

Riemannian 3-Manifolds with a Flatness Condition

by

Ryan Gunderson

Department of Mathematics

Duke University

Date: _____

Approved:

Robert Bryant, Supervisor

Paul Aspinwall

Leslie Saper

Colleen Robles

Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
in the Graduate School of Duke University
2019

ABSTRACT

Riemannian 3-Manifolds with a Flatness Condition

by

Ryan Gunderson

Department of Mathematics
Duke University

Date: _____

Approved:

Robert Bryant, Supervisor

Paul Aspinwall

Leslie Saper

Colleen Robles

An abstract of a dissertation submitted in partial fulfillment of the requirements for
the degree of Doctor of Philosophy in the Department of Mathematics
in the Graduate School of Duke University
2019

Copyright © 2019 by Ryan Gunderson
All rights reserved except the rights granted by the
Creative Commons Attribution-Noncommercial Licence

Abstract

The fundamental point-wise invariant of a Riemannian manifold (M, g) is the Riemann curvature tensor. Many special types of Riemannian manifolds can be characterized by conditions on the Riemann curvature tensor and tensor fields derived from it. Examples include Einstein manifolds and conformally flat manifolds.

Here we restrict ourselves to three dimensions and explore the Riemannian manifolds that arise when imposing conditions on the irreducible components of the first covariant derivative of the Riemann curvature tensor. Specifically, we look at an irreducible component of the covariant derivative which takes the form of a traceless symmetric $(0, 3)$ -tensor field. We classify the local and global structure of manifolds where this tensor field vanishes.

Contents

Abstract	iv
List of Figures	vii
Acknowledgements	viii
1 Introduction	1
2 Background	4
2.1 G -structures	4
2.1.1 Equivalence of G -structures and the Fundamental 1-form . . .	5
2.1.2 Connections on G -structures	7
2.2 Exterior Differential Systems	8
2.3 Representations of $SO(3)$	9
2.4 Maurer-Cartan Form on a Lie Group	12
3 The Structure Equations and Problem Statement	14
3.1 The Structure Equations of a Riemannian Manifold	14
3.1.1 In Dimension 3	15
3.1.2 Prolongation of the Structure Equations in Dimension 3 . . .	17
3.2 The EDS of a 7-less Metric	22
3.3 The Variety of Potential Curvatures	25
4 Construction of Local Solutions	29
4.1 The Conformally Flat Case	31

4.1.1	Description of the Manifolds	31
4.1.2	Computation of the Curvature	33
4.2	The Non-conformally Flat Case	41
4.2.1	Non-zero Differential of Scalar Curvature	42
4.2.2	Vanishing Differential of Scalar Curvature	53
4.3	Remarks on Generality and Symmetry of Germs	58
5	Global Structure of 7-less Manifolds	60
5.1	Conformally Flat 7-less Metrics	60
5.1.1	Maximality of Structure	60
5.1.2	Global invariants	63
5.2	Non-conformally Flat 7-less Metrics	66
5.2.1	Normal Form	66
5.2.2	Maximal Structures	74
6	Conclusions	89
A	Parametrization of Orbits	91
	Bibliography	101
	Biography	102

List of Figures

5.1	Examples of normalized quadratic curves.	73
5.2	Exceptional quadratic curves.	86

Acknowledgements

First and foremost, I must express my sincerest gratitude and deepest thanks to my advisor, Robert Bryant. He has served as a truly fantastic resource in every step of my studies, and, without his guidance and encouragement, my graduate studies would have been nowhere near as educational and enjoyable as they were. He is one of the people I respect most, both as a mathematician and as a person, and I consider myself truly lucky for having had the opportunity to work with him.

Alongside Robert Bryant, I would like to thank Paul Aspinwall, Leslie Saper, and Colleen Robles for serving on both my preliminary and defense committees and for taking the time to review and offer improvements to this dissertation.

I would also like to express my thanks to Robert Bryant, Lenny Ng, Leslie Saper, and Chad Schoen, all of whom were instructors for some of the best courses I took during my time here at Duke.

Finally, I would like to thank some of my fellow students and friends: Gavin and Yuhao, for sharing their research and offering guidance on mine; Chester, who has always been a source of interesting conversations about math, computer science, languages, teaching, and all manner of interesting topics; and lastly Mike, Brendan, Josh, Erin, Kyle, Rob, Matt, Will, Kevin, and Ollie, all of whom have been good friends and resources over the years.

1

Introduction

Let (M, g) be a Riemannian manifold. The fundamental pointwise invariant of M is the Riemann curvature tensor, $Riem$, a $(0, 4)$ -tensor field on M defined in terms of the derivatives of the metric up to second order. This fundamental invariant is in fact decomposable into three irreducible components (in the sense of $O(n)$ -representations),

$$Riem = S + R + W,$$

where S is the scale curvature, R is the traceless Ricci curvature, and W is the Weyl curvature of g [Bes08]. Important classes of Riemannian manifolds are defined by, or equivalent to, pointwise conditions on these irreducible components. For example, an Einstein manifold is defined to be a manifold where the full Ricci tensor is proportional to the metric. This is equivalent to the traceless Ricci curvature vanishing. While conformal flatness is generally not defined in terms of curvature tensors at all, a fundamental result is that a manifold of dimension $n \geq 4$ is conformally flat if and only if the Weyl curvature vanishes [Sha97]. A natural question to ask is, “What do other algebraic relationships between tensor fields derived from the Riemann curvature tensor tell us about the manifold?”

In this dissertation, we examine one such question in dimension $n = 3$. For the remainder of the chapter we will assume M is a Riemannian 3-manifold. In dimension three, the Weyl curvature is always zero, so the decomposition of the full Riemann curvature tensor into irreducible components involves only the components of the Ricci tensor: scalar curvature and traceless Ricci curvature. After components of the Riemann curvature tensor, the next natural source of tensor fields on M is its covariant derivative, $\nabla Riem$. In dimension 3, $\nabla Riem$ decomposes into three irreducible parts [Bes08]:

$$\nabla Riem = D + C + T.$$

The component D is the differential of scalar curvature, a $(0,1)$ -tensor field. A manifold M has $D = 0$ if and only if, M has constant scalar curvature. This class of manifolds is central to the Yamabe problem. The component C is known as the Cotton-York tensor, a traceless symmetric $(0,2)$ -tensor field. In dimension 3, a manifold is conformally flat if and only if the Cotton-York tensor vanishes [Sha97]. The remaining component T is a traceless symmetric $(0,3)$ -tensor. It is largely unstudied and will be our main focus.

The tensor T takes values in a seven-dimensional representation of $O(3)$ and hence has seven independent components. We will refer to it as the ‘7-piece’ of the curvature and refer to manifolds where T vanishes as ‘7-less.’ Up to diffeomorphisms, a Riemannian metric on a three manifold is described by 3 independent functions of three variables [Bry14] while the 7-less condition is a system of 7 third order PDEs in the metric, making the 7-less condition highly over determined. In this dissertation, we classify the local and global structure of 7-less manifolds.

Our work in this dissertation divides into three main parts. In Chapter 3 we derive the structure equations of a 7-less metric. In Chapter 4 we proceed to construct local metrics that realize all possible 7-less germs. In Chapter 5, for each 7-less germ, we

construct a maximal 7-less manifolds containing the germ.

2

Background

2.1 G -structures

Let M be a smooth manifold of dimension n . A *coframe* based at $p \in M$ is an isomorphism $\alpha : T_p M \rightarrow \mathbb{R}^n$. By writing a coframe as $\alpha = \alpha^i e_i$ where the e_i form the standard basis of \mathbb{R}^n , we can understand a coframe as a collection of n linearly independent 1-forms $\alpha^1, \dots, \alpha^n$ on $T_p M$. The *coframe bundle* is the smooth bundle over M formed by the collection of coframes above all points in M and is denoted $\mathcal{F}(M)$. The projection map $\pi : \mathcal{F}(M) \rightarrow M$ takes a coframe to the point in M where it is based.

There is a right action of $\mathrm{GL}(n)$ on $\mathcal{F}(M)$ given by $\alpha \cdot g = g^{-1} \circ \alpha$. This action is both free and transitive on the fibers of π , showing that the fibers of $\mathcal{F}(M)$ are diffeomorphic to $\mathrm{GL}(n)$ and giving $\mathcal{F}(M)$ the structure of a principal right $\mathrm{GL}(n)$ -bundle. A (local) section of this bundle is called a (local) *coframing* on M and corresponds to a choice of n linearly independent 1-forms on (an open subset) M .

Let G be a Lie subgroup of $\mathrm{GL}(n)$. A *smooth G -structure* on M is a smooth principal G -subbundle of $\mathcal{F}(M)$. This can be thought of as a smooth section of

the bundle $\mathcal{F}(M)/G$ of G -orbits. The group G is called the *structure group* of the G -structure.

Many geometric structures can be understood as G -structure for particular structure groups:

Example 2.1 (Riemannian Manifold). *A coframing $\alpha = (\alpha^1, \dots, \alpha^n)$ on a manifold M defines a Riemannian metric by taking the metric to be $g = (\alpha^1)^2 + \dots + (\alpha^n)^2$. Another coframing $\tilde{\alpha} = (\tilde{\alpha}^1, \dots, \tilde{\alpha}^n)$ will give an equivalent metric if and only if $\tilde{\alpha} = \alpha \cdot g$ for some $g : M \rightarrow O(n)$. Thus a Riemannian metric can be understood as a choice of an $O(n)$ -orbit in $\mathcal{F}(M)$ at each point in M forming an $O(n)$ -structure called the orthonormal coframe bundle. The resulting metric will be smooth if and only if the $O(n)$ -subbundle is a smooth subbundle of $\mathcal{F}(M)$.*

Example 2.2 (Almost Symplectic Manifold). *On a $2n$ -manifold, an almost symplectic structure Ω can be understood as the $\mathrm{Sp}(2n, \mathbb{R})$ -structure of coframes*

$$\alpha = (\alpha^1, \dots, \alpha^{2n})$$

in which the almost symplectic form Ω can be written as

$$\Omega = \alpha^1 \wedge \alpha^2 + \alpha^3 \wedge \alpha^4 + \dots + \alpha^{2n-1} \wedge \alpha^{2n}.$$

Example 2.3 (Almost Complex Manifold). *On a $2n$ -manifold, an almost complex structure $J : TM \rightarrow TM$ can be understood as a $\mathrm{GL}(n, \mathbb{C})$ -structure of coframes whose dual frame $(X_1, \dots, X_n, Y_1, \dots, Y_n)$ satisfies $J(X_i) = Y_i$ and $J(Y_i) = -X_i$.*

2.1.1 Equivalence of G -structures and the Fundamental 1-form

Let M and \tilde{M} be smooth manifolds. If $f : M \rightarrow \tilde{M}$ is a diffeomorphism, there is an induced bundle diffeomorphism $f^* : \mathcal{F}(\tilde{M}) \rightarrow \mathcal{F}(M)$ given by pullback.

Let $\mathcal{B} \subseteq \mathcal{F}(M)$ and $\tilde{\mathcal{B}} \subseteq \mathcal{F}(\tilde{M})$ be two G -structures. We say that the two G -structures are *equivalent* if there exists a diffeomorphism $f : M \rightarrow \tilde{M}$ such

that $f^*(\tilde{\mathcal{B}}) = \mathcal{B}$. We call such a diffeomorphism an *equivalence of G -structures*. Equivalence maps correspond to structure preserving maps. For example, in the case of $G = O(n)$, equivalence maps between G -structures are precisely Riemannian isometries. In classifying structures described by G -structures, we often want to mod out by this type of equivalence.

A G -structure \mathcal{B} can be described at a point $p \in M$ by a (generally non-unique) choice of coframe α_p at p . Then $\mathcal{B} \cap \pi^{-1}(p) = \{\alpha \cdot g : g \in G\}$. A smooth G -structure \mathcal{B} can similarly be described locally by a choice of local coframing α . If α and $\tilde{\alpha}$ defined on $U \subseteq M$ are two local coframings, they will describe the same G -structure on U if and only if each pair of pointwise coframings α_p and $\tilde{\alpha}_p$ lie in the same G -orbit. This is the case when $\tilde{\alpha}_p = \alpha_p \cdot g(p)$ for each $p \in U$ where $g : U \rightarrow G$. Thus a local equivalence between $U \subseteq M$ with G -structure \mathcal{B} generated by α and $\tilde{U} \subseteq \tilde{M}$ with G -structure $\tilde{\mathcal{B}}$ generated by $\tilde{\alpha}$ is a diffeomorphism $f : U \rightarrow \tilde{U}$ satisfying $\alpha_{f^{-1}(\tilde{p})} = f^*(\tilde{\alpha}_{\tilde{p}}) \cdot g(\tilde{p})$ for some $g : \tilde{U} \rightarrow G$. This approach to demonstrating equivalence depends on the choice of coframings on the manifolds. We now look at a coframe independent method using the whole of the G -structure.

Let M^n be a smooth manifold with coframe bundle $\mathcal{F}(M)$. The *fundamental 1-form* of $\mathcal{F}(M)$ is an \mathbb{R}^n -valued 1-form ω on $\mathcal{F}(M)$ defined by $\omega(v_\alpha) = \alpha(\pi_*(v_\alpha))$ where $v_\alpha \in T_\alpha \mathcal{F}(M)$. By writing $\omega = \omega^i e_i$, where the e_i form the standard basis for \mathbb{R}^n , we can understand ω as a collection of n linearly independent \mathbb{R} -valued 1-forms $\omega^1, \dots, \omega^n$ on $\mathcal{F}(M)$. As any G -structure \mathcal{B} is a subbundle of $\mathcal{F}(M)$, we may restrict ω to \mathcal{B} by pullback under the inclusion map. The following result is proved in [Gar89].

Theorem 2.4. *Let $G \subseteq \text{GL}(n)$ be a connected Lie group. Let \mathcal{B} and $\tilde{\mathcal{B}}$ be G -structures on manifolds M and \tilde{M} with fundamental 1-forms ω and $\tilde{\omega}$, respectively. The G -structures \mathcal{B} and $\tilde{\mathcal{B}}$ are equivalent if and only if there exists a diffeomor-*

phism $f : \mathcal{B} \rightarrow \tilde{\mathcal{B}}$ such that $f^*(\tilde{\omega}) = \omega$.

This Theorem allows us to reduce the question of equivalence to one regarding differential forms.

2.1.2 Connections on G -structures

The fundamental 1-form of a G -structure gives n linearly independent 1-forms on the bundle $\mathcal{B} \subseteq \mathcal{F}(M)$. It is often useful to extend this to a full coframing on \mathcal{B} .

The vertical tangent space of any coframing $\alpha \in \mathcal{B}$ is the subspace of $T_\alpha \mathcal{B}$ parallel to the fibers of $\pi : \mathcal{B} \rightarrow M$. Explicitly, the vertical tangent space of α is the kernel of the pushforward map $\pi_* : T_\alpha \mathcal{B} \rightarrow T_{\pi(\alpha)} M$. As the action of G on the fibers of \mathcal{B} is both free and transitive, there is a natural isomorphism between the vertical tangent space of each point $\alpha \in \mathcal{F}(M)$ and the Lie algebra \mathfrak{g} of G given explicitly by taking $v \in \mathfrak{g}$ to

$$\left. \frac{d}{dt} \right|_{t=0} (\alpha \cdot \exp(tv)).$$

The inverse map defines \mathfrak{g} valued 1-form η on the vertical tangent space. By passing into a local coordinate system on \mathcal{B} , one can check that η satisfies $R_g^*(\eta) = \text{Ad}_{g^{-1}} \eta$ where R denotes the right action of G on \mathcal{B} and Ad is the adjoint action.

We can extend the definition of η linearly in such a way that this pullback equivariance remains true. The resulting \mathfrak{g} -valued form is called a connection form. There is a correspondence between connection forms and Ehresmann connections defined by horizontal bundles that takes each connection form η to the bundle $\ker(\eta)$ over M .

Fixing a basis $v_1, \dots, v_{\dim(\mathfrak{g})}$ for \mathfrak{g} , we can write a connection form as $\eta = \eta^i v_i$ and so we may understand a connection form as a collection of $\dim(\mathfrak{g})$ linearly independent 1-forms. As the fundamental 1-form ω evaluates to be zero on vertical tangent spaces (and so is semi-basic with respect to the projection π), we see that

the forms $\omega^1, \dots, \omega^n, \eta^1, \dots, \eta^{\dim(\mathfrak{g})}$ are linearly independent and hence form a basis of 1-forms on \mathcal{B} .

2.2 Exterior Differential Systems

Definition 2.5. *Let M be a smooth manifold. Let $\mathcal{I} \subseteq \Omega^*(M)$ be a graded ideal in the algebra of differential forms that is closed under exterior differentiation. We call the pair (M, \mathcal{I}) an exterior differential system on M . An immersed submanifold $i : N \rightarrow M$ is called an integral submanifold of (M, \mathcal{I}) if $i^*(\theta) = 0$ for all $\theta \in \mathcal{I}$.*

The simplest exterior differential system is one generated as an algebraic ideal by a collection of 1-forms:

Definition 2.6. *Let (M^{r+s}, \mathcal{I}) be an exterior differential system. We say \mathcal{I} is Frobenius of rank s if it is generated by s pointwise linearly independent 1-forms $\omega_1, \dots, \omega_s$ satisfying*

$$d\omega_i \equiv 0 \pmod{\omega_1, \dots, \omega_s}$$

for $1 \leq i \leq s$.

The following theorem tells us the local structure of Frobenius systems [BCG⁺12]:

Theorem 2.7 (Frobenius). *Let (M^{r+s}, \mathcal{I}) be a Frobenius system of rank s . Then for sufficiently small neighborhoods, there exists coordinates $x^1, \dots, x^r, y^1, \dots, y^s$ such that \mathcal{I} is generated by dy^1, \dots, dy^s .*

Theorem 2.7 tells us that locally M is foliated by maximal integral manifolds of dimension r given locally by the y^i -constant level sets. The collection of these maximal integral manifolds forms a smooth manifold of dimension s called the *leaf space* of the foliation.

Definition 2.8. *Let $C_{jk}^i = -C_{kj}^i$ and F_i^α be smooth functions on \mathbb{R}^s for $1 \leq i, j, k \leq n$ and $1 \leq \alpha \leq s$. Let M be an n -manifold. We say that a coframing $\omega^1, \dots, \omega^n$*

along with a function $a = (a^1, \dots, a^s) : M \rightarrow \mathbb{R}^s$ is an augmented coframing for the functions C_{jk}^i and F_i^α if

$$\begin{aligned} d\omega &= \frac{1}{2} a^*(C_{jk}^i) \omega^j \wedge \omega^k, \\ da^\alpha &= a^*(F_i^\alpha) \omega^i. \end{aligned} \tag{2.1}$$

We call (2.1) the structure equations of the augmented coframing.

We will later formulate our main problem of finding Riemannian manifolds with a flatness condition as finding augmented coframings for a particular set of functions C_{jk}^i and F_i^α . The following theorem is attributed to Cartan and proved in its following form by Robert Bryant in an unpublished collection of notes:

Theorem 2.9. *Let $C_{jk}^i = -C_{kj}^i$ and F_i^α be smooth functions on \mathbb{R}^s for $1 \leq i, j, k \leq n$ and $1 \leq \alpha \leq s$. Assume that M is an n -manifold with augmented coframing $\omega_1, \dots, \omega_n$ and $a : M \rightarrow \mathbb{R}^s$ and that N is an n -manifold with augmented coframing η_1, \dots, η_n and $b : N \rightarrow \mathbb{R}^s$. If $p \in M$ and $q \in N$ with $a(p) = b(q) \in \mathbb{R}^s$, then there exists a neighborhood $U \subseteq M$ of x and a smooth map $f : U \rightarrow N$ such that $f(q) = q$, $f^*(\eta) = \omega$, and $f^*(b) = a$.*

Theorem 2.9 tells us that an augmented coframing for given C_{jk}^i and F_i^α is determined locally up to diffeomorphism by the value of the augmentation function a at a point. Importantly, the theorem says nothing about the existence of such augmented coframings.

2.3 Representations of $\text{SO}(3)$

Many of the calculations done involve functions and differential forms taking values in representations of $\text{SO}(3)$, the structure group of an oriented Riemannian 3-manifold. Here we introduce some notation to make these calculations more concise as well as understandable.

We begin by describing the irreducible representation of $\text{SO}(3)$. Let $\mathbb{R}[x_1, x_2, x_3]$ be the polynomial ring in three variables. The group $\text{SO}(3)$ acts on this ring via the action of $\text{SO}(3)$ by rotations on \mathbb{R}^3 with variables x_1, x_2, x_3 making $\mathbb{R}[x_1, x_2, x_3]$ into an $\text{SO}(3)$ -representation. Let $\mathcal{V}^n \subseteq \mathbb{R}[x_1, x_2, x_3]$ be the $\text{SO}(3)$ -subrepresentation consisting of homogeneous polynomials of degree n and let $\mathcal{H}^n \subseteq \mathcal{V}^n$ be the $\text{SO}(3)$ -subrepresentation of harmonic homogeneous polynomials of degree n . Each \mathcal{H}^n is an irreducible $\text{SO}(3)$ -representation of dimension $2n + 1$. Furthermore, every irreducible finite dimensional representation of $\text{SO}(3)$ is isomorphic to \mathcal{H}^n for some $n \geq 0$ [TD85].

Let $\rho = x_1^2 + x_2^2 + x_3^2$. Every homogeneous polynomial $p \in \mathcal{V}^n$ of degree n can be written uniquely as

$$p = \sum_{k=0}^{\lfloor n/2 \rfloor} h_{n-2k} \rho^k$$

for some $h_{n-2k} \in \mathcal{H}^{n-2k}$. This gives the decomposition

$$\mathcal{V}^n \cong \bigoplus_{k=0}^{\lfloor n/2 \rfloor} \mathcal{H}^{n-2k}$$

into irreducible representations. The projection maps $\pi_k : \mathcal{V}^n \rightarrow \mathcal{H}^{n-2k}$ with $\pi_k(p) = h_{n-2k}$ can be expressed as polynomials in the operators ρ , multiplication by ρ , and ∇^2 , the Laplacian. The Clebsch-Gordan formulas for $SU(2)$ -representations can be applied for $\text{SO}(3)$ to give the tensor decomposition

$$\mathcal{H}^a \otimes \mathcal{H}^b \cong \mathcal{H}^{a+b} \oplus \mathcal{H}^{a+b-1} \oplus \dots \oplus \mathcal{H}^{|a-b|}. \quad (2.2)$$

Composition with projections into the irreducible components of this decomposition of the tensor product give us bilinear maps, which we denote as

$$\langle \cdot, \cdot \rangle_n : \mathcal{H}^a \times \mathcal{H}^b \rightarrow \mathcal{H}^{a+b-n}.$$

These maps can be given explicitly using the π maps projecting a polynomial into its harmonic parts. We have

$$\langle p, q \rangle_{2k} = \pi_k(pq) \quad (2.3)$$

and

$$\langle p, q \rangle_{2k+1} = \pi_k((p_{x_2}q_{x_3} - p_{x_3}q_{x_2}) \cdot x_1 + (p_{x_3}q_{x_1} - p_{x_1}q_{x_3}) \cdot x_2 + (p_{x_1}q_{x_2} - p_{x_2}q_{x_1}) \cdot x_3). \quad (2.4)$$

We can use these pairings on representations of $\text{SO}(3)$ to define pairings on differential forms taking values in representations of $\text{SO}(3)$. Let M be a smooth manifold and let $\alpha \in \Omega^k(M, \mathcal{H}^a)$, an \mathcal{H}^a -valued k -form, and $\beta \in \Omega^\ell(M, \mathcal{H}^b)$, an \mathcal{H}^b -valued ℓ -form. Define the pairing

$$\langle \cdot, \cdot \rangle_n : \Omega^k(M, \mathcal{H}^a) \times \Omega^\ell(M, \mathcal{H}^b) \rightarrow \Omega^{k+\ell}(M, \mathcal{H}^{a+b-n})$$

by

$$\langle \alpha, \beta \rangle_n(v_1, \dots, v_{k+\ell}) = \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} \text{sgn}(\sigma) \langle \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}), \beta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}) \rangle_n \quad (2.5)$$

where we note that the pairing $\langle \cdot, \cdot \rangle_n$ is the above pairing on $\text{SO}(3)$ representations. These pairings on representation-valued forms satisfy several properties including the antisymmetry identity

$$\langle \beta, \alpha \rangle_n = (-1)^{n+k\ell} \langle \alpha, \beta \rangle_n \quad (2.6)$$

and the Leibniz rule

$$d\langle \alpha, \beta \rangle_n = \langle d\alpha, \beta \rangle_n + (-1)^k \langle \alpha, d\beta \rangle_n. \quad (2.7)$$

Finally, we remark that these pairings satisfy many ‘regrouping’ identities. For example, for $p, q, r \in \mathcal{H}^1$ we have the identity

$$\langle \langle p, q \rangle_1, r \rangle_1 + \langle \langle q, r \rangle_1, p \rangle_1 + \langle \langle r, p \rangle_1, q \rangle_1 = 0. \quad (2.8)$$

These identities are often not obvious and generating a full list is difficult. However, if the degrees of the inputs as well as the number of inputs an identity may involve is fixed, then a complete list can be computed via an exhaustive search. Regardless of these difficulties, we will make use of these identities regularly in our calculations without explicit mention.

2.4 Maurer-Cartan Form on a Lie Group

Let G be a Lie group with associated Lie algebra \mathfrak{g} . Let $L_a : G \rightarrow G$ be the function given by left multiplication by $a \in G$, i.e. $L_a(g) = ag$. Left multiplication by a group element gives a diffeomorphism of G with itself. The *Maurer-Cartan form* on G is the \mathfrak{g} -valued 1-form $\omega : TG \rightarrow \mathfrak{g}$ given by the pushforward

$$\omega(v_a) = (L_{a^{-1}})_*(v_a)$$

where $v_a \in T_aG$ for $a \in G$. This 1-form is left invariant in the sense that $L_a^*(\omega) = \omega$ for all $a \in G$. A choice of coframing on \mathfrak{g} gives rise to a global coframing on G and allows us to understand ω as a collection of $\dim(G)$ \mathbb{R} -valued 1-forms. The Maurer-Cartan form satisfies the *Maurer-Cartan identity* [Sha97]

$$d\omega(X, Y) = -[\omega(X), \omega(Y)] \tag{2.9}$$

for $X, Y \in TG$ and the bracket operation taking place in \mathfrak{g} .

In the case that G is a linear Lie group, i.e., $G \subseteq \text{GL}(n)$, the Lie algebra is a Lie sub-algebra of the $n \times n$ matrices. In this case, the Maurer-Cartan form is given by

$$\omega = A^{-1}dA,$$

where $A : G \hookrightarrow \text{GL}(n)$ is the inclusion of G into $\text{GL}(n)$. The Lie bracket takes the form of matrix commutator, so in this case (2.9) takes the form

$$d\omega(X, Y) = -(\omega(X)\omega(Y) - \omega(Y)\omega(X)).$$

In the case of linear Lie groups, we will write this identity as $d\omega = -\omega \wedge \omega$, where the wedge product is understood as matrix multiplication with wedge product used in place of standard multiplication.

Example 2.10. *Let $G = \text{SO}(3) \subseteq \text{GL}(3)$ be the group of orientation preserving symmetries of the standard inner product on \mathbb{R}^3 . The Lie algebra $\mathfrak{so}(3)$ equals the set of skew-symmetric 3×3 matrices. Hence the Maurer-Cartan form can be understood as a matrix of \mathbb{R} -valued 1-forms $\omega_1, \omega_2, \omega_3$ on $\text{SO}(3)$ arranged as*

$$\omega = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}.$$

The Maurer-Cartan identity takes the form

$$\begin{bmatrix} 0 & d\omega_3 & -d\omega_2 \\ -d\omega_3 & 0 & d\omega_1 \\ d\omega_2 & -d\omega_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \omega_1 \wedge \omega_2 & -\omega_3 \wedge \omega_1 \\ -\omega_1 \wedge \omega_2 & 0 & \omega_2 \wedge \omega_3 \\ \omega_3 \wedge \omega_1 & -\omega_2 \wedge \omega_3 & 0 \end{bmatrix}.$$

Lastly, we have the following theorem regarding Lie algebra valued 1-forms on manifolds [Sha97]:

Theorem 2.11. *Let M be a simply connected manifold. Let G be a Lie group with Lie algebra \mathfrak{g} and Maurer-Cartan form ω . Let α be a \mathfrak{g} -valued 1-form on M satisfying the Maurer-Cartan identity*

$$d\alpha(X, Y) = -[\alpha(X), \alpha(Y)].$$

Then, for each $p \in M$ and $g \in G$, there exists a unique map $f : M \rightarrow G$ such that $f(p) = g$ and $f^(\omega) = \alpha$.*

The map f is called a *developing map*.

