

Derivation of the Time Dependent Gross Pitaevskii Equation with External Fields

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Abstract

Using a new method [11] it is possible to derive mean field equations from the microscopic N body Schrödinger evolution of interacting particles without using BBGKY hierarchies.

Recently this method was used to derive the Hartree equation for singular interactions [5] and the Gross Pitaevskii equation without positivity condition on the interaction [12] where one had to restrict the scaling behavior of the interaction.

In this paper more general scalings shall be considered assuming positivity of the interaction.

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1 Introduction

In this paper we analyze the dynamics of a Bose condensate of N interacting particles when the external trap — described by an external potential A_t — is changed, for example removed.

We are interested in solutions of the N -particle Schrödinger equation

$$i \frac{d}{dt} \Psi_t = H \Psi_t \quad (1)$$

with some symmetric (under exchange of any two variables) Ψ_0 we shall specify below and the Hamiltonian

$$H = - \sum_{j=1}^N \Delta_j + \sum_{1 \leq j < k \leq N} V_\beta(x_j - x_k) + \sum_{j=1}^N A_t(x_j) \quad (2)$$

acting on the Hilbert space $L^2(\mathbb{R}^{3N}, \mathbb{C})$, where $\beta \in \mathbb{R}$ stands for the scaling behavior of the interaction. Note, that Ψ depends on N . For ease of notation this shall not be indicated (as well as for many other N -dependent objects). The V_β scale with the particle number in such a way, that the total interaction energy is (like the total kinetic energy of the N particles) of order one.

For the moment one may think of an interaction which is given by $V_\beta(x) = N^{-1+3\beta} V(N^\beta x)$ for a compactly supported, spherically symmetric, positive potential $V \in L^\infty$. The interactions we shall choose below will be of a more general form.

The A_t describing the trap potential is a time dependent external potential which we shall choose — in contrast to V_β — N -independent. Note, that H conserves symmetry, i.e. for any symmetric function Ψ_0 also $H\Psi_0$ and thus Ψ_t is symmetric.

Assume moreover that the initial wave function Ψ_0 is a condensate in the sense that the reduced one particle marginal density

$$\mu^{\Psi_0} := \int \Psi^*(\cdot, x_2, \dots, x_N) \Psi(\cdot, x_2, \dots, x_N) d^3x_2 \dots d^3x_N$$

converges to $|\varphi_0\rangle\langle\varphi_0|$ in operator norm.

Under these and some additional technical assumptions we shall show that also μ^{Ψ_t} will be a condensate, i.e. that there exist L^2 functions φ_t such that in operator norm

$$\lim_{N \rightarrow \infty} \mu^{\Psi_t} = |\varphi_t\rangle\langle\varphi_t|$$

uniform in t on any compact subset of \mathbb{R}^+ and — under additional decay conditions on φ_t — uniform in $t \in \mathbb{R}^+$.

In addition we shall show that φ_t solves the differential equation

$$i \frac{d}{dt} \varphi_t = (-\Delta + A_t + \overline{V}_{\varphi_t}) \varphi_t \quad (3)$$

with φ_0 as above, where the “mean field” \bar{V}_{φ_t} depends on φ_t itself, so (3) is a non-linear equation.

For different regimes of β different effective mean field potentials will appear: For $\beta = 0$ each particle feels $N^{-1} \sum_{j=2}^N V(x_1 - x_j) \approx \int V(x - y) |\varphi_t|^2(y) d^3y$ interactions as long as the particles are roughly $|\varphi_t|^2$ -distributed. Hence the mean field is given by $\bar{V}_{\varphi_t} = V \star |\varphi_t|^2$. This case is less involved than scalings $0 < \beta \leq 1$, thus it fits best to introduce the new method (see [11]).

For $0 < \beta$ the interaction becomes δ -like. To be able to “average out” the potential it is important to control the microscopic structure of Ψ_t . Assuming that the energy of Ψ_t is small, the microscopic structure is — whenever two particles approach — roughly given by the zero energy scattering state of the potential V_β . Let us for the moment call this zero energy scattering state $f_\beta^N(x)$. Changing to coordinates $y = N^\beta x$ the zero energy scattering state satisfies

$$N^{2\beta}(-\Delta + \frac{1}{2}N^{-1+\beta}V(y))f_\beta^N(y) = 0.$$

For $\beta = 1$ the scaling of the potential is such that the zero energy scattering state $f_\beta^N(x)$ of the potential V_β just scales with y , i.e. $f_1^N(x) = f_1^1(Nx)$. Since $\int V_\beta(x)f_\beta^N(x)dx$ equals 8π times the scattering length of V_β it follows that the mean field is given by $2a|\varphi_t|^2$, where $a/(4\pi)$ is the scattering length of V .

The microscopic structure formed by the wave functions enables us to generalize the interactions when $\beta = 1$ and V is compactly supported: Since the scattering length of the potential is always smaller than the radius of the support of the potentials, the coupling constant of the interaction may grow arbitrarily fast in N in that case. Hence we shall also consider interactions of the form

$$V_{1,\mu}(x) = N^\mu V(N^{-1}x) \tag{4}$$

with $\mu > 2$. In this case the wave function avoids the interaction regions and still the scattering length and thus the effect of each interaction is of order N^{-1} .

For $0 < \beta < 1$ the scaling is “softer” and the microscopic structure disappears as $N \rightarrow \infty$. Thus the mean field is given by $V_{\varphi_t} = \|V\|_1 |\varphi_t|^2$. One can also argue, that for “soft scalings” the scattering length is in good approximation given by the first order Born approximation and thus roughly the L_1 -norm of the interaction divided by 8π .

Note that the cases $\beta < 0$ and $\beta > 1$ are of minor interest: In both cases the interaction becomes negligible. In the case $\beta < 0$ the interactions are more or less constant over the support of Ψ , in the case $\beta > 1$ the radius of the support (and with it the scattering length) of the interaction shrinks faster than N^{-1} . Thus the effective interaction felt by each particle becomes negligible.

A proof for the cases $0 < \beta \leq 1$ without external fields based on a hierarchical method analogous to BBGKY hierarchies can be found in [2, 3]. The simpler, one dimensional case is treated in [1]. We shall give an alternative proof in three dimensions including time dependent external potentials and generalize to scalings of the form (4). Furthermore we shall prove that the convergence holds uniform in time, assuming that φ_t shows sufficient decay behavior.

In recent years there has been a growing number of experiments with Bose Einstein condensates where the influence of the mean field has been analyzed (see for example [6]). In many of these experiments the condensate propagates while an external field is present, for example in the well known atomic laser experiments the condensates are dropped in the gravitational field. Thus a theoretical understanding of the dynamics of Bose Einstein condensates in external fields is appreciated.

2 Formulation of the Problem

2.1 The new method

The method we shall use in this paper is in details explained in [11]. Heuristically speaking it is based on the idea of counting for each time t the relative number of those particles which are not in the state φ_t and estimating the time derivative of that value. To put that onto a rigorous level we need to define some projectors first.

Definition 2.1 Let $\varphi \in L^2(\mathbb{R}^3, \mathbb{C})$.

- (a) For any $1 \leq j \leq N$ the projectors $p_j^\varphi : L^2(\mathbb{R}^{3N}, \mathbb{C}) \rightarrow L^2(\mathbb{R}^{3N}, \mathbb{C})$ and $q_j^\varphi : L^2(\mathbb{R}^{3N}, \mathbb{C}) \rightarrow L^2(\mathbb{R}^{3N}, \mathbb{C})$ are given by

$$p_j^\varphi \Psi = \varphi(x_j) \int \varphi^*(x_j) \Psi(x_1, \dots, x) d^3 x_j \quad \forall \Psi \in L^2(\mathbb{R}^{3N}, \mathbb{C})$$

and $q_j^\varphi = 1 - p_j^\varphi$.

We shall also use the bra-ket notation $p_j^\varphi = |\varphi(x_j)\rangle\langle\varphi(x_j)|$.

- (b) For any $0 \leq k \leq N$ we define the set

$$\mathcal{A}_k := \{(a_1, a_2, \dots, a_N) : a_j \in \{0, 1\} ; \sum_{j=1}^N a_l = k\}$$

and the orthogonal projector P_k^φ acting on $L^2(\mathbb{R}^{3N}, \mathbb{C})$ as

$$P_k^\varphi := \sum_{a \in \mathcal{A}_k} \prod_{j=1}^N (p_j^\varphi)^{1-a_j} (q_j^\varphi)^{a_j} .$$

For negative k and $k > N$ we set $P_k^\varphi := 0$.

- (c) For any function $m : \mathbb{N}^2 \rightarrow \mathbb{R}_0^+$ we define the operator $\hat{m}^\varphi : L^2(\mathbb{R}^{3N}, \mathbb{C}) \rightarrow L^2(\mathbb{R}^{3N}, \mathbb{C})$ as

$$\hat{m}^\varphi := \sum_{j=0}^N m(j, N) P_j^\varphi . \quad (5)$$

We shall also need the shifted operators $\widehat{m}_d^\varphi : L^2(\mathbb{R}^{3N}, \mathbb{C}) \rightarrow L^2(\mathbb{R}^{3N}, \mathbb{C})$ given by

$$\widehat{m}_d^\varphi := \sum_{j=d}^{N+d} m(j+d, N) P_j^\varphi .$$

2.2 Derivation of the Gross-Pitaevskii equation

This paper deals with the case $0 < \beta \leq 1$ only. Then (3) becomes the Gross Pitaevskii equation

$$i \frac{d}{dt} \varphi_t = (-\Delta + A_t) \varphi_t + 2a|\varphi_t|^2 \varphi_t := h^{GP} \varphi_t . \quad (6)$$

Following [11] we shall define a functional $\alpha : L^2(\mathbb{R}^{3N}, \mathbb{C}) \otimes L^2(\mathbb{R}^3, \mathbb{C}) \rightarrow \mathbb{R}^+$ such that

- (a) $\frac{d}{dt} \alpha(\Psi_t, \varphi_t)$ can be estimated by $\alpha(\Psi_t, \varphi_t) + \mathcal{O}(1)$, giving good control of $\alpha(\Psi_t, \varphi_t)$ via Grönwall.
- (b) $\alpha(\Psi, \varphi) \rightarrow 0$ implies convergence the reduced one particle density matrix of Ψ to $|\varphi\rangle\langle\varphi|$ in trace norm.

In the case $\beta = 0$ it turned out that the choice

$$\alpha(\Psi, \varphi) = \left\langle\left\langle \Psi, (\widehat{n}^\varphi)^j \Psi \right\rangle\right\rangle$$

(again $n(k, N) = \sqrt{k/N}$ and $\langle\langle \cdot \rangle\rangle$ is scalar product on $L^2(\mathbb{R}^{3N}, \mathbb{C})$) for arbitrary $j > 0$ does the job (see for example [11] and [5], where the cases $j = 2$ respectively $j = 1$ are treated for different interactions).

Depending on the particular setting slight adjustments of the functional α are sometimes needed to get sufficient control of $\frac{d}{dt} \alpha(\Psi_t, \varphi_t)$. When dealing with interactions which peak very fast as N tends to infinity, adding a functional which takes care of the smoothness of Ψ proves to be helpful. Doing the estimates it turns out that one needs that $\|\nabla_1 q_1^\varphi \Psi\|$ is small (see Lemma 4.4 (d)). With the Grönwall argument in mind the first idea one might have is to add precisely this term to α , but on the other hand the time derivative of $\|\nabla_1 q_1^\varphi \Psi\|$ is hard to control. Therefore we add the difference of the energy per particle of Ψ and the Gross-Pitaevskii-energy of φ to our functional. It is natural to assume that — if $\mu^\Psi \rightarrow |\varphi\rangle\langle\varphi|$ — this difference is initially small and one expects that during time evolution the energy change per particle of Ψ and the energy change of φ are approximately the same.

Therefore we shall need the energy functional $\mathcal{E} : L^2(\mathbb{R}^{3N}, \mathbb{C}) \rightarrow \mathbb{R}$

$$\mathcal{E}(\Psi) = N^{-1} \langle\langle \Psi, H \Psi \rangle\rangle ,$$

as well as the Gross Pitaevskii energy functional $\mathcal{E}^{GP} : L^2(\mathbb{R}^3, \mathbb{C}) \rightarrow \mathbb{R}$

$$\mathcal{E}^{GP}(\varphi) := \langle \nabla \varphi, \nabla \varphi \rangle + \langle \varphi, (A_t + a|\varphi|^2) \varphi \rangle = \langle \varphi, (h^{GP} - a|\varphi|^2) \varphi \rangle . \quad (7)$$

Doing the estimates it turns out that $\|\nabla_1 q_1^\varphi \Psi\|$ is small in terms of the energy difference plus $\langle \Psi, \widehat{n}^\varphi \Psi \rangle$. Therefore we choose α in the following way:

Definition 2.2 Let $n(k, N) := \sqrt{k/N}$. We define for any $N \in \mathbb{N}$ the functional $\alpha : L^2(\mathbb{R}^{3N}, \mathbb{C}) \times L^2(\mathbb{R}^3, \mathbb{C}) \rightarrow \mathbb{R}_0^+$

$$\alpha(\Psi, \varphi) := \langle \Psi, \widehat{n}^\varphi \Psi \rangle + |\mathcal{E}(\Psi) - \mathcal{E}^{GP}(\varphi)| .$$

To get good control of $\langle \Psi_t, \widehat{n}^{\varphi_t} \Psi_t \rangle$, the solutions φ_t of the Gross Pitaevskii equation we shall consider have to satisfy some additional conditions.

Definition 2.3 We define the set of “good” solutions of the Gross-Pitaevskii equation

$$\mathcal{G} := \{ \varphi_t : i \frac{d}{dt} \varphi_t = h^{GP} \varphi_t; \|\varphi_t\|_\infty + \|\Delta \varphi_t\| < \infty \forall t \geq 0 \} .$$

Furthermore we shall — depending on β — need some conditions on the interaction V_β . These conditions shall include the potentials we used in the introduction, i.e. potentials which scale like $V_\beta(x) = N^{-1+3\beta} V(N^\beta x)$ as well as scalings of the form (4) for compactly supported, spherically symmetric, positive potentials $V \in L^\infty$.

Definition 2.4 Let $a > 0$. For any $0 < \beta \leq 1$ we define the auxiliary set

$$\mathcal{U}_\beta := \{ V_\beta \text{ pos. and spher. symm.}, V_\beta(x) = 0 \forall x > RN^{-\beta} \text{ for some } R < \infty \}$$

as well as the set of potentials with appropriate scaling behavior for $0 < \beta < 1$

$$\mathcal{V}_\beta := \{ V_\beta \in \mathcal{U}_\beta : \lim_{N \rightarrow \infty} N^{1-3\beta} \|V_\beta\|_\infty < \infty; \\ \lim_{N \rightarrow \infty} N^\eta | \|NV_\beta\|_1 - 2a | < \infty \text{ for some } \eta > 0 \} ,$$

and for $\beta = 1$

$$\mathcal{V}_1 := \{ V_1 \in \mathcal{U}_1 : \lim_{N \rightarrow \infty} N^\eta | 4\pi N \text{scat}(V_1) - a | < \infty \text{ for some } \eta > 0 \} ,$$

where $\text{scat}(V)$ is the scattering length of the potential V .

With these definitions we arrive at the main Theorem:

Theorem 2.5 Let $0 < \beta \leq 1$, let $V_\beta \in \mathcal{V}_\beta$, let A_t be an external potential with $\sup_{x \in \mathbb{R}^3, t \in \mathbb{R}} |\dot{A}_t| < \infty$. Let $\varphi_t \in \mathcal{G}$ and Ψ_0 be symmetric with $\|\Psi_0\| = 1$. Then there exists a $\eta > 0$ and constants $C_1, C_2 < \infty$ such that

$$\alpha(\Psi_t, \varphi_t) \leq C_1 e^{C_2 (\ln N)^{1/3} \int_0^t \|\varphi_s\|_\infty + \|\nabla \varphi_s\|_{6,loc} + \|\dot{A}_s\|_\infty ds} (\alpha(\Psi_0, \varphi_0) + N^{-\eta}) , \quad (8)$$

where $\|\cdot\|_{6,loc} : L^2(\mathbb{R}^3, \mathbb{C}) \rightarrow \mathbb{R}^+$ is the “local L^6 -norm” given by

$$\|\varphi\|_{6,loc} := \sup_{x \in \mathbb{R}^3} \|\mathbb{1}_{|\cdot-x| \leq 1} \varphi\|_6 .$$

Remark 2.6 (a) Lieb, Seiringer and Yngvason have proven that for the ground state Ψ^{gs} of a trapped Bose gas and the ground state φ^{gs} of the respective Gross-Pitaevskii energy functional $\mathcal{E}(\Psi^{gs}) - \mathcal{E}^{GP}(\varphi^{gs}) \rightarrow 0$ as $N \rightarrow \infty$ [9]. In [7] Lieb and Seiringer show that $\mu^{\Psi^{gs}} \rightarrow |\varphi^{gs}\rangle\langle\varphi^{gs}|$. Hence for the ground state of a trapped Bose gas $\lim_{N \rightarrow \infty} \alpha(\Psi^{gs}, \varphi^{gs}) = 0$.

(b) For all $\eta > 0$ one can find a $N > 0$ such that $(\ln N)^{1/3} < \eta \ln N$. Thus $e^{(\ln N)^{1/3}} \leq C e^{\eta \ln N} = C N^\eta$, so if $\alpha(\Psi_0, \varphi_0) \leq C N^\eta$ for some $\eta > 0$ and if $\int_0^t \|\varphi_s\|_\infty + \|\nabla\varphi_s\|_{6,loc} + \|\dot{A}_s\|_\infty ds < \infty$ it follows that the right hand side of (8) is small.

(c) Using Sobolev $\|\nabla\varphi_s\|_{6,loc} \leq \|\nabla\varphi_s\|_6 \leq \|\Delta\varphi\|$. Thus $\|\nabla\varphi_s\|_{6,loc}$ can be bounded by the square of the Gross-Pitaevskii Energy.

On the other hand $\|\nabla\varphi_s\|_{6,loc} \leq \|\nabla\varphi\|_\infty$. Since we are in the defocussing regime one expects when the potential is turned off that $\|\varphi\|_\infty$ and $\|\nabla\varphi\|_\infty$ decay like $t^{-3/2}$. Whenever $\int_0^\infty \|\varphi_s\|_\infty + \|\nabla\varphi_s\|_{6,loc} + \|\dot{A}_s\|_\infty ds < \infty$ the right hand side of (8) is small uniform in t .

(d) It has been shown in [11] that

$$\lim_{N \rightarrow \infty} \langle \Psi, \hat{n}^\varphi \Psi \rangle = 0$$

implies weak convergence of the reduced one particle density matrix of Ψ against $|\varphi\rangle\langle\varphi|$ and vice versa. For other equivalent definitions of asymptotic 100% condensation see [10].

(e) The set \mathcal{V}_1 includes potentials with scalings of the form (4).

2.3 Skeleton of the Proof

We shall prove the Theorem via Grönwall, so our goal is to show that there exists a $\eta > 0$ such that

$$\frac{d}{dt} \alpha(\Psi_t, \varphi_t) \leq C(\alpha(\Psi_t, \varphi_t) + N^{-\eta}). \quad (9)$$

Therefore we shall define a functional $\alpha' : L^2(\mathbb{R}^{3N}, \mathbb{C}) \otimes L^2(\mathbb{R}^{3N}, \mathbb{C}) \rightarrow \mathbb{R}$ such that $\frac{d}{dt} \alpha(\Psi_t, \varphi_t) \leq \alpha'(\Psi_t, \varphi_t)$. It is convenient to split up $\alpha' = \alpha'_0 + \alpha'_1 + \alpha'_2$ and treat these summands separately (see Definition 3.5 and Lemma 3.6). Then we will show that we can find a respective bound for $\alpha'(\Psi_t, \varphi_t)$. A nice feature of the method we use is that we can avoid propagation estimates on Ψ_t to get (9): Similar as in in [11] one can estimate the functional $\alpha'(\Psi, \varphi)$ uniform in Ψ and φ in terms of $\alpha(\Psi, \varphi)$ and $N^{-\eta}$ times some polynomial in $\|\varphi\|_\infty$, $\|\nabla\varphi\|_{6,loc}$ and $\|\Delta\varphi\| < \infty$. Under the assumption $\varphi_t \in \mathcal{G}$ we get (9).

The proof is organized as follows:

(a) The respective estimates of the $\alpha'_{0,1,2}(\Psi, \varphi)$ shall be given in section 4. The procedure is similar as in [11]. It turns out that

- We get good control of α'_0 for all $0 < \beta \leq 1$.
- We get sufficient control of α_1 for $\beta < 1/3$, only.
- For α_2 some of the estimates are in terms of $\|\nabla_1 q_1 \Psi\|$ and some estimates require that $\beta < 1$.

So the next step will be to show that $\|\nabla_1 q_1 \Psi\|$ is small: For later reference we shall give in section 4.1 an estimate of the interaction energy and an implicit estimate on $\|\nabla_1 q_1 \Psi\|$ which holds for all $0 < \beta \leq 1$. The result will be used in section 4.2 to control $\|\nabla_1 q_1 \Psi\|$ in terms of $\alpha(\Psi, \varphi)$ and $N^{-\eta}$ for some $\eta > 0$ under the restriction $0 < \beta < 1$.

This enables us to finish the proof of the Theorem for $\beta < 1/3$ using Grönwall (section 4.3).

- (b) After that we generalize the proof of the Theorem to the case $\beta < 1$. We already have good control of α'_0 and α'_2 . To make α'_1 controllable we use the microscopic structure to adjust α in such a way that the respective adjusted α'_1 is controllable. Therefore we need some estimates on the microscopic structure of the wave function. These are given in section 5.1. In section 5.2 we adjust α and prove, that the adjustment in fact changes the respective α'_1 such that it is controllable for all $0 < \beta \leq 1$. Then we complete in section 6.4 the proof of the Theorem for $0 < \beta < 1$.
- (c) Our final goal is to treat the case $\beta = 1$. To be able to use our results of the previous sections, we have to generalize our estimates on $\|\nabla_1 q_1 \Psi\|$ to the case $\beta = 1$ first. It turns out that $\|\nabla_1 q_1 \Psi\|$ is in fact not small for $\beta = 1$: Some non-negligible part of the kinetic energy is used to build up the microscopic structure in that case. Nevertheless we are able to control the kinetic energy of $q_1 \Psi$ outside some small set around the positions of the other particles (section 6.1).

In the next section we show that this new estimate is in fact sufficient to recover our old estimates, in particular Lemma 4.4 (d).

Similar as in (b) we now make another adjustment of α using again the microscopic structure. We adjust α in such a way that the respective α'_2 is controllable also for $\beta = 1$ (section 6.3).

Finally we complete the proof of the Theorem (section 6.4).

3 Preliminaries

Notation 3.1 (a) Throughout the paper hats $\hat{\cdot}$ shall solemnly be used in the sense of Definition 2.1 (c). The label n shall always be used for the function $n(k, N) = \sqrt{k/N}$.

- (b) In the following we shall omit the upper index φ on $p_j, q_j, P_j, P_{j,k}$ and $\hat{\cdot}$. It shall be replaced exclusively in a few formulas where their φ -dependence plays an important role.

- (c) We shall need the operator $H^{GP} := \sum_{j=1} h_j^{GP}$, where h_j^{GP} is the Gross Pitaeuskii (6) operator acting on the j^{th} particle.
- (d) In our estimates below we shall need the operator norm $\|\cdot\|_{op}$ defined for any linear operator $f : L^2(\mathbb{R}^{3N}, \mathbb{C}) \rightarrow L^2(\mathbb{R}^{3N}, \mathbb{C})$ by

$$\|f\|_{op} := \sup_{\|\Psi\|=1} \|f\Psi\|.$$

- (e) Constants appearing in estimates will generically be denoted by C . We shall not distinguish constants appearing in a sequence of estimates, i.e. in $X \leq CY \leq CZ$ the constants may differ.

First we need some properties of the objects defined in Definition 2.1

Lemma 3.2 (a) For any weights $m, r : \mathbb{N}^2 \rightarrow \mathbb{R}_0^+$ we have that

$$\widehat{m}\widehat{r} = \widehat{m}\widehat{r} = \widehat{r}\widehat{m} \quad \widehat{m}p_j = p_j\widehat{m} \quad \widehat{m}P_k = P_k\widehat{m}.$$

- (b) Let $n : \mathbb{N}^2 \rightarrow \mathbb{R}_0^+$ be given by $n(k, N) := \sqrt{k/N}$. Then the square of \widehat{n} (c.f. (5)) equals the relative particle number operator of particles not in the state φ , i.e.

$$(\widehat{n})^2 = N^{-1} \sum_{j=1}^N q_j. \quad (10)$$

- (c) For any weight $m : \mathbb{N}^2 \rightarrow \mathbb{R}_0^+$ and any function $f : \mathbb{R}^6 \rightarrow \mathbb{R}$ and any $j, k = 0, 1, 2$

$$\widehat{m}Q_j f(x_1, x_2) Q_k = Q_j f(x_1, x_2) \widehat{m}_{j-k} Q_k,$$

where $Q_0 := p_1 p_2$, $Q_1 \in \{p_1 q_2, q_1 p_2\}$ and $Q_2 := q_1 q_2$.

- (d) For any weight $m : \mathbb{N}^2 \rightarrow \mathbb{R}_0^+$ and any function $f : \mathbb{R}^6 \rightarrow \mathbb{R}$

$$[f(x_1, x_2), \widehat{m}] = [f(x_1, x_2), p_1 p_2 (\widehat{m} - \widehat{m}_2) + p_1 q_2 (\widehat{m} - \widehat{m}_1) + q_1 p_2 (\widehat{m} - \widehat{m}_1)]$$

- (e) Let $f \in L^1$, $g \in L^2$, $h \in L^3$ with $h(x) = 0$ for all $|x| > 0$.

$$\|p_j f(x_j - x_k) p_j\|_{op} \leq \|f\|_1 \|\varphi\|_\infty^2, \quad (11)$$

$$\|g(x_j - x_k) p_j\|_{op} \leq \|g\|_2 \|\varphi\|_\infty, \quad (12)$$

$$\|h(x_j - x_k) \nabla p_j\|_{op} \leq \|h\|_3 \|\nabla \varphi\|_{6,loc}. \quad (13)$$

Proof:

- (a) follows immediately from Definition 2.1, using that p_j and q_j are orthogonal projectors.

(b) Note that $\cup_{k=0}^N \mathcal{A}_k = \{0, 1\}^N$, so $1 = \sum_{k=0}^N P_k$. Using also $(q_k)^2 = q_k$ and $q_k p_k = 0$ we get

$$N^{-1} \sum_{k=1}^N q_k = N^{-1} \sum_{k=1}^N q_k \sum_{j=0}^N P_j = N^{-1} \sum_{j=0}^N \sum_{k=1}^N q_k P_j = N^{-1} \sum_{j=0}^N j P_j$$

and (b) follows.

(c) Using the definitions above we have

$$\widehat{m} Q_j f(x_1, x_2) Q_k = \sum_{l=0}^N m(l) P_l Q_j f(x_1, x_2) Q_k$$

The number of projectors q_j in $P_l Q_j$ in the coordinates $j = 3, \dots, N$ is equal to $l - j$. The p_j and q_j with $j = 3, \dots, N$ commute with $Q_j f(x_1, x_2) Q_k$. Thus $P_l Q_j f(x_1, x_2) Q_k = Q_j f(x_1, x_2) Q_k P_{l-j+k}$ and

$$\begin{aligned} \widehat{m} Q_j f(x_1, x_2) Q_k &= \sum_{l=0}^N m(l) Q_j f(x_1, x_2) Q_k P_{l-j+k} \\ &= \sum_{l=k-j}^{N+k-j} Q_j f(x_1, x_2) m(l+j-k) P_l Q_k = Q_j f(x_1, x_2) \widehat{m}_{j-k} Q_k. \end{aligned}$$

(d) First note that

$$\begin{aligned} &[f(x_1, x_2), \widehat{m}] - [f(x_1, x_2), p_1 p_2 (\widehat{m} - \widehat{m}_2) + p_1 q_2 (\widehat{m} - \widehat{m}_1) + q_1 p_2 (\widehat{m} - \widehat{m}_1)] \\ &= [f(x_1, x_2), q_1 q_2 \widehat{m}] + [f(x_1, x_2), p_1 p_2 \widehat{m}_2 + p_1 q_2 \widehat{m}_1 + q_1 p_2 \widehat{m}_1]. \end{aligned} \quad (14)$$

We shall show that the right hand side is zero. Multiplying the right hand side with $p_1 p_2$ from the left one gets

$$\begin{aligned} &p_1 p_2 f(x_1, x_2) q_1 q_2 \widehat{m} + p_1 p_2 f(x_1, x_2) p_1 p_2 \widehat{m}_2 - p_1 p_2 \widehat{m}_2 f(x_1, x_2) \\ &+ p_1 p_2 f(x_1, x_2) p_1 q_2 \widehat{m}_1 + p_1 p_2 f(x_1, x_2) q_1 p_2 \widehat{m}_1 \end{aligned}$$

Using (c) the latter is zero. Multiplying (14) with $p_1 q_2$ from the left one gets

$$\begin{aligned} &p_1 q_2 f(x_1, x_2) q_1 q_2 \widehat{m} + p_1 q_2 f(x_1, x_2) p_1 p_2 \widehat{m}_2 + p_1 q_2 f(x_1, x_2) p_1 q_2 \widehat{m}_1 \\ &+ p_1 q_2 f(x_1, x_2) q_1 p_2 \widehat{m}_1 - p_1 q_2 \widehat{m}_1 f(x_1, x_2) \end{aligned}$$

Using (c) the latter is zero. Also multiplying with $q_1 p_2$ yields zero due to symmetry in interchanging x_1 with x_2 . Multiplying (14) with $q_1 q_2$ from the left one gets

$$\begin{aligned} &q_1 q_2 f(x_1, x_2) \widehat{m} q_1 q_2 - q_1 q_2 \widehat{m} f(x_1, x_2) + q_1 q_2 f(x_1, x_2) p_1 p_2 \widehat{m}_2 + \\ &q_1 q_2 f(x_1, x_2) p_1 q_2 \widehat{m}_1 + q_1 q_2 f(x_1, x_2) q_1 p_2 \widehat{m}_1 \end{aligned}$$

which is again zero, thus (14).

(e) To show (11) we use the notation $p_j = |\varphi(x_j)\rangle\langle\varphi(x_j)|$

$$\begin{aligned} p_j f(x_j - x_k) p_j &= |\varphi(x_j)\rangle\langle\varphi(x_j)| f(x_j - x_k) |\varphi(x_j)\rangle\langle\varphi(x_j)| \\ &= |\varphi(x_j)\rangle\langle\varphi(x_j)| (f \star |\varphi|^2)(x_k) |\varphi(x_j)\rangle\langle\varphi(x_j)| = p_j (f \star |\varphi|^2)(x_k) . \end{aligned} \quad (15)$$

It follows that

$$\|p_j f(x_j - x_k) p_j\|_{op} \leq \|f\|_1 \|\varphi\|_\infty^2 .$$

With Young we get (11).

