From vortices to instantons on the Euclidean Schwarzschild manifold

ÁKOS NAGY AND GONÇALO OLIVEIRA

ABSTRACT. The first irreducible solution of the $\mathrm{SU}(2)$ self-duality equations on the Euclidean Schwarzschild (ES) manifold was found by Charap and Duff in 1977, only 2 years later than the famous BPST instantons on $\mathbb{R}^4$ were discovered. While soon after, in 1978, the ADHM construction gave a complete description of the moduli spaces of instantons on $\mathbb{R}^4$, the case of the Euclidean Schwarzschild manifold has resisted many efforts for the past 40 years.

By exploring a correspondence between the planar Abelian vortices and spherically symmetric instantons on ES, we obtain: a complete description of a connected component of the moduli space of unit energy $\mathrm{SU}(2)$ instantons; new examples of instantons with non-integer energy (and non-trivial holonomy at infinity); a complete classification of finite energy, spherically symmetric, $\mathrm{SU}(2)$ instantons.

As opposed to the previously known solutions, the generic instanton coming from our construction is not invariant under the full isometry group, in particular not static. Hence disproving a conjecture of Tekin.

1. INTRODUCTION

Motivation and Background. Let G be a compact semisimple Lie group and P a principal G-bundle over an oriented, Riemannian 4-manifold (X^4, g_X). The Hodge-$*$ operator associated to the metric g_X, acting on 2-forms satisfies $*^2 = 1$. Hence the bundle of 2-forms splits into ± 1 eigenspaces for $*$ as $\Lambda^2_X \cong \Lambda^+_X \oplus \Lambda^-_X$. Given a connection D on P, its curvature $\mathcal{F}_D$ is a 2-form with values in the adjoint bundle $\mathfrak{g}_P$, that is a section of $\Lambda^2_X \otimes \mathfrak{g}_P$. Hence, it decomposes as $\mathcal{F}_D = \mathcal{F}_D^+ + \mathcal{F}_D^-$, with $\mathcal{F}_D^\pm$ taking values in $\Lambda^\pm_X \otimes \mathfrak{g}_P$. A connection D is called *self-dual* if $\mathcal{F}_D^- = 0$, or equivalently

$$* \mathcal{F}_D = \mathcal{F}_D.$$

Date: November 15, 2017.
2010 *Mathematics Subject Classification.* 53C07,58D27,70S15,83C57.
Key words and phrases. Euclidean Schwarzschild manifold, instantons, vortices, Kazdan–Warner equation.

Instantons are self-dual connections with finite Yang–Mills energy

$$E_{\mathrm{YM}}(D) = \frac{1}{8\pi^2} \int_X |\mathcal{F}_D|^2 \, \mathrm{dvol}_X,$$

where $|\cdot|$ is computed using the Riemannian metric on $\Lambda_X^{\pm}$ and an Ad-invariant inner product on $\mathfrak{g}_P$. Instantons have since long been of great interest to physicists and they have also been extensively studied by mathematicians for more than 4 decades now [3].

When the underlying manifold X is closed, instantons have quantized energy, determined by characteristic classes. In fact, instantons are global minimizers for E_{YM} on a fixed principal G-bundle. Their moduli spaces are finite dimensional and non-compact with well-understood compactifications; cf. [33, 35].

When X is non-compact, the theory is much less understood. On *asymptotically locally Euclidean* (ALE) *gravitational instantons*, the energy is also quantized. For example, in the case of the Euclidean $\mathbb{R}^4$, the energy is, roughly speaking, classified $\pi_3(\mathrm{SU}(2)) \cong \mathbb{Z}$, and can be thought of as a winding number determined by the instanton on the "sphere at infinity". For the general ALE gravitational instanton, the instanton moduli space can also be described via the generalized ADHM construction; cf. [1, 24]. In particular, the moduli spaces are finite dimensional, non-compact, complete, and hyper-Kähler.

Another important class of Ricci-flat, non-compact, but complete manifolds are the *asymptotically locally flat* (ALF) spaces. In comparison to ALE manifolds, which have quartic volume growth, an ALF manifold has cubic volume growth, being asymptotic to circle bundle over a 3-dimensional cone and constant length fibres. When these spaces are also gravitational instantons, their instanton moduli spaces are usually studied using tools from hyper-Kähler geometry. The construction of such instanton moduli spaces on ALF gravitational instantons is an active area of research and significant progress has been made recently using Cherkis' bow construction [12, 13]. One further remark is that in this case the energy need not be quantized, as exemplified for example in [14, Section 4].

However, there are important ALF manifolds, which are not hyper-Kähler. Probably the most well-known examples are "Wick rotated black hole metrics", the *Euclidean Schwarzschild* (ES) manifold, and its generalizations (e.g. the Euclidean Kerr manifold). These spaces first appeared in the work of Hawking on black hole thermodynamics in Euclidean quantum gravity [21]. The ES and Kerr metric on $\mathbb{R}^2 \times S^2$ are some of the few known examples of Ricci flat, complete, but non-compact 4-manifolds which are not hyper-Kähler; cf. [10, 11][1]. However, Yang–Mills theory on the ES manifold has remained quite unexplored.

In [9], Charap and Duff used a fully isometric ansatz to produce two types of instantons. The first is a single, irreducible, and unit energy solution, which can be described as the spin connection on the anti-chiral spinor bundle of the ES manifold. Thus it is a particular instance of a result due to Atiyah, Hitchin, and Singer on Einstein manifolds [2, Proposition 2.2]. The second type of solutions found by Charap and Duff, turned out to be reducible to $\mathrm{U}(1) \cong \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1)) \subset \mathrm{SU}(2)$, as shown by Etesi and Hausel [15]. These form a family indexed by an integer, n, have energy $2n^2$, and give rise to all $\mathrm{U}(1)$ instantons on the ES manifold. In other words, these are the only self-dual and finite energy solutions of the Euclidean Maxwell equation.

In [16], making use of Gromov–Lawson relative index theorem and the Hausel–Hunsicker–Mazzeo compactification of ALF spaces (cf. [20]), Etesi and Jardim computed the dimension of moduli spaces of certain instantons on ALF spaces. These instantons are required to have strictly faster than quadratic curvature decay on the ES manifold. This condition guarantees integer energy for the instantons by [14]. In particular, the moduli space of unit energy instantons on the ES manifold is 2-dimensional, and non-empty, due to the example Charap and Duff.

In [25], Mosna and Tavares showed, using numerical simulations, that unlike the in compact or ALE cases, the ES manifold carries irreducible $\mathrm{SU}(2)$ instantons of non-integer energy. Their work (numerically) shows the existence of an $\mathrm{SU}(2)$ instanton for each energy value $E \in [1, 2]$, interpolating from the irreducible Charap–Duff instanton ($E = 1$), to the (unique) reducible instanton of energy $E = 2$. These instantons and one other family with finite energy were also studied in [6, 28]. All of these instantons are completely invariant under the action of the group of orientation preserving isometries, $\mathrm{Isom}_+(X, g_X) \cong \mathrm{S}(\mathrm{O}(2) \times \mathrm{O}(3))$, so they are both static and spherically symmetric.

Despite many efforts, e.g. [6, 14, 16, 25, 32], still, very little is known about irreducible, non-Abelian instantons on the ES manifold.

Main Results. First we recast a correspondence between spherically symmetric $\mathrm{SU}(2)$ instantons and planar Abelian vortices. This method was first introduced by Witten in [36], and subsequently studied by Taubes and Garcia-Prada [17, 31]. However, they worked with different

[1]We thank Hans Joachim Hein for bringing these references to our attention.

geometries and with somewhat different applications in mind. In this paper, we prove the following results on the ES manifold:

- A component of the unit energy Etesi–Jardim moduli space is canonically diffeomorphic to $\mathbb{C} \cong \mathbb{R}^2$. The Charap–Duff instanton being the origin of $\mathbb{C}$.
- Any irreducible, spherically symmetric $\mathrm{SU}(2)$ instantons must have energy $E \in (0, 2)$. We give a complete classification of these instantons and their moduli spaces. These moduli spaces are canonically homeomorphic to each other for energies in either $(0, 1)$ or $[1, 2)$.
- Uhlenbeck compactness: For each energy $E \in (0, 2)$, the moduli space of energy $\mathcal{E}$, irreducible, spherically symmetric $\mathrm{SU}(2)$ instantons is compact in the weak L^2_1-topology.

  In particular, this is true for the full moduli space of unit energy instantons. Thus, in that case the compactification is homeomorphic to $S^2 \cong \mathbb{C} \cup \{\infty\}$, hence closed. We also remark here that the holonomy around the "circle at infinity" in the imaginary time direction is $\mathrm{diag}(\exp(2\pi E i), \exp(-2\pi E i)) \in \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1)) \subset \mathrm{SU}(2)$, hence trivial for integer energies.

While these instantons are all spherically symmetric, the generic one is not static, thus not completely invariant under the full group of orientation preserving isometries, $\mathrm{Isom}_+(X, g_X) \cong \mathrm{S}(\mathrm{O}(2) \times \mathrm{O}(3))$. In particular, our analysis recovers all previously known instantons on the ES manifold (the Mosna–Tavares/Radu–Tchrakian–Yang solutions), but also another family of fully isometric, irreducible solutions, which are indexed by energy values $E \in (0, 2)$. These solutions have exactly quadratic curvature decay, and the unit energy one corresponds to the Uhlenbeck compactification point of the Etesi–Jardim moduli space, as described above. We summarize these results in the Main Theorem.

For the rest of the paper, let $\mathcal{M}_{X,E}^{\mathrm{SO}(3)}$ be the moduli space of spherically symmetric $\mathrm{SU}(2)$ instantons of energy $\mathcal{E}$ on the ES manifold, X, and $(\mathcal{M}_{X,E}^{\mathrm{SO}(3)})^* \subseteq \mathcal{M}_{X,E}^{\mathrm{SO}(3)}$ be the subspace of irreducible instantons.