The Structure Equations and Problem Statement

In this chapter, we give the structure equations of a Riemannian 3-manifold with the desired curvature conditions. Once these have been stated, the problem of finding all local metrics with the curvature condition can be reformulated as finding augmented coframings satisfying these structure equations. Finally, using the structure equations we give some additional pointwise conditions on the curvature of these metrics.

3.1 The Structure Equations of a Riemannian Manifold

Let (M^n, g) be a Riemannian manifold of dimension n . As all of our calculations concern the local structure of M , *we will further assume that M is oriented*. Analogous to Example 2.1, the metric g is equivalent to a choice of $\text{SO}(n)$ -structure \mathcal{B} on M of oriented coframes, which we call the *oriented orthonormal coframe bundle*. Let $\omega_1, \dots, \omega_n \in \Omega^1(\mathcal{B})$ be the fundamental 1-forms of \mathcal{B} . The Fundamental Lemma of Riemannian geometry says there is a unique choice of connection 1-forms ϕ_{ij} sat-

isfying $\phi_{ji} = -\phi_{ij}$ and the structure equations

$$d\omega_i = -\phi_{ij} \wedge \omega_j, \quad (3.1)$$

$$d\phi_{ij} = -\phi_{ik} \wedge \omega_{kj} + \frac{1}{2} Riem_{ijkl} \omega_k \wedge \omega_l, \quad (3.2)$$

where the $Riem_{ijkl}$ are functions on \mathcal{B} corresponding to the components of the Riemann curvature tensor in its (0,4) form. Explicitly,

$$\pi^*(Riem) = Riem_{ijkl} \omega_i \otimes \omega_j \otimes \omega_k \otimes \omega_l.$$

The ϕ_{ij} can be regarded as components of an $\mathfrak{so}(n)$ -valued 1-form defining a connection on \mathcal{B} . This connection is called the Levi-Civita connection of (M, g) .

3.1.1 In Dimension 3

Let us now assume that (M^3, g) is a three dimensional Riemannian manifold. The fundamental 1-form $\omega = (\omega_1, \omega_2, \omega_3)$ takes values in $\mathbb{R}^3 \cong \mathcal{H}^1$, the degree 1 harmonic polynomials in three variables. So we will regard the fundamental 1-form ω as an \mathcal{H}^1 -valued form written in components as

$$\omega = \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3. \quad (3.3)$$

The connection form ϕ takes values in the Lie algebra

$$\mathfrak{so}(3) \cong \Lambda^2(\mathbb{R}^3) \cong \Lambda^2(\mathcal{H}^1) \cong \mathcal{H}^1.$$

This isomorphism of $\mathfrak{so}(3)$ with \mathbb{R}^3 allows us to write our connection forms with a single index rather than the standard two indices used in other dimensions. This isomorphism suggests the renaming

$$\phi_1 := \phi_{23}, \quad \phi_2 := \phi_{31}, \quad \phi_3 := \phi_{12}.$$

The replacement of two alternating indices with a single index is referred to as the *omitted index notation*. As we did for the fundamental 1-form, we will also consider

the connection form ϕ as \mathcal{H}^1 -valued, with components

$$\phi = \phi_1 x_1 + \phi_2 x_2 + \phi_3 x_3. \quad (3.4)$$

We would like to write the structure equations (3.1) and (3.2) using pairings on $\text{SO}(3)$. In order to do this we must also express the components of the Riemann curvature tensor as a representation-valued function.

Proposition 3.1. *Let (M^3, g) be an oriented Riemannian 3-manifold with associated $\text{SO}(3)$ -structure \mathcal{B} and let T be a fully symmetric traceless $(0, k)$ -tensor on M . With respect to the coframing $\omega = (\omega_1, \omega_2, \omega_3)$, $\pi^*(T)$ can be expressed uniquely as*

$$\pi^*(T) = T_{i_1 \dots i_k}(\omega) \omega_{i_1} \otimes \dots \otimes \omega_{i_k}, \quad (3.5)$$

where the functions $T_{i_1 \dots i_k} : \mathcal{B} \rightarrow \mathbb{R}$ are fully symmetric in their indicies. The function $\bar{T} : \mathcal{B} \rightarrow \mathcal{V}^k$ defined by

$$\bar{T}(\omega) = T_{i_1 \dots i_k}(\omega) x_{i_1} \dots x_{i_k} \quad (3.6)$$

takes values in \mathcal{H}^k and is $\text{SO}(3)$ -equivariant. Conversely, any $\text{SO}(3)$ -equivariant function $\bar{T} : \mathcal{B} \rightarrow \mathcal{H}^k$ with components defined by (3.6) defines a symmetric $(0, k)$ -tensor on M by (3.5).

Proof. Omitted. □

We will refer to symmetric tensors on M and the associated equivariant functions on \mathcal{B} by the same name, as the meaning should be clear by context.

The standard symmetries of the Riemann curvature tensor show that it takes values in a subrepresentation of $S^2(\Lambda^2(\mathbb{R}^n))$. This is a proper subrepresentation in dimensions greater than three (where there exist non-trivial first Bianchi identities). In dimension three there are no non-trivial first Bianchi identities, and

$$S^2(\Lambda^2(\mathbb{R}^3)) \cong S^2(\Lambda^2(\mathcal{H}^1)) \cong S^2(\mathcal{H}^1) \cong \mathcal{V}^2 \cong \mathcal{H}^2 \oplus \mathcal{H}^0.$$

The isomorphism with \mathcal{V}^2 implies that the components of *Riem* can be expressed using two indices. This corresponds to the fact that in three dimensions the Weyl curvature tensor is zero, leaving only Ricci curvature. The decomposition into $\mathcal{H}^2 \oplus \mathcal{H}^0$ corresponds to the decomposition of Ricci curvature into traceless Ricci curvature and scalar curvature. Using Proposition 3.1, we regard traceless Ricci curvature and scalar curvature as equivariant \mathcal{H}^2 -valued and \mathcal{H}^0 -valued functions, R and S , respectively.

With our fundamental 1-form, connection form, and irreducible curvature components expressed as $\text{SO}(3)$ representation valued forms, the structure equations (3.1) and (3.2) can be expressed as

$$d\omega = \langle \phi, \omega \rangle_1 \tag{3.7}$$

$$d\phi = \frac{1}{2} \langle \phi, \phi \rangle_1 - \frac{5}{4} \langle R, \langle \omega, \omega \rangle_1 \rangle_2 + \frac{1}{12} \langle S, \langle \omega, \omega \rangle_1 \rangle_2. \tag{3.8}$$

3.1.2 Prolongation of the Structure Equations in Dimension 3

We would like to consider conditions on the first covariant differential of the Riemann curvature tensor. To do this, we introduce structure equations for the \mathcal{H}^k -valued functions corresponding to the irreducible components R and S of the Riemann curvature tensor. These structure equations will involve the irreducible components of $\nabla Riem$. The following proposition describes the general form that the derivative of such equivalent \mathcal{H}^k -valued functions on \mathcal{B} take.

Proposition 3.2. *Let $T^{(k)} : \mathcal{B} \rightarrow \mathcal{H}^k$ be an equivariant \mathcal{H}^k valued function on \mathcal{B} .*

1. *If $k > 0$, then $dT^{(k)}$ takes the form*

$$dT^{(k)} = -\langle T^{(k)}, \phi \rangle_1 + \langle F^{(k-1)}, \omega \rangle_0 + \langle F^{(k)}, \omega \rangle_1 + \langle F^{(k+1)}, \omega \rangle_2$$

where $F^{(k-1)}$, $F^{(k)}$, and $F^{(k+1)}$ are unique equivariant functions on \mathcal{B} taking values in \mathcal{H}^{k-1} , \mathcal{H}^k , and \mathcal{H}^{k+1} , respectively.

2. If $k = 0$, then $dT^{(0)}$ takes the form

$$dT^{(0)} = \langle F^{(1)}, \omega \rangle_2$$

where $F^{(1)}$ is a unique equivariant function on \mathcal{B} taking values in \mathcal{H}^1 .

Proof. The equivariance condition is equivalent to the assumption that

$$dT^{(k)} \equiv -\langle T^{(k)}, \phi \rangle_1 \pmod{\omega}.$$

Thus, we see that $dT^{(k)} + \langle T^{(k)}, \phi \rangle_1$ is an equivariant, semi-basic, \mathcal{H}^k -valued 1-form. The forms ω and ϕ give pointwise isomorphisms of $T_\theta \mathcal{B}$ with $\mathcal{H}^1 \oplus \mathcal{H}^1$. We can therefore regard $dT^{(k)} + \langle T^{(k)}, \phi \rangle_1$ as an element of $\mathcal{H}^k \otimes (\mathcal{H}^1)^*$. The non-degenerate pairing $\langle \cdot, \cdot \rangle_2 : \mathcal{H}^1 \times \mathcal{H}^1 \rightarrow \mathbb{R}$ gives the isomorphism of $(\mathcal{H}^1)^*$ with \mathcal{H}^1 . Thus every element of $\mathcal{H}^k \otimes (\mathcal{H}^1)^*$ can be written in the form

$$\sum_i p_i \langle r_i, \cdot \rangle_2 = \sum_i \langle p_i, \langle r_i, \cdot \rangle_2 \rangle_0$$

for some $p_i \in \mathcal{H}^k$ and $r_i \in \mathcal{H}^1$. For $k > 0$, the $\text{SO}(3)$ -pairings satisfy the regrouping relationships

$$\langle a_k, \langle b_1, c_1 \rangle_2 \rangle_0 = \alpha_k \langle \langle a_k, b_1 \rangle_2, c_1 \rangle_0 + \beta_k \langle \langle a_k, b_1 \rangle_1, c_1 \rangle_1 + \gamma_k \langle \langle a_k, b_1 \rangle_0, c_1 \rangle_2$$

for all $a_k \in \mathcal{H}^k$, $b_1, c_1 \in \mathcal{H}^1$, for some non-zero constants $\alpha_k, \beta_k, \gamma_k \in \mathbb{R}$ depending on k . For $k > 0$, we use these regrouping relations to get

$$dT^{(k)} + \langle T^{(k)}, \phi \rangle_1 = \langle F^{(k-1)}, \omega \rangle_0 + \langle F^{(k)}, \omega \rangle_1 + \langle F^{(k+1)}, \omega \rangle_2$$

for some $F^{(k-1)}$, $F^{(k)}$, $F^{(k+1)}$, as desired. A similar regrouping relationship proves this in the case $k = 0$. \square

Using Proposition 3.2, we have the structure equations

$$dR = -\langle R, \phi \rangle_1 + \langle R_1, \omega \rangle_0 + \frac{1}{3} \langle C, \omega \rangle_1 + \frac{7}{9} \langle T, \omega \rangle_2, \quad (3.9)$$

$$dS = 3 \langle D, \omega \rangle_1, \quad (3.10)$$

where R_1 , C , T , and D are functions on \mathcal{B} taking values in \mathcal{H}^1 , \mathcal{H}^2 , \mathcal{H}^3 , and \mathcal{H}^1 , respectively. The unabsorbed coefficients of C and D are chosen to make C represent the standard Cotton-York tensor [Bes08] and D represent the differential of scalar curvature. The unabsorbed coefficient of T is chosen to make an explicit formula for T given later involve fewer fractions. The identity $d^2\phi = 0$ yields the equality

$$R_1 = \frac{1}{10}D.$$

This equality is the manifestation of the classical second Bianchi identities.

The three independent symmetric tensors D , C , and T correspond to the three irreducible component of the covariant differential $\nabla Riem$ in three dimensions. The symmetric $(0,1)$ -tensor D is the differential of the scalar curvature. The symmetric $(0,2)$ -tensor C is known as the Cotton-York tensor and vanishes if and only if g is conformally flat [Bes08].

The remaining symmetric $(0,3)$ -tensor is unnamed. As T has takes values in the seven-dimensional representation \mathcal{H}^3 , we will refer to it as the *7-piece* of the curvature.

Definition 3.3. *A manifold on which the tensor T vanishes will be called 7-less.*

Our focus will be on understanding 7-less manifolds. Later we will give an explicit formula for the components of T , allowing us to check whether a metric is 7-less. To understand the structure equations of a 7-less manifold, we shall assume that the 7-piece is zero and prolong to give structure equations for the remaining irreducible components C and D . Using Proposition 3.2, we have the structure equations

$$dC = -\langle C, \phi \rangle_1 + \langle C_1, \omega \rangle_0 + \langle Z, \omega \rangle_1 + \langle C_3, \omega \rangle_2, \quad (3.11)$$

$$dD = -\langle D, \phi \rangle_1 + \langle X, \omega \rangle_0 + \langle D_1, \omega \rangle_1 + \langle D_2, \omega \rangle_2, \quad (3.12)$$

where C_1 , Z , C_3 , X , D_1 , and D_2 are functions on \mathcal{B} taking values in \mathcal{H}^1 , \mathcal{H}^2 , \mathcal{H}^3 , \mathcal{H}^0 , \mathcal{H}^1 , and \mathcal{H}^2 , respectively. Using the identities $d^2R = 0$ and $d^2S = 0$, we get the

higher order Bianchi identities

$$C_1 = 0, \quad C_3 = 0, \quad D_1 = 0,$$

$$D_2 = \frac{50}{3}Z - \frac{175}{2}\langle R, R \rangle_2 - \frac{25}{3}\langle S, R \rangle_0.$$

The remaining two independent functions Z and X represent a symmetric (0,2)-tensor and a symmetric (0,0)-tensor (a function), respectively. These are the two irreducible components of the second covariant derivative of the full curvature tensor of a 7-less manifold. Using Proposition 3.2 a final time, we have the structure equations

$$dZ = -\langle Z, \phi \rangle_1 + \langle Z_1, \omega \rangle_0 + \langle Z_2, \omega \rangle_1 + \langle Z_3, \omega \rangle_2, \quad (3.13)$$

$$dX = \langle X_1, \omega \rangle_0, \quad (3.14)$$

where $Z_1, Z_2, Z_3,$ and X_1 are functions on \mathcal{B} taking values in $\mathcal{H}^1, \mathcal{H}^2, \mathcal{H}^3,$ and $\mathcal{H}^1,$ respectively. Using the identities $d^2C = 0$ and $d^2D = 0,$ we again get higher order Bianchi identities

$$Z_1 = \frac{1}{4}\langle R, C \rangle_3,$$

$$Z_2 = \frac{1}{6}\langle S, C \rangle_0 + \frac{7}{4}\langle R, C \rangle_2,$$

$$Z_3 = -\frac{7}{12}\langle R, C \rangle_1,$$

$$X_1 = -\frac{4}{3}\langle S, D \rangle_0 - 25\langle R, D \rangle_2 + \frac{100}{3}\langle R, C \rangle_3.$$

These identities show that for a 7-less metric, the third covariant derivative of the curvature is algebraically described in terms of the lower order covariant derivatives.

We have now seen that the metric and connection forms along with the irreducible components of the covariant differentials of the curvature tensor of a 7-less manifold

satisfy the following set of structure equations on the orthonormal coframe bundle \mathcal{B} :

$$\begin{aligned}
d\omega &= \langle \phi, \omega \rangle_1 \\
d\phi &= \frac{1}{2} \langle \phi, \phi \rangle_1 - \frac{5}{4} \langle R, \langle \omega, \omega \rangle_1 \rangle_2 + \frac{1}{12} \langle S, \langle \omega, \omega \rangle_1 \rangle_2, \\
dR &= -\langle R, \phi \rangle_1 + \frac{1}{10} \langle D, \omega \rangle_0 + \frac{1}{3} \langle C, \omega \rangle_1, \\
dS &= 3 \langle D, \omega \rangle_1, \\
dC &= -\langle C, \phi \rangle_1 + \langle Z, \omega \rangle_1, \\
dD &= -\langle D, \phi \rangle_1 + \langle X, \omega \rangle_0 + \langle \frac{50}{3} Z - \frac{175}{2} \langle R, R \rangle_2 - \frac{25}{3} \langle S, R \rangle_0, \omega \rangle_2, \\
dZ &= -\langle Z, \phi \rangle_1 + \frac{1}{4} \langle \langle R, C \rangle_3, \omega \rangle_0 + \langle \frac{1}{6} \langle S, C \rangle_0 + \frac{7}{4} \langle R, C \rangle_2, \omega \rangle_1 - \frac{7}{12} \langle \langle R, C \rangle_1, \omega \rangle_2, \\
dX &= \langle -\frac{4}{3} \langle S, D \rangle_0 - 25 \langle R, D \rangle_2 + \frac{100}{3} \langle R, C \rangle_3, \omega \rangle_0,
\end{aligned} \tag{3.15}$$

For later calculations, it is useful to have explicit formulas for the target curvature component T along with the components $R, S, C, D, Z,$ and X appearing in the structure equations of a 7-less metric. Let F_I be a function on \mathcal{B} . We define the functions $F_{I;1}, F_{I;2}, F_{I;3}$ on \mathcal{B} by the equation

$$dF_I \equiv F_{I;1}\omega_1 + F_{I;2}\omega_2 + F_{I;3}\omega_3 \pmod{\phi_1, \phi_2, \phi_3}.$$

Let $Riem_{ijkl}$ be the components of the full Riemann curvature tensor as a (0,4)-tensor expressed with respect to the coframing $\omega_1, \omega_2, \omega_3$. Let ϵ_{ijk} be the Levi-Civita symbol: ϵ_{ijk} is the signature of the permutation (i, j, k) if all indices are distinct, and zero if any index is repeated. Then the components of the irreducible curvature

tensors above are given by

$$\begin{aligned}
S &= \text{Riem}_{ijij}, && \text{(Scalar Curvature)} \\
R_{ij} &= \text{Riem}_{ikjk} - \frac{1}{3}S\delta_{ij} && \text{(Traceless Ricci Curvature)} \\
D_i &= S_{;i} && \text{(Scalar Curvature Differential)} \\
C_{ij} &= (R_{li;k} - \frac{1}{4}D_k\delta_{il})\epsilon_{klj} && \text{(Cotton-York Tensor)} \\
T_{ijk} &= R_{ij;k} + R_{jk;i} + R_{ki;j} && \text{(3.16)} \\
&\quad - \frac{6}{15}(\delta_{ij}D_k + \delta_{jk}D_i + \delta_{ki}D_j) && \text{(7-part)} \\
Z_{ij} &= \frac{1}{6}(C_{ik;l} - C_{il;k})\epsilon_{klj}, && \text{(used only when } T = 0) \\
X &= \frac{1}{3}\delta_{ij}D_{i;j}. && \text{(used only when } T = 0)
\end{aligned}$$

3.2 The EDS of a 7-less Metric

Definition 3.4. *Define*

$$\mathcal{H}^c := \mathcal{H}_R^2 \oplus \mathcal{H}_S^0 \oplus \mathcal{H}_C^2 \oplus \mathcal{H}_D^1 \oplus \mathcal{H}_Z^2 \oplus \mathcal{H}_X^0 \cong \mathbb{R}^{20},$$

where the subscripts correspond to correspond to the name of the coordinate on the summand. Let M be a 7-less manifold with oriented orthonormal coframe bundle \mathcal{B} . Define the combined curvature function $\psi : \mathcal{B} \rightarrow \mathcal{H}^c$ to be the function whose components are the individual curvature functions: R, S, C, D, Z, X . That is:

$$\psi = (R, S, C, D, Z, X) : \mathcal{B} \rightarrow \mathcal{H}^c.$$

Equation (3.15) describes $d\psi$ in term of the coframing (ω, ϕ) and ψ itself. The following proposition describes the correspondence between 7-less metrics and the structure equations (3.15).

Proposition 3.5. *Let M be a 7-less manifold with oriented coframe bundle \mathcal{B} and associated combined curvature function $\psi : \mathcal{B} \rightarrow \mathcal{H}^c$. Then $((\omega, \phi), \psi)$ forms an*

augmented coframing on \mathcal{B} in the sense of Theorem 2.9 satisfying structure equations (3.15).

Conversely, assume $((\omega, \phi), \psi)$ forms an augmented coframing satisfying the structure equations (3.15) on a neighborhood $U \subseteq \mathbb{R}^6$ of the origin. Then there exists a 7-less manifold M with associated coframe bundle \mathcal{B} with augmented coframing $((\omega_{\mathcal{B}}, \phi_{\mathcal{B}}), \psi_{\mathcal{B}})$ and a local embedding $f : U \rightarrow \mathcal{B}$ such that $f^*(\omega_{\mathcal{B}}) = \omega$, $f^*(\phi_{\mathcal{B}}) = \phi$, and $f^*(\psi_{\mathcal{B}}) = \psi$.

Proof. Given a 7-less manifold, the augmented coframing $((\omega, \phi), \psi)$ in \mathcal{B} satisfies the structure equations (3.15) by definition of being 7-less.

For the converse, observe that form ω describes a Frobenius system on U and hence is foliated by 3-dimensional integral manifolds. Let L be the leaf space of this foliation. By restricting U to a smaller neighborhood of the origin if necessary, we may assume L has the structure of a smooth manifold and that the projection $\pi : U \rightarrow L$ is a smooth submersion. We can compute that the symmetric 2-form $g_{\mathcal{B}} = \omega_1^2 + \omega_2^2 + \omega_3^2$ on U is basic with respect to the projection π . Hence there exists a symmetric 2-form g on L such that $\pi^*(g) = g_{\mathcal{B}}$. The form g is symmetric and non-degenerate, so (L, g) is a Riemannian manifold. Let \mathcal{B} be the oriented coframe bundle of L . At each point $p \in U$, let the horizontal tangent space $H_p \subseteq T_p U$ be the subspace on which ϕ restricts to be zero. The pushforward π_* gives an isomorphism between H_p and $T_{\pi(p)} M$. As ω restricts to each H_p to be non-degenerate, the isomorphism between H_p and $T_{\pi(p)} M$ induces a coframing at $\pi(p) \in M$. This association defines the map $f : U \rightarrow \mathcal{B}$. We can check that f is a local embedding and that $f^*(\omega_{\mathcal{B}}) = \omega$. Both (ω, ϕ) and $(\omega_{\mathcal{B}}, \phi_{\mathcal{B}})$ satisfy (3.1) and (3.2). By the Fundamental Lemma of Riemannian geometry, $f^*(\phi_{\mathcal{B}}) = \phi$. Finally, $f^*(\psi_{\mathcal{B}}) = \psi$ is seen by differentiating the structure equations (3.1) and (3.2). This shows that $(\omega_{\mathcal{B}}, \phi_{\mathcal{B}})$ has the tensor T vanishing since (ω, ϕ) does, so M is 7-less. \square

Proposition 3.5 shows us that any augmented coframing satisfying (3.15) generates a 7-less manifold. However, distinct (up to diffeomorphism) augmented coframings do not necessarily give distinct (up to diffeomorphism) 7-less manifolds.

Since the combined curvature function ψ is $\mathrm{SO}(3)$ -equivariant, it descends to a function $\bar{\psi} : M \rightarrow \mathcal{H}^c/\mathrm{SO}(3)$ making the following diagram commute:

$$\begin{array}{ccc} B & \xrightarrow{\psi} & \mathcal{H}^c \\ \downarrow \pi & & \downarrow \pi \\ M & \xrightarrow{\bar{\psi}} & \mathcal{H}^c/\mathrm{SO}(3) \end{array}$$

Note that $\mathcal{H}^c/\mathrm{SO}(3)$ is not a manifold.

Definition 3.6. *For a 7-less manifold, we call the function $\bar{\psi} : M \rightarrow \mathcal{H}^c/\mathrm{SO}(3)$ the 7-less curvature function.*

Proposition 3.7. *Let (M, g) and (\tilde{M}, \tilde{g}) be two 7-less manifolds. Let $\bar{\psi} : M \rightarrow \mathcal{H}^c/\mathrm{SO}(3)$ and $\tilde{\bar{\psi}} : \tilde{M} \rightarrow \mathcal{H}^c/\mathrm{SO}(3)$ be the 7-less curvature functions of M and \tilde{M} , respectively. If $p \in M$ and $\tilde{p} \in \tilde{M}$ satisfy $\bar{\psi}(p) = \tilde{\bar{\psi}}(\tilde{p})$, then there exists neighborhoods $U \subseteq M$ and $\tilde{U} \subseteq \tilde{M}$ of p and \tilde{p} , respectively, and an isometry $f : U \rightarrow \tilde{U}$ with $f(p) = \tilde{p}$.*

Proof. Let $\mathcal{B} \xrightarrow{\pi} M$ and $\tilde{\mathcal{B}} \xrightarrow{\tilde{\pi}} \tilde{M}$ be the oriented orthonormal coframe bundles of M and \tilde{M} and let $\bar{\psi} : \mathcal{B} \rightarrow \mathcal{H}^c/\mathrm{SO}(3)$ and $\tilde{\bar{\psi}} : \tilde{\mathcal{B}} \rightarrow \mathcal{H}^c/\mathrm{SO}(3)$ be the combined curvature functions of M and \tilde{M} . Since $\bar{\psi}(p) = \tilde{\bar{\psi}}(\tilde{p})$, there exists $\theta \in \pi^{-1}(p) \subseteq \mathcal{B}$ and $\tilde{\theta} \in \tilde{\pi}^{-1}(\tilde{p}) \subseteq \tilde{\mathcal{B}}$ such that $\psi(\theta) = \tilde{\psi}(\tilde{\theta})$.

Let (ω, ϕ) and $(\tilde{\omega}, \tilde{\phi})$ be the standard coframings on \mathcal{B} and $\tilde{\mathcal{B}}$, respectively. The pairs $((\omega, \phi), \psi)$ and $((\tilde{\omega}, \tilde{\phi}), \tilde{\psi})$ form augmented coframings satisfying the structure equations (3.15) and $\psi(\theta) = \tilde{\psi}(\tilde{\theta})$. By Theorem 2.9, there exist neighborhoods $N \subseteq \mathcal{B}$ and $\tilde{N} \subseteq \tilde{\mathcal{B}}$ of θ and $\tilde{\theta}$ and a diffeomorphism $\tilde{f} : N \rightarrow \tilde{N}$ with $\tilde{f}(\theta) = \tilde{\theta}$,

$f^*(\tilde{\omega}) = \omega$, $f^*(\tilde{\phi}) = \phi$, and $f^*(\tilde{\psi}) = \psi$. As ω , $\tilde{\omega}$, ϕ , $\tilde{\phi}$, a and \tilde{a} are equivariant, we can extend N , \tilde{N} , and \tilde{f} so that we may assume N and \tilde{N} are unions of fibers of π and $\tilde{\pi}$, respectively.

Take $U = \pi(N)$ and $\tilde{U} = \tilde{\pi}(\tilde{N})$. As \tilde{f} pulls back the fundamental 1-form of $\tilde{\mathcal{B}}$ to be the fundamental 1-form of \mathcal{B} , by Theorem 2.4, there exists an isometry $f : U \rightarrow \tilde{U}$ such that $f(p) = \tilde{p}$ as desired. \square

Proposition 3.7 shows that the germ of a 7-less metric at a point is uniquely determined up to diffeomorphism by the value of the 7-less curvature function $\bar{\psi}$ at the point.