For (12) we write

$$\begin{aligned} \|g(x_j - x_j) p_j\|_{op}^2 &= \sup_{\|\Psi\|=1} \|g(x_j - x_j) p_j \Psi\|^2 = \\ &= \sup_{\|\Psi\|=1} \langle\langle \Psi, p_j g(x_j - x_j)^2 p_j \Psi \rangle\rangle \\ &\leq \|p_j g(x_j - x_j)^2 p_j\|_{op} . \end{aligned}$$

With (11) we get (12). For (13) we have using Young's inequality

$$\begin{aligned} \|h(x_j - x_k) \nabla p_j\|_{op} &\leq \sup_{y \in \mathbb{R}^3} |\langle \nabla \varphi, h^2(\cdot - y) \nabla \varphi \rangle|^{1/2} \\ &\leq \sup_{y \in \mathbb{R}^3} \|h^2\|_{3/2}^{1/2} \|\mathbf{1}_{|\cdot - y| \leq 1} |\nabla \varphi|^2\|_3^{1/2} \\ &= \|h\|_3 \|\nabla \varphi\|_{6,loc} . \end{aligned}$$

□

When doing the estimates we will encounter wave functions where some of the symmetry is broken (at this point the reader should exemplarily think of the wave function $V_\beta(x_1 - x_2)\Psi$ which is not symmetric under exchange of the variables x_1 and x_3 for example). Therefore we want to formulate some of our results for wave functions which are not symmetric under exchange of any two variables x_j, x_k . This leads to the following definition

Definition 3.3 *We define for any finite set $\mathcal{M} \subset \mathbb{N}$ the space $\mathcal{H}_\mathcal{M} \subset L^2(\mathbb{R}^{3N}, \mathbb{C})$ of functions which are symmetric in all variables but those in \mathcal{M}*

$$\begin{aligned} \Psi \in \mathcal{H}_\mathcal{M} &\Leftrightarrow \Psi(x_1, \dots, x_j, \dots, x_k, \dots, x_N) = \Psi(x_1, \dots, x_k, \dots, x_j, \dots, x_N) \\ &\text{for all } j, k \notin \mathcal{M} . \end{aligned}$$

and the operator norm $\|\cdot\|_\mathcal{M}$ on $\mathcal{H}_\mathcal{M} \rightarrow L^2(\mathbb{R}^{3N}, \mathbb{C})$ by

$$\|A\|_\mathcal{M} := \sup_{\Psi, \chi \in \mathcal{H}_\mathcal{M}; \|\Psi\| = \|\chi\| = 1} \langle\langle \chi, A\Psi \rangle\rangle .$$

With Definition 2.1 we arrive directly at the following Lemma based on combinatorics of the p_j and q_j :

Lemma 3.4 For any $f : \mathbb{N}^2 \rightarrow \mathbb{R}_0^+$ and any finite set $\mathcal{M}_a \subset \mathbb{N}$ with $1 \notin \mathcal{M}_a$ and any finite set $\mathcal{M}_b \subset \mathbb{N}$ with $1, 2 \notin \mathcal{M}_b$ there exists a $C > \infty$ such that

$$\left\| \widehat{f} q_1 \Psi \right\|^2 \leq C \|\widehat{f} \widehat{n} \Psi\|^2 \quad \text{for any } \Psi \in \mathcal{H}_{\mathcal{M}_a}, N > |\mathcal{M}_a| \quad (16)$$

$$\left\| \widehat{f} q_1 q_2 \Psi \right\|^2 \leq C \|\widehat{f} (\widehat{n})^2 \Psi\|^2 \quad \text{for any } \Psi \in \mathcal{H}_{\mathcal{M}_b}, N > |\mathcal{M}_b|. \quad (17)$$

Proof: Let $\Psi \in \mathcal{H}_{\mathcal{M}_a}$ for some finite set $1 \in \mathcal{M}_a \subset \mathbb{N}$. For (16) we can write using symmetry of Ψ and Lemma 3.2 (b)

$$\begin{aligned} \|\widehat{f} \widehat{n} \Psi\|^2 &= \langle \Psi, (\widehat{f})^2 (\widehat{n})^2 \Psi \rangle = N^{-1} \sum_{k=1}^N \langle \Psi, (\widehat{f})^2 q_k \Psi \rangle \\ &\geq N^{-1} \sum_{k \notin \mathcal{M}_a} \langle \Psi, (\widehat{f})^2 q_k \Psi \rangle = \frac{N - |\mathcal{M}_a|}{N} \langle \Psi, (\widehat{f})^2 q_1 \Psi \rangle \\ &= \frac{N - |\mathcal{M}_a|}{N} \|\widehat{f} q_1 \Psi\|^2. \end{aligned}$$

Similarly we have for $\Psi \in \mathcal{H}_{\mathcal{M}_b}$

$$\begin{aligned} \|\widehat{f} (\widehat{n})^2 \Psi\|^2 &= \langle \Psi, (\widehat{f})^2 (\widehat{n})^4 \Psi \rangle \geq N^{-2} \sum_{j, k \notin \mathcal{M}_b} \langle \Psi, (\widehat{f})^2 q_j q_k \Psi \rangle \\ &= \frac{N - |\mathcal{M}|(N - |\mathcal{M}| - 1)}{N^2} \langle \Psi, (\widehat{f})^2 q_1 q_2 \Psi \rangle + \frac{|\mathcal{M}|}{N^2} \langle \Psi, (\widehat{f})^2 q_1 \Psi \rangle \\ &\geq \frac{N - |\mathcal{M}|(N - |\mathcal{M}| - 1)}{N^2} \|\widehat{f} q_1 q_2 \Psi\|^2 \end{aligned}$$

and the Lemma follows. \square

Our next step is to define the functionals α'_j , $j = 0, 1, 2$ which, as explained above, control the time derivative of $\alpha(\Psi_t, \varphi_t)$.

Definition 3.5 Using the notation

$$Z_\beta(x_j, x_k) := V_\beta(x_j - x_k) - \frac{2a}{N-1} |\varphi|^2(x_j) - \frac{2a}{N-1} |\varphi|^2(x_k)$$

we define functionals $\alpha'_{0,1,2} : L^2(\mathbb{R}^{3N}, \mathbb{C}) \rightarrow \mathbb{R}^+$ by

$$\alpha'_0(\Psi, \varphi) = \left| \langle \Psi_t, \dot{A} \Psi_t \rangle - \langle \varphi, \dot{A} \varphi \rangle \right| \quad (18)$$

$$\alpha'_1(\Psi, \varphi) = 2N(N-1) \Im \left(\langle \Psi, Z_\beta(x_1, x_2) p_1 p_2 (\widehat{n} - \widehat{n}_2) \Psi \rangle \right) \quad (19)$$

$$\alpha'_2(\Psi, \varphi) = 4N(N-1) \Im \left(\langle \Psi, Z_\beta(x_1, x_2) p_1 q_2 (\widehat{n} - \widehat{n}_1) \Psi \rangle \right). \quad (20)$$

Lemma 3.6 For any solution of the Schrödinger equation Ψ_t and any solution of the Gross-Pitaevskii equation φ_t we have

$$\left| \frac{d}{dt} \alpha(\Psi_t, \varphi_t) \right| \leq \sum_{j=0}^2 \alpha'_j(\Psi_t, \varphi_t). \quad (21)$$

Proof: For the proof of the Lemma we shall restore the upper index φ_t to pay respect to the time dependence of \widehat{n}^{φ_t} . We have for the time derivative of the first summand of α

$$\begin{aligned} \frac{d}{dt} \langle \Psi_t, \widehat{n}^{\varphi_t} \Psi_t \rangle &= -i \langle \Psi_t, \widehat{n}^{\varphi_t} \Psi_t \rangle + i \langle \Psi_t, \widehat{n}^{\varphi_t} H \Psi_t \rangle \\ &\quad + i \langle \Psi_t, [H_t^{GP}, \widehat{n}^{\varphi_t}] \Psi_t \rangle \\ &= -i \langle \Psi_t, [H - H_t^{GP}, \widehat{n}^{\varphi_t}] \Psi_t \rangle \\ &= -iN(N-1) \langle \Psi_t, [Z_\beta(x_1, x_2), \widehat{n}^{\varphi_t}] \Psi_t \rangle. \end{aligned}$$

Using Lemma 3.2 (d) it follows that the latter equals

$$\begin{aligned} &-iN(N-1) \langle \Psi_t, [Z_\beta(x_1, x_2), p_1 p_2 (\widehat{n}^{\varphi_t} - \widehat{n}_2^{\varphi_t})] \Psi_t \rangle \\ &-2iN(N-1) \langle \Psi_t, [Z_\beta(x_1, x_2), p_1 q_2 (\widehat{n}^{\varphi_t} - \widehat{n}_1^{\varphi_t})] \Psi_t \rangle. \end{aligned}$$

Since Z_β is selfadjoint this is $\alpha'_1 + \alpha'_2$.

For the second summand of α we have

$$\begin{aligned} \frac{d}{dt} (\mathcal{E}(\Psi_t) - \mathcal{E}^{GP}(\varphi_t)) &= \langle \Psi_t, \dot{A}_t(x_1) \Psi_t \rangle - \langle \varphi_t, a \left(\frac{d}{dt} |\varphi_t|^2 \right) \varphi_t \rangle \\ &\quad - \langle \varphi_t, \dot{A}_t \varphi_t \rangle - \langle \varphi_t, [(h^{GP} - a|\varphi_t|^2), h^{GP}] \varphi_t \rangle \\ &= \langle \Psi_t, \dot{A}_t(x_1) \Psi_t \rangle - \langle \varphi_t, \dot{A}_t \varphi_t \rangle + \langle \varphi_t, [a|\varphi_t|^2, h^{GP}] \varphi_t \rangle \\ &\quad - \langle \varphi_t, [a|\varphi_t|^2, h^{GP}] \varphi_t \rangle. \end{aligned} \tag{22}$$

Hence

$$\frac{d}{dt} |\mathcal{E}(\Psi_t) - \mathcal{E}^{GP}(\varphi_t)| \leq \alpha'_0(\Psi, \varphi) \tag{23}$$

which proves the Lemma. □

4 Control of the α' for $\beta < 1/3$

As a first step we shall prove the Theorem for scalings $\beta < 1/3$. It is not surprising that this case is special: $\beta < 1/3$ means that the mean distance of two particle is much smaller than the radius of the support of the interactions as $N \rightarrow \infty$. Thus we are in a regime where for $|\Psi|^2$ -typical configurations many of the interactions overlap and one arrives directly at a mean field picture.

Our goal is now to control the functionals $\alpha'_{0,1,2}$ in such a way, that we can conclude that $\alpha(\Psi_t, \varphi_t)$ is small via Grönwall. Remember that for the φ 's we are interested in $\|\varphi\|_\infty$, and $\|\nabla\varphi\|_{6,loc} \geq \|\Delta\varphi\|$ are bounded (see Definition 2.3). Hence it is sufficient to estimate α'_j , $j = 0, 1, 2$ in terms of $\alpha + N^{-\eta}$ for some $\eta > 0$ times an arbitrary polynomial in $\|\varphi\|_\infty$, $\|\nabla\varphi\|_{6,loc}$ and $\|\Delta\varphi\|$. So we define

Definition 4.1 The set \mathcal{F} of functionals $L^2(\mathbb{R}^3, \mathbb{C}) \rightarrow \mathbb{R}^+$ is given by

$$\begin{aligned} \mathcal{K} \in \mathcal{F} &\Leftrightarrow \text{there exists a polynomial } K : (\mathbb{R}^+)^3 \rightarrow \mathbb{R}^+ \text{ such that} \\ \mathcal{K}(\varphi) &= K(\|\varphi\|_\infty, \|\Delta\varphi\|) . \end{aligned}$$

And we want to show that for some $\mathcal{K} \in \mathcal{F}$, some $\eta > 0$ and $j = 0, 1, 2$

$$\alpha'_j \leq \mathcal{K}(\varphi)(\alpha(\Psi, \varphi) + N^{-\eta})$$

uniform in $(\Psi, \varphi) \in L^2(\mathbb{R}^{3N}, \mathbb{C}) \otimes L^2(\mathbb{R}^3, \mathbb{C})$.

Rewrite $\alpha'_{1,2}$ multiplying Z_β with $1 = p_1p_1 + p_1q_2 + q_1p_2 + q_1q_1$ from the right. Lemma 3.2 (c) shows that

$$p_1p_2Z_\beta(x_1, x_2)p_1p_2(\hat{n} - \hat{n}_2) = p_1p_2(\hat{n} - \hat{n}_2)Z_\beta(x_1, x_2)p_1p_2$$

is selfadjoint and so is $p_1q_2Z_\beta(x_1, x_2)p_1q_2(\hat{n} - \hat{n}_1)$.

The operator $q_1p_2Z_\beta(x_1, x_2)p_1q_2(\hat{n} - \hat{n}_1)$ is invariant under adjunction plus simultaneous exchange of the variable x_1 and x_2 . Thus the sandwiches with Ψ of the respective operators are real and using Lemma 3.2 (c)

$$\alpha'_1(\Psi, \varphi) = 4N(N-1)\Im(\langle\langle \Psi, p_1q_2(\hat{n}_{-1} - \hat{n}_1)Z_\beta(x_1, x_2)p_1p_2\Psi \rangle\rangle) \quad (24)$$

$$+ 2N(N-1)\Im(\langle\langle \Psi, q_1q_2(\hat{n}_{-2} - \hat{n})Z_\beta(x_1, x_2)p_1p_2\Psi \rangle\rangle) \quad (25)$$

$$\alpha'_2(\Psi, \varphi) = 4N(N-1)\Im(\langle\langle \Psi, p_1p_1Z_\beta(x_1, x_2)p_1q_2(\hat{n} - \hat{n}_1)\Psi \rangle\rangle) \quad (26)$$

$$+ 4N(N-1)\Im(\langle\langle \Psi, q_1q_2(\hat{n}_{-1} - \hat{n})Z_\beta(x_1, x_2)p_1q_2\Psi \rangle\rangle) . \quad (27)$$

The task of this section is to estimate all the terms on the right hand sides of (24) and (26) as well as α'_0 . This will be done in Lemma 4.4 below, but let us first give some heuristic arguments why they are small

- From a physical point of view (24) and (26) are the most important. Here we use that in leading order the interaction and the mean field cancel out. Note first that one of the mean field parts in Z_β is zero: $p_jq_j = 0$, thus $p_1q_1a|\varphi(x_1)|p_1p_2 = 0$. For the interaction part in Z_β we use formula (15): $p_1V_\beta(x_1 - x_2)p_1 = V_\beta \star |\varphi|^2(x_2)p_1$. Since V_β is δ -like and its integral is a/N the latter is $\approx a|\varphi|^2(x_2)p_1$, cancelling out most of the mean field part in Z_β . Thus (24) and (26) are small.
- Since there is neither a p_1 nor p_2 on the left side of V_β in (25) the latter seems at first view to grow with N . It is indeed not small for general non-symmetric normalized Ψ : If all the mass of Ψ was concentrated in an area where $x_1 \approx x_2$ (which is of course not possible for symmetric Ψ), then (25) would in fact grow with N . So to estimate (25) we have to use symmetry of Ψ . The trick is to estimate

$$2N\Im \left(\langle\langle (\hat{n}_{-2} - \hat{n})q_1\Psi, \sum_{j=2}^N q_j Z_\beta(x_1, x_j)p_1p_j\Psi \rangle\rangle \right)$$

which is for symmetric Ψ equal to (25). Note that in view of Lemma 3.2 (b) $(\hat{n}_{-2} - \hat{n})q_1\Psi$ is of order N^{-1} . Using Cauchy Schwarz we have to control

$$\begin{aligned} \left\| \sum_{j=2}^N q_j Z_\beta(x_1, x_j) p_1 p_j \Psi \right\|^2 &= \sum_{2 \leq j \leq N} \langle \Psi, p_1 p_j Z_\beta(x_1, x_j) q_j Z_\beta(x_1, x_j) p_1 p_j \Psi \rangle \\ &\quad + 2 \sum_{2 \leq k < j \leq N} \langle \Psi, p_1 p_k Z_\beta(x_1, x_k) q_k q_j Z_\beta(x_1, x_j) p_1 p_j \Psi \rangle \end{aligned}$$

The first line has only N summands. Since $\|V_\beta\|_1$ is of order N^{-1} this line is small if $\|V_\beta\|_\infty \ll 1$ which is the case for $\beta < 1/3$.

For the second line we can write

$$\begin{aligned} \sum_{2 \leq k < j \leq N} \langle \sqrt{Z_\beta(x_1, x_k) p_k} \sqrt{Z_\beta(x_1, x_j) p_1 q_j} \Psi, \\ \sqrt{Z_\beta(x_1, x_j) p_j} \sqrt{Z_\beta(x_1, x_k) p_1 q_k} \Psi \rangle . \end{aligned}$$

Now we have enough projectors p on both sides of the interactions to be able to integrate them against φ (c.f. Lemma 3.2 (e)) and it is clear that it at least does not grow with N . Below we shall show that for $\beta < 1/3$ (25) and (26) are in fact bounded by $\alpha + N^{-\eta}$ for some $\eta > 0$.

- To show that (27) is small one needs to use smoothness of Ψ . We do so in the following way: We introduce a potential $U_{\beta_1, \beta}$ with moderate scaling behavior and the same L^1 norm as V_β . This is done in definition 4.2. The scaling β_1 of $U_{\beta_1, \beta}$ will be chosen such that (27) with V_β replaced by $U_{\beta_1, \beta}$ can be controlled. The difference — (27) with V_β replaced by $V_\beta - U_{\beta_1, \beta}$ — we integrate by parts. It follows that (27) can be controlled in terms of $\|\nabla_1 q_1 \Psi\|$ which is small in view of Lemma 4.6.

Before we prove the Theorem let us first do the preparations needed to control (27) as explained right above, i.e. introduce the smeared out interaction $U_{\beta_1, \beta}$.

Definition 4.2 For any $0 \leq \beta_1 \leq \beta \leq 1$ and any $V_\beta \in \mathcal{V}_\beta$ we define

$$U_{\beta_1, \beta}(\mathbf{x}) := \begin{cases} \frac{3}{4\pi} \|V_\beta\|_1 N^{3\beta_1}, & \text{for } x < N^{-\beta_1}; \\ 0, & \text{else.} \end{cases}$$

and

$$h_{\beta_1, \beta}(x) := \int |x - y|^{-1} (V_\beta(y) - U_{\beta_1, \beta}(y)) d^3 y \quad (28)$$

Lemma 4.3 For any $0 \leq \beta_1 \leq \beta < 1$ and any $V_\beta \in \mathcal{V}_\beta$

$$\Delta h = V_\beta - U_{\beta_1, \beta}, \quad U_{\beta_1, \beta} \in \mathcal{V}_{\beta_1}, \quad (29)$$

$$\|h_{\beta_1, \beta}\| \leq CN^{-1-\beta_1/2}, \quad \|h_{\beta_1, \beta}\|_3 \leq CN^{-1}(\ln N)^{1/3}, \quad (30)$$

$$\|\nabla h_{\beta_1, \beta}\|_1 \leq CN^{-1-\beta_1}, \quad \|\nabla h_{\beta_1, \beta}\| \leq CN^{-1+\beta/2}. \quad (31)$$

Proof: The Lemma gives in fact a well known result of standard electrostatics: V_β can be understood as a given charge density, $U_{\beta_1, \beta}$ was defined in such a way, that the “total charge” is zero. Hence the potential h_{β, β_1} is constant outside the support of $U_{\beta_1, \beta}$ in our case, by definition ($h_{\beta, \beta_1}(x) = 0$ decays like x^{-1} as $x \rightarrow \infty$) this constant is zero.

The first statement of the Lemma is almost trivial

$$\begin{aligned}\Delta h(x) &= \int \Delta |x - y|^{-1} (V_\beta(y) - U_{\beta_1, \beta}(y)) d^3 y \\ &= V_\beta(x) - U_{\beta_1, \beta}(x) .\end{aligned}$$

By definition $U_{\beta_1, \beta} \in \mathcal{U}_{\beta_1}$, $\|U_{\beta_1, \beta}\|_1 = \|V_\beta\|_1$ and $\lim_{N \rightarrow \infty} N^{1-3\beta_1} \|U_{\beta_1, \beta}\|_\infty < \infty$. Since $V_\beta \in \mathcal{V}_\beta$ we get that $\lim_{N \rightarrow \infty} N^{1+\eta} (\|U_{\beta_1, \beta}\|_1 - a/N) < \infty$ implying $U_{\beta_1, \beta} \in \mathcal{V}_{\beta_1}$ which completes the proof of line (29). Since $\|U_{\beta_1, \beta}\|_1 = \|V_\beta\|_1$ and $\Delta h(x) = 0$ for $x > N^{-\beta_1}$ it follows that $h = 0$ for $x > N^{-\beta_1}$. Furthermore $|h_{\beta_1, \beta}(x)| < CN^{-1}|x|^{-1}$ and $|\nabla h_{\beta_1, \beta}(x)| < CN^{-1}|x|^{-2}$, implying the two equations in line (30) as well as the first equation in (31).

To get the second equation in (31) we write for $\nabla h_{\beta_1, \beta}$

$$\begin{aligned}\nabla h_{\beta_1, \beta}(x) &= \int \mathbf{1}_{|x-y| < N^{-\beta}} \frac{x-y}{|x-y|^3} (V_\beta(y) - U_{\beta_1, \beta}(y)) d^3 y \\ &\quad + \int \mathbf{1}_{|x-y| > N^{-\beta}} \frac{x-y}{|x-y|^3} (V_\beta(y) - U_{\beta_1, \beta}(y)) d^3 y\end{aligned}$$

Using Young’s inequality it follows that

$$\begin{aligned}\|\nabla h_{\beta_1, \beta}\|_\infty &\leq \left\| \mathbf{1}_{|\cdot| < N^{-\beta}} \frac{(\cdot)}{|\cdot|^3} \right\|_1 (\|V_\beta\|_\infty + \|U_{\beta_1, \beta}\|_\infty) \\ &\quad + \left\| \mathbf{1}_{|\cdot| > N^{-\beta}} \frac{(\cdot)}{|\cdot|^3} \right\|_\infty (\|V_\beta\|_1 + \|U_{\beta_1, \beta}\|_1) \\ &\leq CN^{-\beta} N^{-1+3\beta} + CN^{2\beta} N^{-1} .\end{aligned}$$

Since $|\nabla h_{\beta_1, \beta}| < CN^{-1}|x|^{-2}$

$$\begin{aligned}\|\nabla h_{\beta_1, \beta}\|^2 &\leq C \int_{N^{-\beta}}^\infty N^{-2}|x|^{-4} dx + CN^{-3\beta} \|\nabla h_{\beta_1, \beta}\|_\infty^2 \\ &\leq CN^{-2+\beta} + CN^{-2+\beta} .\end{aligned}$$

□

We now arrive at the central point of this section which is estimating α'_0 (18), α'_1 ((24) and (25)) and α'_2 ((26) and (27)) following the strategy explained above.

Lemma 4.4 *Let $\mathcal{M} \subset \mathbb{N}$ with $1, 2 \notin \mathcal{M}$ and $m : \mathbb{N}^2 \rightarrow \mathbb{R}^+$ with $m \leq n^{-1}$.*

(a) Let $0 < \beta \leq 1$, $f \in L^\infty$. Then there exists a $C < \infty$ such that

$$|\langle \Psi, f(x_1)\Psi \rangle - \langle \varphi, f(x)\varphi \rangle| \leq 2\|f\|_\infty C\alpha(\Psi, \varphi)$$

for any $\Psi \in \mathcal{H}_M$.

(b) Let $0 < \beta \leq 1$, $V_\beta \in \mathcal{U}_\beta$ with $\lim_{N \rightarrow \infty} N^\eta(N\|V_\beta f_{\beta,1}\|_1 - a) \leq \infty$ for some $\eta > 0$. Then there exists a $\mathcal{K} \in \mathcal{F}$ and a $\eta > 0$ such that

$$N\|p_1 p_2 Z_\beta(x_1, x_2) q_1 p_2 \widehat{m}\|_{\mathcal{M}} \leq \mathcal{K}(\varphi)\|\varphi\|_\infty N^{-\eta}$$

(c) Let $0 < \beta < 1/3$, $V_\beta \in \mathcal{V}_\beta$. Then there exists a $\mathcal{K} \in \mathcal{F}$ and a $\eta > 0$ such that

$$N|\langle \Psi p_1 p_2 Z_\beta(x_1, x_2) \widehat{m} q_1 q_2 \chi \rangle| \leq \mathcal{K}(\varphi)\|\varphi\|_\infty (\langle \Psi, \widehat{n}\Psi \rangle \langle \chi, \widehat{n}\chi \rangle)^{1/2} + N^{-\eta}.$$

for any $\Psi, \chi \in \mathcal{H}_M$.

(d) Let $0 < \beta < 1$, $V_\beta \in \mathcal{V}_\beta$. Then there exists a $\mathcal{K} \in \mathcal{F}$ and a $\eta > 0$ such that

$$\begin{aligned} N|\langle \Psi p_1 q_2 Z_\beta(x_1, x_2) \widehat{m} q_1 q_2 \Psi \rangle| &\leq \mathcal{K}(\varphi)(\|\varphi\|_\infty + (\ln N)^{1/3} \|\nabla \varphi\|_{6,loc}) \\ &\quad (\langle \Psi, \widehat{n}\Psi \rangle + \|\nabla_1 q_1 \Psi\|^2 + N^{-\eta}). \end{aligned}$$

for any symmetric Ψ (i.e. $\Psi \in \mathcal{H}_0$).

Proof:

(a) Using $1 = p_1 + p_2$ and Lemma 3.4

$$\begin{aligned} &|\langle \Psi, f(x_1)\Psi \rangle - \langle \varphi, f(x)\varphi \rangle| \\ &= |\langle \Psi, p_1 f(x_1) p_1 \Psi \rangle - \langle \varphi, f(x)\varphi \rangle + 2\Re(\langle \Psi, q_1 f(x_1) p_1 \Psi \rangle) \\ &\quad + \langle \Psi, q_1 f(x_1) q_1 \Psi \rangle| \\ &\leq (1 - \|p_1 \Psi\|^2) \langle \varphi, f(x)\varphi \rangle + 2 \left| \Re \left(\langle \Psi, q_1 \widehat{n}^{-1/2} f(x_1) \widehat{n}_1^{1/2} p_1 \Psi \rangle \right) \right| \\ &\quad + \langle \Psi, q_1 f(x_1) q_1 \Psi \rangle \\ &\leq \alpha(\Psi, \varphi)\|f\|_\infty + C\alpha(\Psi, \varphi)\|f\|_\infty + \alpha(\Psi, \varphi)\|f\|_\infty. \end{aligned}$$

(b) In view of 3.4

$$\begin{aligned} N\langle \Psi, p_1 p_2 Z_\beta(x_1, x_2) q_1 p_2 \widehat{m}\Psi \rangle &\leq N\|p_1 p_2 Z_\beta(x_1, x_2) q_1 p_2\|_{op} \|q_1 \widehat{m}\Psi\| \\ &\leq CN\|p_1 p_2 Z_\beta(x_1, x_2) q_1 p_2\|_{op}. \end{aligned}$$

$\|p_1 p_2 Z_\beta(x_1, x_2) q_1 p_2\|_{op}$ can be estimated using $p_1 q_1 = 0$ and (15):

$$\begin{aligned} &\|p_1 p_2 (V_\beta(x_1 - x_2) - \frac{2a}{N-1} |\varphi(x_1)|^2 - \frac{2a}{N-1} |\varphi(x_2)|^2) q_1 p_2\|_{op} \\ &= \|p_1 p_2 (V_\beta(x_1 - x_2) - \frac{2a}{N-1} |\varphi(x_1)|^2) p_2\|_{op} \\ &\leq \|p_1 p_2 \left((V_\beta \star |\varphi|^2)(x_1) - \frac{2a}{N-1} |\varphi(x_1)|^2 \right)\|_{op} \end{aligned}$$

We introduce $a_N := \|V_\beta\|_1$. Since $V_\beta \in \mathcal{V}_\beta$ we can find a $\eta > 0$ such that

$$\begin{aligned} &\leq \|p_1 ((V_\beta - a_N \delta) \star |\varphi|^2)(x_1)\|_{op} + CN^{-\eta} \|\varphi\|_\infty^2 \\ &\leq \|\varphi\|_\infty \|(V_\beta - a_N \delta) \star |\varphi|^2\| + CN^{-\eta} \|\varphi\|_\infty^2. \end{aligned}$$

Let $h \in L^\infty$ be given by

$$\Delta h(x) = V_\beta(x) - a_N \delta(x).$$

As above (see Lemma 4.3) we have that $h(x) = 0$ for $x > RN^{-\beta}$, where $RN^{-\beta}$ is the radius of the support of V_β and that $\|\nabla h\|_1 \leq N^{-1-\beta}$. Partial integration and Young's inequality give that

$$\begin{aligned} \|(V_\beta - a_N \delta) \star |\varphi|^2\| &= \|(\nabla h) \star (\nabla |\varphi|^2)\| \\ &\leq \|\nabla h\|_1 \|\nabla |\varphi|^2\| \leq 2 \|\nabla h\|_1 \|\nabla \varphi\| \|\varphi\|_\infty. \end{aligned}$$

Hence

$$\|p_1 p_2 \left((V_\beta \star |\varphi|^2)(x_1) - \frac{2a}{N-1} |\varphi(x_1)|^2 \right) p_2\|_{op} \leq C \|\varphi\|_\infty N^{-\eta} (\|\nabla \varphi\| + \|\varphi\|_\infty) \quad (32)$$

for some $\eta > 0$ and (b) follows.

(c) Let us first find an upper bound for

$$\left\| \sum_{j \in \mathcal{M}} q_j f(x_1, x_j) \widehat{r} p_1 p_j \Psi \right\|^2$$

for general $r : \mathbb{N}^2 \rightarrow \mathbb{R}^+$, $f \in L^2 \cap L^1$ and $\Psi \in \mathcal{H}_\mathcal{M}$ with $1, 2 \notin \mathcal{M}$. Using Lemma 3.4

$$\begin{aligned} &\left\| \sum_{j \notin \mathcal{M}} q_j f(x_1 - x_j) \widehat{r} p_1 p_j \Psi \right\|^2 \quad (33) \\ &= \sum_{j \neq k \notin \mathcal{M}} \langle \widehat{r} p_1 p_j \Psi, f(x_1 - x_j) q_j q_k f(x_1 - x_k) p_k p_1 \widehat{r} \Psi \rangle \\ &\quad + \sum_{j \notin \mathcal{M}} \langle \widehat{r} \Psi, p_1 p_j f(x_1 - x_j) q_j f(x_1 - x_j) p_1 p_j \widehat{r} \Psi \rangle \\ &\leq \sum_{j \neq k \notin \mathcal{M}} \langle q_k \widehat{r} \Psi, p_1 \sqrt{f}(x_1 - x_k) p_j \sqrt{f}(x_1 - x_j) \\ &\quad \sqrt{f}(x_1 - x_k) p_k \sqrt{f}(x_1 - x_j) p_1 q_j \widehat{r} \Psi \rangle \\ &\quad + \sum_{j \notin \mathcal{M}} \langle \widehat{r} \Psi, p_1 p_j f(x_1 - x_j) f(x_1 - x_j) p_1 p_j \widehat{r} \Psi \rangle \\ &\leq N^2 \|\sqrt{f}(x_1 - x_2) p_1\|_{op}^4 \|q_2 \widehat{r} \Psi\|^2 + N \|f^2\|_1 \|\varphi\|_\infty^2 \|\widehat{r}\|_{op}^2 \\ &\leq N^2 \|\varphi\|_\infty^4 \|f\|_1^2 \|\widehat{r} \Psi\|^2 + N \|f\|^2 \|\varphi\|_\infty^2 \sup_{1 \leq k \leq N} |r(k, N)^2|. \end{aligned}$$

We now come back to (c). We define for some $\varepsilon > 0$ we shall specify below the functions $m^{a,b} : \mathbb{N}^2 \rightarrow \mathbb{R}^+$ by

$$m^a(k, N) := m(k, N) \quad \text{for } k < N^{1-\varepsilon}; \quad m^a(k, N) = 0 \quad \text{for } k \geq N^{1-\varepsilon}$$

and $m^b = m - m^a$. Note also that $p_2|\varphi(x_1)|^2q_2 = p_1|\varphi(x_2)|^2q_1 = 0$. It follows that (c) is bounded by

$$N|\langle \Psi, p_1p_2V_\beta(x_1 - x_2)\widehat{m}^a q_1q_2\chi \rangle| + N|\langle \Psi, p_1p_2V_\beta(x_1 - x_2)\widehat{m}^b q_1q_2\chi \rangle|. \quad (34)$$

Defining also $g : \mathbb{N}^2 \rightarrow \mathbb{R}^+$ by $g(k, N) = 1$ for $k < N^{1-\varepsilon}$, $g(k, N) = 0$ for $k \geq N^{1-\varepsilon}$ we have that $m^a = m^a g$ and the first summand in (34) equals

$$\begin{aligned} & N\langle \Psi, \widehat{g}_2 p_1 p_2 V_\beta(x_1 - x_2) q_1 q_2 \widehat{m}^a \chi \rangle \\ &= N(N - |\mathcal{M}| - 1)^{-1} \langle \Psi, \sum_{j \notin \mathcal{M}} \widehat{g}_2 p_1 p_j V_\beta(x_1 - x_j) q_1 q_j \widehat{m}^a \chi \rangle \\ &\leq \left\| \sum_{j \notin \mathcal{M}} \widehat{g}_2 q_j V_\beta(x_1 - x_j) p_1 p_j \Psi \right\| \|\widehat{m}^a q_1 \chi\|. \end{aligned} \quad (35)$$

In view of (33) and Lemma 3.4 the latter is bounded by

$$\begin{aligned} & N\|\varphi\|_\infty^2 \|V_\beta\|_1 \|\widehat{n}\widehat{g}_2\chi\| + N^{1/2} \|V_\beta\| \|\varphi\|_\infty \sup_{1 \leq k \leq N} |g(k, N)| \\ &\leq CN^{-\varepsilon/2} \|\varphi\|_\infty^2 + CN^{-1/2+3\beta/2} \|\varphi\|_\infty. \end{aligned}$$

Using Lemma 3.4 the second summand in (34) is bounded by

$$\begin{aligned} & N\langle \Psi, p_1 p_2 f(x_1, x_2) q_1 q_2 \widehat{m}^b \chi \rangle \\ &= N(N - |\mathcal{M}| - 1)^{-1} \langle \Psi, \sum_{j \notin \mathcal{M}} (\widehat{m}_2^b)^{1/2} p_1 p_j f(x_1, x_j) q_1 q_j (\widehat{m}^b)^{1/2} \chi \rangle \\ &\leq C \left\| \sum_{j \notin \mathcal{M}} q_j f(x_1, x_j) (\widehat{m}_2^b)^{1/2} p_1 p_j \Psi \right\| \|(\widehat{m}^b)^{1/2} q_1 \chi\| \\ &\leq C \left\| \sum_{j \notin \mathcal{M}} q_j f(x_1, x_j) (\widehat{m}_2^b)^{1/2} p_1 p_j \Psi \right\| \sqrt{\alpha(\chi, \varphi)}. \end{aligned} \quad (36)$$

Since $m(k, N) < \sqrt{N/k}$ one has $\sup_{1 \leq k \leq N} |(m^b(k, N))^{1/2}| = N^{\varepsilon/4}$ and

$$\|(\widehat{m}_2^b)^{1/2} \widehat{n} \Psi\|^2 \leq C \|\widehat{n}_2^{1/2} \Psi\|^2 \leq C\alpha(\Psi, \varphi) + CN^{-1/2}.$$

Thus (33) and Lemma 3.4 imply that the second summand in (34) is bounded by

$$\begin{aligned} & C\sqrt{\alpha(\chi, \varphi)} \left(N\|\varphi\|_\infty^2 \|f\|_1 \sqrt{\alpha(\Psi, \varphi)} + N^{1/2} \|f\| \|\varphi\|_\infty N^{\varepsilon/4} \right) \\ &\leq C\|\varphi\|_\infty^2 \sqrt{\alpha(\chi, \varphi) \alpha(\Psi, \varphi)} + CN^{-1/2-3\beta/2+\varepsilon} \|\varphi\|_\infty \end{aligned}$$

Summarizing we have that

$$N|\langle\langle\Psi, p_1 p_2 f(x_1, x_2) \widehat{m} q_1 q_2 \Psi\rangle\rangle| \leq C \|\varphi\|_\infty^2 \sqrt{\alpha(\chi, \varphi) \alpha(\Psi, \varphi)} \\ + C \|\varphi\|_\infty N^{(-1+3\beta)/2+\varepsilon/4} + C \|\varphi\|_\infty^2 N^{-\varepsilon/2}.$$

Choosing $0 < \varepsilon < (-1+3\beta)/2$ and $\eta < \min\{-1+3\beta+2\varepsilon, \varepsilon/2\}$ (c) follows.

