$

IV.2.2 Bound on (168):

$$|(168)| \leq 2|\langle\!\langle \Psi_N, p_1\left(\frac{\hat{\boldsymbol{A}}^2_\kappa(x_1)}{N} - \boldsymbol{A}^2_\kappa(x_1,t)\right)q_1\Psi_N\rangle\!\rangle| \tag{196}$$
$$= 2|\langle\!\langle \Psi_N, p_1\left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} - \boldsymbol{A}_\kappa(x_1,t)\right)\left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} + \boldsymbol{A}_\kappa(x_1,t)\right)q_1\Psi_N\rangle\!\rangle| \tag{197}$$
$$\leq 2|\langle\!\langle \Psi_N, p_1\boldsymbol{\mathcal{A}}^-(x_1,t)\left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} + \boldsymbol{A}_\kappa(x_1,t)\right)q_1\Psi_N\rangle\!\rangle| \tag{198}$$
$$+ 2|\langle\!\langle \Psi_N, p_1\boldsymbol{\mathcal{A}}^+(x_1,t)\left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} + \boldsymbol{A}_\kappa(x_1,t)\right)q_1\Psi_N\rangle\!\rangle| \tag{199}$$

First, we deal with line (198):

$$(198) =2|\langle\langle\Psi_N, p_1 \left(\eta \star \boldsymbol{\mathcal{E}}^-\right)(x_1,t) \left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} + \boldsymbol{A}_\kappa(x_1,t)\right) q_1\Psi_N\rangle\rangle| \tag{200}$$

$$=2|\langle\langle\boldsymbol{\mathcal{E}}^+(y,t)p_1\Psi_N, \eta(y-x_1) \left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} + \boldsymbol{A}_\kappa(x_1,t)\right) q_1\Psi_N\rangle\rangle_{;y}| \tag{201}$$

$$\leq \left|\left|\boldsymbol{\mathcal{E}}^+(y,t)p_1\Psi_N\right|\right|^2_{;y} + \left|\left|\eta(x_1-y) \left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} + \boldsymbol{A}_\kappa(x_1,t)\right) q_1\Psi_N\right|\right|^2_{;y} \tag{202}$$

$$\leq\langle\langle\boldsymbol{\mathcal{E}}^-(y,t)\Psi_N, \boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle\rangle_{;y} + ||\eta||_2^2 \left|\left|\left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} + \boldsymbol{A}_\kappa(x_1,t)\right) q_1\Psi_N\right|\right|^2 \tag{203}$$

Making use of Lemma VI.1 and $(a+b) \leq 2(a^2+b^2)$, we obtain

$$(198) \leq \beta^b + 2\,||\eta||_2^2 \left(||\boldsymbol{A}_\kappa||_\infty^2\,||q_1\Psi_N||^2 + \left|\left|\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} q_1\Psi_N\right|\right|^2\right). \tag{204}$$

$$\tag{205}$$

By means of (89) and Lemma IV.7, this becomes

$$(198) \leq C\Lambda^2 \left(\beta + \frac{\Lambda}{N}\right). \tag{206}$$

The second line is bounded by

$$(199) = 2|\langle\langle\Psi_N, p_1\boldsymbol{\mathcal{A}}^+(x_1,t) \left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} + \boldsymbol{A}_\kappa(x_1,t)\right) q_1\Psi_N\rangle\rangle| \tag{207}$$

$$= 2|\langle\langle\Psi_N, p_1 \left[\left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} + \boldsymbol{A}_\kappa(x_1,t)\right)\boldsymbol{\mathcal{A}}^+(x_1,t) + \frac{\Lambda^2}{4\pi^2 N}\right] q_1\Psi_N\rangle\rangle| \tag{208}$$

$$\leq 2|\langle\langle\Psi_N, p_1 \left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} + \boldsymbol{A}_\kappa(x_1,t)\right)\boldsymbol{\mathcal{A}}^+(x_1,t) q_1\Psi_N\rangle\rangle| \tag{209}$$

$$+ \frac{2\Lambda^2}{4\pi^2 N}|\underbrace{\langle\langle\Psi_N, p_1 q_1\Psi_N\rangle\rangle}_{=0}|, \tag{210}$$

where we used the commutation relation

$$\left[\boldsymbol{\mathcal{A}}^+(x_1,t), \left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} + \boldsymbol{A}_\kappa(x_1,t)\right)\right] = N^{-1}\left[\hat{\boldsymbol{A}}^+_\kappa(x_1), \hat{\boldsymbol{A}}^-_\kappa(x_1)\right] = \frac{\Lambda^2}{4\pi^2 N}. \tag{211}$$

Lemma IV.3 and Lemma VI.1 lead to

$$(199) \leq 2|\langle\!\langle \left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} + \boldsymbol{A}_\kappa(x_1,t) \right) p_1 \Psi_N, \int d^3y \eta(x_1-y) \boldsymbol{\mathcal{E}}^+(y,t) q_1 \Psi_N \rangle\!\rangle| \tag{212}$$

$$= 2|\langle\!\langle \hat{n}\eta(x_1-y) \left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} + \boldsymbol{A}_\kappa(x_1,t) \right) p_1 \Psi_N, \hat{n}^{-1} \boldsymbol{\mathcal{E}}^+(y,t) q_1 \Psi_N \rangle\!\rangle_{;y}| \tag{213}$$

$$\leq \left|\left| \hat{n}^{-1} \boldsymbol{\mathcal{E}}^+(y,t) q_1 \Psi_N \right|\right|_{;y}^2 + \left|\left| \hat{n}\eta(x_1-y) \left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} + \boldsymbol{A}_\kappa(x_1,t) \right) p_1 \Psi_N \right|\right|_{;y}^2 \tag{214}$$

$$\leq \beta^b + \left|\left| \hat{n}\eta(x_1-y) \left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} + \boldsymbol{A}_\kappa(x_1,t) \right) p_1 \Psi_N \right|\right|_{;y}^2 \tag{215}$$

By means of Lemma IV.7, we estimate

$$\left|\left| \hat{n}\eta(x_1-y) \left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} + \boldsymbol{A}_\kappa(x_1,t) \right) p_1 \Psi_N \right|\right|_{;y}^2 \tag{216}$$

$$= \langle\!\langle \eta(x_1-y) \left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} + \boldsymbol{A}_\kappa(x_1,t) \right) p_1 \Psi_N, \hat{n}^2 \eta(x_1-y) \left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} + \boldsymbol{A}_\kappa(x_1,t) \right) p_1 \Psi_N \rangle\!\rangle_{;y} \tag{217}$$

$$= \frac{1}{N} \langle\!\langle \eta(x_1-y) \left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} + \boldsymbol{A}_\kappa(x_1,t) \right) p_1 \Psi_N, q_1 \eta(x_1-y) \left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} + \boldsymbol{A}_\kappa(x_1,t) \right) p_1 \Psi_N \rangle\!\rangle_{;y} \tag{218}$$

$$+ \frac{N-1}{N} \langle\!\langle \eta(x_1-y) \left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} + \boldsymbol{A}_\kappa(x_1,t) \right) p_1 \Psi_N, q_2 \eta(x_1-y) \left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} + \boldsymbol{A}_\kappa(x_1,t) \right) p_1 \Psi_N \rangle\!\rangle_{;y} \tag{219}$$

$$\leq \frac{1}{N} \langle\!\langle \left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} + \boldsymbol{A}_\kappa(x_1,t) \right) p_1 \Psi_N, \left(\int d^3y |\eta|^2(x_1-y) \right) \left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} + \boldsymbol{A}_\kappa(x_1,t) \right) p_1 \Psi_N \rangle\!\rangle \tag{220}$$

$$+ \langle\!\langle \left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} + \boldsymbol{A}_\kappa(x_1,t) \right) p_1 q_2 \Psi_N, \left(\int d^3y |\eta|^2(x_1-y) \right) \left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} + \boldsymbol{A}_\kappa(x_1,t) \right) p_1 q_2 \Psi_N \rangle\!\rangle \tag{221}$$

$$= \frac{1}{N} ||\eta||_2^2 \left|\left| \left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} + \boldsymbol{A}_\kappa(x_1,t) \right) p_1 \Psi_N \right|\right|^2 + ||\eta||_2^2 \left|\left| \left(\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} + \boldsymbol{A}_\kappa(x_1,t) \right) p_1 q_2 \Psi_N \right|\right|^2 \tag{222}$$

$$\leq C\Lambda \left(||\boldsymbol{A}_\kappa||_\infty^2 \beta^a + \left|\left| \frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} p_1 q_2 \Psi_N \right|\right|^2 + N^{-1} \left(||\boldsymbol{A}_\kappa||_\infty^2 + \left|\left| \frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} p_1 \Psi_N \right|\right|^2 \right) \right) \tag{223}$$

$$\leq C\Lambda^2 \left(\beta + \frac{\Lambda}{N} \right). \tag{224}$$

and obtain

$$(199) \leq C\Lambda^2 \left(\beta + \frac{\Lambda}{N} \right). \tag{225}$$

In total this gives,

$$|(168)| \leq (198) + (199) \leq C\Lambda^2 \left(\beta + \frac{\Lambda}{N} \right). \tag{226}$$

IV.2.3 Bound on (169):

Next, we are dealing with the term that arises from the direct interaction. Inserting the identity $1 = p_2 + q_2$ and using the shorthand shorthand notation

$$Z(x_1, x_2) := \frac{N-1}{N} v(x_1 - x_2) - \left(v \star |\varphi|^2\right)(x_1) \tag{227}$$

gives

$$\begin{aligned}
(169) =& -2\mathrm{Im}\langle\langle \Psi_N, p_1 Z(x_1,x_2) q_1 \Psi_N \rangle\rangle && (228)\\
=& -2\mathrm{Im}\langle\langle \Psi_N, p_1 p_2 Z(x_1,x_2) q_1 p_2 \Psi_N \rangle\rangle && (229)\\
& -2\mathrm{Im}\langle\langle \Psi_N, p_1 p_2 Z(x_1,x_2) q_1 q_2 \Psi_N \rangle\rangle && (230)\\
& -2\mathrm{Im}\langle\langle \Psi_N, p_1 q_2 Z(x_1,x_2) q_1 p_2 \Psi_N \rangle\rangle && (231)\\
& -2\mathrm{Im}\langle\langle \Psi_N, p_1 q_2 Z(x_1,x_2) q_1 q_2 \Psi_N \rangle\rangle. && (232)
\end{aligned}$$