3.3 The Variety of Potential Curvatures

In Section 3.2 we showed that a 7-less metric germ corresponds to a point in $\mathcal{H}^c/\text{SO}(3)$ via the function 7-less curvature function, $\bar{\psi}$. In this section, we give necessary conditions on points in $\mathcal{H}^c/\text{SO}(3)$ for such points to correspond to germs of 7-less metrics.

In Section 3.1.2, we made use of the identity $d^2 = 0$ applied to several of the curvature components to derive Bianchi identities. The identities $d^2Z = 0$ and $d^2X = 0$ were not used. Expanding these identities using (3.15) yields the identities $Q_{61} = 0$ and $Q_{63} = 0$ where

$$Q_{61} = 10\langle R, Z \rangle_3 - 3\langle D, C \rangle_2 \in \mathcal{H}_1,$$

$$Q_{63} = 5\langle R, Z \rangle_1 + \langle D, C \rangle_0 \in \mathcal{H}_3.$$

We can further differentiate these zero quantities. The identities $dQ_{61} = 0$ and

$dQ_{63} = 0$ yield identities $Q_{71} = 0$, $Q_{72} = 0$, and $Q_{73} = 0$ where

$$\begin{aligned}
Q_{71} &= 20\langle C, Z \rangle_3 - 18\langle D, Z \rangle_2 + 105\langle\langle R, R \rangle_2, C \rangle_3 + 10\langle\langle S, R \rangle_0, C \rangle_3 \in \mathcal{H}^1, \\
Q_{72} &= -140\langle C, Z \rangle_2 - 2\langle D, Z \rangle_1 - 6\langle C, X \rangle_0 + 105\langle\langle R, R \rangle_0, C \rangle_4 + 105\langle\langle R, R \rangle_2, C \rangle_2 \\
&\quad + 180\langle\langle R, R \rangle_4, C \rangle_0 + 70\langle\langle S, R \rangle_0, C \rangle_2 \in \mathcal{H}^2, \\
Q_{73} &= 30\langle C, Z \rangle_1 + 18\langle D, Z \rangle_0 + 135\langle\langle R, R \rangle_0, C \rangle_3 - 180\langle\langle R, R \rangle_2, C \rangle_1 \\
&\quad + 15\langle\langle S, R \rangle_0, C \rangle_1 \in \mathcal{H}^3.
\end{aligned}$$

The identities $dQ_{71} = 0$, $dQ_{72} = 0$ and $dQ_{73} = 0$ yield identities $Q_{81} = 0$, $Q_{82} = 0$, and $Q_{83} = 0$ where

$$\begin{aligned}
Q_{81} &= -10\langle\langle S, R \rangle_0, Z \rangle_3 + 14\langle\langle R, D \rangle_0, C \rangle_4 - 2\langle\langle R, D \rangle_1, C \rangle_3 + 54\langle\langle R, D \rangle_2, C \rangle_2 \\
&\quad + 3\langle\langle S, D \rangle_0, C \rangle_2 \in \mathcal{H}^1, \\
Q_{82} &= -840\langle Z, Z \rangle_2 - 36\langle X, Z \rangle_0 + 4410\langle\langle R, R \rangle_2, Z \rangle_2 + 420\langle\langle S, R \rangle_0, Z \rangle_2 \\
&\quad + 600\langle\langle R, C \rangle_0, C \rangle_4 + 25\langle\langle R, C \rangle_1, C \rangle_3 - 1080\langle\langle R, C \rangle_2, C \rangle_2 + 21\langle\langle R, D \rangle_0, C \rangle_3 \\
&\quad + 66\langle\langle R, D \rangle_2, C \rangle_1 - 2\langle\langle S, D \rangle_0, C \rangle_1 \in \mathcal{H}^2, \\
Q_{83} &= 90\langle\langle R, R \rangle_0, Z \rangle_3 + 300\langle\langle R, R \rangle_2, Z \rangle_1 + 80\langle\langle S, R \rangle_0, Z \rangle_1 - 210\langle\langle R, C \rangle_0, C \rangle_3 \\
&\quad + 375\langle\langle R, C \rangle_1, C \rangle_2 + 378\langle\langle R, D \rangle_0, C \rangle_2 + 6\langle\langle R, D \rangle_1, C \rangle_1 + 408\langle\langle R, D \rangle_2, C \rangle_0 \\
&\quad + 16\langle\langle S, D \rangle_0, C \rangle_0 \in \mathcal{H}^3.
\end{aligned}$$

The identities $dQ_{81} = 0$, $dQ_{82} = 0$ and $dQ_{83} = 0$ yield identities $Q_{91} = 0$ and $Q_{93} = 0$

where

$$\begin{aligned}
Q_{91} = & -1260\langle\langle R, C \rangle_2, Z \rangle_3 - 300\langle\langle R, C \rangle_3, Z \rangle_2 - 80\langle\langle S, C \rangle_0, Z \rangle_3 - 448\langle\langle R, D \rangle_0, Z \rangle_4 \\
& - 2\langle\langle R, D \rangle_1, Z \rangle_3 + 1242\langle\langle R, D \rangle_2, Z \rangle_2 + 72\langle\langle S, D \rangle_0, Z \rangle_2 + 36\langle\langle R, C \rangle_3, X \rangle_0 \\
& + 12\langle\langle D, C \rangle_1, C \rangle_3 + 9\langle\langle D, D \rangle_0, C \rangle_3 + 1470\langle\langle\langle R, R \rangle_0, R \rangle_3, C \rangle_4 \\
& - 2450\langle\langle\langle R, R \rangle_0, R \rangle_4, C \rangle_3 - 840\langle\langle\langle S, R \rangle_0, R \rangle_2, C \rangle_3 \\
& - 40\langle\langle\langle S, S \rangle_0, R \rangle_0, C \rangle_3 \in \mathcal{H}^1, \\
Q_{93} = & -3960\langle\langle R, C \rangle_0, Z \rangle_3 + 900\langle\langle R, C \rangle_1, Z \rangle_2 + 5280\langle\langle R, C \rangle_2, Z \rangle_1 \\
& + 4800\langle\langle R, C \rangle_3, Z \rangle_0 + 640\langle\langle S, C \rangle_0, Z \rangle_1 + 14904\langle\langle R, D \rangle_0, Z \rangle_2 \\
& - 144\langle\langle R, D \rangle_1, Z \rangle_1 + 12384\langle\langle R, D \rangle_2, Z \rangle_0 + 384\langle\langle S, D \rangle_0, Z \rangle_0 \\
& + 252\langle\langle R, C \rangle_1, X \rangle_0 - 1680\langle\langle C, C \rangle_0, C \rangle_3 + 10800\langle\langle D, C \rangle_0, C \rangle_2 \\
& + 384\langle\langle D, C \rangle_1, C \rangle_1 - 8316\langle\langle\langle R, R \rangle_0, R \rangle_2, C \rangle_3 + 11340\langle\langle\langle R, R \rangle_0, R \rangle_3, C \rangle_2 \\
& - 12775\langle\langle\langle R, R \rangle_0, R \rangle_4, C \rangle_1 + 900\langle\langle\langle S, R \rangle_0, R \rangle_0, C \rangle_3 - 2880\langle\langle\langle S, R \rangle_0, R \rangle_2, C \rangle_1 \\
& + 320\langle\langle\langle S, S \rangle_0, R \rangle_0, C \rangle_1 \in \mathcal{H}^3.
\end{aligned}$$

Finally, we can calculate that dQ_{91} and dQ_{93} are both zero modulo the Q_{ij} quantities showing that the identities $dQ_{91} = 0$ and $dQ_{93} = 0$ yield no new identities.

The components of the identities $Q_{ij} = 0$ are polynomials in the components of the curvature tensors. Thus, these identities generate an ideal over the polynomial ring whose 20 variables are the components of the combined curvature function.

Definition 3.8. *Define $\mathcal{J} \subseteq \mathbb{R}[S, R_{ij}, D_i, C_{ij}, X, Z_{ij}]$ to be the ideal generated by the polynomial components of the Q_{ij} . Let $\mathcal{V} = V(\mathcal{J}) \subseteq \mathcal{H}^c$ be the variety associated to \mathcal{J} .*

Since the generators of \mathcal{J} are given by $\text{SO}(3)$ -invariant pairings, \mathcal{J} and \mathcal{V} are both $\text{SO}(3)$ -invariant. Hence \mathcal{V} is the union of $\text{SO}(3)$ -orbits of \mathcal{H}^c , as expected. The above identities derived from differentiating the structure equations of a 7-less manifold give the following theorem:

Theorem 3.9. *Let M be a 7-less manifold with combined curvature function $\psi : \mathcal{B} \rightarrow \mathcal{H}^c$ and 7-less curvature function $\bar{\psi} : M \rightarrow \mathcal{H}^c/\mathrm{SO}(3)$. Then ψ takes values in \mathcal{V} and $\bar{\psi}$ takes values in $\mathcal{V}/\mathrm{SO}(3)$.*

Construction of Local Solutions

In the previous chapter, we formulated the problem of finding 7-less metrics as finding augmented coframings satisfying some structure equations. The value of the 7-less curvature function $\bar{\psi}$ at a point determined the local structure of the associated 7-less metric. Finally, we gave a subset $\mathcal{V}/\mathrm{SO}(3) \subseteq \mathcal{H}^c/\mathrm{SO}(3)$ that this 7-less curvature function may take values in derived from the variety $\mathcal{V} \subseteq \mathcal{H}^c$ that the combined curvature function must take values on.

While we have shown that it is necessary that the 7-less curvature function $\bar{\psi}$ take values in $\mathcal{V}/\mathrm{SO}(3)$ we have not shown that every point in $\mathcal{V}/\mathrm{SO}(3)$ actually corresponds to the germ of a 7-less metric. A version of the Frobenius theorem can be applied to conclude that, at points where \mathcal{V} is a smooth manifold, a 7-less germ exists such that the 7-less curvature function equals the orbit containing the point. However, at singular points we can draw no such conclusion. Furthermore, this theorem gives only existence of the germs. Explicit description requires integration of a system of ordinary differential equations.

In this chapter, we decompose the variety \mathcal{V} into several parts corresponding to several geometric conditions which we analyze individually.

Definition 4.1. Let $\mathcal{V}_C \subseteq \mathcal{V}$ be the subset of defined by $C = 0$ and $Z = 0$. Let $\mathcal{V}_{NC} = \mathcal{V} \setminus \mathcal{V}_C$ be the subset of \mathcal{V} where either $C \neq 0$ or $Z \neq 0$. Let $\mathcal{V}_{NC}^D \subseteq \mathcal{V}_{NC}$ be the subset where $D = 0$. Let $\mathcal{V}_{NC}^{ND} = \mathcal{V}_{NC} \setminus \mathcal{V}_{NC}^D$ be the subset of \mathcal{V}_{NC} where $D \neq 0$.

These subsets give a partition of \mathcal{V} as

$$\mathcal{V} = \mathcal{V}_C \cup \mathcal{V}_{NC} = \mathcal{V}_C \cup (\mathcal{V}_{NC}^D \cup \mathcal{V}_{NC}^{ND}).$$

Each of the parts of \mathcal{V} will be considered separately. These conditions corresponding to the subsets are:

1. Conformally flat metrics (\mathcal{V}_C),
2. Non-conformally flat metrics (\mathcal{V}_{NC}) :
 - (a) where the gradient of scalar curvature is non-zero (\mathcal{V}_{NC}^{ND}),
 - (b) where the gradient of scalar curvature is zero (\mathcal{V}_{NC}^D), which we will decompose further:
 - i. where the Hessian of scalar curvature has rank equal to one,
 - ii. where the Hessian of scalar curvature has rank not equal to one.

In each case we will construct explicit metrics whose combined curvature functions take values in a part of \mathcal{V} . Once all cases have been covered, all points in $\mathcal{V}/\text{SO}(3)$ will be realized as values of the 7-less curvature function giving the following classification theorem:

Theorem 4.2. *There is a one to one correspondence between diffeomorphism classes of germs of 7-less metrics and points in $\mathcal{V}/\text{SO}(3)$ given by the 7-less curvature function $\bar{\psi}$.*

4.1 The Conformally Flat Case

We begin by constructing conformally flat 7-less metrics. In three dimensions, the conformal class is measured by the Cotton-York tensor, which is an irreducible component of $\nabla Riem$, the first covariant derivative of the Riemannian curvature tensor. We have denoted this tensor as C and it is a component of the combined curvature function ψ .

In three dimensions, a metric is conformally flat if and only if $C = 0$ [Sha97]. Differentiation of this condition gives $dC = 0$ and consequently $Z = 0$. Hence any conformally flat 7-less metric must have the tensors C and Z vanish. This condition corresponds to ψ taking values in \mathcal{V}_C . We observe that when $C = 0$ and $Z = 0$ all of the Q_{ij} expressions generating \mathcal{J} are zero. Hence any point in \mathcal{H}^c where $C = 0$ and $Z = 0$ lies in \mathcal{V} showing

$$\mathcal{V}_C = \mathcal{H}_R^2 \oplus \mathcal{H}_S^0 \oplus 0 \oplus \mathcal{H}_D^1 \oplus 0 \oplus \mathcal{H}_X^0.$$

The subset \mathcal{V}_C forms a subspace of dimension 10 spanned by the components of R , S , D , and X . We will construct a family of conformally flat 7-less metrics whose 7-less curvature functions realize each point in $\mathcal{V}_C/\text{SO}(3)$.

4.1.1 Description of the Manifolds

Let $L = \mathbb{R}^5$ with coordinates $(x_0, x_1, x_2, x_3, x_4)$. Let

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We define a symmetric 2-form on L by

$$g = dx^t Q dx = 2dx_0 dx_4 + dx_1^2 + dx_2^2 + dx_3^2.$$

One can check that g is non-degenerate and has signature $(+, +, +, +, -)$. Thus (L, g) has the structure of a Minkowski space. Next, let P be an arbitrary symmetric 5×5 matrix with entries $p_{ij} = p_{ji}$. We define two quadratic forms on L by $Q(x) = x^t Q x$ and $P(x) = x^t P x$. Let

$$N := Q^{-1}(0)$$

be the null cone of the quadratic form Q .

Definition 4.3. *Let P be an arbitrary quadric form on L as above. Let*

$$\Sigma_P = N \cap P^{-1}(1) \subseteq L.$$

We will refer to manifolds constructed in this way as Σ_P manifolds.

We begin by showing that Σ_P manifolds are in fact Riemannian manifolds.

Proposition 4.4. *For any quadratic form P , $\Sigma_P \subseteq L$ is a 3-manifold. Furthermore, the restriction of g to Σ_P is positive definite making (Σ_P, g) into a Riemannian manifold.*

Proof. First we show Σ_P is indeed a 3-manifold. By the implicit function theorem, $N = Q^{-1}(0)$ is a manifold away from 0 and $P^{-1}(1)$ is a manifold everywhere. If N and $P^{-1}(1)$ intersect transversely at all points, then Σ_P is a smooth manifold. Clearly $0 \notin \Sigma_P$ as $P(0) \neq 1$. As N is a cone, for any nonzero $x \in N$, the radial vector x is tangent to N at x . Since P is a quadratic form, $P(tx) = t^2 P(x)$ for any $t \in \mathbb{R}$ and $x \in \mathbb{R}^5$. Thus, for any $x \in P^{-1}(1)$, the radial vector x is not tangent to $P^{-1}(1)$ as $dP(x) \neq 0$. Now we have that if $x \in \Sigma_P = N \cap P^{-1}(1)$, the vector x is tangent to N but not to $P^{-1}(1)$ and so $T_x N \neq T_x P^{-1}(1)$. Since $T_x N$ and $T_x P^{-1}(1)$ both have co-dimension 1, this implies that N and $P^{-1}(1)$ intersect transversely at x . Hence Σ_P is a smooth manifold of dimension 3 by the implicit function theorem.

Now we show that g restricts to Σ_P to be positive definite. We note that at each point in N , g restricts to be degenerate with three positive eigen-directions and one

null direction that is always parallel to the radial direction. As $T\Sigma_P \subset TN$ at each point and the radial direction is not in $T\Sigma_P$, the restriction of g from N to Σ_P is positive definite. \square

Remark 4.5. *If P_1 and P_2 are two quadratic forms that differ by a multiple of Q , then $\Sigma_{P_1} = \Sigma_{P_2}$. Furthermore, if Σ_{P_1} and Σ_{P_2} are non-empty, $\Sigma_{P_1} = \Sigma_{P_2}$ implies $P_1 - P_2$ is a multiple of Q .*

4.1.2 Computation of the Curvature

We now compute the curvature tensors on Σ_P to see that these manifolds are both 7-less and conformally flat. To do this, we construct the oriented orthonormal coframe bundle \mathcal{B} with its coframing (ω, ϕ) and examine the structure equations. For the remainder of this section, P is an arbitrary fixed quadratic form on L giving a fixed, non-empty manifold Σ_P .

Consider the Lie group

$$SO(Q) = \{A \in \text{GL}(5) : A^tQA = Q, \det(A) = 1\} \subseteq \text{GL}(5).$$

As Q is non-degenerate with signature $(+, +, +, +, -)$, $SO(Q)$ is isomorphic to the indefinite orthogonal group $SO(4, 1)$. We denote the Lie algebra of $SO(Q)$ as $\mathfrak{so}(Q) \subseteq M_{5 \times 5}$. Let $e_0, e_1, e_2, e_3, e_4 : \text{GL}(5) \rightarrow L$ denote the functions projecting into the columns of a matrix.

We now identify the oriented orthonormal coframe bundle of Σ with a subset of $SO(Q)$. Consider an orthonormal coframe $\theta = (\theta_1, \theta_2, \theta_3)_p$ based at $p \in \Sigma_P$. There exist a unique orthonormal frame $f_\theta = (v_1, v_2, v_3)_p$ such that $\theta_i = Q(v_i, \cdot)$, the dual frame. As the radial vector p is not in $T_p\Sigma_P$ we see that the vectors p, v_1, v_2, v_3 are linearly independent.

Since $\Sigma_P \subseteq N = Q^{-1}(0)$, we can calculate that $Q(p, p) = 0$ and the tangent vectors v_i satisfy $Q(p, v_i) = 0$. Finally, as the v_i are orthonormal we have $Q(v_i, v_j) = \delta_{ij}$.

The equations $Q(p, q) = 2$, $Q(p, v_i) = 0$, and $Q(q, q) = 0$ define a unique vector $q \in L$. We define a matrix A_θ associated to the orthonormal coframe by

$$A_\theta = (p, v_1, v_2, v_3, q).$$

By construction, A_θ satisfies $A_\theta^t Q A_\theta = Q$ and consequently has $\det(A_\theta) = \pm 1$. If $\theta = (\theta_1, \theta_2, \theta_3)$, we define $-\theta = (-\theta_1, -\theta_2, -\theta_3)$. We can check that $\det(A_{-\theta}) = -\det(A_\theta)$. The coframes θ that have $\det(A_\theta) = 1$ define an orientation on Σ_P . Thus the map $\theta \mapsto A_\theta$ defines an embedding of the oriented orthonormal frame bundle of Σ_P into $\text{SO}(Q)$. The image of this map is the six dimensional submanifold

$$\mathcal{B} = \{A = (e_0, e_1, e_2, e_3, e_4) \in \text{SO}(Q) : e_0 \in \Sigma_P, e_1, e_2, e_3 \in T_{e_0}\Sigma_P\} \subseteq \text{SO}(Q).$$

With this identification of \mathcal{B} as the oriented orthonormal coframe bundle, the basepoint projection $\pi : \mathcal{B} \rightarrow \Sigma_P$ is $\pi = e_0$, projection into the first column of an element in \mathcal{B} .

We now describe the fundamental 1-form and Levi-Civita connection form on \mathcal{B} as restrictions of the Maurer-Cartan 1-forms of $\text{SO}(Q)$. Differentiation of the defining equations $A^t Q A = Q$ and $\det(A) = 1$ we find that the lie algebra of $\text{SO}(Q)$ is given by

$$\mathfrak{so}(Q) = \left\{ \begin{bmatrix} x_{00} & -x_{14} & -x_{24} & -x_{34} & 0 \\ x_{10} & 0 & x_{12} & -x_{31} & x_{14} \\ x_{20} & -x_{31} & 0 & x_{23} & x_{24} \\ x_{30} & x_{31} & -x_{23} & 0 & x_{34} \\ 0 & -x_{10} & -x_{20} & -x_{30} & -x_{00} \end{bmatrix} : x_{ij} \in \mathbb{R} \right\}.$$

Hence the $\mathfrak{so}(Q)$ -valued Maurer-Cartan form α on $\text{SO}(Q)$ can be expressed in components using \mathbb{R} -valued 1-forms as

$$\alpha = \begin{bmatrix} \tau & -\eta_1 & -\eta_2 & -\eta_3 & 0 \\ \omega_1 & 0 & \phi_3 & -\phi_2 & \eta_1 \\ \omega_2 & -\phi_3 & 0 & \phi_1 & \eta_2 \\ \omega_3 & \phi_2 & -\phi_1 & 0 & \eta_3 \\ 0 & -\omega_1 & -\omega_2 & -\omega_3 & -\tau \end{bmatrix}.$$

As $\text{SO}(Q)$ is a matrix group we can express α as $\alpha = A^{-1}dA$ where $A : \text{SO}(Q) \rightarrow \text{GL}(5)$ is the inclusion of $\text{SO}(Q)$ into $\text{GL}(5)$.

Proposition 4.6. *The restrictions of the forms ω_i and ϕ_i from $\text{SO}(Q)$ to \mathcal{B} form a coframing on \mathcal{B} . Furthermore, the fundamental 1-form on \mathcal{B} is $\omega = (\omega_1, \omega_2, \omega_3)$ and the Levi-Civita connection form on \mathcal{B} is $\phi = (\phi_1, \phi_2, \phi_3)$.*

Proof. The equation $\alpha = A^{-1}dA$ can be rewritten as

$$\begin{aligned}
de_0 &= \tau e_0 + \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3, \\
de_1 &= -\eta_1 e_0 - \phi_3 e_2 + \phi_2 e_3 - \omega_1 e_4, \\
de_2 &= -\eta_2 e_0 + \phi_3 e_1 - \phi_1 e_3 - \omega_2 e_4, \\
de_3 &= -\eta_3 e_0 - \phi_2 e_1 + \phi_1 e_2 - \omega_3 e_4, \\
de_4 &= \eta_1 e_1 + \eta_2 e_2 + \eta_3 e_3 - \tau e_4.
\end{aligned} \tag{4.1}$$

On \mathcal{B} , the function e_0 takes values in Σ_P and so de_0 takes values in $T_{e_0}\Sigma_P$. As e_1, e_2, e_3 form an orthonormal basis for $T_{e_0}\Sigma$, we conclude that when e_0 is restricted to B , de_0 is expressed as a linear combination of e_1, e_2, e_3 alone. This shows that $\tau = 0$ when restricted to \mathcal{B} . As $\tau = 0$ on B we have the expression

$$de_0 = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3$$

when restricted to \mathcal{B} .

The Maurer-Cartan equation $d\alpha = -\alpha \wedge \alpha$ holds on $\text{SO}(Q)$ and implies that

$$d\tau = \eta_1 \wedge \omega_1 + \eta_2 \wedge \omega_2 + \eta_3 \wedge \omega_3.$$

Restricting to \mathcal{B} this equation becomes

$$0 = \eta_1 \wedge \omega_1 + \eta_2 \wedge \omega_2 + \eta_3 \wedge \omega_3$$

since $\tau = 0$ on \mathcal{B} . By Cartan's Lemma [BCG⁺12], this implies that the restrictions of η_i and ω_i satisfy

$$\eta_i = h_{ij}\omega_j \tag{4.2}$$

for some functions $h_{ij} : \mathcal{B} \rightarrow \mathbb{R}$ satisfying $h_{ji} = h_{ij}$.

We conclude that, when restricted to \mathcal{B} , the forms τ and η_i can be expressed as linear combinations of the forms ω_i and ϕ_i . As the forms τ , ω_i , η_i , and ϕ_i form a coframing on $\text{SO}(Q)$ and $B \subseteq \text{SO}(Q)$ is a six dimensional submanifold, this implies that the six forms ω_i and ϕ_i are linearly independent when restricted to \mathcal{B} . Thus the ω_i and ϕ_i form a coframing of \mathcal{B} .

Recall that the projection $\pi : \mathcal{B} \rightarrow \Sigma_P$ is given by $\pi = e_0$. The oriented orthonormal coframe corresponding to a matrix $A \in \mathcal{B}$ is given by the triple

$$\theta = (Q(e_1, \cdot), Q(e_2, \cdot), Q(e_3, \cdot)).$$

The defining condition on the fundamental 1-form ω is

$$\omega(v_\theta) = \theta(\pi_*(v_\theta))$$

for $v_\theta \in T_\theta \mathcal{B}$. We compute

$$\begin{aligned} \theta(\pi_*(v_\theta)) &= \theta(\text{de}_0(v_\theta)) \\ &= \theta((\omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3)(v_\theta)) \\ &= \theta(\omega_1(v_\theta) e_1 + \omega_2(v_\theta) e_2 + \omega_3(v_\theta) e_3) \\ &= \omega_1(v_\theta) \theta(e_1) + \omega_2(v_\theta) \theta(e_2) + \omega_3(v_\theta) \theta(e_3) \\ &= \omega_1(v_\theta) (1, 0, 0) + \omega_2(v_\theta) (0, 1, 0) + \omega_3(v_\theta) (0, 0, 1) \\ &= (\omega_1(v_\theta), \omega_2(v_\theta), \omega_3(v_\theta)). \end{aligned}$$

Hence we conclude $\omega = (\omega_1, \omega_2, \omega_3)$ as desired.

What remains is to show that $\phi = (\phi_1, \phi_2, \phi_3)$ is the Levi-Civita connection form on \mathcal{B} . The defining property of ϕ is that it satisfy the structure equations (3.1) and (3.2), which must be suitably modified to align with the omitted index notation. Making use of the Maurer-Cartan equation $\text{d}\alpha = -\alpha \wedge \alpha$ along with (4.2),

we calculate

$$\begin{aligned}
d\omega_1 &= \phi_2 \wedge \omega_3 - \phi_3 \wedge \omega_2, \\
d\omega_2 &= \phi_3 \wedge \omega_1 - \phi_1 \wedge \omega_3, \\
d\omega_3 &= \phi_1 \wedge \omega_2 - \phi_2 \wedge \omega_1,
\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}
d\phi_1 &= \phi_2 \wedge \phi_3 + \omega_2 \wedge \eta_3 - \omega_3 \wedge \eta_2 \\
&= \phi_2 \wedge \phi_3 + (h_{22} + h_{33})\omega_2 \wedge \omega_3 - h_{12}\omega_3 \wedge \omega_1 - h_{31}\omega_1 \wedge \omega_2, \\
d\phi_2 &= \phi_3 \wedge \phi_1 + \omega_3 \wedge \eta_1 - \omega_1 \wedge \eta_3 \\
&= \phi_3 \wedge \phi_1 - h_{12}\omega_2 \wedge \omega_3 + (h_{33} + h_{11})\omega_3 \wedge \omega_1 - h_{23}\omega_1 \wedge \omega_2, \\
d\phi_3 &= \phi_1 \wedge \phi_2 + \omega_1 \wedge \eta_2 - \omega_2 \wedge \eta_1 \\
&= \phi_1 \wedge \phi_2 - h_{31}\omega_2 \wedge \omega_3 - h_{23}\omega_3 \wedge \omega_1 + (h_{11} + h_{22})\omega_1 \wedge \omega_2,
\end{aligned} \tag{4.4}$$

which are (3.1) and (3.2) in omitted index notation. As ϕ satisfies the structure equations, it is the Levi-Civita connection form on \mathcal{B} . \square

The calculation in (4.4) shows that the components of the Riemann curvature tensor are expressed in terms of the h_{ij} functions. The following lemma shows the relationship between the quadratic form P defining Σ_P and the h_{ij} functions.

