Dynamic Screening in a Long Term Relationship

by

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Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Economics
in the Graduate School of Duke University
2009

Abstract

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Abstract

I characterize optimal multi-period contracts offered by a monopolist to a buyer whose private valuation evolves according to a branching process with privately known transition probability. The optimal contract can be implemented in a simple way, and presents the buyer with a tradeoff between a high initial fixed fee and low future prices. In an interaction with a long time horizon, the relationship will terminate prematurely with probability close to one. Optimal mechanisms are quite different from models in which the transition probability is known, and the buyer's private information is his initial valuation. Optimal contracts resemble the structure of term life insurance contracts, and have features similar to actual interactions between retailers and suppliers.

To Bruce Campbell

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List of Abbreviations and Symbols

Symbols

Put general notes about symbol usage in text here. Notice this text is double-spaced, as required. The text under “Abbreviations” is show single-spaced because of a bug. You can avoid that bug and have double-spaced text by following the directions there.

$\mathbb{X}$	A blackboard bold X. Neat.
$\mathcal{X}$	A caligraphic X. Neat.
$\mathfrak{X}$	A fraktur X. Neat.
$\mathbf{X}$	A boldface X.
X	A sans-serif X. Bad notation.
X	A roman X.

Abbreviations

Long lines in the `symbollist` environment are single spaced, like in the other front matter tables. This text block is single-spaced due to a bug! You must leave a blank line after the `\section*` and again before the `symbollist` to avoid this nastiness. Notice the blank lines around the `\section*{Symbols}}`above.

AR	Aqua Regia, also known as hydrocloric acid plus a splash of nitric acid.
SHORT	Notice the change in alignment caused by the label width between this list and the one above. Also notice that this multiline description is properly spaced.

OMFGTXTMSG4ME	Abbreviations/Symbols in the item are limited to about a quarter of the textwidth, so don’t pack too much in there. You’ll bust the margins and it looks really bad.

1

Dynamic Screening in a Long Term Relationship

1.1 Introduction

Long term contracts are frequently encountered in a variety of situations including trade, employment, finance, venture capital, supply chain management, and insurance. In many interactions in which long term contracts are commonly used, important features of the contracting environment can change through time. For example: a buyer's value for a good, an employee's outside options, a borrower's wealth, the probability of success in a business venture, a producer's cost, or the probability of an insurance claim, all change through time in a stochastic way. In many cases, these kinds of changes are privately observed by one of the contracting parties, creating an adverse selection problem in each period of the interaction; therefore, optimal contracts need to provide incentives for the informed party to reveal what it has learned about ongoing changes to the state of the environment. Optimal long term contracts in this kind of setting have been the subject of several recent papers (see Battaglini 2005, Pavan 2007).

The primary purpose of this paper is to assess the impact of an additional source of private information on the structure and properties of optimal long term contracts. Besides private information about the *state* of the contracting environment, in this analysis one party also has private information about the *manner in which the state evolves.* This information is summarized by a parameter of the stochastic process governing the state's evolution. I analyze this issue in the context of an ongoing trading relationship between a monopolist and a consumer.[1] In this relationship, the monopolist (seller) has all of the bargaining power and can credibly commit to the terms of trade for the entire interaction at the very beginning, but the consumer (buyer) has private information about his valuation for the good in each period and about a parameter of the stochastic process which governs the way his value evolves through time.

As one example of a setting with this type of information structure, consider a long term contract offered by a large retailer to a small producer of some consumer good. Under the terms of the contract, the retailer commits to allocate a certain amount of shelf space to the producer in exchange for a slotting fee. The producer's value for shelf space is influenced by production costs, which are privately known and can decrease over time because of innovations to the production process. Innovations occur stochastically, with the same probability in every period, and this probability is private information. The producer, therefore, has private information about both his current value for shelf space (determined by his cost), and the manner in which this value evolves (determined by the probability of an innovation).

As a second example, consider life insurance contracts. Under the terms of the contract, the insurance provider agrees to pay the beneficiaries specified by the agreement (usually dependents of the policyholder) a certain amount of money in case the policyholder dies, and individuals purchase life insurance in order to provide for their

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[1] This is the setting most frequently encountered in the literature, and describes many applications of the theory.

dependents in case of death. Certainly, the probability of death in a given period is a crucial factor in determining the value of life insurance to the individual. Because the individual has private information about his physical and mental health, family history, lifestyle, stress level, etc. his value for life insurance is private. He may also have private information about how this probability is likely to evolve in the future: for example, the individual may anticipate receiving a series of promotions at work (with a certain probability), each of which will lead to more responsibility, more stress, a more sedentary lifestyle, and more plane travel, all of which increase the chance of death and the therefore the value for life insurance. In this setting again, the buyer has private information about both his value for life insurance in each period and about the manner in which this value evolves through time.

In this paper I characterize profit maximizing N-period contracts for a monopolist facing a buyer whose private valuation evolves according to a branching process with a privately known transition probability. For this type of stochastic process, each transition changes the support of the buyer's future valuations and thus local changes to the valuation have long term consequences. Specifically, I assume that in each period, the buyer's valuation is multiplied by a random shock.[2] The magnitude of the shock can take on one of two values: the good shock, u, with probability λ, and the bad shock, $d < u$ with probability $1 - \lambda$. The quantities u and d are both common knowledge, but whether the shock in any period is good or bad is known only to the buyer. Furthermore, the parameter λ, which determines the distribution of the buyer's value in all future periods, is known only to the buyer at the beginning of the interaction. This parameter captures the buyer's private information about the evolution of his valuations. A buyer with a high value of λ is more likely to experience the good shock in each period, and the distribution for his value at any

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[2] The buyer's valuation in a given period is therefore the product of the shocks he has experienced over the course of the interaction. In the next version of this paper I plan to extend the model to allow for a more general relationship between the buyer's value and the history of shocks.

point in time first order dominates the distribution of a buyer with a lower value of λ.

The seller's goal at the beginning of the interaction is to design a long term contract (or other general sales mechanism) to maximize her expected profit. The techniques of mechanism design allow me to study the general problem in this setting without restricting attention to any particular class of contracts or mechanisms or otherwise limiting the power of the seller to design the interaction in the way that she deems fit. Since I assume that the seller has full commitment power,[3] the Revelation Principle[4] allows me to represent any equilibrium outcome of any game by an incentive compatible direct revelation mechanism. A direct revelation mechanism maps a history of buyer reports about his private information into a probability of sale and an associated payment in each period. A direct revelation mechanism is incentive compatible if, in equilibrium, it provides the buyer with incentives to truthfully report his private information in each period. Since the seller can not force the buyer to participate in the interaction, the mechanism must also satisfy individual rationality; that is, the mechanism must provide the buyer with incentives to voluntarily participate. The seller's problem in its most general form is to design a direct revelation mechanism that maximizes the expected payment she receives from the buyer, given the incentive compatibility and individual rationality requirements.[5]

The main result of this paper is that any profit-maximizing mechanism can be implemented by a contract with a simple structure. The seller commits to a finite menu of *price plans*. Each price plan presents the buyer with a sign-up fee and a price for the good in each period of the interaction. On the basis of λ, the buyer can select any plan in the menu, and then participate or not in future periods as he

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[3] Under any optimal contract the seller will be tempted to renege throughout the interaction. I assume that the seller can credibly committ not to do so.

[4] Baron and Myerson (1982), Myerson (1979)

[5] The seller's cost is normalized to zero.

sees fit. Because the prices for all periods are set by the seller at the very beginning of the interaction, the optimal mechanism can be implemented without eliciting information from the buyer over the lifetime of the contract; the only information that the seller needs to elicit from the buyer is his choice of price plan, which he reports at the beginning of the interaction.

Each of the price plans in the menu begins with a *honeymoon phase* of length H. During the honeymoon phase, the price for the good is the lowest possible valuation that any buyer could have. After the honeymoon phase is over, the price of the good is multiplied by a factor of u in each period.[6] In a price plan with a longer honeymoon phase the price of the good in every period is (weakly) lower than in a plan with a shorter honeymoon phase. In order to select a price plan with a longer honeymoon phase, the buyer must pay a larger sign-up fee. Buyers with high values of λ will optimally choose price plans with longer honeymoons and higher sign-up fees.

Under any price plan, the price of the good grows in the long run by a factor of u, but the buyer's valuation grows by a factor of u only some of the time (with probability λ). In a sufficiently long interaction, with high probability, the buyer's value will eventually drop below the price. Once this happens, the buyer's value in all subsequent periods will remain below the price, and the buyer will never choose to buy the good again, effectively terminating the relationship prematurely. This kind of endogenous early termination would not occur in a model in which the value of λ were known, and the buyer's private information was his first valuation, the standard case considered in the literature. Other differences between the "standard case" and the optimal contract when λ is private will be discussed in sections four and five. Connections between the optimal contract of this paper and features of

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[6] During the honeymoon phase, in period $k = 1, ..., H$,the price of the good is d^k. j periods after the end of the honeymoon phase, that is, in period $j + H$, the price is $d^H u^j$.

real-world interactions and contracts will be discussed in section six.

The analysis in this paper is closely related to the literature on dynamic price discrimination, for example Battaglini (2005), Rustichini and Wolinsky (1995), Roberts (1982),Townsend (1982), Pavan (2007). I depart from the common assumptions of this literature in two ways. First, most of these papers consider stochastic processes for the buyer's valuation that have the same finite support in every period.[7] In this kind of an environment, changes to the buyer's valuation are transitory. Starting from any value in the support, the buyer's value will eventually return to any other state. In my analysis, valuation shocks change the support of the buyer's future valuation and therefore have long run consequences. More substantially, in all of these papers, the buyer's initial valuation is private information but the stochastic process giving rise to this buyer's valuation is assumed to be common knowledge. This assumption that the stochastic process is known is quite strong and has important consequences for the optimal mechanism; the goal of this paper is to analyze the implications of relaxing this assumption by making the stochastic process private information.

Courty and Li (2002) also analyze a price discrimination model in which the buyer is screened sequentially, first in his private information about a parameter of the distribution giving rise to his valuation, and then in the valuation itself. In their model there are multiple periods, but the good is sold only once, not multiple times as it is here. Miravete (2003) looks at this type of sequential screening in the context of contracts for telephone service and discovers significant evidence that sequential screening considerations play a role in the design of these contracts.

This paper is also related to the literature on price experimentation and learning,

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[7] Notable exceptions are Pavan (2007) and Pavan, Segal and Toikka (2008). Pavan analyzes properties of the optimal mechanism and how they depend on the underlying stochastic process, allowing for processes with long run consequences, including the binomial process considered here. Still, in his analysis there is no private information about the process itself.

for example Kennan (2001), Taylor and Loginova (2008). In Taylor and Loginova, the principal tries to learn a parameter of the stochastic process generating the buyer's value by offering different prices over time, and the buyer may strategically reject or accept the seller's offers in order to hide or signal information about his stochastic process. In contrast to the present analysis, in Taylor and Loginova, the principal can not commit to long term contracts and therefore must experiment in order to learn about the buyer's parameter. Finally, this paper is related to the literature on dynamic agency models with moral hazard and adverse selection, for example Sannikov (2007), Dai, Lewis and Lopomo (2006), and Lewis and Sappington (1997). In Lewis and Sappington, an agent is hired by a principal to work on a project for two periods. Neither the agent's ability, nor the agent's effort, are observable to the principal. The principal must therefore design a compensation contract to induce both the right amount of effort on the part of the agent and to screen the agent's ability initially. Lewis and Sappington's main focus is on the effect of the outcome of the project in the first period on the incentives offered to the agent in the second period. In their model, the stochastic evolution of the project is under the control of the agent and the state of the project is observable; in this paper, the evolution of the buyer's valuation is exogenous and private.

One paper that deserves special mention is a working paper by Pavan, Segal, and Toikka (2008). Their paper develops a theory of dynamic mechanism design in a general environment. They allow for settings in which the distribution of the agent's (buyer's) type at each point in time can depend on both his history of past type resolutions[8] and on the history of outcomes prescribed by the mechanism. They also allow for general payoff functions and do not constrain payoffs to be time-separable. They provide extremely useful and widely applicable results for the characterization of incentive compatible mechanisms, including a dynamic analogue of the envelope

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[8] and is thus Non-Markov

formula familiar from static mechanism design, and conditions on the environment for which this envelope formula is also a sufficient condition. Their setting is sufficiently general to explore some of the issues that I consider in this paper; however, the model presented here violates some technical assumptions made in their analysis. Because of this, I derive the results without referring to their work explicitly.

The paper proceeds as follows: in sections two and three, I derive the solution of the model, and discuss some of its the comparative statics properties. In sections three and four I discuss properties of the optimal mechanism and assess the implications of private information about the stochastic process. In section five, I discuss implications of the model in the context of the two previously mentioned examples: supply chain management and life insurance. Section six concludes.

1.2 Two Period Model

I begin the analysis by solving the two period version of the model. This solution will serve as the basis step of the inductive argument which I use to find the solution of the model with arbitrary time horizon. The main features of the optimal contract for the two period case are also present in the optimal contract for the N period case, and the two period model is therefore a good starting point for the analysis.

One seller faces one buyer.[9] The buyer has a unit demand for the seller's good in period $k = 1, 2$. Specifically, if the buyer is allowed to purchase the good in period k with probability q_k [10] at a price t_k, the buyer's period k utility will be

$$v_k q_k - t_k$$

Scalar v_k represents the buyer's period k value for the good. This value evolves

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[9] Because the seller has zero cost of production, you can also imagine that the seller faces a continuum of buyers.

[10] Alternatively, you can think of $q_k \in [0, 1]$ as a quantity of the good.

stochastically through time according to a *recombinant binomial tree process*[11] with transition probability λ. Formally, the process is described by:

$$\begin{aligned} Pr(v_{k+1} = uv_k | v_k) &= \lambda \\ Pr(v_{k+1} = dv_k | v_k) &= 1 - \lambda \\ v_0 &= 1 \\ u &\geq d \geq 0 \end{aligned}$$

Conditional on the buyer's value at time k, there are two possible values at time $k+1$. One possible value is u multiplied by the value today, and the other possible value is d multiplied by the value today. Think of the buyer's value undergoing one of two possible shocks in each period, a good shock or a bad shock. The good shock results in a fixed percentage change in the buyer's valuation, that leaves the buyer with a higher valuation than the percentage change associated with the bad shock. The probability of the buyer experiencing the good shock is the same in each period of the interaction, and is given by parameter λ. Because λ captures the buyer's tendency or propensity to experience a relatively higher value, I sometimes refer to λ as the *propensity parameter*.

I assume that the magnitudes of the good and bad shocks u, d and the starting point of the process, $v_0 = 1$ are common knowledge. The parameter λ is privately known to the buyer time zero, while the seller believes that λ is drawn from a twice differentiable cdf F with support on $[0, 1]$.[12] At the beginning of each period $k = 1, 2$, the buyer privately learns his period k valuation shock, u or d, and thereby his period k valuation. In this model, the buyer has two different kinds of private information. In period zero, the buyer has private information about the evolution

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[11] This process is commonly used in finance to model stock prices. For example: Cox, Ross and Rubinstein. "Option Pricing: A Simplified Approach." Journal of Financial Economics (1979) Vol. 7 pp. 229-263

[12] There are some regularity conditions on F which will be discussed later in the paper.

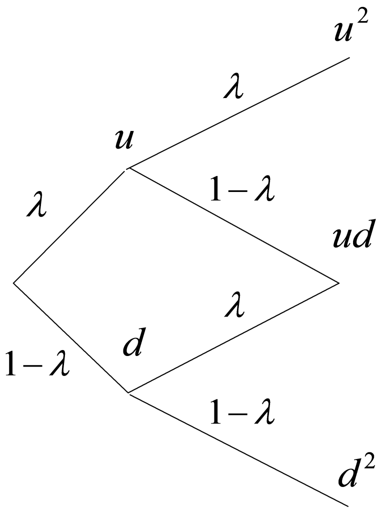

FIGURE 1.1: A way to visualize the stochastic process.

of future demand and in each period, the buyer has private information about that period's valuation shock, or realized demand.

According to the Revelation Principle, any sales procedure to which the seller can commit at time zero, is outcome-equivalent to an *incentive compatible direct revelation mechanism.* In a dynamic environment, a direct revelation mechanism can be described by a family of functions, which map histories of buyer reports about his private information into an outcome for each period. In the trade setting considered here, the outcome in each period is described a probability of delivery of the good (sometimes referred to as an "allocation"), and an associated payment. Therefore, a direct revelation mechanism maps histories of buyer reports about his private information into a delivery probability and an associated payment in each period.

In period zero, the buyer's private information is the parameter λ, and, since the buyer first demands the good in period one, the delivery probability in period zero is zero, but the buyer may be asked to make an initial payment or "sign-up fee." The direct revelation mechanism therefore maps a period zero report of λ, denoted λ', into a payment $T_0(\lambda')$. In period one, the buyer is asked to report his new information, that is, his first-period valuation shock, (either u or d), and on the basis of the reported shock and his reported value of λ, the buyer is assigned a delivery probability and a period one payment. The direct revelation mechanism therefore maps reported history (λ', r_1) into a delivery probability $Q_1(\lambda', r_1)$ and an associated payment $T_1(\lambda', r_1)$. In period two, the buyer is again asked to report his new information, that is, his second period valuation shock, and on the basis of his second reported shock, his first reported shock, and his reported value of λ is again assigned a delivery probability and period two payment.[13] The direct revelation mechanism therefore maps reported history (λ', r_1, r_2) into a delivery probability $Q_2(\lambda', r_1, r_2)$ and an associated payment $T_2(\lambda', r_1, r_2)$.[14]

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[13] Because of two sided commitment, the payment functions can be combined into one lifetime payment, but I find the above representation to be more convenient.

[14] Notice, that the buyer is only asked to report the new information that he has learned in each period, and is not required to re-report all of his past information in every period. Allowing (or requiring) the buyer to re-report his entire history of types in each period does not change the set of incentive compatible mechanisms. The two approaches are equivalent, and I find the first approach to be more convenient because the buyer is only ever asked to report a single piece of information.

Period 0

0.1	Buyer learns λ, seller learns her prior $F(\lambda)$
0.2	Seller announces and commits to a direct revelation mechanism
0.3	Buyer makes a participation decision
0.4	If he chooses to participate, buyer reports λ' to the seller and pays $T_0(\lambda')$

Period 1

1.1	Buyer draws his period one valuation shock $s_1 \in d, u$
1.2	Buyer sends a report of his period one shock $r_1 \in d, u$ to the seller
1.3	Buyer is assigned the object with probability $Q_1(\lambda', r_1)$ and pays $T_1(\lambda', r_1)$

Period 2

2.1	Buyer draws his period two valuation shock $s_2 \in d, u$
2.2	Buyer sends a report of his period two shock $r_2 \in d, u$ to the seller
2.3	Buyer is assigned the object with probability $Q_2(\lambda', r_1, r_2)$ and pays $T_1(\lambda', r_1, r_2)$

Game Ends

The above table presents a detailed timeline of the interaction.

1.2.1 Incentive Compatibility

According to the Revelation Principle, the outcome of any game between buyer and seller can be represented by an outcome of some *incentive compatible* direct revelation mechanism. In a static setting, incentive compatibility simply requires that the buyer has incentives to report his private information truthfully. In a dynamic setting, the buyer is presented with multiple pieces of information over time, and a strategy for the buyer is a complete contingent plan for reporting his private information at every information set, including information sets that would never be reached in an equilibrium of the mechanism. In particular, any mechanism induces an *equilibrium* reporting plan, which is the reporting plan that the buyer anticipates using if asked to participate in the mechanism.

