

Examples of the Local L^2 -Cohomology of Algebraic Varieties

by

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Mark Stern

Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
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ABSTRACT

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Abstract

L^2 -cohomology is a cohomology theory on Riemannian manifolds. It agrees with de Rham cohomology in the compact case, but is often different in the non-compact case. In some informative examples of stratified spaces, L^2 -cohomology of the regular part often agrees with the middle-perversity intersection homology of the stratified space, giving a de Rham-style theorem. Unfortunately, this doesn't always happen; we give a family of examples of real algebraic varieties for which the L^2 -cohomology and middle-perversity intersection homology are not equal.

In both this family and in the case of complex singularities, it often happens that we can decompose the space into regions where the metric looks like that of a multiply-warped product, or like interpolations between such regions. An illuminating class of examples is that of normal complex surface singularities. In this case, the decomposition was begun by Hsiang and Pati and completed by Nagase, and this decomposition played a heavy role in the computation of L^2 -cohomology.

Cheeger, Goresky, and MacPherson conjectured that the intersection cohomology of complex projective varieties and the L^2 -cohomology of their regular part are isomorphic. One hope at the time of the conjecture would be that the proof would shed light on the local structure of complex algebraic singularities. If one instead looks at the local L^2 -cohomology and ask that it is isomorphic to the local intersection

homology, then the conjecture does imply restrictions on the cohomology which is only apparent after a closer look at the local geometry around the singularity.

In this thesis, we calculate the local L^2 -cohomology for several examples of affine real and complex algebraic varieties with isolated singularities with the metric induced with the Euclidean metric. We give examples of real algebraic varieties where the local L^2 -cohomology is not isomorphic to the middle intersection homology. We give another example where the local L^2 -cohomology is not even a subspace of the cohomology of the link. We also calculate the local L^2 -cohomology for a class of weighted homogeneous hypersurfaces; this class of examples includes the A_k -singularities in arbitrary dimension.

Dedicated to my wife.

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1

Introduction

Cheeger studied the L^2 -cohomology of special types of spaces (X, g) called metric cones and metric horns. These are spaces where $X = (0, 1] \times L$ is topologically a cylinder for some compact Riemannian manifold (L, g_L) , possibly with corners, and the metric is given by $g = dr^2 + r^{2c}g_L$ for some $c \geq 1$. In this case, he found the following intriguing result:

Theorem 1.1. [5] *Let $X = (0, 1] \times L$ for a compact manifold L , $n = \dim L$ be odd, and X be a metric cone or horn. Then the L^2 -cohomology of (X, g) is given by*

$$H_{(2)}^k(X, g) = \begin{cases} 0 & k > \frac{n+1}{2} \\ H_{dR}^k(L) & k \leq \frac{n+1}{2}. \end{cases}$$

This truncation of the cohomology of the link is reminiscent of intersection cohomology. In fact, in this case the intersection cohomology with middle perversity is isomorphic to the L^2 -cohomology. This was an exciting discovery. Intersection cohomology had shown itself to be a robust theory of cohomology for spaces with singularities, possessing nice properties which ordinary cohomology did not, such as Poincaré Duality. L^2 -cohomology presented itself as a possible de Rham-style coho-

mology theory which could allow calculations involving intersection homology using the techniques of analysis, in much the same way that standard de Rham theory and Hodge theory are related to ordinary de Rham cohomology.

The question that is naturally posed next was, For which spaces (X, g) are the L^2 -cohomology and intersection cohomology isomorphic? While intersection cohomology is a topological invariant, the L^2 -cohomology depends on the specific metric g chosen, so the question can be restated as, Is there a metric g so that (X, g) has isomorphic L^2 -cohomology and intersection cohomology?

These questions were inevitably turned towards complex algebraic varieties, and Cheeger, Goresky, and MacPherson made the following conjecture:

Conjecture 1.2. *[6] Let X be a complex projective variety, and let g be the metric on the regular part of X which is induced from the Fubini-Study metric on projective space. Then*

$$H_{(2)}^k(X, g) \cong \text{Hom}(IH_k(X), \mathbb{R}).$$

Due to the axiomatic description of intersection homology, it suffices to check this locally only. Thus, they use the analytic results of [5] to prove the conjecture in the case of algebraic varieties with only isolated singularities where the neighborhoods of those singularities are metric cones or metric horns, or metrics quasi-isometric to these.

In [15], Hsiang and Pati claimed to prove this conjecture in the case of surfaces with isolated singularities. Unfortunately, their paper had a important gap which was subsequently filled by [17].¹ Together, the two papers prove the result by break-

¹ Before finding this paper by Nagase, this gap was also filled independently by the author twenty years later, demonstrating the importance of full literature searches.

ing up the neighborhood of a singularity into regions in which the metric is well known, then using the explicit description of the metric to prove the requisite analytic estimates.

There is a common theme in both Cheeger’s proof of the metric horn case and in Hsiang-Pati and Nagase’s proof for algebraic surfaces. Topologically, $X = (0, 1] \times L$. There is the usual homotopy operator which takes a smooth form on X and outputs the value of the form on $\{1\} \times L$. In “low degrees”, this operator is bounded and the resulting form is an L^2 form. In “high degrees”, any form in the domain of d must vanish as r approaches 0. This allows us to “extend” the form to a form on the cone with the vertex included. This space is contractible, and so we can homotope the form to zero. Thus, in low degrees, the L^2 -cohomology of X equals the cohomology of the link L , and in high degrees, the L^2 -cohomology of X vanishes. Of course, there are many details that go into making this program rigorous.

Unfortunately, one can show that the degrees which are considered “low” by this structure do not always necessarily match up with what intersection cohomology would consider “low” degrees. For example, we prove the following Künneth-like theorem for multiply warped products:

Theorem 4.30. *Let $L = L_1 \times \cdots \times L_m$ be the product of compact manifolds, and $X = (0, 1] \times L$. Give X the metric*

$$g = dr^2 + r^{2c_1} g_{L_1} + \cdots + r^{2c_m} g_{L_m}$$

for some constants c_i and metrics g_{L_i} on L_i . Then the L^2 -cohomology of X is isomorphic to

$$H_{(2)}^k(X, g) \cong \bigoplus_{\substack{k_1 + \dots + k_m = k \\ c_1(n_1 - 2k_1) + \dots + c_m(n_m - 2k_m) > -1}} H_{dR}^{k_1}(L_1) \otimes \dots \otimes H_{dR}^{k_m}(L_m).$$

In particular, $H_{(2)}^k(X, g) \subseteq H_{dR}^k(L)$.

Later, in Section 4.5, we show how to realize metrics of this kind as the metric on real algebraic varieties with an isolated singularity. For some of these metrics, there are forms in middle degree or higher which are L^2 when extended independently of r . Correspondingly, the local L^2 -cohomology of these spaces can vary wildly from the middle intersection homology. We also give a relatively simple example of a space whose local L^2 -cohomology is not even a subspace of the cohomology of the link. These examples give us an idea of what could conceivably go wrong.

One of the original hopes of the CGM conjecture was that a proof would shed light on the local geometry of complex singularities. In fact, a proof of the conjecture was put forth by Ohsawa for complex varieties with isolated singularities (see e.g. [18] and [19]). However, the proof involves the degeneration of certain complete metrics, and does not give insight into the local geometry of these singularities. Therefore, one might still look for a more explicit proof of the kind used by Hsiang-Pati and Nagase.

A proof working locally with the Fubini-Study metric (or equivalently, the metric on affine varieties induced by the Euclidean metric) has its difficulties, however, because the metric is incomplete. The proofs of this kind that the author is aware of all involve decomposing the metric into regions where it is quasi-isometric to a particularly nice model metric, then patching the result together. This patching together must be done carefully; standard techniques like the Mayer-Vietoris sequence are often invalid in this context.

We give a proof of this kind for hypersurfaces of the type

$$\{z_0^k + g(z_1, \dots, z_n) = 0\} \subset \mathbb{C}^{n+1},$$

where g is a homogeneous polynomial of degree less than k . This includes, for example, the A_{k+1} singularities. In particular, we prove the following theorem:

Theorem 6.23. *Let $V \subset \mathbb{C}^{n+1}$ be the hypersurface determined by the vanishing of a polynomial*

$$f(z) = z_0^\beta + g(z_1, \dots, z_n),$$

for some integers α and β satisfying $2 \leq \alpha < \beta$ and some homogeneous polynomial g of degree α . Then the local L^2 -cohomology of V is given by

$$H_{(2)}^k(V) = \begin{cases} 0 & k \geq \dim_{\mathbb{C}}(V) \\ H_{dR}^k(L) & k < \dim_{\mathbb{C}}(V). \end{cases}$$

where L is the link of V at the origin.

Our proof involves splitting a neighborhood of V at the origin into regions which are either multiply warped products of a special kind or interpolations between such regions. In Chapter 5, we show familiar bounds and techniques are valid in each of these regions, then in Chapter 6 we patch them together to get our result.

2

Background

2.1 Basic Notation for Riemannian Manifolds

Let X be an n -dimensional, oriented manifold with corners with a Riemannian metric g . Let $\Omega^\bullet(X)$ denote the smooth differential forms on X ; this, along with its exterior derivative d , is a cochain complex. The cohomology $H_{dR}^\bullet(X) = H^\bullet(\Omega^\bullet(X))$ of this cochain complex is called the de Rham cohomology. The de Rham theorem states that $H_{dR}^\bullet(X)$ is isomorphic to the singular cohomology with real coefficients $H^\bullet(X, \mathbb{R})$.

By definition, the metric g defines an inner product on the pointwise tangent spaces $T_x X$, and thus it defines a norm. Dually, there is an inner product and norm on the cotangent space $T_x^* X$. This inner product (resp. norm) can be extended to the wedge product $\bigwedge^k T_x^* X$, which can be thought of as a map from pairs of k -forms ω, ϕ to scalar functions (ω, ϕ) (resp. from k -forms ω to scalar functions $|\omega|$).

The top degree wedge product $\bigwedge^n T_x^* X$ at a point $x \in X$ is a 1-dimensional

real vector space. The orientation on X is a choice of which non-zero elements are positive. There is a unique, positive element whose norm is 1, and this element varies smoothly on X . This defines a form dV which we call the volume form induced by the metric g and the orientation. Using the volume form, the pointwise inner product and norm induce an inner product and norm on global forms via

$$\langle \alpha, \beta \rangle = \int_X (\alpha, \beta) dV, \text{ and}$$

$$\|\alpha\| = \left(\int_X |\alpha|^2 dV \right)^{1/2}.$$

2.2 L^2 -Cohomology

We define $\text{dom}(d) \subset \Omega^\bullet(X)$ to consist of the forms α for which $\|\alpha\| < \infty$ and $\|d\alpha\| < \infty$. Because $\text{dom}(d)$ forms a subcomplex of $\Omega^\bullet(X)$, we can consider its cohomology $H_{(2)}^\bullet(X, g)$. We call this the L^2 -cohomology of (X, g) . This depends not only on the topology of X , but also in an essential way on the metric g .

Example 2.1.

$$\dim H_{(2)}^1((0, 1), dx^2) = 0, \text{ but } \dim H_{(2)}^1((0, \infty), dx^2) = \infty$$

where dx^2 denotes the standard metric on $(0, \infty)$.

Proof. To see the first one, notice that $\|f(x)dx\|^2 = \int_0^1 |f|^2 dx$. Therefore, if $f(x)dx$ is an L^2 form, then

$$\begin{aligned}
\left\| \int_{1/2}^x f(y) dy \right\|^2 &= \int_0^1 \left| \int_{1/2}^x f(y) dy \right|^2 dx \\
&\leq \int_0^{1/2} \left(\int_x^{1/2} |f(y)|^2 dy \right) dx + \int_{1/2}^1 \left(\int_{1/2}^x |f(y)|^2 dy \right) dx \\
&\leq \int_0^1 |f(y)|^2 dy < \infty.
\end{aligned}$$

Therefore, $\int_{1/2}^x f(y) dy$ is an L^2 -form, and $f(x) dx = d\left(\int_{1/2}^x f(y) dy\right)$. Thus, every closed, L^2 1-form is the differential of an L^2 0-form, and $H_{(2)}^1((0, 1), dx^2) = 0$.

Now we consider the other example. For a positive real number $s \in (-1, -\frac{1}{2})$, consider the form $(1+x)^s dx$. This form is L^2 , but there is no L^2 function for which this is the differential. Indeed, if $df = (1+x)^s dx$, then $f(x) = C + \frac{1}{1+s}(1+x)^{s+1}$ for some constant C , and no value of C can make f an L^2 function. Even more than this, any form which is a finite linear combination of such forms is an L^2 form but cannot be the differential of an L^2 function.

Thus, the vector space spanned by $\{(1+x)^s dx\}$ for $s \in (-1, -\frac{1}{2})$ injects into $H_{(2)}^1((0, 1), dx^2)$, so the dimension of $H_{(2)}^1((0, 1), dx^2)$ is infinite. \square

The subcomplex $dom(d)$ is not complete under the norm $\|\cdot\|$; we denote the completion¹ $L^2\Omega^\bullet(X, g)$. The exterior derivative d on $dom(d)$ is not bounded. However, we can extend d to a closed, unbounded operator on $L^2\Omega^\bullet(X, g)$. There are two common ways to do this: the weak extension and the strong extension. The weak extension of d is the adjoint δ_0^* , where δ_0 is the unbounded operator which acts as the codifferential and whose domain consists of compactly supported smooth forms (and

¹ This is the same completion as the one we get by completing the subcomplex of $\Omega^\bullet(X)$ consisting of L^2 forms, ignoring the L^2 condition on $d\omega$.

compactly supported away from any boundary of X). The strong extension of d is the operator whose graph is the closure of the graph of d . It is a classic result that these extensions give the same operator; see [9] and [14]. We will denote the extension as \bar{d} and its domain as $dom(\bar{d})$ and use whichever definition is more convenient as needed.

The subspace $dom(\bar{d})$ is also a cochain complex, and we denote its cohomology as $H_{(2),\#}^\bullet(X, g)$. As [5] points out, the inclusion $dom(d) \hookrightarrow dom(\bar{d})$ is a quasi-isomorphism of complexes. In particular,

$$H_{(2)}^\bullet(X, g) \cong H_{(2),\#}^\bullet(X, g).$$

This can be proven using a smoothing homotopy operator; see Section 12 of [11] for an explicit description of this process. Which complex to use in practice is a delicate question: \bar{d} is a closed operator, which makes some calculations easier, while $dom(d)$ consists of smooth forms, which makes other calculations easier. We will try to be clear about when we are using each complex; as a rough heuristic, $dom(d)$ is sufficient for low degree forms, while we may need to use $dom(\bar{d})$ for high degree forms.

2.3 Quasi-Isometry

Ideally, we'd like to make specific calculations for a few model metrics, then relate the metrics we find back to these model metrics. The metrics we find in practice will not be these exact model metrics, but they will be quasi-isometric to them. Here we introduce the concept of quasi-isometry and look at some of the properties that are preserved under this notion.

Definition 2.2. Take two oriented Riemannian manifolds (possibly with corners) (X, g) and (Y, h) and a diffeomorphism $f : X \rightarrow Y$ between them. Then f is called

a *quasi-isometry* if there exist constants $0 < c < C < \infty$ so that

$$c|v|_{(X,g)} \leq |f_*v|_{(Y,h)} \leq C|v|_{(X,g)}$$

for every $v \in T_x X$ and every $x \in X$. We will say that (X, g) and (Y, h) are *quasi-isometric*. Crucially, the same constants c and C must be used for every $x \in X$.

Remark 2.3. Often in the literature, quasi-isometry means a slightly stronger notion. Namely, the space X is fixed, and two metrics g and h on X are called quasi-isometric if the identity on X is a quasi-isometry between (X, g) and (X, h) as defined here. Our definition will allow us a little more flexibility in our calculations.

Lemma 2.4. *Let $f : (X, g) \rightarrow (Y, h)$ be a quasi-isometry, and let $\dim X = n$.*

1. f^{-1} is a quasi-isometry with constants $\frac{1}{C}$ and $\frac{1}{c}$.
2. For a k -form $\alpha \in \Omega^k(X)$,

$$c^k f^*|\alpha| \leq |f^*\alpha| \leq C^k f^*|\alpha|$$

as functions on X .

3. The map $f^* : \Omega^\bullet(Y) \rightarrow \Omega^\bullet(X)$ induces a bounded isomorphism of complexes $f^* : \text{dom}(d_Y) \rightarrow \text{dom}(d_X)$, and this in turn induces isomorphisms

$$f^* : H_{(2)}^k(Y, h) \rightarrow H_{(2)}^k(X, g).$$

Proof. 1. By definition, f^{-1} is also a diffeomorphism. For a tangent vector $w \in T_y Y$, apply the estimates in the definition to the vector $v = f_*^{-1}w$ to get

$$\begin{aligned} c|f_*^{-1}w|_{(X,g)} &\leq |f_*f_*^{-1}w|_{(Y,h)} \leq C|f_*^{-1}w|_{(X,g)} \\ c|f_*^{-1}w|_{(X,g)} &\leq |w|_{(Y,h)} \leq C|f_*^{-1}w|_{(X,g)} \\ \frac{1}{C}|w|_{(Y,h)} &\leq |f_*^{-1}w|_{(X,g)} \leq \frac{1}{c}|w|_{(Y,h)}. \end{aligned}$$

2. At a point $x \in X$,

$$\begin{aligned}
|f^*\alpha| &= \max_{\substack{(v_1, \dots, v_k) \in \prod^k T_x X \\ |v_j| \leq 1}} (f^*\alpha)(v_1, \dots, v_k) \\
&= \max_{\substack{(v_1, \dots, v_k) \in \prod^k T_x X \\ |v_j| \leq 1}} \alpha(f_*v_1, \dots, f_*v_k) \\
&\leq \max_{\substack{(w_1, \dots, w_k) \in \prod^k T_{f(x)} Y \\ |w_j| \leq C}} \alpha(w_1, \dots, w_k) \\
&= \max_{\substack{(w_1, \dots, w_k) \in \prod^k T_{f(x)} Y \\ |w_j| \leq 1}} \alpha(Cw_1, \dots, Cw_k) \\
&= C^k \max_{\substack{(w_1, \dots, w_k) \in \prod^k T_{f(x)} Y \\ |w_j| \leq 1}} \alpha(w_1, \dots, w_k) \\
&= C^k f^*|\alpha|.
\end{aligned}$$

For the other direction, apply the same reasoning to the quasi-isometry f^{-1} to show

$$|\alpha| = |(f^{-1})^* f^* \alpha| \leq \frac{1}{c^k} (f^{-1})^* |f^* \alpha|,$$

apply f^* to both sides, and rearrange.

3. Because the vector space $\bigwedge^n T_x^* X$ is one dimensional for each $x \in X$, there is some non-vanishing function $p : X \rightarrow \mathbb{R}$ so that

$$f^* dV_Y = p(x) dV_X,$$

where dV_Y and dV_X are the volume forms induced by the metrics h and g , respectively. Applying part 2, we get

$$\begin{aligned}
c^n f^* |dV_Y| &\leq |f^* dV_Y| \leq C^n f^* |dV_Y| \\
c^n f^* |dV_Y| &\leq |p(x) dV_X| \leq C^n f^* |dV_Y| \\
c^n &\leq |p(x)| \leq C^n.
\end{aligned}$$

Assume f is orientation-preserving, so $p(x) > 0$; if not, just insert a minus sign where appropriate below. We have the following estimate for the L^2 norm of $f^*\alpha$:

$$\begin{aligned}
\|f^*\alpha\|_{(X,g)}^2 &= \int_X |f^*\alpha|^2 dV_X \\
&\leq \int_X C^k f^*|\alpha|^2 \frac{p(x)}{c^n} dV_X \\
&= \frac{C^k}{c^n} \int_X f^*|\alpha|^2 f^* dV_Y \\
&= \frac{C^k}{c^n} \int_Y |\alpha|^2 dV_Y \\
&= \frac{C^k}{c^n} \|\alpha\|_{(Y,h)}^2.
\end{aligned}$$

Similarly, we show that $\frac{c^k}{C^n} \|\alpha\|^2 \leq \|f^*\alpha\|^2$. Combining these two estimates, we've shown that $\alpha \in \Omega^k(Y)$ is an L^2 form if and only if $f^*\alpha$ is. Likewise, $\beta \in \Omega^k(X)$ is an L^2 form if and only if $(f^{-1})^*\beta$ is. Furthermore, f^* and $(f^{-1})^*$ are bounded maps. Similarly, $d(f^*\alpha) = f^*d\alpha$ is L^2 if and only if $d\alpha$ is.

Thus, $f^* : \text{dom}(d_Y) \rightarrow \text{dom}(d_X)$ and $(f^{-1})^* : \text{dom}(d_X) \rightarrow \text{dom}(d_Y)$ are well defined and inverses of each other. Both commute with the exterior derivative, so they give an isomorphism of chain complexes, which in turn induces an isomorphism on the cohomology of those complexes. \square

Lastly, we have a lemma on the relationship between bi-Lipschitz maps and quasi-isometries.

Lemma 2.5. *Let A and B be smooth submanifolds of \mathbb{R}^n , and assume that A and B are each given the induced Riemannian metric with respect to the Euclidean metric on*

\mathbb{R}^n . Assume we are given a diffeomorphism $f : A \rightarrow B$ and constants $0 < c < C < \infty$ so that

$$c\|x - y\| \leq \|f(x) - f(y)\| \leq C\|x - y\|$$

for all $x, y \in A$. Then f is a quasi-isometry.

Proof. In this case, $T_x A$ and $T_{f(x)} B$ are subspaces of \mathbb{R}^n , and the metric on them is the usual Euclidean inner product. Thus, we want to show for a tangent vector $v \in T_x A$ that

$$c\|v\| \leq \|f_* v\| \leq C\|v\|.$$

Because $v \in T_x A$, there is a smooth path $\gamma : (-1, 1) \rightarrow A$ so that $\gamma(0) = x$ and $\gamma'(0) = v$. Then

$$v = \lim_{t \rightarrow 0} \frac{\gamma(t) - \gamma(0)}{t}, \text{ and}$$

$$f_* v = \lim_{t \rightarrow 0} \frac{f(\gamma(t)) - f(\gamma(0))}{t}.$$

Thus, we calculate

$$\begin{aligned} \|f_* v\| &= \left\| \lim_{t \rightarrow 0} \frac{f(\gamma(t)) - f(\gamma(0))}{t} \right\| \\ &= \lim_{t \rightarrow 0} \left\| \frac{f(\gamma(t)) - f(\gamma(0))}{t} \right\| \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \|f(\gamma(t)) - f(\gamma(0))\|. \end{aligned}$$

Using our assumption that f is bi-Lipschitz, we get the inequalities

$$c \lim_{t \rightarrow 0} \frac{1}{t} \|\gamma(t) - \gamma(0)\| \leq \lim_{t \rightarrow 0} \frac{1}{t} \|f(\gamma(t)) - f(\gamma(0))\| \leq C \lim_{t \rightarrow 0} \frac{1}{t} \|\gamma(t) - \gamma(0)\|$$

$$c\|v\| \leq \|f_* v\| \leq C\|v\|. \quad \square$$

2.4 The Cheeger-Goresky-MacPherson Conjecture

We will give some motivation in this section for what will come later. Let $V \subset \mathbb{C}\mathbb{P}^n$ be a complex projective variety. There is a natural metric on $\mathbb{C}\mathbb{P}^n$ called the Fubini-Study metric. We can consider the Riemannian manifold V_{reg} which is the non-singular part of V with the metric induced from its embedding in $\mathbb{C}\mathbb{P}^n$.

Conjecture 2.6. *[6] Let V be a complex projective variety, and let g be the metric on the regular part V_{reg} of V which is induced from the Fubini-Study metric on projective space. Then*

$$H_{(2)}^k(V_{reg}, g) \cong \text{Hom}(IH_k(V), \mathbb{R}),$$

where $IH_K(V)$ is the middle perversity intersection homology.²

We will not give a treatment of intersection homology here, nor do we use it later (but one can learn about it more in the article [6] this conjecture comes from, as well as in [8] and [16] or in the original articles [12] and [13]). This is because there is a local description which does not make use of these definitions:

Lemma 2.7. *[6] The Cheeger-Goresky-MacPherson conjecture holds if V has only isolated singular points, and for every point $x \in V$ and every sufficiently small, contractible neighborhood U of x ,*

$$H_{(2)}^k(U, g) = \begin{cases} 0 & k \geq \dim_{\mathbb{C}}(V) \\ H_{dR}^k(L) & k < \dim_{\mathbb{C}}(V) \end{cases},$$

where L is the link of the point x in V .

There is a similar local description in the case of a non-isolated singular point, but we will only consider spaces with isolated singular points in this thesis.

² Note that in [6], they allow forms to take complex values, so in their notation the field of coefficients is \mathbb{C} , not \mathbb{R} .

Let U be a small contractible neighborhood in $\mathbb{C}\mathbb{P}^n$ of a point $x \in V$; then there is a holomorphic map from U to a neighborhood U' of the origin in \mathbb{C}^n . The image of $U \cap V$ under this map is $U' \cap V'$ for some affine algebraic variety $V' \subset \mathbb{C}^n$, and the image of $U \cap V_{reg}$ is the non-singular part V'_{reg} of $U' \cap V'$. Furthermore, this map is a quasi-isometry between $U \cap V_{reg}$ with the metric induced by the Fubini-Study metric and V'_{reg} with the metric induced by the Euclidean metric.

Therefore, to prove the CGM conjecture locally, it suffices to consider the local L^2 -cohomology of affine, complex algebraic varieties with the metric induced by the Euclidean metric. In fact, the conjecture was motivated by the independent discovery of this kind of vanishing in the case of the L^2 -cohomology of metric cones and horns by Cheeger and of the same vanishing results in the case of middle perversity intersection homology by Goresky and MacPherson.

Let us now give some history of the progress of this conjecture. A proof of the conjecture in the case of complex curves can be found in Brüning and Lesch [4]. Hsiang and Pati claimed a proof for normal surfaces in [15], but there was a vital gap. Nagase found and fixed this gap in [17]. The original proof in [15] involves noticing that singularities of normal surfaces are isolated, that the singularities can be blown up so that the preimage of each singularity is the union of divisors with normal crossings, and that the quasi-isometry class of the metric has a particularly nice form with respect to certain blow-ups. Hsiang and Pati called this type of metric a *Cheeger metric*; we discuss a small generalization of these in Chapter 5.

However, they failed to calculate that in a decomposition of the neighborhood as $(0, 1] \times L$ for the link L of the singularity, there are also regions which can be understood as *interpolations of the Cheeger metrics*. The gap was found by Nagase

and corrected in [17]. Similar calculations of the local geometry of algebraic surfaces can be found in the recent work of Birbrair, Neumann, and Pichon [2]. There has also been work in studying what have since been called Hsiang-Pati coordinates for more general varieties; see [20] and [7] for more details.

In a different vein, Saper found a class of complete metrics on V_{reg} when V has isolated singularities (see [22] for the surface case, and [23] for the general dimension case). Ohsawa then proved the CGM conjecture in the case of isolated singularities by expanding this class of complete metrics, finding a sequence of metrics in this class which degenerates to the Fubini-Study metric, and showing the required vanishing properties are preserved through this degeneration (see [19] for details).

One might then consider the matter closed for varieties with isolated singularities. However, this proof technique is somewhat unsatisfying in that it does not give us the same understanding of the metric locally as previous approaches. In particular, one might hope that a proof of the CGM conjecture would give us the analytic tools to work with the L^2 differential forms on V directly, or some understanding of the local geometry. This is certainly the hope of Cheeger, Goresky, and MacPherson in their original paper, in which they write, “The analysis is extremely delicate and it depends on as yet unexplored aspects of the metric structure of X near a singularity,” and “The resulting understanding of the differential geometry of the singularities of X would be extremely interesting in itself.”

Metrics on Topological Cones

3.1 Common Notation

We will find ourselves in the following situation repeatedly. Let $X = (0, 1] \times L$ be a product space, where L is a compact manifold with corners. We call L the *link*. We will have different conventions for coordinates on L depending on what specifically L is, but we will use r to represent the coordinate on $(0, 1]$. Let $\Omega^\bullet(L)$ be the smooth forms on L .

For any form $\alpha \in \Omega^\bullet(X)$, we denote its evaluation at a specific value of r as

$$\alpha(r) \in \Omega^\bullet(L) \oplus dr \wedge \Omega^\bullet(L).$$

We can write any form $\alpha \in \Omega^\bullet(X)$ as a form $\alpha = \kappa_r \alpha + dr \wedge \iota_r \alpha$; this uniquely defines κ_r and ι_r as projection operators.¹ For all $r \in (0, 1]$, $\kappa_r \alpha(r)$ and $\iota_r \alpha(r)$ are forms in $\Omega^\bullet(L)$. We can think of $\kappa_r \alpha(r)$ and $\iota_r \alpha(r)$ as forms on the link via the inclusion $L \cong \{r\} \times L \hookrightarrow X$.

¹ We will use similar notation when we can break the tangent space into a direct sum of more summands.

In this chapter, we consider metrics on X that are quasi-isometric to a metric g of the form

$$g = dr^2 + g_r$$

where for each $r \in (0, 1]$, we have a metric g_r on the link, and g_r depends smoothly² on r . Notice that $\frac{\partial}{\partial r}$ is orthogonal to any tangent vector of the link at all points in X . These metrics appear in practice under mild conditions:

Lemma 3.1. *Take a metric g on $(0, 1] \times L$. Assume there are constants $0 < c < 1 < C < \infty$ so that for all $v \in T_*L$,*

$$\left| g \left(\frac{\partial}{\partial r}, v \right) \right| \leq (1 - c) g \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right)^{1/2} g(v, v)^{1/2}, \text{ and}$$

$$c \leq g \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right) \leq C.$$

*Then the identity map is a quasi-isometry between g and $dr^2 + g_r$, where g_r is the metric g restricted to T_*L .*

Proof. Take an arbitrary tangent vector $a\frac{\partial}{\partial r} + v$, where $v \in T_*L$. We can use our assumptions to get the following lower bound:

$$\begin{aligned} g \left(a\frac{\partial}{\partial r} + v, a\frac{\partial}{\partial r} + v \right) &= a^2 g \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right) + 2a g \left(\frac{\partial}{\partial r}, v \right) + g(v, v) \\ &\geq a^2 g \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right) - 2a(1 - c) g \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right)^{1/2} g(v, v)^{1/2} + g(v, v) \\ &= c \left(a^2 g \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right) + g(v, v) \right) + (1 - c) \left(a g \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right)^{1/2} - g(v, v)^{1/2} \right)^2 \\ &\geq c (a^2 c + g(v, v)) \\ &\geq c^2 (a^2 + g(v, v)). \end{aligned}$$

² By this we mean that in a local coordinate patch of L , g_r can be written as a symmetric matrix whose entries are smooth functions of r .

Similarly, we can use our assumption and the AM-GM inequality to get the following upper bound:

$$\begin{aligned}
g\left(a\frac{\partial}{\partial r} + v, a\frac{\partial}{\partial r} + v\right) &= a^2 g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) + 2a g\left(\frac{\partial}{\partial r}, v\right) + g(v, v) \\
&\leq a^2 g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) + 2ag\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right)^{1/2} g(v, v)^{1/2} + g(v, v) \\
&\leq 2a^2 g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) + 2g(v, v) \\
&\leq 2a^2 C + 2g(v, v) \\
&\leq 2C (a^2 + g(v, v)) .
\end{aligned}$$

Therefore, g is quasi-isometric under the identity to $dr^2 + g_r$, where $g_r(v, v) = g(v, v)$ for $v \in T_*L$. \square

Given that we have many metrics g_r on the link (one for each value of r), there are many relevant L^2 -norms for forms on the link. For a form $\omega \in \Omega^\bullet(L) \oplus dr \wedge \Omega^\bullet(L)$, we denote by $\|\omega\|_r$ the L^2 -norm of ω that we get by using g_r . Then for a form $\alpha \in \Omega^\bullet(X)$, the L^2 -norm can be calculated via

$$\|\alpha\|^2 = \int_0^1 \|\alpha(r)\|_r^2 dr .$$

Let d_L be the exterior derivative on the link L , and let d be the exterior derivative on X . Then for $\alpha \in \Omega(X)$, we have the formula

$$d\alpha = d(\kappa_r \alpha + dr \wedge \iota_r \alpha) = d_L \kappa_r \alpha + dr \wedge \left(\frac{\partial \kappa_r \alpha}{\partial r} - d_L \iota_r \alpha \right) .$$

3.2 Useful Lemmas

The following lemmas will often be useful. They aim to generalize results originally proved for cones and horns in [5].

