ON r-GAPS BETWEEN ZEROS OF THE RIEMANN ZETA-FUNCTION

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Abstract. Under the Riemann Hypothesis, we prove for any natural number r there exist infinitely many large natural numbers n such that $(\gamma_{n+r} - \gamma_n)/(2\pi/\log\gamma_n) > r+\Theta\sqrt{r}$ and $(\gamma_{n+r}-\gamma_n)/(2\pi/\log\gamma_n) < r-\vartheta\sqrt{r}$ for explicit absolute positive constants Θ and ϑ, where γ denotes an ordinate of a zero of the Riemann zeta-function on the critical line. Selberg published announcements of this result several times but did not include a proof. We also suggest a general framework which might lead to stronger statements concerning the vertical distribution of nontrivial zeros of the Riemann zeta-function.

1. Introduction

Let $\zeta(s)$ denote the Riemann zeta-function, and let $\rho = \beta+i\gamma$ denote a nontrivial zero of $\zeta(s)$. Consider the sequence of ordinates of zeros in the upper half plane

$$0 < \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_n \leq \gamma_{n+1} \leq \ldots .$$

It is well known that

$$N(T) := \sum_{|\gamma|\leq T} 1 \sim \frac{T}{2\pi}\log T,$$

from which it follows that the average gap between consecutive zeros is $2\pi/\log\gamma_n$. Assuming the Riemann Hypothesis, $\beta = 1/2$ and $\gamma \in \mathbb{R}$. The main result of this article is a proof of the following theorem.

Theorem. *Assuming the Riemann Hypothesis, for any natural number r there exist infinitely many large n such that*

$$\frac{\gamma_{n+r} - \gamma_n}{2\pi/\log\gamma_n} > r + \Theta\sqrt{r}$$

and

$$\frac{\gamma_{n+r} - \gamma_n}{2\pi/\log\gamma_n} < r - \vartheta\sqrt{r}$$

for the absolute positive constants $\Theta = 0.570717$ and $\vartheta = 0.359222$. Moreover, for r sufficiently large, we may take $\Theta = 0.906498$ and $\vartheta = 0.640989$.

There are discrepancies in the literature regarding the correct statement of this result, which we hope to now clarify. Selberg first stated this result with no proof in [Sel47, p. 199], however there the result appears with the factor $\sqrt{r}$ missing. While the correct $\Theta\sqrt{r}$, $-\vartheta\sqrt{r}$ terms appear in the Acknowledgements section of [Mue82] (there stated as a single absolute constant for both quantities), the statement there

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is missing the assumption of the Riemann Hypothesis. In the errata of Volume 1 of his collected papers [Sel89, p. 355], the statement appearing in Theorem 1 is given with an unspecified positive absolute constant. A proof of Theorem 1 is given by Heath-Brown in [Tit86, p. 246-249] for $r = 1$ using a different method then the one used here. To our knowledge, our proof is the first to appear in the literature for $r > 1$.

For a fixed, positive integer r, let

$$\lambda_r := \limsup_{n\to\infty} \frac{\gamma_{n+r} - \gamma_n}{2\pi/\log\gamma_n} \qquad \text{and} \qquad \mu_r := \liminf_{n\to\infty} \frac{\gamma_{n+r} - \gamma_n}{2\pi/\log\gamma_n}. \tag{1.1}$$

By definition $\lambda_r \geq r$ and similarly $\mu_r \leq r$, however random matrix theory predicts that $\lambda_r = \infty$ and $\mu_r = 0$. To prove Theorem 1, we adapt a method developed by Conrey, Ghosh, and Gonek [CGG84] on gaps between consecutive nontrivial zeros of $\zeta(s)$ in the interval $[0, T]$ for T large. In the method, one compares averages of a well-chosen polynomial of the form

$$A(t) := \sum_{n\leq X} \frac{a^{\pm}(n)}{n^{it}},$$

where $X = T^{1-\delta}$ for some small $\delta > 0$. To adapt for r-gaps, we set

$$M_1 := \int_T^{2T} |A(t)|^2 \, dt$$

and

$$M_2(c_r) := \int_{-\pi c_r/\log T}^{\pi c_r/\log T} \sum_{T\leq\gamma\leq 2T} |A(\gamma+\alpha)|^2 \, d\alpha,$$

where c_r is some nonzero real number. We see that $M_2(c_r)$ is monotonically increasing and

$$M_2(\mu_r) \leq rM_1 \leq M_2(\lambda_r).$$

Therefore, if $M_2(c_r) < rM_1$ for some choice of $a^+(n)$ and c_r then $\lambda_r > c_r$. Similarly, if $M_2(c_r) > rM_1$ for some choice of $a^-(n)$ and c_r then $\mu_r < c_r$.

Connecting their work to a previous result of Montgomery and Odlyzko [MO84], Conrey, Ghosh, and Gonek show

$$\frac{M_2(c_r)}{M_1} = h^{\pm}(c_r) + o(1),$$

where $h(c_r)$ is defined by

$$h^{\pm}(c_r) := c_r \mp \frac{\Re\left(\displaystyle\sum_{kn\leq X} \frac{a^{\pm}(n)\overline{a^{\pm}(kn)}g_{c_r}(k)\Lambda(k)}{kn}\right)}{\displaystyle\sum_{n\leq X}\frac{|a^{\pm}(n)|^2}{n}} \tag{1.2}$$

and

$$g_{c_r}(k) = \frac{2\sin\left(\pi c_r \frac{\log k}{\log T}\right)}{\pi\log k}$$

so that $|g_{c_r}(k)| \leq 2c_r/\log T$. The function $h^{\pm}(c_r)$ was introduced by Montgomery and Odlyzko to study gaps between consecutive zeros of $\zeta(s)$. In particular, they show that if one is able to find c_r such that $h^+(c_r) < r$ then $\lambda_r > c_r$ and such that if $h^-(c_r) > r$ then $\mu_r < c_r$.

