Stochastic Switching in Evolution Equations

by

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Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
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2014

ABSTRACT

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Abstract

We consider stochastic hybrid systems that stem from evolution equations with right-hand sides that stochastically switch between a given set of right-hand sides. To begin our study, we consider a linear ordinary differential equation whose right-hand side stochastically switches between a collection of different matrices. Despite its apparent simplicity, we prove that this system can exhibit surprising behavior.

Next, we construct mathematical machinery for analyzing general stochastic hybrid systems. This machinery combines techniques from various fields of mathematics to prove convergence to a steady state distribution and to analyze its structure.

Finally, we apply the tools from our general framework to partial differential equations with randomly switching boundary conditions. There, we see that these tools yield explicit formulae for statistics of the process and make seemingly intractable problems amenable to analysis.

This thesis is dedicated to my wife.

Contents

List of Abbreviations and Symbols

Symbols

$\mathbb{R}$	the set of real numbers
$\mathbb{R}^d$	the set of d-dimensional real numbers
H	a real separable Hilbert space
L^2	Lebesgue space of square-integrable functions
E	expectation
P	probability measure
$\mathcal{F}$	σ-algebra
(x, y)	inner product of two elements, x and y, of a Hilbert space

Abbreviations

ODE	ordinary differential equation
PDE	partial differential equation

Acknowledgements

First, I earnestly thank my advisors, Professors Michael Reed and Jonathan Mattingly. They introduced these problems to me and provided crucial direction. In addition to their vital guidance in this research, they have helped me to develop as a teacher and a mentor. I am truly grateful for them.

I thank Professors Thomas Beale and James Nolen for their service on my committee and for all that I learned from them in the many classes I took from them.

I thank my undergraduate professors at Carnegie Mellon University; in particular, I thank Professors David Kinderlehrer, Jack Schaeffer, William Hrusa, John Mackey, Elaine Cozzi, and Steven Shreve.

I thank Jack Bookman and Lewis Blake for teaching me so much about how to teach math. I thank Hugh Crumley and Doug James for helping me to grow as a teacher in the Certificate in College Teaching Program in the Duke University Graduate School. I thank all my students and mentees; it has been a great privilege to work with all of them.

I thank all my biological collaborators; in particular, I thank Professor Fred Nijhout from Duke University and Professors Mary Gamble and Megan Hall from Columbia University.

I thank the National Science Foundation and the Howard Hughes Medical Institute for financial support.

Lastly, I thank my wife, my parents, and my siblings for their love and support.

1

Introduction

In this thesis, we consider stochastic hybrid systems that stem from evolution equations with right-hand sides that stochastically switch between a given set of right-hand sides.

Stochastic hybrid systems are a type of stochastic process that are used in many areas of biology (for example, molecular biology [10], [27], [9], ecology [32], epidemiology [15]) and many other applied areas outside of biology [31]. The word "hybrid" is used because these processes involve both continuous dynamics and discrete events. For the example of an evolution equation with a stochastically switching right-hand side, the continuous dynamics are the different right-hand sides and the discrete events are when the right-hand side switches.

In general, a stochastic hybrid system is a continuous-time stochastic process with two components: a continuous component $(X_t)_{t \geqslant 0}$ and a jump component $(J_t)_{t \geqslant 0}$. The jump component J_t is a jump process on a finite set, and for each element of its state space we assign some continuous dynamics to X_t. In between jumps of J_t, the component X_t evolves according to the dynamics associated with the current state of J_t. When J_t jumps, X_t switches to following the dynamics associated with the new state of J_t.

In Chapter 2, we consider the stochastic process driven by an ordinary differential equation whose right-hand side randomly switches between a collection of different linear terms.[1] Explicitly, we consider the process (X_t, J_t) where $X_t \in \mathbb{R}^d$ solves

$$\dot{X}_t = A_{J_t} X_t$$

with J_t a continuous-time Markov jump process on a finite set E and $\{A_j\}_{j\in E}$ a set of real matrices. Despite their apparent simplicity, we prove that these systems can have surprising behavior.

Specifically, we construct planar examples that switch between two matrices where the individual matrices and the average of the two matrices are all stable (all eigenvalues have strictly negative real part), but nonetheless the process goes to infinity at large time for certain values of the switching rate. To state our result precisely, let r scale the rate at which the right-hand side switches by letting J_t have generator rQ, and define the average matrix $\bar{A} := \sum_{j\in E} A_j \pi_j$ where π is the invariant measure of J_t.

Theorem. *There exist matrices $A_0, A_1 \in \mathbb{R}^{2\times 2}$ so that A_0, A_1, and $\bar{A}$ are each stable, but nonetheless $\|X_t\| \to \infty$ almost surely as $t \to \infty$ for some switching rate r.*

We further construct examples in higher dimensions where again A_0, A_1, and $\bar{A}$ are all stable, but $\|X_t\|$ has arbitrarily many transitions between converging to ∞ and converging to 0 as the switching rate varies:

Theorem. *For any positive integer n, there exist matrices A_0, A_1 and n non-overlapping intervals $\{(a_k, b_k)\}_{i=1}^n$ so that*

1. *A_0, A_1, and $\bar{A}$ are each stable.*

2. *If the switching rate $r \notin \bigcup_{i=1}^n (a_i, b_i)$, then $\|X_t\| \to 0$ almost surely as $t \to \infty$.*

[1] The contents of Chapter 2 overlap significantly with [22].

3. *For every* $i \in \{1, \ldots, n\}$, $\|X_t\| \to \infty$ *almost surely as* $t \to \infty$ *for some switching rate* $r \in (a_i, b_i)$.

Our work in Chapter 2 also has important implications for the general study of stochastic hybrid systems. [12], [5], [4], and [1] all study invariant measures for stochastic hybrid systems. Our work shows that the existence of invariant measures may depend on the switching rates in a complicated way. In [16], [17], and [3], the authors provide conditions under which their randomly switched systems behave according to the individual systems for slow switching and according to the average system for fast switching. In Chapter 2, we prove that stochastically switched linear ODEs also obey this principle by proving that if the individual matrices are each stable, then $\lim_{t\to\infty} \|X_t\| = 0$ for sufficiently slow switching rate and if the average matrix is stable, then $\lim_{t\to\infty} \|X_t\| = 0$ for sufficiently fast switching rate. However, the theorem above shows that the transition between the slow and fast switching regimes can be arbitrarily complicated.

In Chapter 3, we develop general mathematical machinery for analyzing stochastic hybrid systems.[2] We are able to cast many stochastic hybrid systems into this framework. This machinery combines techniques from various fields of mathematics, including probability, ergodic theory, and functional analysis, to yield explicit formulae for important statistics of these processes. Our methods are particularly useful for infinite-dimensional processes, such PDEs with randomly switching boundary conditions.

This machinery examines stochastic hybrid systems from the viewpoint of iterated random maps on abstract spaces. We consider a stochastic hybrid system $(X_t, J_t) \in H \times \{0, 1\}$ with H a separable Hilbert space. For each $j \in \{0, 1\}$, we define $\Phi_j^t : H \to H$ to be the flow map for the continuous dynamics associated with state j. Thus X_t

[2] The contents of Chapter 3 overlap significantly with [21].

is constructed by repeatedly applying the flow maps according to the evolution of the jump component J_t.

We prove that if J_t does not depend on X_t and the flow maps Φ_j^t are contracting in some average sense, then X_t converges in distribution as $t \to \infty$. Furthermore, we prove that this limiting distribution is invariant under applications of the flow map Φ_ξ^τ for random variables τ and ξ chosen appropriately. This invariance property is the main tool that yields explicit formulae for statistics of the process.

In Chapter 4, we apply the tools from Chapter 3 to PDEs with randomly switching boundary conditions.[3] There, we see that these tools yield explicit formulae for important statistics of the solution. These tools are especially useful when the PDE switches between boundary conditions of different types, such as switching between Dirichlet and Neumann conditions.

Our work analyzing PDEs with randomly switching boundary conditions was prompted by various biological problems, including questions in cell polarization, neuroscience, immunology, and insect respiration. We conclude Chapter 4 by applying our results to an open problem in insect respiration.

[3] The contents of Chapter 4 overlap significantly with [21].

2

Stochastically switched linear ODEs

To begin our study of stochastic switching in evolution equations, we consider an ordinary differential equation whose right-hand side randomly switches between a collection of different linear terms. Despite their apparent simplicity, we prove that these systems can have surprising behavior. The contents of this chapter overlap significantly with [22].[1]

2.1 Background and setup

We consider the stochastic process $(X_t)_{t\geqslant 0} \in \mathbb{R}^d$ where X_t solves $\dot{X}_t = A_{I_t} X_t$ with I_t a Markov process on a finite set E and $\{A_i\}_{i\in E}$ a set of $d \times d$ real matrices. The stability of this system when the switching process I_t is deterministic has been extensively studied in the past decade; see [2] and [25].

In [6], the authors study the stochastic problem in the plane with I_t a Markov process and $E = \{0, 1\}$. The authors assume both A_0 and A_1 are Hurwitz (all eigenvalues have strictly negative real part) and prove the surprising result that $\|X_t\|$

[1] First published in Communications in Mathematical Sciences in 2014, published by International Press.

may converge to 0 or $+\infty$ as $t \to \infty$ depending on the switching rate as long as an average matrix $\bar{A} = \lambda A_0 + (1-\lambda)A_1$ has a positive eigenvalue for some $\lambda \in (0,1)$.

In this chapter, we show that the assumption that the average matrix has a positive eigenvalue is not necessary to ensure a blowup. Specifically, we construct examples in the plane where A_0, A_1, and $\bar{A} = \lambda A_0 + (1-\lambda)A_1$ are all Hurwitz, but $\|X_t\| \to +\infty$ almost surely as $t \to \infty$ for certain values of the switching rate. This is significant for the general study of switching processes because it shows that the dynamics of the switching process can be very different from both the individual dynamics (in this case, the A_i's) and the averaged dynamics (in this case, $\bar{A}$). These planar examples are also interesting because they have multiple transitions between $\|X_t\|$ going to 0 and going to $+\infty$ at large time as the switching rate varies. Furthermore, we construct examples in higher dimensions that have arbitrarily many such phase transitions.

Recently researchers have devoted considerable attention to randomly switched systems and we now comment on our work in this broader context. [12], [5], [4], and [1] all study invariant measures for such processes. Our work shows that the existence of such invariant measures may depend in a complicated way on the switching rates. In [16], [17], and [3], the authors provide conditions under which their randomly switched systems behave according to the individual systems for slow switching and according to the averaged system for fast switching. We prove that our system also obeys this principle in Theorems 3 and 4. However, we show in Example 2 that the transition between the slow and fast switching regimes can be quite complicated. Furthermore, Example 3 shows that it can be as complicated as we want.

As background for these surprising results, we first prove sufficient conditions to ensure stability for all switching rates in Section 2.2. Furthermore we also show in Section 2.2 that the individual matrices determine the stability for slow switching and that the average matrix determines the stability for fast switching. In Section

2.3 we use these theorems to construct examples that show "medium" switching can induce blowups even when the individual matrices and the average matrix are all Hurwitz.

We conclude this introduction by defining notation. Let $E = \{0, 1, \ldots, n-1\}$ and let $\{A_i\}_{i \in E}$ be a set of $d \times d$ real matrices. For a given switching rate $r > 0$, let $(I_t)_{t \geqslant 0}$ be an irreducible continuous time Markov process with state space E and generator rQ. Under these assumptions, the Markov process on E with generator rQ has a unique invariant probability measure which we denote by π. Furthermore, π is the unique probability vector satisfying $\pi Q = 0$.

Define $(X_t)_{t \geqslant 0}$ to be the solution of

$$X_t = X_0 + \int_0^t A_{I_s} X_s \, ds, \quad (t \geqslant 0). \tag{2.1}$$

Then $(X_t, I_t)_{t \geqslant 0}$ is a Markov process on $\mathbb{R}^d \times E$. Unless otherwise noted, assume throughout that the distribution of the initial condition (X_0, I_0) is some given probability measure on $\mathbb{R}^d \times E$ satisfying $\mathbb{E}\|X_0\| < \infty$. Define the average matrix

$$\bar{A} = \sum_{i \in E} A_i \pi_i.$$

The following description of our process will be useful. Let $\xi_1, \xi_2, \ldots$ denote the succession of states visited by I_t, $\tau_1, \tau_2, \ldots$ the holding times in each state, $N(t)$ the number of switches before t, and $a_t = t - \sum_{k=1}^{N(t)} \tau_k$ the time since the last switch. Observe that we can write X_t as

$$X_t = \exp(A_{\xi_{N(t)+1}} a_t) \exp(A_{\xi_{N(t)}} \tau_{N(t)}) \ldots \exp(A_{\xi_1} \tau_1) X_0. \tag{2.2}$$

2.2 Basic stability theorems

Theorem 1 (normal case). *If A_i is normal and Hurwitz for each $i \in E$, then $\|X_t\| \to 0$ monotonically as $t \to \infty$ almost surely.*

Proof. Since each A_i is normal and Hurwitz, there exists a $\gamma > 0$ so that for each A_i and for every $t > 0$,

$$\|\exp(A_i t)\| \leqslant e^{-\gamma t} < 1.$$

Therefore

$$\begin{aligned} \|X_t\| &= \|\exp(A_{\xi_{N(t)+1}} a_t) \exp(A_{\xi_{N(t)}} \tau_{N(t)}) \dots \exp(A_{\xi_1} \tau_1) X_0\| \\ &\leqslant \|\exp(A_{\xi_{N(t)+1}} a_t)\| \Big(\prod_{k=1}^{N(t)} \|\exp(A_{\xi_k} \tau_k)\| \Big) \|X_0\| \\ &\leqslant e^{-\gamma t} \|X_0\| \to 0 \quad \text{as } t \to \infty. \end{aligned}$$

To see that the convergence is monotonic, let $0 \leqslant s \leqslant t$ and replace X_0 by X_s in the calculation above. □

Theorem 2 (commuting case)**.** *Assume $\{A_i\}_{i \in E}$ is a commuting family of matrices. If $\bar{A}$ is Hurwitz, then $\|X_t\| \to 0$ as $t \to \infty$ almost surely.*

Proof. Since $\bar{A}$ is Hurwitz, there exist positive β and γ so that for each $t \geqslant 0$

$$\|\exp(\bar{A} t)\| \leqslant \beta e^{-\gamma t}.$$

For each $t > 0$, define

$$C_t = \frac{1}{t} \Big(\sum_{k=1}^{N(t)} A_{\xi_k} \tau_k + A_{\xi_{N(t)+1}} a_t \Big) = \sum_{i \in E} A_i \frac{1}{t} \int_0^t 1_{I_s = i} \, ds.$$

Now since $\{A_i\}_{i \in E}$ is a commuting family of matrices, Equation (2.2) becomes

$$\begin{aligned} \|X_t\| &= \|\exp \Big(\sum_{k=1}^{N(t)} A_{\xi_k} \tau_k + A_{\xi_{N(t)+1}} a_t \Big) X_0\| = \|\exp (C_t t) X_0\| \\ &= \|\exp \big(\bar{A} t\big) \exp \big((C_t - \bar{A}) t\big) X_0\| \leqslant \beta e^{-\gamma t} e^{\|C_t - \bar{A}\| t} \|X_0\|, \end{aligned}$$

Since Q is irreducible, $C_t \to \bar{A}$ almost surely as $t \to \infty$ since $\frac{1}{t} \int_0^t 1_{I_s = i} ds \to \pi_i$ almost surely as $t \to \infty$ (see [28], page 126). Thus, $\|X_t\| \to 0$ almost surely as $t \to \infty$. □

Remark. If $\{A_i\}_{i\in E}$ is a commuting family of matrices and each A_i is Hurwitz, then $\bar{A}$ is Hurwitz. This is an immediate consequence of the fact that eigenvalues "add" - in some order - for commuting matrices.

