



# Diffuse interface model for cell interaction and aggregation with Lennard-Jones type potential

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## Abstract

This study introduces a phase-field model designed to simulate the interaction and aggregation of multicellular systems under flow conditions within a bounded spatial domain. The model incorporates a multi-dimensional Lennard-Jones potential to account for short-range repulsion and adhesive bonding between cells. To solve the governing equations while preserving energy law, a second-order accurate  $C^0$  finite element method is employed. The validity of the model is established through numerical tests, and experimental data from cell stretch tests is utilized for model calibration and validation. Additionally, the study investigates the impact of varying adhesion properties in red blood cells. Overall, this work presents a thermodynamically consistent and computationally efficient framework for simulating cell–cell and cell–wall interactions under flow conditions.

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## 1. Introduction

One major type of biological cell–cell interaction is characterized by the direct interactions between cell membranes. These interactions can be purely mechanical or involve cell adhesion molecules (CAM) exposed on cell membranes [1,2]. Two major factors that dominate such interactions are: (i) finite size effects that drive short-range repulsion (pushing) between cells to prevent them from overlapping; and (ii) adhesion (attraction) due to the formation of adhesive bonds with CAM on adjacent cell membranes [1,2]. Cell–cell interaction is an important subject for understanding hemodynamics because structural interactions at the cellular level unambiguously appear in a broad spectrum of blood flow-related problems ranging from red blood cells (RBCs) distribution [3] in the blood vessel, the growth of blood clot [4], blood cell aggregation [5], sickle cell disease [6], tumor cell dynamics [7] and diabetes [2]. It is known that the RBCs aggregate to form rouleaux structure due to the adhesion forces between RBCs [8,9]. Also, the interaction between blood cells and blood vessel wall is a critical initial step in responding to different diseases, such as pathological inflammation and thrombosis [10–12].

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Over the past decades, a large number of mathematical models are proposed to study cell–cell interaction. In particular, some of these models are devoted to studying the plasma membranes interaction using vesicles which largely preserve the plasma membrane lipidome and proteome [13,14]. Within the framework of sharp interface description in which interfaces separating different components of matter are idealized as hypersurfaces with zero thickness, there are a number of studies focusing on modeling cell–cell and cell–blood-vessel interactions under blood flow conditions. Following the seminal work of Peskin [15,16], the immersed boundary method is used to develop a model for platelet aggregation [17]. Cell–wall interaction model [7] is introduced to simulate the adhesion and deformation of tumor cells at the vessel wall. Local and non-local models [18] are described to investigate the invasion and growth of tumor cells. RBC or vesicle aggregation is studied in [8,19–21]. Multi-scale models are introduced in [22,23] in which dissipative particle dynamics (DPD) is used in [22] to establish a blood cell in flow model, and a stochastic cellular potts model (CPM) is introduced in [23] for studying blood clot growth. To account for cell–cell or cell–substrate adhesion, a Lennard-Jones type potential [24–26] is introduced as a one-dimensional function of the distance between the points on different cell membranes and substrates. The potential is a combination of a repulsive part and an attractive part which shows repulsion when cells get too close to each other, and shows attraction when the distance between cells increases. This potential in combination with the DPD method, is utilized to study cell deformation and doublet suspension [25].

The diffuse interface method, also commonly known as the phase-field approach [27] is another popular method to model cell–cell interaction and aggregation. The diffuse interface model replaces the sharp interface description with a thin transition region (diffuse layer) in which microscopic mixing of the macroscopically distinct components of matter is permitted. The phase-field approach has two main advantages. First, it is easy to implement for tracking interfaces in problems that evolve large deformation or topological change of the interfaces; second, phase-field models derived from the energy dissipation law can be made thermodynamically consistent. This makes designing energy-stable numerical schemes that benefit the long time numerical simulation possible [27–30].

The first goal of this paper is to derive a thermodynamically consistent phase-field model for motion, interaction and aggregation of cells or vesicles under flow conditions in a closed spatial domain using an energy variational method. [31–34]. By treating the cell membrane as a diffuse interface, several phase-field vesicles or cells interaction models have been reported recently. Marth et al. [14] proposed a phase-field model for RBCs and white blood cell interactions by using a Gaussian potential for short-range repulsion. [35] introduced an adhesion potential based on the distance to the substrate in the phase-field framework. Gu et al. [36] proposed a potential by using two independent phase-field functions, one simulating the deformation and adhesion process of the vesicle and the other simulating the fixed substrate. Later on, an adhesion potential using a phase-field formulation [37] is constructed to take vesicle–substrate adhesion into account. However, as previously mentioned, cells can exhibit both adhesive and repulsive behaviors simultaneously. This phenomenon is particularly evident during embryonic development, where cells must adhere to one another to form tissues and organs, while also repelling neighboring cells to establish proper boundaries and patterns. This intricate interplay between adhesion and repulsion is crucial for processes such as tissue morphogenesis, cell sorting, and organ formation. The coexistence of adhesion and repulsion enables cells to dynamically interact with their environment, regulate cell–cell contacts, and maintain tissue organization throughout various biological processes. It is important for the model to consistently consider this aspect. In order to achieve this goal, several studies have proposed models that incorporate both adhesion and repulsion. For instance, [38–40], have explored the physical and mechanical aspects of cell interactions, considering the simultaneous presence of adhesion and repulsion forces. These studies provide valuable insights into understanding the complex dynamics and behaviors exhibited by cells when both adhesive and repulsive mechanisms are at play. While the adhesion is modeled as a function of the gradient of two phases; the repulsion is modeled as a function of two phases, respectively. However, the adhesion potential in [36–38] and the repulsion potential in [38,39] are global which means the forces exist even when two vesicles are far away from each other. Although the model described in [40] ensured that the effects of all interaction potentials are local, its adhesion term involves high-order derivatives which limit the utilization of some well-developed numerical schemes such as  $C^0$  finite element schemes.