Letting $r = 1$ in (1.1), it follows from Theorem 1 that $\lambda_1 > 1$ and $\mu_1 < 1$. Quantitative bounds on λ_1 and μ_1 have been obtained using the above approach, with different choices of $a(n)$ leading to improved results. See [BMN10] and subsequently [FW12] for discussions of these choices. The best current quantitative bounds concerning gaps between consecutive zeros of the Riemann zeta function (under the assumption of the Riemann Hypothesis) are $\lambda_1 > 3.18$, due to Bui and Milinovich [BM16] , and $\mu_1 < 0.515396$, due to Preobrazhenskii [Pre16]. We note that the method employed in [BM16], which is based on the work of Hall [Hal99] and different from the method discussed above, is unconditional if one restricts the analysis to critical zeros.

1.1. **Outline of article.** In Section 2 we prove Theorem 1. In Section 3 we offer a general statement, from which Theorem 1 follows, that lead to a Selberg-like theorem on r-gaps where $\Theta\sqrt{r}$ (or $-\vartheta\sqrt{r}$) is replaced by $\Theta\sqrt{r}g(r)$ (or $-\vartheta\sqrt{r}g(r)$), where $g(r) \to \infty$ as $r \to \infty$. This latter endeavor is work in progress.

2. Proof of Theorem 1

For large gaps for any fixed $r \geq 1$, we choose $a(n) = d_\ell(n)$, where d_ℓ is multiplicative and defined on prime powers by

$$d_\ell(p^m) = \frac{\Gamma(m+\ell)}{\Gamma(\ell)m!}.$$

We fix $\ell \geq 1$ and will ultimately choose $\ell = b\sqrt{r}$ for a constant $b > 1$ to be made explicit in the proof. Similarly, for small gaps for any fixed $r \geq 1$, we choose $a(n) = \lambda(n)d_\ell(n)$, where $\lambda(n)$ denotes the Liouville function.

2.1. **Large gaps for any fixed** $r \geq 1$**.** Take $a^+(n) = d_\ell(n)$ for $\ell \geq 1$ an integer to be determined later. In this case the relevant mean-value to compute is well known:

$$\sum_{n \leq x} \frac{d_\ell(n)^2}{n} = C_r(\log x)^{r^2} + O((\log T)^{r^2-1})$$

for fixed $\ell \geq 1$, uniformly for $x \leq T$, where C_r is a constant which will not have an effect in our application. It is shown in [CGG84, p.422] that for this choice of $a(n)$, the equation $M_2(c_r)/M_1 = h^+(c_r) + o(1)$ reduces to

$$h^+(c_r) = c_r - 2\ell \int_0^1 \frac{\sin(\pi c_r v(1-\delta))}{\pi v}(1-v)^{\ell^2}\, dv + O(1/\log T). \tag{2.1}$$

To detect large gaps of the desired size, we must show that $h^+(c_r) < r$ for fixed $r \geq 1$ and $c_r = r + \Theta\sqrt{r}$ with $\Theta > 0$. We estimate the integral as follows. Let

$$\int_0^1 \frac{\sin(\pi c_r(1-\delta)v)}{\pi v}(1-v)^{\ell^2}\, dv = I_1 + I_2,$$

where

$$I_1 := \int_0^{1/c_r} \frac{\sin(\pi c_r(1-\delta)v)}{\pi v}(1-v)^{\ell^2}\, dv$$

and

$$I_2 := \int_{1/c_r}^1 \frac{\sin(\pi c_r(1-\delta)v)}{\pi v}(1-v)^{\ell^2}\, dv.$$

For I_1, we first observe that the integrand is positive in the range of integration and write

$$I_1 \geq I_{1,a} + I_{1,b}, \tag{2.2}$$

say, where

$$I_{1,a} := \int_0^{1/4c_r} \frac{\sin(\pi c_r(1-\delta)v)}{\pi v}(1-v)^{\ell^2}\,dv,$$
$$I_{1,b} := \int_{1/4c_r}^{1/2c_r} \frac{\sin(\pi c_r(1-\delta)v)}{\pi v}(1-v)^{\ell^2}\,dv,$$

and we have discarded the portion of the integral from $1/(2c_r)$ to $1/c_r$. Now we estimate $I_{1,a}$ and $I_{1,b}$. For $I_{1,a}$, we compare $\sin(\pi c_r(1-\delta)v)$ to $2\sqrt{2}c_r(1-\delta)v$ and find

$$I_{1,a} \geq \int_0^{1/4c_r} \frac{2\sqrt{2}c_r(1-\delta)v}{\pi v}(1-v)^{\ell^2}\,dv = \frac{2\sqrt{2}c_r(1-\delta)}{\pi(\ell^2+1)}\left(1-\left(1-\frac{1}{4c_r}\right)^{\ell^2+1}\right). \tag{2.3}$$

Similarly for $I_{1,b}$, we compare $\sin(\pi c_r(1-\delta)v)$ to $(4-2\sqrt{2})c_r(1-\delta)v$ and find

$$I_{1,b} \geq \frac{(4-2\sqrt{2})c_r(1-\delta)}{\pi(\ell^2+1)}\left(\left(1-\frac{1}{2c_r}\right)^{\ell^2+1} - \left(1-\frac{1}{4c_r}\right)^{\ell^2+1}\right). \tag{2.4}$$

By (2.2), (2.3), and (2.4), we find

$$I_1 \geq \frac{2c_r(1-\delta)}{\pi(\ell^2+1)}\left(\sqrt{2} - (2\sqrt{2}-2)\left(1-\frac{1}{4c_r}\right)^{\ell^2+1} - (2-\sqrt{2})\left(1-\frac{1}{2c_r}\right)^{\ell^2+1}\right).$$

Furthermore, since $\exp(-x) \geq 1-x$ for $x \geq 0$, we have

$$I_1 \geq \frac{2c_r(1-\delta)}{\pi(\ell^2+1)}\left\{\sqrt{2} - (2\sqrt{2}-2)\exp\left(\frac{-(\ell^2+1)}{4c_r}\right) - (2-\sqrt{2})\exp\left(\frac{-(\ell^2+1)}{2c_r}\right)\right\}. \tag{2.5}$$