The third term vanishes due to symmetry of the wave function under the interchange of x_1 and x_2 and we are left with

$$\begin{aligned}
|(169)| \leq& 2|\langle\langle \Psi_N, p_1 p_2 Z(x_1,x_2) q_1 p_2 \Psi_N \rangle\rangle| && (233)\\
&+2|\langle\langle \Psi_N, p_1 p_2 Z(x_1,x_2) q_1 q_2 \Psi_N \rangle\rangle| && (234)\\
&+2|\langle\langle \Psi_N, p_1 q_2 Z(x_1,x_2) q_1 q_2 \Psi_N \rangle\rangle|. && (235)
\end{aligned}$$

The first line is the most important. It is small because the direct interaction of the Many-body system is well approximated by the mean-field potential. By means of

$$p_2 Z(x_1,x_2) p_2 = p_2 \left(\frac{N-1}{N} v(x_1 - x_2) - \left(v \star |\varphi|^2\right)(x_1) \right) p_2 \tag{236}$$

$$= p_2 \left(v \star |\varphi|^2\right)(x_1) \left(\frac{N-1}{N} - 1 \right) = -\frac{1}{N} \left(v \star |\varphi|^2\right)(x_1) p_2 \tag{237}$$

one has

$$\begin{aligned}
|(233)| \leq& 2N^{-1} |\langle\langle \Psi_N, p_1 \left(v \star |\varphi|^2\right)(x_1) p_2 q_1 \Psi_N \rangle\rangle| && (238)\\
\leq& 2N^{-1} \left|\left| \left(v \star |\varphi|^2\right)(x_1) p_1 \Psi_N \right|\right| \, ||p_2 q_1 \Psi_N|| \leq 2N^{-1} \left|\left| v \star |\varphi|^2 \right|\right|_\infty \leq CN^{-1}. && (239)
\end{aligned}$$

We note that

$$\begin{aligned}
p_1 Z^2(x_1,x_2) p_1 =& p_1 \langle \varphi, \left(\frac{N-1}{N} v(x_2 - \cdot) - \left(v \star |\varphi|^2\right) \right)^2 \varphi \rangle && (240)\\
\leq& 2p_1 \langle \varphi, \left(v^2(x_2 - \cdot) + \left(v \star |\varphi|^2\right)^2 \right) \varphi \rangle && (241)\\
\leq& 2p_1 \left(\left|\left| v^2 \star |\varphi|^2 \right|\right|_\infty + \left|\left| v \star |\varphi|^2 \right|\right|_\infty^2 \right) \leq C p_1 && (242)
\end{aligned}$$

gives

$$\left|\left| p_1 Z^2(x_1,x_2) p_1 \right|\right|_{\mathrm{op}} \leq C. \tag{243}$$

Then, we introduce the operator $\hat{m} := \left(N^{-1}\sum_{j=2}^{n} q_j\right)^{1/2}$ and its inverse $\hat{m}^{-1}$ [14]. This leads to

$$
\begin{aligned}
|(234)| =& 2|\langle\!\langle \hat{m}Z(x_1,x_2)p_1p_2\Psi_N, \hat{m}^{-1}q_1q_2\Psi_N\rangle\!\rangle| && (244)\\
\leq& \langle\!\langle \Psi_N, p_1p_2Z(x_1,x_2)\hat{m}^2 Z(x_1,x_2)p_1p_2\Psi_N\rangle\!\rangle + \langle\!\langle \Psi_N, q_1q_2\hat{m}^{-2}\Psi_N\rangle\!\rangle && (245)\\
\leq& N^{-1}\langle\!\langle \Psi_N, p_2p_1Z^2(x_1,x_2)p_1p_2\Psi_N\rangle\!\rangle + \langle\!\langle \Psi_N, q_3p_2p_1Z^2(x_1,x_2)p_1p_2q_3\Psi_N\rangle\!\rangle && (246)\\
&+ \frac{N}{N-1}\langle\!\langle \Psi_N, q_1\sum_{j=2}^{N} N^{-1}q_j\hat{m}^{-2}\Psi_N\rangle\!\rangle && (247)\\
\leq& C\left(N^{-1} + ||q_3\Psi_N||^2\right) + 2\langle\!\langle \Psi_N, q_1\Psi_N\rangle\!\rangle \leq C(\beta^a + N^{-1}). && (248)
\end{aligned}
$$

The last term is bounded by

$$
\begin{aligned}
|(235)| =& 2\langle\!\langle Z(x_1,x_2)p_1q_2\Psi_N, q_1q_2\Psi_N\rangle\!\rangle && (249)\\
\leq& \langle\!\langle \Psi_N, q_2p_1Z^2(x_1,x_2)p_1q_2\Psi_N\rangle\!\rangle + ||q_1q_2\Psi_N||^2 && (250)\\
\leq& \left|\left|p_1Z^2(x_1,x_2)p_1\right|\right|_{\text{op}} ||q_2\Psi_N||^2 + \beta^a \leq C\beta^a && (251)
\end{aligned}
$$

and we have

$$
|(169)| \leq C\left(\beta + N^{-1}\right). \tag{252}
$$

IV.3 Bound on $d_t\beta^b$:

Lemma IV.10. *Let v satisfy (A1), $\varphi_t \in L^2(\mathbb{R}^3)$ with $||\varphi_t|| = 1$, $\alpha_0 \in \mathfrak{h}$ such that $(\boldsymbol{A}(0), \boldsymbol{E}(0)) \in (H^3(\mathbb{R}^3) \oplus H^2(\mathbb{R}^3))$, $\Psi_{N0} \in \left(L_s^2\left(\mathbb{R}^{3N}\right) \otimes \mathcal{F}_p\right) \cap \mathcal{D}(H_m^N)$. Let Ψ_{Nt} the unique solution of (4), $(\varphi_t, \boldsymbol{A}(t), \boldsymbol{E}(t))$ the unique solution of (15) and assume $\sup_{t\in[0,T]}\{||\varphi_t||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{A}(t)||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{E}(t)||_{H^2(\mathbb{R}^3)}\} < \infty$ for any $T \in \mathbb{R}^+$. Then*

$$
|d_t\beta^b| \leq C\Lambda^4\left(\beta + \frac{1}{N}\right). \tag{253}
$$

Proof. The canonical commutation relations (9) give

$$
i\left[H_M^N, \frac{a(k,\lambda)}{\sqrt{N}}\right] = -i|k|\frac{a(k,\lambda)}{\sqrt{N}} - \frac{2i}{N}\sum_{j=1}^{N}\frac{\tilde{\kappa}(k)}{\sqrt{2|k|}}\boldsymbol{\epsilon}_\lambda(k)e^{-ikx_j}\left(i\nabla_j + \frac{\hat{\boldsymbol{A}}_\kappa(x_j)}{\sqrt{N}}\right), \tag{254}
$$

$$
i\left[H_M^N, \frac{a^*(k,\lambda)}{\sqrt{N}}\right] = i|k|\frac{a^*(k,\lambda)}{\sqrt{N}} + \frac{2i}{N}\sum_{j=1}^{N}\frac{\tilde{\kappa}(k)}{\sqrt{2|k|}}\boldsymbol{\epsilon}_\lambda(k)e^{ikx_j}\left(i\nabla_j + \frac{\hat{\boldsymbol{A}}_\kappa(x_j)}{\sqrt{N}}\right) \tag{255}
$$

and the Maxwell-Schrödinger system leads to

$$
\partial_t|k|^{1/2}\alpha_t(k,\lambda) = -i|k|^{3/2}\alpha_t(k,\lambda) + \frac{i}{\sqrt{2}}\tilde{\kappa}(k)\boldsymbol{\epsilon}_\lambda(k)(2\pi)^{3/2}\mathcal{F}\mathcal{T}[\boldsymbol{j}](k). \tag{256}
$$

This let us compute

$$d_t\beta^b = \sum_{\lambda=1,2}\int d^3k\ d_t|k|\langle\!\langle\Psi_N, \left(\frac{a^*(k,\lambda)}{\sqrt{N}} - \alpha_t^*(k,\lambda)\right)\left(\frac{a(k,\lambda)}{\sqrt{N}} - \alpha_t(k,\lambda)\right)\Psi_N\rangle\!\rangle \tag{257}$$

$$= \sum_{\lambda=1,2}\int d^3k|k|\langle\!\langle\Psi_N, i\left[H_m^N, \frac{a^*(k,\lambda)}{\sqrt{N}}\right]\left(\frac{a(k,\lambda)}{\sqrt{N}} - \alpha_t(k,\lambda)\right)\Psi_N\rangle\!\rangle \tag{258}$$

$$+ \sum_{\lambda=1,2}\int d^3k|k|\langle\!\langle\Psi_N, \left(\frac{a^*(k,\lambda)}{\sqrt{N}} - \alpha_t^*(k,\lambda)\right) i\left[H_m^N, \frac{a(k,\lambda)}{\sqrt{N}}\right]\Psi_N\rangle\!\rangle \tag{259}$$

$$- \sum_{\lambda=1,2}\int d^3k|k|^{1/2}\langle\!\langle\Psi_N, \left(\partial_t|k|^{1/2}\alpha_t\right)^*(k,\lambda)\left(\frac{a(k,\lambda)}{\sqrt{N}} - \alpha_t(k,\lambda)\right)\Psi_N\rangle\!\rangle \tag{260}$$

$$- \sum_{\lambda=1,2}\int d^3k|k|^{1/2}\langle\!\langle\Psi_N, \left(\frac{a^*(k,\lambda)}{\sqrt{N}} - \alpha_t^*(k,\lambda)\right)\left(\partial_t|k|^{1/2}\alpha_t\right)(k,\lambda)\Psi_N\rangle\!\rangle \tag{261}$$

$$=i\sum_{\lambda=1,2}\int d^3k|k|^2\langle\!\langle\Psi_N, \left(\frac{a^*(k,\lambda)}{\sqrt{N}} - \alpha_t^*(k,\lambda)\right)\left(\frac{a(k,\lambda)}{\sqrt{N}} - \alpha_t(k,\lambda)\right)\Psi_N\rangle\!\rangle \tag{262}$$

$$-i\sum_{\lambda=1,2}\int d^3k|k|^2\langle\!\langle\Psi_N, \left(\frac{a^*(k,\lambda)}{\sqrt{N}} - \alpha_t^*(k,\lambda)\right)\left(\frac{a(k,\lambda)}{\sqrt{N}} - \alpha_t(k,\lambda)\right)\Psi_N\rangle\!\rangle \tag{263}$$

$$+2\sum_{\lambda=1,2}\int d^3k\langle\!\langle\Psi_N, i\sqrt{\frac{|k|}{2}}\tilde{\kappa}(k)\boldsymbol{\epsilon}_\lambda(k)e^{ikx_1}\left(i\nabla_1 + \frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)\left(\frac{a(k,\lambda)}{\sqrt{N}} - \alpha_t(k,\lambda)\right)\Psi_N\rangle\!\rangle \tag{264}$$

$$-2\sum_{\lambda=1,2}\int d^3k\langle\!\langle\Psi_N, \left(\frac{a^*(k,\lambda)}{\sqrt{N}} - \alpha_t^*(k,\lambda)\right) i\sqrt{\frac{|k|}{2}}\tilde{\kappa}(k)\boldsymbol{\epsilon}_\lambda(k)e^{-ikx_1}\left(i\nabla_1 + \frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)\Psi_N\rangle\!\rangle \tag{265}$$

$$+ \sum_{\lambda=1,2}\int d^3k\langle\!\langle\Psi_N, i\sqrt{\frac{|k|}{2}}\tilde{\kappa}(k)\boldsymbol{\epsilon}_\lambda(k)(2\pi)^{3/2}\mathcal{FT}[\boldsymbol{j}]^*(k)\left(\frac{a(k,\lambda)}{\sqrt{N}} - \alpha_t(k,\lambda)\right)\Psi_N\rangle\!\rangle \tag{266}$$

$$- \sum_{\lambda=1,2}\int d^3k\langle\!\langle\Psi_N, \left(\frac{a^*(k,\lambda)}{\sqrt{N}} - \alpha_t^*(k,\lambda)\right) i\sqrt{\frac{|k|}{2}}\tilde{\kappa}(k)\boldsymbol{\epsilon}_\lambda(k)(2\pi)^{3/2}\mathcal{FT}[\boldsymbol{j}](k)\Psi_N\rangle\!\rangle \tag{267}$$