Theorem 3 (slow switching)**.** *Assume A_i is Hurwitz for each $i \in E$. Then there exists a constant $a > 0$ so that if $r < a$, then $\|X_t\| \to 0$ as $t \to \infty$ almost surely.*

Proof. Since each A_i is Hurwitz, there exist $\beta > 1$ and $\gamma > 0$ so that for each A_i and each $t \geqslant 0$

$$\|\exp(A_i t)\| \leqslant \beta e^{-\gamma t}.$$

Therefore from Equation (2.2), we have that

$$\begin{aligned}\|X_t\| &\leqslant \|\exp(A_{\xi_{N(t)+1}} a_t)\| \Big(\prod_{k=1}^{N(t)} \|\exp(A_{\xi_k}\tau_k)\| \Big) \|X_0\| \\ &\leqslant \beta^{N(t)+1} e^{-\gamma t}\|X_0\| = \exp\Big(\Big(\frac{N(t)+1}{t}\log\beta - \gamma\Big)t\Big)\|X_0\|.\end{aligned} \tag{2.3}$$

Next we claim that we have the following almost sure convergence as $K \to \infty$

$$\frac{1}{K}\sum_{k=1}^{K}\tau_k \to (r\sum_{i\in E}\pi_i q_i)^{-1}, \tag{2.4}$$

where q_i is the ith diagonal entry of Q. To see this, let s^i_j denote the duration of the jth visit of the process I_t to state $i \in E$ and let $V_i(K) := \sum_{k=1}^{K} 1_{\xi_k = i}$ denote the number of visits to i before the Kth jump of the process I_t. Then

$$\frac{1}{K}\sum_{k=1}^{K}\tau_k = \sum_{i\in E}\frac{1}{K}\sum_{j=1}^{V_i(K)} s^i_j = \sum_{i\in E}\frac{V_i(K)}{K}\frac{1}{V_i(K)}\sum_{j=1}^{V_i(K)} s^i_j.$$

For each $i \in E$, $\frac{V_i(K)}{K} \to q_i\pi_i/(\sum_{k\in E} q_k\pi_k)$ almost surely as $K \to \infty$ since Q is irreducible. And by the strong law of large numbers, $\frac{1}{V_i(K)}\sum_{j=1}^{V_i(K)} s^i_j \to \frac{1}{rq_i}$ almost surely as $K \to \infty$. Therefore, Equation (2.4) is verified.

By the definition of $N(t)$ we have that $\sum_{k=1}^{N(t)} \tau_k \leqslant t \leqslant \sum_{k=1}^{N(t)+1} \tau_k$. Therefore

$$\frac{\sum_{k=1}^{N(t)} \tau_k}{N(t)} \leqslant \frac{t}{N(t)} \leqslant \frac{\sum_{k=1}^{N(t)+1} \tau_k}{N(t)+1} \frac{N(t)+1}{N(t)}. \tag{2.5}$$

Since each τ_k is almost surely finite, $N(t) \to \infty$ almost surely as $t \to \infty$. It then follows from combining Equations (2.4) and (2.5) that

$$\frac{N(t)}{t} \to r \sum_{i \in E} \pi_i q_i \quad \text{almost surely as } t \to \infty.$$

So if $r < \gamma(2 \log \beta \sum_{i \in E} \pi_i q_i)^{-1}$, then $\|X_t\| \to 0$ almost surely as $t \to \infty$ by Equation (2.3). □

Theorem 4 (fast switching)**.** *Assume $\bar{A}$ is Hurwitz. Then there exists a constant $b > 0$ so that if $r > b$, then $\|X_t\| \to 0$ as $t \to \infty$ almost surely.*

The proof relies on the following lemma. Let $\mathbb{E}_\nu$ denote the expectation with respect to the measure of the process $(I_t)_{t \geqslant 0}$ with I_0 distributed according to ν. Since we will consider processes with different switching rates, let us momentarily make the dependence on the switching rate explicit by letting $(I_t^{(r)})_{t \geqslant 0}$ be the Markov process on E with generator rQ and define $(X_t^{(r)})_{t \geqslant 0}$ with respect to $(I_t^{(r)})_{t \geqslant 0}$ as before. Define $S_t^{(r)}$ to be the operator that maps X_0 to $X_t^{(r)}$. Observe that $S_t^{(r)}$ is a function of $(I_s^{(r)})_{0 \leqslant s \leqslant t}$.

Lemma 1. *For every probability measure ν on E and for every $t > 0$,*

$$\mathbb{E}_\nu \|S_t^{(r)}\| \to \|\exp(\bar{A}t)\| \quad \textit{as } r \to \infty.$$

Proof. Define $\{\xi_i^1\}_{i=1}^\infty$, $\{\tau_i^1\}_{i=1}^\infty$, $\{N^1(t)\}_{t \geqslant 0}$, and $\{a_t^1\}_{t \geqslant 0}$ as before but now with respect to $\{I_t^{(1)}\}_{t \geqslant 0}$. Let the distribution of I_0 be a given probability measure ν on E and for $\lambda > 0$ define

$$\tilde{S}_t^\lambda = \exp\left(A_{\xi_1^1}^T \frac{\tau_1^1}{\lambda}\right) \exp\left(A_{\xi_2^1}^T \frac{\tau_2^1}{\lambda}\right) \ldots \exp\left(A_{\xi_{N^1(\lambda t)}}^T \frac{\tau_{N^1(\lambda t)}}{\lambda}\right) \exp\left(A_{\xi_{N^1(\lambda t)+1}}^T \frac{a_{\lambda t}^1}{\lambda}\right)$$

where we denote the transpose of a matrix B by B^T. Observe that if $r = \lambda$, then $\tilde{S}_t^\lambda$ has been defined so that $(\tilde{S}_t^\lambda)^T$ and $S_t^{(r)}$ are equal in distribution.

By [19], $\tilde{S}_t^\lambda \to \exp(\bar{A}^T t)$ almost surely in the strong operator topology as $\lambda \to \infty$. Since $\mathbb{R}^d$ is finite-dimensional, we actually have that the convergence holds in the uniform operator topology. Since $\|B\| = \|B^T\|$ for every matrix B, it follows that

$$\|\tilde{S}_t^\lambda\| \to \|\exp(\bar{A}t)\| \text{ almost surely as } \lambda \to \infty.$$

Since $\|\tilde{S}_t^\lambda\| \leqslant \exp(\max_i \|A_i\| t)$ for every $\lambda > 0$, the bounded convergence theorem gives

$$\mathbb{E}\|\tilde{S}_t^\lambda\| \to \|\exp(\bar{A}t)\| \text{ as } \lambda \to \infty.$$

Since $\|\tilde{S}_t^\lambda\|$ and $\|S_t^r\|$ are equal in distribution, the proof is complete. □

of Theorem 4. Since $\bar{A}$ is Hurwitz, there exist positive numbers β and γ so that for every $t \geqslant 0$

$$\|\exp(\bar{A}t)\| \leqslant \beta e^{-\gamma t}.$$

Thus we can choose $T > 0$ so that $\|\exp(\bar{A}T)\| < \frac{1}{4}$. By Lemma 1 there exists a $b > 0$ so that if $r > b$, then $\mathbb{E}_i\|S_T^{(r)}\| < \frac{1}{2}$ for each $i \in E$, where $\mathbb{E}_i$ denotes the expectation with respect to the measure of the process $(I_t)_{t \geqslant 0}$ with initial measure $\mathbb{P}(I_0 = i) = 1$.

Let $r > b$ and define the process $\{M_n\}_{n=0}^\infty$ and the filtration $\{\mathcal{F}_n\}_{n=0}^\infty$ by

$$M_n = \|X_{nT}\| \qquad \text{and} \qquad \mathcal{F}_n = \sigma((X_t, I_t) : 0 \leqslant t \leqslant nT).$$

We claim that M_n is a supermartingale with respect to $\mathcal{F}_n$. It's immediate that $M_n \in \mathcal{F}_n$ and $\mathbb{E}|M_n| \leqslant e^{\Lambda n T} < \infty$ for $\Lambda := \max_{i \in E} \|A_i\|$. For $0 \leqslant s \leqslant t$, define $S(s,t)$ to be the operator that maps X_s to X_t. We now check the supermartingale property.

$$\begin{aligned}
\mathbb{E}[M_{n+1}|\mathcal{F}_n] &\leqslant \mathbb{E}[\|S(nT,(n+1)T)\| \, \|X_{nT}\| |\mathcal{F}_n] \\
&= M_n \mathbb{E}[\|S(nT,(n+1)T)\| |\mathcal{F}_n] \\
&= M_n \mathbb{E}_{I_{nT}} \|S(nT,(n+1)T)\| \\
&\leqslant \frac{1}{2} M_n.
\end{aligned}$$

Taking the expectation of the above inequality and iterating yields $\mathbb{E}M_n \leqslant \frac{1}{2^n}\mathbb{E}M_0$. Therefore M_n converges in L^1 to 0 since $M_n \geqslant 0$. Also since $M_n \geqslant 0$, the martingale convergence theorem implies that M_n must converge almost surely. Therefore M_n converges almost surely to 0.

To conclude that $\|X_t\| \to 0$ almost surely, we need to control $\|X_t\|$ at times between multiples of T. This is easily obtained since $\|X_t\|$ cannot grow faster than $e^{\Lambda t}$. Let $\omega \in \Omega$ be such that $M_n(\omega) \to 0$ and let $\epsilon > 0$. There exists $N = N(\omega, \epsilon)$ so that for all $n \geqslant N$,

$$\|M_n(\omega)\| < e^{-\Lambda T}\epsilon.$$

Thus for all $t \geqslant NT$,

$$\|X_t(\omega)\| \leqslant \|(S(t - T\lfloor t/T \rfloor, t)X_{T\lfloor t/T \rfloor})(\omega)\| \leqslant e^{\Lambda T} M_{\lfloor t/T \rfloor}(\omega) < \epsilon.$$

Since this set of ω's has probability one, the proof is complete. □

Example 1. Assume $E = \{0, 1\}$ and $Q = \left(\begin{smallmatrix} -1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$. Define

$$A_0 = \begin{pmatrix} 1 & 4 \\ 0 & -2 \end{pmatrix} \qquad A_1 = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then A_0 and A_1 each have a positive eigenvalue, but $\bar{A} = \frac{1}{2}(A_0 + A_1)$ is Hurwitz. So despite the fact that each individual matrix is unstable, Theorem 4 guarantees that $\|X_t\| \to 0$ almost surely as $t \to \infty$ for sufficiently fast switching rate.

2.3 Medium switching can be complicated

We will now construct a switching example with two matrices, A_0 and A_1, that is surprising for the following two reasons. First, the individual matrices A_0 and A_1 and the average $\bar{A} = \frac{1}{2}(A_0 + A_1)$ are all Hurwitz, but $\|X_t\|$ will still blow up at large time for certain values of the switching rate. In [6], the authors show that $\|X_t\|$ can blow up if the two individual matrices A_0 and A_1 are Hurwitz as long as the average matrix has a positive eigenvalue. Thus our result shows that this assumption on the average matrix is not necessary.

Second, the asymptotic behavior of the following example has multiple "phase transitions" as the switching rate varies. That is, the process goes to zero at large time for both slow and fast switching, but blows up for medium switching.

We also remark that we can choose the negative real part of all the eigenvalues of A_0, A_1, and $\bar{A}$ to have arbitrarily large absolute value.

Example 2. Assume $\mathbb{P}(X_0 = 0) = 0$ and let $E = \{0, 1\}$ and $Q = \left(\begin{smallmatrix} -1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$. We will show the existence of matrices $A_0, A_1 \in \mathbb{R}^{2\times 2}$ and positive numbers $a < b$, so that

1. A_0, A_1 are each Hurwitz.
2. $\bar{A} = \frac{1}{2}(A_0 + A_1)$ is Hurwitz.
3. If $r \notin (a, b)$, then $\|X_t\| \to 0$ almost surely as $t \to \infty$.
4. $\|X_t\| \to \infty$ almost surely as $t \to \infty$ for some value of $r \in (a, b)$.

For positive α and c, we define

$$A_0 = \begin{pmatrix} -\alpha & c \\ 0 & -\alpha \end{pmatrix} \qquad A_1 = \begin{pmatrix} -\alpha & 0 \\ -c & -\alpha \end{pmatrix}.$$

Observe that A_0 and A_1 each have $-\alpha < 0$ as their only eigenvalue. The two eigenvalues of $\bar{A} = \frac{1}{2}(A_0 + A_1)$ are $-\alpha \pm ic/2$. Thus A_0, A_1, and $\bar{A}$ are each Hurwitz. By Theorems 3 and 4, $\|X_t\| \to 0$ as $t \to \infty$ almost surely for sufficiently large r and for sufficiently small r. We will show that $\|X_t\| \to +\infty$ as $t \to \infty$ almost surely for some intermediate values of r.

We use polar coordinates to study the large time behavior of $\|X_t\|$. Our technique follows [6] in this setting and the well known utility of the polar representation when studying Lyapunov exponents (especially in two-dimensions) which dates back to at least [18]. Define the radial process $R_t := \|X_t\|$ and define the angular process U_t as the point on the unit circle S^1 given by X_t/R_t. A short calculation shows that between jumps R_t and U_t satisfy

$$\dot{R}_t = R_t\langle A_{I_t}U_t, U_t\rangle \tag{2.6}$$

$$\dot{U}_t = A_{I_t}U_t - \langle A_{I_t}U_t, U_t\rangle U_t. \tag{2.7}$$

The advantage of this decomposition is that the evolution of the angular process doesn't depend on the radial process. Therefore (U_t, I_t) is a Markov process on $S^1 \times \{0, 1\}$.

Lemma 2. *If we identify $\theta \in \mathbb{R}$ with $(\cos\theta, \sin\theta) \in S^1$, then the unique invariant measure of the angular process U_t is given by*

$$\mu(d\theta, i) = p_i(\theta; r/c)1_{[0,2\pi]}(\theta)\,d\theta$$

where for any parameter $\lambda > 0$, the functions p_0 and p_1 satisfy

$$p_i(\theta;\lambda) = p_{1-i}(\theta + \pi/2;\lambda) = p_i(\theta + \pi;\lambda) \quad \textit{for } \theta \in \mathbb{R}, \tag{2.8}$$

$$\textit{and} \quad p_0(\theta;\lambda) < p_1(\theta;\lambda) \quad \textit{for } \theta \in (-\frac{\pi}{2}, 0). \tag{2.9}$$

Proof. Define the process $\Theta_t \in \mathbb{R}$ to be the lift of $U_t \in S^1$ from the circle to its covering space $\mathbb{R}$. That is to say Θ_t is the unique process so that $U_t = (\cos\Theta_t, \sin\Theta_t)$, Θ_t is continuous in t, and $\Theta_0 \in [0, 2\pi)$. It follows from Equation (2.7) and plugging in our values for A_0 and A_1 that between jumps Θ_t satisfies

$$\dot{\Theta}_t = -c[I_t\cos^2(\Theta_t) + (1 - I_t)\sin^2(\Theta_t)] \leqslant 0.$$

Since $\min_{i\in\{0,1\}} -c[i\cos^2(\theta) + (1-i)\sin^2(\theta)] \leqslant -c/2 < 0$ for all $\theta \in \mathbb{R}$, it follows that $\Theta_t \to -\infty$ as $t \to \infty$ almost surely. Since Θ_t is continuous, we conclude that the Markov process (U_t, I_t) is recurrent and irreducible and must have a unique invariant measure.

If we identify $\theta \in \mathbb{R}$ with $(\cos\theta, \sin\theta) \in S^1$, then the adjoint of generator of the Markov process (U_t, I_t) is

$$(\mathcal{L}^* q)(\theta, i) = \partial_\theta \left(c\left[(1-i)\sin^2(\theta) + i\cos^2(\theta)\right] q(\theta, i)\right) + r(q(\theta, 1-i) - q(\theta, i)).$$

For $\theta \in (-\frac{\pi}{2}, 0)$ and $\lambda > 0$, define

$$\begin{aligned}
H(\theta;\lambda) &= \exp\left(-2\lambda\cot(2\theta)\right)\int_\theta^0 \exp\left(2\lambda\cot(2y)\right)\sec^2(y)\,dy \\
p_0(\theta;\lambda) &= C\csc^2(\theta)\lambda H(\theta) \\
p_1(\theta;\lambda) &= C\sec^2(\theta)\left[1-\lambda H(\theta)\right].
\end{aligned}$$

where

$$C(\lambda) = \left[4\int_{-\frac{\pi}{2}}^0 \sec^2(x) + (\csc^2(x) - \sec^2(x))\lambda H(x)\,dx\right]^{-1}.$$

Define $H(0;\lambda) = 0 = p_0(0;\lambda)$ and $p_1(0;\lambda) = C(\lambda)$. Extend p_1 and p_0 to be defined on the rest of the real line by Equation (2.8). It is easy to check that these three functions are well-defined.

Writing $p_i(\theta;\lambda)$ as $p(\theta, i;\lambda)$, it is easy to check that $\mathcal{L}^* p(\theta, i;\lambda) = 0$ for all $\theta \in \mathbb{R}$ and for $i = \{0, 1\}$. Thus, the measure μ defined in the statement of the lemma is the unique invariant measure for (U_t, I_t).