We now estimate the second integral I_2. Note that

$$|I_2| \leq \frac{1}{\pi}\int_{1/c_r}^1 \frac{\exp(-\ell^2 v)}{v}\,dv \leq \frac{1}{\pi}\int_{1/c_r}^1 \frac{\exp(-\ell^2 v)}{v}\,dv.$$

Thus, by the change of variable $u = \ell^2 v$, we find

$$I_2 \geq \frac{-1}{\pi}\int_{\ell^2/c_r}^{\infty} \frac{\exp(-u)}{u}\,du. \tag{2.6}$$

Combining the estimates in (2.5) and (2.6), we have

$$\begin{aligned} h^+(c_r) \leq c_r - 2\ell\Bigg\{&\frac{2c_r(1-\delta)}{\pi(\ell^2+1)}\left(\sqrt{2} - (2\sqrt{2}-2)\exp\left(\frac{-(\ell^2+1)}{4c_r}\right)\right.\\ &\left.-(2-\sqrt{2})\exp\left(\frac{-(\ell^2+1)}{2c_r}\right)\right) - \frac{1}{\pi}\int_{\ell^2/c_r}^{\infty}\frac{\exp(-u)}{u}\,du\Bigg\} + O\left(\frac{1}{\log T}\right). \end{aligned}$$

In this case, $c_r = r + \Theta\sqrt{r}$ where $\Theta > 0$, and thus $c_r > 1$ for any $r \geq 1$. Thus, letting $\ell = \sqrt{bc_r - 1}$, where $b > 1$ is a real number that will be chosen later, we have

$$\ell = \sqrt{bc_r - 1} \geq \sqrt{br}\sqrt{1-\frac{1}{b}}$$

for any $r \geq 1$. Furthermore, since we always have $c_r > 1$, for any $r \geq 1$ we have

$$\frac{\ell^2}{c_r} > b - 1,$$

and we may again increase the length of integration in I_2 to write

$$\int_{\ell^2/c_r}^{\infty} \frac{\exp(-u)}{u}\, du < \int_{b-1}^{\infty} \frac{\exp(-u)}{u}\, du.$$

Combining these estimates, we find

$$h^+(c_r) < r + \Theta\sqrt{r} - 2\sqrt{br}\sqrt{1-\frac{1}{b}}\Bigg\{\frac{2(1-\delta)}{\pi b}\left(\sqrt{2} - (2\sqrt{2}-2)\exp\left(\frac{-b}{4}\right)\right.$$
$$\left. - (2-\sqrt{2})\exp\left(\frac{-b}{2}\right)\right) - \frac{1}{\pi}\int_{b-1}^{\infty}\frac{\exp -u}{u}\, du\Bigg\} + O\left(\frac{1}{\log T}\right).$$

To show $h^+(c_r) < r$ and prove the theorem, we set

$$\Theta = \max_b \Bigg\{ 2\sqrt{b}\sqrt{1-\frac{1}{b}}\left(\frac{2}{\pi b}\left(\sqrt{2} - (2\sqrt{2}-2)\exp\left(\frac{-b}{4}\right)\right.\right.$$
$$\left.\left. - (2-\sqrt{2})\exp\left(\frac{-b}{2}\right)\right) - \frac{1}{\pi}\int_{b-1}^{\infty}\frac{\exp -u}{u}\, du\right)\Bigg\}.$$

The choice $b = 5.3$ yields $\Theta = 0.570717$. With δ sufficiently small and T sufficiently large, these choices guarantee that $h^+(c_r) < r$, as desired.

2.2. **Small gaps for any fixed $r \geq 1$.** The proof is similar to the proof for large gaps, so we indicate the necessary changes. In this case we take $a^-(n) = \lambda(n)d_\ell(n)$ for $\ell \geq 1$ fixed. It is given in [CGG84, p.422] that this choice of $a^-(n)$, yields

$$h^-(c_r) = c_r + 2\ell\int_0^1 \frac{\sin(\pi c_r v)(1-\delta)}{\pi v}(1-v)^{\ell^2}\, dv + O\left(1/\log T\right). \tag{2.7}$$

To detect small gaps, we must show that $h(c_r) > r$ for fixed $r \geq 1$. We estimate the integral as before and find

$$h^-(c_r) \geq c_r + 2\ell\Bigg\{\frac{2c_r(1-\delta)}{\pi(\ell^2+1)}\left(\sqrt{2} - (2\sqrt{2}-2)\exp\left(\frac{-(\ell^2+1)}{4c_r}\right)\right.$$
$$\left. - (2-\sqrt{2})\exp\left(\frac{-(\ell^2+1)}{2c_r}\right)\right) - \frac{1}{\pi}\int_{\ell^2/c_r}^{\infty}\frac{\exp(-u)}{u}\, du\Bigg\} + O\left(\frac{1}{\log T}\right).$$

We let $\ell = \sqrt{bc_r - 1}$ and $c_r = r - \vartheta\sqrt{r}$, with $\vartheta > 0$. In this case, we do not always have $c_r > 1$. Indeed, since $\vartheta > 0$, if $r = 1$ then $0 < c_r < 1$. However, if we require that $\vartheta \leq 0.5$, the estimate

$$\ell = \sqrt{bc_r - 1} > \sqrt{br}\sqrt{\frac{1}{2} - \frac{1}{b}}$$

holds for any $r \geq 1$. The requirement that $\vartheta \leq 0.5$ also implies

$$\frac{\ell^2}{c_r} \geq b - 2$$

for any $r \geq 1$, and we may increase the length of integration in I_2 to write

$$\int_{\ell^2/c_r}^{\infty} \frac{\exp(-u)}{u}\, du \leq \int_{b-2}^{\infty} \frac{\exp(-u)}{u}\, du.$$