Indeed, faced with a general direct revelation mechanism, the buyer plans to

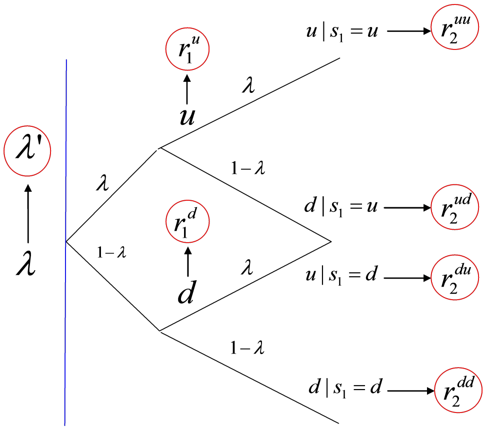

FIGURE 1.2: The buyer's equilibrium reporting plan in an arbitrary mechanism.

optimally report a value of λ' (not necessarily equal to λ) in period zero. Given that he acted as planned in period zero, if he then experiences a valuation shock of u in period one, he plans to optimally report a value of r_1^u, but if his first period valuation shock is d, then he plans to optimally report r_1^d. In period two, given that he acted according to plan in periods zero and one, then if he experienced a good shock in period one, and then another good shock in period two, the buyer plans to optimally report r_2^{uu}; which is not necessarily the same as r_2^{du} his anticipated report of a good shock in period two, given a bad shock in period one. In period two (again assuming everything went according to plan in periods zero and one), if the buyer experiences a bad shock, but experienced a good shock in period one, he will plan to report r_2^{ud}, and he anticipates reporting a bad shock in period two, given a bad shock in period one as r_2^{dd}. The following picture represents such an equilibrium reporting plan:

In a dynamic setting, the Revelation Principle guarantees that any sales procedure has an outcome equivalent representation as a direct revelation mechanism in which truth telling is the buyer's equilibrium reporting plan.[15] Therefore, I can restrict attention to direct revelation mechanisms in which the buyer's equilibrium reporting

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[15] The Revelation Principle does not guarantee that there is no loss of generality in imposing truth-telling constraints following past deviations.

plan is truthful, given that an equilibrium reporting plan which is truthful brings him higher expected utility than any other equilibrium reporting plan.

Mathematically, this leads to the following set of constraints:

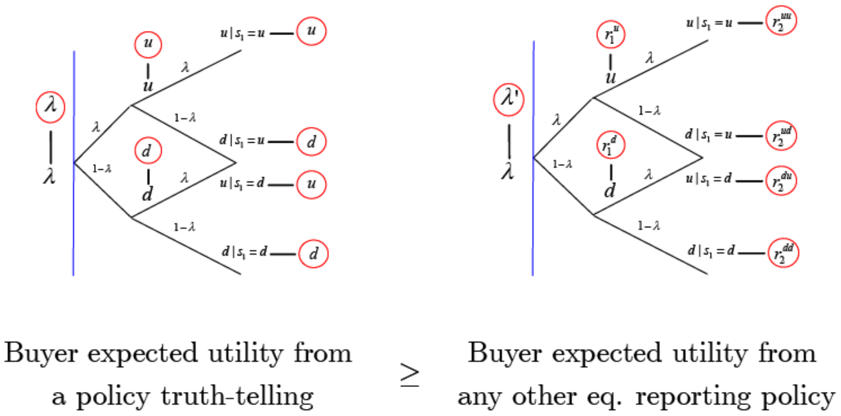

FIGURE 1.3: The buyer's equilibrium reporting plan in an arbitrary mechanism.

$$\forall \lambda, \lambda' \in (0,1),\ r_i^{\cdot} \in \{u, d\}$$

$$\begin{aligned}
-T_0(\lambda) + \lambda\left[Q_1(\lambda,u)\,u - T_1(\lambda,u)\right] + (1-\lambda)\left[Q_1(\lambda,d)\,d - T_1(\lambda,d)\right] +&\\
\lambda^2\left[Q_2(\lambda,u,u)\,u^2 - T_2(\lambda,u,u)\right] +&\\
\lambda(1-\lambda)\left[Q_2(\lambda,u,d)\,ud - T_2(\lambda,u,d)\right] +&\\
\lambda(1-\lambda)\left[Q_2(\lambda,d,u)\,ud - T_2(\lambda,d,u)\right] +&\\
(1-\lambda)^2\left[Q_2(\lambda,d,d)\,d^2 - T_2(\lambda,d,d)\right]&\\
\geq&\\
-T_0(\lambda') + \lambda\left[Q_1(\lambda',r_1^u)\,u - T_1(\lambda',r_1^u)\right] + (1-\lambda)\left[Q_1\left(\lambda',r_1^d\right)d - T_1\left(\lambda',r_1^d\right)\right] +&\\
\lambda^2\left[Q_2(\lambda',r_1^u,r_2^{uu})\,u^2 - T_2(\lambda',r_1^u,r_2^{uu})\right] +&\\
\lambda(1-\lambda)\left[Q_2\left(\lambda',r_1^u,r_2^{ud}\right)ud - T_2\left(\lambda',r_1^u,r_2^{ud}\right)\right] +&\\
\lambda(1-\lambda)\left[Q_2\left(\lambda',r_1^d,r_2^{du}\right)du - T_2\left(\lambda',r_1^d,r_2^{du}\right)\right] +&\\
(1-\lambda)^2\left[Q_2\left(\lambda',r_1^d,r_2^{dd}\right)d^2 - T_2\left(\lambda',r_1^d,r_2^{dd}\right)\right]&
\end{aligned}$$

This set of conditions (IC) is the complete set of incentive compatibility constraints for this problem. This large set of constraints is difficult to work with directly, but it seems reasonable to think that this large set of constraints contains a lot of redundancy. In the next section I focus on reporting policies that deviate from truth-telling in some simple ways, and derive a smaller and much simpler set of necessary constraints. I will then solve the seller's problem using this smaller set of constraints and derive the optimal mechanism for this relaxed problem. In the appendix, I use properties of the solution to the relaxed problem to show that it satisfies conditions (IC). Because the solution to the relaxed problem is feasible in the original problem, it is also optimal in the original problem.

1.2.2 Relaxed Incentive Constraints

The set of inequalities in (IC) is a complicated object that is difficult to work with directly. By considering some simple reporting policies that deviate from truth telling, I derive a set of necessary conditions that is easier to work with. I will use these simpler constraints as the constraints for the relaxed problem. Following the best traditions of dynamic programming, the policies that I consider in the relaxed problem involve single deviations only, and following the best traditions of mechanism design, the deviations are downward, with high types mimicking the reports of low types. These constraints are certainly necessary conditions for incentive compatibility, but at this point in the analysis there is no way of knowing whether they are also sufficient. In fact, there are situations in dynamic mechanism design problems in which the solution to a relaxed problem formulated by considering single deviations only is *not* feasible in the original problem,[16] but in this case the solution to the relaxed problem satisfies all of the (IC) constraints, as verified in the appendix.

Period 2 Constraints.

Suppose the buyer follows a reporting policy that involves only one lie. The lie occurs when the buyer draws a u in period one (which he reports truthfully) followed by another u in period two, which the buyer reports to be a d. Call this reporting policy 1:

Reporting Policy 1				
Period 0	$\lambda' = \lambda$			
Period 1	$r_1^u = u$	$r_1^d = d$		
Period 2	$r_2^{uu} = d$	$r_2^{ud} = d$	$r_2^{du} = u$	$r_2^{dd} = d$

Represented pictorally:

Substituting this reporting policy into IC leads to constraints (DIC2—u)

$$Q_2(\lambda, u, u)u^2 - T_2(\lambda, u, u) \geq Q_2(\lambda, u, d)u^2 - T_2(\lambda, u, d)$$

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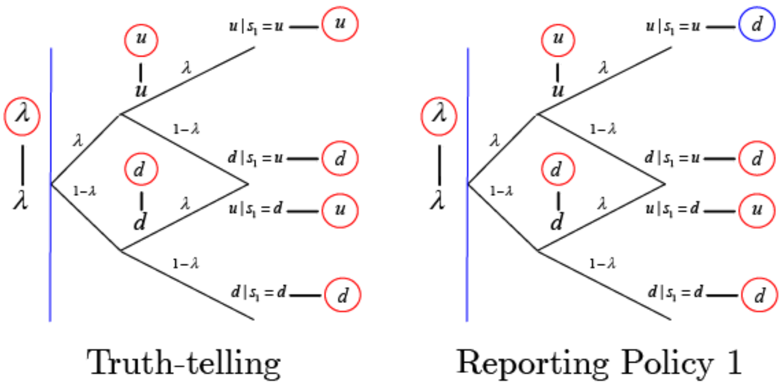
[16] See for example Boleslavsky (2009, working).



Truth-telling Reporting Policy 1

FIGURE 1.4: Reporting Policy One

Suppose again that the buyer follows a reporting policy that involves only one lie. The lie occurs when the buyer draws a d in period one (which he reports truthfully) followed by a u in period two, which the buyer reports to be a d. Call this reporting policy 2:

Reporting Policy 2				
Period 0	$\lambda' = \lambda$			
Period 1	$r_1^u = u$	$r_1^d = d$		
Period 2	$r_2^{uu} = u$	$r_2^{ud} = d$	$r_2^{du} = d$	$r_2^{dd} = d$

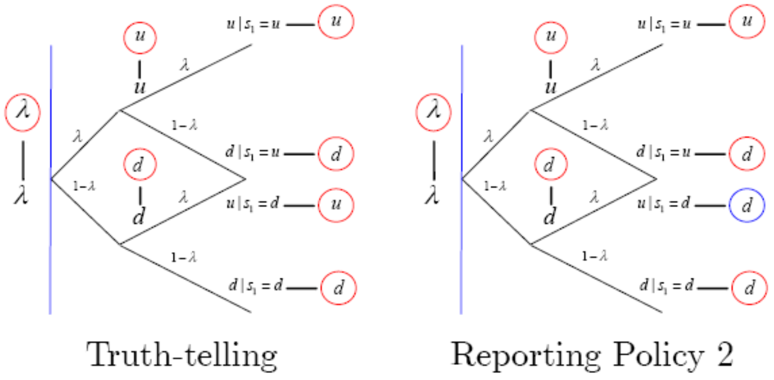
Represented pictorally:

Substituting this reporting policy into IC leads to constraints (DIC2—d)

$$Q_2(\lambda, d, u)ud - T_2(\lambda, d, u) \geq Q_2(\lambda, d, d)ud - T_2(\lambda, d, d)$$

Period 1 Constraint.

Let's turn our attention now to a policy which still involves only one lie. The lie occurs when the buyer draws a u in period one, but reports d instead. Call this reporting policy 3:

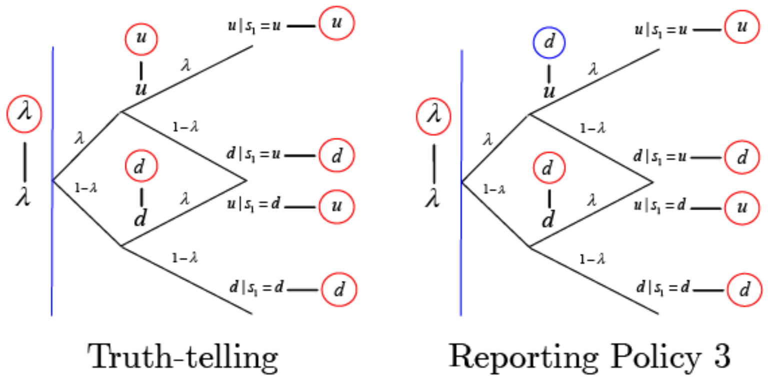


Truth-telling Reporting Policy 2

FIGURE 1.5: Reporting Policy Two

Reporting Policy 3				
Period 0	$\lambda' = \lambda$			
Period 1	$r_1^u = d$	$r_1^d = d$		
Period 2	$r_2^{uu} = u$	$r_2^{ud} = d$	$r_2^{du} = u$	$r_2^{dd} = d$

Represented pictorally:



Truth-telling Reporting Policy 3

FIGURE 1.6: Reporting Policy Three

This leads to constraint (DIC1)

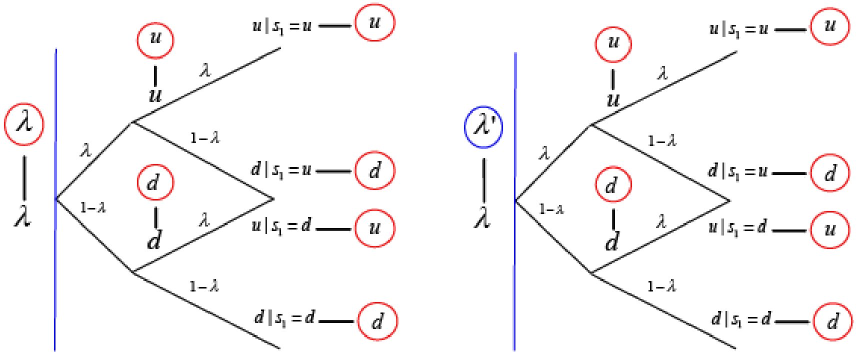
$$
\begin{aligned}
Q_1(\lambda,u)u - T_1(\lambda,u) + \lambda[Q_2(\lambda,u,u)u^2 - T_2(\lambda,u,u)] &+ \\
(1-\lambda)[Q_2(\lambda,u,d)ud - T_2(\lambda,u,d)]& \\
&\geq \\
Q_1(\lambda,d)u - T_1(\lambda,d) + \lambda[Q_2(\lambda,d,u)u^2 - T_2(\lambda,d,u)] &+ \\
(1-\lambda)[Q_2(\lambda,d,d)ud - T_2(\lambda,d,d)]&
\end{aligned}
$$

Period 0 Constraint.

Finally, consider any policy involving deviations only in λ. In a slight abuse of notation, call this reporting policy 4 (which is actually a continuum of reporting policies)

Reporting Policy 3				
Period 0	$\lambda' \neq \lambda$			
Period 1	$r_1^u = u$	$r_1^d = d$		
Period 2	$r_2^{uu} = u$	$r_2^{ud} = d$	$r_2^{du} = u$	$r_2^{dd} = d$

Represented pictorally:



Truth-telling Reporting Policy 4

FIGURE 1.7: Reporting Policy Three

This leads to constraint (DIC λ)

$$\forall \lambda, \lambda' \in (0,1)$$

$$
\begin{aligned}
&-T_0(\lambda) + \lambda\left[Q_1(\lambda,u)\,u - T_1(\lambda,u)\right] + (1-\lambda)\left[Q_1(\lambda,d)\,d - T_1(\lambda,d)\right] + \\
&\lambda^2\left[Q_2(\lambda,u,u)\,u^2 - T_2(\lambda,u,u)\right] + \\
&\lambda(1-\lambda)\left[Q_2(\lambda,u,d)\,ud - T_2(\lambda,u,d)\right] + \\
&\lambda(1-\lambda)\left[Q_2(\lambda,d,u)\,ud - T_2(\lambda,d,u)\right] + \\
&(1-\lambda)^2\left[Q_2(\lambda,d,d)\,d^2 - T_2(\lambda,d,d)\right] \\
&\geq \\
&-T_0(\lambda') + \lambda\left[Q_1(\lambda',u)\,u - T_1(\lambda',u)\right] + (1-\lambda)\left[Q_1(\lambda',d)\,d - T_1(\lambda',d)\right] + \\
&\lambda^2\left[Q_2\left(\lambda',u,u^2\right)u^2 - T_2\left(\lambda',u,u^2\right)\right] + \\
&\lambda(1-\lambda)\left[Q_2(\lambda',u,d)\,ud - T_2(\lambda',u,d)\right] + \\
&\lambda(1-\lambda)\left[Q_2(\lambda',d,u)\,du - T_2(\lambda',d,u)\right] + \\
&(1-\lambda)^2\left[Q_2(\lambda',d,d)\,d^2 - T_2(\lambda',d,d)\right]
\end{aligned}
$$

or

$$\pi(\lambda|\lambda) \geq \pi(\lambda'|\lambda)$$

where

$$
\begin{aligned}
&\pi(\lambda'|\lambda) = \\
&-T_0(\lambda') + \lambda\left[Q_1(\lambda',u)\,u - T_1(\lambda',u)\right] + (1-\lambda)\left[Q_1(\lambda',d)\,d - T_1(\lambda',d)\right] + \\
&\lambda^2\left[Q_2\left(\lambda',u,u^2\right)u^2 - T_2\left(\lambda',u,u^2\right)\right] + \\
&\lambda(1-\lambda)\left[Q_2(\lambda',u,d)\,ud - T_2(\lambda',u,d)\right] + \\
&\lambda(1-\lambda)\left[Q_2(\lambda',d,u)\,du - T_2(\lambda',d,u)\right] + \\
&(1-\lambda)^2\left[Q_2(\lambda',d,d)\,d^2 - T_2(\lambda',d,d)\right]
\end{aligned}
$$

Individual Rationality Constraints.

I assume two-sided commitment, so that a buyer who agrees to participate in period zero must participate for all subsequent periods. This form of participation constraint allows the seller the most freedom in designing the mechanism. The assumption of two-sided commitment can be relaxed for the optimal mechanism, as I discuss later. For now, under two-sided commitment, the seller must provide each type λ a weak incentive to participate, leading to constraint (IR):

$$\forall\lambda, \quad \pi(\lambda|\lambda) \geq 0$$

1.2.3 The Relaxed Problem

Abandoning the entire family of incentive constraints (IC) in favor of the simpler set of necessary constraints derived in the previous section, I now formulate the relaxed version of the seller's problem.

$$\begin{aligned}&\max E_\lambda\{T_0(\lambda) + \lambda T_1(\lambda, u) + (1-\lambda)T_1(\lambda, d) +\\ &\lambda^2 T_2(\lambda, u, u) + (\lambda - \lambda^2)[T_2(\lambda, u, d) + T_2(\lambda, d, u)] + (1-\lambda)^2 T_2(\lambda, d, d)\}\end{aligned}$$

subject to $\forall\lambda, \lambda' \in (0,1)$

$$Q_2(\lambda, u, u)u^2 - T_2(\lambda, u, u) \geq Q_2(\lambda, u, d)u^2 - T_2(\lambda, u, d) \qquad \text{DIC2-u}$$

$$Q_2(\lambda, d, u)ud - T_2(\lambda, d, u) \geq Q_2(\lambda, d, d)ud - T_2(\lambda, d, d) \qquad \text{DIC2-d}$$

$$\begin{aligned}&Q_1(\lambda, u)u - T_1(\lambda, u) +\\ &\lambda[Q_2(\lambda, u, u)u^2 - T_2(\lambda, u, u)] + (1-\lambda)[Q_2(\lambda, u, d)ud - T_2(\lambda, u, d)] \geq \qquad \text{DIC1}\\ &Q_1(\lambda, d)u - T_1(\lambda, d) +\\ &\lambda[Q_2(\lambda, d, u)u^2 - T_2(\lambda, d, u)] + (1-\lambda)[Q_2(\lambda, d, d)ud - T_2(\lambda, d, d)]\end{aligned}$$

$$\pi(\lambda|\lambda) \geq \pi(\lambda'|\lambda) \qquad \text{IC-}\lambda$$

$$\pi(\lambda|\lambda) \geq 0 \qquad \text{IR}$$

Constraints DIC2-u, DIC2-d and DIC1 are inequality constraints, which need not bind for a mechanism to be feasible. To incorporate the possibility that in the optimal mechanism these constraints do not bind, I introduce the functions $\delta_1(\lambda)$, $\delta_2^u(\lambda)$, and $\delta_2^d(\lambda) \geq 0$ which represent the slacks in constraints DIC1, DIC2-u, and DIC2-d.

$$Q_2(\lambda, u, u)u^2 - T_2(\lambda, u, u) \equiv Q_2(\lambda, u, d)u^2 - T_2(\lambda, u, d) + \delta_2^u(\lambda) \qquad \text{DIC2-u}$$

$$Q_2(\lambda, d, u)ud - T_2(\lambda, d, u) \equiv Q_2(\lambda, d, d)ud - T_2(\lambda, d, d) + \delta_2^d(\lambda) \qquad \text{DIC2-d}$$

$$\begin{array}{ll} Q_1(\lambda, u)u - T_1(\lambda, u)+ & \\ \lambda[Q_2(\lambda, u, u)u^2 - T_2(\lambda, u, u)] + (1-\lambda)[Q_2(\lambda, u, d)ud - T_2(\lambda, u, d)] \equiv & \text{DIC1} \\ Q_1(\lambda, d)u - T_1(\lambda, d)+ & \\ \lambda[Q_2(\lambda, d, u)u^2 - T_2(\lambda, d, u)] + (1-\lambda)[Q_2(\lambda, d, d)ud - T_2(\lambda, d, d)] + \delta_1(\lambda) & \end{array}$$

Applying the Envelope Theorem[17] to condition $(IC\ \lambda)$, with $\Pi(\lambda) = \pi(\lambda|\lambda)$, and then substituting the other constraints above gives[18]

$$\begin{array}{rcl} \frac{\partial \Pi}{\partial \lambda} & = & Q_1(\lambda, d)(u-d) + [Q_2(\lambda, u, d) + Q_2(\lambda, d, u)]\lambda(u^2 - ud)+ \\ & & Q_2(\lambda, d, d)2(1-\lambda)(ud - d^2) + \lambda\delta_2^u(\lambda) + (1-\lambda)\delta_2^d(\lambda) + \delta_1(\lambda) \end{array}$$

and therefore,

$$\begin{array}{rcl} \Pi(\lambda) & = & \Pi(0) + \int_0^\lambda Q_1(\lambda, d)(u-d) + [Q_2(\lambda, u, d) + Q_2(\lambda, d, u)]\lambda(u^2 - ud)+ \\ & & Q_2(\lambda, d, d)2(1-\lambda)(ud - d^2) + \lambda\delta_2^u(\lambda) + (1-\lambda)\delta_2^d(\lambda) + \delta_1(\lambda)d\lambda \end{array}$$

Conceptually, $\Pi(\lambda)$ is the amount of surplus the mechanism must give to a buyer with type λ in order to induce him to report λ truthfully, given the relaxed structure of the incentives for truthful reporting of the valuation shocks.