We now check that p_0 and p_1 satisfy Equation (2.9). Let $\lambda > 0$ and observe that for $\theta \in (-\frac{\pi}{2}, 0)$, writing $1 = \sin^2(y)\csc^2(y)$ in the integrand gives

$$\begin{aligned}
H(\theta;\lambda) &= \exp\left(-2\lambda\cot(2\theta)\right)\int_\theta^0 \exp\left(2\lambda\cot(2y)\right)\sec^2(y)\sin^2(y)\csc^2(y)\,dy \\
&< \exp\left(-2\lambda\cot(2\theta)\right)\sin^2(\theta)\int_\theta^0 \exp\left(2\lambda\cot(2y)\right)\sec^2(y)\csc^2(y)\,dy \qquad (2.10) \\
&= \frac{1}{\lambda}\sin^2(\theta),
\end{aligned}$$

since $\sin^2(\theta)$ is strictly decreasing on $(-\frac{\pi}{2}, 0)$ and

$$\frac{d}{dy}\left[\exp\left(2\lambda\cot(2y)\right)\right] = -\lambda\exp\left(2\lambda\cot(2y)\right)\sec^2(y)\csc^2(y).$$

Observe also that for $\theta \in (-\frac{\pi}{2}, 0)$

$$H'(\theta;\lambda) = \lambda H(\theta;\lambda)(\sec^2(\theta) + \csc^2(\theta)) - \sec^2(\theta) = \frac{1}{C}(p_0(\theta;\lambda) - p_1(\theta;\lambda)). \quad (2.11)$$

Combining Equations (2.10) and (2.11), we have that for $\theta \in (-\frac{\pi}{2}, 0)$

$$\frac{1}{C}(p_0(\theta;\lambda) - p_1(\theta;\lambda)) < 0.$$

Thus Equation (2.9) holds. $\square$

Lemma 3. *For $\lambda > 0$, define*

$$G(\lambda) := \int_0^{2\pi} (p_0(\theta;\lambda) - p_1(\theta;\lambda)) \cos(\theta)\sin(\theta)\, d\theta.$$

Then $G(\lambda) > 0$ and

- *If $G\left(\frac{r}{c}\right) > \frac{\alpha}{c}$, then $\|X_t\| \to \infty$ as $t \to \infty$ almost surely.*
- *If $G\left(\frac{r}{c}\right) < \frac{\alpha}{c}$, then $\|X_t\| \to 0$ as $t \to \infty$ almost surely.*

Proof. By Equations (2.8) and (2.9) in the statement of Lemma 2, we have that $(p_0(\theta;\lambda) - p_1(\theta;\lambda))\cos(\theta)\sin(\theta) > 0$ for all θ and thus $G(\lambda) > 0$.

Now by Equation (2.6), we have that

$$\frac{1}{t}\log\left(\frac{R_t}{R_0}\right) = \frac{1}{t}\int_0^t \langle A_{I_s}U_s, U_s\rangle\, ds.$$

Identify $\theta \in \mathbb{R}$ with $e_\theta := (\cos\theta, \sin\theta) \in S^1$. It follows from Lemma 2 and Birkhoff's ergodic theorem that there exists a set $A \in S^1$ with $\mu(A) = 1$ so that if $U_0 \in A$, then

$$\frac{1}{t}\log\left(\frac{R_t}{R_0}\right) \to \int \langle A_i e_\theta, e_\theta\rangle\, \mu(d\theta, i) \quad \text{almost surely as } t \to \infty. \quad (2.12)$$

Define $T_A := \inf\{t \geqslant 0 : U_t \in A\}$ and observe that for any $U_0 \in S^1$, we have that $T_A < \infty$ almost surely since U_t is recurrent. Since T_A is a stopping time, we have that the convergence in Equation (2.12) actually holds for every $U_0 \in S^1$.

Plugging in our choice of A_0 and A_1 and the definition of μ yields

$$\begin{aligned}\int \langle A_i e_\theta, e_\theta\rangle \,\mu(d\theta, i) &= \int_0^{2\pi} \langle A_0 e_\theta, e_\theta\rangle p_0(\theta; r/c)\, d\theta + \int_0^{2\pi} \langle A_1 e_\theta, e_\theta\rangle p_1(\theta; r/c)\, d\theta \\ &= c\int_0^{2\pi} (p_0(\theta; r/c) - p_1(\theta; r/c)) \cos(\theta)\sin(\theta)\, d\theta - \alpha \\ &= cG\left(\frac{r}{c}\right) - \alpha.\end{aligned}$$

Hence if $G\left(\frac{r}{c}\right) > \frac{\alpha}{c}$, then $\lim_{t\to\infty} \frac{1}{t}\log\left(\frac{R_t}{R_0}\right) > 0$ almost surely and thus $\|X_t\| \to \infty$ as $t \to \infty$ almost surely. Similarly if $G\left(\frac{r}{c}\right) < \frac{\alpha}{c}$, then $\|X_t\| \to 0$ as $t \to \infty$ almost surely. □

Since $G\left(\frac{r}{c}\right) > 0$ for every pair of positive numbers r and c, it is immediate that we can choose r, c, and α so that $\|X_t\| \to \infty$ as $t \to \infty$ almost surely.

Remark. Relating this example to the deterministic problem studied in [2], the pair A_0, A_1 defined above fall in to case **S4** with $\mathcal{R} > 1$ of Theorem 1 in [2].

2.3.1 Many transitions between stable and unstable

The following example shows that there exist two matrices such that as the switching rate varies from zero to infinity, the asymptotic behavior of the system will switch between converging to zero and converging to infinity at least any prespecified number of times.

Example 3. Assume $\mathbb{P}(X_0 = 0) = 0$ and let $E = \{0, 1\}$ and $Q = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$. We will show that for any positive integer k, there exist matrices $A_0, A_1 \in \mathbb{R}^{2k\times 2k}$ and positive numbers $a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k$ so that

1. A_0, A_1 are each Hurwitz.

2. $\bar{A} = \frac{1}{2}(A_0 + A_1)$ is Hurwitz.

3. If $r \notin \bigcup_{i=1}^{k}(a_i, b_i)$, then $\|X_t\| \to 0$ almost surely as $t \to \infty$.

4. For every $i \in \{1, \dots, k\}$, $\|X_t\| \to \infty$ almost surely as $t \to \infty$ for some value of $r \in (a_i, b_i)$.

Let k be a given positive integer and define the two block diagonal matrices $A_0, A_1 \in \mathbb{R}^{2k \times 2k}$ by

$$A_0 = \begin{pmatrix} A_0^1 & 0 & \cdots & 0 \\ 0 & A_0^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0^k \end{pmatrix} \qquad A_1 = \begin{pmatrix} A_1^1 & 0 & \cdots & 0 \\ 0 & A_1^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_1^k \end{pmatrix} \tag{2.13}$$

where

$$A_0^i = \begin{pmatrix} -\alpha_i & c_i \\ 0 & -\alpha_i \end{pmatrix} \qquad A_1^i = \begin{pmatrix} -\alpha_i & 0 \\ -c_i & -\alpha_i \end{pmatrix} \tag{2.14}$$

for some positive numbers $\{c_i\}_{i=1}^k$ and $\{\alpha_i\}_{i=1}^k$. It's immediate that A_0, A_1, and $\bar{A}$ are all Hurwitz.

Let X_t denote the $\mathbb{R}^{2k}$-valued process corresponding to (2.13) and $X_t^{(i)}$ the $\mathbb{R}^2$-valued process corresponding to (2.14) for each $i \in \{1, \dots, k\}$. Since the ODEs for $X^{(i)}$ and $X^{(j)}$ are not coupled for $i \neq j$, we have that $X_t = (X_t^{1)}, \dots, X_t^{(k)})$ when viewed as an $(\mathbb{R}^2)^k$-valued process. In particular, one has

$$\|X_t\|^2 = \sum_{i=1}^{k} \|X_t^{(i)}\|^2.$$

Thus $\|X_t\| \to 0$ if and only if $\|X_t^{(i)}\| \to 0$ for every $i \in \{1, \dots, k\}$. Furthermore if $\|X_t^{(i)}\| \to \infty$ for some $i \in \{1, \dots, k\}$, then $\|X_t\| \to \infty$.

The proof proceeds by choosing the parameters α_i and c_i as in Example 2 so that $X^{(i)}$ is unstable for switching rates r in an interval (a_i, b_i) but stable out side of the interval. By arranging so that the collection of intervals $\{(a_j, b_j) : j = 1, \dots, k\}$ are disjoint we will succeed at constructing the desired matrices A_0 and A_1.

More explicitly, it follows from Lemma 3 and Theorems 3 and 4 that we can choose r_1, c_1, α_1, and $a_1 < b_1$ so that

$$G\left(\frac{r_1}{c_1}\right) > \frac{\alpha_1}{c_1} \quad \text{and} \quad G\left(\frac{r}{c_1}\right) < \frac{\alpha_1}{c_1} \text{ if } r \notin (a_1, b_1).$$

Choose $N > \frac{b_1}{a_1}$ and for $i \in \{2, \dots, k\}$ define

$$a_i = \frac{a_1}{N^{i-1}}, \quad b_i = \frac{b_1}{N^{i-1}}, \quad \alpha_i = \frac{\alpha_1}{N^{i-1}}, \quad c_i = \frac{c_1}{N^{i-1}}, \quad r_i = \frac{r_1}{N^{i-1}}.$$

To see that our intervals (a_i, b_i) don't overlap, observe that $a_i < b_i$ for each i and

$$b_i = \frac{b_1}{N^{i-1}} < \frac{Na_1}{N^{i-1}} = a_{i-1}.$$

Next observe that if $r \notin (a_i, b_i)$, then $rN^{i-1} \notin (a_1, b_1)$ and therefore

$$G\left(\frac{r}{c_i}\right) = G\left(\frac{rN^{i-1}}{c_1}\right) < \frac{\alpha_1}{c_1}.$$

Thus, $\|X_t^{(i)}\| \to 0$ almost surely as $t \to \infty$ if $r \notin (a_i, b_i)$.

Finally observe that $r_i \in (a_i, b_i)$ and

$$G\left(\frac{r_i}{c_i}\right) = G\left(\frac{r_1}{c_1}\right) > \frac{\alpha_1}{c_1} = \frac{\alpha_i}{c_i}.$$

Thus, $\|X_t^{(i)}\| \to \infty$ almost surely as $t \to \infty$ if the switching rate is $r_i \in (a_i, b_i)$.

2.4 Conclusions

Stochastically switched linear ODEs are one of the simplest examples of stochastically switched systems. However despite their simplicity, we have shown that their behavior can be quite rich. First, the large time behavior can depend on the switching rate in a very delicate way. Second, this large time behavior can be very different from the large time behavior of both the individual systems and the average system.

3

Abstract setting

We now consider stochastic switching in evolution equations in a general Hilbert space. Under certain assumptions, we prove that the process converges in distribution at large time and we show that the limiting distribution satisfies certain properties. Although applicable to a wide range or stochastic hybrid systems, the contents of this chapter will prove particularly useful when we consider PDEs with randomly switching boundary conditions in Chapter 4. The contents of this chapter overlap significantly with [21].

In Sections 3.1 and 3.2, we construct the process and prove that elements of the limiting distribution satisfy certain invariance properties. In Section 3.3, we prove that the process converges to a steady state distribution. Section 3.4 is devoted to proving the lemmas that were needed to prove this convergence.

3.1 Discrete-time process

We first define the set Ω of all possible switching environments and equip it with a probability measure. Let μ_0 and μ_1 be two probability distributions on the positive real line. Define each switching environment, $\omega \in \Omega$, as the one-sided sequence

$\omega = (\omega_1, \omega_2, \dots)$, where each ω_k is a pair of non-negative real numbers, (τ_0^k, τ_1^k), drawn from $\mu_0 \times \mu_1$. We endow Ω with the infinite product measure generated by $\mu_0 \times \mu_1$. We will use P to denote this measure. To summarize some notation

$$\omega = (\omega_1, \omega_2, \omega_3, \dots) = \Big((\tau_0^1, \tau_1^1), (\tau_0^2, \tau_1^2), (\tau_0^3, \tau_1^3), \dots\Big) \in \Omega.$$

For each $t \geqslant 0$, let $\Phi_t^0(x)$ and $\Phi_t^1(x)$ be two mappings from a separable real Hilbert space H to itself. Assume that for both $i = 0$ and 1 and for each $x \in H$, $E|\Phi_{\tau_i}^i(x)| < \infty$, where τ_i is drawn from μ_i. Assume that $\Phi_t^0(x) = x = \Phi_t^1(x)$ if $t = 0$. Assume that for each $t > 0$ there exists a $K_i(t)$ so that for every $x, y \in H$

$$|\Phi_t^i(x) - \Phi_t^i(y)| \leqslant K_i(t)|x - y| \tag{3.1}$$

for both $i = 0$ and 1. Assume $EK_0(\tau_1)EK_1(\tau_1) < 1$ where τ_0 and τ_1 are drawn from μ_0 and μ_1. Furthermore assume that for each fixed $x \in H$, the mapping

$$t \mapsto \Phi_t^i(x) \in H$$

is continuous for both $i = 0$ and 1.

For each $\omega \in \Omega$, $x \in H$, and natural number k, define the compositions $G_k^\omega : H \to H$ and $F_k^\omega : H \to H$ by

$$G_\omega^k(x) := \Phi^1_{\tau_1^k(\omega)} \circ \Phi^0_{\tau_0^k(\omega)}(x)$$

$$F_\omega^k(x) := \Phi^0_{\tau_0^k(\omega)} \circ \Phi^1_{\tau_1^k(\omega)}(x).$$

For each $\omega \in \Omega$, $x \in H$, and natural number $n > 0$, we define the forward maps φ^n and γ^n and the backward maps φ^{-n} and γ^{-n} by the following compositions of G and F:

$$\begin{aligned} \varphi_\omega^n(x) &= G_\omega^n \circ G_\omega^{n-1} \circ \cdots \circ G_\omega^2 \circ G_\omega^1(x) \\ \gamma_\omega^n(x) &= F_\omega^n \circ F_\omega^{n-1} \circ \cdots \circ F_\omega^2 \circ F_\omega^1(x) \\ \varphi_\omega^{-n}(x) &= G_\omega^1 \circ G_\omega^2 \circ \cdots \circ G_\omega^{n-1} \circ G_\omega^n(x) \\ \gamma_\omega^{-n}(x) &= F_\omega^1 \circ F_\omega^2 \circ \cdots \circ F_\omega^{n-1} \circ F_\omega^n(x). \end{aligned} \tag{3.2}$$

To make our notation complete, we define $\varphi^0(x) = x = \gamma^0(x)$.

These are iterated random functions, (see [13] for a review). Assuming Equation (3.1) and $EK_0(\tau_1)EK_1(\tau_1) < 1$ ensures that G and F are contracting on average. Thus, $\{\varphi^n\}_{n\geqslant 0}$ and $\{\gamma^n\}_{n\geqslant 0}$ are Markov chains with invariant probability distributions given by the distributions of the almost sure limits of φ^{-n} and γ^{-n} as $n \to \infty$, respectively. Moreover, the distributions of the Markov chains φ^n and γ^n converge at a geometric rate to these invariant distributions. These results are immediately attained by applying theorems in [13]. Nonetheless, we prove the following theorem to make our results self-contained.

Theorem 5. *Define*

$$Y_1(\omega) := \lim_{n\to\infty} \varphi_\omega^{-n}(x) \tag{3.3}$$

$$Y_0(\omega) := \lim_{n\to\infty} \gamma_\omega^{-n}(x). \tag{3.4}$$

These limits exist almost surely and are independent of x.

Remark. Random variables such as these are often called "pullbacks" because they take an initial condition x and pull it back to the infinite past.

Proof. We will show the $\varphi^{-n}(x)$ is almost surely Cauchy. Let $x_1, x_2 \in H$ and $n \geqslant m$. Using the triangle inequality, we obtain

$$\begin{aligned}
|\varphi^{-n}(x_1) - \varphi^{-m}(x_2)| &= |G^1 \circ \cdots \circ G^n(x_1) - G^1 \circ \cdots \circ G^m(x_2)| \\
&\leqslant \sum_{i=M+1}^{N} |G^1 \circ \cdots \circ G^i(x_1) - G^1 \circ \cdots \circ G^{i-1}(x_2)| \\
&\leqslant \sum_{i=M+1}^{N} |G^i(x_1) - x_2| \left(\prod_{j=1}^{i-1} K(\tau_0^j) K(\tau_1^j) \right)
\end{aligned}$$

Observe that

$$E\sum_{i=M+1}^{N}|G^i(x_1)-x_2|\left(\prod_{j=1}^{i-1}K(\tau_0^j)K(\tau_1^j)\right)=E|G^1(x_1)-x_2|\sum_{i=M+1}^{N}\left(EK(\tau_0^1)EK(\tau_1^1)\right)^{i-1}$$

$$\leqslant\frac{E|G^1(x_1)-x_2|}{1-EK(\tau_0^1)EK(\tau_1^1)}<\infty.$$

Thus, $\sum_{i=1}^{\infty}|G^1\circ\cdots\circ G^i(x_1)-G^1\circ\cdots\circ G^{i-1}(x_2)|$ converges almost surely. Therefore $\varphi^{-n}(x_1)$ is almost surely Cauchy and thus Y_1 exists almost surely. Note that Y_1 is an H-valued random variable as a limit of H-valued random variables. Since x_1 and x_2 were arbitrary, Y_1 is independent of the x used in its definition. The proof for Y_0 is the same. □

The random variables Y_1 and Y_0 satisfy the following invariance properties.