Motivated by previous works, one novel aspect of the work proposed in this paper is the introduction of a multi-dimensional Lennard-Jones (LJ) type potential within the framework of the phase-field approach for modeling the multiple cell–cell, cell–wall interactions. This new multi-dimensional LJ potential does not involve high-order derivatives, which makes it easier to design a new  $C^0$  finite element scheme in numerical simulation. To the best of the authors' knowledge, this is the first time that both the short-range repulsion by cell finite size effects and

cell–cell, cell–wall adhesion by CAM are taken into consideration consistently in the phase-field framework, and are implemented using a  $C^0$  finite element method. All current phase-field models mainly focus on cell adhesion. Here, our idea of the multi-dimensional LJ type potential enables us to resolve these aforementioned two factors dominating cell–cell, cell–wall interaction in a consistent and simple manner.

The energy variational method [31,32] ensures that energy dissipation law [41,42] is satisfied. This leads to the thermodynamical consistency of the model. All the physics that ones are interested in are taken into consideration through definitions of the energy functional and the dissipation functional, together with the kinematic relations (assumptions) of the dynamic evolution of model state variables.

Numerically simulating these aforementioned phase-field models is challenging. Finite difference method [43], finite element method [35,44–46] and spectral methods [37,47] are proposed with different applications. Despite the fact that most phase-field models follow the energy dissipation law at the continuous level, there have been few works on developing energy-stable numerical schemes for these vesicle or cell models. A decoupled energy-stable scheme [48] is proposed for an Allen Cahn–Navier Stokes (AC–NS) model by introducing an intermediate velocity. An unconditionally energy-stable numerical scheme [44] is introduced for a Cahn Hilliard–Navier Stokes (CH–NS) system. However, there is still room for improving these schemes. E.g., the local inextensibility of the cell membrane is not considered. The discrete energy of numerical solutions computed by these schemes decays in a manner different from the energy dissipation law at the continuous level.

The second goal of this paper is to propose an energy-law-preserving finite element scheme for solving model governing equations, the CH–NS system with the AC general Navier boundary condition (GNBC) following the idea introduced by Shen et al. [32]. There are a few typical energy-law-preserving methods such as midpoint type of schemes (See, e.g. [31,49]), IEQ (See e.g. [50,51]) and SAV See e.g. [50,52]. We shall use a midpoint type of scheme in this paper following [32]. The proposed scheme exactly preserves a discrete counterpart of the energy dissipation property of the continuum model by discretizing the nonlinear term in a specific way. We apply our model to simulate multiple cells and vesicles interacting with each other and the substrate under flow conditions *in silico*, such as RBCs passing a branched blood vessel. We also note that although this scheme is designed for solving the model equations, it can be easily adapted to other CH–NS systems.

The rest of the paper is organized as follows. In Section 2, the thermodynamically consistent model considering cell–cell and cell–wall interaction is derived, and the energy dissipation law of the model is given. In Section 3, an energy-law-preserving scheme is proposed to solve the obtained system. Section 4 is used to present the results of numerical simulations and compare them with the data collected in laboratory experiments. The conclusion is drawn in Section 5.

## 2. Model derivation

Derivation of the model in this paper is based on the energy dissipation law which holds ubiquitously in physical and biophysical systems involving irreversible processes [53–57]. This law states that for an isothermal and closed system the rate of change of the energy of the system is equal to the dissipation of the energy as follows:

$$\frac{d}{dt}E^{\text{total}} = -\Delta \leq 0, \quad (1)$$

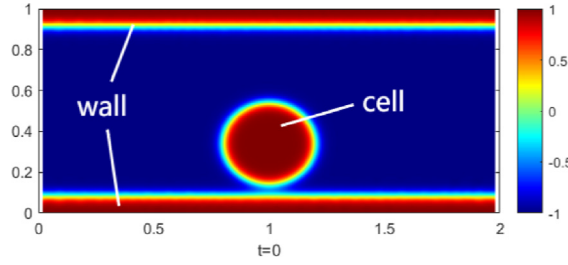
where  $E^{\text{total}}$  is the total energy of the system, which is the sum of the kinetic energy  $\mathcal{K}$  and the Helmholtz free energy  $\mathcal{F}$  of the system.  $\Delta$  is the rate of energy dissipation, which in fact is the entropy production. Eq. (1) can be easily derived via the combination of the First and Second Laws of Thermodynamics.

The choices of the total energy functional and the dissipation functional, together with the kinematic (transport) assumptions of the state variables employed in the system, determine all the physical and mechanical considerations of the problem [58].

### 2.1. Multi-cell interaction system

Energy variational method [31] is adopted for model derivation to ensure the thermodynamics consistency (1) of the derived model. We refer the readers to [31,32,59] for detailed discussions of this method.