I will simplify the problem further. From the expression for the derivative of $\Pi\,(\lambda)$, it is clear that since the allocation functions and slack functions must be non-negative,

---

[17] I use the version of the Envelope Theorem presented in Milgrom and Segal 2002. This version of the theorem requires absolute constinuity and boundedness of the function $\pi\,(\lambda'|\lambda)$ treated as a function of λ only (with λ' held fixed). Because in my case the function $\pi\,(\lambda'|\lambda)$ is a polynomial on the closed interval from zero to one, the conditions are satisfied.

[18] calculations in the appendix

$\frac{\partial \Pi}{\partial \lambda} \geq 0$. Therefore, since $\Pi(\lambda)$ is weakly increasing, if the IR constraint $\Pi(\lambda) \geq 0$ binds at $\lambda = 0$, it is satisfied for all other values of λ, and therefore, individual rationality need only be imposed for the lowest possible value of the propensity parameter. Furthermore, recognizing that the expected payment conditional on λ can be written as the difference between the expected social surplus conditional on λ and the buyer's expected surplus conditional on λ, $\Pi(\lambda)$

$$\begin{aligned}
&E_\lambda\{T_0(\lambda) + \lambda T_1(\lambda, u) + (1-\lambda)T_1(\lambda, d) + \\
&\lambda^2 T_2(\lambda, u, u) + (\lambda - \lambda^2)[T_2(\lambda, u, d) + T_2(\lambda, d, u)] + (1-\lambda)^2 T_2(\lambda, d, d)\} = \\
&\\
&E_\lambda\{\lambda Q_1(\lambda, u)u + (1-\lambda)Q_1(\lambda, d)d + \\
&\lambda^2 Q_2(\lambda, u, u)u^2 + (\lambda - \lambda^2)[Q_2(\lambda, u, d) + Q_2(\lambda, d, u)]ud + (1-\lambda)^2 Q_2(\lambda, d, d)d^2 - \Pi(\lambda)\}
\end{aligned}$$

I rewrite the problem one more time as

$$\begin{aligned}
&\max E_\lambda\{\lambda Q_1(\lambda, u)u + (1-\lambda)Q_1(\lambda, d)d + \\
&\lambda^2 Q_2(\lambda, u, u)u^2 + (\lambda - \lambda^2)[Q_2(\lambda, u, d) + Q_2(\lambda, d, u)]ud + \\
&(1-\lambda)^2 Q_2(\lambda, d, d)d^2 - \Pi(\lambda)\}
\end{aligned}$$

subject to $\forall \lambda \in (0,1)$

$$\begin{aligned}
&\delta_1(\lambda) \geq 0 && \text{C1} \\
&\delta_2^u(\lambda) \geq 0 && \text{C2-u} \\
&\delta_2^d(\lambda) \geq 0 && \text{C2-d}
\end{aligned}$$

$$\begin{aligned}
\Pi(\lambda) \ &= \Pi(0) + \int_0^\lambda Q_1(\lambda, d)(u-d) + [Q_2(\lambda, u, d) + Q_2(\lambda, d, u)]\lambda(u^2 - ud) + && \text{C3} \\
&Q_2(\lambda, d, d)2(1-\lambda)(ud - d^2) + \lambda\delta_2^u(\lambda) + (1-\lambda)\delta_2^d(\lambda) + \delta_1(\lambda)d\lambda
\end{aligned}$$

$$\Pi(0) \geq 0 \qquad \text{C4}$$

Conceptually, the inequalities in C1-C4 of the second version of the relaxed problem correspond to conditions DIC, IC-λ, and IR of the initial formulation. It is clear from constraint C2 that any positive value for $\Pi(0)$, $\delta_1(\lambda)$, $\delta_2^u(\lambda)$ or $\delta_2^d(\lambda)$ only increases the buyer's expected surplus without affecting the social surplus and therefore reduces the principal's profit. Therefore, in any optimal mechanism, $\Pi(0) = 0$ and the slacks $\delta_1(\lambda)$, $\delta_2^u(\lambda)$, and $\delta_2^d(\lambda)$ must all equal zero for all λ (except possibly a set of measure zer). I proceed as in the static case.[19] By reversing the order of integration in the expression for the seller's profit, I obtain an integral for the seller's expected profit which can be maximized by maximizing the integrand pointwise. The integrand is:

$$\begin{aligned}
&\lambda Q_1(\lambda, u)u+\\
&\lambda^2 Q_2(\lambda, u, u)u^2+\\
&\left[(1-\lambda)d - \tfrac{1-F(\lambda)}{f(\lambda)}(u-d)\right] Q_1(\lambda, d)+\\
&\lambda u\left[(1-\lambda)d - \tfrac{1-F(\lambda)}{f(\lambda)}(u-d)\right] Q_2(\lambda, u, d)+\\
&\lambda u\left[(1-\lambda)d - \tfrac{1-F(\lambda)}{f(\lambda)}(u-d)\right] Q_2(\lambda, d, u)+\\
&(1-\lambda)d\left[(1-\lambda)d - 2\tfrac{1-F(\lambda)}{f(\lambda)}(u-d)\right] Q_2(\lambda, d, d)
\end{aligned}$$

and the optimal allocation functions are therefore

$$\begin{aligned}
&Q_1(\lambda, u) = 1\\
&Q_2(\lambda, u, u) = 1\\
&Q_1(\lambda, d) = 1 \quad \text{if} \quad \left[(1-\lambda)d - \tfrac{1-F(\lambda)}{f(\lambda)}(u-d)\right] \geq 0 \quad \text{else } 0\\
&Q_2(\lambda, u, d) = 1 \quad \text{if} \quad \left[(1-\lambda)d - \tfrac{1-F(\lambda)}{f(\lambda)}(u-d)\right] \geq 0 \quad \text{else } 0\\
&Q_2(\lambda, d, u) = 1 \quad \text{if} \quad \left[(1-\lambda)d - \tfrac{1-F(\lambda)}{f(\lambda)}(u-d)\right] \geq 0 \quad \text{else } 0\\
&Q_2(\lambda, d, d) = 1 \quad \text{if} \quad \left[(1-\lambda)d - 2\tfrac{1-F(\lambda)}{f(\lambda)}(u-d)\right] \geq 0 \quad \text{else } 0
\end{aligned}$$

---

[19] calculations in appendix

Let H_k represents some history of reports and j represent the number of d reports in history H_k. The allocation function $Q_k\left(\lambda, H_k\right) = 1$ if and only if

$$(1-\lambda)d - \frac{1-F(\lambda)}{f(\lambda)} j\left(u-d\right) \geq 0$$

This expression is similar to the standard expression for virtual utility. The positive term in this expression, $(1-\lambda)\,d$, represents the benefit of selling the good to a buyer with type λ whose valuation has experienced j downward shocks. By serving this buyer, the seller increases the expected social surplus generated by the mechanism, which contributes positively to his expected profit. On the other hand, in order to serve this buyer the seller must drop the price of the good. The negative term $\frac{1-F(\lambda)}{f(\lambda)} j\left(u-d\right)$ implies that this price drop creates a rent for all buyers with higher λ who have experienced j or fewer downward shocks[20] over the history of the interaction. If the increase in the expected social surplus that selling to this buyer generates is larger than the associated increase in the information rent, then the buyer will be allowed to buy the object; otherwise he will be excluded.

The allocation functions can be rewritten as

$$Q_k\left(\lambda, H_k\right) = \begin{cases} 1 & \text{if} \frac{d}{j(u-d)} \geq \frac{1-F(\lambda)}{f(\lambda)(1-\lambda)} \\ 0 & \text{otherwise} \end{cases}$$

In what follows I assume that $\frac{1-F(\lambda)}{f(\lambda)(1-\lambda)}$ is a non-increasing function. In standard mechanism design problems, we frequently encounter a regularity condition requires that the inverse hazard rate of the prior, $\frac{1-F(\lambda)}{f(\lambda)}$ is non-increasing. This condition ensures that the solution to the relaxed problem (imposing on local incentive constraints) is incentive compatible in the original problem (which requires global that global incentive constraints be satisfied). In this model, I impose a similar condition, requiring that $\frac{1-F(\lambda)}{f(\lambda)(1-\lambda)}$ is non-increasing. Since this term is the ratio of the

---

[20] and thus have a higher value for the good

inverse hazard rate of the prior to the inverse hazard rate of a uniform distribution, I sometimes refer to it as the "relative inverse hazard rate," and to the assumption that the relative inverse hazard rate is decreasing, as "relative regularity." While relative regularity is a stronger condition than ordinary regularity, many commonly used priors have this property.

The allocation functions can be represented graphically in the following way:

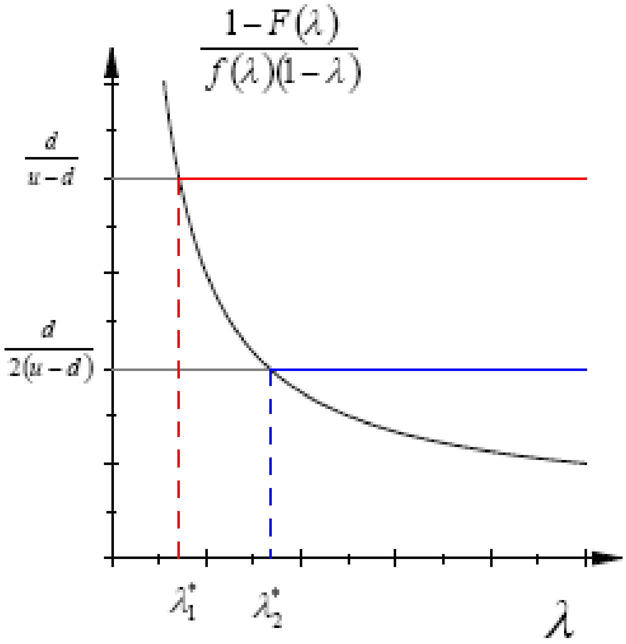


FIGURE 1.8: Graphical Representation of Allocation Functions

In the above figure, the black hyperbola represents the relative inverse hazard rate, $\frac{1-F(\lambda)}{f(\lambda)(1-\lambda)}$, while the horizontal lines represent values $\frac{d}{u-d}$ and $\frac{d}{2(u-d)}$. The intersections between the relative inverse hazard rate and the horizontal lines define cutoff values λ_1^* and λ_2^*, which partition the interval from zero to one into three subintervals:

$$[0,1] = [0,\lambda_1^*) \cup [\lambda_1^*,\lambda_2^*) \cup [\lambda_2^*,1]$$

Consider the first interval.$[0, \lambda_1^*)$. For values of λ in this interval,

$$\frac{d}{u-d} < \frac{1 - F(\lambda)}{f(\lambda)\,(1-\lambda)}$$

and so even one down move reported in history H_k means that the allocation $Q_k\,(\lambda, H_k) = 0$. Still, if there are no down moves reported in history H_k, then $Q_k\,(\lambda, H_k) = 1$. The allocation functions in this interval are therefore:

$$Q_1(\lambda, u) = 1$$

$$Q_2(\lambda, u, u) = 1$$

$$Q_1(\lambda, d) = 0$$

$$Q_2(\lambda, u, d) = 0$$

$$Q_2(\lambda, d, u) = 0$$

$$Q_2(\lambda, d, d) = 0$$

Consider now the second interval $[\lambda_1^*, \lambda_2^*)$. For values of λ in this interval,

$$\tfrac{d}{u-d} \geq \tfrac{1-F(\lambda)}{f(\lambda)(1-\lambda)}$$

$$\tfrac{d}{2(u-d)} < \tfrac{1-F(\lambda)}{f(\lambda)(1-\lambda)}$$

Therefore when λ is in the second interval, following histories with one reported down move, the allocation function is equal to one, but following histories with two reported down moves, the allocation function is equal to zero. The allocation functions in this interval are therefore:

$$Q_1(\lambda, u) = 1$$

$$Q_2(\lambda, u, u) = 1$$

$$Q_1(\lambda, d) = 1$$

$$Q_2(\lambda, u, d) = 1$$

$$Q_2(\lambda, d, u) = 1$$

$$Q_2(\lambda, d, d) = 0$$

Consider finally values of λ in the third interval $[\lambda_2^*, 1]$. For values of λ in this interval

$$\tfrac{d}{u-d} \geq \tfrac{1-F(\lambda)}{f(\lambda)(1-\lambda)}$$

$$\tfrac{d}{2(u-d)} \geq \tfrac{1-F(\lambda)}{f(\lambda)(1-\lambda)}$$

It follows that the allocation functions are one following histories containing up to two down moves. Since (so far) the exposition has confined itself to a two period model, for values of λ in this interval, the allocation functions are equal to one following any history of reports:

$$Q_1(\lambda, u) = 1$$

$$Q_2(\lambda, u, u) = 1$$

$$Q_1(\lambda, d) = 1$$

$$Q_2(\lambda, u, d) = 1$$

$$Q_2(\lambda, d, u) = 1$$

$$Q_2(\lambda, d, d) = 1$$

Using standard techniques, in the appendix, I solve for payment functions that are associated with the allocation functions in each of the three intervals. Because of two-sided commitment, only the lifetime payment conditional on a history of reports is defined uniquely, and there are potentially many ways to decompose this lifetime payment into per-period payments that satisfies incentive compatibility. I choose one such decomposition, which I find intuitive:

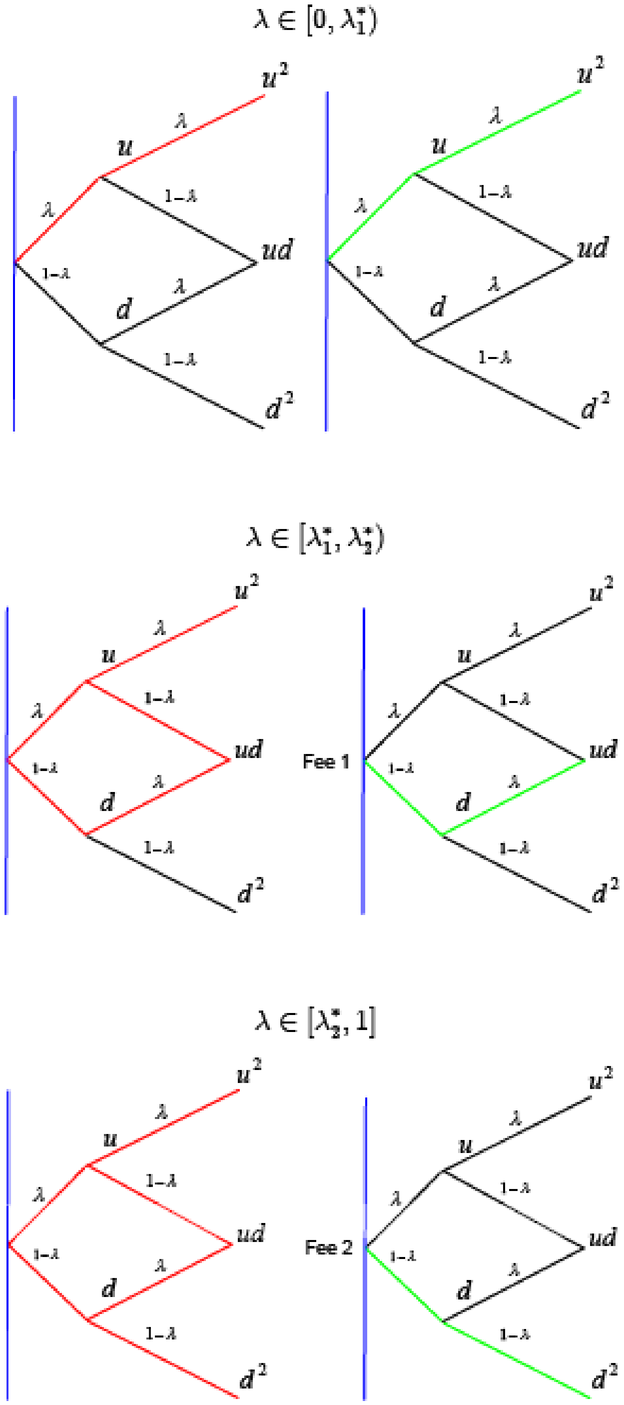
$$\begin{array}{lll}
\lambda \in [0, \lambda_1^*) & \lambda \in [\lambda_1^*, \lambda_2^*) & \lambda \in [\lambda_2^*, 1] \\
T_0(\lambda) = 0 & T_0(\lambda) = \tau_1 & T_0(\lambda) = \tau_2 \\
T_1(\lambda, u) = u & T_1(\lambda, u) = d & T_1(\lambda, u) = d \\
T_1(\lambda, d) = 0 & T_1(\lambda, d) = d & T_1(\lambda, d) = d \\
T_2(\lambda, u, u) = u^2 & T_2(\lambda, u, u) = ud & T_2(\lambda, u, u) = d^2 \\
T_2(\lambda, u, d) = 0 & T_2(\lambda, u, d) = ud & T_2(\lambda, u, d) = d^2 \\
T_2(\lambda, d, u) = 0 & T_2(\lambda, d, u) = ud & T_2(\lambda, d, u) = d^2 \\
T_2(\lambda, d, d) = 0 & T_2(\lambda, d, d) = 0 & T_2(\lambda, d, d) = d^2
\end{array}$$

where

$$\begin{aligned}
\tau_1 &= \lambda_1^*(u-d) + (\lambda_1^*)^2(u^2 - ud) \\
\tau_2 &= (2\lambda_2^* - \lambda_2^{*2})(ud - d^2) + \lambda_1^{*2}(u^2 - ud) + \lambda_1^*(u-d)
\end{aligned}$$

In the following pictures, I represent the allocation and payment functions for buyers whose propensity parameters lie in each of the three intervals discussed previously. In each picture, the left panel depicts the allocation functions, while the right panel depicts the payments. Specifically, in the left panel, the red branches of the tree represent trajectories for the buyers valuation along which the allocation functions are one, while the green branches in the right panel represent the price of the good in each period.

