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On the Betti Numbers of Finite Volume Hyperbolic Manifolds

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Abstract

We obtain strong upper bounds for the Betti numbers of compact complex-hyperbolic manifolds. We use the unitary holonomy to improve the results given by the most direct application of the techniques of [DS17]. We also provide effective upper bounds for Betti numbers of compact quaternionic- and Cayley-hyperbolic manifolds in most degrees. More importantly, we extend our techniques to complete finite volume real- and complex-hyperbolic manifolds. In this setting, we develop new monotonicity inequalities for strongly harmonic forms on hyperbolic cusps and employ a new peaking argument to estimate L^2-cohomology ranks. Finally, we provide bounds on the de Rham cohomology of such spaces, using a combination of our bounds on L^2-cohomology, bounds on the number of cusps in terms of the volume, and the topological interpretation of reduced L^2-cohomology on certain rank one locally symmetric spaces.

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Contents

1 Introduction

In theory, the Selberg trace formula can be used to compute Betti numbers of compact locally symmetric spaces. The expressions produced by the trace formula, however, can be extremely complicated and difficult to understand. DeGeorge and Wallach [DW78] pioneered the use of the trace formula to obtain simpler coarser bounds on Betti numbers and the multiplicity of other representations arising in $L^2(\Gamma\backslash G)$. In particular, the DeGeorge-Wallach bounds did not require extensive evaluations of orbital integrals. This saving in complexity was achieved (with a concomitant loss of precision) by using test functions in the trace formula that were supported in the lift to G of a ball about the origin in G/K that can be identified with an *embedded* ball in $\Gamma\backslash G/K$. Their estimates then reduce to estimates on the growth of matrix coefficients associated to the representation under investigation. Using related techniques, Xue in [Xue91] gave effective estimates on the growth of the first Betti numbers for compact complex-hyperbolic surfaces. Sarnak and Xue [SX91] subsequently pushed beyond the embedded ball barrier for surfaces, allowing test functions supported on the lift to G of balls with radius larger than the injectivity radius of $\Gamma\backslash G/K$. They then use estimates on the test function and the size of the Γ orbits intersecting these balls to bound rather than evaluate terms arising in the Selberg trace formula that are associated to non-identity conjugacy classes in Γ.

In [DS17], we introduced a method for bounding Betti numbers on manifolds without conjugate points and with a negative Ricci curvature upper bound. Like the DeGeorge-Wallach technique, our method gives bounds in terms of the volume of the largest embedded geodesic ball in the manifold. The estimates introduced in [DS17], which we called "Price inequalities", play a similar role to that of the estimates on the growth of matrix coefficients in [DW78], but of course, ours require no locally symmetric hypothesis and easily provide effective bounds on Betti numbers.

Given the recent strong interest in obtaining bounds for cohomology in towers of spaces, both locally symmetric and otherwise, in this paper, we build upon [DS17] to study the Betti numbers of complete finite volume real-hyperbolic and complex-hyperbolic manifolds. We focus first on compact complex hyperbolic manifolds, where we use Kahlerity to obtain optimal bounds within the embedded ball barrier. (We treated compact real-hyperbolic manifolds in [DS17].) We subsequently sharpen our peaking estimates to extend our techniques to complete finite volume real- and complex-hyperbolic manifolds. Along the way, we also provide effective upper bounds for most Betti numbers of compact quaternionic- and Cayley-hyperbolic manifolds.

Before we summarize our results, we introduce a definition.

Definition 1. Let (X^n, g) be a Riemannian manifold. Let $S \subset X$. Define

$$V_{min}(S) = \inf_{x \in S} Vol(B_{\mathrm{inj_x}}(x)),$$

where $\mathrm{inj_x}$ is the injectivity radius at x, and $B_R(x)$ is the geodesic ball of radius R and center x. Set $\mathrm{inj_S} := \inf_{\mathrm{x \in S}} \mathrm{inj_x}$.

Throughout the paper, we let $b^k(M) := \dim_{\mathbb{R}} H^k(M;\mathbb{R})$. We can now state our first result.

Theorem 2. *Let $(M^n = \Gamma\backslash \boldsymbol{H}^n_{\mathbb{C}}, g_{\mathbb{C}})$ be a compact complex-hyperbolic manifold, with $-4 \leq sec_{g_{\mathbb{C}}} \leq -1$. For $k < n$, there exists a positive constant $c(n,k)$ (as in Corollary 41) such that*

$$\frac{b^k(M)}{Vol(M)} \leq c(n,k) V_{min}(M)^{\frac{k-n}{n}}.$$

In [SX91], Sarnak-Xue show that for certain congruence subgroup quotients, we can bound $V_{min}(M)$ below by a power of $Vol(M)$. This gives the following corollary.

Corollary 3. *Let $(M^n = \Gamma\backslash \boldsymbol{H}^n_{\mathbb{C}}, g_{\mathbb{C}})$ be a compact complex-hyperbolic manifold, with $-4 \leq sec_{g_{\mathbb{C}}} \leq -1$, where $\Gamma \leq PU(n,1)$ is a principal congruence subgroup. There exists $c(n,k) > 0$ (independent of the level) such that*

$$\frac{b^k(M)}{Vol(M)} \leq c(n,k) Vol(M)^{\frac{2k-2n}{n^2+2n}}.$$

This result improves and generalizes from $H^{k,0}$ to H^k the results in [Yeu94, Theorem 2.3.1], which seem to be the best published effective bounds for rank $H^{k,0}$ on compact complex-hyperbolic manifolds of dimension bigger than or equal to three. We are unaware of any published results using those techniques in higher dimensions.

For surfaces, our bound on b^1 is sharper than the one derived by Xue in [Xue91], but weaker than the bound given by Sarnak-Xue [SX91] and more recently by Marshall [Mar14]. The bound found by Marshall is a striking $\alpha = \frac{3}{8}$, which is even sharp. These bounds both go beyond the embedded ball barrier in their use of the trace formula, and rely on low dimensional arithmetic information.

In the compact case, our techniques can function as a replacement (or simplification) of estimates of matrix coefficient asymptotic used in [DW78] and[DW79]. In order to compare the two techniques, in the appendix we provide a simple method for estimating the relevant matrix coefficients required to estimate the first Betti number of compact complex-hyperbolic manifolds and show that it reproduces our result. The estimates become somewhat more complex for $k > 1$, and are not included in the appendix. We also include the simpler estimation of the matrix coefficients in the real hyperbolic case, and verify that they reproduce the Betti number estimates we previously derived in [DS17]. With two different techniques measuring two very different quantities, it remains a challenge to understand how to combine them to get stronger information than each provides separately. In the complete finite volume case, the trace formula becomes more complicated, and we are unaware of an extension of the DeGeorge-Wallach argument to this case. The geometric approach, on the other hand, extends to manifolds with rank one cusps to bound the ranks of their reduced L^2-cohomology.

Definition 4. Let (M, g) be a complete Riemannian manifold. Define the vector space of L^2 harmonic k-forms

$$\mathcal{H}^k(M) := \left\{ \omega \in \Lambda^k TM \mid d\omega = 0 = d^*\omega, \text{ and } \int_M \omega \wedge *\omega < \infty \right\},$$

where $*$ is the Hodge star operator. Set

$$b_2^k(M) := \dim_{\mathbb{R}} \mathcal{H}^k(M).$$

If M has finite dimensional L^2-cohomology, then $\mathcal{H}^k(M)$ is isomorphic to the L^2-cohomology in degree k, and then $b_2^k(M)$ is the rank of the k-th L^2-cohomology group of M. Complete finite volume locally symmetric spaces with Rank G = Rank K have finite dimensional L^2-cohomology, see for example [BC83]. Without imposing the equal rank assumption, complete finite volume locally symmetric spaces always have $b_2^k < \infty$, see [BG83]. In this case, $b_2^k(M)$ computes the rank of the *reduced* L^2-cohomology. Thus, it is interesting to extend Theorem 2 to complete finite volume real- and complex-hyperbolic manifolds. We obtain the following results.

Theorem 5. *Let $(M = \Gamma\backslash \boldsymbol{H}^n_{\mathbb{C}}, g_{\mathbb{C}})$ be a complete finite volume complex-hyperbolic manifold, with $-4 \leq sec_{g_{\mathbb{C}}} \leq -1$. Write M as a disjoint union $M = M_0 \cup (\cup_a C_a)$, where C_a are cusps. There exists a constant $a(k,n) > 0$ depending only on dimension and k such that for $k < n$,*

$$b_2^k(M) \leq a(k,n) Vol(M) V_{min}(M_0)^{\frac{k-n}{n}}.$$

In the real-hyperbolic case we obtain the following generalization of Theorem 18 (*cf.* Corollary 116 in [DS17]).

Theorem 6. *Let $(M^n = \Gamma\backslash \boldsymbol{H}^n_{\mathbb{R}}, g_{\mathbb{R}})$ be a complete finite volume real-hyperbolic manifold, with $sec_{g_{\mathbb{R}}} = -1$. Write M as a disjoint union $M = M_0 \cup (\cup_a C_a)$, where C_a are cusps. There exists a constant $\alpha(k,n) > 0$ depending only on dimension and k such that for $k < \frac{n-1}{2}$*

$$b_2^k(M) \leq \alpha(k,n) Vol(M) V_{min}(M_0)^{\frac{2k+1-n}{n-1}}.$$

Finally if $n = 2k+1$, there exists a constant $\alpha(k) > 0$ depending only on k such that

$$b_2^k(M) \leq \alpha(k) \frac{Vol(M)}{\ln(V_{min}(M_0))}.$$

Interestingly, the proof of Theorem 6 requires estimates beyond the embedded ball barrier, but only on cusps, where the counting of lattice points reduces to the study of lattices in euclidean spaces. We also apply Theorem 5 and Theorem 6 to towers of coverings associated to a cofinal filtration of the fundamental group of the base hyperbolic manifolds. We refer to Section 10 for details on the asymptotic behavior of L^2-cohomology along such towers. Specializing to principal congruence subgroups, Theorems 5 and 6 reduce to

Theorem 7. *For $k < \frac{n-1}{2}$, there exists a constant $a(n,k) > 0$ such that for Γ a torsion free principal congruence subgroup of $SO(n,1)$ with its standard $\mathbb{Q}-$rank one rational structure,*

$$b_2^k(\Gamma\backslash \boldsymbol{H}^n_{\mathbb{R}}) \leq a(n,k) Vol(\Gamma\backslash \boldsymbol{H}^n_{\mathbb{R}})^{1-\frac{4(n-1-2k)}{n(n+1)}}. \tag{8}$$

There exists a constant $b(n,k) > 0$ such that for Γ a torsion free principal congruence subgroup of $SU(n,1)$ with its standard $\mathbb{Q}-$rank one rational structure,

$$b_2^k(\Gamma\backslash \boldsymbol{H}^n_{\mathbb{C}}) \leq b(n,k) Vol(\Gamma\backslash \boldsymbol{H}^n_{\mathbb{C}})^{1-\frac{4(n-k)}{(n+1)^2-1}}. \tag{9}$$

For principal congruence subgroup quotients of the the complex 2 ball, we thus have b_2^1 growing at most like the square root of the volume.

Finally, we study the de Rham cohomology of complete finite volume hyperbolic manifolds with cusps along a cofinal tower. This relies on the topological interpretation of L^2-cohomology of locally symmetric varieties (*cf.* [Zuc82]), and on an estimate of the number of cusps along such towers. The problem of estimating the number of cusps in terms of the volume on hyperbolic manifolds with cusps is a well-studied problem in geometric topology; see for example [Kel98], [Par98], [DD15], [BT18] among many other references. Nonetheless, here we need a different point of view on this problem: we consider the asymptotic behavior of the volume normalized number of cusps along a cofinal tower, and this point of view seems to be new. We refer to Section 11 for the background and details.

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2 The Geometry of Geodesic Balls in Rank One Symmetric Spaces

For for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$, let $(\mathbf{H}^n_{\mathbb{K}}, \mathbf{g}_{\mathbb{K}})$ denote the corresponding $\mathbb{K}$- hyperbolic spaces. These spaces have real dimensions respectively n, $2n$, $4n$ and 16. They are symmetric spaces given by the following quotients:

$$\begin{aligned} \mathbf{H}^n_{\mathbb{R}} &= SO(n,1)/SO(n), & \mathbf{H}^n_{\mathbb{C}} &= SU(n,1)/U(n), \\ \mathbf{H}^n_{\mathbb{H}} &= Sp(n,1)/Sp(n)Sp(1), & \mathbf{H}^2_{\mathbb{O}} &= F_{4(-20)}/Spin(9). \end{aligned} \tag{10}$$

Given a point $p \in \mathbf{H}^n_{\mathbb{K}}$, denote by $B_r(p)$ and $S_r(p)$ the geodesic ball and sphere of radius r around p, respectively. In geodesic polar coordinates around the point p,

$$\mathbf{g}_{\mathbb{K}} = dr^2 + g_r, \tag{11}$$

where in the real-hyperbolic case, $g_{\mathbb{R}} = dr^2 + \sinh^2(r) d\sigma^2$, with $d\sigma^2$ the usual round metric on the sphere. In the remaining cases, the metric is best described in terms of a generalized Hopf fibration:

$$S^1 \to S^{2n-1} \to \mathbf{P}^{n-1}_{\mathbb{C}}, \quad S^3 \to S^{4n-1} \to \mathbf{P}^{n-1}_{\mathbb{H}}, \text{ and } S^7 \to S^{15} \to \mathbf{P}^1_{\mathbb{O}},$$

where $\mathbf{P}^{n-1}_{\mathbb{C}}$, $\mathbf{P}^{n-1}_{\mathbb{H}}$ are respectively complex and quaternionic projective spaces; in the octonionic case, we set $\mathbf{P}^1_{\mathbb{O}} := S^8$. For convenience, we summarize these fibrations as follows

$$S^{\dim \mathbb{K} - 1} \to S^{m_{\mathbb{K},n}} \to \mathbf{P}^n_{\mathbb{K}},$$

where $\dim \mathbb{C} = 2$, $\dim \mathbb{H} = 4$, $\dim \mathbb{O} = 8$, and where

$$m_{\mathbb{C},n} = 2n-1, \quad m_{\mathbb{H},n} = 4n-1, \quad m_{\mathbb{O},1} = 15;$$

in the octonionic case, only $n = 1$ occurs. We normalize the max of the sectional curvature of $\mathbf{g}_{\mathbb{K}}$ to be -1. The sectional curvature of $\mathbf{g}_{\mathbb{K}}$ is then negative and quarter-pinched:

$$-4 \leq \sec_{\mathbf{g}_{\mathbb{K}}} \leq -1.$$

We fix this normalization for the rest of this paper.

Decompose the metric on the relevant round unit spheres as $g_{round} = g_v + g_h$, where g_v is the metric restricted to vectors tangent to the generalized Hopf fiber, and g_h is the restriction of the metric to the vectors orthogonal to the fiber. With this notation, we have

$$\mathbf{g}_{\mathbb{K}} = dr^2 + \frac{1}{4}\sinh^2(2r)g_v + \sinh^2(r)g_h. \tag{12}$$

With this metric, the Hopf fibers are totally geodesic and equipped with the standard round metric up to scale; the metric on the horizontal vectors is the pullback from the the base of the standard (rescaled) symmetric metric on $\mathbf{P}^n_{\mathbb{K}}$. Given this explicit description of the metric, we now compute the second fundamental form of the geodesic spheres.

Proposition 13. *The second fundamental form $\boldsymbol{h}_{\mathbb{K}}(r)$ of a geodesic sphere S_r in $(\boldsymbol{H}^n_{\mathbb{K}}, \boldsymbol{g}_{\mathbb{K}})$ has the expression:*

$$\boldsymbol{h}_{\mathbb{K}}(r) = 2\coth(2r)\frac{\sinh^2(2r)}{4}g_v + \coth(r)\sinh^2(r)g_h.$$

Proof. Recall that the Hessian of the smooth distance function centered at p is proportional to the second fundamental form of a geodesic sphere:

$$\mathbf{h}_{\mathbb{K}}(r) = \frac{1}{2}\mathrm{Hess}(r) = \frac{1}{2}L_{\partial_r}(\mathbf{g}_{\mathbb{K}}),$$

where L_{∂_r} is the Lie derivative with respect to the unit length radial vector field. Now, we compute

$$L_{\partial_r}(\mathbf{g}_{\mathbb{K}}) = L_{\partial_r}(g_r),$$

so that the proposition follows from (12). □

Corollary 14. *For $\mathbb{K} \neq \mathbb{R}$, the second fundamental form $\boldsymbol{h}_{\mathbb{K}}(r)$ of a geodesic sphere S_r in $(\boldsymbol{H}^n_{\mathbb{K}}, \boldsymbol{g}_{\mathbb{K}})$ has two distinct eigenvalues*

$$\lambda_1(r) = 2\coth(2r), \quad \lambda_2(r) = \coth(r),$$

with multiplicities $m(\lambda_1, \mathbb{K}) = \dim(\mathbb{K}) - 1$, $m(\lambda_2, \mathbb{K}) = 1 - \dim\mathbb{K} + m_{\mathbb{K},n}$. Finally, the mean curvature $\mathcal{H}_{\mathbb{K},n}(r)$ of a geodesic sphere S_r is

$$\mathcal{H}_{\mathbb{K},n}(r) = 2(\dim(\mathbb{K}) - 1)\coth(2r) + (1 - \dim\mathbb{K} + m_{\mathbb{K},n})\coth(r). \tag{15}$$

In the following lemma, we restate Corollary 14 in a form convenient for subsequent applications to the Price inequality given in Section 3.

Lemma 16. *Let*

$$\lambda_1(r) = ... = \lambda_{\dim(\mathbb{K})-1}(r) > \lambda_{\dim(\mathbb{K})}(r) = ... = \lambda_{m_{\mathbb{K},n}}(r) \tag{17}$$

denote the ordered eigenvalues of $\boldsymbol{h}_{\mathbb{K}}(r)$. *For any integer* $1 \leq k < \dim_{\mathbb{R}} \boldsymbol{H}^n_{\mathbb{K}}$, *we have*

$$\Big(\frac{\mathcal{H}_{\mathbb{K},n}(r)}{2} - \sum_{i=1}^{k} \lambda_i(r)\Big) = \begin{cases} \left(\frac{m_{\mathbb{K},n}}{2} - k\right)\coth(r) + \left(\frac{\dim(\mathbb{K})-1}{2} - k\right)\tanh(r), \\ \text{if } k \leq \dim(\mathbb{K}) - 1; \\ \\ \left(\frac{m_{\mathbb{K},n}}{2} - k\right)\coth(r) - \frac{(\dim(\mathbb{K})-1)}{2}\tanh(r), \\ \text{if } k > \dim(\mathbb{K}) - 1. \end{cases}$$

Proof. It suffices to combine Equations (15) and (17). □

3 On the Betti Numbers of Compact Quotients of $\mathbf{H}^n_{\mathbb{K}}$

In this section, we study Betti numbers of compact locally symmetric rank one spaces via a Price inequality for harmonic forms. We focus on rank one locally symmetric spaces that are *not* real hyperbolic. We recall the following result for compact real-hyperbolic spaces, which we previously treated in [DS17, Corollary 116] (see also [Yeu94, Theorem 2.4.1] for estimates outside the critical degree).

Theorem 18. *Let* $(M^n := \Gamma\backslash \boldsymbol{H}^n_{\mathbb{R}})$ *be a closed real-hyperbolic manifold with* $sec_{g_{\mathbb{R}}} = -1$ *and injectivity radius* $\mathrm{inj_M} \geq 1$. *Given a cofinal filtration* $\{\Gamma_i\}$ *of* Γ, *denote by* $\pi_i : M_i \to M$ *the regular Riemannian cover of* M *associated to* Γ_i. *For any integer* $k < \frac{n-1}{2}$, *there exists a positive constant* $c_1(n,k)$ *such*

$$\frac{b^k(M_i)}{Vol(M_i)} \leq c_1(n,k) V_{min}(M_i)^{-\frac{n-1-2k}{n-1}}. \tag{19}$$

In particular, the sub volume growth of the Betti numbers along the tower of coverings is exponential in the injectivity radius. For the critical degree $k = \frac{n-1}{2}$, *there exists a positive constant* $c_2(n)$ *such that*

$$\frac{b^k(M_i)}{Vol(M_i)} \leq \frac{c_2(n)}{\mathrm{inj_{M_i}}}. \tag{20}$$

Remarks 21. Along a cofinal tower one can show that

$$\mathrm{inj_{M_i}} \to \infty, \quad \text{as} \quad \mathrm{i} \to \infty,$$

see for example [DW78, Theorem 2.1], so that Theorem 18 immediately implies that the normalized Betti numbers go to zero along the tower.

For the rest of this section, we focus on $(\Gamma\backslash\mathbf{H}^n_{\mathbb{K}}, g_{\mathbb{K}})$ with $\mathbb{K} = \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$. For the remainder of the paper Γ will always denote a torsion free discrete subgroup of the isometries of $\mathbf{H}^n_{\mathbb{K}}$. Let inj_Γ denote the injectivity radius of $(\Gamma\backslash\mathbf{H}^n_{\mathbb{K}}, g_{\mathbb{K}})$.