- (d) Let $U_{\beta_1, \beta}$ be given by definition 4.2. As a first step we show that for $0 \leq \beta_1 < \beta < 1$ and for $h_{\beta_1, \beta}$ given by Definition 4.2 there exists a $\mathcal{K} \in \mathcal{F}$ and a $\eta > 0$ such that

$$N|\langle\langle\Psi, p_1 q_2 (V_\beta(x_1 - x_2) - U_{\beta_1, \beta}(x_1 - x_2)) \widehat{m} q_1 q_2 \Psi\rangle\rangle| \quad (37) \\ \leq \mathcal{K}(\varphi) (\|\varphi\|_\infty + \|\nabla \varphi\|_{6, loc} (\ln N)^{1/3}) \\ (\langle\langle\Psi, \widehat{n} \Psi\rangle\rangle + N^{-\beta_1} \|\nabla_1 q_1 \Psi\|^2 + N^{-\eta}).$$

Lemma 4.3 and integration by parts gives

$$N|\langle\langle\Psi, p_1 q_2 (V_\beta(x_1 - x_2) - U_{\beta_1, \beta}(x_1 - x_2)) \widehat{m} q_1 q_2 \Psi\rangle\rangle| \\ = N|\langle\langle\Psi, q_1 p_2 \widehat{m}_1 (\nabla_1 h_{\beta_1, \beta}(x_1 - x_2)) q_2 \nabla_1 q_1 \Psi\rangle\rangle| \quad (38) \\ + N|\langle\langle\nabla_1 q_1 \Psi, p_2 (\nabla_1 h_{\beta_1, \beta}(x_1 - x_2)) q_1 q_2 \widehat{m} \Psi\rangle\rangle|. \quad (39)$$

For (38) we write

$$(38) \leq N(N-1)^{-1} |\langle\langle\Psi, \sum_{k=2}^N q_1 p_k \widehat{m}_1 (\nabla_1 h_{\beta_1, \beta}(x_1 - x_k)) q_k \nabla_1 q_1 \Psi\rangle\rangle| \\ \leq C \|\sum_{k=2}^N q_k (\nabla_1 h_{\beta_1, \beta}(x_1 - x_k)) q_1 p_k \widehat{m}_1 \Psi\| \|\nabla_1 q_1 \Psi\|.$$

For the first factor we have

$$\|\sum_{k=2}^N q_k (\nabla_1 h_{\beta_1, \beta}(x_1 - x_k)) q_1 p_k \widehat{m}_1 \Psi\|^2 \\ \leq 2 |\langle\langle\Psi, \sum_{2 \leq k < j \leq N} q_1 p_j \widehat{m}_1 (\nabla_1 h_{\beta_1, \beta}(x_1 - x_j)) q_j \\ q_k (\nabla_1 h_{\beta_1, \beta}(x_1 - x_k)) q_1 p_k \widehat{m}_1 \Psi\rangle\rangle| \quad (40)$$

$$+ |\langle\langle\Psi, \sum_{2 \leq k \leq N} q_1 p_k \widehat{m}_1 (\nabla_1 h_{\beta_1, \beta}(x_1 - x_k)) \\ q_k (\nabla_1 h_{\beta_1, \beta}(x_1 - x_k)) q_1 p_k \widehat{m}_1 \Psi\rangle\rangle|. \quad (41)$$

Using that $\nabla_1 h_{\beta_1, \beta}(x_1 - x_k) = -\nabla_k h_{\beta_1, \beta}(x_1 - x_k)$ and integrating by

parts gives

$$\begin{aligned}
(40) & \leq N^2 |\langle \Psi, q_3 q_1 p_2 \widehat{m}_1 (\nabla_3 h_{\beta_1, \beta}(x_1 - x_3)) (\nabla_2 h_{\beta_1, \beta}(x_1 - x_2)) q_2 q_1 p_3 \widehat{m}_1 \Psi \rangle| \\
& \leq N^2 |\langle \nabla_2 \nabla_3 q_3 q_1 p_2 \widehat{m}_1 \Psi, h_{\beta_1, \beta}(x_1 - x_2) h_{\beta_1, \beta}(x_1 - x_3) q_2 q_1 p_3 \widehat{m}_1 \Psi \rangle| \\
& \quad + N^2 |\langle \nabla_3 q_3 q_1 p_2 \widehat{m}_1 \Psi, h_{\beta_1, \beta}(x_1 - x_2) h_{\beta_1, \beta}(x_1 - x_3) \nabla_2 q_2 q_1 p_3 \widehat{m}_1 \Psi \rangle| \\
& \quad + N^2 |\langle q_3 q_1 \nabla_2 p_2 \widehat{m}_1 \Psi, h_{\beta_1, \beta}(x_1 - x_2) h_{\beta_1, \beta}(x_1 - x_3) q_2 q_1 \nabla_3 p_3 \widehat{m}_1 \Psi \rangle| \\
& \quad + N^2 |\langle q_3 q_1 p_2 \widehat{m}_1 \Psi, h_{\beta_1, \beta}(x_1 - x_2) h_{\beta_1, \beta}(x_1 - x_3) \nabla_2 \nabla_3 q_2 q_1 p_3 \widehat{m}_1 \Psi \rangle| \\
& \leq N^2 \|\nabla_3 q_3 q_1 \widehat{m}_1 \Psi\| \|h_{\beta_1, \beta}(x_1 - x_2) \nabla_2 p_2\|_{op} \|h_{\beta_1, \beta}(x_1 - x_3) p_3\|_{op} \|q_2 q_1 \widehat{m}_1 \Psi\| \\
& \quad + N^2 \|\nabla_3 q_3 q_1 \widehat{m}_1 \Psi\| \|p_2 h_{\beta_1, \beta}(x_1 - x_2)\|_{op} \|h_{\beta_1, \beta}(x_1 - x_3) p_3\|_{op} \|\nabla_2 q_2 q_1 \widehat{m}_1 \Psi\| \\
& \quad + N^2 \|q_3 q_1 \widehat{m}_1 \Psi\| \|h_{\beta_1, \beta}(x_1 - x_2) \nabla_2 p_2\|_{op} \|h_{\beta_1, \beta}(x_1 - x_3) \nabla_3 p_3\|_{op} \|q_2 q_1 \widehat{m}_1 \Psi\| \\
& \quad + N^2 \|q_3 q_1 \widehat{m}_1 \Psi\| \|h_{\beta_1, \beta}(x_1 - x_2) p_2\|_{op} \|h_{\beta_1, \beta}(x_1 - x_3) \nabla_3 p_3\|_{op} \|\nabla_2 q_2 q_1 \widehat{m}_1 \Psi\|.
\end{aligned}$$

Note that

$$\begin{aligned}
q_1 \nabla_2 q_2 \widehat{m} \Psi &= q_1 p_2 \nabla_2 q_2 \widehat{m} \Psi + q_1 q_2 \nabla_2 q_2 \widehat{m} \Psi \\
&= q_1 p_2 \widehat{m}_1 \nabla_2 q_2 \Psi + q_1 \widehat{m} q_2 \nabla_2 q_2 \Psi.
\end{aligned}$$

With Lemma 3.4 it follows that

$$\|q_1 \nabla_2 q_2 \widehat{m} \Psi\| \leq C \|\nabla_2 q_2 \Psi\|. \quad (42)$$

This and Lemma 3.2 (e) give

$$(40) \leq C(N^2 \|\nabla_1 q_1 \Psi\|^2 \|h\|_2^2 \|\varphi\|_\infty^2 + N^2 \langle \Psi, \widehat{n} \Psi \rangle \|q_1 \Psi\|^2 \|h\|_3^2 \|\nabla \varphi\|_{6, loc}^2).$$

with Lemma 4.3 it follows that

$$(40) \leq C(N^{-\beta_1} \|\varphi\|_\infty^2 \|\nabla_1 q_1 \Psi\|^2 + \langle \Psi, \widehat{n} \Psi \rangle \|q_1 \Psi\|^2 \|\nabla \varphi\|_{6, loc}^2 (\ln N)^{2/3}).$$

For (41) we have

$$\begin{aligned}
(41) & \leq N \|\widehat{m}_1 q_1 \Psi\|^2 \|p_2 (\nabla_1 h_{\beta_1, \beta}(x_1 - x_2))\|_{op}^2 \\
& \leq C N^{1-2+\beta} \|\varphi\|_\infty^2.
\end{aligned}$$

It follows that (38) is bounded by the right hand side of (37).

To control (39) we use once more that

$$\nabla_1 h_{\beta_1, \beta}(x_1 - x_2) = -\nabla_2 h_{\beta_1, \beta}(x_1 - x_2)$$

and integrate by parts

$$\begin{aligned}
(39) & \leq |\langle \nabla_2 p_2 \nabla_1 q_1 \Psi, h_{\beta_1, \beta}(x_1 - x_2) q_1 q_2 \widehat{m} \Psi \rangle| \\
& \quad + |\langle \nabla_1 q_1 \Psi, p_2 h_{\beta_1, \beta}(x_1 - x_2) q_1 \nabla_2 q_2 \widehat{m} \Psi \rangle| \\
& \leq \|\nabla_1 q_1 \Psi\| \|h_{\beta_1, \beta}(x_1 - x_2) \nabla_2 p_2\|_{op} \|q_1 q_2 \widehat{m} \Psi\| \\
& \quad + \|\nabla_1 q_1 \Psi\| \|p_2 h_{\beta_1, \beta}(x_1 - x_2)\|_{op} \|q_1 \nabla_2 q_2 \widehat{m} \Psi\| \\
& \leq C \|\nabla_1 q_1 \Psi\| \left(\|q_1 \Psi\| (\ln N)^{1/3} \|\nabla \varphi\|_{6, loc} + N^{-\beta_1} \|\varphi\|_\infty \|\nabla_1 q_1 \Psi\| \right).
\end{aligned}$$

and (37) follows.

Now (37) together with Lemma 4.3 gives

$$\begin{aligned}
|N\langle\Psi p_1 q_2 Z_\beta(x_1, x_2) \widehat{m} q_1 q_2 \chi\rangle| &\leq CN|\langle\Psi, p_1 q_2(U_{0,\beta}(x_1 - x_2) \widehat{m} q_1 q_2 \Psi)\rangle| \\
&\quad + CN|\langle\Psi, p_1 q_2 \frac{a}{N-1} |\varphi(x_1)|^2 \widehat{m} q_1 q_2 \Psi\rangle| \\
&\quad + CN|\langle\Psi, p_1 q_2(V_\beta(x_1 - x_2) - U_{0,\beta}(x_1 - x_2)) \widehat{m} q_1 q_2 \Psi\rangle| \\
&\leq CN\|q_2 \Psi\| \|p_1 U_{0,\beta}(x_1 - x_2)\|_{op} \|\widehat{m} q_1 q_2 \Psi\| \\
&\quad + C\|q_2 \Psi\| \|\varphi\|_\infty^2 \|\widehat{m} q_1 q_2 \Psi\| \\
&\quad + \mathcal{K}(\varphi)(\|\varphi\|_\infty + \|\nabla\varphi\|_{6,loc(\ln N)^{1/3}})(\langle\Psi, \widehat{n}\Psi\rangle + \|\nabla_1 q_1 \Psi\|^2 + N^{-1/2}).
\end{aligned}$$

Since by definition $N\|U_{0,\beta}\| \leq C$ we get with Lemma 3.4 that $N\|p_1 U_{0,\beta}(x_1, x_2)\|_{op} \leq C\|\varphi\|_\infty$ and (d) follows. \square

4.1 Controlling the smoothness of Ψ

To get good control for the term in Lemma 4.4 (d) we need in addition a bound on $\|\nabla_1 q_1 \Psi\|$ in terms of $\alpha(\Psi, \varphi) + \mathcal{O}(1)$. $\|\nabla_1 q_1 \Psi\|$ is a part of the kinetic energy of Ψ so the idea is to show that the other contributions to the energy $\mathcal{E}(\Psi)$ are in leading order cancelled out by \mathcal{E}^{GP} and thus $\|\nabla_1 q_1 \Psi\| \approx \mathcal{E}(\Psi) - \mathcal{E}^{GP} \leq \alpha(\Psi, \varphi)$. In the next Lemma we estimate as a first step $\|\nabla_1 q_1 \Psi\|$ in terms of α and the difference between interaction and effective mean field. Note that the following Lemma holds for any $0 \leq \beta \leq 1$ and shall be useful when we generalize to $\beta = 1$. We shall use this estimate in the following section to control $\|\nabla_1 q_1 \Psi\|$ in terms of $\alpha(\Psi, \varphi) + \mathcal{O}(1)$ restricting to $0 < \beta < 1$. For later reference we shall also show that the total interaction energy $\|\sqrt{V_\beta(x_1 - x_2)}\Psi\|^2$ stays bounded.

Lemma 4.5 *For any $0 < \beta \leq 1$ there exists a $\mathcal{K} \in \mathcal{F}$ such that*

$$N\|\sqrt{V_\beta(x_1 - x_2)}\Psi\|^2 \leq \alpha(\Psi, \varphi) + \|\nabla\varphi\|^2 + 2\|A\|_\infty + 2a\|\varphi\|_\infty^2 \quad (43)$$

$$\begin{aligned}
\|\nabla_1 q_1 \Psi\|^2 &\leq \langle\Psi, (2a|\varphi(x_1)|^2 - (N-1)V_\beta(x_1 - x_2))\Psi\rangle \\
&\quad + \mathcal{K}(\varphi)\alpha(\Psi, \varphi).
\end{aligned} \quad (44)$$

uniform in $(\Psi, \varphi) \in L^2(\mathbb{R}^{3N}, \mathbb{C}) \otimes L^2(\mathbb{R}^3, \mathbb{C})$.

Proof:

Using symmetry of Ψ

$$\begin{aligned}
\mathcal{E}(\Psi) - \mathcal{E}^{GP}(\varphi) &= \|\nabla_1 \Psi\|^2 - \|\nabla\varphi\|^2 + (N-1)\langle\Psi, V_\beta(x_1 - x_2)\Psi\rangle \\
&\quad + \langle\Psi, A(x_1)\Psi\rangle - \langle\varphi, 2a|\varphi|^2 + A(x_1)\varphi\rangle.
\end{aligned} \quad (45)$$

Since $|\mathcal{E}(\Psi) - \mathcal{E}^{GP}(\varphi)| \leq \alpha(\Psi, \varphi)$

$$(N-1)\|\sqrt{V_\beta(x_1 - x_2)}\Psi\|^2 \leq \alpha(\Psi, \varphi) + \|\nabla\varphi\|^2 + 2\|A\|_\infty + 2a\|\varphi\|_\infty^2$$

and (43) follows.

For (44) note that

$$\begin{aligned} |\langle \nabla_1 q_1 \Psi, \nabla_1 p_1 \Psi \rangle| &= |\langle q_1 \Psi, \widehat{n}_1^{1/2} \widehat{n}_1^{-1/2} \Delta_1 p_1 \Psi \rangle| \\ &= |\langle q_1 \Psi, \widehat{n}_1^{-1/2} \Delta_1 p_1 \widehat{n}_2^{1/2} \Psi \rangle| \leq \|\Delta\varphi\| \|\widehat{n}_1^{-1/2} q_1 \Psi\| \|\widehat{n}_2^{1/2} p_1 \Psi\|. \end{aligned}$$

Thus with Lemma 3.4 and using that $\sqrt{\frac{k+2}{N}} \leq 3\sqrt{\frac{k}{N}}$ we get that

$$|\langle \nabla_1 q_1 \Psi, \nabla_1 p_1 \Psi \rangle| \leq C \|\Delta\varphi\| \alpha(\Psi, \varphi), \quad (46)$$

implying

$$\|\nabla_1 \Psi\|^2 - \|\nabla_1 p_1 \Psi\|^2 \geq \|\nabla_1 q_1 \Psi\|^2 - C \|\Delta\varphi\| \alpha(\Psi, \varphi).$$

Since

$$\|\nabla_1 p_1 \Psi\|^2 = \|p_1 \Psi\|^2 \|\nabla\varphi\|^2 = (1 - \|q_1 \Psi\|^2) \|\nabla\varphi\|^2 \quad (47)$$

it follows that

$$\|\nabla_1 \Psi\|^2 - \|\nabla\varphi\|^2 \geq \|\nabla_1 q_1 \Psi\|^2 - C \|\Delta\varphi\| \alpha(\Psi, \varphi) - \|\nabla\varphi\|^2 \alpha(\Psi, \varphi). \quad (48)$$

Using this and Lemma 4.4 (a) setting $f = A + 2a|\varphi|^2$ we get with (45)

$$\begin{aligned} \mathcal{E}(\Psi) - \mathcal{E}^{GP}(\varphi) &\geq \|\nabla_1 q_1 \Psi\|^2 + \langle \Psi, ((N-1)V_\beta(x_1 - x_2) - 2a|\varphi(x_1)|^2)\Psi \rangle \\ &\quad - C(\|A\|_\infty + 2a\|\varphi\|_\infty^2 + \|\Delta\varphi\| + \|\nabla\varphi\|^2) \alpha(\Psi, \varphi). \end{aligned}$$

Since

$$\|\nabla\varphi\|^2 = \langle \nabla\varphi, \nabla\varphi \rangle = -\langle \varphi, \Delta\varphi \rangle \leq \|\Delta\varphi\|$$

we get (b). □

4.2 Controlling $\|\nabla_1 q_1 \Psi\|$ for $\beta < 1$

With Lemma 4.5 we have found a bound on $\|\nabla_1 q_1 \Psi\|$ for all $0 < \beta \leq 1$ in terms of effective mean field minus interaction. Note that in view of Lemma 4.5 (a) the interaction is bounded. The mean-field term is bounded by $\langle \Psi, |\varphi|^2 \Psi \rangle \leq \|\varphi\|_\infty^2$ and we have for any $0 < \beta \leq 1$

$$\|\nabla_1 q_1 \Psi\|^2 \leq \mathcal{K}(\varphi) (\alpha(\Psi, \varphi) + 1). \quad (49)$$

But — as explained above — we want to show that $\|\nabla_1 q_1 \Psi\|$ is small using that the effective mean field cancels out the leading order of the interaction.

Lemma 4.6 *Let $0 < \beta < 1$, $V_\beta \in \mathcal{V}_\beta$ and $m : \mathbb{N}^2 \rightarrow \mathbb{R}^+$ with $m \leq n^{-1}$. Then there exists a $\eta > 0$ and a $\mathcal{K} \in \mathcal{F}$ such that for any $\Psi \in \mathcal{H}_0$ and any $\varphi \in L^2(\mathbb{R}^3, \mathbb{C})$*

$$(a) \quad N|\langle\langle\Psi, p_1 p_2 V_\beta(x_1 - x_2)(1 - p_1 p_2)\Psi\rangle\rangle| \leq \mathcal{K}(\varphi)(\alpha(\Psi, \varphi) + N^{-\eta}) .$$

$$(b) \quad \langle\langle\Psi(2a|\varphi(x_1)|^2) - (N - 1)V_\beta(x_1 - x_2)\Psi\rangle\rangle \leq \mathcal{K}(\varphi)(\alpha(\Psi, \varphi) + N^{-\eta}) .$$

$$(c) \quad \|\nabla_1 q_1 \Psi\|^2 \leq \mathcal{K}(\varphi)(\alpha(\Psi, \varphi) + N^{-\eta}) , \quad (50)$$

Proof:

(a) Since $1 - p_1 p_1 = p_1 q_2 + q_1 p_2 + q_1 q_2$ it suffices to show that

$$N|\langle\langle\Psi, p_1 p_2 V_\beta(x_1 - x_2) p_1 q_2 \Psi\rangle\rangle| \leq \mathcal{K}(\varphi)(\alpha(\Psi, \varphi) + N^{-\eta}) \quad (51)$$

$$N|\langle\langle\Psi, p_1 p_2 V_\beta(x_1 - x_2) q_1 q_2 \Psi\rangle\rangle| \leq \mathcal{K}(\varphi)(\alpha(\Psi, \varphi) + N^{-\eta}) \quad (52)$$

For (51) we use Lemma 3.2 (c) and (e) as well as Lemma 3.4 to get

$$\begin{aligned} (51) &= N|\langle\langle\Psi, \hat{n}_1^{-1/2} p_1 p_2 V_\beta(x_1 - x_2) \hat{n}_2^{1/2} p_1 q_2 \Psi\rangle\rangle| \\ &\leq N \|\hat{n}_1^{-1/2} \Psi\| \|p_1 V_\beta(x_1 - x_2) p_1\|_{op} \|\hat{n}_2^{1/2} q_2 \Psi\| \\ &\leq C \|\varphi\|_\infty^2 \alpha(\Psi, \varphi) . \end{aligned}$$

For $0 < \beta < 1/3$ (52) is Lemma 4.4 (c) for the special case $\hat{m} = 1$ (recall that $p_1 |\varphi|^2(x_2) p_1 = p_2 |\varphi|^2(x_1) p_2 = 0$). To generalize to $0 < \beta < 1$ we use Definition 4.2

$$\begin{aligned} N|\langle\langle\Psi, p_1 p_2 V_\beta(x_1 - x_2) q_1 q_2 \Psi\rangle\rangle| &\leq N|\langle\langle\Psi, p_1 p_2 U_{1/4, \beta}(x_1 - x_2) q_1 q_2 \Psi\rangle\rangle| \\ &\quad + N|\langle\langle\Psi, p_1 p_2 (V_\beta - U_{1/4, \beta})(x_1 - x_2) q_1 q_2 \Psi\rangle\rangle| \end{aligned}$$

Since $U_{1/4, \beta} \in \mathcal{V}_{1/4}$ (Lemma 4.3) the first summand has the right bound (Lemma 4.4 (c)). To finish the proof of the Lemma we verify the following formula, which shall be also of use later on.

$$N|\langle\langle\Psi, p_1 p_2 (\Delta h_{1/4, \beta})(x_1 - x_2) q_1 q_2 \Psi\rangle\rangle| \leq C(\|\varphi\|_\infty + \|\nabla \varphi\|_\infty) N^{-\eta} \quad (53)$$

to get (52) in full generality. Integrating by parts we get

$$\begin{aligned} N|\langle\langle\Psi, p_1 p_2 (\Delta h_{1/4, \beta}) q_1 q_2 \Psi\rangle\rangle| &\leq N|\langle\langle\Psi, p_1 p_2 (\nabla_1 h_{1/4, \beta}(x_1 - x_2)) \nabla_1 q_1 q_2 \Psi\rangle\rangle| \\ &\quad + N|\langle\langle\nabla_1 p_1 p_2 \Psi, (\nabla_1 h_{1/4, \beta}(x_1 - x_2)) q_1 q_2 \Psi\rangle\rangle| . \end{aligned} \quad (54)$$

To control (54) we use similar ideas as in the proof of Lemma 4.4

$$(54) \leq C \left\| \sum_{j=2}^N q_j (\nabla_1 h_{1/4, \beta}(x_1 - x_j)) p_1 p_j \Psi \right\| \|\nabla_1 q_1 \Psi\| .$$

The second factor is bounded (see (49)). For the first factor we write

$$\begin{aligned} & \left\| \sum_{j=2}^N q_j (\nabla_1 h_{1/4, \beta}(x_1 - x_j)) p_1 p_j \Psi \right\|^2 \\ &= \sum_{j \neq k \neq 1} \langle \Psi, p_1 p_k (\nabla_1 h_{1/4, \beta}(x_1 - x_k)) q_k q_j (\nabla_1 h(x_1 - x_j)) p_1 p_j \Psi \rangle \end{aligned} \quad (56)$$

$$+ \sum_{j=2}^N \langle \Psi, p_1 p_j (\nabla_1 h_{1/4, \beta}(x_1 - x_j))^2 p_1 p_j \Psi \rangle. \quad (57)$$

(56) can be estimated using Lemma 3.2 (e) and Lemma 4.3

$$\begin{aligned} (56) &= \sum_{j \neq k \neq 1} \langle \Psi, p_1 q_j (\nabla_1 h(x_1 - x_j)) p_j p_k (\nabla_1 h_{1/4, \beta}(x_1 - x_k)) p_1 q_k \Psi \rangle \\ &\leq N^2 \|p_1 (\nabla_1 h(x_1 - x_j)) p_j\|_{op}^2 = N^2 \|\varphi\|_\infty^4 \|\nabla_1 h_{1/4, \beta}\|_1^2 \\ &\leq CN^2 \|\varphi\|_\infty^4 N^{-5/2} = C \|\varphi\|_\infty^4 N^{-1/2}. \end{aligned}$$

(57) is bounded by

$$N \|\nabla h_{1/4, \beta}(x_1 - x_j) p_1\|_{op}^2 \leq N \|\nabla h_{1/4, \beta}(x_1 - x_j)\|^2 \|\varphi\|_\infty^2 \leq CN^{-1+\beta} \|\varphi\|_\infty^2.$$

It follows that (54) is bounded by $\mathcal{K}(\varphi)N^{-\eta}$ for some $\eta > 0$.

Integration by parts yields for (55)

$$\begin{aligned} (55) &\leq N |\langle \Delta_1 p_1 p_2 \Psi, h_{1/4, \beta}(x_1 - x_2) q_1 q_2 \Psi \rangle| \\ &\quad + N |\langle \nabla_1 p_1 p_2 \Psi, h_{1/4, \beta}(x_1 - x_2) \nabla_1 q_1 q_2 \Psi \rangle| \\ &\leq N \|\Delta_1 p_1 \Psi\| \|p_2 h_{1/4, \beta}(x_1 - x_2)\|_{op} \\ &\quad + N \|\nabla_1 p_1 \Psi\| \|p_2 h_{1/4, \beta}(x_1 - x_2)\|_{op} \|\nabla_2 q_2 \Psi\| \\ &\leq N \|\Delta \varphi\| \|\varphi\|_\infty \|h_{1/4, \beta}\| \\ &\quad + N \|\nabla \varphi\| \|\varphi\|_\infty \|h_{1/4, \beta}\| \|\nabla_2 q_2 \Psi\| \end{aligned}$$

with Lemma 4.3 and (49) we get (53) and (52) follows.

(b) We get using selfadjointness of the multiplication operators

$$\begin{aligned} & \langle \Psi, (2a|\varphi(x_1)|^2 - (N-1)V_\beta(x_1 - x_2)) \Psi \rangle \\ &= \langle p_1 p_2 \Psi, (2a|\varphi(x_1)|^2 - (N-1)V_\beta(x_1 - x_2)) p_1 p_2 \Psi \rangle \end{aligned} \quad (58)$$

$$+ 2\Re \langle p_1 p_2 \Psi, (2a|\varphi(x_1)|^2 - (N-1)V_\beta(x_1 - x_2)) (1 - p_1 p_2) \Psi \rangle \quad (59)$$

$$- (N-1) \langle (1 - p_1 p_2) \Psi, V_\beta(x_1 - x_2) (1 - p_1 p_2) \Psi \rangle \quad (60)$$

$$+ 2a \langle (1 - p_1 p_2) \Psi, |\varphi(x_1)|^2 (1 - p_1 p_2) \Psi \rangle. \quad (61)$$

(58) is controlled by formula (32).

Using symmetry of Ψ and $p_2|\varphi(x_1)|^2q_2 = 0$, the absolute value of (59) is bounded by

$$2 \left| \left\langle \left\langle p_1 p_2 \widehat{n}_2^{1/2} \Psi, 2a|\varphi(x_1)|^2 \widehat{n}_1^{-1/2} q_1 p_2 \Psi \right\rangle \right\rangle \right| \\ + 2(N-1) |\langle p_1 p_2 \Psi, 2V_\beta(x_1 - x_2)(1 - p_1 p_2) \Psi \rangle|$$

The first line is controlled by Lemma 3.2 (e) with Lemma 3.4 and thus bounded by $\|\varphi\|_\infty^2 \alpha(\Psi, \varphi)$. The second line is controlled by part (a) of the Lemma. Thus we can find a $\mathcal{K} \in \mathcal{F}$ such that

$$(59) \leq \mathcal{K}(\varphi)(\alpha(\Psi, \varphi) + N^{-\eta}).$$

Positivity of V_β implies that line (60) is negative. (61) is bounded by

$$\|(p_1 q_2 + q_1 p_1 + q_1 q_1) \Psi\|^2 \|\varphi\|_\infty^2 \leq C \|\varphi\|_\infty^2 \alpha(\Psi, \varphi)$$

and we get (b).

(c) follows from (b) with Lemma 4.5.

□

4.3 Proof of the Theorem for $0 < \beta < 1/3$

With Lemma 4.4 and Lemma 4.6 we can now estimate α'_j for $j = 0, 1, 2$ for $\beta < 1/3$. We arrive directly at the following Corollary.

Corollary 4.7 *Let $0 < \beta < 1/3$, $V_\beta \in \mathcal{V}_\beta$. Then there exists a $\mathcal{K} \in \mathcal{F}$ and a $\eta > 0$ such that for any symmetric Ψ , any φ and any $j = 0, 1, 2$*

$$|\alpha'_j(\Psi, \varphi)| \leq (\|\varphi\|_\infty + (\ln N)^{1/3} \|\nabla \varphi\|_{\delta, loc} + \|\dot{A}\|_\infty) \mathcal{K}(\varphi)(\alpha(\Psi, \varphi) + N^{-\eta}). \quad (62)$$

The Theorem follows for $0 < \beta < 1/3$ via Grönwall.

5 Generalizing to $1/3 \leq \beta < 1$

For $\beta > 1/3$ the radius of the interactions is much smaller than the mean distance of the particles, so the interactions do not overlap for typical configurations any more. Still the interaction of our N -body system can be approximated by an effective mean field, let us explain why: Whenever two or more particles come very close the wave function is affected on a microscopic length scale by the interaction of the particles. Neglecting three particle interactions the microscopic structure can be constructed from the zero energy scattering states of V_β . This can be made more clear with the following heuristic argument: In principle one could control the time evolution of Ψ_t by generalized eigenfunction expansion. The relevant eigenfunctions are on a microscopic scale approximately given by this zero energy scattering state.

By this microscopic structure the effect of the interactions is smeared out. For the smeared out effective interactions the old mean field argument holds. Therefore the first step when generalizing the Theorem is to say something about the microscopic structure of the wave functions.

Below we shall use our estimates on the microscopic structure in two places. On the one hand it makes a better control on Ψ_t and thus a better control of $\alpha(\Psi_t, \varphi_t)$ for $\beta \geq 1/3$ possible. On the other hand for $\beta = 1$ the interaction energy of Ψ_t can only be controlled in a suitable way when the microscopic structure of Ψ_t is known.

