Optimal Monitoring Schedule in Dynamic Contracts

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Consider a setting in which a principal induces effort from an agent to reduce the arrival rate of a Poisson process of adverse events. The effort is costly to the agent, and unobservable to the principal, unless the principal is monitoring the agent. Monitoring ensures effort but is costly to the principal. The optimal contract involves monetary payments and monitoring sessions that depend on past arrival times. We formulate the problem as a stochastic optimal control model and solve the problem analytically. The optimal schedules of payment and monitoring demonstrate different structures depending on model parameters, and may involve monitoring for a random period of time. Overall, the optimal dynamic contracts are simple to describe, easy to compute and implement, and intuitive to explain.

Key words: Dynamic Contract, Moral Hazard, Principal-agent Model, Optimal Control, Continuous Time, Costly State Verification.

History: Current version: August 26, 2018.

1. Introduction

Adverse events, such as IT system failures at British Airways in May and August of 2017 (Held 2017), massive data breaches at Yahoo in 2013 and 2014 (Goel and Perlroth 2016), or product adulterations described in Babich and Tang (2012), often bring significant damages to an organization or the society. In many situations, better efforts in maintaining and safeguarding a system can reduce the chance of such adverse events. The challenge is that these events may still occur, albeit less frequently, despite the best effort. And efforts are often hard to verify. Furthermore, people in charge of the effort (an agent) often cannot bear the full consequence of an adverse event due to limited liability. In practice, an agent is often a hired employee or subcontractor, who can be paid one way or another, but cannot compensate damages. In order to ensure efforts, a principal, be it a firm or a government, may decide to "keep an eye" on the agent, which ensures that adverse events occur at a lower rate, and are not due to lack of effort should they happen. For example, Accenture served as an outside vendor to support the IT systems for Kasikornbank of Thailand (also known as the Kbank). According to our conversations with Accenture, once in a while, the

in-house IT team at Kbank would show up to watch the Accenture team working. Such monitoring activities are often too costly to conduct at all the time. The principal can also schedule payments that are contingent on arrivals to motivate effort. How should a principal maintain efforts from the agent while minimizing the total payments and monitoring costs? In particular, in a dynamic setting where adverse events stochastically occur over time, what is the optimal schedule to pay and to monitor the agent?

To answer these questions, we study an optimal contract design problem in a dynamic setting, in which a risk-neutral principal faces a Poisson process of costly adverse events. The instantaneous rate of the Poisson process can be reduced by a risk-neutral agent, if the agent exerts effort at that moment. Effort is costly to the agent and observable to the principal only when the principal conducts costly monitoring. The principal, who can commit to a long term contract over an infinite horizon in continuous time, needs to trade-off direct payments to the agent, versus costly monitoring, in order to induce effort.

We formulate this optimal dynamic contract design problem as a continuous time stochastic optimal control model. The model identifies the optimal monitoring and payment schedule among general admissible policies that ensure continued effort from the agent. We are able to provide a complete characterization of the optimal control policy, which varies depending on the monitoring cost. As expected, if the monitoring cost is lower than a threshold, the principal should monitor all the time. In this case, the agent's total future utility (commonly referred to as the "promised utility," see, for example, Spear and Srivastava 1987) is always kept at 0. Interesting structures emerge when the monitoring cost is higher than the threshold. In this case, the optimal monitoring and payment schedules depend critically on the agent's promised utility. In order to better explain the results more intuitively for some readers of this journal, we describe the optimal contract in a quality control setting and call arrivals defects in a production process.

Generally speaking, if the cost is too high for the principal to monitor all the time, the agent needs to be penalized for each arrival of defects when not being monitored. Because we assume that the agent has limited liability and cannot pay the principal, the penalty takes the form of reduced total future utility (that is, again, the promised utility). That is, as long as this penalty is high enough, the agent is willing to exert effort to reduce the arrival rate of defects. There is a minimum level of penalty that induces the effort. The promised utility needs to take a downward jump of this minimum penalty for each defect without monitoring. Between arrivals of defects, the promised utility gradually increases. When the agent is being monitored, this gradual increase only includes the accrued interest from the promised utility, reflecting the agent's time value of money. When the agent is not being monitored, however, this increase also includes the information rent, which equals the minimum penalty level for each arrival times the arrival rate under effort. When the promised

utility reaches an upper bound, the interest becomes so high that the principal stops delaying the cash payment any further. At this point the principal starts a flow of cash payments that exactly offsets the potential increase in the promised utility. The key trade-off that the principal faces is, therefore, when to spend money on monitoring, and when to pay the information rent, either in cash or in promised utility.

The answer to this question depends on the time discount rates of the two players. When the two players share the same discount rate (i.e., when the patience levels are the same), two distinct structures are both optimal. In the first structure, the principal monitors if and only if the promised utility is below the minimum penalty level mentioned above. (See Figure 1 for an illustration.) This is intuitive, because if the promised utility is already below this level, another decrease of the minimum penalty would bring it to negative, which is not allowed due to limited liability. Therefore, the principal cannot use penalty to induce effort anymore, and has to rely on monitoring. The second optimal contract structure involves randomly starting monitoring whenever an arrival brings the promised utility down below the minimum penalty level. At this point, the principal resets the promised utility back to the minimum penalty level with a probability, or starts monitoring forever while keeping the promised utility at zero. (See Figure 3 for an illustration.) The corresponding principal's value function is concave, with a linear piece between zero and the minimum penalty level. (See Figure 2 for an illustration.) According to the optimal contract, the initial promised utility starts from the value function's maximizer.

When the principal's discount rate is smaller than the agent's (i.e., when the principal is more patient than the agent), the optimal contract demonstrates subtle and important differences in the structure, compared with the equal discount case. In particular, the principal still monitors the agent if and only if the promised utility is lower than a threshold. However, this threshold may not be the minimum penalty level anymore. Instead, it could be higher. If the monitoring threshold is indeed higher than the minimum penalty level, the value function is not only concave and nonlinear, but also differentiable at the monitoring threshold. (See Figure 5 for an illustration.) This is akin to the "smooth pasting" phenomenon arising in optimal stopping problems (see, for example, Dixit 1994).

When the monitoring threshold is higher than the minimum penalty level, there is an interesting implication. Because the promised utility only jumps down with a magnitude of the minimum penalty level, if the threshold is strictly higher than this level, the promised utility never reaches zero. In the optimal control model, feasible contracts have to guarantee that the promised utility is non-negative, which is commonly referred to as either the participation or the individual rationality (IR) constraint. Our result implies that this IR constraint may not be binding at optimality, which seems to be in sharp contrast with the vast majority of the mechanism design literature, starting

from Myerson (1981). In order to explain this somewhat puzzling phenomenon, we may consider the time it takes for the promised utility to increase back to the threshold for monitoring to stop. The lower the initial promised utility level below the threshold, the longer it takes for monitoring to stop. When the promised utility decreases towards zero, the time it takes for monitoring to stop approaches infinity. Setting a threshold higher shortens the monitoring time in the worse case, which explains why it is the preferred choice for high monitoring costs.

The aforementioned insights on the movements of the promised utility and payment schedule is not completely new to our model. In fact, both Biais et al. (2010) and Myerson (2015) study similar models as ours, without monitoring. Biais et al. (2010) consider the agent as a firm, and the principal as an investor. Instead of randomization, the principal can change the firm size when the promised utility becomes too low. Myerson (2015), on the other hand, considers a political economy setting, in which the agent's size cannot be adjusted, but the principal can dynamically replace an agent with a new one. Despite these differences, the movements of the promised utility above the minimum penalty level are essentially the same.

The fundamental difference between Biais et al. (2010) and Myerson (2015) is the time discount rate. In Biais et al. (2010), the principal is strictly more patient than the agent, while Myerson (2015) assumes the two players' time discount rates are the same. Equal time discount rate in this setting introduces an "infinite back-loading" issue. That is, the principal always prefers to delay the cash payment to the future while promising to pay the corresponding interest. In order to prevent the problem to become unbounded, Myerson (2015) introduces an exogenous upper bound on the promised utility. For the different discount rate case, Biais et al. (2010) obtain an endogenous upper bound on the promised utility. We study both the different and equal time discount cases. For the different discount case, although it is also not necessary to introduce an exogenous upper bound, we also describe the optimal contract under such a bound, in case the endogenous one is high for the agent to stomach in practice.

The main differences between our results and those in Biais et al. (2010) and Myerson (2015) lie in the monitoring schedule, and the corresponding principal's value function. In the models without monitoring, as long as the promised utility is below the minimum penalty level, the principal needs to either abruptly downsize the firm (Biais et al. 2010), or randomize between replacing the agent or not (Myerson 2015). In either case, the value function is linear when the promised utility is below the minimum penalty level, and smooth pasting does not arise. In our model, as mentioned earlier, the value function is nonlinear when the principal is more patient than the agent. When the monitoring threshold is higher than the minimum penalty level, our paper appears to offer a novel setting where smooth pasting takes place.

Optimal scheduling of monitoring in a dynamic environment is fundamentally an operations problem. There is a recent stream of papers in the operations research/management science literature that study incentive issues related to auditing/monitoring/inspecting. Most of these papers compare a few classes of practically useful mechanisms, or focus on static settings. Babich and Tang (2012), for example, study three mechanisms (deferred payment mechanism, inspection mechanism, and a mechanism that combines the two) for dealing with product adulteration issues when manufacturers cannot control the suppliers' actions. Rui and Lai (2015) study a similar sets of mechanisms in a similar problem setting, with endogenous procurement decisions and more general defect discovery processes. Kim (2015) studies environmental disclosure and inspection policies in a dynamic setting, and compares deterministic versus random inspection schedules. Plambeck and Taylor (2016) and Plambeck and Taylor (2017) also study environmental monitoring and disclosure issues, using static models. Wang et al. (2016) study monetary and inspection instruments to induce the agent to report the occurrence of an adverse environmental event. The paper models this dynamic adverse selection problem as an optimal control model in continuous time and identifies optimal policies.

Monitoring is a way to conduct *costly state verification* under asymmetric information in the economics literature, started from Townsend (1979) for adverse selection issues in a static setting. Dye (1986) extends the idea to moral hazard problems, also in a static setting. In continuous time dynamic settings, Piskorski and Westerfield (2016) study a model in which the underlying uncertainty is a Brownian motion, and the principal checks the agent following a Poisson process, the rate of which is the design issue. If the agent is found shirking, the principal may terminate the contract. Varas et al. (2017) study a two state hidden Markov model, whose instantaneous transition rates are affected by the agent's effort. The principal decides the schedule of inspecting the true state of the Markov chain in order to induce effort.

Our model and analysis is rooted in the continuous time optimal contracting literature. Sannikov (2008) provides the analytical foundation for these types of models. In Sannikov (2008), the agent's effort affects the drift of a Brownian motion. And the optimal contract is solved as the solution of a stochastic optimal control problem. The Brownian motion setup is natural for corporate finance applications (see, for example, DeMarzo and Sannikov 2006, Biais et al. 2007, Fu 2017, to name a few).

Biais et al. (2010) build upon this framework and study continuous time optimal contracting based on Poisson processes, instead of Brownian motions. One important advantage of Poisson process based models is that the optimal control policies are often easier to describe and implement. Myerson (2015), as mentioned earlier, also studies contract design to induce an agent to reduce the arrival rate of adverse events, although the analysis is based on discrete time approximation. More

recently, Sun and Tian (2017) study how to induce an agent to increase the arrival rate of a Poisson process in a continuous time infinite horizon setting. More broadly, a sequence of recent papers also study optimal contracting problems related to Poisson arrivals (see, for example, Mason and Välimäki 2015, Varas 2017, Green and Taylor 2016, Shan 2017, Hidir 2017). Our paper differs from the aforementioned continuous time dynamic contracting literature in the monitoring component.

More generally, dynamic moral hazard problem has also been the focus of some recent papers published in Operations Research. Plambeck and Zenios (2003) study continuous time control of incentive issues in a make-to-stock production system. Li et al. (2012) investigate how to motivate multiple agents (suppliers) in a discrete time dynamic setting. Their model is also based on the promised utility framework originated from Spear and Srivastava (1987).

The rest of this paper is organized as follows. We introduce the model and supporting concepts in Section 2. We then focus on the equal discount rate case in Section 3, where we introduce general structures of optimal contracts and value functions. Built upon the various concepts introduced in Section 3, Section 4 further investigates the case where the principal is more patient than the agent. We then discuss a number of extensions in Section 5, and conclude the paper with potential directions for future research in Section 6.

2. The Model

We consider a principal-agent model in a continuous time setting. The principal faces a Poisson process with arrival rate $\bar{\lambda}$ of adverse events (defects), each costing the principal a value K. For illustration purposes, we use "arrival" and "defect" interchangeably, even though the model is not limited to representing production processes. The principal hires an agent, who can bring down the instantaneous arrival rate to $\lambda = \bar{\lambda} - \Delta\lambda$, if the agent exerts effort at this point in time. Effort is costly to the agent at a constant rate per unit of time. Whenever shirking, the agent receives a flow of benefit income at rate b. The principal does not observe the effort, unless *monitoring* the agent, at a cost rate m per unit of time. Denote a left-continuous counting process $\{N_t\}_{t\geq 0}$ to represent the total number of arrivals up to time t, the rate of which depends on the agent's effort process. Further denote $\Lambda = \{\lambda_t\}_{t\geq 0}$ to represent the agent's effort process. That is, $\lambda_t \in \{\lambda, \bar{\lambda}\}$ at each time epoch t. The principal and the agent are both risk-neutral and discount future cash flows with discount rates r and ρ, respectively. As is often assumed in the literature (see, for example, Biais et al. 2010), $\rho \geq r > 0$; i.e., the principal is no less patient than the agent. We start with $\rho = r$ in Section 3 before moving to consider $\rho > r$ in Section 4.

We assume that the principal has commitment power to issue a long term contract with the agent. The contract specifies a payment and monitoring schedule over time, which depends on past arrival times. The agent has limited liability, which means that monetary transfer from the principal to

the agent is non-negative at any point in time. Therefore, the agent cannot buy out the principal to mitigate misalignment of incentives. Specifically, denote L_t to represent the principal's cumulative payment to the agent up to time t, such that $dL_t = I_t + \ell_t dt$, in which I_t is an instantaneous payment and ℓ_t is a flow payment at time t, with $I_t \geq 0$ and $\ell_t \geq 0$.

Before moving forward, we elaborate three points in the modeling choice. First, for simplicity of exposition, we assume that the cost K for each arrival is a constant in our model. In fact, our results naturally extend to the case where the cost of each arrival is a random variable and K is its mean, as long as the cost random variable is independent to the effort process. Second, in this paper we focus on contracts that always induce effort from the agent. Therefore, in the base model, the principal always pays the effort cost. That is, the benefit b can be interpreted as saved effort cost. Later in Section 5.3 we extend the model and let the principal optimally delay payments to cover the effort cost. Third, the effort level may be more general than binary. An effort cost that is convex in the effort level yields the same results.

Denote process $\mathcal{M} = \{m_t\}_{t \geq 0}$ to represent the monitoring schedule under the contract, in which $m_t \in \{m, 0\}$ captures the monitoring cost at time t. Monitoring during a time interval $(t, t+\delta]$ guarantees the agent's effort during this time interval. Monitoring may start in one of two ways. First, at any point in time t when an arrival occurs, the principal may decide to start monitoring with a probability $y_t \in [0, 1]$. Second, monitoring may also start at time t in a "deterministic" fashion with respect to realizations of past uncertainties. In order to formally characterize past uncertainties, it is convenient to split the counting process $\{N_t\}_{t \geq 0}$ into two processes $\{N_t^s\}_{t \geq 0}$ and $\{N_t^n\}_{t \geq 0}$, which represent the total numbers of arrivals which have and have not triggered monitoring up to time t, respectively. Therefore, $N_t^s + N_t^n = N_t$. Consequently, we define filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}$ such that $\mathcal{F}_t$ captures the entire historical information up to time t specified by the 2-variate counting process $\{N_t^s, N_t^n\}_{t \geq 0}$. Overall, a contract Γ consists of $\mathcal{F}_t$-predictable payment and monitoring processes, L_t and m_t, respectively.

2.1. Agent's Utility

Given contract Γ and the agent's effort process Λ, the agent's total utility is defined as

$$u(\Gamma, \Lambda) = \mathbb{E}^{\Gamma, \Lambda} \left[\int_0^\infty e^{-\rho\tau} (dL_\tau + b \mathbb{I}_{\lambda_\tau = \bar{\lambda}} \, d\tau) \right]. \tag{2.1}$$

It is standard and convenient to work with the agent's continuation utility (also referred to as the *promised utility*, see, for example, Spear and Srivastava 1987). That is, the total discounted utility starting from time t, defined as the following left-continuous process,

$$W_t(\Gamma, \Lambda) = \mathbb{E}^{\Gamma, \Lambda} \left[\int_t^\infty e^{-\rho(\tau - t)} (dL_\tau + b \mathbb{I}_{\lambda_\tau = \bar{\lambda}} \, d\tau) \, \Big| \, \mathcal{F}_t \right]. \tag{2.2}$$

When there is no confusion, we omit (Γ, Λ) and refer to the agent's continuation utility at time t as W_t. Clearly, $W_0 = u(\Gamma, \Lambda)$. As is often assumed in the literature, the agent does not commit to staying in the contract. Therefore, we require the following participation (also referred to as the *individual rationality*, or, IR) constraint

$$W_t \geq 0, \text{ for all } t \geq 0. \tag{IR}$$

Later in the paper, the optimal contract and the principal's value function are all expressed as functions of the agent's promised utility, so that the optimal contract design problem is essentially a stochastic optimal control with W_t being the state variable.

Under any contract, the agent's promised utility W_t must satisfy the following dynamics.

LEMMA 1. *For any contract Γ, there exits $\mathcal{F}_t$-predictable process $\{(H_t^s, H_t^n)\}_{t \geq 0}$, such that*

$$dW_t = \{\rho W_t - b\mathbb{I}_{\lambda_\tau = \bar{\lambda}} + \lambda_t[y_t H_t^s + (1 - y_t)H_t^n]\}dt - H_t^s dN_t^s - H_t^n dN_t^n - dL_t. \tag{PK}$$

In order to satisfy the agent's continued participation (IR), *H_t^s and H_t^n are less than or equal to W_t.*

The condition (PK) stands for "promise keeping," which ensures that W_t is indeed the agent's continuation utility starting from time t. Lemma 1 follows directly from the Martingale Representation Theorem (Theorem T9, Brémaud 1981, page 64), and extends Lemma 1 in Biais et al. (2010) and Lemma 6 in Sun and Tian (2017) to our setting with a multi-variate counting process. Here we provide an heuristic derivation of (PK) following discrete time approximation, which offers an intuitive illustration.

Consider the promised utility at the beginning of a small time interval $[t, t + \delta)$ to be W_t. With probability $\lambda_t \delta y_t$, there is a defect that triggers monitoring to start. In this case, the promised utility moves to $W_t - H_t^s$. With probability $\lambda_t \delta(1 - y_t)$, on the other hand, there is a defect that does not trigger monitoring. In this case, the promised utility moves to $W_t - H_t^n$. Finally, with probability $1 - \lambda_t \delta$, no arrival occurs and the promised utility moves to $W_{t+\delta}$. Taking into consideration the potential benefit from shirking, and ignoring payment for simplicity, we have

$$W_t = b\delta\mathbb{I}_{\lambda_\tau = \bar{\lambda}} + e^{-\rho\delta}\left\{\lambda_t\delta\left[y_t(W_t - H_t^s) + (1 - y_t)(W_t - H_t^n)\right] + (1 - \lambda_t\delta)W_{t+\delta}\right\}.$$

Following the standard procedure of subtracting W_t from and dividing δ on both sides, and letting δ approach zero, we obtain

$$\lim_{\delta \downarrow 0} \frac{W_{t+\delta} - W_t}{\delta} = \rho W_t - b\mathbb{I}_{\lambda_\tau = \bar{\lambda}} + \lambda_t\left[y_t H_t^s + (1 - y_t)H_t^n\right],$$

which explains the continuously changing part of (PK). The additional terms in (PK), $H_t^s dN_t^s$ and $H_t^n dN_t^n$, further capture the jumps in the promised utility due to defects as mentioned just now. Finally, payment dL_t at time t naturally brings down total future payments.

2.2. Incentive Compatibility

In this paper we consider contracts that always induce effort from the agent. Clearly, the principal is willing to monitor the agent if the monitoring cost m is not prohibitively high. Otherwise the principal may be better off letting the agent shirk rather than enforcing effort through monitoring. Therefore, we assume that

$$K\Delta\lambda \geq m, \tag{2.3}$$

which implies that the cost K of a defect is sufficiently high such that even monitoring all the time is worthwhile to lower its arrival rate. Condition (2.3) allows us to transform our problem into an optimal control model over contracts that must always induce effort.[1] Correspondingly, the counting process $\{N_t\}_{t\geq 0}$ admits intensity $\lambda_t = \lambda$ for all $t \geq 0$.

If a contract Γ induces the agent to always exert effort, that is,

$$u(\Gamma,\bar{\Lambda}) \geq u(\Gamma,\Lambda), \text{ for } \bar{\Lambda} := \{\lambda_t = \lambda\}_{t\geq 0} \text{ and } \forall\Lambda, \tag{2.4}$$

then we call contract Γ *incentive compatible*.

Incentive compatibility critically depends on the following ratio between the agent's private benefit b and the difference in arrival rates $\Delta\lambda$,

$$\beta := \frac{b}{\Delta\lambda}. \tag{2.5}$$

Intuitively, should the principal be able to charge the agent an amount β for each defect, the agent would be indifferent between exerting effort or not. – Shirking in a small time interval δ brings the agent a benefit $b\delta$, which is offset by the higher penalty cost $\Delta\lambda\beta\delta$. – Because charging the agent is not allowed in our setting, the principal instead reduces the agent's promised utility by at least β for each defect in order to induce effort.[2] Therefore, the value β is the minimum penalty in the promised utility that induces effort from the agent, as mentioned in the introduction. This is formalized in the following result.

LEMMA 2. *Contract Γ is incentive compatible, i.e., it satisfies (2.4), if and only if the following condition holds:*

$$y_t H_t^s + (1-y_t)H_t^n \geq \beta, \quad \text{if } m_t = 0. \tag{IC}$$

The term $y_t H_t^s + (1-y_t)H_t^n$ is the expected downward jump when there is an arrival at time t. Constraint (IC) implies that this expected downward jump is at least β when the principal does not monitor.

Constraint (IC) further implies that either H_t^s or H_t^n has to be at least β. Therefore, we can simultaneously satisfy constraints (IC) and (IR) after the potential downward jump only if the

promised utility W_t is at least β. This implies that, in order to induce effort, the principal has to monitor the agent, instead of relying on (IC), whenever the promised utility $W_t < \beta$. We summarize this in the following corollary.

COROLLARY 1. *Under any incentive compatible contract, the agent is monitored ($m_t = m$) when $W_t < \beta$.*

When $W_t \geq \beta$, the principal trades off monitoring and enforcing (IC), and should monitor the agent if and only if the shadow cost of the (IC) constraint is higher than the monitoring cost m.

2.3. Principal's Utility

Assume that the principal receives a constant revenue flow of R. Under an incentive compatible contract, the agent always exerts full effort. The corresponding principal's total discounted utility under an incentive compatible contract Γ is

$$\mathbb{E}^{\Gamma,\bar{\Lambda}}\Big[\int_0^\infty e^{-rt}\Big((R-m_t)dt - KdN_t - dL_t\Big)\Big] = \frac{R-K\lambda}{r} - \mathbb{E}^{\Gamma,\bar{\Lambda}}\Big[\int_0^\infty e^{-rt}\Big(m_t dt + dL_t\Big)\Big].$$

Because the term R only shifts the principal's total utility by a constant and is independent of the contract design, without loss of generality, we set

$$R = K\lambda. \tag{2.6}$$

Therefore, the principal's total discounted utility under an incentive compatible contract Γ is

$$U(\Gamma) = -\mathbb{E}^{\Gamma,\bar{\Lambda}}\Big[\int_0^\infty e^{-rt}\Big(m_t dt + dL_t\Big)\Big]. \tag{2.7}$$

Before finishing this section, we present the following verification result, which provides an upper bound of the principal's utility $U(\Gamma)$ over all incentive compatible contracts. This result forms the foundation of proving optimality under various parameter regimes.