Lemma 3.2. *Let $\omega \in \Omega^\bullet(L)$. If there are constants $A > 0$ and $b < 1$ so that $r^b \|\omega\|_r^2 \leq As^b \|\omega\|_s^2$ for all $r, s \in (0, 1]$ satisfying $r \leq s$, then the extension of ω to a form on all X independent of r is an L^2 -form, and*

$$\|\omega\|^2 \leq \frac{A}{1-b} \|\omega\|_1^2.$$

Proof. Since L is compact and ω is smooth, $\|\omega\|_r^2 < \infty$ for all $r \in (0, 1]$. Then, the claim follows from the following estimates:

$$\|\omega\|^2 = \int_0^1 \|\omega(r)\|_r^2 dr \leq \int_0^1 Ar^{-b} \|\omega\|_1^2 dr = \frac{A}{1-b} \|\omega\|_1^2 < \infty. \quad \square$$

Lemma 3.3. *Let $dr \wedge \omega \in \Omega^\bullet(X)$. If there are constants $A > 0$ and $b < 1$ so that $r^b \|\omega(t)\|_r^2 \leq As^b \|\omega(t)\|_s^2$ for all $r, s, t \in (0, 1]$ satisfying $r \leq s$, then*

$$\left\| \int_1^r \omega(s) ds \right\|^2 \leq \frac{A}{2(1-b)} \|\omega\|^2.$$

Proof.

$$\begin{aligned} \left\| \int_1^r \omega(s) ds \right\|^2 &= \int_0^1 \left\| \int_1^r \omega(s) ds \right\|_r^2 dr \\ &\leq \int_0^1 \left(\int_r^1 \frac{r^b}{As^b} \|\omega(s)\|_r^2 ds \right) \left(\int_r^1 \frac{As^b}{r^b} ds \right) dr \\ &\leq \int_0^1 \left(\int_r^1 \|\omega(s)\|_s^2 ds \right) \left(\int_r^1 \frac{As^b}{r^b} ds \right) dr \\ &\leq \|\omega\|^2 \int_0^1 \left(\int_r^1 \frac{As^b}{r^b} ds \right) dr. \end{aligned}$$

Calculation A.1 shows that for $b < 1$ (including $b = -1$, which is calculated as a special case),

$$\int_0^1 \left(\int_r^1 \frac{As^b}{r^b} ds \right) dr = \frac{A}{2(1-b)}. \quad \square$$

Lemma 3.4. *Let $\alpha \in L^2\Omega^\bullet(X)$. If there is a constant $A > 0$ so that $r\|\alpha(t)\|_r^2 \geq As\|\alpha(t)\|_s^2$ for all $r, s, t \in (0, 1]$ satisfying $r \leq s$, then there is a sequence of numbers $\varepsilon_k \in (0, 1)$ so that $\varepsilon_k \rightarrow 0$ and $\|\alpha(\varepsilon_k)\|_r \xrightarrow{\varepsilon_k \rightarrow 0} 0$ for any fixed $r \in (0, 1]$. Furthermore, for any $0 < \delta < 1$, we can find such a sequence $\{\varepsilon_k\}$ so that $\alpha(\varepsilon_k)$ converges to 0 in $L^2\Omega^\bullet([\delta, 1] \times L)$.*

Proof. The proof is adapted from that of Lemma 1.2 in [5]. If α satisfies the conditions of the lemma, then for any $r \in (0, 1]$,

$$\int_0^r t^{-1}\|\alpha(t)\|_r^2 dt \leq \int_0^r \frac{1}{Ar}\|\alpha(t)\|_t^2 dt \leq \frac{1}{Ar}\|\alpha\|^2 < \infty.$$

Thus, the function $f(t) = t^{-1}\|\alpha(t)\|_r^2$ is L^1 on $(0, r)$; because $f(t)$ is smooth on $(0, 1]$, it is also L^1 on $(0, 1]$ and on $(0, \frac{1}{k})$. On the other hand, $(t|\ln(t)|)^{-1}$ is not L^1 on any interval $(0, \frac{1}{k})$. If $f(t) \geq (t|\ln(t)|)^{-1}$ for every value in $(0, \frac{1}{k})$, we'd get a contradiction, so let ε_k be any value in $(0, \frac{1}{k})$ so that $f(\varepsilon_k) \leq (\varepsilon_k|\ln(\varepsilon_k)|)^{-1}$. Thus $\varepsilon_k f(\varepsilon_k) = \|\alpha(\varepsilon_k)\|_r^2 \leq |\ln(\varepsilon_k)|^{-1}$ for all ε_k . Since the right hand converges to zero as $\varepsilon_k \rightarrow 0$, $\|\alpha(\varepsilon_k)\|_r^2$ also converges to zero.

Choose such a sequence for a value $0 < \delta < 1$. Because $\|\alpha(t)\|_r \leq A\|\alpha(t)\|_\delta$ for $r \in (\delta, 1]$, we get

$$\int_\delta^1 \|\alpha(\varepsilon_k)\|_r^2 dr \leq (1 - \delta)\frac{1}{A}\|\alpha(\varepsilon_k)\|_\delta.$$

The righthand side approaches zero, so $\alpha(\varepsilon_k)$ converges to 0 in $L^2\Omega^\bullet([\delta, 1] \times L)$. \square

Lemma 3.5. *Let $dr \wedge \omega \in \Omega^\bullet(X)$ be an L^2 form. If there is a constant $A > 0$ so that $\|\omega(t)\|_r^2 \geq A\|\omega(t)\|_s^2$ for all $r, s, t \in (0, 1]$ satisfying $r \leq s$, then for any fixed $r \in (0, 1]$,*

$$\int_0^r \omega(t) dt = \lim_{\varepsilon \rightarrow 0} \int_\varepsilon^r \omega(t) dt$$

exists as a form on the link in $L^2\Omega^\bullet(L, g_r)$. Furthermore, for any $0 < \delta < 1$ we get that $\int_\varepsilon^r \omega(t) dt$ converges in $L^2\Omega^\bullet([\delta, 1] \times L)$. We also have that $\int_0^r \omega$ is an L^2 form in $L^2\Omega^\bullet(X)$, and

$$\left\| \int_0^r \omega \right\|^2 \leq \frac{1}{A} \|\omega\|^2.$$

Proof. Notice that if $0 < a < b < r \leq 1$,

$$\left\| \int_a^b \omega(t) dt \right\|_r^2 \leq \int_a^b \|\omega(t)\|_r^2 dt \leq \int_a^b \frac{1}{A} \|\omega(t)\|_t^2 dt.$$

By assumption, $\int_0^r \|\omega(t)\|_t^2 dt$ is finite. For any $\eta > 0$, we can choose b small enough so that $\left\| \int_a^b \omega(t) dt \right\|_r < \eta$. Thus, $\int_\varepsilon^r \omega(t) dt$ forms a Cauchy sequence with respect to ε in $L^2\Omega^\bullet(L, g_r)$, so it converges. Also, for a fixed $0 < \delta < 1$, then

$$\int_\delta^1 \left\| \int_a^b \omega(t) dt \right\|_r^2 dr \leq \frac{1}{A} \int_\delta^1 \left\| \int_a^b \omega(t) dt \right\|_\delta^2 dr = (1 - \delta) \left\| \int_a^b \omega(t) dt \right\|_\delta^2.$$

As we can make this value as small as desired by choosing b small enough, we get that $\int_\varepsilon^r \omega(t) dt$ converges to $\int_0^r \omega(t) dt$ in $L^2\Omega^\bullet([\delta, 1] \times L)$ for any $0 < \delta < 1$.

Now that we've shown $\int_0^r \omega$ exists for each value of r , we will show it is in $L^2\Omega^\bullet(X)$:

$$\begin{aligned} \left\| \int_0^r \omega(t) dt \right\|^2 &= \int_0^1 \left(\left\| \int_0^r \omega(t) dt \right\|_r^2 \right) dr \\ &\leq \int_0^1 \left(\int_0^r \|\omega(t)\|_r^2 dt \right) dr \\ &\leq \int_0^1 \left(\int_0^r \frac{1}{A} \|\omega(t)\|_t^2 dt \right) dr \\ &\leq \frac{1}{A} \int_0^1 \left(\int_0^1 \|\omega(t)\|_t^2 dt \right) dr \\ &= \frac{1}{A} \|\omega\|^2. \end{aligned}$$

□

3.3 The Operators K_a and Their Homotopy Formulas

Definition 3.6. For a number $a \in (0, 1)$, define the operator $K_a : \Omega^\bullet(X) \rightarrow \Omega^\bullet(X)$ as

$$K_a(\phi + dr \wedge \omega) = \int_a^r \omega(t) dt.$$

If the limit

$$\lim_{\varepsilon \rightarrow 0} \int_\varepsilon^r \omega(t) dt$$

exists for all r , then we define the operator K_0 as

$$K_0(\phi + dr \wedge \omega) = \lim_{\varepsilon \rightarrow 0} \int_\varepsilon^r \omega.$$

Here the codomain of K_0 is perhaps best described as paths in $L^2\Omega^\bullet(L)$, as $K_0\alpha$ is not necessarily in $\Omega^\bullet(X)$ nor is it necessarily in $L^2\Omega^\bullet(X)$.

Lemma 3.7. *If $a \in (0, 1]$, then given a smooth form $\alpha = \phi + dr \wedge \omega \in \Omega^\bullet(X)$, we have the identity*

$$dK_a\alpha + K_a d\alpha = \alpha - \phi(a).$$

Proof. This is a straightforward calculation:

$$\begin{aligned} dK_a(\phi + dr \wedge \omega) &= d\left(\int_a^r \omega\right) \\ &= d_L\left(\int_a^r \omega\right) + dr \wedge \frac{\partial}{\partial r} \int_a^r \omega \\ &= \int_a^r d_L\omega + dr \wedge \omega. \end{aligned}$$

$$\begin{aligned}
K_a d(\phi + dr \wedge \omega) &= K_a \left(d_L \phi + dr \wedge \left(\frac{\partial \phi}{\partial r} - d_L \omega \right) \right) \\
&= \int_a^r \left(\frac{\partial \phi}{\partial r} - d_L \omega \right) \\
&= \phi(r) - \phi(a) - \int_a^r d_L \omega.
\end{aligned}$$

Adding these two calculations together, we get our result. \square

Definition 3.8. Define the operator K as

$$K(\phi + dr \wedge \omega) = \begin{cases} \int_1^r \omega & \text{for } \deg(\omega) < \dim_{\mathbb{R}}(X)/2 \\ \int_0^r \omega & \text{for } \deg(\omega) \geq \dim_{\mathbb{R}}(X)/2. \end{cases}$$

This operator may not be bounded on any suitable domain, nor is it even necessarily well-defined for the obvious choices of domains. We will always be careful to show it exists before using it.

Notice also that the definition of K depends not on the quasi-isometry class of X , but on the specific way that X is decomposed as $(0, 1] \times L$. Whether or not K is bounded (or even well-defined) cannot be found merely by knowing the quasi-isometry class of X . In fact, [15] makes the mistake of claiming that K is bounded for a given decomposition when it may not be. Part of the contribution of [17] is in finding an alternative decomposition for which K is bounded.

The following lemma will be helpful in showing certain forms are in the domain of \bar{d} .

Lemma 3.9. *Take a sequence of forms $\{\alpha_i\} \subset \Omega^\bullet(X)$ and forms $\alpha, \beta \in L^2 \Omega^\bullet(X)$ so that for all $0 < \varepsilon < 1$, $\{\alpha_i\}$ converges in $L^2 \Omega^\bullet([\varepsilon, 1] \times L)$ to α and $\left\{ d\alpha_i \Big|_{[\varepsilon, 1] \times L} \right\}$*

converges to $\beta|_{[\varepsilon,1] \times L}$ in $L^2\Omega^\bullet([\varepsilon,1] \times L)$. Then $\alpha \in \text{dom}(\bar{d})$ and $\bar{d}\alpha = \beta$.

Proof. We prove this by using the property that \bar{d} is the weak extension of d . Take a compactly supported γ . Then there is some $0 < \varepsilon < 1$ so that $\gamma(r) = 0$ for $r < \varepsilon$. Use the notation $\langle \alpha_1, \alpha_2 \rangle_{[\varepsilon,1]}$ to be the global inner product on $[\varepsilon,1] \times L$ with its induced metric. Then

$$\begin{aligned} \langle \alpha, \delta\gamma \rangle &= \langle \alpha, \delta\gamma \rangle_{[\varepsilon,1]} \\ &= \lim_{i \rightarrow \infty} \langle \alpha_i, \delta\gamma \rangle_{[\varepsilon,1]} \\ &= \lim_{i \rightarrow \infty} \langle d\alpha_i, \gamma \rangle_{[\varepsilon,1]} \\ &= \langle \beta, \gamma \rangle_{[\varepsilon,1]} \\ &= \langle \beta, \gamma \rangle. \end{aligned}$$

Therefore, $\alpha \in \text{dom}(\bar{d})$, and $\bar{d}\alpha = \beta$. □

Theorem 3.10. Give a space $X = (0,1] \times L$ with real dimension $\dim_{\mathbb{R}}(X) = 2n$.

Assume that

1. K as defined in Definition 3.8 exists and is a bounded operator $K : L^2\Omega^\bullet(X) \rightarrow L^2\Omega^\bullet(X)$, and
2. if $\phi \in L^2\Omega^k(L)$ for $k < n$, then $\phi \in L^2\Omega^k(X)$ when considered as a form on X by extending independently of r .

Take some $\alpha = \phi + dr \wedge \omega \in \text{dom}(d)$.

1. If $\deg(\alpha) < n$, then $K\alpha \in \text{dom}(d)$ and

$$d(K\alpha) + K(d\alpha) = \alpha - \phi(1).$$

2. If $\deg(\alpha) > n$ and

$$r\|\alpha(t)\|_r^2 \geq As\|\alpha(t)\|_s^2$$

for some constant $A > 0$ and for all $r, s, t \in (0, 1]$ with $r \leq s$, then $K\alpha \in \text{dom}(\bar{d})$ and

$$K(d\alpha) + \bar{d}(K\alpha) = \alpha.$$

3. If $\text{deg}(\alpha) = n$ and

$$r\|\phi(t)\|_r^2 \geq As\|\phi(t)\|_s^2$$

for some constant $A > 0$ and for all $r, s, t \in (0, 1]$ with $r \leq s$, then there is some $\beta \in L^2\Omega^{n-1}(L)$ so that $\beta + K\alpha \in \text{dom}(\bar{d})$ and

$$K(d\alpha) + \bar{d}(\beta + K\alpha) = \alpha.$$

Proof. This proof is similar to the proof of the specific case of metric horns as given in [5]. Take a form $\alpha \in \text{dom}(d)$.

1. In this case, $K\alpha = \int_1^r \omega$, and we have the calculation

$$dK\alpha + Kd\alpha = \alpha - \phi(1)$$

from Lemma 3.7. Because $\alpha \in \text{dom}(d)$, α and $d\alpha$ are L^2 forms. Because K is bounded, $Kd\alpha$ is an L^2 -form. By assumption 2, $\phi(1)$ is an L^2 form. Thus, $dK\alpha$ is also an L^2 form, and $K\alpha \in \text{dom}(d)$.

2. We can write α as

$$\alpha = \phi + dr \wedge \omega$$

Pick some $0 < \delta < 1$. Let $\{\varepsilon_k\}$ be the sequence found for the given δ in Lemma 3.4 for the form ϕ . $K_{\varepsilon_k}\alpha$ may not be an L^2 -form, but for a fixed r , it does define a smooth form on the link. We can apply Lemma 3.5 to show that the limit

$$K\alpha = \lim_{\varepsilon \rightarrow 0} K_\varepsilon \alpha$$

converges in $L^2\Omega^\bullet([\delta, 1] \times L)$ for any fixed value $0 < \delta < 1$, and $K\alpha$ is an L^2 form on X . Similarly, $K_\varepsilon(d\alpha)$ converges in $L^2\Omega^\bullet([\delta, 1] \times L)$ to $K(d\alpha)$, and $K(d\alpha) \in L^2\Omega(X)$ is an L^2 -form.

We also know that Lemma 3.7 says we have the following homotopy formula on smooth forms

$$d(K_{\varepsilon_k}\alpha) = \alpha - \phi(\varepsilon_k) - K_{\varepsilon_k}(d\alpha).$$

Using Lemmas 3.4 and 3.5, we see that the righthand side converges to $\alpha - K_0(d\alpha)$ in $L^2\Omega^\bullet([\delta, 1] \times L)$ for all $0 < \delta < 1$. Therefore, we satisfy the conditions of Lemma 3.9. This tells us $K_0\alpha$ is in the domain of \bar{d} , and we have the formula

$$\bar{d}(K_0\alpha) = \alpha - K_0(d\alpha),$$

which given our definition of K , is the same as

$$K(d\alpha) + \bar{d}(K\alpha) = \alpha.$$

3. We can write

$$\alpha = \phi + dr \wedge \omega,$$

where $\deg(\phi) = n$ and $\deg(\omega) = n - 1$.

Notice that as smooth forms,

$$K_\varepsilon(d\alpha) = \phi - \phi(\varepsilon) - \int_\varepsilon^r d_L\omega.$$

For fixed r , $\int_\varepsilon^r d_L\omega \xrightarrow{\varepsilon \rightarrow 0} \int_0^r d_L\omega$ converges to a form in $L^2\Omega(L)$ because all the other terms converge as forms on the link. In particular, for $r = 1$,

$\int_{\varepsilon}^1 d_L \omega$ converges. Now, $\int_{\varepsilon}^1 d_L \omega \in \text{Range}(\bar{d}_L)$ for all ε ; because $\text{Range}(\bar{d}_L)$ is closed, $\int_0^1 d_L \omega$ is also in the range of \bar{d}_L . Thus, $\int_0^1 d_L \omega = \bar{d}_L \beta$ for some $\beta \in \text{dom}(\bar{d}_L) \subset L^2 \Omega^{m-1}(L)$.

Now pick some $0 < \delta < 1$. Let $\{\varepsilon_k\}$ be the sequence found for the given δ in Lemma 3.4 for the form ϕ . $K_{\varepsilon_k} \alpha$ may not be an L^2 -form, but for a fixed r , it does define a smooth form on the link. We can apply Lemma 3.5 to show that the limit

$$K\alpha = \lim_{\varepsilon \rightarrow 0} K_{\varepsilon} \alpha$$

converges in $L^2 \Omega^{\bullet}([\delta, 1] \times L)$ for any fixed value $0 < \delta < 1$, and $K\alpha$ is an L^2 form on X . Similarly, $K_{\varepsilon}(d\alpha)$ converges in $L^2 \Omega^{\bullet}([\delta, 1] \times L)$ to $K(d\alpha)$, and $K(d\alpha) \in L^2 \Omega(X)$ is an L^2 -form.

We can also compute the following identity on smooth forms:

$$d(K_1 \alpha) = \alpha - \phi(\varepsilon_k) - K_{\varepsilon_k}(d\alpha) - \int_{\varepsilon_k}^1 d_L \omega.$$

Therefore,

$$\bar{d}(K_1 \alpha + \beta) = \alpha - \phi(\varepsilon_k) - K_{\varepsilon_k}(d\alpha) - \int_0^{\varepsilon_k} d_L \omega.$$

Using Lemmas 3.4 and 3.5, we see that the righthand side converges to $\alpha - K_0(d\alpha)$ in $L^2 \Omega^{\bullet}([\delta, 1] \times L)$ for all $0 < \delta < 1$. Therefore, we satisfy the conditions of Lemma 3.9. This tells us $K\alpha + \beta$ is in the domain of \bar{d} , and we have the formula

$$\bar{d}(\beta + K\alpha) = \alpha - K(d\alpha).$$

$$K(d\alpha) + \bar{d}(\beta + K\alpha) = \alpha. \quad \square$$

Theorem 3.11. *Assume that X satisfies the conditions of Theorem 3.10. Then for $k < n$,*

$$H_{(2)}^k(X) = H_{dR}^k(L).$$

Proof. Consider the subcomplexes $A^\bullet \subset \Omega^\bullet(L)$ and $B^\bullet \subset \text{dom}(d)$ where

$$A^k = \begin{cases} 0 & k \geq n \\ \text{closed forms} & k = n - 1 \\ \Omega^k(L) & k < n - 1 \end{cases}$$

and

$$B^k = \begin{cases} 0 & k \geq n \\ \text{closed forms} & k = n - 1 \\ (\text{dom}(d))^k & k < n - 1 \end{cases}$$

Then $H^k(A^\bullet) = H_{dR}^k(L)$ and $H^k(B^\bullet) = H_{(2)}^k(X)$ for $k < n$. Assumption 2 of Theorem 3.10 says there is an inclusion $\Omega^k(L) \hookrightarrow L^2\Omega^k(X)$ for $k < n$, which implies there is an inclusion

$$A^\bullet \hookrightarrow B^\bullet.$$

Case 1 of Theorem 3.10 says this inclusion induces an isomorphism on cohomology.

□

Multiply-Warped Products

4.1 Hodge Theory for Manifolds with Boundary

In this section, we will recall some standard results on the Hodge Theory of compact manifolds with boundary.

Let (N, g) be an n -dimensional, compact manifold with a smooth boundary, where g is a Riemannian metric on the interior of N , and g extends continuously to the boundary ∂N . There is a tubular neighborhood T of ∂N which is diffeomorphic to $\partial N \times [0, 1)$, and there is a coordinate q on T so that $q|_{\partial N} = 0$ and $\frac{\partial}{\partial q}$ is perpendicular to ∂N .

Definition 4.1. In the neighborhood T , we can write a form as $\alpha = \alpha_N + dq \wedge \alpha_D$. If $\alpha_D|_{\partial N} = 0$, we say α satisfies *Neumann boundary conditions*, and we denote the space of such forms $\Omega_N^\bullet(N)$. If $\alpha_N|_{\partial N} = 0$, we say α satisfies *Dirichlet boundary conditions*, and we denote the space of such forms $\Omega_D^\bullet(N)$.

If the boundary is empty, then $\Omega_N^\bullet(N) = \Omega_D^\bullet(N) = \Omega^\bullet(N)$. Notice that if $j : \partial N \hookrightarrow N$ is the inclusion of the boundary, then α satisfies Dirichlet bound-

ary conditions if and only if $j^*\alpha = 0$, and α satisfies Neumann boundary conditions if and only if $j^*(\star\alpha) = 0$.

In particular, the Hodge star gives a linear isomorphism $\star : \Omega_N^p(N) \rightarrow \Omega_D^{n-p}(N)$.

Definition 4.2. A *strongly harmonic form* is a form ω so that $d\omega = 0$ and $\delta\omega = 0$.

A *weakly harmonic form* is a form ω so that $\Delta\omega = d\delta\omega + \delta d\omega = 0$.

Lemma 4.3. (*Stoke's Theorem, [24], Proposition 2.1.2*)

If α is a k -form and β is a $(k+1)$ -form, then

$$\int_{\partial N} \alpha \wedge \star\beta = \int_N d\alpha \wedge \star\beta - \int_N \alpha \wedge \star\delta\beta.$$

In other words,

$$\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle + \int_{\partial N} \alpha \wedge \star\beta.$$

Remark 4.4. On a compact manifold without boundary, strongly and weakly harmonic forms are the same. However, if $\partial N \neq \emptyset$, then while all strongly harmonic forms are also weakly harmonic, the converse is not necessarily true. For example, linear functions on the interval $[0, 1]$ with the usual metric are weakly harmonic forms, but they are not strongly harmonic unless they are constant.

Theorem 4.5. (*Hodge decomposition for manifolds with boundary, [24], Cor. 2.4.9*)

Let (N, g) be a compact Riemannian manifold, possibly with boundary, so that g is smooth on the interior and extends continuously to the boundary. Then there are the following orthogonal decompositions of the smooth forms

$$\begin{aligned} \Omega^\bullet(N) &= \mathcal{E} \oplus \mathcal{H} \oplus \mathcal{C} \\ &= \mathcal{E} \oplus \mathcal{H}_{ex} \oplus \mathcal{H}_N \oplus \mathcal{C} \\ &= \mathcal{E} \oplus \mathcal{H}_{co} \oplus \mathcal{H}_D \oplus \mathcal{C} \end{aligned}$$

where

\mathcal{E} = forms which are the differential of a Dirichlet form

\mathcal{C} = forms which are the codifferential of a Neumann form

\mathcal{H} = strongly harmonic forms

\mathcal{H}_{ex} = strongly harmonic forms which are exact

\mathcal{H}_N = strongly harmonic forms which are Neumann

\mathcal{H}_{co} = strongly harmonic forms which are coexact

\mathcal{H}_D = strongly harmonic forms which are Dirichlet.

Note that \mathcal{H}_N is orthogonal to exact forms and \mathcal{H}_D is orthogonal to coexact forms.

Theorem 4.6. ([24], Cor. 3.4.8) If $\alpha \in \mathcal{E} \oplus \mathcal{H}_{ex} \oplus \mathcal{C}$, then there is a form $\beta \in \mathcal{E} \oplus \mathcal{H}_{ex} \oplus \mathcal{C}$ which satisfies the equation

$$\Delta\beta = d\delta\beta + \delta d\beta = \alpha$$

and the boundary conditions

$$\beta \in \Omega_N^\bullet(N), \quad d\beta \in \Omega_N^\bullet(N).$$

This solution β is called the solution to the Poisson equation for α with Neumann boundary conditions. Furthermore, there is some constant C independent of α so that

$$\|\beta\| \leq C\|\alpha\|, \text{ and}$$

$$\|\delta\beta\| \leq C\|\alpha\|.$$

Definition 4.7. The Neumann Green's operator $G : \Omega^p(L) \rightarrow \Omega^p(L)$ is the operator which takes a form α to the solution of the Poisson equation for $\alpha - A\alpha$ with Neumann boundary conditions, where $A : \Omega^p(L) \rightarrow \mathcal{H}_N^p$ is the orthogonal projection operator to \mathcal{H}_N^p .

Lemma 4.8. *The operator G satisfies $dG\alpha = Gd\alpha$. This implies δG is a bounded homotopy operator between the identity operator and the projection to the Neumann strongly harmonic forms. In other words,*

$$(d\delta G + \delta Gd)\alpha = \alpha - A\alpha.$$

Proof. First we must show that $dG\alpha$ satisfies $\Delta dG\alpha = d\alpha$:

$$\begin{aligned} d\delta G\alpha + \delta dG\alpha &= \alpha - A\alpha \\ d(d\delta G\alpha + \delta dG\alpha) &= d(\alpha - A\alpha) \\ d\delta dG\alpha &= d\alpha \\ d\delta dG\alpha + \delta ddG\alpha &= d\alpha \\ \Delta dG\alpha &= d\alpha - Ad\alpha \end{aligned}$$

where $Ad\alpha = 0$ as all Neumann strongly harmonic forms are orthogonal to exact forms. Next, we must check that $dG\alpha$ satisfies the boundary conditions. But $dG\alpha \in \Omega_N^\bullet(N)$ by assumption on G , and $d(dG\alpha) = 0 \in \Omega_N^\bullet(N)$ trivially. \square

Remark 4.9. This lemma is why we use Neumann boundary conditions. There are other boundary conditions that would give us similar theorems to Theorem 4.6. These would give us potentially different Green's operators. However, they will not necessarily satisfy $dG\alpha = Gd\alpha$.

Lemma 4.10. 1. *(Symmetry of the Neumann Laplacian) If $\alpha, d\alpha, \beta$, and $d\beta$ are all in $\Omega_N^\bullet(N)$, then*

$$\langle \Delta\alpha, \beta \rangle = \langle \alpha, \Delta\beta \rangle.$$

2. *(Positivity of the Neumann Laplacian) If α and $d\alpha$ are both in $\Omega_N^\bullet(N)$, then*

$$\langle \Delta\alpha, \alpha \rangle \geq 0.$$

Proof. First notice that if $\gamma \in \Omega^p(N)$ and $\xi \in \Omega_N^{p+1}(N)$, then

$$\langle d\gamma, \xi \rangle = \langle \gamma, \delta\xi \rangle + \int_{\partial N} \gamma \wedge \star\xi = \langle \gamma, \delta\xi \rangle.$$

Now we can start our proof.

1. Assume $\alpha, \beta, d\alpha, d\beta \in \Omega_N^\bullet(N)$. Then

$$\begin{aligned} \langle \Delta\alpha, \beta \rangle &= \langle d\delta\alpha + \delta d\alpha, \beta \rangle \\ &= \langle d\delta\alpha, \beta \rangle + \langle \delta d\alpha, \beta \rangle \\ &= \langle \delta\alpha, \delta\beta \rangle + \langle d\alpha, d\beta \rangle \\ &= \langle \alpha, d\delta\beta \rangle + \langle \alpha, \delta d\beta \rangle \\ &= \langle \alpha, \Delta\beta \rangle. \end{aligned}$$

2. Assume $\alpha, d\alpha \in \Omega_N^\bullet(N)$. Then

$$\begin{aligned} \langle \Delta\alpha, \alpha \rangle &= \langle d\delta\alpha + \delta d\alpha, \alpha \rangle \\ &= \langle \delta\alpha, \delta\alpha \rangle + \langle d\alpha, d\alpha \rangle \\ &= \|\delta\alpha\|^2 + \|d\alpha\|^2 \geq 0. \end{aligned} \quad \square$$

4.2 Hodge Theory for Product Metrics

We now switch our attention to spaces of the form $N = N_1 \times N_2 \times \cdots \times N_m$, where each factor N_k is a compact oriented manifold with boundary of dimension n_k . Then N is a manifold with corners. For such N , the tangent bundle of N decomposes as a direct sum of smaller bundles

$$T_*N = \bigoplus_{k=1}^m T_*N_k.$$

This implies that for each vector field $v \in \Gamma(T_*N)$, there is a unique decomposition $v = v_1 + \cdots + v_m \in \Gamma(\bigoplus_{k=1}^m T_*N_k)$.

Assume each N_k comes with a metric g_{N_k} . Then we can create a product metric g by setting

$$g(v, w) = \begin{cases} 0 & v \in T_*N_j, w \in T_*N_k \text{ for } j \neq k \\ g_{N_k}(v, w) & v, w \in T_*N_k \end{cases}$$

and extending bilinearly. The cotangent space can likewise be decomposed as

$$T^*N = \bigoplus_{k=1}^m T^*N_k.$$

Therefore, any 1-form can be written as

$$\alpha = \alpha_1 + \dots + \alpha_m$$

where $\alpha_k \in T^*N_k$ at every point of the link. This induces a decomposition on the space of differential forms on the link $\Omega^\bullet(N)$:

$$\Omega^\bullet(N) = \bigoplus_{0 \leq p_1 \leq n_1} \dots \bigoplus_{0 \leq p_m \leq n_m} \Omega^{(p_1, \dots, p_m)}(N)$$

where at each point $x = (x_1, \dots, x_m)$, each form in $\Omega^{(p_1, \dots, p_m)}(N)$ is spanned by wedges of forms $\phi = \phi_1 \wedge \phi_2 \wedge \dots \wedge \phi_m$ so that $\phi_k \in \bigwedge^{p_k} T_{x_k}^*N_k$. The degree of each form in $\Omega^{(p_1, \dots, p_m)}(N)$ is $\sum_{k=1}^m p_k$.

Each N_k also comes with a volume form dV_k . This allows us to define a volume form on N :

Definition 4.11. We define the volume form dV on N to be

$$dV = dV_1 \wedge \dots \wedge dV_m$$

where dV_j is the volume form on N_j pulled back through the projection map $N \rightarrow N_k$. Notice that this is a choice of orientation, and it may depend on the order of the factors N_1, \dots, N_m .

Lemma 4.12. *Take a form $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_m \in \Omega^{(p_1, \dots, p_m)}(N)$ where α_k is the pullback of a p_k -form on N_k through the projection map $N \rightarrow N_k$. Let $|\alpha|$ and $\|\alpha\|$ be the pointwise and global norms on α induced by g , respectively, and let $|\alpha_k|_{N_k}$ and $\|\alpha_k\|_{N_k}$ be the local and global norms on α_k considered as a form on N_k . Then*

$$\begin{aligned} |\alpha| &= |\alpha_1|_{N_1} \cdots |\alpha_m|_{N_m} \\ \|\alpha\| &= \|\alpha_1\|_{N_1} \cdots \|\alpha_m\|_{N_m} \end{aligned}$$

Proof. We prove the identity for the pointwise norm first, and we prove it by induction on m . First assume $m = 2$. There is an orthonormal basis $\xi_1, \dots, \xi_{n_1}, \beta_1, \dots, \beta_{n_2}$ of $T_x^*N = T_{x_1}^*N_1 \otimes T_{x_2}^*N_2$, where $\xi_k \in T_{x_1}^*N_1$ and $\beta_k \in T_{x_2}^*N_2$ for each k . We can write

$$\alpha_1 = \sum_{|I|=p_1} a_I \xi_{I(1)} \wedge \cdots \wedge \xi_{I(p_1)}.$$

where I is a p_1 -tuple where $I(j) < I(k)$ for $j < k$. The collection $\{\xi_{I(1)} \wedge \cdots \wedge \xi_{I(p_1)}\}_I$ forms an orthonormal basis of $\bigwedge^{p_1} T_{x_1}^*N_1$, so we can write

$$|\alpha_1|^2 = \sum_{|I|=p_1} |a_I|^2.$$

Similarly, we can write

$$\alpha_2 = \sum_{|J|=p_2} b_J \beta_{J(1)} \wedge \cdots \wedge \beta_{J(p_2)},$$

and

$$|\alpha_2|^2 = \sum_{|J|=p_2} |b_J|^2.$$

Then we calculate

$$\begin{aligned} \alpha_1 \wedge \alpha_2 &= \sum_{|I|=p_1} \sum_{|J|=p_2} (a_I \xi_{I(1)} \wedge \cdots \wedge \xi_{I(p_1)}) \wedge (b_J \beta_{J(1)} \wedge \cdots \wedge \beta_{J(p_2)}) \\ &= \sum_{|I|=p_1} \sum_{|J|=p_2} a_I b_J \xi_{I(1)} \wedge \cdots \wedge \xi_{I(p_1)} \wedge \beta_{J(1)} \wedge \cdots \wedge \beta_{J(p_2)}. \end{aligned}$$

As the collection $\{\xi_{I(1)} \wedge \cdots \wedge \xi_{I(p_1)} \wedge \beta_{J(1)} \wedge \cdots \wedge \beta_{J(p_2)}\}_{I,J}$ forms an orthonormal basis for $\bigwedge^{p_1+p_2} T_x^* N$, we get

$$\begin{aligned} |\alpha_1 \wedge \alpha_2|^2 &= \sum_{I,J} |a_I|^2 |b_J|^2 \\ &= \left(\sum_I |a_I|^2 \right) \left(\sum_J |b_J|^2 \right) \\ &= |\alpha_1|^2 |\alpha_2|^2. \end{aligned}$$

This concludes our base case. Now assume we've shown what is desired for all spaces $N_1 \times \cdots \times N_m$. Then $N_1 \times \cdots \times N_m \times N_{m+1} = (N_1 \times \cdots \times N_m) \times N_{m+1}$. By what we showed for $m = 2$,

$$|\alpha_1 \wedge \cdots \wedge \alpha_m \wedge \alpha_{m+1}| = |\alpha_1 \wedge \cdots \wedge \alpha_m| \cdot |\alpha_{m+1}|.$$

By our inductive assumption,

$$|\alpha_1 \wedge \cdots \wedge \alpha_m| = |\alpha_1| \cdots |\alpha_m|.$$

Thus,

$$|\alpha_1 \wedge \cdots \wedge \alpha_{m+1}| = |\alpha_1| \cdots |\alpha_{m+1}|.$$

This concludes the result for the pointwise norm.

