ON THE BETTI NUMBERS OF COMPACT RANK 2 LOCALLY SYMMETRIC SPACES

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Abstract. We obtain upper bounds for the second Betti numbers of compact rank 2 locally symmetric spaces, namely $\Gamma\backslash SL(3)/SO(3)$, $\Gamma\backslash Sp(4)/U(2)$, and $\Gamma\backslash G_{2(2)}/SO(4)$, where Γ is a cocompact, torsion free lattice. We use representation theory and directly apply the techniques of Di Cerbo and Stern in [5]. In the case of $\Gamma\backslash Sp(4)/U(2)$, we also use unitary holonomy, the complex structure operator that arises from it and the (p,q) decomposition of exterior powers to obtain stronger bounds. In particular, the bounds we provide on the Betti numbers of $\Gamma\backslash Sp(4)/U(2)$ and $\Gamma\backslash G_{2(2)}/SO(4)$ are exponential bounds involving injectivity radius. However, the bound we obtained for $\Gamma\backslash SL(3)/SO(3)$ is a weaker polynomial one.

Contents

1. Introduction

Some of the facts discussed in this introduction are adapted from the introduction and appendix A of [4].

There has been great interest in studying bounds on Betti numbers of compact locally symmetric spaces, especially by number theorists and geometric group theorists. For the geometric group theorists, this is in large part due to the Singer conjecture, which predicts that on a closed, aspherical (higher homotopy groups $\pi_n(M)$ vanish when $n > 1$) n-dimensional manifold, the k^{th} L^2 Betti number vanishes if $k \neq \frac{n}{2}$.

In 1967, Matsushima [14] proved the following formula for the Betti numbers of compact locally symmetric Riemannian manifolds:

Let G be a connected semi-simple Lie group, K a maximal compact subgroup of G, and Γ a discrete torsion free cocompact subgroup of G. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K respectively. Write the Cartan decomposition of $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $M := \Gamma\backslash G/K$ denote the corresponding compact locally symmetric space. Let T be an irreducible unitary representation of G, and T_K be the restriction of T to K. Let $N(\Gamma, T)$ denote the multiplicity of T in $L^2(\Gamma\backslash G)$. Suppose that under the adjoint action of K, T_K decomposes into irreducible representations of K as follows: $T_K = \tau_1 \oplus ... \oplus \tau_n$. Let $M(T_K; \tau_i)$ denote the multiplicity of the irreducible representation τ_i of K in $\bigwedge^k \mathfrak{p}^*$. Let $b^k(M)$ denote the k^{th}-Betti number of M. Then the following formula holds:

$$b^k(M) = \sum_{\text{T: T has vanishing Casimir}} N(\Gamma, T)(\sum_{i=1}^{n} M(T_K; \tau_i)).$$

The Selberg trace formula gives a way to evaluate $N(\Gamma, T)$ but the results can be difficult to interpret. DeGeorge and Wallach [7] provided a method which gives coarser bounds on the multiplicities that are easier to interpret. There have also been several published results on bounds on Betti numbers for complex hyperbolic manifolds. In 1989, Xue [16] used the trace formula and estimates on test functions that are supported in an embedded ball in $\Gamma\backslash SU(2,1)/U(2)$, which has complex dimension 2 to prove effective bounds on its first Betti number. Subsequently, these bounds were improved on by Sarnak and Xue [15] and more recently by Marshall [13], who used a technique from number theory, specifically the endoscopy classification of automorphic forms on $U(2,1)$.

In [5], Stern and Di Cerbo introduced a method for bounding Betti numbers on manifolds without conjugate points and with a negative Ricci curvature upper bound. Like the trace formula, their method gives bounds in terms of the volume of the largest embedded geodesic ball. In their second paper [4], they applied these techniques to rank one compact locally symmetric spaces, i.e. real, complex, quaternionic and octonionic hyperbolic manifolds. In particular, on complex hyperbolic space, they used the complex structure operator J arising from unitary holonomy to improve the bound on the first Betti number derived by Xue [16]. However, they were unable to obtain stronger bounds than Marshall [13] in complex dimension 2.

In this thesis, I build upon the methods by Stern and Di Cerbo in [4] and [5], and use techniques in representation theory to study the Betti numbers of compact rank two locally symmetric spaces. The spaces explored are compact quotients of $SL(3)/SO(3)$, $Sp(4)/U(2)$, and $G_{2(2)}/SO(4)$. The dimensions of these spaces are 5, 6, and 8 respectively. As a consequence of Kazhdan's Property (T), the first cohomology of any

irreducible rank two locally symmetric space vanishes, so we focus our computations on second Betti numbers.
Before we summarize our results, we introduce a definition:

Definition 1.1. Let M be a Riemannian manifold. Define the vector space of L^2-*harmonic k-forms* on M by the following:

$$\mathcal{H}^k(M) := \{\omega \in \Gamma(\bigwedge^k TM) |\, d\omega = d^*\omega = 0, \text{ and } \int_M |\omega|^2 d\mu < \infty\}$$

where $d\mu$ is the Riemannian measure on M. Denote the k^{th} Betti number to be $b^k(M) := \dim_{\mathbb{R}} \mathcal{H}^k(M)$.

Denote our compact locally symmetric space by $M := \Gamma \backslash G/K$, where Γ is a cocompact torsion free lattice, G a non-compact semisimple Lie group with Lie algebra $\mathfrak{g}$, and K a maximal compact subgroup of G, with Lie algebra $\mathfrak{k}$.

From Bekka [2] and 5.1.5 of Kra [6], due to Kazhdan Property (T), we have the following fact:

Proposition 1.1. Let $\Gamma\backslash G/K$ be a rank 2 compact locally symmetric space. Then $H^1(\Gamma, \mathbb{C}) = H^1(\Gamma\backslash G/K, \mathbb{C}) = 0$.

So there are no harmonic 1-forms on rank 2 locally symmetric spaces and we focus our attention on harmonic 2-forms.

On our three symmetric spaces we have the Cartan decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. We decompose $\bigwedge^2 \mathfrak{p}^* \otimes_{\mathbb{R}} \mathbb{C}$ (we denote this as $\bigwedge^2 \mathfrak{p}^*$ for short) into irreducible representations of $\mathfrak{k}$ as follows:

$$\bigwedge^2 \mathfrak{p}^* = \begin{cases} \bigwedge_7^2 \mathfrak{p}^* \oplus \bigwedge_3^2 \mathfrak{p}^*, \ \mathfrak{g} = \mathfrak{sl}(3) \\ \bigwedge_3^{2,0} \mathfrak{p}^* \oplus \bigwedge_{\bar{3}}^{0,2} \mathfrak{p}^* \oplus \bigwedge_3^{1,1} \mathfrak{p}^* \oplus \bigwedge_5^{1,1} \mathfrak{p}^* \oplus \langle \omega \rangle, \ \mathfrak{g} = \mathfrak{sp}(4) \\ \bigwedge_{15}^2 \oplus \bigwedge_3^2 \oplus \bigwedge_{3'}^2 \oplus \bigwedge_7^2, \ \mathfrak{g} = \mathfrak{g}_2, \end{cases}$$

where the subscripts denote the dimension of each representation.

This gives us the corresponding decomposition of the space of harmonic 2-forms on M:

$$\mathcal{H}^2 = \begin{cases} \mathcal{H}_7^2 \oplus \mathcal{H}_3^2, \text{ on } \Gamma\backslash SL(3)/SO(3) \\ \mathcal{H}_{3\oplus\bar{3}}^2 \oplus \mathcal{H}_3^{1,1} \oplus \mathcal{H}_5^{1,1} \oplus \mathcal{H}_1^{1,1}, \text{ on } \Gamma\backslash Sp(4)/U(2) \\ \mathcal{H}_{15}^2 \oplus \mathcal{H}_3^2 \oplus \mathcal{H}_{3'}^2 \oplus \mathcal{H}_7^2, \text{ on } \Gamma\backslash G_{2(2)}/SO(4). \end{cases}$$

Throughout the paper, we use the following notation:

On $\Gamma\backslash SL(3)/SO(3)$, denote $b_7^{2,SL(3)} := \dim \mathcal{H}_7^2$, $b_3^{2,SL(3)} := \dim \mathcal{H}_3^2$. On $\Gamma\backslash Sp(4)/U(2)$, denote $b_{3\oplus\bar{3}}^{2,Sp(4)} := \dim \mathcal{H}_{3\oplus\bar{3}}^2$, $b_3^{2,Sp(4)} := \dim \mathcal{H}_3^{1,1}$, $b_5^{2,Sp(4)} := \dim \mathcal{H}_5^{1,1}$. Finally, on $\Gamma\backslash G_{2(2)}/SO(4)$, denote $b_{15}^{2,G_2} := \dim \mathcal{H}_{15}^2$, $b_3^{2,G_2} := \dim \mathcal{H}_3^2$, $b_{3'}^{2,G_2} := \dim \mathcal{H}_{3'}^2$, and $b_7^{2,G_2} := \dim \mathcal{H}_7^2$.

Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{g}$. Fix a Weyl chamber (defined in section 3) $\mathfrak{a}^+ \subset \mathfrak{a}$. Letting $A^+ = \exp \mathfrak{a}^+$, we write the Cartan decomposition at the Lie group level $G = KA^+K$.

Let $f : A^+ \mapsto \mathbb{R}$ be defined by $f(t_1, t_2) = t_1 + t_2$. Define

$$\Omega_r := \{(t_1, t_2) \in A^+ : \, f(t_1, t_2) \leq r\},$$

$$\Omega'_r := \{(t_1, t_2) \in A^+ : \, f(t_1, t_2) = r\}.$$

Then $K\Omega_r$ and $K\Omega'_r$ are sublevel sets and level sets of f respectively on G/K.

Let $n = \dim M$, and let $d(n)$ denote a strictly positive constant that depends on the dimension of M. Let inj_p be the injectivity radius at the point $p \in M$ and set $\mathrm{inj}_M = \inf_{p \in M} \mathrm{inj}_p$.
We then have the following bounds on the second Betti numbers, for Γ a cocompact torsion free lattice.

Theorem 1. Let $M = \Gamma \setminus SL(3)/SO(3)$, and $R = \dfrac{\mathrm{inj}_M}{\sqrt{2}}$. Then there exists a strictly positive constant $d(5) > 0$ such that:

$$\frac{b_7^{2,SL(3)}}{Vol(M)} \leq \frac{2d(5)}{R}.$$

Theorem 2. Let $M = \Gamma \setminus Sp(4)/U(2)$, and $R = \dfrac{\mathrm{inj}_M}{\sqrt{2}}$. Then there exists a positive constant $d(6) > 0$ such that:

$$\begin{cases} \dfrac{b_5^{2,Sp(4)}}{Vol(M)} \leq 4d(6)e^{-R/2} \\ \dfrac{b_3^{2,Sp(4)}}{Vol(M)} \leq d(6)e^{-2R} \\ \dfrac{b_{3\oplus\bar{3}}^{2,Sp(4)}}{Vol(M)} \leq d(6)e^{-2R}. \end{cases}$$

Theorem 3. Let $M = \Gamma \backslash G_{2(2)}/SO(4)$, $R = \dfrac{\mathrm{inj}_M}{\sqrt{2}}$. Then there exists a strictly positive constant $d(8)$ such that:

$$\begin{cases} \dfrac{b_{15}^{2,G_2}}{Vol(M)} \leq \dfrac{81d(8)}{38}e^{-38R/81} \\ \dfrac{b_3^{2,G_2}}{Vol(M)} \leq \dfrac{85d(8)}{16}e^{-32R/81} \\ \dfrac{b_{3'}^{2,G_2}}{Vol(M)} \leq d(8)e^{-2R} \\ \dfrac{b_7^{2,G_2}}{Vol(M)} \leq 34d(8)e^{-R/17}. \end{cases}$$

Remark: Our methods did not yield estimates on the Betti number on $\mathcal{H}^3(\Gamma\backslash G_{2(2)}/SO(4))$ and $\mathcal{H}^2_3(\Gamma\backslash SL(3)/SO(3))$.

2. Cartan Decompositions

In this section, we discuss some preliminaries from Lie Theory required to understand locally symmetric spaces, namely the Cartan decomposition. A full discussion of this section can be found in Helgason [11].

2.1. **Lie Algebras.**

First we consider the Cartan decomposition at the Lie Algebra Level:

Definition 2.1. Let $\mathfrak{g}$ be a Lie algebra over a field K and let $x \in \mathfrak{g}$. The adjoint endomorphism $\mathrm{ad}(x)$ of $\mathfrak{g}$ is given by $\mathrm{ad}(x)(y) = [x, y]$ for all $y \in \mathfrak{g}$. The *Killing form* on $\mathfrak{g}$ is given by

$$B(x, y) = \mathrm{trace}(\mathrm{ad}(x) \circ \mathrm{ad}(y)).$$

Definition 2.2. Let $\mathfrak{g}$ be a semisimple Lie algebra and let $B(\ ,\ )$ be its Killing form. An involution on $\mathfrak{g}$ is a Lie algebra automorphism θ satisfying $\theta^2 = 1$. θ is called a *Cartan involution* on $\mathfrak{g}$ if $-B(\theta X, Y)$ is a positive definite bilinear form for all $X, Y \in \mathfrak{g}$.

An example in our case is the involution $\theta(X) = -X^t$ for $X \in \mathfrak{sl}(n)$.

Definition 2.3. The linear map θ has two eigenvalues ± 1. Let $\mathfrak{k}$ and $\mathfrak{p}$ denote the eigenspaces corresponding to $+1$ and -1 respectively. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is called the *Cartan decomposition* of $\mathfrak{g}$.

We have the following facts about the Cartan pair $(\mathfrak{k}, \mathfrak{p})$:

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k},\ [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p},\ [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

The Killing form B is negative definite on $\mathfrak{p}$ and positive definite on $\mathfrak{k}$.

2.2. **Lie Groups.**

Now, we consider the Cartan decomposition on the Lie Group level. The following definition is taken from Fulton and Harris [8] section 21.1:

Definition 2.4. Let $\mathbb{E}$ be a Euclidean vector space with standard Euclidean inner product $(\ ,\ )$. A *root system* Σ is a set of vectors $R \in \mathbb{E}$ called roots satisfying the following four properties:

(1) R is a finite set spanning $\mathbb{E}$.
(2) $\alpha \in R \implies -\alpha \in R$, but $k \cdot \alpha \notin R$ if k is any real number other than ± 1.
(3) For $\alpha \in R$, the reflection W_α in the hyperplane $\alpha^\perp$ maps R to itself.
(4) For $\alpha, \beta \in R$, the real number $2\dfrac{(\beta, \alpha)}{(\alpha, \alpha)}$ is an integer.

Definition 2.5. Let Σ be a root system and $\alpha_i \in \Sigma$ be the roots. Each connected component of the complement of the set of hyperplanes $\alpha_i^\perp$ is called a *Weyl chamber*.

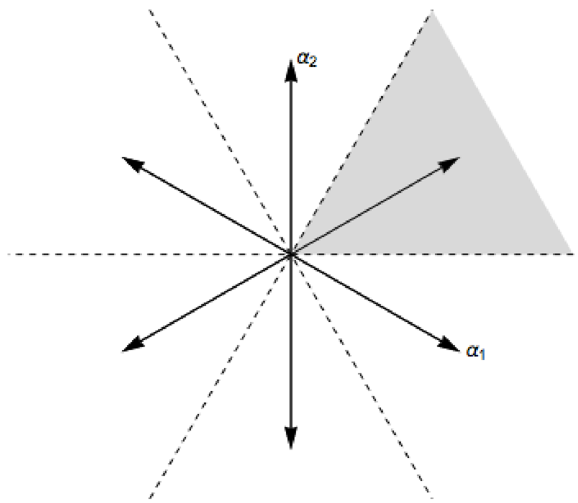

FIGURE 1. Weyl Chambers for A_2 Root System
Image taken from: https://en.wikipedia.org/wiki/Weyl_group

Let G be a non-compact semisimple Lie group and $\mathfrak{g}$ be its Lie algebra. Let θ be a Cartan involution on $\mathfrak{g}$ and $(\mathfrak{k}, \mathfrak{p})$ be the resulting Cartan pair. Let K be a maximal compact subgroup of G with Lie algebra $\mathfrak{k}$.

Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$, and let $A = e^{\mathfrak{a}}$. Fix a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$. Then writing $A^+ = e^{\mathfrak{a}^+}$, we have the following Cartan decomposition at the Lie Group level:

$$G = KA^+K.$$

3. On the Geometry of $\Gamma \backslash SL(3)/SO(3)$

In this section, we study the geometry of compact quotients of SL(3)/SO(3). A key tool in our computations is the Cartan decomposition at the Lie Group level. We also make a quick comparison to hyperbolic 2-space, SL(2)/SO(2).

3.1. **Metric Computation.**

Consider the Cartan decomposition for $G = SL(3)$ with $K = SO(3)$. By Helgason [11], if we let $A^+ = \{\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} : \lambda_1 > \lambda_2 > \lambda_3,\ \lambda_1\lambda_2\lambda_3 = 1, \text{ with } \lambda_1 > \lambda_2 > 1\}$, then $G = KA^+K$.

Now, choose the following basis for $\mathfrak{k} = \mathfrak{so}(3)$, the lie algebra of 3×3 skew symmetric matrices and $\mathfrak{a}$, the Cartan subalgebra of diagonal traceless matrices:

$$\{y_1, y_2, y_3\} = \{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\},$$

$$\{A_1, A_2\} = \{\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}\}.$$

Denote the dual coframe of $\{y_1, y_2, y_3, A_1, A_2\}$ to be $\{\omega^1, \omega^2, \omega^3, d\ln\lambda_1, d\ln\lambda_2\}$. We want to express the Riemannian metric g on $M = SL(3)/SO(3)$ in this coframe.

Recall that the Riemmanian metric g at $p \in M$ is a mapping $g : T_pM \times T_pM \longrightarrow \mathbb{R}$ that is symmetric, bilinear and positive definite. Let $\{X_i\} \subset \mathfrak{X}(M)$ be a local frame on M, where $\mathfrak{X}(M)$ denotes the vector space of smooth vector fields on M. In our case, the frame we picked are the left invariant vector fields $\{y_1, y_2, y_3, A_1, A_2\}$. Let $\{\omega^i\} \subset \Gamma(T^*M)$ be the dual coframe to $\{X_i\}$, in our case $\{\omega_1, \omega_2, \omega_3, d\ln\lambda_1, d\ln\lambda_2\}$. With our frame and dual coframe, denote $g_{ij} := g(X_i, X_j)$. But $g(X_i, X_j) = g_{ij}(\omega^i \otimes \omega^j)(X_i, X_j)$. So we can write g as $g = g_{ij}\omega^i\omega^j$.

Proposition 3.1. The metric on $SL(3)/SO(3)$ is:

$$\hat{g} = \frac{1}{2}g = \frac{(\lambda_2^2 - \lambda_1^2)^2}{(\lambda_1^2\lambda_2^2)}(\omega^1)^2 + \lambda_2^2(\frac{1-\lambda_1^4\lambda_2^2}{\lambda_1^2\lambda_2^2})^2(\omega^2)^2 + \lambda_1^2(\frac{1-\lambda_1^2\lambda_2^4}{\lambda_1^2\lambda_2^2})^2(\omega^3)^2 +$$
$$\frac{4}{\lambda_1^2}d\lambda_1^2 + \frac{4}{\lambda_2^2}d\lambda_2^2.$$

Proof. For any $h \in SL(3)$, we have a map $\varphi : SL(3) \longrightarrow SL(3)/SO(3)$ given by $h \xrightarrow{\varphi} hh^t$. This induces the identification:

$$SL(3)/SO(3) \to \{\text{positive definite, symmetric } 3\times 3 \text{ matrices with determinant } 1\}.$$

Let $p \in M = SL(3)/SO(3)$ and let X, Y be left invariant vector fields on M coming from $\mathfrak{p}$. We have the Riemannian metric $g(X, Y)_p = \mathrm{Tr}(p^{-1}Xp^{-1}Y)$ on $SL(3)/SO(3)$.

Take the following curve in the product space $SO(3) \times A$:

$$\gamma(t) = (k_1e^{tX}, a_1e^{ta}),\ X \in \mathfrak{so}(3),\ a \in \mathfrak{a},\ (k_1, a_1) \in SO(3) \times A,$$

and consider the map $f : SO(3) \times A \longrightarrow SL(3)/SO(3)$ given by $f(k_1, a_1) = k_1a_1^2k_1^{-1}$. First set $a = 0$. Then define $\gamma_{SO(3)}(t) := (k_1e^{tX}, a_1)$. We get

$$df_p(\dot{\gamma}_{SO(3)}) = \frac{d}{dt}|_{t=0}(k_1e^{tX})(a_1^2)(k_1e^{tX})^{-1} = k_1[X, a_1^2]k_1^{-1}.$$

The pullback metric is:

$$g(df_p(\dot{\gamma}_{SO(3)}), df_p(\dot{\gamma}_{SO(3)}))_{f(p)} = g(k_1[X, a_1^2]k_1^{-1}, k_1[X, a_1^2]k_1^{-1})_{k_1a_1^2k_1^{-1}} =$$

$$\mathrm{Tr}((k_1a_1^2k_1^{-1})(k_1[X, a_1^2]k_1^{-1}))(k_1a_1^2k_1^{-1})(k_1[X, a_1^2]k_1^{-1}))) = \mathrm{Tr}((a_1^{-2}[X, a_1^2])^2)(*)$$

Let

$$a_1 = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}.$$

A computation yields:

$$[y_1, a_1^2] = \begin{pmatrix} 0 & \lambda_2^2 - \lambda_1^2 & 0 \\ \lambda_2^2 - \lambda_1^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ [y_2, a_1^2] = \begin{pmatrix} 0 & 0 & \lambda_3^2 - \lambda_1^2 \\ 0 & 0 & 0 \\ \lambda_3^2 - \lambda_1^2 & 0 & 0 \end{pmatrix}$$

$$[y_3, a_1^2] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda_3^2 - \lambda_2^2 \\ 0 & \lambda_3^2 - \lambda_2^2 & 0 \end{pmatrix}.$$

Taking $X = y_1, y_2, y_3$, and substituting this into $(*)$ yields the desired result.

Next, setting $X = 0$, consider $\gamma_A(t) = (k_1, a_1 e^{ta})$.
Then

$$df_p(\dot\gamma_A) = \frac{d}{dt}|_{t=0}(k_1)a_1^2(k_1)^{-1} = 2k_1 a_1^2 a k_1^{-1},$$
$$g(2k_1 a_1^2 a k_1^{-1}, 2k_1 a_1^2 a k_1^{-1})_{k_1 a^2 k_1^{-1}} = 4\mathrm{Tr}(a^2).$$

Set $a = A_1, A_2$ successively, and note that $[d(\ln\lambda)]^2 = \dfrac{1}{\lambda^2} d\lambda^2$.
Finally, a computation yields that $g(df_p(\dot\gamma_{SO(3)}), df_p(\dot\gamma_A)) = 0$. □

3.2. **Lie Derivatives.**

We convert our basis into an orthonormal one:
Take

$$E_1 = \frac{\lambda_1\lambda_2}{\lambda_1^2 - \lambda_2^2} y_1, E_2 = \frac{\lambda_1^2\lambda_2}{\lambda_1^4\lambda_2^2 - 1} y_2, E_3 = \frac{\lambda_1\lambda_2^2}{\lambda_1^2\lambda_2^4 - 1} y_3,$$

$$\partial_{t_1} = \frac{\lambda_1}{2}\partial_{\lambda_1}, \partial_{t_2} = \frac{\lambda_2}{2}\partial_{\lambda_2},$$

giving us an orthonormal frame $\{V_1, ..., V_5\} = \{E_1, E_2, E_3, dt_1, dt_2\}$ and corresponding orthonormal coframe $\{\tilde\omega^1, \tilde\omega^2, \tilde\omega^3, dt_1, dt_2\}$.

Given a 1-form ϕ, let $e(\phi)$ denote exterior multiplication on the left by ϕ. Let $e^*(\phi)$ denote the adjoint operator. With this frame, the Lie derivative along ∂_{t_i} can be written as

$$L_{\partial_{t_i}} = \nabla_{\partial_{t_i}} + e(\tilde\omega^j)e^*(\nabla_{V_j} dt_i).$$

Define the commutation coefficient c_{ij}^k as follows: $[V_i, V_j] = c_{ij}^k V_k$. Since we have an orthonormal frame, $\nabla_{V_j}\tilde\omega^k = -\Gamma_{jm}^k \tilde\omega^m$, with $\Gamma_{ij}^k = \dfrac{1}{2}(c_{ij}^k - c_{ik}^j - c_{jk}^i)$. So this yields the following result:

Proposition 3.2. The Lie derivatives along ∂_{t_1} and ∂_{t_2} are given by:

$$L_{\partial_{t_1}} = \nabla_{\partial_{t_1}} + \frac{\lambda_1^2 + \lambda_2^2}{2(\lambda_1^2 - \lambda_2^2)} e(\tilde\omega^1)e^*(\tilde\omega^1) + \frac{\lambda_1^4\lambda_2^2 + 1}{\lambda_1^4\lambda_2^2 - 1} e(\tilde\omega^2)e^*(\tilde\omega^2)$$
$$+ \frac{\lambda_1^2\lambda_2^4 + 1}{2(\lambda_1^2\lambda_2^4 - 1)} e(\tilde\omega^3)e^*(\tilde\omega^3).$$

$$L_{\partial_{t_2}} = \nabla_{\partial_{t_2}} + \frac{\lambda_1^2+\lambda_2^2}{2(\lambda_2^2-\lambda_1^2)} e(\tilde{\omega}^1)e^*(\tilde{\omega}^1) + \frac{\lambda_1^2\lambda_2^4+1}{2(\lambda_1^2\lambda_2^4-1)} e(\tilde{\omega}^2)e^*(\tilde{\omega}^2)$$
$$+\frac{\lambda_1^4\lambda_2^2+1}{\lambda_1^4\lambda_2^2-1} e(\tilde{\omega}^3)e^*(\tilde{\omega}^3).$$

3.3. **Comparison with Hyperbolic 2-Space.**

For the reader's reference, we make a comparison with a similar (better known) result for hyperbolic 2-space:
Choose the following basis for $\mathfrak{so}(2)$: $y_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, basis for $\mathfrak{a}$: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \frac{1}{\lambda_1} \end{pmatrix}$.
The metric is

$$g = (\frac{1}{\lambda_1^2} - \lambda_1^2)^2(\omega^1)^2 + \frac{4}{\lambda_1^2} d\lambda_1^2.$$

So the Lie derivative is

$$L_{\partial_{t_1}} = \nabla_{\partial_{t_1}} + \frac{\lambda_1^4+1}{\lambda_1^4-1} e(\tilde{\omega}^1)e^*(\tilde{\omega}^1).$$

Take $\lambda_1 = e^{\frac{t_1}{2}}$, this yields

$$L_{\partial_{t_1}} = \nabla_{\partial_{t_1}} + \coth t_1 e(\tilde{\omega}^1)e^*(\tilde{\omega}^1),$$

where $\tilde{\omega} = (\lambda_1^2 - \frac{1}{\lambda_1^2})^{-1}\omega$.

4. Representation Theory of $\mathfrak{so}(3)$

In this section, we provide a short discussion of Casimirs and show that the curvature operator is a $\mathfrak{k}$-Casimir. With this knowledge, we decompose the second exterior power of $\mathfrak{p}^*$ into irreducibles under the adjoint action of $\mathfrak{so}(3)$.

4.1. **Casimirs.**

Definition 4.1. Let $\mathfrak{g}$ be a semisimple Lie Algebra and let B be the Killing form on $\mathfrak{g}$. Let $\{X_i\}_{i=1}^n$ be any basis for $\mathfrak{g}$, and let $\{X^i\}$ be the dual basis with respect to B. Then the *Casimir element* Ω is defined by

$$\Omega := \sum_{i=1}^n X_i X^i$$

Definition 4.2. Let ρ be a representation $\mathfrak{g} \mapsto \mathrm{End(V)}$ for some vector space V. The *Casimir invariant* of ρ is defined by:

$$\rho(\Omega) := \sum_{i=1}^{n} \rho(X_i)\rho(X^i).$$

Note $\rho(\Omega)$ is independent of basis.
For a full discussion of Casimirs, see Hall [9].

We now turn our attention to Casimirs on $\mathfrak{k}$, where G/K is a locally symmetric space, with $\mathfrak{k}$ the Lie Algebra of K.

Recall the following Bochner-Weitzenböck formula:

Let M be a Riemannian manifold. Let $\{e_i\}$ be an orthonormal frame on TM, and $\{\omega^i\}$ be the coframe dual to $\{e_i\}$. Then

$$\Delta = \nabla^*\nabla - e(\omega^j)e^*(\omega^k)R(e_j, e_k)$$

Definition 4.3. $\mathcal{R} := -e(\omega^j)e^*(\omega^k)R(e_j, e_k)$ is called the *curvature operator*.

Proposition 4.1. For $M = G/K$, $\mathcal{R}$ is a $\mathfrak{k}$-Casimir.

Proof. We use the following result from Theorem 4.2 of Helgason [11]:

Lemma 4.1. On a Riemannian symmetric space, at the origin, the Riemannian curvature tensor is given by:

$$R(X,Y)Z = -[[X,Y],Z]$$

for any $X, Y, Z \in \mathfrak{p}$.

Proof of Proposition:
Let $X_i, X_j, X_k, Y_a \in \mathfrak{p}$.

$$[X_i, X_j] = \langle [X_i, X_j], Y_a\rangle Y_a = -Y_a tr[X_i, X_j]Y_a = -Y_a tr X_i[X_j, Y_a] = \langle X_i, ad(Y_a)X_j\rangle Y_a$$

$$e(\omega^i)e^*(\omega^j)\langle [[X_i, X_j], X_k]X_p\rangle e(\omega^p)e^*(\omega^k)$$

$$e(\omega^i)e^*(\omega^j)\langle [Y_a, X_j], X_i\rangle\langle [Y_a, X_k], X_p\rangle e(\omega^p)e^*(\omega^k)$$

$$= e(ad^t(Y_a)\omega^j)e^*(\omega^j)e(ad^t(Y_a)\omega^p)e^*(\omega^k)$$

$$= \sum_a ad(Y_a)ad(Y_a).$$

□

4.2. Decomposition of $\bigwedge^2 \mathfrak{p}^*$ into Irreducibles.

Choose the following orthonormal basis for $\mathfrak{p}$:

$$\{V_1, V_2, V_3, V_4, V_5\} = \{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}\}.$$

Recall from section 2 that $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$. We can thus consider the adjoint action of $\mathfrak{k}$ on $\mathfrak{p}$, giving us the following proposition:

Proposition 4.2. Under $\bigwedge^2$ of the adjoint action of $\mathfrak{so}(3)$ on $\bigwedge^2 \mathfrak{p} \otimes_{\mathbb{R}} \mathbb{C}$, the following decomposition holds: $\bigwedge^2 \mathfrak{p} \otimes_{\mathbb{R}} \mathbb{C} = \bigwedge^2_3 \oplus \bigwedge^2_7$, , where $\bigwedge^2_j$ is a j dimensional $\mathfrak{so}(3)$-irreducible representation.