Lemma 4.7. *Define the function $K : \mathcal{B} \rightarrow M_{5 \times 5}$ by*

$$K(A) = A^t P A. \tag{4.5}$$

Denote the components of K by k_{ij} for $0 \leq i, j \leq 4$. Then $k_{ji} = k_{ij}$ and the functions h_{ij} from (4.2) are given as

$$\begin{aligned}
h_{11} &= k_{11} - k_{40}, & h_{23} &= k_{23}, \\
h_{22} &= k_{22} - k_{40}, & h_{31} &= k_{31}, \\
h_{33} &= k_{33} - k_{40}, & h_{12} &= k_{12}.
\end{aligned} \tag{4.6}$$

Proof. As P is symmetric, $K = A^t P A$ is also symmetric showing $k_{ji} = k_{ij}$. Differentiating K we get

$$\begin{aligned}
dK &= d(A^t P A) \\
&= (dA)^t P A + A^t P (dA) \\
&= (A\alpha)^t P A + A^t P (A\alpha) \\
&= \alpha^t (A^t P A) + (A^t P A)\alpha \\
&= \alpha^t K + K\alpha,
\end{aligned} \tag{4.7}$$

where α is the restriction of the Maurer-Cartan form of $\text{SO}(Q)$ to \mathcal{B} . Recall that on \mathcal{B} the function projecting into the first column, e_0 , takes values in Σ_P . As $\Sigma_P \subseteq P^{-1}(1)$ we see that $k_{00} = 1$ and hence $dk_{00} = 0$. The k_{00} component of equation (4.7) now becomes

$$0 = dk_{00} = 2k_{01}\omega_1 + 2k_{02}\omega_2 + 2k_{03}\omega_3.$$

As the ω_i forms are linearly independent on \mathcal{B} we conclude $k_{01} = 0$, $k_{02} = 0$, and $k_{03} = 0$ and hence $dk_{01} = 0$, $dk_{02} = 0$, and $dk_{03} = 0$. Looking at the k_{01} , k_{02} , and k_{03} components of (4.7) we get

$$\begin{aligned}
0 = dk_{01} &= k_{11}\omega_1 + k_{12}\omega_2 + k_{31}\omega_3 - \eta_1 - k_{04}\omega_1 \\
&= (k_{11} - k_{04} - h_{11})\omega_1 + (k_{12} - h_{12})\omega_2 + (k_{31} - h_{31})\omega_3, \\
0 = dk_{02} &= k_{12}\omega_1 + k_{22}\omega_2 + k_{23}\omega_3 - \eta_2 - k_{04}\omega_2 \\
&= (k_{12} - h_{12})\omega_1 + (k_{22} - k_{04} - h_{22})\omega_2 + (k_{23} - h_{23})\omega_3, \\
0 = dk_{03} &= k_{31}\omega_1 + k_{23}\omega_2 + k_{33}\omega_3 - \eta_3 - k_{04}\omega_3 \\
&= (k_{31} - h_{31})\omega_1 + (k_{23} - h_{23})\omega_2 + (k_{33} - k_{04} - h_{33})\omega_3.
\end{aligned}$$

Again, as the ω_i are linearly independent on \mathcal{B} we can conclude that each of the coefficients in the above expressions are zero which implies (4.6). \square

Recall that components $Riem_{ijkl}$ of the Riemann curvature tensor are equivariant functions on the oriented coframe bundle \mathcal{B} . The structure equations (4.4) and

expressions (4.5) and (4.6) allow us to express the components of the Riemann curvature tensor in term of the defining quadratic form P and the pointwise coframe given as a point in \mathcal{B} :

$$\begin{aligned}
Riem_{2323} &= k_{22} + k_{33} - 2k_{40}, & Riem_{3112} &= -k_{23}, \\
Riem_{3131} &= k_{33} + k_{11} - 2k_{40}, & Riem_{1223} &= -k_{31}, \\
Riem_{1212} &= k_{11} + k_{22} - 2k_{40}, & Riem_{2331} &= -k_{12}.
\end{aligned} \tag{4.8}$$

Equation (4.7) gives an explicit expression of the exterior derivative of each k_{ij} function and hence can be used to compute derivatives of the $Riem_{ijkl}$. Using the formulas given by (3.16) we can explicitly compute the components of the curvature tensors.

Theorem 4.8. *Every Σ_P manifold is both conformally flat and 7-less.*

Proof. Using (4.7) and (3.16) we can compute $C_{ij} = 0$, $Z_{ij} = 0$, and $T_{ijk} = 0$. Thus Σ_P has vanishing Cotton tensor implying conformal flatness in three dimensions. The vanishing of the tensor T is the definition of 7-less. \square

Using (4.7) and (3.16) to compute the remaining curvature components we find

$$\begin{aligned}
R_{11} &= \frac{2}{3}k_{11} - \frac{1}{3}k_{22} - \frac{1}{3}k_{33}, & R_{23} &= k_{23}, & D_1 &= -20k_{41}, \\
R_{22} &= \frac{2}{3}k_{22} - \frac{1}{3}k_{33} - \frac{1}{3}k_{11}, & R_{31} &= k_{31}, & D_2 &= -20k_{42}, \\
R_{33} &= \frac{2}{3}k_{33} - \frac{1}{3}k_{11} - \frac{1}{3}k_{22}, & R_{12} &= k_{12}, & D_3 &= -20k_{43}, \\
S &= 4k_{11} + 4k_{22} + 4k_{33} - 12k_{40}, \\
X &= \frac{20}{3}(-k_{11}^2 - k_{22}^2 - k_{33}^2 - 2k_{23}^2 - 2k_{31}^2 - 2k_{12}^2 \\
&\quad + 2k_{11}k_{40} + 2k_{22}k_{40} + 2k_{33}k_{40} - 3k_{40}^2 + 3k_{44}).
\end{aligned} \tag{4.9}$$

Equation (4.9) gives us an explicit expression for the combined curvature function $\psi : B \rightarrow \mathcal{H}^c$. We are now ready to show that every 7-less germ can be realized by a Σ_P manifold.

Lemma 4.9. *For every point $v \in \mathcal{V}_C$, there exists a quadratic form P with associated manifold Σ_P with oriented orthonormal coframe bundle $\mathcal{B} \subseteq \text{SO}(Q)$ such that $\psi(A) = v$ for some $A \in \mathcal{B}$.*

Proof. By examining (4.9) we see that if the functions k_{ij} for $1 \leq i, j \leq 4$ can be made to take arbitrary values at a point in \mathcal{B} , then any combined curvature in \mathcal{V}_C can be achieved. Consider a quadratic form of form

$$\bar{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & p_{11} & p_{12} & p_{31} & p_{41} \\ 0 & p_{12} & p_{22} & p_{23} & p_{42} \\ 0 & p_{31} & p_{23} & p_{33} & p_{43} \\ 0 & p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix}.$$

The bundle \mathcal{B} associated to the manifold $\Sigma_{\bar{P}}$ contains the identity matrix I . We have $K(I) = I^t \bar{P} I = \bar{P}$ which shows that the function k_{ij} with $1 \leq i, j \leq 4$ can be made arbitrary at I by the correct choice of \bar{P} . In fact, each point in \mathcal{V}_C is the value of the combined curvature function ψ at $I \in B$ for a unique choice of \bar{P} . \square

Descending from the combined curvature function to the 7-less curvature function we get the following.

Theorem 4.10. *For every point $v \in \mathcal{V}_C/\text{SO}(3)$, there exists a quadratic form P with associated manifold Σ_P and point in $p \in \Sigma$ such that $\bar{\psi}(p) = v$.*

Corollary 4.11. *If M is a 7-less manifold with tensors C and Z both zero at a point $p \in M$, then a neighborhood of p is isometric to an open subset of some Σ_P manifold.*

Proof. The 7-less curvature function takes a value in $\mathcal{V}_C/\text{SO}(3)$ at $p \in M$. Theorem 4.10 shows that every point in $\mathcal{V}_C/\text{SO}(3)$ is realized by a point in some Σ_P . The uniqueness result Proposition 3.7 ensures that the desired isometry exists. \square

The classification of conformally flat 7-less germs gives us the following global result:

Proposition 4.12. *Let M be a connected 7-less manifold. If the tensors C and Z are zero at a point, then C and Z are zero on all of M .*

Proof. The conditions $C = 0$ and $Z = 0$ are closed conditions showing that the points where C and Z are zero form a closed set. Corollary 4.11 shows that if C and Z are zero at a point p , there is a local isometry about p with a Σ_P manifold on which C and Z vanish identically. Thus C and Z are zero in a neighborhood of p and so the set of points where C and Z are both zero is also open. \square

Thus, to say that a connected 7-less manifold is not conformally flat is equivalent to the statement that the tensors C and Z never simultaneously vanish and so the 7-less curvature function never takes values in $\mathcal{V}_C/\text{SO}(3)$ and always takes values in $\mathcal{V}_{NC}/\text{SO}(3)$.

4.2 The Non-conformally Flat Case

We now turn our attention to 7-less metrics that are not conformally flat. In particular, we are interested in 7-less germs whose 7-less curvature function takes values in $\mathcal{V}_{NC}/\text{SO}(3)$.

The calculations in this section make extensive use of Gröbner bases calculations to solve a system of equations and derive properties of points in \mathcal{V}_{NC} . In most cases, the calculations themselves are placed in the appendices with only their results stated.

Our analysis of a non-conformally flat 7-less metric breaks into two major cases depending on the behavior of the scalar curvature. The generic case has the differential of scalar curvature, measured by the tensor D , being non-vanishing. This case allows for various normalizations of the problem being possible. The non-generic points where D vanishes turn out to form sub-manifolds of co-dimension at least

one. Some of these non-generic points admit additional symmetry and allow us to construct metrics in a different way.

We will find that in a neighborhood of a generic point in a non-conformally flat 7-less manifold, there is a two dimensional group of local symmetries of the metric. This contrasts with the case of conformally flat 7-less metrics, where the group of local metric symmetries are generically zero dimensional.

4.2.1 Non-zero Differential of Scalar Curvature

We begin our analysis of non-conformally flat 7-less manifolds with the generic points where the tensor D is non-zero and hence have their 7-less curvature function $\bar{\psi}$ take values in $\mathcal{V}_{NC}^{ND}/\text{SO}(3)$.

We begin by giving a parameterization of the sub-variety \mathcal{V}_{NC}^{ND} .

Proposition 4.13. *Every $\text{SO}(3)$ -orbit of \mathcal{V}_{NC}^{ND} contains a point whose components take the form*

$$\begin{aligned}
R_{11} &= \lambda(C_{33} - C_{22}), & Z_{11} &= \mu(C_{33} - C_{22}), \\
R_{22} &= \lambda(5C_{22} + 4C_{33}), & Z_{22} &= \mu(5C_{22} + 4C_{33}), \\
R_{33} &= -\lambda(4C_{22} + 5C_{33}), & Z_{33} &= -\mu(4C_{22} + 5C_{33}), \\
R_{31} &= 0, & Z_{31} &= 0, & C_{31} &= 0, \\
R_{12} &= 0, & Z_{12} &= 0, & C_{12} &= 0, \\
& & Z_{23} &= \frac{1}{3}\kappa(2C_{22} + C_{33})(C_{22} + 2C_{33}), & C_{23} &= 0, \\
D_1 &= 60(\lambda Z_{23} - \mu R_{23}), & D_2 &= 0, & D_3 &= 0, \\
S &= 2\kappa Z_{23} - 6\lambda C_{22} + 6\lambda C_{33} + 54\mu^2, \\
X &= \frac{1}{3}(20(SR_{22} + R_{23}^2 + SR_{33}) + 80(R_{22}^2 + R_{33}^2) + 140R_{22}R_{33} \\
&\quad - 40(Z_{22} - Z_{33})).
\end{aligned} \tag{4.10}$$

where the constants λ , μ , and κ satisfy

$$27\lambda\mu + R_{23}\kappa = 1.$$

Conversely, for every choice of λ, μ, κ , and coordinates $S, R_{ij}, D_i, C_{ij}, X$, and Z_{ij} satisfying the above relations with $D_1 \neq 0$ and either $C_{22} \neq 0$ or $C_{33} \neq 0$ gives a point in \mathcal{V}_{NC}^{ND} .

Proof. See Appendix A. □

Proposition 4.13 gives a complete description of \mathcal{V}_{NC}^{ND} and shows that \mathcal{V}_{NC}^{ND} consists of orbits of two types depending on the value of the C_{ij} . Let p be the point of an orbit described in Proposition 4.13. If p has $C_{22} \neq C_{33}$, then the stabilizer of p is trivial and hence the orbit is homomorphic to $\text{SO}(3)$. If p has $C_{22} = C_{33}$, then the $\text{SO}(2)$ subgroup of $\text{SO}(3)$ action acting on the subspace $\mathbb{R}[x_2, x_3] \subseteq \mathbb{R}[x_1, x_2, x_3]$ is the stabilizer of p and the orbit is homomorphic to $\text{SO}(3)/\text{SO}(2, \mathbb{R}) \cong S^2$.

Definition 4.14. *Let M be a 7-less manifold. A coframe $\theta \in \mathcal{B}$ is called C -adapted if $\psi(\theta) \in \mathcal{H}^c$ is one of the special points described in Proposition 4.13.*

If M is a 7-less manifold that is not conformally flat and $p \in M$ is a point at which the differential of scalar curvature is not zero, then the combined curvature function ψ takes values in \mathcal{V}_{NC}^{ND} at every point in the fiber $\pi^{-1}(p) \subseteq B$. Equivariance of ψ along with Proposition 4.13 tells us that there is a point in $\pi^{-1}(p)$ where the components of ψ satisfy the relationships in Proposition 4.13, showing that there exists a C -adapted coframe at p .

We have the following symmetry property:

Lemma 4.15. *Let M be 7-less and let $p \in M$ be a point with $\bar{\psi}(p) \in \mathcal{V}_{NC}^D/\text{SO}(3)$. Let U be a neighborhood of p where scalar curvature function has a non-vanishing differential and foliates U with connected level sets. Then the 7-less curvature function $\bar{\psi}$ is constant on the level sets of S in U .*

Proof. Let $q \in U$ and let $\omega = (\omega_1, \omega_2, \omega_3)$ be a C -adapted coframe at q . At $\omega \in \mathcal{B}$,

dD_2 and dD_3 take the form

$$dD_2 = F_{2i}\omega_i + D_1\phi_3,$$

$$dD_3 = F_{3i}\omega_i - D_1\phi_2,$$

Let Y_2 and Y_3 at ω be vectors dual to the 1-forms

$$\theta_1 = D_1\omega_2 + F_{32}\phi_2 - F_{22}\phi_3,$$

$$\theta_2 = D_1\omega_3 + F_{33}\phi_2 - F_{23}\phi_3,$$

A direct computation using (3.15) and the relations of Proposition 4.13 shows that $d\psi(Y_2) = 0$ and $d\psi(Y_3) = 0$. The pushforwards $\pi_*(Y_2)$ and $\pi_*(Y_3)$ are linearly independent and $d\psi(Y_2) = 0$ and $d\psi(Y_3) = 0$ imply that $\pi_*(Y_2)$ and $\pi_*(Y_3)$ are tangent to the level set of S at p . Hence $\pi_*(Y_2)$ and $\pi_*(Y_3)$ span the tangent space to the level set of S at p . This shows $d\bar{\psi}$ restricts to zero on the level sets of S . Hence $\bar{\psi}$ is constant on the level sets of S . \square

Lemma 4.15 shows that there should be two linearly independent symmetry vector fields around points where D is non-zero. The following proposition describes the behavior of these symmetries at generic points:

Proposition 4.16. *Let M be a 7-less manifold and let $p \in M$ be a point where $\bar{\psi}(p) \in \mathcal{V}_{NC}^D/\text{SO}(3)$ and pointwise C -adapted coframes have $C_{22} \neq C_{33}$. There exist a neighborhood U of p and two linearly independent Killing fields Y_2 and Y_3 tangent to the level sets of the scalar curvature function S such that $[Y_2, Y_3] = 0$.*

Proof. Let U be a simply connected neighborhood of p such that on U , $D \neq 0$, C -adapted coframes have $C_{22} \neq C_{33}$, and the level sets of S are connected. Let $\omega = (\omega_1, \omega_2, \omega_3)$ be a C -adapted coframe on U and let X_1, X_2, X_3 be the vector fields dual to the ω_i .

We begin by giving expression for the $d\omega_i$. The coframing ω is equivalent to a section $i : U \rightarrow \mathcal{B}$. The defining property of the fundamental 1-form $\omega_{\mathcal{B}}$ on the

oriented orthonormal coframe bundle \mathcal{B} is that $\omega = i^*(\omega_{\mathcal{B}})$. Let

$$\phi = (\phi_1, \phi_2, \phi_3) = i^*(\phi_{\mathcal{B}}), \quad \psi = i^*(\psi_{\mathcal{B}}),$$

where $\phi_{\mathcal{B}}$ and $\psi_{\mathcal{B}}$ are the connection 1-form and combined curvature function on \mathcal{B} , respectively. The pullback of the structure equations (3.15) gives the exterior derivatives of the coframe ω . As ω is C -adapted, the components of ψ satisfy the conditions of Proposition 4.13. In particular, a C -adapted coframe satisfies the identities $D_2 = 0$, $D_3 = 0$, and $C_{23} = 0$. Differentiating these identities using the structure equations (3.15), we get

$$\begin{aligned} 0 &= dC_{23} = F_{1i}(\psi)\omega_i + G_{1i}(\psi)\phi_i, \\ 0 &= dD_2 = F_{2i}(\psi)\omega_i + G_{2i}(\psi)\phi_i, \\ 0 &= dD_3 = F_{3i}(\psi)\omega_i + G_{3i}(\psi)\phi_i, \end{aligned} \tag{4.11}$$

where the $F_{ij}(\psi)$ and $G_{ij}(\psi)$ are polynomial expressions in terms of the components of ψ . The conditions on ψ implied by Proposition 4.13 simplify (4.11) considerably and the assumptions $D_1 \neq 0$ and $C_{22} \neq C_{33}$ ensure that the ϕ_i can be solved for in the system, giving

$$\phi_i = H_{ij}(\psi)\omega_j$$

where $H_{ij}(\psi)$ are some rational functions in the components of ψ . At this point, we can express the $d\omega_i$ in terms of the components of ψ and the ω_i themselves. With these expressions at hand, we can proceed to computing Lie brackets and Lie derivatives.

We claim $[X_2, X_3] = 0$. It is sufficient to show $\omega_i([X_2, X_3]) = 0$ for each ω_i . Using the identity

$$\omega_i([X_2, X_3]) = d(\omega_i(X_3))(X_2) - d(\omega_i(X_2))(X_3) - d\omega_i(X_2, X_3)$$

and the expression for the $d\omega_i$, a direct calculation shows $\omega_i([X_2, X_3]) = 0$ for each ω_i and hence $[X_2, X_3] = 0$.

Next, we compute $\mathcal{L}_{X_2}\omega_i$ and $\mathcal{L}_{X_3}\omega_i$. We can use Cartan's magic formula

$$\mathcal{L}_{X_j}\omega_i = i_{X_j}d(\omega_i) + d(i_{X_j}\omega_i)$$

to compute

$$\begin{aligned} \mathcal{L}_{X_2}\omega_1 &= 0, & \mathcal{L}_{X_2}\omega_2 &= A_{22}(\psi)\omega_1, & \mathcal{L}_{X_2}\omega_3 &= A_{23}(\psi)\omega_1, \\ \mathcal{L}_{X_3}\omega_1 &= 0, & \mathcal{L}_{X_3}\omega_2 &= A_{32}(\psi)\omega_1, & \mathcal{L}_{X_3}\omega_3 &= A_{33}(\psi)\omega_1, \end{aligned}$$

for some rational functions $A_{ij}(\psi)$ that are expression in the components of ψ . As the components of ψ are constant on level sets of S by Lemma 4.15, the A_{ij} can be regarded as functions of S alone. We have

$$\begin{aligned} \mathcal{L}_{X_i}g &= \mathcal{L}_{X_i}(\omega_1^2 + \omega_2^2 + \omega_3^2) \\ &= 2\omega_1\mathcal{L}_{X_i}\omega_1 + 2\omega_2\mathcal{L}_{X_i}\omega_2 + 2\omega_3\mathcal{L}_{X_i}\omega_3 \\ &= 2A_{i2}\omega_1\omega_2 + 2A_{i2}\omega_1\omega_2 \end{aligned}$$

for $i = 2, 3$. In general, the A_{ij} are not zero, so X_2 and X_3 are not Killing fields. Let $f_2(S)$ and $f_3(S)$ be functions of the scalar curvature on U . As f_2 and f_3 are constant on the level sets of S , the differentials can be expressed as

$$df_2 = f_2'(S)\omega_1, \quad df_3 = f_3'(S)\omega_1.$$

Recall the identity

$$\mathcal{L}_{f_i X_j}\omega_k = f_i\mathcal{L}_{X_j}\omega_k + df_i \wedge i_{X_j}\omega_k.$$

Using this identity we get

$$\mathcal{L}_{f_2 X_2 + f_3 X_3}g = 2(f_2 A_{22} + f_2' + f_3 A_{32})\omega_1\omega_2 + 2(f_3 A_{33} + f_3' + f_2 A_{23})\omega_1\omega_3.$$

We see that $\mathcal{L}_{f_2 X_2 + f_3 X_3}g = 0$ if f_2 and f_3 satisfy the system of differential equations

$$\begin{bmatrix} f_2 \\ f_3 \end{bmatrix}' = - \begin{bmatrix} A_{22} & A_{32} \\ A_{23} & A_{33} \end{bmatrix} \begin{bmatrix} f_2 \\ f_3 \end{bmatrix}. \quad (4.12)$$

Let $Y_2 = f_2X_2 + f_3X_3$ where f_2 and f_3 are solutions to (4.12) with initial conditions $(f_2(S(p)), f_3(S(p))) = (1, 0)$ and let $Y_3 = g_2X_2 + g_3X_3$ where g_2 and g_3 are solutions to (4.12) with initial conditions $(g_2(S(p)), g_3(S(p))) = (0, 1)$. By construction, Y_2 and Y_3 are Killing fields. Lastly we must show $[Y_2, Y_3] = 0$. We have

$$\begin{aligned} [Y_2, Y_3] &= [f_2X_2 + f_3X_3, g_2X_2 + g_3X_3] \\ &= [f_2X_2, g_2X_2] + [f_2X_2, g_3X_3] + [f_3X_3, g_2X_2] + [f_3X_3, g_3X_3]. \end{aligned}$$

For each term we have

$$\begin{aligned} [f_iX_i, g_jX_j] &= f_i \cdot dg_j(X_i)X_j - g_j \cdot df_i(X_j)X_i - f_i g_j [X_i, X_j] \\ &= f_i \cdot 0 \cdot X_j - g_j \cdot 0 \cdot X_i - f_i g_j \cdot 0 \\ &= 0. \end{aligned}$$

Thus $[Y_2, Y_3] = 0$ as desired. By continuity, these vector fields are linearly independent in some neighborhood of p . \square

The existence of these commuting Killing fields allows us to construct particularly nice coordinate charts.

Proposition 4.17. *Let M be a 7-less manifold and let $p \in M$ be a point where $\bar{\psi}(p) \in \mathcal{V}_{NC}^D/\mathrm{SO}(3)$ and pointwise C -adapted coframes have $C_{22} \neq C_{33}$. There exists a coordinate chart (x, y, z) about p such that the metric g takes the form*

$$g = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}^t \begin{bmatrix} A(z) & B(z) & 0 \\ B(z) & C(z) & 0 \\ 0 & 0 & (A(z)C(z) - B(z)^2)^{-1} \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}. \quad (4.13)$$

Proof. By assumption $D \neq 0$ in a neighborhood of p and so S can be taken as a coordinate function in a neighborhood of p . Let (x_1, x_2, x_3) be coordinate functions in a neighborhood of p with $x_3 = S$. Write the metric as

$$g = g_{ij}(x_1, x_2, x_3)dx_i dx_j$$

with $g_{ij} = g_{ji}$. By Proposition 4.16, there are commuting Killing fields Y_1 and Y_2 that are tangent to the level sets of $z = S$. Integrating these two vector fields, we can construct new x_1 and x_2 and coordinate functions for which $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$ are Killing fields. As $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$ are Killing fields of the metric components must not depend on x_1 nor x_2 , so our metric in this coordinate system takes the form

$$g = \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}^t \begin{bmatrix} g_{11}(x_3) & g_{12}(x_3) & g_{13}(x_3) \\ g_{21}(x_3) & g_{22}(x_3) & g_{23}(x_3) \\ g_{31}(x_3) & g_{32}(x_3) & g_{33}(x_3) \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}.$$

Define new coordinates $\tilde{x}_1 = x_1 - f_1(z)$, $\tilde{x}_2 = x_2 - f_2(z)$, and $\tilde{x}_3 = x_3$ where f_1 and f_2 are functions satisfying the differential equation

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}' = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}^{-1} \begin{bmatrix} g_{13} \\ g_{23} \end{bmatrix}.$$

In the new coordinate system the metric takes the form

$$g = \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}^t \begin{bmatrix} \tilde{g}_{11}(x_3) & \tilde{g}_{12}(x_3) & 0 \\ \tilde{g}_{21}(x_3) & \tilde{g}_{22}(x_3) & 0 \\ 0 & 0 & \tilde{g}_{33}(x_3) \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}.$$

Finally, we can take coordinates $(x, y, z) = (\tilde{x}_1, \tilde{x}_2, F^{-1}(\tilde{x}_3))$ where $x_3 = F(z)$ is a reparameterization of the x_3 coordinate satisfying the differential equation

$$F'(z) = (\tilde{g}_{33}(F(z))(\tilde{g}_{11}(F(z))\tilde{g}_{22}(F(z)) - \tilde{g}_{12}(F(z))^2))^{-1/2}.$$

In the coordinate system (x, y, z) , the metric takes the form prescribed in (4.13). \square

Proposition 4.17 only guarantees the existence of these coordinate systems in neighborhoods of points where C -adapted coframes have $C_{22} \neq C_{33}$. However, we will see that these types of metrics can produce 7-less germs whose 7-less curvature functions realize points in $\mathcal{V}_{NC}/\text{SO}(3)$ that do not necessarily correspond to all of the conditions of the proposition being satisfied.

We now look at what how the 7-less condition is realized in coordinate systems where the metric takes the form (4.13).

Proposition 4.18. *Let $\mathbb{R}^2 \times \mathbb{R}^1$ have coordinates (x_1, x_2, z) and let g be a symmetric 2-form taking the form*

$$g = \begin{bmatrix} dx_1 \\ dx_2 \\ dz \end{bmatrix}^t \begin{bmatrix} A(z) & B(z) & 0 \\ B(z) & C(z) & 0 \\ 0 & 0 & (A(z)C(z) - B(z)^2)^{-1} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dz \end{bmatrix},$$

for some functions $A(z), B(z), C(z)$. Let $I \subseteq \mathbb{R}$ be a z -interval on which g is positive definite. Let $M = \mathbb{R}^2 \times I \subseteq \mathbb{R}^2 \times \mathbb{R}^1$. The Riemannian manifold (M, g) is 7-less if and only if $A(z), B(z)$, and $C(z)$ are polynomials of degree at most two in z .

Proof. We first note that g is positive definite exactly when the 2×2 matrix

$$\begin{bmatrix} A(z) & B(z) \\ B(z) & C(z) \end{bmatrix}$$

is positive definite. This is when $A > 0$, $C > 0$, and $AC - B^2 > 0$.