Theorem 6. *Let τ_0 and τ_1 be independent draws from μ_0 and μ_1, respectively. Then*

$$Y_0=_d\Phi^0_{\tau_0}(Y_1) \tag{3.5}$$

$$\text{and}\quad Y_1=_d\Phi^1_{\tau_1}(Y_0) \tag{3.6}$$

where $=_d$ denotes equal in distribution.

Proof. We will show that $\Phi^0_{\tau_0}(Y_1)=_d Y_0$. Let $y\in H$ and observe that for any $n\in\mathbb{N}$, we have that

$$\gamma_\omega^{-n}(y)=_d\Phi^0_{\tau_0}\left(\varphi_\omega^{n-1}(\Phi^1_{\tau_1}(y))\right).$$

Taking the limit as $n\to\infty$ yields

$$\lim_{n\to\infty}\gamma^{-n}(y)=_d\lim_{n\to\infty}\Phi^0_{\tau_0}\left(\varphi_\omega^{n-1}(\Phi^1_{\tau_1}(y))\right)=\Phi^0_{\tau_0}\left(\lim_{n\to\infty}\varphi_\omega^{n-1}(\Phi^1_{\tau_1}(y))\right) \tag{3.7}$$

since $\Phi^0_t(x)$ is continuous in x for each t. Recalling that the definitions of Y_0 and Y_1 in Equations (3.3) and (3.4) are independent of x by Theorem 5, we have that Equation (3.7) becomes $Y_0=_d\Phi^0_{\tau_0}(Y_1)$. The proof that $Y_1=_d\Phi^1_{\tau_1}(Y_0)$ is similar. □

Theorem 7. *Suppose there exists a set $S \subset H$ so that for all $t \geqslant 0$, $\Phi_t^i : S \to S$ for $i = 0$ and 1. Then Y_0 and Y_1 are in the closure $\bar{S}$ almost surely.*

Proof. Let $\omega \in \Omega$ be given. If $x \in S$, then $\varphi_\omega^{-n}(x) \in S$ for all $n \geqslant 0$ by assumption. Thus, $\lim_{n\to\infty} \varphi_\omega^{-n}(x) = Y_1 \in \bar{S}$. But by Theorem 5, Y_1 is independent of the initial x used in its definition, so $Y_1 \in \bar{S}$ almost surely. The proof for Y_0 is the same. □

3.2 Continuous-time process

To define the continuous time process, we need more notation. The following notation is standard in renewal theory. For each $\omega \in \Omega$ and natural number n, define

$$S_n := \sum_{k=1}^{n} \tau_0^k + \tau_1^k$$

with $S_0 := 0$. Define $S'_{n+1} := S_n + \tau_0^{n+1}$ for $n \geqslant 0$. Observe that $S'_n < S_n < S'_{n+1} < S_{n+1}$. Define

$$N_t := \max\{n \geqslant 0 : S_n \leqslant t\}.$$

We also define the state process Z_t for $t \geqslant 0$ by

$$Z_t := \begin{cases} 0 & S_{N_t} \leqslant t < S'_{N_t+1} \\ 1 & S'_{N_t+1} \leqslant t. \end{cases} \tag{3.8}$$

Finally, define the elapsed time since the last switch, called the age process, for $t \geqslant 0$ by

$$a_t := Z_t(t - S'_{N_t+1}) + (1 - Z_t)(t - S_{N_t}).$$

We are now ready to define our continuous-time H-valued process. For $u_0 \in H$, $\omega \in \Omega$, and $t \geqslant 0$, define

$$u(t, \omega) = Z_t \Phi_{a_t}^1 \circ \Phi_{\tau_0^{N_t+1}}^0(\varphi^{N_t}(u_0)) + (1 - Z_t)\Phi_{a_t}^0(\varphi^{N_t}(u_0)). \tag{3.9}$$

3.3 Convergence in distribution to mixture of pullbacks

In this section we will find the limiting distribution of $u(t)$ as $t \to \infty$. In order to describe this limiting distribution, we will need to define three more random variables. Define a^0 and a^1 to be two random variables with the following cumulative distribution functions:

$$P(a^0 \leqslant x) = \frac{1}{E\tau_0}\int_0^x \mu_0(y,\infty)\,dy = \frac{E\min(\tau_0,x)}{E\tau_0}$$

$$P(a^1 \leqslant x) = \frac{1}{E\tau_1}\int_0^x \mu_1(y,\infty)\,dy = \frac{E\min(\tau_1,x)}{E\tau_1}.$$

We will see in Lemma 4 that the distributions of a^0 and a^1 can be thought of as the limiting distributions of the age process conditioned on either $Z_t = 0$ or 1. Let ξ be a Bernoulli random variable with parameter $p := \dfrac{E\tau_1}{E\tau_0 + E\tau_1}$, the probability that $Z_t = 1$ at large time. Assume a^0, a^1, and ξ are all chosen to be mutually independent and independent of (τ_0^k, τ_1^k) for every k.

Recall that a measure μ on the real line is said to be arithmetic if there exist a $d > 0$ so that $\mu(\{0, d, 2d, \dots\}) = 1$.

Theorem 8. *If the μ_0 and μ_1 are non-arithmetic, then as $t \to \infty$ we have the following convergence in distribution as $t \to \infty$:*

$$u(t) \to_d \bar{u} := \xi\Phi^1_{a^1}(Y_0) + (1-\xi)\Phi^0_{a^0}(Y_1),$$

where τ_0 is drawn from μ_0 independently from the other random variables.

If the sojourn times, τ_0 and τ_1, are assumed to be exponentially distributed, then we have the following immediate corollary of Theorem 6 and Theorem 8 since the age of a Poisson process is exponentially distributed.

Corollary 1. *If μ_0 and μ_1 are exponential with respective rate parameters r_0 and r_1, then*

$$u(t) \to_d \bar{u} := \xi Y_1 + (1-\xi) Y_0$$

and $p = r_0/(r_0 + r_1)$ *and* τ_0 *is an independent draw from* μ_0*.*

Proof of Theorem 8. In light of Theorem 6, it is sufficient to prove that $u(t)$ converges in distribution to $\Phi^1_{a^1} \circ \Phi^0_{\tau_0}(Y_1) + (1 - \xi)\Phi^0_{a^0}(Y_1)$. We will show that for any A, B, and C Borel measurable subsets of $\mathbb{R}$ and D a Borel subset of H, we have that

$$\begin{aligned} &P(a_t \in A, Z_t\tau_0^{N_t+1} \in B, Z_t \in C, \varphi^{N_t}(u_0) \in D) \\ &\to P(\xi a^1 + (1-\xi)a^0 \in A, \xi\tau_0 \in B, \xi \in C, Y_1 \in D) \quad \text{as } t \to \infty. \end{aligned} \tag{3.10}$$

Once this convergence is shown, the conclusion of our theorem quickly follows. To see this, assume the convergence holds. Define the $(\mathbb{R}^3 \times H)$-valued random variable $X_t := (a_t, Z_t\tau_0^{N_t+1}, Z_t, \varphi^{N_t}(u_0))$. Note that $\mathbb{R}^3 \times H$ is separable since $\mathbb{R}$ and H are each separable. Thus, we can apply Theorem 2.8(i) in [7] to obtain the following convergence in distribution:

$$X_t \to_d (\xi a^1 + (1-\xi)a^0, \xi\tau_0, \xi, Y_1) \quad \text{as } t \to \infty.$$

Define the function $g : \mathbb{R}^3 \times H \to H$ by $g(a, t, z, y) = z\Phi^1_a \circ \Phi^0_t(y) + (1 - z)\Phi^0_t(y)$ and observe that $u(t) = g(a_t, \tau_0^{N_{t+1}}, X_t, \varphi^{N_t})$ and that the limiting random variable in the statement of our theorem is $g(a^1, \tau_0, \xi, Y)$. Since g is continuous, the conclusion of our theorem follows from the continuous mapping theorem. Therefore, it remains only to show the convergence in (3.10).

In what follows, we will make extensive use of indicator functions. For ease of reading, denote the indicator $1_A = 1_A(\omega)$ by $\{A\} = \{A\}(\omega)$.

For each $t \geqslant 0$, define $\mathcal{F}_t$ to be the σ-algebra generated by S_{N_t} and $\{(\tau_0^k, \tau_1^k)\}_{k=N_t+1}^{\infty}$. Since a_t, $\tau_0^{N_t+1}$, and Z_t are measurable with respect to $\mathcal{F}_t$, the tower property of conditional expectation and the triangle inequality give

$$\Big|E\{a_t \in A, Z_t\tau_0^{N_t+1} \in B, Z_t \in C, \varphi^{N_t} \in D\}$$

$$- E\{\xi a^1 + (1-\xi)a^0 \in A, \xi\tau_0 \in B, \xi \in C, Y_1 \in D\}\Big|$$

$$= \Big|E\left[\{a_t \in A, Z_t\tau_0^{N_t+1} \in B, Z_t \in C\}E[\{\varphi^{N_t} \in D\}|\mathcal{F}_t]\right]$$

$$- E\{\xi a^1 + (1-\xi)a^0 \in A, \xi\tau_0 \in B, \xi \in C, Y_1 \in D\}\Big|$$

$$\leqslant \Big|E\left[\{a_t \in A, Z_t\tau_0^{N_t+1} \in B, Z_t \in C\}E[\{\varphi^{N_t} \in D\}|\mathcal{F}_t]\right]$$

$$- E\left[\{a_t \in A, Z_t\tau_0^{N_t+1} \in B, Z_t \in C\}E[\{Y_1 \in D\}]\right]\Big|$$

$$+ \Big|E\left[\{a_t \in A, Z_t\tau_0^{N_t+1} \in B, Z_t \in C\}E[\{Y_1 \in D\}]\right]$$

$$- E\{\xi a^1 + (1-\xi)a^0 \in A, \xi\tau_0 \in B, \xi \in C\}\{Y_1 \in D\}\Big|.$$

By Lemma 9, $E\left[\{\varphi^{N_t} \in D\}|\mathcal{F}_t\right] \to E\left[\{Y_1 \in D\}\right]$ almost surely as $t \to \infty$. Therefore, the first term goes to 0 by the dominated convergence theorem. Since Y_1 is independent of ξ, a^1, a^0, and τ_0, the second term is bounded above by

$$\big|E1\{a_t \in A, Z_t\tau_0^{N_t+1} \in B, Z_t \in C\} - E\{\xi a^1 + (1-\xi)a^0 \in A, \xi\tau_0 \in B, \xi \in C\}\big|. \quad (3.11)$$

To show that (3.11) goes to 0 as $t \to \infty$, we consider the four possible cases for the inclusion of 0 and 1 in C. If both 0 and 1 are not in C, then (3.11) is 0 for all $t \geqslant 0$ since Z_t and ξ are each almost surely 0 or 1.

Suppose $0 \in C$ and $1 \notin C$. Then the indicator function in the first term of (3.11) is only non-zero if $Z_t = 0$. Hence, we can replace $\{Z_t \in C\}$ by $(1 - Z_t)$ and $\{Z_t\tau_0^{N_t+1} \in B\}$ by $\{0 \in B\}$. Similarly, in the second term we replace $\{\xi \in C\}$ by $(1-\xi)$, $\{\xi\tau_0 \in B\}$ by $\{0 \in B\}$, and $\{\xi a^1 + (1-\xi)a^0 \in A\}$ by $\{a^0 \in A\}$. Thus (3.11) becomes

$$(3.11) = |E\{a_t \in A, 0 \in B\}(1 - Z_t) - E\{a^0 \in A, 0 \in B\}(1-\xi)|$$

$$\leqslant |E\{a_t \in A\}(1 - Z_t) - E\{a^0 \in A\}(1-\xi)|.$$

By Lemma 4, this term goes to 0 as $t \to \infty$.

Suppose $1 \in C$ and $0 \notin C$. Then the indicator function in the first term of (3.11) is only non-zero if $Z_t = 1$. Thus after performing similar replacements to those above, (3.11) becomes

$$(3.11) = |E\{a_t \in A, \tau_0^{N_t+1} \in B\}Z_t - E\{a^1 \in A, \tau_0 \in B\}\xi|.$$

Define $\mathcal{F}_t'$ to be the σ-algebra generated by S'_{N_t+1}, $\tau_1^{N_t+1}$, and $\{(\tau_0^k, \tau_1^k)\}_{k=N_t+2}^{\infty}$. Observe that Z_t and a_t are both measurable with respect to $\mathcal{F}_t'$. Therefore, by the tower property of conditional expectation and the triangle inequality we have that

$$\begin{aligned}
&|E\{a_t \in A, \tau_0^{N_t+1} \in B\}Z_t - E\{a^1 \in A, \tau_0 \in B\}\xi| \\
&\leqslant |E\left[\{a_t \in A\}Z_t E\left[\{\tau_0^{N_t+1} \in B\}|\mathcal{F}_t'\right]\right] - E\left[\{a_t \in A\}Z_t E\left[\{\tau_0 \in B\}\right]\right]| \\
&+ |E[\{a_t \in A\}Z_t E[\{\tau_0 \in B\}]] - E\{a^1 \in A, \tau_0 \in B\}\xi|.
\end{aligned}$$

Lemma 10 gives us that $Z_t E[\{\tau_0^{N_t+1} \in B\}|\mathcal{F}'_t] = Z_t E[\{\tau_0^1 \in B\}|\mathcal{F}'_t]$ almost surely and Lemma 11 gives that $E[\{\tau_0^1 \in B\}|\mathcal{F}'_t] \to E[\{\tau_0 \in B\}]$ almost surely as $t \to \infty$. Therefore, the first term goes to 0 as $t \to \infty$ by the dominated convergence theorem. Finally since τ_0 is independent of ξ and a^1, we have the following bound on the second term

$$\begin{aligned}
&|E[\{a_t \in A\}Z_t E[\{\tau_0 \in B\}]] - E\{a^1 \in A, \tau_0 \in B\}\xi| \\
&\leqslant |E\{a_t \in A\}Z_t - E\{a^1 \in A\}\xi|
\end{aligned}$$

This goes to 0 as $t \to \infty$ by Lemma 4.

Finally, if both $0 \in C$ and $1 \in C$, then (3.11) becomes

$$\begin{aligned}
(3.11) &= |E\{a_t \in A, Z_t\tau_0^{N_t+1} \in B\} - E\{\xi a^1 + (1-\xi)a^0 \in A, \xi\tau_0 \in B\}| \\
&\leqslant |E\{a_t \in A, Z_t\tau_0^{N_t+1} \in B\}Z_t - E\{\xi a^1 + (1-\xi)a^0 \in A, \xi\tau_0 \in B\}\xi| \\
&+ |E\{a_t \in A, Z_t\tau_0^{N_t+1} \in B\}(1-Z_t) - E\{\xi a^1 + (1-\xi)a^0 \in A, \xi\tau_0 \in B\}(1-\xi)|.
\end{aligned}$$

We've already shown that each of these terms go to zero as $t \to \infty$, so the proof is complete. □

3.4 The lemmas

We now state and prove all of our lemmas. This first lemma calculates the limiting distribution of the age process. It can be interpreted as first flipping a coin to determine if Z_t is 0 or 1, and then choosing from the limiting distribution of the age conditioned on Z_t.

Lemma 4. *As* $t \to \infty$

$$a_t \to_d \xi a^1 + (1-\xi)a^0.$$

In particular, for any $x \geqslant 0$ *with* $A := (-\infty, x]$*, we have that as* $t \to \infty$

$$|E\{a_t \in A\}Z_t - E\{a^1 \in A\}\xi| + |E\{a_t \in A\}(1-Z_t) - E\{a^0 \in A\}(1-\xi)| \to 0.$$

Proof. Let $x \geqslant 0$ and define $A := (-\infty, x]$. Observe the following bound

$$\begin{aligned} &|E\{a_t \in A\} - E\{\xi a^1 + (1-\xi)a^0 \in A\}| \\ &\leqslant |E\{a_t \in A\}Z_t - E\{a^1 \in A\}\xi| + |E\{a_t \in A\}(1-Z_t) - E\{a^0 \in A\}(1-\xi)|. \end{aligned}$$

We will show that the first term goes to zero. The second term goes to zero by the analogous argument.