5.1 Microscopic Structure

For technical reasons we shall make a smooth spacial cutoff of the zero energy scattering state. We do so by defining — depending on V_β — a potential $W_{\beta_1} \in \mathcal{U}_{\beta_1}$ with softer scaling behavior $\beta_1 < \beta$ in such a way that the potential $V_\beta - W_{\beta_1}$ has scattering length zero, i.e. the zero energy scattering state of $V_\beta - W_{\beta_1}$ is outside the support of W_{β_1} equal to one.

Definition 5.1 *Let $0 < \beta_1 < \beta \leq 1$, $V_\beta \in \mathcal{V}_\beta$ and $a_N/(4\pi)$ be the scattering length of $V_\beta/2$. We define the potential W_{β_1} via*

$$W_{\beta_1}(x) := \begin{cases} a_N N^{3\beta_1}, & \text{for } N^{-\beta_1} < x < R_{\beta_1}; \\ 0, & \text{else.} \end{cases}$$

R_{β_1} is the minimal value which ensures that the scattering length of $V_\beta - W_{\beta_1}$ is zero.

The respective zero energy scattering state shall be denoted by $f_{\beta_1, \beta}$, i.e.

$$\left(-\Delta + \frac{1}{2}(V_\beta - W_{\beta_1}) \right) f_{\beta_1, \beta} = 0, \quad (63)$$

we shall also need

$$g_{\beta_1, \beta} = 1 - f_{\beta_1, \beta}.$$

Lemma 5.2 *For any $0 < \beta_1 < \beta \leq 1$, $V_\beta \in \mathcal{V}_\beta$*

(a)

$$W_{\beta_1} f_{\beta_1, \beta} \in \mathcal{V}_{\beta_1}, \quad \lim_{N \rightarrow \infty} N^\eta |N \|V_\beta f_{\beta_1, \beta}\|_1 - a| < \infty.$$

(b)

$$\begin{aligned} \|g_{\beta_1, \beta}\|_1 &\leq CN^{-1-2\beta_1}, & \|g_{\beta_1, \beta}\| &\leq CN^{-1-\beta_1/2}, \\ \|g_{8/9, 1}\|_3 &\leq CN^{-1}(\ln N)^{1/3} \end{aligned}$$

(c) *For any $x_2 \in \mathbb{R}^3$ and any $\Psi \in L^2(\mathbb{R}^{3N}, \mathbb{C})$*

$$\|\mathbf{1}_{|x_1 - x_2| \leq R_{\beta_1}} \nabla_1 \Psi\|^2 + \frac{1}{2} \langle \langle \Psi, (V_{\beta_1} - W_{\beta_1})(x_1 - x_2) \Psi \rangle \rangle \geq 0.$$

Proof:

Let j_β be the zero energy scattering state of the potential $V_\beta/2$.

Before we prove the different points of the Lemma, let us give some properties of $f_{\beta_1, \beta}$ first.

Since V_β is positive and has compact support of radius r it follows, that $1 > j_\beta(x) \geq 1 - a_N/(4\pi x)$.

Note, that the potential W_{β_1} is zero inside the Ball around zero of radius $N^{-\beta_1}$, hence $f_{\beta_1, \beta}$ is inside this Ball a multiple of j_β , i.e. there exists a K_{β_1} such that

$$K_{\beta_1} f_{\beta_1, \beta}(x) = j_\beta(x) \text{ for } x < N^{-\beta_1} ,$$

in particular the derivative $d_x K_{\beta_1} f_{\beta_1, \beta}(x)$ is positive for $x = N^{-\beta_1}$.

For $x > N^{-\beta_1}$ the $f_{\beta_1, \beta}$ “sees” a negative potential, namely $-W_{\beta_1}$. Due to spherical symmetry $f_{\beta_1, \beta}$ is in that region a linear combination of an in- and an outgoing spherical wave with momentum $k_0 = \sqrt{a_N N^{3\beta_1}}$.

By definition R_{β_1} is the minimal value which ensures that the scattering length of $V_\beta - W_{\beta_1}$ is zero, i.e. the minimal value which ensures that $f_{\beta_1, \beta}$ is constant for $x > R_{\beta_1}$. In other words R_{β_1} is the minimal value satisfying

$$d_{|x|} f_{\beta_1, \beta}(|x|) \Big|_{|x|=R_{\beta_1}} = 0 .$$

It follows that $f_{\beta_1, \beta}$ is a positive function and that $d_{|x|} f_{\beta_1, \beta} \geq 0$.

Finally we have to control the constant K_{β_1} which ensures that $f_{\beta_1, \beta}(x) = 1$ for $x > R_{\beta_1}$: Since W_{β_1} is positive, that $K_{\beta_1} d_x f_{\beta_1, \beta} \leq d_x j_\beta$ and $K_{\beta_1} f_{\beta_1, \beta} \leq j_\beta$.

Since $f_{\beta_1, \beta}(x) = 1$ for $x > R_{\beta_1}$ and $\lim_{x \rightarrow \infty} j_\beta(x) = 1$ we get that $K_{\beta_1} \leq 1$. On the other hand we have, since

$$1 \geq f(N^{-\beta_1}) = j_\beta(N^{-\beta_1})/K_{\beta_1} \geq (1 - a_N/(4\pi N^{-\beta_1}))/K_{\beta_1} \quad (64)$$

that

$$(1 - a_N/(4\pi N^{-\beta_1})) \leq K_{\beta_1} \leq 1 . \quad (65)$$

(a) Using that $f_{\beta_1, \beta} \geq j_\beta$ it follows that

$$|g_{\beta_1, \beta}(x)| \leq a_N/(4\pi x) . \quad (66)$$

Thus

$$\begin{aligned} N \|g_{\beta_1, \beta} V_\beta\|_1 &= N \int_0^{N^{-\beta}} |g_{\beta_1, \beta}(x) V_\beta(x)| x^2 dx + N \int_{N^{-\beta}}^\infty |g_{\beta_1, \beta}(x) V_\beta(x)| x^2 dx \\ &\leq C N N^{-1+3\beta} N^{-1-2\beta} + C N N^{-1+\beta} N^{-1} = C N^{\beta-1} . \end{aligned}$$

Since $V_\beta \in \mathcal{V}_\beta$ it follows that there exists a $\eta > 0$ such that $\lim_{N \rightarrow \infty} N^\eta \|V_\beta\| - a < \infty$ and we get the second statement in (a).

The scattering length of the potential $V_\beta - W_{\beta_1}$ is zero. Thus $\int (V_\beta(x) - W_{\beta_1}(x)) f_{\beta_1, \beta}(x) dx = 0$ and also $\lim_{N \rightarrow \infty} N^\eta (N \|W_{\beta_1} f_{\beta_1, \beta}\|_1 - 2a) \leq \infty$. This implies in particular that R_{β_1} is of order $N^{-\beta_1}$ thus $W_{\beta_1} f_{\beta_1, \beta} \in \mathcal{V}_{\beta_1}$.

(b) Since $g_{\beta_1, \beta}(x) = 0$ for $x > N^{-\beta_1}$ it follows that

$$\begin{aligned}\|g_{\beta_1, \beta}\|_1 &\leq \frac{1}{4\pi} a_N \int_0^{R_{\beta_1}} |x|^{-1} d^3x \leq CN^{-1-2\beta_1} \\ \|g_{\beta_1, \beta}\|_2 &\leq \frac{1}{16\pi^2} a_N^2 \int_0^{R_{\beta_1}} |x|^{-2} d^3x \leq CN^{-2-\beta_1} \\ \|g_{\beta_1, \beta}\|_3 &\leq \frac{1}{64\pi^3} a_N^3 \int_0^{R_{\beta_1}} |x|^{-3} d^3x \leq CN^{-3} \ln N.\end{aligned}$$

(c) To prove (c) we first show that for any $n \in \mathbb{N}$ and any subset $X_n \subset \mathbb{R}^3$ with $|X_n| = n$ which is such that the supports of the potentials $W_{\beta_1}(\cdot - x)$ are pairwise disjoint for any $x \in X_n$ the operator

$$H^{X_n} := -\Delta + \sum_{x_k \in X_n} (V_{\beta_1}(\cdot - x_k) - W_{\beta_1}(\cdot - x_k))$$

is nonnegative.

This one can see in the following way: For any such X_n the zero energy scattering state of H^{X_n} is given by

$$F_{\beta_1, \beta}^{X_n} := \prod_{x_k \in X_n} f_{\beta_1, \beta}(\cdot - x_k).$$

By construction the $f_{\beta_1, \beta}$ are positive, so is $F_{\beta_1, \beta}^{X_n}$. Assume now that H^{X_n} is not nonnegative, i.e. that there exists a ground state $\Psi \in L^2$ of H^{X_n} of negative energy E . Since the phase of the ground state can be chosen such that the ground state is positive we get

$$\langle\langle F_{\beta_1, \beta}^{X_n}, H^{X_n} \Psi \rangle\rangle = \langle\langle F_{\beta_1, \beta}^{X_n}, E \Psi \rangle\rangle < 0. \quad (67)$$

On the other hand we have since $F_{\beta_1, \beta}^{X_n}$ is the zero energy scattering state

$$\langle\langle F_{\beta_1, \beta}^{X_n}, H^{X_n} \Psi \rangle\rangle = \langle\langle H^{X_n} F_{\beta_1, \beta}^{X_n}, \Psi \rangle\rangle = 0.$$

This contradicts (67) and the nonnegativity of H^{X_n} follows.

Having shown that the H^{X_n} are nonnegative we prove (c) by contradiction. Assume that there exists a $\Psi \in \mathcal{D}(H)$ such that

$$\|\mathbb{1}_{|x| \leq R_{\beta_1}} \nabla_1 \Psi\| + \langle\langle \Psi, (V_{\beta_1}(x_1) - W_{\beta_1}(x_1)) \Psi \rangle\rangle = E < 0.$$

Since V_{β_1} and W_{β_1} are spherically symmetric we can assume that Ψ is spherically symmetric and $\Psi(x) = 1$ for $|x| > R_{\beta_1}$. We shall construct now a set of points X_n and a $\chi \in L^2$ such that $\langle\langle \chi, H^{X_n} \chi \rangle\rangle < 0$, contradicting to nonnegativity of H^{X_n} .

For any $R > 0$ let

$$\xi_R(x) := \begin{cases} R^2/x^2, & \text{for } x > R; \\ 1, & \text{else.} \end{cases}$$

Let now X_n be a subset $X_n \subset \mathbb{R}^3$ with $|X_n| = n$ which is such that the supports of the potentials $W_{\beta_1}(\cdot - x_k)$ lie within the Ball around zero with radius R and are pairwise disjoint for any $x_k \in X_n$. Since we are in three dimensions we can choose a n which is of order R^3 .

Let now $\chi_R := \xi_R \prod_{x_k \in X_n} \Psi(x - x_k)$. The energy inside the ball $B_R(0)$ equals En , thus it is negative and of order R^3 . Outside the ball we have only kinetic energy $4 \int_{x > R} R^4/x^6 d^3x$ which is of order R . Choosing R large enough we can find a X_n such that

$$\langle\langle \chi_R, H^{X_n} \chi_R \rangle\rangle$$

is negative, contradicting nonnegativity of H^{X_n} .

□

5.2 First adjustment of the functionals

As mentioned in the introduction we shall use the control on the microscopic structure, i.e. the control of the zero energy scattering state of $(V_\beta - W_{\beta_1})/2$ with some potential W_{β_1} with softer scaling $\beta_1 < \beta$ to get a control of Ψ when β increases. The first idea one might have is to divide Ψ through a function which approximates the microscopic structure (e.g. the product $\prod_{j \neq k} f_{\beta_1, \beta}$ for some suitable $0 < \beta_1 < \beta$), but this is not what we shall do. One reason is that $\left(\prod_{j \neq k} f_{\beta_1, \beta}\right)^{-1}$ gets very large when many particles get very close.

Instead of dividing Ψ through its microscopic structure we equip the projectors with the respective microscopic structure to get the desired estimates. Roughly speaking: The operator $(-\Delta_1 - \Delta_2 + V_\beta(x_1 - x_2))p_1p_2$ is hard to control for large β since $V_\beta(x_1 - x_2)$ is peaked for small $|x_1 - x_2|$. But since $f_{\beta_1, \beta}$ is the zero energy scattering state of $-\Delta_1 + (V_\beta - W_{\beta_1})/2$ it follows that $(-\Delta_1 - \Delta_2 + V_\beta(x_1 - x_2))f_{\beta_1, \beta}p_1p_2\Psi$ is smoother.

To get good estimates we shall incorporate this idea in a very sensible way. How this can be done is easiest explained for a different functional, namely $\tilde{\alpha}(\Psi, \varphi) = \langle\langle \Psi, \hat{n}^2 \Psi \rangle\rangle = \langle\langle \Psi, q_1 \Psi \rangle\rangle$ (see formula (10)). Taking the time derivative and using that $q_1 = 1 - p_1$ one gets among other terms

$$\frac{d}{dt} \tilde{\alpha}(\Psi_t, \varphi_t) = i \sum_{j < k} N^{-1} \langle\langle \Psi, [V_\beta(x_j - x_k), p_1] \Psi \rangle\rangle.$$

Most of the interaction terms commute with p_1 , only $i \sum_{1 < k} N^{-1} \langle\langle \Psi, [V_\beta(x_1 - x_k), p_1] \Psi \rangle\rangle$ remains. Hence the microscopic structure only for the interaction $V_\beta(x_1 - x_k)$ matters.

This insight can also be used for our α_t . Here many of the interactions cancel out due to Lemma 3.6) (d). Looking at Lemma 4.4 and considering $1/3 < \beta < 1$ for the moment, it is (25) which we do not have good control of. Consider the following functional

$$\alpha(\Psi, \varphi) + N(N-1)\Im(\langle \Psi, q_1 q_2 g_{\beta_1, \beta}(x_1 - x_2)(\widehat{n} - \widehat{n}_2)p_1 p_2 \Psi \rangle) . \quad (68)$$

Taking the time derivative of this new functional one gets among other terms a

$$N(N-1)\Im(\langle \Psi, q_1 q_2 [-\Delta_1 - \Delta_2, g_{\beta_1, \beta}(x_1 - x_2)](\widehat{n} - \widehat{n}_2)p_1 p_2 \Psi \rangle) . \quad (69)$$

The commutator equals $(1 - g_{\beta_1, \beta})(V_\beta(x_1 - x_2) - W_{\beta_1}(x_1 - x_2))$ plus mixed derivatives and one sees, that the interactions in (25) are “replaced” by W_{β_1} for the price of new terms that have to be estimated.

Note that this adjustment is not sufficient to get good control of the adjusted functional: Taking the time derivative of this new functional one gets terms where the potential V_β does not cancel. As one shall see below one of these terms is not small for all $\beta < 1$, but still it is better than (25). Therefore we make a similar adjustment as before. We arrive at an iterative adjustment which leads to terms which we get better and better control of. With the iteration we will define below it turns out that five steps are enough to get sufficient control.

Guided by these ideas this section is organized as follows:

- We need different weights m^j for each step of the iterative adjustment. Note that there is some freedom in choosing the starting point of our iteration. It does not necessarily have to be $\alpha(\Psi_0, \varphi_0)$ but it should be close to a multiple of $\alpha(\Psi_0, \varphi_0)$. Hence there are many possible choices for m^j . All important properties the weights have to satisfy in order to generalize the Theorem are stated in Lemma 5.3. We prove the Lemma (i.e. the existence of a weight which satisfies all the important conditions) by construction.

Looking at (68) one can already guess that starting with some weight m^0 , m^1 has to satisfy $m^1(k, N) = m^0(k, N) - m^0(k+2, N)$. This explains (a) of Lemma 5.3. (b) ensures that the starting point of our iteration (which will be $\langle \Psi, \widehat{m}^0 \Psi \rangle$) is in fact close to a multiple of $\alpha(\Psi_0, \varphi_0)$. (b) and (c) of the Lemma are needed for the estimates.

- Having defined the weights m^j we construct some operators $R_{j,k}$, $S_{j,k}$ and $T_{j,k}$ (Definition 5.4). These operators shall then be used to define the functionals $\gamma_{j,k}$ and $\xi_{j,k}$ (Definition 5.6) as well as $\gamma'_{j,k}$ (Definition 5.8) which are the basic elements of the iterative adjustment:

- The $\gamma'_{j,k}$ are defined such, that $\frac{d}{dt}\gamma_{j,k}(\Psi_t, \varphi_t) = \gamma'_{j,k}(\Psi_t, \varphi_t)$ (see Lemma 5.9).
- $\gamma_{0,0}(\Psi, \varphi) = \langle \Psi, \widehat{m}^0 \Psi \rangle$ is the starting point of the iteration. $\xi_{0,0}$ plays the role of α'_1 . Note that $\gamma'_{0,0} - \xi_{0,0}$ is small (Corollary 5.10).

– In the l^{th} step of the adjustment we add $\sum_{j+k=l} \gamma_{j,k}$ to our functional. The respective $\sum_{j+k=l} \gamma'_{j,k}$ cancels out the “old” term we have no sufficient control of (namely $\sum_{j+k=l-1} \xi_{j,k}$) but leads to a “new” term which — as long as $j+k < 5$ — we have no sufficient control of (which is $\sum_{j+k=l} \xi_{j,k}$). All other terms of $\sum_{j+k=l} \gamma'_{j,k}$ are controllable (Lemma 5.11) below.

- Finally we construct functionals Γ and Γ' with $\frac{d}{dt}\Gamma(\Psi_t, \varphi_t) = \Gamma'(\Psi_t, \varphi_t)$ such that $\Gamma'(\Psi, \varphi)$ can be estimated in terms of $\alpha(\Psi, \varphi)$ (Corollary 5.12 (c)) and $\alpha(\Psi, \varphi)$ can be estimated in terms of $\Gamma(\Psi, \varphi)$ (Corollary 5.12 (b)). This allows to estimate $\Gamma(\Psi_t, \varphi_t)$ via Grönwall. Since $\alpha(\Psi, \varphi)$ is controlled by $\Gamma(\Psi, \varphi)$ (Corollary 5.12 (b)) we have good control of $\alpha(\Psi, \varphi)$ for $\beta < 1$.

Lemma 5.3 *There exists a set of weights $\{m^1, m^2, \dots, m^5; m^j : \mathbb{N}^2 \rightarrow \mathbb{R}^+\}$*

(a) *For $0 \leq j < 5$*

$$m^{j+1}(k, N) = m^j(k, N) - m^j(k+2, N)$$

(b) *For any $j > 0$ there exist constants $c_j > 0$ such that for any $k \geq 0$*

$$c_j N^{-j} n^{1-2j}(k+2, N) \leq m^j(k, N) \leq N^{-j} n^{1-2j}(k+2, N). \quad (70)$$

(c) *There exists a $C < \infty$ such that for $k \geq 0$*

$$\begin{aligned} |m^j(k, N) - m^j(k+1, N)| &\leq CN^{-j-1} n^{-1-2j}(k+1, N) \\ |m^j(k, N) - 2m^j(k+1, N) + m^j(k+2, N)| &\leq CN^{-j-2} n^{-3-2j}(k+1, N). \end{aligned}$$

Proof:

Let $m^5(k, N) := N^{-1/2}(k+1)^{-5+1/2}$. We prove the Lemma by constructing m^j for $j \in \{0, 1, \dots, 4\}$: We define the functions m^j for even $N+k$ via

$$\begin{aligned} m^j(N, N) &= (N+2)^{-j} \\ m^j(k, N) &= m^{j+1}(k, N) + m^j(k+2, N). \end{aligned}$$

For odd $N+k$ we set $m^j(k) = (m^j(k-1) + m^j(k+1))/2$.

(a) For even $N+k$ (a) follows by construction. For odd $N+k$ one has

$$\begin{aligned} m^{j+1}(k, N) &= (m^{j+1}(k-1, N) + m^{j+1}(k+1, N))/2 \\ &= (m^j(k-1, N) - m^j(k+1, N))/2 \\ &\quad + (m^j(k+1, N) - m^j(k+3, N))/2 \\ &= m^j(k, N) - m^j(k+2, N). \end{aligned}$$

(b) It suffices to prove (b) for even $N+k$. By construction it follows then also for odd $k+N$. We shall do so via induction over j . For $j=5$ (70) follows directly from the definition of m^5 .

Assume that (70) is satisfied for some $0 < j \leq 5$. By construction we have for even $N+k$ that

$$\begin{aligned} m^{j-1}(k, N) &= m^{j-1}(N, N) + \sum_{\substack{l \text{ even} \\ k \leq l < N}} m^j(l, N) \\ &= (N+2)^{-j} + \sum_{\substack{l \text{ even} \\ k \leq l < N}} m^j(l, N) \end{aligned} \quad (71)$$

By assumption there exist $c_j > 0$, for later use we assume that

$$c_j < \frac{(2j-1)}{3^j}, \quad (72)$$

such that

$$\begin{aligned} c_j \sum_{\substack{l \text{ even} \\ k \leq l < N}} (l+2)^{-j} n(l+2, N) &\leq \sum_{\substack{l \text{ even} \\ k \leq l < N}} |m^j(l, N)| \\ &\leq \sum_{\substack{l \text{ even} \\ k \leq l < N}} (l+2)^{-j} n(l+2, N). \end{aligned}$$

By monotonicity of the function $(\cdot)^{-j+1/2}$ for $j > 0$ it follows that

$$\begin{aligned} c_j \frac{N^{-1/2}}{2} \int_k^{N-2} (x+2)^{-j+1/2} dx &\leq \sum_{\substack{l \text{ even} \\ k \leq l < N}} |m^j(l, N)| \\ &\leq \frac{N^{-1/2}}{2} \int_{k+2}^N (x+2)^{-j+1/2} dx \end{aligned}$$

$$\begin{aligned} \frac{c_j N^{-1/2}}{2j-1} \left((k+2)^{-j+1/2} - N^{-j+1/2} \right) \\ \leq \sum_{\substack{l \text{ even} \\ k \leq l < N}} |m^j(l, N)| \leq \frac{N^{-1/2}}{2j-1} (k+2)^{-j+1/2}. \end{aligned}$$

With (72) and (71) we get for $N > 0$

$$\begin{aligned} \frac{c_j N^{-1/2}}{2j-1} (k+2)^{-j+1/2} \\ \leq \frac{c_j N^{-1/2}}{2j-1} (k+2)^{-j+1/2} - \frac{c_j N^{-j}}{2j-1} + (N+2)^{-j} \\ \leq m^{j-1}(k, N) \leq \frac{N^{-1/2}}{2j-1} (k+2)^{-j+1/2}. \end{aligned}$$

Since $(2j-1)^{-1} < 1$ it follows that (70) holds for $j-1$. Now (b) follows for even $N+k$ via induction.

(c) Let us first prove (c) for even $N+k$. It follow that

$$\begin{aligned} |m^j(k) - m^j(k+1)| &= |m^j(k) - (m^j(k) + m^j(k+2))/2| \\ &= |m^j(k) - m^j(k+2)|/2 = m^{j+1}(k)/2. \end{aligned}$$

With (b) we get the first formula in (c). For the second formula we have

$$\begin{aligned} |m^j(k) - 2m^j(k+1) + m^j(k+2)| \\ = |m^j(k) - (m^j(k) + m^j(k+2)) + m^j(k+2)| = 0. \end{aligned}$$

For odd $N+k$ one has

$$\begin{aligned} |m^j(k) - m^j(k+1)| &= |(m^j(k+1) + m^j(k-1))/2 - m^j(k+1)| \\ &= |m^j(k-1) - m^j(k+1)|/2 = m^{j+1}(k-1) \end{aligned}$$

and

$$\begin{aligned} |m^j(k) - 2m^j(k+1) + m^j(k+2)| \\ = |(m^j(k+1) + m^j(k-1))/2 - 2m^j(k+1) \\ + (m^j(k+1) + m^j(k+3))/2| \\ = \frac{1}{2} |m^j(k-1) - 2m^j(k+1) + m^j(k+3)| \\ = \frac{1}{2} |m^{j+1}(k-1) - m^{j+1}(k+1)| = |m^{j+1}(k)|. \end{aligned}$$

With (b) we get (c). □

Having shown that there exist a weight satisfying the conditions of Lemma 5.3 we use this weight to define some operators that shall be used in our iterative adjustment of α below. Due to symmetry we have some arbitrariness in defining these operators, in particular in choosing the coordinates on which they act. For easier reference below we keep the coordinates $1, \dots, 4$ free, so the operators we shall define next act on the coordinates x_5, x_6, \dots only.

Definition 5.4 For any $j, k \in \mathbb{N}_0$ we define the operators Q_j via

$$Q_j := q_{2j+3}q_{2j+4}g_{1/4,\beta}(x_{2j+3} - x_{2j+4})p_{2j+3}p_{2j+4}$$

and the operators $R_{j,k}$, $R_{j,k}$ and $R_{j,k}$ acting on $L^2(\mathbb{R}^{3N}, \mathbb{C})$ via

$$\begin{aligned} R_{j,k} &:= \frac{N!}{(N-2j-2k)!k!j!} \prod_{l=1}^j Q_l \widehat{m}^{j+k} \prod_{l=1}^k Q_{l+j}^* \\ S_{j,k} &:= \frac{N!}{(N-2j-2k-2)!j!k!} \prod_{l=1}^j Q_l \widehat{m}^{j+k+1} \prod_{l=1}^k Q_{l+j}^* \\ T_{j,k} &:= \frac{N!}{(N-2j-2k-2)!j!k!} \prod_{l=1}^j Q_l (\widehat{m}^{j+k} - \widehat{m}_1^{j+k}) \prod_{l=1}^k Q_{l+j}^* . \end{aligned}$$

For later use it is convenient to define $S_{-1,k} := 0$.

Lemma 5.5 *Let $j, k \in \mathbb{N}_0$, $\mathcal{M} \subset \mathbb{N}$ with $|\mathcal{M} \cap \{5, 6, \dots, 2j+2k+4\}| = M$. Then*

$$\begin{aligned} \|R_{j,k}\|_{\mathcal{M}} &\leq CN^{-(j+k)/8} \|\varphi\|_{\infty}^{j+k} && \text{if } \mathcal{M} \cap \{5, 6, \dots, 2j+2k+4\} = \emptyset , \\ \|R_{j,k}\|_{\mathcal{M}} &\leq N^{-1/2+M/2-(j+k)/8} \|\varphi\|_{\infty}^{j+k} && \text{if } \mathcal{M} \cap \{5, 6, \dots, 2j+2k+4\} \neq \emptyset . \end{aligned}$$

Let furthermore $r : \mathbb{N}^2 \rightarrow \mathbb{R}^+$ with $r \leq n$, then

$$\begin{aligned} \|\widehat{r} S_{j,k}\|_{\mathcal{M}} &\leq CN^{1+M/2-(j+k)/8} \|\varphi\|_{\infty}^{j+k} \\ \|\widehat{r} T_{j,k}\|_{\mathcal{M}} &\leq CN^{1+M/2-(j+k)/8} \|\varphi\|_{\infty}^{j+k} . \end{aligned}$$

Proof: First note that in view of Lemma 3.2 (e) and Lemma 5.2 (b)

$$\|g_{1/4,\beta}(x_1 - x_2)p_1\|_{op} \leq CN^{-1-1/8} \|\varphi\|_{\infty} .$$

Roughly estimating, $\|R_{j,k}\|_{op}$ has $j+k$ such factors, giving a power $N^{-(j+k)(1+1/8)}$. Furthermore we have $2j+2k$ projectors q in the definition of $R_{j,k}$, while m^{j+k} is of order $n(k)k^{-j-k}$. So the projectors q give in view of Lemma 3.4 together with \widehat{m}^{j+k} a factor N^{-j-k} . Since $\frac{N!}{(N-2j-2k)!} < N^{2j+2k}$ one gets by this rough estimate the result above.

In detail: Let $\mathcal{M} \subset \mathbb{N}$. Let $\Psi, \chi \in \mathcal{H}_{\mathcal{M}}$; $\|\Psi\| = \|\chi\| = 1$. Using Lemma 3.2 (c)

$$\begin{aligned} |\langle\langle \Psi, R_{j,k}\chi \rangle\rangle| &= \frac{N!}{j!k!(N-2j-2k)!} \\ &|\langle\langle \Psi, \widehat{n}_1^{-2j+1} \prod_{l=1}^j Q_l \widehat{n}_{2j+1}^{2j-1} \widehat{m}^{j+k} \widehat{n}_{2k+1}^{2k} \prod_{l=1}^k Q_{l+j}^* \widehat{n}_1^{-2k} \chi \rangle\rangle| \end{aligned}$$

With Lemma 5.3 (b) we have that $\|\widehat{n}_{2j+1}^{2j-1} \widehat{m}^{j+k} \widehat{n}_{2k+1}^{2k}\|_{op} \leq CN^{-j-k}$, thus we get with Lemma 3.4

$$\begin{aligned} |\langle\langle \Psi, R_{j,k}\chi \rangle\rangle| &\leq CN^{2j+2k} N^{-j-k} \|\widehat{n}_1^{-2j+1} \prod_{l=5}^{2j+4} q_l \Psi\| \|\widehat{n}_1^{-2k} \prod_{l=5+2j}^{2j+2k+4} q_l \chi\| \\ &\|g_{1/4,\beta}(x_1 - x_2)p_1\|_{op}^{j+k} \end{aligned}$$

The estimate on $\|\widehat{n}_1^{-2j+1} \prod_{l=5}^{2j+4} q_l \Psi\|$ and $\|\widehat{n}^{-2k+1} \prod_{l=5+2j}^{2j+2k+4} q_l \chi\|$ now depends on the symmetry of Ψ respectively χ . If $\mathcal{M} \cap \{5, 6, \dots, 2j+2k+4\} = \emptyset$ we can use Lemma 3.4 for all q_j with $j = 5, \dots, 2j+2k+4$. Then it follows that $\|\widehat{n}_1^{-2j+1} \prod_{l=5}^{2j+4} q_l \Psi\| < C$ and $\|\widehat{n}^{-2k+1} \prod_{l=5+2j}^{2j+2k+4} q_l \chi\| < C$ and thus

$$|\langle \Psi, R_{j,k} \chi \rangle| \leq CN^{-(j+k)/8} \|\varphi\|_\infty^{j+k}.$$

If $\mathcal{M} \subset \mathbb{N}$ with $|\mathcal{M} \cap \{5, 6, \dots, 2j+2k+4\}| = M > 0$ we can define $\mathcal{M}_a := \mathcal{M} \cap \{5, 6, \dots, 2j+4\}$ and $\mathcal{M}_b := \mathcal{M} \cap \{2j+5, 2j+6, \dots, 2j+2k+4\}$ and assume without loss of generality that $|\mathcal{M}_a| > 0$. Then it follows that $\|\widehat{n}_1^{-2j+1} \prod_{l=5}^{2j+4} q_l \Psi\| < CN^{(|\mathcal{M}_a|-1)/2}$ and $\|\widehat{n}^{-2k+1} \prod_{l=5+2j}^{2j+2k+4} q_l \chi\| < CN^{|\mathcal{M}_b|/2}$ and thus

$$|\langle \Psi, R_{j,k} \chi \rangle| \leq CN^{-(j+k)/8+(M-1)/2} \|\varphi\|_\infty^{j+k}.$$

$\widehat{r}S_{j,k}$ can be estimated in a similar way by

$$\begin{aligned} & |\langle \Psi, \widehat{r}S_{j,k} \chi \rangle| \\ & \leq N^{2j+2k+2} |\langle \Psi, \widehat{r} \widehat{n}_1^{-2j-1} \prod_{l=1}^j Q_l \widehat{n}_{2j+1}^{2j+1} \widehat{m}^{j+k+1} \widehat{n}_{2k+1}^{2k} \prod_{l=1}^k Q_{l+j}^* \widehat{n}_1^{-2k} \chi \rangle| \\ & \leq CN^{2j+2k+2-j-k-1} \|\widehat{r} \widehat{n}_1^{-2j-1} \prod_{l=5}^{2j+4} q_l \Psi\| \prod_{l=5+2k}^{2j+2k+4} q_l \chi\| \\ & \quad \|g_{1/4,\beta}(x_1 - x_2) p_1\|_{\mathcal{O}p}^{j+k} \|\widehat{n}_1^{-2k}\|. \end{aligned}$$

Using Lemma 3.4

$$\|\widehat{r} \widehat{n}_1^{-2j-1} \prod_{l=5}^{2j+4} q_l \Psi\| < CN^{|\mathcal{M}_a|/2} \quad \text{and} \quad \|\widehat{n}^{-2k+1} \prod_{l=5+2j}^{2j+2k+4} q_l \chi\| < CN^{|\mathcal{M}_b|/2},$$

thus

$$\begin{aligned} |\langle \Psi, \widehat{r}S_{j,k} \chi \rangle| & \leq CN^{j+k+1} N^{|\mathcal{M}_a|/2} \|\varphi\|_\infty^{j+k} N^{-9(j+k)/8} \\ & = CN^{1+M/2-(j+k)/8} \|\varphi\|_\infty^{j+k}. \end{aligned}$$

For the last equation note that in view of Lemma 5.3 (c) $\widehat{m}^j - \widehat{m}_1^j \leq \widehat{m}^j - \widehat{m}_2^j$, hence we get the same estimate as for $\|\widehat{n}S_{j,k}\|_{\mathcal{M}}$. □

Using the operators defined in Definition 5.4 we now adjust the functional α as explained at the beginning of this section using the functionals $\gamma_{j,k}$ which we shall define next.