Fig. 1 is a schematic illustration showing the setup of the model. Let the problem domain be  $\Omega$ , and its wall boundary be  $\partial\Omega_w$ . For a multi-cellular flow system, the dynamic evolution of the  $i$ th cell (or vesicle) under flow



**Fig. 1.** A schematic showing the model setup. Phase-field functions are used to represent the domain wall boundary and the cell interface, respectively. Values of the phase-field functions are indicated by the color bar.

conditions within  $\Omega$  is tracked by the phase-field function  $\phi_i(\mathbf{x}, t)$ . Notice that  $\phi_i(\mathbf{x}, t) \in [-1, 1]$  with  $\phi_i = 1$  for intracellular space and  $\phi_i = -1$  for extracellular space. The membrane of the cell is identified by  $\phi_i(\mathbf{x}, t) = 0$ . We also introduce a phase-field function  $\phi_w(\mathbf{x}) \in [-1, 1]$  to represent the wall boundary of the domain as shown in Fig. 1. This is for considering cell–wall interaction described below.

**Remark 2.1.** In the case of Fig. 1, the wall effect is considered as a force inside the domain instead of a condition on the boundary. In this particular simulation, the cell is not supposed to contact the wall when moving in the domain which is consistent with the physical fact that the cell would not merge together with the wall. One thing that needs to be emphasized is that adding such an extra term does not conflict with the GNBC. So in the following derivation of the governing equation, we still keep GNBC in the boundary condition.

We assume that the dynamical evolution of the phase-field function  $\phi_i$  is a generalized gradient flow, see Eq. (2a). We also utilize the laws of conservation for describing the dynamics of the linear momentum and the total mass of the system, see Eqs. (2b)–(2c). The cell membrane (interface) is assumed to be inextensible. The equation for local inextensibility of the interface is given by Eq. (2d).

$$\frac{\partial \phi_i}{\partial t} + \nabla \cdot (\mathbf{u} \phi_i) = \nabla \cdot \mathbf{q}_{\phi_i} , \quad (2a)$$

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = \nabla \cdot \boldsymbol{\sigma}_\eta + \mathbf{F}_{\phi_1, \phi_2, \dots, \phi_N} , \quad (2b)$$

$$\nabla \cdot \mathbf{u} = 0 , \quad (2c)$$

$$(\mathcal{P}_i : \nabla \mathbf{u}) \delta_i = 0 . \quad (2d)$$

Here  $\rho$  is the averaged density of the system. In this work,  $\rho$  is assumed to be a constant.  $\mathbf{u}$  is the macroscopic velocity of the system, and  $\boldsymbol{\sigma}_\eta$  is the system's visco-elastic stress yet to be specified.

The unidentified flux  $\mathbf{q}_{\phi_i}$  in Eq. (2a) will be determined by postulating that  $\phi_i$  is driven by gradients in the chemical potential. This leads to the Cahn–Hilliard equation which ensures the conservation of the volume of a cell during its dynamic evolution.  $\mathbf{F}_{\phi_1, \phi_2, \dots, \phi_n}$  in Eq. (2b) is the body force induced by cell–fluid interaction and is yet to be specified as well.

It has been reported that the cell membrane can be stretched by only 2%–4% before rupturing [60,61]. This membrane property is referred to as the local inextensibility of the membrane [32,62]. Eq. (2d) is the diffuse interface approximation of the local inextensibility of the membrane of the  $i$ th cell.  $\delta_i = \frac{1}{2} \gamma^2 |\nabla \phi_i|^2$  is the surface delta function with the diffuse interface thickness  $\gamma$ .  $\mathcal{P}_i$  is the projection operator, and is defined to be  $(I - \mathbf{n}_{i,m} \otimes \mathbf{n}_{i,m})$ .  $\mathbf{n}_{i,m} = \frac{\nabla \phi_i}{|\nabla \phi_i|}$  is the unit outward normal vector of the interface when it is defined as an implicit surface by the phase-field function. This equation is equivalent to

$$\mathcal{P}_i : \nabla \mathbf{u} = 0$$

in the case of sharp interface description of the model problem. In the phase-field framework,  $\mathcal{P}_i : \nabla \mathbf{u} = 0$  is extended to the whole domain by multiplying the scalar function  $\delta_i$  [31] for the convenience of computation.

Following the idea introduced in [32,62], here we add a relaxation term  $\xi \gamma^2 \nabla \cdot (\phi_i^2 \nabla \lambda_i)$  in Eq. (2d) where  $\xi$  is a parameter independent of  $\gamma$ , and  $\lambda_i$  is a function that measures the interface “pressure” induced by the inextensibility

of the membrane of the  $i$ th cell. Thus Eq. (2d) is revised to be:

$$(\mathcal{P}_i : \nabla \mathbf{u}) \delta_i + \xi \gamma^2 \nabla \cdot (\phi_i^2 \nabla \lambda_i) = 0 \quad (3)$$

Eqs. (2a)–(2c) and (3) together constitute the governing equations of the model.