LEMMA 3. *Suppose $F(w)$ is a continuous, concave, and upper-bounded function, with $F'(w) \geq -1$. (If $F(w)$ is not differentiable at a point w, we denote $F'(w)$ to be the average between its left and right derivatives.) Consider any incentive compatible contract Γ, which yields the agent's expected utility $u(\Gamma, \bar{\Lambda}) = w = W_0$, followed by the promised utility process $\{W_t\}_{t\geq 0}$ according to (PK). Define a stochastic process $\{\Psi_t\}_{t\geq 0}$, where*

$$\begin{aligned}\Psi_t := {}& F'(W_t)\rho W_t - rF(W_t) - m_t + \lambda y_t[F'(W_t)H_t^s + F(W_t - H_t^s) - F(W_t)] \\ & + \lambda(1-y_t)[F'(W_t)H_t^n + F(W_t - H_t^n) - F(W_t)].\end{aligned} \tag{2.8}$$

If the process $\{\Psi_t\}_{t\geq 0}$ is non-positive almost surely, then we have $F(w) \geq U(\Gamma)$.

3. Equal Discount Rate

In this section, we consider the case in which $r = \rho$. That is, the principal shares the same discount rate with the agent. In this setting, we need to introduce un upper bound for the agent's continuation utility, $\bar{w}$, as an additional model parameter. This is due to the "infinite back loading" problem identified in Myerson (2015) for the equal discount moral hazard problem without monitoring. Without such an upper bound, the principal would keep increasing the agent's promised utility to infinity without payment. We will provide a comprehensive discussion on the intuitive reason for such an upper bound towards the end of this section. To avoid triviality, the upper bound $\bar{w}$ is set to be above β.[3] It is worth noting that we only need such an upper bound when $r = \rho$. When $r < \rho$, the principal is more patient than the agent, it is no longer necessary to introduce $\bar{w}$, as we will discuss in the next section.

When $r = \rho$, the optimal contract structure is not unique. We start from introducing one particular optimal contract structure, and derive the corresponding principal's value function. After that we introduce another optimal contract structure which yields the same value function to the principal. Interestingly, when $r < \rho$ to be discussed in the next section, either contract structure may be optimal, depending on how high the monitoring cost is.

3.1. Optimal Contract Structure: Deterministic

We first describe the evolution of the promised utility under the optimal contract. As discussed in the previous section, the (IC) condition implies that the principal has to monitor the agent *if* the promised utility W_t is lower than β. In this section, we establish that when $r = \rho$, it is optimal to monitor *if and only if* $W_t < \beta$. Moreover, the principal does not penalize the agent for any arrivals (i.e., $H_t^n = H_t^s = 0$) and does not pay the agent while monitoring (i.e., $dL_t = 0$). Therefore, when $W_t < \beta$, (PK) implies that the promised utility evolves according to

$$dW_t = \rho W_t dt. \tag{3.1}$$

Intuitively, the term $\rho W_t dt$ is the "interest" accrued from the promised utility due to time discounting. Furthermore, whenever $W_t < \beta$ at time t, condition (3.1) implies that the promised utility increases deterministically following the simple exponential curve

$$W_{t+\tau} = W_t e^{\rho\tau}, \tag{3.2}$$

until $W_{t+\tau}$ reaches the threshold β.

When the promised utility $W_t \in [\beta, \bar{w})$, the principal no longer monitors the agent, and the promised utility takes a downward jump of β upon each arrival. That is, following the optimal

contract, $H_t^s = H_t^n = \beta$ in (PK). In this case, the principal still does not pay the agent (i.e., $dL_t = 0$), and (PK) reduces to

$$dW_t = (\rho W_t + \beta\lambda)dt - \beta dN_t. \tag{3.3}$$

Compared with (3.1), the rate of increase in (3.3) is higher. Besides the interest term $\rho W_t dt$, the term $\beta\lambda dt$ is the "information rent" that the agent receives, in the form of faster increase in the promised utility when there is no arrival. To see this intuitively, remember that in order to motivate effort, the principal shall charge the agent utility β for each arrival, which occurs with probability λdt. Because the agent cannot actually pay the principal money at an arrival, when there is no arrival during a period dt, the principal increases the agent's promised utility by $\beta\lambda dt$. This exactly equals the expected decrease of the agent's utility for an arrival during this time period, reflected in the downward jump term βdN_t.

Therefore, if there is no arrival during the time interval $[t, t+\tau]$, then, again, the promised utility increases following the curve

$$W_{t+\tau} = W_t e^{\rho\tau} + \frac{\beta\lambda}{\rho}(e^{\rho\tau} - 1), \tag{3.4}$$

as long as $W_{t+\tau} < \bar{w}$. Compared with (3.2), the additional term involving $\beta\lambda$ is the information rent as discussed earlier.

Without an arrival, the agent's promised utility keeps increasing according to a rate $\rho W_t + \beta\lambda$ until W_t reaches the upper bound $\bar{w}$. At this point, the promised utility cannot increase any more, and stays at $\bar{w}$ until the next arrival. That is, when $W_t = \bar{w}$, the promised utility evolves according to

$$dW_t = -\beta dN_t. \tag{3.5}$$

In order to keep W_t at $\bar{w}$, the principal has to pay the agent a flow rate

$$\ell_t = \rho\bar{w} + \beta\lambda, \tag{3.6}$$

to release the upward pressure that would otherwise occur to the promised utility.

Now we are ready to present a formal definition for a class of contracts that includes the optimal one. Note that in the following definition, we allow the monitoring threshold to be a more general level s, although it is just β as described before. This is because the optimal threshold can be higher than β in the next section when $r < \rho$.

DEFINITION 1. Contract $\Gamma_d(w; s, \bar{w})$ is defined as:

(i) The dynamics of the agent's promised utility, W_t, follows (3.1) for $W_t \in [0, s)$, (3.3) for $W_t \in [s, \bar{w})$, and (3.5) for $W_t = \bar{w}$, starting from $W_0 = w$.

(ii) In terms of payments, the agent is not paid when $W_t < \bar{w}$, and is paid at a flow rate (3.6) when $W_t = \bar{w}$.

(iii) Regarding monitoring, $m_t = m$ if and only if $W_t < s$.

In Definition 1, the subscript "d" in $\Gamma_d(w; s, \bar{w})$ stands for *deterministic*, because monitoring starts in a "deterministic" fashion with respect to the arrival times. That is, the promised utility W_t is adapted to the filtration generated from the arrival process. The parameter s represent the monitoring threshold. Clearly, in this section we are only interested in contract $\Gamma_d(w; \beta, \bar{w})$. In the following section when we discuss $\rho > r$, however, the threshold s could be strictly higher than β, and the upper bound $\bar{w}$ may be replaced with an endogenous one.
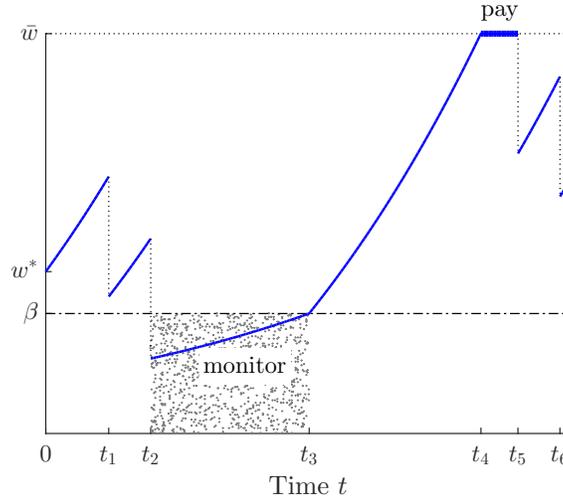

Figure 1 Sample Trajectories of W_t under $\Gamma_d(w^*; \beta, \bar{w})$

Figure 1 provides a sample trajectory under contract $\Gamma_d(w^*; \beta, \bar{w})$. According to this contract, the principal only monitors the agent when the promised utility is below β. This happens during time period $[t_2, t_3)$ in the figure. As long as the promised utility is above β, each arrival induces a downward jump of β, as depicted at time epochs t_1, t_2, t_5, and t_6. The principal pays a flow payment with rate ℓ_t in (3.6) to the agent when the promised utility is at the upper bound $\bar{w}$, during time period $[t_4, t_5)$ in the figure. The promised utility keeps increasing between arrivals, at a lower rate when the agent is being monitored (in the time interval $[t_2, t_3)$ in the figure).

The above description shows that it is fairly easy for the principal to implement this contract over time, keeping track of a single number in a simple way. Furthermore, the contract guarantees incentive compatibility on an intuitive level. Think again in the quality control setting. Monitoring starts when defects arrive rather frequently over a period of time, and downward jumps cannot be compensated by the gradual increase in the promised utility between arrivals. Consequently, the promised utility would eventually fall below the threshold β, triggering the principal to start monitoring. Under monitoring, the promised utility grows rather slowly, which means that payment can only happen far into the future. Therefore, monitoring not only ensures effort at the moment, but also serves as a threat that motivates the agent to exert effort to avoid.

Furthermore, the contract motivates the agent to exert effort besides using the threat of monitoring. When the promised utility is at a level w above β and below $\bar{w}$, payment starts if there is no defects for the next $\frac{1}{\rho} \ln \frac{\rho \bar{w} + \beta \lambda}{\rho w + \beta \lambda}$ period of time (following (3.4) with $W_t = w$ and $W_{t+\tau} = \bar{w}$). Exerting effort increases the chance of the promised utility reaching $\bar{w}$ and the agent receiving payment. Once the flow payment has started, a defect brings the promised utility down from $\bar{w}$ to $\bar{w} - \beta$, and pauses the flow payment for at least a period of time of $\frac{1}{\rho} \ln \frac{\rho \bar{w} + \beta \lambda}{\rho \bar{w} + \beta(\lambda - \rho)}$ (again, following (3.4)). Therefore, once being paid, the agent is willing to exert effort in order to prolong the payment period before it pauses.

3.2. Principal's Value Function

Following the evolution of the promised utility described above, next we heuristically derive the dynamics of the principal's utility as a function of the agent's promised utility using discrete time approximation. Specifically, denote $F(w)$ to represent the principal's total discounted utility when the agent's promised utility is w.

First, for any $w \in [0, \beta)$, over a small time interval with length δ, the principal incurs a monitoring cost $m\delta$, and, following the dynamics (3.1), the agent's promised utility increases to $we^{\rho \delta}$. Therefore, we have the following expression for the principal's utility function,

$$F(w) = -m\delta + e^{-r\delta} F\left(we^{\rho \delta}\right) + o(\delta). \tag{3.7}$$

Assuming $F(w)$ is differentiable on $[0, \beta)$, following standard procedures of subtracting $F(w)$ from and dividing δ on both sides, and letting δ approach 0, we obtain,

$$rF(w) = \rho w F'(w) - m. \tag{3.8}$$

We keep both r and ρ in (3.8) and all the equations in this section, because the value function takes the same expressions when $r < \rho$ in the next section.

Differential equation (3.8) has the following standard solution,

$$F_\theta(w) = \theta w^{\frac{r}{\rho}} - \frac{m}{r}, \tag{L}$$

parameterized with a scalar θ. The tag (L) indicates that the promised utility is *lower* than β. Later in this section, we specify the choice of θ to complete the description of the value function $F_\theta(w)$. When $r = \rho$, as in the setting of this section,

$$F_\theta(w) = \theta w - \frac{m}{r}, \tag{L$_l$}$$

which is a linear function of w, and hence the subscript l in the tag.

Next, for any $w \in [\beta, \bar{w})$, the principal no longer monitors the agent. Following similar heuristic derivations for (3.7), we reach the following delay differential equation (DDE),

$$(\lambda + r)F(w) = \lambda F(w - \beta) + (\rho w + \lambda \beta)F'(w), \tag{H}$$

where the tag (H) indicates that the promised utility is *higher* than β. We denote function $F_\theta(w)$ to be the solution to the DDE (H) on $w \in [\beta, \bar{w})$ with boundary condition (L_l) on $w \in [0, \beta)$.

Finally, for $w = \bar{w}$, the principal pays the agent a flow payment according to (3.6) and keeps the promised utility at $\bar{w}$, until the next arrival. Standard arguments imply that the principal's value function takes the form of,

$$(\lambda + r)F(\bar{w}) = \lambda F(\bar{w} - \beta) - (\rho \bar{w} + \beta \lambda), \tag{U}$$

where the tag (U) stands for *upper* bound. The combination of (H) and (U) implies that $F'(\bar{w}) = -1$. Intuitively, when the slope of the principal's value function is -1, increasing the promised utility further by an amount costs the principal the same amount. This is consistent with the fact that at this point delaying payment while letting the promised utility increase does not yield any further benefit to the principal any more.

In order to specify the optimal value function, we need to determine θ in (L_l). To that end, we introduce function $J(w)$ to be the solution of DDE (H) for $w \geq \beta$ with boundary condition $J(w) = 1$, instead of (L_l), on $w \in [0, \beta)$. Therefore, function $J(w)$ is independent of θ and the monitoring cost m. It is easy to verify that when $\rho = r$, function $F_\theta(w)$, which is the solution to (H) with boundary condition (L_l), can be expressed in terms of $J(w)$ as

$$F_\theta(w) = \theta w - \frac{m}{r}J(w), \quad \text{for } w \leq \bar{w}. \tag{3.9}$$

We further extend the function to $w \geq \bar{w}$ with slope -1, i.e.,

$$F_\theta(w) = F_\theta(\bar{w}) + \bar{w} - w, \text{ for } w > \bar{w}. \tag{3.10}$$

Furthermore, if we define

$$\theta(\bar{w}) = \frac{m}{r}J'(\bar{w}) - 1, \tag{3.11}$$

it is easy to verify that $F'_{\theta(\bar{w})}(\bar{w}) = -1$ so function $F'_{\theta(\bar{w})}(\bar{w})$ is differentiable at $\bar{w}$ with slope -1.

PROPOSITION 1. *We have the following properties regarding the value function $F_{\theta(\bar{w})}(w)$ defined according to* (3.9), (3.10) *and* (3.11):

(i) The value $\theta(\bar{w})$ is bounded. Specifically,

$$-1 \leq \theta(\bar{w}) < \frac{m}{\beta r}. \tag{3.12}$$

(ii) Function $F_{\theta(\bar{w})}(w)$ is linear on $w \in [0, \beta)$ and strictly concave on $w \in [\beta, \bar{w}]$ with $F'_{\theta(\bar{w})}(\bar{w}) = -1$. Moreover, for any $\bar{w}$ and $\tilde{w}$ such that $\beta \leq \tilde{w} < \bar{w}$, we have $F_{\theta(\tilde{w})}(w) < F_{\theta(\bar{w})}(w)$ for any $w \geq 0$.

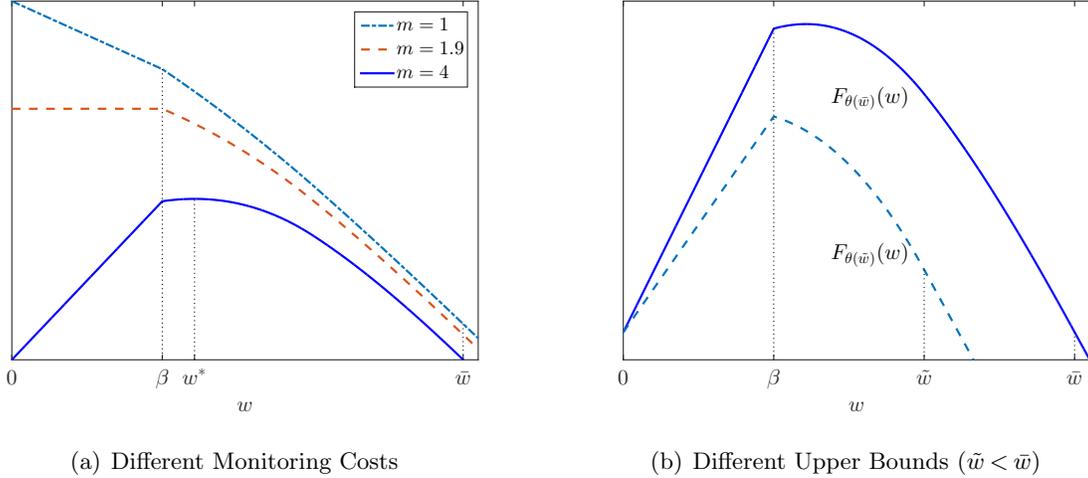(a) Different Monitoring Costs　　　　　　　　(b) Different Upper Bounds ($\tilde{w} < \bar{w}$)

Figure 2　　Principal's Value Function $F_{\theta(\bar{w})}(w)$

Figure 2 depicts the value function with different model parameters. As we can see, all the functions plotted in the two sub-figures are concave, as described in Proposition 1. Therefore, it is easy to find its maximizer

$$w^* = \arg\max_{w \geq 0} F_{\theta(\bar{w})}(w),$$

as depicted in Figure 2(a) for the case of $m = 4$. In order to maximize the total future expected utility, the principal should use contract $\Gamma_d(w^*; \beta, \bar{w})$, starting the contract from promised utility w^*.

Figure 2(a) also demonstrates that the value function decreases with the monitoring cost m, which is intuitive, and consistent with (3.9). According to (3.9) and (3.11), the slope $\theta(\bar{w})$ increases in m, and is positive if and only if $m > r/J'(\bar{w})$. In Figure 2(a), $\theta(\bar{w})$ is positive when $m = 4$, zero when $m = 1.9$, and negative when $m = 1$. If $m < r/J'(\bar{w})$, the slope $\theta(\bar{w}) < 0$ and the maximizer of $F_{\theta(\bar{w})}(w)$ is $w^* = 0$. In this case, the monitoring cost is low enough such that it is optimal for the principal to always monitor while keeping the agent's promised utility at 0. This is depicted in the curve with $m = 1$ in Figure 2(a).

Figure 2(b) further depicts the value function under different exogenous upper bounds on the promised utility. Consistent with Proposition 1(ii), the value function increases with the upper bound. This is also intuitive. From an optimization point of view, the upper bound puts a constraint on the optimal control problem. Relaxing it improves the objective function. From an economic point of view, a higher upper bound allows the principal to delay payments further into the future, and, therefore, improves the principal's utility. This explains the infinite back-loading problem: if allowed, the principal would choose the upper bound to approach infinity.

3.3. Optimal Contract Structure: Randomized

As mentioned in the beginning of this section, the optimal contract structure is not unique. To see this, observe that the principal's value function is linear when $w \in [0, \beta)$. Such a linear function can be achieved via a *randomized* contract structure. That is, randomizing the promised utility between 0 and β when it falls in $[0, \beta)$ yields a linear value function on the interval.

More formally, if an arrival occurs when W_{t-} is in $[\beta, 2\beta)$, the next moment's promised utility lands on 0 with probability $2 - W_{t-}/\beta$, and on β with probability $W_{t-}/\beta - 1$. Technically, this means that in (PK), the jumps are $H_t^s = W_{t-}$ and $H_t^n = W_{t-} - \beta$, and the randomization probability is $y_t = 2 - W_{t-}/\beta$. Therefore, we define the following class of contracts $\Gamma_r(w; \bar{w})$, in which the subscript "r" stands for *randomized.*

DEFINITION 2. Define contract $\Gamma_r(w; \bar{w})$ the same as contract $\Gamma_d(w; \beta, \bar{w})$ in Definition 1, except that the dynamics of the agent's promised utility W_t follows

$$dW_t = \begin{cases} 0, & \text{if } W_t = 0, \\ (\rho W_t + \beta\lambda)dt - W_t dN_t^s - (W_t - \beta)dN_t^n, & \text{if } W_t \in [\beta, \min\{\bar{w}, 2\beta\}), \\ (\rho W_t + \beta\lambda)dt - \beta dN_t, & \text{if } W_t \in [\min\{\bar{w}, 2\beta\}, \bar{w}), \\ -\beta dN_t, & \text{if } W_t = \bar{w}, \end{cases} \tag{3.13}$$
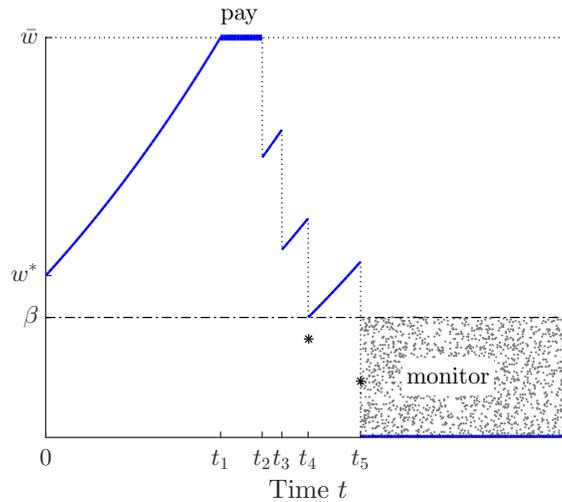
starting from $W_0 = w$.

Figure 3   Sample Trajectories of W_t under $\Gamma_r(w^*; \bar{w})$

Figure 3 depicts a sample trajectory under the randomized contract $\Gamma_r(w^*; \bar{w})$. Arrivals occur at time epochs t_2, t_3, t_4, and t_5. For t_2 and t_3, the promised utility is above 2β before the arrival, which means that the downward jump equals β. Right before time t_4, the promised utility is already below 2β. A downward jump of β would bring the promised utility below β, to the "$*$"

point. Consequently, the principal randomizes the agent's promised utility. On this trajectory, the outcome is β, and, therefore, monitoring does not start. Similar randomization occurs at time t_5 as well. This time, however, the outcome is to set the promised utility to 0. From this point on, the promised utility is maintained at 0 following (3.1), and principal always monitors the agent.

3.4. Proof of Optimality

So far we have not formally established the connection between the value function $F_{\theta(\bar{w})}(w)$ and either contract structure. Now we establish that $F_{\theta(\bar{w})}(w)$ is indeed the optimal value function.

First, the following proposition states that $F_{\theta(\bar{w})}(w)$ is indeed the value function of both contracts $\Gamma_d(w; \beta, \bar{w})$ and $\Gamma_r(w; \bar{w})$.

PROPOSITION 2. *For $F_{\theta(\bar{w})}(w)$ defined according to* (3.9), (3.10) *and* (3.11), *we have*

$$F_{\theta(\bar{w})}(w) = U\big(\Gamma_d(w; \beta, \bar{w})\big) = U\big(\Gamma_r(w; \bar{w})\big).$$

Therefore, starting from w^*, contracts $\Gamma_d(w^*; \beta, \bar{w})$ and $\Gamma_r(w^*; \bar{w})$ both yields the maximum $F_{\theta(\bar{w})}(w^*)$ for the principal.

According to both (3.1) and (3.13), the promise utility stays at 0 forever whenever it falls to 0, according to both contracts $\Gamma_d(w; \beta, \bar{w})$ and $\Gamma_r(w; \bar{w})$. That is, whenever $W_t = 0$, the principal needs to monitor the agent (and endure the monitoring cost) ever after. Following the optimal contract $\Gamma_d(w^*; \beta, \bar{w})$, however, the promised utility hitting zero is a zero measure event. And, starting from a promised utility $w \in (0, \beta)$, dynamics (3.2) implies that the length of a monitoring episode under the optiaml contract $\Gamma_d(w^*; \beta, \bar{w})$ is

$$T_m(w) := \frac{1}{\rho}\left(\ln \beta - \ln w\right), \tag{3.14}$$

which is finite. Superficially, the randomized contract $\Gamma_r(w^*; \bar{w})$ may appear worse due to the possibility of monitoring forever. In fact, when the promised utility before an arrival is in the interval $(\beta, 2\beta)$, under the deterministic contract $\Gamma_d(w^*; \beta, \bar{w})$, the principal has to pay the monitoring cost for a period of time after each arrival. Under the randomized contract $\Gamma_r(w^*; \bar{w})$, however, there is a chance that monitoring does not happen at all, which balances the chance of monitoring ever after. This explains, intuitively, why these two contracts are equivalent to the risk-neutral principal.