For our given $x \geqslant 0$, consider the alternating renewal process that is said to be "on" when $0 \leqslant t - S'_{N_t+1} \leqslant x$ and "off" otherwise. Formally, we define the "on/off" state process

$$b_t = \begin{cases} 1 & \text{if } 0 \leqslant t - S'_{N_t+1} \leqslant x \\ 0 & \text{otherwise.} \end{cases}$$

Observe that the the lengths of time that the process is "on" are $\{\min(\tau_1^k, x)\}_{k=1}^{\infty}$. Similarly, the lengths of time that the process is "off" are τ_0^1 and $\{\tau_0^k + (\tau_1^{k-1} - x)^+\}_{k=2}^{\infty}$, where as usual $(y)^+$ is equal to y if $y \geqslant 0$ and 0 otherwise. Since the distribution of $\min(\tau_1^k, x) + \tau_0^k + (\tau_1^{k-1} - x)^+$ is nonarithmetic, and since $E[\min(\tau_1^k, x) + \tau_0^k + (\tau_1^{k-1} -$

$x)^+] < \infty$, we can apply Theorem 3.4.4 in [29] to obtain

$$\lim_{t\to\infty} P(b_t = 1) = \frac{E\min(\tau_1, x)}{E[\min(\tau_1^k, x) + \tau_0^k + (\tau_1^{k-1} - x)^+]}. \tag{3.12}$$

Informally, this intuitive result states that the probability that the alternating renewal process is "on" at large time is just the expected length of an "on" bout divided by the sum of the expected lengths of an "off" bout and an "on" bout. Since $E[\min(\tau_1^k, x) + \tau_0^k + (\tau_1^{k-1} - x)^+] = E\tau_0 + E\tau_1$ and since the distribution of a^1 is chosen so that $E\tau_1 P(a^1 \leqslant x) = E\min(\tau_1, x)$, Equation (3.12) simplifies to

$$\lim_{t\to\infty} P(b_t = 1) = \frac{E\tau_1 P(a^1 \leqslant x)}{E\tau_0 + E\tau_1}.$$

Therefore

$$E[\{a_t \leqslant x\}Z_t] = P(a_t \leqslant x, Z_t = 1) = P(0 \leqslant t - S_{N_t+1} \leqslant x)$$

$$= P(b_t = 1) \xrightarrow[t\to\infty]{} \frac{E\tau_1 P(a^1 \leqslant x)}{E\tau_0 + E\tau_1} = E[\{a^1 \leqslant x\}\xi].$$

The last equality holds because ξ and a^1 are independent and $E\xi = E\tau_1/(E\tau_0 + E\tau_1)$.

The analogous argument shows that $|E\{a_t \in A\}(1 - Z_t) - E\{a^0 \in A\}(1 - \xi)| \to 0$ as $t \to \infty$ and the proof is complete. $\square$

The next 3 lemmas are general results that are all relatively standard. We return to lemmas directly related to our problem in Lemma 8.

Lemma 5. *Suppose $X_t \to X_\infty$ a.s. as $t \to \infty$ and $X_t \leqslant B$ a.s. where B is a random variable satisfying $EB < \infty$. If $\mathcal{F}_t \subset \mathcal{F}_s$ for $0 \leqslant s \leqslant t$, and $\mathcal{F}_\infty := \cap_{t\geqslant 0}\mathcal{F}_t$, then*

$$E[X_t|\mathcal{F}_t] \to E[X_\infty|\mathcal{F}_\infty] \quad \textit{almost surely as } t \to \infty.$$

Proof. We first show the convergence for an $X = X_t$ independent of t. Let X be any integrable random variable and for $t \leqslant 0$ define

$$M_t := E\left[X|\mathcal{F}_{-t}\right].$$

We claim that $\{M_t\}_{t=0}^{-\infty}$ is a backwards martingale. For $s \leqslant t \leqslant 0$ we have that $\mathcal{F}_{-s} \subset \mathcal{F}_{-t}$ and therefore by the tower property of conditional expectation,

$$E[M_t|\mathcal{F}_{-s}] = E\left[E\left[X|\mathcal{F}_{-t}\right]|\mathcal{F}_{-s}\right] = E\left[X|\mathcal{F}_{-s}\right] = M_s.$$

Since by definition of conditional expectation $M_t \in \mathcal{F}_{-t}$, and since $M_t \leqslant B$ almost surely where $EB < \infty$, we have that M_t is indeed backwards martingale. By the backwards martingale convergence theorem, $M_{-\infty} := \lim_{t\to-\infty} M_t$ exists almost surely and in $L^1(\Omega)$.

We claim that $M_{-\infty} = E\left[X|\mathcal{F}_\infty\right]$. Since for $t \leqslant T \leqslant 0$ we have that $M_t \in \mathcal{F}_{-t} \subset \mathcal{F}_{-T}$, it follows that $M_{-\infty} \in \mathcal{F}_{-T}$. Since $T \leqslant 0$ was arbitrary, $M_{-\infty} \in \mathcal{F}_\infty$.

Let $A \in \mathcal{F}_\infty$. Then

$$\begin{aligned} |EM_{-t}1_A - EM_{-\infty}1_A| &\leqslant E|M_{-t}1_A - M_{-\infty}1_A| \\ &\leqslant E|M_{-t} - M_{-\infty}| \\ &\to 0 \quad \text{as } t \to \infty \end{aligned}$$

since $M_{-t} \to M_{-\infty}$ in $L^1(\Omega)$. But,

$$EM_{-t}1_A = E[E\left[X|\mathcal{F}_t\right]1_A] = E[E\left[X1_A|\mathcal{F}_t\right]] = EX1_A.$$

Therefore $EX1_A = EM_{-\infty}1_A$, and so we conclude that $M_{-\infty} = E\left[X|\mathcal{F}_\infty\right]$.

We now show the convergence for the case where X_t depends on t. Let $T \geqslant 0$ and define $B_T := \sup\{|X_t - X_s| : t, s > T\}$. $B_T \leqslant 2B$, so B_T is integrable. Thus,

$$\limsup_{t\to\infty} E\left[|X_t - X_\infty|\mathcal{F}_t\right] \leqslant \lim_{t\to\infty} E\left[B_T|\mathcal{F}_t\right] = E\left[B_T|\mathcal{F}_\infty\right]$$

By assumption, $B_T \to 0$ a.s. as $T \to \infty$ so by Jensen's inequality

$$|E\left[X_t|\mathcal{F}_t\right] - E\left[X_\infty|\mathcal{F}_t\right]| \leqslant E\left[|X_t - X_\infty\|\mathcal{F}_t\right] \to 0.$$

Therefore,

$$\begin{aligned} |E\left[X_t|\mathcal{F}_t\right] - E\left[X_\infty|\mathcal{F}_\infty\right]| &\leqslant |E\left[X_t|\mathcal{F}_t\right] - E\left[X_\infty|\mathcal{F}_t\right]| \\ &\quad + |E\left[X_\infty|\mathcal{F}_t\right] - E\left[X_\infty|\mathcal{F}_\infty\right]|. \end{aligned}$$

We've just shown that the first term goes to 0, and we've shown that the second term goes to 0 since X_∞ doesn't depend on t, so the proof is complete. □

Lemma 6. *If $X_n \to X_\infty$ a.s. as $n \to \infty$ and $N_t \to \infty$ a.s. as $t \to \infty$, then*

$$X_{N_t} \to X_\infty \quad a.s. \text{ as } t \to \infty$$

Proof. Let $A := \{X_n \nrightarrow X_\infty\}$ and $B := \{N_t \nrightarrow \infty\}$. Then

$$P(X_{N_t} \nrightarrow X_\infty) \leqslant P(A \cup B) \leqslant P(A) + P(B) = 0.$$

□

We now give some standard definitions. Let $(\Omega, \mathcal{F}, P)$ be a probability space. A measurable map $\pi : \Omega \to \Omega$ is said to be **measure preserving** if $P(\pi^{-1}A) = P(A)$ for all $A \in \mathcal{F}$. Let π be a given measure preserving map. A set $A \in \mathcal{F}$ is said to be **π-invariant** if $\pi^{-1}A = A$, where two sets are considered to be equal if their symmetric difference has probability 0. A random variable X is said to be **π-invariant** if $X = X \circ \pi$ almost surely.

Lemma 7. *Let $\pi : \Omega \to \Omega$ be a measure preserving map. If X is π-invariant, then so is every set in its σ-algebra.*

Proof. See, for example, [14] Exercise 7.1.1. □

Lemma 8. *For each $t \geqslant 0$ define $\mathcal{F}_t$ to be the σ-algebra generated by S_{N_t} and $\{(\tau_0^k, \tau_1^k)\}_{k=N_t+1}^{\infty}$. If D is a Borel set of H, then for each $t \geqslant 0$*

$$E\left[\{\varphi^{N_t} \in D\} | \mathcal{F}_t\right] = E\left[\{\varphi^{-N_t} \in D\} | \mathcal{F}_t\right] \quad a.s.$$

Remark. To see why this Lemma should be true, observe that (a) the random variables φ^{N_t} and φ^{-N_t} are equal after a re-ordering of the first N_t-many ω_k's and that (b) the random variables generating $\mathcal{F}_t$ don't depend on the order of the first N_t-many ω_k's.

Proof. Fix a $t \geqslant 0$ and let $A \in \mathcal{F}_t$. By the definition of conditional expectation, we have that

$$\int_\Omega E\left[\{\varphi^{N_t} \in D\}|\mathcal{F}_t\right](\omega)\{A\}(\omega)\,dP = \int_\Omega \{\varphi^{N_t} \in D\}(\omega)\{A\}(\omega)\,dP.$$

Define $\sigma_t : \Omega \to \Omega$ to be the permutation that inverts the order of the first N_t-many ω_k's. That is, $(\sigma_t(\omega))_k = \omega_{N_t-k+1}$ for $k \in \{1, \ldots, N_t\}$ and $(\sigma_t(\omega))_k = \omega_k$ for $k > N_t$. Observe that $N_t(\omega) = N_t(\sigma_t(\omega))$ and thus $\varphi^{N_t}(\omega) = \varphi^{-N_t}(\sigma_t(\omega))$. Also, S_{N_t} and $\{(\tau_0^k, \tau_1^k)\}_{k=N_t+1}^\infty$ are σ_t-invariant, so A is σ_t-invariant by Lemma 7. Thus

$$\int_\Omega \{\varphi^{N_t} \in D\}(\omega)\{A\}(\omega)\,dP = \int_\Omega \{\varphi^{-N_t} \in D\}(\sigma_t(\omega))\{A\}(\sigma_t(\omega))\,dP.$$

Since σ_t is measure preserving and by the definition of conditional expectation,

$$\begin{aligned}\int_\Omega \{\varphi^{-N_t} \in D\}(\sigma_t(\omega))\{A\}(\sigma_t(\omega))\,dP &= \int_\Omega \{\varphi^{-N_t} \in D\}(\omega)\{A\}(\omega)\,dP \\ &= \int_\Omega E\left[\{\varphi^{-N_t} \in D\}|\mathcal{F}_t\right](\omega)\{A\}(\omega)\,dP.\end{aligned}$$

Putting this all together,

$$\int_\Omega E[\{\varphi^{N_t} \in D\}|\mathcal{F}_t]\{A\}\,dP = \int_\Omega E[\{\varphi^{-N_t} \in D\}|\mathcal{F}_t]\{A\}\,dP.$$

Since A was an arbitrary element of $\mathcal{F}_t$, the proof is complete. □

Recall that the random variable Y_1 is defined by $Y_1 := \lim_{n\to\infty} \varphi^{-n}(x)$, and is independent of the choice of $x \in H$, Theorem 5.

Lemma 9. *For each $t \geqslant 0$ define $\mathcal{F}_t$ to be the σ-algebra generated by S_{N_t} and $\{(\tau_0^k, \tau_1^k)\}_{k=N_t+1}^\infty$. If D is a Borel set of H, then with probability one*

$$E\left[\{\varphi^{-N_t} \in D\}|\mathcal{F}_t\right] \to E\{Y_1 \in D\} \quad \text{as } t \to \infty \tag{3.13}$$

$$\text{and} \quad E\left[\{\varphi^{N_t} \in D\}|\mathcal{F}_t\right] \to E\{Y_1 \in D\} \quad \text{as } t \to \infty \tag{3.14}$$

Proof. In light of Lemma 8, it suffices to show the convergence in line (3.13).

Since $\varphi^{-n} \to Y_1$ almost surely as $n \to \infty$ and since $N_t \to \infty$ almost surely as $t \to \infty$, we have that $\varphi^{-N_t} \to Y_1$ almost surely by Lemma 6. Define $\mathcal{F}_\infty := \cap_{t \geqslant 0} \mathcal{F}_t$ and observe that $\mathcal{F}_t \subset \mathcal{F}_s$ for $t \geqslant s \geqslant 0$. Thus, by Lemma 5

$$E\left[\{\varphi^{-N_t} \in D\}|\mathcal{F}_t\right] \to E\left[\{Y_1 \in D\}|\mathcal{F}_\infty\right] \quad \text{almost surely as } t \to \infty.$$

To complete the proof, we will show that for every $A \in \mathcal{F}_\infty$, $P(A) = 0$ or 1. To show this, we will show that $\mathcal{F}_\infty$ is contained in the exchangeable σ-algebra and then apply the Hewitt-Savage zero-one law. Let $n \in \mathbb{N}$, $A \in \mathcal{F}_\infty$, and π_n be an arbitrary permutation of $\omega_1, \ldots, \omega_n$. Define $\pi_t : \Omega \to \Omega$ by

$$\pi_t(\omega) = \begin{cases} \pi_n(\omega) & N_t \geqslant n \\ \omega & N_t < n. \end{cases}$$

Since S_{N_t} and $\{(\tau_0^k, \tau_1^k)\}_{k=N_t+1}^\infty$ are π_t-invariant, then A is π_t-invariant by Lemma 7 as $A \in \mathcal{F}_\infty \subset \mathcal{F}_t$. Therefore

$$P(A\Delta\pi_n^{-1}A, N_t \geqslant n) = P(A\Delta\pi_t^{-1}A, N_t \geqslant n) \leqslant P(A\Delta\pi_t^{-1}A) = 0.$$

Hence

$$\begin{aligned} P(A\Delta\pi_n^{-1}A) &= P(A\Delta\pi_n^{-1}A, N_t \geqslant n) + P(A\Delta\pi_n^{-1}A, N_t < n) \\ &\leqslant P(A\Delta\pi_n^{-1}A, N_t < n) \\ &\leqslant P(N_t < n). \end{aligned}$$

Since t was arbitrary, and because $P(N_t < n) \to 0$ as $t \to \infty$ since $N_t \to \infty$ almost surely, we conclude that $P(A\Delta\pi_n^{-1}A) = 0$. Since π_n was an arbitrary finite permutation, we conclude that $\mathcal{F}_\infty$ is contained in the exchangeable σ-algebra. By the Hewitt-Savage zero-one law, $\mathcal{F}_\infty$ only contains events that have probability 0 or 1. Thus, $\{Y_1 \in D\}$ is trivially independent of $\mathcal{F}_\infty$ and therefore $E\left[\{Y_1 \in D\}|\mathcal{F}_\infty\right] = E\{Y_1 \in D\}$. □

Lemma 10. *For each $t \geqslant 0$, define $\mathcal{F}'_t$ to be the σ-algebra generated by S'_{N_t+1}, $\tau_1^{N_t+1}$, and $\{(\tau_0^k, \tau_1^k)\}_{k=N_t+2}^{\infty}$. Then*

$$Z_t E\left[\{\tau_0^{N_t+1} \in B\}|\mathcal{F}'_t\right] = Z_t E\left[\{\tau_0^1 \in B\}|\mathcal{F}'_t\right] \quad \textit{almost surely.}$$

Remark. Recall that Z_t is either 0 if $S_{N_t} \leqslant t < S'_{N_t+1}$ or 1 if $S'_{N_t+1} \leqslant t$. Hence, this Lemma states that $E\left[\{\tau_0^{N_t+1} \in B\}|\mathcal{F}'_t\right] = E\left[\{\tau_0^1 \in B\}|\mathcal{F}'_t\right]$ if $Z_t = 1$.

Remark. The proof of this Lemma is very similar to the proof of Lemma 8.