From this figure it is clear that whenever the buyer's valuation is greater than or equal to the price, the optimal allocation function is equal to one, and whenever


$\lambda \in [0, \lambda_1^*)$

$\lambda \in [\lambda_1^*, \lambda_2^*)$

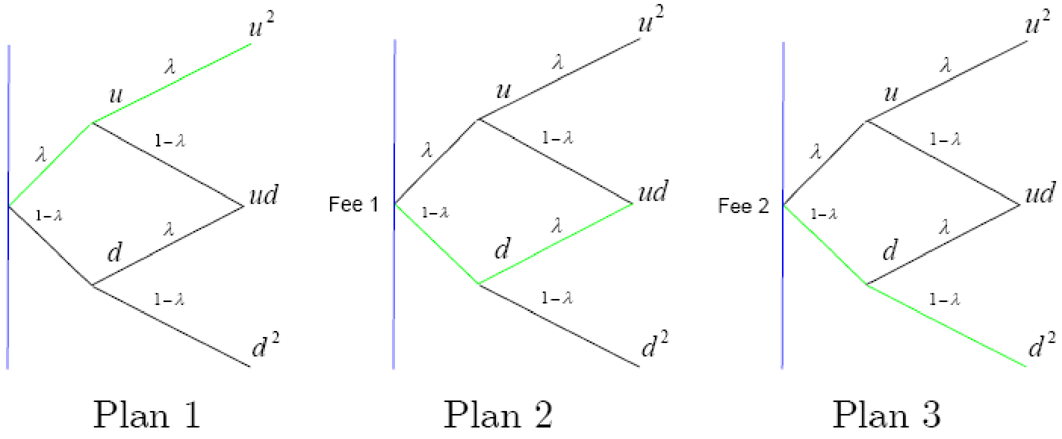
$\lambda \in [\lambda_2^*, 1]$

FIGURE 1.9: Allocations and Payments

the buyer's valuation is strictly below the price, the optimal allocation function is equal to zero. Therefore, in order to implement the optimal mechanism, the seller only needs to commit to a sign up fee and a price for each period, and then let the buyer choose from among the available options. Over the course of the interaction, the buyer can make his own decision about whether or not to purchase the good in each period.

In particular, the seller can implement the optimal mechanism by presenting the buyer with three plans specifying prices for each period:

Plan 1	Plan 2	Plan 3
$t_0 = 0$	$t_0 = \tau_1$	$t_0 = \tau_2$
$t_1 = u$	$t_1 = d$	$t_1 = d$
$t_2 = u^2$	$t_2 = ud$	$t_2 = d^2$



Plan 1 Plan 2 Plan 3

FIGURE 1.10: The seller need only commit to a menu of price plans

Each of the price plans in the menu begins with a "honeymoon phase" of length H, during which the price of the good in period $t_k = d^k$. For plan one, $H = 0$, for plan two, $H = 1$, and for plan 3, $H = 2$. Once the honeymoon phase is over, prices grow by a factor of u in every period. In order to select a plan with a longer honeymoon period, the buyer must pay a larger sign-up fee in period zero. This trade-off between a high initial fixed fee and low future prices is something that is very familiar from every day life, and is the primary means by which the seller separates buyers at the time of contracting. Indeed, in order to justify paying a large fee initially, the buyer should anticipate that his future values will be high enough that the future price discount he purchases will be enough for him to recover the sign up fee. In other words, paying a larger fee for a future price discount is only justified if the buyer has a sufficiently high value of the propensity parameter.

One important implication of the fact that the seller can implement the optimal mechanism through a menu of price plans, is that the seller can do so without eliciting any information from the buyer past period zero. In fact, the only thing that the seller needs to keep track of when implementing the optimal mechanism is the plan chosen by the buyer (or the fixed fee he paid) and the date that the contract took effect.

1.3 N-period model

I tackle the N period problem using an inductive approach. I relate the solution of the $N+1$ period model to the solution of the N period model and use the solution of the two period model as the basis step of the inductive argument. The derivation is heavy on notation and is therefore relegated to the appendix. Speaking in general terms, I find that the optimal allocation functions of the $N+1$ period model, following a report of u in period one, are the same as the optimal allocation functions of the N period model. The optimal allocation functions of the $N+1$ period model, following a report of d in period one, contain an additional distortion not present in the optimal allocation functions of the N period model. Simplifying the distortion and applying an inductive argument, I find that, just as in the two period case, following history of reports H_k which contains j reported down moves, the optimal mechanism calls for the principal to sell the object provided:

$$(1-\lambda)d - \frac{1-F(\lambda)}{f(\lambda)} j\,(u-d) \geq 0$$

or equivalently

$$Q_k\left(\lambda, H_k\right) = \begin{cases} 1 & \text{if} \frac{d}{j(u-d)} \geq \frac{1-F(\lambda)}{f(\lambda)(1-\lambda)} \\ 0 & \text{otherwise} \end{cases}$$

As in the two period case, when the prior is relative regular, the values

$$\lambda_j^* = \inf\left\{\lambda : \frac{d}{j(u-d)} \geq \frac{1-F(\lambda)}{f(\lambda)\,(1-\lambda)}\right\}_{j=1,\ldots,N}$$

define an increasing sequence of cutoffs. A buyer with $\lambda \geq \lambda_j^*$ will be permitted to purchase the object following histories in which his valuation has undergone j (or fewer) downward shocks. By l'Hopital's rule, if F has a derivative of any order that is nonzero at $\lambda = 1$, then $\lim_{\lambda\to1} \frac{1-F(\lambda)}{f(\lambda)(1-\lambda)} = \gamma > 0$. Therefore, under this condition, for any u, d, and γ there will always exist j^* such that $\frac{d}{j^*(u-d)} < \gamma$.[21] This means that for a wide combination of parameters and distribution there exists some finite number of down moves, j^*, such that any buyer, regardless of λ, who experiences j^* or more downward shocks to his valuation will not be allowed to buy the object. Therefore, it is never optimal for the seller to offer an infinite honeymoon period; the optimal menu of price plans offered by the seller always has a finite number of options.

The fact that for any combination of model primitives $u, d, F\left(\lambda\right)$, there is a maximal possible length of the honeymoon phase which is independent of the time horizon

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[21] Notice that j^* depends only on the choice of prior and the parameters u, d, and not on the time horizon of the model.

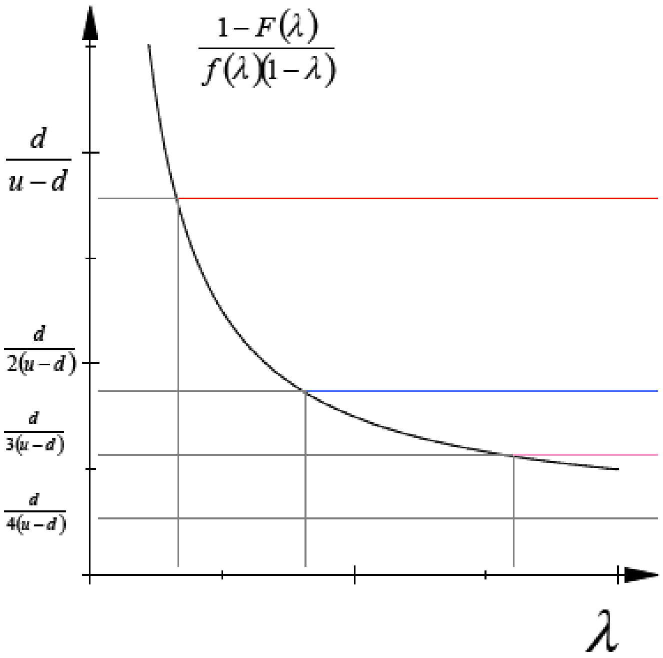


FIGURE 1.11: There are no infinite honeymoons

implies that for large N, early termination will occur with high probability. Indeed, consider some ε, δ and $\widehat{N}$. Because of the law of large numbers, for large enough $\widehat{N}$, a buyer with a value of $\lambda \in [0, 1-\varepsilon]$ will experience more than j^* downward shocks over the first $\widehat{N}$ periods with probability $1-\delta$. Once this happens, all of the buyer's future allocations will be zero, and the buyer will never again be allowed to purchase the object. Therefore, an $\widehat{N}$ can always be found such that with probability arbitrarily close to one, the relationship between the seller and all buyers with $\lambda \in [0, 1-\varepsilon]$ will effectively terminate within the first $\widehat{N}$ periods: the buyer will make no additional payments, and will never again receive the good. The value of $\widehat{N}$ does not depend on N, the number of periods in the interaction. Hence, when the number of periods exceeds $\widehat{N}$, the interaction will terminate prematurely with high probability. Explained more simply, after the honeymoon phase is over, the price will increase in every period, while the buyer's value only increases some of the time (with probability λ in every period). Since the honeymoon period is always finite, given enough time the buyer's value will eventually be overtaken by the price. Once that happens, the buyer will never again purchase the good. The optimal mechanism is therefore not renegotiation proof, and this early termination feature of the mechanism dramatically hurts social welfare.

1.4 Comparative Statics

In this section I will discuss the way in which the optimal mechanism varies with the primitives of the problem: u, d, and $F(\lambda)$. First, let's discuss how the mechanism depends on u and d. Recall that the optimal allocation functions are:

$$Q_k(\lambda, H_k) = \begin{cases} 1 & \text{if} \frac{d}{j(u-d)} \geq \frac{1-F(\lambda)}{f(\lambda)(1-\lambda)} \\ 0 & \text{otherwise} \end{cases}$$

The first simple observation is that the allocation functions are homogeneous of degree zero in u, d. This is the reason why I did not include an explicit discount factor in the model; it would play no role in the results. A second simple observation is that the effect on the optimal mechanism of an increase in u is qualitatively the same as the effect of a decrease in d. As the figure below illustrates, when u increases the mechanism becomes more restrictive. For the smallest value of u, given by u_1, all types in the interval $[\lambda_{j1}^*, 1]$, will experience a honeymoon phase of length (at least) j. When u increases to u_2, the set of types with a honeymoon phase of at least j shrinks to interval $[\lambda_{j2}^*, 1]$. Types in the interval $[\lambda_{j1}^*, \lambda_{j2}^*]$ originally experienced a honeymoon phase of length not less than j, but after the increase in u, these types will experience a honeymoon phase that is less than j. When u grows sufficiently large, for example to u_3, no type will experience a honeymoon phase of length j.

The intuition for this effect is simple. In order to charge a given λ type more when he draws high valuations, the seller must restrict his allocation when he draws

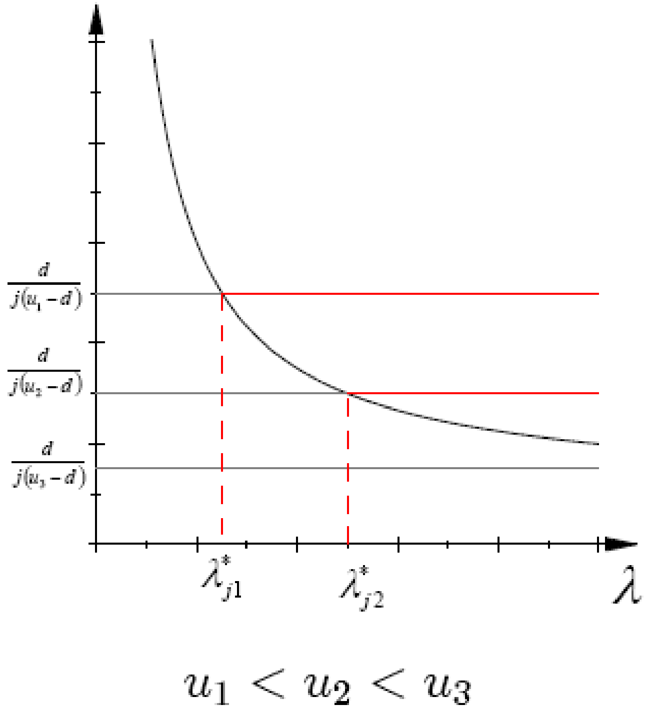


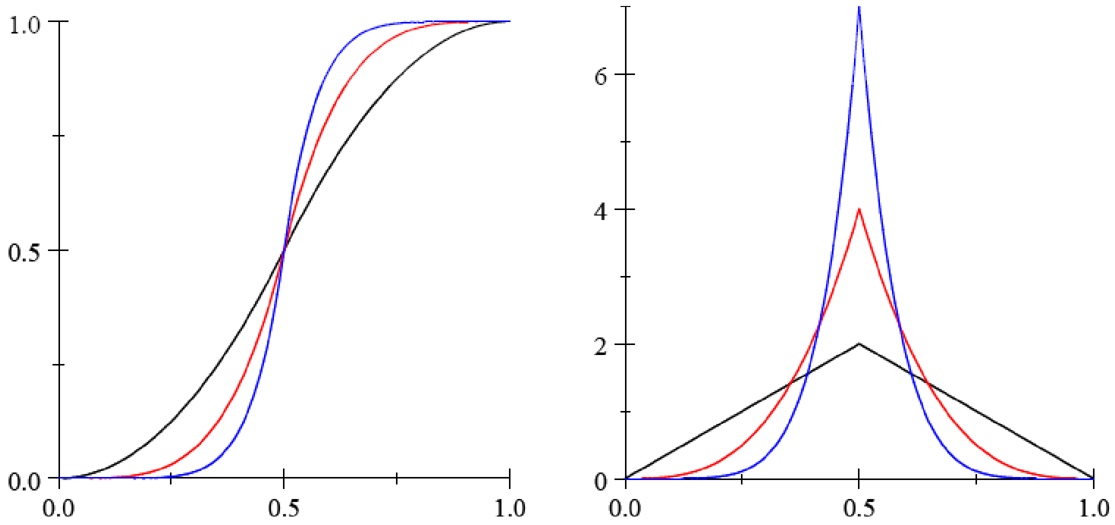
FIGURE 1.12: The effect of an increase in u

low valuations. As u grows relative to d the payoff to the seller from setting higher prices when the buyer draws high valuations grows, and grows more for buyers with high values of λ. Therefore, if u is big, the seller gets more expected profit selling only to buyers with high valuations at high prices, than selling to more buyers at lower prices.

The second comparative static that I analyze is the effect on the optimal mechanism as the prior distribution for λ becomes concentrated at a single value of λ. To that end, consider a family of distributions

$$G_a(\lambda) == \begin{cases} 2^{a-1}\lambda^a & \text{if} \lambda \leq \frac{1}{2} \\ 1 - 2^{a-1}(1-\lambda)^a & \text{if} \lambda \leq \frac{1}{2} \end{cases}$$

As a grows, these distributions approach a degenerate distribution around $\lambda = \frac{1}{2}$, as the following plots of the cdfs and pdfs illustrate



CDFs: $G_2(\lambda)$, $G_4(\lambda)$, and $G_7(\lambda)$ PDFs: $g_2(\lambda)$, $g_4(\lambda)$, and $g_7(\lambda)$

FIGURE 1.13: Family G_a

For this family of priors, the relative hazard rate is given by

$$\frac{1-G_a(\lambda)}{g_a(\lambda)(1-\lambda)} = \begin{cases} \frac{1-2^{a-1}\lambda^a}{a2^{a-1}\lambda^{a-1}(1-\lambda)} & \text{if}\lambda \leq \frac{1}{2} \\ \frac{1}{a} & \text{if}\lambda \leq \frac{1}{2} \end{cases}$$

So, for $a > 1$ the relative inverse hazard rate has a decreasing left hand portion and a flat right hand portion, as the following plot illustrates.

As is evident from the plot, as a grows, the relative inverse hazard rate approaches an L-shape: a vertical line at $\lambda = \frac{1}{2}$, and a horizontal line at height zero for $\lambda > \frac{1}{2}$. Therefore, in the limit, when the buyer reports a value of $\lambda \geq \frac{1}{2}$, he will always receive an allocation of one, regardless of how many down moves he reports, and is charged a price path equivalent to a fixed payment of the full expected social surplus

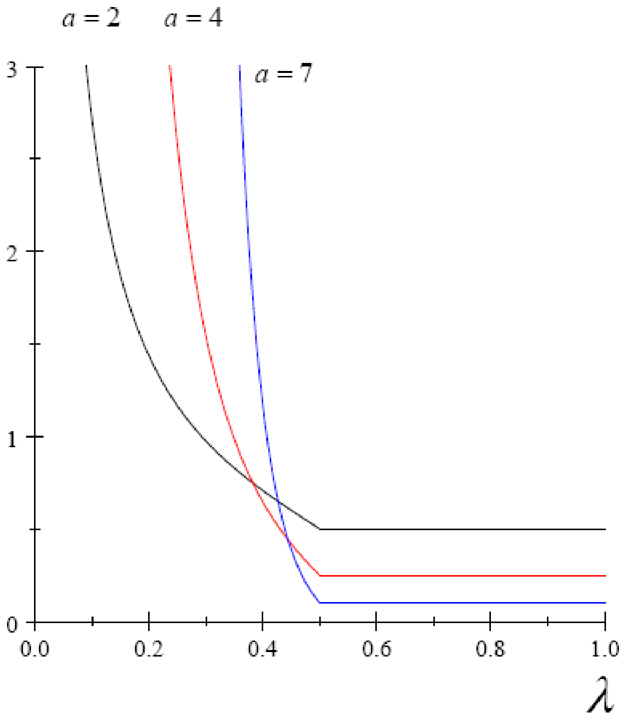


FIGURE 1.14: Relative Inverse hazard rate for G_a

of a buyer with $\lambda = \frac{1}{2}$.[22] When the buyer reports a value of $\lambda < \frac{1}{2}$, the only way that the buyer is allowed to purchase object is when his valuation experiences sequential u shocks.

Consider now what would happen if the value of λ were known to the seller. The seller would maximize the social surplus by always giving the good to the buyer for free and then extract this maximized value of the social surplus as an entry fee. In other words, the seller would charge $t_0 = E\left[\text{Lifetime Utility}|\lambda\right]$ in exchange for always delivering the good. Suppose now that, for some strange reason, the buyer's actual value of the propensity parameter $\widehat{\lambda}$ is bigger than the value of λ that the seller was expecting. In this situation, the buyer would be willing to take the deal

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[22] This may seem at first glance to contradict the claim that there are no infinite honeymoons. Note however, that for any finite a, for which the distribution has the property that the derivative of order a at one is non-zero, the inverse relative hazard rate has a strictly positive limit and there is no infinitie honeymoon. In the limit, however, the assumption underlying the result is violated.