Proof. We have the following commutator relations on $\mathfrak{p}$:

$$[y_1, V_1] = 2V_4, [y_1, V_2] = -V_3, [y_1, V_3] = V_2, [y_1, V_4] = -2V_1, [y_1, V_5] = 0,$$
$$[y_2, V_1] = -V_3, [y_2, V_2] = V_4 + \sqrt{3}V_5, [y_2, V_3] = V_1, [y_2, V_4] = -V_2, [y_2, V_5] = -\sqrt{3}V_2,$$
$$[y_3, V_1] = -V_2, [y_3, V_2] = V_1, [y_3, V_3] = \sqrt{3}V_5 - V_4, [y_3, V_4] = V_3, [y_3, V_5] = -\sqrt{3}V_3.$$

Choose $\langle y_1 \rangle$ as a Cartan for $\mathfrak{so}(3)$. Over $\mathfrak{p} \otimes_{\mathbb{R}} \mathbb{C}$, we then have

$$[y_1, V_1 + iV_4] = -2i(V_1 + iV_4), [y_1, V_2 + iV_3] = i(V_2 + iV_3), [y_1, V_5] = 0.$$

On $\bigwedge^2 \mathfrak{p} \otimes_{\mathbb{R}} \mathbb{C}$, denote the eigenspace of y_1 with complex eigenvalue ci for $c = -3, -2, -1, 0, 1, 2, 3$ by X_c. The y_1-eigenspaces then are:

$$X_3 = span\{(V_1 - iV_4) \wedge (V_2 + iV_3)\},$$
$$X_2 = span\{(V_1 - iV_4) \wedge V_5\},$$
$$X_1 = span\{(V_2 + iV_3) \wedge V_5, (V_1 - iV_4) \wedge (V_2 - iV_3)\},$$
$$X_0 = span\{(V_2 + iV_3) \wedge (V_2 - iV_3), (V_1 + iV_4) \wedge (V_1 - iV_4)\},$$
$$X_{-1} = span\{(V_2 - iV_3) \wedge V_5, (V_1 + iV_4) \wedge (V_2 + iV_3)\},$$
$$X_{-2} = span\{(V_1 + iV_4) \wedge V_5\},$$
$$X_{-3} = span\{(V_1 + iV_4) \wedge (V_2 - iV_3)\}.$$

A lowering operator is an operator that lowers the eigenvalue of another operator. In the case of $\mathfrak{so}(3)$, since $[y_1, y_2 - iy_3] = -i(y_2 - iy_3)$, $y_2 - iy_3$ lowers the y_1-eigenvalue of an eigenvector in $\mathfrak{p} \otimes_{\mathbb{R}} \mathbb{C}$ by i, so $y_2 - iy_3$ is a lowering operator.
Similarly, a raising operator is an operator that raises the eigenvalue of another operator. Since $[y_1, y_2 + iy_3] = i(y_2 + iy_3)$, $y_2 + iy_3$ is a raising operator that raises the y_1−eigenvalue of an eigenvector in $\mathfrak{p} \otimes_{\mathbb{R}} \mathbb{C}$ by i.

Now, we pick a highest weight vector, $(V_1 - iV_4) \wedge (V_2 + iV_3) \in X_3$ and apply the lowering operator $y_2 - iy_3$ successively by its adjoint action, starting with $\mathrm{ad}(y_2 - iy_3)[(V_1 - iV_4) \wedge (V_2 + iV_3)]$. We get the following:

$$X_3 \xrightarrow{y_2 - iy_3} X_2 \xrightarrow{y_2 - iy_3} \text{span}\{2i(V_2 + iV_3) \wedge V_5 - \sqrt{3}(V_1 - iV_4) \wedge (V_2 - iV_3)\} \subset X_1$$

$$\xrightarrow{y_2 - iy_3} \text{span}\{(V_1 + iV_4) \wedge (V_1 - iV_4)\} \subset X_0 \xrightarrow{y_2 - iy_3} \text{span}\{2i(V_2 - iV_3) \wedge V_5 + \sqrt{3}(V_1 + iV_4) \wedge (V_2 + iV_3)\} \subset X_{-1}$$

$$\xrightarrow{y_2-iy_3} X_{-2} \xrightarrow{y_2-iy_3} X_{-3} \xrightarrow{y_2-iy_3} 0$$

Take an orthogonal projection of the vector $2i(V_2+iV_3)\wedge V_5 - \sqrt{3}(V_1-iV_4))$ on the eigenspace X_1. This gives us the vector $i\sqrt{3}(V_2+iV_3)\wedge V_5+(V_1-iV_4)\wedge(V_2-iV_3)\in X_1$. Repeating the same computation, we get:

$$\operatorname{span}\{i\sqrt{3}(V_2+iV_3)\wedge V_5+(V_1-iV_4)\wedge(V_2-iV_3)\}\subset X_1 \xrightarrow{y_2-iy_3} \operatorname{span}\{(V_2+iV_3)\wedge(V_2-iV_3)\}\subset X_0$$
$$\xrightarrow{y_2-iy_3} \operatorname{span}\{\sqrt{3}i(V_2-iV_3)\wedge V_5+(V_1+iV_4)\wedge(V_2+iV_3)\}\subset X_{-1} \xrightarrow{y_2-iy_3} 0.$$

This gives us the required decomposition. □

We now derive a basis for $\bigwedge_7^2 \mathfrak{p}^*$ and $\bigwedge_3^2 \mathfrak{p}^*$ from our computations:

Identifying $\mathfrak{p}$ with its dual $\mathfrak{p}^*$, and taking $\tilde{\omega}^1,\tilde{\omega}^2,\tilde{\omega}^3, dt_1, dt_2$ to be the dual basis to V_j we get the following basis for $\bigwedge_7^2$:

$$\{\tilde{\omega}^1\wedge\tilde{\omega}^2-\tilde{\omega}^3\wedge dt_1, \tilde{\omega}^1\wedge\tilde{\omega}^3+\tilde{\omega}^2\wedge dt_1, \tilde{\omega}^1\wedge dt_2, dt_1\wedge dt_2,$$
$$2\tilde{\omega}^3\wedge dt_2+\sqrt{3}\tilde{\omega}^1\wedge\tilde{\omega}^2+\sqrt{3}\tilde{\omega}^3\wedge dt_1, 2\tilde{\omega}^2\wedge dt_2+\sqrt{3}\tilde{\omega}^1\wedge\tilde{\omega}^3-\sqrt{3}\tilde{\omega}^2\wedge dt_1, \tilde{\omega}^1\wedge dt_1\},$$

and the following basis for $\bigwedge_3^2$:

$$\{\tilde{\omega}^2\wedge\tilde{\omega}^3, \sqrt{3}\tilde{\omega}^2\wedge dt_2-\tilde{\omega}^1\wedge\tilde{\omega}^3+\tilde{\omega}^2\wedge dt_1, \sqrt{3}\tilde{\omega}^3\wedge dt_2-\tilde{\omega}^1\wedge\tilde{\omega}^2-\tilde{\omega}^3\wedge dt_1\}.$$

5. On Betti Numbers of $\Gamma\backslash SL(3)/SO(3)$

In this section, we derive the main results, bounds on Betti numbers of $\Gamma\backslash SL(3)/SO(3)$ using Di Cerbo and Stern's results in [4] and [5].

From the expressions of the Lie derivatives in section 3:

Set

$$Q_1 := \frac{\lambda_1^2+\lambda_2^2}{2(\lambda_1^2-\lambda_2^2)} e(\tilde{\omega}^1)e^*(\tilde{\omega}^1) + \frac{\lambda_1^4\lambda_2^2+1}{\lambda_1^4\lambda_2^2-1} e(\tilde{\omega}^2)e^*(\tilde{\omega}^2) + \frac{\lambda_1^2\lambda_2^4+1}{2(\lambda_1^2\lambda_2^4-1)} e(\tilde{\omega}^3)e^*(\tilde{\omega}^3)$$

and

$$Q_2 := \frac{\lambda_1^2+\lambda_2^2}{2(\lambda_2^2-\lambda_1^2)} e(\tilde{\omega}^1)e^*(\tilde{\omega}^1) + \frac{\lambda_1^2\lambda_2^4+1}{2(\lambda_1^2\lambda_2^4-1)} e(\tilde{\omega}^2)e^*(\tilde{\omega}^2) + \frac{\lambda_1^4\lambda_2^2+1}{\lambda_1^4\lambda_2^2-1} e(\tilde{\omega}^3)e^*(\tilde{\omega}^3)$$

$$H_1 = trQ_1 = \frac{\lambda_1^2+\lambda_2^2}{2(\lambda_1^2-\lambda_2^2)} + \frac{\lambda_1^4\lambda_2^2+1}{\lambda_1^4\lambda_2^2-1} + \frac{\lambda_1^2\lambda_2^4+1}{2(\lambda_1^2\lambda_2^4-1)}$$
$$H_2 = trQ_2 = \frac{\lambda_1^2+\lambda_2^2}{2(\lambda_2^2-\lambda_1^2)} + \frac{\lambda_1^2\lambda_2^4+1}{2(\lambda_1^2\lambda_2^4-1)} + \frac{\lambda_1^4\lambda_2^2+1}{\lambda_1^4\lambda_2^2-1}$$

Let

$$C_1 := \frac{\lambda_1^2+\lambda_2^2}{2(\lambda_1^2-\lambda_2^2)}, C_2 := \frac{\lambda_1^4\lambda_2^2+1}{\lambda_1^4\lambda_2^2-1}, C_3 := \frac{\lambda_1^2\lambda_2^4+1}{2(\lambda_1^2\lambda_2^4-1)}$$

.

Since $\lambda_1>\lambda_2>\lambda_3$ with $\lambda_1, \lambda_1\lambda_2>1$, have $C_1, C_2, C_3>0$.

The Lie derivative with respect to the vector field $\partial_{t_1} + \partial_{t_2}$ is given by

$$L_{\partial_{t_1}+\partial_{t_2}} = \nabla_{\partial_{t_1}+\partial_{t_2}} + (C_2 + C_3)[e(\tilde{\omega}^2)e^*(\tilde{\omega}^2) + e(\tilde{\omega}^3)e^*(\tilde{\omega}^3)].$$

Set:

$$H = H_1 + H_2 = 2(C_2 + C_3),\ Q = (C_2 + C_3)[e(\tilde{\omega}^2)e^*(\tilde{\omega}^2) + e(\tilde{\omega}^3)e^*(\tilde{\omega}^3)],$$

$$\mu_h(r) := \frac{\int_{K\Omega_r'} |[e^*(dt_1) + e^*(dt_2)]h|^2 d\sigma}{\int_{K\Omega_r'} |h|^2 d\sigma},$$

and

$$q_h(r) := \frac{\int_{K\Omega_r} \langle (\frac{H}{2} - Q)h, h\rangle dvol}{\int_{K\Omega_r} |h|^2 dvol},$$

where $d\sigma$ is the volume form on $K\Omega_r'$ and $dvol$ is the volume form on $K\Omega_r$.
In our case,

$$\langle (\frac{H}{2} - Q)h, h\rangle = (C_2 + C_3)|h|^2 - < (C_2 + C_3)[e(\tilde{\omega}^2)e^*(\tilde{\omega}^2) + e(\tilde{\omega}^3)e^*(\tilde{\omega}^3)]h, h > .$$

Let $\pi : G/K \to \Gamma\backslash G/K$ be the universal covering map, and set $M = \Gamma\backslash G/K$. Let $\mathrm{inj}_M = \mathrm{inj}_{\Gamma\backslash G/K} := \inf_{x\in M} \mathrm{inj}_x$. Then if $r < \mathrm{inj}_M$, for all balls of radius r, B_r, the map $B_r \longrightarrow \Gamma\backslash B_r$ is 1-to-1. Therefore, if $K\Omega_R \subset B_r$, where the radius $r < \mathrm{inj}_M$, then $\pi : K\Omega_R \longrightarrow \Gamma\backslash K\Omega_R$ is 1-to-1. So, we compute the values of R for which $K\Omega_R \subset B_r$. Let I_G be the identity element of G and $p \in \Omega_R$. To get the distance between any 2 points, $(t_1, t_2), (t_1', t_2') \in \Omega_R$, use the usual Euclidean metric on $\mathbb{R}^2$, i.e.

$$\mathrm{dist}((t_1, t_2), (t_1', t_2')) = \sqrt{(t_1' - t_1)^2 + (t_2' - t_2)^2}.$$

We then have:

$$\max_{p\in\Omega_R} \mathrm{dist}(I_G, p) \leq$$
$$\max_{(t_1,t_2),(t_1',t_2')\in\Omega_R} \mathrm{dist}((t_1, t_2), (t_1', t_2')) \leq \max_{(t_1,t_2)\in\Omega_R} \sqrt{(t_1 + t_2)^2 + (t_1 + t_2)^2} \leq$$
$$\sqrt{2}R \leq \mathrm{inj}_M.$$

So when $R \in (0, \frac{\mathrm{inj}_M}{\sqrt{2}})$, $K\Omega_R \to \Gamma\backslash K\Omega_R$ is 1-to-1.

For the remainder of the paper we assume that Γ is a cocompact, torsion free lattice.
Then the following lemma holds:

Lemma 5.1. Let $M = \Gamma\backslash G/K$. Given $h \in \mathcal{H}^2(M)$, $r \in (0, \frac{\mathrm{inj}_M}{\sqrt{2}})$,

$$\int_{K\Omega_r'} (\frac{1}{2} - \mu_h(r))|h|^2 d\sigma = \int_{K\Omega_r} q_h(r)|h|^2 dvol.$$

Proof. We follow the computation on page 6 of [5]. This gives us

$$\int_{K\Omega_r} \langle L_{\partial_{t_1}+\partial_{t_2}} h, h\rangle dvol = \int_{K\Omega_r} \langle d([e^*(dt_1) + e^*(dt_2)]h), h\rangle dvol$$

$$= \int_{K\Omega_r} \langle [e^*(dt_1)+e^*(dt_2)]h, d^*h\rangle dvol + \int_{K\Omega_r'} |[e^*(dt_1)+e^*(dt_2)]h|^2 d\sigma = \int_{K\Omega_r'} |[e^*(dt_1)+e^*(dt_2)]h|^2 d\sigma.$$

We also get the alternate expression

$$\begin{aligned}\int_{K\Omega_r} \langle L_{\partial_{t_1}+\partial_{t_2}} h, h\rangle dvol &= \int_{K\Omega_r} \langle (\nabla_{\partial_{t_1}+\partial_{t_2}} + Q)h, h\rangle dvol \\ &= \int_{K\Omega_r} \frac{1}{2}(L_{\partial_{t_1}+\partial_{t_2}} - Q)|h|^2 dvol + \int_{K\Omega_r} \langle Qh, h\rangle dvol \\ &= \int_{K\Omega_r} \langle (Q - \frac{H}{2})h, h\rangle dvol + \int_{K\Omega_r'} \frac{1}{2}|h|^2 d\sigma.\end{aligned}$$

Equating both expressions gives us:

$$\int_{K\Omega_r} q_h(r)|h|^2 dvol = \frac{1}{2}\int_{K\Omega_r'} (1 - 2\mu_h(r))|h|^2 d\sigma$$

as required. □

Corollary 5.1. Let $r \in (0, \frac{\mathrm{inj}_M}{\sqrt{2}})$. If $q_h(r) \geq 0$, then $\frac{1}{2} - \mu_h(r) \geq 0$. In particular, $0 \leq \mu_h(r) \leq \frac{1}{2}$.