Given a 1-form ϕ, let $e(\phi)$ denote exterior multiplication on the left by ϕ. Let $e^*(\phi)$ denote the adjoint operator. Fix a point $p \in M$, and consider a geodesic ball $B_R(p)$, with $0 < R \leq \mathrm{inj}_\Gamma$. Given a local orthonormal frame $\{\frac{\partial}{\partial r}\} \cup \{V_j\}_j$ and coframe $\{dr\} \cup \{\omega^j\}_j$, recall that acting on forms of arbitrary degree, the Lie derivative in the radial direction in $B_R(p)$ can be written as

$$L_{\partial_r} = \{d, e^*(dr)\} = \nabla_{\partial_r} + e(\omega^j)e^*(\nabla_{V_j}dr). \tag{22}$$

The Hopf fibers are framed; so, we will henceforth choose our orthonormal frame so that for $1 \leq j \leq \dim(\mathbb{K}) - 1$, V_j is tangent to the fiber and globally defined in $B_R(p)$. We call such a frame and coframe *adapted*. Next, for any harmonic k-form h on $\Gamma\backslash\mathbf{H}^n_{\mathbb{K}}$, we define $\dim(\mathbb{K})$ auxiliary nonnegative functions that naturally arise in our Price equality. Set

$$\mu_h(r) := \frac{\int_{S_r} |e^*(dr)h|^2 d\sigma}{\int_{S_r} |h|^2 d\sigma}, \tag{23}$$

and

$$\zeta^j_h(r) := \frac{\int_{S_r} |e^*(\omega_j)h|^2 d\sigma}{\int_{S_r} |h|^2 d\sigma}, \tag{24}$$

for $j \in 1, ..., \dim(\mathbb{K}) - 1$, where $d\sigma$ is the Riemannian measure induced on the geodesic sphere S_r by $g_{\mathbb{K}}$. These functions are by definition non negative, bounded from above by one, and well defined for any $0 < r \leq \mathrm{inj}_\Gamma$. (In fact, we only need $\sum_j \zeta^j_h$ in applications; so, it is not really necessary that the tangent space to the fiber be framed.)

With this notation, we can now state and prove our first lemma.

Lemma 25. *Given $h \in \mathcal{H}^k(\Gamma\backslash\boldsymbol{H}^n_{\mathbb{K}})$, $p \in \Gamma\backslash\boldsymbol{H}^n_{\mathbb{K}}$, and $R \in (0, \mathrm{inj}_\Gamma)$, we have*

$$\int_{S_R(p)} \Big(\frac{1}{2} - \mu_h(R)\Big)|h|^2 d\sigma = \int_{B_R(p)} q_h(r)|h|^2 dv$$

with

$$q_h(r) = \frac{\mathcal{H}_{\mathbb{K},n}(r)}{2} - k\coth(r) - \tanh(r) \sum_{j=1}^{\dim(\mathbb{K})-1} \zeta^j_h(r) + \mu_h(r)\coth(r),$$

with $\mathcal{H}_{\mathbb{K},n}(r)$ as in Equation (15).

Proof. In an orthonormal adapted coframe $dr, \omega^1, ..., \omega^{m_{\mathbb{K},n}}$, we have by Proposition 13

$$\begin{aligned}
L_{\partial_r} &= \nabla_r + e(\omega^j)e^*(\nabla_{V_j} dr) \\
&= \nabla_r + (\coth(r) + \tanh(r)) \sum_{j=1}^{\dim(\mathbb{K})-1} e(\omega^j)e^*(\omega^j) + \coth(r) \sum_{j=\dim(\mathbb{K})}^{m_{\mathbb{K},n}} e(\omega^j)e^*(\omega^j) \\
&= \nabla_r + \coth(r)\big\{e(dr)e^*(dr) + \sum_{j=1}^{m_{\mathbb{K},n}} e(\omega^j)e^*(\omega^j)\big\} \\
&- \coth(r)e(dr)e^*(dr) + \tanh(r) \sum_{j=1}^{\dim(\mathbb{K})-1} e(\omega^j)e^*(\omega^j).
\end{aligned}$$

We therefore conclude that

$$\begin{aligned}
\int_{B_R} \langle L_{\partial_r} h, h\rangle dv = \int_{S_R(p)} |e^*(dr)h|^2 d\sigma \\
&= \int_{B_R(p)} \langle \nabla_{\partial_r} h, h\rangle dv + k \int_{B_R} \coth(r)|h|^2 dv \\
&- \int_{B_R} \mu_h(r)\coth(r)|h|^2 dv + \sum_{j=1}^{\dim(\mathbb{K})-1} \int_{B_R} \zeta_h^j(r)\tanh(r)|h|^2 dv, \\
&= -\int_{B_R} \frac{\mathcal{H}_{\mathbb{K},n}(r)}{2}|h|^2 dv + \frac{1}{2}\int_{S_R} |h|^2 d\sigma + k\int_{B_R} \coth(r)|h|^2 dv \\
&- \int_{B_R} \mu_h(r)\coth(r)|h|^2 dv + \sum_{j=1}^{\dim(\mathbb{K})-1} \int_{B_R} \zeta_h^j(r)\tanh(r)|h|^2 dv.
\end{aligned}$$

We rearrange these terms to obtain

$$\begin{aligned}
&\int_{S_R} \Big(\frac{1}{2}|h|^2 - |e^*(dr)h|^2\Big) d\sigma \\
&= \int_{B_R} \frac{\mathcal{H}_{\mathbb{K},n}(r)}{2}|h|^2 dv - k\int_{B_R} \coth(r)|h|^2 dv \\
&+ \int_{B_R} \mu_h(r)\coth(r)|h|^2 dv - \sum_{j=1}^{\dim(\mathbb{K})-1} \int_{B_R} \zeta_h^j(r)\tanh(r)|h|^2 dv,
\end{aligned}$$

and the lemma follows from Equations (52) and (24). □

We now control the positivity of the geometric term $q_h(r)$ appearing in Lemma 25.

Lemma 26. *Given* $h \in \mathcal{H}^k(\Gamma\backslash \boldsymbol{H}^n_{\mathbb{K}})$, $p \in \Gamma\backslash \boldsymbol{H}^n_{\mathbb{K}}$, *and* $r \in (0, \mathrm{inj}_\Gamma)$, *we have*

$$\begin{cases} \boxed{q_h(r) > \dfrac{\dim(\boldsymbol{H}^n_{\mathbb{K}}) - \dim(\mathbb{K})}{2} - k \geq 1,} \\ \\ \textit{if } \dim(\mathbb{K}) - 1 < k < \frac{\dim(\boldsymbol{H}^n_{\mathbb{K}}) - \dim(\mathbb{K})}{2}; \\ \\ \boxed{q_h(r) > \dfrac{\dim(\boldsymbol{H}^n_{\mathbb{K}}) + \dim(\mathbb{K}) - 2}{2} - 2k \geq 1,} \\ \\ \textit{if } k \leq \dim(\mathbb{K}) - 1, \\ \\ \textit{unless} \\ \\ \mathbb{K} = \mathbb{C}, \quad n = 2, \quad k = 1, \\ \mathbb{K} = \mathbb{H}, \quad n = 2, \quad k = 3, \\ \mathbb{K} = \mathbb{O}, \quad n = 1, \quad k = 6, 7. \end{cases}$$

In particular, in all of these cases we have $\mu_h(r) < 1/2$ *for any* $0 < r < \mathrm{inj}_\Gamma$.

Proof. First observe that for any $0 < r < \mathrm{inj}_\Gamma$, $q_h(r)$ satisfies

$$q_h(r) \geq \Big(\frac{\mathcal{H}_{\mathbb{K},n}(r)}{2} - \sum_{i=1}^{k} \lambda_i(r)\Big),$$

where the $\{\lambda_i\}_i$ are the ordered eigenvalues of the second fundamental form of the geodesic sphere S_r as in (17). We then conclude using Lemma 16. □

We can now state a general Price inequality for rank one locally symmetric spaces.

Lemma 27. *Let* $\mathrm{inj}_\Gamma \geq 1$. *There exists a constant* $d_{(\mathbb{K},n,k)} > 0$ *such that for* $h \in$

$\mathcal{H}^k(\Gamma\backslash \boldsymbol{H}^n_{\mathbb{K}})$ *and* $p \in \Gamma\backslash \boldsymbol{H}^n_{\mathbb{K}}$,

$$\begin{cases} \boxed{\int_{B_1(p)} |h|^2 dv \leq d_{(\mathbb{K},n,k)} V_{min}(\Gamma\backslash \boldsymbol{H}^n_{\mathbb{K}})^{-\frac{(\dim(\boldsymbol{H}^n_{\mathbb{K}})-\dim(\mathbb{K})-2k)\mathrm{inj}_\Gamma}{\dim(\boldsymbol{H}^n_{\mathbb{K}})+\dim(\mathbb{K})-2}} \int_{B_{\mathrm{inj}_\Gamma}(p)} |h|^2 dv,} \\ \textit{if } \dim(\mathbb{K})-1 < k < \frac{\dim(\boldsymbol{H}^n_{\mathbb{K}})-\dim(\mathbb{K})}{2}; \\ \boxed{\int_{B_1(p)} |h|^2 dv \leq d_{(\mathbb{K},n,k)} V_{min}(\Gamma\backslash \boldsymbol{H}^n_{\mathbb{K}})^{-\frac{(\dim(\boldsymbol{H}^n_{\mathbb{K}})+\dim(\mathbb{K})-4k)\mathrm{inj}_\Gamma}{\dim(\boldsymbol{H}^n_{\mathbb{K}})+\dim(\mathbb{K})-2}} \int_{B_{\mathrm{inj}_\Gamma}(p)} |h|^2 dv,} \\ \textit{if } k \leq \dim(\mathbb{K})-1, \\ \textit{unless} \\ \mathbb{K}=\mathbb{C}, \quad n=2, \quad k=1, \\ \mathbb{K}=\mathbb{H}, \quad n=2, \quad k=3, \\ \mathbb{K}=\mathbb{O}, \quad n=1, \quad k=6,7. \end{cases}$$

Proof. The integral equality for harmonic forms of Proposition 16 in [DS17] implies

$$\phi_h(\tau) \int_{B_\tau(p)} q_h(s)|h|^2 dv = \int_{B_1(p)} q_h(s)|h|^2 dv \geq \int_{B_1} |h|^2 dv$$

with

$$\phi_h(\tau) = e^{-\int_1^\tau \frac{q_h(s)ds}{\frac{1}{2}-\mu_h(s)}}.$$

By Lemma 26, we know that $\mu_h(s) < 1/2$ for any $s \leq \mathrm{inj}_{\mathbb{K}}$. If we then integrate the lower bounds on $q_h(s)$ given in Lemma 26, we obtain the desired bounds. □

We can now prove an effective bound for Betti numbers of rank one compact locally symmetric spaces.

Theorem 28. *There exists a positive constant* $c_{(\mathbb{K},n,k)}$ *such that if* Γ *is cocompact and*

$\mathrm{inj}_\Gamma \geq 1$,

$$\begin{cases} \boxed{\dfrac{b^k(\Gamma\backslash \boldsymbol{H}^n_{\mathbb{K}})}{Vol(\Gamma\backslash \boldsymbol{H}^n_{\mathbb{K}})} \leq c_{(\mathbb{K},n,k)} V_{min}(\Gamma\backslash \boldsymbol{H}^n_{\mathbb{K}})^{-\frac{(\dim(\boldsymbol{H}^n_{\mathbb{K}})-\dim(\mathbb{K})-2k)\mathrm{inj}_\Gamma}{\dim(\boldsymbol{H}^n_{\mathbb{K}})+\dim(\mathbb{K})-2}},} \\ \\ \textit{if } \dim(\mathbb{K})-1 < k < \frac{\dim(\boldsymbol{H}^n_{\mathbb{K}})-\dim(\mathbb{K})}{2}; \\ \\ \boxed{\dfrac{b^k(\Gamma\backslash \boldsymbol{H}^n_{\mathbb{K}})}{Vol(\Gamma\backslash \boldsymbol{H}^n_{\mathbb{K}})} \leq c_{(\mathbb{K},n,k)} V_{min}(\Gamma\backslash \boldsymbol{H}^n_{\mathbb{K}})^{-\frac{(\dim(\boldsymbol{H}^n_{\mathbb{K}})+\dim(\mathbb{K})-4k)\mathrm{inj}_\Gamma}{\dim(\boldsymbol{H}^n_{\mathbb{K}})+\dim(\mathbb{K})-2}},} \\ \\ \textit{if } k \leq \dim(\mathbb{K})-1, \\ \\ \textit{unless} \\ \\ \mathbb{K}=\mathbb{C}, \quad n=2, \quad k=1, \\ \mathbb{K}=\mathbb{H}, \quad n=2, \quad k=3, \\ \mathbb{K}=\mathbb{O}, \quad n=1, \quad k=6,7. \end{cases}$$

Proof. Once we have the Price inequality in Lemma 27, an estimate on the Betti numbers follows from the general peaking and Moser iteration arguments given in [DS17, Section 5]. □

Theorem 28 sharpens the bounds for negatively curved and quarter-pinched manifolds given by the general theory developed in Section 7 of [DS17]. Indeed, Theorem 28 relies heavily upon the special properties of the locally symmetric metrics on rank one locally symmetric spaces. In the next section, we push this analysis further by exploiting the Kähler properties of the complex-hyperbolic metric.

4 Refined Estimates for Compact Quotients of $\mathbf{H}^n_{\mathbb{C}}$

In this section, we improve the results of Section 3 in the case of complex-hyperbolic manifolds. These further results rely on the fact that, in this case, the metric is not only symmetric but also Kähler. If we regard $\mathbf{H}^n_{\mathbb{C}}$ as the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$, the Kähler form is

$$\omega_{\mathbb{C}} = \frac{i}{2}\overline{\partial}\partial \log(1-|z|^2). \tag{29}$$

The associated Riemannian metric is exactly the metric $\mathbf{g}_{\mathbb{C}}$ discussed in Section 2. Given a compact complex-hyperbolic manifold $\Gamma\backslash\mathbf{H}^n_{\mathbb{C}}$, we denote by J its complex structure operator. Let h be a harmonic $k-$form on $\Gamma\backslash\mathbf{H}^n_{\mathbb{C}}$. Since $\Gamma\backslash\mathbf{H}^n_{\mathbb{C}}$ is Kähler, Jh is also harmonic with

$$|h|(p) = |Jh|(p),$$

for any p. In a geodesic ball, $J\frac{\partial}{\partial r}$ is tangent to the Hopf fiber. Hence we can choose the adapted coframe used in the definition (24) with $\omega^1 = Jdr$. Thus, we find convenient to set

$$\zeta_h(r) := \frac{\int_{S_r(p)} |e^*(Jdr)h|^2 d\sigma}{\int_{S_r(p)} |h|^2 d\sigma}, \tag{30}$$

where $\zeta_h(r) = \zeta_h^1(r)$ in our prior notation. Finally, since

$$|e^*(dr)(Jh)| = |e^*(Jdr)(h)|, \text{ and } \quad |e^*(Jdr)(Jh)| = |e^*(dr)h|,$$

we conclude that

$$\mu_{Jh} = \zeta_h, \quad \zeta_{Jh} = \mu_h. \tag{31}$$

We now derive a strengthened Price equality for complex-hyperbolic manifolds. We start with the following integration by parts formula.

Lemma 32. *Let $n \geq 2$. Given $h \in \mathcal{H}^k(\Gamma\backslash \boldsymbol{H}^n_{\mathbb{C}})$ with $k < n$, $p \in \Gamma\backslash \boldsymbol{H}^n_{\mathbb{C}}$, and $R < \mathrm{inj}_\Gamma$, we have*

$$\int_{S_R(p)} \big(1 - \mu_h(R) - \mu_{Jh}(R)\big)|h|^2 d\sigma = \int_{B_R(p)} q(r)|\alpha|^2 dv$$

with $\mu_h(R) = \mu_{Jh}(R) < 1/2$, and

$$q(r) > 2(n-k-1)\coth(r) + 2\tanh(2r).$$

Proof. For $R \leq \mathrm{inj}_\Gamma$, apply Lemma 25 to both h and Jh in $\mathcal{H}^k(\Gamma\backslash \mathbf{H}^n_{\mathbb{C}})$ to obtain

$$\begin{aligned}\int_{S_R(p)} \Big(\frac{1}{2} - \mu_h(R)\Big)|h|^2 d\sigma = &\int_{B_R} \Big(\frac{\mathcal{H}_{\mathbb{C},n}(r)}{2} - k\coth(r)\Big)|h|^2 dv \\ &+ \int_{B_R} \mu_h(r)\coth(r)|h|^2 dv \\ &- \int_{B_R} \mu_{Jh}(r)\tanh(r)|h|^2 dv,\end{aligned} \tag{33}$$

and

$$\begin{aligned}\int_{S_R(p)} \Big(\frac{1}{2} - \mu_{Jh}(R)\Big)|Jh|^2 d\sigma = &\int_{B_R} \Big(\frac{\mathcal{H}_{\mathbb{C},n}(r)}{2} - k\coth(r)\Big)|Jh|^2 dv \\ &+ \int_{B_R} \mu_{Jh}(r)\coth(r)|Jh|^2 dv \\ &- \int_{B_R} \mu_h(r)\tanh(r)|Jh|^2 dv.\end{aligned} \tag{34}$$

Since $|h|^2 = |Jh|^2$, summing Equations (58) and (88) yields

$$\begin{aligned}
&\int_{S_R(p)} \big(1 - \mu_h(R) - \mu_{Jh}(R)\big)|h|^2 d\sigma \qquad (35)\\
&\quad = \int_{B_R(p)} \big((2n-2k-1)\coth(r) + \tanh(r)\big)|h|^2 dv\\
&\quad + \int_{B_R(p)} \big((\mu_h(r) + \mu_{Jh}(r))\coth(r) - (\zeta_h(r) + \zeta_{Jh}(r))\tanh(r)\big)|h|^2\\
&\quad = \int_{B_R(p)} \big((2n-2k-1)\coth(r) + \tanh(r)\big)|h|^2 dv\\
&\quad + \int_{B_R(p)} (\mu_h(r) + \mu_{Jh}(r))(\coth(r) - \tanh(r))|h|^2 dv\\
&\quad > \int_{B_R} \big((2(n-k-1))\coth(r) + 2\coth(2r)\big)|h|^2 dv.
\end{aligned}$$

The positivity of the right hand side of (35) for nonzero h implies $\mu_h + \mu_{Jh} < 1$. To see that $\mu_h = \mu_{Jh}$, we now take the difference of equations (58) and (88) to obtain

$$\int_{S_R(p)} \Big(\mu_{Jh} - \mu_h\Big)|h|^2 d\sigma = \int_{B_R} (\mu_h - \mu_{Jh}) 2\coth(2r)|h|^2 dv. \qquad (36)$$

Set $F := \int_{B_R}(\mu_h - \mu_{Jh})2\coth(2r)|h|^2 dv$. Then we can rewrite (36) as

$$F' = -2\coth(2r)F. \qquad (37)$$

Hence

$$F(r) = \frac{\sinh(2\sigma)}{\sinh(2r)}F(\sigma). \qquad (38)$$

Since $F(0) = 0$, (38) requires $F(\sigma) = 0$ for all σ, and $\mu_h = \mu_{Jh} < \frac{1}{2}$ follows. □

Following Proposition 16 in [DS17], we have the following Price inequality for complex-hyperbolic manifolds.

Lemma 39. *Let* $\text{inj}_\Gamma \geq 1$. *There exists a constant* $d(n,k) > 0$ *such that if* $h \in \mathcal{H}^k(\Gamma\backslash \boldsymbol{H}^n_{\mathbb{C}})$ *with* $k < n$, $p \in \Gamma\backslash \boldsymbol{H}^n_{\mathbb{C}}$, *and* $\tau < \text{inj}_\Gamma$, *then*

$$\int_{B_1(p)} |h|^2 dv \leq c(n,k) e^{-2(n-k)\tau} \int_{B_\tau(p)} |h|^2 dv. \qquad (40)$$

Proof. We have that

$$\phi(\tau) \int_{B_\tau(p)} q(s)|h|^2 dv = \int_{B_1(p)} q(s)|h|^2 dv \geq \int_{B_1} |h|^2 dv,$$

with

$$\phi(\tau) \leq e^{-\int_1^\tau q(s)ds}.$$

Integrating the lower bound on $q(s)$ given in Lemma 32, we obtain the desired bound. □

We can now give an effective bound for the Betti numbers of compact complex-hyperbolic manifolds.