We can compute the components T_{ijk} of the tensor T with respect to the (possibly non-orthonormal) coframe (dx_1, dx_2, dz) using coordinate expression for the components of the Riemann curvature tensor and Christoffel symbols. The seven

independent components of T are

$$T_{111} = 0,$$

$$T_{112} = 0,$$

$$T_{113} = \frac{1}{10} (5B^2A''' - ACA''' - 8ABB''' + 4A^2C'''),$$

$$T_{221} = 0,$$

$$T_{222} = 0,$$

$$T_{223} = \frac{1}{10} (4C^2A''' - 8BCB''' + 5B^2C''' - ACC'''),$$

$$T_{123} = \frac{1}{10} (4BCA''' - 3B^2B''' - 5ACB''' + 4ABC''').$$

The components that are not identically zero show that the metric is 7-less if and only if A''' , B''' , and C''' satisfy the linear system

$$\begin{bmatrix} 5B^2 - AC & -8AB & 4A^2 \\ 4C^2 & -8BC & 5B^2 - AC \\ 4BC & -3B^2 - 5AC & 4AB \end{bmatrix} \begin{bmatrix} A''' \\ B''' \\ C''' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The determinant of the coefficient matrix of this system is $-75(AC - B^2)^3$, which is non-zero, as $AC - B^2$ is non-zero. Hence the metric is 7-less if and only if

$$A'''(z) = 0, \quad B'''(z) = 0, \quad C'''(z) = 0.$$

This is equivalent to A , B , and C being polynomials of degree at most two. \square

Definition 4.19. We call a metric g on $\mathbb{R}^2 \times \mathbb{R}^1$ block 7-less if it takes the form

$$g = \begin{bmatrix} dx_1 \\ dx_2 \\ dz \end{bmatrix}^t \begin{bmatrix} A(z) & B(z) & 0 \\ B(z) & C(z) & 0 \\ 0 & 0 & (A(z)C(z) - B(z)^2)^{-1} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dz \end{bmatrix},$$

where $A(z)$, $B(z)$, and $C(z)$ are polynomials of degree at most two.

We now look at what points in $\mathcal{V}/\text{SO}(3)$ are realized as values of the 7-less curvature function on block 7-less metrics.

Lemma 4.20. *Let g be a block 7-less metric with*

$$\begin{aligned} A(z) &= 1 + a_1 z + a_2 z^2, \\ B(z) &= b_1 z + b_2 z^2, \\ C(z) &= 1 + c_1 z + c_2 z^2. \end{aligned}$$

With respect to the orthonormal coframe (dz, dx_1, dx_2) at the origin the curvature tensors take the form

$$\begin{aligned} R_{11} &= \frac{1}{3}(-a_2 - c_2), & C_{11} &= \frac{1}{2}(a_2 b_1 - a_1 b_2 + b_2 c_1 - b_1 c_2), \\ R_{22} &= \frac{1}{3}(2c_2 - a_2), & C_{22} &= \frac{1}{2}(a_2 b_1 - a_1 b_2 - 2b_2 c_1 + 2b_1 c_2), \\ R_{33} &= \frac{1}{3}(2a_2 - c_2), & C_{33} &= \frac{1}{2}(b_2 c_1 - b_1 c_2 - 2a_2 b_1 + 2a_1 b_2), \\ R_{23} &= -b_2, & C_{23} &= 0, \\ R_{31} &= 0, & C_{31} &= 0, \\ R_{12} &= 0, & C_{12} &= 0, \\ Z_{11} &= \frac{1}{8}(-2a_2 b_1^2 + 2a_1 b_1 b_2 + a_1 a_2 c_1 + 2b_1 b_2 c_1 - a_2 c_1^2 - a_1^2 c_2 - 2b_1^2 c_2 + a_1 c_1 c_2), \\ Z_{22} &= \frac{1}{8}(4b_1^2 c_2 - 4b_1 b_2 c_1 - 2a_2 b_1^2 + 2a_1 b_1 b_2 + a_1 a_2 c_1 + 2a_2 c_1^2 - a_1^2 c_2 - 2a_1 c_1 c_2), \\ Z_{33} &= \frac{1}{8}(4a_2 b_1^2 - 4a_1 b_1 b_2 - 2a_1 a_2 c_1 + 2b_1 b_2 c_1 - a_2 c_1^2 + 2a_1^2 c_2 - 2b_1^2 c_2 + a_1 c_1 c_2), \\ Z_{23} &= \frac{3}{8}(-a_2 b_1 c_1 + 2a_1 b_2 c_1 - a_1 b_1 c_2), & Z_{31} &= 0, & Z_{12} &= 0, \\ S &= \frac{3}{2}b_1^2 - \frac{3}{2}a_1 c_1 - 2a_2 - 2c_2, \\ X &= \frac{5}{3}(2a_1 b_1 b_2 - a_1 a_2 c_1 + 2b_1 b_2 c_1 - a_2 c_1^2 - a_1^2 c_2 - a_1 c_1 c_2 + 4b_2^2 - 4a_2 c_2). \end{aligned} \tag{4.14}$$

Proof. This calculation can be done directly using the formulas in (3.16). \square

Theorem 4.21. *Every point in $\mathcal{V}_{NC}^{ND}/\text{SO}(3)$ is the value of the 7-less curvature function at some point in a block 7-less manifold.*

Proof. Let g be a block 7-less metric with

$$\begin{aligned} A(z) &= 1 + m_1 a_v z + m_2 a_v z^2, \\ B(z) &= b_1 z + b_2 z^2, \\ C(z) &= 1 + m_1 c_v z + m_2 c_v z^2, \end{aligned}$$

where $a_v, c_v, m_1, m_2, b_1,$ and b_2 are constants.

Lemma 4.20 shows that $\omega = (dz, dx_1, dx_2)$ is a C -adapted coframe at the origin and

$$\begin{aligned} D_1 &= 10(b_1 b_2 - a_v c_v m_1 m_2), \\ C_{22} &= \frac{1}{2}(b_1 m_2 - b_2 m_1)(a_v + 2c_v), \\ C_{33} &= \frac{1}{2}(b_2 m_1 - b_1 m_2)(2a_v + c_v). \end{aligned}$$

In order for the combined curvature function at ω to take values in $V(\mathcal{J})_{NC}^{ND}$ one of C_{22} and C_{33} must be nonzero and D_1 must also be nonzero. Thus we assume a_v or c_v is nonzero, $b_2 m_1 - b_1 m_2$ is nonzero, and $b_1 b_2 - a_v c_v m_1 m_2$ is nonzero.

Using the constants defined in Proposition 4.13 we can calculate

$$\begin{aligned} R_{23} &= -b_2, \\ \lambda &= \frac{2}{9} \cdot \frac{m_2}{b_1 m_2 - b_2 m_1}, \\ \kappa &= \frac{m_1}{b_1 m_2 - b_2 m_1}, \\ \mu &= \begin{cases} \frac{1}{6b_1}, & b_1 \neq 0, \\ 0, & b_1 = 0. \end{cases} \end{aligned}$$

To show that any point in \mathcal{V}_{NC}^{ND} can be realized as a value of the combined curvature function ψ we need only see that the values $C_{22}, C_{33}, R_{23}, \lambda, \mu,$ and κ can be made to assume arbitrary values subject to the condition that not both C_{22} and C_{33} are

zero, D_1 is non-zero, and $27\lambda\mu + R_{23}\kappa = 1$. This can be done directly with careful algebra and considering the two cases: $b_1 \neq 0$ and $b_1 = 0$.

As every point in \mathcal{V}_{NC}^{ND} is the value of the combined curvature function for some choice of coframe in a block 7-less metric, every point in $\mathcal{V}_{NC}^{ND}/\text{SO}(3)$ is the value of the 7-less curvature function for some point in some block 7-less manifold. \square

With Theorem 4.21, we have now constructed germs of 7-less manifolds corresponding to all points in $\mathcal{V}_{NC}^{ND}/\text{SO}(3)$.

4.2.2 Vanishing Differential of Scalar Curvature

We now consider the subset \mathcal{V}_{NC}^D of \mathcal{V}_{NC} which correspond to 7-less germs with $D = 0$.

We further divide into sub-cases based on the behavior of the Hessian of scalar curvature, $\text{Hess}(S)$, which is a symmetric 2-form on a manifold M . In components we have $\text{Hess}(S) = H_{ij}\omega_i\omega_j$ where H_{ij} are functions on the orthonormal coframe bundle. Unlike previous tensor fields, $\text{Hess}(S)$ is not a traceless symmetric tensor, so $H_{ij}x_i x_j$ does not take values in an irreducible representation of $\text{SO}(3)$, but rather in the representation $\mathcal{V}^2 \cong \mathcal{H}^0 \oplus \mathcal{H}^2$ with the projections into the irreducible components being the trace and traceless parts.

Using the notation as describing (3.16), we have the explicit description of the components

$$H_{ij} = D_{i;j}.$$

Note that the functions H_{ij} are functions of the components of the combined curvature function ψ , so we can also understand $\text{Hess}(S)$ as a function on \mathcal{H}^c . We will regard $\text{Hess}(S)$ as a symmetric matrix with entries H_{ij} . We further break \mathcal{V}_{NC}^D up based on the rank of $\text{Hess}(S)$.

Definition 4.22. Let \mathcal{V}_{NC}^{D0} , \mathcal{V}_{NC}^{D1} , \mathcal{V}_{NC}^{D2} , \mathcal{V}_{NC}^{D3} be the subsets of \mathcal{V}_{NC}^D where $\text{Hess}(S)$ has rank 0, 1, 2, and 3, respectively.

We have the partition

$$\mathcal{V}_{NC}^D = \mathcal{V}_{NC}^{D0} \cup \mathcal{V}_{NC}^{D1} \cup \mathcal{V}_{NC}^{D2} \cup \mathcal{V}_{NC}^{D3}.$$

Proposition 4.23. The rank of $\text{Hess}(S)$ at points in \mathcal{V}_{NC}^D is 0, 1, or 2 showing \mathcal{V}_{NC}^{D3} is empty. We also have the following parameterizations of \mathcal{V}_{NC}^{D0} , \mathcal{V}_{NC}^{D1} , and \mathcal{V}_{NC}^{D2} :

- Every $\text{SO}(3)$ -orbit of \mathcal{V}_{NC}^{D0} and \mathcal{V}_{NC}^{D2} contains a point whose components take the form

$$\begin{aligned} R_{11} &= R_{11}, & C_{11} &= C_{11}, & Z_{11} &= \frac{1}{2}SR_{11} + \frac{3}{2}R_{11}^2 - \frac{3}{20}X, \\ R_{22} &= -\frac{1}{2}R_{11}, & C_{22} &= -\frac{1}{2}C_{11}, & Z_{22} &= -\frac{1}{4}SR_{11} - \frac{3}{4}R_{11}^2 + \frac{3}{40}X, \\ R_{33} &= -\frac{1}{2}R_{11}, & C_{33} &= -\frac{1}{2}C_{11}, & Z_{33} &= -\frac{1}{4}SR_{11} - \frac{3}{4}R_{11}^2 + \frac{3}{40}X, \\ R_{23} &= 0, & C_{23} &= 0, & Z_{23} &= 0, \\ R_{31} &= 0, & C_{31} &= 0, & Z_{31} &= 0, \\ R_{12} &= 0, & C_{12} &= 0, & Z_{12} &= 0, \\ D_1 &= 0, & D_2 &= 0, & D_3 &= 0, \end{aligned} \tag{4.15}$$

and has

$$\text{Hess}(S) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{2}X & 0 \\ 0 & 0 & \frac{3}{2}X \end{bmatrix}.$$

Conversely, any choice of components satisfying the above form with $C \neq 0$ or $Z \neq 0$ gives a point in either \mathcal{V}_{NC}^{D0} or \mathcal{V}_{NC}^{D2} .

- Every $\text{SO}(3)$ -orbit of \mathcal{V}_{NC}^{D1} contains a point whose components take the form

$$\begin{aligned}
C_{11} &= \lambda(R_{33} - R_{22}), \\
R_{22} &= -\frac{1}{2}R_{11}, \quad C_{22} = \lambda(5R_{22} + 4R_{33}), \\
R_{33} &= -\frac{1}{2}R_{11}, \quad C_{33} = -\lambda(4R_{22} + 5R_{33}), \\
R_{23} &= 0, \quad C_{23} = \kappa(2R_{22} + R_{33})(R_{22} + 2R_{33}), \\
R_{31} &= 0, \quad C_{31} = 0, \\
R_{12} &= 0, \quad C_{12} = 0, \\
D_1 &= 0, \quad D_2 = 0, \quad D_3 = 0, \quad (4.16) \\
Z_{11} &= -\frac{1}{3}(2\kappa^2 R_{22}^2 + 5\kappa^2 R_{22}R_{33} + 2\kappa^2 R_{33}^2 + 9\lambda^2)(R_{33} + R_{22}), \\
Z_{22} &= \frac{1}{3}(2\kappa^2 R_{22}^2 + 5\kappa^2 R_{22}R_{33} + 2\kappa^2 R_{33}^2 + 9\lambda^2)R_{22}, \\
Z_{33} &= \frac{1}{3}(2\kappa^2 R_{22}^2 + 5\kappa^2 R_{22}R_{33} + 2\kappa^2 R_{33}^2 + 9\lambda^2)R_{33}, \\
Z_{23} &= 0, \quad Z_{31} = 0, \quad Z_{12} = 0, \\
S &= \frac{4}{3}\kappa^2 R_{22}^2 + \frac{10}{3}\kappa^2 R_{22}R_{33} + \frac{4}{3}\kappa^2 R_{33}^2 + 6\lambda^2 - 6R_{22} - 6R_{33}, \\
X &= -\frac{40}{3}R_{22}^2 - \frac{100}{3}R_{22}R_{33} - \frac{40}{3}R_{33}^2,
\end{aligned}$$

and has

$$\text{Hess}(S) = \begin{bmatrix} -40R_{22}^2 - 100R_{22}R_{33} - 40R_{33}^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where λ and κ are constants. Conversely, any choice of components satisfying the above form with $C \neq 0$ or $Z \neq 0$ and $-40(R_{22})^2 - 100R_{22}R_{33} - 40(R_{33})^2 \neq 0$ gives a point in \mathcal{V}_{NC}^{D1} .

Proof. See Appendix A. □

We now construct metrics whose combined curvature functions realize the orbits of \mathcal{V}_{NC}^D as described in Proposition 4.23. The parameterization given for \mathcal{V}_{NC}^{D1} is similar to the parameterization given for \mathcal{V}_{NC}^{ND} . Since orbits in $V(\mathcal{J})_{NC}^{ND}$ were realized by block 7-less metrics we may expect that the orbits of \mathcal{V}_{NC}^{D1} could also be realized by these metrics. This is in fact the case.

Theorem 4.24. *Every point in $\mathcal{V}_{NC}^{D1}/\text{SO}(3)$ is the value of the 7-less curvature function at some point in a block 7-less manifold.*

Proof. Let g be a block 7-less metric with functions

$$\begin{aligned} A(z) &= 1 + ma_v z + a_v z^2, \\ B(z) &= b_1 z, \\ C(z) &= 1 + mc_v z - c_v z^2, \end{aligned}$$

where a_v , c_v , m , and b_1 are constants.

Lemma 4.20 shows that $\omega = (dz, dx_1, dx_2)$ is an orthonormal coframe at the origin and that the conditions stated in the rank 1 case in Proposition 4.23 are satisfied with respect to this coframe. The parameters of the parameterization from the proposition are R_{22} , R_{33} , λ , and κ . Direct computation gives that these parameters are

$$R_{22} = -\frac{1}{3}a_v - \frac{2}{3}c_v, \quad R_{33} = \frac{2}{3}a_v + \frac{1}{3}c_v, \quad \lambda = \frac{1}{2}b_1, \quad \kappa = -\frac{3}{2}m.$$

Hence all values of R_{22} , R_{33} , λ , and κ can be achieved by choosing appropriate constants a_v , c_v , m , and b_1 . Passing to the 7-less curvature function, we get the desired result. \square

The last case of points in $\mathcal{V}/\text{SO}(3)$ we must consider are those orbits in \mathcal{V}_{NC}^{D0} and \mathcal{V}_{NC}^{D2} . The description of these two cases in Proposition 4.23 is symmetric with respect to the rotation by the action of the $SO(2)$ subgroup of $SO(3)$ acting on $\mathbb{R}[y, z] \subseteq \mathbb{R}[x, y, z]$. We might suppose that these non-generic points appear as the limit of generic points of a block 7-less metric.

Let $\mathbb{R}^1 \times \mathbb{R}^2$ have coordinates w on the \mathbb{R}^1 factor and polar coordinates (θ, r) on the \mathbb{R}^2 factor. Consider the covering map

$$f : \mathbb{R}^2 \times (0, \infty) \rightarrow \mathbb{R}^1 \times \mathbb{R}^2$$

given by

$$w = x_1, \quad \theta = x_2, \quad r = \sqrt{2z}.$$

If $\mathbb{R}^2 \times \mathbb{R}^1$ has a block 7-less metric with coefficients $A(z)$, $B(z)$, $C(z)$, then f induces a metric on $(\mathbb{R}^1 \times D_\rho^2) \setminus (\mathbb{R}^1 \times (0, 0))$ of the form

$$g = \begin{bmatrix} dz \\ d\theta \\ dr \end{bmatrix}^t \begin{bmatrix} A(r^2/2) & B(r^2/2) & 0 \\ B(r^2/2) & C(r^2/2) & 0 \\ 0 & 0 & r^2(A(r^2/2)C(r^2/2) - B(r^2/2)^2)^{-1} \end{bmatrix} \begin{bmatrix} dz \\ d\theta \\ dr \end{bmatrix}.$$

If $A(0)C(0) - B(0)^2 = 0$, then the plane $z = 0$ is a coordinate singularity in the coordinate system on $\mathbb{R}^2 \times \mathbb{R}^1$. However, if $z = 0$ is a simple zero of $A(z)C(z) - B(z)^2$, then in coordinates on $\mathbb{R}^1 \times \mathbb{R}^2$ the metric may extend across the line $r = 0$.

Definition 4.25. Let $\mathbb{R}^1 \times \mathbb{R}^2$ have coordinates (w, θ, r) . We call a metric on $\mathbb{R}^1 \times \mathbb{R}^2$ cylindrical 7-less if it takes the form

$$g = \begin{bmatrix} dw \\ d\theta \\ dr \end{bmatrix}^t \begin{bmatrix} A(r^2/2) & B(r^2/2) & 0 \\ B(r^2/2) & C(r^2/2) & 0 \\ 0 & 0 & r^2(A(r^2/2)C(r^2/2) - B(r^2/2)^2)^{-1} \end{bmatrix} \begin{bmatrix} dw \\ d\theta \\ dr \end{bmatrix},$$

where $A(z)$, $B(z)$, and $C(z)$ take the form

$$\begin{aligned} A(z) &= 1 + a_1z + a_2z^2, \\ B(z) &= b_1z + b_2z^2, \\ C(z) &= 2z + c_2z^2. \end{aligned}$$

Proposition 4.26. A cylindrical 7-less metric extends smoothly across the line $r = 0$ and the resulting extension is 7-less at all points. With metric coefficients expressed as in Definition 4.25, the metric along the curve $r = 0$ is $g = dw^2 + dx^2 + dy^2$ where x and y are standard Cartesian coordinates on the \mathbb{R}^2 factor of $\mathbb{R}^1 \times \mathbb{R}^2$.

Proof. A direct computation shows that the metric extends smoothly. By construction, T vanishes at all points with $r > 0$. Another coordinate computation shows that the components of T vanish at $r = 0$. \square

Theorem 4.27. Each point in $\mathcal{V}_{NC}^{D_0}/\text{SO}(3)$ and $\mathcal{V}_{NC}^{D_2}^2/\text{SO}(3)$ is the value of the 7-less curvature function at $r = 0$ for some cylindrical 7-less metric.

Proof. Consider the cylindrical 7-less metric with coefficient functions

$$\begin{aligned} A(z) &= 1 + a_1 z + a_2 z^2, \\ B(z) &= b_2 z^2, \\ C(z) &= 2z + c_2 z^2, \end{aligned}$$

By Proposition 4.26, the metric g extends smoothly across the line $r = 0$ where it takes the form $g = dw^2 + dx^2 + dy^2$, the standard metric. With respect to the orthonormal coframe $\omega = (dw, dx, dy)$ at the origin, the components of the combined curvature function can be computed to satisfy the relations of the rank 0 and 2 case of Proposition 4.23 where

$$R_{11} = \frac{2}{3}c_2, \quad C_{11} = 2b_2, \quad S = -3a_1 - 2c_2, \quad X = -\frac{10}{3}a_1c_2 - \frac{20}{3}a_2.$$

We see that parameters can be made to take arbitrary values, showing that the 7-less curvature function $\bar{\psi}$ can take arbitrary values in $\mathcal{V}_{NC}^{D0}/\text{SO}(3)$ and $\mathcal{V}_{NC}^{D2}/\text{SO}(3)$. \square

With this final case finished, we have constructed germs corresponding to all points in \mathcal{V} finishing the proof of Theorem 4.2.

4.3 Remarks on Generality and Symmetry of Germs

We finish this chapter with some surprising observations. The subset $\mathcal{V}_C/\text{SO}(3)$ is parameterized by $10 - 3 = 7$ parameters. This shows that the space of conformally flat, 7-less germs is ‘7-dimensional.’ To contrast, Proposition 4.13 shows that $\mathcal{V}_{NC}^{ND}/\text{SO}(3)$ is parametrized by 5 parameters and Proposition 4.23 shows that the largest part of $\mathcal{V}_{NC}^D/\text{SO}(3)$ is parameterized by 4 parameters. This shows that the space of non-conformally flat, 7-less germs is ‘5-dimensional.’ Being conformally flat and 7-less is equivalent to the vanishing of tensors C , with 5 independent components, and T , with 7 independent components, making for a total of 12 third order PDEs in the components of the metric rather than they 7 PDEs imposed by just

the 7-less condition. The more strict condition gives rise to a larger family of local structures.

Furthermore, we observed that the generic conformally flat, 7-less germ has no local symmetries of the metric while the generic non-conformally flat germ has a two-dimensional symmetry group. The assumption of more generic curvature, in the form of a non-zero Cotton tensor, leads to more symmetries in the solutions of the 7-less equations.

Global Structure of 7-less Manifolds

In this chapter, we examine some global properties of 7-less manifolds. In particular, we will construct maximal manifolds realizing each 7-less germ and give some global invariants of 7-less metrics.

5.1 Conformally Flat 7-less Metrics

5.1.1 Maximality of Structure

In this section we give a proof that the 7-less manifolds formed by connected components of Σ_P manifolds cannot be extended past their boundary and hence are maximal 7-less manifolds.

Definition 5.1. *Let M be a conformally flat, simply connected, 7-less 3-manifold and let \mathcal{B} the oriented orthonormal coframe bundle of M with coframing $\omega = (\omega_1, \omega_2, \omega_3)$ and $\phi = (\phi_1, \phi_2, \phi_3)$. Let Q be the quadratic form on \mathbb{R}^5 given in Section 4.1.1.*

Define α be the $\mathfrak{so}(Q)$ -valued 1-form on \mathcal{B} given by

$$\alpha = \begin{bmatrix} 0 & -\eta_1 & -\eta_2 & -\eta_3 & 0 \\ \omega_1 & 0 & \phi_3 & -\phi_2 & \eta_1 \\ \omega_2 & -\phi_3 & 0 & \phi_1 & \eta_2 \\ \omega_3 & \phi_2 & -\phi_1 & 0 & \eta_3 \\ 0 & -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{bmatrix},$$

where $\eta_i = h_{ij}\omega_j$ where $h_{ij} = h_{ji}$ with

$$\begin{aligned} h_{11} &= R_{11} + \frac{1}{12}S, & h_{23} &= R_{23}, \\ h_{22} &= R_{22} + \frac{1}{12}S, & h_{31} &= R_{31}, \\ h_{33} &= R_{33} + \frac{1}{12}S, & h_{12} &= R_{12}. \end{aligned}$$

We will call α the $\mathfrak{so}(Q)$ 1-form of M .

Lemma 5.2. *Let M be a conformally flat, simply connected, 7-less manifold with oriented orthonormal coframe bundle \mathcal{B} and let α be the $\mathfrak{so}(Q)$ 1-form of M . Let ω be the Maurer-Cartan form on $\mathrm{SO}(Q)$. Let $\theta \in \mathcal{B}$ and let $g \in \mathrm{SO}(Q)$. There exists a unique immersion $f : \mathcal{B} \rightarrow \mathrm{SO}(Q)$ such that $f(\theta) = g$ and $f^*(\omega) = \alpha$.*

Proof. Using (3.15) we check that the form α satisfies

$$d\alpha = -\alpha \wedge \alpha$$

as a result of M being conformally flat. Let $\tilde{\mathcal{B}}$ be the double cover of \mathcal{B} and let $\alpha_{\tilde{\mathcal{B}}}$ be the $\mathfrak{so}(Q)$ -valued 1-form given by the pullback of α under the projection map. Let $\tilde{\theta} \in \tilde{\mathcal{B}}$ be a point in the fiber above θ . The manifold $\tilde{\mathcal{B}}$ is a $\mathrm{Spin}(3)$ -bundle over M and is therefore simply connected. As $\tilde{\mathcal{B}}$ is simply connected and satisfies the Maurer-Cartan identity, by Theorem 2.11 there is a unique map $\tilde{f} : \tilde{\mathcal{B}} \rightarrow \mathrm{SO}(Q)$ such that $\tilde{f}(\tilde{\theta}) = g$ and $\tilde{f}^*(\omega) = \alpha_{\tilde{\mathcal{B}}}$. This map is an immersion as α has six linearly independent component 1-forms. Integrating α along a non-contractible loop in each

$\mathrm{SO}(3)$ -fiber of \mathcal{B} shows that the map \tilde{f} takes both points in a fiber of $\tilde{\mathcal{B}}$ above a point in \mathcal{B} to the same value in $\mathrm{SO}(Q)$. Hence the map \tilde{f} descends to a map $f : \mathcal{B} \rightarrow \mathrm{SO}(Q)$ with $f(\theta) = g$ and $f^*(\omega) = \alpha$, remaining an immersion. Uniqueness of f follows from the uniqueness of \tilde{f} .

$$\begin{array}{ccccc}
\mathrm{Spin}(3) & \longrightarrow & \tilde{\mathcal{B}} & & \\
\pi \downarrow & & \pi \downarrow & \searrow \tilde{f} & \\
\mathrm{SO}(3) & \longrightarrow & \mathcal{B} & \xrightarrow{f} & \mathrm{SO}(Q) \\
& & \pi \downarrow & & \\
& & M & &
\end{array}$$

□

Lemma 5.3. *Let M be a simply connected, conformally flat, 7-less manifold. Let $p \in M$ and $v_1, v_2, v_3 \in T_p M$ be an orthonormal frame. Let $N \subseteq \mathbb{R}^{4,1}$ be the null-cone of the quadratic form Q . Let $q \in N$ and $w_1, w_2, w_3 \in T_q N$ be orthonormal vectors not parallel to q . There is a unique map $f : M \rightarrow N$ such that $f(p) = q$ and $f_*(v_i) = w_i$.*

Proof. Let \mathcal{B} be the oriented orthonormal coframe bundle of M and let α be the $\mathfrak{so}(Q)$ -valued 1-form on \mathcal{B} . Let ω be the Maurer-Cartan form on $\mathrm{SO}(Q)$. Let θ be the coframe on M dual to (v_1, v_2, v_3) . There is a unique element $A \in \mathrm{SO}(Q)$ whose first four columns are (q, w_1, w_2, w_3) . Let $\tilde{f} : \mathcal{B} \rightarrow \mathrm{SO}(Q)$ be the developing map of α with $\tilde{f}(\theta) = A$. Let $e_1 : \mathrm{SO}(Q) \rightarrow \mathbb{R}^{4,1}$ be the projection onto the first column of a matrix in $\mathrm{SO}(Q)$. We can check that e_1 takes values in the null-cone, N . Next, we can check that the composition $e_1 \circ \tilde{f}$ is constant on the fibers of \mathcal{B} above points in M . Thus $e_1 \circ \tilde{f}$ descends to a well defined function $f : M \rightarrow N$. We can check that $f_*(v_i) = w_i$. Uniqueness of this map follows from the uniqueness of developing maps. □

We can now show that the connected components of Σ_P manifolds give us maxi-

mal 7-less manifolds.