Main Theorem. *Irreducible, spherically symmetric* $\mathrm{SU}(2)$ *instantons on the ES manifold must have energy* $E \in (0, 2)$*, and for these energy values we have*

$$(\mathcal{M}_{X,E}^{\mathrm{SO}(3)})^* = \mathcal{M}_{X,E}^{\mathrm{SO}(3)}.$$

Furthermore, we have following descriptions these moduli spaces:

(1) *For each* $E \in (0,1)$*,* $\mathcal{M}_{X,E}^{\mathrm{SO}(3)}$ *has 1 element, which is invariant under the action of* $\mathrm{Isom}_+(X, g_X)$.

(2) *For each* $E \in [1,2)$*, the moduli spaces are canonically homeomorphic to* $S^2 = \mathbb{C} \cup \{\infty\}$ *in the weak* L_1^2*-topology. The action of* $\mathrm{Isom}_+(X, g_X)$ *on* $\mathcal{M}_{X,E}^{\mathrm{SO}(3)}$ *has exactly two fixed points,* $0, \infty \in S^2$.

Moreover, instantons with unit energy corresponding to $\mathbb{C} \subset \mathcal{M}_{X,1}^{\mathrm{SO}(3)}$ *have exactly cubic curvature decay, all other irreducible, spherically symmetric* $\mathrm{SU}(2)$ *have exactly quadratic curvature decay.*

In the theorems above, and for $E \in [1,2)$, we mention the existence of a canonical homeomorphism (in the weak L_1^2-topology)

$$c : \mathcal{M}_{X,E}^{\mathrm{SO}(3)} \to \mathbb{C} \cup \{\infty\} = S^2,$$

and it is defined as follows: To each spherically symmetric instanton we associate a vortex field (∇_D, Φ_D) on $\Sigma \cong \mathbb{C}$ equipped with a certain complete, finite volume metric. The only gauge invariant "quantity" one vortex fields have are divisors of the Φ fields. In our case, a generalized version of Bradlow's condition [5, Equation 4.10] implies that divisor has to have either degree 0, or 1. Then the map c simply sends the instanton D to ∞, when the degree is zero, and to the zero of Φ_D in $\Sigma \cong \mathbb{C}$, when the degree is 1.

Remark 1.1. (1) *As mentioned in the previous discussion, for* $E \in [1,2)$ *the two parameters of the moduli correspond, via c, to the "location" of the vortex* (∇_D, Φ_D) *associated with an instanton* D*. One of the coordinates corresponds to the "imaginary time" location. In contrast to the case of the BPST instantons on* $\mathbb{R}^4$*, the other coordinate is not a "concentration" scale for bubbling. Intuitively speaking bubbling for instantons would happen at points, but since all the instantons we construct are spherically symmetric and there is no* $\mathrm{SO}(3)$*-orbit in the ES manifold which is a point, there can be no bubbling.*

(2) *In* [16] *it was proven that the moduli space* $\mathcal{M}_{X,1}$ *of unit energy* $\mathrm{SU}(2)$ *instantons on the Euclidean Schwarzschild manifold is* 2*-dimensional. As a consequence, our main theorem above shows that a connected component of* $\mathcal{M}_{X,1}$ *is canonically homeomorphic to* $S^2 = \mathbb{C} \cup \{\infty\}$.

Connections to Physics. In this short section we lay out how our work ties into modern Theoretical Particle Physics.

The existence of instantons on $\mathbb{R}^4$ has deep consequences for quantum Yang–Mills theory on Minkowski space [8, 23, 30]. Indeed it implies that the vacuum is not unique, and instantons can be interpreted as fields configurations that tunnel between the different vacua.

Such an interpretation on more general manifolds requires the instantons to not be static. A theorem of Birkhoff states that a complete, spherically symmetric solution of the vacuum Einstein equations must be isometric to the Schwarzschild spacetime, and thus, static; cf. [4]. More generally, a spherically symmetric solution of the coupled Einstein–Maxwell equations must be equivalent to the Reissner–Nordström electro-vacuum. These result stay true in the Euclidean setting as well. A very broad generalization of Birkhoff's theorem (to arbitrary dimensions and signatures) can be found in [7]. Since a connection is an instanton if and only if it has vanishing stress-energy tensor, this implies that the only Ricci flat, complete, and spherically symmetric solution of the Euclidean Einstein–Maxwell equations must be one of the reducible Charap–Duff instantons, or anti-instantons on the ES manifold.

These results, together with fact that the Charap–Duff instantons are static and the difficulty in finding further examples of instantons led to the conjecture that such non-static instantons did not exist [32].

In the light of the above observations, our results can be interpreted as the existence of tunneling effects in the quantum Einstein–Yang–Mills theory (at finite temperature), and the invalidity of Birkhoff's theorem in the non-Abelian setting. This disproves a conjecture of Tekin [32].

Organization of the paper. In Section 2 we recast the correspondence between spherically symmetric instantons on $\Sigma \times S^2$ and planar Abelian vortices on Σ following mainly Garcia-Prada [18]. In Section 3 we develop the required technical results on the Kazdan–Warner equation. In Section 4 we introduce the basics of the theory of Abelian vortices on non-compact, complete, finite volume Riemann surfaces. In particular we give an existence theorem for vortices using the results of Section 3. Many of the results in Sections 3 and 4 are developed with a little more generality than what is needed later, with possible further applications in mind. Finally in Section 5 we apply these results to prove our results for instantons on the ES manifold.

Acknowledgement. *The authors are grateful for Gábor Etesi and Mark Stern for helpful discussions. We thank Andy Royston for explaining to us the physical implications of our results and Aleksander Doan for comments on an earlier version of this manuscript. The second author wishes to thank Sergey Cherkis and Marcos Jardim for discussions regarding instantons on ALF manifolds.*

2. From instantons to vortices

In this section, following the ideas of [18, 31, 36], we outline the correspondence between Abelian vortices in dimension 2 and spherically symmetric instantons in dimension 4. This technique has been extensively studied recently for product manifolds $S^2 \times \Sigma$, where Σ is a hyperbolic surface, in [27]. There is an analogous (in fact, dual) correspondence between anti-vortices and anti-instantons.

For the rest of the paper, let Σ be an oriented, complete, finite volume surface with the Riemannian metric g_Σ. The corresponding volume form on Σ is denoted by dvol_Σ. Note that dvol_Σ is also the Kähler form of the metric g_Σ. Similarly let g_{S^2} be the round metric on S^2 of volume 2π and dvol_{S^2} the corresponding volume/Kähler form.

Let $(X, g_X) = (\Sigma \times S^2, g_\Sigma \oplus g_{S^2})$ be the product Kähler manifold, p_Σ and p_{S^2} be the projections to Σ and S^2, respectively. Now we have the canonical inclusion $\mathrm{SO}(3) \hookrightarrow \mathrm{Isom}(S^2, g_{S^2}) \hookrightarrow \mathrm{Isom}(X, g_X)$, and we call $\mathrm{SO}(3)$ invariant objects *spherically symmetric*. Since $\mathrm{SO}(3)$ is connected, the pullback of any smooth bundle over X via an element of $\mathrm{SO}(3)$ is always homotopic, hence isomorphic (gauge equivalent) to the original bundle. Thus we can always talk about connections whose gauge equivalence class is spherically symmetric. By an abuse of notation, we also call such connection spherically symmetric.

The following lemma is a summary of the results of [31, Section V] and [18, Section 3].

Lemma 2.1. *Let D be an spherically symmetric $\mathrm{SU}(2)$ connection on the vector bundle $\mathcal{E}$ over X with $\mathcal{F}_D^{0,2} = 0$. Then D induces a holomorphic structure on $\mathcal{E}$ and there is a holomorphic Hermitian line bundle $\mathcal{L} \to \Sigma$, and a non-negative integer n, such that $\mathcal{E}$ has the orthogonal holomorphic decomposition*

$$\mathcal{E} \cong (p_\Sigma^*(\mathcal{L}) \otimes p_{S^2}^*(\mathcal{O}(-n))) \oplus (p_\Sigma^*(\mathcal{L}^{-1}) \otimes p_{S^2}^*(\mathcal{O}(n))). \tag{2.1}$$

Let $\nabla^{\mathcal{L}}$ be the Chern connection of $\mathcal{L}$, and $\nabla^{\mathcal{O}(n)}$ be the Chern connection of $\mathcal{O}(n)$. If $n \neq 1$, then there is a gauge in which D has the form

$$D = \begin{pmatrix} p_\Sigma^*(\nabla^{\mathcal{L}}) \otimes p_{S^2}^*(\nabla^{\mathcal{O}(-n)}) & 0 \\ 0 & p_\Sigma^*(\nabla^{L^{-1}}) \otimes p_{S^2}^*(\nabla^{\mathcal{O}(n)}) \end{pmatrix}.$$

Let α be a unit length, spherically symmetric element of $\mathcal{O}(-2)\otimes\Lambda^{0,1}_{S^2}\cong\underline{\mathbb{C}}$ on S^2. If $n=1$, then there is a smooth section of L^2, φ, and a gauge in which D has the form

$$D=\begin{pmatrix} p_\Sigma^*(\nabla^{\mathcal{L}})\otimes p_{S^2}^*(\nabla^{\mathcal{O}(-1)}) & p_\Sigma^*(\varphi)\otimes p_{S^2}^*(\alpha) \\ -(p_\Sigma^*(\varphi)\otimes p_{S^2}^*(\alpha))^* & p_\Sigma^*(\nabla^{L^{-1}})\otimes p_{S^2}^*(\nabla^{\mathcal{O}(1)}) \end{pmatrix}. \tag{2.2}$$

Recall that the (conformally invariant) Yang–Mills energy of an $\mathrm{SU}(2)$ connection D on X is

$$E_{\mathrm{YM}}(D)=\frac{1}{8\pi^2}\int_X |\mathcal{F}_D|^2\,\mathrm{dvol}_X\,.$$

Similarly, for each $\tau>0$, the Yang–Mills–Higgs energy of a pair, (∇,Φ), of a connection, ∇, and a section Φ on the same Hermitian line bundle over Σ is

$$E_{\mathrm{YMH}}(\nabla,\Phi)=\frac{1}{2\pi}\int_\Sigma (|F_\nabla|^2+|\nabla\Phi|^2+\frac{1}{4}(\tau-|\Phi|^2))\,\mathrm{dvol}_\Sigma\,. \tag{2.3}$$

As shown by Bradlow in [5], for a vortex field (∇,Φ) on a compact Riemann surface Σ the energy (2.3) can be written as

$$E_{\mathrm{YMH}}(\nabla,\Phi)=\frac{i\tau}{2\pi}\int_\Sigma F_\nabla. \tag{2.4}$$

In fact the same formula holds when Σ is complete and of finite volume, as we show in Proposition 4.1. The gauge invariant quantity

$$\deg(\nabla)=\frac{i}{2\pi}\int_\Sigma F_\nabla,$$

is the degree of the connection ∇.