Thus, requiring that $\vartheta \leq 0.5$, we may put these estimates together to write

$$\begin{aligned} h^-(c_r) > r - \vartheta\sqrt{r} + 2\sqrt{br}\sqrt{\frac{1}{2} - \frac{1}{b}} \Bigg\{ \frac{2(1-\delta)}{\pi b} \Bigg(\sqrt{2} - (2\sqrt{2}-2) \exp\left(\frac{-(\ell^2+1)}{4c_r}\right) \\ - (2-\sqrt{2}) \exp\left(\frac{-(\ell^2+1)}{2c_r}\right)\Bigg) - \frac{1}{\pi}\int_{b-2}^{\infty} \frac{\exp(-u)}{u}\, du \Bigg\} + O\left(\frac{1}{\log T}\right). \end{aligned}$$

To show $h^-(c_r) > r$ and thus prove the theorem, we set

$$\begin{aligned} \vartheta = \max_b \Bigg\{ 2\sqrt{b}\sqrt{\frac{1}{2} - \frac{1}{b}} \Bigg(\frac{2}{\pi b} \Bigg(\sqrt{2} - (2\sqrt{2}-2) \exp\left(\frac{-b}{4}\right) \\ - (2-\sqrt{2}) \exp\left(\frac{-b}{2}\right)\Bigg) - \frac{1}{\pi}\int_{b-2}^{\infty} \frac{\exp -u}{u}\, du \Bigg) \Bigg\}. \end{aligned}$$

The choice $b = 6.77$ yields $\vartheta = 0.359222$. (We note that the condition $\vartheta < 0.5$ is satisfied.) With δ sufficiently small and T sufficiently large, these choices guarantee that $h^+(c_r) < r$, as desired.

Remark. In the argument above, if we had not divided the remaining portion of I_1 into two smaller integrals and instead compared $\sin(\pi c_r(1-\delta)v)$ to $2c_r(1-\delta)v$ over the interval $[0, 1/2c_r]$, we would have ultimately found that one can take $\Theta = 0.447$. Instead, by carrying out the analysis on $I_1 > I_{1,a} + I_{1,b}$ (see (2.2)) and estimating $I_{1,a}$ and $I_{1,b}$ separately, we were able to provide the stronger constant $\Theta = 0.570717$. One could thus improve the absolute constants Θ and ϑ by breaking up I_1 into smaller pieces and estimating each piece accordingly. For example, writing $I_1 > I_{1,a'} + I_{1,b'} + I_{1,c'} + I_{1,d'}$ where each integral has equal length of integration, one can obtain $\Theta = 0.593234$, and writing I_1 into sixteen smaller integrals of equal length of integration yields $\Theta = 0.599648$. For small gaps, writing I_1 into sixteen smaller integrals of equal length of integration yields $\vartheta = 0.379674$.

2.3. **The absolute constants Θ and ϑ when r sufficiently large.** We can improve the constants Θ and ϑ if we allow r to be sufficiently large.

2.3.1. *Large gaps for sufficiently large r.* Starting with (2.1), to detect large gaps of the desired size, we must show that $h^+(c_r) < r$ for sufficiently r and $c_r = r + \Theta\sqrt{r}$ with $\Theta > 0$. Choosing $\ell = B\sqrt{r}$, for sufficiently large r, we have

$$h^+(c_r) < c_r - 2B\sqrt{r}\int_0^1 \frac{\sin(\pi r v(1-\delta))}{\pi v}(1-v)^{B^2 r}\, dv + O(1/\log T).$$

By the change of variable $rv = w$, the above inequality becomes

$$\begin{aligned} h^+(c_r) &< c_r - 2B\sqrt{r}\int_0^r \frac{\sin(\pi w(1-\delta))}{\pi w}\left(1-\frac{w}{r}\right)^{B^2 r} dw + O(1/\log T) \\ &< c_r - 2B\sqrt{r}\int_0^r \frac{\sin(\pi w(1-\delta))}{\pi w}\exp\left(-B^2 w\right) dw + O(1/\log T) \\ &= c_r - 2B\sqrt{r}\int_0^\infty \frac{\sin(\pi w(1-\delta))}{\pi w}\exp\left(-B^2 w\right) dw - 2B\sqrt{r}E(r) + O(1/\log T), \end{aligned}$$

where

$$E(r) = \int_r^\infty \frac{\sin(\pi w(1-\delta))}{\pi w}\exp\left(-B^2 w\right) dw.$$

Note that as $r \to \infty$, $\sqrt{r}E(r) \to 0$, so for sufficiently large r this term is negligible. Thus we set

$$\Theta = \max_B \left\{ 2B\int_0^\infty \frac{\sin(\pi w)}{\pi w}\exp\left(-B^2 w\right) dw \right\}.$$

The choice $B = 1.5$ yields $\Theta = 0.906498$. With δ sufficiently small, T and r sufficiently large, these choices guarantee that $h^+(c_r) < r$.

2.3.2. *Small gaps for sufficiently large r.* We begin with (2.7) and let $\ell = B\sqrt{r}$. Since $r - \vartheta\sqrt{r} > r/2$ for sufficiently large r, we have

$$h^-(c_r) > c_r + 2B\sqrt{r}\int_0^1 \frac{\sin(2^{-1}\pi r v(1-\delta))}{\pi v}(1-v)^{B^2 r} dv + O(1/\log T).$$

We follow an analogous argument as in the previous subsection and ultimately set

$$\vartheta = \max_B \left\{ 2B\int_0^\infty \frac{\sin(2^{-1}\pi w)}{\pi w}\exp\left(-B^2 w\right) dw \right\}.$$

The choice $B = 1.06$ yields $\vartheta = 0.640989$. With δ sufficiently small, T and r sufficiently large, these choices guarantee that $h^-(c_r) > r$.