Proposition 5.4. *Let U be a connected component of a Σ_P manifold and let M be a connected, conformally flat, 7-less manifold. If $f : U \rightarrow M$ is an isometric embedding, then $f(U) = M$.*

Proof. Let \tilde{U} and \tilde{M} be the simply connected covers of U and M , respectively. Then f lifts to an isometric immersion $\tilde{f} : \tilde{U} \rightarrow \tilde{M}$. Let $p \in \tilde{U}$ and let $v_1, v_2, v_3 \in T_p \tilde{U}$ be a frame. Let $i : U \rightarrow N$ be the inclusion of U into N . Let $w_j = (i \circ \pi)_*(v_j)$ for $j = 1, 2, 3$. Let $i' : \tilde{M} \rightarrow N$ be the unique map satisfying $(i' \circ \tilde{f})(p) = (i \circ \pi)(p)$ and $(i' \circ \tilde{f})_*(v_j) = (i \circ \pi)_*(v_j)$ for $j = 1, 2, 3$. By uniqueness of the maps from Lemma 5.3,

$$i \circ \pi = i' \circ \tilde{f}.$$

Thus we have the following commutative diagram:

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\tilde{f}} & \tilde{M} \\ & \searrow^{i \circ \pi} & \swarrow_{i'} \\ & & N \end{array}$$

If $f(U) \neq M$, then there exists a path $\gamma : [0, 1] \rightarrow M$ such that $\gamma(0) \in f(U)$ and $\gamma(1) \notin f(U)$. Lifting this path to the simply connected cover we get a path $\tilde{\gamma} : [0, 1] \rightarrow \tilde{M}$ such that $\tilde{\gamma}(0) \in \tilde{f}(\tilde{U})$ and $\tilde{\gamma}(1) \notin \tilde{f}(\tilde{U})$. Let $t_0 \in (0, 1)$ be the first point such that $\tilde{\gamma}(t_0) \notin \tilde{f}(\tilde{U})$. The path $i'(\tilde{\gamma}(t))$ for $t \in [0, t_0)$ must trace a curve in N approaching the boundary of U . This implies $\lim_{t \rightarrow t_0} \|i'(\tilde{\gamma}(t))\| = \infty$ as U has no finite boundary components. This is a contradiction, as γ and i' are both continuous functions. Thus $f(U) = M$. \square

5.1.2 Global invariants

We now construct some global invariants of conformally flat, 7-less manifolds.

Recall that the combined curvature function ψ takes values in

$$\mathcal{V}_{CF} = \mathcal{H}_R^2 \oplus \mathcal{H}_S^0 \oplus 0 \oplus \mathcal{H}_D^1 \oplus 0 \oplus \mathcal{H}_X^0 \subseteq \mathcal{H}^c.$$

The tensors R , S , D , and X determine the 7-less curvature function $\bar{\psi}$. The following are \mathcal{H}^0 -valued (scalar-valued) function on \mathcal{V}_{CF} which are constant on the $\text{SO}(3)$ -orbits:

$$\begin{aligned} f_1 &:= S, \\ f_2 &:= \langle R, R \rangle_4, \\ f_3 &:= X, \\ f_4 &:= \langle \langle R, R \rangle_2, R \rangle_4, \\ f_5 &:= \langle D, D \rangle_2, \\ f_6 &:= \langle \langle R, D \rangle_2, D \rangle_2 \\ f_7 &:= \langle \langle R, D \rangle_2, \langle R, D \rangle_2 \rangle_2. \end{aligned}$$

As the f_i are constant on the $\text{SO}(3)$ orbits, they descend to functions

$$\bar{f}_i : \mathcal{V} / \text{SO}(3) \rightarrow \mathbb{R}.$$

We define

$$\bar{f} = (\bar{f}_1, \dots, \bar{f}_7) : \mathcal{V}_{CF} / \text{SO}(3) \rightarrow \mathbb{R}^7.$$

Given a conformally flat 7-less manifold M , the function

$$F = (F_1, F_2, F_3, F_4, F_5, F_6, F_7) : \bar{f} \circ \bar{\psi} : M \rightarrow \mathbb{R}^7$$

is determined at each point completely by the 7-less germ. We can calculate that at a generic point, the map F will be a local immersion and hence the image will locally be a 3-dimensional submanifold. Therefore, we might expect the image to be the level set of four independent functions.

Thus, we expect there to be a function

$$\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4) : \mathbb{R}^7 \rightarrow \mathbb{R}^4.$$

such that $\Gamma \circ F$ is constant on the connected components of M . These functions will serve as global invariants of the 7-less manifold. We will attempt to find functions which are polynomials in the seven functions f_1, \dots, f_7 . A direct search of polynomials in the f_i yields

$$\Gamma_1 = 2f_1^2 + 1125f_2 + 9f_3,$$

$$\Gamma_2 = 8f_1^3 + 54f_1f_3 - 47250f_4 - 81f_5,$$

$$\begin{aligned} \Gamma_3 = & 45000f_1^2f_2 - 472500f_1f_4 + 540f_1f_5 + 16453125f_2^2 \\ & + 303750f_2f_3 + 405f_3^2 - 20250f_6, \end{aligned}$$

$$\begin{aligned} \Gamma_4 = & 56f_1^5 + 22500f_1^3f_2 + 630f_1^3f_3 - 675f_1^2f_5 - 5062500f_1f_2^2 \\ & + 101250f_1f_2f_3 + 1620f_1f_3^2 + 10125f_1f_6 + 151875f_2f_5 \\ & + 2126250f_3f_4 - 2430f_3f_5 - 759375f_7. \end{aligned}$$

We take $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$. A direct computation using (3.15) shows that on a conformally flat 7-less manifold,

$$d(\Gamma \circ F) = 0.$$

Hence, the function $\Gamma \circ F$ is constant on the connected components of a conformally flat, block 7-less metric. The function $\Gamma : \mathbb{R}^7 \rightarrow \mathbb{R}^4$ has a surjective differential at generic points in \mathbb{R}^7 and hence the generic level set is a 3-manifold.

Remark 5.5. *The function Γ is constant on connected components and is a point-wise invariant of a conformally flat 7-less metric. If two conformally flat 7-less germs give different values for Γ , we can conclude that they do not lie on the same connected component of a conformally flat 7-less manifold.*

The converse of Remark 5.5 is not necessarily true. While equal values of Γ are necessary for germs to be part of the same connected component of a 7-less manifold, this is not sufficient. A simple example is distinct, non-isotopic connected components of a Σ_P manifold. The function Γ is constant on all of Σ_P . More generally, there may be additional distinguishing functions which more finely separate the images of 7-manifolds under F in \mathbb{R}^7 than Γ does. However, examples have not been found.

5.2 Non-conformally Flat 7-less Metrics

5.2.1 Normal Form

In this section, we consider the general block 7-less metric on $\mathbb{R}^2 \times \mathbb{R}^1$:

$$g = \begin{bmatrix} dx_1 \\ dx_2 \\ dz \end{bmatrix}^t \begin{bmatrix} A(z) & B(z) & 0 \\ B(z) & C(z) & 0 \\ 0 & 0 & (A(z)C(z) - B(z)^2)^{-1} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dz \end{bmatrix}, \quad (5.1)$$

with functions

$$\begin{aligned} A(z) &= a_0 + a_1z + a_2z^2, \\ B(z) &= b_0 + b_1z + b_2z^2, \\ C(z) &= c_0 + c_1z + c_2z^2. \end{aligned} \quad (5.2)$$

In general, g will not be a metric or even defined on portions of $\mathbb{R}^2 \times \mathbb{R}^1$. On the subset where g is defined and positive definite, we have seen that it defines a 7-less metric.

Definition 5.6. *Let g be a block 7-less metric. We define the quadratic curve γ_g associated to g to be the curve in $\text{Sym}_2(\mathbb{R})$, the space of symmetric matrices, given by*

$$\gamma_g(z) = \begin{bmatrix} A(z) & B(z) \\ B(z) & C(z) \end{bmatrix}.$$

Our first goal is to apply a change of coordinates that puts a block 7-less metric into a normal form. This is equivalent to putting the associated quadratic curves

in $\text{Sym}_2(\mathbb{R})$ into a normal form using transformation that correspond to changes in coordinates that preserve block 7-less structures.

We begin by looking at coordinate transformations that preserve a block 7-less form.

Proposition 5.7. *Assume the metric g on $\mathbb{R}^2 \times \mathbb{R}^1$ is in a block 7-less form with respect to coordinates (x_1, x_2, z) . Let $(\tilde{x}_1, \tilde{x}_2, \tilde{z})$ be a new set of coordinates of the form:*

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} \tau_{11} & \tau_{12} & 0 \\ \tau_{21} & \tau_{22} & 0 \\ 0 & 0 & \epsilon(\tau_{11}\tau_{22} - \tau_{12}^2)^{-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix} + \begin{bmatrix} x_{10} \\ x_{20} \\ z_0 \end{bmatrix}$$

where $\tau_{11}\tau_{22} - \tau_{12}\tau_{21} \neq 0$ and $\epsilon = \pm 1$. Then g is block 7-less with respect to the new coordinates as well.

Proof. This is a direct computation. □

To simplify our analysis we will only want to consider block 7-less metrics which correspond to non-conformally flat metrics.

Proposition 5.8. *Let g be a block 7-less metric on $\mathbb{R}^2 \times \mathbb{R}^1$ with coefficients expressed in the form (5.2). If the vectors $(a_1, b_1, c_1)^t$ and $(a_2, b_2, c_2)^t$ are linearly dependent, then g is conformally flat.*

Proof. At each point $p \in \mathbb{R}^2 \times \mathbb{R}^1$ where g forms a Riemannian metric, we can apply a linear change of coordinates from Proposition 5.7 taking p to the origin and making $(a_0, b_0, c_0) = (1, 0, 1)$ in the new coordinate system. Such a change preserves the property that $(a_1, b_1, c_1)^t$ and $(a_2, b_2, c_2)^t$ are linearly dependent. Using Lemma 4.20 we see that the linear dependence assumption gives $C = 0$ and $Z = 0$ at p and so the metric is conformally flat. □

Definition 5.9. A block 7-less metric not satisfying the linear dependence relation of Proposition 5.8 will be called a non-degenerate. A quadratic curve in $\text{Sym}_2(\mathbb{R})$ associated to a non-degenerate 7-less metric will be called non-degenerate.

The degenerate block 7-less metrics can be understood by looking at Σ_P manifolds. For the remainder of the section we shall assume that our block 7-less metric is non-degenerate.

Proposition 5.10. A coordinate change on $\mathbb{R}^2 \times \mathbb{R}^1$ preserves a non-degenerate block 7-less metric if and only if it takes the form of the coordinate change in Proposition 5.7.

Proof. Assume that g is a non-degenerate block 7-less metric. Consider the general coordinate change

$$\tilde{x}_1 = \tilde{x}_1(x_1, x_2, z), \quad \tilde{x}_2 = \tilde{x}_2(x_1, x_2, z), \quad \tilde{z} = \tilde{z}(x_1, x_2, z).$$

Assume this coordinate change preserves the block 7-less structure. A coordinate calculation shows that the non-degeneracy assumption implies that $\|R\|^2$, the square norm of the traceless Ricci tensor of g , is a non-constant polynomial in the z -coordinate. As $\|R\|^2$ is constant on z -constant slices of a block 7-less metric, \tilde{z} must be a function of only z . The Jacobian of the coordinate change now takes the form

$$J = \begin{bmatrix} \partial\tilde{x}_1/\partial x_1 & \partial\tilde{x}_2/\partial x_1 & 0 \\ \partial\tilde{x}_1/\partial x_2 & \partial\tilde{x}_2/\partial x_2 & 0 \\ \partial\tilde{x}_1/\partial z & \partial\tilde{x}_2/\partial z & \partial\tilde{z}/\partial z \end{bmatrix}.$$

Regarding the metric g as a matrix, in the new coordinate system the metric takes the form

$$\tilde{g} = (J^{-1})^t g J^{-1}.$$

As both g and \tilde{g} are in block 7-less form, $\det(g) = 1$ and $\det(\tilde{g}) = 1$. Thus $\det(J) = \pm 1$ showing

$$\partial\tilde{z}/\partial z = \pm \det \begin{bmatrix} \partial\tilde{x}_1/\partial x_1 & \partial\tilde{x}_2/\partial x_1 \\ \partial\tilde{x}_1/\partial x_2 & \partial\tilde{x}_2/\partial x_2 \end{bmatrix}^{-1}.$$

Using the Jacobian we have

$$\begin{bmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial z \end{bmatrix} = J \begin{bmatrix} \partial/\partial\tilde{x}_1 \\ \partial/\partial\tilde{x}_2 \\ \partial/\partial\tilde{z} \end{bmatrix}$$

By assumption, \tilde{g} takes the form

$$\tilde{g} = \begin{bmatrix} \tilde{A}(\tilde{z}) & \tilde{B}(\tilde{z}) & & 0 \\ \tilde{B}(\tilde{z}) & \tilde{C}(\tilde{z}) & & 0 \\ 0 & 0 & (\tilde{A}(\tilde{z})\tilde{C}(\tilde{z}) - \tilde{B}(\tilde{z})^2)^{-1} & \end{bmatrix},$$

The equations

$$g(\partial/\partial x_1, \partial/\partial z) = 0, \quad \tilde{g}(\partial/\partial\tilde{x}_1, \partial/\partial\tilde{z}) = 0,$$

take the form

$$\begin{bmatrix} \partial\tilde{x}_1/\partial x_1 & \partial\tilde{x}_2/\partial x_1 \\ \partial\tilde{x}_1/\partial x_2 & \partial\tilde{x}_2/\partial x_2 \end{bmatrix} \begin{bmatrix} \tilde{A}(\tilde{z}) & \tilde{B}(\tilde{z}) \\ \tilde{B}(\tilde{z}) & \tilde{C}(\tilde{z}) \end{bmatrix} \begin{bmatrix} \partial\tilde{x}_1/\partial z \\ \partial\tilde{x}_2/\partial z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (5.3)$$

As the metric is assumed to be non-degenerate, on an open dense subset of \tilde{z} -values,

$$\begin{bmatrix} \tilde{A}(\tilde{z}) & \tilde{B}(\tilde{z}) \\ \tilde{B}(\tilde{z}) & \tilde{C}(\tilde{z}) \end{bmatrix},$$

has full rank. Thus (5.3) shows $\partial\tilde{x}_1/\partial z = 0$ and $\partial\tilde{x}_2/\partial z = 0$ on an open dense subset of $\mathbb{R}^2 \times \mathbb{R}^1$ and hence are zero.

For \tilde{g} to be constant on z -constant slices, the partials

$$\partial\tilde{x}_1/\partial x_1, \quad \partial\tilde{x}_2/\partial x_1, \quad \partial\tilde{x}_1/\partial x_2, \quad \partial\tilde{x}_2/\partial x_2,$$

must have no dependence on x_1 and x_2 . Finally, as g is non-degenerate, to preserve the property that the metric coefficients \tilde{A} , \tilde{B} , and \tilde{C} are polynomials of degree at most two, the above partials must not depend on z . This proves the proposition. \square

Note that for coordinate changes in Proposition 5.7, translations corresponding to non-zero x_{10} and x_{20} do not alter the metric. Thus we will not make use of them in our normalization process.

Definition 5.11. *Let \mathcal{G} denote the group of coordinate transformations described in Proposition 5.7 where $x_{10} = 0$ and $x_{20} = 0$.*

We see that

$$\mathcal{G} \cong \text{GL}(2) \times (\mathbb{R} \rtimes \mathbb{Z}_2)$$

with factors $\text{GL}(2)$ generated by the matrices τ_{ij} , \mathbb{R} generated by z_0 , and \mathbb{Z}_2 generated by ϵ .

The action of \mathcal{G} on block 7-less metrics induces an action on the quadratic curves γ_g . Let the $\text{GL}(2)$ factor of \mathcal{G} act on $\text{Sym}_2(\mathbb{R})$, the codomain of a quadratic curves, by transpose-conjugation and on \mathbb{R} , the domain of the quadratic curves, by rescaling by the inverse of the determinant. Let the $\mathbb{R} \rtimes \mathbb{Z}_2$ factor of \mathcal{G} act on \mathbb{R} , the domain of a quadratic curve, by translation and negation. Then the action of \mathcal{G} on the quadratic curves is given by pre-composition with the action on \mathbb{R} and post-composition with the action on $\text{Sym}_2(\mathbb{R})$:

$$\gamma_{h \cdot g}(z) = h \cdot \gamma_g(h \cdot z)$$

for $h \in \mathcal{G}$.

Let

$$\sigma\left(\begin{bmatrix} A & B \\ B & C \end{bmatrix}\right) = \frac{1}{2}(A + C), \quad \delta\left(\begin{bmatrix} A & B \\ B & C \end{bmatrix}\right) = \frac{1}{2}(A - C), \quad \beta\left(\begin{bmatrix} A & B \\ B & C \end{bmatrix}\right) = B. \quad (5.4)$$

be coordinates on $\text{Sym}_2(\mathbb{R})$. The non-degenerate curves γ_g are parabolas in this coordinate system.

We now look at how each of the factors of $GL(2) = (SL(2) \times \mathbb{R}^+) \rtimes \mathbb{Z}_2$ act on $Sym_2(\mathbb{R})$. We can see that the subgroup $SL(2) \subseteq GL(2)$ acts on $Sym_2(\mathbb{R})$ as the double cover of $SO(1, 2)$ preserving the indefinite bilinear form $d\sigma^2 - d\delta^2 - d\beta^2$. The diagonal subgroup $\mathbb{R}^+ \subseteq GL(2)$ acts on $Sym_2(\mathbb{R})$ via quadratic rescaling. The \mathbb{Z}_2 subgroup generated by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ corresponds to reflection across the $\delta = 0$ plane. With these actions understood, we can give the following normalization theorem:

Proposition 5.12. *Every non-degenerate quadratic curve in $Sym_2(\mathbb{R})$ can be normalized to exactly one of the following using the action of \mathcal{G} :*

1. *Elliptic, negative:*

$$\begin{bmatrix} \sigma(z) \\ \delta(z) \\ \beta(z) \end{bmatrix} = \begin{bmatrix} \sigma_0 & 0 & 0 \\ \delta_0 & \delta_1 & 0 \\ \beta_0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix},$$

with $\delta_0 \geq 0$ and $\delta_1 > 0$.

2. *Hyperbolic, positive:*

$$\begin{bmatrix} \sigma(z) \\ \delta(z) \\ \beta(z) \end{bmatrix} = \begin{bmatrix} \sigma_0 & 0 & \pm 1 \\ \delta_0 & \delta_1 & 0 \\ \beta_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix},$$

with $\beta_0 \geq 0$, $\delta_0 \geq 0$, and $\delta_1 > 0$.

3. *Hyperbolic, negative:*

$$\begin{bmatrix} \sigma(z) \\ \delta(z) \\ \beta(z) \end{bmatrix} = \begin{bmatrix} \sigma_0 & \sigma_1 & 0 \\ \delta_0 & 0 & 1 \\ \beta_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix},$$

with $\beta_0 \geq 0$, $\sigma_1 > 0$.

4. *Hyperbolic, null:*

$$\begin{bmatrix} \sigma(z) \\ \delta(z) \\ \beta(z) \end{bmatrix} = \begin{bmatrix} \sigma_0 & \mp 1 & \pm 1 \\ \delta_0 & 1 & 1 \\ \beta_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix},$$

with $\beta_0 \geq 0$.

5. *Parabolic, negative:*

$$\begin{bmatrix} \sigma(z) \\ \delta(z) \\ \beta(z) \end{bmatrix} = \begin{bmatrix} \epsilon - \eta & 1 & 0 \\ \delta_0 & 0 & 1 \\ \epsilon + \eta & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix}.$$

6. *Parabolic, null:*

$$\begin{bmatrix} \sigma(z) \\ \delta(z) \\ \beta(z) \end{bmatrix} = \begin{bmatrix} \epsilon - \eta & 0 & \pm 1 \\ 0 & 1 & 0 \\ \epsilon + \eta & 0 & \mp 1 \end{bmatrix} \begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix}.$$

Proof. Our approach is to use the $\mathrm{GL}(2)$ factor or \mathcal{G} to normalize the image of the quadratic curve γ and then use the $\mathbb{R} \times \mathbb{Z}_2$ factor to reparameterize making $\gamma(0)$ the vertex of the curve and standardizing the direction of $\gamma'(0)$.

A non-degenerate quadratic curve lies within a plane $P \subseteq \mathrm{Sym}_2(\mathbb{R})$ with a normal vector n . The action of $\mathrm{SL}(2)$ on $\mathrm{Sym}_2(\mathbb{R})$ as $\mathrm{Spin}(1, 2)$ allows us to reposition P to one of the following normal forms depending on whether the length of n is positive, negative, or zero with respect to the quadratic form $d\sigma^2 - d\delta^2 - d\beta^2$:

$$\begin{array}{ll} \text{elliptic: } \|n\| > 0, & \sigma = \text{constant}, \\ \text{hyperbolic: } \|n\| < 0, & \beta = \text{constant} \geq 0, \\ \text{parabolic: } \|n\| = 0, & \sigma + \beta = \text{constant}. \end{array}$$

Elliptic Case: The stabilizer of the $\mathrm{SL}(2)$ -action on the normalized plane is the $\mathrm{SO}(2)$ subgroup with Lie algebra generated by $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, which acts by rotation about the σ -axis. This action allows us to make $\gamma''(z)$ point along the positive β axis. Using the quadratic rescaling action of $\mathbb{R}^+ \subseteq \mathrm{GL}(2)$, we can make $\gamma''(0)$ have length 1 as well. Next, use the \mathbb{Z}_2 factor of $\mathrm{GL}(n)$ to reflect about the $\delta = 0$ make the vertex lie in the $\delta \geq 0$ half-space. Finally, we can use the $\mathbb{R} \times \mathbb{Z}_2$ action on \mathbb{R} to reparameterize making $\gamma(0)$ the vertex and making $\gamma'(0)$ point along the positive δ axis.

The remaining two cases follow a similar analysis of using the $SL(2)$ -stabilizer of the plane to orient $\gamma''(z)$, the action of \mathbb{R}^+ to normalize its length, and finally the \mathbb{Z}_2 action to either make $\gamma''(z)$ point in the positive δ -direction or to place the vertex of γ in the $\delta \geq 0$ half space. Then the $\mathbb{R} \times \mathbb{Z}_2$ action making $\gamma(0)$ the vertex and making $\gamma'(0)$ point in the preferred direction.

In the hyperbolic case we must consider the cases where $\gamma''(z)$ is positive, negative, and zero with respect to $d\sigma^2 - d\delta^2 - d\beta^2$ giving us the three possible normalizations. In the parabolic case, we must consider the cases where $\gamma''(z)$ is negative and zero with respect to $d\sigma^2 - d\delta^2 - d\beta^2$ giving us the two possible normalizations. \square

We note that the metric defined by a block 7-less metric is Riemannian at a z -value if and only if $\gamma_g(z)$ is positive definite. In terms of the cone $\sigma^2 = \delta^2 + \beta^2$, a point $\gamma_g(z)$ is positive definite if and only if it lies in the interior of the upper nappe of this cone.

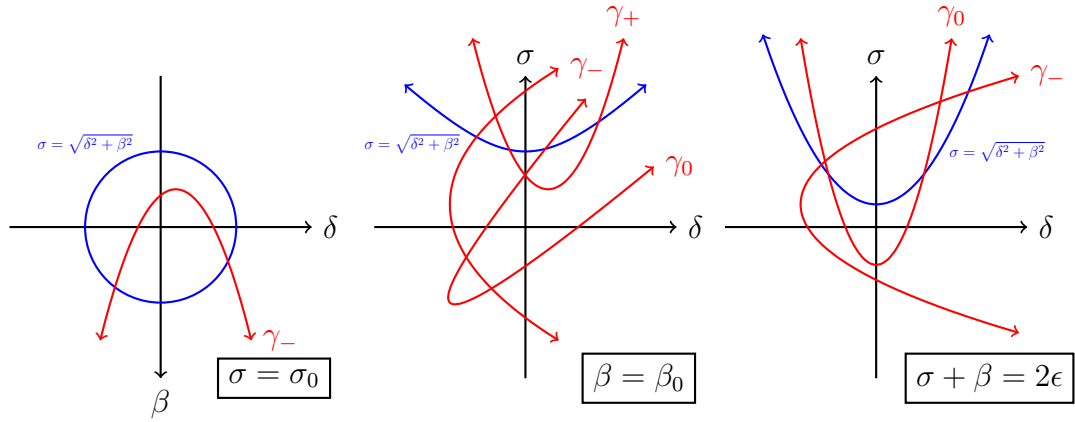


FIGURE 5.1: Examples of normalized quadratic curves in $Sym_2(\mathbb{R})$. Left: The elliptic case with γ_+ in form 1. Middle: The hyperbolic case with γ_+ , γ_- , and γ_+ in forms 2, 3, and 4, respectively. Right: The parabolic case with γ_- and γ_+ in forms 5 and 6, respectively.

The normal forms of quadratic curves give normal forms for non-degenerate block 7-less metrics.

Definition 5.13. *A non-degenerate block 7-less metric is said to be in normal form if the associated non-degenerate curve γ_g is in one of the normal forms in Proposition 5.12.*

The normal form of a block 7-less metric serves as a global invariant of a non-conformally flat 7-less metric. This global invariant can be derived from pointwise values of the 7-less curvature function $\bar{\psi}$. Theorems 4.21, 4.24, and 4.27 allow for constructing a local coordinate system for which the metric is block 7-less (or the limit of block 7-less in the case of cylindrical 7-less). This local coordinate system gives a quadratic curve γ_g which can then be normalized.

5.2.2 Maximal Structures

While the block 7-less and cylindrical 7-less metrics allow us to realize all non-conformally flat 7-less germs, none of these examples are compact and may have boundaries which are at finite distances. In this section, we see that these metrics can be extended across the metric singularity in certain cases. Extending across these singularities allows us to construct maximal examples of non-conformally flat 7-less metrics. All of these examples will turn out to be topologically embeddable as an open dense (non-necessarily proper) subset of S^3 . Most of these examples are either compact, or are complete except at curvature singularities. We do this by beginning with a block 7-less metric on $\mathbb{R}^2 \times \mathbb{R}^1$, forming a quotient with topology $\mathbb{T}^2 \times \mathbb{R}^1$, and then attaching a number S^1 's at the endpoints in such a way that the metric extends smoothly.

We begin by associating to each block 7-less metric on $\mathbb{R}^2 \times \mathbb{R}^1$, a 7-less metric on $\mathbb{T}^2 \times \mathbb{R}^1$.

Definition 5.14. *Let g be a block 7-less metric on $\mathbb{R}^2 \times \mathbb{R}^1$ with coordinates (x_1, x_2, z) .*

Let $\mathbb{T}^2 \times \mathbb{R}^1$ have coordinates $(\theta_1, \theta_2, \rho)$. The map $f : \mathbb{R}^2 \times \mathbb{R}^1 \rightarrow \mathbb{T}^2 \times \mathbb{R}^1$ given by

$$\theta_1 = x_1, \quad \theta_2 = x_2, \quad \rho = z,$$

is a covering map and hence induces a metric g_t on $\mathbb{T}^2 \times \mathbb{R}^1$. We call t_g the torus 7-less metric associated to g .

Clearly every 7-less germ that is realized in a block 7-less metric is also realized in the associated torus 7-less metric.

Remark 5.15. *Two block 7-less metrics that are related by the action of \mathcal{G} can produce non-isometric associated torus 7-less metrics.*

By construction, torus 7-less metrics take the form

$$g = \begin{bmatrix} d\theta_1 \\ d\theta_2 \\ d\rho \end{bmatrix}^t \begin{bmatrix} A(\rho) & B(\rho) & 0 \\ B(\rho) & C(\rho) & 0 \\ 0 & 0 & (A(\rho)C(\rho) - B(\rho)^2)^{-1} \end{bmatrix} \begin{bmatrix} d\theta_1 \\ d\theta_2 \\ d\rho \end{bmatrix},$$

with functions

$$\begin{aligned} A(\rho) &= a_0 + a_1\rho + a_2\rho^2, \\ B(\rho) &= b_0 + b_1\rho + b_2\rho^2, \\ C(\rho) &= c_0 + c_1\rho + c_2\rho^2. \end{aligned} \tag{5.5}$$

Like block 7-less metrics, torus 7-less metrics may only define a Riemannian metric on $\mathbb{T}^2 \times \mathbb{R}^1$ for some values of ρ .