So in order to get estimates, we need to control the positivity of the geometric term $q_h(r)$ using the decomposition of $\mathcal{H}^2 = \mathcal{H}^2_7 \oplus \mathcal{H}^2_3$.

First, consider $\mathcal{H}^2_7$:

From the basis we computed in section 4.2, we have the lower bound $q_h(r) \geq 0$.

We compute the Lie derivative with respect to $(t_1+t_2)(\partial_{t_1}+\partial_{t_2})$, and $(t_1+t_2+1)(\partial_{t_1}+\partial_{t_2})$. A computation yields

$$\begin{aligned}L_{(t_1+t_2)(\partial_{t_1}+\partial_{t_2})} = \nabla_{(t_1+t_2)(\partial_{t_1}+\partial_{t_2})} &+ (t_1+t_2)(C_2+C_3)[e(\tilde{\omega}^2)e^*(\tilde{\omega}^2) + e(\tilde{\omega}^3)e^*(\tilde{\omega}^3)] \\ &+ (e(dt_1)+e(dt_2))(e^*(dt_1)+e^*(dt_2)).\end{aligned}$$

Taking a linear combination of $L_{(t_1+t_2)(\partial_{t_1}+\partial_{t_2})}$ and $L_{\partial_{t_1}+\partial_{t_2}}$ gives:

$$\begin{aligned}L_{(t_1+t_2+1)(\partial_{t_1}+\partial_{t_2})} = \nabla_{(t_1+t_2+1)(\partial_{t_1}+\partial_{t_2})} &+ (t_1+t_2+1)(C_2+C_3)[e(\tilde{\omega}^2)e^*(\tilde{\omega}^2) + e(\tilde{\omega}^3)e^*(\tilde{\omega}^3)] \\ &+ (e(dt_1)+e(dt_2))(e^*(dt_1)+e^*(dt_2)).\end{aligned}$$

Using $L_{(t_1+t_2+1)(\partial_{t_1}+\partial_{t_2})}$, we set

$$\tilde{H} = 2(t_1+t_2+1)(C_2+C_3)) + 2,$$

and

$$\begin{aligned}\tilde{Q} = (t_1+t_2)(C_2+C_3)&[e(\tilde{\omega}^2)e^*(\tilde{\omega}^2) + e(\tilde{\omega}^3)e^*(\tilde{\omega}^3)] \\ &+ (e(dt_1)+e(dt_2))(e^*(dt_1)+e^*(dt_2)).\end{aligned}$$

Let

$$\tilde{q_h}(r) := \frac{\int_{K\Omega_r} \langle (\frac{\tilde{H}}{2} - \tilde{Q})h, h\rangle dvol}{\int_{K\Omega_r} |h|^2 dvol}.$$

From our basis of $\bigwedge^2_7 \mathfrak{p}^*$, we get the lower bound $\tilde{q_h}(r) \geq \frac{1}{4}$.
This gives us the following bound on the cohomology of $\mathcal{H}^2_7$:

Theorem 5.1. Let $M = \Gamma \setminus SL(3)/SO(3)$, and $R = \frac{\text{inj}_M}{\sqrt{2}}$. Then there exists a strictly positive constant $d(5) > 0$ such that:

$$\frac{b_7^{2,SL(3)}}{Vol(M)} \leq \frac{2d(5)}{R}.$$

We need to use Lemma 45 from [5]:

Lemma 5.2. Let (M^n, g) be a closed Riemannian manifold. There exists $h \in \mathcal{H}_g^k(M)$ with $||h||_{L^2} = 1$, such that:

$$\max_{p\in M}|h|^2 \geq \frac{k!(n-k)!b^k(M)}{n!Vol(M)},$$

and Lemma 51 from [5]:

Lemma 5.3. (Moser Inequality) Let $M = \Gamma\backslash G/K$, and $n = \dim M$. Given a harmonic k-form $\alpha \in \mathcal{H}^k(M)$, for any $p \in M$ and $R > 1$, there exists a strictly positive constant $c(n)$ such that

$$||\alpha||^2_{L^\infty(K\Omega_{R/2})} \leq c(n)||\alpha||^2_{L^2(K\Omega_R)}.$$

Define

$$d(n) := \binom{n}{2} c(n).$$

Proof. Proof of Theorem 5.1: First, we need to prove the following monotonicity estimate:

$$\int_{K\Omega_1} |h|^2 dvol \leq \frac{2}{R}\int_{K\Omega_R} |h|^2 dvol.$$

To show this, we adopt a similar computation as Theorem 96 in [5]. We have the following bounds:

$$\frac{1}{4}\int_{K\Omega_1} |h|^2 dvol \leq \int_{K\Omega_1} \tilde{q_h}(r)|h|^2 dvol \leq \int_{K\Omega_R} \tilde{q_h}(r)|h|^2 dvol$$
$$= \int_{K\Omega'_R} (\frac{1}{2} - \mu_h(r))d\sigma \leq \frac{1}{2}\int_{K\Omega'_R} |h|^2 d\sigma.$$

Integrating the above inequality from 0 to R gives us the desired monotonicity estimate.

Applying Lemmas 5.2, 5.3, and our monotonicity estimate implies that there exists a unique $h \in \mathcal{H}_7^2$ such that $||h||_{L^2} = 1$ and

$$\frac{b_7^{2,SL(3)}}{Vol(M)} \leq \binom{5}{2}\max_{p\in M}|h|^2 \leq d(5)\int_{K\Omega_1} |h|^2 dvol \leq \frac{2d(5)}{R}\int_{K\Omega_R} |h|^2 dvol \leq \frac{2d(5)}{R}.$$

□

Remark: We did not get any estimates on the Betti number for $\mathcal{H}_3^2$. This is because for $\tilde{\omega}^2 \wedge \tilde{\omega}^3$, $q_h(r) < 0$.

6. On the Geometry of $\Gamma\backslash Sp(4)/U(2)$

In this section, we study the geometry of compact quotients of $Sp(4)/U(2)$, a 6-dimensional Hermitian locally symmetric space using similar techniques to $SL(3)/SO(3)$.

6.1. **Metric Computation.**

By Helgason [11], an element of the lie algebra $\mathfrak{sp}(4)$ is given by

$$\begin{pmatrix} X_1 & X_2 \\ X_3 & -X_1^t \end{pmatrix}$$

where X_1 is a real 2×2 matrix, X_2, X_3 are symmetric 2×2 matrices.
Since $K = U(2) = SU(2) \rtimes S^1$, $\mathfrak{u}(2) = \mathbb{R} \oplus \mathfrak{su}(2)$.

Consider the following basis for $\mathfrak{u}(2)$:

$$\{X_1, X_2, X_3, X_4\} = \{\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\}$$

where X_1, X_2, X_3 corresponds to $\mathfrak{su}(2) \subset \mathfrak{u}(2)$ and X_4 corresponds to $\mathbb{R} \subset \mathfrak{u}(2)$.

$\mathfrak{a}$ is the abelian subalgebra spanned by:

$$\{A_1, A_2\} = \{\frac{1}{\sqrt{2}}\mathrm{diag}(1,0,-1,0), \frac{1}{\sqrt{2}}\mathrm{diag}(0,1,0,-1)\}$$

Let $G = Sp(4)$, $K = U(2)$, then by Helgason [11], if we let

$$A^+ = \{\mathrm{diag}(\lambda_1, \lambda_2, \frac{1}{\lambda_1}, \frac{1}{\lambda_2}) : \lambda_1 > \lambda_2 > 1\}$$

we get the Cartan decomposition $G = KA^+K$.

Now, let $\{\omega^1, \omega^2, \omega^3, \omega^4, \mathrm{d}\ln\lambda_1, \mathrm{d}\ln\lambda_2\}$ denote the dual coframe to $\{X_1, X_2, X_3, X_4, A_1, A_2\}$. Then by the same computation as in Proposition 1, we have the following result:

Proposition 6.1. The metric on $Sp(4)/U(2)$ is:

$$g = \frac{(\lambda_2^2 - \lambda_1^2)^2}{\lambda_1^2\lambda_2^2}(\omega^1)^2 + \frac{(1-\lambda_1^2\lambda_2^2)^2}{\lambda_1^2\lambda_2^2}(\omega^2)^2 + \frac{1}{2}(\frac{(1-\lambda_1^4)^2}{\lambda_1^4} + \frac{(1-\lambda_2^4)^2}{\lambda_2^4})((\omega^3)^2 + (\omega^4)^2)$$

$$+\frac{4}{\lambda_1^2}d\lambda_1^2 + \frac{4}{\lambda_2^2}d\lambda_2^2.$$

6.2. **Lie Derivatives.**

Convert our basis into an orthonormal one:
Take

$$E_1 = \frac{\lambda_1\lambda_2}{\lambda_1^2-\lambda_2^2}X_1, E_2 = \frac{\lambda_1\lambda_2}{\lambda_1^2\lambda_2^2-1}X_2, E_3 = \frac{\lambda_1^2\lambda_2^2}{\sqrt{\lambda_1^4(1-\lambda_2^4)^2+\lambda_2^4(1-\lambda_1^4)^2}}X_3$$

$$E_4 = \frac{\lambda_1^2\lambda_2^2}{\sqrt{\lambda_1^4(1-\lambda_2^4)^2+\lambda_2^4(1-\lambda_1^4)^2}}, \partial_{t_1} = \frac{\lambda_1}{2}\partial_{\lambda_1}, \ \partial_{t_2} = \frac{\lambda_2}{2}\partial_{\lambda_2}.$$

This gives us an orthonormal frame $\{E_1, E_2, E_3, E_4, \partial_{t_1}, \partial_{t_2}\}$ and the corresponding dual coframe $\{\tilde{\omega}^1, \tilde{\omega}^2, \tilde{\omega}^3, \tilde{\omega}^4, dt_1, dt_2\}$. Following the formulas in section 4.2, we get the following result for the Lie derivatives:

Proposition 6.2. The Lie derivatives along ∂_{t_1} and ∂_{t_2} are given by:

$$L_{\partial_{t_1}} = \nabla_{\partial_{t_1}} + \frac{\lambda_1^2+\lambda_2^2}{2(\lambda_1^2-\lambda_2^2)}e(\tilde{\omega}^1)e^*(\tilde{\omega}^1) + \frac{\lambda_1^2\lambda_2^2+1}{2(\lambda_1^2\lambda_2^2-1)}e(\tilde{\omega}^2)e^*(\tilde{\omega}^2)$$
$$+\frac{\lambda_2^4(\lambda_1^8-1)}{\lambda_2^4(1-\lambda_1^4)^2+\lambda_1^4(1-\lambda_2^4)^2}(e(\tilde{\omega}^3)e^*(\tilde{\omega}^3)+e(\tilde{\omega}^4)e^*(\tilde{\omega}^4)).$$

$$L_{\partial_{t_2}} = \nabla_{\partial_{t_2}} + \frac{\lambda_1^2+\lambda_2^2}{2(\lambda_2^2-\lambda_1^2)}e(\tilde{\omega}^1)e^*(\tilde{\omega}^1) + \frac{\lambda_1^2\lambda_2^2+1}{2(\lambda_1^2\lambda_2^2-1)}e(\tilde{\omega}^2)e^*(\tilde{\omega}^2)$$
$$+\frac{\lambda_1^4(\lambda_2^8-1)}{\lambda_2^4(1-\lambda_1^4)^2+\lambda_1^4(1-\lambda_2^4)^2}(e(\tilde{\omega}^3)e^*(\tilde{\omega}^3)+e(\tilde{\omega}^4)e^*(\tilde{\omega}^4)).$$

Just like in section 5.2, we want to find a decomposition of $\bigwedge^2 \mathfrak{p}^*$ into $\mathfrak{k}$-irreducibles. Since $\Gamma\backslash Sp(4)/U(2)$ is a complex manifold, we want to use the complex structure on the locally symmetric space. We first need to introduce some notions from complex and Kähler geometry.

7. Complex and Kähler Geometry

In this section, we look at some results from complex and Kähler geometry. This discussion is based on Huybrechts [12].

Definition 7.1. Suppose $\{U_\alpha\}$ is an open cover for a manifold M with coordinate maps $x_\alpha : U_\alpha \mapsto \mathbb{C}^n$. M is called *complex* if the maps $x_\beta \circ x_\alpha^{-1}$ can be chosen to be holomorphic.

The Tangent Bundle. Let M be a complex manifold with $dim_{\mathbb{C}}M = n$. Let $\varphi = (z^1, ..., z^n) : U \to \mathbb{C}^n$ be a local holomorphic chart on M. Then we have real coordinates $(x^1..., x^n, y^1, ..., y^n)$ defined by $z^j = x^j + iy^j$ for each $j = 1..., n$. TM is spanned locally by the coordinate vector fields $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\}_{i=1}^n$.
It is useful to think of its complexification $TM \otimes_{\mathbb{R}} \mathbb{C}$. $TM \otimes_{\mathbb{R}} \mathbb{C}$ is locally spanned by

$$\{\frac{1}{2}(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}), \frac{1}{2}(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i})\}_{i=1}^n := \{\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^i}\}_{i=1}^n$$

.
Let

$$T_{1,0} = \langle \frac{\partial}{\partial z^i} \rangle, T_{0,1} = \langle \frac{\partial}{\partial \bar{z}^i} \rangle$$

Similarly, we have

$$T^*M_{\mathbb{C}} = T^{1,0} \oplus T^{0,1}$$

where

$$T^{1,0} = \langle dz^i \rangle, T^{0,1} = \langle d\bar{z}^i \rangle.$$

We have

$$\bigwedge^k T^*M_{\mathbb{C}} = \oplus_{p+q=k} \bigwedge^{p,q} T^*M$$

where $T^{p,q} := \bigwedge^{p,q} T^*M = \langle dz^I \wedge d\bar{z}^J \rangle$, with $|I| = p$, $|J| = q$.

Definition 7.2. The *almost complex structure* J is the endomorphism $J : TM \to TM$ satisfying $J^2 = -1$. J is defined by:

$$J\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i},$$

$$J\frac{\partial}{\partial y_i} = -\frac{\partial}{\partial x_i}.$$

Definition 7.3. Let M be complex. A Riemannian metric g on M is called *Hermitian* if $g(JX, JY) = g(X, Y)$ for any $X, Y \in TM$.

Definition 7.4. Let (M, g) be Hermitian. Set the *fundamental 2-form* to be $\omega(X, Y) := g(JX, Y)$.

Definition 7.5. A Hermitian manifold with fundamental 2-form ω is called *Kähler* if $d\omega = 0$.

Example. $\mathbb{C}^n$ is a Kähler manifold with Kähler form $\omega = dx^1 \wedge dy^1 + ... + dx^n \wedge dy^n$.

8. Representation Theory

In this section, we look at how $\bigwedge^2 \mathfrak{p}^*$ decomposes into irreducibles using the (p,q)-decomposition of the exterior power $\oplus_{p+q=k} \bigwedge^{p,q} T^*M$ described in the last section and by considering the action of $\mathfrak{su}(2)$ on $\mathfrak{p}$.