Corollary 41. *Let* Γ *be co-compact, with* $\mathrm{inj}_\Gamma \geq 1$*. For any positive integer* $k < n$*,*

$$\frac{b^k(\Gamma \backslash \boldsymbol{H}^n_{\mathbb{C}})}{Vol(\Gamma \backslash \boldsymbol{H}^n_{\mathbb{C}})} \leq d(n,k) V_{min}(\Gamma \backslash \boldsymbol{H}^n_{\mathbb{C}})^{\frac{k-n}{n}},$$

where $d(n,k)$ *is a positive constant depending only on the dimension and the degree* k*.*

Proof. Given the Price inequality in Lemma 39, the Betti number estimate follows from the peaking and Moser iteration arguments given in Section 5 of [DS17]. □

Sarnak and Xue [SX91, Equation 21] show that with the metric normalization given by an embedding of a semisimple real Lie group G into $GL(N,\mathbb{R})$ associated to an arithmetic structure, the injectivity radius of $\Gamma_q \backslash G/K$ for Γ_q a level q congruence subgroup, K a maximal compact subgroup of G, satisfies

$$\mathrm{inj}_{\Gamma_\mathrm{q}} \geq 2\ln(\mathrm{q}), \tag{42}$$

Returning now to $G = SU(n,1)$, we rewrite this in a less scale dependent manner as

$$Vol(B_{\mathrm{inj}_{\Gamma_\mathrm{q}}}) \geq \omega_n q^{2n} + O(q^{(2n-1)}),$$

for some constant ω_n depending on the dimension only. By [SX91, Equation 22], there exists a constant $C > 0$, independent of q such that

$$Vol(\Gamma_q \backslash \mathbf{H}^n_{\mathbb{C}}) \leq C q^{n^2+2n}.$$

Hence for q large, we have for some $c > 0$, independent of q:

$$V_{min}(\Gamma_q \backslash \mathbf{H}^n_{\mathbb{C}})) \geq c Vol(\Gamma_q \backslash \mathbf{H}^n_{\mathbb{C}})^{\frac{2}{n+2}}.$$

Consequently,

$$b^k(\Gamma_q \backslash \mathbf{H}^n_{\mathbb{C}}) \leq c(k) Vol(\Gamma_q \backslash \mathbf{H}^n_{\mathbb{C}})^{\frac{n^2+2k}{n^2+2n}}.$$

When $n = 2$, we have

$$b^1(\Gamma_q \backslash \mathbf{H}^2_{\mathbb{C}}) \leq c(1) Vol(\Gamma_q \backslash \mathbf{H}^2_{\mathbb{C}})^{\frac{3}{4}}.$$

5 Peaking Revisited

We now shift our attention to complete finite volume hyperbolic manifolds. As a first step towards treating cusps, we reformulate the peaking equality of [DS17], embedding it in an infinite family of equalities that can be used to estimate Betti numbers. We first record an elementary calculus lemma.

Lemma 43. *Let $\phi : \mathbb{R}^b \to \mathbb{R}$ be a nonzero linear function. Then*

$$b = \frac{\max\{|\phi(h)|^2 : h \in S^{b-1}\}}{\int_{S^{b-1}} |\phi(h)|^2 d\sigma_h}, \tag{44}$$

where $d\sigma_h$ is the usual Riemannian measure on the unit sphere, renormalized to give the sphere unit volume.

Proof. Rotate coordinates so that $\phi(h) = ah_1$. We then have

$$\frac{\max\{|\phi(h)|^2 : h \in S^{b-1}\}}{\int_{S^{b-1}} |\phi(h)|^2 d\sigma_h} = \frac{a^2}{\int_{S^{b-1}} a^2 h_1^2 d\sigma_h} = b,$$

and the proof is complete. □

Let (M, g) be a Riemannian manifold. The unit sphere bundle

$$\pi : S(\Lambda^k TM) \to M$$

of $\Lambda^k TM$ inherits a canonical metric from the metric on M, and a choice of a metric on the spherical fiber. We pick the standard symmetric metric on the spherical fiber and scale it to have volume 1. Let $d\bar{v}$ denote the associated Riemannian measure on $S(\Lambda^k TM)$. We then have:

$$Vol(S(\Lambda^k TM)) = \int_{S(\Lambda^k TM)} 1 d\bar{v} = \int_M 1 dv = Vol(M).$$

Let $S(\mathcal{H}^k(M))$ denote the unit sphere in $\mathcal{H}^k(M)$, where $\mathcal{H}^k(M)$ is endowed with the Hilbert space structure given by the L^2-inner product on harmonic $k-$ forms.

Corollary 45. *For each $\bar{p} \in S(\Lambda^k TM)$ such that $h(\bar{p}) \neq 0$ for some $h \in \mathcal{H}^k(M)$, we have*

$$b_2^k(M) := dim_{\mathbb{R}}(\mathcal{H}^k(M)) = \frac{\max\{|h(\bar{p})|^2 : h \in S(\mathcal{H}^k(M))\}}{\int_{S(\mathcal{H}^k(M))} |h(\bar{p})|^2 d\sigma_h}, \tag{46}$$

where $d\sigma_h$ is the usual Riemannian measure on the unit sphere, renormalized to give the sphere unit volume.

Proof. Given such $\bar{p} \in S(\Lambda^k TM)$, consider the associated non-zero linear function obtained by evaluating $h \in \mathcal{H}^k(M)$ at $\bar{p}$. Apply Lemma 43 to this linear function, and the proof is complete. □

Next, observe that the L^2-norm on k-forms satisfies for all measurable $\Omega \subset M$,

$$\int_\Omega |h|^2(p) dv_p = \binom{n}{k} \int_{\pi^{-1}(\Omega)} |h(\bar{p})|^2 d\bar{v}_{\bar{p}}, \tag{47}$$

as the volume of the fibers is one. Using (47), we write

$$1 = \int_M \int_{S(\mathcal{H}^k(M))} |h|^2(p) dv_p d\sigma_h = \binom{n}{k} \int_{S(\Lambda^k TM)} \int_{S(\mathcal{H}^k(M))} |h(\bar{p})|^2 d\sigma_h d\bar{v}_{\bar{p}}; \tag{48}$$

hence the average value of $\int_{S(\mathcal{H}^k(M))} |h|^2(p)d\sigma_h$ and $\binom{n}{k}\int_{S(\mathcal{H}^k(M))} |h(\bar{p})|^2 d\sigma_h$ is $\frac{1}{Vol(M)}$.

When working on manifolds with cusps, we would like for some $\bar{p} \in S(\Lambda^k TM)$ for which

$$\binom{n}{k}\int_{S(\mathcal{H}^k(M))} |h(\bar{p})|^2 d\sigma_h \geq \frac{1}{Vol(M)}$$

to live above the thick part of the manifold. In fact, when this is the case, Corollary 45 implies

$$b_2^k(M) \leq \max\{|h(\bar{p})|^2 : h \in S(\mathcal{H}^k(M))\}\binom{n}{k} Vol(M),$$

and as $|h(\bar{p})|^2 \leq |h|^2(\pi(\bar{p}))$, we obtain an upper bound on $b_2^k(M)$ by using a Price inequality on geodesic balls and a Moser iteration argument. We refer to Sections 8 and 9.2 for the implementation of this strategy on finite volume hyperbolic manifolds with cusps.

On the other hand, if there is no such p, we obtain additional information, which we record with a lemma.

Lemma 49. *Let $S(\Lambda^k TM) = \pi^{-1}(\Omega_1 \cup \Omega_2)$, where Ω_1 and Ω_2 are measurable subsets of M. Given $\alpha \in (0, 1)$, suppose that*

$$\binom{n}{k}\int_{\pi^{-1}(\Omega_1)}\int_{S(\mathcal{H}^k(M))} |h(\bar{p})|^2 d\sigma_h d\bar{v}_{\bar{p}} < \frac{\alpha Vol(\Omega_1)}{Vol(M)}.$$

Then, the average value of $\binom{n}{k}\int_{S(\mathcal{H}^k(M))} |h(\bar{p})|^2 d\sigma_h$ for $\bar{p} \in \pi^{-1}(\Omega_2)$ is greater than or equal to $\frac{1-\alpha}{Vol(\Omega_2)}$.

Proof. From Equation (48), we have

$$\begin{aligned} 1 &= \int_{S(\Lambda^k TM)}\binom{n}{k}\int_{S(\mathcal{H}^k(M))} |h(\bar{p})|^2 d\sigma_h d\bar{v}_{\bar{p}} \\ &\leq \int_{\pi^{-1}(\Omega_1)}\binom{n}{k}\int_{S(\mathcal{H}^k(M))} |h(\bar{p})|^2 d\sigma_h d\bar{v}_{\bar{p}} + \int_{\pi^{-1}(\Omega_2)}\binom{n}{k}\int_{S(\mathcal{H}^k(M))} |h(\bar{p})|^2 d\sigma_h d\bar{v}_{\bar{p}} \\ &\leq \alpha\frac{Vol(\Omega_1)}{Vol(M)} + \int_{\pi^{-1}(\Omega_2)}\binom{n}{k}\int_{S(\mathcal{H}^k(M))} |h(\bar{p})|^2 d\sigma_h d\bar{v}_{\bar{p}}. \end{aligned} \tag{50}$$

Hence

$$\int_{\pi^{-1}(\Omega_2)}\binom{n}{k}\int_{S(\mathcal{H}^k(M))} |h(\bar{p})|^2 d\sigma_h d\bar{v}_{\bar{p}} \geq (1-\alpha),$$

and the result follows. □

We will use this lemma to balance the relatively small volume of the cusp regions against the smaller injectivity radius at a point in a cusp.

6 L^2-Harmonic Forms on Real-Hyperbolic Cusps

Cusps of complete finite volume real-hyperbolic manifolds are isometric to $[0,\infty)\times F$, equipped with the metric $ds^2+e^{-2s}g_F$, where F is a compact quotient of the $(n-1)$-dimensional Euclidean space, and g_F is a flat metric on F. For simplicity, for most of this section, we consider cusps of complete finite volume real-hyperbolic manifolds that are isometric to $[0,\infty)\times T^{n-1}$, equipped with the metric $ds^2+e^{-2s}g_{T^{n-1}}$, where T^{n-1} is an $(n-1)$-dimensional real torus, and $g_{T^{n-1}}$ is a flat metric on T^{n-1}. This is always the case, if for example, the manifold is the quotient of hyperbolic space by a *neat* arithmetic subgroup. The Price inequality for general cusps follows immediately from this special case, as we show at the end of this section. (Moreover, given a complete finite volume real-hyperbolic manifold, one can always pass to a finite cover where all of the cusps have tori cross sections, see for example [Hum98]. If a base of a tower of manifolds has this property, then so does every manifold in the tower.)

Given a covariant constant orthonormal frame and coframe on T_0, extend it to a $\nabla_{\frac{\partial}{\partial s}}$ covariant constant frame $\{\frac{\partial}{\partial s}\}\cup\{X_i\}_{i=1}^{n-1}$ and coframe $\{ds\}\cup\{\omega^i\}_{i=1}^{n-1}$ on the cusp. Using the explicit form of the hyperbolic metric on the cusp and the flatness of $g_{T^{n-1}}$, it is straightforward to verify that:

$$\nabla_{\partial_s}\partial_s=0,\quad \nabla_{X_i}X_j=\delta_{ij}\frac{\partial}{\partial s},\quad \nabla_{X_i}\frac{\partial}{\partial s}=-X_i.$$

On differential forms, we have therefore the following identity

$$\begin{aligned}L_{\partial_s}&=\nabla_{\partial_s}+\sum_{i=1}^{n-1}e(\omega_j)e^*(\nabla_{X_j}ds)\\&=\nabla_{\partial_s}-\sum_{i=1}^{n-1}e(\omega_j)e^*(\omega_j)-e(ds)e^*(ds)+e(ds)e^*(ds),\end{aligned}$$

so that the pointwise inner product $(L_{\partial_s}\alpha,\alpha)$ satisfies

$$(L_{\partial_s}\alpha,\alpha)=(\nabla_{\partial_s}\alpha,\alpha)-k|\alpha|^2+|i_{\partial_s}\alpha|^2. \tag{51}$$

Given a strongly harmonic L^2 k-form α on a cusp, define for $s\in[0,\infty)$

$$\mu_\alpha(s):=\frac{\int_{T_s}|i_{\partial_s}\alpha|^2d\sigma_s}{\int_{T_s}|\alpha|^2d\sigma_s}. \tag{52}$$

For $0\le s<K$, define Ω_{uK} to be the subset of the cusp identified with $[u,K]\times T^{n-1}$. For $s\in[0,\infty)$, we denote by T_s the cross section $\{s\}\times T^{n-1}$ equipped with the induced metric. With this notation, we have the following:

$$\begin{aligned}\int_{\Omega_{uK}}(L_{\partial_s}\alpha,\alpha)dv&=\int_{\Omega_{uK}}(\{d,i_{\partial_s}\}\alpha,\alpha)dv=\int_{\Omega_{uK}}(d\circ i_{\partial_s}\alpha,\alpha)dv\\&=\int_{T_K}|i_{\partial_s}\alpha|^2d\sigma-\int_{T_u}|i_{\partial_s}\alpha|^2d\sigma.\end{aligned} \tag{53}$$

We also have

$$\int_{\Omega_{uK}} (\nabla_{\partial_s}\alpha, \alpha) dv = \frac{1}{2}\int_{\Omega_{uK}} \partial_s |\alpha|^2 dv \tag{54}$$
$$= \frac{1}{2}\int_{T_K} |\alpha|^2 d\sigma - \frac{1}{2}\int_{T_u} |\alpha|^2 d\sigma + \frac{n-1}{2}\int_{\Omega_{sK}} |\alpha|^2 dv.$$

Using Equations (51), (53), and (54), we have

$$\int_{T_K} |i_{\partial_s}\alpha|^2 d\sigma - \int_{T_u} |i_{\partial_s}\alpha|^2 d\sigma = \Big(\frac{n-1}{2} - k\Big)\int_{\Omega_{sK}} |\alpha|^2 dv + \int_{\Omega_{sK}} |i_{\partial_r}\alpha|^2 dv \tag{55}$$
$$+ \frac{1}{2}\int_{T_K} |\alpha|^2 d\sigma - \frac{1}{2}\int_{T_u} |\alpha|^2 d\sigma.$$

Use Definition (52) to rewrite this as

$$\int_{T_u} \Big(\frac{1}{2} - \mu_\alpha(s)\Big)|\alpha|^2 d\sigma - \int_{T_K} \Big(\frac{1}{2} - \mu_\alpha(K)\Big)|\alpha|^2 d\sigma = \int_{\Omega_{uK}} \Big(\frac{n-1}{2} - k + \mu_\alpha\Big)|\alpha|^2 dv. \tag{56}$$

Specialize now to the case $k < \frac{n-1}{2}$, which makes the right hand side of (56) nonnegative.

Since α is L^2,

$$\int_0^\infty \Big(\int_{T_s} |\alpha|^2 d\sigma\Big) ds = \int_{\Omega_{0\infty}} |\alpha|^2 dv < \infty.$$

Hence there exists a sequence $\{K_i\} \subset (0, \infty)$ going to ∞, such that

$$\lim_{i\to\infty} \int_{T_{K_i}} |\alpha|^2 d\sigma = 0,$$

and since $\mu_\alpha(r)$ is bounded, we conclude that

$$\lim_{i\to\infty} \int_{T_{K_i}} \Big(\frac{1}{2} - \mu_\alpha(K_i)\Big)|\alpha|^2 d\sigma = 0.$$

Thus, for any $u \in [0, \infty)$ we have

$$\int_{T_u} \Big(\frac{1}{2} - \mu_\alpha(s)\Big)|\alpha|^2 d\sigma = \int_{\Omega_{u\infty}} \Big(\frac{n-1}{2} - k + \mu_\alpha\Big)|\alpha|^2 dv. \tag{57}$$

This gives us the following useful corollary.

Corollary 58. *For any $u \in [0, \infty)$ we have*

$$\int_{T_u} \Big(\frac{1}{2} - \mu_\alpha(u)\Big)|\alpha|^2 d\sigma \geq 0. \tag{59}$$

In particular, $0 \leq \mu_\alpha(u) < \frac{1}{2}$. Moreover, $\int_{T_u} \Big(\frac{1}{2} - \mu_\alpha(u)\Big)|\alpha|^2 d\sigma$ is decreasing.

Corollary 58 and Equation (57) enables us to estimate the decay of $\int_{\Omega_{u\infty}} |\alpha|^2 dv$ as u increases. Define

$$g(s) := \int_{\Omega_{s\infty}} \Big(\frac{n-1}{2} - k + \mu_\alpha\Big)|\alpha|^2 dv.$$

For some differentiable $\phi(u)$ to be determined, multiply Equation (57) by $\phi'(u)$ and integrate from σ to τ to obtain:

$$\begin{aligned}\int_{\Omega_{\sigma\tau}} \phi'(s)\Big(\frac{1}{2} - \mu_\alpha\Big)|\alpha|^2 dv &= \phi(\tau)g(\tau) - \phi(\sigma)g(\sigma) - \int_\sigma^\tau \phi(s)g'(s)ds \\ &= \phi(\tau)g(\tau) - \phi(\sigma)g(\sigma) + \int_{\Omega_{\sigma\tau}} \phi(s)\Big(\frac{n-1}{2} - k + \mu_\alpha\Big)|\alpha|^2 dv.\end{aligned} \tag{60}$$

Choose

$$\phi(s) := e^{\int_0^s \frac{[\frac{(n-1)}{2} - k + \mu_\alpha(u)]}{1/2 - \mu_\alpha(u)} du} \geq e^{(n-1-2k)s}. \tag{61}$$

With this choice, the two volume integrals in Equation (60) cancel, reducing Equation 60 to

$$\phi(\sigma)g(\sigma) = \phi(\tau)g(\tau),$$

which we expand as

$$\phi(\sigma)\int_{\Omega_{\sigma\infty}} \Big(\frac{n-1}{2} - k + \mu_\alpha\Big)|\alpha|^2 dv = \phi(\tau)\int_{\Omega_{\tau\infty}} \Big(\frac{n-1}{2} - k + \mu_\alpha\Big)|\alpha|^2 dv. \tag{62}$$

Setting

$$C_\alpha := \int_{\Omega_{0\infty}} \Big(\frac{n-1}{2} - k + \mu_\alpha\Big)|\alpha|^2 dv,$$

gives for any $K > 0$,

$$\int_{\Omega_{K\infty}} \Big(\frac{n-1}{2} - k + \mu_\alpha\Big)|\alpha|^2 dv = \frac{C_\alpha}{\phi(K)} \leq C_\alpha e^{-(n-1-2k)s}.$$

We summarize this discussion with a proposition which is the cusp analog of the Price inequality for geodesic balls proved in [DS17]; so, it seems natural to refer to the monotonicity inequality in Proposition 63 as a *cuspidal* Price inequality.

Proposition 63. *Let α be an L^2-harmonic k-form, $k < \frac{n-1}{2}$, on an n-dimensional real-hyperbolic manifold. For any $s > 0$, on any cusp we have*

$$\int_{\Omega_{s\infty}} |\alpha|^2 dv \leq c_{n,k} e^{-(n-1-2k)s} \int_{\Omega_{0\infty}} |\alpha|^2 dv, \tag{64}$$

where $c_{n,k} := \frac{n+1-2k}{n-1-2k}$.

Proof. The argument detailed above proves this proposition for L^2-harmonic k-forms, $k < (n-1)/2$, on a real-hyperbolic cusp with torus cross section. For a general real-hyperbolic cusp $C := [0, \infty) \times F$, write $F = \Lambda\backslash\mathbb{R}^{n-1}$, with Λ crystallographic, and let

$\Lambda' \leq \Lambda$ be a finite index abelian subgroup. The existence of such a subgroup follows from Bieberbach's theorem, see for example Chapter 3 in [Wol11]. Let $T^{n-1} := \Lambda' \backslash \mathbb{R}^{n-1}$ and consider the associated regular Riemannian cover

$$p' : T^{n-1} \times [0, \infty) \longrightarrow F \times [0, \infty).$$

Given α on C, consider its pull-back $p'^{*}\alpha$ to $T^{n-1} \times [0, \infty)$. We conclude using the multiplicativity under covers of both sides in the inequality in (64). $\square$

7 L^2-Harmonic Forms on Complex-Hyperbolic Cusps

Up to finite cover, cusps of finite volume n-dimensional complex-hyperbolic manifold are diffeomorphic to $[0, \infty) \times N$, where N is a circle bundle with connection over a $2n-1$ torus (*cf.* Remark 72). In other words, the cross-section N is a *nilmanifold*. The metric on the product is given by

$$g = ds^2 + e^{-4s} g_F + e^{-2s} g_H, \tag{65}$$

where g_F is a metric on the circle fiber extended to N by the connection, and g_H is the pull-back to N of a flat metric on the T^{2n-1} base. Let $N_s := \{s\} \times N$ with the induced metric. Let Ω_{uK} denote the subset of the cusp corresponding to $[u, K] \times N$. Let $\{\frac{\partial}{\partial s}\} \cup \{e_j\}_{j=1}^{2n-1}$ and $\{ds\} \cup \{\omega^j\}_{j=1}^{2n-1}$ be a local orthonormal frame and dual frame with e_1 tangent to the circle fiber. With this notation, acting on $k-$forms we have

$$\begin{aligned} L_{\frac{\partial}{\partial s}} &= \nabla_{\frac{\partial}{\partial s}} - \sum_{j=2}^{2n-1} e(\omega^j) e^*(\omega^j) - 2e(\omega^1)e^*(\omega^1) \\ &= \nabla_{\frac{\partial}{\partial s}} - k \cdot Id - e(\omega^1)e^*(\omega^1) + e(ds)e^*(ds), \end{aligned} \tag{66}$$

where Id denotes the identity operator on k-forms. Let J denote the complex structure operator. Then $Jds = \pm\omega^1$. Let α be a strongly harmonic k-form. Then Jh is also a strongly harmonic k-form, and defining μ_α as in (52), we have the equality

$$\mu_{J\alpha}(s) = \frac{\int_{N_s} |i_{e_1}\alpha|^2 d\sigma}{\int_{N_s} |\alpha|^2 |d\sigma}. \tag{67}$$

Arguing as in the preceding section, we deduce

$$\int_{\Omega_{u\infty}} (n - k + \mu_\alpha - \mu_{J\alpha}) |\alpha|^2 dv = \int_{N_u} \Big(\frac{1}{2} - \mu_\alpha\Big) |\alpha|^2 d\sigma \tag{68}$$

Summing (68) with the corresponding equation for $J\alpha$ yields

$$\int_{\Omega_{u\infty}} (2n - 2k)|\alpha|^2 dv = \int_{N_u} (1 - \mu_\alpha - \mu_{J\alpha}) |\alpha|^2 d\sigma. \tag{69}$$

Arguing as in the proof of Proposition 63 yields the following *cuspidal* Price inequality for complex hyperbolic spaces.