Proof. If ω is such that $Z_t = 0$, then the equality is trivially satisfied. Let $A \in \mathcal{F}'_t$. Since $\{\omega \in \Omega : Z_t(\omega) = 1\} \in \mathcal{F}'_t$, we have by the definition of conditional expectation that

$$\int_\Omega E\left[\{\tau_0^{N_t+1} \in B\}|\mathcal{F}'_t\right]\{A \cap \{Z_t = 1\}\}\, dP = \int_\Omega \{\tau_0^{N_t+1} \in B\}(\omega)\{A \cap \{Z_t = 1\}\}(\omega)\, dP.$$

Define $\sigma_t : \Omega \to \Omega$ by

$$(\sigma_t(\omega))_k = \begin{cases} (\tau_0^{N_t+1}, \tau_1^1) & \text{if } k = 1 \text{ and } Z_t = 1 \\ (\tau_0^1, \tau_1^{N_t+1}) & \text{if } k = N_t + 1 \text{ and } Z_t = 1 \\ \omega_k & \text{otherwise.} \end{cases}$$

That is, σ_t switches τ_0^1 and $\tau_0^{N_t+1}$ if $Z_t = 1$ and otherwise does nothing. Since S'_{N_t+1}, $\tau_1^{N_t+1}$, and $\{(\tau_0^k, \tau_1^k)\}_{k=N_t+2}^{\infty}$ are all σ_t-invariant, we have that A is σ_t-invariant by Lemma 7. Also observe that $\{Z_t = 1\}$ is σ_t-invariant. Thus

$$\int_\Omega \{\tau_0^{N_t+1} \in B\}(\omega)\{A \cap \{Z_t = 1\}\}(\omega)\, dP = \int_\Omega \{\tau_0^1 \in B\}(\sigma_t(\omega))\{A \cap \{Z_t = 1\}\}(\sigma_t(\omega))\, dP.$$

Since σ_t is measure preserving, and by the definition of conditional expectation, we have that

$$\begin{aligned}\int_\Omega \{\tau_0^1 \in B\}(\sigma_t(\omega))\{A \cap \{Z_t = 1\}\}(\sigma_t(\omega))\, dP &= \int_\Omega \{\tau_0^1 \in B\}(\omega)\{A \cap \{Z_t = 1\}\}(\omega)\, dP \\ &= \int_\Omega E\left[\{\tau_0^1 \in B\}|\mathcal{F}'_t\right]\{A \cap \{Z_t = 1\}\}\, dP.\end{aligned}$$

Putting all this together,

$$\int_\Omega E[\{\tau_0^{N_t+1} \in B\}|\mathcal{F}_t']\{A \cap \{Z_t = 1\}\}\,dP = \int_\Omega E[\{\tau_0^1 \in B\}|\mathcal{F}_t']\{A \cap \{Z_t = 1\}\}\,dP.$$

This implies that $E[\{\tau_0^{N_t+1} \in B\}|\mathcal{F}'_t] = E[\{\tau_0^1 \in B\}|\mathcal{F}'_t]$ almost surely on $\{Z_t = 1\}$. To see this, let $\epsilon > 0$ define $\Lambda := \{\omega \in \Omega : E[\{\tau_0^{N_t+1} \in B\}|\mathcal{F}'_t] - E[\{\tau_0^1 \in B\}|\mathcal{F}'_t] \geqslant \epsilon\}$. This set is in $\mathcal{F}_t'$, so by the above calculation we have that

$$0 = \int_{\Lambda \cap \{Z_t=1\}} E[\{\tau_0^{N_t+1} \in B\}|\mathcal{F}'_t] - E[\{\tau_0^1 \in B\}|\mathcal{F}'_t]\,dP \geqslant \epsilon P(\Lambda \cap \{Z_t = 1\}).$$

So $P(\Lambda \cap \{Z_t = 1\}) = 0$. The same argument with $\Lambda' := \{\omega \in \Omega : E[\{\tau_0^1 \in B\}|\mathcal{F}_t'] - E[\{\tau_0^{N_t+1} \in B\}|\mathcal{F}_t'] \geqslant \epsilon\}$ completes the proof of the claim. Therefore, $Z_t E[\{\tau_0^{N_t+1} \in B\}|\mathcal{F}_t'] = Z_t E[\{\tau_0^1 \in B\}|\mathcal{F}_t']$ almost surely. $\square$

Lemma 11. *For each $t \geqslant 0$, define $\mathcal{F}_t'$ to be the σ-algebra generated by S'_{N_t+1}, $\tau_1^{N_t+1}$, and $\{(\tau_0^k, \tau_1^k)\}_{k=N_t+2}^{\infty}$. Then*

$$E\left[\{\tau_0^1 \in B\}|\mathcal{F}'_t\right] \to E\{\tau_0 \in B\} \quad \textit{almost surely as } t \to \infty.$$

Remark. The proof of this Lemma is very similar to the proof of Lemma 9.

Proof. Define $\mathcal{F}'_\infty := \cap_{t \geqslant 0} \mathcal{F}_t'$ and observe that $\mathcal{F}'_s \supset \mathcal{F}'_t$ for $0 \leqslant s \leqslant t$. Thus, by Lemma 5

$$E\left[\{\tau_0^1 \in B\}|\mathcal{F}'_t\right] \to E\left[\{\tau_0^1 \in B\}|\mathcal{F}'_\infty\right] \quad \text{almost surely.}$$

We claim that for each $A \in \mathcal{F}'_\infty$, $P(A) = 0$ or 1. To show this, we will show that $\mathcal{F}'_\infty$ is contained in the exchangeable σ-algebra and then apply the Hewitt-Savage zero-one law. Let $n \in \mathbb{N}$, $A \in \mathcal{F}'_\infty$, and π_n be an arbitrary permutation of $(\tau_0^1, \tau_1^1), \ldots, (\tau_0^n, \tau_1^n)$.

Define $\pi_t : \Omega \to \Omega$ by

$$\pi_t(\omega) = \begin{cases} \pi_n(\omega) & N_t \geqslant n \\ \omega & N_t < n. \end{cases}$$

Since S'_{N_t+1}, $\tau_1^{N_t+1}$, and $\{(\tau_0^k, \tau_1^k)\}_{k=N_t+2}^{\infty}$ are π_t-invariant, then A is π_t-invariant by Lemma 7 as $A \in \mathcal{F}'_\infty \subset \mathcal{F}'_t$. Therefore

$$P(A\Delta\pi_n^{-1}A, N_t \geqslant n) = P(A\Delta\pi_t^{-1}A, N_t \geqslant n) \leqslant P(A\Delta\pi_t^{-1}A) = 0.$$

Hence

$$\begin{aligned} P(A\Delta\pi_n^{-1}A) &= P(A\Delta\pi_n^{-1}A, N_t \geqslant n) + P(A\Delta\pi_n^{-1}A, N_t < n) \\ &\leqslant P(A\Delta\pi_n^{-1}A, N_t < n) \\ &\leqslant P(N_t < n). \end{aligned}$$

Since t was arbitrary, we conclude that $P(A\Delta\pi_n^{-1}A) = 0$ because $P(N_t < n+1) \to 0$ as $t \to \infty$ since $N_t \to \infty$ almost surely. Since π_n was an arbitrary finite permutation, we conclude that $\mathcal{F}'_\infty$ is contained in the exchangeable σ-algebra. By the Hewitt-Savage zero-one law, $\mathcal{F}'_\infty$ only contains events that have probability 0 or 1. Thus, $\{\tau_0^1 \in C\}$ is trivially independent of $\mathcal{F}'_\infty$ and so we conclude $E\left[\{\tau_0^1 \in C\} | \mathcal{F}'_\infty\right] = E\{\tau_0^1 \in C\} = E\{\tau_0 \in C\}$. $\square$

4

PDEs with randomly switching boundary conditions

We now use our results from Chapter 3 to study partial differential equations (PDEs) with randomly switching boundary conditions. Our results apply to a range of specific problems, so in Section 4.1 we explain how to cast a problem in our framework and give an example. In Section 4.2 we collect assumptions and in Section 4.3 we prove theorems about the mean of the process. In Sections 4.4 and 4.5 we apply the results from Section 4.3 and Chapter 3 to the heat equation on an interval with a randomly switching boundary condition. In Section 4.7, we use our results to analyze a model of insect respiration. There, we will see that switching between boundary conditions of different types can have surprising biological implications. The contents of this chapter overlap significantly with [21].

4.1 Motivating examples and general setup

Our results can be applied to the following type of stochastic switching problem. Suppose we are given a strongly elliptic second order differential operator L on a domain $D \subset \mathbb{R}^d$ with smooth coefficients which do not depend on t. Assume the domain D is bounded with a smooth boundary and a pair of boundary conditions, (a) and (b). We then consider the stochastic process $u(t, x)$ that solves

$$\partial_t u = Lu \quad \text{in } D \tag{4.1}$$

subject to boundary conditions that switch at random times between (a) and (b). We allow (a) and (b) to be different types. For example, one can be Dirichlet and the other Neumann. For the sake of presentation, we assume (a) are homogenous, but our analysis is easily modified to include the case where (a) are not homogenous.

We formulate this problem in the setting of Chapter 3 as alternating flows on the Hilbert space $L^2(D)$. We define

$$Au := Lu \quad \text{if } u \in D(A)$$

where $D(A)$ is chosen so that A generates the contraction C_0-semigroup that maps an initial condition to the solution of Equation (4.1) at time t subject to boundary conditions (a). Similarly, we define

$$Bu := Lu \quad \text{if } u \in D(B)$$

where $D(B)$ is chosen so that B generates the contraction C_0-semigroup that maps an initial condition to the solution of Equation (4.1) at time t subject to the homogenous version of boundary conditions (b). We then choose $c \in D(L)$ to satisfy boundary conditions (b) and $Lc = 0$. Then the H-valued process defined in Equation (3.9) in Chapter 3 with $\Phi_t^1(f) = e^{At} f$ and $\Phi_t^0(f) = e^{Bt}(f - c) + c$ corresponds to this PDE stochastic switching problem.

We now give a specific example of possible boundary conditions.

Example 4. Consider the solution to Equation (4.1) that switches between

$$u|_{\partial D} = 0 \qquad \text{and} \qquad u|_{\partial D} = b > 0.$$

To cast this problem in the setting of Chapter 3, we define $D(A) = H_0^1(D) \cap H^2(D)$ and $A = L$ on $D(A)$. Further we define $B = A$ on $D(B) = D(A)$ and we choose $c \in D(L)$ so that $Lc = 0$ and $c = b$ on ∂D.

4.2 Assumptions

We now formalize the setup from Section 4.1. Let H be a real separable Hilbert space and let A and B be two self-adjoint operators on H, each with strictly negative spectrum. Hence, A and B generate contraction C_0-semigroups, denoted respectively by e^{At} and e^{Bt}. Assume $A = B$ on $D(A) \cap D(B)$. Assume there exists an operator L on $D(L) \supset D(A) \cup D(B)$ so that

$$Lu = \begin{cases} Au & \text{if } u \in D(A) \\ Bu & \text{if } u \in D(B). \end{cases}$$

Suppose $c \in D(L)$ satisfies $Lc = 0$.

Recalling notation from Chapter 3, let the holding time distributions, μ_0 and μ_1, be continuous. Let $u(t, \omega)$ be the H-valued process defined in Equation (3.9) in Chapter 3 with

$$\Phi_t^1(f) = e^{At} f$$
$$\Phi_t^0(f) = e^{Bt}(f - c) + c.$$

Observe that for each $t > 0$, the maps Φ_t^0 and Φ_t^1 are contractions. Also observe that for each fixed $x \in H$, $\Phi_t^0(x)$ and $\Phi_t^1(x)$ are continuous in t.

Assume that there exists a deterministic $M = M(u_0)$ so that with probability one, $\|u(t)\| \leqslant M$ for each $t \geqslant 0$, where $\|x\| := \sqrt{(x, x)}$ with $(\cdot, \cdot)$ denoting the inner product in H.

For every $0 < s \leqslant t$, define $\eta(s, t)$ to be the random variable that gives the number of switches that occur on the interval (s, t). Formally, we define $\eta(s, t)$ by taking the supremum over partitions σ of the interval (s, t), $s = \sigma_0 < \sigma_1 < \cdots < \sigma_k < \sigma_{k+1} = t$,

$$\eta(s, t)(\omega) := \sup_{\sigma} \sum_{i=0}^{k} |Z_{\sigma_{i+1}}(\omega) - Z_{\sigma_i}(\omega)|,$$

where Z_t is as in Equation (3.8). Assume that μ_0 and μ_1 are such that for every $t > 0$, we have that as $s \to 0$,

$$P(\eta(t+s,t) = 1) = O(s)$$
$$P(\eta(t+s,t) \geqslant 2) = o(s).$$

4.3 The mean satisfies the PDE

In what follows, fix $\phi \in D(A) \cap D(B)$, which will serve as our test function. Observe that each realization of our stochastic process satisfies the weak form of the PDE away from switching times. That is, for a given $\omega \in \Omega$, if t is not a switching time, then

$$\frac{d}{dt}(\phi, u(t,\omega)) = (L\phi, u(t,\omega)). \tag{4.2}$$

The following theorem states that the mean satisfies the weak form of the PDE as well.

Theorem 9. *For every $t > 0$, we have that*

$$\frac{d}{dt}(\phi, Eu(t)) = (L\phi, Eu(t)).$$

To prove this theorem, we will need the following lemma which states that our process satisfies a certain weak uniform continuity condition.

Lemma 12. *For every $\epsilon > 0$ and non-negative integer n, there exists a $\delta(\epsilon/n) > 0$ so that if $|t - s| < \delta(\epsilon/n)$, then*

$$|(\phi, u(t,\omega) - u(s,\omega))1_{\eta(s,t)=n}| < \epsilon \quad a.s.$$

Proof. Observe that if there are no switches between s and t and $Z_s = 0$, then

$$|(\phi, u(t,\omega) - u(s,\omega))| = \left|(\phi, [e^{A(t-s)} - I]u(s,\omega))\right| \leqslant \|e^{A(t-s)}\phi - \phi\|M,$$

since A is self-adjoint and $\|u(t)\| \leqslant M$ a.s. by assumption. Similarly, if there are no switches between s and t and $Z_s = 1$, then

$$|(\phi, u(t,\omega) - u(s,\omega))| = \left|(\phi, [e^{B(t-s)} - I][u(s,\omega) - c])\right| \leqslant \|e^{B(t-s)}\phi - \phi\|(M + \|c\|).$$

Let $\epsilon > 0$ and let n be a non-negative integer. Since e^{At} and e^{Bt} are both C_0-semigroups, we can choose a $\delta > 0$ so that if $0 \leqslant t < \delta$, then

$$\max\{\|e^{At}\phi - \phi\|, \|e^{Bt}\phi - \phi\|\} < \frac{\epsilon}{n(M + \|c\|)}.$$

Let $\omega \in \Omega$ be given and assume $|t - s| < \delta$. If ω is such that $\eta(s,t)(\omega) \neq n$, then the result is immediate. Suppose $\eta(s,t)(\omega) = n$. Let $\{\sigma_k\}_{k=1}^n$ be the n switching times between s and t and let $s = \sigma_0 < \sigma_1 < \cdots < \sigma_n < \sigma_{n+1} = t$. Then

$$\begin{aligned} |(\phi, u(t,\omega) - u(s,\omega))| &\leqslant \sum_{k=0}^{n} |(\phi, u(\sigma_{k+1},\omega) - u(\sigma_k,\omega))| \\ &\leqslant \sum_{k=0}^{n} \max\{\|e^{At}\phi - \phi\|, \|e^{Bt}\phi - \phi\|\}(M + \|c\|) < \epsilon \end{aligned}$$

□

Proof of Theorem 9. We seek to differentiate $E(\phi, u(t))$ with respect to t. Define

$$f(t,\omega) = (\phi, u(t,\omega)).$$

Let $h_n \to 0$ as $n \to \infty$. For a given $t_0 > 0$, define the difference quotient

$$\begin{aligned} g_n(\omega) :=& \frac{1}{h_n}\left(f(t_0 + h_n,\omega) - f(t_0,\omega)\right) \\ =& \frac{1}{h_n}\left(f(t_0 + h_n,\omega) - f(t_0,\omega)\right) 1_{\eta(t_0+h_n,t_0)=0} \\ &+ \frac{1}{h_n}\left(f(t_0 + h_n,\omega) - f(t_0,\omega)\right) 1_{\eta(t_0+h_n,t_0)=1} \\ &+ \frac{1}{h_n}\left(f(t_0 + h_n,\omega) - f(t_0,\omega)\right) 1_{\eta(t_0+h_n,t_0)\geqslant 2} \\ =& (1) + (2) + (3). \end{aligned}$$

We will handle each the these three terms differently.