The following theorem, together with Proposition 2, establishes the optimality of value function $F_{\theta(\bar{w})}(w)$ and both contracts $\Gamma_d(w^*; \beta, \bar{w})$ and $\Gamma_r(w^*; \bar{w})$.

THEOREM 1. *For any incentive compatible contract Γ which yields an agent's utility $w \leq \bar{w}$, we have*

$$U(\Gamma) \leq F_{\theta(\bar{w})}(w) \leq F_{\theta(\bar{w})}(w^*). \tag{3.15}$$

Therefore, contracts $\Gamma_d(w^*; \beta, \bar{w})$ and $\Gamma_r(w^*; \bar{w})$ are both optimal, which yield expected utility $F_{\theta(\bar{w})}(w^*)$ to the principal.

The first inequality in (3.15) follows from Lemma 3. The second inequality simply follows from w^* being the maximizer of $F_{\theta(\bar{w})}(w)$. Therefore, Theorem 1 and Proposition 2 imply that the principal's expected utility generated from contracts $\Gamma_d(w^*; \beta, \bar{w})$ and $\Gamma_r(w^*; \bar{w})$ is higher than those generated from any other incentive compatible contracts. Hence, these two contracts are both optimal. Finally, it is worth pointing out that various combinations of the contracts $\Gamma_d(w^*; \beta, \bar{w})$ and $\Gamma_r(w^*; \bar{w})$ are also optimal. That is, whenever in the interval $(0, \beta)$, the promised utility w can either continuously increase following (3.1), or randomly jump between 0 and β, following (3.13), no matter how it behaved in this interval before.

4. Different Discount Rates

In this section, we consider the case of $\rho > r$. That is, the principal is more patient than the agent. One important distinction compared with the case of $\rho = r$ is that there exists a finite upper bound $\bar{w}^*$ on the promised utility, which is implied endogenously under the optimal contract. Therefore, we no longer need to introduce the exogenous upper bound as in the previous section. Nevertheless, when $\rho > r$ we can still include an exogenous upper bound $\bar{w}$ for the promised utility in the model, which is discussed in Section 4.3.
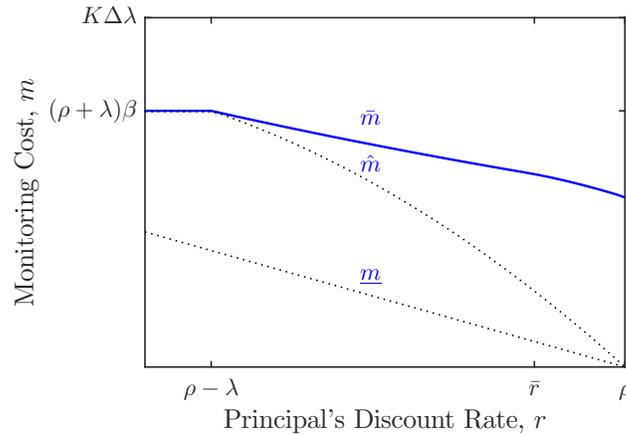

Figure 4 Split of the Low and High Monitoring Cost

For the main parts of this section (Sections 4.1 and 4.2), we show that the structure of the optimal contract changes with model parameters, especially the monitoring cost, as illustrated in Figure 4. In this figure, we vary the principal's discount rate r (as the x-axis) and the monitoring cost m (as the y-axis), while keeping other model parameters fixed. In the following two subsections, we

show that if the monitoring cost m is above a threshold $\bar{m}$ (the solid curve), the optimal contract takes a structure similar to the deterministic contract defined in the previous section. If m is below $\bar{m}$, on the other hand, it is optimal for the principal to always monitor the agent. Figure 4 depicts two additional dotted curves $\hat{m}$ and $\underline{m}$ in this region. We defer the detailed discussion on them to Section 4.2.

Here, we first define the threshold $\bar{m}$ as

$$\bar{m} := \inf_{w > \beta} \frac{r}{J'(w)}. \tag{4.1}$$

For the case of $\rho = r$, equation (3.11) implies that $\theta(\bar{w}) < 0$ for any $\bar{w} > \beta$ when $m < \bar{m}$, which implies that the value function is decreasing, and, therefore, it is optimal to always monitor the agent. Later in Section 4.2, we show that it is still optimal to always monitor the agent when $m < \bar{m}$ when $\rho > r$. Next, we first study the case of $m \geq \bar{m}$.

4.1. High Monitoring Cost

In this subsection, we investigate the case in which the monitoring cost is above the threshold $\bar{m}$.

Recall Corollary 1, the principal needs to monitor when the promised utility w is lower than β. When $m \geq \bar{m}$, the principal may still need to monitor the agent even if w is higher than β. That is, the optimal contract is similar to the deterministic contract Γ_d in the previous section, in which monitoring occurs whenever the promised utility is below a threshold $\alpha \geq \beta$.

Following the same heuristic derivation in Section 3.2, define function $F_{\theta,\alpha}(w)$ to be the solution of DDE (H) for $w \in [\alpha, \infty)$ with boundary condition (L) for $w \in [0, \alpha)$. Next, we identify the optimal value function in a two steps. First, we specify α for a given parameter θ. Then, we establish θ and the endogenous upper bound $\bar{w}^*$.

First, we identify the threshold α for a given parameter θ. A key property of such an α is to induce "smooth pasting" (Dixit and Pindyck 1994) of value function $F_{\theta,\alpha}(w)$ when the monitoring state changes. That is, the left and right derivatives, $F'_{\theta,\alpha}(\alpha_-)$ and $F'_{\theta,\alpha}(\alpha_+)$, respectively, are set to be equal to each other if possible. To this end, it is convenient to define the following function,

$$\begin{aligned} f(\alpha) &:= \left(F'_{\theta,\alpha}(\alpha_-) - F'_{\theta,\alpha}(\alpha_+) \right) (\rho\alpha + \beta\lambda) \\ &= m - \lambda\theta\alpha^{\frac{r}{\rho}} \left[\left(1 - \frac{r\beta}{\rho\alpha} \right) - \left(1 - \frac{\beta}{\alpha} \right)^{\frac{r}{\rho}} \right], \end{aligned} \tag{4.2}$$

in which $F'_{\theta,\alpha}(\alpha_-)$ and $F'_{\theta,\alpha}(\alpha_+)$ are obtained from (L) and (H) with switching point α, respectively. Therefore, we may set $f(\alpha) = 0$ to achieve $F'_{\theta,\alpha}(\alpha_-) = F'_{\theta,\alpha}(\alpha_+)$.

LEMMA 4. *Function $f(\alpha)$ is increasing in α on $[\beta, \infty)$, and $\lim_{\alpha \to \infty} f(\alpha) = m$.*

In order to find the threshold α by solving the equation $f(\alpha) = 0$, denote f^{-1} to represent the inverse function of the monotone function f, and, for any θ, define

$$\alpha_\theta := \begin{cases} \beta, & \text{if } f(\beta) \geq 0, \\ f^{-1}(0), & \text{if } f(\beta) < 0. \end{cases} \tag{4.3}$$

PROPOSITION 3. *(i) We have*

$$f(\alpha_\theta) \geq 0 \ \text{ and } \ (\alpha_\theta - \beta)f(\alpha_\theta) = 0. \tag{4.4}$$

Therefore, if $\alpha_\theta = \beta$, we have $F'_{\theta,\alpha_\theta}(\alpha_{\theta-}) \geq F'_{\theta,\alpha_\theta}(\alpha_{\theta+})$, while if $\alpha_\theta > \beta$, we have $F'_{\theta,\alpha_\theta}(\alpha_{\theta-}) = F'_{\theta,\alpha_\theta}(\alpha_{\theta+})$.

   (ii) Furthermore, if $\alpha_\theta > \beta$, we have $F''_{\theta,\alpha_\theta}(\alpha_{\theta+}) < F''_{\theta,\alpha_\theta}(\alpha_{\theta-}) < 0$.

   (iii) Finally, for any $\alpha \in [\beta, \alpha_\theta)$, $F'_{\theta,\alpha}(\alpha_-) < F'_{\theta,\alpha}(\alpha_+)$; for $\alpha \in (\alpha_\theta, \infty)$, $F'_{\theta,\alpha}(\alpha_-) > F'_{\theta,\alpha}(\alpha_+)$.

Proposition 3(i) and (ii) imply that function $F_{\theta,\alpha_\theta}(w)$ is locally concave at α. This is important for us to show the (global) concavity, and optimality, of this function. Proposition 3(iii) further helps us to establish Proposition 4(ii) to be presented later.

   After characterizing α, we now describe the optimal θ and the endogenous upper bound $\bar{w}^*$.

LEMMA 5. *(i) Function $F_{\theta,\alpha_\theta}(w)$ is supermodular in $(\theta, w) \in \mathbb{R} \times \mathbb{R}^+$. Therefore, derivative $F'_{\theta,\alpha_\theta}(w)$ increases in θ for any w.*

   (ii) For any given parameter $m \geq \bar{m}$, there exist positive quantities $\bar{\theta}$ and $\bar{w}^*$, such that

$$\inf_{w > \alpha_{\bar\theta}} F'_{\bar\theta,\alpha_{\bar\theta}}(w) = -1 \ \text{ and } \ \bar{w}^* := \inf\left\{ \arg\inf_{w > \alpha_{\bar\theta}} F'_{\bar\theta,\alpha_{\bar\theta}}(w) \right\}, \ \text{respectively.} \tag{4.5}$$

Furthermore, we have

$$0 \leq \bar{\theta} < \frac{m}{r}\beta^{-\frac{r}{\rho}} \ \text{ and } \ \bar{w}^* \in [\alpha_{\bar\theta}, \infty). \tag{4.6}$$

Lemma 5(i) implies that the value $\bar{\theta}$ as defined in (4.5) is unique, and can be identified using binary search. Lemma 5(ii) further indicates that $\bar{\theta}$ is upper and lower bounded. The lower bound 0 implies that the value function is non-decreasing on $[0, \alpha_{\bar\theta}]$. Therefore, the maximizer of the value function is non-negative. This further implies that if the monitoring cost is higher than $\bar{m}$, it would be too costly for the principal to always monitor the agent. The upper bound can be used in a binary search algorithm to find $\bar{\theta}$. Finally, (4.6) also indicates that the endogenous upper bound $\bar{w}^*$ is indeed finite.

   Now we are ready to define the following value function $F(w)$ based on $F_{\bar\theta,\alpha_{\bar\theta}}(w)$,

$$F(w) := \begin{cases} F_{\bar\theta,\alpha_{\bar\theta}}(w), & \text{if } w \leq \bar{w}^*, \\ F_{\bar\theta,\alpha_{\bar\theta}}(\bar{w}^*) - (w - \bar{w}^*), & \text{otherwise.} \end{cases} \tag{4.7}$$

Here are some key properties of the value function that are essential for proving its optimality.

PROPOSITION 4. *For $m \geq \bar{m}$, we have:*

(i) *Function $F(w)$ is strictly concave on $w \in [0, \bar{w}^*]$, with $F'(w) = -1$ for $w \geq \bar{w}^*$.*

(ii) *For any $w \in (\alpha_{\bar{\theta}}, \bar{w}^*]$, we have $rF(w) > \rho w F'(w) - m$.*

Proposition 4(i) is a standard property that often arises in the dynamic contracting literature; it is the foundation for proving that $F(w)$ is the optimal value function. Proposition 4(ii), however, appears unique to our setting, and requires a novel proof based on Proposition 3. Comparing this differential inequality with the differential equation (3.8), it is clear that for any promised utility w above the threshold $\alpha_{\bar{\theta}}$, the principal is better off not to monitor. This condition is critical in proving optimality of the threshold structure in our contract.

Based on the calculation of threshold $\alpha_{\bar{\theta}}$ and upper bound $\bar{w}^*$ in Lemma 5, we can establish that the optimal contract is $\Gamma_d(w^*; \alpha_{\bar{\theta}}, \bar{w}^*)$ following Definition 1, in which w^* is a maximizer of function $F(w)$ defined in (4.7).

(a) Function $F(w)$ (b) A Sample Trajectory of W_t under $\Gamma_d(w; \alpha_{\bar{\theta}}, \bar{w}^*)$
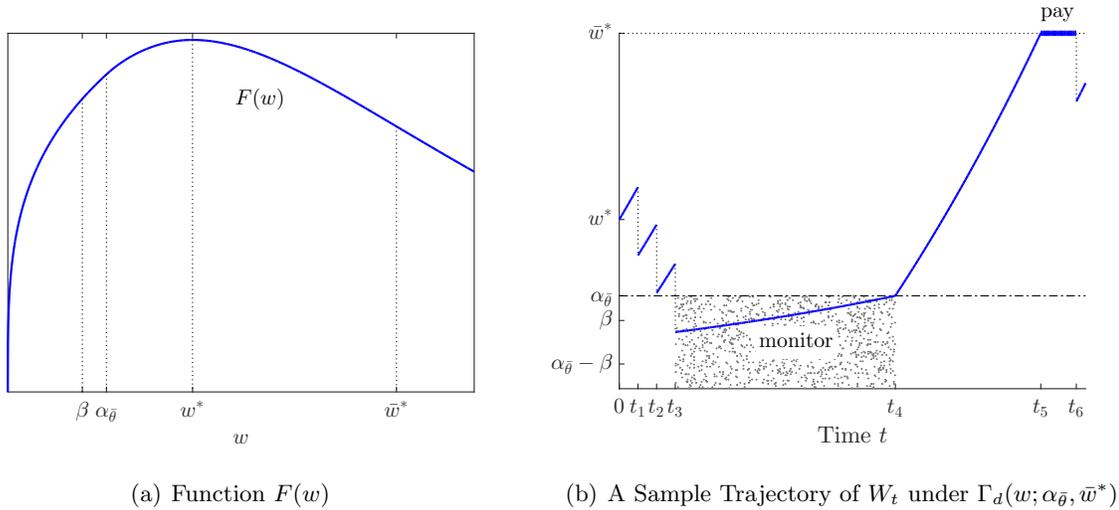
Figure 5   High Monitoring Cost (i.e., $m \geq \bar{m}$)

Figure 5(a) provides a sample sketch of the principal's value function. Under this particular parameter setting, we have $\alpha_{\bar{\theta}} > \beta$, and, therefore, according to Proposition 3, the value function demonstrates "smooth pasting" property at $\alpha_{\bar{\theta}}$.

Figure 5(b) presents a sample trajectory of the promised utility according to the optimal contract, using the same parameter values as in Figure 5(a). As we can see, in this particular example, we have $\alpha_{\bar{\theta}} > \beta$ and $w^* > \alpha_{\bar{\theta}}$. Therefore, the principal starts the contract with the initial promised utility w^*, without monitoring the agent. As long as W_t is above $\alpha_{\bar{\theta}}$, the promised utility W_t takes a downward jump of β for each arrive (at time t_1, t_2, t_3, and t_6 in the figure). In this sample trajectory, the promised utility drops below $\alpha_{\bar{\theta}}$ at time t_3. The principal starts monitoring the agent

at this point while the promised utility W_t cumulates interest and increases along the exponential curve (3.2) until it reaches $\alpha_{\bar{\theta}}$, regardless of arrivals in the interval $[t_3, t_4)$. When the promised utility climbs back to $\alpha_{\bar{\theta}}$ (at time t_4), it keeps increasing along the other exponential curve (3.4) as long as there is no arrival. A flow payment starts when W_t reaches $\bar{w}$ at time t_5, and stops when another arrival (at t_6 in this figure) drops W_t to below $\bar{w}^*$ again.

Similar to Proposition 2 and Theorem 1 for the equal discount case, the following result establishes the optimality for the case of $\rho > r$.

THEOREM 2. *For $m \geq \bar{m}$ and any incentive compatible contract Γ that yields an agent's utility w, we have $U(\Gamma) \leq F(w) \leq F(w^*) = U\big(\Gamma_d(w^*; \alpha_{\bar{\theta}}, \bar{w}^*)\big)$. Therefore, contract $\Gamma_d(w^*; \alpha_{\bar{\theta}}, \bar{w}^*)$ is the optimal contract, which yields utilities $F(w^*)$ for the principal and w^* for the agent.*

It is worth noting that under the optimal contract $\Gamma_d(w^*; \alpha_{\bar{\theta}}, \bar{w}^*)$, it is possible that the agent's promised utility never reaches 0. That is, constraint (IR) is never binding. In fact, as long as $\alpha_{\bar{\theta}} > \beta$, a downward jump induced by an arrival at most brings the promised utility W_t down to $\alpha_{\bar{\theta}} - \beta$, and never lower. (Figure 5 depicts such a case.) This phenomenon contrasts sharply with long held insights in the optimal mechanism/contract design literature, where the individual rationality constraint is generally binding. Therefore, a curious reader may wonder if the agent is over paid under our definition of contract $\Gamma_d(w^*; \alpha_{\bar{\theta}}, \bar{w}^*)$ when the monitoring threshold $\alpha_{\bar{\theta}} > \beta$.

In fact, similar to (3.14), for any monitoring threshold $\alpha > \beta$, starting from the lowest possible promised utility level $\alpha - \beta$, it takes time $\dfrac{1}{\rho} \ln \dfrac{\alpha}{\alpha - \beta}$ for monitoring to stop. This time increases as α decreases, and approaches infinity as α decreases to β. Therefore, the higher the value of α, the shorter the monitoring time period, during which the principal has to endure the monitoring cost at rate m. This explains why for high monitoring cost m (as in the current case), the principal is willing to set a threshold α higher than β, in order to avoid long episodes of monitoring the agent. Even though the agent's promised utility is maintained at strictly positive levels, this strategy yields lower monitoring costs than keeping the threshold at β. This phenomenon highlights the tradeoff that the principal faces between payments to the agent and monitoring costs.

4.2. Low Monitoring Cost

We now consider the case when the monitoring cost m is lower than $\bar{m}$. First, it is helpful to consider the case of $m = \bar{m}$. Following the previous subsection, it is easy to verify that $\bar{\theta} = 0$ and $\alpha_{\bar{\theta}} = \beta$. In this case, the function $F(w)$ is linear (in fact, constant) for $w \in [0, \beta)$. If we decrease m further and still follow contract Γ_d, then Lemma 5 yields a negative $\bar{\theta}$. Loosely speaking, the corresponding value function is decreasing, and, therefore, the optimal contract is to always monitor the agent and keep the promised utility at 0. Indeed, this is the structure of the optimal contract.

More rigorously, contract Γ_d with a starting promised utility 0 and monitoring threshold 0 is, in fact, not optimal. A value function following (L) with a negative $\bar{\theta}$ is convex, instead of concave. A non-concave value function cannot be optimal, because it can be improved through concavification with randomization.

In fact, the exact form of the optimal value function varies with model parameters when $m < \bar{m}$. In Figure 4, two dotted curves, $\hat{m}$ and $\underline{m}$, further divide the $m < \bar{m}$ area into three regions, each corresponding to a distinct value function form. Here we provide the expressions for $\hat{m}$ and $\underline{m}$ as

$$\underline{m} := (\rho - r)\beta \ \text{ and } \ \hat{m} := \begin{cases} \beta(\rho - r)(2\lambda + r)/\lambda, & \text{if } r > \rho - \lambda, \\ (\rho + \lambda)\beta, & \text{if } r \leq \rho - \lambda. \end{cases} \quad (4.8)$$

The simplest value function is a linear function,

$$F(w) = -\frac{m}{r} - w, \quad (4.9)$$

which is optimal when $m < \underline{m}$. In this case, the monitoring cost is so low that the principal should simply pay off any positive promised utility immediately and start monitoring the agent forever.

If $m \in [\underline{m}, \hat{m})$, the optimal value function is a slightly more complex piecewise linear function of the following form,

$$F(w) = \begin{cases} -\dfrac{m}{r} - \left[1 - \dfrac{(\rho - r)\beta - m}{(\lambda + r)\beta}\right]w, & \text{if } w \leq \beta, \\ F(\beta) - (w - \beta), & \text{if } w > \beta. \end{cases} \quad (4.10)$$

The randomized contract $\Gamma_r(w; \beta)$ following Definition 2 achieves the value function. That is, similar to Proposition 2, we can show that $F(w) = U\big(\Gamma_r(w; \beta)\big)$. In other words, if, for whatever reason, the initial promised utility is $w > \beta$, then the principal pays the agent $w - \beta$ to bring the promised utility down to β, and then keeps it there while paying a flow of interest and information rent to the agent, until the first arrival. Upon the arrival, the principal start monitoring the agent forever and no longer pays any information rent or interest. Because the value function is decreasing, its maximizer is 0. Therefore, the optimal contract $\Gamma_r(0; \beta)$ effectively starts monitoring from the very beginning.

If $m \in [\hat{m}, \bar{m})$, however, the optimal value function is more complex. It is the solution to DDE (H) for $w \in [\beta, \bar{w}^*]$ with boundary condition (L$_l$) for $w \in [0, \beta)$, where θ in (L$_l$) and $\bar{w}^*$ are defined as the following,

$$\inf_{w > \beta} F'_{\bar{\theta}}(w) = -1 \ \text{ and } \ \bar{w}^* = \inf\left\{\arg\inf_{w > \beta} F'_{\bar{\theta}}(w)\right\}. \quad (4.11)$$

Therefore, function $F(w)$, define as

$$F(w) = \begin{cases} F_{\bar{\theta}}(w), & \text{if } w \leq \bar{w}^*, \\ F_{\bar{\theta}}(\bar{w}^*) + \bar{w}^* - w, & \text{if } w > \bar{w}^*, \end{cases} \quad (4.12)$$

is linear on $[0, \beta)$ and nonlinear on $[\beta, \bar{w}^*)$, and takes a slope of -1 on $[\bar{w}^*, \infty)$. Furthermore, the next result further characterizes $\bar{w}^*$ and $\bar{\theta}$.

PROPOSITION 5. *For $m \in [\hat{m}, \bar{m})$, and $\bar{\theta}$ and $\bar{w}^*$ defined in (4.11), we have*

$$\bar{w}^* \geq 2\beta \quad and \quad -1 + \frac{\rho - r}{\lambda} < \bar{\theta} \leq 0. \tag{4.13}$$

The randomized contract $\Gamma_r(w; \bar{w}^*)$ achieves the value function. That is, the proof of Proposition 2 already establishes that $F(w) = U\big(\Gamma_r(w; \bar{w}^*)\big)$. Again, the value function is decreasing. Therefore, following contract $\Gamma_r(0; \bar{w}^*)$, the principal monitors the agent from the beginning forever.
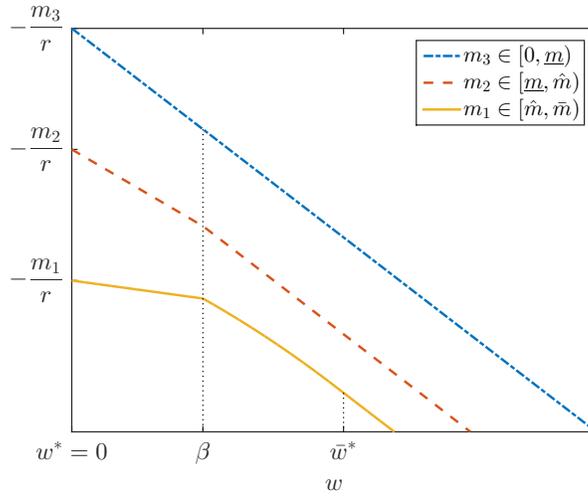

Figure 6 Value Functions with Low Monitoring Costs

Figure 6 depicts the optimal value functions for different monitoring costs. As we can see from the figure, for the monitoring cost $m_3 \in [0, \underline{m})$, the value function is a straight line with slope -1. If we further increase the monitoring cost to $m_2 \in [\underline{m}, \hat{m})$, the value function becomes a piecewise linear function. If the monitoring cost further increases to $m_1 \in [\hat{m}, \bar{m})$, the value function is non-linear in the interval $[\beta, \bar{w}^*)$, with an endogenous $\bar{w}^* > 2\beta$. Furthermore, the value function decreases with the monitoring cost.

Now we are ready to show our main result of this section.