Proposition 70. *Let α be a strongly L^2 harmonic k-form, $k < n$, on an n-dimensional complex-hyperbolic space. For any $s > 0$, on each cusp, we have*

$$\int_{\Omega_{s\infty}} |\alpha|^2 dv \leq e^{-2(n-k)s} \int_{\Omega_{0\infty}} |\alpha|^2 dv. \tag{71}$$

Proof. Proceed similarly to Equation 60 in the proof of Proposition 63, and in this case define for $s > 0$

$$g(s) := \int_{\Omega_{s\infty}} 2(n-k)|\alpha|^2 dv.$$

Thus, following the argument detailed in Section 6, this proposition is proved for complex-hyperbolic cusps with nilmanifolds cross sections. (Unlike in Proposition 63, in this case the constant $c_{n,k}$ can be taken to be one.) For general cusps with infranilmanifolds cross sections, we conclude using the multiplicativity under covers of both sides in the inequality in (71). □

Remark 72. As with real-hyperbolic manifolds, one can always pass to a finite cover of a complete finite volume complex-hyperbolic manifold for which all cusps have nilmanifold cross sections, see for example [Hum98]. In the arithmetic case, it suffices to pass to a suitable congruence subgroup. This fact, however, is not needed for our estimates.

8 L^2-Betti Number Estimates away from Middle Degree

We continue to assume Γ is always discrete and torsion free. Given Γ of co-finite volume, $b_2^k(\Gamma\backslash\mathbf{H}^n_{\mathbb{C}})$ is *always* finite, see Section 1 for more details and references. In this section, we derive effective estimates for b_2^k on such ball quotients. We start with a proposition.

Proposition 73. *Let Γ be co-finite volume. Decompose as a disjoint union*

$$\Gamma\backslash\boldsymbol{H}^n_{\mathbb{C}} = M_0 \cup (\cup_j C_j),$$

where each C_j is a cusp, parameterized by $[0,\infty) \times N^j$. Let Ω^j_{uK} denote the subset of C_j parameterized by $[u, K) \times N^j$. Suppose that:

$$\int_{\pi^{-1}(M_0)} \binom{2n}{k} \int_{S(\mathcal{H}^k(\Gamma\backslash\boldsymbol{H}^n_{\mathbb{C}}))} |h(\bar{p})|^2 d\sigma_h d\bar{v}_{\bar{p}} < \frac{Vol(M_0)}{2Vol(\Gamma\backslash\boldsymbol{H}^n_{\mathbb{C}})}. \tag{74}$$

Then for $k < n$ the average value of

$$\binom{2n}{k} \int_{S(\mathcal{H}^k(\Gamma\backslash\boldsymbol{H}^n_{\mathbb{C}}))} |h(\bar{p})|^2 d\sigma_h \quad \text{on} \quad \pi^{-1}(\cup_j \Omega^j_{01})$$

is greater than

$$\frac{1}{4\sum_j Vol(\Omega^j_{01})}.$$

Proof. Combining (74) and (47) with Lemma 49, we obtain

$$\sum_j \int_{\Omega^j_{0\infty}} \int_{S(\mathcal{H}^k(\Gamma\backslash \mathbf{H}^n_{\mathbb{C}}))} |h|^2(p) d\sigma_h dv_p \geq \frac{1}{2}. \tag{75}$$

Using the Price inequality given in Proposition 70, for each j we have:

$$\int_{S(\mathcal{H}^k(\Gamma\backslash \mathbf{H}^n_{\mathbb{C}})))} \int_{\Omega^j_{1\infty}} |h|^2(p) dv_p d\sigma_h \leq e^{-2(n-k)} \int_{S(\mathcal{H}^k(\Gamma\backslash \mathbf{H}^n_{\mathbb{C}})))} \int_{\Omega^j_{0\infty}} |h|^2(p) dv_p d\sigma_h,$$

so that

$$\int_{S(\mathcal{H}^k(\Gamma\backslash \mathbf{H}^n_{\mathbb{C}})))} \int_{\Omega^j_{01}} |h|^2(p) dv_p d\sigma_h \geq (1 - e^{-2(n-k)}) \int_{S(\mathcal{H}^k(\Gamma\backslash \mathbf{H}^n_{\mathbb{C}})))} \int_{\Omega^j_{0\infty}} |h|^2(p) dv_p d\sigma_h.$$

Combining this last estimate with (75) yields

$$\sum_j \int_{\Omega^j_{01}} \int_{S(\mathcal{H}^k(\Gamma\backslash \mathbf{H}^n_{\mathbb{C}}))} |h(p)|^2 d\sigma_h dv_p \geq \frac{1 - e^{-2(n-k)}}{2} > \frac{1}{4},$$

and the result follows. □

We can now estimate from above the dimension of the space of L^2-harmonic forms on a complete finite volume complex-hyperbolic manifold.

Corollary 76. *Let Γ be co-finite volume. Decompose as a disjoint union*

$$\Gamma\backslash \boldsymbol{H}^n_{\mathbb{C}} = M_0 \cup (\cup_j C_j),$$

where each C_j is a cusp parameterized by $[0,\infty) \times N^j$. Let Ω^j_{uK} denote the subset of C_j parameterized by $[u,K) \times N^j$. Then for some $c_{n,k} > 0$,

$$b^k_2(\Gamma\backslash \boldsymbol{H}^n_{\mathbb{C}}) \leq c_{n,k}\big[Vol(\Gamma\backslash \boldsymbol{H}^n_{\mathbb{C}}) V_{min}(M_0)^{\frac{k-n}{n}} + Vol(\cup_j C_j) V_{min}(\cup_j \Omega^j_{01})^{\frac{k-n}{n}}\big].$$

Proof. First, recall that by Corollary 45, if $\bar{p} \in S(\Lambda^k T(\Gamma\backslash \mathbf{H}^n_{\mathbb{C}}))$ is such that $h(\bar{p}) \neq 0$ for some $h \in \mathcal{H}^k(\Gamma\backslash \mathbf{H}^n_{\mathbb{C}})$ then:

$$b^k_2(\Gamma\backslash \mathbf{H}^n_{\mathbb{C}}) = \frac{\max\{|h(\bar{p})|^2 : h \in S(\mathcal{H}^k(\Gamma\backslash \mathbf{H}^n_{\mathbb{C}}))\}}{\int_{S(\mathcal{H}^k(\Gamma\backslash \mathbf{H}^n_{\mathbb{C}}))} |h(\bar{p})|^2 d\sigma_h}.$$

If there exists a point $\bar{p} \in \pi^{-1}(M_0)$, such that

$$\binom{2n}{k} \int_{S(\mathcal{H}^k(\Gamma\backslash \mathbf{H}^n_{\mathbb{C}}))} |h(\bar{p})|^2 d\sigma_h \geq \frac{1}{2Vol(\Gamma\backslash \mathbf{H}^n_{\mathbb{C}})},$$

then

$$b^k_2(\Gamma\backslash \mathbf{H}^n_{\mathbb{C}}) \leq \max\{|h(\bar{p})|^2 : h \in S(\mathcal{H}^k(\Gamma\backslash \mathbf{H}^n_{\mathbb{C}}))\} \binom{n}{k} \cdot 2Vol(\Gamma\backslash \mathbf{H}^n_{\mathbb{C}}),$$

and the usual Price and Moser tell us that there exists a constant $d_{n,k} > 0$ such that

$$b_2^k(\Gamma\backslash\mathbf{H}_{\mathbb{C}}^n) \leq d_{n,k} Vol(\Gamma\backslash\mathbf{H}_{\mathbb{C}}^n) V_{min}(M_0)^{\frac{k-n}{n}}.$$

Suppose now that for every $\bar{p} \in \pi^{-1}(M_0)$ we have

$$\binom{2n}{k} \int_{S(\mathcal{H}^k(\Gamma\backslash\mathbf{H}_{\mathbb{C}}^n))} |h(\bar{p})|^2 d\sigma_h < \frac{1}{2Vol(\Gamma\backslash\mathbf{H}_{\mathbb{C}}^n)}.$$

This implies that

$$\binom{2n}{k} \int_{\pi^{-1}(M_0)} \int_{S(\mathcal{H}^k(\Gamma\backslash\mathbf{H}_{\mathbb{C}}^n))} |h(p)|^2 d\sigma_h d\bar{v}_{\bar{p}} < \frac{Vol(M_0)}{2Vol(\Gamma\backslash\mathbf{H}_{\mathbb{C}}^n)}.$$

Proposition 73 then implies the existence of $\bar{p} \in \pi^{-1}(\cup_j \Omega_{01}^j)$ with

$$\binom{2n}{k} \int_{S(\mathcal{H}^k(\Gamma\backslash\mathbf{H}_{\mathbb{C}}^n))} |h(\bar{p})|^2 d\sigma_h \geq \frac{1}{4\sum_j Vol(\Omega_{01}^j)}.$$

Thus

$$b_2^k(\Gamma\backslash\mathbf{H}_{\mathbb{C}}^n) \leq \max\{|h(\bar{p})|^2 : h \in S(\mathcal{H}^k(\Gamma\backslash\mathbf{H}_{\mathbb{C}}^n))\}\binom{2n}{k} \cdot 4\sum_j Vol(\Omega_{01}^j). \tag{77}$$

By Price and Moser, we then have for some constant $e_{n,k} > 0$

$$b_2^k(\Gamma\backslash\mathbf{H}_{\mathbb{C}}^n) \leq e_{n,k} V_{min}(\cup_j \Omega_{01}^j)^{\frac{k-n}{n}} \Big(4\sum_j Vol(\Omega_{01}^j)\Big).$$

Setting $c_{n,k} := \max\{d_{n,k}, e_{n,k}\}$ completes the proof. □

Similarly, we have the following proposition and corollary.

Proposition 78. *Let Γ be co-finite volume. Decompose as a disjoint union*

$$\Gamma\backslash\boldsymbol{H}_{\mathbb{R}}^n = M_0 \cup (\cup_j C_j),$$

where each C_j is a cusp, parameterized by $[0,\infty) \times N^j$. Let Ω_{uK}^j denote the subset of C_j parameterized by $[u,K) \times N^j$. Suppose that:

$$\int_{\pi^{-1}(M_0)} \binom{n}{k} \int_{S(\mathcal{H}^k(\Gamma\backslash\boldsymbol{H}_{\mathbb{R}}^n))} |h(\bar{p})|^2 d\sigma_h d\bar{v}_{\bar{p}} < \frac{Vol(M_0)}{2Vol(\Gamma\backslash\boldsymbol{H}_{\mathbb{R}}^n)}.$$

For $k < (n-1)/2$ set

$$R_{n,k} := \frac{1}{n-1-2k} \ln \frac{2(n+1-2k)}{n-1-2k}.$$

Then the average value $\binom{n}{k}\int_{S(\mathcal{H}^k(\Gamma\backslash\boldsymbol{H}_{\mathbb{R}}^n))} |h(\bar{p})|^2 d\sigma_h$ on $\pi^{-1}(\cup_j \Omega_{0R_{n,k}}^j)$ is greater than

$$\frac{1}{4\sum_j Vol(\Omega_{0R_{n,k}}^j)}.$$

Proof. Combining the hypotheses with (47) and Lemma 49, we obtain

$$\sum_j \int_{\Omega^j_{0\infty}} \int_{S(\mathcal{H}^k(\Gamma\backslash \mathbf{H}^n_{\mathbb{R}}))} |h|^2(p) d\sigma_h dv_p \geq \frac{1}{2}. \tag{79}$$

Using the Price inequality on real hyperbolic cusps proved in Proposition 63, for each j we have:

$$\int_{S(\mathcal{H}^k(\Gamma\backslash \mathbf{H}^n_{\mathbb{R}})))} \int_{\Omega^j_{R_{n,k}\infty}} |h|^2(p) dv_p d\sigma_h \leq \frac{1}{2} \int_{S(\mathcal{H}^k(\Gamma\backslash \mathbf{H}^n_{\mathbb{R}})))} \int_{\Omega^j_{0\infty}} |h|^2(p) dv_p d\sigma_h,$$

so that

$$\int_{S(\mathcal{H}^k(\Gamma\backslash \mathbf{H}^n_{\mathbb{R}})))} \int_{\Omega^j_{0R_{n,k}}} |h|^2(p) dv_p d\sigma_h \geq \frac{1}{2} \int_{S(\mathcal{H}^k(\Gamma\backslash \mathbf{H}^n_{\mathbb{R}})))} \int_{\Omega^j_{0\infty}} |h|^2(p) dv_p d\sigma_h.$$

Combining this last estimate with (79) yields

$$\sum_j \int_{\Omega^j_{0R_{n,k}}} \int_{S(\mathcal{H}^k(\Gamma\backslash \mathbf{H}^n_{\mathbb{C}}))} |h(p)|^2 d\sigma_h dv_p > \frac{1}{4},$$

and the result follows. □

We can now estimate outside the critical degree the dimension of the space of L^2-harmonic forms on a complete finite volume real-hyperbolic manifold.

Corollary 80. *Let Γ be co-finite volume. Decompose as a disjoint union*

$$\Gamma\backslash \boldsymbol{H}^n_{\mathbb{R}} = M_0 \cup (\cup_j C_j),$$

where each C_j is a cusp parameterized by $[0,\infty) \times T^{n-1}_j$. Let Ω^j_{uK} denote the subset of C_j parameterized by $[u,K) \times T^{n-1}_j$. For any $k < (n-1)/2$, there exists some $a_{n,k} > 0$ such that

$$b^k_2(\Gamma\backslash \boldsymbol{H}^n_{\mathbb{R}}) \leq c_{n,k}\big[Vol(\Gamma\backslash \boldsymbol{H}^n_{\mathbb{R}}) V_{min}(M_0)^{\frac{2k+1-n}{n-1}} + Vol(\cup_j C_j) V_{min}(\cup_j \Omega^j_{0R_{n,k}})^{\frac{2k+1-n}{n-1}}\big].$$

Proof. As in the proof of Corollary 76, if there exists a point $\bar{p} \in \pi^{-1}(M_0)$ such that

$$\binom{n}{k} \int_{S(\mathcal{H}^k(\Gamma\backslash \mathbf{H}^n_{\mathbb{R}}))} |h(\bar{p})|^2 d\sigma_h \geq \frac{1}{2Vol(\Gamma\backslash \mathbf{H}^n_{\mathbb{R}})},$$

the usual Price and Moser implies that there exists a constant $d_{n,k} > 0$ such that

$$b^k_2(\Gamma\backslash \mathbf{H}^n_{\mathbb{R}}) \leq d_{n,k} Vol(\Gamma\backslash \mathbf{H}^n_{\mathbb{C}}) V_{min}(M_0)^{\frac{2k+1-n}{n-1}}.$$

Suppose now that for any $\bar{p} \in \pi^{-1}(M_0)$ we have

$$\binom{n}{k} \int_{S(\mathcal{H}^k(\Gamma\backslash \mathbf{H}^n_{\mathbb{C}}))} |h(\bar{p})|^2 d\sigma_h < \frac{1}{2Vol(\Gamma\backslash \mathbf{H}^n_{\mathbb{R}})}.$$

By Proposition 73, there exists $\bar{p} \in \pi^{-1}(\cup_j \Omega^j_{0R_{n,k}})$ with

$$\binom{n}{k} \int_{S(\mathcal{H}^k(\Gamma\backslash \mathbf{H}^n_{\mathbb{R}}))} |h(\bar{p})|^2 d\sigma_h \geq \frac{1}{4\sum_j Vol(\Omega^j_{0R_{n,k}})}.$$

By Price and Moser, for some constant $e_{n,k} > 0$,

$$b^k_2(\Gamma\backslash \mathbf{H}^n_{\mathbb{R}}) \leq e_{n,k} V_{min}(\cup_j \Omega^j_{0R_{n,k}})^{\frac{2k+1-n}{n-1}} \Big(4\sum_j Vol(\Omega^j_{0R_{n,k}})\Big).$$

The proof follows upon setting $a_{n,k} := \max\{d_{n,k}, e_{n,k}\}$. □

9 Critical Degree for Real-Hyperbolic Manifolds

Corollary 80 does not cover the case $b^k_2(\Gamma\backslash \mathbf{H}^n_{\mathbb{R}})$ when $n = 2k+1$. For this critical degree case, Equation (57) reduces to

$$\int_{T^a_s} \Big(\frac{1}{2} - \mu_\alpha(s)\Big)|\alpha|^2 d\sigma = \int_{\Omega^a_{s\infty}} \mu_\alpha |\alpha|^2 dv > 0, \tag{81}$$

which again implies $0 \leq \mu(s) < \frac{1}{2}$. Unfortunately, this identity is difficult to use without having control of $\mu(s)$. Thus, we Fourier expand to obtain additional information and control on the size of an L^2-harmonic form of critical degree on a hyperbolic cusp.

9.1 Fourier Primitives

Consider a standard hyperbolic cusp, say C_a, of a complete finite volume real-hyperbolic manifold. This is isometric to $[0,\infty) \times T^a$, where $T^a = \Lambda_a \backslash \mathbb{R}^{n-1}$ is an $(n-1)$-dimensional real torus associated to a full rank lattice of translations Λ_a acting on $\mathbb{R}^{n-1}$. Moreover, C_a is naturally equipped with the metric $g_{-1} = ds^2 + e^{-2s} g_{T^a}$. Let $\{t^k\}_{k=1}^{n-1}$ be Euclidean coordinates on $\mathbb{R}^{n-1}$. Given a strongly harmonic L^2-form h of degree k on C_a, we Fourier expand

$$h = \sum_{v\in\Lambda^*_a} h^v, \tag{82}$$

where for any k

$$L_{\frac{\partial}{\partial t^k}} h^v = 2\pi i v_k h^v.$$

For $v \neq 0$, set

$$b^v := \sum_k \frac{v_k}{2\pi i |v|^2} i_{\frac{\partial}{\partial t^k}} h^v.$$

Then

$$db^v = \sum_k \frac{v_k}{2\pi i |v|^2} L_{\frac{\partial}{\partial t^k}} h^v = h^v. \tag{83}$$

For any $R_1 > R_2 > 0$, by (83) we have

$$\begin{aligned}\int_{\Omega^a_{R_1R_2}} |h|^2 dv &= \int_{\Omega^a_{R_1R_2}} \sum_{v\in\Lambda_a^*\backslash\{0\}} \langle db^v, h^v\rangle dv + \int_{\Omega^a_{R_1R_2}} |h^0|^2 dv \\ &= \int_{T^a_{R_2}} \sum_{v\in\Lambda_a^*\backslash\{0\}} \langle ds\wedge b^v, h^v\rangle dv - \int_{T^a_{R_1}} \sum_{v\in\Lambda_a^*\backslash\{0\}} \langle ds\wedge b^v, h^v\rangle dv \\ &+ \int_{\Omega^a_{R_1R_2}} |h^0|^2 dv, \end{aligned} \tag{84}$$

where in the second equality we used the fact that $d^*h^v = 0$ for all $v \in \Lambda_a^*$. Since $h \in L^2$, taking the limit as $R_2 \to \infty$ in Equation (84) yields the estimate

$$\begin{aligned}\int_{\Omega^a_{R_1\infty}} |h-h^0|^2 dv &= -\int_{T^a_{R_1}} \sum_{v\in\Lambda_a^*\backslash\{0\}} \langle ds\wedge b^v, h^v\rangle dv \\ &\leq \int_{T^a_{R_1}} \sum_{v\in\Lambda_a^*\backslash\{0\}} \frac{e^{-R_1}}{2\pi|v|}|h^v|^2 dv. \end{aligned} \tag{85}$$

Setting

$$\boxed{\delta_{\Lambda_a} := \inf_{v\in\Lambda_a^*\backslash\{0\}} |v|,}$$

we have for any $R > 0$:

$$e^R \int_{\Omega^a_{R\infty}} |h-h^0|^2 dv \leq \frac{1}{2\pi\delta_{\Lambda_a}} \int_{T^a_R} |h-h^0|^2 d\sigma. \tag{86}$$

Integrating (86) from R_0 to ∞ yields

$$\int_{\Omega^a_{R_0\infty}} e^s|h-h^0|^2 dv \leq \left(e^{R_0} + \frac{1}{2\pi\delta_\Lambda}\right) \int_{\Omega^a_{R_0\infty}} |h-h^0|^2 d\sigma. \tag{87}$$

Next, we need to give a bound on the zero mode h^0. It has the form

$$h^0 = h^0_{0I}(s)dt^I + h^0_{1J}(s)ds\wedge dt^J,$$

where we use multi-index notation $dt^I = dt^{i_1}\wedge\cdots\wedge dt^{i_{|I|}}$. Since $dh^0 = 0$, $h^0_{0I}(s)$ is constant. Next, let $*$ be the Hodge operator associated to the hyperbolic metric acting on $k = (n-1)/2$-forms. We can write

$$*h^0 = \pm h^0_{0I} ds\wedge dt^{I^c} \pm e^{-2s}h^0_{1J}(s)dt^{J^c},$$

where by I^c and J^c we indicate the multi-index complement in $\mathbb{R}^{n-1}$. Since $d^*h^0 = \pm(*d*)h^0 = 0$, we have that $d(*h^0) = 0$. This implies $e^{-2s}h^0_{1J}(s)$ is constant so that

$$h^0_{1J}(s) = h^0_{1J}(0)e^{-2s}.$$

Hence for $k = \frac{n-1}{2}$, we have the equality

$$|h^0(s)|^2 = |h^0_{0I}(0)|^2 e^{\frac{(n-1)}{2}s} + |h^0_{1J}(0)|^2 e^{\frac{(n+1)}{2}s},$$

and we conclude that

$$\int_{C_a} |h^0|^2 dv < \infty \quad \Leftrightarrow \quad h^0_{0I}(0) = 0, \quad h^0_{1J}(0) = 0, \quad \Rightarrow \quad h^0 = 0.$$

Equation (87) now simplifies to

$$\int_{\Omega^a_{R_0\infty}} e^s |h|^2 dv \leq \Big(e^{R_0} + \frac{1}{2\pi\delta_{\Lambda_a}}\Big) \int_{\Omega^a_{R_0\infty}} |h|^2 dv.$$

We summarize this discussion with a lemma.