The wall boundary conditions on the top and bottom of the domain, denoted by  $\partial\Omega_w$ , are given as follows:

$$\begin{cases} \mathbf{u} \cdot \mathbf{n} = 0, \\ \mathbf{u}_\tau \cdot \boldsymbol{\tau}_k = f_{\tau_k}, \\ \frac{\partial \phi_i}{\partial t} + \mathbf{u} \cdot \nabla_\Gamma \phi_i = J_{\Gamma_i}, \\ \nabla \lambda_i \cdot \mathbf{n} = 0, \\ \mathbf{q}_{\phi_i} \cdot \mathbf{n} = 0. \end{cases} \quad (4)$$

On the boundary  $\partial\Omega_w$ , an Allen–Cahn (4)<sub>3</sub> type boundary condition is employed for  $\phi_i$ .  $\nabla_\Gamma = \nabla - \mathbf{n}(\mathbf{n} \cdot \nabla)$  is the surface gradient operator, and  $\mathbf{u}_\tau = \mathbf{u} - (\mathbf{u} \cdot \mathbf{n})\mathbf{n}$  is the fluid slip velocity with respect to the wall where  $\boldsymbol{\tau}_i, i = 1, 2$  are the tangential directions of the wall surface.  $\mathbf{n}$  is the unit outward normal vector of the wall.  $f_{\tau_i}$  is the slip velocity of the fluid on the wall along the  $\boldsymbol{\tau}_i$  direction.  $J_\Gamma$  represents the Allen–Cahn type of relaxation on the wall by using the phase-field method and is yet to be identified. We note that these boundary conditions are also used in [31,63].

The total energy functional  $E_{total}$  of the multi-cellular system is assumed to be the sum of the kinetic energy  $E_{kin}$  in the macroscale, elastic energy  $E_{cell}$  of the cell membranes, cell–cell interaction energy  $E_{int}$  and cell–wall adhesion energy  $E_w$  in the microscale. Therefore

$$\begin{aligned} E_{total} &= \underbrace{E_{kin}}_{\text{Macroscale}} + \underbrace{E_{cell} + E_{int} + E_w}_{\text{Microscale}} \\ &= \int_\Omega \left( \frac{1}{2} \rho |\mathbf{u}|^2 \right) dx + \sum_{i=1}^N \int_\Omega \frac{\hat{\kappa}_B}{2\gamma} \left| \frac{f(\phi_i)}{\gamma} \right|^2 dx + \sum_{i=1}^N \frac{\mathcal{M}_s}{2} \frac{(S(\phi_i(t)) - S(\phi_i(t=0)))^2}{S(\phi_i(t=0))} \\ &\quad + \int_\Omega H dx + \sum_{i=1}^N \int_\Omega f_w(\phi_i) ds. \end{aligned} \quad (5)$$

The  $E_{cell}$  consists of an elastic energy  $\sum_{i=1}^N \int_\Omega \frac{\hat{\kappa}_B}{2\gamma} \left| \frac{f(\phi_i)}{\gamma} \right|^2 dx$  and a penalty  $\sum_{i=1}^N \frac{\mathcal{M}_s}{2} \frac{(S(\phi_i(t)) - S(\phi_i(t=0)))^2}{S(\phi_i(t=0))}$  for globally conserving surface areas of the cells.  $\hat{\kappa}_B$  is the bending modulus of the membrane, and  $\gamma$  is the thickness of the diffuse interface representation of the cell membrane. The membrane elastic energy density is given by

$$f(\phi_i) = \frac{\delta G}{\delta \phi_i} = -\gamma^2 \Delta \phi_i + (\phi_i^2 - 1)\phi_i, \quad (6)$$

with

$$G(\phi_i) = \frac{\gamma^2 |\nabla \phi_i|^2}{2} + \frac{(1 - \phi_i^2)^2}{4}. \quad (7)$$

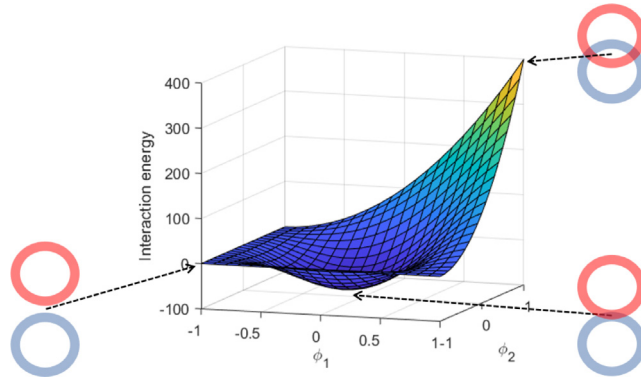
The function  $S(\phi_i) = \int_\Omega \frac{G(\phi_i)}{\gamma} dx$  is used to measure the surface area of the cell [29,31,62].  $\mathcal{M}_s$  is the penalty constant.

The term  $H$  denotes the interaction energy density induced by the interaction of cells. There are many different previous works to define interaction potential  $H$ . See [37,46,64].