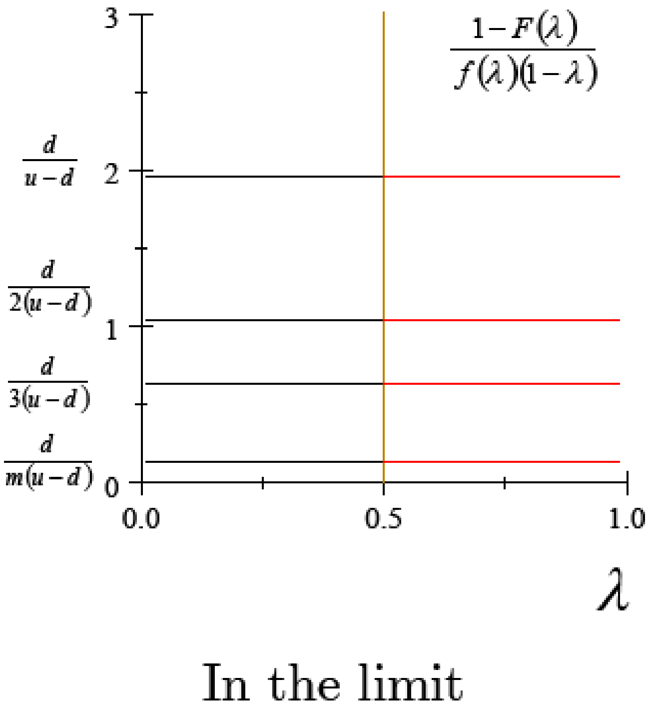

FIGURE 1.15: In the limit

offered by the seller and would experience an allocation of one in every period. If, however, the true propensity $\widehat{\lambda}$ were below the value the buyer was expecting, the buyer would not take the deal and would always be left with a zero allocation, never paying the seller anything.

Comparing the two mechanisms, as the prior becomes concentrated around a particular value of λ, the mechanism in which λ is screened becomes a ”robust” version of the full extraction mechanism. It treats all types larger than the one which the seller believes the buyer to be in the same way as the full extraction mechanism, but, unlike the full extraction mechanism, allows types below the expected type to purchase the good for full value, giving the seller an additional opportunity to collect a payment from these buyers.

In order to highlight the impact of private information about λ on the structure and properties of the optimal mechanism, it is useful to briefly consider the standard case considered in the literature. In this standard case, the value of λ is common knowledge, but the buyer has private information at the time of contracting about his initial valuation for the good.

1.5 Benchmark: The Standard Case

This benchmark is the general model presented in Pavan 2007 adapted to the recombinant binomial process of this paper. Suppose everything about the model is the same, except for two things. First, the value of λ is known to the seller; second, contracting happens at time one, when the buyer knows his first period valuation. For simplicity I will discuss a two-period model, but the discussion extends trivially to N periods.

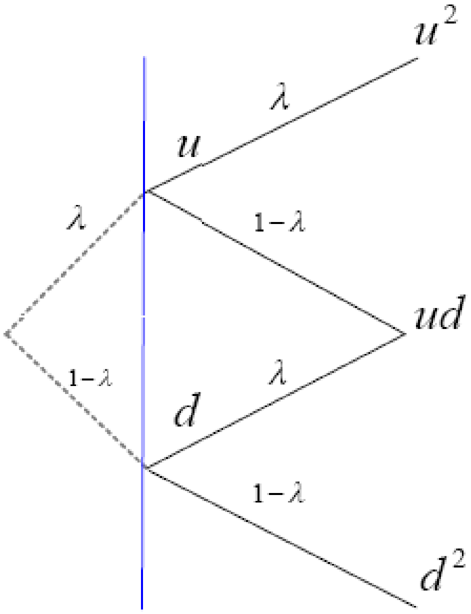

Figure 1.16: Benchmark Model

This model is very closely related to a static price discrimination model with two types, for example Mussa and Rosen (1978). Indeed, in order to receive the good for the duration of the contract, a buyer who drew a value of u in period one would be willing to pay $T^u = u\left(\lambda u + (1-\lambda)\, d\right)$. On the other hand, a buyer who drew a value of d in period one would only be willing to pay $T^d = d\left(\lambda u + (1-\lambda)\, d\right)$.

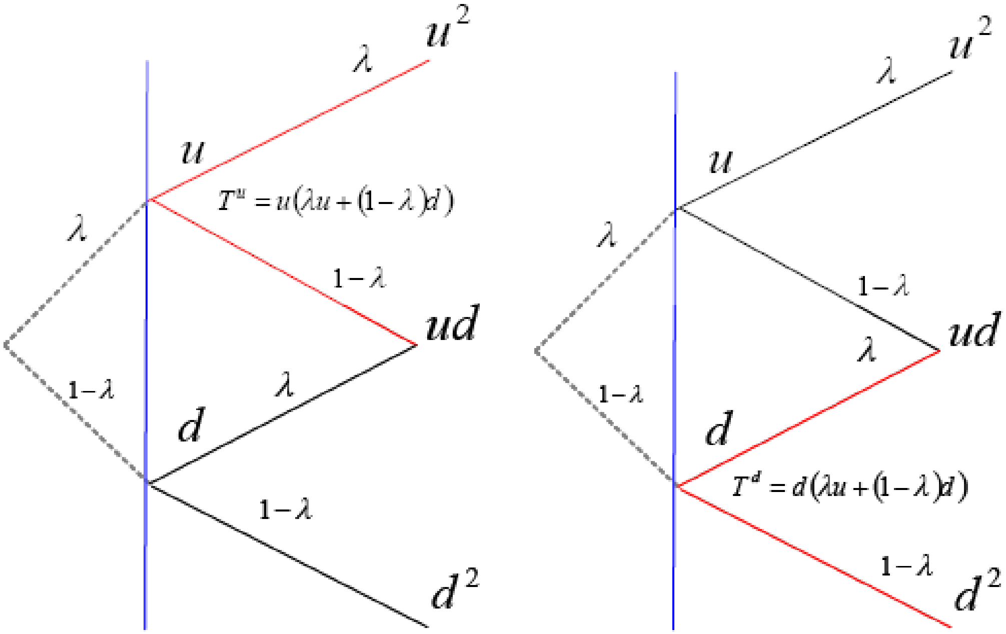

Figure 1.17: Benchmark Model

The seller therefore has two options. Option one is to post a price of T^u and serve only buyers who draw a u initially. This leads to a profit of λT^u. The seller's other option is to post a price of T^d and serve all buyers, leading to a profit of T^d.

Therefore, it is easy to see that if $\lambda u \geq d$ then the seller will post the high price T^u and only u buyers will be served, while if $d > \lambda u$, then the seller will post price T^d and include all buyers.

Before moving on to a discussion of the optimal mechanism when λ is unknown and a comparison of the optimal mechanisms in the two settings, I would like to point out that the "standard case" mechanism is neither incentive compatible nor individ-

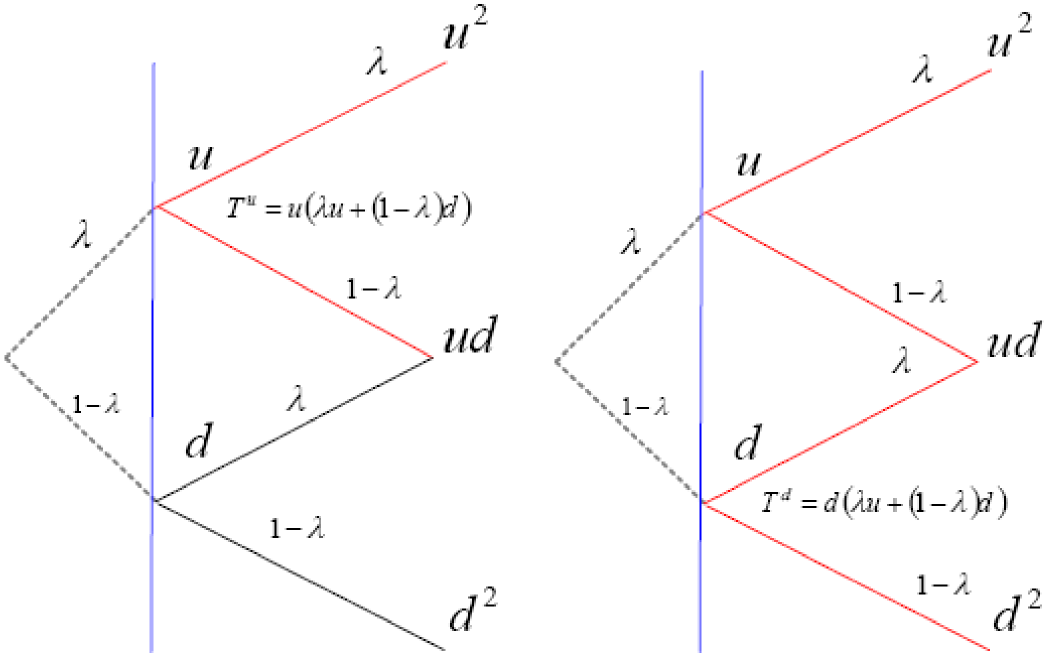

FIGURE 1.18: Benchmark Model

ually rational for a buyer whose true propensity parameter $\widehat{\lambda}$ is less than the value the mechanism designer believes it to be. Indeed, if $\widehat{\lambda} < \lambda$, then a buyer who draws a value of d would rather quit and get zero than pay $T^d = d\left(\lambda u + (1-\lambda)\,d\right).$ Similarly, if the buyer drew a u, he would not be willing to pay $T^u = u\left(\lambda u + (1-\lambda)\,d\right)$ but would be willing to pay $T^d = d\left(\lambda u + (1-\lambda)\,d\right)$. It is therefore crucial for the optimality of this mechanism that the seller know λ *exactly,* quite a strong assumption. The goal of this paper is to assess the implications of abandoning this assumption by introducing seller uncertainty about λ.

1.6 Discussion

To summarize some previous discussion, with private λ, the seller can implement the optimal mechanism by committing to a finite menu of price plans, and allowing the buyer to choose an option from the menu at time zero. In future periods, the decision about whether or not to purchase the good is left to the buyer, and the seller does not need to elicit any information from the buyer at any time past time zero. Each price plan in the menu can be described by the length of its honeymoon phase H and the sign-up fee. Recall that during the honeymoon phase the price in each period is the lowest possible valuation, d^k, while after the honeymoon phase is over, the price of the good is multiplied by a factor of u in each period. Price plans with longer honeymoon phases require a larger sign up fee. A buyer with a high value of the propensity parameter will choose a plan with a longer honeymoon phase and higher sign-up fee.

I mentioned in the section on individual rationality that the assumption of two-sided commitment could be relaxed. In some settings covered by this type of long term contract model, it is difficult to imagine that the seller could force the buyer to be bound by the terms of the contract for its entire duration. In principle, the buyer could refuse to participate in any particular period. While the seller is bound by the contract terms which are easily enforced by a court, forcing a buyer to participate through the court system could entail significant challenges (not the least of which could be locating the buyer). Because of this, many models of long run contracting allow for the possibility of buyer non-participation in some period. The common assumption includes a kind of punishment for non-participation: if the buyer does not participate in some period, the contract is invalidated, and the buyer is left with zero payoff in all future periods. This assumption is referred to as "one-sided commitment." In fact, this monicker is a bit misleading. The implication of the name is that only the seller is fully committed, while the buyer is not committed at all. In fact the seller is only committed so long as the buyer has participated in the past. One could imagine a "weak one-sided commitment" in which the seller is fully committed to the terms of the contract, regardless of the actions of the buyer. In other words, under weak one-sided commitment, the buyer should have incentives to participate in each period, even if the future contract terms are still in place following a period of non-participation. If a contract satisfies "weak" one sided commitment, it certainly satisfies one-sided commitment, and if a contract satisfies one-sided commitment then it satisfies two-sided commitment. The reverse chain of implication does not necessarily hold.

Although in the derivation of the optimal contract I assumed two-sided commitment, I find that the optimal contract satisfies weak one-sided commitment. Indeed, once the buyer has accepted a price plan and paid the sign-up fee, the buyer would never have a strict incentive to avoid participation in any period, even though he could skip a period and have the future terms of the contract remain in place. This is a consequence of the fact that when the allocation function in some period is equal to one, the seller's valuation is higher than the price of the good, and when it is zero, the buyer's valuation is below the price of the good.[23] The mechanism thus never forces the buyer to purchasing the good for more than it is worth to him by threatening to eliminate all of his future surplus if he doesn't comply.

A natural question to ask is whether the optimal contract can ever be implemented by a sale of the production technology, rather than a quantity of the good in each period. This type of solution frequently occurs in both one-shot and long term contracting models.[24] If the seller were to sell the production technology to the buyer, then, because there is zero cost of production, the buyer would always choose to produce the good. Therefore, selling the production technology following some history of reports is equivalent to offering the buyer an allocation of one in all future periods. Recall from the discussion of the N-period model, that for any combination of model primitives $(u, d, F(\lambda))$ there exists some j^* such that for any value of λ,

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[23] This can be seen most clearly in the panels depicting the allocation and payment functions presented in section 2.3.

[24] Mussa and Rosen (1978), Battaglini (2005)

$Q_k(\lambda, H_k) = 0$ if H_k contains j^* or more reported down shocks. Therefore, if the time horizon N exceeds j^*, then selling the production to the buyer can not be optimal *at the time of contracting*, because there is no type λ who receives an allocation of one in all future periods for every combination of reports. This is markedly different from what happens in the standard case. The standard case contract is essentially equivalent to a take it or leave it offer for the sale of the production technology at time zero.

Another way to state this distinction is to say that in the standard case there is *no premature termination* of the relationship. If the buyer pays the price in the initial period, he will always receive the good in the future; if he doesn't he won't, but there is never a circumstance in which he receives the good for a certain number of periods, but not after. This is in stark contrast to the model with privately known λ, in which premature termination happens with high probability for long time horizons.

Although with a long time horizon it can not be optimal to sell the production technology at the time of contracting, it can be optimal following certain histories. Suppose that the buyer has a value of λ in the interval from $[\lambda_j^*, \lambda_{j+1}^*)$, so that his allocation is equal to one as long as he has reported j or fewer down shocks. If, for example, the buyer has $j-1$ periods remaining in the interaction and has reported no down moves, he will receive an allocation of one in all future periods, and following this history the seller could just sell him the production technology. Notice, however, that all histories with this property must be within last j^* periods of the interaction.[25] When the time horizon is sufficiently long, the interaction will end with arbitrarily high probability within the first $\widehat{N}$ periods, where $\widehat{N}$ is independent of N. Therefore, when $N > \widehat{N} + j$, the histories following which sale of the production technology is

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[25] In period $k \in \{1, ..., N - j^*\}$, for every type λ, and every history of reports, a future sequence of j^* reported d shocks will lead to all future allocation functions set to zero. Therefore, there is no history within the first $N - j^*$ periods, following which a sale of the production technology is optimal.

optimal, occur with probability arbitrarily close to zero.

One similarity between the optimal mechanisms in the two settings is that they can both be implemented without eliciting information past time zero. In the standard case, the only relevant piece of information for the mechanism is the buyer's first valuation, whereas in my setting, the only relevant piece of information is the buyer's propensity parameter. This kind of result is encountered in other long term contracting papers, for example in the literature on the ratchet effect. Still, it may seem strange that in the optimal mechanism there is no cross-checking between the buyer's initial reported value of λ and his reported shocks over the lifetime of the contract. After all, since these shocks are reported truthfully, the seller can form a running estimate of the buyer's true value λ using the buyer's reported shocks, and because of two sided commitment, if this estimate deviates sufficiently from the value of λ reported initially, the seller can punish the buyer. An extreme version of this idea is for the seller always to sell the object to the buyer and to charge the buyer the expected value of his lifetime utility, given his reported value of λ, as an entry fee. The seller would ask the buyer to report his valuation shocks, and if the running estimate of λ given the reported shocks were too far from the initially reported value, then the seller would punish the buyer. The problem with this scheme is that because the seller commits to always sell the object, independent of the buyer's reported valuations, the seller has no way to ensure that the buyer will report the valuation shocks truthfully. Indeed, under the proposed scheme, the buyer can always choose a sequence of reports that is consistent with his initially reported value of λ at no cost. The buyer can therefore initially report any value of λ that he wishes and avoid punishment. This would not be the case in a setting similar to that of Strausz (2006), in which the seller receives exogenous information that is correlated with the buyer's private information. In such a setting the seller can achieve full extraction by cross-checking the buyer's reports with his exogenous information.

1.7 Implications for Real World Situations

1.7.1 Supply Chain Management

To connect the model to the supplier-retailer example mentioned in the introduction, imagine that the monopolist represents a large retailer and the buyer represents a supplier. The retailer buys goods from the supplier to sell to consumers, or alternatively, the retailer sells shelf space to suppliers. Without the retailer's shelf space, the supplier can not sell his goods to consumers. The profit the supplier would earn if he could sell his goods directly to consumers is his value for the shelf space. In each period of the interaction, there is a chance that the supplier will reduce his production costs, thereby increasing this value. Indeed, in any period, there is a probability λ that the supplier's production cost will decrease. In the model, an the occurrence of an innovation is interpreted as the good valuation shock, leading to an increase in the value by a factor of $u > 1$. If the supplier does not discover a cost reducing innovation, that is experiences the bad shock, then the value of shelf space remains the same, so $d = 1$. The retailer can not directly observe the supplier's production cost, and the supplier's value for shelf space is private information in each period. Moreover, the retailer does not have good information about the supplier's R & D department, and therefore the probability of the supplier reducing his cost in any period is also private information.

Under the optimal mechanism the retailer will charge a supplier with a given λ a flat price for stocking the good for the first H periods, where H is the length of the honeymoon phase associated with the supplier's value of λ. In periods H through N, the price that the retailer will charge the supplier will grow by a factor of u in each period. In other words, in the first H periods of the interaction, the retailer will pay the supplier a fixed amount to purchase his goods, but in each of the remaining periods, the retailer will pay a smaller and smaller price for the supplier's goods. In a sufficiently long interaction, the drop in price will eventually become too great for the supplier, and the supplier will stop selling his goods to the retailer.

The optimal mechanism matches some features of actual interactions between a large retailer and its suppliers. Walmart, the largest retailer in the world, frequently exercises considerable market power in setting the terms of trade with its suppliers. Consider the following quotes from "The Walmart You Didn't Know"[26]

> The retailer has a clear policy for suppliers: On basic products that don't change, the price Wal-Mart will pay [suppliers] must drop year after year.
>
> One year, because of costs, [an umbrella manufacturer] went to Wal-Mart and asked for a 5% price increase. Wal-Mart said, 'We were expecting a 5% decrease. We're off by 10%. Go back and sharpen your pencil.' The umbrella man scrimped and came back with a 2% increase–...and he was out entirely.

From this we see that the interaction has two features that are predicted by the model. First, the price that Walmart pays suppliers drops through time. Second, early termination of the business relationship occurs frequently. Of course, both of these features may have other plausible explanations, for example, competition among suppliers. Still, this behavior on the part of the retailer is consistent with the behavior predicted by the model with privately known λ, and is *not* consistent with the standard case in the literature. However, there is still no reason to think from the quotes presented above that Walmart is explicitly screening producers for λ. In order to make such a claim, we would need some reason to think that producers with better R & D capabilities would be treated more favorably, perhaps by being offered a "honeymoon" or grace period in which the price Walmart pays does not drop (or does not drop as much). There is nothing in the above discussion to suggest that this is the case.

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[26] http://www.fastcompany.com/magazine/77/walmart.html

1.7.2 Term Life Insurance

As mentioned in the introduction, in term life insurance contracts, the buyer's valuation for life insurance in any period is influenced by his probability of death in the period, which evolves stochastically through time. Typically, we think of this probability as increasing through time. If this probability increases in a given period, then the buyer's value for life insurance also increases, and so it is reasonable in this setting to propose that $u > 1$ while $d = 1$.