Lemma 88. *Let C_a be a standard real-hyperbolic cusp of dimension n, and let $h \in L^2$ be a strongly harmonic form of degree $(n-1)/2$ on it. For any $R_0 \in [0, \infty)$, we have the integral inequality:*

$$\int_{\Omega^a_{R_0\infty}} e^{s-R_0} |h|^2 dv \leq \Big(1 + \frac{e^{-R_0}}{2\pi\delta_{\Lambda_a}}\Big) \int_{\Omega^a_{R_0\infty}} |h|^2 dv. \tag{89}$$

Remark 90. We remark that Lemma 88 holds true for non-standard real-hyperbolic cusps as well. To see this, let C be a real-hyperbolic cusp diffeomorphic to $[0, \infty) \times F$, where $F = \Lambda\backslash\mathbb{R}^{n-1}$ is an $(n-1)$-dimensional flat manifold associated to a full rank lattice Λ acting on $\mathbb{R}^{n-1}$. The lattice Λ does *not* need to be a lattice of translations in $\mathbb{R}^{n-1}$, but by Bieberbach's theorem (*cf.* Chapter 3 in [Wol11]), we can always find a finite index lattice of translations of minimal index $\Lambda_a \leq \Lambda$s such that $[\Lambda : \Lambda_a] \leq C_{n-1}$, where C_{n-1} is a positive constant depending on the dimension only. Consider then the associated Riemannian cover of index $[\Lambda : \Lambda_a]$

$$p : T^a \times [0, \infty) \longrightarrow F \times [0, \infty),$$

where T^a is the $(n-1)$-torus associated to the lattice of translations Λ_a. Given an L^2 strongly harmonic form h on C, p^*h is L^2 and strongly harmonic on $[0, \infty) \times T_a$, with the pulled back metric. Apply Lemma 88 to p^*h and use the multiplicativity of the integrals to conclude

$$\int_{\Omega_{R_0\infty}} e^{s-R_0} |h|^2 dv \leq \Big(1 + \frac{e^{-R_0}}{2\pi\delta_{\Lambda_a}}\Big) \int_{\Omega_{R_0\infty}} |h|^2 dv, \tag{91}$$

where now $\Omega_{R_0\infty} = [R_0, \infty) \times F$.

We now combine the inequality in (89) and Remark 90 with the peaking argument presented in Section 5 to obtain the following corollary.

Corollary 92. *Let $(\Gamma\backslash \boldsymbol{H}^n_{\mathbb{R}}, g_{\mathbb{R}})$ be a complete finite volume real-hyperbolic manifold of odd dimension $n = 2k+1$ for some integer $k \geq 1$. For $R_0 \in [0,\infty)$, on any cusp $C_a = [0,\infty)\times F$ (not necessarily standard) we have*

$$\int_{\Omega^a_{(R_0+1)\infty}} \int_{S(\mathcal{H}^k(\Gamma\backslash \boldsymbol{H}^n_{\mathbb{R}}))} |h|^2 d\sigma_h dv \\ \leq \frac{e^{-R_0}}{2\pi\delta_{\Lambda_a}}\Big(e - 1 - \frac{e^{-R_0}}{2\pi\delta_{\Lambda_a}}\Big)^{-1} \int_{\Omega^a_{R_0(R_0+1)}} \int_{S(\mathcal{H}^k(\Gamma\backslash \boldsymbol{H}^n_{\mathbb{R}}))} |h|^2 d\sigma_h dv,$$

where Λ_a is a lattice of translation of minimal index in Λ, where $F := \Lambda\backslash\mathbb{R}^{n-1}$.

Proof. If the cusp has a torus cross section, by Lemma 88 we have

$$\int_{\Omega^a_{(R_0+1)\infty}} e^{s-R_0}|h|^2 dv \leq \Big(1 + \frac{e^{-R_0}}{2\pi\delta_{\Lambda_a}}\Big) \int_{\Omega^a_{R_0\infty}} |h|^2 dv - \int_{\Omega^a_{R_0(R_0+1)}} e^{s-R_0}|h|^2 dv,$$

so that

$$e\int_{\Omega^a_{(R_0+1)\infty}} |h|^2 dv \leq \Big(1+\frac{e^{-R_0}}{2\pi\delta_{\Lambda_a}}\Big)\Big(\int_{\Omega^a_{R_0(R_0+1)}} |h|^2 dv + \int_{\Omega^a_{(R_0+1)\infty}} |h|^2 dv\Big) - \int_{\Omega^a_{R_0(R_0+1)}} |h|^2 dv,$$

which implies

$$\Big(e - 1 - \frac{e^{-R_0}}{2\pi\delta_{\Lambda_a}}\Big)\int_{\Omega^a_{(R_0+1)\infty}} |h|^2 dv \leq \frac{e^{-R_0}}{2\pi\delta_{\Lambda_a}} \int_{\Omega^a_{R_0(R_0+1)}} |h|^2 dv.$$

We then conclude by integrating over $S(\mathcal{H}^k(\Gamma\backslash \mathbf{H}^n_{\mathbb{R}}))$. If the cusp is not standard, let $F = \Lambda\backslash\mathbb{R}^{n-1}$ be the associated flat cross-section, and let Λ_a be an abelian subgroup of minimal index in Λ, and conclude the argument by appealing to Remark 90. □

When dealing with lattices that are close to a dilation of $\mathbb{Z}^{n-1}$ (as one might expect to find in congruence subgroup cases) the preceding estimate can be used to extend our Betti number estimates to the cusped case. In more general lattices, more care is required balancing the size of δ_{Λ_a}, volumes, and local injectivity radii.

9.2 Cusps and Lattices

Let $(M^n := \Gamma\backslash \mathbf{H}^n_{\mathbb{R}}, g_{\mathbb{R}})$ be a complete finite volume real-hyperbolic manifold with the property that all of its cusps are *standard*. Recall that one can always arrange this to be the case by passing to a finite regular cover, see for example [Hum98]. Also, assume the dimension to be $n = 2k+1$ for some integer $k \geq 1$. As in Section 5, we let

$$\pi : S(\Lambda^k TM) \to M$$

be the unit sphere bundle, where the total space is equipped with the metric induced from $g_{\mathbb{R}}$ and the standard metric on the spherical fiber whose volume is normalized to one. We denote by $d\bar{v}$ the associated Riemannian measure on the total space.

Next, decompose M^n as the disjoint union

$$M = M_0 \cup (\cup_a C_a),$$

where each C_a is a cusp parametrized by $[0, \infty) \times T^a$. Set

$$M_k := M_0 \cup (\cup_a ([0, k) \times T^a)), \quad C_{a,k} := [k, \infty) \times T^a.$$

Thus, we have

$$Vol(C_{a,k}) = e^{-k(n-1)} Vol(C_{a,0}).$$

Let $\nu(M)$ be the smallest integer such that

$$\binom{n}{k} \int_{\pi^{-1}(M_{\nu(M)})} \int_{S(\mathcal{H}^k(M))} |h(\bar{p})|^2 d\sigma_h d\bar{v}_{\bar{p}} \geq \frac{\sum_{j=1}^{\nu(M)+1} j^{-2}}{2\zeta(2)}. \tag{93}$$

Here $\zeta(2)$ is the Riemann zeta function evaluated at 2, introduced to normalize the right hand side of (93) to be smaller than $\frac{1}{2}$. Equation (48) ensures that $\nu(M)$ is well-defined. If $\nu(M) > 0$, we have

$$\binom{n}{k} \int_{\pi^{-1}(M_{\nu(M)-1})} \int_{S(\mathcal{H}^k(M))} |h(\bar{p})|^2 d\sigma_h d\bar{v}_{\bar{p}} < \frac{\sum_{j=1}^{\nu(M)} j^{-2}}{2\zeta(2)}, \tag{94}$$

which then implies

$$\binom{n}{k} \int_{\pi^{-1}(M_{\nu(M)} \setminus M_{\nu(M)-1})} \int_{S(\mathcal{H}^k(M))} |h(\bar{p})|^2 d\sigma_h d\bar{v}_{\bar{p}} \geq \frac{(\nu(M)+1)^{-2}}{2\zeta(2)}.$$

Hence there exists $\bar{p} \in \pi^{-1}(M_{\nu(M)} \setminus M_{\nu(M)-1})$ such that

$$\binom{n}{k} \int_{S(\mathcal{H}^k(M))} |h(\bar{p})|^2 d\sigma_h \geq \frac{(\nu(M)+1)^{-2}}{2\zeta(2) Vol(M_{\nu(M)} \setminus M_{\nu(M)-1})}. \tag{95}$$

At the cost of decreasing constants on the right hand side of (95), it is convenient to further restrict the location of $\bar{p}$ satisfying (a modified) (95). To effect this, define the index set $I(\nu(M))$ so that

$$\boxed{a \in I(\nu(M)) \quad \Longleftrightarrow \quad \frac{e^{2-\nu(M)}}{2\pi\delta_{\lambda_a}} < \frac{1}{4(\nu(M)+1)^2}.}$$

Next, we claim that

$$\binom{n}{k} \int_{\pi^{-1}((M_{\nu(M)} \setminus M_{\nu(M)-1}) \cap (\cup_{a \in I(\nu(M))} C_a))} \int_{S(\mathcal{H}^k(M))} |h(\bar{p})|^2 d\sigma_h d\bar{v}_{\bar{p}} < \frac{1}{8(\nu(M)+1)^2}. \tag{96}$$

To verify this, use Corollary 92 with $R_0 = \nu(M) - 2$ to obtain

$$\sum_{a \in I(\nu(M))} \int_{\pi^{-1}(\Omega^a_{(\nu(M)-1)\infty})} \int_{S(\mathcal{H}^k(M))} |h(\bar{p})|^2 d\sigma_h d\bar{v}_{\bar{p}} < \frac{1}{4(\nu(M)+1)^2} \sum_{a \in I(\nu(M))} \int_{\pi^{-1}(\Omega^a_{(\nu(M)-2)(\nu(M)-1)})} \int_{S(\mathcal{H}^k(M))} |h(\bar{p})|^2 d\sigma_h d\bar{v}_{\bar{p}}. \tag{97}$$

Seeking a contradiction, we note that if Equation (96) does not hold, then Equation (97) implies

$$\sum_{a \in I(\nu(M))} \binom{n}{k} \int_{\pi^{-1}(\Omega^a_{(\nu(M)-2)(\nu(M)-1)})} \int_{S(\mathcal{H}^k(M))} |h(\bar{p})|^2 d\sigma_h d\bar{v}_{\bar{p}} \geq \frac{4(\nu(M)+1)^2}{8(\nu(M)+1)^2} = \frac{1}{2}.$$

This contradicts Equation (94), and we have verified that (96) holds. Hence, by Equation (48) we also know that

$$\binom{n}{k} \int_{\pi^{-1}((M_{\nu(M)} \setminus M_{\nu(M)-1}) \cap ((\cup_{a \in I(\nu(M))} C_a)^c)} \int_{S(\mathcal{H}^k(M))} |h(\bar{p})|^2 d\sigma_h d\bar{v}_{\bar{p}} \geq \frac{1}{8(\nu(M)+1)^2},$$

and consequently there exists $\bar{p} \in \pi^{-1}(M_{\nu(M)} \setminus M_{\nu(M)-1}) \cap (\cup_{a \in I(\nu(M))} C_a)^c)$ such that

$$\binom{n}{k} \int_{S(\mathcal{H}^k(M))} |h(\bar{p})|^2 d\sigma_h \geq \frac{(\nu(M)+1)^{-2}}{8Vol(M_{\nu(M)} \setminus M_{\nu(M)-1})}. \tag{98}$$

Applying Corollary 45 yields

$$\begin{aligned} b_2^k(M) &= \frac{\max_{h \in S^{b-1}} |h(\bar{p})|^2}{\int_{S^{b-1}} |h(\bar{p})|^2 d\sigma_h} \\ &\leq 8\binom{n}{k} (\nu(M)+1)^2 Vol(M_{\nu(M)} \setminus M_{\nu(M)-1}) \max_{h \in S(\mathcal{H}^k(M))} |h(\bar{p})|^2 \\ &\leq 8\binom{n}{k} e^{-(n-1)\nu(M)} (\nu(M)+1)^2 Vol(M_1 \setminus M_0) \max_{h \in S(\mathcal{H}^k(M))} |h(\bar{p})|^2, \end{aligned} \tag{99}$$

where in the last inequality we used the explicit form of the hyperbolic metric on cusps. Now, the injectivity radius at $\pi(\bar{p})$ is bigger than or equal to $e^{-\nu(M)} \mathrm{inj}_{M_0}$, where we set

$$\boxed{\mathrm{inj}_{M_0} := \inf_{x \in M_0} \mathrm{inj_x},}$$

cf. Definition 1. Hence if $e^{-\nu(M)} \mathrm{inj_{M_0}} \geq 1$, we have

$$\max_{h \in S(\mathcal{H}^k(M))} |h(p)|^2 \leq \frac{K_n}{e^{-\nu(M)} \mathrm{inj_{M_0}}},$$

for some constant K_n depending on the dimension n and $g_{\mathbb{R}}$ only. This follows from Price inequalities for harmonic forms on real-hyperbolic manifolds (see [DS17, Corollary 108]). Thus, under this assumption on $e^{-\nu(M)} \mathrm{inj}_{M_0}$, we have

$$b_2^k(M) \leq D_n (\nu(M)+1)^2 e^{-(n-2)\nu(M)} \frac{Vol(M_1 \setminus M_0)}{\mathrm{inj_{M_0}}}. \tag{100}$$

In particular, under this restrictive injectivity assumption, we find that

$$\frac{b_2^{\frac{n-1}{2}}(M)}{Vol(M)} \leq \frac{L(n)}{\mathrm{inj_{M_0}}} \tag{101}$$

for some positive constant $L(n)$ depending on the dimension only. This is analogous to Equation (120) in Theorem 18 for compact real-hyperbolic manifolds, where the global injectivity radius of the compact hyperbolic manifold is replaced with the injectivity radius, inj_{M_0}, of the thick part of the manifold with cusps.

We now study the remaining case $e^{-\nu(M)}\mathrm{inj_{M_0}} < 1$. Consider the torus $T^a = \Lambda_a \backslash \mathbb{R}^{n-1}$. We recall some invariants of lattices of translations acting on Euclidean spaces. For $i \in \{1, ..., n-1\}$, set

$$\lambda_i(\Lambda_a) := \inf\{r : \dim \operatorname{span} \Lambda_a \cap \bar{B}(0,r) \geq i\}, \tag{102}$$

Then $\lambda_1(\Lambda_a)$ is the injectivity radius of T^a, and $\delta_{\Lambda_a} = \lambda_1(\Lambda_a^*)$. We recall the lattice relation for a rank $n-1$ lattice:

$$\delta_{\Lambda_a}\lambda_{n-1}(\Lambda_a) \geq 1. \tag{103}$$

To see this, let $v \in \Lambda_a^*$ such that $|v| = \delta_{\Lambda_a} = \lambda_1(\Lambda_a^*)$. Consider any set $w_1, ..., w_{n-1}$ of $n-1$ linearly independent vectors in $(\Lambda_a^*)^* = \Lambda_a$. By construction of the dual lattice, there exists $i \in \{1, ..., n-1\}$ such that $\langle w_i, v\rangle \neq 0$, so that

$$\langle w_i, v\rangle \in \mathbb{Z} \quad \Rightarrow \quad |w_i| \geq \frac{1}{|v|},$$

and the proof of (103) is complete. Next, for $y \in \Omega^a_{(\nu(M)-1)\nu(M)}$ with $a \notin I(\nu(M))$, we have by definition

$$\frac{1}{\delta_{\Lambda_a}} \geq \frac{2\pi e^{(\nu(M)-2)}}{4(\nu(M)+1)^2}.$$

Then by (103), we have

$$\lambda_{n-1}(\Lambda_a) \geq \frac{e^{\nu(M)}}{5(\nu(M)+1)^2}. \tag{104}$$

Let $\tilde{C}_a$ denote the universal cover of C_a, and let $q : \tilde{C}_a \to C_a$ denote the projection map. Recall that our near term goal is to estimate from above $|h(\bar{p})|^2$ for

$$\pi(\bar{p}) \in C_a \cap (M_{\nu(M)} \setminus M_{\nu(M)-1}), \quad a \notin I(\nu(M)),$$

so that we can exploit (99). We will use elliptic estimates to obtain this pointwise bound from an integral bound for $|h|^2$ in a geodesic ball about $\pi(\bar{p})$. Such estimates become worse as the radius of the ball decreases. Hence we will lift h to $\tilde{C}_a$, where we can take the radius to be large. We next bound the L^2-norm of the lift in terms of the L^2-norm of h. This entire discussion is nontrivial only when the ambient lattices are far from square.