The first two lines cancel and with help of commutation relation $\left[\nabla_1, \boldsymbol{\epsilon}_\lambda(k)e^{ikx_1}\right] = 0$ (recall Definition (8)) we conlude (265) = (264)* and (267) = (266)*. This gives rise to

$$d_t\beta^b =$$

$$+4\mathrm{Re}\sum_{\lambda=1,2}\int d^3k\langle\!\langle\Psi_N, i\sqrt{\frac{|k|}{2}}\tilde{\kappa}(k)\boldsymbol{\epsilon}_\lambda(k)e^{ikx_1}\left(i\nabla_1 + \frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)\left(\frac{a(k,\lambda)}{\sqrt{N}} - \alpha_t(k,\lambda)\right)\Psi_N\rangle\!\rangle \tag{268}$$

$$+2\mathrm{Re}\sum_{\lambda=1,2}\int d^3k\langle\!\langle\Psi_N, i\sqrt{\frac{|k|}{2}}\tilde{\kappa}(k)\boldsymbol{\epsilon}_\lambda(k)(2\pi)^{3/2}\mathcal{FT}[\boldsymbol{j}]^*(k)\left(\frac{a(k,\lambda)}{\sqrt{N}} - \alpha_t(k,\lambda)\right)\Psi_N\rangle\!\rangle. \tag{269}$$

Inserting the identity $1 = p_1 + q_1$ and using the relation

$$\sum_{\lambda=1,2}\int d^3k i\sqrt{\frac{|k|}{2}}\tilde{\kappa}(k)\boldsymbol{\epsilon}_\lambda(k)e^{ikx_1}\left(\frac{a(k,\lambda)}{\sqrt{N}} - \alpha_t(k,\lambda)\right) = \left(\kappa\star\boldsymbol{\mathcal{E}}^+\right)(x_1,t) \tag{270}$$

this reads

$$d_t\beta^b = +4\mathrm{Re}\langle\!\langle\Psi_N, p_1\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\kappa(y-x_1)p_1\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle\!\rangle_{;y} \tag{271}$$

$$+2\mathrm{Re}\langle\!\langle \Psi_N, p_1\left(\kappa(y-x_1)i\nabla_1 + i\nabla_1\kappa(y-x_1)\right)p_1\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle\!\rangle_{;y} \tag{272}$$

$$+2\mathrm{Re}\langle\!\langle \Psi_N, \left(\int d^3z\kappa(y-z)\boldsymbol{j}(z)\right)\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle\!\rangle_{;y} \tag{273}$$

$$+4\mathrm{Re}\langle\!\langle \Psi_N, q_1\kappa(y-x_1)i\nabla_1 p_1\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle\!\rangle_{;y} \tag{274}$$

$$+4\mathrm{Re}\langle\!\langle \Psi_N, q_1\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\kappa(y-x_1)p_1\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle\!\rangle_{;y} \tag{275}$$

$$+4\mathrm{Re}\langle\!\langle \Psi_N, \left(i\nabla_1 + \frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)\kappa(y-x_1)q_1\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle\!\rangle_{;y} \tag{276}$$

$$\tag{277}$$

With the relations

$$p_1\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\kappa(y-x_1)p_1 = p_1\int d^3z|\varphi|^2(z)\frac{\hat{\boldsymbol{A}}_\kappa(z)}{\sqrt{N}}\kappa(y-z), \tag{278}$$

$$p_1\left(\kappa(y-x_1)i\nabla_1 + i\nabla_1\kappa(y-x_1)\right)p_1 = -2p_1\int d^3z\kappa(y-z)\mathrm{Im}[\varphi^*\nabla\varphi](z), \tag{279}$$

$$\boldsymbol{j} = 2\left(\mathrm{Im}(\varphi^*\nabla\varphi) - |\varphi|^2\boldsymbol{A}_\kappa\right) \tag{280}$$

we obtain

$$d_t\beta^b = -4\mathrm{Re}\int d^3z|\varphi|^2(z)\langle\!\langle \Psi_N, q_1\frac{\hat{\boldsymbol{A}}_\kappa(z)}{\sqrt{N}}\kappa(y-z)\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle\!\rangle_{;y} \tag{281}$$

$$+4\mathrm{Re}\int d^3z\mathrm{Im}[\varphi^*\nabla\varphi](z)\langle\!\langle \Psi_N, q_1\kappa(y-z)\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle\!\rangle_{;y} \tag{282}$$

$$+4\mathrm{Re}\int d^3z|\varphi|^2(z)\langle\!\langle \Psi_N, \kappa(y-z)\left(\frac{\hat{\boldsymbol{A}}_\kappa(z)}{\sqrt{N}} - \boldsymbol{A}_\kappa(z,t)\right)\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle\!\rangle_{;y} \tag{283}$$

$$+4\mathrm{Re}\langle\!\langle \Psi_N, q_1\kappa(y-x_1)i\nabla_1 p_1\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle\!\rangle_{;y} \tag{284}$$

$$+4\mathrm{Re}\langle\!\langle \Psi_N, q_1\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\kappa(y-x_1)p_1\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle\!\rangle_{;y} \tag{285}$$

$$+4\mathrm{Re}\langle\!\langle \Psi_N, \left(-i\nabla_1 - \frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)\kappa(y-x_1)q_1\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle\!\rangle_{;y}. \tag{286}$$

Again we estimate each line separately:

$$|(281)| \leq 4\left|\int d^3z|\varphi|^2(z)\langle\!\langle \Psi_N, q_1\frac{\hat{\boldsymbol{A}}_\kappa(z)}{\sqrt{N}}\kappa(y-z)\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle\!\rangle_{;y}\right| \tag{287}$$

$$\leq 4\int d^3y\int d^3z|\varphi|^2(z)|\kappa(y-z)||\langle\!\langle \frac{\hat{\boldsymbol{A}}_\kappa(z)}{\sqrt{N}}q_1\Psi_N, \boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle\!\rangle| \tag{288}$$

$$\leq 4\int d^3y\int d^3z|\varphi|^2(z)\left|\left|\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\right|\right||\kappa(y-z)|\left|\left|\frac{\hat{\boldsymbol{A}}_\kappa(z)}{\sqrt{N}}q_1\Psi_N\right|\right| \tag{289}$$

$$\leq 2\int d^3y\int d^3z|\varphi|^2(z)\left|\left|\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\right|\right|^2 \tag{290}$$

$$+2\int d^3z|\varphi|^2(z)\left|\left|\frac{\hat{\boldsymbol{A}}_\kappa(z)}{\sqrt{N}}q_1\Psi_N\right|\right|^2\left(\int d^3y|\kappa(y-z)|^2\right) \tag{291}$$

$$=2\langle\!\langle\Psi_N,\boldsymbol{\mathcal{E}}^-(y,t)\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle\!\rangle_{;y}+2\,||\kappa||_2^2\int d^3z|\varphi|^2(z)\left|\left|\frac{\hat{\boldsymbol{A}}_\kappa(z)}{\sqrt{N}}q_1\Psi_N\right|\right|^2. \tag{292}$$

By means of Lemma IV.7 and (89) this becomes

$$|(281)|\leq 2\beta^b+C\Lambda^4\left(\beta+\frac{\Lambda}{N}\right)\leq C\Lambda^4\left(\beta+\frac{\Lambda}{N}\right). \tag{293}$$

Similarly,

$$|(282)|\leq 4\int d^3y\int d^3z|\kappa(y-z)||\varphi(z)||\nabla\varphi(z)||\langle\!\langle q_1\Psi_N,\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle\!\rangle| \tag{294}$$

$$\leq 4\int d^3y\int d^3z|\kappa(y-z)||\varphi(z)||\nabla\varphi(z)|\left|\left|\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\right|\right|\,||q_1\Psi_N|| \tag{295}$$

$$\leq 2\int d^3y\int d^3z|\nabla\varphi(z)|^2\left|\left|\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\right|\right|^2 \tag{296}$$

$$+2\int d^3z|\varphi(z)|^2\,||q_1\Psi_N||^2\left(\int d^3y|\kappa(y-z)|^2\right) \tag{297}$$

$$\leq 2\,||\nabla\varphi||_2^2\,\beta^b+2\,||\kappa||_2^2\,\beta^a\leq C\Lambda^3\beta \tag{298}$$

and

$$|(283)|\leq 4\int d^3y\int d^3z|\varphi|^2(z)|\kappa(y-z)||\langle\!\langle\boldsymbol{\mathcal{A}}(z,t)\Psi_N,\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle\!\rangle| \tag{299}$$

$$\leq 4\int d^3y\int d^3z|\varphi|^2(z)|\kappa(y-z)|\,||\boldsymbol{\mathcal{A}}(z,t)\Psi_N||\,\left|\left|\boldsymbol{\mathcal{E}}^+(y,t)\right|\right| \tag{300}$$

$$\leq 2\int d^3z|\varphi|^2(z)\,||\boldsymbol{\mathcal{A}}(z,t)\Psi_N||^2\int d^3y|\kappa(y-z)|^2 \tag{301}$$

$$+2\int d^3z|\varphi|^2(z)\int d^3y\left|\left|\boldsymbol{\mathcal{E}}^+(y,t)\right|\right|^2 \tag{302}$$

$$\leq 2\beta^b+2\,||\kappa||_2^2\int d^3z|\varphi|^2(z)\,||\boldsymbol{\mathcal{A}}(z,t)\Psi_N||^2. \tag{303}$$