The model predicts several things. First, the optimal contract can be implemented without eliciting information from the buyer over the lifetime of the contract. Second, the optimal mechanism can be implemented by the seller committing to a finite menu of price plans. Each price plan is characterized by a combination of honeymoon phase and sign-up fee. During the honeymoon phase the price is constant, and after the honeymoon phase, the price will increase in each period. Plans with longer honeymoon phases require larger sign-up fees. The first three entries of the menu can be represented graphically in the following way:

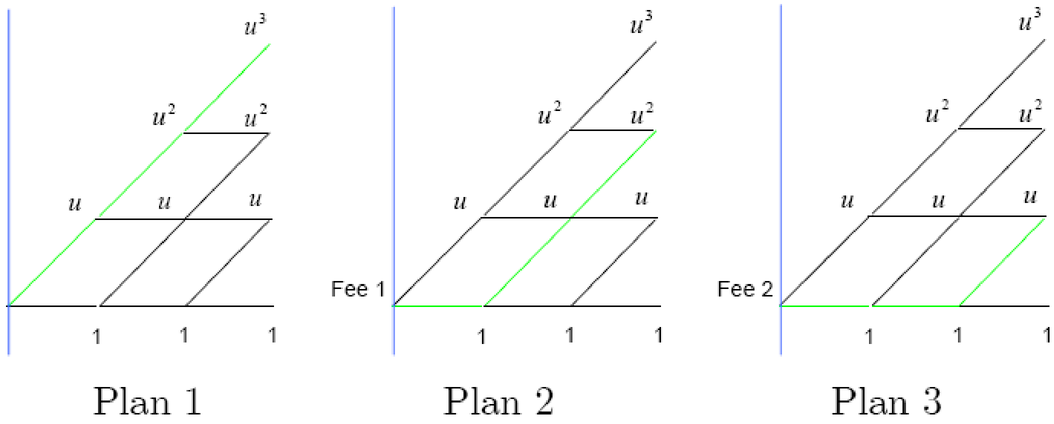
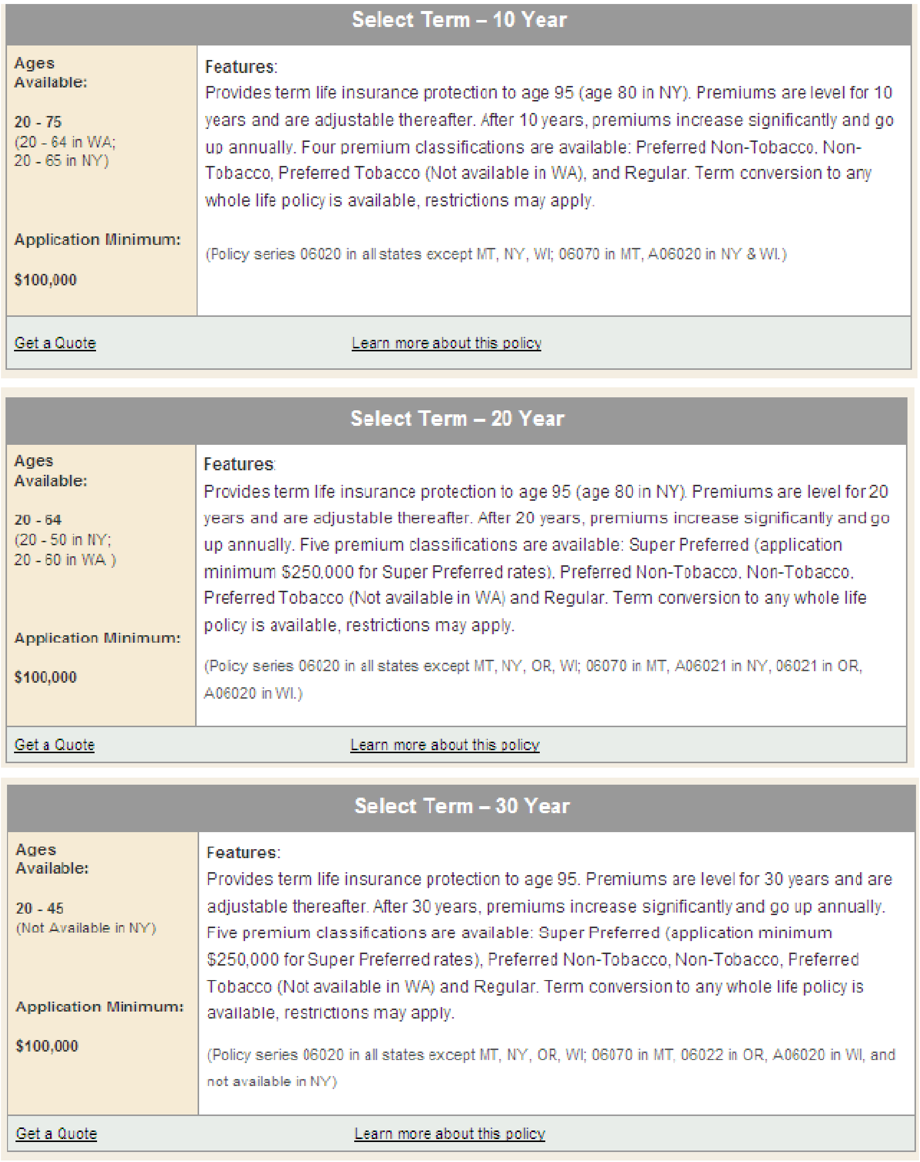


FIGURE 1.19: Life Insurance

Consider the following term life insurance plans, taken directly from State Farm's website:[27]



Select Term – 10 Year

Ages Available:

20 - 75
(20 - 64 in WA;
20 - 65 in NY)

Application Minimum:

$100,000

Features:
Provides term life insurance protection to age 95 (age 80 in NY). Premiums are level for 10 years and are adjustable thereafter. After 10 years, premiums increase significantly and go up annually. Four premium classifications are available: Preferred Non-Tobacco, Non-Tobacco, Preferred Tobacco (Not available in WA), and Regular. Term conversion to any whole life policy is available, restrictions may apply.

(Policy series 06020 in all states except MT, NY, WI; 06070 in MT, A06020 in NY & WI.)

Get a Quote Learn more about this policy

Select Term – 20 Year

Ages Available:

20 - 64
(20 - 50 in NY;
20 - 60 in WA)

Application Minimum:

$100,000

Features:
Provides term life insurance protection to age 95 (age 80 in NY). Premiums are level for 20 years and are adjustable thereafter. After 20 years, premiums increase significantly and go up annually. Five premium classifications are available: Super Preferred (application minimum $250,000 for Super Preferred rates), Preferred Non-Tobacco, Non-Tobacco, Preferred Tobacco (Not available in WA) and Regular. Term conversion to any whole life policy is available, restrictions may apply.

(Policy series 06020 in all states except MT, NY, OR, WI; 06070 in MT, A06021 in NY, 06021 in OR, A06020 in WI.)

Get a Quote Learn more about this policy

Select Term – 30 Year

Ages Available:

20 - 45
(Not Available in NY)

Application Minimum:

$100,000

Features:
Provides term life insurance protection to age 95. Premiums are level for 30 years and are adjustable thereafter. After 30 years, premiums increase significantly and go up annually. Five premium classifications are available: Super Preferred (application minimum $250,000 for Super Preferred rates), Preferred Non-Tobacco, Non-Tobacco, Preferred Tobacco (Not available in WA) and Regular. Term conversion to any whole life policy is available, restrictions may apply.

(Policy series 06020 in all states except MT, NY, OR, WI; 06070 in MT, 06022 in OR, A06020 in WI, and not available in NY)

Get a Quote Learn more about this policy

FIGURE 1.20: Life Insurance Plans

State farm, therefore, offers buyers a choice of a period of time in which premiums will be level, followed by an increasing premium thereafter. Plans with a longer phase of flat prices cost more than plans with a shorter phase. Interestingly (and perhaps surprisingly without this theory) State Farm does not elicit any kind of

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[27] http://www.statefarm.com/insurance/life_annuity/life/term/termtabl.asp

health information while the contract is in effect. The structure of the contract offered by State Farm, therefore closely matches the structure predicted by the theory. In this case, we can see that there is reason to think that State Farm is explicitly screening the buyer's private information about the evolution of his future value for life insurance.

1.8 Conclusion

In this paper I study contracting in the context of a long term buyer/seller relationship. The buyer's value evolves over time and, unlike other analyses, the buyer has private information about the manner in which this value evolves at the time of contracting. This information is given by the buyer's "propensity parameter" which is the probability that the buyer will experience a good valuation shock in each period. The decision about whether or not to sell to a buyer with a particular propensity parameter and valuation involves balancing two effects; on the one hand, selling to a buyer with propensity λ and a given valuation improves the seller's profit because it allows him to collect a payment from the buyer; on the other hand, it reduces the price that the seller can charge to all buyers who have higher propensities and higher valuations. In order to balance information rent and expected social surplus, the optimal mechanism limits the number of "down" reports that any particular type λ, can make before the relationship is terminated.

The seller can implement the optimal mechanism by committing to a finite menu of price plans, and allowing the buyer to choose an option from the menu at time zero. In future periods, the decision about whether or not to purchase the good is left to the buyer, and the seller does not need to elicit any information from the buyer at any time past time zero. Each price plan in the menu can be described by the length of its honeymoon phase H and the sign-up fee. During the honeymoon phase the price in each period is the lowest possible valuation, d^k, while after the honeymoon phase is over, the price of the good is multiplied by a factor of u in each period. Price plans with longer honeymoon phases require a larger sign up fee. A buyer with a high value of the propensity parameter will choose a plan with a longer honeymoon phase and higher sign-up fee.

An interesting feature of the results is that, for long time horizons, the interaction will end prematurely with probability close to one. This happens because the tolerance assigned to a particular value of λ is independent of the time horizon, and as N grows, by the law of large numbers, the probability that the number of down moves exceeds the fixed tolerance approaches one. Long term interactions of the kind considered in this paper are therefore essentially finite in duration. This result does not rely on cost considerations on the part of the seller or competition among buyers; it arises purely from rent extraction given the nature of the buyer's private information. The structure of the optimal contract, and this early termination feature resemble aspects of actual contracts used in the real world.

The model of this paper can be modified in several interesting ways. First, in the current model the buyer's valuation in each period is given by the product of the shocks he has experienced over the duration of the contract. By allowing for a more general dependence of value on the history of shocks, a more general theory could be developed; one which connects the properties of the underlying valuation process to the structure of the contract. Second, in the current model, the evolution of the buyer's valuation is exogenous; endogenizing this value could bring interesting insights applicable to the literature on experience goods. For example, if the model were modified so that probability that the buyer's valuation grows in each period depends on both his propensity and his allocation, (and can therefore be influenced by the seller), this model could be interpreted as a model of a habit-forming or addictive good in which the buyer has private information about his inherent susceptibility to addiction. If, on the other hand, the buyer could influence the probability of the good shock through time, this could be interpreted as a model of explicit investment in R & D. Furthermore, the assumption that the buyer knows his propensity parameter with certainty can also be modified with interesting results. If instead of the buyer's ex ante type being the probability λ itself, but rather a belief about λ, then information about the buyer's shocks would also reveal information about his underlying run parameter λ. In such a model, the short run shocks would serve both as a source of rent for the buyer and as information about his true underlying type, and manipulating reports would alter the principal's beliefs about the buyer's underlying type.[28] Finally, throughout this paper, the monopolist was granted full commitment power. The full commitment assumption is particularly strong in a long run contract environment. Because she will be tempted to renege on the contract after learning λ, full commitment power is essential for the seller to implement the optimal contracts of this paper. Weakening the full commitment assumption in this context could also be an interesting line of future research.

---

[28] I analyze a simple model with these features in a related work

Appendix A

Technical Details

A.1 Calculation of the Information Rent in the Two Period Model

The constraints of the Relaxed Problem are:

$\forall \lambda, \lambda' \in (0,1)$

$$Q_2(\lambda,u,u)u^2 - T_2(\lambda,u,u) \geq Q_2(\lambda,u,d)u^2 - T_2(\lambda,u,d) \qquad \text{DIC2-u}$$

$$Q_2(\lambda,d,u)ud - T_2(\lambda,d,u) \geq Q_2(\lambda,d,d)ud - T_2(\lambda,d,d) \qquad \text{DIC2-d}$$

$$\begin{aligned}&Q_1(\lambda,u)u - T_1(\lambda,u)+\\ &\lambda[Q_2(\lambda,u,u)u^2 - T_2(\lambda,u,u)] + (1-\lambda)[Q_2(\lambda,u,d)ud - T_2(\lambda,u,d)] \geq \qquad \text{DIC1}\\ &Q_1(\lambda,d)u - T_1(\lambda,d)+\\ &\lambda[Q_2(\lambda,d,u)u^2 - T_2(\lambda,d,u)] + (1-\lambda)[Q_2(\lambda,d,d)ud - T_2(\lambda,d,d)]\end{aligned}$$

$$\arg\max_{\lambda'} \pi(\lambda'|\lambda) = \lambda \qquad \text{IC-}\lambda$$

$$\pi(\lambda|\lambda) \geq 0 \qquad \text{IR}$$

where

$$
\begin{aligned}
\pi(\lambda'|\lambda) \quad = \quad & \\
& -T_0\left(\lambda'\right)+\lambda\left[Q_1\left(\lambda', u\right) u-T_1\left(\lambda', u\right)\right]+(1-\lambda)\left[Q_1\left(\lambda', d\right) d-T_1\left(\lambda', d\right)\right]+ \\
& \lambda^2\left[Q_2\left(\lambda', u, u^2\right) u^2-T_2\left(\lambda', u, u^2\right)\right]+ \\
& \lambda(1-\lambda)\left[Q_2\left(\lambda', u, d\right) u d-T_2\left(\lambda', u, d\right)\right]+ \\
& \lambda(1-\lambda)\left[Q_2\left(\lambda', d, u\right) d u-T_2\left(\lambda', d, u\right)\right]+ \\
& (1-\lambda)^2\left[Q_2\left(\lambda', d, d\right) d^2-T_2\left(\lambda', d, d\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \\
& -T(\lambda')+ \\
& \lambda\{ \\
& Q_1(\lambda', u)u - T_1(\lambda', u) + \\
& \lambda[Q_2(\lambda', u, u)u^2 - T_2(\lambda', u, u)] + \\
& (1-\lambda)[Q_2(\lambda', u, d)ud - T_2(\lambda', u, d)]\} + \\
& (1-\lambda)\{ \\
& Q_1(\lambda', d)d - T_1(\lambda', d) + \\
& \lambda[Q_2(\lambda', d, u)ud - T_2(\lambda', d, u)] + \\
& (1-\lambda)[Q_2(\lambda', d, d)d^2 - T_2(\lambda', d, d)]\}
\end{aligned}
$$

This second "recursive" formulation is key to the solution of the model. Apply the Envelope Theorem (as in Milgrom and Segal 2002) to condition IC λ with $\Pi(\lambda) = \pi(\lambda|\lambda)$

$$\frac{d\Pi}{d\lambda} = \frac{\partial \pi}{\partial \lambda}]_{\lambda'=\lambda}$$

$$\begin{aligned}
\frac{\partial \pi}{\partial \lambda}]_{\lambda'=\lambda} &= \\
&= Q_1(\lambda,u)u - T_1(\lambda,u) + \\
&\quad \lambda[Q_2(\lambda,u,u)u^2 - T_2(\lambda,u,u)] + (1-\lambda)[Q_2(\lambda,u,d)ud - T_2(\lambda',u,d)] - \\
&\quad Q_1(\lambda,d)d - T_1(\lambda,d) + \\
&\quad \lambda[Q_2(\lambda,d,u)ud - T_2(\lambda,d,u)] + (1-\lambda)[Q_2(\lambda,d,d)d^2 - T_2(\lambda,d,d)] \\
&\quad \lambda\frac{d}{d\lambda}\{Q_1(\lambda',u)u - T_1(\lambda',u) + \\
&\quad \lambda[Q_2(\lambda',u,u)u^2 - T_2(\lambda',u,u)] + (1-\lambda)[Q_2(\lambda',u,d)ud - T_2(\lambda',u,d)]\}_{\lambda'=\lambda} + \\
&\quad (1-\lambda)\frac{d}{d\lambda}\{Q_1(\lambda',d)d - T_1(\lambda',d) + \\
&\quad \lambda[Q_2(\lambda',d,u)ud - T_2(\lambda',d,u)] + (1-\lambda)[Q_2(\lambda',d,d)d^2 - T_2(\lambda',d,d)]\}_{\lambda'=\lambda}
\end{aligned}$$

Simplifying this expression gives:

$$\begin{aligned}
\frac{\partial \pi}{\partial \lambda}]_{\lambda'=\lambda} &= \\
&= Q_1(\lambda,u)u - T_1(\lambda,u) + \\
&\quad \lambda[Q_2(\lambda,u,u)u^2 - T_2(\lambda,u,u)] + (1-\lambda)[Q_2(\lambda,u,d)ud - T_2(\lambda',u,d)] - \\
&\quad Q_1(\lambda,d)d - T_1(\lambda,d) + \\
&\quad \lambda[Q_2(\lambda,d,u)ud - T_2(\lambda,d,u)] + (1-\lambda)[Q_2(\lambda,d,d)d^2 - T_2(\lambda,d,d)] \\
&\quad \lambda\{Q_2(\lambda',u,u)u^2 - T_2(\lambda',u,u) - [Q_2(\lambda',u,d)ud - T_2(\lambda',u,d)\} + \\
&\quad (1-\lambda)\{Q_2(\lambda',d,u)ud - T_2(\lambda',d,u) - [Q_2(\lambda',d,d)d^2 - T_2(\lambda',d,d)]\}
\end{aligned}$$

Substituting the DIC constraints gives:

$$
\begin{aligned}
\frac{\partial \pi}{\partial \lambda}]_{\lambda'=\lambda} &= \\
&= Q_1(\lambda,d)u - T_1(\lambda,d) + \\
&\quad \lambda[Q_2(\lambda,d,u)u^2 - T_2(\lambda,d,u)] + (1-\lambda)[Q_2(\lambda,d,d)ud - T_2(\lambda,u,d)] + \delta_1(\lambda) - \\
&\quad \{Q_1(\lambda,d)d - T_1(\lambda,d) + \\
&\quad \lambda[Q_2(\lambda,d,u)ud - T_2(\lambda,d,u)] + (1-\lambda)[Q_2(\lambda,d,d)d^2 - T_2(\lambda,d,d)]\} + \\
&\quad \lambda\{Q_2(\lambda,u,d)u^2 - T_2(\lambda,u,d) + \delta_2^u(\lambda) - [Q_2(\lambda,u,d)ud - T_2(\lambda,u,d)]\} + \\
&\quad (1-\lambda)\{Q_2(\lambda,d,d)ud - T_2(\lambda,d,d) + \delta_2^d(\lambda) - [Q_2(\lambda,d,d)d^2 - T_2(\lambda,d,d)]\}
\end{aligned}
$$

Cancelling the payment functions and simplifying gives:

$$
\begin{aligned}
&\lambda Q_1(\lambda,u)u + \\
&\lambda^2 Q_2(\lambda,u,u)u^2 + \\
&\left[(1-\lambda)d - \tfrac{1-F(\lambda)}{f(\lambda)}(u-d)\right] Q_1(\lambda,d) + \\
&\lambda u \left[(1-\lambda)d - \tfrac{1-F(\lambda)}{f(\lambda)}(u-d)\right] Q_2(\lambda,u,d) + \\
&\lambda u \left[(1-\lambda)d - \tfrac{1-F(\lambda)}{f(\lambda)}(u-d)\right] Q_2(\lambda,d,u) + \\
&(1-\lambda)d \left[(1-\lambda)d - 2\tfrac{1-F(\lambda)}{f(\lambda)}(u-d)\right] Q_2(\lambda,d,d)
\end{aligned}
$$