THEOREM 3. *For $m < \bar{m}$ and any incentive compatible contract Γ that yields an agent's utility w, we have $U(\Gamma) \leq F(w) \leq F(0) = -m/r$, in which concave function $F(w)$ is defined as (4.9) for $m \in [0, \underline{m})$, (4.10) for $m \in [\underline{m}, \hat{m})$, and (4.12) for $m \in [\hat{m}, \bar{m})$, where $\bar{\theta}$ and $\bar{w}^*$ are defined in (4.11). Therefore, it is optimal for the principal to always monitor the agent.*

4.3. Exogenous Upper Bound

In certain practical settings, the principal may not be able to allow the promised utility to grow too high before paying the agent. This is especially true if the agent's discount rate is close to the

principal's, in which case the endogenous upper bound $\bar{w}^*$, although finite, tends to be very large. Therefore, in this subsection we allow the model to include an exogenous upper bound $\bar{w}$ on the promised utility. That is, our optimization problem has an additional constraint $w \leq \bar{w}$, similar to the case of $\rho = r$.

It is clear that if the exogenous upper bound $\bar{w}$ is higher than the endogenous $\bar{w}^*$, then the constraint $w \leq \bar{w}$ is not binding and it has no effect on the optimal contract. Therefore, we focus on the situation where, after computing $\bar{w}^*$ without considering $\bar{w}$, the principal realizes that $\bar{w} < \bar{w}^*$.

An immediate observation is that the threshold $\bar{m}$, which separates the high and low monitoring cost regions, needs to change from (4.1) to the following,

$$\bar{m}(\bar{w}) := \inf_{w \in (\beta, \bar{w}]} \frac{r}{J'(w)}. \tag{4.14}$$

Obviously, this new threshold $\bar{m}(\bar{w})$ increases in the upper bound $\bar{w}$, and, therefore, is greater than or equal to $\bar{m}$ defined in (4.1). Therefore, the principal may choose to always monitor the agent for higher monitoring costs comparing with the base model without $\bar{w}$. This is intuitive not only mathematically, but also practically. The upper bound pushes the principal to start payments "prematurely." Given the trade-off between payments and monitoring costs, such a pressure makes monitoring more favorable.

Finally, thresholds $\hat{m}$ and $\underline{m}$ do not change with the upper bound $\bar{w}$. The main results of this section only require slight changes to accommodate the upper bound $\bar{w}$. For example, in specifying the monitoring threshold $\alpha_{\bar{\theta}}$ and optimal value functions, (4.5) and (4.11) are changed to $F'_{\bar{\theta}, \alpha_{\bar{\theta}}}(\bar{w}) = -1$ and $F'_{\bar{\theta}}(\bar{w}) = -1$, respectively. The optimal contract for high monitoring cost is $\Gamma_d(w^*; \alpha_{\bar{\theta}}, \bar{w})$, in which w^* is the maximizer of the corresponding updated value function. For the low monitoring cost case, contract $\Gamma_r(0; \bar{w})$ achieves the optimal value function in place of $\Gamma_r(0; \bar{w}^*)$.

5. Discussions and Extensions

In this section we discuss a number of issues beyond our base model.

5.1. Computation

It is worth pointing out that the optimal value functions and contracts presented in Sections 3 and 4 are very easy to compute. For the value function $F_{\theta(\bar{w})}(w)$ of Section 3, we only need to first solve the function $J(w)$ using, for example, the standard shooting method starting from $J(w) = 1$ for $w \in [0, \beta)$, following DDE (H). After obtaining the function $J(w)$, we obtain the slope $\theta(\bar{w})$ using (3.11). Then the value function $F_{\theta(\bar{w})}(w)$ is readily available following (3.9). The exact definition of the optimal contract follows the initial promised utility w^*, which is a maximizer of $F_{\theta(\bar{w})}(w)$.

After obtaining the optimal contract, implementing it over time becomes very easy, as we have already discussed in Section 3.

When the principal is more patient than the agent ($\rho > r$), threshold $\bar{m}$ defined in (4.1) is easy to compute. In fact, it has closed form expressions if r and p do not differ too much, as shown in the following result.

PROPOSITION 6. *(a) For $r \in (0, \rho - \lambda]$, we have $\bar{m} = (\rho + \lambda)\beta$;*

(b) For $r \in (\rho - \lambda, \bar{r}]$, we have

$$\bar{m} = (\rho + \lambda)\beta \left[1 - \frac{\beta \rho}{\beta(2\rho + \lambda)} \right]^{\frac{\lambda + r}{\rho} - 1},$$

in which $\bar{r}$ is the unique solution to the following equation on $[\rho - \lambda, \rho]$,

$$\left[1 - \frac{\beta \rho}{\beta(2\rho + \lambda)} \right]^{\frac{\lambda + \bar{r}}{\rho} - 1} = 1 - \frac{\rho - \bar{r}}{\lambda}. \tag{5.1}$$

If the monitoring cost is higher than the threshold $\bar{m}$, the computation is slightly more complex than the case with equal discount rates. In order to specify the optimal value function $F_{\theta, \alpha_{\bar{\theta}}}(w)$, we also need to search for the slope $\bar{\theta}$ through a binary search. In Algorithm 1 we provide a pseudo code for the arguably more complex case of $\rho > r$ and $m > \bar{m}(\bar{w})$ with an exogenous upper bound $\bar{w}$.

The logic behind Steps 6 and 7 of Algorithm 1 follows from Lemma 5 and Proposition 7 below.

PROPOSITION 7. *Function $F_{\theta, \alpha_\theta}(w)$ is strictly concave on $w \in [0, \hat{w}]$ for $\hat{w}$ defined as the following,*

$$\hat{w} := \begin{cases} \inf\{\arg\inf_{w > \alpha_\theta} F'_{\theta, \alpha_\theta}(w)\}, & \text{if } \theta \geq \bar{\theta}, \\ \inf\{w : w \geq \alpha_\theta \text{ and } F'_{\theta, \alpha_\theta}(w) < -1\}, & \text{if } \theta < \bar{\theta}. \end{cases} \tag{5.2}$$

Following the definition of $\bar{\theta}$ in (4.5), Lemma 5(i) implies that if $\theta \geq \bar{\theta}$, we must have $F_{\theta, \alpha_\theta}(w) \geq -1$ for all $w \geq \alpha_\theta$. Therefore, the existence of a point $\hat{w}$ such that $F_{\theta, \alpha_\theta}(\hat{w}) < -1$ must imply that $\theta < \bar{\theta}$. Consequently, value θ serves as a lower bound θ_l for $\bar{\theta}$. Furthermore, Proposition 7 guarantees that if $\theta < \bar{\theta}$, for any $w \leq \hat{w}$, we must have $F''_{\theta, \alpha_\theta}(w) < 0$. Therefore, the search does not stop prematurely at a point following Steps 8 and 9.

The logic behind Steps 8 and 9 also follows from Proposition 7 together with Lemma 5(i). In particular, Proposition 7 implies that for $\theta > \bar{\theta}$, as soon as we observe a point $\bar{w}$ with $F''_{\theta, \alpha_\theta}(\hat{w}) = 0$ for the first time, the point $\hat{w}$ must be the minimum of derivative $F'_{\theta, \alpha_\theta}(w)$ over the entire interval $[\alpha_\theta, \infty)$. Hence, if $F'_{\theta, \alpha_\theta}(\hat{w}) > -1$, we must have $\theta > \bar{\theta}$.

Overall, the algorithm involves a binary search for $\bar{\theta}$, and solving for the value function given any current choice of θ. This computation, again, is very easy to implement. Overall, the easy computation and simple contract structures make our results easily implementable in practice.

Algorithm 1

1: Let $Stopping \leftarrow 0$, $\theta_l \leftarrow 0$, and $\theta_h \leftarrow m\dfrac{\beta^{-r/\rho}}{r}$ following (4.6)

2: **while** $Stopping = 0$ **do**

3:     Let $\theta \leftarrow (\theta_l + \theta_h)/2$

4:     Compute α_θ according to (4.3), in which the function f is defined in (4.2)

5:     Use the shooting method to compute function $F_\theta(w)$ following DDE (H) for $w \geq \alpha_\theta$ with boundary condition (L) on $w \in [0, \alpha_\theta)$, until a point $\hat{w} \in [\alpha_\theta, \bar{w}]$ that must satisfies one of the following cases:

6:     **if** $F'_{\theta,\alpha_\theta}(\hat{w}) < -1$ **then**

7:         Let $\theta_l \leftarrow \theta$

8:     **else if** $(\hat{w} < \bar{w}$ **and** $F''_{\theta,\alpha_\theta}(\hat{w}) \geq 0)$ **or** $\hat{w} = \bar{w}$ **then**

9:         **if** $(F'_{\theta,\alpha_\theta}(\hat{w}) > -1$ **and** $\hat{w} < \bar{w})$ **then**

10:             Let $\theta_h \leftarrow \theta$

11:         **else if** $(F'_{\theta,\alpha_\theta}(\hat{w}) = -1$ **or** $\hat{w} = \bar{w})$ **then**

12:             Let $\bar{w}^* \leftarrow \hat{w}$, $\bar{\theta} \leftarrow \theta$ and $Stopping \leftarrow 1$

13:         **end if**

14:     **end if**

15: **end while**

5.2. Discrete V.S. Continuous Time Models

Although the results that we have obtained in this paper are easy to compute and implement, one may still desire a discrete time model, which, arguably, is more familiar to many in the Operations Research community. Unfortunately, a discrete time model losses some of the salient features of the simplicity in the optimal solution from the continuous time model. The optimal solution may become cumbersome and computation challenging. Therefore, in our opinion, the advantage of a continuous time model in this setting is well justified.

Consider, for example, the case of $\rho > r$ and $m > \bar{m}$ again for a discrete time version of the model. (The exact value of $\bar{m}$ in a discrete time approximation is no longer as simple as (4.1), but one can still imagine its existence.) In a discrete time model, we no longer have "smooth pasting." Consequently, we loose the essentially closed form solutions from the continuous time model, and have to resort to numerically solving dynamic programs.

To elaborate our point, here we present a discrete time model that corresponds to our problem. Consider a discrete time model with time interval δ. During this time period, the probability of an arrival, the monitoring cost, and the time discount factors for the principal and the agent are $\lambda\delta$,

$m\delta$, $e^{-\delta r}$, and $e^{-\delta \rho}$, respectively. The following dynamic program, in which F_d represents the value function, captures the optimal contract design problem of finding an optimal incentive compatible contract,

$$F_d(w) = \max_{(q,w_m,w_a,w_n,l_m,l_a,l_n) \in \Pi(w)} (1-q) \left\{ \lambda \delta \left[-l_a + e^{-r\delta} F_d(w_a) \right] + (1 - \lambda \delta) \left[-l_n + e^{-r\delta} F_d(w_n) \right] \right\}$$
$$+ q \left[-m\delta - l_m + e^{-r\delta} F_d(w_m) \right]. \qquad (5.3)$$

Here decision variable q represents the probability of starting monitoring; w_m, w_a and w_n the promised utilities of monitoring, if an arrival occurs without monitoring, and if no arrival occurs without monitoring, respectively; and l_m, l_a and l_n the payments while monitoring, and with and without arrivals when not monitoring, respectively. The feasible region $\Pi(w)$ is summarized as the following,

$$q \left(l_m + e^{-\rho\delta} w_m \right) + (1-q) \left[\lambda \delta \left(l_a + e^{-\rho\delta} w_a \right) + (1 - \lambda \delta) \left(l_n + e^{-\rho\delta} w_n \right) \right] = w, \qquad (\text{PK}_\delta)$$

$$\left(l_a + e^{-\rho\delta} w_a \right) - \left(l_n + e^{-\rho\delta} w_n \right) \geq \beta, \qquad (\text{IC}_\delta)$$

$$w_a, w_n, w_m \geq 0 \qquad (\text{IR}_\delta)$$

$$l_a, l_n, l_m \geq 0$$

$$0 \leq q \leq 1$$

Here the constraints (PK_δ), (IC_δ), and (IR_δ) correspond to discrete time versions of the constraints (PK), (IC), and (IR), respectively. In particular, the incentive compatible constraint (IC_δ) is derived from the following inequality, reflecting that the agent is better off exerting effort when not being monitored,

$$q \left(l_m + e^{-\rho\delta} w_m \right) + (1-q) \left[\lambda \delta \left(l_a + e^{-\rho\delta} w_a \right) + (1 - \lambda \delta) \left(l_n + e^{-\rho\delta} w_n \right) \right]$$
$$\geq q \left(l_m + e^{-\rho\delta} w_m \right) + (1-q) \left[b\delta + \bar{\lambda} \delta \left(l_a + e^{-\rho\delta} w_a \right) + (1 - \bar{\lambda}\delta) \left(l_n + e^{-\rho\delta} w_n \right) \right].$$

Note that the feasible region $\Pi(w)$ is non-convex. Therefore, if one follows the value iteration algorithm to numerically solve the dynamic program (5.3), the optimization for each state w is a non-concave maximization. Furthermore, the optimal contract structure may be more complex than the continuous time model. In particular, the switch between monitoring and non-monitoring may involve the randomization variable $q \in (0,1)$. Note that this randomization variable helps maintaining concavity of the value function. If we do not even introduce such a randomization decision, then the value function is the maximum between two continuation functions representing monitoring and no monitoring, respectively, and almost certainly not concave.

As a side note, one may see through the intuition of smooth pasting in the continuous time model from this discrete time approximation. The two terms $-m\delta - l_m + e^{-r\delta} F_d(w_m)$ and

$\lambda\delta\left[-l_a + e^{-r\delta}F_d(w_a)\right] + (1 - \lambda\delta)\left[-l_n + e^{-r\delta}F_d(w_n)\right]$ in (5.3) represent the continuation value of monitoring and no-monitoring, respectively. Even if both terms are concave functions of w, the maximum between the two may not be concave. Therefore, the optimal decision q yields a concave upper envelop for these two curves. That is, there is an interval of w, on which the optimal value function is linear, and tangent to both curves. As the time interval δ approaches 0, however, the aforementioned interval diminishes to a single point, $\alpha_{\bar{\theta}}$. Smooth pasting occurs because the two curves share such a common tangent line. For a discrete time model, unfortunately, the interval may not diminish to a single point. In this case, if the promised utility w falls in this interval, the optimal q randomizes the next period's promised utility.

5.3. Costly Effort

In the base model, the agent is able to receive a benefit flow of b when shirking. The practical justification of this standard modeling choice is that effort is costly to the agent. Knowing that the incentive compatible contract motivates full effort from the agent, the principal pays off the effort cost flow, which is outside contract specifications in our model. Therefore, when shirking, the benefit is the saved effort cost.

In some situations, the agent's effort cost is not necessarily financial. We may think about it as a level of "pain" that the agent has to endure. The pain needs to be compensated, but not necessarily immediately. That is, when the effort cost b is not paid off immediately, it needs to be added to the promised utility.

Specifically, we denote b to represent the agent's effort cost per unit of time. Correspondingly, we revise the deterministic contract Γ_d to the following.

DEFINITION 3. Contract $\Gamma'_d(w; s, \bar{w})$ is defined as:

(i) The dynamics of the agent's promised utility, W_t, follows

$$dW_t = \begin{cases} (\rho W_t + b)dt, & \text{if } W_t \in [0, s) \\ (\rho W_t + \beta\lambda + b)dt - \beta_t dN_t, & \text{if } W_t \in [s, \bar{w}), \\ -\beta dN_t, & \text{if } W_t = \bar{w}, \end{cases} \tag{5.4}$$

starting from $W_0 = w$.

(ii) In terms of payments, the agent is not paid when $W_t < \bar{w}$. When $W_t = \bar{w}$, the agent is paid a flow with rate

$$\ell_t = \rho\bar{w} + \beta\lambda + b. \tag{5.5}$$

(iii) Regarding monitoring, $m_t = m$ if and only if $W_t < s$.

Comparing the promised utility dynamics (5.4) with (3.1) and (3.3), or payment (5.5) with (3.6), there is an extra term b. This term reflects that the effort cost is not paid immediately, but added

to the promised utility, as discussed above. Furthermore, note that even if $W_t = 0$, the dynamics (5.4) implies that the promised utility is not staying at 0 ever after. Instead, because the principal does not pay off the effort cost immediately, the increase in the promised utility eventually brings it above the threshold s, at which point the principal stops monitoring. Therefore, not paying for the effort cost b immediately allows the principal to avoid monitoring the agent forever.

The change in the randomized contract is more subtle. For the moment, let us imagine that, starting from promised utility $W_t = 0$, the promised utility is allowed to climb up following the dynamic $dW_t = (\rho W_t + b)dt$. A moment later, the promised utility becomes strictly positive, although very small. At this point, the promised utility already needs to be randomized to either 0 or β. In the continuous time limit, such a randomization process corresponds to letting the promised utility staying at 0 for an exponentially distributed period of time before jumping to β. This exponentially distributed random time introduces another counting process $\{Q_t\}_{t\geq 0}$. And the mean of the exponentially distributed time needs to be $b/\beta = \Delta\lambda$, to ensure that the upward jump in agent's promised utility reflects the expected effort cost during this period of time.

Therefore, we revise the randomized contract Γ_r to the follow definition.

DEFINITION 4. Define contract $\Gamma'_r(w; \bar{w})$ the same as contract $\Gamma'_d(w; \beta, \bar{w})$ in Definition 3, except that the dynamics of the agent's promised utility W_t follows

$$dW_t = \begin{cases} \beta dQ_t, & \text{if } W_t = 0, \\ (\rho W_t + \beta\lambda + b)dt - W_t dN_t^s - (W_t - \beta)dN_t^n, & \text{if } W_t \in [\beta, \min\{\bar{w}, 2\beta\}), \\ (\rho W_t + \beta\lambda + b)dt - \beta dN_t, & \text{if } W_t \in [\min\{\bar{w}, 2\beta\}, \bar{w}), \\ -\beta dN_t, & \text{if } W_t = \bar{w}, \end{cases} \quad (5.6)$$

starting from $W_0 = w$.

Figure 7 depicts a sample path for the contract $\Gamma'_r(w; \bar{w}^*)$. Compared with Figure 3 of the randomized contract $\Gamma_r(w; \bar{w})$, we observe that monitoring episodes do not last forever. Instead, the length of each monitoring episode is an exponential random variable. The agent's effort cost during the random monitoring episode is compensated in the form of the sudden increase in the promised utility at the end of each episode.

In order to establish the optimality of contract Γ'_r, the constraint (PK) needs to be revised to

$$dW_t = \{\rho W_t + b\mathbb{I}_{\lambda_\tau = \lambda} + \lambda_t[y_t H_t^s + (1 - y_t)H_t^n + q_t H_t^q]\}dt - H_t^s dN_t^s - H_t^n dN_t^n - H_t^q dQ_t - dL_t. \quad (\text{PK}')$$

Compared with (PK), here the promised utility is increased with a rate b if the agent exerts effort. In addition, the contract has another control variable q_t, which represents the rate at which monitoring stops. In the contract $\Gamma'_r(w; \bar{w})$ defined above, the rate is $q_t = \Delta\lambda$. The jump H_t^q occurs when the random monitoring period ends. In the contract $\Gamma'_r(w; \bar{w})$, we have $H_t^q = \beta$. For more
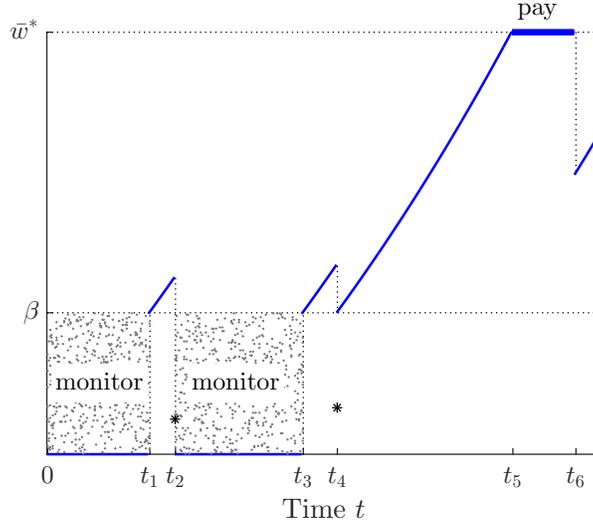
Figure 7   A Sample Trajectory of W_t under Contract $\Gamma'_r(w; \bar{w}^*)$

general incentive compatible contracts, of course, q_t and H^q_t do not have to take values according to $\Gamma'_r(w; \bar{w})$.

As a side note, we could have introduced control q_t and H^q_t from the very beginning of the paper, which allows for a more general class of contracts. In fact, in this more general class, both contracts $\Gamma_d(w; s, \bar{w})$ and $\Gamma_r(w; \bar{w})$ correspond to $q_t = 0$ and $H^q_t = 0$. However, such a generalization neither improves the optimal contracts, nor is intuitive. Therefore, we only introduce them here when they become necessary, because without them the optimal contract cannot be attained in the original more restrictive contract class. Finally, the constraint (IC) does not change with the current revision of the model. Therefore, incentive compatibility does not restrict the choice of q_t and H^q_t. This is intuitive because they are not directly related to arrivals.

Now we are ready to describe the optimal value functions. Overall speaking, equations (L), (L_l), (H), and (U) are revised to the following equations, respectively:

$$F_\theta(w) = \theta \big(w + b/\rho\big)^{r/\rho} - m/r, \tag{L$'$}$$

$$F_\theta(w) = \theta \big(w + b/r\big) - m/r, \tag{L$'_l$}$$

$$(\lambda + r)F(w) = \lambda F(w - \beta) + (\rho w + \lambda\beta + b)F'(w), \tag{H$'$}$$

$$(\lambda + r)F(w) = \lambda F(w - \beta) - (\rho\bar{w} + \lambda\beta + b). \tag{U$'$}$$

As we can see, if we set $b = 0$, equations (L$'$), (L$'_l$), (H$'$), and (U$'$) reduce to (L), (L_l), (H), and (U), respectively. The different cases of the value function and optimal contracts of this section follow closely to the main body of the paper otherwise. Furthermore, contracts Γ'_d and Γ'_r are optimal in place of Γ_d or Γ_r, respectively. We leave any remaining details for readers to figure out.

6. Concluding Remarks and Future Research Directions

This paper studies the optimal monitoring and payment mechanism to induce an agent's effort in order to reduce the arrival rate of adverse events. We formulate the problem as a continuous time stochastic optimal control model, and fully characterize its optimal solution. Interestingly, the structures of the optimal solution vary depending on model parameters, in particular, the monitoring cost. The variations of the contract structures highlight the trade off that the principal faces, between the monitoring cost and direct payment to the agent. The optimal contract structures are simple to describe, easy to compute and implement, and intuitive to explain. In particular, the key to computing the optimal contract only involves numerically solving a delay differential equation combined with a single dimensional search.

We hope that this paper motivates more studies of monitoring/inspection scheduling in dynamic settings. A directly related problem is to identify the optimal monitoring contract to motivate an agent to *increase* the arrival rate of a Poisson process of "good" events, similar to the setting in Sun and Tian (2017). Here we list a number of other potential extensions for future research.

Fixed Cost to Start Monitoring In our paper we did not consider a fixed cost of start monitoring. If such a fixed cost exists, we expect that there are two different thresholds. When the promised utility falls below the lower threshold, monitoring starts. And monitoring stops after the promised utility increases to be above the higher threshold. The intuition for such a "control band" structure is similar to the (s, S) policy in inventory control with fixed ordering cost.

General Cost of Arrival When we introduce the model, we claim our results extend naturally to the case of random cost for each arrival, as long as the random cost is not associated with the effort process. In this case we just need to use K to represent the mean cost per arrival. More generally, however, the effort process may affect the random cost. For example, the agent's effort may affect not only the rate of arrival, but also the distribution of the cost. In this case, the optimal contract needs to take advantage of the information contained in the magnitude of the cost of an arrival. Such a setting is much more complex than the one studied in this paper. Even without monitoring, the dynamic contracting problem with multiple signal types has not been well understood. We suspect that the general optimal contract could be so complex such that a fruitful way to proceed is to explore approximations of the optimal contracts that are easy to compute and implement.