Remark 1. *In Lemma 3.5 we show that the degree is invariant under smooth, bounded, and complex (thus not necessarily unitary) gauge transformations. Note that when Σ is non-compact the degree need not be an integer.*

In the next theorem the bundle $\mathcal{E}$ is assumed to have the form (2.1).

Theorem 2.2. *If D is an spherically symmetric $\mathrm{SU}(2)$ instanton, then the following hold:*

- *D is flat if and only if $n=0$. In this case, the Yang–Mills energy of D is zero.*
- *D is reducible with holonomy $\mathrm{S}(\mathrm{U}(1)\oplus\mathrm{U}(1))$, if and only if $n\geqslant 1$ and $\Phi=0$. In this case $\nabla^{\mathcal{L}}$ is a Hermitian–Yang–Mills connection on $\mathcal{L}$ with constant n and the Yang–Mills energy of D is $2n^2$.*

- *If D is irreducible, then $n = 1$. In this case, if ∇ is the Hermitian connection induced by $\nabla^{\mathcal{L}}$ on L^2 and $\Phi = 2\varphi$, then (∇, Φ) is a vortex field with $\tau = 4$ and the Yang–Mills energy of D is the degree of ∇.*

Proof. If $n = 0$, then (by omitting the pullbacks) the curvature of D has the form

$$\mathcal{F}_D = \begin{pmatrix} F_{\nabla^{\mathcal{L}}} & 0 \\ 0 & -F_{\nabla^{\mathcal{L}}} \end{pmatrix}.$$

If D is an instanton, then $\nabla^{\mathcal{L}}$ is flat, and hence so is D. The Yang–Mills energy of a flat connection is zero, which proves the first part of the theorem.

If $n > 1$, or $n = 1$ and $\varphi = 0$, then D is clearly reducible to (at least) $\mathrm{S}(\mathrm{U}(1) \oplus \mathrm{U}(1))$. Let Λ be the contraction of 2-forms on Σ with dvol_Σ. If D is an instanton, then straightforward computation, using that $F_{\nabla^{\mathcal{O}(n)}} = -in\,\mathrm{dvol}_{S^2}$, shows that $i\Lambda F_{\nabla^{\mathcal{L}}} = n$. The computation of the Yang–Mills energy is also straightforward, which proves the second part of the theorem.

When $n = 1$, let ∇ be the connection induced by $\nabla^{\mathcal{L}}$ on L^2. The curvature of D has the form

$$\mathcal{F}_D = \begin{pmatrix} F_{\nabla^{\mathcal{L}}} + i(1 - |\varphi|^2)\,\mathrm{dvol}_{S^2} & \nabla\varphi \wedge \alpha \\ -(\nabla\varphi \wedge \alpha)^* & -F_{\nabla^{\mathcal{L}}} - i(1 - |\varphi|^2)\,\mathrm{dvol}_{S^2} \end{pmatrix}. \tag{2.5}$$

Now D is an instanton if and only if the following equations hold

$$iF_{\nabla^{\mathcal{L}}} = (1 - |\varphi|^2)\,\mathrm{dvol}_\Sigma, \tag{2.6a}$$

$$(\nabla - i *_\Sigma \nabla)\varphi = 0, \tag{2.6b}$$

Since $\nabla - i *_\Sigma \nabla = \nabla^{0,1} = \overline{\partial}_\nabla$, equation (2.6b) is equivalent to φ being holomorphic with respect to the holomorphic structure induced by ∇. Note that every connection on $\mathcal{L}^2$ is uniquely induced by one on $\mathcal{L}$, and $F_\nabla = 2F_{\nabla^{\mathcal{L}}}$. Let $\Phi = 2\varphi$ and Λ be the contraction of 2-forms with dvol_Σ. Now equations (2.6a) and (2.6b) can be rewritten as

$$2i\Lambda F_\nabla = 4 - |\Phi|^2, \tag{2.7a}$$

$$\overline{\partial}_\nabla \Phi = 0. \tag{2.7b}$$

The equations (2.7a) and (2.7b) are the vortex equations on $\mathbb{C}$ for $\tau = 4$. Using the spherical symmetry, we get that

$$\begin{aligned} E_{\mathrm{YM}}(D) &= \frac{1}{8\pi^2} \int\limits_{\Sigma\times S^2} \left(2|F_\nabla|^2 + 2|\nabla\Phi|^2 + 2(1-|\Phi|^2)^2\right) \mathrm{dvol}_\Sigma \wedge \mathrm{dvol}_{S^2} \\ &= \frac{1}{2\pi}\frac{1}{4} \int\limits_{\Sigma} \left(|F_\nabla|^2 + |\nabla\Phi|^2 + \frac{1}{4}(4-|\Phi|^2)^2\right) \mathrm{dvol}_\Sigma, \end{aligned} \tag{2.8}$$

and using equation (2.4) for $\tau = 4$, it shows that

$$E_{\mathrm{YM}}(D) = \deg(\nabla). \tag{2.9}$$

□

Remark 2.3. *The instantons in the second bullet of Theorem 2.2 are exactly the Abelian instantons in* [9, 15].

3. The Kazdan–Warner equation

In Section 4 we use the Kazdan–Warner equation to find solutions of the vortex equation. By now, there are several references on the Kazdan–Warner equation. The most related to our applications is [34], albeit the uniqueness claimed in their main theorem is not addressed anywhere in the paper. Hence, and for completeness, we include a proof below of the existence and uniqueness of solutions of the Kazdan–Warner equations, albeit on more restricted class of manifolds than in [34]. Furthermore, our technique provides decay estimates for the solution, an issue that was also not considered in that reference. Many of the existence results of this section are not new and for a great part we follow the methods of [22]. The main result of the section is Theorem 3.1. We also prove Lemmas 3.5 and 3.6 which we use in other parts of this paper as well.

For the rest of this section let $\overline{\Sigma}$ be a closed surface, and Σ be the complement of a finite set of points in $\overline{\Sigma}$. Let g_Σ be a complete, finite volume metric on Σ that conformally extends over $\overline{\Sigma}$. Fix such a conformal factor, κ. Let Δ be the Laplacian on (Σ, g_Σ), an $\overline{\Delta}$ be the Laplacian on $(\overline{\Sigma}, g_{\overline{\Sigma}} = \kappa g_\Sigma)$. By an abuse of notation, extensions (and restrictions) of objects are denoted by the same symbols.

Theorem 3.1. *Let g, h be smooth functions on Σ, such that $\int_\Sigma g > 0$, $h \geqslant 0$ and not identically zero, and g, h extends smoothly over $\overline{\Sigma}$. Then there is a smooth and bounded function f on Σ*

which solves the Kazdan–Warner equation

$$\Delta f = g - he^{2f}, \tag{3.1}$$

Moreover, f is unique among bounded functions, smoothly extends over $\overline{\Sigma}$, and there is a constant $C = C(\Sigma, g_\Sigma)$, such that

$$\|f\|_{L^\infty} \leqslant C(\|g\|_{L^\infty} + \|h\|_{L^\infty}).$$

Remark 3.2. *Since the (isolated) points of $\overline{\Sigma} \backslash \Sigma$ are the "ends" of Σ, we conclude the the limit of f along any end exists.*

To prove Theorem 3.1 we need to prove a few smaller results.

Proposition 3.3. *Let g, h be smooth functions on $\overline{\Sigma}$, with $\int_{\overline{\Sigma}} g > 0$ and $h \geqslant \varepsilon > 0$. Then there is a unique function f, solving the Kazdan–Warner equation*

$$\overline{\Delta} f = g - he^{2f}.$$

Proof. The proof follows from finding a sub and supersolution. We start with the construction of a subsolution and consider the function defined by

$$\widehat{g} = g - \frac{1}{\mathrm{Vol}(\overline{\Sigma}, g_{\overline{\Sigma}})} \int\limits_{\overline{\Sigma}} g \, \mathrm{dvol}_{\overline{\Sigma}} .$$

This is obviously bounded and of zero average, hence there is u such that $\overline{\Delta} u = \widehat{g}$. Then we set $f_- = u - c_-$ and find c_- so that this is a subsolution. For that notice that, as u is bounded and $\int_{\overline{\Sigma}} g > 0$, there is a $c_- > 0$ sufficiently big so that $he^{2u}e^{-2c_-} - \frac{1}{\mathrm{Vol}(\overline{\Sigma}, g_{\overline{\Sigma}})} \int_{\overline{\Sigma}} g < 0$. For such a c_- we have

$$\begin{aligned}
\overline{\Delta} f_- &= g - he^{2f_-} + he^{2f_-} - \frac{1}{\mathrm{Vol}(\overline{\Sigma}, g_{\overline{\Sigma}})} \int\limits_{\overline{\Sigma}} g \, \mathrm{dvol}_{\overline{\Sigma}} \\
&= g - he^{2f_-} + he^{2u}e^{-2c_-} - \frac{1}{\mathrm{Vol}(\overline{\Sigma}, g_{\overline{\Sigma}})} \int\limits_{\overline{\Sigma}} g \, \mathrm{dvol}_{\overline{\Sigma}} \\
&\leqslant g - he^{2f_-},
\end{aligned}$$

so that f_- is a subsolution.