□

3. Other choices of $a(n)$

In this section we indicate a possible path towards producing a Selberg-like theorem on r-gaps where $\pm A\sqrt{r}$ is replaced by $\pm A\sqrt{r}g(r)$, with $g(r)$ a suitable function with $g(r) \to \infty$ as $r \to \infty$. The idea is to make the choice of $a(n)$ more flexible by letting $a^\pm(n) = b^\pm(n)(f_1 * f_2 * \cdots * f_\ell)(n)$ where $b^+(n) = 1$ and $b^-(n) = \lambda(n)$ and each $f_j(n) = F_j(\log n/\log X)$ is a smooth function with sufficient decay. Recall that to detect large and small gaps, we must estimate the ratio given in (1.2), namely

$$\frac{N(a^\pm(n), c_r, X)}{D(a^\pm(n), X)} := \frac{\Re\left(\displaystyle\sum_{kn\le X}\frac{a^\pm(n)a^\pm(kn)g_{c_r}(k)\Lambda(k)}{kn}\right)}{\displaystyle\sum_{n\le X}\frac{|a^\pm(n)|^2}{n}}. \tag{3.1}$$

We claim that

$$N(a^{\pm}(n), c_r, X) \approx \pm 2(\log X)^{\ell^2} \int_0^1 \int \cdots \int_{V'} \frac{\sin(c\pi\nu)}{\pi\nu}$$
$$\times \prod_{j=1}^{\ell} f_j\left(\sum_{k=1}^{\ell}\theta_{jk}\right)\left\{\sum_{K=1}^{\ell}\prod_{k=1}^{\ell} f_k\left(\sum_{j=1}^{\ell}\theta_{jk}+\delta_{Kk}\nu\right)\right\}\prod_{1\le j,k\le \ell} d\theta_{jk}\,d\nu$$

where the region $V' \subset [0,1]^{\ell^2}$ is determined by $0 \le \sum_{1\le j,k\le\ell}\theta_{jk} \le 1-\nu$, and δ_{Kk} denotes the Kronecker delta symbol. Furthermore we claim that
(3.2)

$$D(a^{\pm}(n), X) \approx (\log X)^{\ell^2} \int \cdots \int_V \prod_{j=1}^{\ell} f_j\left(\sum_{1\le k\le \ell}\theta_{jk}\right)\prod_{k=1}^{\ell} f_k\left(\sum_{1\le j\le\ell}\theta_{jk}\right)\prod_{1\le j,k\le\ell} d\theta_{jk}$$

where the region $V \subset [0,1]^{\ell^2}$ is determined by $0 \le \sum_{1\le j,k\le\ell}\theta_{jk} \le 1$.

Remark. Theorem 1 is also a consequence of this approach. Indeed, the choice of $f_j = f_k = 1$ for all $1 \le j,k \le \ell$ yields (2.1) when $b^+(n) = 1$ and (2.7) when $b^-(n) = \lambda(n)$. Then one may continue the argument in Section 2, choosing $\ell = \lfloor\sqrt{br-1}\rfloor$ for $b > 1$ a constant.

3.1. **The denominator.** We first give a sketch of how to arrive at the expression for $D(a_n^{\pm}, X)$ given in 3.2, which starts by considering

$$\sum_{n\le X}\frac{|b^{\pm}(n)(f_1 * f_2 * \cdots * f_\ell)(n)|^2}{n}.$$

Since $|b^{\pm}(n)|^2 = 1$, we open up the product and find

$$\sum_{n\le X}\frac{|(f_1 * f_2 * \cdots * f_\ell)(n)|^2}{n} = \sum_{n\le X}\sum_{\substack{n=m_1\cdots m_\ell\\ n=m_{\ell+1}\cdots m_{2\ell}}}\prod_{1\le j,k\le\ell}\frac{f_j(m_j)f_k(m_{\ell+k})}{m_j}. \tag{3.3}$$

Writing each $f(m) = F_j\left(\frac{\log m}{\log X}\right)$ in terms of its Fourier transform

$$f(m) = F\left(\frac{\log m}{\log X}\right) = \int_{-\infty}^{\infty}\widehat{F}(u)m^{-iu/L}\,du,$$

where $L := \log X$, we may write the right-hand side of (3.3) as

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{1\le j,k\le\ell}\widehat{F}_j(u_j)\widehat{F}_{\ell+k}(u_{\ell+k})$$
$$\times\sum_{n\le X}\sum_{\substack{n=m_1\cdots m_\ell\\ n=m_{\ell+1}\cdots m_{2\ell}}}\frac{1}{m_1^{1/2+iu_1/L}\cdots m_{2\ell}^{1/2+iu_{2\ell}/L}}\prod_{1\le j\le 2\ell} du_j.$$

To treat the innermost sum, we appeal to Theorem 2.4.1 of [CFK+05, p.59] to write

$$\sum_{\substack{n\le X\\ n=m_1\cdots m_\ell\\ n=m_{\ell+1}\cdots m_{2\ell}}} \frac{1}{m_1^{1/2+iu_1/L}\cdots m_{2\ell}^{1/2+iu_{2\ell}/L}}$$
$$= \frac{1}{2\pi i}\int_{(c)} \frac{X^s}{s} \prod_{1\le j,k\le \ell} \zeta\left(s+1+\frac{i(u_j+u_{\ell+k})}{L}\right) A_\ell(s/2;u)\, ds,$$

where $A_\ell(s/2;u)$ is an arithmetic factor which can be ignored since it should cancel out in the ratio (3.1). If our functions f are suitably chosen, we can perform a contour shift and replace each zeta-factor with its Laurent expansion, finding

$$\frac{1}{2\pi i}\int_{(c)} \frac{X^s}{s} \prod_{1\le j,k\le \ell} \zeta\left(s+1+\frac{i(u_j+u_{\ell+k})}{L}\right) ds$$
$$\approx \frac{1}{2\pi i}\int_{(c')} \frac{X^s}{s} \prod_{1\le j,k\le \ell} \frac{1}{s+i(u_j+u_{\ell+k})/L}\, ds$$
$$\approx \frac{1}{2\pi i}\int_{(c')} \frac{X^s}{s} \prod_{1\le j,k\le \ell} \left(\int_0^1 (y_{jk})^{s+i(u_j+u_{\ell+k})/L-1}\, dy_{jk}\right) ds.$$