Let g_t be a torus 7-less metric and let $I \subseteq \mathbb{R}$ be a maximal ρ -interval on which g_t (equivalently g) defines a Riemannian metric. Each of the endpoints of a maximal interval I can be either finite or infinite giving us three cases to consider:

1. Both endpoints are infinite: $I = (-\infty, \infty)$.
2. Exactly one of the endpoints is infinite: $I = (a, \infty)$.
3. Both endpoints are finite $I = (a, b)$.

Definition 5.16. Let g be a block 7-less metric with associated quadratic curve $\gamma_g(z)$ in $\text{Sym}_2(\mathbb{R})$. Let

$$q_g(z) = \det(\gamma_g(z))$$

We will call $q_g(z)$ the rank polynomial of g . Similarly, we will also refer to $q_g(\rho)$ as the rank polynomial of the associated torus 7-less metric.

Remark 5.17. If g is a non-degenerate block 7-less metric, then the degree of the rank polynomial is at least 2.

If a block 7-less metric defines a Riemannian metric on a maximal z -interval $I = (a, b)$, then each of a and b must either be infinite or a zero of the rank polynomial of the metric. We begin by examining the behavior of a block/torus 7-less metric in the case that I has an infinite endpoint.

Proposition 5.18. Let g be a non-degenerate block 7-less metric. The limit

$$\lim_{z \rightarrow \pm\infty} \|R(z)\|^2$$

of the square norm of the traceless Ricci curvature tensor of g diverges.

Proof. A coordinate computation shows that $\|R(z)\|^2$ is a polynomial in z . The non-degeneracy condition on g implies that $\deg(\|R(z)\|^2) \geq 2$. As $\deg(\|R(z)\|^2) > 0$, $\lim_{z \rightarrow \pm\infty} \|R(z)\|^2$ diverges. \square

Proposition 5.18 shows that a curvature singularity occurs at infinity for non-degenerate block/torus 7-less metrics. Thus, if the maximal z/ρ -interval I contains an endpoint at infinity, there will be no way to extend the metric across the boundary at infinity.

While torus 7-less metrics always have curvature singularities at infinity, the following proposition shows that some of these metrics are geodesically complete in the direction of infinity while others have curvature singularities at finite distances.

Proposition 5.19. *Let g be a non-degenerate block 7-less metric and assume g defines a Riemannian metric on the z -interval (a, ∞) . Then g is complete in the direction of $z = \infty$ if and only if $\deg(q_g(z)) = 2$.*

Proof. It suffices to look at the length of the curve $c(t) = (0, 0, t)$ for $t \in (a + 1, \infty)$.

The length of the curve c is

$$L_c = \int_{a+1}^{\infty} \frac{dz}{\sqrt{q_g(z)}}.$$

This integral converges and L_c is finite exactly when $\deg(q_g) > 2$. □

We now look at the limiting behavior of block 7-less metrics at finite z -values. A coordinate calculation shows that curvature singularities never develop at finite z -values. The following proposition describes the cases when a block 7-less metric is complete in the direction of coordinate singularity at a finite z -value.

Proposition 5.20. *Let g be a non-degenerate block 7-less metric. Assume that g defines a Riemannian metric on the z -interval (a, b) and assume that $z = a$ is a root of the rank polynomial $q_g(z)$. Then g is complete in the direction of $z = a$ if and only if $z = a$ is a root of $q(z)$ if multiplicity at least two.*

Proof. It suffices again to look at the length of the curve $c(t) = (0, 0, t)$ for $t \in (a, a + \epsilon)$. The curve has length

$$L_c = \int_a^{a+\epsilon} \frac{dz}{\sqrt{q(z)}}.$$

This integral converges and L_c is finite exactly when $z = a$ is a simple root of $q_g(z)$. □

Propositions 5.18 and 5.20 show that a torus 7-less metric on a maximal subset $\mathbb{T}^2 \times (a, b)$ will not be able to be extended at infinite endpoints or endpoints which

are non-simple roots of the rank polynomial $q_g(\rho)$. This leaves us to consider only the cases where $q_g(\rho)$ has simple roots at the endpoints. We have already seen that cylindrical 7-less metrics serve as extension of block 7-less metrics at simple roots if the coefficients of block 7-less metric take a particular form (see Definition 4.25). We now give a similar, more general, construction for torus 7-less metrics.

Definition 5.21. *Let $\mathbb{T}^2 \times \mathbb{R}^1$ have coordinates $(\theta_1, \theta_2, \rho)$ and let $\mathbb{T}^1 \times \mathbb{R}^2$ have coordinates ϕ_0 on the \mathbb{T}^1 factor and polar coefficients (ϕ_1, r) on the \mathbb{R}^2 factor. For $\rho_0 \in \mathbb{R}$, define the maps*

$$i_{L1;\rho_0}, i_{L2;\rho_0} : \mathbb{T}^2 \times (\rho_0, \infty) \rightarrow \mathbb{T}^1 \times \mathbb{R}^2$$

and

$$i_{R1;\rho_0}, i_{R2;\rho_0} : \mathbb{T}^2 \times (-\infty, \rho_0) \rightarrow \mathbb{T}^1 \times \mathbb{R}^2$$

by

$$\begin{aligned} i_{L1;\rho_0}(\theta_1, \theta_2, \rho) &= (\phi_0, \phi_1, \rho) = (\theta_1, \theta_2, \sqrt{\rho - \rho_0}), \\ i_{L2;\rho_0}(\theta_1, \theta_2, \rho) &= (\phi_0, \phi_1, \rho) = (\theta_2, \theta_1, \sqrt{\rho - \rho_0}), \\ i_{R1;\rho_0}(\theta_1, \theta_2, \rho) &= (\phi_0, \phi_1, \rho) = (\theta_1, \theta_2, \sqrt{\rho_0 - \rho}), \\ i_{R2;\rho_0}(\theta_1, \theta_2, \rho) &= (\phi_0, \phi_1, \rho) = (\theta_2, \theta_1, \sqrt{\rho_0 - \rho}). \end{aligned}$$

If we view $\mathbb{T}^1 \times \mathbb{R}^2$ as a solid torus sitting in $\mathbb{R}^3 \subseteq S^3$, these four maps serve as embedding of portions of the thickened torus $\mathbb{T}^2 \times \mathbb{R}^1$ into the solid torus in S^3 where the tori formed by ρ -constant slices of $\mathbb{T}^2 \times \mathbb{R}$ degenerate to an S^1 as $\rho \rightarrow \rho_0$. In this way, we can use these maps to ‘attach’ the S^1 corresponding to $\mathbb{T}^1 \times \{0, 0\}$ to a torus 7-less structure. The maps $i_{L1;\rho_0}$ and $i_{R1;\rho_0}$ have the S^1 ’s form by ρ -constant, θ_1 -constant slices degenerating to points on the newly attached S^1 and the maps $i_{L2;\rho_0}$ and $i_{R2;\rho_0}$ have the S^1 ’s form by ρ -constant, θ_2 -constant slices degenerating to points on the newly attached S^1 .

Let g_t be a torus 7-less metric and let $I = (a, b)$ be a maximal ρ -interval on which g_t defines a Riemannian metric. If a is finite, then the maps $i_{L1;a}$ and $i_{L2;a}$ induce 7-less metrics on the portion of $\mathbb{T}^1 \times \mathbb{R}^2$ with $0 < r < b - a$. The following proposition gives the condition for these induced metrics to extend across the S^1 defined by $r = 0$.

Proposition 5.22. *Let g_t be a torus 7-less metric with coefficient functions (see (5.5))*

$$\begin{aligned} A(\rho) &= a_0 + a_1(\rho - a) + a_2(\rho - a)^2, \\ B(\rho) &= b_0 + b_1(\rho - a) + b_2(\rho - a)^2, \\ C(\rho) &= c_0 + c_1(\rho - a) + c_2(\rho - a)^2. \end{aligned}$$

Then the metric induced by $i_{L1;a}$ on the portion of $\mathbb{T}^1 \times \mathbb{R}^2$ with $r > 0$ extends smoothly across the S^1 defined by $r = 0$ if and only if

$$b_0 = 0, \quad c_0 = 0, \quad a_0 c_1^2 = 4.$$

If these conditions are satisfied, the metric induced along $r = 0$ will take the form

$$g = a_0 d\phi_0^2 + c_1 dx^2 + c_1 dy^2.$$

The resulting metric on $\mathbb{T}^1 \times \mathbb{R}^2$ is 7-less at all points.

Proof. The metric induced by $i_{L1;a}$ is

$$\begin{bmatrix} d\phi_0 \\ d\phi_1 \\ d\rho \end{bmatrix}^t \begin{bmatrix} \tilde{A}(r^2) & \tilde{B}(r^2) & 0 \\ \tilde{B}(r^2) & \tilde{C}(r^2) & 0 \\ 0 & 0 & 4r^2(\tilde{A}(r^2)\tilde{C}(r^2) - \tilde{B}(r^2)^2)^{-1} \end{bmatrix} \begin{bmatrix} d\phi_0 \\ d\phi_1 \\ d\rho \end{bmatrix},$$

where

$$\begin{aligned} \tilde{A}(r) &= a_0 + a_1 r + a_2 r^2, \\ \tilde{B}(r) &= b_0 + b_1 r + b_2 r^2, \\ \tilde{C}(r) &= c_0 + c_1 r + c_2 r^2. \end{aligned}$$

A coordinate computation mimicking that of Proposition 4.26 shows that the metric extends across $r = 0$ if and only if the conditions of the proposition are satisfied and the components of the tensor T vanish at $r = 0$. \square

Statements similar to Proposition 5.22 exist for maps $i_{L2;\rho_0}$, $i_{R1;\rho_0}$, and $i_{R2;\rho_0}$. The equivalent statements for $i_{R1;\rho_0}$ and $i_{R2;\rho_0}$ have the roles of the $A(z)$ and $C(z)$ coefficients reversed.

With Proposition 5.22 we are ready to describe the process for constructing maximal 7-less structures. For a given non-conformally flat 7-less germ we proceed in the following steps:

1. Construct a block 7-less metric g which realizes the germ (or a cylindrical 7-less structure and use the block 7-less metric which limits to the 7-less germ).
2. Determine the maximal z -interval I containing the germ on which g defines a Riemannian metric.
3. If possible, apply a coordinate change in \mathcal{G} so that, at each of the finite endpoints of the interval I corresponding to a simple root of the rank polynomial q_g , the metric takes a form described in Proposition 5.22.
4. To the associated torus 7-less metric, attach an S^1 at each end of $\mathbb{T}^2 \times I$ corresponding to simple roots of the rank polynomial q_g using change of coordinate maps $i_{L1;\rho_0}$, $i_{L2;\rho_0}$, $i_{R1;\rho_0}$, and $i_{R2;\rho_0}$. The 7-less metric extends to these new S^1 's by Proposition 5.22.

Assuming that Step 3 can be done, the results of this process will yield one of the following 7-less structures:

- S1. If neither endpoint corresponds to a simple root of the rank polynomial $q_g(\rho)$, no S^1 's are attached. The resulting topology is $\mathbb{T}^2 \times \mathbb{R}$.
- S2. If a single endpoint corresponds to a simple root of the rank polynomial $q_g(\rho)$, a single S^1 is attached. The resulting topology is $\mathbb{T}^1 \times \mathbb{R}^2$, the solid torus.

S3. If both endpoints correspond to simple roots of the rank polynomial $q_q(\rho)$, two S^1 's are attached.

- (a) If the attaching maps used are $i_{L1;a}$ and $i_{R2;b}$ or $i_{L2;a}$ and $i_{L1;b}$, the resulting topology is S^3 .
- (b) If the attaching maps used are $i_{L1;a}$ and $i_{R1;b}$ or $i_{L2;a}$ and $i_{L2;b}$, the resulting topology is $S^2 \times \mathbb{T}^1$.

In all completed cases, infinite endpoints give curvature singularities as ρ approaches an endpoint at $\pm\infty$, the metric is complete in the direction of any finite endpoint, and the metric may or may not be complete in the direction of an infinite endpoints depending on the degree of the rank polynomial $q_p(\rho)$. All boundary points located at finite distances have curvature singularities and hence these structures are maximal.

We now look at when Step 3 can be accomplished: Putting a block 7-less metric in a form where the metric can be extended past appropriate singularities. There are two cases we must consider: where one endpoint of the maximal interval is a simple root of the rank polynomial, and where both endpoints are simple roots of the rank polynomial.

The following proposition describes what we will see to be the generic configuration in the case of a finite maximal interval with two simple roots:

Proposition 5.23. *Let g be a non-degenerate block 7-less metric and let $I = (a, b)$ be a maximal z -interval on which g defines a Riemannian metric where a and b are simple zeros of the rank polynomial q_g (implying $\text{rank}(\gamma_g(a)) = 1$ and $\text{rank}(\gamma_g(b)) = 1$). Further assume $\ker(\gamma_g(a)) \neq \ker(\gamma_g(b))$. Then there exists a change of coordinates giving an isometric block 7-less metric \tilde{g} on a corresponding maximal interval $\tilde{I} = (\tilde{a}, \tilde{b})$ such that \tilde{g}_t has coefficients that*

1. satisfy the conditions of Proposition 5.22 with $\rho_0 = \tilde{a}$ and
2. satisfy the conditions of Proposition 5.22 with $\rho_0 = \tilde{b}$ (with the roles of $A(\rho)$ and $B(\rho)$ reversed).

Proof. Write the coefficients of the metric g as Taylor polynomials centered at $z = a$ as

$$\begin{aligned} A(z) &= a_0 + a_1(z-a) + a_2(z-a)^2, \\ B(z) &= b_0 + b_1(z-a) + b_2(z-a)^2, \\ C(z) &= c_0 + c_1(z-a) + c_2(z-a)^2. \end{aligned}$$

The assumption of that $z = a$ is a simple root of the rank polynomial implies that $\text{rank}(\gamma_g(a)) = 1$. Let $v = (\alpha, \beta) \in \ker(\gamma_g(a))$ with $\alpha^2 + \beta^2 = 1$. The change of coordinates

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} \beta & -\alpha & 0 \\ \alpha & \beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix} \quad (5.6)$$

puts the expansion of the coefficients of g about $z = a$ into the form

$$\begin{aligned} A(z) &= a_0 + a_1(z-a) + a_2(z-a)^2, \\ B(z) &= b_1(z-a) + b_2(z-a)^2, \\ C(z) &= c_1(z-a) + c_2(z-a)^2. \end{aligned}$$

The rank polynomial of g does not change as a result of the coordinate change. Now write the coefficients of the metric g (following the coordinate change (5.6)) as Taylor polynomials centered at $z = b$ as

$$\begin{aligned} A(z) &= a'_0 + a'_1(z-b) + a'_2(z-b)^2, \\ B(z) &= b'_0 + b'_1(z-b) + b'_2(z-b)^2, \\ C(z) &= c'_0 + c'_1(z-b) + c'_2(z-b)^2. \end{aligned}$$

The assumption that $z = b$ is a simple root of $\gamma_g(z)$ implies that $\text{rank}(\gamma_g(z)) = 1$. Let $v' = (\alpha', \beta') \in \ker(\gamma_g(b))$. The assumption that $\ker(\gamma_g(a)) \neq \ker(\gamma_g(b))$ implies

that $\alpha' \neq 0$. Thus, by rescaling, we can choose $\alpha' = 1$. The coordinate change

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} 1 & \beta' & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix} \quad (5.7)$$

puts the expansion of the coefficients of g about $z = b$ into the form

$$\begin{aligned} A(z) &= a'_1(z-b) + a'_2(z-b)^2, \\ B(z) &= b'_1(z-b) + b'_2(z-b)^2, \\ C(z) &= c'_0 + c'_1(z-b) + c'_2(z-b)^2, \end{aligned}$$

while preserving the form of the expansion of the coefficients about $z = a$. Again, the rank polynomial of g does not change as a result of the coordinate change.

The assumption that $z = a$ is a simple root of q_g implies that $c_1 \neq 0$ and the assumption that $z = b$ is a simple root of q_g implies $a'_1 \neq 0$. Furthermore, the assumption that g defines a Riemannian metric on (a, b) implies that $a_0 > 0$, $c'_0 > 0$, $c_1 > 0$, and $a'_1 < 0$.

The change of coordinates

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & (m_1 m_2)^{-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix} \quad (5.8)$$

changes the interval on which the metric g defines a Riemannian metric to $\tilde{I} = (m_1 m_2 a, m_1 m_2 b)$. The coefficients of the expansions of the metric about $m_1 m_2 a$ (written as $\tilde{a}_i, \tilde{b}_i, \tilde{c}_i$) and the coefficients of the expansions of the metric about $m_1 m_2 b$ (written as $\tilde{a}'_i, \tilde{b}'_i, \tilde{c}'_i$) are expressed in terms of the current coefficients as:

$$\tilde{a}_0 = \frac{1}{m_1^2} a_0, \quad \tilde{c}_1 = \frac{m_1}{m_2} c_1, \quad \tilde{c}'_0 = \frac{1}{m_2^2} c'_0, \quad \tilde{a}'_1 = \frac{m_2}{m_1} a'_1.$$

By choosing $m_2 = \sqrt{a_0} c_1 / 2$ and $m_1 = \sqrt{c'_0} a'_1 / 2$, the resulting coefficients will satisfy $\tilde{a}_0 \tilde{c}_1^2 = 4$ and $\tilde{c}'_0 (\tilde{a}'_1)^2 = 4$. Combining the three changes of coordinates from (5.6), (5.7), and (5.8) gives the desired block 7-less form. \square

The block 7-less metrics resulting from Proposition 5.23 give Riemannian metrics on the maximal interval (\tilde{a}, \tilde{b}) that allow for attaching an S^1 and extending the metric at the \tilde{a} endpoint using the map $i_{L1;\tilde{a}}$ and attaching an S^1 and extending the metric at the \tilde{b} endpoint using the map $i_{R2;\tilde{b}}$. The resulting manifold is a topological S^3 .

Following the same proof, if only one endpoint of a maximal interval is a simple root of the rank polynomial, the expansion about that one endpoint can still be put into a form satisfying 5.22. The block metrics following from this normalization will allow for attaching an S^1 and extending the metric using either $i_{L1;\rho_0}$ or $i_{R1;\rho_0}$ depending which end of the interval the simple root occurs at. In particular, it is always possible to complete Step 3 in the above construction of maximal 7-less structures if there is only a single endpoint of the maximal interval that corresponds to a simple root of the rank polynomial.

We now turn to the question of which non-degenerate block 7-less metrics satisfy the conditions of Proposition 5.23.

Proposition 5.24. *Let $K \subseteq \text{Sym}_2(\mathbb{R})$ be the cone defined by $\sigma^2 = \delta^2 + \beta^2$. If g is a block 7-less metric and $I = (a, b)$ is a finite maximal z -interval on which g defines a Riemannian metric, the pair (g, I) satisfies the conditions of Proposition 5.23 if and only if the associated quadratic curve γ_g intersects the upper nappe of the cone K transversely at $z = a$ and $z = b$ and these points of intersection do not lie in a common 1-dimensional subspace.*

Proof. In term of the coordinates (σ, δ, β) on $\text{Sym}_2(\mathbb{R})$ we have

$$q_g(z) = \det(\gamma_g(z)) = \sigma(\gamma_g(z))^2 - \delta(\gamma_g(z))^2 - \beta(\gamma_g(z))^2.$$

The zeros of the rank polynomial q_g are precisely the z -values where $\gamma_g(z)$ intersects K . The order of a zero of q_g is the order of the meeting of the quadratic curve γ_g and K at the point corresponding to the zero. Hence simple roots of $\gamma_g(z)$ correspond to transverse intersection of $\gamma_g(z)$ and K .

The points on the K are matrices that do not have full rank. Points on $K \setminus \{0\}$ have rank 1 and the origin is the zero matrix with rank zero. The matrices in a 1-dimensional subspace of $\text{Sym}_2(\mathbb{R})$ are all multiples of one another. Hence all non-zero matrices on a 1-dimensional subspace contained within K have the same 1-dimensional kernel. It is straightforward to check that two nonzero elements of K have the same kernel if and only if they lie in a common 1-dimensional subspace.

Finally, we recall that a block 7-less metric g defines a Riemannian metric precisely when $\gamma_g(z)$ is positive definite. A matrix in $\text{Sym}_2(\mathbb{R})$ is positive definite when its coordinates satisfy $\sigma^2 > \delta^2 + \beta^2$, that is, matrices sitting with the upper nappe of the cone K . Thus, any transverse intersections must occur on the upper nappe of the cone. \square

The conditions laid out in Proposition 5.24 are invariant properties of a quadratic curve under the coordinates changes in the group \mathcal{G} . Thus, to determine whether a block 7-less metric satisfies the conditions of Proposition 5.24, it suffices to look at the normal form of its associated quadratic curve given in Proposition 5.12. All six of the normal forms allow for two distinct transverse intersection of the quadratic curve γ_g with the cone K connected by a portion of γ_g sitting in the upper nappe of K . In the elliptic and parabolic cases, these intersections will always occur in distinct 1-dimensional subspaces of K . However, in the hyperbolic cases with $\beta_0 = 0$, intersections with K will occur in only two 1-dimensional subspaces: the line spanned by $(\sigma, \delta, \beta) = (1, 1, 0)$ and the line spanned by $(\sigma, \delta, \beta) = (1, -1, 0)$. In these cases, it is possible to get two transverse intersections along the line $\sigma = \delta$.

The condition $\beta = 0$ corresponds to the assumption that the coefficient function $B(z)$ of the block 7-less metric is zero. Thus any block 7-less metric of this form will

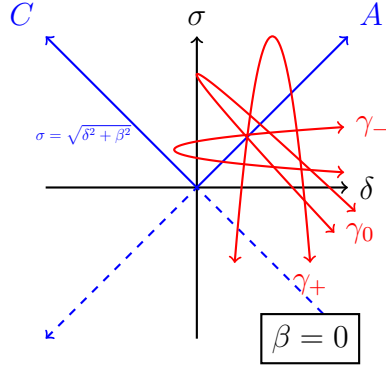


FIGURE 5.2: Examples of quadratic curves corresponding to block 7-less metrics that define Riemannian metrics on finite intervals ending in simple roots of the rank polynomial but not satisfying the conditions of Proposition 5.24. Note that the lines $\sigma = \delta$ and $\sigma = -\delta$ form axes for the functions A and C corresponding to the coefficients of the associated block 7-less metric.

have coefficient functions

$$\begin{aligned} A(z) &= a_0 + a_1 z + a_2 z^2, \\ B(z) &= 0, \\ C(z) &= c_0 + c_1 z + c_2 z^2. \end{aligned}$$

The intersections of the quadratic curve with the upper nappe of the cone will both occur along the line $\sigma = \delta$ corresponding to points where the coefficient function $C(z)$ is zero. We see that $c = 0$ at z -values:

$$\begin{aligned} a &= \frac{-c_1 - \sqrt{c_1^2 - 4c_0c_2}}{2c_2}, \\ b &= \frac{-c_1 + \sqrt{c_1^2 - 4c_0c_2}}{2c_2}. \end{aligned}$$

The expansions of the metric about these points are

$$\begin{aligned} A(z) &= a_0 + a_1(z - a) + a_2(z - a)^2, \\ B(z) &= 0, \\ C(z) &= c_1(z - a) + c_2(z - a)^2, \end{aligned}$$

and

$$\begin{aligned} A(z) &= a'_0 + a'_1(z-b) + a'_2(z-b)^2, \\ B(z) &= 0, \\ C(z) &= c'_1(z-b) + c'_2(z-b)^2, \end{aligned}$$

where

$$a_0 = \frac{a_2c_1^2 + 2a_0c_2^2 - (2a_2c_0 + a_1c_1)c_2 - (a_2c_1 - a_1c_2)\sqrt{c_1^2 - 4c_0c_2}}{2c_2^2}$$

$$a'_0 = \frac{a_2c_1^2 + 2a_0c_2^2 - (2a_2c_0 + a_1c_1)c_2 + (a_2c_1 - a_1c_2)\sqrt{c_1^2 - 4c_0c_2}}{2c_2^2}$$

$$c_1 = \sqrt{c_1^2 - 4c_0c_2},$$

$$c'_1 = -\sqrt{c_1^2 - 4c_0c_2}.$$

Proposition 5.22 tells us that an S^1 can be attached onto the associated torus 7-less metric at both ends if $a_0c_1^2 = 4$ and $a'_0(c'_1)^2 = 4$. If $a_0c_1^2 \neq 4$, a coordinate transformation of the type (5.8) can be applied which results in rescaling $a_0c_1^2$ by a factor of m_2^2 . However, such a coordinate change also rescales $a'_0(c'_1)^2$ by the same factor m_2^2 . Thus, both endpoints can be put into a form where an S^1 can be attached and the metric smoothly extended if and only if $a_0c_1^2 = a'_0(c'_1)^2$. We see

$$a_0c_1^2 - a'_0(c'_1)^2 = \frac{(a_2c_1 - a_1c_2)(c_1^2 - 4c_0c_2)^{3/2}}{c_2^2}.$$

The assumption that the block 7-less metric is non-degenerate implies that $a_2c_1 - a_1c_2 \neq 0$ and the assumption that the quadratic curve intersects the line $\sigma = \delta$ implies $c_2 \neq 0$. Thus $a_0c_1^2 = a'_0(c'_1)^2$ if and only if

$$c_0 = \frac{c_1^2}{4c_2}.$$

This implies that the coefficient function $C(z)$ takes the form

$$C(z) = \frac{(2c_2z + c_1)^2}{4c_2}.$$

This shows that the assumption that both ends of the torus metric can be extended across S^1 's implies that the quadratic curve intersects the cone K with multiplicity two and not transversely at two points.

While we see that we are unable to choose a torus 7-less structure which can be extended across both endpoints, we are still able to normalize and attach an S^1 at one of the endpoints. This still results in a maximal structure. Consider the case of attaching an S^1 at the left endpoint of a maximal interval (a, b) . Consider the S^1 's formed in the torus 7-less metric defined by $\rho = \rho_0$ and $\theta_2 = \theta_{20}$. As the metric degenerates as $\rho_0 \rightarrow b$, the length of these S^1 limits to zero implying that any extension at the endpoint $\rho = b$ must have each these S^1 collapsing to a point. This implies that the only extension at the $\rho = b$ boundary must be attaching an S^1 via the map $i_{R1;b}$. However, we saw that the metric cannot be extended across this new S^1 due to the metric developing into that of a 'pointed cone.'

6

Conclusions

Here we summarize our results and pose some related questions that may be of interest in future research.

In this dissertation, we studied 3-manifolds with 7-less metrics, which are defined by the condition $T = 0$. This serves a first step in understanding what geometric interpretation the tensor field T has. Two natural next steps in this exploration would be constructing metrics where:

1. T is covariant constant, i.e. $\nabla T = 0$. Some rudimentary calculations show that the local solutions to this equation are parameterized by at a finite number of constants.
2. T is restricted to taking values in special $SO(3)$ -orbits of \mathcal{H}^3 : for instance, those cubic polynomials that are the traceless part of a symmetric cube of a polynomial in \mathcal{H}^1 .

The 7-less equation can also be generalized to higher dimensions. In higher dimensions, the covariant derivative decomposed into three irreducible components: the differential of scalar curvature, the Cotton tensor, and a traceless symmetric 3-

tensor [Bes08]. This traceless symmetric 3-tensor serves as the analog of T in higher dimensions. Some natural first questions to ask would be:

1. In dimension 3, the $T = 0$ condition leads to a finite dimensional set of local solutions. Is this the case in higher dimensions?
2. In dimension 3, when the Cotton tensor vanishes along side the 7-piece, the local solutions are less symmetric than when the Cotton tensor is non-vanishing. Does a similar property emerge in higher dimensions? If so, is this a consequence of the vanishing of the Cotton tensor itself or the conformal flatness implied by the vanishing of the Cotton tensor?
3. Do higher dimensional analogs of Σ_P manifolds have an interesting relationship to the higher dimensional analog of the 7-part? It is simple to see that higher dimensional Σ_P manifolds are conformally flat and so this may play into the previous question.