In order to estimate these integrals on $\tilde{C}_a$, we need to bound the number of elements in $q^{-1}(y) \cap B(o, \frac{R}{2})$, $R \leq 1$ to be determined, for any $o \in \tilde{C}_a \cap q^{-1}(M_{\nu(M)} \setminus M_{\nu(M)-1})$. Since this intersection is empty unless there exists $y_0 \in q^{-1}(y)$ such that $B(o, \frac{R}{2}) \subset B(y_0, R)$, it suffices to estimate $q^{-1}(y) \cap B(y_0, R)$. Write for $y \in T^a_s$, $s \in [\nu(M) - 1 - \frac{R}{2}, \nu(M) + \frac{R}{2}]$,

$$q^{-1}(y) \cap B(y_0, R) = \{y_0 + v : v \in \Lambda_a \cap B(0, e^s R)\}.$$

Here we recall that Λ_a is a lattice in T_0, and the metric rescales the lattice by a factor of e^{-s} in T^a_s. For any R such that $e^s R < \lambda_{n-1}(\Lambda_a)$, we have $\{v \in \Lambda_a \cap B(0, e^s R)\}$ is a subset of an $(n-2)$-dimensional linear subspace, say Σ^a_{n-2}. So, we now restrict R to satisfy

$$e^{\nu(M)+1} R < \lambda_{n-1}(\Lambda_a).$$

Hence we can estimate the cardinality of the set $\{v \in \Lambda_a \cap B(0, e^s R)\}$ as follows. Define the larger set $S := \Lambda_a \cap B(0, \lambda_{n-1}(\Lambda_a))$. Given two distinct vectors v and w in S, we have

$$B\Big(v, \frac{\lambda_1(\Lambda_a)}{2}\Big) \cap B\Big(w, \frac{\lambda_1(\Lambda_a)}{2}\Big) = \emptyset$$

so that

$$\begin{aligned}
&\sum_{v \in S} vol\Big(B\Big(v, \frac{\lambda_1(\Lambda_a)}{2}\Big) \cap \Sigma^a_{n-2}\Big) \leq vol\Big(B\Big(0, e^s R + \frac{\lambda_1(\Lambda_a)}{2}\Big) \cap \Sigma^a_{n-2}\Big) \\
&\Longrightarrow |S| \lambda_1(\Lambda_a)^{n-2} \leq C_n \Big(e^{\nu(M)} R + \frac{\lambda_1(\Lambda_a)}{2}\Big)^{n-2},
\end{aligned}$$

for some positive constant C_n. We therefore have the estimate

$$|q^{-1}(y) \cap B(y_0, R)| \leq C_n \lambda_1(\Lambda_a)^{2-n} \Big(e^{\nu(M)} R + \frac{\lambda_1(\Lambda_a)}{2}\Big)^{n-2}. \tag{105}$$

Let $h \in \mathcal{H}^{\frac{n-1}{2}}(M)$. Standard elliptic estimates (see for example [DS17, Lemma 51]) give the following estimate for some $a(n) > 0$, for $x \in C_a$, and for $x_0 \in q^{-1}(x)$:

$$|h|^2(x) = |q^* h|^2(x_0) \leq a(n) \Big(1 + \frac{2}{R}\Big)^n \|q^* h\|^2_{L^2\left(B_{\frac{R}{2}}(x_0)\right)}.$$

By (105), for $x \in C_a \cap (M_{\nu(M)} \setminus M_{\nu(M)-1})$, the map $q : B_{\frac{R}{2}}(x_0) \to C_a$ is at most

$$\Big[C_n \lambda_1(\Lambda_a)^{2-n} \Big(e^{\nu(M)} R + \frac{\lambda_1(\Lambda_a)}{2}\Big)^{n-2}\Big] \quad \text{to} \quad 1,$$

where by $[\cdot]$ we denote the integer part. Therefore

$$\|q^* h\|^2_{L^2\left(B_{\frac{R}{2}}(x_0)\right)} \leq C_n \lambda_1(\Lambda_a)^{2-n} \Big(e^{\nu(M)} R + \frac{\lambda_1(\Lambda_a)}{2}\Big)^{n-2} \|h\|^2_{L^2(M)},$$

so that there exists some $D(n) > 0$ such that

$$|h(x)|^2 \leq D(n) \Big(1 + \frac{2}{R}\Big)^n \Big(e^{\nu(M)} \lambda_1(\Lambda_a)^{-1} R + \frac{1}{2}\Big)^{n-2} \|h\|^2_{L^2(M)}. \tag{106}$$

Now combining Equations (99) and (106), we obtain for some constant $D_2(n) > 0$,

$$b_2^k(M) \tag{107}$$
$$\leq D_2(n)e^{-\nu(M)}(\nu(M)+1)^2 Vol(M_1 \setminus M_0)\Big(1+\frac{2}{R}\Big)^n \Big(\lambda_1(\Lambda_a)^{-1}R + \frac{e^{-\nu(M)}}{2}\Big)^{n-2}.$$

We now choose

$$R := \frac{1}{20(\nu(M)+1)^2}.$$

By Equation (104) this R satisfies our earlier constraint: $e^{\nu(M)+1}R < \lambda_{n-1}(\Lambda_a)$. Hence Equation (107) yields

$$b_2^k(M) \tag{108}$$
$$\leq D_2(n)e^{-\nu(M)} Vol(M_1 \setminus M_0)\Big(1+40(\nu(M)+1)^2\Big)^{n+1}\Big(\frac{\lambda_1(\Lambda_a)^{-1}}{20(\nu(M)+1)^2} + \frac{e^{-\nu(M)}}{2}\Big)^{n-2}.$$

We summarize this discussion with a theorem.

Theorem 109. *Let M be a $2k+1$ dimensional, complete finite volume real-hyperbolic manifold. Write M as a disjoint union*

$$M = M_0 \cup (\cup_a C_a),$$

where each C_a is a cusp. Then there exists $B(k) > 0$ depending only on k so that if $\mathrm{inj}_{\mathrm{M_0}} \geq 1$ *then*

$$b_2^k(M) \leq B(k)\frac{Vol(M)}{\mathrm{inj}_{\mathrm{M_0}}}. \tag{110}$$

Proof. If $\nu(M) = 0$, we have

$$\binom{n}{k}\int_{\pi^{-1}(M_0)}\int_{S(\mathcal{H}^k(M))} |h(\bar{p})|^2 d\sigma_h d\bar{v}_{\bar{p}} \geq \frac{1}{2\zeta(2)} > \frac{1}{4},$$

so that there exists $\bar{p} \in \pi^{-1}(M_0)$ such that $\binom{n}{k}\int_{S(\mathcal{H}^k(M))}|h(p)|^2 d\sigma_h \geq \frac{1}{4Vol(M_0)}$. We now apply Lemma 25 to obtain for $h \in S(\mathcal{H}^k(M))$:

$$\int_{S_r(\pi(\bar{p}))}\Big(\frac{1}{2}-\mu_h\Big)|h|^2 d\sigma = \int_{B_r(\pi(\bar{p}))}\mu_h|h|^2 dv, \tag{111}$$

for any $r \leq inj_{\pi(\bar{p})}$. Since $\mu_h(0) < \frac{1}{2}$ by [DS17, Lemma 18], Equation (111) implies $\mu_h(r) \in (0, \frac{1}{2})$ for all r. Consequently, $\int_{S_r(\pi(\bar{p}))}\Big(\frac{1}{2}-\mu_h\Big)|h|^2 d\sigma$ is monotonically increasing. Then we have for $\frac{1}{2} \leq \tau < \mathrm{inj}_{\mathrm{M_0}}$

$$\int_{B_\tau(\pi(\bar{p}))}|h|^2 dv \geq \int_{\frac{1}{2}}^{\tau}\int_{S_r(\pi(\bar{p}))}|h|^2 d\sigma dr > 2\int_{\frac{1}{2}}^{\tau}\int_{S_r(\pi(\bar{p}))}\Big(\frac{1}{2}-\mu_h\Big)|h|^2 d\sigma dr$$
$$> 2\int_{\frac{1}{2}}^{\tau}\int_{S_{\frac{1}{2}}(\pi(\bar{p}))}\Big(\frac{1}{2}-\mu_h\Big)|h|^2 d\sigma dr = 2\Big(\tau - \frac{1}{2}\Big)\int_{B_{\frac{1}{2}}(\pi(\bar{p}))}\mu_h|h|^2 dv. \tag{112}$$

In the proof of [DS17, Corollary 108], we show that for $r \in [0,1]$, $\mu_h(r) \geq \beta(n)$, for some $\beta(n) > 0$ independent of h and M. Hence Equation (112) coupled to the elliptic estimate given in [DS17, Lemma 51] yield for $\nu(M) = 0$, and for some positive constants $c(n)$ and $d(n)$,

$$b_2^k(M) \leq c(n)Vol(M_0)|h(\pi(\bar{p}))|^2 \leq d(n)Vol(M_0) \int_{B_{\frac{1}{2}}(\pi(\bar{p}))} |h|^2 dv$$

$$\leq \frac{2d(n)\beta(n)^{-1}}{(\mathrm{inj_{M_0}} - \frac{1}{2})} \int_{B_{\mathrm{inj_{M_0}}}(\pi(\bar{p}))} |h|^2 dv \cdot Vol(M_0) \leq \frac{2d(n)\beta(n)^{-1}}{(\mathrm{inj_{M_0}} - \frac{1}{2})} \cdot Vol(M_0).$$

Finally if $\nu(M) > 0$, then Equation (108) bounds from above b_2^k with a constant multiple of $\frac{Vol(M_0)}{\mathrm{inj_{M_0}}}$, and the result follows. □

10 L^2-Cohomology on a Tower of Coverings

We now study the growth of L^2-cohomology on towers of real- and complex-hyperbolic manifolds with cusps.

10.1 Fattening the Thick Part

We show that, up to finite cover, we can fatten the injectivity radius of the thick part of the quotient of a Hadamard manifold with residually finite fundamental group. This is the analog for a manifold with cusps of Theorem 2.1 in [DW78].

Let (M, g) be a complete finite volume Riemannian manifold such that $-b^2 \leq \sec_g \leq -a^2$, with $a, b \neq 0$. Denote by $(\widetilde{M}, \tilde{g})$ the Riemannian universal cover of (M, g). Let $\Gamma = \pi_1(M)$ be the associated lattice in $\mathrm{Iso}(\widetilde{M})$, so that $M = \Gamma \backslash \widetilde{M}$. Assume that Γ is residually finite. Let $\{\Gamma_k\}_{k \in \mathbb{N}}$ be a cofinal filtration of $\Gamma =: \Gamma_0$ by finite index normal subgroups. We define the continuous Γ-invariant function $d_\Gamma : \widetilde{M} \to [0, \infty)$ by

$$d_\Gamma(p) = \min\{d_{\tilde{g}}(p, \gamma p) : \gamma \in \Gamma, \gamma \neq 1\}.$$

Thus $d_\Gamma(p)$ is twice the injectivity radius of $\omega(p)$ in M, where $\omega : \widetilde{M} \to M$ is the universal covering map. Given $\delta > 0$, define the closed subset $\widetilde{M_\delta} \subset \widetilde{M}$ by

$$\widetilde{M_\delta} := \{q \in \widetilde{M} \quad | \quad d_\Gamma(q) \geq \delta\}.$$

Also, define the associated closed subset of M:

$$M_\delta := \{p \in M \quad | \quad 2 \cdot \mathrm{inj_p} \geq \delta\}.$$

Let $\{q_k : M_k \to M\}_{k \in \mathbb{N}}$ be the sequence of regular Riemannian coverings of M associated to the cofinal filtration $\{\Gamma_k\}_{k \in \mathbb{N}}$. For each k, define the numerical invariant

$$r_{k,\delta} := \min\{d(z, \gamma_k z) \quad | \quad z \in \widetilde{M_\delta}, \quad \gamma_k \in \Gamma_k, \quad \gamma_k \neq 1\}.$$

Notice that by definition we have

$$r_{0,\delta} = \delta, \text{ and } \quad r_{k,\delta'} \leq r_{k,\delta}$$

for any $\delta' \leq \delta$ and for any k.

Lemma 113. *For any $k \geq 0$, denote by $\omega_k : (\widetilde{M}, \tilde{g}) \to (M_k, q_k^*(g))$ the Riemannian universal covering map. For any $z \in \widetilde{M}_\delta$, we have that*

$$\omega_k : B\Big(z; \frac{r_{k,\delta}}{2}\Big) \cap \widetilde{M}_\delta \to \omega_k\Big(B\Big(z; \frac{r_{k,\delta}}{2}\Big) \cap \widetilde{M}_\delta\Big) \tag{114}$$

is an isometry. Finally, we have

$$\lim_{k\to\infty} r_{k,\delta} = \infty. \tag{115}$$

Proof. If Equation (115) does not hold, there exist infinite sequences $z_k \in \widetilde{M}_\delta$ and $\gamma_k \in \Gamma_k \setminus \{1\}$ such that $d(z_k, \gamma_k z_k) \leq 2N$ for some positive constant N. Let D be a fundamental domain for M_δ. Thus, $D \subset \widetilde{M}_\delta$ is a connected open set such that $\omega : D \to M_\delta$ is injective and $\omega : \bar{D} \to M_\delta$ is surjective, where $\bar{D}$ is the closure of D in $\widetilde{M}_\delta$. Thus for all k, there exists $g_k \in \Gamma$ such that $g_k z_k \in \bar{D}$. Define $z'_k = g_k z_k$ and $\gamma'_k = g_k \gamma_k g_k^{-1}$. Since Γ_k is a normal subgroup of Γ, we have that $\gamma'_k \in \Gamma_k$. By compactness of $\bar{D}$, there exists a subsequence $\{z'_{k_j}\}$ converging to a point $\bar{z} \in \bar{D}$. Now since

$$d(z'_k, \gamma'_k z'_k) = d(g_k z_k, g_k \gamma_k z_k) = d(z_k, \gamma_k z_k),$$

we have that

$$d(\bar{z}, \gamma'_{k_j} \bar{z}) \leq 2d(\bar{z}, z'_{k_j}) + 2N.$$

Since $d(\bar{z}, z'_{k_j}) \to 0$, we conclude that, up to a subsequence, $\gamma'_{k_j} \bar{z}$ converges to a point $w \in \overline{B(\bar{z}; 2N)} \cap \widetilde{M}_\delta$ for some $\epsilon > 0$. This implies that

$$\omega(\bar{z}) = \omega(\gamma'_{k_j} \bar{z}) \longrightarrow \omega(w).$$

Thus, there exists $\gamma \in \Gamma$ such that $\gamma w = \bar{z}$. We therefore conclude

$$(\gamma'_{k_j} \cdot \gamma) w = \gamma'_{k_j} \bar{z} \longrightarrow w.$$

Now the action of Γ on $\widetilde{M}$ is properly discontinuous, so that $\gamma'_{k_j} \cdot \gamma = \{1\}$ for all j sufficiently large. Thus, we must have $\gamma = \{1\}$ which then implies the contradiction $\gamma'_{k_j} = \{1\}$. The proof of (115) is then complete. Equation (114) simply follows from the definition of $r_{k,\delta}$. □

We can now present the analog for manifold with cusps of Theorem 2.1 in [DW78]

Theorem 116. *Let $(M := \Gamma\backslash\widetilde{M}, g)$ be a complete finite volume quotient of a Hadamard manifold $(\widetilde{M}, \tilde{g})$. Assume $\Gamma = \pi_1(M)$ is residually finite. Given a cofinal filtration $\{\Gamma_i\}$ of Γ, denote by $\omega_i : M_i \to M$ the regular Riemannian cover of M associated to Γ_i equipped with the pull-back metric, say g_i. Decompose M as a disjoint union*

$$M = M_0 \cup (\cup_j C_j),$$

where M_0 is compact manifold with boundary and each C_j is a cusp. Define $M_0^i := \omega_i^{-1}(M_0)$, and let $\mathrm{inj}_{\mathrm{M_0^i}} := \inf_{\mathrm{x}\in\mathrm{M_0^i}} \mathrm{inj}_{\mathrm{g_i}}(\mathrm{x})$. We then have

$$\lim_{i\to\infty} \mathrm{inj}_{\mathrm{M_0^i}} = \infty. \tag{117}$$

Proof. Let $\{\delta_n\}$ be a sequence of real numbers converging to zero, and let $\{M_n\}$ be a sequence of closed subset in M defined by

$$M_n := \{p \in M \quad | \quad 2 \cdot \mathrm{inj_p} \geq \delta_\mathrm{n}\}.$$

We then have that

$$\lim_{n\to\infty} d_n = \infty,$$

where $d_n := dist(\partial M_0, \partial M_n)$. Given an integer $n \geq 1$, by Lemma 113 there exists an integer k_n such that

$$r_{k,\delta_n} \geq d_n,$$

for any $k \geq k_n$. We therefore have that

$$\mathrm{inj}_{\mathrm{M_0^k}} \geq \mathrm{d_n},$$

for any $k \geq k_n$. By letting $n \to \infty$ and recalling that by construction $d_n \to \infty$, we conclude that Equation (117) is satisfied. □

10.2 Asymptotic Behavior of L^2-Cohomology

In this section, we study the L^2-cohomology of complete finite volume hyperbolic manifolds on towers of coverings. We first extend Theorem 18 to real-hyperbolic manifolds with cusps.

Theorem 118. *Let $(M^n := \Gamma\backslash \boldsymbol{H}^n_{\mathbb{R}}, g_{\mathbb{R}})$, with Γ co-finite volume. Given a cofinal filtration $\{\Gamma_i\}$ of Γ, denote by $\pi_i : M_i \to M$ the regular Riemannian cover of M associated to Γ_i. Decompose M^n as a disjoint union*

$$M = M_0 \cup (\cup_j C_j)$$

where M_0 is a compact manifold with boundary and each C_j is a cusp. Define $M_0^i := \pi_i^{-1}(M_0)$, and assume $\mathrm{inj}_{\mathrm{M_0^i}} := \inf_{\mathrm{x}\in\mathrm{M_0^i}} \mathrm{inj}_{\mathrm{g_i}}(\mathrm{x}) \geq 1$. For any integer $1 \leq k < \frac{n-1}{2}$, there exists a positive constant $c_1(n,k)$ such that

$$\frac{b_2^k(M_i)}{Vol(M_i)} \leq c_1(n,k) V_{min}(M_0^i)^{-\frac{(n-1-2k)}{n-1}}. \tag{119}$$

In particular, the sub volume growth of the Betti numbers along the tower of coverings is exponential in $\mathrm{inj}_{\mathrm{M_0^i}}$. *For* $n = 2k+1$ *there is a positive constant* $c_2(n,k)$ *such that*

$$\frac{b_2^k(M_i)}{Vol(M_i)} \leq \frac{c_2(n,k)}{\mathrm{inj}_{\mathrm{M_0^i}}}. \tag{120}$$

Proof. Combine Corollary 80 and Theorem 109 with Theorem 116. □

We have the following analogous result for complex-hyperbolic manifolds with cusps.

Theorem 121. *Let* $(M := \Gamma\backslash \boldsymbol{H}^n_{\mathbb{C}}, g_{\mathbb{C}})$ *be complete finite volume. Given a cofinal filtration* $\{\Gamma_i\}$ *of* Γ, *denote by* $\pi_i : M_i \to M$ *the regular Riemannian cover of* M *associated to* Γ_i. *Decompose* M *as a disjoint union*

$$M = M_0 \cup (\cup_j C_j)$$

where M_0 *is a compact manifold with boundary and each* C_j *is a cusp. Define* $M_0^i := \pi_i^{-1}(M_0)$, *and assume* $\mathrm{inj}_{\mathrm{M_0^i}} := \inf_{\mathrm{x}\in \mathrm{M_0^i}} \mathrm{inj}_{\mathrm{g_i}}(\mathrm{x}) \geq 1$. *For any integer* $1 \leq k < n$, *there exists a positive constant* $c_1(n,k)$ *such that*

$$\frac{b_2^k(M_i)}{Vol(M_i)} \leq c(n,k) V_{min}(M_0^i)^{\frac{k-n}{n}}. \tag{122}$$

In particular, the growth of $\frac{b_2^k(M_i)}{Vol(M_i)}$ *along the tower of coverings is exponential in* $\mathrm{inj}_{\mathrm{M_0^i}}$.

Proof. Combine Corollary 76 with Theorem 116. □

10.3 Noncompact congruence subgroup quotients

In this subsection, we adapt and sharpen the arguments of [SX91, Sections 2 and 4] to the noncompact case. We realize $SO(n,1)$ and $SU(n,1)$ as the stabilizers of integral quadratic (respectively hermitian) forms of signature $(n,1)$: $Q(x) := \sum_{1\leq i,j\leq n} b_{ij}x_ix_j$, and $H(z) := \sum_{1\leq i,j\leq n} h_{ij}z_i\bar{z}_j$. Let I_{n+1} denote the identity matrix in GL_{n+1}. Let b and h denote the Gram matrices of the forms Q and H. Let $\Gamma_{\mathbb{R}}(n,q)$ and $\Gamma_{\mathbb{C}}(n,q)$ denote the arithmetic subgroups of $SO(n,1)$, respectively $SU(n,1)$ defined by

$$\Gamma_{\mathbb{R}}(n,q) := SO(n,1) \cap \{g \in GL(n+1,\mathbb{Z}) : g \equiv I_{n+1} \pmod{q}\},$$

and

$$\Gamma_{\mathbb{C}}(n,q) := SU(n,1) \cap \{g \in GL(n+1,\mathbb{Z}+i\mathbb{Z}) : g \equiv I_{n+1} \pmod{q}\}.$$

Let $|w|_\infty$ denote the supremum norm for a matrix a.

Lemma 123. *If* $\gamma \in \Gamma_{\mathbb{R}}(n,q) \setminus \{I_{n+1}\}$ *and* $q^2 \geq 2|b|_\infty$, *then the largest eigenvalue of* $\gamma^t\gamma$ *is greater than or equal to* $(\frac{q^2-2|b|_\infty}{2det(b)})^2$, *respectively* $(\frac{q^2-2|h_\infty|}{2det(h)})^2$. *If* $\mu \in \Gamma_{\mathbb{C}}(n,q) \setminus \{I_{n+1}\}$, *and* $q^2 \geq 2|h|_\infty$, *then the largest eigenvalue of* $\mu^t\mu$ *is greater than or equal to* $(\frac{q^2-2|h|_\infty}{2det(h)})^2$.