Another way of thinking about arrivals is not to consider them simply as defects, as we do in this paper, but as breakdowns of a production process (machine). That is, the agent is a maintenance team, whose effort reduces the arrival rate of breakdowns. The cost therefore corresponds to the lost revenue when the machine is down. The breakdown time can be random. Even without monitoring,

such a model has not been studied in the literature, and one of the authors of this paper has been working on a related problem in an on-going research project. It would be interesting to consider combining such a model with monitoring as a potential future research direction.

Agent More Patient than Principal   Following long traditions of the dynamic contracting literature, we assume that the principal is no less patient than the agent. This is true in most practical settings, where the principal, as the contract designer, often possesses more resources than the agent. One may wonder what happens if the agent is more patient than the principal, as $\rho < r$. In this case, delaying payments is even more beneficial to the principal, because the interest the principal collects during the delay is higher than what is demanded by the agent. As a result, we believe that one still need to introduce the exogenous upper bound on the promised utility to make sure that an optimal solution exists. In order to prove optimality, we still need to establish concavity of the value function. This appears to be quite challenging when $\rho < r$. Neither the proof techniques in this paper nor the ones in Biais et al. (2010) work when $\rho < r$. We suspect one needs to carefully go through the discrete time model similar to the one in Section 5.2 and follow a proof logic of Biais et al. (2007) and Sun and Tian (2017) to show that uniform convergence between the discrete time value function and the continuous time one. Given the length and complexity of this paper, and the relative lack of practical motivation for this technically interesting setting, we consider it outside the scope of this paper.

Periodic Monitoring Schedules   The optimal monitoring and payment schedules are dynamically adjusted following changes of the promised utility. In certain situations the principal may prefer a more "regular" schedule, for example, monitoring the agent during the first week of every month. It is worth pointing out that such a more regular schedule, while potentially easier to implement for certain practitioners, has a number of moving parts to be determined, which may not be easy. For example, how to compute the optimal cycle length and length of monitoring period during each cycle? How to combine the optimal payment schedule coupled with such an periodic monitoring schedule? We leave the investigation of such easy/inuitive contract structures to future research.

Imperfect Monitoring   Monitoring in our setting can be perceived as another signal on the agent's effort level that the principal can pay to obtain, besides the arrivals. In this paper we assume that this signal perfectly reflects the effort level. More generally, one can consider imperfect monitoring. That is, the agent's effort changes the statistic of another stochastic process, which is observable to the principal only if the principal pays for it. For example, in the quality control setting mentioned throughout the paper, the arrivals represent customer complains. The principal may choose to monitor by conducting costly customer surveys, and collect praises from customers.

The arrivals of praises constitutes the second Poisson process that is observable to the principal only if the principal pays for this information. Our model sheds light on tackling more complex and general incentive systems such as these.

Endnotes

1. Condition (2.3) is an economic condition, rather than a mathematical condition, which does not affect our results or proofs mathematically.

2. Without the limited liability constraint, even if the agent cannot buy out the entire enterprise, the principal can simply charge the agent a cash amount of β to induce effort. Therefore, the limited liability constraint prevents the model from becoming trivial.

3. If $\bar{w} < \beta$, the only (IC) contract is to always monitor the agent.

Acknowledgments

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Appendix

A. Proofs in Section 2

Proof of Lemma 1. For a generic contract Γ and effort process Λ, following Equations (2.1) and (2.2), we define the agent's total expected utility conditioned on the information available at time t as

$$
\begin{aligned}
u_t(\Gamma, \Lambda) &:= \mathbb{E}^{\Gamma, \Lambda}\Big[\int_0^\infty e^{-\rho\tau}(dL_\tau + b\,\mathbb{I}_{\lambda_\tau = \bar{\lambda}}\,d\tau)\,\Big|\,\mathcal{F}_t \Big] \\
&= \int_0^t e^{-\rho\tau}(dL_\tau + b\,\mathbb{I}_{\lambda_\tau = \bar{\lambda}}\,d\tau) + e^{-\rho t}W_t(\Gamma, \Lambda).
\end{aligned} \tag{A.1}
$$

Therefore, $u_0(\Gamma, \Lambda) = u(\Gamma, \Lambda)$. Moreover, it is easy to verify that process $\{u_t\}_{t \geq 0}$ is an $\mathcal{F}_t$-martingale by conditional expectation's tower property. Define processes

$$
M_t^{s,\Lambda} := \int_0^t y_\tau \lambda_\tau\,d\tau - N_t^s
$$

$$
\text{and } M_t^{n,\Lambda} := \int_0^t (1 - y_\tau)\lambda_\tau\,d\tau - N_t^n. \tag{A.2}
$$

Following the Martingale Representation Theorem (see, for example, Theorem T9 of Brémaud 1981, page 64), there exist $\mathcal{F}_t$-predictable processes $\{H_t^s(\Gamma, \Lambda)\}_{t \geq 0}$ and $\{H_t^n(\Gamma, \Lambda)\}_{t \geq 0}$ such that

$$
u_t(\Gamma, \Lambda) = u_0(\Gamma, \Lambda) + \int_0^t e^{-\rho\tau}\Big[H_\tau^s(\Gamma, \Lambda)\,dM_\tau^{s,\Lambda} + H_\tau^n(\Gamma, \Lambda)\,dM_\tau^{n,\Lambda} \Big], \quad \forall t \geq 0. \tag{A.3}
$$

On the one hand, (A.1) implies

$$
du_t = e^{-\rho t}\Big[dL_t + b\,\mathbb{I}_{\lambda_t = \bar{\lambda}}\,dt - \rho W_t(\Gamma, \Lambda)\,dt + dW_t(\Gamma, \Lambda) \Big]. \tag{A.4}
$$

On the other hand, (A.3) implies

$$
\begin{aligned}
du_t &= e^{-\rho t}\Big[H_t^s(\Gamma, \Lambda)\,dM_t^{s,\Lambda} + H_t^n(\Gamma, \Lambda)\,dM_t^{n,\Lambda} \Big] \\
&= e^{-\rho t}\Big[H_t^s(\Gamma, \Lambda)\big(y_t \lambda_t\,dt - dN_t^s \big) + H_t^n(\Gamma, \Lambda)\big((1 - y_t)\lambda_t\,dt - dN_t^n \big) \Big],
\end{aligned} \tag{A.5}
$$

where the second equality follows from the definitions in (A.2). Combining (A.4) and (A.5) yields (PK). □

Proof of Lemma 2. This result corresponds to Proposition 1 in Biais et al. (2010). Denote $\mathcal{F}_t$-measurable random variable $\tilde{u}_t(\Gamma, \Lambda, \bar{\Lambda})$ to represent the agent's utility under effort process Λ before time t and effort process $\bar{\Lambda}$ afterwards. We have

$$
\tilde{u}_t(\Gamma, \Lambda, \bar{\Lambda}) = \int_0^t e^{-\rho\tau}(dL_\tau + b\,\mathbb{I}_{\lambda_\tau = \bar{\lambda}}\,d\tau) + e^{-\rho t}W_t(\Gamma, \bar{\Lambda}). \tag{A.6}
$$

Consider any sample trajectory of $\{N_t^s, N_t^n\}_{t \geq 0}$ and effort process Λ and $\bar{\Lambda}$,

$$
\tilde{u}_t(\Gamma, \Lambda, \bar{\Lambda}) = u_t(\Gamma, \bar{\Lambda}) + \int_0^t e^{-\rho\tau} b\,\mathbb{I}_{\lambda_\tau = \bar{\lambda}}\,d\tau
$$

$$\begin{aligned}
&= u_0(\Gamma, \bar{\Lambda}) + \int_0^t e^{-\rho\tau} \Big[H_\tau^s(\Gamma, \bar{\Lambda})\, dM_\tau^{s,\bar{\Lambda}} + H_\tau^n(\Gamma, \bar{\Lambda})\, dM_\tau^{n,\bar{\Lambda}} + b\,\mathbb{I}_{\lambda_\tau = \bar{\lambda}}\, d\tau \Big] \\
&= u_0(\Gamma, \bar{\Lambda}) + \int_0^t e^{-\rho\tau} \Big[H_\tau^s(\Gamma, \bar{\Lambda})\, dM_\tau^{s,\Lambda} + H_\tau^n(\Gamma, \bar{\Lambda})\, dM_\tau^{n,\Lambda} \Big] \\
&\quad - \int_0^t e^{-\rho\tau} \Big[y_\tau H_\tau^s(\Gamma, \bar{\Lambda}) + (1 - y_\tau) H_\tau^n(\Gamma, \bar{\Lambda}) - \beta \Big] \Delta\lambda\, \mathbb{I}_{\lambda_\tau = \bar{\lambda}}\, d\tau,
\end{aligned}$$

where the first equality follows from (A.1) and (A.6), the second equality from (A.3), the third equality from (A.2) and the definition of β in (2.5).

Consider any two times $t' < t$,

$$\begin{aligned}
\mathbb{E}[\tilde{u}_t(\Gamma, \Lambda, \bar{\Lambda}) \,|\, \mathcal{F}_{t'}] &= u_0(\Gamma, \bar{\Lambda}) + \int_0^{t'} e^{-\rho\tau} \Big[H_\tau^s(\Gamma, \bar{\Lambda})\, dM_\tau^{s,\Lambda} + H_\tau^n(\Gamma, \bar{\Lambda})\, dM_\tau^{n,\Lambda} \Big] \\
&\quad - \int_0^{t'} e^{-\rho\tau} \Big[y_\tau H_\tau^s(\Gamma, \bar{\Lambda}) + (1 - y_\tau) H_\tau^n(\Gamma, \bar{\Lambda}) - \beta \Big] \Delta\lambda\, \mathbb{I}_{\lambda_\tau = \bar{\lambda}}\, d\tau \\
&\quad - \mathbb{E}\Big[\int_{t'}^t e^{-\rho\tau} [y_\tau H_\tau^s(\Gamma, \bar{\Lambda}) + (1 - y_\tau) H_\tau^n(\Gamma, \bar{\Lambda}) - \beta] \Delta\lambda\, \mathbb{I}_{\lambda_\tau = \bar{\lambda}}\, d\tau \,\Big|\, \mathcal{F}_{t'} \Big] \\
&= \tilde{u}_{t'}(\Gamma, \Lambda, \bar{\Lambda}) - \Delta\lambda\, \mathbb{E}\Big[\int_{t'}^t e^{-\rho\tau} [y_\tau H_\tau^s(\Gamma, \bar{\Lambda}) + (1 - y_\tau) H_\tau^n(\Gamma, \bar{\Lambda}) - \beta] \mathbb{I}_{\lambda_\tau = \bar{\lambda}}\, d\tau \,\Big|\, \mathcal{F}_{t'} \Big].
\end{aligned}$$
$$(A.7)$$

(i) On the one hand, if (IC) holds under contract Γ, (A.7) suggests that

$$\mathbb{E}[\tilde{u}_t(\Gamma, \Lambda, \bar{\Lambda}) \,|\, \mathcal{F}_{t'}] \le \tilde{u}_{t'}(\Gamma, \Lambda, \bar{\Lambda}),$$

which implies that the process $\{\tilde{u}_t\}_{t \ge 0}$ is a super-martingale. Taking $t' = 0$ and letting $t \to \infty$, we have

$$u(\Gamma, \Lambda) = \lim_{t \to \infty} \Big\{ \mathbb{E}[\tilde{u}_t(\Gamma, \Lambda, \bar{\Lambda}) \,|\, \mathcal{F}_0] \Big\} \le \tilde{u}_0(\Gamma, \Lambda, \bar{\Lambda}) = u(\Gamma, \bar{\Lambda}). \tag{A.8}$$

That is, effort process $\bar{\Lambda}$ dominates any other process Λ under contract Γ, or, Γ is incentive compatible if (IC) holds.

(ii) On the other hand, suppose (IC) does not hold, or, $y_\tau H_\tau^s(\Gamma, \bar{\Lambda}) + (1 - y_\tau) H_\tau^n(\Gamma, \bar{\Lambda}) < \beta$ for $m_\tau = 0$ over a subset of $[0, t]$ with positive measure. Define an effort process $\Lambda = \{\lambda_\tau\}_{\tau \ge 0}$, such that $\lambda_\tau = \lambda$ for $\forall \tau \in (t, \infty)$, and for $\forall \tau \in [0, t]$:

$$\lambda_\tau = \begin{cases} \bar{\lambda} & \text{if } y_\tau H_\tau^s(\Gamma, \bar{\Lambda}) + (1 - y_\tau) H_\tau^n(\Gamma, \bar{\Lambda}) < \beta \text{ and } m_\tau = 0, \\ \lambda & \text{otherwise.} \end{cases}$$

Clearly, we must have

$$-\mathbb{E}\Big[\int_0^t e^{-\rho\tau} [y_\tau H_\tau^s(\Gamma, \bar{\Lambda}) + (1 - y_\tau) H_\tau^n(\Gamma, \bar{\Lambda}) - \beta] \Delta\lambda\, \mathbb{I}_{\Lambda_\tau = \bar{\lambda}}\, d\tau \,\Big|\, \mathcal{F}_{t'} \Big] > 0.$$

As a result, taking $t' = 0$ and letting $t \to \infty$ in (A.7), we obtain

$$u(\Gamma, \Lambda) = \lim_{t \to \infty} \Big\{ \mathbb{E}[\tilde{u}_t(\Gamma, \Lambda, \bar{\Lambda}) \,|\, \mathcal{F}_0] \Big\} > \tilde{u}_0(\Gamma, \Lambda, \bar{\Lambda}) = u(\Gamma, \bar{\Lambda}).$$

This implies that effort process Λ dominates $\bar{\Lambda}$, or, contract Γ is *not* incentive compatible when (IC) does not hold. $\square$

Proof of Lemma 3. Following Itô's change of variable formula with function F (see, for example, Theorem 14.3.2 of Cohen and Elliott 2015), for any $\tau \geq 0$, we have:

$$
\begin{aligned}
e^{-r\tau}F(W_\tau) &= F(w) + \int_0^\tau \left[e^{-rt}dF(W_t) - re^{-rt}F(W_t)dt \right] \\
&= F(w) + \int_0^\tau e^{-rt}(m_t dt + dL_t) + \int_0^\tau e^{-rt}\mathcal{A}_t,
\end{aligned}
\tag{A.9}
$$

where

$$
\begin{aligned}
\mathcal{A}_t &:= dF(W_t) - rF(W_t)dt - m_t dt - dL_t \\
&= F'(W_t)\left[\left(\rho W_t + \lambda(y_t H_t^s + (1-y_t)H_t^n) \right)dt - \ell_t dt \right] - rF(W_t)dt \\
&\quad + F(W_t - H_t^s dN_t^s - H_t^n dN_t^n - I_t) - F(W_t) - m_t dt - dL_t.
\end{aligned}
$$

Further define

$$
\begin{aligned}
\mathcal{B}_t &:= [F(W_t - H_t^s) - F(W_t)](dN_t^s - \lambda y_t dt) \\
&\quad + [F(W_t - H_t^n) - F(W_t)](dN_t^n - \lambda(1-y_t)dt).
\end{aligned}
\tag{A.10}
$$

Because function $F(w)$ is concave and $F'(w) \geq -1$, we have

$$
\begin{aligned}
\mathcal{A}_t &\leq F'(W_t)\left(\rho W_t + \lambda(y_t H_t^s + (1-y_t)H_t^n) \right)dt + F(W_t - H_t^s dN_t^s - H_t^n dN_t^n) \\
&\quad - F'(W_t)\ell_t dt - F'(W_t - H_t^s dN_t^s - H_t^n dN_t^n)I_t - F(W_t) - rF(W_t)dt - m_t dt - dL_t \\
&\leq F'(W_t)\left(\rho W_t + \lambda(y_t H_t^s + (1-y_t)H_t^n) \right)dt - rF(W_t)dt - m_t dt + F(W_t - H_t^s dN_t^s - H_t^n dN_t^n) - F(W_t) \\
&= F'(W_t)\left(\rho W_t + \lambda(y_t H_t^s + (1-y_t)H_t^n) \right)dt - rF(W_t)dt - m_t dt + [F(W_t - H_t^s) - F(W_t)]dN_t^s \\
&\quad + [F(W_t - H_t^n) - F(W_t)]dN_t^n \\
&= \mathcal{B}_t + \Psi_t dt.
\end{aligned}
$$

Therefore, if $\Psi_t \leq 0$, we must have $\mathcal{A}_t \leq \mathcal{B}_t$ almost surely. Taking the expectation on both sides of (A.9), we immediately have

$$
F(w) \geq \mathbb{E}^{\Gamma, \bar{\Lambda}}\left[e^{-r\tau}F(W_\tau) - \int_0^\tau e^{-rt}\left(m_t dt + dL_t \right) \right],
$$

where we use the fact that $\int_0^\tau e^{-rt}\mathcal{B}_t$ is a martingale.

Taking $\tau \to \infty$, the above inequality reduces to

$$
F(w) \geq -\mathbb{E}^{\Gamma, \bar{\Lambda}}\left[\int_0^\infty e^{-rt}\left(m_t dt + dL_t \right) \right] = U(\Gamma).
$$

This completes the proof. $\square$

B. Proofs in Section 3

We first present the following technical lemma.

LEMMA 6. *For any $\alpha \geq \beta$, starting with any boundary condition $F(w)$ that is continuous for $w \in [0, \alpha]$, DDE (H) uniquely determines a continuous function $F(w)$ on $w \in [0, \infty)$. Furthermore, function $F(w)$ is increasing on $[\alpha, \infty)$ if either of the following two conditions holds.*

(i) Function $F(w)$ is positive and non-decreasing on $[0, \alpha]$;

(ii) Function $F(w)$ is increasing within $[0, \alpha]$ and $F(\alpha) \geq 0$.

Proof. Solving DDE (H) with well-defined boundary conditions over $w \in [0, \alpha]$ is equivalent to solving a sequence of initial value problems over interval $[\alpha + k\beta, \alpha + (k+1)\beta]$ where $k = 0, 1, \ldots$. This sequence of initial value problems satisfy the Cauchy-Lipschitz Theorem (see, for example, Theorem 1.1 of Hartman 1982); therefore, a unique and differentiable solution is guaranteed over $w \in (\alpha, \infty)$.

(i) If $F(w)$ is positive and non-decreasing for $w \in [0, \alpha]$, suppose it is non-increasing for some $w \geq \alpha$. Let

$$\hat{w} := \min\{w | F'(w_+) \leq 0 \text{ and } w \geq \alpha\}.$$

DDE (H) implies

$$rF(\hat{w}) + \lambda[F(\hat{w}) - F(\hat{w} - \beta)] \leq 0,$$

which is impossible, because $F(\hat{w}) \geq F(\alpha) > 0$ as assumed, and $F(\hat{w}) \geq F(\hat{w} - \beta)$ for $w \in [\alpha, \hat{w}]$ from the definition of $\hat{w}$. Therefore, we must have $F'(w) > 0$ for $\forall w \in [\alpha, \infty)$. Part (ii) can be proven similarly. □

LEMMA 7. *Consider the case $r = \rho$. Function $J(w)$, which is the solution of DDE (H) with boundary condition $J(w) = 1$ on $w \in [0, \beta)$, is increasing and strictly convex on $w \in [\beta, \infty)$.*

Proof. Note that following DDE (H), function $J(w)$ is differentiable for $w > \beta$, and is twice-differentiable except at $w = 2\beta$. Taking derivatives on both sides of DDE (H) yields

$$J''(w) = \frac{(\lambda + r - \rho)J'(w) - \lambda J'(w - \beta)}{\rho w + \beta \lambda}, \quad \text{for } w \in (\beta, 2\beta) \cup (2\beta, \infty). \tag{B.1}$$

In particular, there is

$$J(w) = \frac{r}{\lambda + r}\left(\frac{\rho w + \lambda \beta}{\rho \beta + \lambda \beta}\right)^{\frac{\lambda + r}{\rho}} + \frac{\lambda}{\lambda + r}, \text{ for } w \in (\beta, 2\beta), \tag{B.2}$$

whose first and second derivatives are

$$J'(w) = \frac{r(\rho w + \lambda \beta)^{\frac{\lambda + r - \rho}{\rho}}}{(\rho \beta + \lambda \beta)^{\frac{\lambda + r}{\rho}}}$$

$$J''(w) = \frac{r(\rho w + \lambda \beta)^{\frac{\lambda + r - 2\rho}{\rho}}}{(\rho \beta + \lambda \beta)^{\frac{\lambda + r}{\rho}}}(\lambda + r - \rho). \tag{B.3}$$

When $r = \rho$, the closed-form expression in (B.2) is clearly convex, i.e., $J(w)$ is convex for $w \in [\beta, 2\beta)$. Consider the point $w = 2\beta$, (B.1) and (B.3) yield

$$J''(2\beta_+) = \frac{\lambda r[(2\beta\rho + \beta\lambda)^{\frac{\lambda}{\rho}} - (\beta\rho + \beta\lambda)^{\frac{\lambda}{\rho}}]}{(2\beta\rho + \beta\lambda)(\beta\rho + \beta\lambda)^{\frac{\lambda + r}{\rho}}} > 0.$$

Therefore, we only need to show $J''(w) > 0$ for all $w > 2\beta$. We prove by contradiction. Suppose, on the contrary, there exists some $w > 2\beta$, such that $J''(w) \leq 0$. Define

$$\hat{w} := \min\{w | J'(w) \leq J'(w - \beta) \text{ and } w > 2\beta\}.$$

By construction, we have $J''(w) > 0$ for all $w \in (\beta, \hat{w})$. First, $\hat{w}$ cannot be in $(2\beta, 3\beta]$, because otherwise we would have $\hat{w} - \beta \leq 2\beta$, and,

$$J'(\hat{w}) > J'(2\beta) \geq J'(\hat{w} - \beta),$$

which contradicts the definition of $\hat{w}$. Second, $\hat{w}$ cannot be greater than 3β either, because otherwise we have

$$J'(\hat{w}) = J'(\hat{w} - \beta) + \int_0^\beta J''(\hat{w} - \beta + x)dx > J'(\hat{w} - \beta),$$

which, again, contradicts the definition of $\hat{w}$. Therefore, we must have $J''(w) > 0$ for all $w > 2\beta$. Last but not the least, monotonicity of $J(w)$ follows directly from Lemma 6, which completes the proof. $\qquad\square$

Proof of Proposition 1. (i) Given that $\theta(\bar{w}) = \frac{m}{r}J'(\bar{w}) - 1$ and $J'(\bar{w}) \geq 0$ (from Lemma 7), we have $\theta(\bar{w}) \geq -1$. We prove $\theta(\bar{w}) < \frac{m}{\beta r}$ by contradiction. Suppose, on the contrary, we have $\theta(\bar{w}) \geq \frac{m}{\beta r}$. It's easy to verify that function $F_{\theta(\bar{w})}(w)$ can be decomposed as

$$F_{\theta(\bar{w})}(w) = \left(\theta(\bar{w}) - \frac{m}{r\beta}\right)G_1(w) + \frac{m}{r}G_2(w), \text{ for } w \geq 0,$$

in which functions $G_1(w)$ and $G_2(w)$ are the solution of DDE (H) with boundary conditions being

$$G_1(w) = w \text{ and } G_2(w) = \frac{w}{\beta} - 1, \quad \forall w \in [0, \beta),$$

respectively. Since $G_1(w)$ and $G_2(w)$ are both increasing on $[0, \beta]$ and nonnegative at $w = \beta$, we know that they are both increasing on $[\beta, \infty)$ by Lemma 6. As such, $F_{\theta(\bar{w})}(\bar{w})$ is increasing on $[\beta, \infty)$ as well, which contradicts to $F'_{\theta(\bar{w})}(w) = -1$. Therefore, contradiction is established and we must have $\theta_{\bar{w}} < \frac{m}{\beta r}$.