Finally, we have given constructions for maximal 7-less manifolds here. The conformally flat cases have an algebraic construction in terms of varieties in an ambient Minkowski space. We contrast that with the non-conformally flat case where most of our construction was done using local coordinates. The results of these coordinate constructions were 7-less manifolds that embed topologically as dense submanifolds in S^3 . The block 7-less metrics used in these constructions seem to be algebraic in nature as the coefficients of the metric are quadric polynomials. This leads us to wonder if there is an algebraic construction of the non-conformally flat 7-less manifolds similar to the construction of the conformally flat ones.

A

Parametrization of Orbits

Here we provide an outline for the calculations needed for proving Propositions 4.13 and 4.23 that give parameterizations of the orbits of \mathcal{V}_{NC} . We recall that \mathcal{V}_{NC} is a subset of \mathcal{V} , which is a variety in

$$\mathcal{H}^c = \mathcal{H}_R^2 \oplus \mathcal{H}_S^0 \oplus \mathcal{H}_C^2 \oplus \mathcal{H}_D^1 \oplus \mathcal{H}_Z^2 \oplus \mathcal{H}_X^0 \cong \mathbb{R}^{20}$$

generated by the 50 independent components of the Q_{ij} quantities found in Section 3.3, the set of which we call \mathcal{Q} . Using the trace-free property of R , C , and Z , we see that a list of coordinates on \mathcal{H}^c is

$$\begin{aligned} & (S, R_{22}, R_{33}, R_{23}, R_{31}, R_{12}, D_1, D_2, D_3, C_{22}, \\ & C_{33}, C_{23}, C_{31}, C_{12}, X, Z_{22}, Z_{33}, Z_{23}, Z_{31}, Z_{12}) \end{aligned} \tag{A.1}$$

and that every element of \mathcal{Q} can be written using these 20 components alone. To parameterize the orbits of \mathcal{V}_{NC} we now just need to solve the system of equations given by the vanishing of the elements of \mathcal{Q} . To do this, we will make various statements about certain quantities vanishing being implied by the element of \mathcal{Q} being zero. To check that these claims are true, it suffices to check that the claimed

vanishing quantity is in the ideal generated by \mathcal{Q} , which we write as $\langle \mathcal{Q} \rangle$. This can be done by computing a Gröbner basis for the ideal.

Remark A.1. *While any monomial order will work to test membership of the ideal, we find that the graded reverse lexicographic order with variable ordered as in (A.1) tends to be most computationally efficient with any ‘auxiliary’ variables being added to the end of the list.*

(Proof of Proposition 4.13). Using the action of $\text{SO}(3)$, we see that every orbit in \mathcal{V}_{NC}^{ND} has a point where

$$D_2 = 0, \quad D_3 = 0, \quad C_{23} = 0.$$

The assumption that D is non-vanishing implies that $D_1 \neq 0$ at such a point. We make these substitutions into \mathcal{Q} , reducing to 17 variables and giving an updated ideal $\langle \mathcal{Q} \rangle$. Next, we find

$$D_1^2 C_{31}, D_1^2 C_{12} \in \langle \mathcal{Q} \rangle.$$

As $D_1 \neq 0$, we conclude $C_{31} = 0$ and $C_{12} = 0$. Substituting into \mathcal{Q} , we are left with 15 variables. Next, we find

$$D_1^2 Z_{31}, D_1^2 Z_{12} \in \langle \mathcal{Q} \rangle$$

and we conclude $Z_{31} = 0$ and $Z_{12} = 0$. Substituting into \mathcal{Q} , we are left with 13 variables. We now claim that not both C_{22} and C_{33} are zero. If we make the assumption that $C_{22} = 0$ and $C_{33} = 0$ and substitute into \mathcal{Q} , we find that

$$Z_{22} D_1, Z_{33} D_1, Z_{23} D_1 \in \langle \mathcal{Q} \rangle.$$

and we find $Z_{22} = 0$, $Z_{33} = 0$, and $Z_{23} = 0$. This is a contradiction, as we assumed that one of C or Z was non-zero. Hence we conclude that one of C_{22} or C_{33} is non-zero. Next, we find that

$$Z_{22} R_{31}, Z_{22} R_{12}, Z_{33} R_{31}, Z_{33} R_{12}, Z_{23} R_{31}, Z_{23} R_{12} \in \langle \mathcal{Q} \rangle.$$

We claim $R_{31} = 0$ and $R_{12} = 0$. If we make the assumption that one of R_{31} or R_{12} is nonzero, we find that $Z_{22} = 0$, $Z_{33} = 0$, $Z_{23} = 0$. substituting into \mathcal{Q} , we would then find

$$C_{22}D_1, C_{33}D_1 \in \langle \mathcal{Q} \rangle.$$

and conclude $C_{22} = 0$ and $C_{33} = 0$. This is a contradiction, as we have seen that one of C_{22} or C_{33} is non-zero. Hence we conclude that $R_{31} = 0$ and $R_{12} = 0$. We make these substitutions into \mathcal{Q} , reducing to 11 variables. Next, we introduce the auxiliary variable \bar{X} and let

$$X = \frac{20}{3}(SR_{22} + 4R_{22}^2 + R_{23}^2 + SR_{33} + 7R_{22}R_{33} + 4R_{33}^2 - 2Z_{22} - 2Z_{33}) - \bar{X}.$$

Making this substitution into \mathcal{Q} , thereby eliminating X , we find that

$$\bar{X}Z_{23}, \bar{X}Z_{22}D_1, \bar{X}Z_{33}D_1 \in \langle \mathcal{Q} \rangle.$$

If $\bar{X} \neq 0$, then $Z_{22} = 0$, $Z_{33} = 0$, and $Z_{23} = 0$. Making these substitutions into \mathcal{Q} , we could then find

$$C_{22}D_1, C_{33}D_1 \in \langle \mathcal{Q} \rangle$$

and we would conclude $C_{22} = 0$ and $C_{33} = 0$, a contradiction. Hence $\bar{X} = 0$ and therefore

$$X = \frac{20}{3}(SR_{22} + 4R_{22}^2 + R_{23}^2 + SR_{33} + 7R_{22}R_{33} + 4R_{33}^2 - 2Z_{22} - 2Z_{33}).$$

We make this substitution into \mathcal{Q} , reducing to 10 variables. Next, we find

$$D_1(4C_{22}Z_{22} + 5C_{33}Z_{22} + 5C_{22}Z_{33} + 4C_{33}Z_{33}) \in \langle \mathcal{Q} \rangle.$$

This implies that $4C_{22}Z_{22} + 5C_{33}Z_{22} + 5C_{22}Z_{33} + 4C_{33}Z_{33} = 0$ and hence the matrix

$$\begin{bmatrix} Z_{22} & 5C_{22} + 4C_{33} \\ -Z_{33} & 4C_{22} + 5C_{33} \end{bmatrix}$$

has rank 1. As not both of C_{22} and C_{33} are zero, we see that the second column of this matrix is non-zero and hence the first column is a multiple of the second. This gives

$$Z_{22} = \mu(5C_{22} + 4C_{33}), \quad Z_{33} = -\mu(4C_{22} + 5C_{33}),$$

for some number, μ . Similarly, we find,

$$D_1(4R_{22}C_{22} + 5R_{33}C_{22} + 5R_{22}C_{33} + 4R_{33}C_{33}) \in \langle \mathcal{Q} \rangle$$

and conclude

$$R_{22} = \lambda(5C_{22} + 4C_{33}), \quad R_{33} = -\lambda(4C_{22} + 5C_{33}),$$

for some number λ . Making these substitutions into \mathcal{Q} reduces us to 8 variables.

Next, we find

$$C_{22}(60R_{23}\mu - 60Z_{23}\lambda - D_1), C_{33}(60R_{23}\mu - 60Z_{23}\lambda - D_1) \in \langle \mathcal{Q} \rangle.$$

As not both C_{22} and C_{33} are zero we find $D_1 = 60(R_{23}\mu - Z_{23}\lambda)$. Making this substitution into \mathcal{Q} reduces us to 7 variables. Note that as $D_1 \neq 0$ we have $\lambda \neq 0$ or $\mu \neq 0$ as well as $R_{23} \neq 0$ or $Z_{23} \neq 0$. Next, we find that

$$\lambda C_{22}M, \lambda C_{33}M, \mu C_{22}M, \mu C_{33}M \in \langle \mathcal{Q} \rangle$$

where M is any of the 2×2 minors of the matrix

$$\begin{bmatrix} (2C_{22} + C_{33})(C_{22} + 2C_{33}) & 3Z_{23} \\ R_{23} & 1 - 27\lambda\mu \\ 2Z_{23} & 6C_{22}\lambda - 6C_{33}\lambda - 54\mu^2 + S \end{bmatrix}.$$

Thus this matrix has rank 1. As not both of R_{23} and Z_{23} are zero, we see that the left column is non-zero and hence the second column is a multiple of the first. So we have

$$Z_{23} = \frac{\kappa}{3}(2C_{22} + C_{33})(C_{22} + 2C_{33}), \quad S = 2Z_{23}\kappa - 6C_{22}\lambda + 6C_{33}\lambda + 54\mu^2,$$

for some number κ . Making these substitution reduces us to 6 variables. Finally, we can check

$$\langle \mathcal{Q} \rangle = \langle 27\lambda\mu + R_{23}\kappa - 1 \rangle.$$

This implies that the reaming components satisfy $27\lambda\mu + R_{23}\kappa = 1$.

Combining these substitutions we get the parameterization given in 4.10. \square

(*Proof of Proposition 4.23*). At every point in \mathcal{V}_{NC}^D we have

$$D_1 = 0, \quad D_2 = 0, \quad D_3 = 0.$$

Making these substitution reduces us to 17 variables. Using the action of $\text{SO}(3)$, we see that every orbit in \mathcal{V}_{NC}^D has a point where

$$Z_{23} = 0, \quad Z_{31} = 0, \quad Z_{12} = 0.$$

Making these substitution reduces us to 14 variables. Next, we find

$$Z_{22}R_{23}, Z_{22}R_{31}, Z_{22}R_{12}, Z_{33}R_{23}, Z_{33}R_{31}, Z_{33}R_{12} \in \langle \mathcal{Q} \rangle.$$

If one of Z_{22} or Z_{33} is non-zero, this implies $R_{23} = 0$, $R_{31} = 0$, and $R_{12} = 0$. If $Z_{22} = 0$ and $Z_{33} = 0$, then we can use the action of $\text{SO}(3)$ to find a point in the orbit where $R_{23} = 0$, $R_{31} = 0$, and $R_{12} = 0$. In either case, we conclude that the orbit contains a point satisfying the additional relations $R_{23} = 0$, $R_{31} = 0$, and $R_{12} = 0$. Making these substitutions reduces us to 11 variables. At this point, we can check that H , the matrix expression of the Hessian of S , is diagonal. Introducing new variables \bar{X} , \bar{Z}_{22} , and \bar{Z}_{33} and making the substitutions

$$X = \frac{10}{3}(-SR_{22} + 2R_{22}^2 - SR_{33} + 8R_{22}R_{33} + 2R_{33}^2 + 2Z_{22} + 2Z_{33}) + \bar{X},$$

$$Z_{22} = \frac{1}{2}SR_{22} + R_{22}^2 - 2R_{22}R_{33} - 2R_{33}^2 - \frac{1}{20}(\bar{X} + \bar{Z}_{33} - 2\bar{Z}_{22}),$$

$$Z_{33} = \frac{1}{2}SR_{33} - 2R_{22}^2 - 2R_{22}R_{33} + R_{33}^2 - \frac{1}{20}(\bar{X} + \bar{Z}_{22} - 2\bar{Z}_{33}),$$

the components of the matrix H take the form

$$\begin{aligned} H_{11} &= \bar{X}, & H_{22} &= \bar{Z}_{22}, & H_{33} &= \bar{Z}_{33}, \\ H_{23} &= 0, & H_{32} &= 0, & H_{31} &= 0. \end{aligned}$$

With this change of variables, statements about the rank of H can be made in terms of vanishing or non-vanishing of \bar{X} , \bar{Z}_{22} , and \bar{Z}_{33} .

Case: $\text{rank}(H) = 3$. If $\text{Rank}(H) = 3$, then we have that $\bar{X} \neq 0$, $\bar{Z}_{22} \neq 0$, and $\bar{Z}_{33} \neq 0$. We find

$$\bar{Z}_{22}C_{23}, \bar{Z}_{22}C_{12}, \bar{Z}_{33}C_{23}, \bar{Z}_{33}C_{31}, \bar{X}C_{31}, \bar{X}C_{12} \in \langle \mathcal{Q} \rangle,$$

implying $C_{23} = 0$, $C_{31} = 0$, and $C_{12} = 0$. Making these substitutions we next find

$$\bar{X}\bar{Z}_{22}(2C_{22} + C_{33}), \bar{X}\bar{Z}_{33}(C_{22} + 2C_{33}) \in \langle \mathcal{Q} \rangle,$$

implying $C_{22} = 0$, and $C_{33} = 0$. Making these substitutions we next find

$$\bar{X}\bar{Z}_{22}\bar{Z}_{33}(40R_{22}^2 - 20SR_{33} + 160R_{22}R_{33} + 40R_{33}^2 + 4\bar{X} - 20SR_{22} - 2\bar{Z}_{22} - 2\bar{Z}_{33}) \in \langle \mathcal{Q} \rangle,$$

implying $\bar{X} = \frac{1}{2}(10SR_{22} - 20R_{22}^2 + 10SR_{33} - 80R_{22}R_{33} - 20R_{33}^2 + \bar{Z}_{22} + \bar{Z}_{33})$. Making this substitution we find

$$\bar{Z}_{22}\bar{Z}_{33}(-10SR_{22} - 60R_{22}^2 + 10SR_{33} + 60R_{33}^2 - 3\bar{Z}_{22} + 3\bar{Z}_{33}) \in \langle \mathcal{Q} \rangle,$$

implying $-10SR_{22} - 60R_{22}^2 + 10SR_{33} + 60R_{33}^2 - 3\bar{Z}_{22} + 3\bar{Z}_{33} = 0$. Back substituting we find that this implies that the original Z_{22} and Z_{33} are both zero. Thus $C = 0$ and $Z = 0$. This is a contradiction, as we are considering only points where one of these are non-zero. This shows that H never has rank 3 at points in V_{NC}^D .

Case: $\text{rank}(H) = 2$. If $\text{Rank}(H) = 2$, then, without loss of generality, we have that $\bar{X} = 0$, $\bar{Z}_{22} \neq 0$, and $\bar{Z}_{33} \neq 0$ leaving us with 10 free variables. We again have

$$\bar{Z}_{22}C_{23}, \bar{Z}_{22}C_{12}, \bar{Z}_{33}C_{23}, \bar{Z}_{33}C_{31} \in \langle \mathcal{Q} \rangle,$$

giving $C_{23} = 0$, $C_{31} = 0$, and $C_{12} = 0$. After making these substitutions, we are left with 7 free variables. Next, we find

$$\bar{Z}_{33}\bar{Z}_{22}(C_{22} - C_{33}) \in \langle \mathcal{Q} \rangle.$$

This shows $C_{22} = C_{33}$. To preserve symmetry, we use the ‘traceless’ relationship and introduce a new variable C_{11} defined by $C_{11} + C_{22} + C_{33} = 0$. Then, we have $C_{22} = -\frac{1}{2}C_{11}$ and $C_{33} = -\frac{1}{2}C_{11}$. Making these substitutions we are left with 6 free variables. We find

$$\bar{Z}_{22}\bar{Z}_{33}(-10SR_{22} - 60R_{22}^2 + 10SR_{33} + 60R_{33}^2 - 3\bar{Z}_{22} + 3\bar{Z}_{33}) \in \langle \mathcal{Q} \rangle.$$

This gives $\bar{Z}_{22} - \bar{Z}_{33} = \frac{1}{3}(-10SR_{22} - 60R_{22}^2 + 10SR_{33} + 60R_{33}^2)$. We again introduce a variable \bar{Z}_{11} satisfying $\bar{Z}_{11} + \bar{Z}_{22} + \bar{Z}_{33} = 0$. This gives

$$\bar{Z}_{22} = -\frac{1}{2}\bar{Z}_{11} + \frac{1}{6}(-10SR_{22} - 60R_{22}^2 + 10SR_{33} + 60R_{33}^2),$$

$$\bar{Z}_{33} = -\frac{1}{2}\bar{Z}_{11} - \frac{1}{6}(-10SR_{22} - 60R_{22}^2 + 10SR_{33} + 60R_{33}^2).$$

Making these substitutions we are left with 5 free variables. We now claim $R_{22} = R_{33}$. Assume this is not the case. We find

$$(R_{33} - R_{22})(5SR_{22} - 10R_{22}^2 + 5SR_{33} - 40R_{22}R_{33} - 10R_{33}^2 + \bar{Z}_{11}), (R_{22} - R_{33})C_{11}^2 \in \langle \mathcal{Q} \rangle.$$

This implies $C_{11} = 0$ and $\bar{Z}_{11} = -5SR_{22} + 10R_{22}^2 - 5SR_{33} + 40R_{22}R_{33} + 10R_{33}^2$. Back substituting we find that this implies that the original Z_{22} and Z_{33} are both zero. Thus $C = 0$ and $Z = 0$. This is a contradiction. Thus $R_{22} = R_{33}$. Introducing another variable R_{11} satisfying $R_{11} + R_{22} + R_{33} = 0$, we have $R_{22} = -\frac{1}{2}R_{11}$ and $R_{33} = -\frac{1}{2}R_{11}$. Making this final substitutions we are left with 4 free variables and find $\langle \mathcal{Q} \rangle = 0$ implying there are no additional relations.

Case: $\text{rank}(H) = 1$. If $\text{Rank}(H) = 1$, then, without loss of generality, we have that $\bar{X} \neq 0$, $\bar{Z}_{22} = 0$, and $\bar{Z}_{33} = 0$ leaving us with 9 free variables. We check

$$\bar{X}C_{31}, \bar{X}C_{12} \in \langle \mathcal{Q} \rangle$$

implying $C_{31} = 0$ and $C_{12} = 0$. Making these substitution we are left with 7 free variables. To simplify some calculations, we introduce two new variables, \bar{R}_{22} and \bar{R}_{33} , defined by

$$R_{33} = \frac{1}{3}(2\bar{R}_{33} - \bar{R}_{22}), \quad R_{22} = \frac{1}{3}(2\bar{R}_{22} - \bar{R}_{33}).$$

We now see that the 2×2 minors of the matrix

$$\begin{bmatrix} -30C_{22}\bar{R}_{22} - 60C_{33}\bar{R}_{22} + 60C_{22}\bar{R}_{33} + 30C_{33}\bar{R}_{33} & \bar{X} \\ 3C_{22} + 6C_{33} & \bar{R}_{33} \\ -6C_{22} - 3C_{33} & \bar{R}_{22} \end{bmatrix}$$

are in $\langle \mathcal{Q} \rangle$ implying that the matrix has rank one. As $\bar{X} \neq 0$, we see that the right column is non-zero and hence the left column is a multiple of the right. This gives

$$C_{22} = \lambda(2\bar{R}_{22} + \bar{R}_{33}), \quad C_{33} = -\lambda(\bar{R}_{22} + 2\bar{R}_{33}),$$

for some constant λ . Making these substitution we are left with 6 free variables. We now claim $20\bar{R}_{22}\bar{R}_{33} + \bar{X} = 0$. Assume this is not the case. We find

$$(\bar{R}_{22} - \bar{R}_{33})(20\bar{R}_{22}\bar{R}_{33} + \bar{X}), \bar{X}\lambda(20\bar{R}_{22}\bar{R}_{33} + \bar{X}), C_{23}(20\bar{R}_{22}\bar{R}_{33} + \bar{X}) \in \langle \mathcal{Q} \rangle$$

implying $\lambda = 0$, $C_{23} = 0$, $\bar{R}_{22} = -\frac{1}{2}\bar{R}_{11}$, and $\bar{R}_{33} = -\frac{1}{2}\bar{R}_{11}$ for a new variable \bar{R}_{11} .

We then find

$$\bar{X}(5S\bar{R}_{11} + 5\bar{R}_{11}^2 + 3\bar{X}) \in \langle \mathcal{Q} \rangle,$$

implying $\bar{X} = -\frac{5}{3}(S\bar{R}_{11} + \bar{R}_{11}^2)$. Back substituting we find $C = 0$ and $Z = 0$, a contradiction. Thus, we conclude $20\bar{R}_{22}\bar{R}_{33} + \bar{X} = 0$ and so $\bar{X} = -20\bar{R}_{22}\bar{R}_{33}$.

Making this assumption we are left with 5 free variables. The assumption that $\bar{X} \neq 0$ implies $\bar{R}_{22} \neq 0$ and $\bar{R}_{33} \neq 0$. Next, we check that the determinant of the matrix

$$\begin{bmatrix} -3(6\lambda^2 - S - 2\bar{R}_{22} - 2\bar{R}_{33}) & 2C_{23} \\ C_{23} & \bar{R}_{22}\bar{R}_{33} \end{bmatrix}$$

is in $\langle \mathcal{Q} \rangle$ implying that the matrix has rank 1. Since $\bar{R}_{22}\bar{R}_{33} \neq 0$, the right column is non-zero and hence the left column is a multiple of the right. This gives

$$C_{23} = \kappa \bar{R}_{22} \bar{R}_{33}, \quad S = \frac{2}{3} \kappa^2 \bar{R}_{22} \bar{R}_{33} + 6\lambda^2 - 2\bar{R}_{22} - 2\bar{R}_{33},$$

for some constant κ . Making these substitutions we are left with 3 free variables and find $\langle \mathcal{Q} \rangle = 0$ implying there are no additional relations.

Case: $\text{rank}(H) = 0$. In this case we take a different approach. Rather than using the action of $\text{SO}(3)$ to find a point with $Z_{23} = 0$, $Z_{31} = 0$, and $Z_{12} = 0$, we use the action to find a point where $C_{23} = 0$, $C_{31} = 0$, and $C_{12} = 0$. Making these substitutions along with substitutions $D_1 = 0$, D_2 , and $D_3 = 0$, we are left with 14 free variables. As $\text{rank}(H) = 0$, all 6 independent components of H must be zero. This gives

$$Z_{22} = \frac{1}{2}SR_{22} + R_{22}^2 + R_{12}^2 + R_{23}^2 - 2R_{31}^2 - 2R_{33}^2 - 2R_{22}R_{33} - \frac{3}{20}X,$$

$$Z_{33} = \frac{1}{2}SR_{33} + R_{33}^2 + R_{23}^2 + R_{31}^2 - 2R_{12}^2 - 2R_{22}^2 - 2R_{22}R_{33} - \frac{3}{20}X,$$

$$Z_{23} = \frac{1}{2}SR_{23} + 3R_{22}R_{23} + 3R_{12}R_{31} + 3R_{23}R_{33},$$

$$Z_{31} = \frac{1}{2}SR_{31} + 3R_{12}R_{23} - 3R_{22}R_{31},$$

$$Z_{12} = \frac{1}{2}SR_{12} + 3R_{23}R_{31} - 3R_{12}R_{33},$$

$$X = 0.$$

Making these substitutions we have 8 free variables. We claim that $R_{23} = 0$, $R_{31} = 0$, $R_{12} = 0$, and $R_{22} = R_{33}$. We consider the cases where $C \neq 0$ and $C = 0$. Assume $C \neq 0$, that is, $C_{22} \neq 0$ or $C_{33} \neq 0$. We have

$$C_{22}^2 R_{23}^3, C_{22}^2 R_{31}^3, C_{22}^2 R_{12}^3, C_{33}^2 R_{23}^3, C_{33}^2 R_{31}^3, C_{33}^2 R_{12}^3 \in \langle \mathcal{Q} \rangle,$$

implying $R_{23} = 0$, $R_{31} = 0$, and $R_{12} = 0$. Making these substitutions we find

$$(C_{22} - C_{33})(C_{22} + 2C_{33})(2C_{22} + C_{33}) \in \langle \mathcal{Q} \rangle.$$

With the variable C_{11} defined by $C_{11} + C_{22} + C_{33} = 0$, the above relation implies that $C_{22} = C_{33}$, $C_{33} = C_{11}$, or $C_{11} = C_{22}$. We can use the action of $\text{SO}(3)$ to exchange variables and assume $C_{22} = C_{33}$ giving $C_{22} = -\frac{1}{2}C_{11}$ and $C_{33} = -\frac{1}{2}C_{11}$ with $C_{11} \neq 0$. substituting we find

$$(R_{22} - R_{33})C_{11}^2 \in \langle \mathcal{Q} \rangle$$

implying $R_{22} = R_{33}$. Next, we consider the case where $C = 0$. This give $C_{22} = 0$ and $C_{33} = 0$. In this case, we can use the action of $\text{SO}(3)$ to find a point in the orbit where $R_{23} = 0$, $R_{31} = 0$, and $R_{12} = 0$. We then find

$$(R_{22} - R_{33})(R_{22} + 2R_{33})(2R_{22} + R_{33}) \in \langle \mathcal{Q} \rangle.$$

With R_{11} defined by $R_{11} + R_{22} + R_{33} = 0$, this equation implies $R_{22} = R_{33}$, $R_{33} = R_{11}$, or $R_{11} = R_{22}$. Using the action of $\text{SO}(3)$, we can exchange variables and assume $R_{22} = R_{33}$. Thus, regardless of the value of C , we find that these is a point in the orbit $R_{23} = 0$, $R_{31} = 0$, $R_{12} = 0$, and $R_{22} = R_{33}$, which we can write as $R_{22} = -\frac{1}{2}R_{11}$ and $R_{33} = -\frac{1}{2}R_{11}$. Making these substitution we are left with 4 free variables. Next, we claim that $C_{22} = C_{33}$. We check that

$$(C_{22} - C_{33})R_{11}^2 \in \langle \mathcal{Q} \rangle.$$

If $C_{22} \neq C_{33}$, then $R_{11} = 0$ implying that $R_{11} = 0$ giving $R = 0$. We check

$$(C_{22} - C_{33})(C_{22} + 2C_{33})(2C_{22} + C_{33}) \in \langle \mathcal{Q} \rangle.$$

As explained above, we can use this relation and the action of $\text{SO}(3)$ to find a point where $C_{22} = C_{33}$. Thus, we have $C_{22} = C_{33}$ giving $C_{22} = -\frac{1}{2}C_{11}$ and $C_{33} = -\frac{1}{2}C_{11}$. Making these substitutions we are left with 3 free variables find $\langle \mathcal{Q} \rangle = 0$ implying there are no additional relations.

In each case, making the substitutions gives the relations in (4.15) and (4.16). \square

Bibliography

- [BCG⁺12] Robert L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, and P. A. Griffiths. *Exterior differential systems*, volume 18 of *Mathematical Sciences Research Institute Publications*. Springer-Verlag New York Inc., 2012.

- [Bes08] A. L. Besse. *Einstein Manifolds*. Classics in Mathematics. Springer, 2008.

- [Bry14] Robert L. Bryant. Notes on exterior differential systems, 2014.

- [Gar89] Robert B. Gardner. *The method of equivalence and its applications*. Society for Industrial and Applied Mathematics, 1989.

- [Sha97] R. W. Sharpe. *Differential geometry: Cartan's generalization of Kleins Erlangen program*, volume 166 of *Graduate Texts in Mathematics*. Springer, 1997.

- [TD85] Bröcker Theodor and Tammo Tom Dieck. *Representations of compact Lie groups*, volume 98 of *Graduate Texts in Mathematics*. Springer, 1985.

Biography

Ryan Gunderson attended the University of Nebraska - Lincoln from 2009 to 2013 graduating with a Bachelor of Science in mathematics. He then attended Duke University from 2013 to 2019 working under the supervision of Robert Bryant and graduating with a PhD in mathematics. Following graduation he will work at Susquehanna International Group in Bala Cynwyd, Pennsylvania.