A.2 Detailed Solution of the Relaxed Problem

In text I show that the relaxed problem can be written as:

$\max E_\lambda\{\lambda Q_1(\lambda,u)u + (1-\lambda)Q_1(\lambda,d)d +$
$\lambda^2 Q_2(\lambda,u,u)u^2 + (\lambda-\lambda^2)[Q_2(\lambda,u,d)+Q_2(\lambda,d,u)]ud +$
$(1-\lambda)^2 Q_2(\lambda,d,d)d^2 - \Pi(\lambda)\}$

subject to $\forall\lambda \in (0,1)$

$\delta_1(\lambda) \geq 0$ C1
$\delta_2^u(\lambda) \geq 0$ C2-u
$\delta_2^d(\lambda) \geq 0$ C2-d

$\Pi(\lambda) = \Pi(0) + \int_0^\lambda Q_1(\lambda,d)(u-d) + [Q_2(\lambda,u,d)+Q_2(\lambda,d,u)]\lambda(u^2-ud)+$ C3
$Q_2(\lambda,d,d)2(1-\lambda)(ud-d^2) + \lambda\delta_2^u(\lambda) + (1-\lambda)\delta_2^d(\lambda) + \delta_1(\lambda)d\lambda$

$\Pi(0) \geq 0$ C4

Obviously, $\Pi(0) = \delta_1(\lambda) = \delta_2(\lambda) = \delta_3(\lambda) = 0$.

$$E_\lambda\{\lambda Q_1(\lambda,u)u + (1-\lambda)Q_1(\lambda,d)d + \lambda^2 Q_2(\lambda,u,u)u^2 +$$
$$(\lambda-\lambda^2)[Q_2(\lambda,u,d)+Q_2(\lambda,d,u)]ud + (1-\lambda)^2 Q_2(\lambda,d,d)d^2 - \Pi(\lambda)\} =$$

$$E_\lambda\{\lambda Q_1(\lambda,u)u + (1-\lambda)Q_1(\lambda,d)d + \lambda^2 Q_2(\lambda,u,u)u^2 +$$
$$(\lambda-\lambda^2)[Q_2(\lambda,u,d)+Q_2(\lambda,d,u)]ud + (1-\lambda)^2 Q_2(\lambda,d,d)d^2 -$$
$$\int_0^\lambda Q_1(s,d)(u-d) +$$
$$[Q_2(s,u,d)+Q_2(s,d,u)]s(u^2-ud) +$$
$$Q_2(s,d,d)2(1-s)(ud-d^2)ds\} =$$

$$\int_0^1 \{\lambda Q_1(\lambda,u)u + (1-\lambda)Q_1(\lambda,d)d + \lambda^2 Q_2(\lambda,u,u)u^2 +$$

$$(\lambda-\lambda^2)[Q_2(\lambda,u,d)+Q_2(\lambda,d,u)]ud +$$

$$(1-\lambda)^2 Q_2(\lambda,d,d)d^2\} f(\lambda)d\lambda -$$

$$\int_0^1 \int_0^\lambda \{Q_1(s,d)(u-d) +$$

$$[Q_2(s,u,d)+Q_2(s,d,u)]s(u^2-ud) +$$

$$Q_2(s,d,d)2(1-s)(ud-d^2)\} f(\lambda)ds d\lambda =$$

$$\int_0^1 \{\lambda Q_1(\lambda,u)u + (1-\lambda)Q_1(\lambda,d)d + \lambda^2 Q_2(\lambda,u,u)u^2 +$$

$$(\lambda-\lambda^2)[Q_2(\lambda,u,d)+Q_2(\lambda,d,u)]ud +$$

$$(1-\lambda)^2 Q_2(\lambda,d,d)d^2\} f(\lambda)d\lambda -$$

$$\int_0^1 \int_s^1 \{Q_1(s,d)(u-d) +$$

$$[Q_2(s,u,d)+Q_2(s,d,u)]s(u^2-ud) +$$

$$Q_2(s,d,d)2(1-s)(ud-d^2)\} f(\lambda)d\lambda ds =$$

$$\int_0^1 \{\lambda Q_1(\lambda,u)u + (1-\lambda)Q_1(\lambda,d)d + \lambda^2 Q_2(\lambda,u,u)u^2 +$$

$$(\lambda-\lambda^2)[Q_2(\lambda,u,d)+Q_2(\lambda,d,u)]ud +$$

$$(1-\lambda)^2 Q_2(\lambda,d,d)d^2\} f(\lambda)d\lambda -$$

$$\int_0^1 (1-F(s))\{Q_1(s,d)(u-d) +$$

$$[Q_2(s,u,d)+Q_2(s,d,u)]s(u^2-ud) +$$

$$Q_2(s,d,d)2(1-s)(ud-d^2)\} ds =$$

$$\begin{aligned}\int_0^1 \lambda Q_1(\lambda,u)u + (1-\lambda)Q_1(\lambda,d)d + \lambda^2 Q_2(\lambda,u,u)u^2 + \\ (\lambda-\lambda^2)[Q_2(\lambda,u,d)+Q_2(\lambda,d,u)]ud + \\ (1-\lambda)^2 Q_2(\lambda,d,d)d^2 - \\ \frac{1-F(\lambda)}{f(\lambda)}\{Q_1(\lambda,d)(u-d) + \\ [Q_2(\lambda,u,d)+Q_2(\lambda,d,u)]\lambda(u^2-ud) + \\ Q_2(\lambda,d,d)2(1-\lambda)(ud-d^2)\}f(\lambda)d\lambda =\end{aligned}$$

This integral can be maximized by maximizing the integrand. Combining terms, the integrand becomes

$$\begin{aligned}&\lambda Q_1(\lambda,u)u + \\ &\lambda^2 Q_2(\lambda,u,u)u^2 + \\ &\left[(1-\lambda)d - \tfrac{1-F(\lambda)}{f(\lambda)}(u-d)\right]Q_1(\lambda,d) + \\ &\lambda u\left[(1-\lambda)d - \tfrac{1-F(\lambda)}{f(\lambda)}(u-d)\right]Q_2(\lambda,u,d) + \\ &\lambda u\left[(1-\lambda)d - \tfrac{1-F(\lambda)}{f(\lambda)}(u-d)\right]Q_2(\lambda,d,u) + \\ &(1-\lambda)d\left[(1-\lambda)d - 2\tfrac{1-F(\lambda)}{f(\lambda)}(u-d)\right]Q_2(\lambda,d,d)\end{aligned}$$

A.3 Calculation of Payment Functions

In this section, I derive a set of optimal payment functions associated with the optimal allocation functions found in the previous section. To accomplish this, I use the optimal allocation functions to calculate $\Pi(\lambda)$, the expected utility of a buyer with type λ, the integral derived in the body of the paper. I then take the difference between the social surplus generated by a buyer of type λ and the expected utility of a buyer with type λ, $\Pi(\lambda)$. This difference is the expected lifetime payment under the optimal mechanism of a the buyer with type λ. Because I have assumed two-sided commitment, there are many decompositions of this expected lifetime payment function into payment functions for each state that satisfy the constraints of the relaxed problem. I propose one particular way of making this decomposition, which I find natural and intuitive. Let the expected social surplus of a buyer with propensity λ be denoted $ESS(\lambda)$.

Case I

Recall that for $\lambda \leq \lambda_1^*$

$$Q_1(\lambda, u) = 1$$

$$Q_2(\lambda, u, u) = 1$$

$$Q_1(\lambda, d) = 0$$

$$Q_2(\lambda, u, d) = 0$$

$$Q_2(\lambda, d, u) = 0$$

$$Q_2(\lambda, d, d) = 0$$

Applying the formulas, it is easy to see that

$$\Pi(\lambda) = 0$$

$$ESS(\lambda) = \lambda u + \lambda^2 u^2$$

It follows that the total expected payment conditional on λ, which is the difference between $ESS(\lambda)$ and $\Pi(\lambda)$ is equal to $\bar{t}(\lambda) = \lambda u + \lambda^2 u^2$

I decompose this total lifetime payment into per-period payment functions that satisfy the constraints of the relaxed problem:

$$T_0(\lambda) = 0$$

$$T_1(\lambda, u) = u$$

$$T_1(\lambda, d) = 0$$

$$T_2(\lambda, u, u) = u^2$$

$$T_2(\lambda, u, d) = 0$$

$$T_2(\lambda, d, u) = 0$$

$$T_2(\lambda, d, d) = 0$$

Case II

Recall that for $\lambda_1^* \leq \lambda \leq \lambda_2^*$

$$Q_1(\lambda, u) = 1$$

$$Q_2(\lambda, u, u) = 1$$

$$Q_1(\lambda, d) = 1$$

$$Q_2(\lambda, u, d) = 1$$

$$Q_2(\lambda, d, u) = 1$$

$$Q_2(\lambda, d, d) = 0$$

Applying the formulas, it is easy to see that

$$\begin{aligned} \Pi(\lambda) &= \int_{\lambda_1^*}^{\lambda} (u-d) + 2s(u^2 - ud)ds \\ &= (u-d)(\lambda - \lambda_1^*) + (\lambda^2 - \lambda_1^{*2})(u^2 - ud) \\ ESS(\lambda) &= \lambda u + (1-\lambda)d + \lambda^2 u^2 + 2(\lambda - \lambda^2)ud \end{aligned}$$

It follows that the total expected payment conditional on λ, which is the difference between $ESS(\lambda)$ and $\Pi(\lambda)$ is equal to

$$\begin{aligned}\bar{t}(\lambda) &= (u^2-ud)\lambda_1^{*2}+(u-d)\lambda_1^{*}+ \\ &\quad \lambda d+(1-\lambda)d+\lambda^2 ud+2(\lambda-\lambda^2)ud\end{aligned}$$

I decompose this total lifetime payment into per-period payment functions that satisfy the constraints of the relaxed problem:

$$T_0(\lambda)=\lambda_1^*(u-d)+(\lambda_1^*)^2(u^2-ud)$$

$$T_1(\lambda,u)=d$$

$$T_1(\lambda,d)=d$$

$$T_2(\lambda,u,u)=ud$$

$$T_2(\lambda,u,d)=ud$$

$$T_2(\lambda,d,u)=ud$$

$$T_2(\lambda,d,d)=0$$

Case III

Recall that for $\lambda_2^*\leq\lambda$

$$Q_1(\lambda,u)=1$$

$$Q_2(\lambda,u,u)=1$$

$$Q_1(\lambda,d)=1$$

$$Q_2(\lambda,u,d)=1$$

$$Q_2(\lambda,d,u)=1$$

$$Q_2(\lambda,d,d)=1$$

Applying the formulas, it is easy to see that

$$\begin{aligned}
\Pi(\lambda) &= \textstyle\int_{\lambda_1^*}^{\lambda}(u-d)+2s(u^2-ud)ds+\int_{\lambda_2^*}^{\lambda}2(1-s)(ud-d^2)ds \\
&= (u-d)(\lambda-\lambda_1^*)+(\lambda^2-\lambda_1^{*2})(u^2-ud)+ \\
&\quad\ (2(\lambda-\lambda_2^*)-(\lambda^2-\lambda_2^{*2}))(ud-d^2) \\
ESS(\lambda) &= \lambda u+(1-\lambda)d+\lambda^2u^2+2(\lambda-\lambda^2)ud+(1-\lambda)^2d^2
\end{aligned}$$

It follows that the total expected payment conditional on λ, which is the difference between $ESS(\lambda)$ and $\Pi(\lambda)$ is equal to

$$\begin{aligned}
\bar{t}(\lambda) &= (2\lambda_2^*-\lambda_2^{*2})(ud-d^2)+ \\
&\quad\ \lambda_1^{*2}(u^2-ud)+\lambda_1^*(u-d)+d+d^2
\end{aligned}$$

I decompose this total lifetime payment into per-period payment functions that satisfy the constraints of the relaxed problem:

$$T_0(\lambda)=(2\lambda_2^*-\lambda_2^{*2})(ud-d^2)+\lambda_1^{*2}(u^2-ud)+\lambda_1^*(u-d)$$

$$T_1(\lambda,u)=d$$

$$T_1(\lambda,d)=d$$

$$T_2(\lambda,u,u)=d^2$$

$$T_2(\lambda,u,d)=d^2$$

$$T_2(\lambda,d,u)=d^2$$

$$T_2(\lambda,d,d)=d^2$$

A.4 Verification of Incentive Compatibility

I have derived the optimal mechanism with in the class of mechanisms defined by the relaxed constraints. However, the relaxed constraints are only necessary constraints, derived from the infinite family of constraints IC. I now turn to the question of whether the optimal mechanism for the relaxed problem is incentive compatible in the sense of condition IC.

First, notice that for any $\lambda, r_1, r_2 \in \{u, d\}$ the optimal mechanism satifies the following conditions:

$$\begin{aligned}
Q_1(\lambda,u)u - T_1(\lambda,u) &\geq Q_1(\lambda,r_1)u - T_1(\lambda,r_1) \\
Q_1(\lambda,d)d - T_1(\lambda,d) &\geq Q_1(\lambda,r_1)d - T_1(\lambda,r_1) \\
Q_2(\lambda,u,u)u^2 - T_2(\lambda,u,u) &\geq Q_2(\lambda,r_1,r_2)u^2 - T_2(\lambda,r_1,r_2)] \\
Q_2(\lambda,u,d)ud - T_2(\lambda,u,d) &\geq Q_2(\lambda,r_1,r_2)ud - T_2(\lambda,r_1,r_2) \\
Q_2(\lambda,d,u)ud - T_2(\lambda,d,u) &\geq Q_2(\lambda,r_1,r_2)ud - T_2(\lambda,r_1,r_2) \\
Q_2(\lambda,d,d)d^2 - T_2(\lambda,d,d) &\geq Q_2(\lambda,r_1,r_2)d^2 - T_2(\lambda,r_1,r_2)
\end{aligned}$$

Therefore, multiplying each constraint by the appropriate coefficient implies that for any λ, λ' any r_1^u, r_1^d, $r_2^{u\ u}, r_2^{ud}, r_2^{du}, r_2^{dd} \in \{u, d\}$

$$\begin{aligned}
\lambda[Q_1(\lambda',u)u - T_1(\lambda',u)] &\geq \lambda[Q_1(\lambda',r_1^u)u - T_1(\lambda',r_1^u)] \\
(1-\lambda)[Q_1(\lambda',d)d - T_1(\lambda',d)] &\geq (1-\lambda)[Q_1(\lambda',r_1^d)d - T_1(\lambda',r_1^d)] \\
\lambda^2[Q_2(\lambda',u,u)u^2 - T_2(\lambda',u,u)] &\geq \lambda^2[Q_2(\lambda',r_1^u,r_2^{u\ u})u^2 - T_2(\lambda',r_1^u,r_2^{u\ u})] \\
\lambda(1-\lambda)[Q_2(\lambda',u,d)ud - T_2(\lambda',u,d)] &\geq \lambda(1-\lambda)[Q_2(\lambda',r_1^u,r_2^{ud})ud - T_2(\lambda',r_1^u,r_2^{ud})] \\
\lambda(1-\lambda)[Q_2(\lambda',d,u)ud - T_2(\lambda',d,u)] &\geq \lambda(1-\lambda)[Q_2(\lambda',r_1^d,r_2^{du})ud - T_2(\lambda',r_1^d,r_2^{du})] \\
(1-\lambda)^2[Q_2(\lambda',d,d)d^2 - T_2(\lambda',d,d)] &\geq (1-\lambda)^2[Q_2(\lambda',r_1^d,r_2^{dd})d^2 - T_2(\lambda',r_1^d,r_2^{dd})]
\end{aligned}$$

Adding the inequalities and subtracting $T_0(\lambda')$ from both sides gives the following condition

$$\begin{aligned} & \pi(\lambda'|\lambda) \\ & \geq \\ & -T_0(\lambda') + \lambda\left[Q_1(\lambda', r_1^u)u - T_1(\lambda', r_1^u)\right] + (1-\lambda)\left[Q_1\left(\lambda', r_1^d\right)d - T_1\left(\lambda', r_1^d\right)\right] + \\ & \lambda^2\left[Q_2(\lambda', r_1^u, r_2^{uu})u^2 - T_2(\lambda', r_1^u, r_2^{uu})\right] + \\ & \lambda(1-\lambda)\left[Q_2\left(\lambda', r_1^u, r_2^{ud}\right)ud - T_2\left(\lambda', r_1^u, r_2^{ud}\right)\right] + \\ & \lambda(1-\lambda)\left[Q_2\left(\lambda', r_1^d, r_2^{du}\right)du - T_2\left(\lambda', r_1^d, r_2^{du}\right)\right] + \\ & (1-\lambda)^2\left[Q_2\left(\lambda', r_1^d, r_2^{dd}\right)d^2 - T_2\left(\lambda', r_1^d, r_2^{dd}\right)\right] \end{aligned}$$

It is also easy to verify that even though I only used the local version of condition IC-λ (in the form of the Envelope Condition) that the global form of constraint IC-λ holds as well, so that for all $\lambda, \pi(\lambda|\lambda) \geq \pi(\lambda'|\lambda)$. This implies that

$$\begin{aligned} & \pi(\lambda|\lambda) \\ & \geq \\ & -T_0(\lambda') + \lambda\left[Q_1(\lambda', r_1^u)u - T_1(\lambda', r_1^u)\right] + (1-\lambda)\left[Q_1\left(\lambda', r_1^d\right)d - T_1\left(\lambda', r_1^d\right)\right] + \\ & \lambda^2\left[Q_2(\lambda', r_1^u, r_2^{uu})u^2 - T_2(\lambda', r_1^u, r_2^{uu})\right] + \\ & \lambda(1-\lambda)\left[Q_2\left(\lambda', r_1^u, r_2^{ud}\right)ud - T_2\left(\lambda', r_1^u, r_2^{ud}\right)\right] + \\ & \lambda(1-\lambda)\left[Q_2\left(\lambda', r_1^d, r_2^{du}\right)du - T_2\left(\lambda', r_1^d, r_2^{du}\right)\right] + \\ & (1-\lambda)^2\left[Q_2\left(\lambda', r_1^d, r_2^{dd}\right)d^2 - T_2\left(\lambda', r_1^d, r_2^{dd}\right)\right] \end{aligned}$$

which is nothing but condition (IC).

A.5 Analysis of the N-period Model

The general approach I will use for the analysis of the N period model is to relate the solution of the N period model to the solution of the $N-1$ period model. To this end, I introduce the following notation. Consider the model above extended to N periods. First, let $M_k = (m_1, m_2, .., m_k)$ represent a sequence of messages of length k. Similarly, let $H_k = (h_1, h_2, .., h_k)$ represent a sequence of past types for the agent. To each sequence of past types $H_k = (h_1, h_2, .., h_k)$ assign a scalar $\overline{h}_k = \Pi_{i=1...k} h_i$, which represents the agent's valuation for the good at the end of period k, given that his sequence of types is H_k.