Linearity and $(a+b)^2\leq 2(a^2+b^2)$ lead to

$$|(283)|\leq 2\beta^b+4\,||\kappa||_2^2\int d^3z|\varphi|^2(z)\left(\left|\left|\boldsymbol{\mathcal{A}}^+(z,t)\Psi_N\right|\right|^2+\left|\left|\boldsymbol{\mathcal{A}}^-(z,t)\Psi_N\right|\right|^2\right). \tag{304}$$

By means of the commutation relation

$$\left[\boldsymbol{\mathcal{A}}^+(z,t),\boldsymbol{\mathcal{A}}^-(z,t)\right]=\frac{1}{N}\left[\hat{\boldsymbol{A}}_\kappa^+(z),\hat{\boldsymbol{A}}_\kappa^-(z)\right]=\frac{\Lambda^2}{4\pi^2N} \tag{305}$$

we calculate

$$\int d^3z|\varphi|^2(z)\left|\left|\boldsymbol{\mathcal{A}}^-(z,t)\Psi_N\right|\right|^2=\int d^3z|\varphi|^2(z)\langle\!\langle\Psi_N,\boldsymbol{\mathcal{A}}^+(z,t)\boldsymbol{\mathcal{A}}^-(z,t)\Psi_N\rangle\!\rangle \tag{306}$$

$$=\int d^3z|\varphi|^2(z)\langle\!\langle\Psi_N,\left(\boldsymbol{\mathcal{A}}^-(z,t)\boldsymbol{\mathcal{A}}^+(z,t)+\frac{1}{4\pi^2}\frac{\Lambda^2}{N}\right)\Psi_N\rangle\!\rangle \tag{307}$$

$$=\int d^3z|\varphi|^2(z)\left|\left|\boldsymbol{\mathcal{A}}^+(z,t)\Psi_N\right|\right|^2+\frac{1}{4\pi^2}\frac{\Lambda^2}{N} \tag{308}$$

and obtain

$$|(283)|\leq 2\beta^b+||\kappa||_2^2\frac{\Lambda^2}{\pi^2N}+8\,||\kappa||_2^2\int d^3z|\varphi|^2(z)\left|\left|\boldsymbol{\mathcal{A}}^+(z,t)\Psi_N\right|\right|^2. \tag{309}$$

Due to (86) one has

$$\int d^3z|\varphi|^2(z)\left|\left|\boldsymbol{\mathcal{A}}^+(z,t)\Psi_N\right|\right|^2=\int d^3z|\varphi|^2(z)\langle\!\langle\Psi_N,\boldsymbol{\mathcal{A}}^-(z,t)\boldsymbol{\mathcal{A}}^+(z,t)\Psi_N\rangle\!\rangle \tag{310}$$

$$=\int d^3z|\varphi|^2(z)\langle\!\langle\Psi_N,\int d^3y\eta(z-y)\boldsymbol{\mathcal{E}}^-(y,t)\int d^3l\eta(z-l)\boldsymbol{\mathcal{E}}^+(l,t)\Psi_N\rangle\!\rangle \tag{311}$$

$$\leq\int d^3y\int d^3z\int d^3l|\varphi|^2(z)|\eta(z-y)||\eta(z-l)||\langle\!\langle\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N,\boldsymbol{\mathcal{E}}^+(l,t)\Psi_N\rangle\!\rangle| \tag{312}$$

$$\leq\frac{1}{2}\int d^3z|\varphi|^2(z)\int d^3l\left|\left|\boldsymbol{\mathcal{E}}^+(l,t)\Psi_N\right|\right|^2\int d^3y|\eta(z-y)|^2 \tag{313}$$

$$+\frac{1}{2}\int d^3z|\varphi|^2(z)\int d^3y\left|\left|\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\right|\right|^2\int d^3l|\eta(z-l)|^2 \tag{314}$$

$$\leq||\eta||_2^2\int d^3y\left|\left|\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\right|\right|^2\leq||\eta||_2^2\,\beta^b\leq C\Lambda\beta^b, \tag{315}$$

which gives

$$|(283)|\leq 2\beta^b+||\kappa||_2^2\frac{\Lambda^2}{\pi^2N}+C\Lambda\,||\kappa||_2^2\,\beta^b\leq C\Lambda^4\left(\beta+\frac{\Lambda}{N}\right). \tag{316}$$

The next terms are bounded by

$$|(284)|\leq 4|\langle\!\langle\kappa(y-x_1)q_1\Psi_N,i\nabla_1p_1\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle\!\rangle_{;y}| \tag{317}$$

$$\leq 2\langle\!\langle\kappa(y-x_1)q_1\Psi_N,\kappa(y-x_1)q_1\Psi_N\rangle\!\rangle_{;y} \tag{318}$$

$$+2\langle\!\langle i\nabla_1p_1\boldsymbol{\mathcal{E}}^+(y,t),i\nabla_1p_1\boldsymbol{\mathcal{E}}^+(y,t)\rangle\!\rangle_{;y} \tag{319}$$

$$=2\langle\!\langle q_1\Psi_N,\left(\int d^3y|\kappa(y-x_1)|^2\right)q_1\Psi_N\rangle\!\rangle \tag{320}$$

$$+2\langle\!\langle\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N,p_1\left(-\Delta_1\right)p_1\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle\!\rangle_{;y} \tag{321}$$

$$=2\,||\kappa||_2^2\,\langle\!\langle\Psi_N,q_1\Psi_N\rangle\!\rangle+2\,||\nabla\varphi||_2^2\,\langle\!\langle\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N,p_1\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle\!\rangle_{;y} \tag{322}$$

$$\leq 2\,||\kappa||_2^2\,\beta^a+2\,||\nabla\varphi||_2^2\,\langle\!\langle\Psi_N,\boldsymbol{\mathcal{E}}^-(y,t)\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle\!\rangle_{;y} \tag{323}$$

$$\leq C\Lambda^3\beta. \tag{324}$$

and

$$\begin{aligned}
|(285)| \leq & 4|\langle\!\langle \kappa(y-z_1)\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}q_1\Psi_N, p_1\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle\!\rangle_{;y}| && (325)\\
\leq & 2\langle\!\langle \kappa(y-x_1)\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}q_1\Psi_N, \kappa(y-x_1)\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}q_1\Psi_N\rangle\!\rangle && (326)\\
& +2\langle\!\langle \boldsymbol{\mathcal{E}}^+(y,t)\Psi_N, \boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle\!\rangle_{;y} && (327)\\
= & 2\langle\!\langle \frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}q_1\Psi_N, \left(\int d^3y|\kappa(y-x_1)|^2\right)\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}q_1\Psi_N\rangle\!\rangle && (328)\\
& +2\langle\!\langle \Psi_N, \boldsymbol{\mathcal{E}}^-(y,t)\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle\!\rangle_{;y} && (329)\\
\leq & 2\beta^b + 2\left|\left|\kappa\right|\right|_2^2\left|\left|\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}q_1\Psi_N\right|\right|^2 \leq C\Lambda^4\left(\beta+\frac{\Lambda}{N}\right). && (330)
\end{aligned}$$

where we made use of Lemma IV.7.

$$\begin{aligned}
|(286)| \leq & 4|\langle\!\langle \Psi_N, \left(-i\nabla_1 - \frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)\kappa(y-x_1)q_1\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle\!\rangle_{;y}| && (331)\\
= & 4|\langle\!\langle \hat{n}\kappa(y-x_1)\left(-i\nabla_1 - \frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)\Psi_N, \hat{n}^{-1}q_1\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle\!\rangle_{;y}| && (332)\\
\leq & 2\left|\left|\hat{n}\kappa(y-x_1)\left(-i\nabla_1 - \frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)\Psi_N\right|\right|_{;y}^2 + 2\left|\left|\hat{n}^{-1}\boldsymbol{\mathcal{E}}^+(y,t)q_1\Psi_N\right|\right|_{;y}^2. && (333)
\end{aligned}$$

By means of Lemma VI.1 and Lemma IV.8, this can be bounded by

$$|(286)| \leq C\Lambda^3\left(\beta+\frac{\Lambda}{N}\right) + 2\langle\!\langle \Psi_N, \boldsymbol{\mathcal{E}}^-(y,t)\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle\!\rangle_{;y} \leq \; C\Lambda^3\left(\beta+\frac{\Lambda}{N}\right). \quad (334)$$

IV.4 Bound on $d_t\beta$:

The Maxwell-Schrödinger equations are a conserved system. Consequently, the energy of the effective system does not change during the time evolution

$$\mathcal{E}_M[\varphi_t,\alpha_t] = \mathcal{E}_M[\varphi_0,\alpha_0]. \quad (335)$$

Likewise β^c is a constant of motion, because due to the self-adjointness of the Pauli-Fierz Hamiltonian there is a strongly continuous unitary group $\{e^{-itH_m^N}\}_{t\in\mathbb{R}}$ such that $\Psi_{Nt} = e^{-itH_m^N}\Psi_{N0}$ and

$$\begin{aligned}
\beta^c[\Psi_{Nt},\varphi_t,\alpha_t] = & \left|\left|\left(\frac{H_m^N}{N} - \mathcal{E}_M[\varphi_t,\alpha_t]\right)\Psi_{Nt}\right|\right|^2 && (336)\\
= & \left|\left|\left(\frac{H_m^N}{N} - \mathcal{E}_M[\varphi_0,\alpha_0]\right)e^{-itH_m^N}\Psi_{N0}\right|\right|^2 && (337)\\
= & \left|\left|e^{-itH_m^N}\left(\frac{H_m^N}{N} - \mathcal{E}_M[\varphi_0,\alpha_0]\right)\Psi_{N0}\right|\right|^2 = \beta^c[\Psi_{N0},\varphi_0,\alpha_0]. && (338)
\end{aligned}$$

Altogether, the functional β is bounded by

Lemma IV.11. *Let v satisfy (A1), $\varphi_t \in L^2(\mathbb{R}^3)$ with $||\varphi_t|| = 1$, $\alpha_0 \in \mathfrak{h}$ such that $(\boldsymbol{A}(0), \boldsymbol{E}(0)) \in (H^3(\mathbb{R}^3) \oplus H^2(\mathbb{R}^3))$, $\Psi_{N0} \in \left(L_s^2\left(\mathbb{R}^{3N}\right) \otimes \mathcal{F}_p\right) \cap \mathcal{D}(H_m^N)$. Let Ψ_{Nt} the unique solution of (4), $(\varphi_t, \boldsymbol{A}(t), \boldsymbol{E}(t))$ the unique solution of (15) and assume $\sup_{t\in[0,T]}\{||\varphi_t||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{A}(t)||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{E}(t)||_{H^2(\mathbb{R}^3)}\} < \infty$ for any $T \in \mathbb{R}^+$.*

$$|d_t\beta| \leq C\Lambda^4 \left(\beta + \frac{1}{N}\right). \tag{339}$$

Proof. Lemma IV.9 and Lemma IV.10.