We now look for a supersolution of the form $f_+ = u + c_+$. Again, we compute

$$\begin{aligned}\overline{\Delta} f_+ &= g - he^{2f_+} + he^{2f_+} - \frac{1}{\mathrm{Vol}(\overline{\Sigma}, g_{\overline{\Sigma}})} \int\limits_{\overline{\Sigma}} g \,\mathrm{dvol}_{\overline{\Sigma}} \\ &\geqslant g - he^{2f_+} + \varepsilon e^{2u} e^{2c_+} - \frac{1}{\mathrm{Vol}(\overline{\Sigma}, g_{\overline{\Sigma}})} \int\limits_{\overline{\Sigma}} g \,\mathrm{dvol}_{\overline{\Sigma}} \\ &\geqslant g - he^{2f_+}.\end{aligned}$$

by choosing c_+ big enough so that $\varepsilon e^{2u} e^{2c_+} - \frac{1}{\mathrm{Vol}(\overline{\Sigma}, g_{\overline{\Sigma}})} \int_{\overline{\Sigma}} g > 0$.

So under these hypothesis there is a solution of the Kazdan–Warner equation satisfying

$$f_- \leqslant f \leqslant f_+.$$

For the uniqueness suppose one is given to solutions f_1 and f_2. Then setting $F = f_1 - f_2$ gives us

$$\overline{\Delta} F = he^{2f_2}(1 - e^{2F}).$$

Moreover, as $\overline{\Sigma}$ is compact F attains a maximum and a minimum. At the maximum we must have

$$0 \leqslant \overline{\Delta} F = he^{2f_2}(1 - e^{2F}),$$

so that F must be non-positive at the maximum. Similarly we show that F must be non-negative at the minimum, and so we must have $F = 0$ and thus $f_1 = f_2$. □

This can be used as an intermediate result towards the proof of the following stronger version of the existence theorem.

Proposition 3.4. *Let g, h smooth functions on $\overline{\Sigma}$, so that $\int_{\overline{\Sigma}} g > 0$ and $h \geqslant 0$ is not identically zero. Then there is a unique function f, solving the Kazdan–Warner equation*

$$\Delta f = g - he^{2f}. \tag{3.2}$$

Proof. Again, we use the method sub and supersolutions, following [22].

In order to find a subsolution we first modify the equation and solve instead

$$\overline{\Delta} f_- = g - (h + 1)e^{2f_-}.$$

This satisfies the hypothesis of Proposition 3.3 and so admits a solution f_-. To see that such f_- is a subsolution to equation (3.2) simply observe that

$$\overline{\Delta} f_- = g - (h+1)e^{2f_-} = g - he^{2f_-} - e^{2f_-} \leqslant g - he^{2f_-}.$$

Finding a supersolution is slightly more involved. First let $\lambda > 0$ be large enough so that

$$\int_{\overline{\Sigma}} (g - h\lambda)\, \mathrm{dvol}_{\overline{\Sigma}} < 0.$$

This can clearly be made from the hypothesis that $h \geqslant 0$ is not identically. Then we solve the equation

$$\overline{\Delta} u = (g - h\lambda) - \frac{1}{\mathrm{Vol}(\overline{\Sigma}, g_{\overline{\Sigma}})} \int_{\overline{\Sigma}} (g - h\lambda)\, \mathrm{dvol}_{\overline{\Sigma}}.$$

Then we let $f_+ = u + c_+$ and find $c_+ > 0$ so that this is an supersolution.

$$\overline{\Delta} f_+ = g - h\lambda - \frac{1}{\mathrm{Vol}(\overline{\Sigma}, g_{\overline{\Sigma}})} \int_{\overline{\Sigma}} (g - h\lambda)\, \mathrm{dvol}_{\overline{\Sigma}} \geqslant g - he^{2f_+}(\lambda e^{-2u} e^{-2c_+}),$$

and by making $c_+ > 0$ large enough we can make $\lambda e^{-2u} e^{-2c_+} \leqslant 1$, and thus $\overline{\Delta} f_+ \geqslant g - he^{2f_+}$.

The uniqueness follows again by supposing that we have two solutions f_1 and f_2. Then $F = f_1 - f_2$ satisfies

$$\overline{\Delta} F = he^{2f_2}(1 - e^{2F}) \leqslant -2he^{2f_2} F, \tag{3.3}$$

where we used the convexity of the exponential function. Hence

$$(\overline{\Delta} + 2he^{2f_2})F \leqslant 0.$$

The maximum principle yields that any maximum of F must be non-positive. On the other hand integrating equation (3.3) we get

$$0 = \int_{\overline{\Sigma}} he^{2f_2}(1 - e^{2F})\, \mathrm{dvol}_{\overline{\Sigma}},$$

which for a non-positive F requires $F = 0$. □

The following result will be used to show the uniqueness part in Theorem 3.1, and it will also be useful in later sections.

Lemma 3.5. *Let f be a smooth, uniformly bounded function on Σ. Then*

$$\int_\Sigma \Delta f \, \mathrm{dvol}_\Sigma = 0.$$

Proof. Let $p \in \Sigma$, $r > 0$, and define $\chi_r : \Sigma \to \mathbb{R}$ to be a compactly supported function which equals to 1 on the geodesic ball of radius r about p, $B_r(p)$, vanishes in the complement of $B_{2r}(p)$ and there is a constant $c > 0$, such that

$$|\nabla \chi_r| \leqslant c r^{-1}, \qquad |\nabla^2 \chi_r| \leqslant c r^{-2}.$$

Then using χ_r we can write

$$\int_\Sigma \Delta f \, \mathrm{dvol}_\Sigma = \lim_{r\to\infty} \int_\Sigma \chi_r \Delta f \, \mathrm{dvol}_\Sigma = \lim_{r\to\infty} \int_\Sigma f \Delta \chi_r \, \mathrm{dvol}_\Sigma,$$

where in the last equality we integrated by parts using the fact that there are no boundary terms as χ_r, $\nabla\chi_r$ have compact support. Moreover, by construction we have $|\Delta\chi_r| \leqslant c r^{-2}$ and is supported in $B_{2r}(p) \backslash B_r(p)$. Without loss of generality, we can assume, that $\int_\Sigma \Delta f$ is non-negative, otherwise we can use $-f$. We conclude that

$$0 \leqslant \int_\Sigma \Delta f \, \mathrm{dvol}_\Sigma \leqslant c \|f\|_{L^\infty} \lim_{r\to\infty} \frac{\mathrm{Vol}(B_{2r}(p)\backslash B_r(p))}{r^2} = 0,$$

as $\mathrm{Vol}(B_{2r}(p)\backslash B_r(p))$ is uniformly bounded by above (in fact it decays). □

Next we prove the main result of this section.

Proof of Theorem 3.1. By assumption $\overline{\Sigma}$ is closed and equipped with a smooth non-negative function κ, which only vanishes on $\overline{\Sigma}\backslash\Sigma$, and $g_{\overline{\Sigma}} = \kappa g_\Sigma$ extends to a smooth metric on $\overline{\Sigma}$. Then we have that $\overline{\Delta} = \kappa^{-2}\Delta$. As the functions g, h on Σ are smooth and bounded, we conclude that κg, κh extend to smooth functions on $\overline{\Sigma}$. Hence, the Kazdan–Warner equation (3.1) extends naturally to an equation on $\overline{\Sigma}$ given by

$$\overline{\Delta} f = \kappa g - \kappa h e^{2f},$$

for a function f on $\overline{\Sigma}$. This satisfies the hypothesis of Proposition 3.4 and so we obtain a unique smooth solution f. By construction, f is smooth and bounded, and solves the Kazdan–Warner equation on Σ. The uniqueness part can be proven exactly as in the proof of Proposition 3.4 but carrying out the integration over Σ using dvol_{g_Σ} rather than over $\overline{\Sigma}$. This requires the knowledge that the integral over Σ of the Laplacian of a bounded function vanishes, which follows from Lemma 3.5.

The proof of the claim about the L^∞-norm of f is standard, and can be found in [34]. $\square$

Finally, we present below a comparisons result for the Kazdan–Warner equation. This will be useful for proving the claim about the Uhlenbeck compactness in Main Theorem.