Definition 5.6 For any $j, k > 0$ with $j+k \leq 5$ we define

$$\begin{aligned} \gamma_{j,k}(\Psi, \varphi) & := \langle \Psi, R_{j,k} \Psi \rangle \\ \xi_{j,k}(\Psi, \varphi) & := \langle \Psi, Z_\beta(x_1, x_2) p_1 p_2 S_{j,k} \Psi \rangle \text{ for } j, k \geq 0 \end{aligned}$$

As explained above we have after the l^{th} step of our iteration a remainder $\sum_{j+k=l} \xi_{j,k}$. We wish to show that this remainder is controllable after sufficiently many steps of iteration. It turns out that five steps are enough:

Proposition 5.7 (*Control of the remainder*) *There exists a $\mathcal{K} \in \mathcal{F}$ such that*

$$|\xi_{j,k}| \leq CN^{1/2-(j+k)/8} \|\varphi\|_{\infty} \mathcal{K}(\varphi).$$

Proof: Let us first prove the following three formulas which shall also be of use below.

$$\|\sqrt{|Z_{\beta}(x_1, x_2)|} p_1\|_{op} \leq CN^{-1/2} \|\varphi\|_{\infty} \quad (73)$$

$$\|\sqrt{|Z_{\beta}(x_1, x_2)|} \Psi\| \leq CN^{-1/2} \left(1 + \sqrt{\alpha(\Psi, \varphi)} + \|\nabla\varphi\| + \|\varphi\|_{\infty}\right) \quad (74)$$

$$\|p_1 Z_{\beta}(x_1, x_2) \Psi\| \leq CN^{-1} \|\varphi\|_{\infty} \left(1 + \sqrt{\alpha(\Psi, \varphi)} + \|\nabla\varphi\| + \|\varphi\|_{\infty}\right) \quad (75)$$

(73) follows from Lemma 4.4 (a) together with Lemma 3.2 (e):

$$\begin{aligned} \|\sqrt{|Z_{\beta}(x_1, x_2)|} p_1\|_{op}^2 &= \|p_1 Z_{\beta}(x_1, x_2) p_1\|_{op} \\ &\leq \|p_1 V_{\beta}(x_1 - x_2) p_1\|_{op} + \frac{2a}{N-1} \|p_1(|\varphi(x_1)|^2 + |\varphi(x_2)|^2) p_1\|_{op} \\ &\leq CN^{-1} \|\varphi\|_{\infty}^2. \end{aligned}$$

(74) follows from Lemma 4.5 together with Lemma 4.4 (a)

$$\begin{aligned} \|\sqrt{|Z_{\beta}(x_1, x_2)|} \Psi\|^2 &\leq \langle \Psi, (V_{\beta}(x_1 - x_2) + \frac{2a}{N-1} |\varphi(x_1)|^2 + \frac{2a}{N-1} |\varphi(x_2)|^2) \Psi \rangle \\ &\leq N^{-1} \alpha(\Psi, \varphi) + CN^{-1} (1 + \|\nabla\varphi\|^2 + \|\varphi\|_{\infty}^2) \end{aligned}$$

and (75) is a direct consequence of (73) and (74).

It follows that

$$\begin{aligned} |\xi_{j,k}| &\leq \|p_1 Z_{\beta}(x_1, x_2) \Psi\| \|\widehat{n}_1^{-1}\|_{op} \|\widehat{n}_1 S_{j,k}\|_{\{1,2\}} \\ &\leq CN^{1/2-(j+k)/8} \|\varphi\|_{\infty}^{j+k+1} \left(1 + \sqrt{\alpha(\Psi, \varphi)} + \|\nabla\varphi\| + \|\varphi\|_{\infty}\right). \end{aligned}$$

□

Definition 5.8 *For any $j, k > 0$ let the functional $\gamma'_{j,k} : L^2(\mathbb{R}^{3N}, \mathbb{C}) \otimes L^2(\mathbb{R}^3, \mathbb{C}) \rightarrow \mathbb{R}^+$ be given by*

$$\gamma'_{j,k} := -2\mathfrak{S} \left(\sum_{j+k=l} \gamma_{j,k}^a + \gamma_{j,k}^b + \gamma_{j,k}^c + \frac{1}{4} \gamma_{j,k}^d + \gamma_{j,k}^e + \gamma_{j,k}^f - \xi_{j-1,k} - \frac{1}{2} \xi_{j,k} \right).$$

where the different summands are

(a) *The mixed derivative term*

$$\begin{aligned}\gamma_{j,k}^a(\Psi, \varphi) &:= j \langle \Psi, q_1 q_2 [H, g(x_1 - x_2)] p_1 p_2 S_{j-1,k} \Psi \rangle \\ &\quad + j \langle \Psi, q_1 q_2 ((W_{1/4} - V_\beta) f_{1/4,\beta}(x_1 - x_2) p_1 p_2 S_{j-1,k} \Psi) \rangle.\end{aligned}$$

(b) *The smoothed out interaction term*

$$\begin{aligned}\gamma_{j,k}^b(\Psi, \varphi) &:= -j \langle \Psi, q_1 q_2 ((W_{1/4} - V_\beta) f_{1/4,\beta}(x_1 - x_2) p_1 p_2 S_{j-1,k} \Psi) \rangle \\ &\quad - \xi_{j-1,k}(\Psi, \varphi) + j \langle \Psi, [Z_\beta(x_5, x_6), R_{j,k}] \Psi \rangle\end{aligned}$$

(c) *Three particle interactions*

$$\gamma_{j,k}^c(\Psi, \varphi) := (N - 2j - 2k) j \langle \Psi, [Z_\beta(x_1, x_5), R_{j,k}] \Psi \rangle$$

(d) *Interaction terms of the correction first type*

$$\begin{aligned}\gamma_{j,k}^d(\Psi, \varphi) &:= (N - 2j - 2k)(N - 2j - 2k - 1) \langle \Psi, [Z_\beta(x_1, x_2), R_{j,k}] \Psi \rangle \\ &\quad - \xi_{j,k}(\Psi, \varphi) + \xi_{k,j}^*(\Psi, \varphi)\end{aligned}$$

(e) *Interaction terms of the correction second type*

$$\gamma_{j,k}^e(\Psi, \varphi) = j(j-1) \langle \Psi, [Z_\beta(x_5, x_7), R_{j,k}] \Psi \rangle$$

(f) *Interaction terms of the correction third type*

$$\gamma_{j,k}^f(\Psi, \varphi) = jk \Im (\langle \Psi, [Z_\beta(x_5, x_{2j+5}), R_{j,k}] \Psi \rangle)$$

Lemma 5.9 *For all $0 \leq l \leq 5$*

$$\sum_{j+k=l} \frac{d}{dt} \gamma_{j,k}(\Psi_t, \varphi_t) = \sum_{j+k=l} \gamma'_{j,k}(\Psi_t, \varphi_t).$$

Proof: First note, that the $R_{j,k}$ are time dependent, since the operators \widehat{m}^j depend on φ_t . The time derivative of Q_j is

$$\dot{Q}_j = -i[H^{GP}, Q_j] + iq_{2j+3}q_{2j+4}[H^{GP}, g_{1/4,\beta}(x_{2j+3} - x_{2j+4})]p_{2j+3}p_{2j+4}$$

thus by symmetry

$$\begin{aligned}\langle \Psi_t, (\dot{R}_{j,k}) \Psi_t \rangle &= -i \langle \Psi_t, [H^{GP}, R_{j,k}] \Psi_t \rangle \\ &\quad + ij \langle \Psi_t, q_1 q_2 [H^{GP}, g_{1/4,\beta}(x_1 - x_2)] p_1 p_2 S_{j-1,k} \Psi_t \rangle \\ &\quad + ik \langle \Psi_t, S_{j,k-1} p_1 p_2 [H^{GP}, g_{1/4,\beta}(x_1 - x_2)] q_1 q_2 \Psi_t \rangle.\end{aligned}$$

Note that after exchanging some variables the adjoint of $S_{j,k}$ equals $S_{k,j}$. Using symmetry and changing the label $k \rightarrow j$ in the last line

$$\begin{aligned} \hat{\gamma}_{j,k}(\Psi_t, \varphi_t) = & i \langle \Psi_t, [H - H^{GP}, R_{j,k}] \Psi_t \rangle \\ & + ij \langle \Psi_t, q_1 q_2 [H^{GP}, g_{1/4, \beta}(x_1 - x_2)] p_1 p_2 S_{j-1, k} \Psi_t \rangle \\ & - ij \langle \Psi_t, q_1 q_2 [H^{GP}, g_{1/4, \beta}(x_1 - x_2)] p_1 p_2 S_{j-1, k} \Psi_t \rangle^* . \end{aligned}$$

So the Lemma follows once we have shown that

$$\begin{aligned} \sum_{j+k=l} \gamma'_{j,k}(\Psi, \varphi) = & \sum_{j+k=l} i \langle \Psi, [\sum_{1 \leq l < m \leq N} Z_\beta(x_l, x_m), R_{j,k}] \Psi \rangle \\ & - 2j \Im (\langle \Psi, q_1 q_2 [H^{GP}, g_{1/4, \beta}(x_1 - x_2)] p_1 p_2 S_{j-1, k} \Psi \rangle) . \end{aligned} \quad (76)$$

As above we want to get rid of the sum $1 \leq l < m \leq N$ using that many summands are equal because of symmetry. $R_{j,k} = \frac{N!}{(N-2j-2k)!} A_j \hat{m}^{j+k} B_{j,k}$ breaks some of the symmetry but it is still symmetric in exchanging any two variables with indices in $\mathcal{M}_a = \{5, 6, \dots, 2j+4\}$ as well as in exchanging any two variables with indices in $\mathcal{M}_b = \{2j+5, 2j+6, \dots, 2j+2k+4\}$ and in exchanging any two variables with indices in $\mathcal{M}_c = \{1, 2, 3, 4, 2j+2k+5, 2j+2k+6, \dots, N\}$.

We arrive at three different cases for the variable x_l : $x_l \in \mathcal{M}_a$, $x_l \in \mathcal{M}_b$ and $x_l \in \mathcal{M}_c$. For the case $x_l \in \mathcal{M}_c$ we arrive at three different cases for the variable x_m . For the case $x_l \in \mathcal{M}_a$ more symmetry is broken via the factor $q_l q_{l \pm 1} g_{1/4, \beta}(x_l - x_{l \pm 1}) p_l p_{l \pm 1}$ (+ if l is odd, - if l is even) appearing in $R_{j,k}$ (see definition 5.4). Similar for the case the case $x_l \in \mathcal{M}_b$. Hence we arrive at the following eight different summands:

$$\begin{aligned} & i \langle \Psi, [\sum_{1 \leq l < m \leq N} Z_\beta(x_l, x_m), R_{j,k}] \Psi \rangle \\ = & \frac{i}{2} (N - 2j - 2k)(N - 2j - 2k - 1) \langle \Psi, [Z_\beta(x_1, x_2), R_{j,k}] \Psi \rangle \\ & + i(N - 2j - 2k)j \langle \Psi, [Z_\beta(x_1, x_5), R_{j,k}] \Psi \rangle \\ & + ij \langle \Psi, [Z_\beta(x_5, x_6), R_{j,k}] \Psi \rangle \\ & + i(N - 2j - 2k)k \langle \Psi, [Z_\beta(x_1, x_{2j+5}), R_{j,k}] \Psi \rangle \\ & + ik \langle \Psi, [Z_\beta(x_{2j+5}, x_{2j+6}), R_{j,k}] \Psi \rangle \\ & + ij(2j - 1) \langle \Psi, [Z_\beta(x_5, x_7), R_{j,k}] \Psi \rangle \\ & + ij2k \langle \Psi, [Z_\beta(x_5, x_{2j+5}), R_{j,k}] \Psi \rangle \\ & + ik(2k - 1) \langle \Psi, [Z_\beta(x_{2j+5}, x_{2j+7}), R_{j,k}] \Psi \rangle . \end{aligned}$$

Using that after exchanging some variables the adjoint of $R_{j,k}$ equals $R_{k,j}$

$$\begin{aligned}
& i \langle \langle \Psi, [\sum_{1 \leq l < m \leq N} Z_\beta(x_l, x_m), R_{j,k}] \Psi \rangle \rangle \\
&= \frac{i}{4} (N - 2j - 2k)(N - 2j - 2k - 1) \langle \langle \Psi, [Z_\beta(x_1, x_2), R_{j,k}] \Psi \rangle \rangle \\
&\quad - \frac{i}{4} (N - 2j - 2k)(N - 2j - 2k - 1) \langle \langle \Psi, [Z_\beta(x_1, x_2), R_{k,j}] \Psi \rangle \rangle^* \\
&\quad + i(N - 2j - 2k)j \langle \langle \Psi, [Z_\beta(x_1, x_5), R_{j,k}] \Psi \rangle \rangle \\
&\quad + ij \langle \langle \Psi, [Z_\beta(x_5, x_6), R_{j,k}] \Psi \rangle \rangle \\
&\quad - i(N - 2j - 2k)k \langle \langle \Psi, [Z_\beta(x_1, x_5), R_{k,j}] \Psi \rangle \rangle^* \\
&\quad - ik \langle \langle \Psi, [Z_\beta(x_5, x_6), R_{k,j}] \Psi \rangle \rangle^* \\
&\quad + ij(2j - 1) \langle \langle \Psi, [Z_\beta(x_5, x_7), R_{j,k}] \Psi \rangle \rangle \\
&\quad + ijk \langle \langle \Psi, [Z_\beta(x_5, x_{2j+5}), R_{j,k}] \Psi \rangle \rangle \\
&\quad - ijk \langle \langle \Psi, [Z_\beta(x_{2j+5}, x_5), R_{k,j}] \Psi \rangle \rangle^* \\
&\quad - ik(2k - 1) \langle \langle \Psi, [Z_\beta(x_5, x_7), R_{k,j}] \Psi \rangle \rangle^*
\end{aligned}$$

It follows that $\sum_{j+k=l} \gamma'_{j,k}(\Psi, \varphi)$ equals

$$\begin{aligned}
&= -\frac{1}{2} \sum_{j+k=l} (N - 2j - 2k)(N - 2j - 2k - 1) \Im(\langle \langle \Psi, [Z_\beta(x_1, x_2), R_{j,k}] \Psi \rangle \rangle) \\
&\quad - 2 \sum_{j+k=l} (N - 2j - 2k)j \Im(\langle \langle \Psi, [Z_\beta(x_1, x_5), R_{j,k}] \Psi \rangle \rangle) \\
&\quad - 2 \sum_{j+k=l} j \Im(\langle \langle \Psi, [Z_\beta(x_5, x_6), R_{j,k}] \Psi \rangle \rangle) \\
&\quad - 2 \sum_{j+k=l} j(2j - 1) \Im(\langle \langle \Psi, [Z_\beta(x_5, x_7), R_{j,k}] \Psi \rangle \rangle) \\
&\quad - 2 \sum_{j+k=l} jk \Im(\langle \langle \Psi, [Z_\beta(x_5, x_{2j+5}), R_{j,k}] \Psi \rangle \rangle) \\
&\quad - 2 \sum_{j+k=l} j \Im(\langle \langle \Psi, q_1 q_2 [H, g_{1/4, \beta}(x_1 - x_2)] p_1 p_2 S_{j-1, k} \Psi \rangle \rangle) .
\end{aligned}$$

Adding

$$-2 \sum_{j+k=l} j \Im(\langle \langle \Psi, q_1 q_2 ((W_{1/4} - V_\beta) f_{1/4, \beta}(x_1 - x_2) p_1 p_2 S_{j-1, k} \Psi) \rangle \rangle)$$

to the last line and subtracting it from the third line, as well as subtracting $-2 \sum_{j+k=l} \Im(\xi_{j-1, k})$ from the third line and adding it to the total and subtracting $-2 \sum_{j+k=l} \Im(\xi_{j, k})/2 = -2 \sum_{j+k=l} \Im((\xi_{j, k} - \xi_{k, j}^*)/4)$ from the first line and adding it to the total gives that the right hand side of (76) equals

$$-2 \Im \left(\sum_{j+k=l} \frac{1}{4} \gamma_{j, k}^d + \gamma_{j, k}^c + \gamma_{j, k}^b + \gamma_{j, k}^e + \gamma_{j, k}^f + \gamma_{j, k}^a + \xi_{j-1, k} + \frac{1}{2} \xi_{j, k} \right)$$

which proves the Lemma. □

Having proven that the functionals γ' can be understood as the time derivative of the functionals γ our next step is to control the functionals γ . To start with $\gamma_{0,0}$.

Corollary 5.10 *Let $\beta < 1$. Then there exists a $\mathcal{K} \in \mathcal{F}$ and a $\eta > 0$ such that for $\xi := \gamma'_{0,0} - \Im(\xi_{0,0})$*

$$\xi(\Psi, \varphi) \leq \mathcal{K}(\varphi)(\|\varphi\|_\infty + (\ln N)^{1/3} \|\nabla\varphi\|_{6,loc})(\langle \Psi, \widehat{n}\Psi \rangle + \|\nabla_1 q_1 \Psi\|^2 + N^{-\eta}).$$

Proof: By Definition 5.8 one sees, that for $\gamma'_{0,0} - \Im(\xi_{0,0})$ only (d) remains. And here the problematic term in fact cancels out In view of Lemma 3.2 (d) and Lemma 5.3 (a)

$$\begin{aligned} \xi(\Psi, \varphi) &= N(N-1) (\langle \Psi, [Z_\beta(x_1, x_2), \widehat{m}^0] \Psi \rangle - \langle \Psi, [Z_\beta(x_1, x_2), p_1 p_2 \widehat{m}^1] \Psi \rangle) \\ &= 2N(N-1) \langle \Psi, [Z_\beta(x_1, x_2), p_1 q_2 (\widehat{m}^0 - \widehat{m}_1^0)] \Psi \rangle \\ &= 2N(N-1) \Im (\langle \Psi, p_1 p_2 Z_\beta(x_1, x_2) p_1 q_2 (\widehat{m}^0 - \widehat{m}_1^0) \Psi \rangle) \\ &\quad + 2N(N-1) \Im (\langle \Psi, q_1 q_2 Z_\beta(x_1, x_2) p_1 q_2 (\widehat{m}^0 - \widehat{m}_1^0) \Psi \rangle). \end{aligned} \tag{77}$$

Since $\beta < 1$ this is controlled by Lemma 4.4 (b) and (d) using the bounds from Lemma 5.3 (c). □

Lemma 5.11 *For any $1/3 \leq \beta \leq 1$, $l > 0$ there exists a $\mathcal{K} \in \mathcal{F}$, $\eta > 0$ such that*

$$\begin{aligned} \sum_{j+k=l} \gamma'_{j,k}(\Psi, \varphi) + 2\Im(\xi_{j-1,k}(\Psi, \varphi)) + \Im(\xi_{j,k}(\Psi, \varphi)) \\ \leq (\|\varphi\|_\infty + (\ln N)^{1/3} \|\nabla\varphi\|_{6,loc}) \mathcal{K}(\varphi) (\alpha(\Psi, \varphi) + N^{-\eta}). \end{aligned}$$

Proof:

To prove the Lemma we shall estimate the imaginary parts of $\gamma_{j,k}^a, \gamma_{j,k}^b, \dots, \gamma_{j,k}^f$ separately.

(a) The commutator in $\gamma_{j,k}^a$ equals

$$\begin{aligned} [H, g_{1/4,\beta}(x_1 - x_2)] &= -[H, f_{1/4,\beta}(x_1 - x_2)] \\ &= [\Delta_1 + \Delta_2, f_{1/4,\beta}(x_1 - x_2)] \\ &= (\Delta_1 + \Delta_2) f_{1/4,\beta}(x_1 - x_2) \\ &\quad + (\nabla_1 f_{1/4,\beta}(x_1 - x_2)) \nabla_1 - (\nabla_2 f_{1/4,\beta}(x_1 - x_2)) \nabla_2 \\ &= -(W_{1/4} - V_\beta) f_{1/4,\beta}(x_1 - x_2) \\ &\quad + (\nabla_1 g_{1/4,\beta}(x_1 - x_2)) \nabla_1 - (\nabla_2 g_{1/4,\beta}(x_1 - x_2)) \nabla_2. \end{aligned} \tag{78}$$

This and integration by parts gives

$$\begin{aligned}
|\gamma_{j,k}^a(\Psi, \varphi)| &\leq 2j |\langle \Psi, q_1 q_2 (\nabla_1 g_{1/4, \beta}(x_1 - x_2)) \nabla_1 p_1 p_2 S_{j-1, k} \Psi \rangle| \\
&\leq 2j |\langle \nabla_1 q_1 q_2 \widehat{n}_1^{-1} \Psi, (g_{1/4, \beta}(x_1 - x_2)) \nabla_1 p_1 p_2 \widehat{n}_3 S_{j-1, k} \Psi \rangle| \\
&\quad + 2j |\langle \Psi, \widehat{n}_1^{-1} q_1 q_2 (g_{1/4, \beta}(x_1 - x_2)) \Delta_1 p_1 p_2 \widehat{n}_3 S_{j-1, k} \Psi \rangle| \\
&\leq 2j \|\nabla_1 q_1 q_2 \widehat{n}_1^{-1} \Psi\| \|\nabla \varphi\| \|g_{1/4, \beta}(x_1 - x_2)\| p_2 \|_{op} \|\widehat{n}_3 S_{j-1, k}\|_{\{1, 2\}} \\
&\quad + 2j \|\widehat{n}_1^{-1} q_1 \Psi\| \|\Delta \varphi\| \|g_{1/4, \beta}(x_1 - x_2)\| p_2 \|_{op} \|q_2 S_{j-1, k}\|_{\{1, 2\}}.
\end{aligned}$$

Using (42), Lemma 3.2 (e) and Lemma 3.4 the latter is bounded by

$$CN^{-(j+k)/8} \|\varphi\|_\infty^{j+k} (\|\nabla \varphi\| + \|\Delta \varphi\|).$$

(b) Using symmetry it follows that

$$\langle \Psi, Z_\beta(x_5, x_6) R_{j, k} \Psi \rangle = \langle \Psi, Z_\beta(x_1, x_2) q_1 q_2 g_{1/4, \beta}(x_1 - x_2) p_1 p_2 S_{j-1, k} \Psi \rangle$$

It follows that

$$\begin{aligned}
\gamma_{j,k}^b(\Psi, \varphi) &= -j \langle \Psi, q_1 q_2 ((W_{1/4} - V_\beta) f_{1/4, \beta}(x_1 - x_2) p_1 p_2 S_{j-1, k} \Psi) \rangle \\
&\quad - j \langle \Psi, Z_\beta(x_1, x_2) p_1 p_2 S_{j-1, k} \Psi \rangle \\
&\quad + j \langle \Psi, Z_\beta(x_1, x_2) q_1 q_2 g_{1/4, \beta}(x_1 - x_2) p_1 p_2 S_{j-1, k} \Psi \rangle \\
&\quad - j \langle \Psi, R_{j, k} Z_\beta(x_5, x_6) \Psi \rangle \\
&= -j \langle \Psi, q_1 q_2 ((W_{1/4} - V_\beta) f_{1/4, \beta}(x_1 - x_2) p_1 p_2 S_{j-1, k} \Psi) \rangle \\
&\quad - j \langle \Psi, Z_\beta(x_1, x_2) f_{1/4, \beta}(x_1 - x_2) p_1 p_2 S_{j-1, k} \Psi \rangle \\
&\quad - j \langle \Psi, Z_\beta(x_1, x_2) (1 - q_1 q_2) g_{1/4, \beta}(x_1 - x_2) p_1 p_2 S_{j-1, k} \Psi \rangle \\
&\quad - j \langle \Psi, R_{j, k} Z_\beta(x_5, x_6) \Psi \rangle
\end{aligned}$$

and thus

$$\begin{aligned}
&|\sum_{j+k=l} \Im(\gamma_{j,k}^b(\Psi, \varphi))| \\
&\leq \sum_{j+k=l} j |\langle \Psi, q_1 q_2 (W_{1/4}(x_1 - x_2) - \frac{4a}{N-1} |\varphi(x_1)^2|) \\
&\quad f_{1/4, \beta}(x_1 - x_2) p_1 p_2 S_{j-1, k} \Psi \rangle| \tag{79}
\end{aligned}$$

$$+ |\sum_{j+k=l} j \Im \langle \Psi, p_1 p_2 Z_\beta(x_1, x_2) f_{1/4, \beta}(x_1 - x_2) p_1 p_2 S_{j-1, k} \Psi \rangle| \tag{80}$$

$$+ \sum_{j+k=l} 2j |\langle \Psi, p_1 q_2 Z_\beta(x_1, x_2) f_{1/4, \beta}(x_1 - x_2) p_1 p_2 S_{j-1, k} \Psi \rangle| \tag{81}$$

$$+ \sum_{j+k=l} j |\langle \Psi, Z_\beta(x_1, x_2) (1 - q_1 q_2) g_{1/4, \beta}(x_1 - x_2) p_1 p_2 S_{j-1, k} \Psi \rangle| \tag{82}$$

$$+ \sum_{j+k=l} j |\langle \Psi, R_{j, k} Z_\beta(x_5, x_6) \Psi \rangle|. \tag{83}$$

If $(j, k) = (1, 0)$ we get in view of Lemma 4.4 (c) (recall than $Nm^0 \leq Cn^{-1}$) that there exists a $\mathcal{K} \in \mathcal{F}$

$$\begin{aligned}
|(79)| &\leq \sum_{j+k=l} jN^2 |\langle \Psi, q_1 q_2 W_{1/4} f_{1/4, \beta}(x_1 - x_2) p_1 p_2 \widehat{m}^0 \Psi \rangle| \\
&\quad + \sum_{j+k=l} jN^2 |\langle \Psi, q_1 q_2 \frac{4a}{N-1} |\varphi(x_1)|^2 g_{1/4, \beta}(x_1 - x_2) p_1 p_2 \widehat{m}^0 \Psi \rangle| \\
&\leq \mathcal{K}(\varphi) \|\varphi\|_\infty (\langle \Psi, \widehat{n} \Psi \rangle + CN^{-\eta}) \|\varphi\|_\infty^3.
\end{aligned}$$

For $(j+k) > 1$ we shall use Lemma 4.4 (c) to control (79). Note, that although for $\chi := S_{j-1, k} \Psi$ some symmetry is broken, still $\chi \in \mathcal{H}_{\mathcal{M}}$ for $\mathcal{M} = \{5, 6, \dots, 2j+2k+4\}$. We write

$$\begin{aligned}
|(79)| &= \sum_{j+k=l} j |\langle \Psi, \widehat{n}_1^{-1} q_1 q_2 (W_{1/4}(x_1 - x_2) - \frac{4a}{N-1} |\varphi(x_1)|^2) \\
&\quad f_{1/4, \beta}(x_1 - x_2) \widehat{n}_3 p_1 p_2 S_{j-1, k} \Psi \rangle| \\
&\leq \sum_{j+k=l} j \|\widehat{n}_1^{-1} q_1 q_2 W_{1/4} f_{1/4, \beta}(x_1 - x_2) p_1 p_2\|_{\mathcal{M}} \|\widehat{n}_3 S_{j-1, k}\|_{\{1, 2\}} \\
&\quad + \sum_{j+k=l} \frac{2aj}{N-1} \|\widehat{n}_1^{-1} q_1 \Psi\| \|\varphi\|_\infty^2 \|\widehat{n}_3 S_{j-1, k}\|_{\{1, 2\}}.
\end{aligned}$$

Due to Lemma Lemma 5.2 $W_{1/4} f_{1/4, \beta} \in \mathcal{V}_{1/4}$. Thus Lemma 4.4 (c) and Lemma 5.5 imply that |(79)| has the right bound.

Note that $S_{j, k}$ becomes after exchanging some of the variables the adjoint of $S_{k, j}$ selfadjoint, thus (80) = 0 for all l .