For the global identity, we use Tonelli's Theorem:

$$\begin{aligned} \|\alpha\|^2 &= \int_{N_1 \times \cdots \times N_m} |\alpha|^2 dV \\ &= \int_{N_1 \times \cdots \times N_m} |\alpha_1|^2 \cdots |\alpha_m|^2 dV_1 \wedge \cdots \wedge dV_m \\ &= \left(\int_{N_1} |\alpha_1|^2 dV_1 \right) \cdots \left(\int_{N_m} |\alpha_m|^2 dV_m \right) \\ &= \|\alpha_1\|_{N_1}^2 \cdots \|\alpha_m\|_{N_m}^2, \end{aligned}$$

This concludes the proof. □

By an abuse of notation, denote by $\Omega^j(N_k) \subset \Omega^j(N)$ the pullback of the j -forms on N_k through the projection map $\pi_k : N \rightarrow N_k$. Then consider the injection

$$E : \bigoplus_{p_1 + \dots + p_m = p} \Omega^{p_1}(N_1) \otimes \dots \otimes \Omega^{p_m}(N_m) \hookrightarrow \Omega^{(p_1, \dots, p_m)}(N)$$

obtained via

$$\alpha_1 \otimes \dots \otimes \alpha_m \mapsto \pi_1^* \alpha_1 \wedge \dots \wedge \pi_m^* \alpha_m.$$

Give the left hand side the norm

$$\|\alpha_1 \otimes \dots \otimes \alpha_m\| = \left(\int_{N_1} |\alpha_1|^2 dV_1 \right)^{1/2} \dots \left(\int_{N_m} |\alpha_m|^2 dV_m \right)^{1/2}.$$

By Lemma 4.12, our injection E is an isometry onto its image.

Lemma 4.13. *The induced map \hat{E} on the completions*

$$\hat{E} : \bigoplus_{p_1 + \dots + p_m = p} L^2 \Omega^{p_1}(N_1) \otimes \dots \otimes L^2 \Omega^{p_m}(N_m) \rightarrow L^2 \Omega^{(p_1, \dots, p_m)}(N)$$

where the lefthand side is tensor product of Hilbert spaces is an isomorphism of Hilbert spaces.

Proof. Because E is injective, so is \hat{E} . By the BLT Theorem (see [21]), \hat{E} is bounded. If we can show that \hat{E} is surjective, then we'll be done by the Inverse Mapping Theorem. We shall do this now.

At any point $(x_1, \dots, x_m) \in N_1 \times \dots \times N_m$, there is a neighborhood $U = U_1 \times \dots \times U_m$, where each $U_k \subseteq N_k$ is a neighborhood of x_k such that there are coordinate functions $\{u_{kj}\}_{j=1}^{n_k}$ where du_{kj} span T^*U_k at each point on U_k . By an abuse of notation, we also allow u_{kj} and du_{kj} to be functions and 1-forms on U respectively by pulling back through π_k . Thus, $\bigcup_{k=1}^m \{du_{kj}\}_{j=1}^{n_k}$ forms a basis of T^*U at each point in U , and the collection of all possible wedges

$$\left\{ du_{1j_{11}} \wedge \dots \wedge du_{1j_{1p_1}} \wedge du_{2j_{21}} \wedge \dots \wedge du_{mj_{mp_m}} \right\}_{1 \leq j_{ki} \leq n_k}$$

locally spans $\bigwedge^{p_1+\dots+p_m} T^*U$.

Because each N_k is compact, we can cover N_k by a finite collection $\left\{U_k^{\ell_k}\right\}_{\ell_k}$ of such neighborhoods, and we can find a partition of unity $\left\{\chi_k^{\ell_k}\right\}_{\ell_k}$ on N_k subordinate to this cover. Then the collection $\left\{U^{(\ell_1,\dots,\ell_m)}\right\} = \left\{U_1^{\ell_1} \times \dots \times U_m^{\ell_m}\right\}_{(\ell_1,\dots,\ell_m)}$ of such neighborhoods covers N . If we extend each $\chi_k^{\ell_k}$ independently of the other variables, and set $\chi^{(\ell_1,\dots,\ell_m)} = \prod_{k=1}^m \chi_k^{\ell_k}$, the collection $\left\{\chi^{(\ell_1,\dots,\ell_m)}\right\}_{(\ell_1,\dots,\ell_m)}$ is a partition of unity subordinate to the cover $\left\{U^{(\ell_1,\dots,\ell_m)}\right\}$. To see this, first note that it is clear each $\chi^{(\ell_1,\dots,\ell_m)}$ is compactly supported in $U^{(\ell_1,\dots,\ell_m)}$, and that

$$\sum_{(\ell_1,\dots,\ell_m)} \chi^{(\ell_1,\dots,\ell_m)} = \sum_{(\ell_1,\dots,\ell_m)} \left(\prod_k \chi_k^{\ell_k} \right) = \prod_k \left(\sum_{\ell_k} \chi_k^{\ell_k} \right) = \prod_k 1 = 1.$$

Therefore, we can write any (p_1, \dots, p_m) -form α as

$$\alpha = \sum_{(\ell_1,\dots,\ell_m)} \chi^{(\ell_1,\dots,\ell_m)} \alpha,$$

and we can write $\chi^{(\ell_1,\dots,\ell_m)} \alpha$ in local coordinates as

$$\chi^{(\ell_1,\dots,\ell_m)} \alpha = \sum_{1 \leq j_{ki} \leq n_k} \chi^{(\ell_1,\dots,\ell_m)} a_J du_{1j_{11}} \wedge \dots \wedge du_{1j_{1p_1}} \wedge du_{2j_{21}} \wedge \dots \wedge du_{mj_{mp_m}}.$$

Because α is an L^2 form, the functions $\chi^{(\ell_1,\dots,\ell_m)} a_J$ are L^2 functions. It is a well known theorem that $L^2(N_1 \times \dots \times N_m, g_{N_1} + g_{N_2} + \dots + g_{N_m})$ is isomorphic to $L^2(N_1, g_1) \otimes \dots \otimes L^2(N_m, g_m)$, where the tensor product is a tensor product of Hilbert spaces; see Theorem II.10 of [21] for more details. Thus, we can write the functions as the infinite sum of separable functions:

$$\chi^{(\ell_1,\dots,\ell_m)} a_J = \sum_{i=1}^{\infty} b_{1i} b_{2i} \dots b_{mi},$$

where b_{ki} is the pullback of a function on N_k . Therefore, we can write α as the infinite sum

$$\alpha = \sum_{i=1}^{\infty} \sum_{(\ell_1, \dots, \ell_m)} \sum_{1 \leq j_{ki} \leq n_k} (b_{1i} du_{1j_{11}} \wedge \dots \wedge du_{1j_{1p_1}}) \wedge \dots \wedge (b_{mi} du_{1j_{m1}} \wedge \dots \wedge du_{mj_{mp_m}}) .$$

Each of the terms inside the k^{th} pair of parentheses is the pullback of a form on N_k , so each of the terms in this sum is in the image of E . Therefore, α is in the image of \widehat{E} . \square

This lemma will help us a lot in the definition of some of the operators in this chapter.

Definition 4.14. The exterior derivative d maps $\Omega^{(p_1, \dots, p_m)}(N)$ into $\bigoplus_{j=1}^m \Omega^{(p_1, \dots, p_j+1, \dots, p_m)}(N)$.

Define the *partial differential*

$$d_{N_k} : \Omega^{(p_1, \dots, p_m)}(N) \rightarrow \Omega^{(p_1, \dots, p_k+1, \dots, p_m)}(N)$$

to be the corresponding component of d . Define d_{N_k} on all of $\Omega^p(N)$ by extending \mathbb{R} -linearly.

Lemma 4.15. If $\alpha = \alpha_1 \wedge \dots \wedge \alpha_m$ where α_k is the pullback of a p_k -form on N_k , then

$$d_{N_k} \alpha = (-1)^{p_1 + \dots + p_{k-1}} \alpha_1 \wedge \dots \wedge \alpha_{k-1} \wedge d_{N_k} \alpha_k \wedge \alpha_{k+1} \wedge \dots \wedge \alpha_m .$$

Proof. By standard properties of the exterior derivative of wedge products, we have

$$d\alpha = \sum_{j=1}^m (-1)^{p_1 + \dots + p_{j-1}} \alpha_1 \wedge \dots \wedge \alpha_{j-1} \wedge d\alpha_j \wedge \alpha_{j+1} \wedge \dots \wedge \alpha_m .$$

Now $d_{N_j} \alpha_i = 0$ for $i \neq j$, because α_i is constant in all the N_j directions. Therefore,

$$d\alpha = \sum_{j=1}^m (-1)^{p_1 + \dots + p_{j-1}} \alpha_1 \wedge \dots \wedge \alpha_{j-1} \wedge d_{N_j} \alpha_j \wedge \alpha_{j+1} \wedge \dots \wedge \alpha_m .$$

The only one of these terms that resides in $\Omega^{(p_1, \dots, p_k+1, \dots, p_m)}(N)$ is

$$(-1)^{p_1 + \dots + p_{k-1}} \alpha_1 \wedge \dots \wedge \alpha_{k-1} \wedge d_{N_k} \alpha_k \wedge \alpha_{k+1} \wedge \dots \wedge \alpha_m. \quad \square$$

We can define partial analogs to other geometric operators as well.

Definition 4.16.

1. We define the partial codifferential $\delta_{N_k} : \Omega^p(N) \rightarrow \Omega^{p-1}(N)$ to be

$$\delta_{N_k} = (-1)^{n(p-1)+1} \star d_{N_k} \star,$$

where \star is the usual Hodge star on all of N .

2. We define the partial Laplacian Δ_{N_k} to be

$$\Delta_{N_k} = \delta_{N_k} d_{N_k} + d_{N_k} \delta_{N_k}.$$

These also have particularly nice forms when applied to separable forms.

Lemma 4.17. *If $\alpha = \alpha_1 \wedge \dots \wedge \alpha_m$ where α_k is the pullback of a p_k -form on N_k , then*

$$\delta_{N_k} \alpha = (-1)^{p_1 + \dots + p_{k-1}} \alpha_1 \wedge \dots \wedge \alpha_{k-1} \wedge \delta_{N_k} \alpha_k \wedge \alpha_{k+1} \wedge \dots \wedge \alpha_m, \text{ and}$$

$$\Delta_{N_k} \alpha = \alpha_1 \wedge \dots \wedge \alpha_{k-1} \wedge \Delta_{N_k} \alpha_k \wedge \alpha_{k+1} \wedge \dots \wedge \alpha_m.$$

Proof. We first prove the identity for δ_{N_1} when $m = 2$. Take a (p_1, p_2) -form $\alpha_1 \wedge \alpha_2$ on $N_1 \times N_2$. Define $\star_1 \alpha_1$ to be the Hodge star of α_1 on (N_1, g_1) , and similarly define $\star_2 \alpha_2$ to be the Hodge star of α_2 on (N_2, g_2) . Notice that if $\eta_1 \wedge \eta_2$ is another separable (p_1, p_2) -form, then

$$\begin{aligned} (\eta_1 \wedge \eta_2) \wedge ((-1)^{p_2(n_1-p_1)} \star_1 \alpha_1 \wedge \star_2 \alpha_2) &= (\eta_1 \wedge \star_1 \alpha_1) \wedge (\eta_2 \wedge \star_2 \alpha_2) \\ &= (\eta_1, \alpha_1) \cdot (\eta_2, \alpha_2) dV_1 \wedge dV_2 \\ &= (\eta_1 \wedge \eta_2, \alpha_1 \wedge \alpha_2) dV \end{aligned}$$

Therefore, $\star(\alpha_1 \wedge \alpha_2) = (-1)^{p_2(n_1-p_1)} \star_1 \alpha_1 \wedge \star_2 \alpha_2$.

We calculate using our above observation and Lemma 4.15 that

$$\begin{aligned} \star d_{N_1} \star (\alpha_1 \wedge \alpha_2) &= \star \left((-1)^{p_2(n_1-p_1)} (d_{N_1} \star_1 \alpha_1) \wedge \star_2 \alpha_2 \right) \\ &= (-1)^{p_2(n_1-p_1)+(n_2-p_2)(p_1-1)} (\star_1 d_{N_1} \star_1 \alpha_1) \wedge (\star_2 \star_2 \alpha_2) \\ &= (-1)^{p_2(n_1-p_1)+(n_2-p_2)(p_1-1)+(n_2-p_2)p_2} (\star_1 d_{N_1} \star_1 \alpha_1) \wedge \alpha_2 \end{aligned}$$

And setting $c = p_2(n_1-p_1) + (n_2-p_2)(p_1-1) + (n_2-p_2)p_2 + n(p-1) + 1 + n_1(p_1-1) + 1$, we get

$$\delta_{N_1}(\alpha_1 \wedge \alpha_2) = (-1)^c (\delta_{N_1} \alpha_1) \wedge \alpha_2.$$

We now can check that $(-1)^c = 1$, and we've shown

$$\delta_{N_1}(\alpha_1 \wedge \alpha_2) = (\delta_{N_1} \alpha_1) \wedge \alpha_2.$$

Now take a general m . Write $M = N_2 \times \cdots \times N_m$ and $\beta = \alpha_2 \wedge \cdots \wedge \alpha_m$, so $N = N_1 \times M$ and $\alpha = \alpha_1 \wedge \beta$. Applying the two factor case we've already shown, we get

$$\begin{aligned} \delta_{N_1}(\alpha_1 \wedge \cdots \wedge \alpha_m) &= \delta_{N_1}(\alpha_1 \wedge \beta) \\ &= (\delta_{N_1} \alpha_1) \wedge \beta \\ &= (\delta_{N_1} \alpha_1) \wedge \alpha_2 \wedge \cdots \wedge \alpha_m. \end{aligned}$$

Lastly, if $k \neq 1$, we can write

$$\begin{aligned} \delta_{N_k}(\alpha_1 \wedge \cdots \wedge \alpha_m) &= (-1)^{p_k(p_1+\cdots+p_{k-1})} \delta_{N_k}(\alpha_k \wedge \alpha_1 \wedge \cdots \wedge \alpha_{k-1} \wedge \alpha_{k+1} \wedge \cdots \wedge \alpha_m) \\ &= (-1)^{p_k(p_1+\cdots+p_{k-1})} (\delta_{N_k} \alpha_k) \wedge \alpha_1 \wedge \cdots \wedge \alpha_{k-1} \wedge \alpha_{k+1} \wedge \cdots \wedge \alpha_m \\ &= (-1)^{p_k(p_1+\cdots+p_{k-1})+(p_k-1)(p_1+\cdots+p_{k-1})} \alpha_1 \wedge \cdots \wedge \alpha_{k-1} \wedge \delta_{N_k} \alpha_k \wedge \alpha_{k+1} \wedge \cdots \wedge \alpha_m \\ &= (-1)^{(p_1+\cdots+p_{k-1})} \alpha_1 \wedge \cdots \wedge \alpha_{k-1} \wedge \delta_{N_k} \alpha_k \wedge \alpha_{k+1} \wedge \cdots \wedge \alpha_m. \end{aligned}$$

This ends the proof for δ_{N_k} . Now we turn our attention to Δ_{N_k} . Luckily, this is much easier, using what we've already done and Lemma 4.15:

$$\begin{aligned}
\Delta_{N_k}(\alpha_1 \wedge \cdots \wedge \alpha_m) &= (d_{N_k} \delta_{N_k} + \delta_{N_k} d_{N_k})(\alpha_1 \wedge \cdots \wedge \alpha_m) \\
&= \alpha_1 \wedge \cdots \wedge \alpha_{k-1} \wedge ((d_{N_k} \delta_{N_k} + \delta_{N_k} d_{N_k})\alpha_k) \wedge \alpha_{k+1} \wedge \cdots \wedge \alpha_m \\
&= \alpha_1 \wedge \cdots \wedge \alpha_{k-1} \wedge \Delta_{N_k} \alpha_k \wedge \alpha_{k+1} \wedge \cdots \wedge \alpha_m. \quad \square
\end{aligned}$$

Lemma 4.18. *Assume $\{\phi_j\} \subset \Omega^\bullet(N)$ is a sequence of smooth forms which converge in L^2 to a smooth form $\phi \in \Omega^\bullet(N)$, and take an integer k satisfying $1 \leq k \leq m$. If the sequence $\{d_{N_k} \phi_j\}$ converges in L^2 , it converges to $d_{N_k} \phi$. Similarly, if $\{\delta_{N_k} \phi_j\}$ converges in L^2 , it converges to $\delta_{N_k} \phi$.*

Proof. Take such a sequence of smooth forms $\{\phi_j\}$. It is a standard result that the exterior derivative on a compact Riemannian manifold with boundary is a closable operator; see [10] for details. But each of the spaces $\Omega^{(p_1, \dots, p_m)}(N)$ are orthogonal to the others, and the operator d_{N_k} is the composition of d with the orthogonal projection into one of these spaces. Therefore, d_{N_k} is closable, so the sequence $\{d_{N_k} \phi_j\}$ converges in L^2 to $d_{N_k} \phi$. Because the Hodge star is an isometry between Hilbert spaces, $\delta_{N_k} = (-1)^{n(p-1)+1} \star d_{N_k} \star$ is also a closable operator, so $\{\delta_{N_k} \phi_j\}$ converges in L^2 to $\delta_{N_k} \phi$. \square

We now prove some properties about these partial operators. The following geometric identities hold:

Lemma 4.19. *For $j, k \in \{1, \dots, m\}$,*

1. $d = d_{N_1} + \cdots + d_{N_m}$
2. $\delta = \delta_{N_1} + \cdots + \delta_{N_m}$
3. $d_{N_k} d_{N_j} = -d_{N_j} d_{N_k}$

4. $\delta_{N_k} \delta_{N_j} = -\delta_{N_j} \delta_{N_k}$
5. $\delta_{N_k} d_{N_j} = -d_{N_j} \delta_{N_k}$ if $j \neq k$
6. $\Delta = \Delta_{N_1} + \cdots + \Delta_{N_m}$
7. $d_{N_j} \Delta_{N_k} = \Delta_{N_k} d_{N_j}$ if $j \neq k$.

Proof. 1. This directly follows from the definition of the d_{N_k} .

2. This is a straightforward calculation:

$$\begin{aligned}
\delta &= (-1)^{n(p-1)+1} \star d \star \\
&= (-1)^{n(p-1)+1} \star (d_{N_1} + \cdots + d_{N_m}) \star \\
&= (-1)^{n(p-1)+1} \star d_{N_1} \star + \cdots + (-1)^{n(p-1)+1} \star d_{N_m} \star \\
&= \delta_{N_1} + \cdots + \delta_{N_m} .
\end{aligned}$$

3. First we show this for a separable form $\alpha_1 \wedge \cdots \wedge \alpha_m$. First, if $j = k$, then

$$\begin{aligned}
d_{N_k} d_{N_k} (\alpha_1 \wedge \cdots \wedge \alpha_m) &= (-1)^{p_1 + \cdots + p_{k-1}} d_{N_k} (\alpha_1 \wedge \cdots \wedge \alpha_{k-1} \wedge d_{N_k} \alpha_k \wedge \alpha_{k+1} \wedge \cdots \wedge \alpha_m) \\
&= \alpha_1 \wedge \cdots \wedge \alpha_{k-1} \wedge d_{N_k} d_{N_k} \alpha_k \wedge \alpha_{k+1} \wedge \cdots \wedge \alpha_m \\
&= 0 .
\end{aligned}$$

Now assume that $j \neq k$, and assume without loss of generality that $j < k$.

Then

$$\begin{aligned}
d_{N_k} d_{N_j} (\alpha_1 \wedge \cdots \wedge \alpha_m) &= (-1)^{p_1 + \cdots + p_{j-1}} d_{N_k} (\alpha_1 \wedge \cdots \wedge \alpha_{j-1} \wedge d_{N_j} \alpha_j \wedge \alpha_{j+1} \wedge \cdots \wedge \alpha_m) \\
&= (-1)^{2p_1 + \cdots + 2p_{j-1} + p_j + \cdots + p_{k-1} + 1} \alpha_1 \wedge \cdots \wedge d_{N_j} \alpha_j \wedge \\
&\quad \cdots \wedge d_{N_k} \alpha_k \wedge \cdots \wedge \alpha_m .
\end{aligned}$$

Notice the extra 1 in the exponent of the leading -1 , due to the fact that d_{N_j}

raises the degree of the j^{th} term by 1. On the other hand,

$$\begin{aligned}
d_{N_j} d_{N_k} (\alpha_1 \wedge \cdots \wedge \alpha_m) &= (-1)^{p_1 + \cdots + p_{k-1}} d_{N_j} (\alpha_1 \wedge \cdots \wedge \alpha_{k-1} \wedge d_{N_k} \alpha_k \wedge \alpha_{k+1} \wedge \cdots \wedge \alpha_m) \\
&= (-1)^{2p_1 + \cdots + 2p_{j-1} + p_j + \cdots + p_{k-1}} \alpha_1 \wedge \cdots \wedge d_{N_j} \alpha_j \wedge \\
&\quad \cdots \wedge d_{N_k} \alpha_k \wedge \cdots \wedge \alpha_m .
\end{aligned}$$

Therefore, $d_{N_j}d_{N_k} = -d_{N_k}d_{N_j}$ on separable forms, and by linearity this is also true on finite sums of separable forms. Take an arbitrary smooth form $\alpha \in \Omega^\bullet(N)$. By Lemma 4.13, we can find a sequence of finite sums of separable forms $\{\phi_i\}$ which converges to α in L^2 . By applying Lemma 4.18 twice, we get that the sequence $\{(d_{N_k}d_{N_j} + d_{N_j}d_{N_k})\phi_i\}$ converges to $(d_{N_k}d_{N_j} + d_{N_j}d_{N_k})\alpha$; as the sequence consists of only zero terms, we get that $(d_{N_k}d_{N_j} + d_{N_j}d_{N_k})\alpha = 0$ as well.

4. This is also a straightforward calculation:

$$\begin{aligned}
\delta_{N_k}\delta_{N_j} &= ((-1)^{np+1} \star d_{N_k} \star) ((-1)^{n(p-1)+1} \star d_{N_j} \star) \\
&= (-1)^{np+1}(-1)^{n(p-1)+1}(-1)^{n(n-p)+1} \star d_{N_k} ((-1)^{n(n-p)+1} \star \star) d_{N_j} \star \\
&= (-1)^{np-1} \star d_{N_k} d_{N_j} \star \\
&= (-1)^{np-1} \star (-d_{N_j}d_{N_k}) \star \\
&= -\delta_{N_j}\delta_{N_k}.
\end{aligned}$$

5. This proof will go almost identically to the one for part 3. First we show this for a separable form $\alpha_1 \wedge \cdots \wedge \alpha_m$. Assume that $j < k$; the proof for $k < j$ is the same, except the extra 1 in the exponent of the leading (-1) will appear in the second calculation instead of the first. Then

$$\begin{aligned}
\delta_{N_k}d_{N_j}(\alpha_1 \wedge \cdots \wedge \alpha_m) &= (-1)^{p_1+\cdots+p_{j-1}}\delta_{N_k}(\alpha_1 \wedge \cdots \wedge \alpha_{j-1} \wedge d_{N_j}\alpha_j \wedge \alpha_{j+1} \wedge \cdots \wedge \alpha_m) \\
&= (-1)^{2p_1+\cdots+2p_{j-1}+p_j+\cdots+p_{k-1}+1}\alpha_1 \wedge \cdots \wedge d_{N_j}\alpha_j \wedge \\
&\quad \cdots \wedge \delta_{N_k}\alpha_k \wedge \cdots \wedge \alpha_m.
\end{aligned}$$

Notice the extra 1 in the exponent of the leading -1 , due to the fact that d_{N_j} raises the degree of the j^{th} term by 1. On the other hand,

$$\begin{aligned}
d_{N_j}\delta_{N_k}(\alpha_1 \wedge \cdots \wedge \alpha_m) &= (-1)^{p_1+\cdots+p_{k-1}}d_{N_j}(\alpha_1 \wedge \cdots \wedge \alpha_{k-1} \wedge \delta_{N_k}\alpha_k \wedge \alpha_{k+1} \wedge \cdots \wedge \alpha_m) \\
&= (-1)^{2p_1+\cdots+2p_{j-1}+p_j+\cdots+p_{k-1}}\alpha_1 \wedge \cdots \wedge d_{N_j}\alpha_j \wedge \\
&\quad \cdots \wedge \delta_{N_k}\alpha_k \wedge \cdots \wedge \alpha_m.
\end{aligned}$$

Therefore, $d_{N_j}\delta_{N_k} = -\delta_{N_k}d_{N_j}$ on separable forms, and by linearity this is also true on finite sums of separable forms. Take an arbitrary smooth form $\alpha \in \Omega^\bullet(N)$. By Lemma 4.13, we can find a sequence of finite sums of separable forms $\{\phi_i\}$ which converges to α in L^2 . By applying Lemma 4.18 twice, we get that the sequence $\{(\delta_{N_k}d_{N_j} + d_{N_j}\delta_{N_k})\phi_i\}$ converges to $(\delta_{N_k}d_{N_j} + d_{N_j}\delta_{N_k})\alpha$; as the sequence consists of only zero terms, we get that $(\delta_{N_k}d_{N_j} + d_{N_j}\delta_{N_k})\alpha = 0$ as well.

6. We finish with one more simple calculation, using what we've already proven:

$$\begin{aligned}
\Delta &= d\delta + \delta d \\
&= (d_{N_1} + \cdots + d_{N_m})(\delta_{N_1} + \cdots + \delta_{N_m}) + (\delta_{N_1} + \cdots + \delta_{N_m})(d_{N_1} + \cdots + d_{N_m}) \\
&= \sum_{j,k=1}^m d_{N_j}\delta_{N_k} + \delta_{N_j}d_{N_k} \\
&= \sum_j (d_{N_j}\delta_{N_j} + \delta_{N_j}d_{N_j}) + \sum_{j \neq k} (d_{N_j}\delta_{N_k} + \delta_{N_k}d_{N_j}) \\
&= \Delta_{N_1} + \cdots + \Delta_{N_m}.
\end{aligned}$$

7. This follows directly from parts 3 and 5, using the fact that $\Delta_{N_k} = d_{N_k}\delta_{N_k} + \delta_{N_k}d_{N_k}$. \square

Definition 4.20. Let $G_{N_k} : \Omega^\bullet(N_k) \rightarrow \Omega^\bullet(N_k)$ be the Neumann Green's operator; we can extend G_{N_k} to finite sums of separable forms by defining

$$G_{N_k}(\alpha_1 \wedge \cdots \wedge \alpha_m) = \alpha_1 \wedge \cdots \wedge \alpha_{k-1} \wedge G_{N_k}\alpha_k \wedge \alpha_{k+1} \wedge \cdots \wedge \alpha_m.$$

and extending by linearity. Similarly, let $A_k : \Omega^\bullet(N_k) \rightarrow \Omega^\bullet(N)$ be the projection onto the Neumann strongly harmonic forms of N_k , $\mathcal{H}_N(N_k)$, and define A_k on finite sums of separable forms by defining

$$A_k(\alpha_1 \wedge \cdots \wedge \alpha_m) = \alpha_1 \wedge \cdots \wedge \alpha_{k-1} \wedge A_k\alpha_k \wedge \alpha_{k+1} \wedge \cdots \wedge \alpha_m.$$

Theorem 4.21. *The operators G_{N_k} , $\delta_{N_k}G_{N_k}$, and A_k on $\Omega^{p_1}(N_1) \otimes \cdots \otimes \Omega^{p_m}(N_m)$ are bounded operators and extend uniquely to bounded operators on $L^2\Omega^\bullet(N)$. The operators G_{N_k} and $\delta_{N_k}G_{N_k}$ have the same bound as the Neumann Green's operator for (N_k, g_{N_k}) , while the operator A_k has the bound 1.*

Proof. By Lemma 4.12,

$$\begin{aligned} \|G_{N_k}(\alpha_1 \wedge \cdots \wedge \alpha_m)\| &= \|\alpha_1\| \cdots \|\alpha_{k-1}\| \|G_{N_k}\alpha_k\| \|\alpha_{k+1}\| \cdots \|\alpha_m\| \\ &\leq C_k \|\alpha\|. \end{aligned}$$

where C_k is the bound for the Neumann Green's operator for (N_k, g_{N_k}) . By Lemma 4.13 and the BLT Theorem, G_{N_k} extends uniquely to a bounded operator with the same bound on all of $L^2\Omega^\bullet(N)$.

The same exact argument applies to $\delta_{N_k}G_k$, replacing each instance of G_{N_k} with $\delta_{N_k}G_{N_k}$, and it applies also to A_k , by replacing the instances of G_{N_k} with A_k and the instances of C_k with 1. \square

Proposition 4.22. *1. G_{N_k} , $\delta_{N_k}G_{N_k}$, and A_k all map $\Omega^\bullet(N)$ into $\Omega^\bullet(N)$ (i.e. the image of a smooth form through these operators is again smooth).*

2. $d_{N_j}G_{N_k} = G_{N_k}d_{N_j}$ for all j .

3. $d_{N_k}A_k = 0$ and $\delta_{N_k}A_k = 0$.

4. $d_{N_j}A_k = A_kd_{N_j}$ and $\delta_{N_j}A_k = A_k\delta_{N_j}$ for $j \neq k$.

5. $dG_{N_k} = G_{N_k}d$.

6. $dA_k = A_kd$.

7. We have the following homotopy formula

$$d(\delta_{N_k}G_{N_k}\alpha) + \delta_{N_k}G_{N_k}(d\alpha) = \alpha - A_k\alpha$$

Proof. In the proof of Lemma 4.13, we show that we can write any smooth form as the finite sum of forms which can be written as

$$\alpha = a du_{kj_1} \wedge \cdots \wedge du_{kj_{p_k}} \wedge \beta,$$

where a is a function which is compactly supported in some coordinate neighborhood $U_1 \times \cdots \times U_m$, $du_{kj_1}, \cdots, du_{kj_{p_k}}$ are a collection of the coordinates pulled back from N_k , and β is a wedge of a collection of the coordinates pulled back from the other factors N_j for $j \neq k$. We define $\gamma = a du_{kj_1} \wedge \cdots \wedge du_{kj_{p_k}}$ so that $\alpha = \gamma \wedge \beta$.

We show that $G_{N_k}\alpha$, $\delta_{N_k}\alpha$, and $A_k\alpha$ are all smooth forms; then because these are linear operators and any form can be written as a finite sum of such pieces, we will be done. In fact, by the definitions of A_{N_k} and G_{N_k} and by Lemma 4.17, it suffices to show that $G_{N_k}\gamma$ and $\delta_{N_k}G_{N_k}\gamma$ are smooth.

Notice that γ can be thought of as a function from $N_1 \times \cdots \times N_{k-1} \times N_{k+1} \times \cdots \times N_m$ into $\Omega^{p_k}(N_k)$. Then A_k is the pointwise projection to the Neumann strongly harmonic forms; by the Appendix at the end of Chapter 2 in [24], this means that because γ is smooth, then $A_k\gamma$ is smooth.

Now we look at G_{N_k} and $\delta_{N_k}G_{N_k}$. By elliptic regularity, $G_{N_k}\gamma$ and $\delta_{N_k}G_{N_k}\gamma$ are smooth forms on N_k for every choice of values $x_j \in U_j$, $j \neq k$. In other words, the partial derivatives of $G_{N_k}\gamma$ and $\delta_{N_k}G_{N_k}\gamma$ which are composed of the operators $\frac{\partial}{\partial x_{k_i}}$ all exist. Because G_{N_k} and $\delta_{N_k}G_{N_k}$ are linear operators which are independent of u_{j_i} for $j \neq k$, they commute with $\frac{\partial}{\partial u_{j_i}}$. Therefore, iteratively all partial derivatives of $G_{N_k}\gamma$ and $\delta_{N_k}G_{N_k}\gamma$ exist.

Now to prove parts 2 through 4, we first note that by Lemma 4.18 and the fact

that compositions and linear combinations of closable operators are closed, that if $B : \Omega^\bullet(N) \rightarrow \Omega^\bullet(N)$ is an operator obtained by linear combinations of composition of finitely many $d_{N_1}, \dots, d_{N_m}, \delta_{N_1}, \dots, \delta_{N_m}$ and bounded operators, then if $\{\phi_i\}$ is a sequence of smooth forms which converges in L^2 to another smooth form ϕ , then the sequence of smooth forms $\{B(\phi_i)\}$ converges in L^2 to $B\phi$.

Then notice that we can rearrange each of the identities in parts 2 through 4 to be such an operator, we can verify this operator is zero on finite sums of separable forms, and we can apply Lemma 4.13 to see that the operator is zero everywhere.

For example, set $B = d_{N_j}G_{N_k} - G_{N_k}d_{N_j}$. Assuming first that $j < k$, we get by Lemma 4.15 and the definition of G_{N_k} that

$$\begin{aligned} B(\alpha_1 \wedge \dots \wedge \alpha_m) &= (d_{N_j}G_{N_k} - G_{N_k}d_{N_j})(\alpha_1 \wedge \dots \wedge \alpha_m) \\ &= (-1)^{(p_1 + \dots + p_{j-1})} \alpha_1 \wedge \dots \wedge d_{N_j} \alpha_j \wedge \dots \wedge G_{N_k} \alpha_k \wedge \dots \wedge \alpha_m \\ &\quad - (-1)^{(p_1 + \dots + p_{j-1})} \alpha_1 \wedge \dots \wedge d_{N_j} \alpha_j \wedge \dots \wedge G_{N_k} \alpha_k \wedge \dots \wedge \alpha_m \\ &= 0. \end{aligned}$$

Now assuming that $j = k$, we get by Lemma 4.15, the definition of G_{N_k} , and 4.19 that

$$\begin{aligned} B(\alpha_1 \wedge \dots \wedge \alpha_m) &= (d_{N_k}G_{N_k} - G_{N_k}d_{N_k})(\alpha_1 \wedge \dots \wedge \alpha_m) \\ &= (-1)^{(p_1 + \dots + p_{k-1})} \alpha_1 \wedge \dots \wedge d_{N_k} G_{N_k} \alpha_k \wedge \dots \wedge \alpha_m \\ &\quad - (-1)^{(p_1 + \dots + p_{j-1})} \alpha_1 \wedge \dots \wedge G_{N_k} d_{N_k} \alpha_k \wedge \dots \wedge \alpha_m \\ &= (-1)^{(p_1 + \dots + p_{k-1})} \alpha_1 \wedge \dots \wedge (d_{N_k} G_{N_k} \alpha_k - G_{N_k} d_{N_k} \alpha_k) \wedge \dots \wedge \alpha_m \\ &= 0. \end{aligned}$$

Part 5 comes from part 2 and the fact that $d = d_{N_1} + \dots + d_{N_m}$. Part 6 comes from parts 3 and 4, the fact that $d = d_{N_1} + \dots + d_{N_m}$, and the fact that the Neumann

strongly harmonic forms on N_k are orthogonal to exact forms, i.e. $A_k d_{N_k} = 0$.