We apply Grönwall's lemma

$$\begin{aligned}\beta(t) \leq& \beta(0)e^{\Lambda^4 \int_0^t dsC(s)} + N^{-1}\int_0^t ds\Lambda^4 C(s)e^{\Lambda^4 \int_s^t d\tau C(\tau)} &(340)\\ \leq& e^{\Lambda^4 \int_0^t dsC(s)}\left(\beta(0) + N^{-1}\Lambda^4 \int_0^t dsC(s)\right) &(341)\end{aligned}$$

and obtain

$$\beta(t) \leq N^{-\min\{1,\delta\}}C_1 e^{\Lambda^4 \int_0^t dsC(s)}\left(1 + \Lambda^4 \int_0^t dsC(s)\right) \tag{342}$$

for $\beta(0) \leq C_1 N^{-\delta}$. This proves Theorem IV.1 with $C_2(t) = \int_0^t dsC(s)$.

V Initial conditions

In this section, we investigate the validity of $\beta(0) \leq CN^{-\delta}$. We will show that it holds for product states of the form (12). Nevertheless, we expect it to be valid also for states with weak correlations.

Lemma V.1. *Let $\varphi_0 \in H^3(\mathbb{R}^3)$ with $||\varphi_0|| = 1$ and $\alpha_0 \in \mathfrak{h}$ such that $(\boldsymbol{A}(0), \boldsymbol{E}(0)) \in (H^3(\mathbb{R}^3) \oplus H^2(\mathbb{R}^3))$. For $\Psi_{N0} = \varphi_0^{\otimes N} \otimes W(\sqrt{N}\alpha_0)\Omega$ we have*

$$\beta[\Psi_{N0}, \varphi_0, \alpha_0] \leq C\Lambda^4 N^{-1}. \tag{343}$$

By Theorem IV.1 and subsection III.1 this proves Theorem I.1. Before we prove Lemma V.1, we state some preliminaries that turn out to be useful later.

Preliminaries for the proof

First, we recall some well known properties of Weyl operators (13).

Lemma V.2. *Let $f, g \in \mathfrak{h} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$.*

(i) $W(f)$ is a unitary operator and

$$W^*(f) = W^{-1}(f) = W(-f). \tag{344}$$

(ii) We have

$$W^*(f)a(k,\lambda)W(f) = a(k,\lambda) + f(k,\lambda), \tag{345}$$
$$W^*(f)a^*(k,\lambda)W(f) = a^*(k,\lambda) + f^*(k,\lambda). \tag{346}$$

(iii) From (ii) we see that coherent states are eigenvectors of annihilation operators

$$a(k,\lambda)W(f)\Omega = f(k,\lambda)W(f)\Omega \;\Rightarrow\; a(g)W(f)\Omega = \langle g, f\rangle_{\mathfrak{h}} W(f)\Omega. \tag{347}$$

Next, we state the expected values of higher moments of the vector potential and the field energy.

Lemma V.3. *Let* $\Psi_N \in \mathcal{D}(H_m^N)$, $\alpha_0 \in \mathfrak{h}$ *such that* $(||\boldsymbol{A}(0)||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{E}(0)||_{H^2(\mathbb{R}^3)}) < \infty$ *and* $\boldsymbol{E}_\kappa^+(x,t)$, $\mathcal{E}_f(t)$, $\mathcal{E}_{f^2}(t)$ *be defined by* (82), (26), (27). *Define*

$$\gamma_{il}^{\perp\Lambda}(x) := \int d^3k \frac{|\tilde{\kappa}|^2}{|k|} e^{ikx} \left(\delta_{il} - \frac{k_i k_l}{|k|^2}\right) \quad \text{with} \quad \left|\left|\gamma_{il}^{\perp\Lambda}\right|\right|_2^2 \leq \frac{2\Lambda}{\pi}. \tag{348}$$

Then, one has

$$\langle W(\sqrt{N}\alpha_0)\Omega, \frac{\hat{\boldsymbol{A}}_\kappa(x)}{\sqrt{N}} W(\sqrt{N}\alpha_0)\Omega\rangle_{\mathcal{F}_p} = \boldsymbol{A}_\kappa(x,0), \tag{349}$$

$$\langle W(\sqrt{N}\alpha_0)\Omega, \frac{\hat{\boldsymbol{A}}_\kappa^2(x)}{N} W(\sqrt{N}\alpha_0)\Omega\rangle_{\mathcal{F}_p} = \boldsymbol{A}_\kappa^2(x,0) + \frac{\Lambda^2}{4\pi^2 N}, \tag{350}$$

$$\langle W(\sqrt{N}\alpha_0)\Omega, \frac{\hat{\boldsymbol{A}}_\kappa^i(x)\hat{\boldsymbol{A}}_\kappa^j(y)}{N} W(\sqrt{N}\alpha_0)\Omega\rangle_{\mathcal{F}_p} = \boldsymbol{A}_\kappa^i(x,0)\boldsymbol{A}_\kappa^j(y,0) + \frac{\gamma_{ij}^{\perp\Lambda}(x-y)}{2N}, \tag{351}$$

$$\langle W(\sqrt{N}\alpha_0)\Omega, \frac{H_f}{N} W(\sqrt{N}\alpha_0)\Omega\rangle_{\mathcal{F}_p} = \mathcal{E}_f(0), \tag{352}$$

$$\langle W(\sqrt{N}\alpha_0)\Omega, \frac{H_f^2}{N^2} W(\sqrt{N}\alpha_0)\Omega\rangle_{\mathcal{F}_p} = \mathcal{E}_f^2(0) + \frac{1}{N}\mathcal{E}_{f^2}(0), \tag{353}$$

$$\langle W(\sqrt{N}\alpha_0)\Omega, \frac{\hat{\boldsymbol{A}}_\kappa(x)H_f}{N^{3/2}} W(\sqrt{N}\alpha_0)\Omega\rangle_{\mathcal{F}_p} = \boldsymbol{A}_\kappa(x,0)\mathcal{E}_f(0) - \frac{i\boldsymbol{E}_\kappa^+(x,0)}{N}, \tag{354}$$

$$\langle W(\sqrt{N}\alpha_0)\Omega, \frac{\hat{\boldsymbol{A}}_\kappa^2(x)H_f}{N^2} W(\sqrt{N}\alpha_0)\Omega\rangle_{\mathcal{F}_p} = \boldsymbol{A}_\kappa^2(x)\mathcal{E}_f(0) + \tag{355}$$

$$+\frac{1}{N}\left(\frac{\Lambda^2}{4\pi^2}\mathcal{E}_f(0) - 2i\boldsymbol{A}_\kappa(x,0)\boldsymbol{E}_\kappa^+(x,0)\right). \tag{356}$$

Proof. The proof is a simple application of the canonical commutation relations (9) and part (ii) from Lemma V.2.

Proof of Lemma V.1. In the following, we show that β is small for states of product type. For β^a this is easily seen by

$$\beta^a[\Psi_{N0}, \varphi_0] = \langle\!\langle \Psi_{N0}, q_1 \otimes \mathbb{1}_{\mathcal{F}_p}\Psi_{N0}\rangle\!\rangle = \langle \varphi_0, \varphi_0\rangle_{L^2(\mathbb{R}^3)} - \langle \varphi_0, \varphi_0\rangle_{L^2(\mathbb{R}^3)}^2 = 0, \tag{357}$$

where we used that the scalar product factorizes for product states and q_1 only acts on the Hilbert space of the first charged particle. Lemma V.2 let us conclude

$$\frac{a(k,\lambda)}{\sqrt{N}} W(\sqrt{N}\alpha_0)\Omega = \alpha_0(k,\lambda)W(\sqrt{N}\alpha_0)\Omega \tag{358}$$

and we obtain

$$\beta^b[\Psi_{N0},\alpha_0] = \tag{359}$$

$$= \sum_{\lambda=1,2}\int d^3k|k|\langle\langle\left(\frac{a(k,\lambda)}{\sqrt{N}}-\alpha_0(k,\lambda)\right)\Psi_{N0},\left(\frac{a(k,\lambda)}{\sqrt{N}}-\alpha_0(k,\lambda)\right)\Psi_{N0}\rangle\rangle \tag{360}$$

$$= \sum_{\lambda=1,2}\int d^3k|k|\langle\langle(\alpha_0(k,\lambda)-\alpha_0(k,\lambda))\Psi_{N0},(\alpha_0(k,\lambda)-\alpha_0(k,\lambda))\Psi_{N0}\rangle\rangle = 0. \tag{361}$$

To show that the product structure suppresses the fluctuations of the energy per particle around its mean value is more elaborate. However, the idea of the proof simple and in the spirit of the law of large numbers from probability theory. We bound β^c by

$$\beta^c = \langle\langle\left(\frac{H_m^N}{N}-\mathcal{E}_M\right)\Psi_N,\left(\frac{H_m^N}{N}-\mathcal{E}_M\right)\Psi_N\rangle\rangle \tag{362}$$

$$\leq |\langle\langle\frac{H_m^N}{N}\Psi_N,\frac{H_m^N}{N}\Psi_N\rangle\rangle-\mathcal{E}_M^2| + 2\mathcal{E}_M|\mathcal{E}_M-\langle\langle\Psi_N,\frac{H_m^N}{N}\Psi_N\rangle\rangle| \tag{363}$$

and show that

(i) $|\langle\langle\Psi_{N0},\frac{H_m^N}{N}\Psi_{N0}\rangle\rangle-\mathcal{E}_M[\varphi_0,\alpha_0]| \leq C\Lambda^2N^{-1}$

(ii) $|\langle\langle\frac{H_m^N}{N}\Psi_{N0},\frac{H_m^N}{N}\Psi_{N0}\rangle\rangle-\mathcal{E}_M^2[\varphi_0,\alpha_0]| \leq C\Lambda^4N^{-1}$

holds for states of product type.