As before, let $\pi_N(\lambda'|\lambda)$, represent the agent's expected utility in an N period mechanism when he reports λ' in period 0 given that his true type is λ, and he plans to make truthful reports in the future. As an extension of this, let $\pi_N^k(\lambda', M_{N-k}|\lambda, H_{N-k})$ represent the buyer's expected utility in an N period mechanism from following a policy of truthful reporting in periods $k+1$ through N, given a past sequence of reports λ', and true past types λ, H_k. So for example,

$$\pi_2^1(\lambda', d|\lambda, u) = \lambda[Q_2(\lambda', d, u)u^2 - T_2(\lambda', d, u)] + (1-\lambda)[Q_2(\lambda', d, d)ud - T_2(\lambda', d, d)]$$

$$\begin{aligned}\pi_3^1(\lambda', d|\lambda, u) &= \lambda[Q_2(\lambda', d, u)u^2 - T_2(\lambda', d, u)] + (1-\lambda)[Q_2(\lambda', d, d)ud - T_2(\lambda', d, d)] + \\ &\quad \lambda^2[Q_3(\lambda', d, u, u)u^3 - T_3(\lambda', d, u, u)] + \\ &\quad \lambda(1-\lambda)[Q_3(\lambda', d, u, d)u^2 d - T_3(\lambda', d, u, d)] + \\ &\quad \lambda(1-\lambda)[Q_2(\lambda', d, d, u)u^2 d - T_3(\lambda', d, d, u)] + \\ &\quad (1-\lambda)^2[Q_3(\lambda', d, d, d)d^3 - T_3(\lambda', d, d, d)]\end{aligned}$$

As a convention, let $\pi_N^N(\bullet|\bullet) = 0$.

It is also useful to introduce notation for the two separate pieces that compose $\pi_N^k(\lambda', M_{N-k}|\lambda, H_{N-k})$, the expected allocation and the expected payment. Therefore, let $ESS_N^k(\lambda', M_{N-k}|\lambda, H_{N-k})$ represent the agent's expected allocation, and $EP_N^k(\lambda', M_{N-k}|\lambda, H_{N-k})$ represent the agent's expected payment in an N period mechanism from following a policy of truthful reporting in periods $k+1$ through N, given a past sequence of reports λ', M_{N-k}, and true past types λ, H_k. Continuing the example above,

$$\begin{aligned}
ESS_2^1(\lambda', d|\lambda, u) &= \lambda Q_2(\lambda', d, u)u^2 + (1-\lambda)Q_2(\lambda', d, d)ud \\
EP_2^1(\lambda', d|\lambda, u) &= \lambda T_2(\lambda', d, u) + (1-\lambda)T_2(\lambda', d, d) \\
ESS_3^1(\lambda', d|\lambda, u) &= \lambda Q_2(\lambda', d, u)u^2 + (1-\lambda)Q_2(\lambda', d, d)ud + \\
&\quad \lambda^2 Q_3(\lambda', d, u, u)u^3 + \\
&\quad \lambda(1-\lambda)Q_3(\lambda', d, u, d)u^2 d + \\
&\quad \lambda(1-\lambda)Q_2(\lambda', d, d, u)u^2 d + \\
&\quad (1-\lambda)^2 Q_3(\lambda', d, d, d)d^3 \\
EP_3^1(\lambda', d|\lambda, u) &= \lambda T_2(\lambda', d, u) + (1-\lambda)T_2(\lambda', d, d) + \\
&\quad \lambda^2 T_3(\lambda', d, u, u) + \\
&\quad \lambda(1-\lambda)T_3(\lambda', d, u, d) + \\
&\quad \lambda(1-\lambda)T_3(\lambda', d, d, u) + \\
&\quad (1-\lambda)^2 T_3(\lambda', d, d, d)
\end{aligned}$$

One final piece of notation is $S_N(\lambda, M_k)$, which is just the social surplus generated in an N period model when the agent truthfully reports type λ, with a sequence of messages M_k appended to the history at the beginning of each allocation function. For example,

$$
\begin{aligned}
S_1(\lambda, d) &= \lambda Q_2(\lambda, d, u)u + (1-\lambda)Q_2(\lambda, d, d)d
\end{aligned}
$$

$$
\begin{aligned}
S_3^1(\lambda, d) &= \lambda Q_2(\lambda, d, u)u + (1-\lambda)Q_2(\lambda, d, d)d + \\
&\quad \lambda^2 Q_3(\lambda', d, u, u)u^2 + \\
&\quad \lambda(1-\lambda)Q_3(\lambda, d, u, d)ud + \\
&\quad \lambda(1-\lambda)Q_3(\lambda, d, d, u)ud + \\
&\quad (1-\lambda)^2 Q_3(\lambda, d, d, d)d^2
\end{aligned}
$$

Comparing the above notation, it is obvious that

$$ESS_N^k(\lambda, M_{N-k}|\lambda, H_{N-k}) = S_{N-k}(M_{N-k})\overline{h}_{N-k}$$

The law of total expectation also implies the following recursive relationships:

$$
\begin{aligned}
\pi_N(\lambda'|\lambda) &= \lambda[Q_1(\lambda', u)u - T_1(\lambda', u) + \pi_N^1(\lambda', u|\lambda, u)] + \\
&\quad (1-\lambda)[Q_1(\lambda', d)d - T_1(\lambda', d) + \pi_N^1(\lambda', d|\lambda, d)]
\end{aligned}
$$

$$
\begin{aligned}
&\lambda[Q_{k+1}(\lambda', M_k, u)\overline{h}_k u - T_{k+1}(\lambda', M_k, u) + \pi_N^{k+1}(\lambda', M_k, u|\lambda, H_k, u)] + \\
&(1-\lambda)[Q_{k+1}(\lambda', M_k, d)\overline{h}_{N-k} d - T_{k+1}(\lambda', M_k, d) + \pi_N^{k+1}(\lambda', M_k, d|\lambda, H_k, d)]
\end{aligned}
$$

Using this notation, it is easy to write down the relaxed family of IC constraints. Just as in the two-period model analyzed in the body of the paper, the relaxed constraints prevent only single deviations from truth-telling. In period 0, we get the N period analog of constraint IC-λ.

$$\pi_N(\lambda|\lambda) \geq \pi_N(\lambda'|\lambda)$$

In periods 1 through N we obtain the N period analogues of constraints DIC2-u,DIC2-d,DIC1. Specifically,

$$Q_1(\lambda, u)u - T_1(\lambda, u) + \pi_N^1(\lambda, u|\lambda, u) \geq Q_1(\lambda, d)u - T_1(\lambda, d) + \pi_N^1(\lambda, d|\lambda, u)$$

$$Q_k(\lambda, H_{k-1}, u)\overline{h}_{k-1}u - T_k(\lambda, H_{k-1}, u) + \pi_N^k(\lambda, H_{k-1}, u|\lambda, H_{k-1}, u) \geq$$
$$Q_k(\lambda, H_{k-1}, d)\overline{h}_{k-1}u - T_k(\lambda, H_{k-1}, d) + \pi_N^k(\lambda, H_{k-1}, d|\lambda, H_{k-1}, u)$$

In the course of solving the two period problem, I showed that in the optimal mechanism, the downward incentive compatibility constraints for values are binding. Extending this to the N period case, I formulate the principal's N period problem.

$\max E_\lambda\{EP_N(\lambda|\lambda)\}$

subject to $\forall \lambda, \lambda' \in (0, 1)$

$Q_1(\lambda, u)u - T_1(\lambda, u) + \pi_N^1(\lambda, u|\lambda, u) = Q_1(\lambda, d)u - T_1(\lambda, d) + \pi_N^1(\lambda, d|\lambda, u)$

for $k = 2, ..., N,$ and all $H_{k-1} \in \times_{i=1}^{k-1}(u, d)$
$Q_k(\lambda, H_{k-1}, u)\overline{h}_{k-1}u - T_k(\lambda, H_{k-1}, u) + \pi_N^k(\lambda, H_{k-1}, u|\lambda, H_{k-1}, u) =$
$Q_k(\lambda, H_{k-1}, d)\overline{h}_{k-1}u - T_k(\lambda, H_{k-1}, d) + \pi_N^k(\lambda, H_{k-1}, d|\lambda, H_{k-1}, u)$

$\pi_N(\lambda|\lambda) \geq \pi_N(\lambda'|\lambda)$

$\pi_N(\lambda|\lambda) \geq 0$

I proceed in the standard way, applying the Envelope Theorem to constraint IC-λ.

$$\begin{aligned}\arg\max_{\lambda'} \pi_N(\lambda'|\lambda) &= \lambda \\ \Pi_N(\lambda) \equiv \max_{\lambda'} \pi_N(\lambda'|\lambda) &= \pi(\lambda|\lambda) \\ \frac{d\Pi_N}{d\lambda} &= \frac{\partial \pi_N}{\partial \lambda}]_{\lambda'=\lambda}\end{aligned}$$

Using the recursive relationship,

$$\begin{aligned}\pi_N(\lambda'|\lambda) &= \lambda[Q_1(\lambda', u)u - T_1(\lambda', u) + \pi_N^1(\lambda', u|\lambda, u)] + \\ &\quad (1-\lambda)[Q_1(\lambda', d)d - T_1(\lambda', d) + \pi_N^1(\lambda', d|\lambda, d)]\end{aligned}$$

it is easy to see that:

$$\begin{aligned}\frac{d\Pi_N}{d\lambda} &= Q_1(\lambda, u)u - T_1(\lambda, u) + \pi_N^1(\lambda, u|\lambda, u) - \\ &\quad [Q_1(\lambda, d)d - T_1(\lambda, d) + \pi_N^1(\lambda, d|\lambda, d)] + \\ &\quad \lambda\frac{d}{d\lambda}\{\pi_N^1(\lambda', u|\lambda, u)\}_{\lambda'=\lambda} + \\ &\quad (1-\lambda)\frac{d}{d\lambda}\{\pi_N^1(\lambda', d|\lambda, d)\}_{\lambda'=\lambda}\end{aligned}$$

Substituting the identities

$$\begin{aligned}ESS_N^1(\lambda, d|\lambda, u) &= S_{N-1}(d)u \\ ESS_N^1(\lambda, d|\lambda, d) &= S_{N-1}(d)d\end{aligned}$$

gives the following expression:

$$\begin{aligned}\frac{d\Pi_N}{d\lambda} &= Q_1(\lambda,d)(u-d)+S_{N-1}(d)(u-d)+\\ &[Q_1(\lambda,d)d-T_1(\lambda,d)+\pi_N^1(\lambda,d|\lambda,d)]+\\ &\lambda\frac{d}{d\lambda}\{\pi_N^1(\lambda',u|\lambda,u)\}_{\lambda'=\lambda}+(1-\lambda)\frac{d}{d\lambda}\{\pi_N^1(\lambda',d|\lambda,d)\}_{\lambda'=\lambda}\end{aligned}$$

For the remainder of the analysis let $R_N(\lambda)\equiv\frac{d\Pi_N}{d\lambda}$, and $R_{N-k}(\lambda,M_k)$ denote the function $R_{N-k}(\lambda)$ with sequence of messages M_k appended to the beginning of every history of reports. For example, in the analysis of the two-period case, I show that

$$\begin{aligned}R_2(\lambda) &= Q_1(\lambda,d)(u-d)+\lambda Q_2(\lambda,d,u)(u^2-ud)+\lambda Q_2(\lambda,u,d)(u^2-ud)+\\ &2(1-\lambda)Q_2(\lambda,d,d)(ud-d^2)\end{aligned}$$

The expression for $R_N(\lambda)$ derived above contains the somewhat mysterious terms $\frac{d}{d\lambda}\{\pi_N^1(\lambda',u|\lambda,u)\}_{\lambda'=\lambda}$ and $\frac{d}{d\lambda}\{\pi_N^1(\lambda',d|\lambda,d)\}_{\lambda'=\lambda}$. These terms, look similar to $R_{N-1}(\lambda)=\frac{d}{d\lambda}\{\pi_{N-1}(\lambda'|\lambda)\}_{\lambda'=\lambda}$, though not identical. Still, they look similar enough that one might suspect that there is a connection; in fact, there is connection, proved at the end of the analysis.

The constraints of the relaxed problem imply that for $1\leq k\leq N-1$, and any $N\geq 2$ $\frac{d}{d\lambda}\{\pi_N^k(\lambda',H_k|\lambda,H_k)\}_{\lambda'=\lambda}=\overline{h}_k R_{N-k}(\lambda,H_k)$.

Making these substitutions implied by the lemma, gives a recursive formulation for the information rent term, relating the information rent in period N to the information rent in period $N-1$.

$$R_N(\lambda)=Q_1(\lambda,d)(u-d)+S_{N-1}(d)(u-d)+\lambda u R_{N-1}(\lambda,u)+(1-\lambda)dR_{N-1}(\lambda,d)$$

The seller's profit function, after substituting the ICCs, is the expected social surplus, minus the buyer's expected information rent, which after changing the order of integration is reduced to the expected social surplus minus the inverse hazard rate multiplied by the information rent term.

$$\begin{aligned}\Pi_N^P(\lambda) &= ESS_N(\lambda) - \\ &\quad \frac{1-F(\lambda)}{f(\lambda)} \\ &\quad \{Q_1(\lambda,d)(u-d) + S_{N-1}(d)(u-d) + \lambda u R_{N-1}(\lambda,u) + (1-\lambda)dR_{N-1}(\lambda,d)\}\end{aligned}$$

Expanding $ESS_N(\lambda)$ one period recursively, I find that

$$\begin{aligned}
\Pi_N^P(\lambda) &= \lambda\left[Q_1(\lambda,u)\,u + ESS_{N-1}(\lambda,u|\lambda,u)\right] + \\
&\quad (1-\lambda)\left[Q_1(\lambda,d)\,d + ESS_{N-1}(\lambda,d|\lambda,d)\right] - \\
&\quad \frac{1-F(\lambda)}{f(\lambda)} \\
&\quad \{Q_1(\lambda,d)(u-d) + S_{N-1}(d)(u-d) + \lambda u R_{N-1}(\lambda,u) + (1-\lambda)dR_{N-1}(\lambda,d)\}
\end{aligned}$$

$$\begin{aligned}
&= Q_1(\lambda,u)\,\lambda u + Q_1(\lambda,d)\left\{(1-\lambda)\,d - \frac{1-F(\lambda)}{f(\lambda)}(u-d)\right\} + \\
&\quad \lambda ESS_{N-1}(\lambda,u|\lambda,u) - \frac{1-F(\lambda)}{f(\lambda)}\lambda u R_{N-1}(\lambda,u) + \\
&\quad (1-\lambda)\,ESS_{N-1}(\lambda,d|\lambda,d) - \\
&\quad \frac{1-F(\lambda)}{f(\lambda)}\left\{S_{N-1}(d)\,(u-d) + (1-\lambda)dR_{N-1}(\lambda,d)\right\}
\end{aligned}$$

$$\begin{aligned}
&= Q_1(\lambda,u)\,\lambda u + Q_1(\lambda,d)\left\{(1-\lambda)\,d - \frac{1-F(\lambda)}{f(\lambda)}(u-d)\right\} + \\
&\quad \lambda u\left\{S_{N-1}(\lambda,u) - \frac{1-F(\lambda)}{f(\lambda)}R_{N-1}(\lambda,u)\right\} + \\
&\quad (1-\lambda)\,dS_{N-1}(\lambda,d) - \frac{1-F(\lambda)}{f(\lambda)}\left\{S_{N-1}(\lambda,d)\,(u-d) + (1-\lambda)dR_{N-1}(\lambda,d)\right\}
\end{aligned}$$

There are several things to notice about this expression. First, in any model of arbitrary length, $Q_1(\lambda,u) = 1$ and $Q_1(\lambda,d) = 1$ if and only if the coefficient $\left\{(1-\lambda)\,d - \frac{1-F(\lambda)}{f(\lambda)}(u-d)\right\}$ is positive.

Second, the term $\left\{S_{N-1}(\lambda,u) - \frac{1-F(\lambda)}{f(\lambda)}R_{N-1}(\lambda,u)\right\}$ is the same as $\Pi_{N-1}^P(\lambda)$ with the addition of a reported u at the beginning of every allocation function that appears in this expression. Therefore, the optimal allocation functions in an N period model following a report of u in period one are the same as the optimal allocation functions in an $N-1$ period model. To determine the allocation functions of the N period model following a report of d in period one, consider the piece of the N period information rent that involves these functions:

$$(1-\lambda)\,dS_{N-1}(\lambda,d)-\frac{1-F(\lambda)}{f(\lambda)}\left\{S_{N-1}(\lambda,d)(u-d)+(1-\lambda)dR_{N-1}(\lambda,d)\right\}$$

I will use an inductive argument. Let σ_k^{N-1} be the coefficient on the allocation function following history H_k in the expression S_{N-1}, and let ρ_k^{N-1} be the coefficient on the same allocation function in expression R_{N-1}.Assume that $\frac{\rho_k^{N-1}}{\sigma_k^{N-1}}=j\frac{u-d}{d(1-\lambda)}$ where j is the number of d reports in history H_k The case of $N=2$ can be verified by direct examination of the formulas derived in the body of the paper.

The the above expression implies that the coefficient on the allocation function for history (d,H_k) in the expression for the principal's N period profit is

$$(1-\lambda)\,d\sigma_k-\frac{1-F(\lambda)}{f(\lambda)}\left\{\sigma_k(u-d)+(1-\lambda)\,d\rho_k\right\}$$

which, following the inductive assumption, is just

$$\begin{aligned}(1-\lambda)\,d\sigma_k-\frac{1-F(\lambda)}{f(\lambda)}\left\{\sigma_k(u-d)+j(1-\lambda)\,d\frac{u-d}{d(1-\lambda)}\sigma_k\right\} &=\\ (1-\lambda)\,d\sigma_k-\frac{1-F(\lambda)}{f(\lambda)}\left\{\sigma_k(u-d)+j(u-d)\,\sigma_k\right\} &=\\ \sigma_k\left[(1-\lambda)\,d-\frac{1-F(\lambda)}{f(\lambda)}\left\{(j+1)(u-d)\right\}\right]\end{aligned}$$

This implies that $\rho_{k+1}^{N}=\sigma_k^{N-1}(j+1)(u-d)$. Direct computation shows that $\sigma_{k+1}^{N}=(1-\lambda)^{k+1}\lambda^{N-k-1}d^{k+1}u^{N-k-1}$ and $\sigma_k^{N-1}=(1-\lambda)^{k}\lambda^{N-1-k}d^{k}u^{N-1-k}$. It follows that $\frac{\sigma_k^{N-1}}{\sigma_{k+1}^N} = \frac{1}{(1-\lambda)d}$. Hence, $\frac{\rho_{k+1}^N}{\sigma_{k+1}^N} = \frac{\sigma_k^{N-1}}{\sigma_{k+1}^N}(j+1)(u-d) = (j+1)\frac{(u-d)}{d(1-\lambda)}$, which verifies the inductive assumption.

The above argument shows that if H_k contains j reports of d, then in the N period model the coefficient on $Q_{k+1}(d, H_k)$ is proportional to $\left[(1-\lambda)d - \frac{1-F(\lambda)}{f(\lambda)}(j+1)(u-d)\right]$. Combining this fact with the fact that the coefficients on the allocation functions in the N period model $Q_{k+1}(u, H_k) = Q_k(H_k)$ from the $N-1$ period model brings me to the following result: In a model with any number of periods,

$$Q_k(\lambda, H_k) = 1 \Leftrightarrow (1-\lambda)d - \frac{1-F(\lambda)}{f(\lambda)}j(u-d) \geq 0$$

where j is the number of reported downs in history H_k. Verification of incentive compatibility is similar to the case of two periods.

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Biography

1. Full name: Raphael Boleslavsky
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Ralph was born on a cold winter night in Philadelphia, 1979. Since then he has enjoyed the cold. He was an underachiever in high school, mostly because he was bored. His parents are glad that he turned out alright. He is one of the only people to defeat both Godzilla and the Incredible Hulk in single combat.