8.1. Decomposition of $\bigwedge^2 \mathfrak{p}^*$.

We first note that $Sp(4)/U(2)$ has $U(2)$ holonomy, making it a Kähler manifold. We thus get the following (p,q)-decomposition:

$$\bigwedge^2 \mathfrak{p}^* \otimes_{\mathbb{R}} \mathbb{C} = \bigwedge^{2,0} \mathfrak{p}^* \otimes_{\mathbb{R}} \mathbb{C} \oplus \bigwedge^{1,1} \mathfrak{p}^* \otimes_{\mathbb{R}} \mathbb{C} \oplus \bigwedge^{0,2} \mathfrak{p}^* \otimes_{\mathbb{R}} \mathbb{C}$$

where $\dim \bigwedge^{2,0} \mathfrak{p}^* \otimes_{\mathbb{R}} \mathbb{C} = \bigwedge^{0,2} \mathfrak{p}^* \otimes_{\mathbb{R}} \mathbb{C} = 3$ and $\dim \bigwedge^{1,1} \mathfrak{p}^* \otimes_{\mathbb{R}} \mathbb{C} = 9$.
For simplicity, we will just refer to $\mathfrak{p}^* \otimes_{\mathbb{R}} \mathbb{C}$ as $\mathfrak{p}^*$.

Choose the following orthonormal basis on $\mathfrak{p}$:

$$\{V_1, V_2, V_3, V_4, V_5, V_6\} = \{\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$
$$\frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\mathrm{diag}(1,0,-1,0), \frac{1}{\sqrt{2}}\mathrm{diag}(0,1,0,-1)\}$$

We first derive the J operator for $\mathfrak{p}$. An element of $\mathfrak{p}$ is of the form

$$\begin{pmatrix} A & B \\ B & -A \end{pmatrix},$$

A, B are real 2 by 2 matrices.

We claim that $J = \frac{1}{2}\begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$, since if we set $\frac{\partial}{\partial x} := \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}$ and $\frac{\partial}{\partial y} := \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}$, we get:

$$[J, \begin{pmatrix} A & B \\ B & -A \end{pmatrix}] = \begin{pmatrix} B & -A \\ -A & -B \end{pmatrix},$$

so J satisfies $J\frac{\partial}{\partial x} = \frac{\partial}{\partial y}$, and $J\frac{\partial}{\partial y} = -\frac{\partial}{\partial x}$.
We then have the following relations:

$$[J, V_1] = -V_2, [J, V_3] = \frac{1}{\sqrt{2}}(-V_5 + V_6), [J, V_4] = \frac{1}{\sqrt{2}}(V_5 + V_6).$$

If we set

$$\frac{\partial}{\partial z^1} := \frac{1}{2}(V_1 + iV_2), \frac{\partial}{\partial z^2} := \frac{1}{2}(V_3 - \frac{i}{\sqrt{2}}(-V_5 + V_6)), \frac{\partial}{\partial z^3} := \frac{1}{2}(V_4 - \frac{i}{\sqrt{2}}(V_5 + V_6)),$$

then $\{\frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^2} + \frac{\partial}{\partial z^3}, \frac{\partial}{\partial z^2} - \frac{\partial}{\partial z^3}\}$ is a basis for $T_{1,0}$.

Letting $\{\tilde{\omega}^1, \tilde{\omega}^2, \tilde{\omega}^3, \tilde{\omega}^4, \tilde{\omega}^5, \tilde{\omega}^6\}$ denote the dual basis to $\{V_1, V_2, V_3, V_4, V_5, V_6\}$, with $\tilde{\omega}^5 = dt_1$, $\tilde{\omega}^6 = dt_2$, we have the following basis for $\bigwedge^{1,0} \mathfrak{p}^*$:

$$\{\tilde{\omega}^1 + i\tilde{\omega}^2, \tilde{\omega}^3 + \tilde{\omega}^4 - i\sqrt{2}\tilde{\omega}^6, \tilde{\omega}^3 - \tilde{\omega}^4 + i\sqrt{2}\tilde{\omega}^5\},$$

and $\bigwedge^{0,1} \mathfrak{p}^*$:

$$\{\tilde{\omega}^1 - i\tilde{\omega}^2, \tilde{\omega}^3 + \tilde{\omega}^4 + i\sqrt{2}\tilde{\omega}^6, \tilde{\omega}^3 - \tilde{\omega}^4 - i\sqrt{2}\tilde{\omega}^5\}.$$

This gives us the following basis for $\bigwedge^{2,0} \mathfrak{p}^*$:

$$\{(\tilde{\omega}^1+i\tilde{\omega}^2)\wedge(\tilde{\omega}^3+\tilde{\omega}^4-i\sqrt{2}\tilde{\omega}^6), (\tilde{\omega}^1+i\tilde{\omega}^2)\wedge(\tilde{\omega}^3-\tilde{\omega}^4+i\sqrt{2}\tilde{\omega}^5), (\tilde{\omega}^3+\tilde{\omega}^4-i\sqrt{2}\tilde{\omega}^6)\wedge(\tilde{\omega}^3-\tilde{\omega}^4+i\sqrt{2}\tilde{\omega}^5)\}$$

and similarly the following basis for $\bigwedge^{0,2}\mathfrak{p}^*$:

$$\{(\tilde\omega^1-i\tilde\omega^2)\wedge(\tilde\omega^3+\tilde\omega^4+i\sqrt{2}\tilde\omega^6),(\tilde\omega^1-i\tilde\omega^2)\wedge(\tilde\omega^3-\tilde\omega^4-i\sqrt{2}\tilde\omega^5),(\tilde\omega^3+\tilde\omega^4+i\sqrt{2}\tilde\omega^6)\wedge(\tilde\omega^3-\tilde\omega^4-i\sqrt{2}\tilde\omega^5)\}.$$

We have that $\bigwedge^{1,1}\mathfrak{p}^* = \bigwedge_0^{1,1}\mathfrak{p}^*\oplus\langle\omega\rangle$ where $\langle\omega\rangle^\perp = \bigwedge_0^{1,1}\mathfrak{p}^*$. A basis for $\bigwedge_0^{1,1}\mathfrak{p}^*$ is given by:

$$\{2(\tilde\omega^1\wedge\tilde\omega^2)+\sqrt{2}[(\tilde\omega^3+\tilde\omega^4)\wedge\tilde\omega^6],(\tilde\omega^3+\tilde\omega^4)\wedge\tilde\omega^6+(\tilde\omega^3-\tilde\omega^4)\wedge\tilde\omega^5,(\tilde\omega^1+i\tilde\omega^2)\wedge(\tilde\omega^3+\tilde\omega^4+i\sqrt{2}\tilde\omega^6),$$
$$(\tilde\omega^1+i\tilde\omega^2)\wedge(\tilde\omega^3-\tilde\omega^4-i\sqrt{2}\tilde\omega^5),(\tilde\omega^3+\tilde\omega^4-i\sqrt{2}\tilde\omega^6)\wedge(\tilde\omega^1-i\tilde\omega^2),(\tilde\omega^3+\tilde\omega^4-i\sqrt{2}\tilde\omega^6)\wedge(\tilde\omega^3-\tilde\omega^4-i\sqrt{2}\tilde\omega^5),$$
$$(\tilde\omega^3-\tilde\omega^4+i\sqrt{2}\tilde\omega^5)\wedge(\tilde\omega^1-i\tilde\omega^2),(\tilde\omega^3-\tilde\omega^4+i\sqrt{2}\tilde\omega^5)\wedge(\tilde\omega^3+\tilde\omega^4+i\sqrt{2}\tilde\omega^6)\}$$

8.2. Action of $\mathfrak{su}(2)$ on $\bigwedge_0^{1,1}\mathfrak{p}^*$.

In this subsection, we consider $\bigwedge^2$ of the adjoint action of $\mathfrak{su}(2)$ on $\bigwedge_0^{1,1}\mathfrak{p}^*$, and prove that $\bigwedge_0^{1,1}\mathfrak{p}^* = \mathbf{3}\oplus\mathbf{5}$ under this action.

Consider the following basis for $\mathfrak{su}(2)$, embedded in $\mathfrak{u}(2)$:

$$\{X_1,X_2,X_3\} = \{\frac{1}{2}\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix},\frac{1}{2}\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix},\frac{1}{2}\begin{pmatrix}0&0&-1&0\\0&0&0&1\\1&0&0&0\\0&-1&0&0\end{pmatrix}\}$$

Let $\langle X_3\rangle$ be the Cartan for $\mathfrak{su}(2)$. On $\mathfrak{p}\otimes_{\mathbb{R}}\mathbb{C}$, denote the eigenspace of X_3 with complex eigenvalue ci for $c=-1,0,-1$ by $\tilde{X}_c$. The X_3-eigenspaces on $\mathfrak{p}\otimes_{\mathbb{R}}\mathbb{C}$ are then:

$$W_0 = \operatorname{span}\{V_1,V_2\}$$
$$W_1 = \operatorname{span}\{V_3+V_4-i\sqrt{2}V_6, V_3-V_4-i\sqrt{2}V_5\}$$
$$W_{-1} = \operatorname{span}\{V_3+V_4+i\sqrt{2}V_6, V_3-V_4+i\sqrt{2}V_5\},$$

where a lowering operator is X_1-iX_2.

Apply $\operatorname{ad}(X_1-iX_2)$ to the basis of $\bigwedge_0^{1,1}\mathfrak{p}\otimes_{\mathbb{R}}\mathbb{C}$, starting with the highest weight vector $(V_3+V_4-i\sqrt{2}V_6)\wedge(V_3-V_4-i\sqrt{2}V_5)$. Identify $\mathfrak{p}$ with its dual $\mathfrak{p}^*$. A similar computation to section 5.2 gives us $\bigwedge_0^{1,1}\mathfrak{p}^* = \mathbf{5}\oplus\mathbf{3}$, with the following bases.

Basis for $\mathbf{5}$:

$$\{(\tilde\omega^3+\tilde\omega^4-i\sqrt{2}\tilde\omega^6)\wedge(\tilde\omega^3-\tilde\omega^4-i\sqrt{2}\tilde\omega^5),(\tilde\omega^1+i\tilde\omega^2)\wedge(\tilde\omega^3-\tilde\omega^4-i\sqrt{2}\tilde\omega^5)+(\tilde\omega^1-i\tilde\omega^2)\wedge(\tilde\omega^3+\tilde\omega^4-i\sqrt{2}\tilde\omega^6)$$
$$4(\tilde\omega^1\wedge\tilde\omega^2)-\sqrt{2}(\tilde\omega^3-\tilde\omega^4)\wedge\tilde\omega^5+\sqrt{2}(\tilde\omega^3+\tilde\omega^4)\wedge\tilde\omega^6,$$
$$(\tilde\omega^1-i\tilde\omega^2)\wedge(\tilde\omega^3-\tilde\omega^4+i\sqrt{2}\tilde\omega^5)+(\tilde\omega^1+i\tilde\omega^2)\wedge(\tilde\omega^3+\tilde\omega^4+i\sqrt{2}\tilde\omega^6)$$
$$(\tilde\omega^3+\tilde\omega^4+i\sqrt{2}\tilde\omega^6)\wedge(\tilde\omega^3-\tilde\omega^4+i\sqrt{2}\tilde\omega^5)\}.$$

Basis for $\mathbf{3}$:

$$\{(\tilde\omega^1+i\tilde\omega^2)\wedge(\tilde\omega^3-\tilde\omega^4-i\sqrt{2}\tilde\omega^5)-(\tilde\omega^1-i\tilde\omega^2)\wedge(\tilde\omega^3+\tilde\omega^4+i\sqrt{2}\tilde\omega^6),$$

$$(\tilde\omega^3 - \tilde\omega^4) \wedge \tilde\omega^5 + (\tilde\omega^3 + \tilde\omega^4) \wedge \tilde\omega^6,$$
$$(\tilde\omega^1 - i\tilde\omega^2) \wedge (\tilde\omega^3 - \tilde\omega^4 + i\sqrt{2}\tilde\omega^5) - (\tilde\omega^1 + i\tilde\omega^2) \wedge (\tilde\omega^3 + \tilde\omega^4 - i\sqrt{2}\tilde\omega^6)\}.$$

Remark: $\bigwedge^{2,0}$ and $\bigwedge^{0,2}$ are irreducible under $\bigwedge^2$ of the adjoint action of $\mathfrak{su}(2)$.

8.3. **A Basis of 2-forms for Irreducible Summands.**

We summarize the results from our computations of bases vectors below:

(i) $\langle\omega\rangle = 2\tilde\omega^1 \wedge \tilde\omega^2 - \sqrt{2}\tilde\omega^3 \wedge \tilde\omega^6 - \sqrt{2}\tilde\omega^4 \wedge \tilde\omega^6 + \sqrt{2}\tilde\omega^3 \wedge \tilde\omega^5 - \sqrt{2}\tilde\omega^4 \wedge \tilde\omega^5$.

(ii) Basis for $\bigwedge^{2,0} \oplus \bigwedge^{0,2}$:

$$\{\tilde\omega^1\wedge\tilde\omega^3+\tilde\omega^1\wedge\tilde\omega^4+\sqrt{2}\tilde\omega^2\wedge\tilde\omega^6,\ \tilde\omega^2\wedge\tilde\omega^3+\tilde\omega^2\wedge\tilde\omega^4-\sqrt{2}\tilde\omega^1\wedge\tilde\omega^6,\ \tilde\omega^1\wedge\tilde\omega^3-\tilde\omega^1\wedge\tilde\omega^4-\sqrt{2}\tilde\omega^2\wedge\tilde\omega^5,$$
$$\sqrt{2}\tilde\omega^1 \wedge \tilde\omega^5 + \tilde\omega^2 \wedge \tilde\omega^3 - \tilde\omega^2 \wedge \tilde\omega^4,\ \tilde\omega^3 \wedge \tilde\omega^4 + \tilde\omega^5 \wedge \tilde\omega^6,$$
$$\tilde\omega^3 \wedge \tilde\omega^5 + \tilde\omega^4 \wedge \tilde\omega^5 + \tilde\omega^3 \wedge \tilde\omega^6 - \tilde\omega^4 \wedge \tilde\omega^6\}.$$

(iii) Basis for **5** in $\bigwedge_0^{1,1}$:

$$\{\tilde\omega^3 \wedge \tilde\omega^4 - \tilde\omega^5 \wedge \tilde\omega^6,\ \tilde\omega^3 \wedge \tilde\omega^5 - \tilde\omega^3 \wedge \tilde\omega^6 + \tilde\omega^4 \wedge \tilde\omega^5 + \tilde\omega^4 \wedge \tilde\omega^6,$$
$$2\tilde\omega^1 \wedge \tilde\omega^3 + \sqrt{2}\tilde\omega^2 \wedge \tilde\omega^5 - \sqrt{2}\tilde\omega^2 \wedge \tilde\omega^6,\ 2\tilde\omega^2 \wedge \tilde\omega^4 + \sqrt{2}\tilde\omega^1 \wedge \tilde\omega^6 - \sqrt{2}\tilde\omega^1 \wedge \tilde\omega^5,$$
$$4\tilde\omega^1 \wedge \tilde\omega^2 - \sqrt{2}\tilde\omega^3 \wedge \tilde\omega^5 + \sqrt{2}\tilde\omega^4 \wedge \tilde\omega^5 + \sqrt{2}\tilde\omega^3 \wedge \tilde\omega^6 + \sqrt{2}\tilde\omega^4 \wedge \tilde\omega^6\}.$$

(iv) Basis for **3** in $\bigwedge_0^{1,1}$:

$$\{2\tilde\omega^1 \wedge \tilde\omega^4 - \sqrt{2}\tilde\omega^2 \wedge \tilde\omega^5 - \sqrt{2}\tilde\omega^2 \wedge \tilde\omega^6,\ 2\tilde\omega^2 \wedge \tilde\omega^3 - \sqrt{2}\tilde\omega^1 \wedge \tilde\omega^5 + \sqrt{2}\tilde\omega^1 \wedge \tilde\omega^6,$$
$$\tilde\omega^3 \wedge \tilde\omega^5 - \tilde\omega^4 \wedge \tilde\omega^5 + \tilde\omega^3 \wedge \tilde\omega^6 + \tilde\omega^4 \wedge \tilde\omega^6\}.$$

9. On Betti Numbers of $\Gamma\backslash Sp(4)/U(2)$

In this section, we compute bounds on the normalized second Betti numbers. Consider the Lie derivative with respect to the vector field $\partial_{t_1} + \partial_{t_2}$.