Lemma 3.6. *Let $K \subset \Sigma$ be a compact set, g_1, g_2, h_1, and h_2 be as in Theorem 3.1, and f_1, f_2 be the corresponding solutions to the Kazdan–Warner equation. Assume, that the L^∞-norm g_1, g_2, h_1, and h_2 are all bounded by some constant B, and that there is an $\varepsilon, \delta > 0$ and $p \in \Sigma$, such that $h_i|_{B_p(\varepsilon)} > \varepsilon$, for $i = 1, 2$. Then there is a constant $c = c(K, B, p, \varepsilon)$, such that the following holds:*

$$\|f_1 - f_2\|_{L^2_2(K)} \leqslant c(\|g_1 - g_2\|_{L^2(\Sigma)} + \|h_1 - h_2\|_{L^2(\Sigma)}). \tag{3.4}$$

Proof. Pick and fix a compact set $K' \subset \Sigma$ such that $K \subset U \subset K'$ for some open set U. First we prove that there is a constant $c' = c'(K', B, p, \varepsilon)$, such that the following holds:

$$\|f_1 - f_2\|_{L^\infty(K')} \leqslant c(\|g_1 - g_2\|_{L^2(\Sigma)} + \|h_1 - h_2\|_{L^2(\Sigma)}). \tag{3.5}$$

Let $F = f_1 - f_2$. Then, using the convexity of the exponential function, we get

$$\Delta F = (g_1 - g_2) + e^{2f_2}(h_2 - h_1 e^{2F}) \leqslant |g_1 - g_2| + e^{2f_2}|h_2 - h_1| - 2h_1 e^{2f_2} F. \tag{3.6}$$

Rearranging this we have, for some constant $c_1 = c_1(B)$, that

$$(\Delta + 2h_1 e^{2f_2})F \leqslant |g_1 - g_2| + e^{2f_2}|h_2 - h_1| \leqslant c_1(|g_1 - g_2| + |h_2 - h_1|).$$

Let G_1 be the Green's operator of the positive definite elliptic operator $\Delta + h_1 e^{2f_2}$. Standard elliptic theory tells us that G_1 is an integral operator with a positive kernel, thus

$$F \leqslant c_1 G_1(|g_1 - g_2| + |h_2 - h_1|).$$

By the hypotheses and Theorem 3.1, we get that there is a constant $c_2 = c_2(\varepsilon, p, B)$, such that $2h_1 e^{2f_2} \geqslant c_2$ on $B_p(\varepsilon)$. Let G now be the Green's operator of the positive definite elliptic operator $\Delta + c_2 \chi_{B_p(\varepsilon)}$. Since $c_2 \chi_{B_p(\varepsilon)} \leqslant 2h_1 e^{2f_2}$ everywhere on Σ, we get that $G \geqslant G_1$, thus

$$F \leqslant c_1 G(|g_1 - g_2| + |h_2 - h_1|).$$

Now G (composed with the restriction to K) is a continuous linear operator from $L^2(\Sigma)$ to $L^2_2(K')$. The continuous embedding $L^2_2(K') \hookrightarrow L^\infty(K')$, gives us a constant $c_3 = c_3(\varepsilon, p, B, K)$

such that

$$F|_K \leqslant c_3(\|g_1 - g_2\|_{L^2(\Sigma)} + \|h_2 - h_1\|_{L^2(\Sigma)})).$$

Reversing the roles of f_1 and f_2 we get a similar lower bound, and the existence of c'.

Using inequality (3.5), we now prove inequality (3.4). By the hypotheses and equation (3.6), we have that there is a constant $c_4 = c_4(\varepsilon, p, B)$, such that

$$|\Delta F| \leqslant c_4(|g_1 - g_2| + e^{2f_2}|h_2 - h_1| + |F|). \tag{3.7}$$

Thus, by elliptic regularity, we get that there is a constant $c_5 = c_5(\varepsilon, p, B, K, K')$, such that

$$\|F\|_{L^2_2(K)} \leqslant c_5(\|\Delta F\|_{L^2(K')} + \|F\|_{L^2(K')}). \tag{3.8}$$

Combining inequalities (3.5), (3.7) and (3.8), we get inequality (3.4). □

4. Vortices on complete, non-compact, finite volume Riemann surfaces

Before proving the main results, we give a short detour to recall some facts about Abelian vortices on oriented, finite volume, complete, but non-compact Riemannian surfaces. We emphasize here that we do not require (lower) bounds on injectivity radius. This setup when we use the third bullet of Theorem 2.2 to construct instantons on the ES manifold.

The first result is an application of Lemma 3.5 and is a simple extension of the results in [5], for complete finite volume Riemannian surfaces. The only complication, comparing to the result of that reference, comes from the boundary terms in the integration by parts, which can be dealt using Lemma 3.5.

Proposition 4.1. *Let (Σ, g_Σ) be a complete, finite volume Riemannian surface, with a Hermitian line bundle $(\mathcal{N}, H)$. Let (∇, Φ) be a pair of a Hermitian connection on $\mathcal{N}$ and a smooth section of $\mathcal{N}$, with $E_{\mathrm{YMH}}(\nabla, \Phi) < \infty$. Then*

$$2\pi E_{\mathrm{YMH}}(\nabla, \Phi) = \|i\Lambda F_\nabla - \frac{1}{2}(\tau - |\Phi|^2)\|^2_{L^2} + 2\|\overline{\partial_\nabla}\Phi\|^2_{L^2} + \tau \int\limits_\Sigma i\Lambda F_\nabla \,\mathrm{dvol}_\Sigma\,.$$

In particular, if (∇, Φ) is a finite energy τ-vortex we have

$$E_{\mathrm{YMH}}(\nabla, \Phi) = \tau \deg(\nabla) \leqslant \frac{\tau^2}{4\pi}\mathrm{Vol}(\Sigma, g_\Sigma),$$

with equality if and only if $\Phi = 0$.

Proof. The first step is to use the Kähler identities to compute

$$\begin{aligned}\int_\Sigma i\Lambda F_\nabla |\Phi|^2 \,\mathrm{dvol}_\Sigma &= \int_\Sigma H(\Phi, i\Lambda(\overline{\partial}_\nabla \partial_\nabla \Phi + \partial_\nabla \overline{\partial}_\nabla \Phi)\,\mathrm{dvol}_\Sigma \\ &= \int_\Sigma H(\Phi, \partial_\nabla^* \partial_\nabla \Phi - \overline{\partial}_\nabla^* \overline{\partial}_\nabla \Phi)\,\mathrm{dvol}_\Sigma \\ &= \|\partial_\nabla \Phi\|_{L^2}^2 - \|\overline{\partial}_\nabla \Phi\|_{L^2}^2, \end{aligned} \tag{4.1}$$

with the last step conditional on the boundary terms in the integration vanishing. This boundary term is of the form

$$\lim_{r\to\infty} \int_{\partial \overline{B_r(p)}} H(\Phi, *\nabla\Phi) = \frac{1}{2} \int_\Sigma \Delta |\Phi|^2 \,\mathrm{dvol}_\Sigma,$$

which vanishes by combining the finite energy assumption and [19, Lemma 1.1.A.]. Now notice that $|F_\nabla| = |i\Lambda F_\nabla|$ and so

$$\begin{aligned}\|i\Lambda F_\nabla - \tfrac{1}{2}(\tau - |\Phi|^2)\|_{L^2}^2 &= \|F_\nabla\|_{L^2}^2 + \frac{1}{4}\|\tau - |\Phi|^2\|_{L^2}^2 - \langle i\Lambda F_\nabla, (\tau - |\Phi|^2)\rangle_{L^2} \\ &= \|F_\nabla\|_{L^2}^2 + \frac{1}{4}\|\tau - |\Phi|^2\|_{L^2}^2 - \int_\Sigma i\Lambda F_\nabla (\tau - |\Phi|^2)\,\mathrm{dvol}_\Sigma \\ &= \|F_\nabla\|_{L^2}^2 + \frac{1}{4}\|\tau - |\Phi|^2\|_{L^2}^2 + \|\nabla\Phi\|_{L^2}^2 - 2\pi\tau \deg(\nabla) - 2\|\overline{\partial}_\nabla \Phi\|_{L^2}^2, \end{aligned}$$

where in the last equality we have used equation (4.1). Rearranging gives the formula in the statement.

In the case when (∇, Φ) is a τ-vortex of finite energy, it is proven in [31, Lemma 3.1] that $|\Phi|^2 \leqslant \tau$ and so the assumption that $|\Phi|$ is bounded holds immediately and the first part of the result applies. In that case we further have that the two first terms in the equation for the energy vanish and we are left with the term involving the degree.

The fact that the energy is uniformly bounded from above by $\frac{\tau^2}{2}\mathrm{Vol}(\Sigma, g_\Sigma)$ is a consequence of the τ-vortex equation as follows

$$E_{\mathrm{YMH}}(D) = \tau \deg(\nabla) = \tau \frac{i}{2\pi} \int_\Sigma F_\nabla = \tau \frac{i}{2\pi} \int_\Sigma \frac{1}{2i}(\tau - |\Phi|^2)\,\mathrm{dvol}_\Sigma \leqslant \frac{\tau^2}{4\pi} \mathrm{Vol}(\Sigma, g_\Sigma).$$

□

We give now the main consequence of Theorem 3.1.

Corollary 4.2. *Let (Σ, g_Σ) be a complete, finite volume, Riemannian surface satisfying the assumptions of Theorem 3.1 and $\mathcal{N}$ complex line bundle over Σ. Fix an Hermitian structure H on $\mathcal{N}$ and let ∇^H be an H-Hermitian connection and Φ_0 be a $\overline{\partial}_{\nabla^H}$-holomorphic section of $\mathcal{N}$.*

If $i\Lambda F_{\nabla^H}$ and $|\Phi_0|_H^2$ are bounded, then there is a unique, smooth, and bounded f so that

$$(\nabla, \Phi) = (\nabla^H + \partial f - \overline{\partial} f, e^f \Phi_0) \tag{4.2}$$

is a τ-vortex if and only if Bradlow's condition holds (cf. [5, Equation 4.10]*):*

$$\tau > \frac{1}{\mathrm{Vol}(\Sigma, g_\Sigma)} \int_\Sigma 2i\Lambda F_{\nabla^H} \,\mathrm{dvol}_\Sigma = 4\pi \frac{\deg(\nabla^H)}{\mathrm{Vol}(\Sigma, g_\Sigma)}. \tag{4.3}$$

Moreover, the degree of the resulting vortex is the same as the degree of ∇^H, that is

$$\deg(\nabla) = \deg(\nabla^H). \tag{4.4}$$

Proof. We start by recalling the connection between vortices and the solutions of the Kazdan–Warner equation (cf. [5, 34]). There is a vortex field configuration (∇, Φ), in the complex gauge orbit of $(\overline{\partial}_{\nabla^H}, \Phi_0)$, if and only if (∇, Φ) has the form (4.2) with an f that satisfies

$$2i\Lambda F_\nabla = 2i\Lambda F_{\nabla^H} + 2\Delta f = \tau - |\Phi_0|_H^2 e^{2f}. \tag{4.5}$$

Setting $g = \frac{\tau}{2} - i\Lambda F_{\nabla^H}$ and $h = \frac{1}{2}|\Phi_0|_H^2$, equation (4.5) is equivalent to the Kazdan–Warner equation (3.1). According to Theorem 3.1, one can uniquely solve equation (4.5) if $i\Lambda F_{\nabla^H}$ and $|\Phi_0|_H^2$ are bounded, and $\int_\Sigma g > 0$, which is equivalent to equation (4.3). The resulting solution f is smooth and uniformly bounded, hence using the Kähler identities and Lemma 3.5 proves equation (4.4). □

Remark 4.3. *Notice that if instead of using (∇^H, Φ_0) in the previous result we use $(\nabla^H, \widehat{\Phi}_0 = k\Phi_0)$ for some constant k we must instead solve the Kazdan–Warner equation*

$$2i\Lambda F_{\nabla^H} + 2\Delta \widehat{f} = \tau - k^2 |\Phi_0|_H^2 e^{2\widehat{f}},$$

for $\widehat{f}$. The resulting (unique bounded) solution is given by $\widehat{f} = f - \log(k^2)$, where f denotes the solution of the original Kazdan–Warner equation. Therefore, the resulting vortices obtained from applying the previous corollary to $(\nabla^H, \widehat{\Phi}_0)$ and (∇^H, Φ_0) are the same.