Proof. We treat the $\Gamma_{\mathbb{R}}(n,q)$ case. The $\Gamma_{\mathbb{C}}(n,q)$ case is identical. Let $\gamma \in \Gamma_{\mathbb{R}}(n,q)$. Write $\gamma = I_{n+1} + qh_1$, h_1 integral. Then

$$b = \gamma^t b \gamma = (I_{n+1} + qh_1^t) b (I_{n+1} + qh_1) = b + q(bh_1 + h_1^t b) + q^2 h_1^t b h_1. \tag{124}$$

Then we can write $bh_1 = A_1 + qbh_2$, with A_1 skew symmetric and (half-) integral and bh_2 symmetric and (half-) integral. Inserting this into (124) yields

$$2bh_2 = A_1 b A_1 - q(h_2^t A_1 - A_1 h_2) - q^2 h_2^t b h_2. \tag{125}$$

Since A_1 is skew and b is symmetric and invertible, $A_1 b A_1 \neq 0$ unless $A_1 = 0$. In particular, if the integral matrix $A_1 b A_1$ vanishes, then $h_2 \neq 0$ by the nontriviality of γ. On the other hand, we see from (125) that h_2 is nonzero if A_1 is nonzero. In either case, h_2 cannot be zero, and we have

$$b\gamma + (b\gamma)^t = 2b + 2q^2 b h_2.$$

Hence $|b|_\infty |\gamma|_\infty \geq |b\gamma|_\infty \geq \frac{q^2}{2} - |b|_\infty$, and the claim follows easily. □

Let $M_{\mathbb{R}}(n,q) := \Gamma_{\mathbb{R}}(n,q)\backslash \mathbf{H}^n_{\mathbb{R}}$, and $M_{\mathbb{C}}(n,q) := \Gamma_{\mathbb{C}}(n,q)\backslash \mathbf{H}^n_{\mathbb{C}}$. Arguing as in [SX91, Section 2], we see the injectivity radius at the identity coset is greater than or equal to $2\ln q - G$, where G is a constant depending only on the relevant Gram matrix. Fix decompositions

$$M_{\mathbb{R}}(n,1) = M_{\mathbb{R}}(n,1)_0 \cup_j C^1_j, \text{ and } M_{\mathbb{C}}(n,1) = M_{\mathbb{C}}(n,1)_0 \cup_j \tilde{C}^1_j,$$

where the C^1_j and $\tilde{C}^1_j$ are cusps, and $M_{\mathbb{R}}(n,1)_0$ and $M_{\mathbb{C}}(n,1)_0$ are compact orbifolds with boundary. Fix similar decompositions

$$M_{\mathbb{R}}(n,q) = M_{\mathbb{R}}(n,q)_0 \cup_j C^q_j, \text{ and } M_{\mathbb{C}}(n,1) = M_{\mathbb{C}}(n,1)_0 \cup_j \tilde{C}^q_j,$$

where $M_{\mathbb{R}}(n,q)_0$ and $M_{\mathbb{C}}(n,q)_0$ project to $M_{\mathbb{R}}(n,1)_0$ and $M_{\mathbb{C}}(n,1)_0$ and the cusps project to the cusps under the natural map. Let δ and $\tilde{\delta}$ denote the diameters of $M_{\mathbb{R}}(n,1)_0$ and $M_{\mathbb{C}}(n,1)_0$, respectively. Then

$$\mathrm{inj}_{\mathrm{M}_{\mathbb{R}}(\mathrm{n,q})_0} \geq 2\ln(\mathrm{q}) - \mathrm{G} - \delta \text{ and } \mathrm{inj}_{\mathrm{M}_{\mathbb{C}}(\mathrm{n,q})_0} \geq 2\ln(\mathrm{q}) - \mathrm{G} - \tilde{\delta}. \tag{126}$$

Theorem 127. *For $k < \frac{n-1}{2}$, there exists a constant $a(n,k) > 0$ such that for $\Gamma_{\mathbb{R}}(n,q)$ torsion free,*

$$b^k_2(M_{\mathbb{R}}(n,q)) \leq a(n,k) Vol(M_{\mathbb{R}}(n,q))^{1-\frac{4(n-1-2k)}{n(n+1)}}. \tag{128}$$

There exists a constant $b(n,k) > 0$ such that for $\Gamma_{\mathbb{C}}(n,q)$ torsion free,

$$b^k_2(M_{\mathbb{R}}(n,q)) \leq b(n,k) Vol(M_{\mathbb{R}}(n,q))^{1-\frac{4(n-k)}{(n+1)^2-1}}. \tag{129}$$

Proof. The theorem is a consequence of Theorems 118 and 121. First use the injectivity radius estimates (126) to estimate for positive constants α_j independent of q, $V_{min}(M_{\mathbb{R}}(n,q)_0) \geq \alpha_1 q^{2n-2}$ and $V_{min}(M_{\mathbb{C}}(n,q)_0) \geq \alpha_2 q^{4n}$. Then note that for positive constants β_j independent of q, $Vol(M_{\mathbb{R}}(n,q)) \leq \beta_1 q^{\frac{n(n+1)}{2}}$ and $Vol(M_{\mathbb{C}}(n,q)) \leq \beta_2 q^{(n+1)^2-1}$. These latter estimates can be obtained by embedding $\Gamma_{\mathbb{K}}(n,q)\backslash\Gamma_{\mathbb{K}}(n,1)$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$ into the corresponding orthogonal (respectively unitary) groups with coefficients in a finite field. Orders for these groups are computed, for example, in [Oes84, Section 1.6]. □

We remark that the restriction to torsion free $\Gamma_{\mathbb{K}}(n,q)$ is made simply for convenience and is not essential.

11 Normalized Number of Cusps and Topological Interpretation

In Section 10, we analyzed the growth of L^2-cohomology on towers of coverings of complete finite volume real- and complex-hyperbolic manifolds. We now address the problem of understanding the growth of de Rham cohomology on towers of coverings of such manifolds. This study requires an estimate on the number of cusps of hyperbolic manifolds along towers of coverings. This is a geometric problem of independent interest.

11.1 Normalized Number of Cusps

We start with some general remarks concerning complete finite volume quotients of Hadamard manifolds. Given a finite volume non-compact manifold (M,g) with pinched sectional curvature,

$$-b^2 \leq \sec_g \leq -a^2 < 0. \tag{130}$$

we define the volume normalized number of cusps (or simply the *normalized number of cusps*) to be:

$$\boxed{R_g(M) := \frac{N_c(M)}{Vol(M)},} \tag{131}$$

where $N_c(M)$ denotes the the number of cusps of M. By foundational work of Eberlein [Ebe80], N_c (the number of topological ends) is necessarily finite. In particular $\pi_1(M)$ is finitely presented. On the other hand, counterexamples of Gromov [Gro78] show this is not the case if the pinching condition is weakened to $a = 0$, see also Example 2 page 459 in [Ebe80]. This is true already in dimension $n = 3$, and in all of these examples the fundamental group is infinitely generated.

Now, Eberlein's result simply tells us that $N_c(M) < \infty$, but the actual upper bound may depend upon (M,g) itself. With some extra work, one can show that the upper bound depends upon the Margulis constant associated to (M,g). Such coarse bounds are usually far from optimal, and the Margulis constant may be hard to estimate. In many concrete geometric topology questions, it is then useful to derive effective uniform upper bounds on R_g for certain specific classes of complete finite volume quotients of Hadamard manifolds. This concrete problem is particularly well-studied in the case of finite volume quotients of $(\mathbf{H}^n_{\mathbb{R}}, g_{\mathbb{R}})$ and $(\mathbf{H}^n_{\mathbb{C}}, g_{\mathbb{C}})$. Interestingly, one may study this problem with a wildly different set of techniques ranging from topology to complex algebraic geometry. We refer the interested reader to the classical references of Parker [Par98] and Kellerhals [Kel98], and to the more recent literature in [Hwa06], [DD14], [DD15], [DD17], and [BT18]. In all of these instances, one seeks the *smallest* explicitly computable number $c(n, \mathbb{K})$ such that

$$R_{g_{\mathbb{K}}}(\Gamma\backslash\mathbf{H}^n_{\mathbb{K}}) \leq c(n, \mathbb{K}),$$

for *all* non-uniform torsion free lattices $\Gamma \leq \mathrm{Iso}(\mathbf{H}^n_{\mathbb{K}})$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$.

When $\mathbb{K} = \mathbb{R}$ and $n = 3$, there is an intriguing connection between normalized Betti numbers and $R_{g_{\mathbb{R}}}$. Namely, it is a consequence of basic 3-manifold theory (*cf.* [Hat00]) that for any complete finite volume $\Gamma\backslash\mathbf{H}^3_{\mathbb{R}}$

$$b_1(\Gamma\backslash\mathbf{H}^3_{\mathbb{R}}) \geq N_c(\Gamma\backslash\mathbf{H}^3_{\mathbb{R}}) \quad \Longrightarrow \quad 0 < R_{g_{\mathbb{R}}}(\Gamma\backslash\mathbf{H}^3_{\mathbb{R}}) \leq \frac{b_1(\Gamma\backslash\mathbf{H}^3_{\mathbb{R}})}{Vol(\Gamma\backslash\mathbf{H}^3_{\mathbb{R}}))}. \tag{132}$$

Thus, given a cofinal filtration $\{\Gamma_i\}$ of Γ, combining Lück's approximation theorem [Lüc94] with Dodziuk's vanishing [Dod79] we obtain

$$\lim_{i\to\infty} \frac{b_1(\Gamma_i\backslash\mathbf{H}^3_{\mathbb{R}})}{Vol(\Gamma_i\backslash\mathbf{H}^3_{\mathbb{R}}))} = 0 \quad \Longrightarrow \quad \lim_{i\to\infty} R_{g_{\mathbb{R}}}(\Gamma_i\backslash\mathbf{H}^3_{\mathbb{R}}) = 0. \tag{133}$$

We observe that in order to apply the main approximation theorem in [Lüc94], we simply need $\pi_1(\Gamma\backslash\mathbf{H}^3_{\mathbb{R}})$ to be finitely presented and residually finite, and this is certainly the case as $Vol(\Gamma\backslash\mathbf{H}^3_{\mathbb{R}})$ is assumed to be finite.

We next generalize this result to show that the normalized number of cusps goes *monotonically* to zero along a tower associated to a cofinal filtration of the fundamental group of any finite volume hyperbolic manifold. Now for general rank one locally symmetric spaces, one cannot expect a bound as in Equation (132) to hold. Indeed [DiS17] gives an explicit sequence of complex hyperbolic surfaces with cusps with first Betti number equal to 2 and diverging number of cusps. Thus, unlike the case of hyperbolic 3-manifolds, the convergence to zero of the normalized number of cusps cannot be derived from Lück's approximation and vanishing of L^2-Betti numbers. Our approach also has the advantage of producing an *effective* estimate on the rate of convergence of $R_{g_{\mathbb{K}}}$. The following lemma is stated for complete finite volume quotients of rank one symmetric spaces of non-compact type, but it can be generalized to complete finite volume quotients of Hadamard manifolds with residually finite fundamental group. We do not develop that generalization here.

Lemma 134. *Let $(M^n := \Gamma\backslash\boldsymbol{H}^n_{\mathbb{K}}, g_{\mathbb{K}})$ be a complete finite volume hyperbolic manifold with $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$. Given a cofinal filtration $\{\Gamma_i\}$ of Γ, denote by $\omega_i : M_i \to M$ the regular Riemannian cover of M associated to Γ_i equipped with the pull-back metric say g_i. Decompose M^n as a disjoint union*

$$M^n = M_0 \cup (\cup_j C_j),$$

where M_0 is a compact manifold with boundary and each C_j is a cusp. Define $M_0^i := \omega_i^{-1}(M_0)$, and let $\mathrm{inj}_{\mathrm{M_0^i}} := \inf_{\mathrm{x} \in \mathrm{M_0^i}} \mathrm{inj}_{\mathrm{g_i}}(\mathrm{x})$. We then have

$$R_{g_{\mathbb{K}}}(M_i) := \frac{N_c(M_i)}{Vol(M_i)} \leq Vol(B_{\frac{1}{2}\mathrm{inj}_{\mathrm{M_0^i}}})^{-1}. \tag{135}$$

where $Vol(B_{\frac{1}{2}\mathrm{inj}_{\mathrm{M_0^i}}})$ is the volume of a ball of radius $\frac{1}{2}\mathrm{inj}_{\mathrm{M_0^i}}$ in $(\boldsymbol{H}^n_{\mathbb{K}}, g_{\mathbb{K}})$. Moreover, we have the monotonicity

$$R_{g_{\mathbb{K}}}(M_i) \geq R_{g_{\mathbb{K}}}(M_{i+1}),$$

for any i.

Proof. For any cusp C_a^i, there is a point $z_a \in C_a^i$ such that

$$\mathrm{inj}_{\mathrm{z_a}} = \frac{\mathrm{inj}_{\mathrm{M_0^i}}}{2}.$$

Note that z_a is in the interior of C_a^i. In particular, for any point $q \in \partial C_a^i$, we have

$$d_{g_i}(q, z_a) \geq |\mathrm{inj_q} - \mathrm{inj}_{\mathrm{z_a}}| > \frac{\mathrm{inj}_{\mathrm{M_0^i}}}{2}.$$

We then have

$$Vol(M_i) > \sum_a Vol(B_{\frac{1}{2}\mathrm{inj}_{\mathrm{M_0^i}}}(z_a)) = N_c(M_i) Vol(B_{\frac{1}{2}\mathrm{inj}_{\mathrm{M_0^i}}}),$$

which implies (135).

To show the monotonicity of $R_{g_{\mathbb{K}}}$ along the tower, we argue as follows. Let

$$\tau : M' \to M$$

be *any* covering of M of degree say $\kappa \geq 2$. As each C_j is covered by at most κ disjoint cusps in M', we compute:

$$R_{g_{\mathbb{K}}}(M') = \frac{N_c(M')}{Vol(M')} \leq \frac{\kappa \cdot N_c(M)}{\kappa \cdot Vol(M)} = R_{g_{\mathbb{K}}}(M),$$

completing the proof. □

Remark 136. As shown in Lemma 134, $R_{g_{\mathbb{K}}}$ is non-increasing in a tower of coverings. In particular, hyperbolic manifolds that do *not* finitely cover any other hyperbolic manifold tend to have large $R_{g_{\mathbb{K}}}$. Thus, hyperbolic manifolds with *minimal* volume are good candidates on which to test the sharpness of any possible cusp count. Interestingly, this is exactly the case for the *sharp* cusp count for complex-hyperbolic surfaces presented in [DD14]. Indeed, such a bound is saturated by an arithmetic 4-cusped complex-hyperbolic surface of minimal volume constructed by Hirzebruch in [Hir84].

11.2 Bounds for Cohomology on Towers

We can now derive bounds for the de Rham cohomology of real- and complex-hyperbolic manifolds. Recall that for a complete finite volume hyperbolic manifold, we have the following topological interpretation for the L^2-cohomology groups.

Theorem 137. *[Zuc82, Theorems 6.2 and 6.9] Let M^n be a complete finite volume real- or complex-hyperbolic manifold of real dimension n. We have the following isomorphisms:*

$$\mathcal{H}^k(M^n) = \begin{cases} H^k(M^n;\mathbb{R}), & \text{if} \quad k < \frac{n-1}{2}; \\ Im\big(H^k_c(M^n;\mathbb{R}) \to H^k(M^n;\mathbb{R})\big), & \text{if} \quad k = \frac{n-1}{2}, \frac{n}{2}, \frac{n+1}{2}; \\ H^k_c(M^n), & \text{if} \quad k > \frac{n+1}{2}; \end{cases}$$

where by $H^k_c(M^n)$ we denote the cohomology with compact support in degree k of M^n.

Combining Theorem 137 with our previous results, we obtain a satisfactory understanding of the Betti numbers of complex-hyperbolic manifolds and even dimensional real-hyperbolic manifolds in a tower of coverings obtained by a cofinal filtration. For odd dimensional real-hyperbolic manifolds, we need further investigation in the critical degree. In particular, we need to keep into account the normalized number of cusps coefficient $R_{g_{\mathbb{R}}}$, introduced in Section 11.1.

Let $(M^n := \Gamma\backslash\mathbf{H}^n_{\mathbb{R}}, g_{\mathbb{R}})$ be a complete finite volume real-hyperbolic manifold. Given a cofinal filtration $\{\Gamma_i\}$ of Γ, let $\pi_i : M_i \to M$ denote the regular Riemannian cover of M associated to Γ_i. Decompose M^n as a disjoint union

$$M = M_0 \cup (\cup_j C_j),$$

where M_0 is a compact manifold with boundary and each C_j is a cusp. Define $M^i_0 := \pi_i^{-1}(M_0)$. It follows that:

$$M^i_0 = M_i \setminus \bigcup_{s=1}^{N_c(M_i)} C^i_s, \quad \partial M^i_0 = \bigcup_{s=1}^{N_c(M_i)} \partial C^i_s.$$

From the long exact sequence in cohomology, we have

$$... \to H^k(M^i_0, \partial M^i_0;\mathbb{R}) \to H^k(M_i;\mathbb{R}) \to H^k(\cup_s C^i_s;\mathbb{R}) \to H^{k+1}(M^i_0, \partial M^i_0;\mathbb{R}) \to ...,$$

so that

$$\begin{aligned} \dim_{\mathbb{R}} H^k(M_i;\mathbb{R}) &\leq \dim_{\mathbb{R}} Im\big(H^k(M^i_0, \partial M^i_0;\mathbb{R}) \to H^k(M_i;\mathbb{R})\big) + L(n)N_c(M_i) \\ &= \dim_{\mathbb{R}} Im\big(H^k_c(M_i;\mathbb{R}) \to H^k(M_i;\mathbb{R})\big) + L(n)N_c(M_i). \end{aligned} \tag{138}$$

for some positive constant $L(n) > 0$ *independent* of the index i. This follows from the fact that every cusp cross section is flat, and that by Bieberbach's theorem there are only finitely many diffeomorphism types of flat manifolds with given dimension. Suppose now $n = 2m+1$. By Theorem 137 we have:

$$\dim_{\mathbb{R}} Im\big(H^m_c(M_i;\mathbb{R}) \to H^m(M_i;\mathbb{R})\big) = \dim_{\mathbb{R}} \mathcal{H}^m_{g_{\mathbb{R}}}(M_i),$$

and so we obtain:

$$\frac{\dim_{\mathbb{R}} H^k(M_i;\mathbb{R})}{Vol(M_i)} \leq L(n) R_{g_{\mathbb{R}}}(M_i) + \frac{b^k_2(M_i)}{Vol(M_i)}. \tag{139}$$

We are now ready to prove the following proposition.

Proposition 140. *Let $(M^n := \Gamma\backslash \boldsymbol{H}^n_{\mathbb{R}}, g_{\mathbb{R}})$ be a complete finite volume real-hyperbolic manifold of dimension $n = 2k+1$, for some positive integer k. Given a cofinal filtration $\{\Gamma_i\}$ of Γ, let $\pi_i : M_i \to M$ denote the regular Riemannian cover of M associated to Γ_i. Set $g_i := \pi_i^* g_{\mathbb{R}}$. Decompose M^n as a disjoint union*

$$M^n = M_0 \cup (\cup_j C_j),$$

where M_0 is a compact manifold with boundary and each C_j is a cusp. Define $M^i_0 := \pi_i^{-1}(M_0)$, and let $\mathrm{inj}_{\mathrm{M}^i_0} := \inf_{\mathrm{x}\in \mathrm{M}^i_0} \mathrm{inj}_{\mathrm{g_i}}(\mathrm{x})$. There exists a constant $\lambda(k)$ such that

$$\frac{\dim_{\mathbb{R}} H^k(M_i;\mathbb{R})}{Vol(M_i)} \leq \frac{\lambda(k)}{\ln(V_{min}(M^i_0))}. \tag{141}$$

In particular, the sub volume growth of $\dim_{\mathbb{R}} H^k(M_i;\mathbb{R})$ is at least of the order $\mathrm{inj}^{-1}_{\mathrm{M}^i_0}$.

Proof. Combining Equation (139), Lemma 134 and Theorem 118 we obtain

$$\frac{\dim_{\mathbb{R}} H^k(M_i;\mathbb{R})}{Vol(M_i)} \leq L(n) Vol(B_{\frac{1}{2}\mathrm{inj}_{\mathrm{M}^i_0}})^{-1} + \frac{c_3(n,k)}{\mathrm{inj}_{\mathrm{M}^i_0}},$$

so that we can find another constant $\lambda(k) > 0$ so that Equation (141) is satisfied. In particular, by Theorem 116 we have

$$\lim_{i\to\infty} \frac{\dim_{\mathbb{R}} H^k(M_i;\mathbb{R})}{Vol(M_i)} = 0,$$

with a decay of the order at least $\mathrm{inj}^{-1}_{\mathrm{M}^i_0}$. □

12 Appendix

12.1 Comparison with the Trace Formula

Let G be a semisimple Lie group, $K \subset G$ a maximal compact subgroup, and $\Gamma \subset G$ a discrete torsion free cocompact subgroup. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the lie algebras of G and K respectively. Write the Cartan decomposition of $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $M := \Gamma\backslash G/K$ denote the corresponding compact locally symmetric space. In [Mat67], Matsushima established the following link between cohomology and representation theory.

Theorem 142 (Matsushima)**.** *For an irreducible unitary representation ω of G, let ω_K denote the restriction of ω to K, $N(\Gamma,\omega)$ denote the multiplicity of ω in $L^2(\Gamma\backslash G)$, and $m_k(\omega_K)$ denote the multiplicity of ω_K in $\Lambda^k\mathfrak{p}^*$. Then the Betti numbers are given by the following finite sum:*

$$b^k(M) = \sum_{\omega:\omega \text{ has vanishing casimir}} m_k(\omega_K)N(\Gamma,\omega). \tag{143}$$

The Selberg trace formula provides a mechanism for expressing the $N(\Gamma,\omega)$ in terms orbital integrals, which may, however, be difficult to interpret. In [DW78] DeGeorge-Wallach introduced a method for estimating these multiplicities from above, which we now recall.