We have:

$$L_{\partial_{t_1}+\partial_{t_2}} = \nabla_{\partial_{t_1}+\partial_{t_2}} + \frac{\lambda_1^2\lambda_2^2+1}{\lambda_1^2\lambda_2^2-1}e(\tilde\omega^2)e^*(\tilde\omega^2) + \frac{\lambda_1^4(\lambda_2^8-1)+\lambda_2^4(\lambda_1^8-1)}{\lambda_2^4(1-\lambda_1^4)^2+\lambda_1^4(1-\lambda_2^4)^2}[e(\tilde\omega^3)e^*(\tilde\omega^3)+e(\tilde\omega^4)e^*(\tilde\omega^4)]$$

Set

$$H = \frac{\lambda_1^2\lambda_2^2+1}{\lambda_1^2\lambda_2^2-1} + 2\left(\frac{\lambda_1^4(\lambda_2^8-1)+\lambda_2^4(\lambda_1^8-1)}{\lambda_2^4(1-\lambda_1^4)^2+\lambda_1^4(1-\lambda_2^4)^2}\right),$$

$$Q = \frac{\lambda_1^2\lambda_2^2+1}{\lambda_1^2\lambda_2^2-1}e(\tilde\omega^2)e^*(\tilde\omega^2) + \frac{\lambda_1^4(\lambda_2^8-1)+\lambda_2^4(\lambda_1^8-1)}{\lambda_2^4(1-\lambda_1^4)^2+\lambda_1^4(1-\lambda_2^4)^2}[e(\tilde\omega^3)e^*(\tilde\omega^3)+e(\tilde\omega^4)e^*(\tilde\omega^4)],$$

and as before

$$q_h(r) := \frac{\int_{K\Omega_r} \langle (\frac{H}{2} - Q)h, h\rangle}{\int_{K\Omega_r} |h|^2}.$$

A computation yields the following lower bounds on $q_h(r)$:

$$q_h(r) \geq \begin{cases} 1 \text{ on } \mathcal{H}^2_3,\ \mathcal{H}^2_{3\oplus\bar{3}} \\ \frac{1}{4} \text{ on } \mathcal{H}^2_5. \end{cases}$$

This gives us the following result on bounds on the second Betti numbers:

Theorem 9.1. Let $M = \Gamma \setminus Sp(4)/U(2)$, and $R = \frac{\text{inj}_M}{\sqrt{2}}$. Then there exists a positive constant $d(6) > 0$ such that:

$$\begin{cases} \frac{b_5^{2,Sp(4)}}{Vol(M)} \leq 4d(6)e^{-R/2} \\ \frac{b_3^{2,Sp(4)}}{Vol(M)} \leq d(6)e^{-2R} \\ \frac{b_{3\oplus\bar{3}}^{2,Sp(4)}}{Vol(M)} \leq d(6)e^{-2R}. \end{cases}$$

To prove Theorem 9.1, we first need the following proposition, taken from Proposition 16 in [5]:

Proposition 9.1. Let (M, g) be a Riemannian manifold. Let $R = \frac{\text{inj}_M}{\sqrt{2}}$. Suppose $\sigma < \tau \leq R$. Define $\phi(\tau) := \exp\left(-\int_\sigma^\tau \frac{q_h(s)ds}{\frac{1}{2} - \mu_h(s)}\right)$.
Then for any strongly harmonic $k-$form $h \in \mathcal{H}^k(M)$ and for any $\sigma < \tau \leq R$, we have the equality:

$$\phi(\sigma)\int_{K\Omega_\sigma} q_h(r)|h|^2 dvol = \phi(\tau)\int_{K\Omega_\tau} q_h(r)|h|^2 dvol.$$

Proof. Proof of Theorem 9.1: We compute bounds for $\mathcal{H}^2_{3\oplus\bar{3}}$ and $\mathcal{H}^2_3$. From the lower bounds on $q_h(r)$, and the previous proposition, we get the following monotonicity estimates:

$$\phi(\tau)\int_{K\Omega_\tau} q_h(r)|h|^2 dvol = \int_{K\Omega_1} q_h(r)|h|^2 dvol \geq \int_{K\Omega_1} |h|^2 dvol.$$

We know that $\mu_h(r) \leq \frac{1}{2}$ since $q_h(r) \geq 0$. Integrating the lower bounds on $q_h(r)$, and applying Proposition 9.1 gives:

$$d(6)\int_{K\Omega_1} |h|^2 dvol \leq d(6)e^{-2R}\int_{K\Omega_R} |h|^2 dvol.$$

Applying Lemmas 6.2 and 6.3, we obtain the following desired bound:

$$\frac{b_3^{2,Sp(4)}}{Vol(M)} \leq \binom{6}{2}\max_{p\in M}|h|^2 \leq d(6)\int_{K\Omega_1} |h|^2 dvol \leq d(6)e^{-2R}\int_{K\Omega_R} |h|^2 dvol \leq d(6)e^{-2R}.$$

Repeat the same computation for $b^{2,Sp(4)}_{3\oplus\bar{3}}$ and $b^{2,Sp(4)}_{5}$. □

10. On G_2 and its Lie Algebra

In this section, we introduce the exceptional Lie group G_2 and introduce its real forms and Lie algebra.

10.1. **Real Forms of G_2.**

This discussion is based on Baez [1] and Harvey [10]:

G_2, an exceptional Lie group, has 2 real forms, the compact form $G_{2(-14)}$, and the non-compact form $G_{2(2)}$. An explicit description of these forms are as follows:

$$G_{2(-14)} = \mathrm{Aut}(\mathbb{O}) \subset \mathrm{SO}(\mathrm{Im}(\mathbb{O})) = \mathrm{SO}(7)$$

$$G_{2(2)} = \mathrm{Aut}(\tilde{\mathbb{O}}) \subset \mathrm{SO}(\mathrm{Im}(\tilde{\mathbb{O}})) = \mathrm{SO}(3,4),$$

where $\mathbb{O}$ refers to the octonions and $\tilde{\mathbb{O}}$ refers to the split Cayley octonions, defined by the following multiplication:

Suppose we have $a, b, c, d \in \mathbb{H}$, then

$$\begin{cases} (a,b)(c,d) = (ac + \bar{d}b, da + b\bar{c}), \text{ on } \tilde{\mathbb{O}} \\ (a,b)(c,d) = (ac - \bar{d}b, da + b\bar{c}), \text{ on } \mathbb{O}. \end{cases}$$

Equivalently, let $e_1, ..., e_7$ denote the standard basis of $\mathbb{R}^7$, and $\omega^1, ..., \omega^7$ be the corresponding dual basis. Write $\omega^{ijk} := \omega^i \wedge \omega^j \wedge \omega^k$.

Definition 10.1. Define

$$\phi := \omega^{123} + \omega^{145} + \omega^{167} + \omega^{246} - \omega^{257} - \omega^{347} - \omega^{356},$$

then

$$G_{2(-14)} := \{g \in \mathrm{GL}(7, \mathbb{R}) | g^*(\phi) = \phi\}$$

Definition 10.2. Define

$$\tilde{\phi} := -\omega^{123} + \omega^{145} + \omega^{167} + \omega^{246} - \omega^{257} - \omega^{347} - \omega^{356},$$

then

$$G_{2(2)} := \{g \in \mathrm{GL}(7, \mathbb{R}) | g^*(\tilde{\phi}) = \tilde{\phi}\}.$$

Remark: ϕ and $\tilde{\phi}$ differ by a single minus sign in one of the terms.

10.2. **Lie Algebra of $G_{2(2)}$.**

For the rest of the paper, we focus on the non-compact split form $G_{2(2)}$. We give an explicit description of its Lie algebra, denoted by $\mathfrak{g}_2$. This discussion is based on Bor [3].

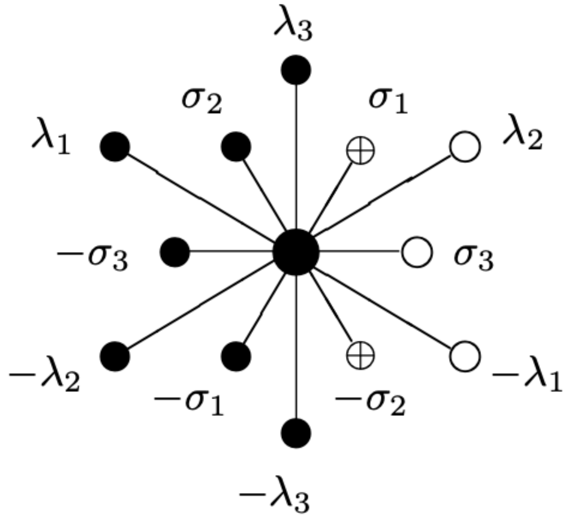

FIGURE 2. Root Diagram of $\mathfrak{g}_2$

Let $\mathfrak{g}_2$ be the lie algebra of $G_{2(2)}$. An explicit matrix realization that first appeared in Elie Cartan's 1894 thesis is given by the set of 7×7 matrices of the form:

$$\begin{pmatrix} A & \Omega_{\mathbf{c}} & -2\mathbf{b} \\ \Omega_{\mathbf{b}} & -A^t & -2\mathbf{c} \\ \mathbf{c}^t & \mathbf{b}^t & 0 \end{pmatrix}$$

where $A \in \mathfrak{sl}(3, \mathbb{R})$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^3$, and for each $\mathbf{u} = (u_1, u_2, u_3)^t \in \mathbb{R}^3$, let $\Omega_{\mathbf{u}}$ denote the skew-symmetric 3×3 matrix

$$\Omega_{\mathbf{u}} = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}$$

Its root diagram is given in Figure 2, which is taken from [3]. Furthermore, $G_{2(2)}$ contains a 6-dimensional maximal compact subgroup K, a double cover of $\mathrm{SO}(3) \times \mathrm{SO}(3)$, in which we typically make the identification $K \cong \mathrm{SO}(4)$, with lie algebra $\mathfrak{k}$ given by

$$\begin{pmatrix} \Omega_{\mathbf{a}} & \Omega_{\mathbf{b}} & -2\mathbf{b} \\ \Omega_{\mathbf{b}} & \Omega_{\mathbf{a}} & -2\mathbf{b} \\ \mathbf{b}^t & \mathbf{b}^t & 0 \end{pmatrix}$$

We will use this matrix realization in our computations, and later show in a proposition that $\mathfrak{k} \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

11. On the Geometry of $\Gamma \backslash G_{2(2)}/\mathrm{SO}(4)$

In this section, we study the geometry of compact quotients of $G_{2(2)}/\mathrm{SO}(4)$, an 8-dimensional rank 2 locally symmetric space using similar techniques as the previous cases.

11.1. **Metric Computation.**

Choose the following basis for $\mathfrak{k}$:

$$X_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, X_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_3 = \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_4 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, X_5 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$X_6 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and the following basis for the abelian subalgebra $\mathfrak{a}$ of $\mathfrak{g}_2$:

$$A_1 = \frac{1}{2}\mathrm{diag}(1,-1,0,-1,1,0,0),\ A_2 = \frac{1}{2\sqrt{3}}\mathrm{diag}(1,1,-2,-1,-1,2,0)$$

Now, let $G = G_{2(2)}$, $K = SO(4)$, then if we set $A^+ = \{\mathrm{diag}(\lambda_1, \lambda_2, 1/\lambda_1\lambda_2, 1/\lambda_1, 1/\lambda_2, \lambda_1\lambda_2, 1) : \lambda_1 > \lambda_2 > 1\}$, then $G = KA^+K$.

Denote the dual coframe of $\{X_1, ..., X_6, A_1, A_2\}$ by $\{\omega^1, ..., \omega^6, \mathrm{d}\ln\lambda_1, \mathrm{d}\ln\lambda_2\}$. By a similar computation as Proposition 4.1, we have the following metric:

Proposition 11.1. The metric on $G_{2(2)}/SO(4)$ is:

$$g = \frac{(\lambda_1^2\lambda_2^4-1)^2}{\lambda_1^2\lambda_2^4}(\omega^1)^2 + \frac{(\lambda_1^4\lambda_2^2-1)^2}{\lambda_1^4\lambda_2^2}(\omega^2)^2 + \frac{(\lambda_1^2-\lambda_2^2)^2}{\lambda_1^2\lambda_2^2}(\omega^3)^2 + \frac{13(\lambda_1^2-1)^2}{12\lambda_1^2}(\omega^4)^2$$
$$+\frac{13(\lambda_2^2-1)^2}{12\lambda_2^2}(\omega^5)^2 + \frac{13(\lambda_1^2\lambda_2^2-1)^2}{12\lambda_1^2\lambda_2^2}(\omega^6)^2 + \frac{4}{\lambda_1^2}d\lambda_1^2 + \frac{4}{\lambda_2^2}d\lambda_2^2.$$

11.2. **Lie Derivatives.**

Now, we convert our basis into an orthonormal one.
Take:

$$E_1 = \frac{\lambda_1\lambda_2^2}{\lambda_1^2\lambda_2^4-1}X_1, E_2 = \frac{\lambda_2\lambda_1^2}{\lambda_2^2\lambda_1^4-1}X_2, E_3 = \frac{\lambda_1\lambda_2}{\lambda_1^2-\lambda_2^2}X_3,$$

$$E_4 = \sqrt{\frac{12}{13}}\frac{\lambda_1}{\lambda_1^2-1}X_4, E_5 = \sqrt{\frac{12}{13}}\frac{\lambda_2}{\lambda_2^2-1}X_5, E_6 = \sqrt{\frac{12}{13}}\frac{\lambda_1\lambda_2}{\lambda_1^2\lambda_2^2-1}X_4,$$

$$\partial_{t_1} = \frac{\lambda_1}{2}\partial_{\lambda_1},\ \partial_{t_2} = \frac{\lambda_2}{2}\partial_{\lambda_2}.$$

Let $\{\tilde{\omega}^1, ..., \tilde{\omega}^6, dt_1, dt_2\}$ be the corresponding dual coframe. We get the following Lie derivatives:

Proposition 11.2. The Lie derivatives along ∂_{t_1} and ∂_{t_2} are given by:

$$L_{\partial_{t_1}} = \nabla_{\partial_{t_1}} + \frac{\lambda_1^2\lambda_2^4+1}{2(\lambda_1^2\lambda_2^4-1)}e(\tilde{\omega}^1)e^*(\tilde{\omega}^1) + \frac{\lambda_1^4\lambda_2^2+1}{\lambda_1^4\lambda_2^2-1}e(\tilde{\omega}^2)e^*(\tilde{\omega}^2) + \frac{\lambda_1^2+\lambda_2^2}{2(\lambda_1^2-\lambda_2^2)}e(\tilde{\omega}^3)e^*(\tilde{\omega}^3) +$$
$$\frac{\lambda_1^2+1}{2(\lambda_1^2-1)}e(\tilde{\omega}^4)e^*(\tilde{\omega}^4) + \frac{\lambda_1^2\lambda_2^2+1}{2(\lambda_1^2\lambda_2^2-1)}e(\tilde{\omega}^6)e^*(\tilde{\omega}^6).$$

$$L_{\partial_{t_2}} = \nabla_{\partial_{t_2}} + \frac{\lambda_1^2\lambda_2^4+1}{\lambda_1^2\lambda_2^4-1}e(\tilde{\omega}^1)e^*(\tilde{\omega}^1) + \frac{\lambda_1^4\lambda_2^2+1}{2(\lambda_1^4\lambda_2^2-1)}e(\tilde{\omega}^2)e^*(\tilde{\omega}^2) + \frac{\lambda_1^2+\lambda_2^2}{2(\lambda_2^2-\lambda_1^2)}e(\tilde{\omega}^3)e^*(\tilde{\omega}^3) +$$
$$\frac{\lambda_2^2+1}{2(\lambda_2^2-1)}e(\tilde{\omega}^5)e^*(\tilde{\omega}^5) + \frac{\lambda_1^2\lambda_2^2+1}{2(\lambda_1^2\lambda_2^2-1)}e(\tilde{\omega}^6)e^*(\tilde{\omega}^6).$$

12. Representation Theory

In this section, we look at how $\bigwedge^2 \mathfrak{p}^*$ decomposes into irreducibles considering the action of $\mathfrak{k}$ on $\mathfrak{p}$.