We first consider (1). Assume that ω is such that $t_0 \neq S_k(\omega)$ for all k. Observe that

$$\frac{1}{h_n}\left(f(t_0+h_n,\omega)-f(t_0,\omega)\right) \to \frac{d}{dt}f(t_0,\omega) = (L\phi, u(t_0)).$$

Also observe that for such an ω, we have that $1_{\eta(t_0+h_n,t_0)=0}(\omega) = 1$ for n sufficiently large. Since this set of ω's has probability 1, we conclude that

$$(1) = \frac{1}{h_n}\left(f(t_0+h_n,\omega)-f(t_0,\omega)\right) 1_{\eta(t_0+h_n,t_0)=0} \to (L\phi, u(t)) \quad \text{a.s.}$$

We will now apply the bounded convergence theorem to (1). Define $B := \|L\phi\|M$. We will show that for each n,

$$|(1)| = |\frac{1}{h_n}\left(f(t_0+h_n,\omega)-f(t_0,\omega)\right) 1_{\eta(t_0+h_n,t_0)=0}| \leqslant B \quad \text{a.s.}$$

Let n and ω be given. If $\eta(t_0+h_n,t_0)(\omega) \neq 0$, then the bound is trivially satisfied. If $\eta(t_0+h_n,t_0)(\omega) = 0$, then $f(t,\omega)$ is differentiable in t for all $t \in (t_0, t_0+h_n)$. Therefore we can employ the mean value theorem to obtain the upper bound

$$\left|\frac{1}{h_n}\left(f(t_0+h_n,\omega)-f(t_0,\omega)\right)\right| \leqslant \sup_{t\in(t_0,t_0+h_n)} \left|\frac{d}{dt}f(t_0,\omega)\right| = \sup_{t\in(t_0,t_0+h_n)} |(L\phi, u(t_0,\omega))| \leqslant B,$$

since $\|u(t)\| \leqslant M$ by assumption. Thus B is an almost sure bound for (1) and so by the bounded convergence theorem, $E(1) \to E(L\phi, u(t))$ as $n \to \infty$.

To complete the proof, we need only show that (2) and (3) both tend to 0 in mean as $n \to \infty$. We first work on (2). Observe that

$$\begin{aligned} E|(2)| &= E\left|\frac{1}{h_n}\left(f(t_0+h_n,\omega)-f(t_0,\omega)\right) 1_{\eta(t_0+h_n,t_0)=1}\right| \\ &\leqslant \frac{1}{h_n}\operatorname{ess\,sup}_\omega \left|(f(t_0+h_n,\omega)-f(t_0,\omega))1_{\eta(t_0+h_n,t_0)=1}\right| E\left(1_{\eta(t_0+h_n,t_0)=1}\right). \end{aligned}$$

By the Lemma 12, $\operatorname{ess\,sup}_\omega \left|(f(t_0+h_n,\omega)-f(t_0,\omega))1_{\eta(t_0+h_n,t_0)=1}\right| \to 0$ as $n \to \infty$. Since by assumption $P\left(\eta(t_0+h_n,t_0)=1\right) = O(h_n)$, we conclude that $E|(2)| \to 0$ as $n \to \infty$.

Finally, we show that $E(3) \to 0$ as $n \to \infty$. By the assumption that $\|u(t)\| \leqslant M$,

$$E|(2)| = E\left|\frac{1}{h_n}\left(f(t_0+h_n,\omega) - f(t_0,\omega)\right) 1_{\eta(t_0+h_n,t_0)\geqslant 2}\right|$$

$$\leqslant \frac{2M}{h_n} P\left(\eta(t_0+h_n,t_0) \geqslant 2\right).$$

Since by assumption $P\left(\eta(t_0+h_n,t_0) \geqslant 2\right) = o(h_n)$, we conclude that $E|(3)| \to 0$ as $n \to \infty$.

Therefore

$$\frac{E(\phi, u(t_0+h_n)) - E(\phi, u(t_0))}{h_n} = Eg_n$$

$$\to E(L\phi, u(t)) \text{ as } n \to \infty.$$

Since h_n was an arbitrary sequence tending to 0 and t_0 was an arbitrary positive number, we conclude that $\frac{d}{dt}E(\phi, u(t)) = E(L\phi, u(t))$ for all $t > 0$.

Since taking the inner product against ϕ or $L\phi$ are both bounded linear operators on H, we can exchange expectation with inner product to obtain

$$\frac{d}{dt}(\phi, Eu(t)) = \frac{d}{dt}E(\phi, u(t)) = E(L\phi, u(t)) = (L\phi, Eu(t)).$$

□

We now show that the mean at large time satisfies the homogeneous PDE.

Theorem 10. *Let $\bar{u}$ have the limiting distribution of $u(t)$ as $t \to \infty$, as in Theorem 8. Then $Eu(t) \to E\bar{u}$ weakly in H as $t \to \infty$ and $E\bar{u}$ satisfies*

$$(L\phi, E\bar{u}) = 0.$$

Remark. We will often assume the differential operator L and the domain D to be sufficiently regular so that $E\bar{u}$ is actually a C^∞ function satisfying $LE\bar{u} = 0$ pointwise.

Proof. By Theorem 8, $Eg(u(t)) \to Eg(\bar{u}(t))$ as $t \to \infty$ for every continuous and bounded $g : H \to \mathbb{R}$. For any $\eta \in H$, the function $(\eta, \cdot) : H \to \mathbb{R}$ is continuous and since by assumption, $\|u(t)\| \leqslant M$ a.s., we have that

$$E(\eta, u(t)) \to E(\eta, \bar{u}) \quad \text{as } t \to \infty.$$

Since taking the inner product against η is a bounded linear operator on H, we can exchange expectation with inner product to obtain $(\eta, Eu(t)) = E(\eta, u(t))$ and $E(\eta, \bar{u}) = (\eta, E\bar{u})$. So, $Eu(t) \to E\bar{u}$ weakly in H as $t \to \infty$.

Of course it follows that in particular

$$(\phi, Eu(t)) \to (\phi, E\bar{u}) \quad \text{and} \quad (L\phi, Eu(t)) \to (L\phi, E\bar{u}) \quad \text{as } t \to \infty.$$

By Theorem 9, $\frac{d}{dt}(\phi, Eu(t)) = (L\phi, Eu(t))$. Thus, $(\phi, Eu(t))$ and $\frac{d}{dt}(\phi, Eu(t))$ both converge as $t \to \infty$ and so we conclude that $\frac{d}{dt}(\phi, Eu(t))$ must actually converge to 0. Hence, $(L\phi, E\bar{u}) = 0$. □

4.4 Dirichlet/Dirichlet switching

In this section and the next, we apply our results from Section 4.3 to the heat equation on the interval $[0, L]$. We impose an absorbing Dirichlet boundary condition at $x = 0$ and a stochastically switching boundary condition at $x = L$. In this section, we consider switching between two Dirichlet boundary conditions at $x = L$. In the next section, we consider switching between a Dirichlet and a Neumann boundary condition at $x = L$.

Consider the stochastic process that solves

$$\partial_t u = D\Delta u \quad \text{in } (0, L) \tag{4.3}$$

and at exponentially distributed times switches between the boundary conditions

$$\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} u(0,t) &= 0 \\ u(L,t) &= b > 0. \end{aligned}$$

To cast this problem in the setting of previous sections, we set our Hilbert space to be $L^2[0, L]$ and define the operator

$$D(A) := H_0^1(0, L) \cap H^2(0, L)$$

$$Au := \Delta u \quad \text{if } u \in D(A).$$

We set $B = A$ on $D(B) = D(A)$, and $c = \frac{b}{L}x \in L^2[0, L]$. Let our switching time distributions, μ_0 and μ_1, be exponential with respective rate parameters r_0 and r_1. Let $u(t, \omega)$ be the H-valued process defined in Equation (3.9) of Chapter 3 with $\Phi_t^1(f) = e^{At}f$ and $\Phi_t^0(f) = e^{Bt}(f - c) + c$.

We are interested in studying the large time distribution of $u(t)$. By Theorem 8, we have that $u(t)$ converges in distribution as $t \to \infty$. Let $\bar{u}$ have this limiting distribution. By the definition of Y_0 and Y_1, it is immediate that $\bar{u}$ is almost surely smooth, and using Theorem 7, it is immediate that for each $x \in [0, L]$, $\bar{u}(x) \leqslant \frac{b}{L}x$.

4.4.1 Expectation

We will now use the tools from Chapter 3 and Section 4.3 to find the large time expectation of the solution to this stochastic switching problem. We will see in Section 4.4.2 that it is possible to find this expectation using only the results of Chapter 3 since we are switching between two Dirichlet boundary conditions. However, the results of Section 4.3 will be needed when we consider switching between Dirichlet and Neumann boundary conditions in Section 4.5. Thus, we go through the calculation in this section in order to compare to the calculations in Section 4.5.

It is immediate that all of the assumptions in Section 4.2 are satisfied, except for one; we need to check that there exists a deterministic M so that $\|u(t)\| \leqslant M$ almost surely for all $t \geqslant 0$. We show that and more in the following Lemma. Recall that Y_1 and Y_0 are the pullbacks defined in Equations (3.3) and (3.4) in Chapter 3.

Lemma 13. *Under the assumptions of the current section, we have that*

$$\|u(t)\| \leqslant \sqrt{L}\max\{\|u_0\|_\infty, b\},$$

where $\|\cdot\|_\infty$ *denotes the* $L^\infty[0,L]$ *norm. Furthermore,*

$$\|Y_1\|_\infty \leqslant b \quad \text{and} \quad \|Y_0\|_\infty \leqslant b \quad \text{almost surely.}$$

Proof. First note that $\|c\|_\infty = \|\frac{b}{L}x\|_\infty = b$. If $f \in L^2[0,L]$, then by the maximum principle, we have that for any $t \geqslant 0$

$$\|e^{At}f\|_\infty \leqslant \|f\|_\infty \quad \text{and} \quad \|e^{Bt}(f-c)+c\|_\infty \leqslant \max\{b, \|f\|_\infty\}. \tag{4.4}$$

Hence, $\max\{\|u(t)\|_\infty, b\}$ is non-increasing in t and so the bound on $\|u(t)\|$ is proven.

Since $S := \{f \in L^2[0,L] : \|f\|_\infty \leqslant b\}$ is a closed set in $L^2[0,L]$, Equation (4.4) and Theorem 7 give the desired bounds on $\|Y_1\|_\infty$ and $\|Y_0\|_\infty$. □

As in Theorem 8, let $\bar{u}$ have the limiting distribution of $u(t)$ as $t \to \infty$. Then by Theorem 10, $E\bar{u} \in L^2[0,L]$ satisfies

$$(\Delta\phi, E\bar{u}) = 0$$

for each $\phi \in C_0^\infty(0,L)$. By the regularity of Δ on $[0,L]$, we have that $E\bar{u}$ is the linear function

$$(E\bar{u})(x) = sx + d$$

for some $s, d \in \mathbb{R}$. By Corollary 1 of Chapter 3, we have that

$$sx + d = pEY_1 + (1-p)EY_0 \tag{4.5}$$

where $p = r_0/(r_0+r_1)$. We will use Equation (4.5) to determine s and d. While it can be shown that EY_0 and EY_1 are smooth functions, we will instead use test functions and take limits to avoid evaluating EY_0 and EY_1 at specific points in $[0,L]$.

Let $\{\phi_n\}_{n=1}^\infty \in C_0^\infty(0,L)$ be such that $\|\phi_n\|_{L^1} = 1$ for each n and

$$\lim_{n\to\infty}(\phi_n, f) = f(L)$$

for each $f \in C[0, L]$. Since the inner product with ϕ_n is a bounded linear transformation in $L^2[0, L]$, we can interchange expectation with inner product to obtain

$$sL + d = \lim_{n\to\infty} (\phi_n, E\bar{u}) = \lim_{n\to\infty} [(\phi_n, pEY_1 + (1-p)EY_0)] \tag{4.6}$$

$$= \lim_{n\to\infty} [pE(\phi_n, Y_1) + (1-p)E(\phi_n, Y_0)]. \tag{4.7}$$

We want to exchange the limit with the expectation. To do this, first observe that $Y_1(x)$ and $Y_0(x)$ are each almost surely continuous functions of $x \in [0, L]$ with $Y_1(L) = 0$ and $Y_0(L) = b$ almost surely. Thus,

$$\lim_{n\to\infty} (\phi_n, Y_0) = b \quad \text{and} \quad \lim_{n\to\infty} (\phi_n, Y_1) = 0 \quad \text{almost surely.}$$

Using Lemma 13 and the assumption that $\|\phi_n\|_{L^1} = 1$ for each n, Holder's inequality gives

$$|(\phi_n, Y_0)| \leqslant b \quad \text{and} \quad |(\phi_n, Y_1)| \leqslant b \quad \text{almost surely.}$$

So we apply the bounded convergence theorem to Equation 4.7 to obtain

$$sL + d = pE \lim_{n\to\infty} (\phi_n, Y_1) + (1-p)E \lim_{n\to\infty} (\phi_n, Y_0)$$

$$= p0 + (1-p)b = (1-p)b.$$

The same argument shows that $d = 0$ and so we conclude that

$$E\bar{u} = (1-p)\frac{b}{L}x.$$

We comment on this result in Section 4.6

4.4.2 Statistics of Fourier coefficients

Switching between two Dirichlet boundary conditions is significantly simpler than switching between boundary conditions of different types. This is due to the fact that the two solution operators that we use when switching between two Dirichlet boundary conditions employ the same semigroup and thus the same orthonormal basis. Hence, we only need to consider the projections of the stochastic process in this basis.

In this example, this orthonormal basis is $\left\{\sqrt{2/L}\sin(xk\pi/L)\right\}_{k=1}^{\infty}$ and the corresponding eigenvalues are $\{-D(k\pi/L)^2\}_{k=1}^{\infty}$. Denote this basis by $\{b_k\}_{k=1}^{\infty}$ and the eigenvalues by $\{-\beta_k\}_{k=1}^{\infty}$. Observe that for each k, the Fourier coefficient $u_k(t) := (b_k, u(t)) \in \mathbb{R}$ is the solution to a one-dimensional ODE with right-hand side that switches between $-\beta_k u_k$ and $-\beta_k(u_k - c_k)$, where $c_k = (b_k, c)$. We remark that Theorem 7 gives that $(b_k, \bar{u})$ is almost surely between 0 and c_k on the real line since this set is invariant under both the solution operator to each of those one-dimensional ODEs.

In [8], the authors considered such one-dimensional ODE stochastic switching problems. From their results, we obtain the distribution of the Fourier coefficients of Y_0 and Y_1. In particular, we have that for each k,

$$\frac{(Y_0, b_k)}{(b_k, c)} \sim \text{Beta}\left(\frac{r_1}{\beta_k} + 1, \frac{r_0}{\beta_k}\right) \quad \text{and} \quad \frac{(Y_1, b_k)}{(b_k, c)} \sim \text{Beta}\left(\frac{r_1}{\beta_k}, \frac{r_0}{\beta_k} + 1\right). \tag{4.8}$$

From this, we immediately obtain statistics of the individual Fourier coefficients of Y_0 and Y_1, such as expectations

$$E(Y_0, b_k) = \frac{r_1 + \beta_k}{r_1 + r_0 + \beta_k}(c, b_k) \quad \text{and} \quad E(Y_1, b_k) = \frac{r_1}{r_1 + r_0 + \beta_n}(c, b_k).$$

We also obtain variances

$$\text{Var}(Y_0, b_k) = \frac{\beta_k r_0(\beta_k + r_1)}{(\beta_k + r_0 + r_1)^2(2\beta_k + r_0 + r_1)}(c, b_k)^2$$

$$\text{Var}(Y_1, b_k) = \frac{\beta_k r_1(\beta_k + r_0)}{(\beta_k + r_0 + r_1)^2(2\beta_k + r_0 + r_1)}(c, b_k)^2.$$

However, knowing the distributions of the individual Fourier coefficients of Y_0 or Y_1 is of course not enough to obtain their joint distributions. Nonetheless, we can use

Theorem 6 from Chapter 3 to obtain joint statistics. To illustrate, we will calculate $E(Y_0, b_n)(Y_0, b_m)$. Theorem 6 gives

$$E(Y_0, b_n)(Y_0, b_m) = E(e^{B\tau_0}(e^{B\tau_1}Y_0 - c) + c, b_n)(e^{B\tau_0}(e^{B\tau_1}Y_0 - c) + c, b_m),$$

where τ_0 and τ_1 are independent exponential random variables with rates r_0 and r_1. After recalling some basic facts about exponential random variables and making some algebraic manipulations, we obtain that $E(Y_0, b_n)(Y_0, b_m)$ is equal to

$$\frac{(\beta_m + \beta_n + r_1)((\beta_m + \beta_n)(\beta_m + r_1)(\beta_n + r_1) + (2\beta_m\beta_n + (\beta_m + \beta_n)r_1)r_0)}{(\beta_m + \beta_n)(\beta_m + r_1 + r_0)(\beta_n + r_1 + r_0)(\beta_m + \beta_n + r_1 + r_0)}(c, b_m)(c, b_n).$$

From this, we can readily compute the covariance of (Y_0, b_n) and (Y_0, b_m). Other joint statistics of (Y_0, b_n), (Y_0, b_m), (Y_1, b_n), and (Y_1, b_m) are found in analogous ways.