Finally, part 7 comes from the following calculation, in which we use Lemmas 4.8 and 4.19:

$$\begin{aligned}
d(\delta_{N_k} G_{N_k} \alpha) + \delta_{N_k} G_{N_k} (d\alpha) &= (d_{N_1} + \cdots + d_{N_m}) \delta_{N_k} G_{N_k} \alpha + \delta_{N_k} G_{N_k} (d_{N_1} + \cdots + d_{N_m}) \\
&= \sum_{j=1}^m (d_{N_j} \delta_{N_k} G_{N_k}) \alpha + (\delta_{N_k} G_{N_k} d_{N_j}) \alpha \\
&= (d_{N_k} \delta_{N_k} G_{N_k} + \delta_{N_k} d_{N_k} G_{N_k}) \alpha + \sum_{j \neq k} (d_{N_j} \delta_{N_k} G_{N_k} - d_{N_j} \delta_{N_k} G_{N_k}) \alpha \\
&= (d_{N_k} \delta_{N_k} G_{N_k} + \delta_{N_k} d_{N_k} G_{N_k}) \alpha.
\end{aligned}$$

When applying $(d_{N_k} \delta_{N_k} G_{N_k} + \delta_{N_k} d_{N_k} G_{N_k})$ to a separable form, we get

$$\begin{aligned}
(d_{N_k} \delta_{N_k} G_{N_k} + \delta_{N_k} d_{N_k} G_{N_k})(\alpha_1 \wedge \cdots \wedge \alpha_m) &= \alpha_1 \wedge \cdots \wedge (d_{N_k} \delta_{N_k} G_{N_k} + \delta_{N_k} d_{N_k} G_{N_k}) \alpha_k \wedge \cdots \wedge \alpha_m \\
&= \alpha_1 \wedge \cdots \wedge (\alpha_k - A_k \alpha_k) \wedge \cdots \wedge \alpha_m \\
&= \alpha - A_k \alpha.
\end{aligned}$$

If we take an arbitrary form α , we can approximate it by a sequence of finite sums of separable forms $\{\alpha_i\}$. Then the sequence $\{(d_{N_k} \delta_{N_k} G_{N_k} + \delta_{N_k} d_{N_k} G_{N_k}) \alpha_i - \alpha_i + A_k \alpha_i\}$ must converge to $(d_{N_k} \delta_{N_k} G_{N_k} + \delta_{N_k} d_{N_k} G_{N_k}) \alpha - \alpha + A_k \alpha$. As the sequence consists of only zeros, we've shown the identity. \square

Proposition 4.23. *Take a sequence of Riemannian manifolds with boundary (N_k, g_{N_k}) of dimension n_k , respectively, where each metric extends continuously to the boundary (the boundary may be empty). Consider the product manifold $N = N_1 \times \cdots \times N_m$ with the product metric $g = g_{N_1} + \cdots + g_{N_m}$. There exist linear operators $A, B : \Omega^\bullet(N) \rightarrow \Omega^\bullet(N)$ so that*

$$\alpha = A\alpha + dB\alpha + Bd\alpha \text{ and}$$

$$A\alpha \in \mathcal{H}_N(N)$$

where $\mathcal{H}_N(N)$ is the space of Neumann strongly harmonic forms in N . Furthermore, if C_k is the bound for the Neumann Green's operator G_{N_k} on (N_k, g_{N_k}) , then we have the bounds

$$\|A\alpha\|^2 \leq \|\alpha\|^2, \text{ and}$$

$$\|B\alpha\|^2 \leq \left(\sum_{k=1}^m C_k \right) \|\alpha\|^2.$$

Proof. Take a form $\alpha \in \Omega^\bullet(N)$. Let A_k and G_{N_k} be the operators defined in Definition 4.20. Define the operators B_k inductively via

$$B_1 = \delta_{N_1} G_{N_1} \text{ and}$$

$$B_k = \delta_{N_k} G_{N_k} + A_k B_{k-1}.$$

We will inductively show the homotopy formulas

$$\alpha - A_k A_{k-1} \cdots A_1 \alpha = dB_k \alpha + B_k d\alpha.$$

The base case is part 7 of the previous proposition applied when $k = 1$. If it holds for B_{k-1} , then we use parts 6 and 7 along with our inductive hypothesis to say

$$\begin{aligned} dB_k \alpha + B_k d\alpha &= d(\delta_{N_k} G_{N_k} + A_k B_{k-1}) \alpha + (\delta_{N_k} G_{N_k} + A_k B_{k-1}) d\alpha \\ &= d(\delta_{N_k} G_{N_k} \alpha) + \delta_{N_k} G_{N_k} (d\alpha) + A_k dB_{k-1} \alpha + A_k B_{k-1} d\alpha \\ &= \alpha - A_k \alpha + A_k (\alpha - A_{k-1} A_{k-2} \cdots A_1 \alpha) \\ &= \alpha - A_k A_{k-1} \cdots A_1 \alpha. \end{aligned}$$

Set $A = A_m \cdots A_1$ and $B = B_m$. Then we have the homotopy formula

$$dB\alpha + B d\alpha = \alpha - A\alpha.$$

Using parts 3 and 4 of the previous proposition, it is clear that $d_{N_k} A\alpha = 0$ and $\delta_{N_k} A\alpha = 0$ for all k , so $\alpha \in \mathcal{H}(N)$. Furthermore, A_k maps forms into forms with

Neumann boundary conditions on N_k , subsequent operators A_j do not change that, and $\alpha \in \mathcal{H}_N(N)$ if and only if α satisfies the Neumann boundary conditions for each factor N_k . Therefore, $A\alpha \in \mathcal{H}_N(N)$.

Note also that

$$\|A_k\alpha\| \leq \|\alpha\|$$

because it is an orthogonal projection, and so inductively we see that

$$\|A\alpha\| \leq \|\alpha\|.$$

Similarly, we have the bound

$$\begin{aligned} \|B_k\alpha\|^2 &= \|\delta_{N_k}G_{N_k}\alpha + A_kB_{k-1}\alpha\|^2 \\ &\leq C_k\|\alpha\|^2 + \|A_kB_{k-1}\alpha\|^2 \\ &\leq C_k\|\alpha\|^2 + \|B_{k-1}\alpha\|^2. \end{aligned}$$

Inductively, we can see that

$$\|B\alpha\|^2 \leq (C_1 + \cdots + C_m)\|\alpha\|^2. \quad \square$$

It will be helpful now to have a characterization of $\mathcal{H}_N(N)$.

Theorem 4.24. (*Harmonic Künneth Theorem*)

Every Neumann strongly harmonic form α can be written as a sum

$$\alpha = \sum_i \alpha_{1i} \wedge \cdots \wedge \alpha_{mi},$$

where each α_{ki} is the pullback of a Neumann strongly harmonic form on N_k . In other words,

$$\mathcal{H}_N^p(N) = \bigoplus_{p_1 + \cdots + p_m = p} \mathcal{H}_N^{p_1}(N_1) \wedge \cdots \wedge \mathcal{H}_N^{p_m}(N_m).$$

Proof. Pick values $x_j \in N_j$ for $j \neq 1$. Then choose a coordinate neighborhood $U_j \subset N_j$ containing each x_j . The neighborhood $N_1 \times U_2 \times \cdots \times U_m$ is a neighborhood of the fiber $N_1 \times \{x_2\} \times \cdots \times \{x_m\}$. We can write the restriction of any form in this neighborhood as

$$\alpha = \sum_I \alpha_I \wedge \beta_I,$$

where $\alpha \in \Omega^{(p_1, 0, \dots, 0)}(U)$ and β_I is the wedge of $(p_2 + \cdots + p_m)$ many 1-forms du_{ji} , where u_{ji} is the pullback of a coordinate function on N_j for $j \neq 1$. Since $d\alpha = 0$ and $\delta\alpha = 0$, we know $d_{N_1}\alpha = 0$ and $\delta_{N_1}\alpha = 0$. Therefore, $\alpha_I(y_2, \dots, y_m) \in \mathcal{H}_N(N_1)$ is a Neumann strongly harmonic form for each value $(y_2, \dots, y_m) \in U_2 \times \cdots \times U_m$. By Theorem 2.2.7 of [24], $\mathcal{H}_N^{p_1}(N_1)$ is finite dimensional. Choose a basis $\{h_{1i}\}$ of $\mathcal{H}_N^{p_1}(N_1)$. Then we can write α_I uniquely as a linear combination of elements of this basis; the coefficients $a_{1i,I}$ will be smooth functions on U which are constant on each fiber $N_1 \times \{x_2\} \times \cdots \times \{x_m\}$. Then we can rewrite α as

$$\begin{aligned} \alpha &= \sum_I \left(\sum_{i=1}^{\dim \mathcal{H}_N^{p_1}(N_1)} a_{1i,I} h_{1i} \wedge \beta_I \right) \\ &= \sum_{i=1}^{\dim \mathcal{H}_N^{p_1}(N_1)} h_{1i} \wedge \left(\sum_{I_{j(1) \neq 1}} a_{1i,I} \beta_I \right). \end{aligned}$$

Because these coefficients are chosen uniquely, these local characterizations patch together to give us a global decomposition of α as

$$\alpha = \sum_{i=1}^{\dim \mathcal{H}_N^{p_1}(N_1)} h_{1i} \wedge \gamma_i,$$

where the forms γ_i are each Neumann, harmonic, and constant on each fiber $N_1 \times \{x_2\} \times \cdots \times \{x_m\}$. After applying this procedure iteratively, we get our desired result. \square

4.3 Hodge Theory for Scaled Metrics

In this section, we examine the Hodge theory of a compact Riemannian manifold with boundary when the metric is scaled. Let N be an n -dimensional compact Riemannian manifold with boundary, with a smooth metric g which extends continuously to the boundary. Denote $(\cdot, \cdot)_\lambda, \langle \cdot, \cdot \rangle_\lambda, |\cdot|_\lambda, \|\cdot\|_\lambda, dV_\lambda, \star_\lambda, \delta_\lambda, \Delta_\lambda,$ and G_λ to be, respectively, the pointwise inner product, global inner product, pointwise norm, global norm, volume form, Hodge star, codifferential, Laplacian, and Neumann Green's operator of $(N, \lambda^2 g)$ for some positive real number $\lambda > 0$. A commonly used choice will be $\lambda = r^{2c}$ for some $c \geq 1$, but this will not always be the case.

Lemma 4.25. *Let $\alpha, \beta \in \Omega^k(N)$. We have the following formulas:*

1. $|\alpha|_\lambda = \lambda^{-k} |\alpha|_1$
2. $(\alpha, \beta)_\lambda = \lambda^{-2k} (\alpha, \beta)_1$
3. $dV_\lambda = \lambda^n dV_1$
4. $\langle \alpha, \beta \rangle_\lambda = \lambda^{(-2k+n)} \langle \alpha, \beta \rangle_1$
5. $\|\alpha\|_\lambda = \lambda^{(-k+n/2)} \|\alpha\|_1$
6. $\star_\lambda \alpha = \lambda^{(-2k+n)} \star_1 \alpha$
7. $\delta_\lambda \alpha = \lambda^{-2} \delta_1 \alpha$
8. $\Delta_\lambda \alpha = \lambda^{-2} \Delta_1 \alpha$
9. $G_\lambda \alpha = \lambda^2 G_1 \alpha$.

Proof. 1. The identity map is a quasi-isometry from (N, g) to $(N, \lambda^2 g)$ where $|v|_\lambda = \lambda |v|_1$ (so we can use λ as both the upper and lower bound). Then applying Lemma 2.4.2, we get our result.

2. If α or β are zero at the given point, then the identity is trivial. Otherwise,

$$\left(\frac{\alpha}{|\alpha|_\lambda}, \frac{\beta}{|\beta|_\lambda} \right)_\lambda = 1 = \left(\frac{\alpha}{|\alpha|_1}, \frac{\beta}{|\beta|_1} \right)_1, \text{ so}$$

$$(\alpha, \beta)_\lambda = \frac{|\alpha|_\lambda |\beta|_\lambda}{|\alpha|_1 |\beta|_1} (\alpha, \beta)_1 = \lambda^{-2k} (\alpha, \beta)_1 .$$

3. We know dV_λ is a positive scalar multiple of dV_1 because the orientation on N doesn't change. The correct scaling factor comes from part 1.

$$\begin{aligned} 4. \quad \langle \alpha, \beta \rangle_\lambda &= \int_N (\alpha, \beta)_\lambda dV_\lambda \\ &= \int_N \lambda^{-2k+n} (\alpha, \beta)_1 dV_1 \\ &= \lambda^{-2k+n} \langle \alpha, \beta \rangle_1 . \end{aligned}$$

$$5. \quad \|\alpha\|_\lambda = (\langle \alpha, \alpha \rangle_\lambda)^{1/2} = \lambda^{(-k+n/2)} (\langle \alpha, \alpha \rangle_1)^{1/2} = \lambda^{(-k+n/2)} \|\alpha\|_1 .$$

6. For an arbitrary $\eta \in \Omega^k(N)$, we have

$$\begin{aligned} \eta \wedge \lambda^{(-2k+n)} \star_1 \alpha &= (\eta, \lambda^{(-2k+n)} \alpha)_1 dV_1 \\ &= \lambda^{-2k} (\eta, \alpha)_1 \lambda^n dV_1 \\ &= (\eta, \alpha)_\lambda dV_\lambda \\ &= \eta \wedge \star_\lambda \alpha . \end{aligned}$$

This gives us our identity.

7.
$$\begin{aligned}\delta_\lambda \alpha &= (-1)^{n(k-1)+1} \star_\lambda d \star_\lambda \alpha \\ &= (-1)^{n(k-1)+1} \lambda^{(-2(n-k+1)+n)} \star_1 d (\lambda^{(-2k+n)} \star_1 \alpha) \\ &= \lambda^{-2} (-1)^{n(k-1)+1} \star_1 d \star_1 \alpha \\ &= \lambda^{-2} \delta_1 \alpha.\end{aligned}$$
8.
$$\Delta_\lambda \alpha = (d\delta_\lambda \alpha + \delta_\lambda d)\alpha = \lambda^{-2}(d\delta_1 \alpha + \delta_1 d)\alpha = \lambda^{-2} \Delta_1 \alpha.$$
9.
$$\Delta_\lambda \lambda^2 G_1 \alpha = \lambda^{-2} \Delta_1 \lambda^2 G_1 \alpha = \alpha - A\alpha.$$

Furthermore, $\lambda^2 G_1 \alpha \in \Omega_N^\bullet(N)$ and $d(\lambda^2 G_1 \alpha) = \lambda^2 d(G_1 \alpha) \in \Omega_N^\bullet(N)$, so it satisfies the boundary conditions. \square

Corollary 4.26. *The operators $\delta_\lambda G_\lambda$ and A_λ , where A_λ is the projection to the Neumann strongly harmonic forms, are constant in λ . Furthermore, there is a constant C , independent of λ , so that*

$$\|\delta G \alpha\|_\lambda \leq C \lambda \|\alpha\|_\lambda, \text{ and}$$

$$\|A \alpha\|_\lambda \leq \|\alpha\|_\lambda.$$

Proof. Lemma 4.25 tells us $\delta_\lambda G_\lambda = \lambda^{-2} \delta_1 \lambda^2 G_1 = \delta_1 G_1$ is constant. Since $\delta_1 G_1$ is bounded on the link (N, g) , we have on a k -form ϕ ,

$$\|\delta_1 G_1 \alpha\|_\lambda = \lambda^{-(k-1)+n/2} \|\delta_1 G_1 \alpha\|_1 \leq \lambda^{-(k-1)+n/2} C \|\alpha\|_1 \leq \lambda C \|\alpha\|_r.$$

Notice that the codifferential on $(N, \lambda^2 g)$ is a scalar multiple of its counterpart on (N, g) . This implies the Hodge decomposition of Theorem 4.5 is constant in λ . Thus, the projection operator onto the Neumann strongly harmonic forms is constant, and the bound is always 1 for an orthogonal projection. \square

4.4 L^2 -Cohomology of Multiply-Warped Products

Let $X = (0, 1] \times L$ be a topological cone where the link $L = L_1 \times \cdots \times L_m$ is the product of compact manifolds with boundary. Then the tangent space of X decomposes as

$$T_*X = T_*(0, 1] \oplus T_*L_1 \oplus \cdots \oplus T_*L_m.$$

Say there are metrics g_{L_k} on T_*L_k for all k , and the metric on X is given by

$$g = dr^2 + g_r = dr^2 + r^{2c_1}g_{L_1} + \cdots + r^{2c_m}g_{L_m}.$$

Then we call this a *multiply-warped product*.

Lemma 4.27. 1. *The space of Neumann strongly harmonic forms \mathcal{H}_N on (L, g_r) is constant¹ in r .*

2. *The operators $A_{L,r}$ and $B_{L,r}$ obtained from applying Proposition 4.23 to the link L with the metric g_r are constant in r .*

3. *The operators*

$$B(\phi + dr \wedge \omega) = B_{L,1}\phi - dr \wedge B_{L,1}\omega \text{ and}$$

$$A(\phi + dr \wedge \omega) = A_{L,1}\phi + dr \wedge A_{L,1}\omega$$

are bounded operators on $L^2\Omega^k(X)$ satisfying the homotopy formula

$$Bd\alpha + dB\alpha = \alpha - A\alpha.$$

Proof. 1. Assume $\alpha \in \mathcal{H}_N(L, g_r)$. We will show that for $s \in (0, 1)$, $\alpha \in \mathcal{H}_N(L, g_s)$.

Note that α is also a weakly harmonic form, so by Lemma 4.19 and 4.25,

$$\begin{aligned} 0 &= \Delta_r \alpha \\ &= \Delta_{L_1,r} \alpha + \cdots + \Delta_{L_m,r} \alpha \\ &= (r^{-2c_1} \Delta_{L_1} \alpha + \cdots + r^{-2c_m} \Delta_{L_m} \alpha). \end{aligned}$$

¹ Note that this does not necessarily imply that the entire Hodge Decomposition is constant in r . In particular, the image of the codifferential might change.

Because $\langle r^{-2c_k} \Delta_{L_k} \alpha, \alpha \rangle \geq 0$ for each k by Lemma 4.10.2, we have $r^{-2c_k} \Delta_{L_k} \alpha = 0$. Because $r, s \neq 0$, we get for each k that $s^{-2c_k} \Delta_{L_k} \alpha = 0$. Therefore,

$$\Delta_{L,s} \alpha = s^{-2c_1} \Delta_{L_1} \alpha + \cdots + s^{-2c_m} \Delta_{L_m} \alpha = 0.$$

In other words, α is a weakly harmonic form on (L, g_s) . Notice also the Neumann boundary conditions $\alpha \in \Omega_N^\bullet(L)$ and $d\alpha \in \Omega_N^\bullet(L)$ do not depend on r .

Lastly, a weakly harmonic form α satisfying $\alpha \in \Omega_N^\bullet(L)$ and $d\alpha \in \Omega_N^\bullet(L)$ is a strongly harmonic form by Proposition 3.4.5 of [24], so $\alpha \in \mathcal{H}_N(L, g_s)$. Thus, $\mathcal{H}_N(L, g_r)$ does not depend on r .

2. The operators B_k from the proof of Proposition 4.23 are of the form $\delta_{N_k} G_{N_k} + A_k B_{k-1}$. From Corollary 4.26, we see that $\delta_{N_k} G_{N_k}$ and A_k do not depend on r . Therefore, B_k does not depend on r ; in particular, B_m does not depend on r , which is what we call here $B_{L,r}$, nor does $A_m \cdots A_1$, which is what we call here $A_{L,r}$.
3. From Corollary 4.26 we learn that the corresponding bounds C_k used in the proof of Proposition 4.23 can be chosen to be independent² of r . Then we have

$$\begin{aligned} \|B\alpha\|^2 &= \int_0^1 \|B\alpha\|_r^2 dr \\ &\leq \int_0^1 \left(\sum_{k=1}^m C_k \right) \|\alpha\|_r^2 dr \\ &= \left(\sum_{k=1}^m C_k \right) \|\alpha\|^2. \end{aligned}$$

² We can do even better, but independent will suffice here.

Similarly,

$$\begin{aligned}
\|A\alpha\|^2 &= \int_0^1 \|A\alpha\|_r^2 dr \\
&\leq \int_0^1 \|\alpha\|_r^2 dr \\
&= \|\alpha\|^2.
\end{aligned}$$

As for the homotopy formula, this is a straightforward calculation since everything is smooth:

$$\begin{aligned}
dB(\phi + dr \wedge \omega) &= d(B_{L,1}\phi - dr \wedge B_{L,1}\omega) \\
&= d_L B_{L,1}\phi + dr \wedge \left(\frac{\partial B_{L,1}\phi}{\partial r} + d_L B_{L,1}\omega \right), \text{ and} \\
Bd(\phi + dr \wedge \omega) &= B \left(d_L \phi + dr \wedge \left(\frac{\partial \phi}{\partial r} - d_L \omega \right) \right) \\
&= B_{L,1} d_L \phi - dr \wedge \left(B_{L,1} \frac{\partial \phi}{\partial r} - B_{L,1} d_L \omega \right).
\end{aligned}$$

Since $\frac{\partial}{\partial r}$ commutes with $B_{L,1} = \delta_{L,1} G_1$, we get

$$\begin{aligned}
dB\alpha + Bd\alpha &= d_L B_{L,1}\phi + B_{L,1} d_L \phi + dr \wedge (d_L B_{L,1}\omega + B_{L,1} d_L \omega) \\
&= \phi - A\phi + dr \wedge (\omega - A\omega) \\
&= \alpha - A\alpha.
\end{aligned}$$

□

Lemma 4.28. For $\alpha \in \Omega^{(p_1, \dots, p_m)}(L)$,

$$\|\alpha\|_r^2 = r^{c_1(n_1 - 2p_1) + \dots + c_m(n_m - 2p_m)} \|\alpha\|_1^2.$$

Proof. First consider a form $\alpha = \alpha_1 \wedge \dots \wedge \alpha_m$ where α_k is a pullback of a p_k -form on L_k through the projection map $L \rightarrow L_k$. Lemma 4.12 says that pointwise, we have

$$|\alpha|_{(L, g_r)}^2 = |\alpha_1|_{(L_1, r^{2c_1} g_{L_1})}^2 \cdots |\alpha_m|_{(L_m, r^{2c_m} g_{L_m})}^2.$$

By Lemma 4.25, we have

$$|\alpha_k|_{(L_k, r^{2c_k} g_{L_k})}^2 = r^{-2c_k p_k} |\alpha_k|_{(L_k, g_{L_k})}^2.$$

Combining these two facts, we get

$$|\alpha|_{(L, g_r)}^2 = r^{-2(c_1 p_1 + \dots + c_m p_m)} |\alpha_1|_{(L_1, g_1)}^2 \cdots |\alpha_m|_{(L_m, g_m)}^2 = r^{-2(c_1 p_1 + \dots + c_m p_m)} |\alpha|_{(L, g_1)}^2.$$

We can locally write a general α as a sum $\sum_j \alpha_{1j} \wedge \cdots \wedge \alpha_{mj}$, where the collection $\{\alpha_{1j} \wedge \cdots \wedge \alpha_{mj}\}$ are mutually orthogonal. Then

$$\begin{aligned} |\alpha|_{(L, g_r)}^2 &= \sum_j |\alpha_{1j} \wedge \cdots \wedge \alpha_{mj}|_{(L, g_r)}^2 \\ &= \sum_j r^{-2(c_1 p_1 + \dots + c_m p_m)} |\alpha_{1j} \wedge \cdots \wedge \alpha_{mj}|_{(L, g_1)}^2 \\ &= r^{-2(c_1 p_1 + \dots + c_m p_m)} |\alpha|_{(L, g_1)}^2. \end{aligned}$$

Finally, we can calculate

$$\begin{aligned} \|\alpha\|_r^2 &= \int_L |\alpha|_{(L, g_r)}^2 dV_r \\ &= \int_L r^{-2(c_1 p_1 + \dots + c_m p_m)} |\alpha|_{(L, g_1)}^2 r^{c_1 n_1 + \dots + c_m n_m} dV_1 \\ &= r^{c_1(n_1 - 2p_1) + \dots + c_m(n_m - 2p_m)} \int_L |\alpha|_{(L, g_1)}^2 dV_1 \\ &= r^{c_1(n_1 - 2p_1) + \dots + c_m(n_m - 2p_m)} \|\alpha\|_1^2. \quad \square \end{aligned}$$

Corollary 4.29. *Let $\phi \in \Omega^k(L)$, and extend it to a form on all of $\Omega^k(X)$ independently of r . This extension is an L^2 form if and only if*

$$\phi \in \bigoplus_{\substack{k_1 + \dots + k_m = k \\ c_1(n_1 - 2k_1) + \dots + c_m(n_m - 2k_m) > -1}} \Omega^{(k_1, \dots, k_m)}(L).$$

Proof. Take $\phi \in \Omega^{(k_1, \dots, k_m)}(L)$, and set $c = c_1(n_1 - 2k_1) + \dots + c_m(n_m - 2k_m)$. Then Lemma 4.28 says $\|\phi\|_r = r^c \|\phi\|_1$. If $c > -1$, then Lemma 3.2 says that ϕ extends

to an L^2 form on X . On the other hand, if $c \leq -1$, then Lemma 3.4 says that if ϕ extends to an L^2 form on X , then $\phi = 0$. \square

Now we are ready to state our version of the Künneth Theorem for multiply-warped products:

Theorem 4.30. *Let $L = L_1 \times \cdots \times L_m$ be the product of compact manifolds, and $X = (0, 1] \times L$. Give X the metric*

$$g = dr^2 + r^{2c_1} g_{L_1} + \cdots + r^{2c_m} g_{L_m}$$

for some constants c_i and metrics g_{L_i} on L_i . Then the L^2 -cohomology of X is isomorphic to

$$H_{(2)}^k(X, g) \cong \bigoplus_{\substack{k_1 + \cdots + k_m = k \\ c_1(n_1 - 2k_1) + \cdots + c_m(n_m - 2k_m) > -1}} H_{dR}^{k_1}(L_1) \otimes \cdots \otimes H_{dR}^{k_m}(L_m).$$

In particular, $H_{(2)}^k(X, g) \subseteq H_{dR}^k(L)$.

Proof. Take $\alpha \in \text{dom}(d)$ so that $d\alpha = 0$. Use Lemma 4.27 to write

$$\alpha = A\phi + dr \wedge A\omega + dB\alpha.$$

We can write $A\omega = \eta_{low} + \eta_{high}$, where

$$\begin{aligned} \eta_{low}(r) &\in \bigoplus_{\substack{k_1 + \cdots + k_m = k \\ c_1(n_1 - 2k_1) + \cdots + c_m(n_m - 2k_m) > 0}} \Omega^{(k_1, \dots, k_m)}(L) \\ \eta_{high}(r) &\in \bigoplus_{\substack{k_1 + \cdots + k_m = k \\ c_1(n_1 - 2k_1) + \cdots + c_m(n_m - 2k_m) \leq 0}} \Omega^{(k_1, \dots, k_m)}(L) \end{aligned}$$

for all $r \in (0, 1)$. Due to Theorem 4.24, both η_{low} and η_{high} are also strongly harmonic forms; in particular, $d_L \eta_{low} = 0$ and $d_L \eta_{high} = 0$. Using Lemmas 3.3, 3.5, and 4.28, we find that

$$\int_1^r \eta_{low} \quad \text{and} \quad \int_0^r \eta_{high}$$

are L^2 forms. By Lemma 3.9 and the fact that $d\left(\int_\varepsilon^r \eta_{high}\right) = dr \wedge \eta_{high}$ for all $\varepsilon > 0$, we get

$$\bar{d}\left(\int_0^r \eta_{high}\right) = dr \wedge \eta_{high}.$$

Combining these facts and the fact that $d\left(\int_1^r \eta_{low}\right)$, we get

$$\alpha = A\phi + \bar{d}\left(\int_1^r \eta_{low} + \int_0^r \eta_{high} + H\alpha\right).$$

Thus, any class in $H_{(2)}^k(X, g)$ can be represented as a Neumann strongly harmonic form $A\alpha \in \mathcal{H}_N^k(L)$ extended independently of r . Since $A\alpha$ is independent of r yet is still an L^2 form, Corollary 4.29 implies

$$\begin{aligned} A\alpha \in \mathcal{H}_N^k(L) \bigcap & \bigoplus_{\substack{k_1 + \dots + k_m = k \\ c_1(n_1 - 2k_1) + \dots + c_m(n_m - 2k_m) > -1}} \Omega^{(k_1, \dots, k_m)}(L) \\ \in & \bigoplus_{\substack{k_1 + \dots + k_m = k \\ c_1(n_1 - 2k_1) + \dots + c_m(n_m - 2k_m) > -1}} \mathcal{H}_N^{k_1}(L_1) \wedge \dots \wedge \mathcal{H}_N^{k_m}(L_m). \end{aligned}$$

Combined with the Hodge Theorem, this finishes the proof. \square

4.5 Examples

Example 4.31. Let $X = (0, 1] \times L$ for some n -dimensional compact manifold L with boundary. Assume X is equipped with a metric g given by

$$g = dr^2 + r^{2c}g_L$$

for some metric g_L on L and real constant $c \geq 1$. Then Cheeger shows³ in [5] (and we've reproven in this chapter) that

$$H_{(2)}^k(X) = \begin{cases} 0 & k \geq \frac{n}{2} + \frac{1}{2c} \\ H_{dR}^k(L) & k < \frac{n}{2} + \frac{1}{2c} \end{cases}$$

³ He includes the condition that \bar{d}_L on degree $(n-1)/2$ forms has closed range, but this condition is satisfied in the case of a compact link; see [9] for details.

In particular, if n is odd or $H_{(2)}^{n/2}(L) = 0$, then the L^2 -cohomology is isomorphic to the middle intersection homology.

Example 4.32. Let $X = (0, 1] \times L_1 \times L_2$, where $\dim L_1 = 1$, $\dim L_2 = n$ for some even n , and L_1 and L_2 are both compact manifolds with boundary. Assume X is equipped with a metric g given by

$$g = dr^2 + r^2 g_{L_1} + r^{2c} g_{L_2}.$$

for some metrics g_{L_1} and g_{L_2} on L_1 and L_2 , respectively, and some real constant $c \geq 1$. Then $c_1 = 1$, $c_2 = c$, $n_1 = 1$, and $n_2 = n$. We must check when

$$(1 - 2p_1) + c(n - 2p_2) > -1.$$

If $p_1 = 0$, then this happens when $p_2 > \frac{n}{2}$. If $p_1 = 1$, then this happens when $p_2 \geq \frac{n}{2}$. Thus, overall, this happens when $p = p_1 + p_2 > \frac{n}{2}$. Thus, we've shown

$$H_{(2)}^k(X) = \begin{cases} 0 & k \geq \frac{n+2}{2} \\ H_{dR}^k(L) & k < \frac{n+2}{2} \end{cases}$$

The requirement that n is even is key.

Metrics of this kind are called Cheeger metrics in [15], though we will extend this definition to slightly more general spaces in the next chapter. They play an important role in the calculation of the local L^2 -cohomology of complex algebraic varieties.

Example 4.33. In this section, we'll construct a series of real affine algebraic varieties with an isolated singularity whose local L^2 -cohomology at the singularity is not isomorphic to its local middle perversity intersection homology. A similar question was looked at in [1], and in fact our family of examples extends theirs.

Let (n_1, \dots, n_m) be a tuple of natural numbers $n_k \geq 1$, and let (c, \dots, c_m) be a tuple of rational numbers $c_k \geq 1$; write in lowest terms $c_k = p_k/q_k$, where the p_k and q_k are positive integers. Let $X \subset \mathbb{R}^{1+\sum_{k=1}^m (n_k+1)}$ be the real algebraic variety cut out by the system of equations

$$\begin{aligned} z^{2p_1} &= \left(\sum_{j=0}^{n_1} x_{1j}^2 \right)^{q_1} \\ &\vdots \\ z^{2p_m} &= \left(\sum_{j=0}^{n_m} x_{mj}^2 \right)^{q_m}. \end{aligned}$$

Then X is a real algebraic variety with an isolated singularity at the origin. The link L of X can be found in this case by intersecting X with a union of disjoint hyperplanes defined by $|z| = 1$, which gives the equations

$$\begin{aligned} z &= \pm 1 \\ 1 &= \sum_{j=0}^{n_1} x_{1j}^2 \\ &\vdots \\ 1 &= \sum_{j=0}^{n_m} x_{mj}^2. \end{aligned}$$

Therefore, we see that the link is the union of products

$$L = (N_1 \times \cdots \times N_m) \bigcup (N_1 \times \cdots \times N_m),$$

where N_k is an n_k -sphere. We also see that we have a diffeomorphism

$$(0, 1] \times L \xrightarrow{\psi} X \cap \{|z| \leq 1\}$$

given by

$$\psi(r, \pm 1, y_{11}, \dots, y_{1n_1}, y_{21}, \dots, y_{mn_m}) = (\pm r, r^{c_1} y_1, \dots, r^{c_1} y_{1n_1}, r^{c_2} y_{21}, \dots, r^{c_m} y_{mn_m}).$$

Lemma 4.34. *This diffeomorphism is a quasi-isometry between the metric on $X \cap \{0 < |z| \leq 1\}$ induced by the Euclidean metric and the multiply-warped product metric on $(0, 1] \times L$ given by*

$$g = dr^2 + r^{2c_1}g_{N_1} + \cdots + r^{2c_m}g_{N_m}.$$

Proof. At a point $(r, \pm 1, y_{11}, \cdots, y_{1n_1}, y_{21}, \cdots, y_{mn_m}) = (\pm r, r^{c_1}y_1, \cdots, r^{c_1}y_{1n_1}, r^{c_2}y_{21}, \cdots, r^{c_m}y_{mn_m})$, the pushforward of $\frac{\partial}{\partial r}$ is

$$\psi_* \frac{\partial}{\partial r} = (\pm 1, c_1 r^{c_1-1} y_{11}, \cdots, c_m r^{c_m-1} y_{mn_m}),$$

while the pushforward of $v = (0, v_{11}, \cdots, v_{mn_m}) \in T_*L$ is

$$\psi_* v = (0, r^{c_1} v_{11}, \cdots, r^{c_m} v_{mn_m}).$$

If $v \in T_*N_k$, then $v_{ij} = 0$ for $i \neq k$, $(v_{k1}, \cdots, v_{kn_k})$ is orthogonal to $(y_{k1}, \cdots, y_{kn_k})$, and $\psi_* v = r^{c_k} v$. Thus, $\psi_* v$ is orthogonal to $\psi_* \frac{\partial}{\partial r}$ and any $w \in T_*N_j$ for $j \neq k$, while $\langle \psi_* v, \psi_* w \rangle = r^{2c_k} \langle v, w \rangle$ for $v, w \in T_*N_k$. Therefore, the pullback metric on $(0, 1] \times L$ through ψ is given by

$$\left\| \psi_* \frac{\partial}{\partial r} \right\|^2 dr^2 + r^{2c_1} g_{N_1} + \cdots + r^{2c_m} g_{N_m}.$$

Thus,

$$\left\| \psi_* \frac{\partial}{\partial r} \right\|^2 = 1 + \sum_{k=1}^m \sum_{j=1}^{n_k} c_k^2 r^{2(c_k-1)} y_{kj}^2 = 1 + \sum_{k=1}^m c_k^2 r^{2(c_k-1)}.$$

This gives us the bounds

$$1 \leq \left\| \psi_* \frac{\partial}{\partial r} \right\|^2 \leq 1 + \sum_{k=1}^m c_k^2.$$

Applying Lemma 3.1, we get our result. □

Therefore, X is quasi-isometric to a multiply-warped product. We know its L^2 -cohomology by applying Lemma 2.4 and Theorem 4.30. Generically the L^2 -cohomology of these varieties will be different from the middle perversity intersection homology.