Let χ denote the characteristic function of the identity component of $\pi^{-1}(B_{\mathrm{inj_M}}(\Gamma K)) \subset G$, where $\pi : G \to M$ is the natural projection. Let ω be an irreducible unitary representation of G on a Hilbert space H_ω. Let $v \in H_\omega$ be a unit vector. Let $N(\Gamma,\omega)$ denote the multiplicity of ω in $L^2(\Gamma\backslash G)$. Then DeGeorge-Wallach use the trace formula to show that [DW78, Corollary 3.2]

$$\frac{N(\Gamma,\omega)}{Vol(\Gamma\backslash G)} \le \frac{1}{\int_G \chi(g)\langle\omega(g)v,v\rangle^2 dv}. \tag{144}$$

The role of χ is to ensure that only the (conjugacy class of the) identity enters the trace formula. Hence lower bounds on the absolute value of the matrix coefficient $\langle\omega(g)v,v\rangle$ become upper bounds on the normalized multiplicity $\frac{N(\Gamma,\omega)}{Vol(\Gamma\backslash G)}$.

For the geometer, perhaps the most interesting instance of H_ω is the subspace of $L^2(\Gamma\backslash G)$ generated by L^2-harmonic forms of suitable K-invariance type. Let h be a harmonic m-form on M with L^2-norm one. Let $\pi : \Gamma\backslash G \to M$ be the quotient map. Using left invariant vector fields to define a canonical trivialization of $\Lambda^m T^*(\Gamma\backslash G)$, the metric induces a pairing for all $x, y \in \Gamma\backslash G$,

$$\langle\cdot,\cdot\rangle : \Lambda^m T^*_x(\Gamma\backslash G) \times \Lambda^m T^*_y(\Gamma\backslash G) \to \mathbb{R}.$$

Let σ denote the adjoint representation of K on $\Lambda^m\mathfrak{p}^*$. Viewed as a $\Lambda^m\mathfrak{p}^*$ valued function, π^*h satisfies for all $k \in K$

$$\pi^*h(gk) = \sigma^{-1}(k)h(g). \tag{145}$$

Choosing $v \in H_\omega$ to be π^*h, the matrix coefficient becomes

$$\langle\omega(g)v,v\rangle = \int_{\Gamma\backslash G} \langle\pi^*h(xg), \pi^*h(x)\rangle dv_x, \tag{146}$$

and lower bounds on this inner product lead to upper bounds on $b^k(M)$ via (143).

The estimate given in [DS17], on the other hand, essentially reduces to

$$\frac{b^m(M)}{Vol(M)} \le C sup_{p\in M, h\neq 0} \frac{\int_{B_1(p)} |h|^2 dv}{\int_{B_{\mathrm{inj_M}}(p)} |h|^2 dv},$$

for some constant C depending only on the curvature of M, if we assume that $\mathrm{inj_M} > 1$. Hence lower bounds on the growth of norms of harmonic forms is once again the key component of the estimate. In this more geometric approach, the restriction to embedded balls in M rather than larger balls in the universal cover is not a function of the Selberg trace formula, but is instead a simple mechanism to ensure that $\int_{B_R(\tilde{p})\subset\tilde{M}} |\omega^* h|^2 dv \leq \int_M |h|^2 dv$, where $\omega : \tilde{M} \to M$ is the universal cover. The role of the Price inequality - like the differential equations governing the matrix coefficients - is to provide a mechanism for estimating the growth of $\int_{B_r(p)} |h|^2 dv$ as a function of r.

In the remainder of this appendix we present a direct analysis of matrix coefficients arising in the estimation of the first Betti numbers of compact quotients of complex-hyperbolic spaces and all but middle degree Betti numbers of compact quotients of real-hyperbolic spaces. The same method, albeit with additional complications, can be used to estimate the matrix coefficients required to bound all the Betti numbers outside middle degree for compact complex-hyperbolic spaces, but we do not include those computations here, as they are significantly longer than the method used in Section 4 and simply reproduce the results given there. There is, of course, a vast literature on the estimation of matrix coefficients, but we found our approach in these cases so much simpler than the general theory, that we felt it might be useful, especially to the non-expert, to include it here.

12.2 Real rank one generalities

Let G be a real rank one semisimple Lie group. Fix a maximal $\mathbb{R}$-split torus $A \subset G$, and a Weyl chamber $\mathfrak{a}^+$. Denote the positive restricted roots of A by λ and 2λ. Let $m(\lambda)$ and $m(2\lambda)$ denote their multiplicities. Choose an element $u \in \mathfrak{a}^+$ so that $\lambda(u) = 1$. Let U denote the covector metrically dual to u. Write $a(t) := exp(tu)$.

Using the KAK decomposition of G, we reduce the analysis of the matrix coefficient (146) to the study of the function

$$\phi(t) := \int_{\Gamma\backslash G} \langle \pi^* h(xa(t)), \pi^* h(x)\rangle dv_x. \tag{147}$$

In particular, it is easy to show that for some $c(G, m) > 0$ depending only on G and m, with $\langle \omega(g)v, v\rangle$ as in (146),

$$\int_{B_R} \langle \omega(g)v, v\rangle^2 dg \geq c(G, m) \int_0^R \phi(t)^2 \sinh(t)^{m(\lambda)+m(2\lambda)} \cosh(t)^{m(2\lambda)} dt. \tag{148}$$

Combining (143), (144), and (148), we obtain for some $C(G, m) > 0$, depending only on G and m,

$$\frac{b^m(\Gamma\backslash G/K)}{Vol(\Gamma\backslash G/K)} \leq \frac{C(G, m)}{\int_0^{\mathrm{inj_M}} \phi(t)^2 \sinh(t)^{m(\lambda)+m(2\lambda)} \cosh(t)^{m(2\lambda)} dt}. \tag{149}$$

We prove the following results.

Proposition 150. *Let $G = SU(n,1)$ and $m = 1$. For $t > 1$, there are constants $c_1(n), c_2(n) > 0$ such that*

$$\frac{c_1(n)}{\sinh(t)} \leq \phi(t) \leq \frac{c_2(n)}{\sinh(t)}. \tag{151}$$

Proposition 152. *Let $G = SO(n,1)$ and $m < \frac{n-1}{2}$. For $t > 1$, there is a constant $c(n,m) > 0$ such that*

$$\frac{c(n,m)}{\sinh^m(t)} \leq \phi(t). \tag{153}$$

For $m = \frac{n-1}{2}$, and $t > 1$, there is a constant $c_0(n) > 0$ such that

$$\frac{c_0(n)}{t} \leq \phi(t). \tag{154}$$

As a corollary we recover (with different constants) our prior sections' Betti number bounds for compact real- and complex-hyperbolic manifolds (but only in degree 1 in the complex-hyperbolic case here).

12.3 $G = SU(n,1)$

In this section we prove Proposition 150.

In order to estimate ϕ, we must also understand

$$\psi(t) := \int_{\Gamma\backslash G} \langle (e(U)e^*(U) + e(JU)e^*(JU))\pi^* h(xa(t)), \pi^* h(x)\rangle dv_x.$$

ϕ and ψ satisfy differential equations following from the fact that $0 = dh = d^*h = dJh = d^*Jh$. Choose an orthonormal basis of $\mathfrak{p}$ of the form $\{X_j\}_{j=1}^{2n-2} \cup \{u, Ju\}$, where the X_j are in the span of the $\pm\lambda$ root spaces. Ju is in the span of the $\pm 2\lambda$ root spaces. Let $\{\omega^j\}_{j=1}^{2n-2} \cup \{U, JU\}$ denote the dual frame. With respect to the framing by left invariant vector fields

$$d\pi^* h = \left(e(\omega^j)X_j + e(U)u + e(JU)Ju\right)\pi^* h, \tag{155}$$

and

$$d^*\pi^* h = -\left(e^*(\omega^j)X_j + e^*(U)u + e^*(JU)Ju\right)\pi^* h. \tag{156}$$

For a proof of the these identities, see for example [MM63, Equations (4.12) and (6.3)]. Henceforth, we will simply write h instead of $\pi^* h$. Hence we have

$$\begin{aligned} 0 &= \int_{\Gamma\backslash G} \langle e^*(U)(e(\omega^j)X_j + e(U)u + e(JU)Ju)h(xa(t)), h(x)\rangle dv_x \\ &= \frac{d}{dt}\int_{\Gamma\backslash G} \langle e^*(U)e(U)h(xa(t)), h(x)\rangle dv_x + \\ &\quad \int_{\Gamma\backslash G} \langle e^*(U)(e(\omega^j)X_j + e(JU)Ju)h(xa(t)), h(x)\rangle dv_x, \end{aligned} \tag{157}$$

and

$$
\begin{aligned}
0 &= \int_{\Gamma\backslash G} \langle e(U)(e^*(\omega^j)X_j + e^*(U)u + e^*(JU)Ju)h(xa(t)), h(x)\rangle dv_x \\
&= \frac{d}{dt}\int_{\Gamma\backslash G} \langle e(U)e^*(U)h(xa(t)), h(x)\rangle dv_x + \\
&\quad \int_{\Gamma\backslash G} \langle e(U)(e^*(\omega^j)X_j + e^*(JU)Ju)h(xa(t)), h(x)\rangle dv_x. \qquad (158)
\end{aligned}
$$

We have two additional equations obtained by replacing h by Jh. To generate differential equations for ϕ and ψ from (157) and (158), we follow the first step of the strategy indicated in [Kna86, Chapter VIII Section 7] and illustrated in a special case in [Kna86, Lemma 8.15]. Write

$$
X_j = \coth(t)Y_j - \frac{1}{\sinh(t)}a^{-1}(t)Y_j a(t), \quad Ju = \coth(t)Y_{2\lambda} - \frac{1}{\sinh(t)}a^{-1}(t)Y_{2\lambda}a(t). \qquad (159)
$$

Here Y_j and $Y_{2\lambda} \in \mathfrak{k}$ are defined as follows. Write $X_j = N_j + N_j^*$, where N_j is in the λ root space. Then $Y_j = N_j - N_j^*$. $Y_{2\lambda}$ is similarly defined. Combining (159) and (145), we have

$$
\begin{aligned}
&\int_{\Gamma\backslash G} \langle e^*(U)e(\omega^j)X_j h(xa(t)), h(x)\rangle dv_x \\
&= \coth(t)\int_{\Gamma\backslash G} \langle e^*(U)e(\omega^j)Y_j h(xa(t)), h(x)\rangle dv_x \\
&- \frac{1}{\sinh(t)}\int_{\Gamma\backslash G} \langle e^*(U)e(\omega^j)a^{-1}(t)Y_j a(t)h(xa(t)), h(x)\rangle dv_x \\
&= -\coth(t)\int_{\Gamma\backslash G} \langle e^*(U)e(\omega^j)\sigma(Y_j)h(xa(t)), h(x)\rangle dv_x \\
&- \frac{1}{\sinh(t)}\int_{\Gamma\backslash G} \langle e^*(U)e(\omega^j)\frac{d}{ds}_{|s=0} h(x\exp(sY_j)a(t)), h(x)\rangle dv_x \\
&= -\coth(t)\int_{\Gamma\backslash G} \langle e^*(U)e(\omega^j)\sigma(Y_j)h(xa(t)), h(x)\rangle dv_x \\
&- \frac{1}{\sinh(t)}\int_{\Gamma\backslash G} \langle e^*(U)(e(\omega^j)\frac{d}{ds}_{|s=0} h(xa(t)), h(x\exp(-sY_j))\rangle dv_x \\
&= -\coth(t)\int_{\Gamma\backslash G} \langle e^*(U)e(\omega^j)\sigma(Y_j)h(xa(t)), h(x)\rangle dv_x \\
&- \frac{1}{\sinh(t)}\int_{\Gamma\backslash G} \langle \sigma(Y_j)e^*(U)(e(\omega^j)h(xa(t)), h(x)\rangle dv_x. \qquad (160)
\end{aligned}
$$

We may similarly replace the Ju derivatives with $-\coth(2t)\sigma(Y_{2\lambda})$ and $\frac{1}{\sinh(2t)}\sigma(Y_{2\lambda})$ terms. The adjoint representation satisfies

$$
\sigma(Y_j) = e(u)e^*(X_j) - e(X_j)e^*(u) + e(JX_j)e^*(Ju) - e(Ju)e^*(JX_j). \qquad (161)
$$

$$\sigma(Y_{2\lambda}) = 2e(u)e^*(Ju) - 2e(Ju)e^*(u) - e(JX_j)e^*(X_j). \tag{162}$$

Taking the closed and coclosed equations (157) and (158) and the corresponding equations with h replaced by Jh, replacing X_j and Ju by combinations of $\sigma(Y_j)$ and $\sigma(Y_{2\lambda})$ as indicated in (160), and substituting in expressions (161) and (162) for $\sigma(Y_j)$ and $\sigma(Y_{2\lambda})$, yields a system of 4 differential equations, from which we obtain

$$0 = (\phi - \psi)' + \Big(\coth(t) + \frac{1}{\sinh(t)}\Big)(\phi - \psi) - (n-1)\Big[\coth(t) + \frac{1}{\sinh(t)}\Big]\psi(t). \tag{163}$$

and

$$0 = \psi'(t) + [(2n-2)\coth(t) + 2\tanh(t)]\psi(t) - \frac{2}{\sinh(t)}(\phi(t) - \psi(t)), \tag{164}$$

with initial conditions $\phi(0) = 1$, and $0 < \psi(0) < 1$. From these equations, we see that $\phi(t) - \psi(t)$ cannot become zero while $\psi(t)$ is positive and ψ cannot vanish while $\phi - \psi$ is positive. Hence $\phi(t) > \psi(t) > 0$ for all t.

Next, we combine the equations and introduce integrating factors to write

$$\begin{aligned}&\Big[\sinh(t)\tanh\Big(\frac{t}{2}\Big)^{2n-1}(\phi - n\psi)\Big]' \\ &= (2n-2)\Big[(n-1)\frac{\cosh(t)-1}{\sinh(t)} + \tanh(t)\Big]\sinh(t)\tanh\Big(\frac{t}{2}\Big)^{2n-1}\psi.\end{aligned} \tag{165}$$

Hence for any t, we have $\phi(t) \geq n\psi(t)$. We also have the expression

$$\begin{aligned}\Big(\frac{\sinh(t)}{\tanh(\frac{t}{2})}\phi\Big)' &= -\Big[(n-2)\frac{\cosh(t)-1}{\sinh(t)} + 2\tanh(t)\Big]\frac{\sinh(t)}{\tanh(\frac{t}{2})}\psi(t) \\ &\geq -\Big[(n-2)\frac{\cosh(t)-1}{\sinh(t)} + 2\tanh(t)\Big]\frac{1}{n}\frac{\sinh(t)}{\tanh(\frac{t}{2})}\phi(t) \\ &\geq -C_n\frac{\sinh(t)}{\tanh(\frac{t}{2})}\phi(t),\end{aligned} \tag{166}$$

where $C_n := \sup_t \frac{1}{n}\big[(n-2)\frac{\cosh(t)-1}{\sinh(t)} + 2\tanh(t)\big]$. Hence $\frac{\sinh(t)}{\tanh(\frac{t}{2})}\phi(t) \geq e^{-tC_n}$, and $\phi(t) \leq \frac{\tanh(\frac{t}{2})}{\sinh(t)}$. Finally from the expression

$$\Big[\sinh(t)\tanh\Big(\frac{t}{2}\Big)(\phi - \psi)\Big]' = (n-1)[\cosh(t)+1]\tanh\Big(\frac{t}{2}\Big)\psi(t), \tag{167}$$

we see that

$$\sinh(t)\tanh\Big(\frac{t}{2}\Big)(\phi - \psi) \geq \sinh(1)\tanh\Big(\frac{1}{2}\Big)(\phi(1) - \psi(1)) \geq \frac{n-1}{n}\tanh^2\Big(\frac{1}{2}\Big)e^{-C_n}.$$

In particular, for $t > 1$,

$$\frac{\tanh(\frac{t}{2})}{\sinh(t)} \leq \phi(t) \leq \frac{c(n)}{\sinh(t)},$$

proving Proposition 150.

12.4 $G = SO(n,1)$

In this subsection, we let h be an L^2-norm one harmonic k-form on a compact real-hyperbolic n-manifold $M = \Gamma\backslash SO(n,1)/SO(n)$. Let π denote the projection map $\pi : \Gamma\backslash SO(n,1) \to M$. Define the matrix coefficient

$$f(t) := \int_{\Gamma\backslash SO(n,1)} \langle \pi^* h(xa(t)), \pi^* h(x)\rangle dv_x,$$

with $a(t) := \exp(tu)$, u defined as in the preceding subsection. Define also the auxiliary function

$$b(t) := \int_{\Gamma\backslash SO(n,1)} \langle e(u)e^*(u)\pi^* h(xa(t)), \pi^* h(x)\rangle dv_x.$$

Computing as in the previous subsection, one derives the following coupled equations for f and b:

$$(f-b)' + k\coth(t)(f-b) - \frac{n-k}{\sinh(t)}b = 0, \tag{168}$$

$$b' + (n-k)\coth(t)b - \frac{k}{\sinh(t)}(f-b) = 0, \tag{169}$$

with initial conditions $f(0) = 1$, $0 < b(0) < 1$. Introduce integrating factors to rewrite these equations as

$$\frac{d}{dt}(\sinh^k(t)(f-b)) = (n-k)\sinh^{k-1}(t)b,$$

$$\frac{d}{dt}(\sinh^{n-k}(t)b) = k\sinh^{n-k-1}(t)(f-b).$$

From these equations, we see that $f > b > 0$ for all t, and for $t \geq 1$,

$$f(t) \geq \frac{\sinh^k(1)(f(1)-b(1))}{\sinh^k(t)}. \tag{170}$$

Next, rewrite Equation 168 as

$$\begin{aligned} 0 &= \Big(f - \frac{n}{k}b + \frac{n-k}{k}b\Big)' + k\coth(t)\Big(f - \frac{n}{k}b + \frac{n-k}{k}b\Big) - \frac{n-k}{\sinh(t)}b \\ &= \Big(f - \frac{n}{k}b\Big)' + \frac{n-k}{k}b' + k\coth(t)\Big(f - \frac{n}{k}b\Big) + (n-k)\coth(t)b - \frac{n-k}{\sinh(t)}b, \end{aligned}$$

so that by substituting the identity for b' given in (169) we obtain

$$\Big(f - \frac{n}{k}b\Big)' + \frac{n-k+k\cosh(t)}{\sinh(t)}\Big(f - \frac{n}{k}b\Big) = \frac{(n-k)(n-2k)(\cosh(t)-1)}{k\sinh(t)}b > 0.$$

This also shows that

$$b \leq \frac{k}{n} f. \tag{171}$$

Now, by summing Equations (168) and (169) we obtain

$$f' = -k \coth(t) f + \frac{n-2k}{\sinh(t)} b - (n-2k)\coth(t) b + \frac{k}{\sinh(t)} f.$$

By using this expression for f', and after some manipulations with hyperbolic functions one finds

$$\Big(\frac{\sinh^k(t)}{\tanh^k(\frac{t}{2})} f\Big)' + \frac{\sinh^k(t)}{\tanh^k(\frac{t}{2})} \frac{(n-2k)(\cosh(t)-1)}{\sinh(t)} b = 0,$$

and then

$$\Big(\frac{\sinh^k(t)}{\tanh^k(\frac{t}{2})} f\Big)' + \frac{2(n-2k)\cosh^2(\frac{t}{2})\sinh^{k-1}(t)}{\tanh^{k-2}(\frac{t}{2})} b = 0. \tag{172}$$

Next, for $2k < n$, we use (171) to replace (172) by a differential inequality

$$\Big(\frac{\sinh^k(t)}{\tanh^k(\frac{t}{2})} f\Big)' \geq -\frac{2(n-2k)\cosh^2(\frac{t}{2})\sinh^{k-1}(t)}{\tanh^{k-2}(\frac{t}{2})} \frac{k}{n} f.$$

Since

$$\lim_{t\to 0} \frac{\sinh^k(t)}{\tanh^k(\frac{t}{2})} = 2^k,$$

integration from 0 to t yields

$$f(t) \geq 2^k \frac{\tanh^k(\frac{t}{2})}{\sinh^k(t)} e^{-\int_0^t \frac{2(n-2k)\cosh^2(\frac{s}{2})\tanh^2(\frac{s}{2})}{\sinh(s)} \frac{k}{n} ds}.$$

In particular, we have

$$f(1) \geq 2^k \frac{\tanh^k(\frac{1}{2})}{\sinh^k(1)} e^{-\int_0^1 \frac{2(n-2k)\cosh^2(\frac{s}{2})\tanh^2(\frac{s}{2})}{\sinh(s)} \frac{k}{n} ds},$$

so that by evaluating (171) at $t = 1$ and using the estimate in (170) then yields

$$f(t) \geq \frac{c}{\sinh^k(t)},$$

with

$$c = e^{-\int_0^1 \frac{2(n-2k)\cosh^2(\frac{s}{2})\tanh^2(\frac{s}{2})}{\sinh(s)} \frac{k}{n} ds} \frac{(n-k)2^k \tanh^k(\frac{1}{2})}{n}.$$

Hence, for some constant $c(n,k) > 0$, we have

$$\int_0^R f(t)^2 \sinh^{n-1}(t) dt \geq \begin{cases} c(n,k) e^{(n-1-2k)R}, & \text{if} \quad n-1 > 2k; \\ \\ c(n,k) R, & \text{if} \quad n-1 = 2k. \end{cases} \tag{173}$$

In the notation used in Section 12.2 we have $\phi = f$, so that a proof of Proposition 152 follows from the estimate in Equation 173.

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