5. Application to the Euclidean Schwarzschild manifold

We start this section by introducing, without proofs, the ES manifold. The base manifold is $X = \mathbb{R}^2 \times S^2$. Let $(t, \rho) \in (0, 2\pi) \times (0, \infty)$ be the usual polar coordinates on $\mathbb{R}^2$, and g_{S^2} the round metric of volume 2π on S^2, as in Section 2. Then, for each *mass* $m \in \mathbb{R}_+$, the Riemannian metric on X on the dense chart on which t and ρ are defined is

$$g_X = 8m^2\left(\frac{2\rho}{\rho+1}dt^2 + \frac{\rho+1}{2\rho}d\rho^2 + (\rho+1)^2 g_{S^2}\right). \tag{5.1}$$

While the metric (5.1) seems singular on the *bolt* at $\rho = 0$ (the *"Euclidean event horizon"*), the change of coordinate $x^2 = \rho$ results in an expression which extends smoothly over the whole manifold as a complete and Ricci-flat Riemannian metric for all $m > 0$. Traditionally, in theoretical physics, g_X is given in the coordinates $\tau = 4mt$ and $r = 2m(\rho + 1)$. On this chart we have

$$g_X = \left(1 - \tfrac{2m}{r}\right)d\tau^2 + \frac{1}{1 - \frac{2m}{r}}dr^2 + 2r^2 g_{S^2}.$$

The isometry group of (X, g_X) is $\mathrm{O}(2) \times \mathrm{O}(3)$, where $\mathrm{O}(2)$ acts on $\mathbb{R}^2$ and $\mathrm{O}(3)$ acts on S^2, the usual ways. The subgroup of orientation preserving isometries is

$$\mathrm{Isom}_+(X, g_X) \cong \mathrm{S}(\mathrm{O}(2) \times \mathrm{O}(3)).$$

While $\mathrm{SO}(3)$ acts isometrically on the ES manifold, the metric is a wrapped product, hence the results of Section 4 cannot be used for g_X. However, using the smooth, spherically symmetric conformal factor $\frac{1}{8m^2(\rho+1)^2}$ we get a non-wrapped metric $\widetilde{g}_X$ to which Theorem 2.2 applies. Since the self-duality equation is a conformally invariant, instantons for g_X and $\widetilde{g}_X$ are the same, and the Yang–Mills energy is invariant.

In our case $\Sigma = \mathbb{R}^2$ and g_Σ is given by

$$g_\Sigma = \frac{2\rho}{(\rho+1)^3}dt^2 + \frac{1}{2\rho(\rho+1)}d\rho^2. \tag{5.2}$$

Straightforward computations show that g_Σ is complete, has volume 2π. Its Gauß curvature is $-4 + \frac{12}{\rho+1}$, thus Σ is asymptotically a hyperbolic cylinder. There is technical issue with Σ: its injectivity radius is not uniformly bounded below, thus one cannot use global Sobolev embeddings. The scalar Laplacian of Σ takes the form of

$$\Delta_\Sigma = -\frac{(\rho+1)^3}{2\rho}\partial_\chi^2 - 2\rho(\rho+1)\partial_\rho^2 - 2\partial_\rho.$$

A dense chart of Σ is given by the coordinate z such that

$$z = \sqrt{\rho}\exp(\tfrac{\rho}{2} - it) \in \mathbb{C}, \tag{5.3}$$

which extends to a global holomorphic chart $\Sigma \to \mathbb{C}$.

Given its relevance for the proof of our main results we shall separately state the following observation.

Lemma 5.1. *The Riemannian surface (Σ, g_Σ) is complete, of finite volume and satisfies the hypothesis of Theorem 3.1. In particular, Corollary 4.2 applies to (Σ, g_Σ).*

Proof. The fact that g_Σ is complete of finite volume is proven in the discussion in the beginning of Section 5. We now address the question of whether g_Σ satisfies the hypothesis of Theorem 3.1. First recall that $\Sigma \cong \mathbb{C}$ is a biholomorphism. Moreover, in terms of the function z in equation (5.3), the metric can be written as

$$g_\Sigma = \frac{2}{(1+\rho)^3 e^\rho}|dz|^2, \tag{5.4}$$

we are in the conditions of [29, Proposition 2.4], which proves the assumptions of Theorem 3.1 hold. Indeed, for future reference we note that the function ρ satisfies

$$\Delta_\Sigma \rho = -2.$$

□

Before beginning the proof of our main result, we also note that, using equations (5.3) and (5.4), (Σ, g_Σ) can be conformally compactified the smooth Riemannian manifold $(S^2 = \Sigma \cup \{\infty\}, \widetilde{g})$, where $\widetilde{g}|_\Sigma = \kappa g_\Sigma$ with

$$\kappa = \frac{(\rho+1)^3 e^\rho}{(1+|z|^2)^2} = \frac{(\rho+1)^3 e^\rho}{(1+\rho e^\rho)^2}. \tag{5.5}$$

5.1. **Proof of the main results.** Since $H^4(X;\mathbb{Z})$ is trivial, there is a unique $\mathrm{SU}(2)$ principal bundle on X, and since $\pi_1(X)$ is trivial, there is a unique flat $\mathrm{SU}(2)$ connection on that bundle. Etesi and Hausel proved in [15] that for each positive integer n, there is a unique reducible spherically symmetric $\mathrm{SU}(2)$ instanton, with holonomy equal to $\mathrm{S}(\mathrm{U}(1) \oplus \mathrm{U}(1)) \cong \mathrm{U}(1)$ and energy equal to $2n^2$ on X.

Using Theorem 2.2 and Corollary 4.2 we get a classification of the irreducible, spherically symmetric $\mathrm{SU}(2)$ instantons on the ES manifold.

Corollary 5.2 (Energy restriction)**.** *Any finite energy, irreducible, spherically symmetric* $\mathrm{SU}(2)$ *instanton D, on the ES manifold must satisfy* $E_{\mathrm{YM}}(D) \in (0, 2)$.

Proof. By Lemma 2.1, any spherically symmetric $\mathrm{SU}(2)$ connection, D, of the form (2.2) is irreducible if and only if $n = 1$ and Φ does not vanish identically. In this case, Φ is also non-zero, and by Theorem 2.2, D is an instanton if and only if the equations (2.7a) and (2.7b), hold on $\mathbb{R}^2$, with the metric given by equation (5.2). Then (∇, Φ) is a 4-vortex on Σ with $\Phi \neq 0$, and it follows from equation (2.9) that the (necessarily positive) energy of D satisfies

$$0 < E_{\mathrm{YM}}(D) = \deg(\nabla) = \frac{1}{2\pi} \int\limits_{\Sigma} i\Lambda F_{\nabla} \, \mathrm{dvol}_{\Sigma} < \frac{1}{2\pi} \frac{4}{2} \mathrm{Vol}(\Sigma) = 2. \tag{5.6}$$

□

It will be useful to have the following lemma stated separately.

Lemma 5.3. *Let D be an spherically symmetric* $\mathrm{SU}(2)$ *connection on the ES manifold, and* (∇, Φ) *be the corresponding field configuration on Σ as in Theorem 2.2. Then* $|\mathcal{F}_D|^2_{g_X}$ *descends to Σ as a smooth function, and moreover the following equation holds*

$$|\mathcal{F}_D|^2_{g_X} = \frac{|F_{\nabla}|^2_{g_\Sigma} + |\nabla\Phi|^2_{g_\Sigma} + \frac{1}{4}(4 - |\Phi|^2_H)^2}{16 m^4 (\rho + 1)^4} \tag{5.7}$$

Proof. This follows from a simple computation using the formula for $\mathcal{F}_D$ given in the proof of Theorem 2.2 and the formula for the metric on the ES manifold given in equation (5.2). □

Remark 5.4. *Since the 4-vortex fields* (∇, Φ)*, that we construct in this paper are all bounded, equation* (5.7) *gives at least quadratic curvature decay for all spherically symmetric* $\mathrm{SU}(2)$ *instantons on the ES manifold.*

Continuing to use Theorem 2.2, our goal now is to fix a complex line bundle equipped with the Hermitian structure H, then use Corollary 4.2 to produce 4-vortices on Σ. The next result identifies these and gives a restriction on Φ of any possible 4-vortex (∇, Φ).