First, we give a proposition:

Proposition 12.1. $\mathfrak{k} \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

Proof. We have the following commutation relations:

$$[X_1, X_2] = X_3,\ [X_1, X_3] = -X_2,\ [X_1, X_4] = 0,\ [X_1, X_5] = X_6,\ [X_1, X_6] = -X_5$$
$$[X_2, X_3] = X_1,\ [X_2, X_4] = -X_6,\ [X_2, X_5] = 0,\ [X_2, X_6] = X_4$$
$$[X_3, X_4] = X_5,\ [X_3, X_5] = -X_4,\ [X_3, X_6] = 0$$
$$[X_4, X_5] = 3X_3 + 2X_6,\ [X_4, X_6] = -3X_2 - 2X_5$$
$$[X_5, X_6] = 3X_1 + 2X_4$$

Set:

$$\tilde{X}_1 = \frac{1}{4}(X_1 + X_4),\ \tilde{X}_2 = \frac{1}{4}(X_2 + X_5),\ \tilde{X}_3 = \frac{1}{4}(X_3 + X_6), \text{ and}$$

$$Y_1 = \frac{3}{4}X_1 - \frac{1}{4}X_4,\ Y_2 = \frac{3}{4}X_2 - \frac{1}{4}X_5,\ Y_3 = \frac{3}{4}X_3 - \frac{1}{4}X_6,$$

then we get:

$$[\tilde{X}_1, \tilde{X}_2] = \tilde{X}_3,\ [\tilde{X}_2, \tilde{X}_3] = \tilde{X}_1,\ [\tilde{X}_3, \tilde{X}_1] = \tilde{X}_2, \text{ and}$$

$$[Y_1, Y_2] = Y_3,\ [Y_2, Y_3] = Y_1,\ [Y_3, Y_1] = Y_2,$$

with $[\tilde{X}_i, Y_j] = 0$ for all i, j. □

We can now take the Cartan subalgebra of $\mathfrak{k}$ to be $\text{span}(\tilde{X}_3, Y_3)$.

12.1. **Action of $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ on $\mathfrak{p} \otimes_{\mathbb{R}} \mathbb{C}$.**

The matrix realization of $\mathfrak{p}$ is given by

$$\begin{pmatrix} A & \Omega_{\mathbf{b}} & 2\mathbf{b} \\ \Omega_{\mathbf{b}}^t & -A^t & 2\mathbf{b} \\ \mathbf{b}^t & -\mathbf{b}^t & 0 \end{pmatrix}$$

where A is a symmetric traceless 3×3 matrix, $\mathbf{b} \in \mathbb{R}^3$.

Choose the following orthonormal basis for $\mathfrak{p}$:

$$V_1 = \begin{pmatrix} 0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&-1&0 \\ 0&0&0&0&-1&0&0 \\ 0&0&0&0&0&0&0 \end{pmatrix}, V_2 = \begin{pmatrix} 0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0 \\ 0&0&0&0&0&-1&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&-1&0&0&0 \\ 0&0&0&0&0&0&0 \end{pmatrix},$$

$$V_3 = \begin{pmatrix} 0&1&0&0&0&0&0 \\ 1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&-1&0&0 \\ 0&0&0&-1&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \end{pmatrix},$$

$$V_4 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0&0&0&0&0&0&2 \\ 0&0&0&0&0&-1&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&-2 \\ 0&0&1&0&0&0&0 \\ 0&-1&0&0&0&0&0 \\ 1&0&0&-1&0&0&0 \end{pmatrix}, V_5 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&2 \\ 0&0&0&-1&0&0&0 \\ 0&0&-1&0&0&0&0 \\ 0&0&0&0&0&0&-2 \\ 1&0&0&0&0&0&0 \\ 0&1&0&0&-1&0&0 \end{pmatrix},$$

$$V_6 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$V_7 = \frac{1}{2}\mathrm{diag}(1,-1,0,-1,1,0,0),\ V_8 = \frac{1}{2\sqrt{3}}\mathrm{diag}(1,1,-2,-1,-1,2,0).$$

Let $V_{(a,b)}$ be the eigenspace corresponding to the set of matrices $M \in \mathfrak{p} \otimes_{\mathbb{R}} \mathbb{C}$ such that $[\tilde{X}_3, M] = aiM$, and $[Y_3, M] = biM$.

A computation then yields the following $(\tilde{X}_3, Y_3)$-eigenspaces for $\mathfrak{p} \otimes_{\mathbb{R}} \mathbb{C}$:

$$\begin{gathered} V_{(1,3)} = \mathrm{span}\{V_3 + iV_7\},\ V_{(-1,3)} = \mathrm{span}\{(V_1 - \sqrt{3}V_4) + i(V_2 + \sqrt{3}V_5)\}, \\ V_{(-1,1)} = \mathrm{span}\{V_6 + iV_8\},\ V_{(-1,-1)} = \mathrm{span}\{(\sqrt{3}V_1 + V_4) - i(\sqrt{3}V_2 - V_5)\}, \\ V_{(-1,-3)} = \mathrm{span}\{V_3 - iV_7\}, V_{(1,-3)} = \mathrm{span}\{(V_1 - \sqrt{3}V_4) - i(V_2 + \sqrt{3}V_5)\}, \\ V_{(1,-1)} = \mathrm{span}\{V_6 - iV_8\},\ V_{(1,1)} = \mathrm{span}\{(\sqrt{3}V_1 + V_4) + i(\sqrt{3}V_2 - V_5)\}. \end{gathered}$$

The weight diagram is as follows:

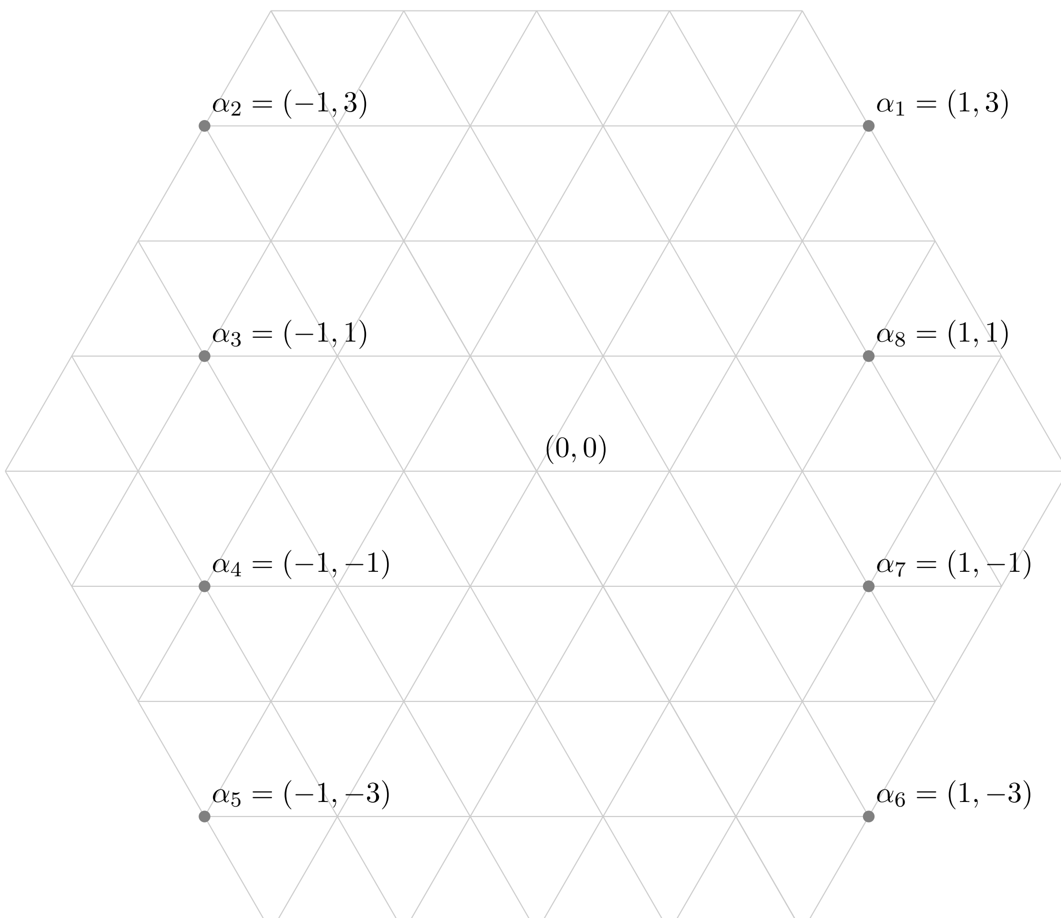

Figure 3: Weights of the Action of $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ on $\mathfrak{p} \otimes_{\mathbb{R}} \mathbb{C}$.

12.2. Decomposition of $\bigwedge^2 \mathfrak{p}^*$ into Irreducible Summands.

From the above weight diagram, we get the following weight diagram for $\bigwedge^2$ of the adjoint action of $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ on $\bigwedge^2 \mathfrak{p}$:

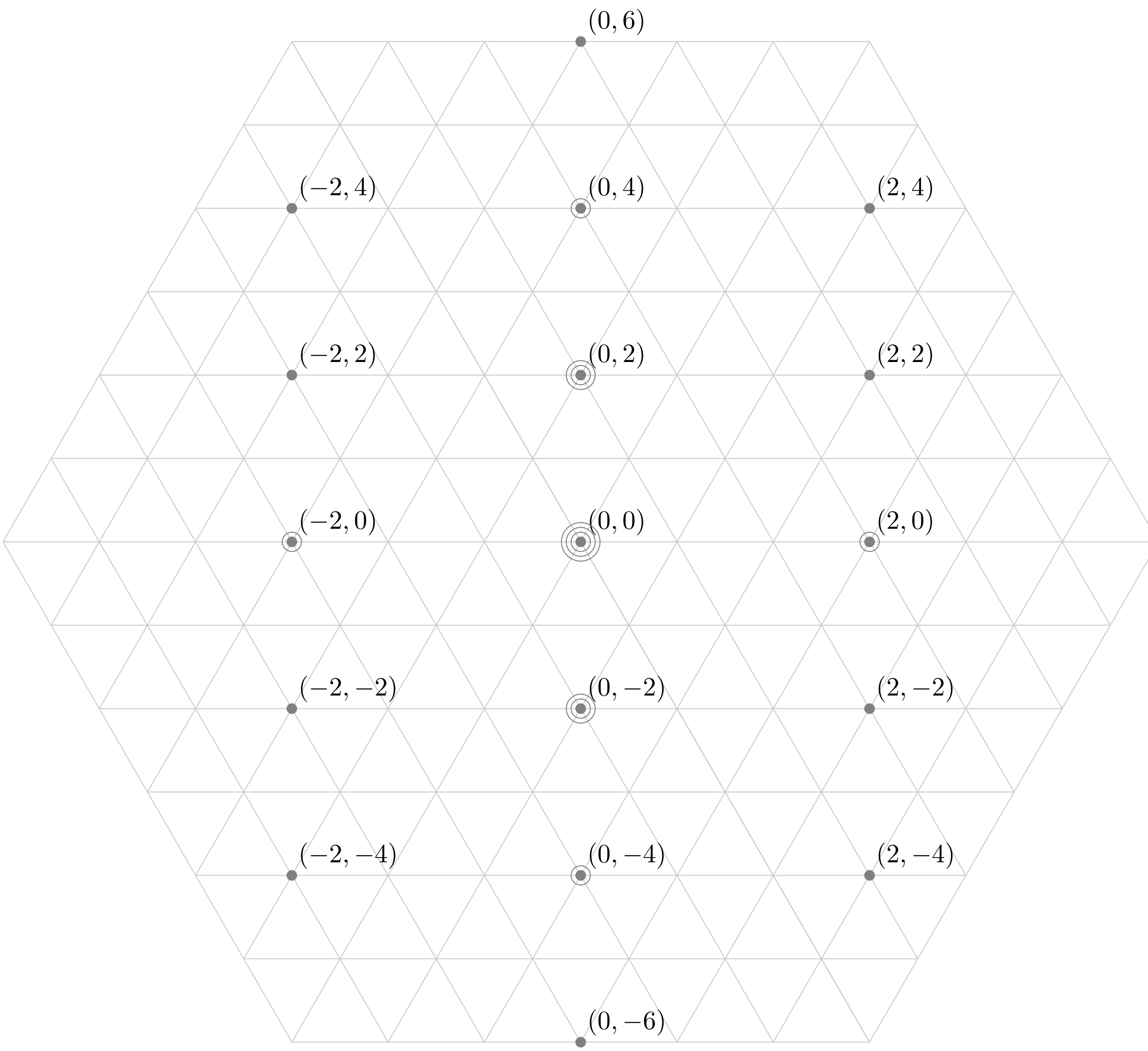

Figure 4: Weights of $\bigwedge^2$ of the Adjoint Action of $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ on $\bigwedge^2 \mathfrak{p} \otimes_{\mathbb{R}} \mathbb{C}$.

This gives us the following decomposition:

$$\bigwedge^2 \mathfrak{p} \otimes_{\mathbb{R}} \mathbb{C} = (3,5) \oplus (1,7) \oplus (1,3) \oplus (3,1) = \mathbf{15} \oplus \mathbf{7} \oplus \mathbf{3'} \oplus \mathbf{3}.$$

We get the corresponding decomposition of the space of harmonic 2-forms:

$$\mathcal{H}^2(\Gamma \backslash G_{2(2)}/SO(4)) = \mathcal{H}^2_{15} \oplus \mathcal{H}^2_{7} \oplus \mathcal{H}^2_{3'} \oplus \mathcal{H}^2_{3}.$$

We now compute a basis for each of these irreducible summands.