To illustrate this in a specific case, set $m = 2$, $(n_1, n_2) = (3, 2)$, and $(c_1, c_2) = (1, 4)$. The link is the union of two copies of $S^3 \times S^2$. The L^2 -cohomology will have rank 2 in degree 3 due to the top degree of S^3 while the middle perversity intersection homology will be trivial; conversely, the intersection homology will have rank 2 in degree 2 due to the top degree of S^2 , while the L^2 -cohomology will be trivial.

Example 4.35. Though we showed in Theorem 4.30 that the L^2 -cohomology of a product metric is always a subspace of the cohomology of link, this is not always the case for more general spaces. Here, we construct an example of a multiply-warped fiber bundle with an L^2 form which is exact but not L^2 -exact.

Let M be the unit tangent sphere bundle of the real 2-sphere. Then M is 3-dimensional, and M can be described as a fiber bundle

$$\begin{array}{ccc} S^1 & \longrightarrow & M \\ & & \downarrow \pi \\ & & S^2 \end{array}$$

The tangent space of M decomposes globally as $T_*M = T_*S^1 \oplus T_*S^2$. In fact, T_*S^1 is a natural subspace, but T_*S^2 is not. However, we can choose the subspace of parallel vectors if we fix a metric on M .

Now let $X = (0, 1] \times M$, pick such a decomposition, and put a metric g_r on T_*M for each r as follows:

$$g_r = r^2 g_{S^2} + r^7 g_{S^1}.$$

Then put a metric g on all of X like so:

$$\begin{aligned} g &= dr^2 + g_r \\ &= dr^2 + r^2 g_{S^2} + r^7 g_{S^1}. \end{aligned}$$

There is a natural map $T^*S^2 \hookrightarrow T^*M$, and the pullback through any fiber map j^* gives an isomorphism $T^*S^1 = (T^*S^2)^\perp$. T^*S^1 is a real line bundle on a simply connected space, so it is trivial. Pick a nonvanishing section τ ; we can choose τ so that $|\tau| = 1$ everywhere and pick the sign of τ so that $\int_{S^1} j^*\tau$ is positive for every fiber $j : S^1 \hookrightarrow M$.

It turns out τ has several desirable properties. First of all, τ pulls back to a generator of the 1-dimensional cohomology of any fiber, i.e. τ is a global angular form. This means that the differential of τ is the pullback of the Euler class, i.e. $d\tau = \pi^*e$. For this fiber bundle, e is twice the volume form; this is Exercise 11.18 in [3]. In particular, π^*e is a nonzero 2-form. Because π^*e vanishes on $T_*S^1 \otimes T_*S^2$ and τ vanishes on T_*S^2 , we can calculate their norms, when considered as forms on X by extending independently of r :

$$\begin{aligned} \|\tau\|^2 &= \int_0^1 r^2 r^{-7/2} \|\tau\|_1^2 dr = \int_0^1 r^{-1.5} \|\tau\|_1^2 dr = \infty \text{ and} \\ \|\pi^*e\|^2 &= \int_0^1 r^{2-4} r^{7/2} \|\pi^*e\|_1^2 dr = \int_0^1 r^{1.5} \|\pi^*e\|_1^2 dr < \infty. \end{aligned}$$

Thus, τ is *not* an L^2 form, but $\pi^*(e)$ is.

Theorem 4.36. $\pi^*(e)$ is an L^2 form which is exact, but it is not the differential of any L^2 -form.

Proof. All that's left to prove is that $\pi^*(e)$ is not the differential of an L^2 form. Take any 1-form $\omega \in \Omega^\bullet(X)$ satisfying $d\omega = \pi^*e$. Then $d(\omega - \tau) = 0$. Since $H^1(M) = 0$, this says that $\omega - \tau = df$, where f is a function on M . We can write

$$df = s dr + t\tau + \eta$$

where s and t are smooth functions and $\eta \in \Gamma(T^*S^2)$. Then if $j : S^1 \rightarrow M$ is an inclusion of a fiber,

$$\int_{S^1} j^*t\tau = \int_{S^1} j^*(s dr + t\tau + \eta) = \int_{S^1} j^*df = 0.$$

Over an arbitrary fiber,

$$\int_{S^1} j^*|1+t|^2\tau \geq \left| \int_{S^1} j^*(1+t)\tau \right|^2 = \left| \int_{S^1} j^*\tau + 0 \right|^2 \geq c$$

for some positive constant c , since $\left| \int_{S^1} j^*\tau \right|^2$ is a continuous, positive function on the compact space S^2 . We can use this result to show that $\|\omega\|_r^2 \geq cr^{-1.5}$:

$$\begin{aligned} \|\omega\|_r^2 &= \int_M (|s dr|^2 + |(1+t)\tau|^2 r^{-7} + |\eta|^2 r^2) dV_r \\ &\geq \int_M |1+t|^2 \frac{r^{-1.5}}{2} \tau \wedge \pi^*e \\ &= \int_{S^2} \left(\int_F j^*(|(1+t)|^2\tau) \right) \frac{r^{-1.5}}{2} e \\ &\geq \int_{S^2} \frac{cr^{-1.5}}{2} e \\ &= cr^{-1.5}. \end{aligned}$$

Therefore,

$$\|\omega\| = \int_0^1 \|\omega\|_r^2 dr \geq \int_0^1 cr^{-1.5} dr = \infty,$$

and π^*e is not the differential of any L^2 -form. □

Analysis of Model Metrics

5.1 Analysis of Metrics of Cheeger Type

Consider an $(n + 2)$ -dimensional manifold $X = (0, 1] \times L$ so that n is even, and L is a fiber bundle with base S^1 and fiber N

$$\begin{array}{ccc} N & \longrightarrow & L \\ & & \downarrow \\ & & S^1 \end{array}$$

where (N, g_N) is an n -dimensional, compact Riemannian manifold (possibly with boundary). Then the tangent space $T_x X$ decomposes pointwise into $T(0, 1] \oplus TS^1 \oplus TN$, and we assume there is a value $c \geq 1$ so that the metric on X is given by

$$dr^2 + r^2 d\theta^2 + r^{2c} g_N,$$

where r and θ are coordinates on $(0, 1]$ and S^1 respectively. We will call metrics of this kind *metrics of Cheeger type* (the terminology first appeared in [15], although our definition is a little broader).

We can write any $\alpha \in \Omega^k(L)$ as $\alpha = \kappa_\theta \alpha + d\theta \wedge \iota_\theta \alpha$, which allows us to split $\Omega^k(L)$ into

$$\Omega^k(L) = \Omega^{(0,k)}(L) \oplus \Omega^{(1,k-1)}(L)$$

where

$\Omega^{(0,k)}$ has no $d\theta$ part

$\Omega^{(1,k-1)}$ has a $d\theta$ part.

Lemma 5.1. 1. Take $\omega \in \Omega^{(0,k)}(L)$. Then $\|\omega\|_r^2 = r^{2c(n-k-1)+1} \|\omega\|_1^2$.

2. Take $\omega \in \Omega^{(1,k-1)}(L)$. Then $\|\omega\|_r^2 = r^{2c(n-k)-1} \|\omega\|_1^2$.

3. For $k < \frac{n+2}{2}$ and $\omega \in \Omega^k(L)$, $r^{-1} \|\omega\|_r^2 \leq s^{-1} \|\omega\|_s^2$ for $r, s \in (0, 1]$ and $r \leq s$.

4. For $k \geq \frac{n+2}{2}$ and $\omega \in \Omega^k(L)$, $r \|\omega\|_r^2 \geq s \|\omega\|_s^2$ for $r, s \in (0, 1]$ and $r \leq s$.

Proof. If a collection of vectors forms an orthonormal basis of a vector space V with an inner product g , then the corresponding collection of covectors forms an orthonormal basis on the dual space V^* with the inner product that g induces on V^* ; this is the definition of the induced inner product. Similarly, an orthonormal basis of the wedge product $\bigwedge^k V$ is given by k -fold wedge products of the elements of the orthonormal basis of V .

At a point $x \in L$ and using the metric $d\theta^2 + g_N$, we can find an orthonormal basis $g_1 = \frac{\partial}{\partial \theta}, v_1, \dots, v_n$ of L where $v_k \in T_x N \subset T_x L$. If we instead use the metric $g_r = r^2 d\theta^2 + r^{2c} g_N$, then the orthonormal basis can instead be given by

$$r^{-1} \frac{\partial}{\partial \theta}, r^{-c} v_1, \dots, r^{-c} v_n.$$

If we choose dual vectors $d\theta, \omega_1, \dots, \omega_n$ for the vectors associated to g_1 , then the dual vectors for the metric g_r is given by

$$rd\theta, r^c\omega_1, \dots, r^c\omega_n.$$

Thus, if $|rd\theta|_r = 1$, then we must have $|d\theta|_r = r^{-1}$. Likewise, $|\omega_k| = r^{-c}$. Then the orthonormal basis for $\bigwedge^k T_x^*L$ with the metric induced by g_r is given by the collection

$$\{r d\theta \wedge r^c\omega_{J(1)} \wedge \dots \wedge r^c\omega_{J(k-1)}\}_J \cup \{r^c\omega_{I(1)} \wedge \dots \wedge r^c\omega_{I(k)}\}_I.$$

In particular, the volume form is given at a point by $r^{1+cn} d\theta \wedge \omega_1 \wedge \dots \wedge \omega_n$. Therefore, we have

$$|d\theta \wedge \omega_{J(1)} \wedge \dots \wedge \omega_{J(k-1)}|_r = r^{-1-c(k-1)} \quad \text{and}$$

$$|\omega_{I(1)} \wedge \dots \wedge \omega_{I(k)}|_r = r^{-ck}.$$

We can write a form $\omega \in \Omega^{(0,k)}(L)$ at x as

$$\omega = \sum_I a_I \omega_{I(1)} \wedge \dots \wedge \omega_{I(k)}.$$

Therefore, $|\omega|_r^2 = \sum_I |a_I|^2 r^{-2ck} = r^{-2ck} |\omega|_1^2$, and we can calculate

$$\begin{aligned} \|\omega\|_r^2 &= \int_L |\omega|_r^2 dV_r \\ &= \int_L r^{-2ck} |\omega|_r^2 r^{1+cn} dV_1 \\ &= r^{1+c(n-2k)} \|\omega\|_1^2. \end{aligned}$$

This gives us part 1 of our Lemma. Similarly, if $\omega \in \Omega^{(1,k-1)}(L)$, we have

$$\begin{aligned} \|\omega\|_r^2 &= \int_L |\omega|_r^2 dV_r \\ &= \int_L r^{-2-2c(k-1)} |\omega|_r^2 r^{1+cn} dV_1 \\ &= r^{-1+c(n-2(k-1))} \|\omega\|_1^2. \end{aligned}$$

This gives us part 2. For parts 3 and 4, merely notice that if $k < \frac{n+2}{2}$, then $k \leq \frac{n}{2}$, and

$$-1 + c(n - 2(k - 1)) \geq 1 + c(n - 2k) \geq 1.$$

On the other hand, if $k \geq \frac{n+2}{2}$, then

$$1 + c(n - 2k) \leq -1 + c(n - 2(k - 1)) \leq -1.$$

Then we can apply parts 1 and 2. □

Parts 3 and 4 of Lemma 5.1 will allow us to use the Lemmas 3.2 to 3.5 from Section 3.1. One important theorem is the following:

Theorem 5.2. *The operator K from Definition 3.8 is well-defined and bounded. Furthermore, if $\omega \in \Omega^p(L)$ for $p < \frac{n+2}{2}$, then ω extends to an L^2 form on X .*

Proof. Using parts 3 and 4 of Lemma 5.1, we can apply Lemmas 3.2, 3.3, and 3.5. □

5.2 Analysis of Interpolations of Metrics of Cheeger Type

In this section, we will introduce the notion of an interpolation of metrics of Cheeger type.

Definition 5.3. Consider an $(n + 3)$ -dimensional manifold with corners

$$X = (0, 1] \times [-1, 2] \times S^1 \times N$$

where N is an n -dimensional compact manifold, possibly with corners. Then T_*X decomposes into $T_*(0, 1] \oplus T_*[-1, 2] \oplus T_*S^1 \oplus T_*N$. For $\xi > \gamma \geq 1$, give X the metric

$$g = \begin{cases} dr^2 + r^{2\xi}dq^2 + r^2d\theta^2 + r^{2\xi}g_N & -1 < q < 0 \\ dr^2 + r^{2\gamma}dq^2 + r^2d\theta^2 + (qr^\gamma + (1 - q)r^\xi)^2 g_N & 0 < q < 1 \\ dr^2 + r^{2\gamma}dq^2 + r^2d\theta^2 + r^{2\gamma}g_N & 1 < q < 2 \end{cases} \quad (5.1)$$

where dr generates $T^*(0, 1]$, dq generates $T^*[1, 2]$, $d\theta$ generates T^*S^1 , and g_N is some metric on N . We will write $\tilde{N} = S^1 \times N$ as a shorthand.

Remark 5.4. Some care is needed in dealing with this metric because it is not continuous. However, it extends to a smooth metric on each of the three sections $-1 \leq q \leq 0$, $0 \leq q \leq 1$, and $1 \leq q \leq 2$. When we need results from previous sections, we will always be able to apply them to one of these three sections.

We call such a space an *interpolation of metrics of Cheeger type*. Note that we require a product, and not merely a fiber bundle as in the previous section.

Because all the information for this metric is encoded in the middle section where $0 \leq q \leq 1$, we will also describe the above metric as an *interpolation of metrics of Cheeger type with middle part*

$$dr^2 + r^{2\gamma} dq^2 + (qr^\gamma + (1 - q)r^\xi)^2 g_N.$$

The metric g is a metric of Cheeger type for $-1 \leq q \leq 0$, a different metric of Cheeger type for $1 \leq q \leq 2$, and the region $0 \leq q \leq 1$ sort of interpolates between the two metrics on either side (although note how the norm of dq scales with r slightly differently for $-1 \leq q \leq 0$ than in the other two regions).

Because the tangent space of the link decomposes, the space of differential forms on the link $L = [-1, 2] \times \tilde{N}$ also decomposes:

$$\Omega^p(L) = \bigoplus_{p_q + p_\theta + p_N = p} \Omega^{(p_q, p_\theta, p_N)}(L)$$

where a form in $\Omega^{(p_q, p_\theta, p_N)}(L)$ is the linear combination of wedges of p_q many dq , p_θ many $d\theta$, and p_N many 1-forms from T^*N . We also define $\Omega^{(p_q, p_\theta, p_N)}(X) \subset \Omega^\bullet(X)$

to be the subspace of forms $\alpha \in \Omega^\bullet(X)$ so that $\alpha(r) \in \Omega^{(p_q, p_\theta, p_N)}(L)$ for all $r \in (0, 1]$.

Because we have such an explicit description of the metric, we can write down precise formulas for $\|\omega\|_r$ and $\|\omega\|$ as defined in Section 3.1. Also, it will be helpful in estimates to consider the portion of $\|\omega\|$ which comes from each of the three regions.

Definition 5.5. Let $\omega \in \Omega^{(p_q, p_\theta, p_N)}(X)$. We define

$$\|\omega(q, r)\|_{q,r} = \left(\int_{\tilde{N}} |\omega(q, r)|^2 r^{1-2p_\theta} r^{\xi(1-2p_q)} r^{\xi(n-2p_N)} dV_{\tilde{N}} \right)^{1/2} \quad \text{for } -1 < q < 0$$

$$\|\omega(q, r)\|_{q,r} = \left(\int_{\tilde{N}} |\omega(q, r)|^2 r^{1-2p_\theta} r^{\gamma(1-2p_q)} (qr^\gamma + (1-q)r^\xi)^{n-2p_N} dV_{\tilde{N}} \right)^{1/2} \quad \text{for } 0 < q < 1$$

$$\|\omega(q, r)\|_{q,r} = \left(\int_{\tilde{N}} |\omega(q, r)|^2 r^{1-2p_\theta} r^{\gamma(1-2p_q)} r^{\gamma(n-2p_N)} dV_{\tilde{N}} \right)^{1/2} \quad \text{for } 1 < q < 2.$$

We also define

$$\|\omega\|_{-1 \leq q \leq 0}^2 = \int_0^1 \int_{-1}^0 \|\omega(q, r)\|_{q,r}^2 dq dr$$

$$\|\omega\|_{0 \leq q \leq 1}^2 = \int_0^1 \int_0^1 \|\omega(q, r)\|_{q,r}^2 dq dr$$

$$\|\omega\|_{1 \leq q \leq 2}^2 = \int_0^1 \int_1^2 \|\omega(q, r)\|_{q,r}^2 dq dr$$

$$\|\omega\|_r^2 = \int_{-1}^2 \|\omega(q, r)\|_{q,r}^2 dq.$$

Notice

$$\|\omega\|^2 = \|\omega\|_{-1 \leq q \leq 0}^2 + \|\omega\|_{0 \leq q \leq 1}^2 + \|\omega\|_{1 \leq q \leq 2}^2.$$

Theorem 5.6. Consider a p -form $\omega \in \Omega^{(p_q, p_\theta, p_N)}(L)$ with degree $p = p_q + p_\theta + p_N$.

1. If $n - 2p_N < 0$, $p < \frac{n+3}{2}$, and $n(\xi - \gamma) < 2$, then there is some $b < 1$ so that

$$r^b \|\omega\|_r^2 \leq s^b \|\omega\|_s^2$$

for $0 < r \leq s \leq 1$.

2. If $n - 2p_N \geq 0$ and $p < \frac{n+3}{2}$, then there is some $b < 1$ so that

$$r^b \|\omega\|_r^2 \leq s^b \|\omega\|_s^2$$

for $0 < r \leq s \leq 1$.

3. If $n - 2p_N \leq 0$ and $p \geq \frac{n+3}{2}$, then

$$r \|\omega\|_r^2 \geq s \|\omega\|_s^2$$

for $0 < r \leq s \leq 1$.

4. If $n - 2p_N > 0$, $p \geq \frac{n+3}{2}$, and ω is independent of q for $0 \leq q \leq 1$, then

$$r \|\omega\|_r^2 \geq \frac{s}{n - 2p_N + 1} \|\omega\|_s^2$$

for $0 < r \leq s \leq 1$.

5. If $n - 2p_N > 0$, $p \geq \frac{n+3}{2}$, and $n(\xi - \gamma) < 1$, then

$$\|\omega\|_r^2 \geq \|\omega\|_s^2$$

for $0 < r \leq s \leq 1$.

For the following proof, set

$$c_0 = \frac{1 - 2p_\theta + \xi(1 - 2p_q) + \xi(n - 2p_N)}{2},$$

$$c_1 = \frac{1 - 2p_\theta + \gamma(1 - 2p_q) + \xi(n - 2p_N)}{2}, \text{ and}$$

$$c_2 = \frac{1 - 2p_\theta + \gamma(1 - 2p_q) + \gamma(n - 2p_N)}{2}.$$

Proof. Assume $0 < r \leq s \leq 1$. For $-1 < q < 0$,

$$\|\omega(q)\|_{q,r}^2 = \left(\frac{r}{s}\right)^{2c_0} \|\omega(q)\|_{q,s}^2,$$

and for $1 < q < 2$,

$$\|\omega(q)\|_{q,r}^2 = \left(\frac{r}{s}\right)^{2c_2} \|\omega(q)\|_{q,s}^2.$$

For $0 < q < 1$, we have

$$\begin{aligned} \|\omega(q)\|_{q,r}^2 &= \int_{\tilde{N}} |\omega(q)|^2 r^{1-2p_\theta} r^{\gamma(1-2p_q)} (qr^\gamma + (1-q)r^\xi)^{n-2p_N} dV_{\tilde{N}} \\ &= \left(\frac{r}{s}\right)^{1-2p_\theta+\gamma(1-2p_q)} \left(\frac{qr^\gamma + (1-q)r^\xi}{qs^\gamma + (1-q)s^\xi}\right)^{n-2p_N} \|\omega(q)\|_{q,s}^2. \end{aligned}$$

By Lemma A.2, if $n - 2p_N > 0$, then

$$\left(\frac{r}{s}\right)^{2c_1} \|\omega(q)\|_{q,s}^2 \leq \|\omega(q)\|_{q,r}^2 \leq \left(\frac{r}{s}\right)^{2c_2} \|\omega(q)\|_{q,s}^2,$$

if $n - 2p_N = 0$, then

$$\|\omega(q)\|_{q,r}^2 = \left(\frac{r}{s}\right)^{2c_2} \|\omega(q)\|_{q,s}^2,$$

and if $n - 2p_N < 0$, then

$$\left(\frac{r}{s}\right)^{2c_2} \|\omega(q)\|_{q,s}^2 \leq \|\omega(q)\|_{q,r}^2 \leq \left(\frac{r}{s}\right)^{2c_1} \|\omega(q)\|_{q,s}^2.$$

If $p < \frac{n+3}{2}$, then $p \leq \frac{n+1}{2}$ since it must be an integer, and using this, one can check that both $c_0, c_2 \geq \frac{1}{2}$. Furthermore, if $n(\xi - \gamma) < 2$, then

$$\begin{aligned} 2c_1 &= 1 - 2p_\theta + \gamma(1 - 2p_q) + \xi(n - 2p_N) \\ &= 1 - 2p_\theta + \gamma(1 - 2p_q) + \gamma(n - 2p_N) + (\xi - \gamma)(n - 2p_N) \\ &\geq 2c_2 - n(\xi - \gamma) \\ &> 1 - 2 = -1. \end{aligned}$$

On the other hand, if $p \geq \frac{n+3}{2}$, then one can check that both $c_0, c_2 \leq -\frac{1}{2}$. Now we have enough to check our theorem case-by-case.

1. Assume that $n - 2p_N < 0$, $p < \frac{n+3}{2}$, and $n(\xi - \gamma) < 1$. If we set $b = -\min\{2c_0, 2c_1, 2c_2\} < 1$, then

$$r^b \|\omega(q)\|_{q,r}^2 \leq s^b \|\omega(q)\|_{q,s}^2.$$

By integrating in q , we get what is desired.

2. Assume that $n - 2p_N \geq 0$ and $p < \frac{n+3}{2}$. If we set $b = -\min\{2c_0, 2c_2\} \leq 0$, then

$$r^b \|\omega(q)\|_{q,r}^2 \leq s^b \|\omega(q)\|_{q,s}^2.$$

By integrating in q , we get what is desired.

3. Assume that $n - 2p_N \leq 0$ and $p \geq \frac{n+3}{2}$. Then $\max\{2c_0, 2c_2\} \leq -1$, so

$$r \|\omega(q)\|_{q,r}^2 \geq s \|\omega(q)\|_{q,s}^2.$$

By integrating in q , we get what is desired.

4. Assume that $n - 2p_N > 0$, $p \geq \frac{n+3}{2}$, and ω is independent of q for $0 \leq q \leq 1$. Then we must have $p_\theta = 1$, $p_q = 1$, and $p_N = \frac{n-1}{2}$. This gives us $c_0 = c_2 = -1$, so for $-1 < q < 0$ or $1 < q < 2$, we get

$$r \|\omega(q)\|_{q,r}^2 = s \|\omega(q)\|_{q,s}^2.$$

Because ω is independent of q for $0 < q < 1$,

$$\begin{aligned}
\int_0^1 \|\omega\|_{q,r}^2 dq &= \int_0^1 \int_{\tilde{N}} |\omega(q)|^2 r^{1-2p_\theta} r^{\gamma(1-2p_q)} (qr^\gamma + (1-q)r^\xi)^{n-2p_N} dV_{\tilde{N}} dq \\
&= \int_{\tilde{N}} |\omega|^2 \left(\int_0^1 r^{1-2p_\theta} r^{\gamma(1-2p_q)} (qr^\gamma + (1-q)r^\xi)^{n-2p_N} dq \right) dV_{\tilde{N}} \\
&= \int_{\tilde{N}} |\omega|^2 \left(r^{1-2p_\theta} r^{\gamma(1-2p_q)} \frac{r^{\gamma(n-2p_N+1)} - r^{\xi(n-2p_N+1)}}{r^\gamma - r^\xi} \right) dV_{\tilde{N}} \\
&= \int_{\tilde{N}} |\omega|^2 \left(r^{1-2p_\theta} r^{\gamma(1-2p_q)} \sum_{j=0}^{n-2p_N} r^{j\gamma} r^{(n-2p_N-j)\xi} \right) dV_{\tilde{N}} \\
&= \left(\frac{r}{s} \right)^{1-2p_\theta+\gamma(1-2p_q)} \left(\frac{\sum_{j=0}^{n-2p_N} r^{j\gamma+(n-2p_N-j)\xi}}{\sum_{j=0}^{n-2p_N} s^{j\gamma+(n-2p_N-j)\xi}} \right) \int_0^1 \|\omega\|_{q,s}^2 dq \\
&\geq \frac{1}{n-2p_N+1} \left(\frac{r}{s} \right)^{1-2p_\theta+\gamma(1-2p_q)} \left(\frac{r}{s} \right)^{\gamma(n-2p_N)} \int_0^1 \|\omega\|_{q,s}^2 dq \\
&= \frac{1}{n-2p_N+1} \left(\frac{r}{s} \right)^{2c_2} \int_0^1 \|\omega\|_{q,s}^2 dq \\
&\geq \frac{1}{n-2p_N+1} \left(\frac{s}{r} \right) \int_0^1 \|\omega\|_{q,s}^2 .
\end{aligned}$$

We can put all this together to get

$$\begin{aligned}
r\|\omega\|_r^2 &= r \int_{-1}^2 \|\omega(q)\|_{q,r}^2 dq \\
&\geq \frac{s}{n-2p_N+1} \int_{-1}^2 \|\omega(q)\|_{q,s}^2 dq \\
&= \frac{s}{n-2p_N+1} \|\omega\|_s^2
\end{aligned}$$

5. Assume that $n-2p_N > 0$, $p \geq \frac{n+3}{2}$, and $n(\xi-\gamma) < 2$. Then we must have $p_\theta = 1$, $p_q = 1$, and $p_N = \frac{n-1}{2}$. This gives us $2c_0 = 2c_2 = -1$, and $2c_1 = -1 - \gamma + \xi < 0$, so

$$\|\omega(q)\|_{q,r}^2 \geq \|\omega(q)\|_{q,s}^2 .$$

By integrating in q , we get what is desired. \square

Remark 5.7. Case 1 consists of forms with no dq or $d\theta$ part, because the conditions on p_N and p imply that $p_N = p$. Similarly, Case 4 consists of forms which are linear combinations of $dq \wedge d\theta \wedge \phi$, where ϕ is an $\frac{n-1}{2}$ -form on the link, because the conditions on p_N and p imply $p = p_N + 2$.

Theorem 5.8. *Assume that $n(\xi - \gamma) < 1$. Then the operator K from Definition 3.8 is well-defined and bounded. Furthermore, if $\omega \in \Omega^p(L)$ for $p < \frac{n+3}{2}$, then ω extends to an L^2 form on X .*

Proof. Using cases 1, 2, 3, and 5 of Theorem 5.6, we can apply Lemmas 3.2, 3.3, and 3.5. \square

Unfortunately, we do not quite have enough control to apply Theorem 3.10 and calculate the L^2 -cohomology directly. If $n(\xi - \gamma) < 1$, we are pretty close though! For $\alpha \in \text{dom}(d)$ and $\deg(\alpha) < \frac{n+3}{2}$, we can use Theorem 3.10 to show that $K\alpha \in \text{dom}(d)$ and

$$d(K\alpha) + K(d\alpha) = \alpha - \phi(1).$$

Likewise, for $\alpha \in \text{dom}(d)$ and $\deg(\alpha) > \frac{n+3}{2}$, we can use Theorem 3.10 to show that $K\alpha \in \text{dom}(\bar{d})$ and

$$\bar{d}(K\alpha) + K(d\alpha) = \alpha.$$

Thus, we can see that if $n(\xi - \gamma) < 1$, then

$$H_{(2)}^k(X) = \begin{cases} 0 & k > \frac{n+3}{2} \\ H_{dR}^k(L) & k < \frac{n+3}{2} \end{cases}.$$

The only finicky degree is the middle degree, where unfortunately the bound from case 5 of Theorem 5.6 does not give us quite enough information to guarantee the conclusions of Lemma 3.4. We must do a little work to homotope any form in $\text{dom}(d)$

of degree $(n + 3)/2$ to the sum of a form with $n - 2p_N \leq 0$ (case 3) and a form with $n - 2p_N = 1$ which is independent of q for $0 \leq q \leq 1$ (case 4).

5.3 Decomposing Into Interpolations With Smaller Eccentricity

The value $n(\xi - \gamma)$ which appeared in the statement of Theorem 5.6 plays an important role; we will call it the *eccentricity* of the interpolation. For some of our results to hold, we'll need $n(\xi - \gamma)$ to be small enough. The following theorem shows how to split an interpolation into the union of two distinct interpolations, each with half the eccentricity. We can then repeat this process as often as we'd like until the eccentricity is as low as we'd like; in particular, we can continue until the eccentricity in each region is less than 1, which is sufficient for Theorem 5.6.

Theorem 5.9. *Consider an interpolation of metrics of Cheeger type with exponents γ and ξ as given by Equation 5.1. Set $s = \frac{1}{2}(\xi - \gamma)$. Then the region $-1 \leq q \leq \frac{1}{2}r^s$ is quasi-isometric to an interpolation of metrics of Cheeger type with $\gamma + s$ and ξ , and the region $\frac{1}{2}r^s \leq q \leq 2$ is quasi-isometric to an interpolation of metrics of Cheeger type with γ and $\gamma + s$.*

First we will prove a lemma.

Lemma 5.10. *If $a, b, c_\theta, c_N : X \rightarrow [0, \infty)$ are positive real functions and a is bounded from above by some constant C , then the identity is a quasi-isometry between the metric*

$$(1 + a^2) dr^2 + ab(dr dp + dp dr) + b^2 dp^2 + c_\theta d\theta^2 + c_N g_N$$

and the metric

$$dr^2 + b^2 dp^2 + c_\theta d\theta^2 + c_N g_N .$$

Proof. By assumption, $a < C$ for some constant C , so

$$1 \leq 1 + a^2 \leq 1 + C^2 .$$

Furthermore,

$$\begin{aligned} ab &\leq \frac{a}{(1+a^2)^{1/2}} (1+a^2)^{1/2} b \\ &\leq \frac{C}{(1+C^2)^{1/2}} (1+a^2)^{1/2} b \end{aligned}$$

where the last inequality comes from Lemma A.2 after defining $q = \frac{a^2}{C^2}$ and rearranging

$$\frac{a}{(1+a^2)^{1/2}} = \left(\frac{(1-q) \cdot 0 + qC^2}{(1-q) + q(1+C^2)} \right)^{1/2}.$$

Then we can apply Lemma 3.1 to get our desired result. \square

Now let's go ahead and prove Theorem 5.9.

Proof. We'll first prove the theorem for the region $-1 \leq q \leq \frac{1}{2}r^s$.