Lemma 5.5. *Let $\mathcal{N}$ be a holomorphic Hermitian vector bundle and (∇, Φ) a 4-vortex on $\mathcal{N} \to \Sigma$. Then $\mathcal{N}$ is isomorphic to the trivial holomorphic line bundle $\underline{\mathbb{C}}$ equipped with the Hermitian structure* $H(\Phi, \Psi) = \overline{\Phi}\Psi e^{-\alpha(z, \overline{z})}$*, for a real valued $\alpha(z, \overline{z})$ satisfying*

$$\deg(\nabla^H) = \frac{i}{2\pi} \int\limits_{\Sigma} \partial\overline{\partial}\alpha < 2.$$

Under such an isomorphism Φ *is a polynomial in* z *of degree at most 1.*

Proof. Note that any bundle over Σ has to be topologically trivial. Moreover, since $H^1(\Sigma, \mathcal{O}_\Sigma)$ is trivial, any holomorphic structure is equivalent to the standard one which we now fix and denote by $\overline{\partial}$. Holomorphic sections, then simply correspond to entire functions in the coordinate $z \in \mathbb{C}$ from equation (5.3). Now, set H_0 to be the Hermitian structure whose associated Chern connection ∇^{H_0} is the trivial flat connection, that is given two complex valued functions Φ, Ψ we have $H_0(\Phi, \Psi) = \overline{\Phi}\Psi$. Any other Hermitian structure can be written as $H = e^{-\alpha(z,\overline{z})} H_0$, for a real valued function α. The associated Chern connection is $\nabla^H = \nabla^{H_0} - \partial\alpha$. Note that by construction ∇^{H_0} and ∇^H induce the same holomorphic structure, and

$$\deg(\nabla^H) = \frac{i}{2\pi} \int_\Sigma \partial\overline{\partial}\alpha.$$

The first thing to notice is that, from [31, Lemma 3.1], any finite energy τ-vortex field, (∇, Φ), satisfies $|\Phi|_H^2 \leqslant \tau$ (recall from Theorem 2.2 that in in our case $\tau = 4$). Hence, its growth rate must be (at most) $|\Phi(z)|^2 = O(e^{\alpha(z,\overline{z})})$. Standard complex analytic arguments show, that when $|\Phi(z)|$ is not $O(|z|^n)$, then the degree $\deg(\nabla^H)$ is greater than n. Thus, by equation (5.6), we have that $|\Phi(z)|$ must grow slower than $|z|^2$. We now apply a complex gauge transformation to (∇, Φ) so that the resulting connection is compatible with $\overline{\partial}$. This has the consequence that Φ changes to Φ_0, which is a holomorphic function differing from Φ by multiplication with a non-zero uniformly bounded function. Hence, we must also have that $|\Phi_0(z)|$ grows strictly slower than z^2, being either a non-zero constant, or a first order polynomial. □

We now put all the previous work together to finalize the proof the Main Theorem.

The proof of the Main Theorem. In part, what is left to do is to give a converse to Lemma 5.5 using Corollary 4.2 in the case of $\mathcal{N} = \mathcal{L}^2$. For that we shall start with a polynomial Φ_0, as in Lemma 5.5, and an H-Hermitian connection ∇^H on the trivial bundle over Σ. Then we must show, using Corollary 4.2, that in a complex gauge orbit of (∇^H, Φ_0) there is a unique 4-vortex. We split the proof in two cases according to whether Φ_0 is a constant or a degree 1 polynomial in z. In fact, having in mind Remark 4.3 we may pick this polynomial to be monic.

First we suppose $\Phi_0 = 1$. Then in the notation of Lemma 5.5, and with no loss of generality, we may let $\alpha = E\rho$, with $E \geqslant 0$. Then $i\Lambda\partial\overline{\partial}\alpha$ and $|\Phi_0|_H^2$ are both bounded and

$$\deg(\nabla^H) = E.$$

Hence, Corollary 4.2 applies and gives a 4-vortex (∇^E, Φ^E) provided that

$$E < \frac{4\mathrm{Vol}(\Sigma, g_\Sigma)}{4\pi} = 2.$$

This in turn gives rise, via the third bullet of Theorem 2.2, to an instanton D on Schwarzschild whose energy is given by $E_{\mathrm{YM}}(D) = E$.

When $E = 0$, the corresponding instanton is the unique flat connection, hence reducible. To analyse the case when $E > 0$, recall that $\Phi^E = e^f \Phi_0$ for a uniformly bounded function f. Hence, when $E > 0$, the corresponding vortex (∇^E, Φ^E), is such that

$$|\Phi^E|_H^2 = O(e^{-E\rho}).$$

This means, by the 4-vortex equation (2.7a), that is $2i\Lambda F_{\nabla^E} = 4 - |\Phi^E|_H^2$, we have that $i\Lambda F_{\nabla^E} \to 2 \neq 0$, as $\rho \to \infty$. Thus, further using equation (5.7) in Lemma 5.3 we conclude that the corresponding instantons have quadratic curvature decay.

We now turn to the second case, that is when $\Phi_0(z)$ is a degree 1-polynomial, say $\Phi_0(z) = z - z_0$, for some $z_0 \in \mathbb{C}$. Again, with no loss of generality, we set $\alpha = E\rho + \ln(\rho + 1)$, then we have

$$\begin{aligned} i\Lambda F_{\nabla^H} &= 2(E-1) + \frac{4}{\rho+1}, \\ |\Phi_0|_H^2 &= O(e^{-(E-1)\rho}), \\ \deg(\nabla^H) &= E. \end{aligned}$$

Thus, as long as $E \in [1, 2)$, the conditions of Corollary 4.2 hold true. Moreover, using equation (5.3) and the fact that f is bounded, we get that the corresponding vortex (∇^E, Φ^E) satisfies

$$|\Phi^E|_H^2 = O(e^{-(E-1)\rho}).$$

Hence, as in the previous case, equation (2.7a) implies that $i\Lambda F_{\nabla^E} \to 2 \neq 0$, as $\rho \to \infty$, if $E > 1$. Once again, this shows that for $E \in (1, 2)$ the corresponding instantons have exactly quadratic curvature decay. When $E = 1$, then the corresponding vortex (∇, Φ), is such that

$$|\Phi|_H^2 = \frac{|z - z_0|^2}{(\rho + 1)e^\rho} e^{2f}, \tag{5.8}$$

where f is the smooth and bounded solution of the Kazdan–Warner equation (4.5). By the results of Section 3, f extends smoothly over $S^2 = \Sigma \cup \{\infty\}$. Hence, $f(\infty) = \lim_{\rho\to\infty} f(\rho)$ exists, and in the same limit we have $|\Phi|_H^2 \to e^{2f(\infty)}$. Now recall that the metric κg_Σ, where κ is given by equation (5.5) also extends over S^2 smoothly, and since $\kappa = O((\rho+1)e^{-\rho})$, so is $|df|_{g_\Sigma}$. In particular, f converges exponentially fast, and for large ρ,

$$|\partial_\rho f| = O\left(\frac{e^{-\rho}}{\rho+1}\right). \tag{5.9}$$

Moreover, as every other term equation (4.5) converges uniformly in the limit when $\rho \to \infty$, so does Δf. Denote the limits by l. For each $\varepsilon > 0$, pick ρ_ε, such that $\rho > \rho_\varepsilon >$ implies $l - \varepsilon \leqslant \Delta f$. Let $\Sigma_\varepsilon = \{\rho > \rho_\varepsilon\}$. Then, using Lemma 3.5 and equation (5.9), we get

$$(l-\varepsilon)\frac{2\pi}{\rho_\varepsilon + 1} \leqslant \int_{\Sigma_\varepsilon} (l-\varepsilon)\,\mathrm{dvol}_\Sigma \leqslant \int \Delta f\,\mathrm{dvol}_\Sigma \leqslant \int_{\partial\Sigma_\varepsilon} *df = O\left(\frac{e^{-\rho_\varepsilon}}{\rho_\varepsilon+1}\right).$$

As $\varepsilon \to 0^+$, we can assume that $\rho_\varepsilon \to \infty$, thus the limit is non-positive. Similar argument shows that it is also non-negative, hence $l = 0$. Using this, equations (4.5) and (5.8), and that f converges exponentially fast to $f(\infty)$ as $\rho \to \infty$, we get that $|\Delta f|$ also has exponential decay. By equations (2.5) and (4.5) this gives cubic curvature decay for the corresponding instanton.

Next we address the claim about the Uhlenbeck compactness of $\mathcal{M}_{X,E}^{\mathrm{SO}(3)}$. Without loss of generality, it is enough to investigate the convergence properties of sequences, $\{[D_n]\}_{n\in\mathbb{N}}$, of instantons in $\mathcal{M}_{X,E}^{\mathrm{SO}(3)}\backslash\{\infty\}$. Call (∇^n, Φ^n) the vortex fields corresponding to D_n. Using the action of the isometry group, $\mathrm{SO}(2)$, of Σ, and the compactness of the circle, we can also assume, that the divisor of (∇^n, Φ^n) has coordinates $t = 0$, and $\rho = \rho_n \in [0, \infty)$. To prove that $\mathcal{M}_{X,E}^{\mathrm{SO}(3)}$ is connected and compact in the weak L_1^2-topology, and the claim about the canonical map, it is enough to prove that if $\rho_n \to \rho_\infty \in [0, \infty]$, then there is a sequence of gauge transformations, γ_n, such that the sequence, $\{\gamma_n(D_n)\}_{n\in\mathbb{N}}$, converges weakly in L_1^2 to the spherically symmetric $\mathrm{SU}(2)$ instanton, whose corresponding vortex field has divisor at $t = 0$ and $\rho = \rho_\infty$, where $\rho_n = \infty$ means the divisor "at infinity" in S^2. Since $\{[D_n]\}_{n\in\mathbb{N}}$ is bounded in L_1^2, it is enough to show pointwise convergence of $\mathcal{F}_{D_n}$ on compact sets, and by construction, it is enough to prove the analogous claim on the level of vortices. Let H and ∇^H be as above, and for each $n \in \mathbb{N}$ write $\nabla^n = \nabla^H + \partial f_n - \overline{\partial} f_n$ and $\Phi^n = e^{f_n}\frac{z-\rho_n}{\sqrt{1+\rho_n^2}}$. The corresponding g_n and h_n are

$$g_n = g = 2 - i\Lambda F_{\nabla^H}, \qquad h_n = \frac{|z-\rho_n|^2}{(1+\rho_n^2)(\rho+1)e^{E\rho}}.$$

Since $\{\rho_n\}_{n\in\mathbb{N}}$ is either convergent or goes to infinity, we can pick a point $p \in \Sigma$ and ε as in Lemma 3.6. Let

$$h_\infty = \begin{cases} \frac{|z-\rho_\infty|^2}{(1+\rho_\infty^2)(\rho+1)e^{E\rho}}, & \text{when } \rho_\infty \neq \infty, \\ \frac{1}{(\rho+1)e^{E\rho}}, & \text{when } \rho_\infty = \infty. \end{cases}$$

Since $\{h_n\}_{n\in\mathbb{N}}$ is convergent to h_∞ in L^2 of any compact sets, we get local L^2_2 convergence of $\{f_n\}_{n\in\mathbb{N}}$ to f_∞, which in turn implies the weak L^2_1 convergence of $\{D_n\}_{n\in\mathbb{N}}$ to D_∞.