Inserting this estimate into the full expression and reordering terms, we find

$$\sum_{n\le X} \frac{|(f_1*f_2*\cdots*f_\ell)(n)|^2}{n} \approx \frac{1}{2\pi i}\int_{(c')}\int_0^1\cdots\int_0^1 \frac{\left(x\prod_{1\le j,k\le\ell} y_{jk}\right)^s}{s}$$
$$\times \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \prod_{1\le j\le \ell} \widehat{F}_j(u_j)\left(\prod_{1\le k\le \ell} y_{jk}\right)^{iu_j/L} \prod_{1\le k\le\ell} \widehat{F}_k(u_{\ell+k})$$
$$\times \left(\prod_{1\le j\le\ell} y_{jk}\right)^{iu_{\ell+k}/L} \frac{\prod_{1\le j\le 2\ell} du_j \prod_{1\le j,k\le\ell} dy_{jk}}{\prod_{1\le j,k\le \ell} y_{jk}} ds$$

$$\approx \int_0^1\cdots\int_0^1 \prod_{1\le j\le\ell} f_j\left(\frac{-\log\prod_{1\le k\le\ell} y_{jk}}{L}\right)\prod_{1\le k\le\ell} f_k\left(\frac{-\log\prod_{1\le j\le\ell} y_{jk}}{L}\right)$$
$$\times\left(\frac{1}{2\pi i}\int_{(c')} \frac{\left(X\prod_{1\le j,k\le\ell} y_{jk}\right)^s}{s}\right)\frac{\prod_{1\le j,k\le\ell} dy_{jk}}{\prod_{1\le j,k\le\ell} y_{jk}}.$$

Using

$$\frac{1}{2\pi i}\int_{(c')} \frac{\left(x\prod_{1\le j,k\le\ell} y_{jk}\right)^s ds}{s} = \begin{cases} 1 & \text{if } \prod_{1\le j,k\le\ell} y_{jk} > \frac{1}{X}\\ 0 & \text{if } \prod_{1\le j,k\le\ell} y_{jk} < \frac{1}{X}\end{cases}$$

and the change of variable

$$y_{jk} = X^{-\theta_{jk}} \qquad \Rightarrow \qquad dy_{jk} = -X^{-\theta_{jk}}\log X\, d\theta_{jk}, \tag{3.4}$$

we have

$$\sum_{n\le X}\frac{|a^{\pm}(n)|^2}{n}$$
$$\approx (\log X)^{\ell^2}\int\cdots\int_V \prod_{j=1}^{\ell} f_j\left(\sum_{1\le k\le \ell}\theta_{jk}\right)\prod_{k=1}^{\ell} f_k\left(\sum_{1\le j\le \ell}\theta_{jk}\right)\prod_{1\le j,k\le \ell} d\theta_{jk},$$

where the region $V\subset[0,1]^{\ell^2}$ is given by $V=\{\theta_{jk}:0\le\sum_{1\le j,k\le\ell}\theta_{jk}\le 1\}$.

3.2. **The numerator** $N(a^{\pm}(n),c_r,X)$**.** For the numerator of (3.1), we may focus on the contribution of the primes (see, for example [BMN10, p. 4171]). Thus we wish to find a closed form of the expression

$$\sum_{np\le X}\frac{a^{\pm}(n)\overline{a^{\pm}(np)}}{np}\sin\left(\frac{c_r\pi\log p}{L}\right)$$

where again $a^{\pm}(n)=b^{\pm}(n)(f_1*\cdots f_\ell)(n)$ and $L=\log X$. Note that if $b^{+}(n)=1$ then $b^{+}(p)=1$, and if $b^{-}(n)=\lambda(n)$ then $b^{-}(p)=-1$. Thus, opening up the product and passing to the Fourier transform for each function f as before, we have

$$\frac{2}{\pi}\sum_{np\le X}\frac{a^{\pm}(n)a^{\pm}(np)}{np}\sin\left(\frac{c\pi\log p}{L}\right)$$
$$\approx \pm\frac{2}{\pi}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{1\le j\le\ell}\widehat{F}_j(u_j)\prod_{1\le k\le\ell}\widehat{F}_k(u_{\ell+k})\sum_{p\le X}\frac{\sin(c\pi\log p/L)}{p^{1/2}}$$
$$\times\sum_{\substack{m_1\cdots m_\ell\le X/p\\ m_{\ell+1}\cdots m_{2\ell}=m_1\cdots m_\ell p}}\frac{\prod_{1\le j\le 2\ell}du_j}{m_1^{1/2+iu_1/L}\cdots m_{2\ell}^{1/2+iu_{2\ell}/L}}$$
$$\approx \pm\frac{2}{\pi}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{1\le j\le\ell}\widehat{F}_j(u_j)\prod_{1\le k\le\ell}\widehat{F}_k(u_{\ell+k})\sum_{p\le X}\frac{\sin(c\pi\log p/L)}{p^{1/2}}$$
$$\times\frac{1}{2\pi i}\int_{(c)}\frac{(X/p)^s}{s}\sum_{m_1\cdots m_\ell p=m_{\ell+1}\cdots m_{2\ell}}\frac{ds}{m_1^{(s+1)/2+iu_1/L}\cdots m_{2\ell}^{(s+1)/2+iu_{2\ell}/L}}.$$

This new inner sum requires more care, due to the presence of the prime p in the summation condition. Since $m_1\cdots m_\ell p=m_{\ell+1}\cdots m_{2\ell}$, we consider the case where $p|m_{\ell+1},p|m_{\ell+2},\ldots$ or $p|m_{2\ell}$. (Of course, p could divide any combination of two or more of the $m_{\ell+k}$'s, which would need to be handled in an error term.) If $p|m_{\ell+k}$, define

$$\frac{m_{\ell+k}}{p}:=m'_{\ell+k}.$$

Then

$$\sum_{m_1\cdots m_\ell p = m_{\ell+1}\cdots m_{2\ell}} \frac{1}{m_1^{s+1/2+iu_1/L}\cdots m_{2\ell}^{s+1/2+iv_\ell/L}} \sim \sum_{k=1}^{\ell} \frac{1}{p^{1/2+iu_{\ell+k}/L+s}}$$
$$\times \sum_{\substack{m_1\cdots m_\ell \\ = \\ m_{\ell+1}\cdots m'_{\ell+k}\cdots m_{2\ell}}} \frac{1}{m_1^{s+1/2+iu_1/L}\cdots (m'_{\ell+k})^{s+1/2+iu_{\ell+k}/L}\cdots m_{2\ell}^{s+1/2+iu_{2\ell}/L}}.$$