We can define a piecewise-smooth homeomorphism

$$(0, 1] \times [-1, 2] \times \tilde{N} \longrightarrow \left\{ (r, q, x) \in (0, 1] \times [-1, 2] \times \tilde{N} \mid -1 \leq q \leq \frac{1}{2}r^s \right\}$$

$$(r, p, x) \xrightarrow{f} (r, \psi(r, p), x)$$

where

$$\psi(r, p) = \begin{cases} p & -1 \leq p \leq 0 \\ \frac{1}{3}pr^s & 0 \leq p \leq 1 \\ \frac{1}{6}(p+1)r^s & 1 \leq p \leq 2 \end{cases}.$$

Because this map is piecewise smooth, we can pull back the metric to get a metric on $(0, 1] \times [-1, 2] \times \tilde{N}$. To calculate this, we first pull back dq :

$$f^*dq = \begin{cases} dp & -1 < p < 0 \\ \frac{1}{3}(r^s dp + spr^{s-1} dr) & 0 < p < 1 \\ \frac{1}{6}[r^s dp + (p+1)sr^{s-1} dr] & 1 < p < 2 \end{cases}$$

Substituting our values for q and dq into Equation 5.1, we first find that on $-1 < p < 0$,

$$f^*g = dr^2 + r^{2\xi}dp^2 + r^2d\theta^2 + r^{2\xi}g_{\tilde{N}}.$$

This is already what we want in this region. Next, we find that on $0 < p < 1$,

$$\begin{aligned} f^*g = & \left(1 + \frac{s^2}{9}p^2r^{2(\gamma+s)-2}\right) dr^2 + \frac{1}{9}spr^{2(\gamma+s)-1}(dp dr + dr dp) \\ & + \frac{1}{9}r^{2(\gamma+s)}dp^2 + r^2d\theta^2 + \left(\frac{1}{3}pr^{\gamma+s} + \left(1 - \frac{1}{3}pr^s\right)r^\xi\right)^2 g_{\tilde{N}}. \end{aligned}$$

We apply Lemma 5.10 to see that f^*g is quasi-isometric to the metric

$$dr^2 + \frac{1}{9}r^{2(\gamma+s)}dp^2 + r^2d\theta^2 + \left(\frac{1}{3}pr^{\gamma+s} + \left(1 - \frac{1}{3}pr^s\right)r^\xi\right)^2 g_{\tilde{N}}.$$

Rewrite $\frac{1}{3}pr^{\gamma+s} + \left(1 - \frac{1}{3}pr^s\right)r^\xi = p\left(\frac{1}{3}r^{\gamma+s} - \frac{1}{3}r^{\xi+s} + r^\xi\right) + (1-p)r^\xi$, so that we can use Lemma A.2 to see

$$\frac{1}{3} \leq \frac{1}{3} - \frac{1}{3}r^{2s} + r^s \leq \frac{\frac{1}{3}pr^{\gamma+s} + \left(1 - \frac{1}{3}pr^s\right)r^\xi}{pr^{\gamma+s} + (1-p)r^\xi} \leq 1.$$

Therefore, we can use Lemma A.3 to see that the identity is a quasi-isometry between f^*g and the metric

$$dr^2 + r^{2(\gamma+s)}dp^2 + r^2d\theta^2 + (pr^{\gamma+s} + (1-p)r^\xi)^2 g_{\tilde{N}}.$$

Similarly, we find that on $1 < p < 2$,

$$\begin{aligned} f^*g = & \left(1 + \frac{s^2}{36}(p+1)^2r^{2(\gamma+s)-2}\right) dr^2 + \frac{1}{36}s(p+1)r^{2(\gamma+s)-1}(dp dr + dr dp) \\ & + \frac{1}{36}r^{2(\gamma+s)}dp^2 + r^2d\theta^2 + \left(\frac{1}{6}(p+1)r^{\gamma+s} + \left(1 - \frac{1}{6}(p+1)r^s\right)r^\xi\right)^2 g_{\tilde{N}}. \end{aligned}$$

We apply Lemma 5.10 to see that f^*g is quasi-isometric to the metric

$$dr^2 + \frac{1}{36}r^{2(\gamma+s)}dp^2 + r^2d\theta^2 + \left(\frac{1}{6}(p+1)r^{\gamma+s} + \left(1 - \frac{1}{6}(p+1)r^s\right)r^\xi\right)^2 g_{\tilde{N}}.$$

Rewrite

$$\frac{\frac{1}{6}(p+1)r^{\gamma+s} + \left(1 - \frac{1}{6}(p+1)r^s\right)r^\xi}{r^{\gamma+s}} = \frac{1}{6}(p+1) + r^s \left(1 - \frac{1}{6}(p+1)r^s\right).$$

Therefore,

$$\frac{1}{3} \leq \frac{\frac{1}{6}(p+1)r^{\gamma+s} + \left(1 - \frac{1}{6}(p+1)r^s\right)r^\xi}{r^{\gamma+s}} \leq 2.$$

Therefore, we can use Lemma A.3 to see that f^*g is quasi-isometric to

$$dr^2 + r^{2(\gamma+s)}dp^2 + r^2d\theta^2 + r^{2(\gamma+s)}g_{\tilde{N}}.$$

Now we prove the theorem for the region $\frac{1}{2}r^s \leq q \leq 2$. We can define a piecewise-smooth homeomorphism

$$(0, 1] \times [-1, 2] \times \tilde{N} \longrightarrow \left\{ (r, q, x) \in (0, 1] \times [-1, 2] \times \tilde{N} \mid \frac{1}{2}r^s \leq q \leq 2 \right\}$$

$$(r, p, x) \xrightarrow{f} (r, \psi(r, p), x)$$

where

$$\psi(r, p) = \begin{cases} \left(\frac{2}{3} + \frac{1}{6}p\right)r^s & -1 \leq p \leq 0 \\ \frac{2}{3}r^s(1-p) + p & 0 \leq p \leq 1 \\ p & 1 \leq p \leq 2. \end{cases}$$

Because this map is piecewise smooth, we can pull back the metric to get a metric on $(0, 1] \times [-1, 2] \times \tilde{N}$. To calculate this, we first pull back dq :

$$f^*dq = \begin{cases} \frac{1}{6}r^s dp + \left(\frac{2}{3} + \frac{1}{6}p\right)sr^{s-1} dr & -1 < p < 0 \\ \left(1 - \frac{2}{3}r^s\right) dp + \frac{2}{3}(1-p)sr^{s-1} dr & 0 < p < 1 \\ dp & 1 < p < 2. \end{cases}$$

Substituting our values for q and dq into Equation 5.1, we first find that on $-1 <$

$p < 0$,

$$f^*g = \left(1 + \left(\frac{2}{3} + \frac{1}{6}p\right)^2 s^2 r^{2(\gamma+s)-2}\right) dr^2 + \frac{1}{6} \left(\frac{2}{3} + \frac{1}{6}p\right) sr^{2(\gamma+s)-1} (dp dr + dr dp) \\ + \frac{1}{36} r^{2(\gamma+s)} dp^2 + r^2 d\theta^2 + \left(\left(\frac{2}{3} + \frac{1}{6}p\right) r^{\gamma+s} + \left(1 - \left(\frac{2}{3} + \frac{1}{6}p\right) r^s\right) r^\xi\right)^2 g_{\tilde{N}}.$$

We apply Lemma 5.10 to see that f^*g is quasi-isometric to the metric

$$dr^2 + \frac{1}{36} r^{2(\gamma+s)} dp^2 + r^2 d\theta^2 + \left(\left(\frac{2}{3} + \frac{1}{6}p\right) r^{\gamma+s} + \left(1 - \left(\frac{2}{3} + \frac{1}{6}p\right) r^s\right) r^\xi\right)^2 g_{\tilde{N}}.$$

Rewrite

$$\frac{\left(\frac{2}{3} + \frac{1}{6}p\right) r^{\gamma+s} + \left(1 - \left(\frac{2}{3} + \frac{1}{6}p\right) r^s\right) r^\xi}{r^{\gamma+s}} = \frac{2}{3} + \frac{1}{6}p + \left(1 - \left(\frac{2}{3} + \frac{1}{6}p\right) r^s\right) r^\xi.$$

Therefore,

$$\frac{1}{2} \leq \frac{\left(\frac{2}{3} + \frac{1}{6}p\right) r^{\gamma+s} + \left(1 - \left(\frac{2}{3} + \frac{1}{6}p\right) r^s\right) r^\xi}{r^{\gamma+s}} \leq 2.$$

Therefore, we can use Lemma A.3 to see that f^*g is quasi-isometric to

$$dr^2 + r^{2(\gamma+s)} dp^2 + r^2 d\theta^2 + r^{2(\gamma+s)} g_{\tilde{N}}.$$

Now we analyze the case where $0 < p < 1$. Substituting our values for q and dq into Equation 5.1, we find that

$$f^*g = \left(1 + \left(\frac{2}{3}(1-p)\right)^2 s^2 r^{2(\gamma+s)-2}\right) dr^2 + \left(1 - \frac{2}{3}r^s\right) \frac{2}{3}(1-p) sr^{2\gamma+s-1} (dp dr + dr dp) \\ + \left(1 - \frac{2}{3}r^s\right)^2 r^{2\gamma} dp^2 + r^2 d\theta^2 + (qr^\gamma + (1-q)r^\xi)^2 g_{\tilde{N}}.$$

where $q = \frac{2}{3}r^s(1-p) + p$. We apply Lemma 5.10 to see that f^*g is quasi-isometric to the metric

$$dr^2 + \left(1 - \frac{2}{3}r^s\right)^2 r^{2\gamma} dp^2 + r^2 d\theta^2 + (qr^\gamma + (1-q)r^\xi)^2 g_{\tilde{N}}.$$

Rewrite

$$\frac{qr^\gamma + (1-q)r^\xi}{pr^\gamma + (1-p)r^{\gamma+s}} = \frac{pr^\gamma + (1-p)\left(\frac{2}{3}r^{\gamma+s} + r^\xi - \frac{2}{3}r^{\xi+s}\right)}{pr^\gamma + (1-p)r^{\gamma+s}}.$$

We can use Lemma A.2 to see

$$\frac{2}{3} \leq \frac{qr^\gamma + (1-q)r^\xi}{pr^\gamma + (1-p)r^{\gamma+s}} \leq 1.$$

Therefore, we can use Lemma A.3 to see that f^*g is quasi-isometric to

$$dr^2 + r^{2\gamma}dp^2 + r^2d\theta^2 + (pr^\gamma + (1-p)r^{\gamma+s})^2 g_{\tilde{N}}.$$

Lastly, we analyze the case where $1 < p < 2$. Substituting our values for q and dq into Equation 5.1, we find that

$$f^*g = dr^2 + r^2 d\theta^2 + r^{2\gamma} dp^2 + r^{2\gamma} g_{\tilde{N}}.$$

As this is already what we want, we are finished. \square

Remark 5.11. In some contexts, there may be other, more natural, ways of decreasing the eccentricity. For example, in the case of complex algebraic surfaces, [17] remarks that this can be accomplished by taking extra blow-ups and redefining the flow in a natural way.

5.4 Homotoping to Forms Independent of q

Some forms in $\text{dom}(d)$ may not be in the Cases 1-4 of Theorem 5.6, so we will work to find a cohomologous form which is a sum of forms in these cases for every element of $\text{dom}(d)$. To do that, we will first use a harmonic projection operator associated to N , and then we will average in the q direction. Harmonic projection and averaging operators are a common tool when calculating L^2 -cohomology in other contexts; both [22] and [25] use similar homotopies, as did we in Chapter 4.

First, we define a the harmonic projection homotopy in the N directions. This will proceed very similarly to Theorem 2.29 in [25]. Let $d_N, \delta_N(q, r), G_N(q, r)$, and $A_N(q, r)$ be the operators on $\Omega^\bullet(L)$ induced by the metric g_r and which are associated with the factor N as defined in Definitions 4.14, 4.16, and 4.20.

Lemma 5.12. *1. The space $\mathcal{H}_N(N, (qr^\gamma + (1 - q)r^\xi)^2 g_N)$ of Neumann strongly harmonic forms is constant in q and r .*

2. The operators A_N and $\delta_N G_N$ are constant in r .

Proof. Setting $\lambda = (qr^\gamma + (1 - q)r^\xi)$, these are both consequences of Corollary 4.26 in the section on scaled metrics. \square

Definition 5.13. Define the operators $A, H : \Omega^\bullet(X) \rightarrow \Omega^\bullet(X)$ by

$$A(\phi + dr \wedge \omega) = A_N \phi + dr \wedge A_N \omega$$

$$H(\phi + dr \wedge \omega) = \delta_N G_N \phi - dr \wedge \delta_N G_N \omega.$$

Theorem 5.14. *The operators A and H satisfy the following properties:*

- 1. H, A , and $dq \wedge H$ are all bounded,*
- 2. $d_N(A\alpha) = 0$,*
- 3. A is a chain map, i.e. $dA = Ad$, and*
- 4. $dH\alpha + Hd\alpha = \alpha - A\alpha$.*

Proof. 1. From Corollary 4.26 we learn that the corresponding bounds C_k used in the proof of Proposition 4.23 can be chosen to be independent of r . Then by combining Proposition 4.22 and Corollary 4.26, we have for $\omega \in \Omega^\bullet(L)$ that‘

$$\|\delta_N G_N \omega\|_{q,r} \leq \begin{cases} Cr^\xi \|\omega\|_{q,r} & -1 < q < 0 \\ C(qr^\gamma + (1 - q)r^\xi) \|\omega\|_{q,r} & 0 < q < 1 \\ Cr^\gamma \|\omega\|_{q,r} & 1 < q < 2, \text{ and} \end{cases}$$

$$\|A\omega\|_{q,r} \leq \|\omega\|_{q,r}.$$

Since $(qr^\gamma + (1-q)r^\xi) \leq r^\gamma$, we get

$$\|dq \wedge \delta_N G_N \omega\|_{q,r} \leq C \|\omega\|_{q,r}, \text{ and}$$

$$\|\delta_N G_N \omega\|_{q,r} \leq C \|\omega\|_{q,r}.$$

Therefore, writing $\alpha = \phi + dr \wedge \omega$, we get

$$\begin{aligned} \|H\alpha\|^2 &= \int_0^1 \int_{-1}^2 \|H\alpha\|_{q,r}^2 dq dr \\ &= \int_0^1 \int_{-1}^2 (\|\delta_N G_N \phi\|_{q,r}^2 + \|\delta_N G_N \omega\|_{q,r}^2) dq dr \\ &\leq C^2 \int_0^1 \int_{-1}^2 (\|\phi\|_{q,r}^2 + \|\omega\|_{q,r}^2) dq dr \\ &= C^2 \int_0^1 \int_{-1}^2 \|\alpha\|_{q,r}^2 dq dr \\ &= C^2 \|\alpha\|^2. \end{aligned}$$

Similarly, we get

$$\begin{aligned} \|dq \wedge H\alpha\|^2 &= \int_0^1 \int_{-1}^2 \|dq \wedge H\alpha\|_{q,r}^2 dq dr \\ &= \int_0^1 \int_{-1}^2 (\|dq \wedge \delta_N G_N \phi\|_{q,r}^2 + \|dq \wedge \delta_N G_N \omega\|_{q,r}^2) dq dr \\ &\leq C^2 \int_0^1 \int_{-1}^2 (\|\phi\|_{q,r}^2 + \|\omega\|_{q,r}^2) dq dr \\ &= C^2 \int_0^1 \int_{-1}^2 \|\alpha\|_{q,r}^2 dq dr \\ &= C^2 \|\alpha\|^2, \text{ and} \end{aligned}$$

$$\begin{aligned}
\|A\alpha\|^2 &= \int_0^1 \int_{-1}^2 \|A\alpha\|_{q,r}^2 dq dr \\
&= \int_0^1 \int_{-1}^2 (\|A_N\phi\|_{q,r}^2 + \|A_N\omega\|_{q,r}^2) dq dr \\
&\leq \int_0^1 \int_{-1}^2 (\|\phi\|_{q,r}^2 + \|\omega\|_{q,r}^2) dq dr \\
&= \int_0^1 \int_{-1}^2 \|\alpha\|_{q,r}^2 dq dr \\
&= \|\alpha\|^2.
\end{aligned}$$

2. This is part 3 of Proposition 4.22.

3. By Lemma 4.19, we have

$$d = d_r + d_\theta + d_q + d_N.$$

Then $Ad = dA$ follows by statement 8 of Proposition 4.22.

4. The homotopy formula follows from a straightforward calculation since everything is smooth:

$$\begin{aligned}
dH(\phi + dr \wedge \omega) &= d(\delta_N G_N \phi - dr \wedge \delta_N G_N \omega) \\
&= d_L \delta_N G_N \phi + dr \wedge \left(\frac{\partial \delta_N G_N \phi}{\partial r} + d_L \delta_N G_N \omega \right), \text{ and}
\end{aligned}$$

$$\begin{aligned}
Hd(\phi + dr \wedge \omega) &= H \left(d_L \phi + dr \wedge \left(\frac{\partial \phi}{\partial r} - d_L \omega \right) \right) \\
&= \delta_N G_N d_L \phi - dr \wedge \left(\delta_N G_N \frac{\partial \phi}{\partial r} - \delta_N G_N d_L \omega \right).
\end{aligned}$$

Since $\delta_N G_N$ is constant in r by Lemma 5.12, $\frac{\partial}{\partial r}$ commutes with $\delta_N G_N$. We can

combine our calculations and use statement 7 of Proposition 4.22 to get

$$\begin{aligned}
dH\alpha + Hd\alpha &= d_L\delta_N G_N \phi + \delta_N G_N d_L\phi + dr \wedge (d_L\delta_N G_N \omega + \delta_N G_N d_L\omega) \\
&= \phi - A_N\phi + dr \wedge (\omega - A_N\omega) \\
&= \alpha - A\alpha.
\end{aligned}$$

□

The following subspace is somewhat problematic on these interpolations, so we will find a way of dealing with it.

Definition 5.15. We define M to be the subspace of $\Omega^\bullet(X)$ given by

$$M = \bigoplus_{p_N < \frac{n}{2}} \Omega^{(p_\theta, p_q, p_N)}(X) \oplus dr \wedge \Omega^{(p_\theta, p_q, p_N)}(X).$$

Notice that M is not a subcomplex.

Definition 5.16. Take a form $\alpha \in M$, and define the operator $G : M \rightarrow M$ as

$$G\alpha = 2 \int_1^{3/2} \left(\int_c^q \iota_q \alpha(r, q') dq' \right) dc.$$

The following lemma will be useful.

Lemma 5.17. *If $\alpha \in M$ is an L^2 form which is independent of q , then there is a constant $A > 0$ so that*

$$\|\alpha\|_{-1 \leq q \leq 0} \leq A \|\alpha\|_{0 \leq q \leq 1},$$

$$\|\alpha\|_{-1 \leq q \leq 0} \leq \|\alpha\|_{1 \leq q \leq 2}, \text{ and}$$

$$\|\alpha\|_{0 \leq q \leq 1} \leq \|\alpha\|_{1 \leq q \leq 2}.$$

Proof. Because α is constant in q , we can write $\alpha(q, r) = \alpha(r)$. Set

$$\psi(r) = \int_{\tilde{N}} |\alpha(r)|^2 dV_{\tilde{N}},$$

and then we can rewrite the definitions from Definition 5.5 as

$$\begin{aligned}\|\alpha\|_{-1 \leq q \leq 0} &= \int_0^1 \int_{-1}^0 \psi(r) r^{1-2p_\theta} r^\xi (1-2p_q+n-2p_N) dq dr, \\ \|\alpha\|_{0 \leq q \leq 1} &= \int_0^1 \int_0^1 \psi(r) r^{1-2p_\theta} r^{\gamma(1-2p_q)} (qr^\gamma + (1-q)r^\xi)^{n-2p_N} dq dr, \text{ and} \\ \|\alpha\|_{1 \leq q \leq 2} &= \int_0^1 \int_1^2 \psi(r) r^{1-2p_\theta} r^{\gamma(1-2p_q+n-2p_N)} dq dr.\end{aligned}$$

Notice that if $k \geq 0$, then

$$r^{\xi k} \leq (qr^\gamma + (1-q)r^\xi)^k \leq r^{\gamma k}.$$

Because $n - 2p_N > 0$ and $1 - 2p_q \geq -1$, it must be that $1 - 2p_q + n - 2p_N \geq 0$. Thus we get our second and third inequalities directly.

For our first inequality, we must use that α is constant in q to allow us to integrate in q to see

$$\begin{aligned}\int_0^1 (qr^\gamma + (1-q)r^\xi)^{n-2p_N} dq &= \frac{r^{\gamma(n-2p_N+1)} - r^{\xi(n-2p_N+1)}}{(r^\gamma - r^\xi)(n - 2p_N + 1)} \\ &= \frac{1}{n - 2p_N + 1} (r^{\gamma(n-2p_N)} + r^{\gamma(n-2p_N-1)}r^\xi + \dots + r^{\xi(n-2p_N)}) \\ &\geq \frac{1}{n - 2p_N + 1} r^{\gamma(n-2p_N)}\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\alpha\|_{-1 \leq q \leq 0} &= \int_0^1 \int_{-1}^0 \psi(r) r^{1-2p_\theta} r^{\xi(1-2p_q+n-2p_N)} dq dr \\
&= \int_0^1 \psi(r) r^{1-2p_\theta} r^{\xi(1-2p_q+n-2p_N)} dr \\
&\leq \int_0^1 \psi(r) r^{1-2p_\theta} r^{\gamma(1-2p_q+n-2p_N)} dr \\
&\leq (n-2p_N+1) \int_0^1 \int_0^1 \psi(r) r^{1-2p_\theta} r^{\gamma(1-2p_q)} (qr^\gamma + (1-q)r^\xi)^{n-2p_N} dq dr \\
&\leq (n+1) \|\alpha\|_{0 \leq q \leq 1}.
\end{aligned}$$

Setting $A = n + 1$, this completes our proof. \square

Theorem 5.18. *The operators G and $dq \wedge G$ are bounded on $M \cap L^2\Omega^\bullet(X)$.*

Proof. First notice that $\kappa_q G\alpha = G\alpha$, so if $dq \wedge G$ is bounded, so is G .

Next, let α be a form in $L^2\Omega^{(p_\theta, p_q, p_N)}(X)$. Then referring to Definition 5.5 we have

$$\begin{aligned}
\|dq \wedge \iota_q \alpha\|_{-1 \leq q \leq 0}^2 &= \int_0^1 \int_{-1}^0 \int_{\tilde{N}} |\iota_q \alpha|^2 r^{1-2p_\theta} r^{-\xi} r^{\xi(n-2p_N)} dV_{\tilde{N}} dq dr \\
\|dq \wedge \iota_q \alpha\|_{0 \leq q \leq 1}^2 &= \int_0^1 \int_0^1 \int_{\tilde{N}} |\iota_q \alpha|^2 r^{1-2p_\theta} r^{-\gamma} (qr^\gamma + (1-q)r^\xi)^{n-2p_N} dV_{\tilde{N}} dq dr \\
\|dq \wedge \iota_q \alpha\|_{1 \leq q \leq 2}^2 &= \int_0^1 \int_1^2 \int_{\tilde{N}} |\iota_q \alpha|^2 r^{1-2p_\theta} r^{-\gamma} r^{\gamma(n-2p_N)} dV_{\tilde{N}} dq dr \\
\|dq \wedge \iota_q \alpha\|^2 &= \|dq \wedge \iota_q \alpha\|_{-1 \leq q \leq 0}^2 + \|dq \wedge \iota_q \alpha\|_{0 \leq q \leq 1}^2 + \|dq \wedge \iota_q \alpha\|_{1 \leq q \leq 2}^2.
\end{aligned}$$

We will show the following inequalities

$$\|dq \wedge G\alpha\|_{1 \leq q \leq 2}^2 \leq \|dq \wedge \iota_q \alpha\|_{1 \leq q \leq 2}^2 \quad (5.2)$$

$$\|dq \wedge G\alpha\|_{0 \leq q \leq 1}^2 \leq \|dq \wedge \iota_q \alpha\|_{0 \leq q \leq 1}^2 + \|dq \wedge \iota_q \alpha\|_{1 \leq q \leq 2}^2 \quad (5.3)$$

$$\|dq \wedge G\alpha\|_{-1 \leq q \leq 0}^2 \leq \|dq \wedge \iota_q \alpha\|_{-1 \leq q \leq 0}^2 + A \|dq \wedge \iota_q \alpha\|_{0 \leq q \leq 1}^2 + \|dq \wedge \iota_q \alpha\|_{1 \leq q \leq 2}^2. \quad (5.4)$$

which together imply

$$\|dq \wedge G\alpha\|^2 \leq (2 + A)\|dq \wedge \iota_q \alpha\|^2 \leq (2 + A)\|\alpha\|^2.$$

This will complete the proof of Theorem 5.18. First, we show that for $1 \leq q \leq 2$ and $1 \leq c \leq 2$,

$$\begin{aligned} \left\| dq \wedge \int_c^q \iota_q \alpha(r, q') dq' \right\|_{1 \leq q \leq 2}^2 &= \int_0^1 \int_1^2 \int_{\tilde{N}} \left| \int_c^q \iota_q \alpha dq' \right|^2 r^{1-2p_\theta} r^{-\gamma} r^{\gamma(n-2p_N)} dV_{\tilde{N}} dq dr \\ &\leq \int_0^1 \int_1^2 \int_{\tilde{N}} \left(\int_1^2 |\iota_q \alpha|^2 dq' \right) r^{1-2p_\theta} r^{-\gamma} r^{\gamma(n-2p_N)} dV_{\tilde{N}} dq dr \\ &= \int_1^2 \|dq \wedge \iota_q \alpha\|_{1 \leq q \leq 2}^2 dq \\ &= \|dq \wedge \iota_q \alpha\|_{1 \leq q \leq 2}^2, \end{aligned}$$

for $-1 \leq q \leq 0$,

$$\begin{aligned} \left\| dq \wedge \int_0^q \iota_q \alpha(r, q') dq' \right\|_{-1 \leq q \leq 0}^2 &= \int_0^1 \int_{-1}^0 \int_{\tilde{N}} \left| \int_0^q \iota_q \alpha dq' \right|^2 r^{1-2p_\theta} r^{-\gamma} r^{\gamma(n-2p_N)} dV_{\tilde{N}} dq dr \\ &\leq \int_0^1 \int_{-1}^0 \int_{\tilde{N}} \left(\int_{-1}^0 |\iota_q \alpha|^2 dq' \right) r^{1-2p_\theta} r^{-\gamma} r^{\gamma(n-2p_N)} dV_{\tilde{N}} dq dr \\ &= \int_{-1}^0 \|dq \wedge \iota_q \alpha\|_{-1 \leq q \leq 0}^2 dq \\ &= \|dq \wedge \iota_q \alpha\|_{-1 \leq q \leq 0}^2, \end{aligned}$$

and for $0 \leq q \leq 1$,

$$\begin{aligned}
& \left\| dq \wedge \int_1^q \iota_q \alpha(r, q') dq' \right\|_{0 \leq q \leq 1}^2 \\
&= \int_0^1 \int_0^1 \int_{\tilde{N}} \left| \int_1^q \iota_q \alpha(r, q') dq' \right|^2 r^{1-2p_\theta} r^{-\gamma} (qr^\gamma + (1-q)r^\xi)^{n-2p_N} dV_{\tilde{N}} dq dr \\
&\leq \int_0^1 \int_0^1 \int_{\tilde{N}} \left(\int_q^1 |\iota_q \alpha|^2 (q'r^\gamma + (1-q')r^\xi)^{n-2p_N} dq' \right) \left(\int_q^1 (q'r^\gamma + (1-q')r^\xi)^{-n+2p_N} dq' \right) \\
&\quad r^{1-2p_\theta} r^{-\gamma} (qr^\gamma + (1-q)r^\xi)^{n-2p_N} dV_{\tilde{N}} dq dr \\
&= \int_0^1 \int_0^1 \int_{\tilde{N}} \left(\int_q^1 |\iota_q \alpha|^2 (q'r^\gamma + (1-q')r^\xi)^{n-2p_N} dq' \right) \left(\int_q^1 \left(\frac{qr^\gamma + (1-q)r^\xi}{q'r^\gamma + (1-q')r^\xi} \right)^{n-2p_N} dq' \right) \\
&\quad r^{1-2p_\theta} r^{-\gamma} dV_{\tilde{N}} dq dr \\
&\leq \int_0^1 \int_0^1 \int_{\tilde{N}} \left(\int_q^1 |\iota_q \alpha|^2 (q'r^\gamma + (1-q')r^\xi)^{n-2p_N} dq' \right) \left(\int_q^1 1 dq' \right) r^{1-2p_\theta} r^{-\gamma} dV_{\tilde{N}} dq dr \\
&\leq \int_0^1 \int_0^1 \int_{\tilde{N}} \left(\int_0^1 |\iota_q \alpha|^2 r^{1-2p_\theta} r^{-\gamma} (q'r^\gamma + (1-q')r^\xi)^{n-2p_N} dq' \right) dV_{\tilde{N}} dq dr \\
&\leq \int_0^1 \|dq \wedge \iota_q \alpha\|_{0 \leq q \leq 1}^2 dq \\
&= \|dq \wedge \iota_q \alpha\|_{0 \leq q \leq 1}^2.
\end{aligned}$$

Now we notice that for $1 \leq c \leq 2$,

$$dq \wedge \int_c^q \iota_q \alpha = \begin{cases} dq \wedge \left(\int_c^q \iota_q \alpha \right) & \text{for } 1 \leq q \leq 2 \\ dq \wedge \left(\int_1^q \iota_q \alpha + \int_c^1 \iota_q \alpha \right) & \text{for } 0 \leq q \leq 1 \\ dq \wedge \left(\int_0^q \iota_q \alpha + \int_1^0 \iota_q \alpha + \int_c^1 \iota_q \alpha \right) & \text{for } -1 \leq q \leq 0. \end{cases}$$

Combining these calculations and applying Lemma 5.17, we get for $1 \leq q \leq 2$ and $1 \leq c \leq 2$,

$$\left\| dq \wedge \int_c^q \iota_q \alpha \right\|_{1 \leq q \leq 2} \leq \|dq \wedge \iota_q \alpha\|_{1 \leq q \leq 2},$$

for $0 \leq q \leq 1$ and $1 \leq c \leq 2$,

$$\begin{aligned}
\left\| dq \wedge \int_c^q \iota_q \alpha \right\|_{0 \leq q \leq 1} &\leq \left\| dq \wedge \int_1^q \iota_q \alpha \right\|_{0 \leq q \leq 1} + \left\| dq \wedge \int_c^1 \iota_q \alpha \right\|_{0 \leq q \leq 1} \\
&\leq \left\| dq \wedge \int_1^q \iota_q \alpha \right\|_{0 \leq q \leq 1} + \left\| dq \wedge \int_c^1 \iota_q \alpha \right\|_{1 \leq q \leq 2} \\
&\leq \|dq \wedge \iota_q \alpha\|_{0 \leq q \leq 1} + \|dq \wedge \iota_q \alpha\|_{1 \leq q \leq 2},
\end{aligned}$$

and for $-1 \leq q \leq 0$ and $1 \leq c \leq 2$,

$$\begin{aligned}
\left\| dq \wedge \int_c^q \iota_q \alpha \right\|_{-1 \leq q \leq 0} &\leq \left\| dq \wedge \int_0^q \iota_q \alpha \right\|_{-1 \leq q \leq 0} + \left\| dq \wedge \int_1^0 \iota_q \alpha \right\|_{-1 \leq q \leq 0} + \left\| dq \wedge \int_c^1 \iota_q \alpha \right\|_{-1 \leq q \leq 0} \\
&\leq \left\| dq \wedge \int_0^q \iota_q \alpha \right\|_{-1 \leq q \leq 0} + A \left\| dq \wedge \int_1^0 \iota_q \alpha \right\|_{0 \leq q \leq 1} + \left\| dq \wedge \int_c^1 \iota_q \alpha \right\|_{1 \leq q \leq 2} \\
&\leq \|dq \wedge \iota_q \alpha\|_{-1 \leq q \leq 0} + A \|dq \wedge \iota_q \alpha\|_{0 \leq q \leq 1} + \|dq \wedge \iota_q \alpha\|_{1 \leq q \leq 2}.
\end{aligned}$$

Lastly, for $1 \leq q \leq 2$,

$$\begin{aligned}
\|dq \wedge G\alpha\|_{1 \leq q \leq 2} &= \left\| dq \wedge 2 \int_1^{3/2} \int_c^q \iota_q \alpha(q') dq' dc \right\|_{1 \leq q \leq 2} \\
&\leq 2 \int_1^{3/2} \left\| dq \wedge \int_c^q \iota_q \alpha \right\|_{1 \leq q \leq 2} dc \\
&\leq 2 \int_1^{3/2} \|dq \wedge \alpha\|_{1 \leq q \leq 2} dc \\
&= \|dq \wedge \alpha\|_{1 \leq q \leq 2},
\end{aligned}$$

for $0 \leq q \leq 1$,

$$\begin{aligned}
\|dq \wedge G\alpha\|_{0 \leq q \leq 1} &= \left\| dq \wedge 2 \int_1^{3/2} \int_c^q \iota_q \alpha(q') dq' dc \right\|_{0 \leq q \leq 1} \\
&\leq 2 \int_1^{3/2} \left\| dq \wedge \int_c^q \iota_q \alpha \right\|_{0 \leq q \leq 1} \\
&\leq 2 \int_1^{3/2} \left(\|dq \wedge \alpha\|_{0 \leq q \leq 1} + \|dq \wedge \alpha\|_{1 \leq q \leq 2} \right) dc \\
&= \|dq \wedge \alpha\|_{0 \leq q \leq 1} + \|dq \wedge \alpha\|_{1 \leq q \leq 2} ,
\end{aligned}$$

and for $-1 \leq q \leq 0$,

$$\begin{aligned}
\|dq \wedge G\alpha\|_{-1 \leq q \leq 0} &= \left\| dq \wedge 2 \int_1^{3/2} \int_c^q \iota_q \alpha(q') dq' dc \right\|_{-1 \leq q \leq 0} \\
&\leq 2 \int_1^{3/2} \left\| dq \wedge \int_c^1 \iota_q \alpha \right\|_{-1 \leq q \leq 0} \\
&\leq 2 \int_1^{3/2} \left(\|dq \wedge \alpha\|_{-1 \leq q \leq 0} + A \|dq \wedge \alpha\|_{0 \leq q \leq 1} + \|dq \wedge \alpha\|_{1 \leq q \leq 2} \right) dc \\
&= \|dq \wedge \alpha\|_{-1 \leq q \leq 0} + A \|dq \wedge \alpha\|_{0 \leq q \leq 1} + \|dq \wedge \alpha\|_{1 \leq q \leq 2} ,
\end{aligned}$$

These are the inequalities 5.2 – 5.4, so we are done. \square

Theorem 5.19. *Assume that both $\alpha, d\alpha \in M$ and $\alpha \in \text{dom}(d)$. Define the operator*

$$B\alpha = 2 \int_1^{3/2} \kappa_q \alpha(r, c) dc .$$

Then

1. B is a bounded operator, and
2. $G\alpha$ is in $\text{dom}(d)$ and it satisfies the homotopy formula

$$dG\alpha + Gd\alpha = \alpha - B\alpha .$$

Proof. First we show B is bounded. Because $B\alpha$ is independent of q , by Lemma 5.17 we get

$$\begin{aligned}
\|B\alpha\|^2 &\leq 3 \|B\alpha\|_{1 \leq q \leq 2}^2 \\
&= 3 \left\| 2 \int_1^{3/2} \kappa_q \alpha(r, c) dc \right\|_{1 \leq q \leq 2}^2 \\
&= 3 \int_0^1 \int_1^2 \int_{\tilde{N}} \left| 2 \int_1^{3/2} \kappa_q \alpha(r, c) dc \right|^2 r^{1-p_\theta + \gamma(1-p_\gamma) + \gamma(n-p_N)} dV_{\tilde{N}} dq dr \\
&\leq 3 \int_0^1 \int_1^2 \int_{\tilde{N}} \left(2 \int_1^{3/2} |\kappa_q \alpha(r, c)|^2 dc \right) r^{1-p_\theta + \gamma(1-p_\gamma) + \gamma(n-p_N)} dV_{\tilde{N}} dq dr \\
&\leq 6 \int_1^2 \|\kappa_q \alpha\|_{1 \leq q \leq 2}^2 dq \\
&\leq 6 \|\alpha\|_{1 \leq q \leq 2}^2.
\end{aligned}$$

To get the formula on smooth forms, we calculate (using the shorthand $d_{other} = d_r + d_\theta + d_N$):

$$\begin{aligned}
dG\alpha &= (d_q + d_{other}) 2 \int_1^{3/2} \int_c^q \iota_q \alpha(r, q') dq' dc \\
&= 2 \int_1^{3/2} \iota_q \alpha(r, q) dc + d_{other} \left(2 \int_1^{3/2} \int_c^q \iota_q \alpha(r, q') dq' dc \right) \\
&= \iota_q \alpha + d_{other} \left(2 \int_1^{3/2} \int_c^q \iota_q \alpha(r, q') dq' dc \right), \text{ and}
\end{aligned}$$

$$\begin{aligned}
Gd\alpha &= G(d_q + d_{other})(\kappa_q\alpha + dq \wedge \iota_q\alpha) \\
&= G(d_{other}\kappa_q\alpha + dq \wedge \left(\frac{\partial\kappa_q\alpha}{\partial q} - d_{other}\iota_q\alpha\right)) \\
&= 2 \int_1^{3/2} \int_c^q \left(\frac{\partial\kappa_q\alpha}{\partial q}(r, q') - d_{other}\iota_q\alpha(r, q')\right) dq' dc \\
&= 2 \int_1^{3/2} (\kappa_q\alpha - \kappa_q\alpha(r, c)) dc - 2 \int_1^{3/2} \int_c^q d_{other}\iota_q\alpha(r, q') dq' dc \\
&= \kappa_q\alpha - B\alpha - d_{other} \left(2 \int_1^{3/2} \int_c^q \iota_q\alpha(r, q') dq' dc\right).
\end{aligned}$$

Adding these together, we get our formula. Finally, we've shown that $G\alpha$ is L^2 , $Gd\alpha$ is L^2 , and $B\alpha$ is L^2 , so $dG\alpha$ must be L^2 . This tells us $G\alpha \in \text{dom}(d)$. \square

Now we've come to the Main Theorem of this section.