What we have shown so far is that the map given by our construction above is a continuous surjection from S^2 to $\mathcal{M}^{\mathrm{SO}(3)}_{X,E}$. In the final step of this proof, we show that it is also injective, that is the instantons corresponding to different points (degree 1 effective divisors) are not gauge equivalent.

Let (∇, Φ) be the 4-vortex fields corresponding to $z_0 \in S^2$. By the spherical symmetry of D, the Yang–Mills energy density $\frac{1}{8\pi^2}|\mathcal{F}_D|^2$ descends to Σ. Moreover, using equations (2.7a), (2.7b) and (5.7), we have the following equation on Σ

$$(\Delta_\Sigma + 4)|\Phi|^2 = 16 - 16m^4(\rho+1)^4|\mathcal{F}_D|^2$$

Let G be the Green's function of the non-degenerate, positive, elliptic operator $\Delta_\Sigma + 4$. Now, using that $\int_\Sigma G(-,x)\,\mathrm{dvol}_\Sigma(x) = \frac{1}{4}$, we have

$$|\Phi|^2 = 4 - 16m^4 \int_\Sigma G(-,x)(\rho(x)+1)^4|\mathcal{F}_D(x)|^2\,\mathrm{dvol}_\Sigma(x). \tag{5.10}$$

Since the right hand side of equation (5.10) is gauge invariant, so is the left hand side. The equation $\Phi = 0$ holds exactly at z_0, by the construction, thus the instanton corresponding to different points in S^2 are gauge inequivalent. This concludes the proof. □

Remark 5.6. *We end this section with a few important remarks:*

(1) *The Charap–Duff instanton appears in this picture as the origin of* $\mathbb{C} \subset \mathcal{M}^{\mathrm{SO}(3)}_{X,1}$.

(2) *All the instantons corresponding to the points* $\{0,\infty\} \in \mathbb{C}\cup\{\infty\} \cong \mathcal{M}^{\mathrm{SO}(3)}_{X,E}$ *are also static, that is they are invariant under the whole isometry group* $\mathrm{S}(\mathrm{O}(2)\times\mathrm{O}(3))$.

(3) *When* $D \notin \{0,\infty\}$*, the left hand side of equation* (5.10) *is not static, so neither is* $\mathbb{E}_{\mathrm{YM}}(D)$*. Thus all instanton in* $\mathcal{M}^{\mathrm{SO}(3)}_{X,E}\backslash\{0,\infty\}$ *are* not static. *This disproves a conjecture of Tekin in* [32].

(4) *In this picture, and for* $E \in [1,2)$ *the canonical map*

$$\mathcal{M}^{SO(3)}_{X,E} \to \mathbb{C}\cup\{\infty\} = S^2,$$

is nothing but the map which to an spherically symmetric instanton D assigns the divisor of the associated vortex (∇, Φ).

6. Directions for further work

In this final section we mention a few research directions that our work opens up.

Does $\mathcal{M}_{X,1}$ have other connected components? The (connected component of the) isometry group, $\mathrm{SO}(2)\times\mathrm{SO}(3)$, acts on the trivial $\mathrm{SU}(2)$-bundle over the ES manifold. It induces an action on the space of connections, which preserves the self-duality equations. Thus, inducing an action on $\mathcal{M}_{X,1}$. We shall explain below how, by studying this action, one may be able to conclude that the universal cover $\widetilde{\mathcal{M}}_{X,1}$ of the Uhlenbeck compactified moduli space is homeomorphic to a disjoint union of 2-spheres S^2's

$$\widetilde{\mathcal{M}}_{X,1} \cong S^2 \sqcup \ldots \sqcup S^2,$$

and determine the number of connected components.

From the results of Etesi-Jardim in [16], any connected component of the Uhlenbeck compactified moduli space must be 2-dimensional. Our results state that the instantons in $\mathcal{M}_{X,1}$ fixed by $\{1\} \times \mathrm{SO}(3)$ form a connected component homeomorphic to S^2, hence $\{1\} \times \mathrm{SO}(3)$ must act without fixed points on any other component of $\mathcal{M}_{X,1}$. Supposing this action to be continuous, we conclude that any such component (if non-empty) must be homeomorphic to (a quotient of) S^2, with $\{1\} \times \mathrm{SO}(3)$ acting transitively.

The L^2-metric on the moduli space. It follows from the proof of the Main Theorem, that the spherically symmetric part of the unit energy Etesi–Jardim moduli space is connected (albeit non-compact) in the strong L^2_1-topology. Then, simple computation which we omit here, shows that the canonical map from Σ is in fact a biholomorphism with respect to the complex structure (5.3) and the L^2-complex structure on the moduli space.

A further question, which seems to be more difficult, is that of computing the L^2-metric in the moduli space. One can immediately see, that this metric is compatible with the L^2-complex structure, thus it differs from g_Σ by a conformal transformation. Since we established a connection with $\mathcal{M}^{\mathrm{SO}(3)}_{X,1}$ and a vortex moduli space on Σ, the methods of [26] could potentially be used to understand (at least) the asymptotic behavior of the L^2-metric. This computation could shed light on the L^2-cohomology of the moduli space, and thus serve as an example for the several Sen-type conjectures.

Application to other geometries. It is plausible that similar techniques, exploring the correspondence between instantons and vortices, may be successfully applied to study instantons on other geometries. Obvious candidates come from physics: the Euclidean Reissner–Nordström–de Sitter manifold describes a charged black hole of mass $m \in \mathbb{R}_+$, electric charge $Q \in \mathbb{R}$, and cosmological constant $\Lambda \in \mathbb{R}$. The base manifold is still $\mathbb{R}^2 \times S^2$, with the Riemannian metric

$$g_{RNdS} = \left(1 - \tfrac{2m}{r} + \tfrac{Q^2}{r^2} - \tfrac{\Lambda r^2}{3}\right) d\tau^2 + \frac{1}{1 - \frac{2m}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}} dr^2 + 2r^2 g_{S^2}.$$

The metric g_{NRdS} is an *electro-vacuum*, that is it solves the Euclidean Einstein–Maxwell equations with cosmological constant equal to Λ. The (identity component of the) isometry group of g_{RNdS} is also $\mathrm{SO}(2) \times \mathrm{SO}(3)$, and when $Q = \Lambda = 0$, then $(\mathbb{R}^2 \times S^2, g_{RNdS})$ reduces to the ES manifold. Note that when charge is non-zero, then g_{NRdS} is not Einstein, and when the cosmological constant is non-zero, then g_{NRdS} is not ALF.

However, our method cannot be used on the Euclidean Kerr manifold, which corresponds to a “spinning” black hole with non-zero angular momentum, and it is a non-compact, complete, Ricci flat ALF manifold with $\mathrm{SO}(2) \times \mathrm{SO}(2)$ isometry only.

Relation to BPS monopoles and Hitchin’s equation. In connection with the first research direction mentioned above is the question of studying static, that is $\mathrm{SO}(2) \times \{1\}$ invariant instantons. This could not only lead to the discovery of new instantons, but also serve as a way to determine the number of connected components of $\mathcal{M}_{X,1}$, as mentioned before.

Simple computations show that these static instantons can be interpreted as BPS monopoles on a “cylinder" $\mathbb{R} \times S^2$ equipped with a metric which is asymptotically conical at one end and asymptotically hyperbolic in the other.

One could also impose $\mathrm{SO}(2) \times \mathrm{SO}(2) \subset \mathrm{S}(\mathrm{O}(2) \times \mathrm{O}(3)) \cong \mathrm{Isom}_+(X, g_X)$ invariance. In this case, the instantons correspond to solutions of an equation similar to Hitchin’s equations.

Understanding monopoles and Hitchin systems (in the sense described above) has the potential to be applicable to the Euclidean Kerr manifold, unlike the methods of this paper.

Non-minimal Yang–Mills and Ginzburg–Landau fields. As in [31, Section V.], the correspondence between instantons and vortices extends to a pairing between spherically symmetric solutions of the 4-dimensional (pure) $\mathrm{SU}(2)$ Yang–Mills equations and the 2-dimensional (critically coupled) Abelian Ginzburg–Landau equations, and this pairing preserves energy, in the sense of equation (2.8). Thus one can get information about non-minimal Yang–Mills fields on the ES manifold via understanding the Ginzburg–Landau equations on Σ, or vice versa.

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(Ákos Nagy) The Fields Institute, 222 College Street, Toronto, Ontario, Canada, M5T 3J1 & Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

E-mail address: contact@akosnagy.com

URL: akosnagy.com

(Gonçalo Oliveira) IMPA, Estrada Dona Castorina, 110, Jardim BotÃćnico | CEP 22460-320, Rio de Janeiro, RJ - Brasil

E-mail address: goliveira@impa.br

URL: w3.impa.br/~goliveira