(i) The mean value of the energy per particle

For ease of notation we denote $\boldsymbol{A}_\kappa(\cdot,0)$, $\boldsymbol{E}_\kappa^+(\cdot,0)$, $\mathcal{E}_f(0)$, $\mathcal{E}_{f^2}(0)$ by $\boldsymbol{A}_\kappa(\cdot)$, $\boldsymbol{E}_\kappa(\cdot)$, $\mathcal{E}_f$, $\mathcal{E}_{f^2}$ in the following. The mean value of the energy per particle is given by

$$\langle\langle\Psi_{N0},\frac{H_m^N}{N}\Psi_{N0}\rangle\rangle = \langle\langle\Psi_{N0},\frac{1}{N}\sum_{j=1}^N\left(-i\nabla_j-\frac{\hat{\boldsymbol{A}}_\kappa(x_j)}{\sqrt{N}}\right)^2\Psi_{N0}\rangle\rangle \tag{364}$$

$$+\langle\langle\Psi_{N0},\frac{1}{2N^2}\sum_{j\neq k}v(x_j-x_k)\Psi_{N0}\rangle\rangle \tag{365}$$

$$+\langle\langle\Psi_{N0},\frac{H_f}{N}\Psi_{N0}\rangle\rangle. \tag{366}$$

Due to symmetry and the product structure of Ψ_{N0} this becomes

$$\langle\langle\Psi_{N0},\frac{H_m^N}{N}\Psi_{N0}\rangle\rangle = \langle\varphi_0,(-\Delta)\varphi_0\rangle \tag{367}$$

$$+2i\langle\varphi_0,\langle W(\sqrt{N}\alpha_0)\Omega,\frac{\hat{\boldsymbol{A}}_\kappa}{\sqrt{N}}W(\sqrt{N}\alpha_0)\Omega\rangle_{\mathcal{F}_p}\nabla\varphi_0\rangle \tag{368}$$

$$+\langle\varphi_0,\langle W(\sqrt{N}\alpha_0)\Omega,\frac{\hat{\boldsymbol{A}}_\kappa^2}{N}W(\sqrt{N}\alpha_0)\Omega\rangle_{\mathcal{F}_p}\varphi_0\rangle \tag{369}$$

$$+\frac{N(N-1)}{2N^2}\langle\varphi_0,\left(v\star|\varphi_0|^2\right)\varphi_0\rangle \tag{370}$$

$$+\langle W(\sqrt{N}\alpha_0)\Omega,\frac{H_f}{N}W(\sqrt{N}\alpha_0)\Omega\rangle_{\mathcal{F}_p}. \tag{371}$$

Lemma V.3 gives

$$\langle\!\langle \Psi_{N0}, \frac{H_m^N}{N}\Psi_{N0}\rangle\!\rangle = ||(-i\nabla - \boldsymbol{A}_\kappa)\,\varphi_0||^2 + \frac{1}{2}\langle\varphi_0, \left(v\star|\varphi_0|^2\right)\varphi_0\rangle + \mathcal{E}_f \tag{372}$$

$$+\frac{\Lambda^2}{4\pi^2 N} - \frac{1}{2N}\langle\varphi_0, \left(v\star|\varphi_0|^2\right)\varphi_0\rangle \tag{373}$$

and we obtain

$$\langle\!\langle \Psi_{N0}, \frac{H_m^N}{N}\Psi_{N0}\rangle\!\rangle = \mathcal{E}_M[\varphi_0, \alpha_0] + \frac{\Lambda^2}{4\pi^2 N} - \frac{1}{2N}\langle\varphi_0, \left(v\star|\varphi_0|^2\right)\varphi_0\rangle. \tag{374}$$

(ii) The second moment of the energy per particle

Next, we are showing that the second moment of the energy per particle approximately equals the energy of the effective system squared. We split the double sum, arising from the second moment of the Many-body Hamiltonian into its diagonal and remaining part. The diagonal only consists of N constituents and has a subleading contribution for large N. On the contrary, there are N^2 elements from the off-diagonal, giving rise to $\mathcal{E}_M^2$. In order to keep track of the estimates, we break the second moment of the energy per particle as well as the effective energy squared into pieces.

$$\langle\!\langle \frac{H_m^N}{N}\Psi_{N0}, \frac{H_m^N}{N}\Psi_{N0}\rangle\!\rangle = \tag{375}$$

$$= \frac{1}{N^2}\sum_{j,k}\langle\!\langle \left(-i\nabla_j - \frac{\hat{\boldsymbol{A}}_\kappa(x_j)}{\sqrt{N}}\right)^2\Psi_{N0}, \left(-i\nabla_k - \frac{\hat{\boldsymbol{A}}_\kappa(x_k)}{\sqrt{N}}\right)^2\Psi_{N0}\rangle\!\rangle \tag{376}$$

$$+\frac{1}{4N^4}\sum_{i\neq j, k\neq l}\langle\!\langle v(x_i - x_j)\Psi_{N0}, v(x_k - x_l)\Psi_{N0}\rangle\!\rangle \tag{377}$$

$$+\frac{1}{N^2}\langle\!\langle \Psi_{N0}, H_f^2\Psi_{N0}\rangle\!\rangle \tag{378}$$

$$+\frac{1}{N^3}\sum_{j,k\neq l}\mathrm{Re}\langle\!\langle \left(-i\nabla_j - \frac{\hat{\boldsymbol{A}}_\kappa(x_j)}{\sqrt{N}}\right)^2\Psi_{N0}, v(x_k - x_l)\Psi_{N0}\rangle\!\rangle \tag{379}$$

$$+\frac{2}{N^2}\sum_{j}\mathrm{Re}\langle\!\langle \left(-i\nabla_j - \frac{\hat{\boldsymbol{A}}_\kappa(x_j)}{\sqrt{N}}\right)^2\Psi_{N0}, H_f\Psi_{N0}\rangle\!\rangle \tag{380}$$

$$+\frac{1}{N^3}\sum_{j\neq k}\mathrm{Re}\langle\!\langle v(x_j - x_k)\Psi_{N0}, H_f\Psi_{N0}\rangle\!\rangle, \tag{381}$$

and

$$\mathcal{E}_M^2[\varphi_0, \alpha_0] = \langle\varphi_0, (-i\nabla - \boldsymbol{A}_\kappa)^2\,\varphi_0\rangle^2 \tag{382}$$

$$+1/4\langle\varphi_0, (v\star|\varphi_0|^2)\varphi_0\rangle^2 \tag{383}$$

$$+\mathcal{E}_f^2 \tag{384}$$

$$+\langle\varphi_0, \left((-i\nabla - \boldsymbol{A}_\kappa)^2\right)\varphi_0\rangle\langle\varphi_0, (v\star|\varphi_0|^2)\varphi_0\rangle \tag{385}$$

$$+2\langle\varphi_0, (-i\nabla - \boldsymbol{A}_\kappa)^2\,\varphi_0\rangle\mathcal{E}_f \tag{386}$$

$$+\langle\varphi_0, (v\star|\varphi_0|^2)\varphi_0\rangle\mathcal{E}_f. \tag{387}$$

In the following, we estimate the difference of the corresponding expressions. As a result we obtain

$$|\langle\!\langle \frac{H_m^N}{N}\Psi_{N0}, \frac{H_m^N}{N}\Psi_{N0}\rangle\!\rangle - \mathcal{E}_M^2[\varphi_0,\alpha_0]| \leq C\Lambda^4 N^{-1}. \tag{388}$$

$|(376) - (382)| \leq C\Lambda^4/N$:

The off-diagonal part of (376) is given by

$$\langle\!\langle \left(-i\nabla_1 - \frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)^2 \Psi_{N0}, \left(-i\nabla_2 - \frac{\hat{\boldsymbol{A}}_\kappa(x_2)}{\sqrt{N}}\right)^2 \Psi_{N0}\rangle\!\rangle \tag{389}$$

$$= \langle\!\langle (-\Delta_1)\,\Psi_{N0}, (-\Delta_2)\,\Psi_{N0}\rangle\!\rangle \tag{390}$$

$$+2i\langle\!\langle (-\Delta_1)\,\Psi_{N0}, \frac{\hat{\boldsymbol{A}}_\kappa(x_2)}{\sqrt{N}}\nabla_2\Psi_{N0}\rangle\!\rangle + 2i\langle\!\langle \frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\nabla_1\Psi_{N0}, (-\Delta_2)\,\Psi_{N0}\rangle\!\rangle \tag{391}$$

$$+\langle\!\langle (-\Delta_1)\,\Psi_{N0}, \frac{\hat{\boldsymbol{A}}^2_\kappa(x_2)}{N}\Psi_{N0}\rangle\!\rangle + \langle\!\langle \frac{\hat{\boldsymbol{A}}^2_\kappa(x_1)}{N}\Psi_{N0}, (-\Delta_2)\,\Psi_{N0}\rangle\!\rangle \tag{392}$$

$$-4\langle\!\langle \frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\nabla_1\Psi_{N0}, \frac{\hat{\boldsymbol{A}}_\kappa(x_2)}{\sqrt{N}}\nabla_2\Psi_{N0}\rangle\!\rangle \tag{393}$$

$$+2i\langle\!\langle \frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\nabla_1\Psi_{N0}, \frac{\hat{\boldsymbol{A}}^2_\kappa(x_2)}{N}\Psi_{N0}\rangle\!\rangle + 2i\langle\!\langle \frac{\hat{\boldsymbol{A}}^2_\kappa(x_1)}{N}\Psi_{N0}, \frac{\hat{\boldsymbol{A}}_\kappa(x_2)}{\sqrt{N}}\nabla_2\Psi_{N0}\rangle\!\rangle \tag{394}$$

$$+\langle\!\langle \frac{\hat{\boldsymbol{A}}^2_\kappa(x_1)}{N}\Psi_{N0}, \frac{\hat{\boldsymbol{A}}^2_\kappa(x_2)}{N}\Psi_{N0}\rangle\!\rangle. \tag{395}$$

By means of Lemma V.3 we have

$$(390) + (391) = \langle\varphi_0, (-\Delta)\,\varphi_0\rangle^2 + 4i\langle \boldsymbol{A}_\kappa\varphi_0, \nabla\varphi_0\rangle\langle\varphi_0, (-\Delta)\,\varphi_0\rangle, \tag{396}$$

$$(392) = 2\langle\varphi_0, \boldsymbol{A}_\kappa^2\varphi_0\rangle\langle\varphi_0, (-\Delta)\,\varphi_0\rangle + \frac{\Lambda^2}{2\pi^2 N}\left|\left|\nabla\varphi_0\right|\right|^2, \tag{397}$$

$$(393) = -4\langle\varphi_0, \boldsymbol{A}_\kappa\nabla\varphi_0\rangle^2 - \tag{398}$$

$$-\frac{2}{N}\int d^3x \int d^3y \varphi_0^*(x)\varphi_0^*(y)\gamma_{kl}^{\perp\Lambda}(x-y)\left(\nabla^k\varphi_0\right)(x)\left(\nabla^l\varphi_0\right)(y). \tag{399}$$