Applying Theorem 2.4.1 of [CFK$^+$05] to the innermost sum, we find

$$\frac{2}{\pi}\sum_{np\le X} \frac{a^{\pm}(n)a^{\pm}(np)}{np}\sin\left(\frac{c\pi\log p}{L}\right) \approx \pm\frac{2}{\pi}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{1\le j\le \ell}\widehat{F}_j(u_j)\prod_{1\le k\le \ell}\widehat{F}_k(u_{\ell+k})$$
$$\times \sum_{p\le X}\frac{\sin(c\pi\log p/L)}{p^{1/2}}\frac{1}{2\pi i}\int_{\sigma}\frac{(X/p)^s}{s}\left(\sum_{1\le k\le \ell}\frac{1}{p^{1/2+iu_{\ell+k}/L+s}}\right)$$
$$\times \prod_{1\le j,k\le \ell}\zeta\left(1+s+\frac{i(u_j+u_{\ell+k})}{L}\right)ds\prod_{1\le j\le 2\ell}du_j.$$

We now turn our attention to the sum over primes

$$\sum_{p\le X}\frac{\sin(c\pi\log p/L)}{p^{1/2}}\left(\sum_{1\le k\le \ell}\frac{1}{p^{1/2+iu_{\ell+k}/L+s}}\right) = \sum_{1\le k\le \ell}\sum_{p\le X}\frac{\sin(c\pi\log p/L)}{p^{1+iu_{\ell+k}/L+s}}.$$

Applying the Prime Number Theorem with remainder, along with the change of variable $p = X^{\nu}$, where $0\le \nu\le 1$, we have

$$\sum_{1\le k\le \ell}\sum_{p\le X}\frac{\sin(c\pi\log p/L)}{p^{1+iu_{\ell+k}/L+s}} \approx \sum_{1\le k\le \ell}\int_0^1\frac{\sin(c\pi\nu)}{\nu X^{\nu(s+iu_{\ell+k}/L)}}\,d\nu + O(1/L)$$
$$\approx \int_0^1\frac{\sin(c\pi\nu)}{\nu X^{\nu s}}\sum_{1\le k\le \ell}\frac{d\nu}{X^{\nu iu_{\ell+k}/L}}.$$

Therefore

$$\sum_{np\le X}\frac{a^{\pm}(n)a^{\pm}(np)}{np}\sin\left(\frac{c\pi\log p}{L}\right) \approx \pm\frac{2}{\pi}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{1\le j\le \ell}\widehat{F}_j(u_j)\prod_{1\le k\le \ell}\widehat{F}_k(u_{\ell+k})$$
$$\times\int_0^1\frac{\sin(c\pi\nu)}{\nu}\left(\sum_{1\le k\le \ell}\frac{1}{X^{\nu iu_{\ell+k}/L}}\right)\frac{1}{2\pi i}\int_{\sigma}\frac{(X^{1-\nu})^s}{s}$$
$$\times\prod_{1\le j,k\le \ell}\zeta\left(1+s+\frac{i(u_j+u_{\ell+k})}{L}\right)d\nu ds\prod_{1\le j\le 2\ell}du_j.$$

To finish, we follow the previous line of thought, using the Laurent series of $\zeta(s)$ and the same change of variable as in (3.4) as before to conclude

$$\begin{aligned}
&\sum_{np\le X}\frac{a^{\pm}(n)a^{\pm}(np)}{np}\sin\left(\frac{c\pi\log p}{L}\right)\\
&\quad\approx\pm\frac{2}{\pi}(\log X)^{\ell^2}\int_0^1\int\cdots\int_{X^{\nu-1}\le\prod y_{jk}\le 1}\frac{\sin(c\pi\nu)}{\nu}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{1\le j\le\ell}\widehat{F}_j(u_j)\\
&\qquad\times\prod_{1\le k\le\ell}\widehat{F}_k(u_{\ell+k})\prod_{1\le j,k\le\ell}y_{jk}^{i(u_j+u_{\ell+k})/L}\sum_{1\le k\le\ell}\frac{1}{X^{\nu i u_{\ell+k}/L}}\\
&\qquad\quad\times\frac{\prod_{1\le j\le 2\ell}du_j\,d\nu\prod_{1\le j,k\le\ell}dy_{jk}\,ds}{\prod_{1\le j,k\le\ell}y_{jk}}\\
&\quad\approx\pm\frac{2}{\pi}(\log X)^{\ell^2}\int_0^1\int_{V'}\frac{\sin(c\pi\nu)}{\nu}\\
&\qquad\quad\times\prod_{j=1}^{\ell}f_j\left(\sum_{k=1}^{\ell}\theta_{jk}\right)\left\{\sum_{K=1}^{\ell}\prod_{k=1}^{\ell}f_k\left(\sum_{j=1}^{\ell}\theta_{jk}+\delta_{Kk}\nu\right)\right\}\prod_{1\le j,k\le\ell}d\theta_{jk}\,d\nu.
\end{aligned}$$

We note that in the application to Selberg's Theorem, choosing each $f_j=f_k=1$ yields the additional factor ℓ appearing in (2.1) and (2.7), which comes from the sum over $1\le K\le\ell$ in the last line above.

4. Numeric Tests

We compare three choices for $a^{\pm}(n)$ in the case that $\ell=2$ and $r=1$ and show that the general convolution presented in Section 3 yields numerically stronger results in this test case. In particular we test

(4.1)
$$a^{\pm}(n)=\begin{cases} b^{\pm}(n)d_\ell(n) & \text{“no smoothing”}\\ b^{\pm}(n)d_\ell(n)F\left(\frac{\log n}{\log T}\right) & \text{“smoothing after the convolution”}\\ b^{\pm}(n)(f_1*\cdots*f_\ell)(n)\text{ with }f_j(n)=F_j\left(\frac{\log n}{\log T}\right) & \text{“smoothing within the convolution”}\end{cases}.$$

As before, $b^+(n)=1$ is used when testing for large gaps, $b^-(n)=\lambda(n)$ is used when testing for small gaps, and F,F_j denote smooth functions.