4.4.3 L^2-variance

To further illustrate the statistics that we can compute when the PDE switches between two Dirichlet boundary conditions, we now calculate the L^2-variance at large time. By Corollary 1,

$$E\|\bar{u}\|^2 = pE\|Y_1\|^2 + (1 - p)E\|Y_0\|^2.$$

As above, denote the orthonormal set of eigenvectors of B by $\{b_k\}_{k=1}^{\infty}$ and corresponding eigenvalues by $\{-\beta_k\}_{k=1}^{\infty}$. It follows from Theorem 6 that

$$E(b_k, Y_0)^2 = E(b_k, e^{B\tau_0}(e^{A\tau_1}Y_0 - c) + c)^2.$$

This simplifies to

$$\mathbb{E}(b_k, Y_0)^2 = \frac{(r_1 + \beta_k)(r_1 + 2\beta_k)}{(r_0 + r_1 + \beta_k)(r_0 + r_1 + 2\beta_k)}(b_k, c)^2, \tag{4.9}$$

since $A = B$ and $Ee^{-\beta\tau} = \frac{r}{r+\beta}$ if $\beta > 0$ and τ is exponentially distributed with rate r. Similarly,

$$\mathbb{E}(b_k, Y_1)^2 = \frac{r_1(r_1 + \beta_k)}{(r_0 + r_1 + \beta_k)(r_0 + r_1 + 2\beta_k)}(b_k, c)^2. \tag{4.10}$$

Note that Equations (4.9) and (4.10) could also be obtained from Equation 4.8. Adding Equations (4.9) and (4.10) and summing over k yields

$$E\|\bar{u}\|^2 = \sum_{k=1}^{\infty} \frac{r_1(r_1+\beta_k)}{(r_0+r_1)(r_0+r_1+\beta_k)}(b_k, c)^2$$

Now $\beta_k = D(k\pi/L)^2$, $b_k(x) = \sqrt{2/L}\sin(xk\pi/L)$, and $(b_k, \frac{b}{L}x)^2 = (-1)^{k+1}\frac{b\sqrt{2L}}{k\pi}$. Hence,

$$\begin{aligned} E\|\bar{u}\|^2 &= \frac{2Lb^2 r_1}{\pi^2(r_0+r_1)} \sum_{k=1}^{\infty} \frac{r_1 + D(k\pi/L)^2}{(r_0+r_1+D(k\pi/L)^2)k^2} \\ &= \frac{b^2 r_1[L^2 r_1(r_0+r_1) + 3Dr_0(\gamma\coth(\gamma)-1)]}{3L(r_0+r_1)^3} \end{aligned}$$

where $\gamma = L\sqrt{r_0 + r_1/D}$. Since $E\bar{u} = (1-p)\frac{b}{L}x$, we conclude

$$\begin{aligned} E\|\bar{u} - E\bar{u}\|^2 &= E\|\bar{u}\|^2 - \frac{L}{3}\left(\frac{br_1}{r_0+r_1}\right)^2 \\ &= \frac{b^2 D r_1 r_0(\gamma\coth(\gamma)-1)}{L(r_0+r_1)^3}. \end{aligned}$$

4.5 Dirichlet/Neumann switching

Again we consider the stochastic process that solves

$$\partial_t u = D\Delta u \quad \text{in } (0, L) \tag{4.11}$$

and at exponentially distributed times switches between two sets of boundary conditions. But now suppose the boundary conditions switch between

$$\begin{aligned} u(0,t) &= 0 \\ u_x(L,t) &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} u(0,t) &= 0 \\ u(L,t) &= b > 0. \end{aligned}$$

Again we cast this problem in the setting of previous sections, setting our Hilbert space to be $L^2[0, L]$. We define B, c, μ_0, and μ_1 as above in Section 4.4 case, but now we define

$$D(A) := \left\{ \phi \in H^2(0,L) : \frac{\partial \phi}{\partial \mathbf{n}}(L) = 0 = \phi(0) \right\}$$

$$Au := \Delta u \quad \text{if } u \in D(A).$$

Let $u(t,\omega)$ be the H-valued process defined in Equation (3.9) of Chapter 3 with $\Phi_t^1(f) = e^{At}f$ and $\Phi_t^0(f) = e^{Bt}(f-c)+c$. Denote the orthonormal set of eigenvectors of A by $\{a_k\}_{k=1}^{\infty}$ and corresponding eigenvalues by $\{-\alpha_k\}_{k=1}^{\infty}$.

We are interested in studying the large time distribution of $u(t)$. By Theorem 8, we have that $u(t)$ converges in distribution as $t \to \infty$. Let $\bar{u}$ have this limiting distribution. By the definition of Y_0 and Y_1, it is immediate that $\bar{u}$ is almost surely smooth, and using Theorem 7, it is immediate that for each $x \in [0,L]$, $\bar{u}(x) \leqslant \frac{b}{L}x$.

4.5.1 Expectation

We will need the following Lemma which is exactly analogous to Lemma 13.

Lemma 14. *Under the assumptions of the current section, we have that*

$$\|u(t)\| \leqslant L\bigg(\max\{\|u_0\|_\infty, b\} \bigg)^2,$$

where $\|\cdot\|_\infty$ *denotes the* $L^\infty[0,L]$ *norm. Furthermore,*

$$\|Y_1\|_\infty \leqslant b \quad \textit{and} \quad \|Y_0\|_\infty \leqslant b \quad \textit{almost surely.}$$

Proof. The proof is the same as the proof of Lemma 13. □

As in Section 4.4, we have that

$$(E\bar{u})(x) = sx + d = pEY_1 + (1-p)EY_0,$$

where $p = r_0/(r_0 + r_1)$. Since $Y_1(x)$ and $Y_0(x)$ are each almost surely continuous functions of $x \in [0,L]$ with $Y_1(0) = 0 = Y_0(0)$ almost surely, $d = 0$ by the same argument that we usedin Section 4.4.

We now find the slope s. Since $\sum_{k=1}^{n}(a_k, EY_1)a_k$ converges to EY_1 in $L^2[0,L]$ as $n \to \infty$, we have that for any $\phi \in C_0^\infty(0,L)$

$$(\phi, sx) = (\phi, pEY_1) + (1-p)(\phi, EY_0) \tag{4.12}$$

$$= p\left(\phi, \sum_{k=1}^{\infty}(a_k, EY_1)a_k\right) + (1-p)(\phi, EY_0). \tag{4.13}$$

We will need the following Proposition which is an immediate corollary of Theorem 6 and Corollary 1.

Proposition 1. *Under the assumptions of Section 4.5, we have that for each $k \in \mathbb{N}$*

$$E[e^{-\alpha_k \tau_1}](a_k, EY_0) = (a_k, EY_1).$$

This Proposition combined with $sx = pEY_1 + (1-p)EY_0$ yields

$$s(a_k, x) = p(a_k, EY_1) + (1-p)\frac{(a_k, EY_1)}{E[e^{-\alpha_k \tau_1}]}.$$

Rearranging terms gives

$$(a_k, EY_1) = E[e^{-\alpha_k \tau_1}]\frac{s(a_k, x)}{pE[e^{-\alpha_k \tau_1}] + (1-p)}.$$

Plugging this into Equation (4.13) gives

$$(\phi, sx) = p\left(\phi\ ,\ \sum_{k=1}^{\infty} E[e^{-\alpha_k \tau_1}]\frac{s(a_k, x)}{pE[e^{-\alpha_k \tau_1}] + (1-p)}a_k\right) + (1-p)(\phi, EY_0)$$

Solving for s, we find that

$$s = (1-p)(\phi, EY_0)\left((\phi, x) - p\left(\phi\ ,\ \sum_{k=1}^{\infty} E[e^{-\alpha_k \tau_1}]\frac{(a_k, x)}{pE[e^{-\alpha_k \tau_1}] + (1-p)}a_k\right)\right)^{-1}$$

Let $\{\phi_n\}_{n=1}^{\infty} \in C_0^\infty(0,L)$ be such that $\|\phi_n\|_{L^1} = 1$ for each n and $\lim_{n\to\infty}(\phi_n, f) = f(L)$ for each $f \in C[0,L]$. Observe that $\lim_{n\to\infty}(\phi_n, x) = L$ and using Lemma 14 and the same argument as in Section 4.4, we have that

$$\lim_{n\to\infty}(\phi_n, EY_0) = b.$$

Now, we want to show that

$$\lim_{n\to\infty}\left(\phi_n\ ,\ \sum_{k=1}^{\infty}E[e^{-\alpha_k\tau_1}]\frac{(a_k,x)}{pE[e^{-\alpha_k\tau_1}]+(1-p)}a_k\right) = \sum_{k=1}^{\infty}E[e^{-\alpha_k\tau_1}]\frac{(a_k,x)}{pE[e^{-\alpha_k\tau_1}]+(1-p)}a_k(L). \tag{4.14}$$

To do this, we need to show that $\sum_{k=1}^{\infty}E[e^{-\alpha_k\tau_1}]\frac{(a_k,x)}{pE[e^{-\alpha_k\tau_1}]+(1-p)}a_k(x)$ is a continuous function of x. To show this, we need to show that the series converges uniformly in x. This is easily obtained. Note that $\alpha_k = \frac{D(2k-1)^2\pi^2}{4L^2}$ and $a_k(x) = \sqrt{\frac{2}{L}}\sin\left(\sqrt{\frac{\alpha_k}{D}}x\right) = \sqrt{\frac{2}{L}}\sin\left(\frac{(2k-1)\pi x}{2L}\right)$. Hence, $E[e^{-\alpha_k\tau_1}] = \frac{r_1}{r_1+\alpha_k} \leqslant 1$ and $pEe^{-\alpha_k\tau_1}+(1-p) \geqslant 1-p$. Furthermore, $\|a_k\|_\infty \leqslant \sqrt{\frac{2}{L}}$ and

$$(a_k,x) = \int_0^L a_k(x)x\,dx = \frac{4\sqrt{2}L^{3/2}}{\pi^2}\frac{(-1)^{k+1}}{(2k-1)^2} = \frac{\sqrt{\frac{2}{L}}D(-1)^{k+1}}{\alpha_k}.$$

So for any $N\in\mathbb{N}$

$$\left\|\sum_{k=N}^{\infty}E[e^{-\alpha_k\tau_1}]\frac{(a_k,x)}{pE[e^{-\alpha_k\tau_1}]+(1-p)}a_k(x)\right\|_\infty \leqslant \sum_{k=N}^{\infty}|(a_k,x)|\,||a_k(x)||_\infty = \sum_{k=N}^{\infty}\frac{16L}{\pi^2(2k-1)^2}\to 0 \quad \text{as } N\to\infty.$$

Hence, (4.14) is verified.

Thus,

$$s = \frac{(1-p)b}{L-p\sum_{k=1}^{\infty}E[e^{-\alpha_k\tau_1}]\frac{(a_k,x)}{pEe^{-\alpha_k\tau_1}+(1-p)}a_k(L)}.$$

Using the assumptions on μ_0, μ_1, α_k and a_k, this simplifies to

$$s = \frac{b}{L}\left(1+\frac{\rho}{\gamma}\tanh(\gamma)\right)^{-1} \tag{4.15}$$

where $\gamma = L\sqrt{(r_0+r_1)/D}$ and $\rho = r_0/r_1$.

4.6 Comparison

In both the Dirichlet/Dirichlet and Dirichlet/Neumann switching considered above, the large time expectation is a linear function. The slope of this function is very simple expression in the Dirichlet/Dirichlet case. It depends only on the proportion of time the system has a particular boundary condition.

However in the Dirichlet/Neumann case, the slope of the function also depends on the rate of switching between the two boundary conditions. Observe that if we keep the ratio r_0/r_1 fixed, and let the overall switching rate $r_0 + r_1$ go to 0, then the slope for the Dirichlet/Neumann case in Equation (4.15) approaches the same slope as in the Dirichlet/Dirichlet case, namely $\left(\frac{r_1}{r_0+r_1}\right)\frac{b}{L}$. And if we keep the ratio r_0/r_1 fixed, and let the overall switching rate $r_0 + r_1$ go to infinity, then the slope for the Dirichlet/Neumann case in Equation (4.15) approaches $\frac{b}{L}$. The biological implications of this result are discussed in the following section.

4.7 Application to insect physiology

Essentially all insects breathe via a network of tubes that allows oxygen and carbon dioxide to diffuse to and from their cells [30]. Air enters and exits this network through valve-like holes (called spiracles) in the exoskeleton. These spiracles regulate air flow by opening and closing. Surprisingly, spiracles have three distinct phases of activity, each typically lasting for hours. There is a completely closed phase, a completely open phase, and a flutter phase in which the spiracles rapidly open and close [24].

Insect physiologists have proposed at least five major hypotheses to explain the purpose of this behavior [11]. In order to address these competing hypotheses, physiologists would like to understand how much cellular oxygen uptake decreases as a result of the spiracles closing.

To answer this question, we consider the following model. We represent a tube by the interval $[0, L]$ and model the oxygen concentration at a point $x \in [0, L]$ at time t by the function $u(x, t)$. As diffusion is the primary mechanism for oxygen movement in the tubes (see [26]), the function u satisfies the heat equation with some diffusion coefficient D. We impose an absorbing boundary condition at the left endpoint of the interval to represent cellular oxygen absorption where the tube meets the insect tissue. The right endpoint represents the spiracle, and since the spiracle opens and closes, the boundary condition here switches between a no flux boundary condition, $u_x(L, t) = 0$ (spiracle closed) and a Dirichlet boundary condition, $u(L, t) = b > 0$ (spiracle open). We suppose that the spiracle switches from open to closed and from closed to open with exponential rates r_0 and r_1 respectively.

Then, the oxygen concentration $u(x, t)$ is the same process described above in Secion 4.5. Using the results from that section, if we let $\rho = r_0/r_1$ and $\gamma = L\sqrt{(r_0 + r_1)/D}$, then the expected oxygen flux to the cells at large time is given by

$$\frac{bD}{L}\left(1 + \frac{\rho}{\gamma}\tanh(\gamma)\right)^{-1}.$$

This formula is noteworthy because it shows that the cellular oxygen uptake not only depends on the average proportion of time the spiracle is open, but it also depends on the overall rate of opening and closing. In particular, note that if we keep the ratio ρ fixed, but let γ become large, then the oxygen uptake approaches $\frac{bD}{L}$.

In biological terms, the insect can have its spiracles open an arbitrarily small proportion of time, and yet receive essentially just as much oxygen as if its spiracles were always open.

More work needs to be done to make this model more applicable to the actual biological problem. In addition to oxygen flux, there are other variables that are affected by the opening and closing of spiracles. For example, the carbon dioxide concentration in the tubes and the amount of water loss due to evaporation are affected by the opening and closing of spiracles. Future work would incorporate these variables into the model. In addition, this model assumed that the tubes do not branch. Hence, future work would extend this model to the consider branching tubes.

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Biography

Sean David Lawley was born in Cocoa Beach, Florida on June 27, 1987. He received his B.S. with University Honors in Computational Finance from Carnegie Mellon University in May 2009. In May 2011, Sean earned his M.A. in Mathematics from Duke University. In May 2014, Sean earned his Ph.D. in Mathematics from Duke University.

Sean joined the Phi Beta Kappa Honor Society and the Phi Kappa Phi Honor Society in 2009. He was awarded the L.P. and Barbara Smith Award for Teaching Excellence from the Duke University Mathematics Department in 2013.

Sean was supported by a National Science Foundation Research Training Grant from 2010-2014 and by a grant from the Howard Hughes Medical Institute during the summers of 2010 and 2013.

Sean has authored articles in stochastic dynamical systems ([22], [21]) and pharmacokinetic modeling applied to arsenic detoxification ([20], [23]).

In July 2014, Sean will begin a three-year Assistant Professor Lecturer position in the Mathematics Department at the University of Utah. This position will be partially supported by a National Science Foundation Mathematical Biology Research Training Grant.