(ii) We only needs to show that $F'_{\theta(\bar{w})}(\beta_+) \leq F'_{\theta(\bar{w})}(\beta_-)$ since afterwards, concavity of $F_{\theta(\bar{w})}(w)$ follows immediately from the strict convexity of function $J(w)$ in Lemma 7 and the decomposition in (3.9). If $r = \rho$, according to (H), we have

$$F'_{\theta(\bar{w})}(\beta_+) = \frac{(\lambda + r)F_{\theta(\bar{w})}(\beta) - \lambda F_{\theta(\bar{w})}(0)}{(r + \lambda)\beta} = \frac{\lambda}{\lambda + r}\theta(\bar{w}) \leq \theta(\bar{w}) = F'_{\theta(\bar{w})}(\beta_-). \tag{B.4}$$

Next, by the definition of $\theta(\bar{w})$, we know that $\theta(\bar{w})$ is strictly increasing in $\bar{w}$ (recall the convexity of $J(w)$). Therefore, for any $\tilde{w} \in [\beta, \bar{w})$, we have $\theta(\bar{w}) > \theta_{(}\tilde{w})$. For any $w \geq 0$, decomposition (3.9) implies that

$$F_{\theta(\bar{w})}(w) - F_{\theta(\tilde{w})}(w) = [\theta(\bar{w}) - \theta(\tilde{w})]w > 0.$$

This completes the proof. $\qquad\square$

LEMMA 8. *Consider a concave function $F(w)$ that satisfies equations (H), (U), and (3.10). For any $w \geq \beta$, the following function $\Phi(w, x)$ is increasing in $x \in (-\infty, 0]$ and decreasing in $x \in [0, \infty)$,*

$$\Phi(w, x) := F'(w)x + F(w - x). \tag{B.5}$$

Proof. Taking the first derivative of $\Phi(w, x)$ with respect to x yields

$$\frac{\partial \Phi(w, x)}{\partial x} = F'(w) - F'(w - x).$$

Because $F(w)$ is concave, we know that $\dfrac{\partial \Phi(w, x)}{\partial x} \geq 0$ when $x \leq 0$ and $\dfrac{\partial \Phi(w, x)}{\partial x} \leq 0$ when $x \geq 0$. That is, for any $w \geq \beta$, $\Phi(w, x)$ is increasing in $x \in (-\infty, 0]$ and decreasing in $x \in [0, \infty)$. $\qquad\square$

Proof of Proposition 2. Starting with any promised utility $W_0 = w \in [0, \bar{w}]$, consider the process $\{W_t\}_{t \geq 0}$ according to (PK) in which the counting processes $\{(N_t^s, N_t^n)\}_{t \geq 0}$ are generated from the effort process $\bar{\Lambda}$ under contracts $\Gamma_d(w; \beta, \bar{w})$ or $\Gamma_r(w; \bar{w})$. Clearly, we must have $0 \leq W_t \leq \bar{w}$ for $\forall t$.

(i) First, we consider contract $\Gamma_d(w; \beta, \bar{w})$ defined in Definition 1. Following Itô's Formula for jump processes, we have

$$dF_{\theta(\bar{w})}(W_t) = [F_{\theta(\bar{w})}(W_t - I_t) - F_{\theta(\bar{w})}(W_t)] + F'_{\theta(\bar{w})}(W_t)\Big\{\rho W_t + \lambda_t[y_t H_t^s + (1 - y_t)H_t^n] - \ell_t\Big\}dt$$
$$+ [F_{\theta(\bar{w})}(W_t - H_t^s) - F_{\theta(\bar{w})}(W_t)]dN_t^s + [F_{\theta(\bar{w})}(W_t - H_t^n) - F_{\theta(\bar{w})}(W_t)]dN_t^n. \quad \text{(B.6)}$$

Following the dynamics of W_t according to Definition 1, we have

$$dF_{\theta(\bar{w})}(W_t) = F'_{\theta(\bar{w})}(W_t)\Big(\rho W_t \mathbb{I}_{W_t < \beta} + (\rho W_t + \lambda\beta)\mathbb{I}_{\beta \leq W_t < \bar{w}}\Big)dt$$
$$+ \Big[F_{\theta(\bar{w})}(W_t - \beta) - F_{\theta(\bar{w})}(W_t)\Big]\mathbb{I}_{\beta \leq W_t \leq \bar{w}}dN_t. \quad \text{(B.7)}$$

For any $\tau \geq 0$, we have:

$$e^{-r\tau}F_{\theta(\bar{w})}(W_\tau) = F_{\theta(\bar{w})}(w) + \int_0^\tau F_{\theta(\bar{w})}(W_t)\,de^{-rt} + \int_0^\tau e^{-rt}dF_{\theta(\bar{w})}(W_t)$$
$$= F_{\theta(\bar{w})}(w) + \int_0^\tau e^{-rt}\Big[F'_{\theta(\bar{w})}(W_t)\Big(\rho W_t\mathbb{I}_{W_t < \beta} + (\rho W_t + \beta\lambda)\mathbb{I}_{\beta \leq W_t < \bar{w}}\Big) - rF_{\theta(\bar{w})}(W_t)\Big]dt$$
$$+ \int_0^\tau e^{-rt}\Big[F_{\theta(\bar{w})}(W_t - \beta) - F_{\theta(\bar{w})}(W_t)\Big]\mathbb{I}_{\beta \leq W_t \leq \bar{w}}dN_t. \quad \text{(B.8)}$$

From Equations(3.8) and (H), we know that $F_{\theta(\bar{w})}(w)$ satisfies

$$F'_{\theta(\bar{w})}(W_t)\rho W_t\mathbb{I}_{W_t < \beta} = [rF_{\theta(\bar{w})}(W_t) + m]\mathbb{I}_{W_t < \beta}, \text{ and}$$
$$F'_{\theta(\bar{w})}(W_t)(\rho W_t + \beta\lambda)\mathbb{I}_{\beta \leq W_t < \bar{w}} = \Big[(\lambda + r)F_{\theta(\bar{w})}(W_t) - \lambda F_{\theta(\bar{w})}(W_t - \beta)\Big]\mathbb{I}_{\beta \leq W_t < \bar{w}}.$$

Substituting the above equations into (B.8), we obtain

$$e^{-r\tau}F_{\theta(\bar{w})}(W_\tau) = F_{\theta(\bar{w})}(w) + \int_0^\tau e^{-rt}\Big[F_{\theta(\bar{w})}(W_t - \beta) - F_{\theta(\bar{w})}(W_t)\Big]\mathbb{I}_{\beta \leq W_t \leq \bar{w}}dN_t$$
$$+ \int_0^\tau e^{-rt}\Big[(\lambda + r)F_{\theta(\bar{w})}(W_t) - \lambda F_{\theta(\bar{w})}(W_t - \beta)\Big]\mathbb{I}_{\beta \leq W_t < \bar{w}}dt$$
$$+ \int_0^\tau e^{-rt}\Big[(rF_{\theta(\bar{w})}(W_t) + m)\mathbb{I}_{W_t < \beta} - rF_{\theta(\bar{w})}(W_t)\Big]dt$$
$$= F_{\theta(\bar{w})}(w) + \int_0^\tau e^{-rt}\Big(m\mathbb{I}_{W_t < \beta} + (\rho\bar{w} + \beta\lambda)\mathbb{I}_{W_t = \bar{w}}\Big)dt + \Omega_\tau, \quad \text{(B.9)}$$

where the second equality utilizes Equation (U), and the process $\{\Omega_\tau\}_{\tau \geq 0}$, defined as

$$\Omega_\tau := \int_0^\tau e^{-rt}\Big[F_{\theta(\bar{w})}(W_t - \beta) - F_{\theta(\bar{w})}(W_t)\Big]\mathbb{I}_{\beta \leq W_t \leq \bar{w}}(dN_t - \lambda dt),$$

is a martingale. Taking expectation on both sides of (B.9) and letting $\tau \to \infty$, we have

$$F_{\theta(\bar{w})}(w) = -\mathbb{E}^{\Gamma_d(w;\beta,\bar{w}),\bar{\Lambda}}\Big[\int_0^\infty e^{-rt}\Big(m\mathbb{I}_{W_t < \beta} + (\rho\bar{w} + \beta\lambda)\mathbb{I}_{W_t = \bar{w}}\Big)dt\Big]$$
$$= -\mathbb{E}^{\Gamma_d(w;\beta,\bar{w}),\bar{\Lambda}}\Big[\int_0^\infty e^{-rt}\Big(m_t\,dt + dL_t\Big)\Big] = U(\Gamma_d(w; \beta, \bar{w})),$$

where the second equality follows from Definition 1.

(ii) Next, we consider contract $\Gamma_r(w; \bar{w})$ defined in Definition 2. Following Itô's Formula for jump process, we have

$$
\begin{aligned}
dF_{\theta(\bar{w})}(W_t) = {}& F'_{\theta(\bar{w})}(W_t)(\rho W_t + \lambda\beta)\mathbb{I}_{\beta \leq W_t < \bar{w}} dt \\
& + \Big\{ [F_{\theta(\bar{w})}(\beta) - F(W_t)] dN_t^n + [F_{\theta(\bar{w})}(0) - F_{\theta(\bar{w})}(W_t)]dN_t^s \Big\}\mathbb{I}_{\beta \leq W_t \leq \min\{\bar{w}, 2\beta\}} \\
& + [F_{\theta(\bar{w})}(W_t - \beta) - F_{\theta(\bar{w})}(W_t)]\mathbb{I}_{\min\{\bar{w}, 2\beta\} < W_t \leq \bar{w}} dN_t^n.
\end{aligned}
\tag{B.10}
$$

For any time $\tau \geq 0$, we have

$$
\begin{aligned}
e^{-r\tau} F_{\theta(\bar{w})}(W_\tau) ={}& F_{\theta(\bar{w})}(w) + \int_0^\tau F_{\theta(\bar{w})}(W_t) de^{-rt} + \int_0^\tau e^{-rt} dF_{\theta(\bar{w})}(W_t) \\
={}& F_{\theta(\bar{w})}(w) + \int_0^\tau e^{-rt}\Big(F'_{\theta(\bar{w})}(W_t)(\rho W_t + \lambda\beta)\mathbb{I}_{\beta \leq W_t < \bar{w}} - rF_{\theta(\bar{w})}(W_t)\Big) dt \\
& + \int_0^\tau e^{-rt}\Big\{ [F_{\theta(\bar{w})}(\beta) - F_{\theta(\bar{w})}(W_t)]\mathbb{I}_{\beta \leq W_t \leq \min\{\bar{w}, 2\beta\}} dN_t^n \\
& \qquad\qquad + [F_{\theta(\bar{w})}(0) - F_{\theta(\bar{w})}(W_t)]\mathbb{I}_{\beta \leq W_t \leq \min\{\bar{w}, 2\beta\}} dN_t^s \\
& \qquad\qquad + [F_{\theta(\bar{w})}(W_t - \beta) - F_{\theta(\bar{w})}(W_t)]\mathbb{I}_{\min\{\bar{w}, 2\beta\} < W_t \leq \bar{w}} dN_t \Big\} \\
={}& F_{\theta(\bar{w})}(w) + \int_0^\tau e^{-rt}\Big(m\mathbb{I}_{W_t = 0}\, dt + (\rho W_t + \beta\lambda)\mathbb{I}_{W_t = \bar{w}}\, dt\Big) + \Omega_\tau,
\end{aligned}
\tag{B.11}
$$

where the last equality follows from Equation (L), which implies that $F_{\theta(\bar{w})}(0)(2 - \frac{W_t}{\beta}) + F_{\theta(\bar{w})}(\beta)(\frac{W_t}{\beta} - 1) = F_{\theta(\bar{w})}(W_t - \beta)$, and the process $\{\Omega_\tau\}_{\tau \geq 0}$, defined as

$$
\begin{aligned}
\Omega_\tau := \int_0^\tau e^{-rt} \Big\{ & \Big(F_{\theta(\bar{w})}(0) - F_{\theta(\bar{w})}(W_t)\Big)\Big(dN_t^s - (2 - \frac{W_t}{\beta})\lambda dt\Big)\mathbb{I}_{\beta \leq W_t \leq \min\{\bar{w}, 2\beta\}} \\
& + \Big(F_{\theta(\bar{w})}(\beta) - F_{\theta(\bar{w})}(W_t)\Big)\Big(dN_t^n - (\frac{W_t}{\beta} - 1)\lambda dt\Big)\mathbb{I}_{\beta \leq W_t \leq \min\{\bar{w}, 2\beta\}} \\
& + \Big(F_{\theta(\bar{w})}(W_t - \beta) - F_{\theta(\bar{w})}(W_t)\Big)\Big(dN_t - \lambda dt\Big)\mathbb{I}_{\min\{\bar{w}, 2\beta\} < W_t \leq \bar{w}} \Big\},
\end{aligned}
$$

is a martingale.

Taking expectation on both sides of (B.11) and letting $\tau \to \infty$, we have

$$
F_{\theta(\bar{w})}(w) = -\mathbb{E}^{\Gamma_r(w; \bar{w}), \Lambda}\Big[\int_0^\infty e^{-rt}\Big(m\mathbb{I}_{W_t = 0}\, dt + (\rho\bar{w} + \beta\lambda)\mathbb{I}_{W_t = \bar{w}}\, dt\Big)\Big] = U(\Gamma_r(w; \bar{w})).
$$

This completes the proof. $\qquad\square$

Proof of Theorem 1. By Proposition 1, we know that $F_{\theta(\bar{w})}(w)$ is concave and $F'_{\theta(\bar{w})}(w) \geq -1$. Therefore, we only need to show $\Psi_t \leq 0$ holds almost surely (recall Lemma 3).

From Equation (2.8), we have

$$
\begin{aligned}
\Psi_t \leq {}& \lambda\Big[F'_{\theta(\bar{w})}(W_t)\Big(y_t H_t^s + (1 - y_t)H_t^n\Big) + y_t F_{\theta(\bar{w})}(W_t - H_t^s) + (1 - y_t)F_{\theta(\bar{w})}(W_t - H_t^n) - F_{\theta(\bar{w})}(W_t)\Big] \\
& + F'_{\theta(\bar{w})}(W_t)\rho W_t - rF_{\theta(\bar{w})}(W_t) - m_t \\
\leq {}& \lambda\Phi\Big(W_t, y_t H_t^s + (1 - y_t)H_t^n\Big) + F'_{\theta(\bar{w})}(W_t)\rho W_t - (\lambda + r)F_{\theta(\bar{w})}(W_t) - m_t,
\end{aligned}
\tag{B.12}
$$

in which function Φ is defined in (B.5).

(i) When $W_t < \beta$, we know that the principal monitors the agent (i.e., $m_t = m$). From Equation (3.8), we have

$$\Psi_t \leq \rho W_t F'_{\theta(\bar{w})}(W_t) - r F_{\theta(\bar{w})}(W_t) - m = 0.$$

(ii) When $\beta \leq W_t \leq \bar{w}$, substituting (H) into inequality (B.12) yields

$$\Psi_t \leq \lambda \Phi\Big(W_t, y_t H_t^s + (1 - y_t) H_t^n\Big) - \lambda \beta F'_{\theta(\bar{w})}(W_t) - \lambda F_{\theta(\bar{w})}(W_t - \beta) - m_t.$$

• If the principal does not monitor at time t (i.e., $m_t = 0$), we must have $y_t H_t^s + (1 - y_t) H_t^n \geq \beta$. Lemma 8 implies

$$\Psi_t \leq \lambda \Phi(W_t, \beta) - \lambda \beta F'_{\theta(\bar{w})}(W_t) - \lambda F_{\theta(\bar{w})}(W_t - \beta) = 0.$$

• If the principal monitors at time t (i.e., $m_t = m$), Lemma 8 implies

$$\begin{aligned}
\Psi_t &\leq \lambda \Phi(W_t, 0) - \lambda \beta F'_{\theta(\bar{w})}(W_t) - \lambda F_{\theta(\bar{w})}(W_t - \beta) - m \\
&= -r F_{\theta(\bar{w})}(W_t) + r W_t F'_{\theta(\bar{w})}(W_t) - m \\
&\leq -r F_{\theta(\bar{w})}(\beta) + r \beta F'_{\theta(\bar{w})}(\beta) - m \\
&= -r \Big(\theta(\bar{w})\beta - \frac{m}{r}\Big) + r \beta \theta \bar{w} \frac{\lambda}{\lambda + r} - m \\
&= r \beta \theta(\bar{w}) \Big(\frac{\lambda}{\lambda + r} - 1\Big) \leq 0,
\end{aligned}$$

where the second inequality follows the fact that $-r F(W_t) + r W_t F'(W_t)$ is decreasing in W_t.

To sum up, we must have $\Psi_t \leq 0$. This completes the proof. □

C. Proofs in Section 4

To prove Proposition 6 and Lemma 5, we first characterize the structural properties of function $J(w)$ for the case $r < \rho$.

LEMMA 9. *Consider the case $r < \rho$. For any $\alpha \geq \beta$, define function $J(w)$ to be the solution of DDE (H) with boundary condition $J(w) = 1$ on $w \in [0, \alpha)$. $J(w)$ exhibits the following properties.*
   (i) When $r \leq \rho - \lambda$, $J(w)$ is concave on $[\alpha, \infty)$.
   (ii) When $\rho - \lambda < r < \bar{r}$, $J(w)$ is convex on $[\alpha, \alpha + \beta]$ and concave on $[\alpha + \beta, \infty)$.
   (iii) We have $\limsup\limits_{w \to \infty} J'(w) = 0$.

Proof. (i) Note that following DDE (H), function $J(w)$ is differentiable for $w > \alpha$, and is twice-differentiable except at $w = \alpha + \beta$. Taking derivatives on both sides of DDE (H) yields

$$J''(w) = \frac{(\lambda + r - \rho)J'(w) - \lambda J'(w - \beta)}{\rho w + \beta \lambda}, \quad \text{for } w \in (\alpha, \alpha + \beta) \cup (\alpha + \beta, \infty). \tag{C.1}$$

Because $J(w)$ is non-decreasing (recall Lemma 6), we have $J'(w) \geq 0$ and $J'(w - \beta) \geq 0$ for $w \in (\alpha, \infty)$. As such, we must have $J''(w) \leq 0$ if $r \leq \rho - \lambda$, except $w = \alpha + \beta$, at which point $J(w)$ is differentiable. Therefore, $J(w)$ is concave on (α, ∞) and continuity of function $J(w)$ extends concavity to $w \in [\alpha, \infty)$.

(ii) Starting from boundary conditions $J(w) = 1$ for $w \in [0, \alpha)$, DDE (H) yields the following:

$$J(w) = \frac{r}{\lambda + r} \Big(\frac{\rho w + \lambda \beta}{\rho \beta + \lambda \beta}\Big)^{\frac{\lambda + r}{\rho}} + \frac{\lambda}{\lambda + r}, \quad \text{for } w \in (\alpha, \alpha + \beta), \tag{C.2}$$

whose first and second derivatives are

$$J'(w) = \frac{r(\rho w + \lambda \beta)^{\frac{\lambda + r - \rho}{\rho}}}{(\rho \beta + \lambda \beta)^{\frac{\lambda + r}{\rho}}}$$

$$J''(w) = \frac{r(\rho w + \lambda \beta)^{\frac{\lambda + r - 2\rho}{\rho}}}{(\rho \beta + \lambda \beta)^{\frac{\lambda + r}{\rho}}} (\lambda + r - \rho). \tag{C.3}$$

Therefore, if $r > \rho - \lambda$, we must have $J''(w) > 0$, i.e., $J(w)$ is convex on $[\alpha, \alpha + \beta]$ since function $J(w)$ is continuous.

Given that the right-hand-side of (5.1) is increasing whereas the left-hand-side is decreasing in $\bar{r}$, it is readily shown that Equation (5.1) has a unique solution within $(\rho - \lambda, \rho)$. Moreover, for $r \in (\rho - \lambda, \bar{r})$, we have

$$\left(\frac{2\rho + \lambda}{\rho + \lambda} \right)^{\frac{\lambda + r}{\rho} - 1} < \frac{\lambda}{\lambda + r - \rho},$$

which implies that $J''((\alpha + \beta)_+) < 0$, i.e., we have $(\lambda + r - \rho)J'(\alpha + \beta) < \lambda J'(\alpha_+)$.

To prove that $J(w)$ is concave on $[\alpha + \beta, \infty)$, we only need to show $J''(w) \le 0$ for all $w > \alpha + \beta$. Suppose, on the contrary, there exists a $w > \alpha + \beta$ such that $J''(w) > 0$. Define

$$\hat{w} := \min\{w | (\lambda + r - \rho)J'(w) \ge \lambda J'(w - \beta) \text{ and } w > \alpha + \beta\}.$$

• If $\hat{w} \in (\alpha + \beta, \alpha + 2\beta]$, we have $J'(\alpha + \beta) \ge J'(\hat{w})$ because $J(w)$ is concave within $[\alpha + \beta, \hat{w}]$. We also have $J'(\beta_+) < J'(\hat{w} - \beta)$ because $J(w)$ is convex on $[\alpha, \alpha + \beta]$. Therefore,

$$(\lambda + r - \rho)J'(\hat{w}) \le (\lambda + r - \rho)J'(\alpha + \beta) < \lambda J'(\beta_+) < \lambda J'(\hat{w} - \beta).$$

• If $\hat{w} > \alpha + 2\beta$, then both $J(\hat{w})$ and $J(\hat{w} - \beta)$ are twice continuously differentiable. As such,

$$J'(\hat{w}) = J'(\hat{w} - \beta) + \int_0^\beta J''(\hat{w} - \beta + x)dx < J'(\hat{w} - \beta),$$

which implies $(\lambda + r - \rho)J'(\hat{w}) < \lambda J'(\hat{w} - \beta)$.

That is, $(\lambda + r - \rho)J'(\hat{w}) \ge \lambda J'(\hat{w} - \beta)$ cannot be true for either case. Therefore, $J(w)$ is concave on $[\alpha + \beta, \infty)$.

(iii) For notational convenience, we let $\ell := \limsup_{w \to \infty} J'(w)$. First we show that ℓ cannot be positive infinity. Suppose, on the contrary, $\ell = \limsup_{w \to \infty} J'(w) = \infty$. Then there exists an increasing divergent sequence $\{w_n\}_{n \ge 1}$ in $(\alpha + \beta, \infty)$ such that $\lim_{w \to \infty} J'(w_n) = \infty$ and

$$w_n = \arg\max_{w \in [0, w_n]} \{J'(w)\}.$$

Then for each $n \ge 1$ by mean value theorem, there exists $\hat{w}_n \in (w_n - \beta, w_n)$, such that

$$(\rho w_n + \beta \lambda)J'(w_n) = \lambda[J(w_n) - J(w_n - \beta)] + rJ(w_n)$$
$$= \lambda \beta J'(\hat{w}_n) + rJ(w_n).$$

Rearranging the equation above, one gets

$$J'(\hat{w}_n) = \frac{w_n}{\lambda \beta}[\rho J'(w_n) - \frac{r}{w_n}J(w_n)] + J'(w_n). \tag{C.4}$$

Since $J(0) = 1$ and $\{J'(w_n)\}_n$ is an increasing sequence, there is $J(w_n) - 1 \leq w_n J'(w_n)$ by construction. For such n, from (C.4) and the fact $J(w_n) \leq w_n J'(w_n)$, there is

$$J'(\hat{w}_n) \geq \frac{(\rho - r) w_n J'(w_n)}{\lambda \beta}.$$

Since $J'(w_n) > 0$, an immediate result follows the previous inequality is that

$$\frac{J'(\hat{w}_n)}{J'(w_n)} \geq \frac{(\rho - r) w_n}{\lambda \beta},$$

which goes to infinity as n goes to infinity. Therefore, we can obtain $J'(\hat{w}_n) > J'(w_n)$ eventually, which contradicts to the definition of w_n since $\hat{w}_n < w_n$. Thus, we have that ℓ must be finite.