Theorem 5.20. *Let X be an interpolation of Cheeger metrics. Given an $(\frac{n+3}{2})$ -form $\alpha \in \text{dom}(d)$ so that $d\alpha = 0$, there is a form $\gamma \in \text{dom}(d)$ so that the following holds:*

1. $\gamma(q) = 0$ for $q < -3/4$ and $q > 7/4$.
2. Set $\beta = \alpha - d\gamma$. Then there is a constant $A > 0$ so that

$$r\|\kappa_r\beta(t)\|_r^2 \geq As\|\kappa_r\beta(t)\|_s^2$$

for all $r, s, t \in (0, 1]$ and $r \leq s$.

Proof. Let $\chi_1(q)$ and $\chi_2(q)$ be smooth functions which depend only on q such that

$$\chi_1(q) = \begin{cases} 1 & -1/2 < q < 3/2 \\ 0 & q < -3/4 \text{ and } q > 7/4 \end{cases} \text{ and } \chi_2(q) = \begin{cases} 1 & -1/4 < q < 5/4 \\ 0 & q < -1/2 \text{ and } q > 3/2 \end{cases}.$$

Let H and A be the operators from Definition 5.13, and let G and B be the operators from Theorem 5.19. Note that we can orthogonally write $A\alpha = \zeta_1 + \zeta_2$ so that

$\zeta_1 \in M$ where M is the subspace of Definition 5.15 and ζ_2 is in case 3 of Theorem 5.6. Remember that $d_N A\alpha = 0$ which implies both $d_N \zeta_0 = 0$ and $d_N \zeta_1 = 0$, and thus $d\zeta_1 \in M$. Note that ζ_1 and ζ_1 are both L^2 .

By using the homotopy formulas of Theorems 5.14 and 5.19, we can show

$$\alpha = \beta + d\gamma$$

where

$$\gamma = \chi_1 H\alpha + \chi_2 G\zeta_1, \text{ and}$$

$$\beta = (1 - \chi_1)\alpha - d\chi_1 \wedge H\alpha + \chi_1 \zeta_2 + (\chi_1 - \chi_2)\zeta_1 + \chi_2 B\zeta_1 - d\chi_2 \wedge G\zeta_1.$$

Clearly $\gamma = 0$ for $q < -3/4$ and $q > 7/4$. $\gamma \in L^2$ because H is bounded, G is bounded on M , and χ_1 and χ_2 are bounded functions. $d\gamma \in L^2$ because $\alpha \in L^2$ and each of the terms of β is L^2 , which we can see by using the fact that χ_1 and χ_2 are bounded functions and $dq \wedge H$, $dq \wedge G$, and B are bounded operators on their appropriate domains. Thus, γ and β are in $\text{dom}(d)$. (As a side remark, be careful: the individual terms of β are *not* necessarily each in $\text{dom}(d)$.)

Furthermore, each term of β either satisfies $n - 2p_N \leq 0$ or is independent of q on $0 \leq q \leq 1$ (several of these terms satisfy this trivially because they vanish completely on this interval). Thus, β satisfies the desired estimate by Theorem 5.6. \square

6

A Special Case of Weighted Homogenous Hypersurfaces

6.1 Preliminaries on Weighted Homogeneous Polynomials

Definition 6.1. Given a tuple of natural numbers $(\alpha_0, \alpha_1, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^{n+1}$, the weighted degree of a monomial $z_0^{d_0} z_1^{d_1} \cdots z_n^{d_n}$ is $\sum_{k=0}^n \alpha_k d_k$. A *weighted homogeneous polynomial* $F(z_0, z_1, \dots, z_n)$ for the weight $(\alpha_0, \alpha_1, \dots, \alpha_n)$ is a polynomial which is the sum of monomials of equal weighted degree. Then the weighted degree of the polynomial is the common weighted degree of its monomials.

Example 6.2. Any homogeneous polynomial is a weighted homogeneous polynomial for the weight $(1, 1, \dots, 1)$.

Example 6.3. $F(z_0, z_1, z_2) = z_0^3 + z_1^2 + z_2^2$ is a weighted homogeneous polynomial for the weight $(2, 3, 3)$, and the weighted degree is 6.

Remark 6.4. If $F(z_0, \dots, z_n)$ is a weighted homogeneous polynomial for some weight $(\hat{\alpha}_0, \dots, \hat{\alpha}_n)$, it is also weighted homogeneous for a weight $(\alpha_0, \dots, \alpha_n)$ for which $\gcd(\alpha_0, \dots, \alpha_n) = 1$.

We will restrict our attention to weighted homogeneous polynomials of the following form

$$F(z_0, z_1, \dots, z_n) = z_0^{\hat{\beta}} + g(z_1, \dots, z_n)$$

where $\hat{\beta} \in \mathbb{Z}_{\geq 3}$ and $g(z_1, \dots, z_n)$ is a (standard) homogeneous polynomial of degree $\hat{\alpha}$ satisfying $2 \leq \hat{\alpha} < \hat{\beta}$. Then F is a weighted homogeneous polynomial for the weight $(\alpha, \beta, \dots, \beta)$, where $\alpha = \hat{\alpha}/\gcd(\hat{\alpha}, \hat{\beta})$ and $\beta = \hat{\beta}/\gcd(\hat{\alpha}, \hat{\beta})$. We will denote the set where F vanishes the variety

$$V = \left\{ z \in \mathbb{C}^{n+1} \mid F(z) = 0 \right\}.$$

Example 6.5. For $m > 1$, the A_m polynomials $F(z_0, \dots, z_n) = z_0^{m+1} + z_1^2 + \dots + z_n^2$ are a class of examples satisfying the above conditions. When m is even, $\alpha = 2, \beta = m+1$ and when m is odd, $\alpha = 1, \beta = (m+1)/2$.

Lemma 6.6. *V has an isolated singularity at the origin.*

Proof. As the vanishing set of the polynomial F , V has a singularity at a point z if and only if $F(z) = 0$ and $dF(z) = 0$. We calculate $dF = [\hat{\beta}z_0^{\hat{\beta}-1}, dg]$. Because g is homogeneous of degree ≥ 2 , $dg(z_1, \dots, z_n) = 0$ if and only if $z_1 = \dots = z_n = 0$. Therefore, $dF(z) = 0$ if and only if $z_0 = z_1 = \dots = z_n = 0$. Since $F(0, \dots, 0) = 0$, the origin is the lone singularity of V . \square

Using this lemma, the regular part of V is given by $V_{reg} = V - \{0\}$. We endow V_{reg} with the metric g induced from the Euclidean metric on \mathbb{C}^{n+1} .

Definition 6.7. Define $z_{1 \rightarrow n} = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $|z_{1 \rightarrow n}| = (|z_1|^2 + \dots + |z_n|^2)^{1/2}$.

Definition 6.8. The *weighted flow* is the action on \mathbb{C}^{n+1} by the multiplicative group $(0, \infty)$ via

$$\lambda \star (z_0, \dots, z_n) = (\lambda z_0, \lambda^{\beta/\alpha} z_1, \dots, \lambda^{\beta/\alpha} z_n).$$

Clearly the flow restricts to an action on V .

We want to calculate the local L^2 -cohomology of V at the origin. To do this, we can take any bounded, open neighborhood $U' \subset \mathbb{C}^{n+1}$ of the origin that we like and calculate $H_{(2)}^\bullet(V_{reg} \cap U', g)$. We would usually need to ensure that the neighborhood U' was small enough, but V is weighted homogeneous, so for a small enough value of λ , the map $z \mapsto \lambda \star z$ gives a bi-Lipschitz diffeomorphism between the neighborhood we choose and a neighborhood of arbitrarily small size. Then we can apply Lemmas 2.5 and 2.4 to see that this map induces an isomorphism on the L^2 -cohomology.

As we can choose whichever open neighborhood of the origin that we like, we define $U' = \left\{ z \in \mathbb{C}^{n+1} \mid |z_0| \leq \eta \text{ and } |z_{1 \rightarrow n}| \leq 1 \right\}$ and $U = V_{reg} \cap U'$. We will choose η “small enough”, which will be made more explicit in the next section. Then in Section 6.4, we calculate $H_{(2)}^\bullet(U, g)$.

Remark 6.9. One could use the weighted flow to decompose a sufficiently nice, bounded neighborhood of the origin into $(0, 1] \times L$ in the following way. First, define L to be the intersection of V with the boundary of the neighborhood. Second, define the map from $(0, 1] \times L$ into V as

$$(r, z) \mapsto r \star z.$$

There is a large problem with this approach, however: this map is not a quasi-isometry between a metric of the form

$$dr^2 + g_r$$

and the metric on V_{reg} . Therefore, very little of our previous analysis is applicable. We will spend the next two sections finding a different map which is a quasi-isometry between a metric $dr^2 + g_r$ and the metric on V_{reg} .

6.2 Finding Good Flow Lines

Definition 6.10. Define \tilde{L} and N to be

$$\tilde{L} = \{z \in V \mid |z_{1 \rightarrow n}| = 1, |z_0| \leq \delta\}$$

$$N = \{z \in V \mid |z_{1 \rightarrow n}| = 1, z_0 = 0\} \subset \tilde{L}.$$

Lemma 6.11. *For δ small enough, \tilde{L} is diffeomorphic to $B \times N \subseteq \mathbb{C}^{n+1}$ where B is a closed 2-dimensional disk of radius δ . Furthermore, we can choose the diffeomorphism $\psi : \tilde{L} \rightarrow B \times N$ so that $\psi(z) = (z_0, \psi_N(z))$ for some $\psi_N : \tilde{L} \rightarrow N$.*

Proof. First, we notice that

$$V \cap \{|z_{1 \rightarrow n}|^2 = c\}$$

is smooth for all $c > 0$. One can do this by first observing that by Sard's theorem applied to $s(z) = |z_{1 \rightarrow n}|^2$ restricted to V_{reg} , they can't all be singular. Then notice that all of these are homeomorphic to each other by using the weighted flow. So, in particular, this space is smooth for $c = 1$, and we denote this space as

$$M = V \cap \{|z_{1 \rightarrow n}|^2 = 1\}.$$

The vectors $(1, 0, \dots, 0)$ and $(i, 0, \dots, 0)$ are tangent to $\{|z_{1 \rightarrow n}|^2 = 1\}$ at every point because $|z_{1 \rightarrow n}|^2$ does not depend on z_0 . These vectors are also tangent to V when $z_0 = 0$, because V is the vanishing set of f , and the differential of f at such a point is given by

$$\begin{aligned} df &= (\hat{\beta} z_0^{\hat{\beta}-1}, dg) \\ &= (0, dg). \end{aligned}$$

Therefore, these vectors are tangent to M at points where $z_0 = 0$. This implies the differential of the map

$$z_0 : M \rightarrow \mathbb{C}$$

is surjective at any point satisfying $z_0 = 0$. Therefore, we can find a $\delta > 0$ so that the differential of this map is surjective at any point satisfying $|z_0| \leq \delta$.

In particular, the map

$$z_0 : \tilde{L} \rightarrow B$$

is a submersion. It's also true that \tilde{L} is compact and that this map is surjective. Therefore, Ehresmann's Lemma says that z_0 is a locally trivial fiber bundle. Since it is a locally trivial fiber bundle over the contractible space B , it is globally trivial. Therefore, we can find a map $\psi_N : \tilde{L} \rightarrow N$ which trivializes the fiber bundle, i.e. that makes the map $\psi(z) = (z_0, \psi_N(z)) : \tilde{L} \rightarrow B \times N$ a diffeomorphism. \square

Definition 6.12. Define the subset $W \subset V_{reg}$ as follows:

$$W = \left\{ z \in V_{reg} \mid |z_0| \leq \delta |z_{1 \rightarrow n}|^{\alpha/\beta}, |z_{1 \rightarrow n}| \leq 1 \right\}.$$

Lemma 6.13. *There is a quasi-isometry $\psi : W \rightarrow X$ where*

$$X = \left\{ z \in \mathbb{C}^{n+1} - \{0\} \mid \left(0, \frac{z_{1 \rightarrow n}}{|z_{1 \rightarrow n}|} \right) \in N, |z_0| \leq \delta |z_{1 \rightarrow n}|^{\alpha/\beta}, |z_{1 \rightarrow n}| \leq 1 \right\}.$$

Further, we can choose ψ so that $\psi(z) = (z_0, \psi_N(z))$ for some $\psi_N : X \rightarrow N$ and so that $\psi(\lambda \star z) = \lambda \star \psi(z)$ for $\lambda \in (0, 1]$.

Proof. By Lemma 6.11, there exists a diffeomorphism

$$\bar{\psi} : \tilde{L} \rightarrow X \cap \{|z_{1 \rightarrow n}| = 1\} = B \times N,$$

where B is the closed 2-dimensional disk of radius δ , so that

$$\bar{\psi}(\bar{z}) = (\bar{z}_0, \bar{\psi}_N(\bar{z}))$$

for some $\bar{\psi}_N : \tilde{L} \rightarrow N$. Note that both W and X are invariant under the weighted flow by $\lambda \in (0, 1]$.

Given $z = (z_0, z_{1 \rightarrow n}) \in W$, there is a unique $\bar{z} \in \tilde{L}$ and $s \in (0, 1]$ so that $z = s \star \bar{z}$.

In fact,

$$s = |z_{1 \rightarrow n}|^{\alpha/\beta} \text{ and } \bar{z} = s^{-1} \star z = \left(\frac{z_0}{|z_{1 \rightarrow n}|^{\alpha/\beta}}, \frac{z_{1 \rightarrow n}}{|z_{1 \rightarrow n}|} \right).$$

We define the map $\psi : W \rightarrow X$ via

$$\psi(z) = \psi(s \star \bar{z}) = s \star \bar{\psi}(\bar{z}).$$

Since $\bar{\psi}(\bar{z}) = (\bar{z}_0, \bar{\psi}_N(\bar{z}))$, we see that $\psi(z) = (z_0, \psi_N(z))$, where

$$\psi_N(z) = |z_{1 \rightarrow n}| \bar{\psi}_N(|z_{1 \rightarrow n}|^{-\alpha/\beta} \star z).$$

Notice that $|\psi_N(z)| = |z_{1 \rightarrow n}|$, which in turn implies that $\|\psi(z)\| = \|z\|$.

Then ψ is a bijection between W and X . By Lemma 2.5, it now suffices to show that ψ is bi-Lipschitz. Notice that for all $b \in B$, $\bar{\psi}_N$ restricts to a diffeomorphism between $\{z \in \tilde{L} \mid z_0 = b\}$ and N , which is bi-Lipschitz as it is a diffeomorphism between compact manifolds. Further, we can choose the same lower and upper constants a and A , respectively, for each value of b , because b ranges over a compact set. We can choose these constants so that $0 < a < 1 < A < \infty$.

Now we show ψ is Lipschitz. Consider first $z, x \in W$ so that $|z_{1 \rightarrow n}| = |x_{1 \rightarrow n}|$. Set $s = |z_{1 \rightarrow n}|^{\alpha/\beta}$. Then

$$\begin{aligned} |\psi_N(z) - \psi_N(x)|^2 &= |s^{\beta/\alpha} \bar{\psi}_N(\frac{1}{s} \star z) - s^{\beta/\alpha} \bar{\psi}_N(\frac{1}{s} \star x)|^2 \\ &= s^{2\beta/\alpha} |\bar{\psi}_N(\frac{1}{s} \star z) - \bar{\psi}_N(\frac{1}{s} \star x)|^2 \\ &\leq A^2 s^{2\beta/\alpha} |s^{-\beta/\alpha} z_{1 \rightarrow n} - s^{-\beta/\alpha} x_{1 \rightarrow n}|^2 \\ &\leq A^2 |z_{1 \rightarrow n} - x_{1 \rightarrow n}|^2. \end{aligned}$$

If $|z_{1 \rightarrow n}| \neq |x_{1 \rightarrow n}|$, then we assume without loss of generality that $|z_{1 \rightarrow n}| < |x_{1 \rightarrow n}|$ and we set $t < 1$ so that $|z_{1 \rightarrow n}| = t^{\beta/\alpha}|x_{1 \rightarrow n}|$. Then $x_{1 \rightarrow n}$ is on a sphere of radius $|x_{1 \rightarrow n}|$ and $z_{1 \rightarrow n}$ is on a sphere of radius $t^{\beta/\alpha}|x_{1 \rightarrow n}|$, so the distance between $x_{1 \rightarrow n}$ and $z_{1 \rightarrow n}$ is at least the difference between the radii, i.e. $|x_{1 \rightarrow n} - z_{1 \rightarrow n}| \geq (1 - t^{\beta/\alpha})|x_{1 \rightarrow n}|$. We also have the bound $|t^{\beta/\alpha}x_{1 \rightarrow n} - z_{1 \rightarrow n}| \leq |x_{1 \rightarrow n} - z_{1 \rightarrow n}|$, which we get from Lemma A.4. We put this together with our previous result to get

$$\begin{aligned}
|\psi_N(z) - \psi_N(x)|^2 &\leq |\psi_N(z) - t^{\beta/\alpha}\psi_N(x)|^2 + |t^{\beta/\alpha}\psi_N(x) - \psi_N(x)|^2 \\
&= |\psi_N(z) - \psi_N(t \star x)|^2 + (1 - t^{\beta/\alpha})^2 |\psi_N(x)|^2 \\
&= |\psi_N(z) - \psi_N(t \star x)|^2 + (1 - t^{\beta/\alpha})^2 |x_{1 \rightarrow n}|^2 \\
&\leq A^2 |z_{1 \rightarrow n} - t^{\beta/\alpha}x_{1 \rightarrow n}|^2 + |z_{1 \rightarrow n} - x_{1 \rightarrow n}|^2 \\
&= (A^2 + 1) |z_{1 \rightarrow n} - x_{1 \rightarrow n}|^2.
\end{aligned}$$

Now we finally have

$$\begin{aligned}
\|\psi(z) - \psi(x)\|^2 &= |z_0 - x_0|^2 + |\psi_N(z) - \psi_N(x)|^2 \\
&\leq |z_0 - x_0|^2 + (A^2 + 1) |z_{1 \rightarrow n} - x_{1 \rightarrow n}|^2 \\
&\leq (A^2 + 1) \|z - x\|^2.
\end{aligned}$$

This gives us that ψ is Lipschitz.

Next we show the other direction. Consider first $z, x \in W$ satisfying $|z_{1 \rightarrow n}| = |x_{1 \rightarrow n}|$. Set $s = |z_{1 \rightarrow n}|^{\alpha/\beta}$. Then

$$\begin{aligned}
|\psi_N(z) - \psi_N(x)|^2 &= |s^{\beta/\alpha}\bar{\psi}_N(\frac{1}{s} \star z) - s^{\beta/\alpha}\bar{\psi}_N(\frac{1}{s} \star x)|^2 \\
&= s^{2\beta/\alpha} |\bar{\psi}_N(\frac{1}{s} \star z) - \bar{\psi}_N(\frac{1}{s} \star x)|^2 \\
&\geq a^2 s^{2\beta/\alpha} |s^{-\beta/\alpha} z_{1 \rightarrow n} - s^{-\beta/\alpha} x_{1 \rightarrow n}|^2 \\
&= a^2 |z_{1 \rightarrow n} - x_{1 \rightarrow n}|^2.
\end{aligned}$$

If $|z_{1 \rightarrow n}| \neq |x_{1 \rightarrow n}|$, then we assume without loss of generality that $|z_{1 \rightarrow n}| < |x_{1 \rightarrow n}|$ and we set t so that $|z_{1 \rightarrow n}| = t^{\beta/\alpha}|x_{1 \rightarrow n}|$. Set $s = |z_{1 \rightarrow n}|$. Notice that by prior reasoning,

$$\begin{aligned} |\psi_N(z) - t^{\beta/\alpha}\psi_N(x)|^2 &\leq |\psi_N(z) - \psi_N(x)|^2, \text{ and} \\ (1 - t^{\beta/\alpha})|\psi_N(x)| &\leq |\psi_N(x) - \psi_N(z)|. \end{aligned}$$

Putting this together, we get

$$\begin{aligned} |z_{1 \rightarrow n} - x_{1 \rightarrow n}|^2 &\leq |z_{1 \rightarrow n} - t^{\beta/\alpha}x_{1 \rightarrow n}|^2 + |t^{\beta/\alpha}x_{1 \rightarrow n} - x_{1 \rightarrow n}|^2 \\ &\leq \frac{1}{a^2}|\psi_N(z) - \psi_N(t \star x)|^2 + (1 - t^{\beta/\alpha})^2|x_{1 \rightarrow n}|^2 \\ &= \frac{1}{a^2}|\psi_N(z) - t^{\beta/\alpha}\psi_N(x)|^2 + (1 - t^{\beta/\alpha})^2|\psi_N(x)|^2 \\ &\leq \frac{1}{a^2}|\psi_N(z) - \psi_N(x)|^2 + |\psi_N(x) - \psi_N(z)|^2 \\ &\leq \left(\frac{1}{a^2} + 1\right)|\psi_N(z) - \psi_N(x)|^2. \end{aligned}$$

Now we finally have

$$\begin{aligned} \|\psi(z) - \psi(x)\|^2 &= |z_0 - x_0|^2 + |\psi_N(z) - \psi_N(x)|^2 \\ &\geq |z_0 - x_0|^2 + \left(\frac{a^2}{a^2 + 1}\right)|z_{1 \rightarrow n} - x_{1 \rightarrow n}|^2 \\ &\geq \left(\frac{a^2}{a^2 + 1}\right)\|z - x\|^2. \end{aligned}$$

This concludes the proof. □

Definition 6.14. Choose $\eta = \delta/2$, where δ is chosen small enough for Lemma 6.11 to work. Then define

$$\begin{aligned} U &= \left\{ z \in V_{reg} \mid |z_0| \leq \eta, |z_{1 \rightarrow n}| \leq 1 \right\}, \text{ and} \\ L &= \left\{ z \in U \mid |z_0| = \eta \text{ or } |z_{1 \rightarrow n}| = 1 \right\} = \partial U. \end{aligned}$$

Let

$$\begin{aligned}\hat{U} &= U \cap W \\ &= \left\{ z \in U \mid |z_0| \leq \delta |z_{1 \rightarrow n}|^{\alpha/\beta} \right\}.\end{aligned}$$

Then ψ restricts to a quasi-isometry from \hat{U} to \hat{X} where

$$\hat{X} = \left\{ z \in \mathbb{C}^{n+1} \mid \left(0, \frac{z_{1 \rightarrow n}}{|z_{1 \rightarrow n}|} \right) \in N, |z_{1 \rightarrow n}| \leq 1, |z_0| \leq \min\{\eta, \delta |z_{1 \rightarrow n}|^{\alpha/\beta}\} \right\}.$$

Define a map $(r, \phi) : U \rightarrow (0, 1] \times L$ via

$$\begin{aligned}r(z) &= \begin{cases} |z_{1 \rightarrow n}| & \text{if } |z_0| \leq \eta |z_{1 \rightarrow n}|, \\ |z_0|/\eta & \text{if } |z_0| \geq \eta |z_{1 \rightarrow n}|, \end{cases} \\ \phi(z) &= \begin{cases} \psi^{-1} \left(\frac{1}{r} \psi(z) \right) & \text{if } \frac{a+2}{3} \left(\frac{|z_0|}{\eta} \right) \leq |z_{1 \rightarrow n}|, \\ \psi^{-1} \left(\frac{z_0}{r}, (qr + (1-q)r^{\beta/\alpha})^{-1} \psi_N(z) \right) & \text{if } \frac{2a+1}{3} \left(\frac{|z_0|}{\eta} \right)^{\beta/\alpha} \leq |z_{1 \rightarrow n}| \leq \frac{a+2}{3} \left(\frac{|z_0|}{\eta} \right), \\ \left(\frac{z_0}{r}, \frac{z_{1 \rightarrow n}}{r^{\beta/\alpha}} \right) & \text{if } |z_{1 \rightarrow n}| \leq \frac{2a+1}{3} \left(\frac{|z_0|}{\eta} \right)^{\beta/\alpha}, \end{cases}\end{aligned}$$

where

$$q = \frac{|z_{1 \rightarrow n}| - \frac{2a+1}{3} r^{\beta/\alpha}}{\frac{a+2}{3} r - \frac{2a+1}{3} r^{\beta/\alpha}} \quad \text{for} \quad \frac{2a+1}{3} \left(\frac{|z_0|}{\eta} \right)^{\beta/\alpha} \leq |z_{1 \rightarrow n}| \leq \frac{a+2}{3} \left(\frac{|z_0|}{\eta} \right)$$

and

$$a = 2^{-\beta/\alpha}.$$

Note that $r = \frac{|z_0|}{\eta}$ in the region where q is defined above, and also notice that the map ψ is not used in the definition of ϕ for z outside of \hat{U} . Lastly, notice that r and ϕ are continuous and piecewise smooth, and (r, ϕ) is invertible, with a continuous, piecewise smooth inverse. In fact, we can define this map explicitly.

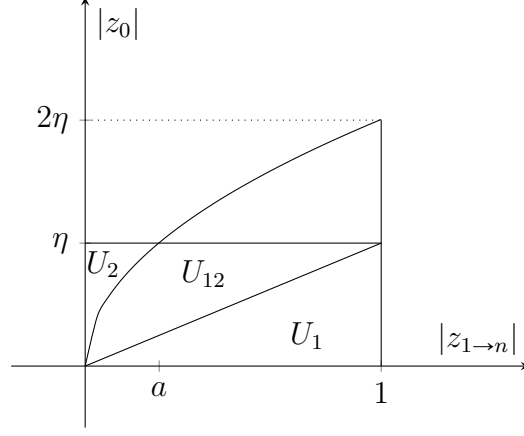


FIGURE 6.1: Decomposition of \hat{U}

First, we point out a decomposition $L = L_1 \cup L_{12} \cup L_2$ where

$$\begin{aligned} L_1 &= \left\{ z \in L \mid |z_{1 \rightarrow n}| = 1 \right\} \\ L_{12} &= \left\{ z \in L \mid |z_0| = \eta, z \in \hat{U} \right\} \\ L_2 &= \left\{ z \in L \mid |z_0| = \eta, z \notin \hat{U} \right\}. \end{aligned}$$

We can extend the definition of q to all of L_{12} via

$$q = \frac{|z_{1 \rightarrow n}| - \frac{a+2}{3}r}{1 - \frac{a+2}{3}r} + 1 \quad \text{for} \quad \frac{a+2}{3} \frac{|z_0|}{\eta} \leq |z_{1 \rightarrow n}| \leq 1$$

and

$$q = \frac{|z_{1 \rightarrow n}| - ar^{\beta/\alpha}}{\frac{1+2a}{3}r^{\beta/\alpha} - ar^{\beta/\alpha}} - 1 \quad \text{for} \quad a \left(\frac{|z_0|}{\eta} \right)^{\beta/\alpha} \leq |z_{1 \rightarrow n}| \leq \frac{2a+1}{3} \left(\frac{|z_0|}{\eta} \right)^{\beta/\alpha}.$$

There is a diffeomorphism $L_{12} \rightarrow S^1 \times [-1, 2] \times N$ given by

$$\xi \mapsto \left(\arg(\xi_0), q, \bar{\psi}_N \left(\frac{1}{|\xi_{1 \rightarrow n}|^{\alpha/\beta}} \star \xi \right) \right),$$

where q is given by the three equations above.

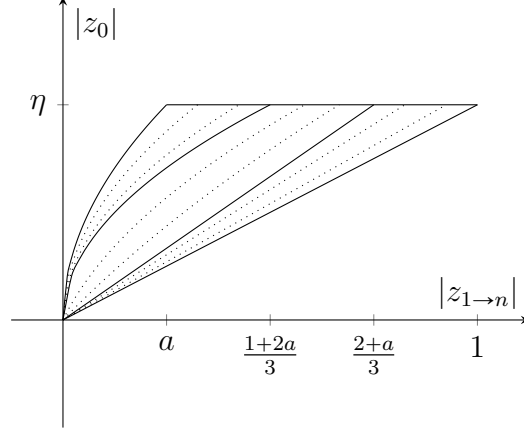


FIGURE 6.2: Flow on U_{12}

Now we can more easily write down the inverse $(r, \phi)^{-1} : (0, 1] \times L \rightarrow U$:

$$(r, \phi)^{-1}(r, \xi) = \begin{cases} \psi^{-1}(r\psi(\xi)) & \xi \in L_1 \\ \psi^{-1}(r\psi(\xi)) & \xi \in L_{12}, 1 \leq q \leq 2 \\ \psi^{-1}(r\xi_0, (qr + (1-q)r^{\beta/\alpha})\psi_N(\xi)) & \xi \in L_{12}, 0 \leq q \leq 1 \\ (r\xi_0, r^{\beta/\alpha}\xi_{1 \rightarrow n}) & \xi \in L_{12}, -1 \leq q \leq 0 \\ (r\xi_0, r^{\beta/\alpha}\xi_{1 \rightarrow n}) & \xi \in L_2 \end{cases}$$

In the next section, it will be useful to note that $(r\xi_0, r^{\beta/\alpha}\xi_{1 \rightarrow n}) = r \star \xi = \psi^{-1}(r \star \psi(\xi)) = \psi^{-1}(r\xi_0, r^{\beta/\alpha}\psi_N(\xi_{1 \rightarrow n}))$. This also makes it clearer that $(r, \phi)^{-1}$ is indeed continuous.

6.3 Calculating the Metric up to Quasi-Isometry

Ideally, we would precisely calculate the metric on $(0, 1] \times L$ which we get by pullback through $(r, \phi)^{-1}$. However, this would get very ugly. It suffices to find a metric on $(0, 1] \times L$ which is quasi-isometric to this pullback metric under the identity map on $(0, 1] \times L$ (this last condition, that a quasi-isometry between them is given by the identity, is crucial to several of our later arguments). This will be the goal of this

section.

Let's start by separating $U = U_1 \cup U_{12} \cup U_2$, where $U_1 = (r, \phi)^{-1}((0, 1] \times L_1)$, and so on. We will deal with each of U_1 , U_{12} , and U_2 in turn. Notice that (r, ϕ) restricted to $\hat{U} = U_1 \cup U_{12}$ factors through \hat{X} via

$$\begin{array}{ccc} \hat{U} & \xrightarrow{(r, \phi)} & (0, 1] \times (L_1 \cup L_{12}) \\ & \searrow \psi & \nearrow h^{-1} \\ & \hat{X} & \end{array}$$

for the diffeomorphism $h : (0, 1] \times L_1 \cup L_{12} \rightarrow \hat{X}$ given by

$$h(r, \xi) = \psi((r, \phi)^{-1}(r, \xi))$$

$$= \begin{cases} r\psi(\xi) & \xi \in L_1 \\ r\psi(\xi) & \xi \in L_{12}, 1 \leq q \leq 2 \\ (r\xi_0, (qr + (1-q)r^{\beta/\alpha})\psi_N(\xi)) & \xi \in L_{12}, 0 \leq q \leq 1 \\ r \star \psi(\xi) & \xi \in L_{12}, -1 \leq q \leq 0 \end{cases}$$

Lemma 6.15. *If $f : (X, g_X) \rightarrow (Y, g_Y)$ is a quasi-isometry and $p : Z \rightarrow X$ is a diffeomorphism, then the identity on Z gives a quasi-isometry between $(Z, (f \circ p)^* g_Y)$ and $(Z, p^* g_X)$.*

Proof.

$$\|(f \circ p)^* g_Y(v)\| = \|(f^* g_Y)(p_* v)\| \leq C \|g_X(p_* v)\| = C \|(p^* g_X)(v)\|$$

The proof of the other direction is basically identical. \square

The actual metric on $(0, 1] \times L$ is given by $((r, \phi)^{-1})^* g_U$, where g_U is the metric on U induced by the standard Euclidean metric. However, since ψ is a quasi-isomorphism, this lemma says we can instead consider the metric on $(0, 1] \times (L_1 \cup L_{12})$ given by $h^* g_{\hat{X}}$, where $g_{\hat{X}}$ is the metric on \hat{X} . First we consider $(0, 1] \times L_1$.

Theorem 6.16. *The identity on $(0, 1] \times L_1$ is a quasi-isometry between the metric $((r, \theta)^{-1})^* g_U$ and the conical metric*

$$dr^2 + r^2 g_{L_1},$$

where $g_{L_1} = h^* g_{h(L_1)}$ is the pullback of the metric on $h(L_1) \subset \hat{X}$ induced by the standard Euclidean metric.