In order to deal with the last two lines, we use the relations

$$\langle W(\sqrt{N}\alpha_0)\Omega, \frac{\hat{\boldsymbol{A}}^2_\kappa(x)\hat{\boldsymbol{A}}^i_\kappa(y)}{N^{3/2}}W(\sqrt{N}\alpha_0)\Omega\rangle_{\mathcal{F}_p} = \boldsymbol{A}^2_\kappa(x)\boldsymbol{A}^i_\kappa(y) + \frac{1}{N}\frac{\Lambda^2}{4\pi^2}\boldsymbol{A}^i_\kappa(y) \tag{400}$$

$$+\frac{1}{N}\sum_{j=1}^{3}\gamma_{ij}^{\perp\Lambda}(x-y)\hat{\boldsymbol{A}}^j_\kappa(x), \tag{401}$$

and

$$\langle W(\sqrt{N}\alpha_0)\Omega, \frac{\hat{\boldsymbol{A}}_\kappa^2(x)\hat{\boldsymbol{A}}_\kappa^2(y)}{N^2} W(\sqrt{N}\alpha_0)\Omega\rangle_{\mathcal{F}_p} = \boldsymbol{A}_\kappa^2(x)\boldsymbol{A}_\kappa^2(y)+ \tag{402}$$

$$+\frac{1}{N}\frac{\Lambda^2}{4\pi^2}\left(\boldsymbol{A}_\kappa^2(x)+\boldsymbol{A}_\kappa^2(y)\right)+\frac{2}{N}\sum_{k,l=1}^{3}\gamma_{kl}^{\perp\Lambda}(x-y)\boldsymbol{A}_\kappa^k(x)\boldsymbol{A}_\kappa^l(y) \tag{403}$$

$$+\frac{1}{N^2}\left(\sum_{k,l=1}^{3}|\gamma_{kl}^{\perp\Lambda}(x-y)|^2+\frac{\Lambda^4}{(2\pi)^4}\right), \tag{404}$$

which can also be obtained by the canonical commutation relations (9) and Lemma V.2. Consequently, we have

$$(394) = 4i\langle\varphi_0, \boldsymbol{A}_\kappa^2\varphi_0\rangle\langle\varphi_0, \boldsymbol{A}_\kappa\nabla\varphi_0\rangle \tag{405}$$

$$+\frac{4i}{N}\left(\frac{\Lambda^2}{4\pi^2}\langle\varphi_0, \boldsymbol{A}_\kappa\nabla\varphi_0\rangle+\int d^3x\int d^3y\varphi_0^*(x)\varphi_0^*(y)\gamma_{kl}^{\perp\Lambda}\boldsymbol{A}_\kappa^k(x)\varphi(x)\left(\nabla^l\varphi\right)(y)\right), \tag{406}$$

$$(395) = \langle\varphi_0, \boldsymbol{A}_\kappa^2\varphi_0\rangle^2+\frac{2}{N}\frac{\Lambda^4}{4\pi^2}\langle\varphi_0, \boldsymbol{A}_\kappa^2\varphi_0\rangle+ \tag{407}$$

$$+\frac{2}{N}\int d^3x\int d^3y\varphi_0^*(x)\varphi_0^*(y)\gamma_{kl}^{\perp\Lambda}(x-y)\boldsymbol{A}_\kappa^k(x)\boldsymbol{A}_\kappa^l(y)\varphi(x)\varphi(y) \tag{408}$$

$$+\frac{1}{N^2}\left(\frac{\Lambda^4}{(2\pi)^4}+\int d^3x\int d^3y\varphi_0^*(x)\varphi_0^*(y)\sum_{k,l}|\gamma_{k,l}^{\perp\Lambda}(x-y)|^2\varphi(x)\varphi(y)\right). \tag{409}$$

and

$$|(389)-\langle\varphi_0, \left(-i\nabla-\boldsymbol{A}_\kappa\right)^2\varphi_0\rangle^2| \leq C\frac{\Lambda^4}{N} \tag{410}$$

because all error terms are bounded by $C\Lambda^4/N$ under the assumptions of Lemma V.1. Since the diagonal part of (376) is of order N^{-1}, this implies

$$|(376)-(382)| \leq C\frac{\Lambda^4}{N}. \tag{411}$$

$|(377)-(383)| \leq C/N$:

Due to symmetry and $v(-x)=v(x)$ line (377) reads

$$\frac{1}{4N^4}\sum_{i\neq j,k\neq l}\langle\!\langle v(x_i-x_j)\Psi_{N0}, v(x_k-x_l)\Psi_{N0}\rangle\!\rangle \tag{412}$$

$$=\frac{1}{4}\langle\!\langle v(x_1-x_2)\Psi_{N0}, v(x_3-x_4)\Psi_{N0}\rangle\!\rangle \tag{413}$$

$$-\frac{6N^2-11N+6}{N^3}\langle\!\langle v(x_1-x_2)\Psi_{N0}, v(x_3-x_4)\Psi_{N0}\rangle\!\rangle \tag{414}$$

$$+\frac{N-1}{2N^3}\langle\!\langle v(x_1-x_2)\Psi_{N0}, v(x_1-x_2)\Psi_{N0}\rangle\!\rangle \tag{415}$$

$$+\frac{(N-1)(N-2)}{N^3}\langle\!\langle v(x_1-x_2)\Psi_{N0}, v(x_1-x_3)\Psi_{N0}\rangle\!\rangle. \tag{416}$$

The product structure of the initial state gives

$$\langle\!\langle v(x_1-x_2)\Psi_{N0}, v(x_3-x_4)\Psi_{N0}\rangle\!\rangle = \langle \varphi_0, \left(v\star|\varphi_0|^2\right)\varphi_0\rangle^2, \tag{417}$$

$$||v(x_1-x_2)\Psi_{N0}||^2 = \langle \varphi_0, \left(v^2\star|\varphi_0|^2\right)\varphi_0\rangle \tag{418}$$

and we conclude

$$|\frac{1}{4N^4}\sum_{i\neq j,k\neq l}\langle\!\langle v(x_i-x_j)\Psi_{N0}, v(x_k-x_l)\Psi_{N0}\rangle\!\rangle - \frac{1}{4}\langle \varphi_0, \left(v\star|\varphi_0|^2\right)\varphi_0\rangle^2| \leq \tag{419}$$

$$\leq\frac{6}{N}|\langle\!\langle v(x_1-x_2)\Psi_{N0}, v(x_3-x_4)\Psi_{N0}\rangle\!\rangle| + \frac{1}{N}\,||v(x_1-x_2)\Psi_{N0}||^2 \tag{420}$$

$$+\frac{1}{N}|\langle\!\langle v(x_1-x_2)\Psi_{N0}, v(x_1-x_3)\Psi_{N0}\rangle\!\rangle| \tag{421}$$

$$\leq\frac{8}{N}\,||v(x_1-x_2)\Psi_{N0}||^2 = \frac{8}{N}\langle \varphi_0, \left(v^2\star|\varphi_0|^2\right)\varphi_0\rangle. \tag{422}$$

$|(378)-(384)| \leq C/N$:

This bound results from Lemma V.3 because

$$\frac{1}{N^2}\langle\!\langle \Psi_{N0}, H_f^2\Psi_{N0}\rangle\!\rangle = \frac{1}{N^2}\langle W(\sqrt{N}\alpha_0)\Omega, H_f^2 W(\sqrt{N}\alpha_0)\Omega\rangle_{\mathcal{F}_p} = \mathcal{E}_f^2 + \frac{1}{N}\mathcal{E}_{f^2}. \tag{423}$$

$|(379)-(385)| \leq C\Lambda^2/N$:

Line (379) simplifies to

$$\frac{1}{N^3}\sum_{j,k\neq l}\mathrm{Re}\langle\!\langle \left(-i\nabla_j - \frac{\hat{\boldsymbol{A}}_\kappa(x_j)}{\sqrt{N}}\right)^2\Psi_{N0}, v(x_k-x_l)\Psi_{N0}\rangle\!\rangle \tag{424}$$

$$=\frac{(N-1)(N-2)}{N^2}\mathrm{Re}\langle\!\langle \left(-i\nabla_1 - \frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)^2\Psi_{N0}, v(x_2-x_3)\Psi_{N0}\rangle\!\rangle \tag{425}$$

$$+\frac{2(N-1)}{N^2}\mathrm{Re}\langle\!\langle \left(-i\nabla_1 - \frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)^2\Psi_{N0}, v(x_2-x_1)\Psi_{N0}\rangle\!\rangle \tag{426}$$

$$=\left(1-\frac{3(N-2)}{N^2}\right)\langle\varphi_0, \left(-i\nabla - \boldsymbol{A}_\kappa\right)^2\varphi_0\rangle\langle \varphi_0, \left(v\star|\varphi_0|^2\right)\varphi_0\rangle \tag{427}$$

$$+\frac{(N-1)(N-2)\Lambda^2}{4\pi^2N^3}\langle \varphi_0, \left(v\star|\varphi_0|^2\right)\varphi_0\rangle \tag{428}$$

$$+\frac{2(N-1)}{N^2}\mathrm{Re}\langle\!\langle \left(-i\nabla_1 - \frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)^2\Psi_{N0}, v(x_2-x_1)\Psi_{N0}\rangle\!\rangle. \tag{429}$$

Consequently the estimate follows because $||v(x_1-x_2)\Psi_{N0}|| = \langle\varphi_0, (v\star|\varphi_0|^2)\,\varphi_0\rangle$ and $\left|\left|\left(-i\nabla_1 - \frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)^2\Psi_{N0}\right|\right|$ are finite under the assumptions of Lemma V.1.

$|(380) - (386)| \leq C\Lambda^2/N$**:**

Similar to the previous calculations we obtain

$$\frac{2}{N^2}\sum_{j=1}^{N}\mathrm{Re}\langle\!\langle\left(-i\nabla_j - \frac{\hat{\boldsymbol{A}}_\kappa(x_j)}{\sqrt{N}}\right)^2\Psi_{N0}, H_f\Psi_{N0}\rangle\!\rangle \tag{430}$$

$$=2\mathrm{Re}\langle\!\langle\left(-\Delta_1 + 2i\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}} + \frac{\hat{\boldsymbol{A}}_\kappa^2(x_1)}{N} + V_{ex}(x_1)\right)\Psi_{N0}, \frac{H_f}{N}\Psi_{N0}\rangle\!\rangle \tag{431}$$

$$=2\langle\varphi_0, (-i\nabla - \boldsymbol{A}_\kappa)^2\varphi_0\rangle\mathcal{E}_f \tag{432}$$

$$+\frac{2}{N}\mathrm{Re}\left(\frac{\Lambda^2}{4\pi^2}\mathcal{E}_f - 4\langle\nabla\varphi_0, \boldsymbol{E}_\kappa^+(x)\varphi_0\rangle - 2i\langle\boldsymbol{A}_\kappa\varphi_0, \boldsymbol{E}_\kappa^+\varphi_0\rangle\right). \tag{433}$$

By means of

$$|\langle\boldsymbol{A}_\kappa\varphi_0, \boldsymbol{E}_\kappa^+\varphi\rangle| \leq ||\boldsymbol{A}_\kappa||_\infty\, ||\varphi_0||_\infty\, \left|\left|\boldsymbol{E}_\kappa^+\right|\right|, \tag{434}$$

$$|\langle\nabla\varphi_0, \boldsymbol{E}_\kappa^+\varphi_0\rangle| \leq ||\nabla\varphi_0||\, ||\varphi_0||_\infty\, \left|\left|\boldsymbol{E}_\kappa^+\right|\right|, \tag{435}$$

and

$$\left|\left|\boldsymbol{E}_\kappa^+\right|\right|_2^2 = \frac{1}{2}\sum_{\lambda=1,2}\int_{|k|\leq\Lambda} d^3k|k||\alpha_0(k,\lambda)|^2 \leq \mathcal{E}_f \tag{436}$$

the inequality follows.