Here we begin with considering mechanical interaction between two cells identified by phase-field functions  $\phi_1$  and  $\phi_2$ , respectively. Recall that  $\phi_i = 1$  represents the intracellular space, and  $\phi_i = -1$  represents the extracellular space of the  $i$ th cell, respectively. Whether there exists mechanical interaction between the two cells can be determined by measuring the overlap (i.e., occupying the same physical space) of the intracellular spaces of these two cells. To account for adhesion and repulsion between  $\phi_1$  and  $\phi_2$ , we propose the following multi-dimensional Lennard-Jones type interaction energy density

$$H_{12} = Q_1(\phi_1 + 1)^2(\phi_2 + 1)^2 - Q_2(\phi_1^2 - 1)^2(\phi_2^2 - 1)^2. \quad (8)$$

The first term in Eq. (8) accounts for the repulsion and achieves the maximum when two cells overlap at a spatial location  $\mathbf{x}$ , i.e.,  $\phi_1(\mathbf{x}) = \phi_2(\mathbf{x}) = 1$ . The second term represents the adhesion, and is nonzero only when the diffuse



**Fig. 2.** Interaction energy with respect to  $\phi_1$  and  $\phi_2$  at a space point. The energy status of different overlapping condition of the cell phase at the point are pointed out as well. ( $Q_1 = 50$ ,  $Q_2 = 400$ ).

layers representing the membranes of the two cells overlap, i.e., at the location of overlap  $\mathbf{x}$ ,  $-1 < \phi_1(\mathbf{x}) < 1$  and  $-1 < \phi_2(\mathbf{x}) < 1$ . This is used to mimic the adhesive interaction mediated by forming adhesive bonds with CAM on adjacent cell membranes, or via forming binding bonds between CAM and cell-extracellular matrix. In the real world, the formation and dissociation of the bonds are stochastic in nature and depend on the concentrations of agonists [2,65]. Here we let the  $Q_1$  and  $Q_2$  be constants for simplicity. We note that the formation and dissociation of binding bonds occur on an order of micro-second time scale. And our model concerns the dynamics of cells on a second order time scale. Thus it is reasonable to neglect the stochastic effects of CAM binding.

Fig. 2 plots the energy landscape of the interaction potential at a spatial point due to the presence of the phases  $\phi_1$  and  $\phi_2$ . The energy is equal to 0 when these two phases do not touch or overlap i.e.,  $\phi_1(\mathbf{x}) = \phi_2(\mathbf{x}) = -1$ . When they start to overlap, the energy first decreases, which means that the attraction force between these two phases dominates. Then the energy increases, which indicates the repulsive force dominates. This prevents the two phases from occupying the same physical space. So the interacting potential energy behaves conceptually similar to a Lennard-Jones potential. We remark that in [36] a potential of the form  $(\phi_1^2 - 1)^2(\phi_2^2 - 1)$  is used, where only attractive force is included.

Generalization of  $H_{12}$  for the multiple cell interactions leads to the following interaction energy density definition:

$$H = \sum_{i=1}^N \sum_{i < j} H_{ij} = \sum_{i=1}^N \sum_{i < j} [Q_1(\phi_i + 1)^2(\phi_j + 1)^2 - Q_2(\phi_i^2 - 1)^2(\phi_j^2 - 1)^2]. \tag{9}$$

Notice that Eq. (9) is consistent with [40] when the system reaches thermodynamic equilibrium. This also proves the feasibility of our model. Following the above interaction potential definition, the cell–wall interaction energy density is defined by setting  $\phi_2 \equiv \phi_w$  in Eq. (8),

$$f_w(\phi_i) = Q_{w1}(\phi_i + 1)^2(\phi_w + 1)^2 - Q_{w2}(\phi_i^2 - 1)^2(\phi_w^2 - 1)^2, \tag{10}$$

where  $Q_{w1}$  is repulsive energy density coefficient and  $Q_{w2}$  is adhesion energy density coefficient. Notice that the cell–wall energy  $E_w$  is defined on the bulk region of the domain, and is not on the boundary.  $f_w$  is non-zero only when the two phases  $\phi_i$  and  $\phi_w$  overlap. This can be interpreted as that the attraction force is present only when the cell makes contact with (or is close enough to) the wall, and the binding bond is formed.

With the total energy of the system given, the chemical potential  $\mu_i$  for each phase  $\phi_i$  is calculated as follows

$$\begin{aligned} \mu_i &= \frac{\delta(E_{cell} + E_{int} + E_w)}{\delta\phi_i} \\ &= \frac{\hat{\kappa}_B}{\gamma^3} g(\phi_i) + \frac{\mathcal{M}_s}{\gamma} \frac{S(\phi_i) - S(\phi_{i,0})}{S(\phi_{i,0})} f(\phi_i) + \frac{\partial H}{\partial \phi_i} + \frac{\partial f_w(\phi_i)}{\partial \phi_i}, \end{aligned} \tag{11}$$

where  $g(\phi_i) = -\gamma^2 \Delta f(\phi_i) + (3\phi_i^2 - 1)f(\phi_i)$ .

The dissipation functional of the system consists of the dissipation introduced by fluid viscosity, friction near the wall, and interfacial mixing due to the diffuse interface representation [32]:

$$\begin{aligned} \Delta = & \int_{\Omega} 2\eta |\mathbf{D}_{\eta}|^2 d\mathbf{x} + \sum_{i=1}^N \int_{\Omega} \frac{1}{M_{\phi}} |q_{\phi_i}|^2 d\mathbf{x} + \int_{\partial\Omega_w} \beta_s |\mathbf{u}_{\tau}|^2 ds \\ & + \sum_{i=1}^N \left( \int_{\partial\Omega_w} \kappa_{\Gamma} |J_{\Gamma_i}|^2 ds + \int_{\Omega} \xi |\gamma \phi_i \nabla \lambda_i|^2 ds \right), \end{aligned} \quad (12)$$

where  $\mathbf{D}_{\eta} = \frac{\nabla \mathbf{u} + (\nabla \mathbf{u})^T}{2}$ ,  $\eta$  is the viscosity of the fluid,  $\beta_s$  is the fluid–wall friction coefficient,  $M_{\phi}$  and  $\kappa_{\Gamma}$  are the mobility of the phases on the bulk and the boundary of the domain. We note that in general, the viscosity  $\eta$  could be a function of all phase-field phases  $\eta = \eta(\phi_w, \phi_1, \dots, \phi_n)$ .