For (81) we use Lemma 4.4 (b) and Lemma 5.5

$$\begin{aligned}
|(81)| &\leq \sum_{j+k=l} 2j |\langle \Psi, \widehat{n}_1^{-1} p_1 q_2 (V_\beta f_{1/4, \beta}(x_1 - x_2) - \frac{2a}{N-1} |\varphi(x_1)|^2 \\
&\quad - \frac{2a}{N-1} |\varphi(x_2)|^2) p_1 p_2 \widehat{n}_3 S_{j-1, k} \Psi \rangle| \\
&\quad + 2 |\langle \Psi, \widehat{n}_1^{-1} p_1 q_2 g_{1/4, \beta}(x_1 - x_2) (\frac{2a}{N-1} |\varphi(x_1)|^2 \\
&\quad + \frac{2a}{N-1} |\varphi(x_2)|^2) p_1 p_2 \widehat{n}_3 S_{j-1, k} \Psi \rangle|. \\
&\leq C \|\widehat{n}_1^{-1} p_1 q_2 (V_\beta f_{1/4, \beta}(x_1 - x_2) - \frac{2a}{N-1} |\varphi(x_2)|^2) p_1 p_2\|_{\mathcal{M}} \|\widehat{n}_3 S_{j-1, k}\|_{\{1, 2\}} \\
&\quad + C \|\widehat{n}_1^{-1} q_2 \Psi\| \|p_1 g_{1/4, \beta}(x_1 - x_2)\|_{op} N^{-1} \|\varphi\|_\infty^2 \|\widehat{n}_3 S_{j-1, k}\|_{\{1, 2\}} \\
&\leq CN^{-(j+k)/8} \|\varphi\|_\infty^{1+j+k} (\|\varphi\|_\infty + 1).
\end{aligned}$$

For (82) note, that $(1 - q_1 q_2) = p_1 p_2 + p_1 q_2 + q_1 p_2$. Since all factors in (82) are symmetric in exchanging x_1 with x_2 it is sufficient to control

$$\langle \Psi, Z_\beta(x_1, x_2) p_1 r_2 g_{1/4, \beta}(x_1 - x_2) p_1 p_2 S_{j-1, k} \Psi \rangle \quad (84)$$

for $r_2 \in \{p_2, q_2\}$ to get good control of |(82)|. Using (75) line (84) is bounded by

$$\begin{aligned} & \|p_1 Z_\beta(x_1, x_2) \Psi\| \|p_1 g_{1/4, \beta}(x_1 - x_2) p_1\|_{op} \|\widehat{n}_1^{-1}\|_{op} \|\widehat{n}_1 S_{j-1, k}\|_{\{1, 2\}} \\ & \leq C N^{-2-1/2+1/2} N^{1-(j+k)/8} \|\varphi\|_\infty^{2+j+k} \leq C N^{-1} \|\varphi\|_\infty^{2+j+k} . \end{aligned}$$

For (83) we use that $p_j = p_j^2$, thus

$$(83) = \sum_{j+k=l} j \langle \Psi, R_{j, k} p_5 p_6 Z_\beta(x_5 - x_6) \Psi \rangle$$

With Lemma 5.5 it follows that

$$\begin{aligned} |(83)| & \leq \sum_{j+k=l} \|R_{j, k}\|_{\{5, 6\}} \|p_1 Z_\beta(x_1, x_2) \Psi\| \\ & \leq C N^{-1/2+1-(j+k)/8} \|\varphi\|_\infty^{j+k} N^{-1} \|\varphi\|_\infty \\ & \leq C \|\varphi\|_\infty^{j+k+1} N^{-1/2} . \end{aligned}$$

(c) Using $p_5^2 = p_5$ we have $R_{j, k} = R_{j, k} p_5$, thus

$$|\gamma_{j, k}^c(\Psi, \varphi)| \leq N j |\langle \Psi, R_{j, k} p_5 Z_\beta(x_1, x_5) \Psi \rangle| \quad (85)$$

$$+ N j |\langle \Psi, Z_\beta(x_1, x_5) R_{j, k} \Psi \rangle| . \quad (86)$$

For (85) we use Lemma 5.5 and (75)

$$\begin{aligned} |(85)| & \leq N j \|R_{j, k}\|_{\{1, 5\}} \|p_5 V_\beta(x_1 - x_5) \Psi\| \\ & \leq C \|\varphi\|_\infty^{j+k+1} N^{-(j+k)/8} . \end{aligned}$$

For (86) we write using $q_3 = p_1 q_3 + q_1 q_3 = p_1 q_3 + q_1 - q_1 p_3$

$$(86) = N |\langle \Psi, Z_\beta(x_1, x_3) q_3 q_4 g_{1/4, \beta}(x_3 - x_4) p_3 p_4 S_{j-1, k} \Psi \rangle|$$

$$\leq N |\langle \Psi, Z_\beta(x_1, x_3) p_1 \widehat{n}_1^{-1} q_3 q_4 g_{1/4, \beta}(x_3 - x_4) p_3 p_4 \widehat{n}_3 S_{j-1, k} \Psi \rangle| \quad (87)$$

$$+ N |\langle \Psi, Z_\beta(x_1, x_3) q_1 q_4 g_{1/4, \beta}(x_3 - x_4) p_3 p_4 S_{j-1, k} \Psi \rangle| \quad (88)$$

$$+ N |\langle \Psi, Z_\beta(x_1, x_3) q_1 p_3 q_4 g_{1/4, \beta}(x_3 - x_4) p_3 p_4 S_{j-1, k} \Psi \rangle| . \quad (89)$$

For (87) note that $p_1 Z_\beta(x_1, x_3) \Psi \in \mathcal{H}_{\{1, 3\}}$, so with Lemma 3.2 (b)

$$\|\widehat{n}_1^{-1} q_4 p_1 Z_\beta(x_1 - x_3) \Psi\| \leq C \|p_1 Z_\beta(x_1 - x_3) \Psi\| \leq C N^{-1} \|\varphi\|_\infty .$$

Using also Lemma 5.5 we get

$$(87) \leq C \|\varphi\|_\infty \|g_{1/4,\beta}(x_3 - x_4)p_3\|_{op} \|\widehat{n}_3 S_{j-1,k}\|_{\{1,3,4\}} \\ \leq C \|\varphi\|_\infty^{j+k+1} N^{-(j+k)/8} .$$

For (88) we use Lemma 5.5

$$|(88)| \leq N \|\sqrt{|Z_\beta(x_1, x_3)|} \Psi\| \\ q_4 g_{1/4,\beta}(x_3 - x_4) p_4 \sqrt{|Z_\beta(x_1, x_3)|} p_3 S_{j-1,k} \widehat{n}_1 \widehat{n}_1^{-1} q_1 \Psi \| \\ \leq C N \|\sqrt{|Z_\beta(x_1, x_3)|} \Psi\| \|g_{1/4,\beta}(x_3 - x_4) p_4\|_{op} \\ \|\sqrt{|Z_\beta(x_1, x_3)|} p_3\|_{op} \|\widehat{n}_1 S_{j-1,k}\|_{\{1,3,4\}} \|\widehat{n}_1^{-1} q_1 \Psi\| \\ = C N^{-(j+k)/8} \|\varphi\|_\infty^{j+k+1} .$$

Again using Lemma 5.5 we get for the last term in (c)

$$(89) \leq N \|p_3 Z_\beta(x_1, x_3) \Psi\| \|g_{1/4,\beta}(x_3 - x_4) p_4\|_{op} \\ \|\widehat{n}_1 S_{j-1,k}\|_{\{1,3,4\}} \|\widehat{n}_1^{-1} q_1 \Psi\| \\ \leq C \|\varphi\|_\infty^{j+k+1} N^{-(j+k)/8} .$$

(d) Using Lemma 3.2 (d) we get that

$$(N - 2j - 2k)(N - 2j - 2k - 1) [Z_\beta(x_1, x_2), R_{j,k}] \\ = [Z_\beta(x_1, x_2), p_1 p_2 S_{j,k}] + [Z_\beta(x_1, x_2), p_1 q_2 T_{j,k}] \\ + [Z_\beta(x_1, x_2), q_1 p_2 T_{j,k}] .$$

Hence

$$\gamma_{j,k}^d(\Psi, \varphi) = -2 \langle \Psi, [Z_\beta(x_1, x_2), p_1 q_2 T_{j,k}], \Psi \rangle$$

thus

$$|\gamma_{j,k}^d(\Psi, \varphi)| \leq 2 \langle \Psi, Z_\beta(x_1, x_2) p_1 T_{j,k} \widehat{n}_1 \widehat{n}_1^{-1} q_2 \Psi \rangle \\ + 2 \langle \Psi, q_2 \widehat{n}_1^{-1} \widehat{n}_1 T_{j,k} p_1 Z_\beta(x_1, x_2) \Psi \rangle$$

Lemma 5.5 gives

$$|\gamma_{j,k}^d(\Psi, \varphi)| \leq C \|\widehat{n}_1 T_{j,k}\|_{\{1,2\}} \|p_1 Z_\beta(x_1, x_2) \Psi\| \|\widehat{n}_1^{-1} q_2 \Psi\| \\ \leq C \|\varphi\|_\infty^{j+k+1} N^{1-(j+k)/8-1} .$$

(e) Again using $R_{j,k} = R_{j,k}p_5$ as well as $q_1 = 1 - p_1$

$$|\gamma_{j,k}^e(\Psi, \varphi)| \leq j(j-1) |\langle \Psi, R_{j,k}p_5 Z_\beta(x_5, x_7)\Psi \rangle| \quad (90)$$

$$+ (j-1) |\langle \Psi, Z_\beta(x_1, x_5)p_1 q_2 g_{1/4,\beta}(x_1 - x_2)p_1 p_2 S_{j-1,k}\Psi \rangle| \quad (91)$$

$$+ (j-1) |\langle \Psi, Z_\beta(x_1, x_5)q_2 g_{1/4,\beta}(x_1 - x_2)p_1 p_2 S_{j-1,k}\Psi \rangle|. \quad (92)$$

Lemma 5.5 gives

$$\begin{aligned} |(90)| &\leq C \|R_{j,k}\|_{\{5,7\}} \|p_5 Z_\beta(x_5, x_7)\Psi\| \\ &\leq C \|\varphi\|_\infty^{j+k+1} N^{1/2-(j+k)/8-1} \end{aligned}$$

as well as

$$\begin{aligned} |(91)| &\leq C \|p_1 Z_\beta(x_1, x_5)\Psi\| \|p_1 g_{1/4,\beta}(x_1 - x_2)p_1\|_{op} \\ &\quad \|\widehat{n}_1 S_{j-1,k}\|_{\{1,2,5\}} \|\widehat{n}_1^{-1}\|_{op} \\ &\leq C \|\varphi\|_\infty^{j+k+2} N^{-1-1-1/2+3/2-(j-1+k)/8+1/2} \end{aligned}$$

and

$$\begin{aligned} |(92)| &= -(j-1) \langle \sqrt{Z_\beta(x_1, x_5)}\Psi, \\ &\quad q_2 g_{1/4,\beta}(x_1 - x_2)p_2 \sqrt{Z_\beta(x_1, x_5)}p_1 S_{j-1,k}\Psi \rangle \\ |(92)| &\leq C \|\sqrt{Z_\beta(x_1, x_5)}\Psi\| \|g_{1/4,\beta}(x_1 - x_2)p_2\|_{op} \\ &\quad \|\sqrt{Z_\beta(x_1, x_5)}p_1\|_{op} \|\widehat{n}_1 S_{j-1,k}\|_{\{1,2,5\}} \|\widehat{n}_1^{-1}\|_{op} \\ &\leq C \|\varphi\|_\infty^{j+k+1} N^{-1/2-1-1/8-1/2+3/2-(j-1+k)/8+1/2}. \end{aligned}$$

(f) Using as above $R_{j,k} = R_{j,k}p_5 = p_{2j+5}R_{j,k}$

$$\begin{aligned} |\gamma_{j,k}^f(\Psi, \varphi)| &\leq jk |\langle \Psi, R_{j,k}p_5 Z_\beta(x_5, x_{2j+5})\Psi \rangle| \\ &\quad + jk |\langle \Psi, Z_\beta(x_5, x_{2j+5})p_{2j+5}R_{j,k}\Psi \rangle| \\ &\leq C \|R_{j,k}\|_{\{5,2j+5\}} \|p_5 Z_\beta(x_5, x_{2j+5})\Psi\| \\ &\leq C \|\varphi\|_\infty^{j+k+1} N^{1/2-(j+k)/8-1}. \end{aligned}$$

□

5.3 Proof of the Theorem for $1/3 \leq \beta < 1$

Summarizing the last section we get the following Corollary, which directly gives the Theorem.

Corollary 5.12 *Let $0 < \beta < 1$. There exists a functional $\Gamma : L^2(\mathbb{R}^{3N}, \mathbb{C}) \otimes L^2(\mathbb{R}^3, \mathbb{C}) \rightarrow \mathbb{R}^+$, a functional $\Gamma' : L^2(\mathbb{R}^{3N}, \mathbb{C}) \otimes L^2(\mathbb{R}^3, \mathbb{C}) \rightarrow \mathbb{R}$ and a $c > 0$ such that*

(a)

$$\left| \frac{d}{dt} \Gamma(\Psi_t, \varphi_t) \right| \leq |\Gamma'(\Psi_t, \varphi_t)| .$$

(b)

$$c\alpha(\Psi, \varphi) - CN^{-\eta} \leq \Gamma(\Psi, \varphi) \leq \alpha(\Psi, \varphi) + CN^{-\eta}$$

uniform in Ψ, φ

(c) There exists a functional $\mathcal{K} \in \mathcal{F}$ such that

$$|\Gamma'(\Psi, \varphi)| \leq (\|\varphi\|_\infty + (\ln N)^{1/3} \|\nabla \varphi\|_{6,loc} + \|\dot{A}\|_\infty) \mathcal{K}(\varphi) (\alpha(\Psi, \varphi) + N^{-\eta})$$

uniform in Ψ, φ .

Proof: Set

$$\Gamma(\Psi, \varphi) := \sum_{j+k \leq 5} 2^{-j-k} \gamma_{j,k}(\Psi, \varphi) + |\mathcal{E}(\Psi) - \mathcal{E}^{GP}(\varphi)| \quad \text{and}$$

$$\Gamma'(\Psi, \varphi) := \sum_{j+k \leq 5} 2^{-j-k} \gamma'_{j,k}(\Psi, \varphi) + \frac{d}{dt} |\mathcal{E}(\Psi) - \mathcal{E}^{GP}(\varphi)| .$$

(a) follows from Lemma 5.9 with (23).

(b)

$$\Gamma(\Psi, \varphi) = \langle \Psi, \hat{m}^0 \Psi \rangle + |\mathcal{E}(\Psi) - \mathcal{E}^{GP}(\varphi)| + \sum_{1 \leq j+k \leq 5} \langle \Psi, R_{j,k} \Psi \rangle + \Gamma(\Psi, \varphi)$$

In view of Lemma 5.3 we have that

$$c_0 \alpha(\Psi, \varphi) \leq \langle \Psi, \hat{m}^0 \Psi \rangle + |\mathcal{E}(\Psi) - \mathcal{E}^{GP}(\varphi)| \leq \alpha(\Psi, \varphi) .$$

The other summands are in view of Lemma 5.5 bounded by $CN^{-\eta}$ and (b) follows.

(c) Recall that $\xi_{-1,k} = 0$, thus

$$\begin{aligned} \Gamma'(\Psi, \varphi) &= \sum_{j+k \leq 5} 2^{-j-k} (\gamma'_{j,k}(\Psi, \varphi) + 2\mathfrak{I}(\xi_{j-1,k}) - \mathfrak{I}(\xi_{j,k})) \\ &\quad + \sum_{j+k=5} 2^{-5} \mathfrak{I}(\xi_{j,k}) + \frac{d}{dt} |\mathcal{E}(\Psi) - \mathcal{E}^{GP}(\varphi)| \end{aligned}$$

The first line is controlled by Lemma 5.11. The second line is bounded by Proposition 5.7 and (23).

□

From (b) and (c) it follows that

$$\Gamma'(\Psi, \varphi) \leq (\|\varphi\|_\infty + (\ln N)^{1/3} \|\nabla \varphi\|_{6,loc} + \|\dot{A}\|_\infty) \mathcal{K}(\varphi) (\Gamma(\Psi, \varphi) + N^{-\eta})$$

and we get via Grönwall

$$\Gamma(\Psi_t, \varphi_t) \leq e^{\int_0^t (\|\varphi_s\|_\infty + (\ln N)^{1/3} \|\nabla \varphi_s\|_{6,loc} + \|\dot{A}_s\|_\infty) \mathcal{K}(\varphi_s) ds} (\Gamma(\Psi_0, \varphi_0) + N^{-\eta}) .$$

For $\varphi \in \mathcal{G}$ we have that $\sup_{s \in \mathbb{R}} \{\mathcal{K}(\varphi_s)\} < \infty$. Again using (b) we get the bound on $\alpha(\Psi_t, \varphi_t)$ as stated in Theorem 2.5.

6 Generalizing to $\beta = 1$

Lemma 5.11 holds for $\beta = 1$. When generalizing the Theorem to $\beta = 1$ below one “only” has to adjust Γ in Corollary 5.12 such that the ξ in Corollary 5.10 becomes controllable.

Recall that (c.f. Corollary 5.10)

$$\xi = -2N(N-1) \Im \left(\langle \langle \Psi, Z_\beta(x_1, x_2) p_1 q_2 (\hat{m}^0 - \hat{m}_1^0) \Psi \rangle \rangle \right) .$$

The method we use is similar as above: We add a functional γ to Γ such that the interaction term in ξ is smoothed out by the cost of additional terms. It turns out that all the additional terms are controllable so in contrast to section 5.2 this first adjustment will be sufficient.

The “new” interaction term will scale moderately and can — using Lemma 4.4 (d) — be controlled in terms of $\|q_1 \nabla_1 \Psi\|$. But we only got good control of $\|\nabla_1 q_1 \Psi\|$ for $\beta < 1$. Hence we have to generalize our estimates on $\|\nabla_1 q_1 \Psi\|$ to $\beta = 1$ first.

6.1 Controlling $\|\nabla_1 q_1 \Psi\|$ for $\beta = 1$

For $\beta = 1$ a relevant part of the kinetic energy is used to form the microscopic structure. That part of the kinetic energy is concentrated around the scattering centers. Hence $\|\nabla_1 q_1 \Psi\|$ will in fact *not* be small.

The microscopic structure is — neglecting three particle interactions — given by Lemma 5.2. So we shall first cutoff three particle interactions, i.e. we define a cutoff function which does not depend on x_1 and cuts off all parts of the wave function where two particles x_j, x_k with $j \neq k, j, k \neq 1$ come to close (see \mathcal{B}_1 in Definition 6.1).

After that we shall cutoff that part of the kinetic energy which is used to form the microscopic structure. The latter is concentrated around the scattering centers (i.e. on the sets $\bar{\mathcal{A}}_j$ given by Definition 6.1).

Then we show that $\|\mathbf{1}_{\bar{\mathcal{A}}_1} \nabla_1 q_1 \Psi\|$ is small (see Lemma 6.4 below).

Having good control on $\|\mathbf{1}_{\bar{\mathcal{A}}_1} \nabla_1 q_1 \Psi\|$ instead of $\|\nabla_1 q_1 \Psi\|$ Lemma 4.4 (d) has of course to be changed appropriately. This will be done in Lemma 6.5.

Definition 6.1 For any $j, k = \mathbb{N}$ let

$$a_{j,k} := \{(x_1, x_2, \dots, x_N) \in \mathbb{R}^{3N} : |x_j - x_k| < N^{-26/27}\} \quad (93)$$

$$\overline{\mathcal{A}}_j := \bigcup_{k \neq j} a_{j,k} \quad \mathcal{A}_j := \mathbb{R}^{3N} \setminus \overline{\mathcal{A}}_j \quad \overline{\mathcal{B}}_j := \bigcup_{k, l \neq j} a_{k,l} \quad \mathcal{B}_j := \mathbb{R}^{3N} \setminus \overline{\mathcal{B}}_j .$$

Proposition 6.2 (a)

$$\|\mathbb{1}_{\overline{\mathcal{A}}_j} p_1\|_{op} \leq C \|\varphi\|_\infty N^{-17/18} ,$$

(b)

$$\|\mathbb{1}_{\overline{\mathcal{A}}_j} \Psi\| \leq CN^{-17/27} \|\nabla_j \Psi\| ,$$

(c)

$$\|\mathbb{1}_{\overline{\mathcal{B}}_j} \Psi\| \leq CN^{-7/54} \|\nabla_j \Psi\| .$$

Proof:

(a)

$$\begin{aligned} \|\mathbb{1}_{\overline{\mathcal{A}}_j} p_1\|_{op} &\leq \|\varphi\|_\infty \|\mathbb{1}_{\overline{\mathcal{A}}_j}\| \\ &= \|\varphi\|_\infty |\overline{\mathcal{A}}_j|^{1/2} \leq \|\varphi\|_\infty N^{(1-26/9)/2} \end{aligned}$$

(b) Using Hölder and Sobolev under the x_k -integration we get

$$\begin{aligned} \|\mathbb{1}_{\overline{\mathcal{A}}_j} \Psi\|^2 &= \|\mathbb{1}_{\overline{\mathcal{A}}_j} \Psi^2\|_1 \leq \|\mathbb{1}_{\overline{\mathcal{A}}_j}\|_{3/2} \|\Psi^2\|_3 \leq |\overline{\mathcal{A}}_j|^{2/3} \|\nabla_j \Psi\|^2 \\ &\leq \|\nabla_1 \Psi\|^2 (NN^{-26/9})^{2/3} \leq N^{-34/27} \|\nabla_1 \Psi\|^2 . \end{aligned}$$

Since $\|\nabla_1 \Psi\| < C$ (b) follows.

(c) We use that $\overline{\mathcal{B}}_j \subset \bigcup_{k=1}^N \overline{\mathcal{A}}_k$. Hence one can find pairwise disjoint sets $\mathcal{C}_k \subset \mathcal{A}_k$, $k = 1, \dots, N$ such that $\overline{\mathcal{B}}_j \subset \bigcup_{k=1}^N \mathcal{C}_k$. Since the sets \mathcal{C}_k are pairwise disjoint, the $\mathbb{1}_{\mathcal{C}_k} \Psi$ are pairwise orthogonal and we get

$$\|\mathbb{1}_{\overline{\mathcal{B}}_j} \Psi\|^2 = \sum_{k=1}^N \|\mathbb{1}_{\mathcal{C}_k} \Psi\|^2 \leq \sum_{k=1}^N \|\mathbb{1}_{\overline{\mathcal{A}}_k} \Psi\|^2 \leq CN^{-7/27} \|\nabla_j \Psi\|^2 .$$

□

Next we prepare estimates of some energy terms we shall need below.

Corollary 6.3 Let $V_1 \in \mathcal{V}_1$, $0 < \beta_1 < 1$. Then there exist a $\mathcal{K} \in \mathcal{F}$ and a $\eta > 0$ such that for all $\Psi \in \mathcal{H}_0$ and all $\varphi \in L^2(\mathbb{R}^3, \mathbb{C})$

$$\langle\langle \Psi, (2a|\varphi(x_1)|^2 - (N-1)\mathbb{1}_{\mathcal{B}_1} W_{\beta_1}(x_1 - x_2)) \Psi \rangle\rangle \leq \mathcal{K}(\varphi)(\alpha(\Psi, \varphi) + N^{-\eta})$$

Proof:

$$\begin{aligned} & \langle\langle \Psi, (2a|\varphi(x_1)|^2 - (N-1)\mathbb{1}_{\mathcal{B}_1}W_{\beta_1}(x_1-x_2))\Psi \rangle\rangle \\ &= \langle\langle \mathbb{1}_{\mathcal{B}_1}\Psi, p_1p_2(2a|\varphi(x_1)|^2 - (N-1)W_{\beta_1}(x_1-x_2))\mathbb{1}_{\mathcal{B}_1}p_1p_2\Psi \rangle\rangle \end{aligned} \quad (94)$$

$$+ 2a\langle\langle \Psi, |\varphi(x_1)|^2 - \mathbb{1}_{\mathcal{B}_1}p_1p_2|\varphi(x_1)|^2p_1p_2\mathbb{1}_{\mathcal{B}_1}\Psi \rangle\rangle \quad (95)$$

$$+ 2(N-1)\langle\langle \Psi, \mathbb{1}_{\mathcal{B}_1}p_1p_2W_{\beta_1}(x_1-x_2)(1-p_1p_2)\mathbb{1}_{\mathcal{B}_1}\Psi \rangle\rangle \quad (96)$$

$$- (N-1)\langle\langle \Psi, \mathbb{1}_{\mathcal{B}_1}(1-p_1p_2)W_{\beta_1}(x_1-x_2)(1-p_1p_2)\mathbb{1}_{\mathcal{B}_1}\Psi \rangle\rangle. \quad (97)$$

Using (15) and (32) we get that (94) is bounded by

$$|(94)| \leq \mathcal{K}(\varphi)(\alpha(\Psi, \varphi) + N^{-\eta}).$$

Due to Lemma 3.4

$$|\langle\langle \Psi, |\varphi(x_1)|^2\Psi \rangle\rangle - \langle\varphi, |\varphi^2|\varphi\rangle| \leq C\|\varphi\|_\infty^2\alpha(\varphi, \varphi).$$

On the other hand we have

$$\langle\langle \Psi, \mathbb{1}_{\mathcal{B}_1}p_1p_2|\varphi(x_1)|^2p_1p_2\mathbb{1}_{\mathcal{B}_1}\Psi \rangle\rangle = \langle\varphi, |\varphi^2|\varphi\rangle\|p_1p_2\mathbb{1}_{\mathcal{B}_1}\Psi\|^2$$

and with Proposition 6.2 that

$$\begin{aligned} |\|p_1p_2\mathbb{1}_{\mathcal{B}_1}\Psi\|^2 - 1| &\leq |\|p_1p_2\mathbb{1}_{\mathcal{B}_1}\Psi\|^2 - \|p_1p_2\Psi\|^2| + |\|p_1p_2\Psi\|^2 - 1| \\ &\leq CN^{-7/54} + \alpha(\Psi, \varphi). \end{aligned}$$

Thus

$$|(95)| \leq \mathcal{K}(\varphi)(\alpha(\Psi, \varphi) + N^{-\eta}).$$

For (96) we use that the support of $W_{\beta_1}(x_1-x_2)$ and $\overline{\mathcal{A}}_1$ are disjoint, thus

$$(96) = -2(N-1)\langle\langle \Psi, \mathbb{1}_{\mathcal{B}_1}p_1p_2W_{\beta_1}(x_1-x_2)p_1p_2\mathbb{1}_{\mathcal{B}_1}\mathbb{1}_{\overline{\mathcal{A}}_1}\Psi \rangle\rangle \quad (98)$$

$$+ 2(N-1)\langle\langle \Psi, \mathbb{1}_{\mathcal{B}_1}\mathbb{1}_{\overline{\mathcal{A}}_1}p_1p_2W_{\beta_1}(x_1-x_2)(1-p_1p_2)\mathbb{1}_{\mathcal{B}_1}\mathbb{1}_{\mathcal{A}_1}\Psi \rangle\rangle \quad (99)$$

$$+ 2(N-1)\langle\langle \Psi, \mathbb{1}_{\mathcal{B}_1}\mathbb{1}_{\mathcal{A}_1}p_1p_2W_{\beta_1}(x_1-x_2)(1-p_1p_2)\mathbb{1}_{\mathcal{B}_1}\mathbb{1}_{\mathcal{A}_1}\Psi \rangle\rangle. \quad (100)$$

Using Proposition 6.2 and Lemma 3.2 (e) we have

$$|(98)| \leq 2N\|p_1W_{\beta_1}(x_1-x_2)p_1\|_{op}\|\mathbb{1}_{\overline{\mathcal{A}}_1}\Psi\| \leq C\|\varphi\|_\infty^2N^{-17/27}.$$

For (99) we have using Proposition 6.2 and Lemma 3.2 (e)

$$\begin{aligned} |99| &\leq 2N\|\mathbb{1}_{\overline{\mathcal{A}}_1}\Psi\|\|\mathbb{1}_{\overline{\mathcal{A}}_1}p_1\|_{op}\|p_1W_{\beta_1}(x_1-x_2)\|_{op} \\ &\leq CN N^{-17/24}\|\varphi\|_\infty N^{-17/18}\|\varphi\|_\infty\|W_{\beta_1}\| \\ &\leq CN^{-11/72+3/2(\beta-1)}\|\varphi\|_\infty^2. \end{aligned}$$

Note that $\mathbb{1}_{\mathcal{B}_1}\mathbb{1}_{\mathcal{A}_1}\mathbb{R}^{3N} \setminus \bigcup_{j \neq k} a_{j,k}$, thus $\mathbb{1}_{\mathcal{B}_1}\mathbb{1}_{\mathcal{A}_1}\Psi \in \mathcal{H}_\emptyset$. Therefore (100) is controlled by Lemma 4.6 (a) and also bounded by the right hand side of the Corollary.

Having good control of (94), (95) and (96) and using that (97) is negative the Lemma follows.

□

Lemma 6.4 *Let $V_1 \in \mathcal{V}_1$. Then there exists a $\eta > 0$ and a $\mathcal{K} \in \mathcal{F}$ such that for any $\Psi \in \mathcal{H}_0$ and any $\varphi \in L^2(\mathbb{R}^3, \mathbb{C})$*

(a)

$$\|\mathbf{1}_{\mathcal{A}_1} \nabla_1 q_1 \Psi\|^2 \leq \mathcal{K}(\varphi)(\alpha(\Psi, \varphi) + N^{-\eta})$$

(b)

$$(N-1) \|\mathbf{1}_{\overline{\mathcal{B}}_1} \sqrt{V_1}(x_1 - x_2) \Psi\|^2 \leq \mathcal{K}(\varphi)(\alpha(\Psi, \varphi) + N^{-\eta})$$

(c)

$$\|\mathbf{1}_{\overline{\mathcal{B}}_1} \nabla_1 q_1 \Psi\|^2 \leq \mathcal{K}(\varphi)(\alpha(\Psi, \varphi) + N^{-\eta})$$

Proof:

(a)+(b) For any $0 < \beta_1 < 1$ we have

$$\begin{aligned} & \|\nabla_1 q_1 \Psi\|^2 + \langle \Psi, ((N-1)V_1(x_1 - x_2) - 2a|\varphi(x_1)|^2)\Psi \rangle \\ &= \|\mathbf{1}_{\mathcal{A}_1} \nabla_1 q_1 \Psi\|^2 + \|\mathbf{1}_{\overline{\mathcal{B}}_1} \mathbf{1}_{\overline{\mathcal{A}}_1} \nabla_1 q_1 \Psi\|^2 + \|\mathbf{1}_{\mathcal{B}_1} \mathbf{1}_{\overline{\mathcal{A}}_1} \nabla_1 q_1 \Psi\|^2 \\ & \quad + (N-1) \|\mathbf{1}_{\overline{\mathcal{B}}_1} \sqrt{V_1}(x_1 - x_2) \Psi\|^2 \\ & \quad + \langle \Psi, \sum_{j \neq 1} \mathbf{1}_{\mathcal{B}_1} (V_1 - W_{\beta_1})(x_1 - x_j) \Psi \rangle \\ & \quad + \langle \Psi, \left(\sum_{j \neq 1} \mathbf{1}_{\mathcal{B}_1} W_{\beta_1}(x_1 - x_j) - 2a|\varphi(x_1)|^2 \right) \Psi \rangle . \end{aligned}$$

Using that $q_1 = 1 - p_1$ and symmetry gives (after reordering)

$$= (N-1) \|\mathbf{1}_{\overline{\mathcal{B}}_1} \sqrt{V_1}(x_1 - x_2) \Psi\|^2 + \|\mathbf{1}_{\mathcal{A}_1} \nabla_1 q_1 \Psi\|^2 \quad (101)$$

$$+ \|\mathbf{1}_{\overline{\mathcal{B}}_1} \mathbf{1}_{\overline{\mathcal{A}}_1} \nabla_1 q_1 \Psi\|^2 + \|\mathbf{1}_{\mathcal{B}_1} \mathbf{1}_{\overline{\mathcal{A}}_1} \nabla_1 p_1 \Psi\|^2 \quad (102)$$

$$- 2\Re(\langle \nabla_1 q_1 \Psi, \mathbf{1}_{\mathcal{B}_1} \mathbf{1}_{\overline{\mathcal{A}}_1} \nabla_1 p_1 \Psi \rangle) \quad (103)$$

$$+ \|\mathbf{1}_{\mathcal{B}_1} \mathbf{1}_{\overline{\mathcal{A}}_1} \nabla_1 \Psi\|^2 + \langle \Psi, \sum_{j \neq 1} \mathbf{1}_{\mathcal{B}_1} (V_1 - W_{\beta_1})(x_1 - x_j) \Psi \rangle \quad (104)$$

$$+ \langle \Psi, \left(\sum_{j \neq 1} \mathbf{1}_{\mathcal{B}_1} W_{\beta_1}(x_1 - x_j) - 2a|\varphi(x_1)|^2 \right) \Psi \rangle \quad (105)$$

Proposition 6.2 (b) and (49) yields that for some $\mathcal{K} \in \mathcal{F}$

$$\begin{aligned} |(103)| & \leq 2 |\Re(\langle \nabla_1 q_1 \Psi, \mathbf{1}_{\overline{\mathcal{A}}_1} \nabla_1 p_1 \mathbf{1}_{\mathcal{B}_1} \Psi \rangle)| \\ & \leq \|\nabla_1 q_1 \Psi\| \|\nabla \varphi\| \|\mathbf{1}_{\mathcal{B}_1} \Psi\| \\ & \leq \mathcal{K}(\varphi) N^{-7/54} . \end{aligned}$$

Choosing $\beta_1 < 1$ large enough the support of the potentials $V_1(x_1 - x_j)$ and $W_{\beta_1}(x_1 - x_j)$ are subsets of $\bar{\mathcal{A}}_1 := \mathbb{R}^{3N} \setminus \mathcal{A}_1$ (c.f. Definition 6.1). Furthermore we have that the support of the potentials

$$\mathbb{1}_{\mathcal{B}_1}(V_1(x_1 - x_j) - W_{\beta_1}(x_1 - x_j))$$

are pairwise disjoint for different j . It follows with Lemma 5.2 (c) that (104) is positive.

Corollary 6.3 gives a bound on (105) it follows that

$$(101) + (102) \leq \mathcal{K}(\varphi)(\alpha(\Psi, \varphi) + N^{-\eta}).$$

Since all the summands in (101) and (102) are positive, it follows that each of them is bounded by $\mathcal{K}(\varphi)(\alpha(\Psi, \varphi) + N^{-\eta})$ and we get (a), (b) as well as

$$\|\mathbb{1}_{\bar{\mathcal{B}}_1} \mathbb{1}_{\bar{\mathcal{A}}_1} \nabla_1 q_1 \Psi\|^2 \leq \mathcal{K}(\varphi)(\alpha(\Psi, \varphi) + N^{-\eta}). \quad (106)$$

(c) (106) and (a) give

$$\begin{aligned} \|\mathbb{1}_{\bar{\mathcal{B}}_1} \nabla_1 q_1 \Psi\|^2 &\leq \|\mathbb{1}_{\bar{\mathcal{B}}_1} \mathbb{1}_{\bar{\mathcal{A}}_1} \nabla_1 q_1 \Psi\|^2 + \|\mathbb{1}_{\bar{\mathcal{B}}_1} \mathbb{1}_{\mathcal{A}_1} \nabla_1 q_1 \Psi\|^2 \\ &\leq \|\mathbb{1}_{\bar{\mathcal{B}}_1} \mathbb{1}_{\bar{\mathcal{A}}_1} \nabla_1 q_1 \Psi\|^2 + \|\mathbb{1}_{\mathcal{A}_1} \nabla_1 q_1 \Psi\|^2 \\ &\leq \mathcal{K}(\varphi)(\alpha(\Psi, \varphi) + N^{-\eta}). \end{aligned}$$

□

6.2 Generalizing Lemma 4.4 (d)

For $\beta = 1$ our plan is again to smoothen out the interaction using the microscopic structure and use Lemma 4.4 for the smoothed interaction terms. To be able to do so we first have to estimate Lemma 4.4 (d) in terms of $\|\mathbb{1}_{\mathcal{A}_1} \nabla_1 q_1 \Psi\|$ (Remember that for $\beta = 1$ $\|\nabla_1 q_1 \Psi\|$ is not small).

Lemma 6.5 *Let for $0 < \beta < 1$ $V_\beta \in \mathcal{V}_\beta$, $g \in L^2$ and $m : \mathbb{N}^2 \rightarrow \mathbb{R}^+$ with $m \leq n^{-1}$. Then there exists a $\mathcal{K} \in \mathcal{F}$ and a $\eta > 0$ such that for any $\Psi \in \mathcal{M}$ with $\{1, 2\} \subset \mathcal{M}$*

$$\begin{aligned} N \langle \Psi q_1 p_2 V_\beta(x_1 - x_2) \hat{m} q_1 q_2 \Psi \rangle & \quad (107) \\ &\leq \mathcal{K}(\varphi)(\|\varphi\|_\infty + (\ln N)^{1/3} \|\nabla \varphi\|_{6,loc}) (\langle \Psi, \hat{n} \Psi \rangle + \|\mathbb{1}_{\mathcal{A}_1} \nabla_1 q_1 \Psi\|^2 + N^{-\eta}) \end{aligned}$$

Proof:

We first prove the Lemma for some small $0 < \beta$ and generalize to all $0 < \beta < 1$ thereafter.