$|(381) - (387)| \leq C/N$**:**

Making use of symmetry and Lemma V.3 one has

$$\frac{1}{N^3}\sum_{j\neq k}\mathrm{Re}\langle\!\langle v(x_k - x_k)\Psi_{N0}, H_f\Psi_{N0}\rangle\!\rangle = \frac{N-1}{N}\langle\!\langle v(x_1 - x_2)\Psi_{N0}, \frac{H_f}{N}\Psi_{N0}\rangle\!\rangle \tag{437}$$

$$=\left(1 - \frac{1}{N}\right)\langle\varphi_0, \left(v\star|\varphi_0|^2\right)\varphi_0\rangle\mathcal{E}_f. \tag{438}$$

This shows the last inequality and altogether we obtain

$$|\langle\!\langle\frac{H_m^N}{N}\Psi_{N0}, \frac{H_m^N}{N}\Psi_{N0}\rangle\!\rangle - \mathcal{E}_M^2[\varphi_0, \alpha_0]| \leq C\Lambda^4 N^{-1}, \tag{439}$$

which proves Lemma V.1. □

VI Appendix

Lemma VI.1. *Let* $\varphi_t \in H^3(\mathbb{R}^3)$ *and* $\Psi_N \in \left(L_s^2\left(\mathbb{R}^{3N}\right) \otimes \mathcal{F}_p\right) \cap \mathcal{D}(H_m^N)$*. For the operator* $\hat{n} := \left(\sum_{j=1}^N q_j\right)^{1/2}$*, its inverse* $\hat{n}^{-1}$ *and* $\hat{\boldsymbol{G}} \in \left\{\boldsymbol{\mathcal{A}}^+, \boldsymbol{\mathcal{A}}^-, \boldsymbol{\mathcal{E}}^+, \boldsymbol{\mathcal{E}}^-\right\}$ *one has*

$$\left|\left|\hat{\boldsymbol{G}}(x_1,t)p_1\Psi_N\right|\right|^2 \leq C\langle\langle\hat{\boldsymbol{G}}(y,t)\Psi_N, \hat{\boldsymbol{G}}(y,t)\Psi_N\rangle\rangle_{;y}, \tag{440}$$

$$\left|\left|\hat{\boldsymbol{G}}(x_1,t)\nabla_1 p_1\Psi_N\right|\right|^2 \leq C\langle\langle\hat{\boldsymbol{G}}(y,t)\Psi_N, \hat{\boldsymbol{G}}(y,t)\Psi_N\rangle\rangle_{;y}, \tag{441}$$

$$\left|\left|\hat{n}^{-1}\hat{\boldsymbol{G}}(y,t)q_1\Psi_N\right|\right|^2_{;y} = \langle\langle\hat{\boldsymbol{G}}(y,t)\Psi_N, \hat{\boldsymbol{G}}(y,t)\Psi_N\rangle\rangle_{;y}, \tag{442}$$

$$\left|\left|\hat{\boldsymbol{G}}(y,t)\nabla_1 p_1\Psi_N\right|\right|^2 \leq C\langle\langle\hat{\boldsymbol{G}}(y,t)\Psi_N, \hat{\boldsymbol{G}}(y,t)\Psi_N\rangle\rangle_{;y} \tag{443}$$

Moreover,

$$\langle\langle\Psi_N, \boldsymbol{\mathcal{E}}^-(y,t)\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle\rangle_{;y} \leq \beta^b. \tag{444}$$

Proof. First, we show the second inequality

$$\left|\left|\hat{\boldsymbol{G}}(x_1,t)\nabla_1 p_1\Psi_N\right|\right|^2 = \langle\langle\hat{\boldsymbol{G}}(x_1,t)\cdot\nabla_1 p_1\Psi_N, \hat{\boldsymbol{G}}(x_1,t)\cdot\nabla_1 p_1\Psi_N\rangle\rangle \tag{445}$$

$$= \sum_{i,l=1}^3 \langle\langle\hat{\boldsymbol{G}}^i(x_1,t)\nabla_1^i p_1\Psi_N, \hat{\boldsymbol{G}}^l(x_1,t)\nabla_1^l p_1\Psi_N\rangle\rangle \tag{446}$$

$$= \sum_{i,l=1}^3 \langle\langle\Psi_N, p_1\left(-\nabla_1^i\right)\left(\hat{\boldsymbol{G}}^i(x_1,t)\right)^*\hat{\boldsymbol{G}}^l(x_1,t)\nabla_1^l p_1\Psi_N\rangle\rangle. \tag{447}$$

Making use of

$$p_1\left(-\nabla_1^i\right)\left(\hat{\boldsymbol{G}}^i(x_1,t)\right)^*\hat{\boldsymbol{G}}^l(x_1,t)\nabla_1^l p_1 \tag{448}$$

$$= p_1\int d^3y\varphi^*(y)\left(-\nabla_y^i\right)\left(\hat{\boldsymbol{G}}^i(y,t)\right)^*\hat{\boldsymbol{G}}^l(y,t)\left(\nabla^l\varphi\right)(y) \tag{449}$$

$$= p_1\int d^3y\left(\nabla^i\varphi\right)^*(y)\left(\hat{\boldsymbol{G}}^i(y,t)\right)^*\hat{\boldsymbol{G}}^l(y,t)\left(\nabla^l\varphi\right)(y) \tag{450}$$

one has

$$\left|\left|\hat{\boldsymbol{G}}(x_1,t)\nabla_1 p_1\Psi_N\right|\right|^2 \tag{451}$$

$$= \sum_{i,l=1}^3\int d^3y\left(\nabla^i\varphi\right)^*(y)\left(\nabla^l\varphi\right)(y)\langle\langle\Psi_N, p_1\left(\hat{\boldsymbol{G}}^i(y,t)\right)^*\hat{\boldsymbol{G}}^l(y,t)\Psi_N\rangle\rangle \tag{452}$$

$$\leq \sum_{i,l=1}^3\int d^3y|\nabla^i\varphi||\nabla^l\varphi||\langle\langle\hat{\boldsymbol{G}}^i(y,t)\Psi_N, \hat{\boldsymbol{G}}^l(y,t)\Psi_N\rangle\rangle| \tag{453}$$

$$\leq \sum_{i,l=1}^3\int d^3y\frac{1}{2}\left(||\nabla\varphi||_\infty^2\left|\left|\hat{\boldsymbol{G}}^l(y,t)\Psi_N\right|\right|^2 + ||\nabla\varphi||_\infty^2\left|\left|\hat{\boldsymbol{G}}^i(y,t)\Psi_N\right|\right|^2\right) \tag{454}$$

$$\leq 3\,||\nabla\varphi||_\infty^2\,\langle\langle\hat{\boldsymbol{G}}(y,t)\Psi_N, \hat{\boldsymbol{G}}(y,t)\Psi_N\rangle\rangle_{;y}. \tag{455}$$

The third relation follows from

$$\left|\left|\hat{n}^{-1}\hat{\boldsymbol{G}}(y,t)q_1\Psi_N\right|\right|^2_{;y} =\langle\!\langle\hat{\boldsymbol{G}}(y,t)\Psi_N,\hat{n}^{-2}q_1\hat{\boldsymbol{G}}(y,t)\Psi_N\rangle\!\rangle_{;y} \tag{456}$$

$$=\langle\!\langle\hat{\boldsymbol{G}}(y,t)\Psi_N,\hat{n}^{-2}\left(N^{-1}\sum_{j=1}^{N}q_j\right)\hat{\boldsymbol{G}}(y,t)\Psi_N\rangle\!\rangle_{;y} \tag{457}$$

$$=\langle\!\langle\hat{\boldsymbol{G}}(y,t)\Psi_N,\hat{\boldsymbol{G}}(y,t)\Psi_N\rangle\!\rangle_{;y}. \tag{458}$$

The first and fourth inequality are obtained analogously. Moreover, we have

$$\langle\!\langle\Psi_N,\boldsymbol{\mathcal{E}}^-(y,t)\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle\!\rangle_{;y} = \frac{1}{2}\sum_{\lambda=1,2}\int d^3k\tilde{\kappa}(k)|k|^{1/2}\boldsymbol{\epsilon}_\lambda(k)\sum_{\mu=1,2}\int d^3l\tilde{\kappa}(l)\times \tag{459}$$

$$|l|^{1/2}\boldsymbol{\epsilon}_\mu(l)\langle\!\langle\left(\frac{a(k,\lambda)}{\sqrt{N}}-\alpha_t(k,\lambda)\right)\Psi_N,\left(\frac{a(l,\mu)}{\sqrt{N}}-\alpha_t(l,\mu)\right)\Psi_N\rangle\!\rangle\int d^3ye^{i(l-k)y} \tag{460}$$

$$=\frac{(2\pi)^3}{2}\int d^3l|\tilde{\kappa}(k)|^2|k|\sum_{\lambda,\mu}\boldsymbol{\epsilon}_\lambda(k)\boldsymbol{\epsilon}_\mu(k)\times \tag{461}$$

$$\langle\!\langle\left(\frac{a(k,\lambda)}{\sqrt{N}}-\alpha_t(k,\lambda)\right)\Psi_N,\left(\frac{a(k,\lambda)}{\sqrt{N}}-\alpha_t(k,\lambda)\right)\Psi_N\rangle\!\rangle \tag{462}$$

$$=\frac{1}{2}\sum_{\lambda=1,2}\int_{|k|\leq\Lambda}|k|\langle\!\langle\left(\frac{a(k,\lambda)}{\sqrt{N}}-\alpha_t(k,\lambda)\right)\Psi_N,\left(\frac{a(k,\lambda)}{\sqrt{N}}-\alpha_t(k,\lambda)\right)\Psi_N\rangle\!\rangle\leq\beta^b. \tag{463}$$

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