The specific expressions of the flux and stress functions in the kinematic Eqs. (2a)–(2c), (3) and (4) are obtained by taking the time derivative of the total energy functional and comparing with the defined dissipation functional. The time derivative of the total energy goes:

$$\begin{aligned} \frac{dE_{total}}{dt} &= \frac{d}{dt} E_{kin} + \frac{d}{dt} E_{cell} + \frac{d}{dt} E_w \\ &\equiv I_1 + I_2 + I_3. \end{aligned} \quad (13)$$

Taking the time derivative of  $E_{kin}$ , together with the conservation law of momentum (2b), incompressibility of the fluid (2c) and local inextensibility of the membrane (3) yields

$$\begin{aligned} I_1 &= \frac{d}{dt} \int_{\Omega} \frac{\rho |\mathbf{u}|^2}{2} d\mathbf{x} \\ &= \int_{\Omega} \frac{1}{2} \frac{\partial \rho}{\partial t} |\mathbf{u}|^2 d\mathbf{x} + \int_{\Omega} \rho \frac{d\mathbf{u}}{dt} \cdot \mathbf{u} d\mathbf{x} + \int_{\Omega} \nabla \cdot (\rho \mathbf{u}) \frac{|\mathbf{u}|^2}{2} d\mathbf{x} \\ &= \int_{\Omega} (\nabla \cdot \boldsymbol{\sigma}_{\eta}) \cdot \mathbf{u} d\mathbf{x} + \int_{\Omega} \mathbf{F} \cdot \mathbf{u} d\mathbf{x} + \sum_{i=1}^N \int_{\Omega} \lambda_i \delta_i \mathcal{P}_i : \nabla \mathbf{u} d\mathbf{x} \\ &\quad + \sum_{i=1}^N \int_{\Omega} \xi \gamma^2 \lambda_i \nabla \cdot (\phi_i^2 \nabla \lambda_i) d\mathbf{x} - \int_{\Omega} p I : \nabla \mathbf{u} d\mathbf{x} \\ &= - \int_{\Omega} ((\boldsymbol{\sigma}_{\eta} + pI) : \nabla \mathbf{u}) d\mathbf{x} + \int_{\Omega} \mathbf{F} \cdot \mathbf{u} d\mathbf{x} - \sum_{i=1}^N \int_{\Omega} \nabla \cdot (\lambda_i \delta_i \mathcal{P}_i) \cdot \mathbf{u} d\mathbf{x} \\ &\quad - \sum_{i=1}^N \int_{\Omega} \xi \gamma^2 \phi_i^2 (\nabla \lambda_i)^2 d\mathbf{x} + \int_{\partial\Omega_w} ((\boldsymbol{\sigma}_{\eta} + \sum_{i=1}^N \lambda_i \delta_i \mathcal{P}_i) \cdot \mathbf{n}) \cdot \mathbf{u}_{\tau} dS, \end{aligned} \quad (14)$$

where the slip boundary condition is used. Here the pressure  $p$  is a Lagrangian multiplier and is introduced to ensure the incompressibility of the fluid.

Taking the time derivative of  $E_{cell}$  together with the transport assumption of each field-phase phase representing cells (2a) yields

$$\begin{aligned} I_2 &= \frac{d}{dt} \int_{\Omega} \frac{\hat{\kappa}_B}{2\gamma} \sum_{i=1}^N \left| \frac{f(\phi_i)}{\gamma} \right|^2 d\mathbf{x} + \frac{d}{dt} \int_{\Omega} H d\mathbf{x} + \sum_{i=1}^N \frac{d}{dt} \frac{\mathcal{M}_s}{2} \frac{(S(\phi_i) - S(\phi_{i,0}))^2}{S(\phi_{i,0})} \\ &= \int_{\Omega} \frac{\hat{\kappa}_B}{\gamma} \sum_{i=1}^N \frac{f_i}{\gamma^2} \left( -\gamma^2 \Delta \left( \frac{\partial \phi_i}{\partial t} \right) + (3\phi_i^2 - 1) \frac{\partial \phi_i}{\partial t} \right) d\mathbf{x} + \int_{\Omega} \sum_{i=1}^N \frac{\partial H}{\partial \phi_i} \frac{\partial \phi_i}{\partial t} d\mathbf{x} \\ &\quad + \sum_{i=1}^N \frac{d}{dt} \frac{\mathcal{M}_s}{2} \frac{(S(\phi_i) - S(\phi_{i,0}))^2}{S(\phi_{i,0})} \\ &= \int_{\Omega} \frac{\hat{\kappa}_B}{\gamma^3} \sum_{i=1}^N \left( -\gamma^2 \Delta f_i + (3\phi_i^2 - 1) f_i + \frac{\partial H}{\partial \phi_i} + \frac{\mathcal{M}_s}{\gamma} \frac{S(\phi_i) - S(\phi_{i,0})}{S(\phi_{i,0})} f(\phi_i) \right) \frac{\partial \phi_i}{\partial t} d\mathbf{x} \end{aligned}$$