Proof. By Lemma 6.15, it suffices to compare the conical metric with $h^* g_{\hat{X}}$ on $(0, 1] \times L_1$. Remember that

$$h(r, \xi) = r\psi(\xi).$$

We calculate two pushforward vectors: $h_* \frac{\partial}{\partial r}$ and $h_* v$ where v is a vector in $T_\xi L_1$.

$$h_* \frac{\partial}{\partial r} = \frac{d}{dt} (r + t)\psi(\xi) \Big|_{t=0} = \psi(\xi).$$

For v , let $\gamma : (-1, 1) \rightarrow L_1$ be a curve with $\gamma(0) = \xi$ and $\gamma'(0) = v$. Then

$$h_* v = \frac{d}{dt} r\psi(\gamma(t)) \Big|_{t=0} = r(\psi \circ \gamma)'(0) = r\psi_* v.$$

Remember from the proof of Lemma 6.13 that

$$\|\psi(\xi)\|^2 = \|\xi\|^2 = |\xi_0|^2 + |\xi_{1 \rightarrow n}|^2,$$

and $\xi \in L_1$, so $|\xi_{1 \rightarrow n}| = 1$ and $|\xi_0| \leq \eta$. Therefore,

$$1 \leq h^* g_{\hat{X}} \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right) \leq \eta^2 + 1, \text{ and}$$

$$h^* g_{\hat{X}}(v, v) = r^2 \|\psi_* v\|.$$

Also, since $\gamma(t) = (\gamma_0(t), \gamma_{1 \rightarrow n}(t)) \in L_1$ for all t , $|\gamma_{1 \rightarrow n}(t)| = 1$ for all t , and $|\psi_N(\gamma(t))| = |\gamma_{1 \rightarrow n}(t)| = 1$. Therefore, $\psi_N \circ \gamma$ lies entirely in the unit sphere, so

if we write $(\psi \circ \gamma)'(0) = (\gamma'_0(0), (\psi_N \circ \gamma)'(0))$, we learn $(\psi_N \circ \gamma)'(0)$ is orthogonal to $(\psi_N \circ \gamma)(0) = \psi_N(\xi)$. Therefore,

$$\begin{aligned} \left| h^* g_{\hat{X}} \left(\frac{\partial}{\partial r}, v \right) \right| &= \langle \psi(\xi), r(\psi \circ \gamma)'(0) \rangle. \\ &= r |\xi_0| |v_0| \\ &\leq \eta r |v_0| \\ &\leq (1 - c) \|\psi(\xi)\| \|\psi_* v\| \end{aligned}$$

for $c = 1 - \eta$. Therefore, we've satisfied the conditions of Lemma 3.1, so $h^* g_{\hat{X}}$ is quasi-isometric to

$$dr^2 + g_r$$

for g_r the metric on $\{r\} \times L$ induced by $h^* g_{\hat{X}}$. But

$$\begin{aligned} g_r(v, v) &= \|h_* v\|^2 \\ &= \|r(\psi \circ \gamma)'(0)\|^2 \\ &= r^2 \|(\psi \circ \gamma)'(0)\|^2 \\ &= r^2 g_1(v, v) \end{aligned}$$

and we have shown the identity is a quasi-isometry between $h^* g_{\hat{X}}$ and

$$dr^2 + r^2 g_{L_1}. \quad \square$$

Theorem 6.17. *The identity on $(0, 1] \times L_{12}$ is a quasi-isometry between the metric $((r, \theta)^{-1})^* g_U$ and the interpolation of Cheeger metrics*

$$\begin{cases} dr^2 + r^{2\beta/\alpha} dq^2 + r^2 d\theta^2 + r^{2\beta/\alpha} g_{h^{-1}(N)} & -1 \leq q \leq 0 \\ dr^2 + r^2 dq^2 + r^2 d\theta^2 + (qr + (1 - q)r^{\beta/\alpha})^2 g_{h^{-1}(N)} & 0 \leq q \leq 1 \\ dr^2 + r^2 dq^2 + r^2 d\theta^2 + r^2 g_{h^{-1}(N)} & 1 \leq q \leq 2 \end{cases},$$

where $g_{h^{-1}(N)} = h^* g_N$ is the pullback of the metric g_N induced by the standard Euclidean metric on the subspace N as given in Definition 6.10.

Proof. By Lemma 6.15, it suffices to compare our desired metric with $h^*g_{\hat{X}}$ on $(0, 1] \times L_{12}$. First we show this for $1 \leq q \leq 2$. In this region, $h(r, \xi) = r\psi(\xi)$. Similar to the previous theorem, we calculate the pushforward vectors $h_*\frac{\partial}{\partial r}$ and h_*v where v is a vector in $T_\xi L_{12}$.

$$h_*\frac{\partial}{\partial r} = \frac{d}{dt}(r+t)\psi(\xi)\Big|_{t=0} = \psi(\xi).$$

For v , let $\gamma : (-1, 1) \rightarrow L_{12}$ be a curve with $\gamma(0) = \xi$ and $\gamma'(0) = v$. Then

$$h_*v = \frac{d}{dt}r\psi(\gamma(t))\Big|_{t=0} = r(\psi \circ \gamma)'(0) = r\psi_*v.$$

This time we get slightly different bounds

$$\eta^2 \leq \|\psi(\xi)\|^2 \leq \eta^2 + 1.$$

And this time, because $\gamma_0(t)$ is constant, $\gamma'_0(0) = 0$. By this and by Cauchy-Schwarz, we get

$$\begin{aligned} \left| \left\langle h_*\frac{\partial}{\partial r}, h_*v \right\rangle \right| &\leq |\xi_{1 \rightarrow n}| |(\psi_*v)_{1 \rightarrow n}| r \\ &\leq \frac{|\xi_{1 \rightarrow n}|}{(|\xi_0|^2 + |\xi_{1 \rightarrow n}|^2)^{1/2}} \|\xi\| \|\psi_*v\| r \\ &\leq (1 - c) \|h_*\frac{\partial}{\partial r}\| \|h_*v\| \end{aligned}$$

where $c = 1 - \frac{1}{(1+\eta^2)^{1/2}}$; we can get the inequality

$$\frac{|\xi_{1 \rightarrow n}|}{(|\xi_0|^2 + |\xi_{1 \rightarrow n}|^2)^{1/2}} = \frac{1}{\left(1 + \frac{|\xi_0|^2}{|\xi_{1 \rightarrow n}|^2}\right)^{1/2}} \leq \frac{1}{(1 + \eta^2)^{1/2}} = 1 - c$$

because $|\xi_0| = \eta$ and $|\xi_{1 \rightarrow n}| \leq 1$. Therefore, we can apply Lemma 3.1 to get that $h^*g_{\hat{X}}$ is quasi-isometric to

$$dr^2 + g_r.$$

And once again,

$$\begin{aligned}
g_r(v, v) &= \|h_*v\|^2 \\
&= \|r(\psi \circ \gamma)'(0)\|^2 \\
&= r^2\|(\psi \circ \gamma)'(0)\|^2 \\
&= r^2g_1(v, v).
\end{aligned}$$

Therefore, for $1 < q < 2$, we get that $h^*g_{\hat{X}}$ is quasi-isometric to

$$dr^2 + r^2g_{L_{12}}.$$

Next, consider the region where $-1 \leq q \leq 0$.

$$h : (0, 1] \times S^1 \times [-1, 0] \times N \rightarrow U_{12}$$

is given by

$$h(r, \theta, q, x) = \left(r\eta e^{i\theta}, r^{\beta/\alpha} \left(a + \frac{(1+q)(1-a)}{3} \right) x \right).$$

For convenience, set $z = h(r, \theta, q, x)$. Notice that $|z_0| = r\eta$ and $|z_{1 \rightarrow n}|$ varies between $ar^{\beta/\alpha}$ and $\left(\frac{1+2a}{3}\right)r^{\beta/\alpha}$ as q ranges between -1 and 0 . Also remember that $\|x\| = 1$, which implies that $\langle x, v \rangle = 0$ for $v \in T_xN$, viewing both x and v as in \mathbb{C}^n .

Using this, we can calculate the pullback metric through h to be given by

$$\begin{aligned}
&\left(\eta^2 + (\beta/\alpha)^2 r^{2(\beta/\alpha-1)} \left(a + \frac{(1+q)(1-a)}{3} \right)^2 \right) dr^2 + \eta^2 r^2 d\theta^2 + r^{2\beta/\alpha} \left(\frac{1-a}{3} \right)^2 dq^2 \\
&+ r^{2\beta/\alpha} \left(a + \left(\frac{(1+q)(1-a)}{3} \right) \right)^2 g_N + r^{2\beta/\alpha-1} \frac{\beta}{\alpha} \left(\frac{1-a}{3} \right) \left(a + \frac{(1+q)(1-a)}{3} \right) (dq dr + dr dq).
\end{aligned}$$

We calculate

$$\eta^2 \leq \eta^2 + (\beta/\alpha)^2 r^{2(\beta/\alpha-1)} \left(a + \frac{(1+q)(1-a)}{3} \right)^2 \leq \eta^2 + (\beta/\alpha)^2.$$

Furthermore, after noticing (by using Lemma A.2) that

$$\frac{(\beta/\alpha)^2 \left(a + \frac{(1+q)(1-a)}{3} \right)^2 r^{2(\beta/\alpha-1)}}{\eta^2 + (\beta/\alpha)^2 r^{2(\beta/\alpha-1)} \left(a + \frac{(1+q)(1-a)}{3} \right)^2} \leq \frac{(\beta/\alpha)^2}{\eta^2 + (\beta/\alpha)^2} < 1$$

and setting $c = 1 - \frac{\beta/\alpha}{(\eta^2 + (\beta/\alpha)^2)^{1/2}}$, we get

$$\begin{aligned} h^* g_X \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial q} \right) &= r^{2\beta/\alpha-1} (\beta/\alpha) \left(\frac{1-a}{3} \right) \left(a + \frac{(1+q)(1-a)}{3} \right) \\ &= \frac{r^{\beta/\alpha-1} (\beta/\alpha) \left(a + \frac{(1+q)(1-a)}{3} \right)}{\left(\eta^2 + (\beta/\alpha)^2 r^{2(\beta/\alpha-1)} \left(a + \frac{(1+q)(1-a)}{3} \right) \right)^{1/2}} \left\| h_* \frac{\partial}{\partial r} \right\| \left\| h_* \frac{\partial}{\partial q} \right\| \\ &\leq (1-c) \left\| h_* \frac{\partial}{\partial r} \right\| \left\| h_* \frac{\partial}{\partial q} \right\|. \end{aligned}$$

Therefore, we've satisfied the conditions of Lemma 3.1, so $h^* g_{\hat{X}}$ on this piece is quasi-isometric to

$$dr^2 + \eta^2 r^2 d\theta^2 + r^{2\beta/\alpha} \left(\frac{1-a}{3} \right)^2 dq^2 + r^{2\beta/\alpha} \frac{\beta}{\alpha} \left(a + q \left(\frac{1-a}{3} \right) \right)^2 g_N.$$

Note that

$$(\beta/\alpha)a \leq \frac{\beta}{\alpha} \left(a + q \left(\frac{1-a}{3} \right) \right) \leq (\beta/\alpha) \frac{1+2a}{3},$$

so using Lemma A.3, we finally get that $h^* g_{\hat{X}}$ is quasi-isometric to

$$dr^2 + r^2 d\theta^2 + r^{2\beta/\alpha} dq^2 + r^{2\beta/\alpha} g_N.$$

Lastly, we will calculate the metric for $0 \leq q \leq 1$. The map h is given by

$$h(r, \theta, q, x) = (r\eta e^{i\theta}, (qr + (1-q)r^{\beta/\alpha}) A(q)x)$$

for a function $A(q) = \frac{(1+q)(1-a)}{3} + a$. Notice that $\frac{1+2a}{3} \leq A \leq \frac{2+a}{3}$ for all q in this region, and that $A'(q) = \frac{(1-a)}{3}$. Once again, we have that $\|x\| = 1$ and $\langle x, v \rangle = 0$ for $v \in T_x N$, as before. We calculate the pullback metric through h to be given by

$$\begin{aligned}
& \left(\eta^2 + \left(q + (1-q) \frac{\beta}{\alpha} r^{\beta/\alpha-1} \right)^2 A^2 \right) dr^2 + \left((r - r^{\beta/\alpha})A + (qr + (1-q)r^{\beta/\alpha}) \frac{(1-a)}{3} \right)^2 dq^2 \\
& + \eta^2 r^2 d\theta^2 + (qr + (1-q)r^{\beta/\alpha})^2 A^2 g_N \\
& + \left(q + (1-q) \frac{\beta}{\alpha} r^{\beta/\alpha-1} \right) A \left((r - r^{\beta/\alpha})A + (qr + (1-q)r^{\beta/\alpha}) \frac{(1-a)}{3} \right) (dr dq + dq dr).
\end{aligned}$$

We calculate

$$\eta^2 \left(\frac{1+2a}{3} \right)^2 \leq \eta^2 + \left(q + (1-q) \frac{\beta}{\alpha} r^{\beta/\alpha-1} \right)^2 A^2 \leq \eta^2 + (\beta/\alpha)^2.$$

Furthermore, we get by applying Lemma A.2

$$\begin{aligned}
h^* g_{\hat{X}} \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial q} \right) &= \left(q + (1-q) \frac{\beta}{\alpha} r^{\beta/\alpha-1} \right) A \left((r - r^{\beta/\alpha})A + (qr + (1-q)r^{\beta/\alpha}) \frac{(1-a)}{3} \right) \\
&= \left(\frac{(q + (1-q) \frac{\beta}{\alpha} r^{\beta/\alpha-1})^2 A^2}{\eta^2 + (q + (1-q) \frac{\beta}{\alpha} r^{\beta/\alpha-1})^2 A^2} \right)^{1/2} \left\| h_* \frac{\partial}{\partial r} \right\| \left\| h_* \frac{\partial}{\partial q} \right\| \\
&\leq \left(\frac{1}{\eta^2 + 1} \right)^{1/2} \left\| h_* \frac{\partial}{\partial r} \right\| \left\| h_* \frac{\partial}{\partial q} \right\|.
\end{aligned}$$

Thus, we can apply Lemma 3.1 to see that $h^* g_{\hat{X}}$ on this piece is quasi-isometric to

$$dr^2 + \eta^2 r^2 d\theta^2 + \left((r - r^{\beta/\alpha})A + (qr + (1-q)r^{\beta/\alpha}) \frac{(1-a)}{3} \right)^2 dq^2 + (qr + (1-q)r^{\beta/\alpha})^2 A^2 g_N.$$

Note (using Lemma A.2 for the second set of bounds) that

$$\min \left\{ \left(\frac{1}{2} \right)^{\beta/\alpha-1} \frac{1-a}{3}, \frac{A}{2} \right\} \leq \frac{(r - r^{\beta/\alpha})A + (qr + (1-q)r^{\beta/\alpha}) \frac{(1-a)}{3}}{r} \leq 1, \text{ and}$$

$$\frac{2a+1}{3} \leq A \leq \frac{a+2}{3}.$$

(We get the top lefthand bounds from first considering the case where $r > \frac{1}{2}$ for the left bound, then the case where $r < \frac{1}{2}$ for the right bound.) Therefore we can apply Lemma A.3 to show that $h^*g_{\hat{X}}$ on this piece is quasi-isometric to

$$dr^2 + r^2 d\theta^2 + r^2 dq^2 + (qr + (1 - q)r^{\beta/\alpha})^2 g_N. \quad \square$$

Lastly, we compute the metric for U_2 .

Theorem 6.18. *There is a Riemannian manifold with boundary $(N', g_{N'})$ where the interior of N' is a complex $(n - 1)$ -manifold so that T_*L_2 decomposes pointwise as $T_*S^1 \oplus T_*N'$, and the metric on U_2 with respect to this decomposition is quasi-isometric to*

$$dr^2 + r^2 d\theta^2 + r^{2\beta/\alpha} g_{N'}$$

where r and θ are the coordinates on $(0, 1]$ and S^1 , respectively.

Proof. We define $N' = \{z \in L_2 \mid z_0 = \eta\}$. Then we consider the bundle

$$\begin{array}{ccc} N' & \longrightarrow & L_2 \\ & & \downarrow \\ & & S^1 \end{array}$$

where the projection $L_2 \twoheadrightarrow S^1$ is given by $z \mapsto \frac{z_0}{|z_0|}$. This may not be a product, but we can see it really is a fiber bundle by defining a variant of the weighted flow

$$\begin{aligned} \mu : S^1 &\rightarrow L \\ \mu(\theta) &= (e^{i\alpha\theta} z_0, e^{i\beta\theta} z_{1 \rightarrow n}) \end{aligned}$$

and noticing that we can use this to flow N' into the other fibers.

Notice that that the tangent space of L_2 breaks apart pointwise in an orthogonal decomposition as

$$T_*L_2 = T_*S^1 \oplus T_*N'.$$

This decomposition is not natural, so by $T_*S^1 \subset T_*L$ we mean the pushforward of T_*S^1 through the map μ .

If g is a metric on N' and $d\theta$ is the pullback of the coordinate form on S^1 , then the metric on L_2 is given by

$$d\theta^2 + g_{N'}.$$

Now $(r, \phi)^{-1}$ maps the product $(0, 1] \times L_2$ onto U_2 via

$$\begin{aligned} (0, 1] \times L_2 &\rightarrow U_2 \\ (r, \xi) &\mapsto r \star \xi. \end{aligned}$$

We get the following computations of pushforwards for $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial \theta}$, and v for a given vector $v = \gamma'(0)$ which is the gradient of a curve γ which lies entirely in the fiber of the above bundle, i.e. which lies in $T_*N \subset T_*L$:

$$\begin{aligned} (r, \phi)_*^{-1} \frac{\partial}{\partial r} &= \left(\xi_0, \frac{\beta}{\alpha} r^{\beta/\alpha-1} \xi_{1 \rightarrow n} \right) \\ (r, \phi)_*^{-1} \frac{\partial}{\partial \theta} &= (i\alpha r \xi_0, i\beta r^{\beta/\alpha} \xi_{1 \rightarrow n}) \\ (r, \phi)_*^{-1} v &= (0, r^{\beta/\alpha} \gamma'_{1 \rightarrow n}(0)) = r \star v. \end{aligned}$$

By using these to pull back the metric to $(0, 1] \times L_2$, we get

$$\left(\eta^2 + \left(\frac{\beta}{\alpha} \right)^2 r^{2(\beta/\alpha-1)} |\xi_{1 \rightarrow n}|^2 \right) (dr^2 + \alpha^2 r^2 d\theta^2) + r^{2\beta/\alpha} g_{N'}.$$

There are no cross terms because if a vector v is tangent to N' at a point $\xi_{1 \rightarrow n}$, then $\langle v, \xi_{1 \rightarrow n} \rangle = 0$ since N' is a subset of the sphere.

Because the first coefficient satisfies

$$\eta^2 \leq \eta^2 + \left(\frac{\beta}{\alpha} \right)^2 r^{2(\beta/\alpha-1)} |\xi_{1 \rightarrow n}|^2 \leq \eta^2 + \left(\frac{\beta}{\alpha} \right)^2,$$

we can use Lemma A.3 to see that our metric is quasi-isometric to

$$dr^2 + r^2 d\theta^2 + r^{2\beta/\alpha} g_{N'}. \quad \square$$

Remark 6.19. One should be careful when comparing the metrics of $(0, 1] \times L_2$ and $(0, 1] \times L_{12}$ when $-1 < q < 0$. They seem similar, in that both take the form of

$$dr^2 + r^2 d\theta^2 + r^{2\beta/\alpha} g_M$$

for some manifold M . However, there are important differences! Most importantly and confusingly, the $d\theta$ is *not* the same in both cases. On $(0, 1] \times L_{12}$, the pushforward of $\frac{\partial}{\partial \theta}$ through h equals $\left. \frac{d}{dt}(|z_0|e^{it}, z_1 \rightarrow n) \right|_{t=0}$. On the other hand, on $(0, 1] \times L_2$, we have $h_* \frac{\partial}{\partial \theta} = \left. \frac{d}{dt}(e^{it} \star z) \right|_{t=0}$.

6.4 L^2 -Cohomology of V

In this last section, we calculate the L^2 -cohomology of a punctured neighborhood of the entire variety V .

Lemma 6.20. *Let Y , Y_1 , and Y_2 be Riemannian manifolds (perhaps with corners) so that $Y = Y_1 \cup Y_2$ and the metrics of Y_1 and Y_2 are the restrictions of the metric of Y . Let α and $\{\alpha_j\}_{j=1}^\infty$ be forms in $L^2\Omega^k(Y)$. Assume that the restrictions of the α_j converge to α on both Y_1 and Y_2 . Then the sequence $\{\alpha_j\}$ converges to α on the whole space Y .*

Proof. If the sequence converges on both Y_1 and Y_2 , then

$$\|\alpha|_{Y_i} - \alpha_j|_{Y_i}\|_{Y_i}^2 \rightarrow 0$$

for $i = 1, 2$. Then

$$\|\alpha - \alpha_j\|_Y^2 \leq \|\alpha|_{Y_1} - \alpha_j|_{Y_1}\|_{Y_1}^2 + \|\alpha|_{Y_2} - \alpha_j|_{Y_2}\|_{Y_2}^2 \rightarrow 0. \quad \square$$

Theorem 6.21. *Let K be defined as in Definition 3.8. Then K exists and is bounded.*

Proof. Theorems 5.2 and 5.8 prove this theorem when restricting K to metrics of either Cheeger type or Interpolations between metrics of Cheeger type (at least in the case of low eccentricity, which we can always arrange using Theorem 5.9). In Sections 6.2 and 6.3, we give a diffeomorphism $V = (0, 1] \times L$ and split V into pieces $U_1 = (0, 1] \times L_1$, $U_{12} = (0, 1] \times L_{12}$, $U_2 = (0, 1] \times L_2$, where the identities on U_1 and U_2 give quasi-isometries between the induced metric and Cheeger metrics, and the identity on U_{12} gives a quasi-isometry between the induced metric and an interpolation between Cheeger metrics.

Since $K_\varepsilon \alpha$ converges in L^2 on each of the three pieces, $K_\varepsilon \alpha$ converges in L^2 on the whole thing by Lemma 6.20.

Further, K is bounded, since there are constants C_{U_1} , $C_{U_{12}}$ and C_{U_2} so that

$$\begin{aligned} \|K\alpha\|^2 &= \|K\alpha\|_{U_1}^2 + \|K\alpha\|_{U_{12}}^2 + \|K\alpha\|_{U_2}^2 \\ &\leq C_{U_1} \|\alpha\|_{U_1}^2 + C_{U_{12}} \|\alpha\|_{U_{12}}^2 + C_{U_2} \|\alpha\|_{U_2}^2 \\ &\leq \max\{C_{U_1}, C_{U_{12}}, C_{U_2}\} \|\alpha\|^2. \end{aligned} \quad \square$$

Lemma 6.22. *1. Let $\omega \in \Omega^k(L)$ be a k -form for $k < \dim_{\mathbb{C}}(V)$. Then there are constants $A > 0$ and $b < 1$ so that*

$$r^b \|\omega\|_r^2 \leq A s^b \|\omega\|_s^2$$

for all $r, s \in (0, 1]$ satisfying $r \leq s$.

2. Let $\omega \in \text{dom}(d)$ be a k -form for $k > \dim_{\mathbb{C}}(V)$. Then there is a constant A so that

$$r \|\omega\|_r^2 \geq s \|\omega\|_s^2$$

for all $r, s \in (0, 1]$.

3. Let $\alpha \in \text{dom}(d)$ be a k -form for $k = \dim_{\mathbb{C}}(V)$. Then there are forms β and $\gamma \in \text{dom}(d)$ and some constant $A > 0$ so that $\beta = \alpha + d\gamma$ and

$$r\|\kappa_r\beta(t)\|_r^2 \geq sA\|\kappa_r\beta(t)\|_s^2$$

for all $r, s, t \in (0, 1]$ and $r \leq s$.

Proof. Because we can use Lemma 5.1 and Theorem 5.6 to prove 1 and 2 for the regions U_1 , U_{12} and U_2 , we can combine these estimates to get 1 and 2 for all of V .

For case 3, we first restrict α to U_{12} and use Theorem 5.20 to get a form γ so that the form $\beta = \alpha + d\gamma$ satisfies

$$r\|\kappa_r\beta(t)\|_{r,U_{12}}^2 \geq sA\|\kappa_r\beta(t)\|_{s,U_{12}}^2$$

for all $r, s, t \in (0, 1]$ and $r \leq s$. Because γ vanishes on a neighborhood of the boundary of U_{12} by Theorem 5.20, we can extend it to a form on all of V . Then we use Lemma 5.1 to get the estimate for β in the regions U_1 and U_2 . Combining these three estimates, we get the end result. \square

Theorem 6.23. *The L^2 -cohomology of V is given by*

$$H_{(2)}^k(V) = \begin{cases} 0 & k \geq \dim_{\mathbb{C}}(V) \\ H_{dR}^k(L) & k < \dim_{\mathbb{C}}(V). \end{cases}$$

Proof. First, assume $k < \dim_{\mathbb{C}}(V)$. By Theorem 6.21, Lemma 6.22, and Lemma 3.2 we satisfy the conditions of Theorem 3.11, so $H_{(2)}^k(V) = H_{dR}^k(L)$.

Next, assume $k > \dim_{\mathbb{C}}(V)$. Take a class $[\alpha] \in H_{(2)}^k(V)$ and choose a representative α . Then $\alpha \in \text{dom}(d)$. Again by Theorem 6.21, Lemma 6.22 and Lemma 3.2, we satisfy the conditions of Theorem 3.10. Thus $K\alpha$ is in $\text{dom}(\bar{d})$ and

$$\alpha = \bar{d}(K\alpha).$$

After using a smoothing homotopy operator if necessary, we see that $[\alpha] = 0$. Since $[\alpha]$ was arbitrary, $H_{(2)}^k(V) = 0$.

Last, assume $k = \dim_{\mathbb{C}}(V)$. Take a class $[\alpha] \in H_{(2)}^k(V)$ and choose a representative α . We can use Lemma 6.22 to find forms β and γ in $\text{dom}(d)$ and a constant $A > 0$ so that $\beta = \alpha + d\gamma$ and

$$r\|\kappa_r\beta(t)\|_r^2 \geq sA\|\kappa_r\beta(t)\|_s^2$$

for all $r, s, t \in (0, 1]$ and $r \leq s$. Then we can use Theorem 3.10 to say there is some $\xi \in L^2\Omega^{k-1}(L)$ so that $\xi + K\beta \in \text{dom}(\bar{d})$ and

$$\beta = \bar{d}(\xi + K\beta).$$

After using a smoothing homotopy operator if necessary, we see that $[\alpha] = [\beta] = 0$. Since $[\alpha]$ was arbitrary, $H_{(2)}^k(V) = 0$. □

Conclusion

Theorem 6.23 gives us that the local L^2 -cohomology of certain specific varieties is isomorphic to the intersection homology. These include, for example, the A_k simple singularities. A natural next line of inquiry is to ask if the same methods as shown here might work for other weighted homogeneous varieties (or to start, even other Brieskorn varieties).

We see three major issues with extending this work, although none seem insurmountable. The first potential difficulty is that the analog of the interpolations of Cheeger metrics as defined in Section 5.2 might involve more general fiber products, and not merely true products as we assumed. Indeed, for the other simple singularities (namely the D_k , E_6 , E_7 , and E_8 singularities), most of the work in this thesis will still apply, with this one exception. Some preliminary work has suggested that the arguments here may be able to be adapted to these cases as well.

Another, slightly larger issue is that for more complicated singularities, we may need to “interpolate” between three or more model metrics. For example, in the

work of [7], they are able to calculate the metric near each divisor of a suitable relation; however, near the intersection of these divisors, the metric will look like an interpolation between them. This requires a bit more care, and it is not obvious that the averaging arguments of Section 5.4 will still apply. However, again some preliminary calculations by the author suggest that this too might be overcome.

The largest hurdle to extending this work to general weighted homogeneous polynomials is that locally some of the metrics will not look like Cheeger metrics; instead, one might get relatively arbitrary multiply-warped products. As we saw in Chapter 4, these will not necessarily have the requisite properties: K may not be bounded in the correct degrees, there may be high degree forms independent of r which are L^2 , and other difficulties. This does not necessarily mean the CGM conjecture is false, however. For most such varieties, we expect there to be some topological reason why any L^2 closed form of a certain appearance must be L^2 -exact, and why any cohomology class of the link is representable by an L^2 form. Indeed, one major hope is that work in this area will shine some light on possible restrictions that the CGM conjecture would place on the local geometry and topology of complex algebraic varieties.

Appendix A

Extra Calculations

Calculation A.1. For $b < 1$, we have the equality

$$\int_0^1 \left(\int_r^1 \frac{As^b}{r^b} ds \right) dr = \frac{A}{2(1-b)}.$$

Proof. We break this into the $b \neq -1$ case and the $b = -1$ case.

Case 1: ($b \neq -1$)

$$\begin{aligned} \int_0^1 \left(\int_r^1 \frac{As^b}{r^b} ds \right) dr &= \int_0^1 \frac{A}{r^b} \left(\frac{1 - r^{b+1}}{b+1} \right) dr \\ &= \int_0^1 \frac{A}{b+1} (r^{-b} - r) dr \\ &= \frac{A}{b+1} \left(\left(\frac{1}{1-b} - \frac{1}{2} \right) - (0 - 0) \right) \\ &= \frac{A}{2(1-b)}. \end{aligned}$$

Case 2: ($b = -1$)

First, verify that $\frac{d}{dr} \left(\frac{1}{2}r^2 \ln r - \frac{1}{4}r^2 \right) = r \ln r$. Then

$$\begin{aligned}
\int_0^1 \left(\int_r^1 \frac{As^b}{r^b} ds \right) dr &= \int_0^1 \left(\int_r^1 \frac{As^{-1}}{r^{-1}} ds \right) dr \\
&= \int_0^1 Ar (\ln 1 - \ln r) dr \\
&= \int_0^1 -Ar \ln r dr \\
&= -A \left[\left(\frac{1}{2} \ln 1 - \frac{1}{4} \right) - (0 - 0) \right] \\
&= \frac{A}{4} \\
&= \frac{A}{2(1-b)}. \quad \square
\end{aligned}$$

Lemma A.2. *Let $0 \leq a_1, b_1$ be nonnegative real numbers and $0 < a_2, b_2$ be positive real numbers. Then*

$$\begin{aligned}
\max_{0 \leq q \leq 1} \frac{qa_1 + (1-q)b_1}{qa_2 + (1-q)b_2} &= \max \left\{ \frac{a_1}{a_2}, \frac{b_1}{b_2} \right\}, \text{ and} \\
\min_{0 \leq q \leq 1} \frac{qa_1 + (1-q)b_1}{qa_2 + (1-q)b_2} &= \min \left\{ \frac{a_1}{a_2}, \frac{b_1}{b_2} \right\}.
\end{aligned}$$

If $0 < c$, then we also have the inequality

$$\frac{qa_1}{qa_2 + c} \leq \frac{a_1}{a_2 + c}.$$

Proof. Denote $f(q) = \frac{qa_1 + (1-q)b_1}{qa_2 + (1-q)b_2}$. f is a smooth function, so its extrema can only occur at values where $f'(q) = 0$ or at the endpoints. We can calculate

$$f'(q) = \frac{a_1b_2 - a_2b_1}{(qa_2 + (1-q)b_2)^2}.$$

Thus, $f(q)$ is either strictly monotonic in q or constant. Either way, its extrema are at the endpoints, which gives us the first part of our lemma.

To get the last inequality, we can rewrite

$$\frac{qa_1}{qa_2 + c} = \frac{qa_1 + (1 - q) \cdot 0}{q(a_2 + c) + (1 - q)c},$$

and apply the first part of this lemma. \square

Lemma A.3. *Let (X_1, g_1) and (X_2, g_2) be Riemannian manifolds with boundary. Let $X = X_1 \times X_2$, and let $f, h : X \rightarrow \mathbb{R}$ be smooth functions. Construct the metrics g_f and g_h on X via*

$$g_f = g_1 + f g_2, \text{ and}$$

$$g_h = g_1 + h g_2.$$

Assume lastly that there are constants $0 < c \leq 1 \leq C < \infty$ so that $c \leq \frac{f(x)}{h(x)} \leq C$ for every $x \in X$. Then the identity map is a quasi-isometry between (X, g_f) and (X, g_h) .

Proof. Given a point $x = (x_1, x_2)$, we can decompose any tangent vector $v \in T_x X$ as $v = v_1 + v_2$, where $v_1 \in T_{x_1} X_1$ and $v_2 \in T_{x_2} X_2$. Then $g_f(v, v) = g_1(v_1, v_1) + f(x)g_2(v_2, v_2)$, and $g_h(v, v) = g_1(v_1, v_1) + h(x)g_2(v_2, v_2)$. Clearly

$$\begin{aligned} g_f(v, v) &= g_1(v_1, v_1) + f(x)g_2(v_2, v_2) \\ &\leq g_1(v_1, v_1) + C h(x)g_2(v_2, v_2) \\ &\leq C (g_1(v_1, v_1) + h(x)g_2(v_2, v_2)) \\ &= C g_h(v, v). \end{aligned}$$

Similarly,

$$\begin{aligned} g_f(v, v) &= g_1(v_1, v_1) + f(x)g_2(v_2, v_2) \\ &\geq g_1(v_1, v_1) + c h(x)g_2(v_2, v_2) \\ &\geq c (g_1(v_1, v_1) + h(x)g_2(v_2, v_2)) \\ &= c g_h(v, v). \end{aligned}$$

\square

Lemma A.4. *Let $x, z \in \mathbb{R}^n$, and let $x, z \neq 0$. Then the scalar $t \in (-\infty, \infty)$ which minimizes the value $\|tx - z\|$ is $t = \frac{\|z\|}{\|x\|} \cos \theta$, where θ is the angle between x and z . Furthermore, if $\|x\| > \|z\|$, then*

$$\left\| \frac{\|z\|}{\|x\|} x - z \right\| \leq \|x - z\|.$$

Proof. We prove this using the law of cosines. We can calculate

$$\begin{aligned} f(t) &= \|tx - z\|^2 \\ &= \|tx\|^2 + \|z\|^2 - 2\|tx\| \|z\| \cos \theta \\ &= t^2 \|x\|^2 - t(2\|x\| \|z\| \cos \theta) + \|z\|^2 \end{aligned}$$

As this is a smooth function on a bounded set, we find the minimum (if it exists) by setting the derivative equal to zero. We get

$$f'(t) = 2t\|x\|^2 - 2\|x\| \|z\| \cos \theta,$$

and so f has a critical value at $t = \frac{\|z\|}{\|x\|} \cos \theta$, is increasing in t for larger t , and is decreasing in t for smaller t . If $\|x\| \geq \|z\|$, then for any θ , $\frac{\|z\|}{\|x\|} \cos \theta \leq \frac{\|z\|}{\|x\|} \leq 1$, which gives us our second statement. \square

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Biography

Joshua David Cruz graduated from Washington State University with an undergraduate degree in Mathematics. His undergraduate thesis was titled “Creating Barcodes for Shapes.” This work was done under the advisement of Dr. Bala Krishnamoorthy and Dr. Kevin Vixie.

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