Let $\{\tilde{\omega}^1, ..., \tilde{\omega}^8\}$ denote the dual coframe to $\{V_1, ..., V_8\}$, where $\tilde{\omega}^7 = dt_1$ and $\tilde{\omega}^8 = dt_2$. Then, we get

(i) Basis for (3,5):

$$\{-\sqrt{3}\tilde{\omega}^1\wedge\tilde{\omega}^3+\tilde{\omega}^3\wedge\tilde{\omega}^4+\sqrt{3}\tilde{\omega}^2\wedge\tilde{\omega}^7-\tilde{\omega}^5\wedge\tilde{\omega}^7,\ \sqrt{3}\tilde{\omega}^1\wedge\tilde{\omega}^7+\tilde{\omega}^4\wedge\tilde{\omega}^7+\sqrt{3}\tilde{\omega}^2\wedge\tilde{\omega}^3+\tilde{\omega}^3\wedge\tilde{\omega}^5,$$
$$4\tilde{\omega}^1\wedge\tilde{\omega}^4+4\tilde{\omega}^2\wedge\tilde{\omega}^5+\tilde{\omega}^3\wedge\tilde{\omega}^6-\tilde{\omega}^7\wedge\tilde{\omega}^8,\ -4\tilde{\omega}^1\wedge\tilde{\omega}^5+4\tilde{\omega}^2\wedge\tilde{\omega}^4+\tilde{\omega}^3\wedge\tilde{\omega}^8-\tilde{\omega}^6\wedge\tilde{\omega}^7,$$
$$\tilde{\omega}^1\wedge\tilde{\omega}^6-\sqrt{3}\tilde{\omega}^4\wedge\tilde{\omega}^6-\tilde{\omega}^2\wedge\tilde{\omega}^8-\sqrt{3}\tilde{\omega}^5\wedge\tilde{\omega}^8,\ \tilde{\omega}^1\wedge\tilde{\omega}^8-\sqrt{3}\tilde{\omega}^4\wedge\tilde{\omega}^8+\tilde{\omega}^2\wedge\tilde{\omega}^6+\sqrt{3}\tilde{\omega}^5\wedge\tilde{\omega}^6,$$
$$\tilde{\omega}^1\wedge\tilde{\omega}^4-\tilde{\omega}^2\wedge\tilde{\omega}^5,\ \sqrt{3}\tilde{\omega}^1\wedge\tilde{\omega}^2+2\tilde{\omega}^1\wedge\tilde{\omega}^5+\tilde{\omega}^2\wedge\tilde{\omega}^4+\sqrt{3}\tilde{\omega}^4\wedge\tilde{\omega}^5,$$
$$\tilde{\omega}^1\wedge\tilde{\omega}^3+\sqrt{3}\tilde{\omega}^3\wedge\tilde{\omega}^4+\tilde{\omega}^2\wedge\tilde{\omega}^7+\sqrt{3}\tilde{\omega}^5\wedge\tilde{\omega}^7-\sqrt{3}\tilde{\omega}^1\wedge\tilde{\omega}^6-\tilde{\omega}^4\wedge\tilde{\omega}^6-\sqrt{3}\tilde{\omega}^2\wedge\tilde{\omega}^8+\tilde{\omega}^5\wedge\tilde{\omega}^8,$$
$$-\tilde{\omega}^1\wedge\tilde{\omega}^7+\sqrt{3}\tilde{\omega}^4\wedge\tilde{\omega}^7+\tilde{\omega}^2\wedge\tilde{\omega}^3-\sqrt{3}\tilde{\omega}^3\wedge\tilde{\omega}^5-\sqrt{3}\tilde{\omega}^1\wedge\tilde{\omega}^8-\tilde{\omega}^4\wedge\tilde{\omega}^8+\sqrt{3}\tilde{\omega}^2\wedge\tilde{\omega}^6-\tilde{\omega}^5\wedge\tilde{\omega}^6,$$
$$\tilde{\omega}^3\wedge\tilde{\omega}^6+\tilde{\omega}^7\wedge\tilde{\omega}^8,\ \tilde{\omega}^6\wedge\tilde{\omega}^7+\tilde{\omega}^3\wedge\tilde{\omega}^8,$$
$$-\sqrt{3}\tilde{\omega}^1\wedge\tilde{\omega}^3+\tilde{\omega}^3\wedge\tilde{\omega}^4-\sqrt{3}\tilde{\omega}^2\wedge\tilde{\omega}^7+\tilde{\omega}^5\wedge\tilde{\omega}^7+(1-2\sqrt{3})\tilde{\omega}^1\wedge\tilde{\omega}^6-(2+\sqrt{3})\tilde{\omega}^4\wedge\tilde{\omega}^6+(1+2\sqrt{3})\tilde{\omega}^2\wedge\tilde{\omega}^8+$$
$$(-2+\sqrt{3})\tilde{\omega}^5\wedge\tilde{\omega}^8,$$
$$\sqrt{3}\tilde{\omega}^2\wedge\tilde{\omega}^3+\tilde{\omega}^3\wedge\tilde{\omega}^5-\sqrt{3}\tilde{\omega}^1\wedge\tilde{\omega}^7-\tilde{\omega}^4\wedge\tilde{\omega}^7+(1-2\sqrt{3})\tilde{\omega}^2\wedge\tilde{\omega}^6+(2+\sqrt{3})\tilde{\omega}^5\wedge\tilde{\omega}^6-(1+2\sqrt{3})\tilde{\omega}^1\wedge\tilde{\omega}^8+$$
$$(-2+\sqrt{3})\tilde{\omega}^4\wedge\tilde{\omega}^8,$$
$$\tilde{\omega}^3\wedge\tilde{\omega}^7+10\tilde{\omega}^1\wedge\tilde{\omega}^2-2\sqrt{3}\tilde{\omega}^1\wedge\tilde{\omega}^5-2\sqrt{3}\tilde{\omega}^2\wedge\tilde{\omega}^4-6\tilde{\omega}^4\wedge\tilde{\omega}^5-\tilde{\omega}^6\wedge\tilde{\omega}^8\}$$

(ii) Basis for $(1,7)$:

$$\{-\tilde{\omega}^1\wedge\tilde{\omega}^3-\sqrt{3}\tilde{\omega}^3\wedge\tilde{\omega}^4+\tilde{\omega}^2\wedge\tilde{\omega}^7+\sqrt{3}\tilde{\omega}^5\wedge\tilde{\omega}^7,\ -\tilde{\omega}^1\wedge\tilde{\omega}^7+\sqrt{3}\tilde{\omega}^4\wedge\tilde{\omega}^7-\tilde{\omega}^2\wedge\tilde{\omega}^3+\sqrt{3}\tilde{\omega}^3\wedge\tilde{\omega}^5,$$
$$4\tilde{\omega}^1\wedge\tilde{\omega}^4+4\tilde{\omega}^2\wedge\tilde{\omega}^5-\tilde{\omega}^3\wedge\tilde{\omega}^6+\tilde{\omega}^7\wedge\tilde{\omega}^8,\ -4\tilde{\omega}^1\wedge\tilde{\omega}^5+4\tilde{\omega}^2\wedge\tilde{\omega}^4-\tilde{\omega}^3\wedge\tilde{\omega}^8+\tilde{\omega}^6\wedge\tilde{\omega}^7,$$
$$-\sqrt{3}\tilde{\omega}^1\wedge\tilde{\omega}^3+\tilde{\omega}^3\wedge\tilde{\omega}^4-\sqrt{3}\tilde{\omega}^2\wedge\tilde{\omega}^7+\tilde{\omega}^5\wedge\tilde{\omega}^7+(2\sqrt{3}-1)\tilde{\omega}^1\wedge\tilde{\omega}^6+(-2+\sqrt{3})\tilde{\omega}^4\wedge\tilde{\omega}^6+(\sqrt{3}-1)\tilde{\omega}^2\wedge\tilde{\omega}^8+$$
$$(-2+\sqrt{3})\tilde{\omega}^5\wedge\tilde{\omega}^8,$$
$$\sqrt{3}\tilde{\omega}^2\wedge\tilde{\omega}^3+\tilde{\omega}^3\wedge\tilde{\omega}^5-\sqrt{3}\tilde{\omega}^1\wedge\tilde{\omega}^7-\tilde{\omega}^4\wedge\tilde{\omega}^7-(2\sqrt{3}-1)\tilde{\omega}^2\wedge\tilde{\omega}^6+(2+\sqrt{3})\tilde{\omega}^5\wedge\tilde{\omega}^6-(2\sqrt{3}-1)\tilde{\omega}^1\wedge\tilde{\omega}^8-$$
$$(2+\sqrt{3})\tilde{\omega}^4\wedge\tilde{\omega}^8,$$
$$-\tilde{\omega}^3\wedge\tilde{\omega}^7-8\tilde{\omega}^1\wedge\tilde{\omega}^2+4\sqrt{3}\tilde{\omega}^1\wedge\tilde{\omega}^5+4\sqrt{3}\tilde{\omega}^2\wedge\tilde{\omega}^4+3\tilde{\omega}^6\wedge\tilde{\omega}^8\}$$

(iii) Basis for (3,1):

$$\{\tilde{\omega}^1\wedge\tilde{\omega}^3+\sqrt{3}\tilde{\omega}^3\wedge\tilde{\omega}^4+\tilde{\omega}^2\wedge\tilde{\omega}^7+\sqrt{3}\tilde{\omega}^5\wedge\tilde{\omega}^7+\sqrt{3}\tilde{\omega}^1\wedge\tilde{\omega}^6+\tilde{\omega}^4\wedge\tilde{\omega}^6+\sqrt{3}\tilde{\omega}^2\wedge\tilde{\omega}^8-\tilde{\omega}^5\wedge\tilde{\omega}^8,$$
$$\tilde{\omega}^2\wedge\tilde{\omega}^3+\sqrt{3}\tilde{\omega}^3\wedge\tilde{\omega}^5-\tilde{\omega}^1\wedge\tilde{\omega}^7+\sqrt{3}\tilde{\omega}^4\wedge\tilde{\omega}^7+\sqrt{3}\tilde{\omega}^1\wedge\tilde{\omega}^8+\tilde{\omega}^4\wedge\tilde{\omega}^8-\sqrt{3}\tilde{\omega}^2\wedge\tilde{\omega}^6+\tilde{\omega}^5\wedge\tilde{\omega}^6,$$
$$-2\tilde{\omega}^1\wedge\tilde{\omega}^2+2\sqrt{3}\tilde{\omega}^1\wedge\tilde{\omega}^5+2\sqrt{3}\tilde{\omega}^2\wedge\tilde{\omega}^4-2\tilde{\omega}^4\wedge\tilde{\omega}^5-\tilde{\omega}^3\wedge\tilde{\omega}^7-\tilde{\omega}^6\wedge\tilde{\omega}^8\}$$

(iv) Basis for (1,3):

$$\{-\sqrt{3}\tilde{\omega}^1\wedge\tilde{\omega}^3+\tilde{\omega}^3\wedge\tilde{\omega}^4-\sqrt{3}\tilde{\omega}^2\wedge\tilde{\omega}^7+\tilde{\omega}^5\wedge\tilde{\omega}^7,\ \sqrt{3}\tilde{\omega}^2\wedge\tilde{\omega}^3+\tilde{\omega}^3\wedge\tilde{\omega}^5-\sqrt{3}\tilde{\omega}^1\wedge\tilde{\omega}^7-\tilde{\omega}^4\wedge\tilde{\omega}^7,$$
$$\tilde{\omega}^3\wedge\tilde{\omega}^7+3\tilde{\omega}^1\wedge\tilde{\omega}^2-\sqrt{3}\tilde{\omega}^1\wedge\tilde{\omega}^5-\sqrt{3}\tilde{\omega}^2\wedge\tilde{\omega}^4-\tilde{\omega}^4\wedge\tilde{\omega}^5\}$$

13. On Betti Numbers of $\Gamma\backslash G_{2(2)}/\mathrm{SO}(4)$

In this section, we compute the bounds on second Betti numbers of $M = \Gamma\backslash G_{2(2)}/\mathrm{SO}(4)$.

Consider the vector field $\partial_{t_1} + \partial_{t_2}$. Then:

$$L_{\partial_{t_1}+\partial_{t_2}} = \nabla_{\partial_{t_1}+\partial_{t_2}} + \frac{3}{2}\frac{\lambda_1^2\lambda_2^4+1}{\lambda_1^2\lambda_2^4-1}e(\tilde{\omega}^1)e^*(\tilde{\omega}^1) + \frac{3}{2}\frac{\lambda_1^4\lambda_2^2+1}{\lambda_1^4\lambda_2^2-1}e(\tilde{\omega}^2)e^*(\tilde{\omega}^2) + \frac{\lambda_1^2+1}{2(\lambda_1^2-1)}e(\tilde{\omega}^4)e^*(\tilde{\omega}^4) + \frac{\lambda_2^2+1}{2(\lambda_2^2-1)}e(\tilde{\omega}^5)e^*(\tilde{\omega}^5) + \frac{\lambda_1^2\lambda_2^2+1}{\lambda_1^2\lambda_2^2-1}e(\tilde{\omega}^6)e^*(\tilde{\omega}^6).$$

Set:

$$Q = \frac{3}{2}\frac{\lambda_1^2\lambda_2^4+1}{\lambda_1^2\lambda_2^4-1}e(\tilde{\omega}^1)e^*(\tilde{\omega}^1) + \frac{3}{2}\frac{\lambda_1^4\lambda_2^2+1}{\lambda_1^4\lambda_2^2-1}e(\tilde{\omega}^2)e^*(\tilde{\omega}^2) + \frac{\lambda_1^2+1}{2(\lambda_1^2-1)}e(\tilde{\omega}^4)e^*(\tilde{\omega}^4) + \frac{\lambda_2^2+1}{2(\lambda_2^2-1)}e(\tilde{\omega}^5)e^*(\tilde{\omega}^5) + \frac{\lambda_1^2\lambda_2^2+1}{\lambda_1^2\lambda_2^2-1}e(\tilde{\omega}^6)e^*(\tilde{\omega}^6),$$

and

$$H = \frac{3}{2}\frac{\lambda_1^2\lambda_2^4+1}{\lambda_1^2\lambda_2^4-1} + \frac{3}{2}\frac{\lambda_1^4\lambda_2^2+1}{\lambda_1^4\lambda_2^2-1} + \frac{\lambda_1^2+1}{2(\lambda_1^2-1)} + \frac{\lambda_2^2+1}{2(\lambda_2^2-1)} + \frac{\lambda_1^2\lambda_2^2+1}{\lambda_1^2\lambda_2^2-1}.$$

Setting $q_h(r) := \dfrac{\int_{K\Omega_r}\langle(\frac{H}{2}-Q)h,h\rangle}{\int_{K\Omega_r}|h|^2}$ as before, a computation yields the following lower bounds for $q_h(r)$:

$$q_h(r) \geq \begin{cases} \frac{19}{81} \text{ on } \mathcal{H}^2_{15} \\ \frac{16}{85} \text{ on } \mathcal{H}^2_{7} \\ 1 \text{ on } \mathcal{H}^2_{3'} \\ \frac{1}{34} \text{ on } \mathcal{H}^2_{3}. \end{cases}$$

Then the same computation as Theorem 9.1 yield the following theorem:

Theorem 13.1. Let $M = \Gamma \setminus G_{2(2)}/SO(4)$, $R = \frac{\mathrm{inj}_M}{\sqrt{2}}$. Then there exists a strictly positive constant $d(8)$ such that:

$$\begin{cases} \frac{b_{15}^{2,G_2}}{Vol(M)} \leq \frac{81d(8)}{38}e^{-38R/81} \\ \frac{b_{3}^{2,G_2}}{Vol(M)} \leq \frac{85d(8)}{16}e^{-32R/81} \\ \frac{b_{3'}^{2,G_2}}{Vol(M)} \leq d(8)e^{-2R} \\ \frac{b_{7}^{2,G_2}}{Vol(M)} \leq 34d(8)e^{-R/17}. \end{cases}$$

Remark: Our method did not yield estimates on the Betti number of $\mathcal{H}^3(\Gamma\backslash G_{2(2)}/SO(4))$, the space of harmonic 3 forms on $\Gamma\backslash G_{2(2)}/SO(4)$.

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