$$\begin{aligned}
 & - \int_{\partial\Omega_w} \frac{\hat{\kappa}_B}{\gamma} \sum_{i=1}^N f_i \frac{\partial}{\partial t} (\partial_n \phi_i) ds + \int_{\partial\Omega_w} \frac{\hat{\kappa}_B}{\gamma} \sum_{i=1}^N \partial_n f_i \frac{\partial \phi_i}{\partial t} ds \\
 & + \sum_{i=1}^N \int_{\partial\Omega_w} \mathcal{M}_s \frac{S(\phi_i) - S(\phi_{i,0})}{S(\phi_{i,0})} \gamma \partial_n \phi_i \frac{\partial \phi_i}{\partial t} ds \\
 = & \sum_{i=1}^N \left( \int_{\Omega} \mu_i \frac{\partial \phi_i}{\partial t} dx + \int_{\partial\Omega_w} \frac{\hat{\kappa}_B}{\gamma} f_i \frac{\partial}{\partial t} (\partial_n \phi_i) ds + \int_{\partial\Omega_w} \mathcal{M}_s \frac{S(\phi_i) - S(\phi_{i,0})}{S(\phi_{i,0})} \gamma \partial_n \phi_i \frac{\partial \phi_i}{\partial t} ds \right) \\
 = & \sum_{i=1}^N \left( - \int_{\Omega} q_{\phi_i} \cdot \nabla \mu_i dx - \int_{\Omega} \mu_i \mathbf{u} \cdot \nabla \phi_i dx + \int_{\partial\Omega_w} \frac{\hat{\kappa}_B}{\gamma} \partial_n f_i \frac{\partial \phi_i}{\partial t} ds \right. \\
 & \left. + \int_{\partial\Omega_w} \mathcal{M}_s \frac{S(\phi_i) - S(\phi_{i,0})}{S(\phi_{i,0})} \gamma \partial_n \phi_i \frac{\partial \phi_i}{\partial t} ds \right), \tag{15}
 \end{aligned}$$

where the Allan-Cahn boundary condition (4)<sub>3</sub> for each phase is used.

Computing  $\frac{d}{dt} E_w$  yields

$$I_3 = \sum_{i=1}^N \left( \int_{\partial\Omega} \frac{\partial f_w(\phi_i)}{\partial \phi_i} \frac{\partial \phi_i}{\partial t} ds \right). \tag{16}$$

By combining Eqs. (14), (15) and (16), we have:

$$\begin{aligned}
 \frac{d}{dt} E_{total} = & - \int_{\Omega} ((\sigma_{\eta} + pI) : \nabla \mathbf{u}) dx + \int_{\Omega} (\mathbf{F} - \sum_{i=1}^N \mu_i \nabla \phi_i - \sum_{i=1}^N \nabla \cdot (\lambda_i \delta_i \mathcal{P}_i)) \cdot \mathbf{u} dx \\
 & - \sum_{i=1}^N \int_{\Omega} q_{\phi_i} \cdot \nabla \mu_i dx + \sum_{i=1}^N \int_{\Omega} \xi \gamma^2 \phi_i^2 (\nabla \lambda_i)^2 dx \\
 & + \int_{\partial\Omega_w} ((\sigma_{\eta} + \sum_{i=1}^N \lambda_i \delta_i \mathcal{P}_i) \cdot \mathbf{n} - \sum_{i=1}^N \hat{L}_i \nabla_{\Gamma} \phi_i) \cdot \mathbf{u}_{\tau} ds + \sum_{i=1}^N \int_{\partial\Omega_w} \hat{L}_i J_{\Gamma_i} ds, \tag{17}
 \end{aligned}$$

where  $\hat{L}_i = \frac{\hat{\kappa}_B}{\gamma} \partial_n f_i + \mathcal{M}_s \frac{S(\phi_i) - S(\phi_{i,0})}{S(\phi_{i,0})} \gamma \partial_n \phi_i$ .

The energy dissipation law (1) is employed to close the system. In more details, we equal Eq. (17) with the negative of the energy dissipation functional defined in Eq. (12) to ensure the energy dissipation law (1) is satisfied. This gives rise to the following definitions of the flux and stress functions in the kinematic relations (2a)–(2c), (3) and (4).

$$\begin{cases} \sigma_{\eta} = 2\eta \mathbf{D}_{\eta} - pI, & \text{in } \Omega, \\ q_{\phi_j} = M_{\phi_j} \nabla \mu_j, & \text{in } \Omega, \\ \mathbf{F} = \sum_{i=1}^N (\mu_i \nabla \phi_i + \nabla \cdot (\lambda_i \delta_i \mathcal{P}_i)), & \text{in } \Omega, \\ J_{\Gamma_i} = -\kappa_{\Gamma_i}^{-1} \hat{L}_i, & \text{on } \partial\Omega_w, \\ u_{\tau_j} = \beta_s^{-1} (-\mathbf{n} \cdot (\sigma_{\eta} + \sum_{i=1}^N \lambda_i \delta_i \mathcal{P}_i) \cdot \boldsymbol{\tau}_j) + \sum_{i=1}^N \hat{L}_i \partial_{\tau_j} \phi_i, \quad j = 1, 2, & \text{on } \partial\Omega_w. \end{cases} \tag{18}$$