In view of Definition 4.2

$$\begin{aligned}
N|\langle\langle\Psi, q_1 p_2 V_\beta(x_1 - x_2) \widehat{m} q_1 q_2 \Psi\rangle\rangle| &= N|\langle\langle\Psi, q_1 p_2 U_{0,\beta}(x_1 - x_2) \widehat{m} q_1 q_2 \Psi\rangle\rangle| \\
&\quad + N|\langle\langle\Psi, \widehat{m}_1 q_1 p_2 (\Delta h_{0,\beta})(x_1 - x_2) q_1 q_2 \Psi\rangle\rangle| \\
&\leq N|\langle\langle\Psi, q_1 p_2 U_{0,\beta}(x_1 - x_2) \widehat{m} q_1 q_2 \Psi\rangle\rangle| \tag{108}
\end{aligned}$$

$$+ N|\langle\langle\Psi, q_1 p_2 \widehat{m}_1 (\nabla_1 h_{0,\beta}(x_1 - x_2)) q_2 \mathbb{1}_{\overline{\mathcal{A}}_1} \nabla_1 q_1 \Psi\rangle\rangle| \tag{109}$$

$$+ N|\langle\langle\mathbb{1}_{\overline{\mathcal{A}}_1} \nabla_1 q_1 \Psi, p_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) q_1 q_2 \widehat{m} \Psi\rangle\rangle| \tag{110}$$

$$+ N|\langle\langle\Psi, q_1 p_2 \widehat{m}_1 (\nabla_1 h_{0,\beta}(x_1 - x_2)) q_2 \mathbb{1}_{\mathcal{A}_1} \nabla_1 q_1 \Psi\rangle\rangle| \tag{111}$$

$$+ N|\langle\langle\mathbb{1}_{\mathcal{A}_1} \nabla_1 q_1 \Psi, p_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) q_1 q_2 \widehat{m} \Psi\rangle\rangle|. \tag{112}$$

For (108) we have

$$(108) \leq N \|q_1 \Psi\| \|U_{0,\beta}\|_\infty \|\widehat{m} q_1 q_2 \Psi\| \leq CN \alpha(\Psi, \varphi) \|V_\beta\|_1.$$

For (109) and (110).

$$\begin{aligned}
(109) + (110) &\leq N \|\mathbb{1}_{\overline{\mathcal{A}}_1} q_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) \widehat{m}_1 q_1 p_2 \Psi\| \|\nabla_1 q_1 \Psi\| \\
&\quad + N \|\nabla_1 q_1 \Psi\| \|\mathbb{1}_{\overline{\mathcal{A}}_1} p_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) q_1 q_2 \widehat{m} \Psi\|
\end{aligned}$$

Using Proposition 6.2 and Lemma 4.3

$$\begin{aligned}
(109) + (110) &\leq CN^{10/27} \|\nabla_1 q_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) \widehat{m}_1 q_1 p_2 \Psi\| \|\nabla_1 q_1 \Psi\| \\
&\quad + CN^{10/27} \|\nabla_1 q_1 \Psi\| \|\nabla_1 p_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) q_1 q_2 \widehat{m} \Psi\| \\
&\leq CN^{10/27} \|q_2 (\Delta_1 h_{0,\beta}(x_1 - x_2)) \widehat{m}_1 q_1 p_2 \Psi\| \\
&\quad + CN^{10/27} \|q_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) \nabla_1 \widehat{m}_1 q_1 p_2 \Psi\| \\
&\quad + CN^{10/27} \|p_2 (\Delta_1 h_{0,\beta}(x_1 - x_2)) q_1 q_2 \widehat{m} \Psi\| \\
&\quad + CN^{10/27} \|p_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) \nabla_1 q_1 q_2 \widehat{m} \Psi\|
\end{aligned}$$

Note that for any $m \leq Cn^{-1}$ and any $\Psi \in \mathcal{H}_{\{1\}}$

$$\begin{aligned}
\|\nabla_1 q_1 q_2 \widehat{m} \Psi\| &\leq \|p_1 \nabla_1 q_1 q_2 \widehat{m} \Psi\| + \|q_1 \nabla_1 q_1 q_2 \widehat{m} \Psi\| \\
&= \|\widehat{m}_1 q_2 p_1 \nabla_1 q_1 \Psi\| + \|\widehat{m} q_2 q_1 \nabla_1 q_1 \Psi\| \\
&\leq C \|\nabla_1 q_1 \Psi\| \tag{113}
\end{aligned}$$

and similarly

$$\|\nabla_1 q_1 p_2 \widehat{m} \Psi\| \leq C \|\widehat{m}\|_{op} \|\nabla_1 q_1 \Psi\|.$$

Since $\|\nabla_1 q_1 \Psi\|$ is bounded (see (49))

$$\begin{aligned}
(109) + (110) &\leq CN^{10/27} \|\varphi\|_\infty (\|\Delta_1 h_{0,\beta}\| \\
&\quad + \|\nabla_1 h_{0,\beta}\| N^{-1/2} + \|\Delta_1 h_{0,\beta}\| + \|\nabla_1 h_{0,\beta}\|)
\end{aligned}$$

Using the bounds on $\|\Delta_1 h_{0,\beta}\|$ and $\|\nabla_1 h_{0,\beta}\|$ from Lemma 4.3 it follows that for β small enough we can find a $\eta > 0$ and a $\mathcal{K} \in \mathcal{F}$ such that

$$(109) + (110) \leq \mathcal{K}(\varphi) \|\varphi\|_\infty N^{-\eta}.$$

$$\begin{aligned} (111) &\leq \frac{N}{N-1} \left| \left\langle \Psi, \sum_{k=2}^N q_1 p_k \widehat{m}_1(\nabla_1 h_{0,\beta}(x_1 - x_k)) q_k \mathbb{1}_{\mathcal{A}_1} \nabla_1 q_1 \Psi \right\rangle \right| \\ &\leq \frac{N}{N-1} \left\| \sum_{k=2}^N q_k(\nabla_1 h_{0,\beta}(x_1 - x_k)) \widehat{m}_1 q_1 p_k \Psi \right\| \|\mathbb{1}_{\mathcal{A}_1} \nabla_1 q_1 \Psi\| \end{aligned}$$

Due to (49) $\|\mathbb{1}_{\mathcal{A}_1} \nabla_1 q_1 \Psi\|$ is bounded. For the other factor we write

$$\begin{aligned} &\left\| \sum_{k=2}^N q_k(\nabla_1 h_{0,\beta}(x_1 - x_k)) \widehat{m}_1 q_1 p_k \Psi \right\|^2 \\ &= \sum_{2 \leq j < k \leq N} \left\langle \Psi, \widehat{m}_1 q_1 p_j(\nabla_1 h_{0,\beta}(x_1 - x_j)) q_j q_k(\nabla_1 h_{0,\beta}(x_1 - x_k)) \widehat{m}_1 q_1 p_k \Psi \right\rangle \end{aligned} \quad (114)$$

$$+ \sum_{k=2}^N \left\langle \Psi, \widehat{m}_1 q_1 p_k(\nabla_1 h_{0,\beta}(x_1 - x_k)) q_k(\nabla_1 h_{0,\beta}(x_1 - x_k)) \widehat{m}_1 q_1 p_k \Psi \right\rangle. \quad (115)$$

For (114) we use that for any $2 \leq k \leq N$ $\nabla_1 h_{0,\beta}(x_1 - x_k) = \nabla_k h_{0,\beta}(x_1 - x_k)$. Partial integrations yield

$$\begin{aligned} (114) &\leq \sum_{2 \leq j < k \leq N} \left| \left\langle \nabla_k \nabla_j p_j \widehat{m}_1 q_k \Psi, q_1 h_{0,\beta}(x_1 - x_j) h_{0,\beta}(x_1 - x_k) \widehat{m}_1 q_1 p_k q_j \Psi \right\rangle \right| \\ &\quad + \sum_{2 \leq j < k \leq N} \left| \left\langle \nabla_j p_j \widehat{m}_1 q_k \Psi, q_1 h_{0,\beta}(x_1 - x_j) h_{0,\beta}(x_1 - x_k) \nabla_k \widehat{m}_1 q_1 p_k q_j \Psi \right\rangle \right| \\ &\quad + \sum_{2 \leq j < k \leq N} \left| \left\langle \nabla_k \widehat{m}_1 q_k \Psi, q_1 p_j h_{0,\beta}(x_1 - x_j) h_{0,\beta}(x_1 - x_k) \nabla_j \widehat{m}_1 q_1 p_k q_j \Psi \right\rangle \right| \\ &\quad + \sum_{2 \leq j < k \leq N} \left| \left\langle \widehat{m}_1 q_k \Psi, q_1 p_j h_{0,\beta}(x_1 - x_j) h_{0,\beta}(x_1 - x_k) \nabla_j \nabla_k \widehat{m}_1 q_1 p_k q_j \Psi \right\rangle \right|. \end{aligned}$$

Due to symmetry the first and the fourth line are equal and

$$\begin{aligned}
(114) &\leq 2N^2 |\langle \nabla_2 p_2 \nabla_3 \widehat{m}_1 q_3 \Psi, q_1 h_{0,\beta}(x_1 - x_3) p_3 h_{0,\beta}(x_1 - x_2) \widehat{m}_1 q_1 q_2 \Psi \rangle| \\
&\quad + N^2 |\langle \nabla_2 \widehat{m}_1 p_2 q_3 \Psi, q_1 h_{0,\beta}(x_1 - x_2) h_{0,\beta}(x_1 - x_3) \nabla_3 \widehat{m}_1 q_1 p_3 q_2 \Psi \rangle| \\
&\quad + N^2 |\langle \nabla_3 \widehat{m}_1 q_3 \Psi, q_1 h_{0,\beta}(x_1 - x_3) p_3 p_2 h_{0,\beta}(x_1 - x_2) \nabla_2 \widehat{m}_1 q_1 q_2 \Psi \rangle| \\
&\leq 2N^2 \|p_3 h_{0,\beta}(x_1 - x_3) \nabla_3 \widehat{m}_1 q_1 q_3 \Psi\| \|h_{0,\beta}(x_1 - x_2) \nabla_2 p_2\|_{op} \|\widehat{m}_1 q_1 q_2 \Psi\| \\
&\quad + N^2 \|h_{0,\beta}(x_1 - x_2) \nabla_2 p_2\|_{op}^2 \|\widehat{m}_1 q_1 q_3 \Psi\|^2 \\
&\quad + N^2 \|p_2 h_{0,\beta}(x_1 - x_2) \nabla_2 \widehat{m}_1 q_1 q_2 \Psi\|^2 \\
&\leq CN \|p_2 h_{0,\beta}(x_1 - x_2) \nabla_2 \widehat{m}_1 q_1 q_2 \Psi\| \|\nabla \varphi\|_{6,loc} (\ln N)^{1/3} \sqrt{\alpha(\Psi, \varphi)} \\
&\quad + C \|\nabla \varphi\|_{6,loc}^2 (\ln N)^{2/3} \alpha(\Psi, \varphi) \\
&\quad + N^2 \|p_2 h_{0,\beta}(x_1 - x_2) \nabla_2 \widehat{m}_1 q_1 q_2 \Psi\|^2.
\end{aligned}$$

Let us next control the factor $\|p_2 h_{0,\beta}(x_1 - x_2) \nabla_2 \widehat{m}_1 q_1 q_2 \Psi\|$. Using $1 = \mathbb{1}_{\mathcal{A}_1} + \mathbb{1}_{\overline{\mathcal{A}}_1}$ and exchanging the variables x_1 and x_2 we have

$$\begin{aligned}
\|p_2 h_{0,\beta}(x_1 - x_2) \nabla_2 \widehat{m}_1 q_1 q_2 \Psi\| &\leq \|p_1 h_{0,\beta}(x_1 - x_2)\|_{op} \|\nabla_1 q_1 q_2 \widehat{m}_1 \mathbb{1}_{\mathcal{A}_1} \Psi\| \\
&\quad + \|p_1 h_{0,\beta}(x_1 - x_2) \nabla_1\|_{op} \|q_2 \widehat{m}_1 \mathbb{1}_{\overline{\mathcal{A}}_1} q_1 \Psi\|.
\end{aligned}$$

Using formula (113) on the first and Lemma 3.4 on the second summand (note, that $\mathbb{1}_{\mathcal{A}_1} \Psi \in \mathcal{H}_{\{1\}}$) we get

$$\begin{aligned}
\|p_2 h_{0,\beta}(x_1 - x_2) \nabla_2 \widehat{m}_1 q_1 q_2 \Psi\| &\leq C \|p_1 h_{0,\beta}(x_1 - x_2)\|_{op} \|\nabla_1 q_1 \mathbb{1}_{\mathcal{A}_1} q_1 \Psi\| \\
&\quad + \|p_1 h_{0,\beta}(x_1 - x_2) \nabla_1\|_{op} \|\mathbb{1}_{\overline{\mathcal{A}}_1} \Psi\|.
\end{aligned}$$

The first summand is in view of Lemma 3.2 (e) and Lemma 4.3 bounded by

$$CN^{-1} \|\varphi\|_{\infty} \|\mathbb{1}_{\mathcal{A}_1} \nabla_1 q_1 \Psi\|.$$

Partial integration, Proposition (6.2) and again Lemma 3.2 (e) with Lemma 4.3 give for the second summand

$$\begin{aligned}
&\|p_1 h_{0,\beta}(x_1 - x_2) \nabla_1\|_{op} \|\mathbb{1}_{\overline{\mathcal{A}}_1} q_1 \Psi\| \\
&\leq CN^{-17/27} (\|h_{0,\beta}(x_1 - x_2) \nabla_1 p_1\|_{op} + \|(\nabla_1 h_{0,\beta}(x_1 - x_2)) p_1\|_{op}) \|\nabla_1 q_1 \Psi\| \\
&\leq CN^{-17/27} \left(N^{-1} \|\nabla \varphi\|_{6,loc} (\ln N)^{1/3} + N^{-1} \|\varphi\|_{\infty} \right) \|\nabla_1 q_1 \Psi\|
\end{aligned}$$

Using (49) and choosing β sufficiently small there exists a $\eta > 0$ and a $\mathcal{K} \in \mathcal{F}$ such that

$$\begin{aligned}
N \|p_2 h_{0,\beta}(x_1 - x_2) \nabla_2 \widehat{m}_1 q_1 q_2 \Psi\| & \\
&\leq \mathcal{K}(\varphi) (\|\varphi\|_{\infty} + (\ln N)^{1/3} \|\nabla \varphi\|_{6,loc}) (\|\mathbb{1}_{\mathcal{A}_1} \nabla_1 q_1 \Psi\|^2 + N^{-\eta}).
\end{aligned} \tag{116}$$

Thus (114) is bounded by the right hand side of (107). For (115) we have

$$\begin{aligned}
(115) &\leq N \|\widehat{m}_1 q_1 \Psi\|^2 \|p_2 (\nabla_1 h_{0,\beta}(x_1 - x_2))\|_{op}^2 \\
&\leq CN \|\varphi\|_{\infty}^2 \|\nabla_1 h_{0,\beta}\|^2 \leq CN^{-1/2+2\beta} \|\varphi\|_{\infty}^2
\end{aligned}$$

For small enough β that there exists a $\eta > 0$ such that (115) $\leq CN^{-\eta}\|\varphi\|_\infty^2$ and thus (111) is bounded by the right hand side of (107) for some $\mathcal{K} \in \mathcal{F}$.

Again using that $\nabla_1 h_{0,\beta}(x_1 - x_2) = -\nabla_2 h_{0,\beta}(x_1 - x_2)$ and partial integration yields for (112)

$$\begin{aligned} (112) &\leq N|\langle\langle h_{0,\beta}(x_1 - x_2)\nabla_2 p_2 \mathbb{1}_{\mathcal{A}_1} \nabla_1 q_1 \Psi, q_1 q_2 \widehat{m} \Psi \rangle\rangle| \\ &\quad + N|\langle\langle \mathbb{1}_{\mathcal{A}_1} \nabla_1 q_1 \Psi, p_2 h_{0,\beta}(x_1 - x_2) q_1 \nabla_2 q_2 \widehat{m} \Psi \rangle\rangle| \\ &\leq N\|h_{0,\beta}(x_1 - x_2)\nabla_2 p_2\|_{op}\|\mathbb{1}_{\mathcal{A}_1} \nabla_1 q_1 \Psi\|\|q_1 q_2 \widehat{m} \Psi\| \\ &\quad + N\|\mathbb{1}_{\mathcal{A}_1} \nabla_1 q_1 \Psi\|\|p_2 h_{0,\beta}(x_1 - x_2) q_1 \nabla_2 q_2 \widehat{m} \Psi\|. \end{aligned}$$

Using (116) it follows that $N(112) \leq C$ which completes the proof of the Lemma for $0 < \beta \leq \beta_0$ for some $0 < \beta_0 < 1$.

To prove the Lemma for $\beta_0 < \beta < 1$ we write

$$\begin{aligned} |\langle\langle \Psi, q_1 p_2 V_\beta(x_1 - x_2) \widehat{m} q_1 q_2 \Psi \rangle\rangle| &= |\langle\langle \Psi, q_1 p_2 U_{\beta_0,\beta} \widehat{m} q_1 q_2 \Psi \rangle\rangle| \\ &\quad + N\langle\langle \Psi p_1 q_2 (V_\beta(x_1 - x_2) - U_{\beta_0,\beta}(x_1 - x_2)) \widehat{m} q_1 q_2 \Psi \rangle\rangle. \end{aligned}$$

In view of Lemma 4.3 $U_{\beta_0,\beta} \in \mathcal{V}_{\beta_0}$. The case $\beta = \beta_0$ has just been shown, so the first summand is bounded by the right hand side of (107). The second summand is controlled by (37) and the Lemma follows in full generality. \square

6.3 Second adjustment: Making α'_2 controllable

Next we adjust the functional Γ from Corollary 5.12 such that ξ (defined in Corollary 5.10) becomes controllable.

Recall that (see (77))

$$\xi = -2N(N-1)\Im(\langle\langle \Psi, Z_\beta(x_1, x_2) p_1 q_2 (\widehat{m}^0 - \widehat{m}_1^0) \Psi \rangle\rangle). \quad (117)$$

Following the ideas in section 5.2 we define now a functional which smoothens the “bad” interaction term in ξ .

Definition 6.6 Let $V_1 \in \mathcal{V}_1$, let $\widehat{m} = \widehat{m}^0 - \widehat{m}_1^0$.

We define the functional $\gamma : L^2(\mathbb{R}^{3N}, \mathbb{C}) \otimes L^2(\mathbb{R}^3, \mathbb{C}) \rightarrow \mathbb{R}^+$ by

$$\gamma(\Psi, \varphi) := -N(N-1)\Im(\langle\langle \Psi, q_1 q_2 g_{8/9,1}(x_1 - x_2) \widehat{m} p_1 q_2 \Psi \rangle\rangle)$$

and the functional $\gamma' : L^2(\mathbb{R}^{3N}, \mathbb{C}) \otimes L^2(\mathbb{R}^3, \mathbb{C}) \rightarrow \mathbb{R}$ by

$$\gamma'(\Psi, \varphi) := \gamma^a + \gamma^b + \gamma^c + \gamma^d + \xi,$$

where the different summands are

(a) The mixed derivative term

$$\begin{aligned} \gamma^a &= -N(N-1)\Im(\langle\langle \Psi, q_1 q_2 [H, g_{8/9,1}(x_1 - x_2)] \widehat{m} p_1 q_2 \Psi \rangle\rangle) \\ &\quad - N(N-1)\Im(\langle\langle \Psi, q_1 q_2 (W_{8/9} - V_\beta) f_{8/9,1}(x_1 - x_2) \widehat{m} p_1 q_2 \Psi \rangle\rangle). \end{aligned}$$

(b) *The new interaction term*

$$\begin{aligned}\gamma^b &= -\xi + N(N-1)\mathfrak{S}(\langle\langle\Psi, [Z_\beta(x_1, x_2), q_1q_2g_{8/9,1}(x_1-x_2)\widehat{m}p_1q_2]\Psi\rangle\rangle) \\ &\quad + N(N-1)\mathfrak{S}(\langle\langle\Psi, q_1q_2(W_{8/9}-V_\beta)f_{8/9,1}(x_1-x_2)\widehat{m}p_1q_2\Psi\rangle\rangle)\end{aligned}$$

(c) *Three particle interactions*

$$\begin{aligned}\gamma^c &= N(N-1)(N-2) \\ &\quad \mathfrak{S}(\langle\langle\Psi, [Z_\beta(x_1, x_3) + Z_\beta(x_2, x_3), q_1q_2g_{8/9,1}(x_1-x_2)\widehat{m}p_1q_2]\Psi\rangle\rangle)\end{aligned}$$

(d) *Interaction terms of the correction*

$$\begin{aligned}\gamma^d &= N(N-1)(N-2)(N-3) \\ &\quad \mathfrak{S}(\langle\langle\Psi, q_1q_2g_{8/9,1}(x_1-x_2)p_1q_2[Z_\beta(x_3, x_4), \widehat{m}]\Psi\rangle\rangle)\end{aligned}$$

Lemma 4.4 (a) together with Lemma 5.2 (a) imply directly

Corollary 6.7 *There exists a $\eta > 0$ such that*

$$|\gamma(\Psi, \varphi)| \leq CN^{-\eta}\|\varphi\|_\infty.$$

Next we show that in fact γ' satisfies $\gamma'(\Psi_t, \varphi_t) = \frac{d}{dt}\gamma(\Psi_t, \varphi_t)$:

Lemma 6.8

$$\frac{d}{dt}\gamma(\Psi_t, \varphi_t) = \gamma'(\Psi_t, \varphi_t)$$

Proof:

$$\begin{aligned}\frac{d}{dt}\gamma(\Psi_t, \varphi_t) &= -N(N-1)\mathfrak{S}(\langle\langle\Psi, q_1q_2[H, g_{8/9,1}(x_1-x_2)]\widehat{m}p_1q_2\Psi\rangle\rangle) \\ &\quad + N(N-1)\mathfrak{S}(\langle\langle\Psi, [H - H^{GP}, q_1q_2g_{8/9,1}(x_1-x_2)\widehat{m}p_1q_2]\Psi\rangle\rangle)\end{aligned}$$

Using symmetry

$$\begin{aligned}&= -N(N-1)\mathfrak{S}(\langle\langle\Psi, q_1q_2[H, g_{8/9,1}(x_1-x_2)]\widehat{m}p_1q_2\Psi\rangle\rangle) \\ &\quad + N(N-1)\mathfrak{S}(\langle\langle\Psi, [Z_\beta(x_1, x_2), q_1q_2g_{8/9,1}(x_1-x_2)\widehat{m}p_1q_2]\Psi\rangle\rangle) \\ &\quad + N(N-1)(N-2)\mathfrak{S}(\langle\langle\Psi, [Z_\beta(x_1, x_3) + Z_\beta(x_2, x_3), q_1q_2g_{8/9,1}(x_1-x_2)\widehat{m}p_1q_2]\Psi\rangle\rangle) \\ &\quad + N(N-1)(N-2)(N-3)\mathfrak{S}(\langle\langle\Psi, q_1q_2g_{8/9,1}(x_1-x_2)p_1q_2[Z_\beta(x_3, x_4), \widehat{m}]\Psi\rangle\rangle).\end{aligned}$$

□

Subtracting

$$N(N-1)\mathfrak{S}(\langle\langle\Psi, q_1q_2(W_{8/9}-V_\beta)f_{8/9,1}(x_1-x_2)\widehat{m}p_1q_2\Psi\rangle\rangle)$$

from the first line and adding it to the second line, as well as subtracting ξ from the second line and adding it to the total gives that the right hand side of (76) equals $\gamma^a + \gamma^b + \gamma^c + \gamma^d + \xi$ which proves the Lemma.

Lemma 6.9 *Let $\beta = 1$. There exists a $\mathcal{K} \in \mathcal{F}$ such that*

$$\gamma'(\Psi, \varphi) - \xi \leq (\|\varphi\|_\infty + (\ln N)^{1/3} \|\nabla \varphi\|_{6,loc}) \mathcal{K}(\varphi) (\alpha(\Psi, \varphi) + N^{-\eta}) .$$

Proof:

- (a) Using (78) and $\nabla_1 g_{8/9,1}(x_1 - x_2) = -\nabla_2 g_{8/9,1}(x_1 - x_2)$ and integrating by parts we have

$$\begin{aligned} |\gamma^a(\Psi, \varphi)| &\leq N^2 |\langle \Psi, q_1 q_2 \widehat{m}_{-1} (\nabla_2 g_{8/9,1}(x_1 - x_2)) \nabla_2 p_1 q_2 \Psi \rangle| \\ &\quad + N^2 |\langle \Psi, q_1 q_2 \widehat{m}_{-1} (\nabla_1 g_{8/9,1}(x_1 - x_2)) \nabla_1 p_1 q_2 \Psi \rangle| \\ &\leq 2N^2 |\langle \Psi, q_1 q_2 \widehat{m}_{-1} g_{8/9,1}(x_1 - x_2) \nabla_1 \nabla_2 p_1 q_2 \Psi \rangle| \quad (118) \\ &\quad + N^2 |\langle \nabla_1 \widehat{m}_{-1} q_1 q_2 \Psi, g_{8/9,1}(x_1 - x_2) \nabla_2 p_1 q_2 \Psi \rangle| \quad (119) \\ &\quad + N^2 |\langle \nabla_2 q_1 q_2 \Psi, g_{8/9,1}(x_1 - x_2) \nabla_1 \widehat{m}_{-1} p_1 q_2 \Psi \rangle| . \quad (120) \end{aligned}$$

We use that for any $\chi \in L^2(\mathbb{R}^{3N}, \mathbb{C})$ by Hölder- and Sobolev's inequality

$$\begin{aligned} \|\mathbf{1}_{\{(x_1-x_2) \leq RN^{-8/9}\}} \chi\|^2 &= \langle \chi, \mathbf{1}_{\{(x_1-x_2) \leq RN^{-8/9}\}} \chi \rangle \\ &\leq \|\mathbf{1}_{\{(x_1-x_2) \leq RN^{-8/9}\}}\|_{3/2} \|\chi^2\|_3 \\ &\leq CN^{-16/9} \|\nabla_1 \chi\|^2 . \end{aligned}$$

This, (42), Lemma 5.3 and Lemma 3.2 (e) give

$$\begin{aligned} (118) &\leq N^2 \|\mathbf{1}_{\{(x_1-x_2) \leq RN^{-8/9}\}} \widehat{m}_{-1} q_1 q_2 \Psi\| \\ &\quad \|g_{8/9,1}(x_1 - x_2) \nabla_1 p_1\|_{op} \|\nabla_2 q_2 \Psi\| \\ &\leq CN^{2-8/9} \|q_2 \nabla_1 \widehat{m}_{-1} q_1 \Psi\| \|g_{8/9,1}\|_3 \|\nabla \varphi\|_{6,loc} \\ &\leq CN^{-8/9} (\ln N)^{1/3} \|\nabla \varphi\|_{6,loc} . \end{aligned}$$

For (119) we get

$$\begin{aligned} (119) &\leq N^2 \|\widehat{m}_{-1} \nabla_1 q_1 q_2 \Psi\| \|g_{8/9,1}(x_1 - x_2) p_1\|_{op} \|\nabla_2 q_2 \Psi\| \\ &\leq CN^{-4/9} \|\varphi\|_\infty . \end{aligned}$$

To control (120) we use symmetry and write

$$\begin{aligned} (120) &= \frac{N^2}{N-1} |\langle \nabla_2 q_2 \Psi, \sum_{j \neq 2} q_j g_{8/9,1}(x_j - x_2) \nabla_j \widehat{m}_{-1} p_j q_2 \Psi \rangle| \\ &\leq \frac{N^2}{N-1} \|\nabla_2 q_2 \Psi\| \left\| \sum_{j=2}^N q_j g_{8/9,1}(x_j - x_1) \nabla_j \widehat{m}_{-1} p_j q_1 \Psi \right\| \end{aligned}$$

As above $\|\nabla_2 q_2 \Psi\| \leq C$. For the second factor we write

$$\begin{aligned} & \left\| \sum_{j=2}^N q_j g_{8/9,1}(x_j - x_1) \nabla_j \widehat{m} p_j q_1 \Psi \right\|^2 \\ &= \sum_{j=2}^N \langle \nabla_j p_j \widehat{m} q_1 \Psi, g_{8/9,1}(x_j - x_1) q_j g_{8/9,1}(x_j - x_1) \nabla_j p_j \widehat{m} q_1 \Psi \rangle \end{aligned} \quad (121)$$

$$+ \sum_{j \neq k=2}^N \langle \nabla_j p_j \widehat{m} q_1 \Psi, g_{8/9,1}(x_j - x_1) q_j q_k g_{8/9,1}(x_k - x_1) \nabla_k p_k \widehat{m} q_1 \Psi \rangle. \quad (122)$$

For (121) we use symmetry, Lemma 3.2 (e) and Lemma 3.4 (recall that in view of Lemma 5.3 $m(k, N) \leq CN^{-1}n^{-1}(k+1, N)$) and get

$$\begin{aligned} (121) &\leq (N-1) \langle \nabla_2 p_2 \widehat{m} q_1 \Psi, g_{8/9,1}(x_j - x_1) g_{8/9,1}(x_2 - x_1) \nabla_2 p_2 \widehat{m} q_1 \Psi \rangle \\ &\leq N \|g_{8/9,1}(x_2 - x_1) \nabla_2 p_2\|_{op}^2 \|\widehat{m} q_1 \Psi\|^2 \\ &\leq N^{-3} (\ln N)^{2/3} \|\nabla \varphi\|_{6,loc}^2. \end{aligned}$$

For (122) we get

$$\begin{aligned} |(122)| &\leq N^2 \langle \nabla_2 p_2 \mathbf{1}_{\{(x_1-x_3) \leq RN^{-8/9}\}} \widehat{m} q_1 q_3 \Psi, \\ &\quad g_{8/9,1}(x_2 - x_1) g_{8/9,1}(x_3 - x_1) \nabla_3 p_3 \mathbf{1}_{\{(x_1-x_2) \leq RN^{-8/9}\}} q_1 q_2 \Psi \rangle \\ &\leq N^2 \|g_{8/9,1}(x_2 - x_1) \nabla_2 p_2\|_{op}^2 \|\mathbf{1}_{\{(x_1-x_2) \leq RN^{-8/9}\}} \widehat{m} q_1 q_2 \Psi\|^2. \end{aligned}$$

Since $\{(x_1 - x_2) \leq RN^{-8/9}\} \subset \mathcal{A}_\infty$ we get with Proposition 6.2 and (113)

$$\begin{aligned} \|\mathbf{1}_{\{(x_1-x_2) \leq RN^{-8/9}\}} \widehat{m} q_1 q_2 \Psi\| &\leq CN^{-17/27} \|\nabla_1 \widehat{m} q_1 q_2 \Psi\| \\ &\leq CN^{-1-17/27} \|\nabla_1 q_1 \Psi\| \end{aligned}$$

It follows that

$$|(122)| \leq CN^{-2-34/27} \|\nabla \varphi\|_{6,loc}^2 (\ln N)^{2/3} \|\nabla_1 q_1 \Psi\|^2,$$

thus

$$\left\| \sum_{j=2}^N q_j g_{8/9,1}(x_j - x_1) \nabla_j \widehat{m} p_j q_1 \Psi \right\| \leq CN^{-3/2} \|\nabla \varphi\|_{6,loc} (\ln N)^{1/3} (1 + \|\nabla_1 q_1 \Psi\|).$$

With (49) it follows that also |(120)| has the right bound and (a) follows.