Consider a new increasing and divergent sequence $\{w_n\}_{n \geq 1}$ in $(\alpha + \beta, \infty)$ such that $\lim_{n \to \infty} J'(w_n) = \ell$. Then for all $n \geq 1$, we can find a constant D such that $J(w_n) \leq \ell w_n + D$. Let $\hat{w}_n \in (w_n - \beta, w_n)$, by substituting $J(w_n) \leq \ell w_n + D$ into the differential equations of $J(w)$ for all $n \geq 1$, we have

$$\rho J'(w_n) - rl \leq \frac{\lambda \beta [J'(\hat{w}_n) - J'(w_n)] + rD}{w_n}.$$

By letting n goes to infinity, there is

$$(\rho - r) l \leq \lambda \beta \liminf_{n \to \infty} \frac{J'(\hat{w}_n)}{w_n}.$$

If $l > 0$, the above inequality implies that ℓ must go to infinity; this contradicts to the fact that l is finite. Therefore, we have $l \leq 0$. Given that $J'(w) \geq 0$ for all w (recall Lemma 6), we must have $\ell = 0$. $\qquad\square$

Proof of Lemma 4. Taking the first derivative with respect to α, we have

$$f'(\alpha) = -\frac{r \lambda \theta}{\rho} \alpha^{\frac{r}{\rho} - 1} \left[1 + \frac{\rho - r}{\rho \alpha} \beta - \left(1 - \frac{\beta}{\alpha} \right)^{\frac{r}{\rho} - 1} \right].$$

Consider a function

$$h(x) := 1 + (\rho - r) \beta x - \left(1 - \rho \beta x \right)^{\frac{r}{\rho} - 1}, \quad x \in \left[0, \frac{1}{\rho \beta} \right]. \tag{C.5}$$

We know that $h(0) = 0$ and

$$h'(x) = (\rho - r) \beta \left[1 - \left(1 - \rho \beta x \right)^{\frac{r}{\rho} - 2} \right] < 0,$$

where the inequality holds because $r < \rho$. Therefore, we must have $h(x) < 0, \; \forall x \in \left(0, 1/(\rho \beta) \right]$. Consequently,

$$f'(\alpha) = -\frac{r \lambda \theta}{\rho} \alpha^{\frac{r}{\rho} - 1} h\left(\frac{1}{\rho \alpha} \right) > 0,$$

implying that function $f(\alpha)$ is strictly increasing in $\alpha \in [\beta, \infty)$. Moreover, we have

$$\lim_{\alpha \to \infty} f(\alpha) = m - \lambda \theta \lim_{\alpha \to \infty} \frac{1 - \frac{r \beta}{\rho \alpha} - \left(1 - \frac{\beta}{\alpha} \right)^{\frac{r}{\rho}}}{\alpha^{-\frac{r}{\rho}}}$$

$$= m - \lambda\theta \lim_{\alpha\to\infty} \frac{\frac{r\beta}{\rho\alpha^2} - \frac{r\beta}{\rho\alpha^2}\left(1 - \frac{\beta}{\alpha}\right)^{\frac{r}{\rho}-1}}{-\frac{r}{\rho}\alpha^{-\frac{r}{\rho}-1}}$$

$$= m + \lambda\theta\beta \lim_{\alpha\to\infty} \frac{1 - \left(1 - \frac{\beta}{\alpha}\right)^{\frac{r}{\rho}-1}}{\alpha^{1-\frac{r}{\rho}}} = m.$$

This completes the proof. □

Proof of Proposition 3. (i) The definition of α_θ directly implies that $f(\alpha_\theta) \geq 0$. From the definition of $F_{\theta,\alpha}(w)$, for any $\alpha \geq \beta$, we have

$$F'_{\theta,\alpha}(\alpha_-) = \frac{r\theta}{\rho}\alpha^{\frac{r}{\rho}-1} \text{ and } F''_{\theta,\alpha}(\alpha_-) = \frac{r(r-\rho)}{\rho^2}\theta\alpha^{\frac{r}{\rho}-2} < 0.$$

From DDE (H), we have

$$F'_{\theta,\alpha}(\alpha_+) = \frac{1}{\rho\alpha + \lambda\beta}\Big[(\lambda+r)F_{\theta,\alpha}(\alpha) - \lambda F_{\theta,\alpha}(\alpha - \beta)\Big],$$
$$= \frac{1}{\rho\alpha + \lambda\beta}\Big[(\lambda+r)\theta\alpha^{\frac{r}{\rho}} - \lambda\theta(\alpha - \beta)^{\frac{r}{\rho}} - m\Big].$$

As such,

$$F'_{\theta,\alpha}(\alpha_-) - F'_{\theta,\alpha}(\alpha_+) = \frac{1}{\rho\alpha + \beta\lambda}f(\alpha). \tag{C.6}$$

Consider the case $\alpha = \alpha_\theta$. If $f(\beta) \geq 0$, the definition of α_θ implies that $f(\alpha) = f(\beta) \geq 0$, i.e., $F'_{\theta,\alpha_\theta}(\alpha_{\theta-}) \geq F'_{\theta,\alpha_\theta}(\alpha_{\theta+})$. Otherwise we must have $f(\alpha_\theta) = 0$ and $F'_{\theta,\alpha_\theta}(\alpha_{\theta-}) = F'_{\theta,\alpha_\theta}(\alpha_{\theta+})$. Therefore we have (4.4).

(ii) When $\alpha_\theta > \beta$, we must have $f(\alpha_\theta) = 0$. From DDE (H) we have

$$F''_{\theta,\alpha_\theta}(\alpha_{\theta+}) = \frac{1}{\rho\alpha_\theta + \lambda\beta}\Big[(\lambda + r - \rho)F'_{\theta,\alpha_\theta}(\alpha_{\theta+}) - \lambda F'_{\theta,\alpha_\theta}(\alpha_\theta - \beta)\Big],$$
$$= \frac{\theta r}{\rho(\rho\alpha_\theta + \lambda\beta)}\Big[(\lambda + r - \rho)\alpha_\theta^{\frac{r}{\rho}-1} - \lambda(\alpha_\theta - \beta)^{\frac{r}{\rho}-1}\Big].$$

As such,

$$F''_{\theta,\alpha_\theta}(\alpha_{\theta-}) - F''_{\theta,\alpha_\theta}(\alpha_{\theta+}) = \frac{\theta r\lambda\alpha_\theta^{\frac{r}{\rho}-1}}{\rho(\rho\alpha_\theta + \beta\lambda)}\Big[\Big(1 - \frac{\beta}{\alpha_\theta}\Big)^{\frac{r}{\rho}-1} - 1 - \frac{\beta(\rho - r)}{\rho\alpha_\theta}\Big]$$
$$= -\frac{\theta r\lambda\alpha_\theta^{\frac{r}{\rho}-1}}{\rho(\rho\alpha_\theta + \beta\lambda)}h\Big(\frac{1}{\rho\alpha_\theta}\Big) > 0,$$

where function $h(x)$ is defined in (C.5). The above inequality holds because $h(x) < 0$ for $\forall x \in \Big(0, 1/(\rho\beta)\Big]$.

(iii) Consider an arbitrary $\alpha \geq \beta$. If $\alpha < \alpha_\theta$, we must have $f(\alpha) < f(\alpha_\theta) = 0$; which, together with (C.6), implies $F'_{\theta,\alpha}(\alpha_-) < F'_{\theta,\alpha}(\alpha_+)$. Otherwise if $\alpha > \alpha_\theta$, we must have $f(\alpha) > f(\alpha_\theta) \geq 0$, i.e., $F'_{\theta,\alpha}(\alpha_-) > F'_{\theta,\alpha}(\alpha_+)$. □

Proof of Lemma 5. We first show that for any $w \geq 0$, derivative $F'_{\theta,\alpha_\theta}(w_+)$ is increasing in θ. To do so, we define $g(w,\theta) = F_{\theta,\alpha_\theta}(w)$, and show $\dfrac{\partial g(w,\theta)}{\partial\theta}$ is well-defined and strictly increasing in w.

- For $\forall w \in [0, \alpha_\theta)$, we have $\dfrac{\partial g(w,\theta)}{\partial \theta} = w^{\frac{r}{\rho}}$, which is strictly increasing in w.
- For $w = \alpha_\theta$,

$$\frac{\partial g(w, \theta_+)}{\partial \theta}\bigg|_{w=\alpha_\theta} = \lim_{\varepsilon \downarrow 0} \frac{F_{\theta+\varepsilon, \alpha_\theta}(\alpha_\theta) - F_{\theta, \alpha_\theta}(\alpha_\theta)}{\varepsilon} + \frac{F_{\theta, \alpha_{\{\theta+\varepsilon\}}}(\alpha_\theta) - F_{\theta, \alpha_\theta}(\alpha_\theta)}{\varepsilon} \times \frac{d\alpha_\theta}{d\theta}$$

$$= \lim_{\varepsilon \downarrow 0} \frac{(\theta+\varepsilon)\alpha_\theta^{\frac{r}{\rho}} - \theta \alpha_\theta^{\frac{r}{\rho}}}{\varepsilon} = \alpha_\theta^{\frac{r}{\rho}} = \frac{\partial g(w, \theta_-)}{\partial \theta}\bigg|_{w=\alpha_\theta},$$

where the second equality follows from $F_{\theta, \alpha_{\{\theta+\varepsilon\}}}(\alpha_\theta) = F_{\theta, \alpha_\theta}(\alpha_\theta)$ because $\alpha_{\theta+\varepsilon} \geq \alpha_\theta$ for any $\varepsilon > 0$.

- For $w > \alpha_\theta$, note that $g(w, \theta) = F_{\theta, \alpha_\theta}(w)$ is a solution to DDE (H), parameterized by θ. That is, the following equality also holds:

$$(\lambda + r)g(w, \theta) = \lambda g(w - \beta, \theta) + (\rho w + \lambda \beta)\frac{\partial g(w, \theta)}{\partial w}.$$

Taking derivatives *w.r.t* θ on both sides of the above equation, we have

$$(\lambda + r)\frac{\partial g(w, \theta)}{\partial \theta} = \lambda \frac{\partial g(w - \beta, \theta)}{\partial \theta} + (\rho w + \lambda \beta)\frac{\partial g(w, \theta)}{\partial w \partial \theta},$$

which implies that $\dfrac{\partial g(w, \theta)}{\partial \theta}$ satisfies DDE (H) as well. Since $\dfrac{\partial g(0, \theta)}{\partial \theta} > 0$ and $\dfrac{\partial g(w, \theta)}{\partial \theta}$ is increasing on $w \in [0, \alpha_\theta]$, we immediately know that $\dfrac{\partial g(w, \theta)}{\partial \theta}$ is increasing on $w \in [\alpha_\theta, \infty)$ from Lemma 6.

Summarizing the above three cases, we conclude that $\dfrac{\partial g(w, \theta)}{\partial \theta}$ is well-defined and strictly increasing in w. Therefore, derivative $F'_{\theta, \alpha_\theta}(w_+)$ is increasing in θ. As a result, we know that $\inf_w F'_{\theta, \alpha_\theta}(w_+)$ is increasing in θ. On the one hand, when θ is sufficiently large (i.e., $\theta \to \infty$), we know that α_θ approaches ∞ as well. As such,

$$\lim_{\theta \to \infty} \left\{ \inf_{w > \beta} F'_{\theta, \alpha_\theta}(w) \right\} = \lim_{\theta \to \infty} \left\{ \inf_{w > \beta} \left[\frac{r\theta}{\rho} w^{\frac{r}{\rho}-1} \right] \right\} > 0.$$

On the other hand, when $\theta = 0$, we have $\alpha_\theta = \beta$. It is clear that

$$F_{0,\beta}(w) = -\frac{m}{r} J(w) \quad \text{for} \quad \forall w \geq 0.$$

As such,

$$\inf_{w > \beta} F'_{0,\beta}(w) = -\frac{m}{r} \sup_{w > \beta} J'(w) \leq -1,$$

where the last inequality holds because $J(w)$ is nondecreasing and $m \geq \bar{m}$.

Therefore, there exists a unique $\bar{\theta} \geq 0$, such that $\inf_w F'_{\bar{\theta}, \alpha_{\bar{\theta}}}(w_+) = -1$.

Using similar techniques as in the proof of Proposition 1(i), we can show that $\bar{\theta} < \frac{m}{r}\beta^{-\frac{r}{\rho}}$. We only need to modify the definition of functions $G_1(w)$ and $G_2(w)$ as:

$$G_1(w) = w^{\frac{r}{\rho}} \text{ and } G_2(w) = \left(\frac{w}{\beta}\right)^{\frac{r}{\rho}} - 1, \text{ for } w \in [0, \alpha_{\bar{\theta}}].$$

We omit the detailed proof to avoid redundancy.

Finally, we show that $\bar{w}^*$, which is determined by (4.5), is finite. Notice that

$$\bar{\theta} = -\frac{1 - mJ'(\bar{w}^*)/r}{G'_1(\bar{w}^*)}.$$

If, on the contrary, $\bar{w}^* \to \infty$, the above equation implies that $\bar{\theta} < 0$ because $\limsup_{w\to\infty} J'(w) = 0$ [recall Lemma 9(iii)] and $G_1(w)$ is increasing (recall Lemma 6). This contradicts to the fact that $\bar{\theta} > 0$. Therefore, $\bar{w}^*$ must be finite. $\qquad\square$

To prove Proposition 4, we first show a more general result, as presented in the following Lemma.

LEMMA 10. *For any $\theta \geq \bar{\theta}$, define $\hat{w} := \inf\{\arg\inf_{w>\beta} F'_{\theta,\alpha_\theta}(w)\}$. Then function $F_{\theta,\alpha_\theta}(w)$ is strictly concave on $w \in [0, \hat{w}]$.*

Proof. Firstly, $F_{\theta,\alpha_\theta}(w)$ is nondecreasing and concave in $[0, \alpha_\theta)$ because $\theta \geq 0$ (recall Lemma 5). Next, from Proposition 3(i) we know that $F'_{\theta,\alpha_\theta}(\alpha_{\theta-}) \geq F'_{\theta,\alpha_\theta}(\alpha_{\theta+})$. Therefore, in order to show that $F(w)$ is concave, we only need to show that $F_{\theta,\alpha_\theta}(w)$ is concave on $[\alpha_\theta, \hat{w}]$.

Recall that $F_{\theta,\alpha_\theta}(w)$ is continuous on $[0, \hat{w}]$ and $F'_{\theta,\alpha_\theta}(w)$ is continuous on $(\alpha_\theta, \hat{w}]$. By the definition of $\bar{\theta}$, $\bar{w}$ in (4.5) and given the fact that $\theta \geq \bar{\theta}$, we know that $F'_{\theta,\alpha_\theta}(\hat{w}) \geq -1$. Therefore, $F_{\theta,\alpha_\theta}(w) + w$ is increasing on $[0, \hat{w}]$.

We prove by contradiction. Suppose, on the contrary, there exists some $w \in (\alpha_\theta, \hat{w})$, such that $F''_{\theta,\alpha_\theta}(w) \geq 0$ (i.e., $F'_{\theta,\alpha_\theta}(w)$ is increasing at w). Define

$$w_1 := \min\{w | F''_{\theta,\alpha_\theta}(w) \geq 0 \text{ and } \alpha_\theta \leq w < \hat{w}\}.$$

Furthermore, $F'_{\theta,\alpha_\theta}(w)$ must be decreasing on some interval within $(w_1, \hat{w})$, in order for $F'_{\theta,\alpha_\theta}(w)$ to drop to -1 at $\hat{w}$. As such, define

$$w_2 := \inf\{w | F''_{\theta,\alpha_\theta}(w) < 0 \text{ and } w_1 < w < \hat{w}\}.$$

We claim that $F''_{\theta,\alpha_\theta}(w)$ must be continuous at w_2. To show this, recall that only when $\alpha_\theta = \beta$ and $F'_{\theta,\alpha_\theta}(\beta_-) > F'_{\theta,\alpha_\theta}(\beta_+)$, would $F''_{\theta,\alpha_\theta}(w)$ be discontinues at point $w = \beta$ or $w = 2\beta$. Suppose, on the contrary, $F''_{\theta,\alpha_\theta}(w)$ is not continuous at w_2, then we must have $\alpha_\theta = \beta$ and $w_2 = 2\beta$. As such, differentiating (H) would yield

$$
\begin{aligned}
F''_{\theta,\alpha_\theta}(\{2\beta\}_-) &= \frac{(\lambda + r - \rho)F'_{\theta,\alpha_\theta}(2\beta) - \lambda F'_{\theta,\alpha_\theta}(\beta_-)}{2\rho\beta + \beta\lambda} \\
&< \frac{(\lambda + r - \rho)F'_{\theta,\alpha_\theta}(2\beta) - \lambda F'_{\theta,\alpha_\theta}(\beta_+)}{2\rho\beta + \beta\lambda} = F''_{\theta,\alpha_\theta}(\{2\beta\}_+),
\end{aligned}
$$

where the inequality follows from Proposition 3(i), and the assumption that $F'_{\theta,\alpha_\theta}(w)$ is not continuous at β.

Because $F'_{\theta,\alpha_\theta}(w)$ is increasing on $w \in [w_1, w_2]$, we must have $F''_{\theta,\alpha_\theta}(\{2\beta\}_-) > 0$. Therefore,

$$F''_{\theta,\alpha_\theta}(2\beta_+) > F''_{\theta,\alpha_\theta}(2\beta_-) > 0,$$

which is in contradiction to the definition of w_2. Therefore, $F''_{\theta,\alpha_\theta}(w)$ must be continuous at w_2, and as a result, we must have $F''_{\theta,\alpha_\theta}(w_2) = 0$. Consequently, DDE (H) yields

$$(\rho w_2 + \beta\lambda)F'''_{\theta,\alpha_\theta}(w_2) = -\lambda F''_{\theta,\alpha_\theta}(w_2 - \beta) \leq 0,$$

which implies that $F''_{\theta,\alpha_\theta}(w_2 - \beta) \geq 0$. Therefore, we must have $w_2 - \beta \geq w_1$ because $F'_{\theta,\alpha_\theta}(w)$ is decreasing for $w < w_1$. As such, $F'_{\theta,\alpha_\theta}(w)$ is increasing on $[w_2 - \beta, w_2]$. Differentiating (H) at w_2 yields

$$\lambda[F'_{\theta,\alpha_\theta}(w_2) - F'_{\theta,\alpha_\theta}(w_2 - \beta)] = (\rho - r)F'_{\theta,\alpha_\theta}(w_2) \geq 0,$$

which implies that $F'_{\theta,\alpha_\theta}(w_2) \geq 0$.

On the one hand, we have

$$
\begin{aligned}
\rho w_2 + \lambda\beta[F'_{\theta,\alpha_\theta}(w_2)+1] &\leq (\rho w_2 + \lambda\beta)[F'_{\theta,\alpha_\theta}(w_2)+1] \\
&= \lambda[F_{\theta,\alpha_\theta}(w_2) - F_{\theta,\alpha_\theta}(w_2-\beta)] + rF_{\theta,\alpha_\theta}(w_2) + \rho w_2 + \beta\lambda \\
&\leq \lambda\beta F'_{\theta,\alpha_\theta}(w_2) + rF_{\theta,\alpha_\theta}(w_2) + \rho w_2 + \beta\lambda,
\end{aligned}
\tag{C.7}
$$

where the second inequality holds because $F_{\theta,\alpha_\theta}(w)$ is convex within $w \in [w_2-\beta, w_2]$. Rearranging inequality (C.7), yields $F_{\theta,\alpha_\theta}(w_2) \geq 0$, which leads to

$$
F_{\theta,\alpha_\theta}(\hat{w}) + \hat{w} > F_{\theta,\alpha_\theta}(w_2) + w_2 \geq w_2 > 0,
\tag{C.8}
$$

since $F_{\theta,\alpha_\theta}(w) + w$ is increasing on $[0, \hat{w}]$.

On the other hand, Equation (U), in which $\bar{w}$ is set as $\hat{w}$, is equivalent to

$$
\lambda[F_{\theta,\alpha_\theta}(\hat{w}) - F_{\theta,\alpha_\theta}(\hat{w}-\beta)] + rF_{\theta,\alpha_\theta}(\hat{w}) + \rho\hat{w} + \beta\lambda = 0.
$$

Because $F_{\theta,\alpha_\theta}(\hat{w}) > F_{\theta,\alpha_\theta}(\hat{w}-\beta) - \beta$, we must have $rF_{\theta,\alpha_\theta}(\hat{w}) + \rho\hat{w} < 0$. Consequently,

$$
F_{\theta,\alpha_\theta}(\hat{w}) + \hat{w} < -\frac{\rho-r}{r}\hat{w} < 0,
$$

which is in contradiction to (C.8). Therefore, $F''_{\theta,\alpha_\theta}(w) < 0$ for $\forall w \in (\alpha_{\bar{\theta}}, \hat{w})$. To summarize, $F_{\theta,\alpha_\theta}(w)$ is strictly concave within $[0, \hat{w}]$. □

Proof of Proposition 4. (i) First note that function $F_{\bar{\theta},\alpha_{\bar{\theta}}}(w)$ is strictly concave on $w \in [0, \alpha_{\bar{\theta}})$. The concavity proof on $w \in [\alpha_{\bar{\theta}}, \bar{w}^*]$ is a special case of the concavity proof in Lemma 10 with $\theta = \bar{\theta}$ and $\hat{w} = \bar{w}^*$. Therefore, we omit the proof of this part to avoid redundancy.

(ii) We prove by contradiction. Suppose, on the contrary, there exists a $w \in (\alpha_{\bar{\theta}}, \bar{w}^*]$, such that

$$
rF_{\bar{\theta},\alpha_{\bar{\theta}}}(w) \leq \rho w F'_{\bar{\theta},\alpha_{\bar{\theta}}}(w) - m.
$$

Let

$$
\hat{w} := \min\{w | rF_{\bar{\theta},\alpha_{\bar{\theta}}}(w) \leq \rho w F'_{\bar{\theta},\alpha_{\bar{\theta}}}(w) - m \text{ and } \alpha_{\bar{\theta}} < w \leq \bar{w}^*\}.
\tag{C.9}
$$

Therefore, we must have the following relationship,

$$
rF_{\bar{\theta},\alpha_{\bar{\theta}}}(w) > \rho w F'_{\bar{\theta},\alpha_{\bar{\theta}}}(w) - m, \quad \text{for } w \in (\alpha_{\bar{\theta}}, \hat{w}).
\tag{C.10}
$$

For notational convenience, we let $\hat{z} := F_{\bar{\theta},\alpha_{\bar{\theta}}}(\hat{w})$. We first demonstrate that $F_{\bar{\theta},\infty}(\hat{w}) > \hat{z}$ by showing $F'_{\bar{\theta},\alpha_{\bar{\theta}}}(w) < F'_{\bar{\theta},\infty}(w)$ for all $w \in (\alpha_{\bar{\theta}}, \hat{w}]$ using contradiction. Suppose there exists $\tilde{w}$ such that

$$
\tilde{w} := \min\{w | F'_{\bar{\theta},\alpha_{\bar{\theta}}}(w) \geq F'_{\bar{\theta},\infty}(w) \text{ and } \alpha_{\bar{\theta}} < w \leq \hat{w}\}.
$$

On one hand, when $\alpha_{\bar{\theta}} = \beta$, from Proposition 3(i), we have $F'_{\bar{\theta},\alpha_{\bar{\theta}}}(\alpha_{\bar{\theta}+}) \leq F'_{\bar{\theta},\alpha_{\bar{\theta}}}(\alpha_{\bar{\theta}-})$. On the other hand, when $\alpha_{\bar{\theta}} > \beta$, from Proposition 3(ii), we have $F''_{\bar{\theta},\alpha_{\bar{\theta}}}(\alpha_{\bar{\theta}+}) < F''_{\bar{\theta},\alpha_{\bar{\theta}}}(\alpha_{\bar{\theta}-}) < 0$. Either of the above cases leads to the result that $F'_{\bar{\theta},\alpha_{\bar{\theta}}}(w) < F'_{\bar{\theta},\infty}(w)$ for all $w \in (\alpha_{\bar{\theta}}, \tilde{w})$, according to the definition of $\tilde{w}$. Thus, the following relationship holds as well:

$$
F_{\bar{\theta},\alpha_{\bar{\theta}}}(w) < F_{\bar{\theta},\infty}(w), \quad w \in (\alpha_{\bar{\theta}}, \tilde{w}].
\tag{C.11}
$$

Then considering (C.10) at $w = \tilde{w}$, we have

$$
rF_{\bar{\theta},\alpha_{\bar{\theta}}}(\tilde{w}) > \rho\tilde{w} F'_{\bar{\theta},\infty}(\tilde{w}) - m = rF_{\bar{\theta},\infty}(\tilde{w}),
$$

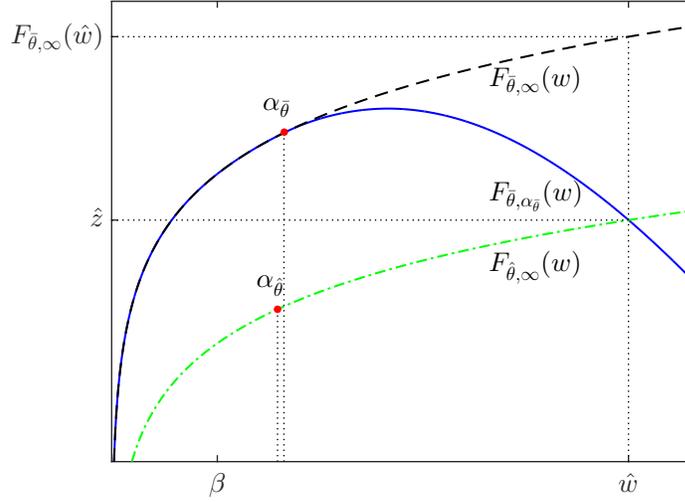**Figure 8**   Illustration of $F_{\bar{\theta},\alpha_{\bar{\theta}}}(w)$, $F_{\bar{\theta},\infty}(w)$, and $F_{\hat{\theta},\infty}(w)$.

where the first inequality follows from $F'_{\bar{\theta},\alpha_{\bar{\theta}}}(\tilde{w}) \geq F'_{\bar{\theta},\infty}(\tilde{w})$ and the next equality follows from (3.8). Clearly, this is a contradiction to the relationship in (C.11). Therefore, we must have $F'_{\bar{\theta},\alpha_{\bar{\theta}}}(w) < F'_{\bar{\theta},\infty}(w)$ for all $w \in (\alpha_{\bar{\theta}}, \hat{w})$. As a result, we have $F_{\bar{\theta},\hat{w}}(\hat{w}) = F_{\bar{\theta},\infty}(\hat{w}) > \hat{z}$.