4.1. **Large gaps when $\ell=2$ and $r=1$.**

We search for the largest value of c for which $h^+(c)<1$.

4.1.1. *No smoothing.* Taking $a^+(n)=d_2(n)$, we have

$$h^+(c)=c-4\int_0^1\frac{\sin(\pi cv)}{\pi v}(1-v)^4\,dv,$$

and find that

$$h^+(2.33)=1.00084\qquad h^+(2.32)=0.993539.$$

This guarantees neighboring zeros at least 2.32 times the average spacing.[1]

4.1.2. *Smoothing after the convolution.* Taking $a^+(n) = d_2(n)F\left(\frac{\log n}{\log T}\right)$, we have

$$h^+(c) = c - \frac{4\int_0^1 \frac{\sin(\pi vc)}{\pi v}\left(\int_0^{1-v} u^3F(u)F(u+v)\ du\right)\ dv}{\int_0^1 u^3F(u)^2\ du}\ .$$

Now with

$$F(u) = 1 + 3.43656u - 4.17743u^2$$

we have $h(2.509, F) = 0.99937$ which guarantees a large gap between neighboring pairs of zeros at least of size 2.509 times the average.[2]

4.1.3. *Smoothing within the convolution.* Now we have

$$h^+(c) = c - \left(\int_0^1 \frac{2\sin(\pi vc)}{\pi v}\int_{w+x+y+z\le 1-v}\big(F(w+x)G(y+z)F(w+y+v)G(x+z) + F(w+x)G(y+z)F(w+y)G(x+z+v)\big)\ dw\ dx\ dy\ dz\ dv\right) \times\left(\int_{w+x+y+z\le 1} F(w+x)G(y+z)F(w+y)G(x+z)\ dw\ dx\ dy\ dz\right)^{-1}.$$

We find that with $F(u) = G(u) = 1 + 10.8753u - 17.1047u^2$, we have

$$h^+(2.59) = 0.992863$$

thus guaranteeing a large gap of at least 2.59 times the average.

TABLE 1. In the case $\ell = 2$ and $r = 1$ and using the coefficients defined in (4.1), the table below compares values c for which $h^+(c) < 1$.

$a^+(n)$	value of c	polynomial
no smoothing	2.32	NA
smoothing after the convolution	2.509	$F(u) = 1 + 3.4365u - 4.17743u^2$
smoothing within the convolution	2.59	$F(u) = G(u) = 1 + 10.8753u - 17.1047u^2$

4.2. **Small gaps when $\ell = 2$ and $r = 1$.** We search for the smallest c such that $h^-(c) > 1$.

[1]This approach is used in [CGG84] with $\ell = 2.2$ and $r = 1$ to obtain a large gap between neighboring pairs of zeros least 2.337 times the average spacing.

[2]Bui, Ng, and Milinovich [BMN10] were the first to use this approach. There, with $\ell = 3$, $r = 1$, and $F(u) = 1 + 11x + 42x^2 + 26x^3 - 75x^4$ the authors obtain a large gap between neighboring pairs of zeros least 2.6950 times the average spacing.

4.2.1. *No smoothing.* Taking $a^-(n) = \lambda(n)d_2(n)$ we have

$$h^-(c) = c + 4\int_0^1 \frac{\sin(\pi cv)}{\pi v}(1-v)^4 \, dv,$$

and we find that

$$h^-(0.561) = 0.999017 \qquad h^-(0.562) = 1.0076$$

This guarantees neighboring zeros at most 0.561 times the average spacing.[3]

4.2.2. *Smoothing after the convolution.* Taking $a^-(n) = \lambda(n)d_2(n)F\left(\frac{\log n}{\log T}\right)$, we have

$$h^-(c,F) = c + \frac{4\int_0^1 \frac{\sin(\pi vc)}{\pi v}\left(\int_0^{1-v} u^3F(u)F(u+v)\,du\right)dv}{\int_0^1 u^3F(u)^2\,du}.$$

Now with

$$F(u) = 1 - 1.33567u + 0.466338u^2$$

we have $h^-(0.5299, F) = 1.00008$ which guarantees a small gap between neighboring pairs of zeros at most of size 0.5299 times the average.[4]

4.2.3. *Smoothing within the convolution.* Now we have

$$h^-(c,F,G) = c + \left(\int_0^1 \frac{2\sin(\pi vc)}{\pi v}\int_{w+x+y+z\le 1-v}\big(F(w+x)G(y+z)F(w+y+v)G(x+z) + F(w+x)G(y+z)F(w+y)G(x+z+v)\big)\,dw\,dx\,dy\,dz\,dv\right)$$
$$\times\left(\int_{w+x+y+z\le 1} F(w+x)G(y+z)F(w+y)G(x+z)\,dw\,dx\,dy\,dz\right)^{-1}.$$

We find that with $F(u) = G(u) =???$, we have

$$h^-(0.5199, F, G) = 0.996807$$

thus guaranteeing a small gap of at most 0.5199 times the average.

Table 2. In the case $\ell = 2$ and $r = 1$ and using the coefficients defined in (4.1), the table below compares values c for which $h^-(c) > 1$.

$a^-(n)$	value of c	polynomial
no smoothing	0.561	NA
smoothing after the convolution	0.5299	$F(u) = 1 - 1.33567u - 0.466338u^2$
smoothing within the convolution	0.5199	$F(u) = G(u) = 1 - 2.11368u + 1.41003u^2$

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[3]In [CGG84], this approach is used with $\ell = 1.1$ and $r = 1$ to obtain a small gap between neighboring pairs of zeros least 0.5172 times the average spacing.

[4]Bui, Ng, and Milinovich [BMN10] use $\ell = 1.23$, $r = 1$, and $F(u) = 1 + 0.99x - 0.42x^2$ to obtain a small gap between neighboring pairs of zeros most 0.5155 times the average spacing.

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