(b) For γ^b we can write in view of (117) and using $q_1 |\varphi(x_2)|^2 p_1 = 0$

$$\gamma^b(\Psi, \varphi) \leq N^2 |\langle \Psi, q_1 q_2 \widehat{m}_{-1} g_{8/9,1}(x_1 - x_2) p_1 q_2 Z_1(x_1, x_2) \Psi \rangle| \quad (123)$$

$$+ N^2 |\langle \Psi, Z_1(x_1, x_2) (q_1 q_2 - 1) g_{8/9,1}(x_1 - x_2) p_1 q_2 \widehat{m} \Psi \rangle| \quad (124)$$

$$+ N^2 |\langle \Psi, q_1 q_2 (W_{8/9}(x_1 - x_2) - V_1(x_1 - x_2) + Z_1(x_1, x_2)) f_{8/9,1}(x_1 - x_2) \widehat{m} p_1 q_2 \Psi \rangle|. \quad (125)$$

For (123) we have

$$\begin{aligned} |(123)| &\leq N^2 \|\widehat{m}_{-1} q_2 \Psi\| \|g_{8/9,1}(x_1 - x_2) p_1\|_{op} \|p_1 Z_1(x_1, x_2) \Psi\| \\ &\leq CN^{-4/9} \|\varphi\|_\infty^2. \end{aligned}$$

For (124)

$$\begin{aligned} (124) &\leq N^2 |\langle \Psi, Z_1(x_1, x_2) p_1 p_2 g_{8/9,1}(x_1 - x_2) \widehat{m} p_1 q_2 \Psi \rangle| \\ &\quad + N^2 |\langle \Psi, Z_1(x_1, x_2) p_1 q_2 g_{8/9,1}(x_1 - x_2) \widehat{m} p_1 q_2 \Psi \rangle| \\ &\quad + N^2 |\langle \Psi, q_1 q_2 g_{8/9,1}(x_1 - x_2) \widehat{m} p_1 q_2 \Psi \rangle| \\ &\leq N^2 \|p_1 Z_1(x_1, x_2) \Psi\| \|p_1 g_{8/9,1}(x_1 - x_2) p_1\|_{op} \|\widehat{m} q_2 \Psi\| \\ &\quad + N^2 \|p_1 Z_1(x_1, x_2) \Psi\| \|p_1 g_{8/9,1}(x_1 - x_2) p_1\|_{op} \|\widehat{m} q_2 \Psi\| \\ &\quad + N^2 \|p_1 Z_1(x_1, x_2) \Psi\| \|g_{8/9,1}(x_1 - x_2) p_1\|_{op} \cdot \|\widehat{m} q_2 \Psi\| \end{aligned}$$

Recall that $m(k, N) \leq CN^{-1} n(k+1, N)^{-1}$, thus

$$(124) \leq CN^{2-2-16/9} \|\varphi\|_\infty^3 + CN^{2-2-4/9} \|\varphi\|_\infty^2.$$

For (125) we have

$$\begin{aligned} (125) &\leq N |\langle \Psi, q_1 q_2 W_{8/9}(x_1 - x_2) f_{8/9,1}(x_1 - x_2) N \widehat{m} p_1 q_2 \Psi \rangle| \\ &\quad + \frac{N^2}{N-1} |\langle \Psi, q_1 q_2 \widehat{m}_{-1} (2a|\varphi(x_1)|^2 + 2a|\varphi(x_2)|^2) f_{8/9,1}(x_1 - x_2) p_1 q_2 \Psi \rangle|. \end{aligned}$$

We get with Lemma 6.5 that the first line is bounded by

$$\mathcal{K}(\varphi) (\|\varphi\|_\infty + (\ln N)^{1/3} \|\nabla \varphi\|_{6,loc}) (\langle \Psi, \widehat{n} \Psi \rangle + \|\mathbf{1}_{\mathcal{A}_1} \nabla_1 q_1 \Psi\|^2 + N^{-\eta}).$$

For the second line recall that $\|f_{8/9,1}\|_\infty = 1$, thus it is controlled by

$$CN \|q_1 q_2 \widehat{m}_{-1} \Psi\| \|\varphi\|_\infty^2 \|q_2 \Psi\| \leq C \|\varphi\|_\infty^2 \langle \Psi, \widehat{n} \Psi \rangle.$$

(c) Using $q_2 = 1 - p_2$ the left hand side of (c) is bounded by

$$|\gamma^c(\Psi, \varphi)| \leq N^3 |\Im \langle \Psi, [Z_1(x_2, x_3), q_1 q_2 \widehat{m}_{-1} g_{8/9,1}(x_1 - x_2) p_1 p_2] \Psi \rangle| \quad (126)$$

$$+ N^3 |\Im \langle \Psi, [Z_1(x_2, x_3), q_1 q_2 \widehat{m}_{-1} g_{8/9,1}(x_1 - x_2) p_1] \Psi \rangle| \quad (127)$$

$$+ N^3 |\Im \langle \Psi, [Z_1(x_1, x_3) q_1 q_2 g_{8/9,1}(x_1 - x_2) \widehat{m} p_1 q_2] \Psi \rangle|. \quad (128)$$

Using symmetry (126) can be controlled like in the proof of Lemma 5.9 (c). For easier reference we repeat the formulas here: Using $1 = p_3 + q_3$

and $q_1 = 1 - p_1$

$$\begin{aligned}
|(126)| &\leq N^3 |\langle \Psi, q_1 q_2 \widehat{m}_{-1} g_{8/9,1}(x_1 - x_2) p_1 p_2 Z_1(x_2, x_3) \Psi \rangle| \\
&\quad + N^3 |\langle \Psi, Z_1(x_2, x_3) p_3 q_1 q_2 \widehat{m}_{-1} g_{8/9,1}(x_1 - x_2) p_1 p_2 \Psi \rangle| \\
&\quad + N^3 |\langle \Psi, Z_1(x_2, x_3) p_1 q_2 g_{8/9,1}(x_1 - x_2) \widehat{m} p_1 p_2 q_3 \widehat{m} \Psi \rangle| \\
&\quad + N^3 |\langle \Psi, \sqrt{Z_1(x_2, x_3)} q_2 g_{8/9,1}(x_1 - x_2) p_2 \sqrt{Z_1(x_2, x_3)} p_1 q_3 \widehat{m} \Psi \rangle| \\
&\leq N^3 \|\widehat{m}_{-1} q_1 \Psi\| \|g_{8/9,1}(x_1 - x_2) p_1\|_{op} \|p_1 Z_1(x_2, x_3) \Psi\| \\
&\quad + N^3 \|q_2 \widehat{m}_{-1} p_3 Z_1(x_2, x_3) \Psi\| \|g_{8/9,1}(x_1 - x_2) p_1\|_{op} \\
&\quad + N^3 \|p_1 Z_1(x_2, x_3) \Psi\| \|g_{8/9,1}(x_1 - x_2) p_1\|_{op} \|q_3 \widehat{m} \Psi\| \\
&\quad + N^3 \|\sqrt{Z_1(x_2, x_3)} \Psi\| \|g_{8/9,1}(x_1 - x_2) p_2\|_{op} \|\sqrt{Z_1(x_2, x_3)} p_1\|_{op} \|q_3 \widehat{m} \Psi\| \\
&\leq CN^3 \|\varphi\|_\infty^2 N^{-1-1-4/9-1} = CN^{-4/9} \|\varphi\|_\infty^2.
\end{aligned}$$

Using $q_2 = 1 - p_2$ we can write for (127)

$$(127) \leq N^3 |\mathfrak{S} \langle \Psi, [Z_1(x_2, x_3), q_1 \widehat{m}_{-1} g_{8/9,1}(x_1 - x_2) p_1] \Psi \rangle| \quad (129)$$

$$+ N^3 |\mathfrak{S} \langle \Psi, q_1 p_2 \widehat{m}_{-1} g_{8/9,1}(x_1 - x_2) p_1 Z_1(x_2, x_3) \Psi \rangle| \quad (130)$$

$$+ N^3 |\mathfrak{S} \langle \Psi, Z_1(x_2, x_3) q_1 p_2 \widehat{m}_{-1} g_{8/9,1}(x_1 - x_2) p_1 \Psi \rangle|. \quad (131)$$

Using that $q_1 g_{8/9,1}(x_1 - x_2) p_1$ commutes with $Z_1(x_2, x_3)$ and then Lemma 3.2 (d) we have

$$\begin{aligned}
(129) &\leq N^3 |\mathfrak{S} \langle \Psi, [Z_1(x_2, x_3), \widehat{m}_{-1}] q_1 g_{8/9,1}(x_1 - x_2) p_1 \Psi \rangle| \\
&\leq N^3 |\mathfrak{S} \langle \Psi, [Z_1(x_2, x_3), p_2 p_3 (\widehat{m}_{-1} - \widehat{m}_1)] q_1 g_{8/9,1}(x_1 - x_2) p_1 \Psi \rangle| \\
&\quad + N^3 |\mathfrak{S} \langle \Psi, [Z_1(x_2, x_3), p_2 q_3 (\widehat{m}_{-1} - \widehat{m}_1)] q_1 g_{8/9,1}(x_1 - x_2) p_1 \Psi \rangle| \\
&\quad + N^3 |\mathfrak{S} \langle \Psi, [Z_1(x_2, x_3), q_2 p_3 (\widehat{m}_{-1} - \widehat{m}_1)] q_1 g_{8/9,1}(x_1 - x_2) p_1 \Psi \rangle| \\
&\leq N^3 |\mathfrak{S} \langle \Psi, Z_1(x_2, x_3) p_2 p_3 (\widehat{m}_{-1} - \widehat{m}_1) q_1 g_{8/9,1}(x_1 - x_2) p_1 \Psi \rangle| \\
&\quad + N^3 |\mathfrak{S} \langle \Psi, q_1 (\widehat{m}_{-1} - \widehat{m}_1) p_2 p_3 Z_1(x_2, x_3) g_{8/9,1}(x_1 - x_2) p_1 \Psi \rangle| \\
&\quad + N^3 |\mathfrak{S} \langle \Psi, Z_1(x_2, x_3) p_2 q_3 (\widehat{m}_{-1} - \widehat{m}_1) q_1 g_{8/9,1}(x_1 - x_2) p_1 \Psi \rangle| \\
&\quad + N^3 |\mathfrak{S} \langle \Psi, q_1 (\widehat{m}_{-1} - \widehat{m}_1) p_2 q_3 Z_1(x_2, x_3) g_{8/9,1}(x_1 - x_2) p_1 \Psi \rangle| \\
&\quad + N^3 |\mathfrak{S} \langle \Psi, Z_1(x_2, x_3) q_2 p_3 (\widehat{m}_{-1} - \widehat{m}_1) q_1 g_{8/9,1}(x_1 - x_2) p_1 \Psi \rangle| \\
&\quad + N^3 |\mathfrak{S} \langle \Psi, q_1 (\widehat{m}_{-1} - \widehat{m}_1) q_2 p_3 Z_1(x_2, x_3) g_{8/9,1}(x_1 - x_2) p_1 \Psi \rangle|
\end{aligned}$$

Using Lemma 3.4

$$\begin{aligned}
&\leq CN^2 \|p_2 Z_1(x_2, x_3) \Psi\| \|g_{8/9,1}(x_1 - x_2) p_1\|_{op} \\
&\quad + CN^2 \|p_2 \sqrt{Z_1(x_2, x_3)}\|_{op} \|g_{8/9,1}(x_1 - x_2) p_1\|_{op} \|\sqrt{Z_1(x_2, x_3)} \Psi\| \\
&\quad + CN^2 \|p_2 Z_1(x_2, x_3) \Psi\| \|g_{8/9,1}(x_1 - x_2) p_1\|_{op} \\
&\quad + CN^2 \|p_2 \sqrt{Z_1(x_2, x_3)}\|_{op} \|g_{8/9,1}(x_1 - x_2) p_1\|_{op} \|\sqrt{Z_1(x_2, x_3)} \Psi\| \\
&\quad + CN^2 \|p_3 Z_1(x_2, x_3) \Psi\| \|g_{8/9,1}(x_1 - x_2) p_1\|_{op} \\
&\quad + CN^2 \|p_3 \sqrt{Z_1(x_2, x_3)}\|_{op} \|g_{8/9,1}(x_1 - x_2) p_1\|_{op} \|\sqrt{Z_1(x_2, x_3)} \Psi\| \\
&\leq CN^{2-2-4/9} \|\varphi\|_\infty^2 = CN^{-4/9} \|\varphi\|_\infty^2 .
\end{aligned}$$

Using Lemma 3.4 (130) is bounded by

$$\begin{aligned}
(130) &= N^3 |\Im \langle \Psi, q_1 \widehat{m}_{-1} p_2 \sqrt{Z_1(x_2, x_3)} g_{8/9,1}(x_1 - x_2) p_1 \sqrt{Z_1(x_2, x_3)} \Psi \rangle| \\
&\leq CN^2 \|p_2 \sqrt{Z_1(x_2, x_3)}\|_{op} \|g_{8/9,1}(x_1 - x_2) p_1\|_{op} \|\sqrt{Z_1(x_2, x_3)} \Psi\| \\
&\leq CN^{-4/9} \|\varphi\|_\infty^2 .
\end{aligned}$$

For (131) we have again with Lemma 3.4

$$(131) \leq CN^2 \|p_2 Z_1(x_2, x_3) \Psi\| \|g_{8/9,1}(x_1 - x_2) p_1\|_{op} \leq CN^{-4/9} \|\varphi\|_\infty^2 .$$

Having controlled (126) and all terms in (127) we split up (128) using $1 = p_3 + q_3$ and $q_1 = 1 - p_1$

$$(128) = N^3 |\langle \Psi, q_1 q_2 \widehat{m}_{-1} g_{8/9,1}(x_1 - x_2) p_1 q_2 Z_1(x_1, x_3) \Psi \rangle| \quad (132)$$

$$\begin{aligned}
&+ \frac{N^3}{N-1} |\langle \Psi, a(|\varphi(x_1)|^2 + |\varphi(x_3)|^2) q_1 q_2 g_{8/9,1}(x_1 - x_2) \widehat{m} p_1 q_2 \Psi \rangle| \\
&\quad (133)
\end{aligned}$$

$$+ N^3 |\langle \Psi, V_1(x_1, x_3) q_1 q_2 g_{8/9,1}(x_1 - x_2) \widehat{m} p_1 q_2 p_3 \Psi \rangle| \quad (134)$$

$$+ N^3 |\langle \Psi, V_1(x_1, x_3) p_1 q_2 g_{8/9,1}(x_1 - x_2) \widehat{m} p_1 q_2 q_3 \Psi \rangle| \quad (135)$$

$$+ N^3 |\langle \Psi, V_1(x_1, x_3) p_2 g_{8/9,1}(x_1 - x_2) \widehat{m} p_1 q_2 q_3 \Psi \rangle| \quad (136)$$

$$+ N^3 |\langle \Psi, V_1(x_1, x_3) q_2 g_{8/9,1}(x_1 - x_2) \widehat{m} p_1 q_2 q_3 \mathbf{1}_{\mathcal{B}_2} \Psi \rangle| \quad (137)$$

$$+ N^3 |\langle \Psi, V_1(x_1, x_3) q_2 g_{8/9,1}(x_1 - x_2) \widehat{m} p_1 q_2 q_3 \mathbf{1}_{\mathcal{B}_2} \Psi \rangle| . \quad (138)$$

Using Lemma 3.4 (132), (133), (134), (135) and (136) are bounded by

$$\begin{aligned}
&CN^3 \|\widehat{m}_{-1} q_2 \Psi\|; \|g_{8/9,1}(x_1 - x_2) p_1\|_{op} \|p_1 Z_1(x_1, x_3) \Psi\| , \\
&CN^2 \|\varphi\|_\infty^2 \|g_{8/9,1}(x_1 - x_2) p_1\|_{op} \|\widehat{m} q_2 \Psi\| , \\
&N^3 \|p_3 V_1(x_1, x_3) \Psi\| \|g_{8/9,1}(x_1 - x_2) p_1\|_{op} \|\widehat{m} q_2 \Psi\| , \\
&N^3 \|p_1 V_1(x_1, x_3) \Psi\| \|g_{8/9,1}(x_1 - x_2) p_1\|_{op} \|\widehat{m} q_2 \Psi\| , \\
&N^3 \|\sqrt{V_1}(x_1, x_3) \Psi\| \|p_2 g_{8/9,1}(x_1 - x_2)\|_{op} \|\sqrt{V_1}(x_1, x_3) p_1\|_{op} \|\widehat{m} q_2 \Psi\| .
\end{aligned}$$

All these are smaller than $C\|\varphi\|_\infty^2 N^{-4/9}$.

Next we turn to (137). Since the support of $g_{8/9,1}(x_1 - x_2)$ is a subset of $\overline{\mathcal{B}}_3$ we get that (137) is bounded by

$$N^3 \|\mathbf{1}_{\overline{\mathcal{B}}_3} \sqrt{V_1}(x_1, x_3) \Psi\| \|g_{8/9,1}(x_1 - x_2) \sqrt{V_1}(x_1, x_3) \widehat{m} p_1 q_2 q_3 \mathbf{1}_{\overline{\mathcal{B}}_2} \Psi\|$$

The first factor is controlled by Lemma 6.4 (b)

$$(137) \leq N^{5/2} (\mathcal{K}(\varphi)(\alpha(\Psi, \varphi) + N^{-\eta}))^{1/2} \|g_{8/9,1}(x_1 - x_2) \sqrt{V_1}(x_1, x_3) \widehat{m} p_1 q_2 q_3 \mathbf{1}_{\overline{\mathcal{B}}_2} \Psi\|.$$

For the remaining factor we use for any fixed x_1, x_2, \dots, x_N Hölder and Sobolev under the x_2 -integral. Setting $\chi := \sqrt{V_1}(x_1, x_3) p_1 q_2 q_3 \widehat{m} \mathbf{1}_{\overline{\mathcal{B}}_2} \Psi$

$$\begin{aligned} \|g_{8/9,1}(x_1 - x_2) \chi\|^2 &\leq \|g_{8/9,1}^2\|_{3/2} \|\chi\|_3^2 \|\chi\|_3 \text{ in } x_2 \\ &= \|g_{8/9,1}\|_3^2 \|\chi\|_6^2 \text{ in } x_2 \leq \|g_{8/9,1}\|_3^2 \|\nabla_2 \chi\|^2. \end{aligned}$$

With Lemma 5.2 and (113) we get

$$\begin{aligned} &\|g_{8/9,1}(x_1 - x_2) \sqrt{V_1}(x_1, x_3) \widehat{m} p_1 q_2 q_3 \mathbf{1}_{\overline{\mathcal{B}}_2} \Psi\| \\ &\leq \|g_{8/9,1}\|_3 \|\nabla_2 \sqrt{V_1}(x_1, x_3) p_1 q_2 q_3 \widehat{m} \mathbf{1}_{\overline{\mathcal{B}}_2} \Psi\| \\ &\leq CN^{-1} (\ln N)^{1/3} \|\sqrt{V_1}(x_1, x_3) p_1\|_{op} \|\nabla_2 q_2 q_3 \widehat{m} \mathbf{1}_{\overline{\mathcal{B}}_2} \Psi\| \\ &\leq CN^{-5/2} (\ln N)^{1/3} \|\nabla_2 q_2 \mathbf{1}_{\overline{\mathcal{B}}_2} \Psi\| \end{aligned}$$

Since

$$\begin{aligned} \|\nabla_2 q_2 \mathbf{1}_{\overline{\mathcal{B}}_2} \Psi\| &\leq 2\|\nabla_2 \mathbf{1}_{\overline{\mathcal{B}}_2} \Psi\| + 2\|\nabla_2 p_2 \mathbf{1}_{\overline{\mathcal{B}}_2} \Psi\| \\ &\leq 2\|\nabla_2 \mathbf{1}_{\overline{\mathcal{B}}_2} \Psi\| + 2\|\nabla \varphi\| \|\mathbf{1}_{\overline{\mathcal{B}}_2} \Psi\| \end{aligned}$$

we get with Lemma 6.4 (c) and Proposition 6.2 that the latter is bounded by $(\mathcal{K}(\varphi)(\alpha(\Psi, \varphi) + N^{-\eta}))^{1/2}$.

Using symmetry (138) is bounded by

$$\begin{aligned}
& CN^2 |\langle \Psi, V_1(x_1, x_3) \sum_{j=4}^N g_{8/9,1}(x_1 - x_j) \widehat{m} p_1 q_3 q_j \mathbf{1}_{\mathcal{B}_j} \Psi \rangle| \\
&= CN^2 |\langle \Psi, \sqrt{V_1}(x_1, x_3) (p_1 p_3 + p_1 q_3 + q_1 p_3 + q_1 q_3) \rangle| \quad (139)
\end{aligned}$$

$$\begin{aligned}
& \sqrt{V_1}(x_1, x_3) \sum_{j=4}^N g_{8/9,1}(x_1 - x_j) \widehat{m} p_1 q_3 q_j \mathbf{1}_{\mathcal{B}_j} \Psi \rangle| \\
&= CN^2 |\langle \Psi, \sqrt{V_1}(x_1, x_3) (p_1 p_3 \widehat{m}_1 + p_1 q_3 \widehat{m} + q_1 p_3 \widehat{m} + q_1 q_3 \widehat{m}_{-1}) \\
& \quad \sum_{j=4}^N g_{8/9,1}(x_1 - x_j) \sqrt{V_1}(x_1, x_3) p_1 q_3 q_j \mathbf{1}_{\mathcal{B}_j} \Psi \rangle| \\
&\leq CN^3 \|\sqrt{V_1}(x_1, x_3) \Psi\| \|\widehat{m}_1\|_{op} \|p_1 \sqrt{g_{8/9,1}}(x_1 - x_2)\|_{op} \quad (140)
\end{aligned}$$

$$\begin{aligned}
& \|p_3 \sqrt{V_1}(x_1, x_3)\|_{op} \|\sqrt{g_{8/9,1}}(x_1 - x_2) p_1\|_{op} \\
&+ CN^2 \|(p_1 q_3 \widehat{m} + q_1 p_3 \widehat{m} + q_1 q_3 \widehat{m}_{-1}) \sqrt{V_1}(x_1, x_3) \Psi\| \quad (141) \\
& \|\sum_{j=4}^N g_{8/9,1}(x_1 - x_j) \sqrt{V_1}(x_1, x_3) p_1 q_3 q_j \mathbf{1}_{\mathcal{B}_j} \Psi\|.
\end{aligned}$$

(140) is bounded by

$$CN^{1/2-16/9} \|\varphi\|_{\infty}^3.$$

Using Lemma 3.4 and Lemma 4.5 the first factor of (141) is bounded by $CN^{-3/2}$. Using the abbreviation $\chi_j = \sqrt{V_1}(x_1, x_3) p_1 q_3 q_j \mathbf{1}_{\mathcal{B}_j} \Psi$ we can write for the second factor

$$\begin{aligned}
\|\sum_{j=4}^N g_{8/9,1}(x_1 - x_j)\|^2 &\leq N^2 \langle \chi_4 g_{8/9,1}(x_1 - x_4) g_{8/9,1}(x_1 - x_5) \chi_5 \rangle \\
&+ N \langle \chi_4 (g_{8/9,1}(x_1 - x_4))^2 \chi_4 \rangle
\end{aligned}$$

Since $g_{8/9,1}(x_1 - x_5)$ and $g_{8/9,1}(x_1 - x_4)$ commute we get

$$\leq N^2 \|g_{8/9,1}(x_1 - x_4) \chi_5\|^2 + N \|g_{8/9,1}(x_1 - x_4) \chi_4\|^2$$

Using this and $q_3 = 1 - p_3$ it follows that

$$\begin{aligned}
(141) &\leq CN^{3/2} \|g_{8/9,1}(x_1 - x_4) \sqrt{V_1}(x_1, x_3) p_1 q_5 \mathbf{1}_{\mathcal{B}_5} \Psi\| \quad (142) \\
&+ CN^{3/2} \|g_{8/9,1}(x_1 - x_4) \sqrt{V_1}(x_1, x_3) p_1 p_3 q_5 \mathbf{1}_{\mathcal{B}_5} \Psi\| \\
&+ CN \|g_{8/9,1}(x_1 - x_4) \sqrt{V_1}(x_1, x_3) p_1 q_3 q_4 \mathbf{1}_{\mathcal{B}_4} \Psi\|
\end{aligned}$$

Note that for large enough N the function $\sqrt{V_1}(x_1, x_3) g_{8/9,1}(x_1 - x_4)$ is different from zero only inside the set $a_{3,4}$. Since \mathcal{B}_5 and $a_{3,4}$ are by

definition disjoint it follows that

$$\begin{aligned}
& g_{8/9,1}(x_1 - x_4)V_1(x_1, x_3)g_{8/9,1}(x_1 - x_5)p_1q_5\mathbb{1}_{\mathcal{B}_5}\Psi \\
& = g_{8/9,1}(x_1 - x_4)V_1(x_1, x_3)g_{8/9,1}(x_1 - x_5)p_1q_5\mathbb{1}_{a_{3,4}}\mathbb{1}_{\mathcal{B}_5}\Psi \\
& = 0.
\end{aligned}$$

Thus the second summand in (142) is zero. Using as above Hölder and Sobolev under the x_2 -integral of the third summand in (142) we get that (141) is bounded by

$$\begin{aligned}
& CN^{-23/18}\|\varphi\|_\infty^3 + CN^{3/2}\|\sqrt{V_1}(x_1, x_3)p_3\|_{op}\|g_{8/9,1}(x_1 - x_4)p_1\|_{op} \\
& + N\|g_{8/9,1}\|_3^2\|\nabla_4\sqrt{V_1}(x_1, x_3)g_{8/9,1}(x_1 - x_4)\widehat{m}p_1q_3q_4\mathbb{1}_{\mathcal{B}_4}\Psi\|^2 \\
& \leq CN^{-4/9}\|\varphi\|_\infty^2 + CN^{-1}(\ln N)^{2/3}\|\sqrt{V_1}(x_1, x_3)p_1\|_{op}^2\|\nabla_4\widehat{m}q_3q_4\mathbb{1}_{\mathcal{B}_4}\Psi\|^2 \\
& \leq CN^{-4/9}\|\varphi\|_\infty^2(\ln N)^{2/3}.
\end{aligned}$$

(d) Using symmetry and Lemma 3.2 (d) γ^d is bounded by

$$\begin{aligned}
\gamma^d & \leq N^4\mathfrak{S}(\langle\langle\Psi, q_1q_2g_{8/9,1}(x_1 - x_2)p_1q_2[Z_1(x_3, x_4), p_3p_4(\widehat{m} - \widehat{m}_2)]\Psi\rangle\rangle) \\
& + 2N^4\mathfrak{S}(\langle\langle\Psi, q_1q_2g_{8/9,1}(x_1 - x_2)p_1q_2[Z_1(x_3, x_4), p_3q_4(\widehat{m} - \widehat{m}_1)]\Psi\rangle\rangle) \\
& \leq N^4\mathfrak{S}(\langle\langle\Psi, Z_1(x_3, x_4)p_3p_4q_1q_2\widehat{n}_1^{-2}g_{8/9,1}(x_1 - x_2)p_1q_2\widehat{n}_2^2(\widehat{m} - \widehat{m}_2)\Psi\rangle\rangle) \\
& + N^4\mathfrak{S}(\langle\langle\Psi, q_1q_2\widehat{n}_1^{-2}g_{8/9,1}(x_1 - x_2)p_1q_2\widehat{n}_2^2(\widehat{m} - \widehat{m}_2)p_3p_4Z_1(x_3, x_4)\Psi\rangle\rangle) \\
& + 2N^4\mathfrak{S}(\langle\langle\Psi, Z_1(x_3, x_4)p_3q_4q_1q_2\widehat{n}_1^{-2}g_{8/9,1}(x_1 - x_2)p_1q_2\widehat{n}_2^2(\widehat{m} - \widehat{m}_1)\Psi\rangle\rangle) \\
& + 2N^4\mathfrak{S}(\langle\langle\Psi, q_1q_2\widehat{n}_1^{-2}g_{8/9,1}(x_1 - x_2)p_1q_2\widehat{n}_2^2(\widehat{m} - \widehat{m}_1)p_3q_4Z_1(x_3, x_4)\Psi\rangle\rangle)
\end{aligned}$$

With Lemma 3.4 it follows that

$$\begin{aligned}
\gamma^d & \leq CN^4\|p_3Z_1(x_3, x_4)\Psi\|\|g_{8/9,1}(x_1 - x_2)p_1\|_{op}\|\widehat{n}\widehat{n}_2^2(\widehat{m} - \widehat{m}_2)\Psi\| \\
& + CN^4\|g_{8/9,1}(x_1 - x_2)p_1\|_{op}\|\widehat{n}\widehat{n}_2^2(\widehat{m} - \widehat{m}_2)\|_{op}\|p_3Z_1(x_3, x_4)\Psi\| \\
& + CN^4\|p_3Z_1(x_3, x_4)\Psi\|\|g_{8/9,1}(x_1 - x_2)p_1\|_{op}\|\widehat{n}\widehat{n}_2^2(\widehat{m} - \widehat{m}_1)\Psi\| \\
& + CN^4\|g_{8/9,1}(x_1 - x_2)p_1\|_{op}\|\widehat{n}\widehat{n}_2^2(\widehat{m} - \widehat{m}_1)\|_{op}\|p_3Z_1(x_3, x_4)\Psi\|.
\end{aligned}$$

Recall that $\widehat{m} = \widehat{m}^0 - \widehat{m}_1^0$. Due to Lemma 70

$$m(k) - m(k+1) = m^0(k) - 2m^0(k+1) + m^0(k+2) \leq CN^{-2}n(k+1)^{-3}$$

and

$$\begin{aligned}
m(k) - m(k+2) & = m^0(k) - m^0(k+1) - m^0(k+2) + m^0(k+3) \\
& \leq CN^{-2}n(k+1)^{-3}.
\end{aligned}$$

It follows that

$$\gamma^d \leq CN^4\|p_3Z_1(x_3, x_4)\Psi\|\|g_{8/9,1}(x_1 - x_2)p_1\|_{op}N^{-2}.$$

In view of (75) we get that also γ^d is bounded by the right hand side of (6.9). \square

6.4 Proof of the Theorem for $\beta = 1$

Corollary 6.10 *Let $\beta = 1$. There exists a functional $\Gamma : L^2(\mathbb{R}^{3N}, \mathbb{C}) \otimes L^2(\mathbb{R}^3, \mathbb{C}) \rightarrow \mathbb{R}^+$, a functional $\Gamma' : L^2(\mathbb{R}^{3N}, \mathbb{C}) \otimes L^2(\mathbb{R}^3, \mathbb{C}) \rightarrow \mathbb{R}$ and a constant $c > 0$ such that*

(a)

$$\left| \frac{d}{dt} \Gamma(\Psi_t, \varphi_t) \right| \leq |\Gamma'(\Psi_t, \varphi_t)|.$$

(b)

$$c\alpha(\Psi, \varphi) - CN^{-\eta} \leq \Gamma(\Psi, \varphi) \leq \alpha(\Psi, \varphi) + CN^{-\eta}$$

uniform in Ψ, φ

(c) *There exists a functional $\mathcal{K} \in \mathcal{F}$ such that*

$$|\Gamma'(\Psi, \varphi)| \leq (\|\varphi\|_\infty + (\ln N)^{1/3} \|\nabla \varphi\|_{6,loc} + \|\dot{A}\|_\infty) \mathcal{K}(\varphi) (\alpha(\Psi, \varphi) + N^{-\eta})$$

uniform in Ψ, φ .

Proof: Set

$$\Gamma(\Psi, \varphi) := \gamma(\Psi, \varphi) + \sum_{j+k \leq 5} 2^{-j-k} \gamma_{j,k}(\Psi, \varphi) + |\mathcal{E}(\Psi) - \mathcal{E}^{GP}(\varphi)| \quad \text{and}$$

$$\Gamma'(\Psi, \varphi) := \gamma'(\Psi, \varphi) + \sum_{j+k \leq 5} 2^{-j-k} \gamma'_{j,k}(\Psi, \varphi) + \frac{d}{dt} |\mathcal{E}(\Psi) - \mathcal{E}^{GP}(\varphi)|.$$

(a) follows from Lemma 5.9 with Lemma 6.8 and (23).

(b) follows from Corollary 5.12 (b) together with Corollary 6.7.

(c) Remember that $\xi := \gamma'_{0,0} - \mathfrak{S}(\xi_{0,0})$ (see Corollary 5.10). Thus

$$\begin{aligned} \Gamma'(\Psi, \varphi) &= \gamma'(\Psi, \varphi) - \xi(\Psi, \varphi) \\ &+ \sum_{1 \leq j+k \leq 5} 2^{-j-k} (\gamma'_{j,k}(\Psi, \varphi) + 2\mathfrak{S}(\xi_{j-1,k}) - \mathfrak{S}(\xi_{j,k})) \\ &+ \sum_{j+k=5} 2^{-5} \mathfrak{S}(\xi_{j,k}) + \frac{d}{dt} |\mathcal{E}(\Psi) - \mathcal{E}^{GP}(\varphi)| \end{aligned}$$

The first line is controlled by Lemma 6.9. The second line by Lemma 5.11.

The third line is bounded by Proposition 5.7 and (23). □

From (b) and (c) it follows that

$$\Gamma'(\Psi, \varphi) \leq (\|\varphi\|_\infty + (\ln N)^{1/3} \|\nabla \varphi\|_{6,loc} + \|\dot{A}\|_\infty) \mathcal{K}(\varphi) (\Gamma(\Psi, \varphi) + N^{-\eta})$$

and we get via Grönwall

$$\Gamma(\Psi_t, \varphi_t) \leq e^{\int_0^t (\|\varphi_s\|_\infty + (\ln N)^{1/3} \|\nabla \varphi_s\|_{6,loc} + \|\dot{A}_s\|_\infty) \mathcal{K}(\varphi_s) ds} (\Gamma(\Psi_0, \varphi_0) + N^{-\eta}).$$

For $\varphi \in \mathcal{G}$ we have that $\sup_{s \in \mathbb{R}} \{\mathcal{K}(\varphi_s)\} < \infty$. Again using (b) we get the bound on $\alpha(\Psi_t, \varphi_t)$ as stated in Theorem 2.5.

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