Let

$$\hat{\theta} := \left(\hat{z} + \frac{m}{r}\right)\hat{w}^{-\frac{r}{\rho}}.$$

Then, we must have

$$\hat{\theta} < \left(F_{\bar{\theta},\hat{w}}(\hat{w}) + \frac{m}{r}\right)w^{-\frac{r}{\rho}} = \bar{\theta}.$$

On the one hand, from Equation (C.9), we have:

$$F'_{\bar{\theta},\alpha_{\bar{\theta}}}(\hat{w}) \geq \frac{r\hat{z} + m}{\rho\hat{w}} = F'_{\hat{\theta},\hat{w}}(\hat{w}_-). \tag{C.12}$$

On the other hand, given that function $f(\alpha)$ is strictly increasing, we have $\alpha_{\hat{\theta}} < \alpha_{\bar{\theta}}$. Therefore, $\hat{w} > \alpha_{\bar{\theta}} > \alpha_{\hat{\theta}}$. By Proposition 3(i) we have

$$F'_{\hat{\theta},\hat{w}}(\hat{w}_+) < F'_{\hat{\theta},\hat{w}}(\hat{w}_-). \tag{C.13}$$

From DDE (H), we know that

$$F'_{\hat{\theta},\hat{w}}(\hat{w}_+) = \frac{1}{\rho\hat{w} + \beta\lambda}\Big[(\lambda + r)F_{\hat{\theta},\hat{w}}(\hat{w}) - \lambda F_{\hat{\theta},\hat{w}}(\hat{w} - \beta)\Big],$$

$$F'_{\bar{\theta},\alpha_{\bar{\theta}}}(\hat{w}) = \frac{1}{\rho\hat{w} + \beta\lambda}\Big[(\lambda + r)F_{\bar{\theta},\alpha_{\bar{\theta}}}(\hat{w}) - \lambda F_{\bar{\theta},\alpha_{\bar{\theta}}}(\hat{w} - \beta)\Big].$$

Because $F_{\hat{\theta},\hat{w}}(\hat{w}) = F_{\bar{\theta},\alpha_{\bar{\theta}}}(\hat{w})$ and $F_{\hat{\theta},\hat{w}}(\hat{w} - \beta) < F_{\bar{\theta},\alpha_{\bar{\theta}}}(\hat{w} - \beta)$, we must have

$$F'_{\hat{\theta},\hat{w}}(\hat{w}_+) > F'_{\bar{\theta},\alpha_{\bar{\theta}}}(\hat{w}). \tag{C.14}$$

Combining (C.13) and (C.14) yields,

$$F'_{\hat{\theta},\hat{w}}(\hat{w}_-) > F'_{\hat{\theta},\hat{w}}(\hat{w}_+) > F'_{\bar{\theta},\alpha_{\bar{\theta}}}(\hat{w}),$$

which contradicts to (C.12). Therefore, there does not exist a $w > \alpha_{\bar{\theta}}$ such that (C.9) holds, which completes the proof. □

To prove Theorem 2, we first present the following Lemma.

LEMMA 11. *Consider a concave function $F(w)$ that satisfies* (H) *with $\alpha \geq \beta$ and* (U), *in which $\bar{w}$ is set as $\bar{w}^*$, at $w = \bar{w}^* \geq \alpha$, and is linear with slope -1 on $[\bar{w}^*, \infty)$. For any $w \geq \bar{w}^*$,*

$$\psi(w) := \lambda F(w-\beta) - (\lambda+r)F(w) - (\rho w + \lambda\beta) \leq 0. \tag{C.15}$$

Proof. Taking the first derivative of $\psi(w)$ yields

$$
\begin{aligned}
\psi'(w) &= \lambda F'(w-\beta) - (\lambda+r)F'(w) - \rho \\
&= \lambda F'(w-\beta) + \lambda + r - \rho \\
&\leq \lambda F'(\bar{w}-\beta) + \lambda + r - \rho,
\end{aligned}
$$

where the second equality follows from $F'(w) = -1$ for $w \geq \bar{w}^*$ and the last inequality follows from concavity of $F(w)$. Condition (H) implies that

$$\lambda F'(\bar{w}^* - \beta) = (\lambda + r - \rho)F'(\bar{w}^*) - (\rho\bar{w}^* + \beta\lambda)F''(\bar{w}^*_+).$$

Note that from the definition of $\bar{w}^*$ in (4.5), we have

$$
\begin{cases}
F''(\bar{w}^*_+) > 0, & \text{if } F''(\bar{w}^*_-) < F''(\bar{w}^*_+), \\
F''(\bar{w}^*_+) = 0, & \text{if } F''(\bar{w}^*_-) = F''(\bar{w}^*_+).
\end{cases} \tag{C.16}
$$

Therefore,

$$\psi'(w) \leq (\lambda + r - \rho)[1 + F'(\bar{w}^*)] - (\rho\bar{w}^* + \beta\lambda)F''(\bar{w}^*_+) \leq 0,$$

where the last inequality follows (C.16). Therefore, $\psi(w)$ is decreasing in $w \in [\bar{w}^*, \infty)$. As such, for any $w \geq \bar{w}^*$,

$$\psi(w) \leq \psi(\bar{w}^*) = \lambda F(\bar{w}^* - \beta) - (\lambda+r)F(\bar{w}^*) - (\rho\bar{w}^* + \lambda\beta) = 0,$$

where we have used equality (U). This completes the proof. □

Proof of Theorem 2. The proof is parallel to those of Proposition 2 and Theorem 1. Note that the proof for $F(w^*) = U(\Gamma_d(w^*; \alpha_{\bar{\theta}}, \bar{w}^*))$ is exactly the same as that of Proposition 2 under contract $\Gamma_d(w^*; \beta, \bar{w})$ except that the switching point is $\alpha_{\bar{\theta}}$ instead of β. We omit the proof of this part to avoid redundancy. In the following we only show that $U(\Gamma) \leq F(w)$ for any incentive compatible contract Γ.

From Proposition 4, we know that $F(w)$ is concave and $F'(w) \geq -1$. Therefore, we only need to show $\Psi_t \leq 0$ holds almost surely (recall Lemma 3).

Recall the inequality (B.12). We consider the following four cases. (i) When $W_t < \beta$, we know that the principal monitors the agent (i.e., $m_t = m$). Following (3.8), we have

$$\Psi_t \leq \rho W_t F'(W_t) - rF(W_t) - m = 0.$$

(ii) When $\beta \leq W_t \leq \alpha_{\bar{\theta}}$, substituting (3.8) into inequality (B.12) yields

$$\Psi_t \leq m - m_t + \lambda\Phi\Big(W_t, y_t H^s_t + (1 - y_t)H^n_t\Big) - \lambda F(W_t).$$

• If the principal does not monitor at time t (i.e., $m_t = 0$), condition (IC) indicates that $y_t H_t^s + (1 - y_t)H_t^n \geq \beta$. Following Lemma 8, we have,

$$\Psi_t \leq m - m_t + \lambda\Phi\Big(W_t, \beta\Big) - \lambda F(W_t)$$
$$= m + \lambda\bar{\theta}W_t^{\frac{r}{\rho}}\Big[\Big(\frac{\beta r}{\rho W_t} - 1\Big) + \Big(1 - \frac{\beta}{W_t}\Big)^{\frac{r}{\rho}}\Big] = f(W_t) < 0,$$

in which we have used Equation (L) and the fact that $f(W_t)$ is increasing with $f(\alpha_{\bar{\theta}}) = 0$.

• If the principal conducts monitoring at time t (i.e., $m_t = m$), considering Lemma 8, inequality (B.12) yields

$$\Psi_t \leq \lambda\Phi(W_t, 0) - \lambda F(W_t) = \lambda F(W_t) - \lambda F(W_t) = 0.$$

(iii) When $\alpha_{\bar{\theta}} < W_t < \bar{w}^*$, substituting (H) into inequality (B.12) yields

$$\Psi_t \leq \lambda\Phi\Big(W_t, y_t H_t^s + (1 - y_t)H_t^n\Big) - \lambda\beta F'(W_t) - \lambda F(W_t - \beta) - m_t.$$

• If the principal does not monitor at time t (i.e., $m_t = 0$), we must have $y_t H_t^s + (1 - y_t)H_t^n \geq \beta$. Lemma 8 implies

$$\Psi_t \leq \lambda\Phi(W_t, \beta) - \lambda\beta F'(W_t) - \lambda F(W_t - \beta) = 0.$$

• If the principal conducts monitoring at time t (i.e., $m_t = m$), Lemma 8 implies

$$\Psi_t \leq \lambda\Phi(W_t, 0) - \lambda\beta F'(W_t) - \lambda F(W_t - \beta) - m$$
$$= -rF(W_t) + \rho W_t F'(W_t) - m < 0,$$

where the last inequality follows from Proposition 4(ii).

(iv) When $W_t \geq \bar{w}^*$, we must have $F'(W_t) = -1$, and inequality (B.12) reduces to

$$\Psi_t \leq \lambda\Phi\Big(W_t, y_t H_t^s + (1 - y_t)H_t^n\Big) - \rho W_t - (\lambda + r)F(W_t) - m_t.$$

• If the principal does not monitor at time t (i.e., $m_t = 0$), condition (IC) and Lemma 8 imply

$$\Psi_t \leq \lambda\Phi(W_t, \beta) - \rho W_t - (\lambda + r)F(W_t) = \psi(W_t) \leq 0.$$

• If the principal conducts monitoring at time t (i.e., $m_t = m$), Lemma 8 implies

$$\Psi_t \leq \lambda\Phi(W_t, 0) - \rho W_t - (\lambda + r)F(W_t) - m$$
$$= -rF(W_t) - \rho W_t - m$$
$$\leq -rF(\alpha_{\bar{\theta}}) - \rho\alpha_{\bar{\theta}} - m,$$

where the second inequality holds because $-rF(w) - \rho w$ is decreasing in w. As such, by considering Equation (3.8), we have

$$\Psi_t \leq -\rho\alpha_{\bar{\theta}}[1 + F'(\alpha_{\bar{\theta}-})] \leq 0.$$

To summarize, we know that $\Psi_t \leq 0$ holds for all the possible cases. This completes the proof.

$\square$

Proof of Proposition 5. First, following DDE (H), for any θ, we have the following closed-form solution for $F_\theta(w)$ for $w \in [\beta, 2\beta]$:

$$F_\theta(w) = \mathcal{K}_\theta \Big(\rho w + \beta \lambda \Big)^{\frac{\lambda+r}{\rho}} - \frac{\lambda m}{r(\lambda+r)} + \frac{\theta \lambda \Big[\beta(\rho-r) + (\lambda+r)w \Big]}{(\lambda+r)(\lambda+r-\rho)}, \tag{C.17}$$

where the θ-dependent parameter $\mathcal{K}_\theta$ is defined as

$$\mathcal{K}_\theta = -\frac{1}{\lambda+r} \Big[m + \frac{\theta\beta(\rho-r)(2\lambda+r)}{\lambda+r-\rho} \Big] \Big(\beta(\rho+\lambda) \Big)^{-\frac{\lambda+r}{\rho}}. \tag{C.18}$$

As such, when

$$\theta \geq \underline{\theta} := -\frac{\lambda+r-\rho}{\lambda},$$

we have

$$\mathcal{K}_\theta \Big(\beta(\rho+\lambda) \Big)^{\frac{\lambda+r}{\rho}} \leq -\frac{1}{\lambda+r} \Big[m - \frac{(\rho-r)\beta(2\lambda+r)}{\lambda} \Big] < 0, \tag{C.19}$$

where the second inequality holds because $m > \hat{m}$.

Second, we show that for any $w \geq 0$, derivative $F'_\theta(w)$ is increasing in θ. To do so, consider the following decomposition of $F_\theta(w)$,

$$F_\theta(w) = \theta G(w) - \frac{m}{r} J(w),$$

where function $G(w)$ satisfies DDE (H) with boundary condition $G(w) = w$ for all $w \in [0, \beta)$, and function $J(w)$ is defined in Lemma 5. Note that because $G'(w) > 0$ for $w \in (0, \beta)$, Lemma 6 implies that $G'(w) > 0$ for all $w \in (\beta, \infty)$, which further implies that

$$\frac{\partial F'_\theta(w)}{\partial \theta} = G'(w) > 0.$$

On the one hand, when $\theta = 0$, it is clear that

$$F_0(w) = -\frac{m}{r} J(w), \text{ for } w \geq 0,$$

and, therefore,

$$\inf_{w \geq \beta} F'_0(w) = -\frac{m}{r} \sup_{w \geq \beta} J'(w_+) \geq -\frac{\bar{m}}{r} \sup_{w \geq \beta} J'(w_+) = -1,$$

where the inequality holds because $J(w)$ is nondecreasing and $m \leq \bar{m}$.

On the other hand, when $\theta = \underline{\theta}$, (C.17) implies that

$$F'_{\underline{\theta}}(\beta_+) = \mathcal{K}_{\underline{\theta}}(\lambda+r)(\rho\beta+\lambda\beta)^{\frac{\lambda+r}{\rho}-1} - 1 < -1,$$

where the inequality holds because $\mathcal{K}_{\underline{\theta}} < 0$. Therefore,

$$\inf_{w > \beta} F'_\theta(w) \leq F'_\theta(\beta_+) < -1.$$

As such, there exists a unique $\bar{\theta} \in (\underline{\theta}, 0]$, such that $\inf_{w > \beta} F'_{\bar{\theta}}(w) = -1$.

From (C.19), we know that $\mathcal{K}_{\bar{\theta}} < 0$. Given that $r > \rho - \lambda$ when $m \in [\hat{m}, \bar{m})$, $F_{\bar{\theta}}(w)$ is strictly concave within $[\beta, 2\beta]$, i.e., $F'_{\bar{\theta}}(w)$ is strictly decreasing in $(\beta, 2\beta)$. By the definitions of $\bar{\theta}$ and $\bar{w}$, we must have $\bar{w} \geq 2\beta$. This completes the proof. $\qquad \square$

Proof of Theorem 3. We first remark that function $F(w)$ is concave. This is obviously true if $m \in [0, \underline{m})$ or $m \in [\underline{m}, \hat{m})$. For $m \in [\hat{m}, \bar{m})$, the concavity proof of $F(w)$ is similar to those of Proposition 1 and Proposition 4, except the slight difference in showing that $F'(\beta_-) \geq F'(\beta_+)$. We omit the detailed proof to avoid redundancy.

Next, we show that $U(\Gamma) \leq F(w)$ for any incentive compatible contract Γ and $\forall w \geq 0$. To do so, we only need to show $\Psi_t \leq 0$ holds almost surely (recall Lemma 3). Given that $F(w)$ takes different forms, depending on the value of m, we consider three cases in the following.

Case (a) If $m \in [0, \underline{m})$, given that $F(w)$ is linear and $F'(w) = -1$ for all $w \geq 0$ [recall Equation (4.9)], we have

$$\Psi_t = -\rho W_t - rF(W_t) - m_t = -(\rho - r)W_t + (m - m_t).$$

Therefore:

(a.1) If $m_t = m$, we have $\Psi_t \leq 0$ because $\rho \geq r$.

(a.2) If $m_t = 0$, Corollary 1 implies $W_t \geq \beta$; consequently,

$$\Psi_t = -(\rho - r)W_t + m \leq -(\rho - r)\beta + m \leq 0,$$

where the inequality holds because $m \leq \underline{m}$.

Therefore, $\Psi_t \leq 0$ holds almost surely. This completes the proof for $m \in [0, \underline{m})$.

Case (b) If $m \in [\underline{m}, \hat{m})$, recall that $F(w)$ takes the piece-wise linear form of (4.10). We consider the following two cases.

(b.1) When $W_t < \beta$, the principal monitors the agent (i.e., $m_t = m$). Following inequality (B.12), we have

$$
\begin{aligned}
\Psi_t &\leq \rho W_t F'(W_t) - rF(W_t) - m \\
&= -\rho W_t \left[1 - \frac{(\rho - r)\beta - m}{(\lambda + r)\beta} \right] + r \left[1 - \frac{(\rho - r)\beta - m}{(\lambda + r)\beta} \right] W_t \\
&= -(\rho - r)W_t \left[1 - \frac{(\rho - r)\beta - m}{(\lambda + r)\beta} \right] \leq 0,
\end{aligned}
$$

where the second inequality holds because $m \geq \underline{m}$.

(b.2) When $W_t \geq \beta$, we have $F'(W_t) = -1$. If the principal does not monitor at time t (i.e., $m_t = 0$), considering the (IC) condition and Lemma 11, we have,

$$
\begin{aligned}
\Psi_t &\leq \lambda \Phi(W_t, \beta) + F'(W_t)\rho W_t - (\lambda + r)F(W_t) \\
&= \lambda F(W_t - \beta) - (\lambda + r)F(W_t) - (\rho W_t + \beta\lambda) = \psi(W_t),
\end{aligned}
$$

where function $\psi(w)$ is decreasing in $w \in [\beta, \infty)$. As such, we have

$$\Psi_t \leq \psi(\beta) = \lambda F(0) - (\lambda + r)F(\beta) - (\rho\beta + \beta\lambda) = 0.$$

If the principal conducts monitoring at time t (i.e., $m_t = m$), Lemma 8 implies that

$$
\begin{aligned}
\Psi_t &\leq \rho W_t F'(W_t) - rF(W_t) - m \\
&= -\rho W_t - rF(W_t) - m \\
&\leq -\rho\beta - rF(\beta) - m < 0.
\end{aligned}
$$

In summary, we always have $\Psi_t \leq 0$, which completes the proof for $m \in [\underline{m}, \hat{m})$.

Case (c) If $m \in [\hat{m}, \bar{m})$, we consider the following three cases.

(c.1) When $W_t < \beta$, the principal must monitor the agent (i.e., $m_t = m$). From inequality (B.12) we have

$$
\begin{aligned}
\Psi_t &\leq \rho W_t F'(W_t) - r F(W_t) - m \\
&= \rho W_t \bar{\theta} - r\Big(\bar{\theta} W_t - \frac{m}{r}\Big) - m = \bar{\theta}(\rho - r)W_t \leq 0,
\end{aligned}
$$

where we have used Equation (L_l) and the fact that $\bar{\theta} \leq 0$.

(c.2) When $\beta \leq W_t < \bar{w}^*$, substituting Equation (H) into inequality (B.12) yields

$$
\Psi_t \leq \lambda \Phi\Big(W_t, y_t H_t^s + (1 - y_t)H_t^n\Big) - \lambda \beta F'(W_t) - \lambda F(W_t - \beta) - m_t.
$$

If the principal does not monitor at time t (i.e., $m_t = 0$), we must have $y_t H_t^s + (1 - y_t)H_t^n \geq \beta$. By Lemma 8, we have,

$$
\Psi_t \leq \lambda \Phi(W_t, \beta) - \lambda \beta F'(W_t) - \lambda F(W_t - \beta) = 0.
$$

If the principal conducts monitoring at time t (i.e., $m_t = m$), by Lemma 8, we have

$$
\begin{aligned}
\Psi_t &\leq \lambda \Phi(W_t, 0) - \lambda \beta F'(W_t) - \lambda F(W_t - \beta) - m \\
&= -r F(W_t) + \rho W_t F'(W_t) - m \\
&\leq -r F(\beta) + \rho \beta F'(\beta_+) - m \\
&\leq -r F(\beta) + \rho \beta F'(\beta_-) - m \leq 0.
\end{aligned}
$$

(c.3) When $W_t \geq \bar{w}^*$, we must have $F'(W_t) = -1$, and inequality (B.12) reduces to

$$
\Psi_t \leq \lambda \Phi\Big(W_t, y_t H_t^s + (1 - y_t)H_t^n\Big) - \rho W_t - (\lambda + r)F(W_t) - m_t.
$$

If the principal does not monitor at time t (i.e., $m_t = 0$), by (IC) condition and Lemma 8, we have,

$$
\begin{aligned}
\Psi_t &\leq \lambda \Phi(W_t, \beta) - \rho W_t - (\lambda + r)F(W_t) \\
&= \lambda F(W_t - \beta) - (\lambda + r)F(W_t) - (\rho W_t + \beta \lambda) \leq 0,
\end{aligned}
$$

where the last inequality follows from Lemma 8.

If the principal conducts monitoring at time t (i.e., $m_t = m$), by Lemma 8, we have

$$
\begin{aligned}
\Psi_t &\leq \lambda \Phi(W_t, 0) - \rho W_t - (\lambda + r)F(W_t) - m \\
&= -r F(W_t) - \rho W_t - m \\
&\leq -r F(\beta) - \rho \beta - m,
\end{aligned}
$$

where the second inequality holds because $-r F(w) - \rho w$ is decreasing in w. As such, considering (L_l), we have

$$
\begin{aligned}
\Psi_t &\leq -r\Big(\bar{\theta}\beta - \frac{m}{r}\Big) - \rho \beta - m \\
&\leq -r\beta \frac{\lambda + r - \rho}{\lambda} - \rho \beta \\
&= -(\rho - r)\frac{(r + \lambda)\beta}{\lambda} < 0,
\end{aligned}
$$

in which the second inequality holds because $\bar{\theta} \geq -1 + (\rho - r)/\lambda$.

In summary, for any contract Γ, we have $\Psi_t \leq 0$ almost surely. This completes the proof. $\qquad \square$

D. Proofs in Section 5

Proof of Proposition 6. (i) From Lemma 9(i), we know that if $r \leq \rho - \lambda$, $J'(w)$ is decreasing on $[\beta, \infty)$. Therefore,

$$\bar{m} = \frac{r}{J'(\beta_+)} = (\rho + \lambda)\beta.$$

From Lemma 9(ii), we know that if $\rho - \lambda < r < \bar{r}$, $J'(w)$ is increasing on $[\beta, 2\beta]$ and decreasing on $[2\beta, \infty)$. Therefore,

$$\bar{m} = \frac{r}{J'(2\beta)} = \frac{[(\rho + \lambda)\beta]^{\frac{\lambda+r}{\rho}}}{[(2\rho + \lambda)\beta]^{\frac{\lambda+r}{\rho} - 1}}.$$

This completes the proof. $\qquad\qquad\square$

Proof of Proposition 7. This proposition follows the same logic as Lemma 10. Therefore, the proof is omitted. $\qquad\qquad\square$