The viscosity, length, velocity, time, bulk, and boundary chemical potentials in the equations are scaled by their corresponding characteristic values  $\eta_0$ ,  $L$ ,  $U$ ,  $\frac{L}{U}$ ,  $\frac{\eta_0 U}{L}$  and  $\eta_0 U$ , respectively. Write  $Q_{w_1}$ ,  $Q_{w_2}$ ,  $Q_1$ ,  $Q_2$  into  $Q_0 q_{w_1}$ ,  $Q_0 q_{w_2}$ ,  $Q_0 q_1$ ,  $Q_0 q_2$ , where  $Q_0$  is the character energy density. To this end, the proposed diffuse interface model for describing the cell–wall, cell–cell interaction, and aggregation in dimensionless form is composed of the

following equations:

$$\begin{cases} Re(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}) + \nabla P = \nabla \cdot (2\eta \mathbf{D}) + \sum_i \mu_i \nabla \phi_i + \sum_i \nabla \cdot (\lambda_i \delta_{\epsilon_i} \mathcal{P}_i), & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0, & \text{in } \Omega, \\ \frac{\partial \phi_i}{\partial t} + \mathbf{u} \cdot \nabla \phi_i = -\mathcal{M} \Delta \mu_i, & \text{in } \Omega, \\ \mu_i = \kappa_B g(\phi_i) + \mathcal{M}_s \frac{S(\phi_i) - S(\phi_{i,0})}{S(\phi_{i,0})} f_i + \alpha \frac{\partial H}{\partial \phi_i} + \alpha \frac{\partial f_w(\phi_i)}{\partial \phi_i}, & \text{in } \Omega, \\ f_i = -\epsilon \Delta \phi_i + \frac{(\phi_i^2 - 1)}{\epsilon} \phi_i, \quad g(\phi_i) = -\Delta f_i + \frac{1}{\epsilon^2} (3\phi_i^2 - 1) f_i, & \text{in } \Omega, \\ \delta_{\epsilon_i} (\mathcal{P}_i : \nabla \mathbf{u}) + \xi \epsilon^2 \nabla \cdot (\phi_i^2 \nabla \lambda_i) = 0, & \text{in } \Omega, \end{cases} \quad (19)$$

with the boundary conditions

$$\begin{cases} \kappa \dot{\phi}_i + L(\phi_i) = 0, & \text{on } \partial \Omega_w, \\ L(\phi_i) = \kappa_B \partial_n f(\phi_i) + \epsilon \mathcal{M}_s \frac{S(\phi_i) - S(\phi_{i,0})}{S(\phi_{i,0})} \partial_n \phi_i, & \text{on } \partial \Omega_w, \\ -l_s^{-1} \mathbf{u}_{\tau_i} = \boldsymbol{\tau}_i \cdot (2\eta \mathbf{D}_\eta + \sum_i \lambda_i \delta_{\epsilon_i} \mathcal{P}_i) \cdot \mathbf{n} - \sum_i L(\phi_i) \partial_{\tau_i} \phi_i, \quad i = 1, 2, & \text{on } \partial \Omega_w, \\ f_i = 0, & \text{on } \partial \Omega_w, \\ \partial_n \lambda_i = 0, \quad \partial_n \mu_i = 0, & \text{on } \partial \Omega_w, \end{cases} \quad (20)$$

where  $S(\phi_i) = \int_{\Omega} \frac{\epsilon}{2} |\nabla \phi_i|^2 + \frac{1}{4\epsilon} (\phi_i^2 - 1)^2 dx$  and  $\delta_{\epsilon_i} = \frac{1}{2} \epsilon^2 |\nabla \phi_i|^2$ .

The dimensionless constants appeared in Eqs. (19)–(20) are given by  $\epsilon = \frac{\gamma}{L}$ ,  $Re = \frac{\rho_0 UL}{\eta_0}$ ,  $\mathcal{M} = \frac{M_\phi \eta_0}{L^2}$ ,  $\kappa_B = \frac{\hat{\kappa}_B}{L^2 \eta_0 U}$ ,  $k = \frac{\hat{\kappa}_B}{\eta_0 L}$ ,  $l_s = \frac{\eta_0}{\beta_s L}$ ,  $\alpha = \frac{Q_0}{\eta_0 U}$ ,  $\mathcal{M}_s = \frac{M_s}{\eta_0 U}$ .

### 3. Energy-law-preserving scheme

We define the Sobolev spaces as follows [31,66]

$$\begin{aligned} \mathbf{W}^{1,3} &= (W^{1,3})^2, \\ \mathbf{W}_N^{1,\frac{3}{2}} &= (W^{1,\frac{3}{2}})^N, \\ \mathbf{W}_N^{1,3} &= (W^{1,3})^N, \\ \mathbf{W}_N^{1,\frac{3}{2}}(\Omega) &= \{ \Lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)^T \}, \\ \mathbf{W}^{1,3}(\Omega) &= \{ \mathbf{u} = (u_x, u_y)^T \in \mathbf{W}^{1,3} \mid \mathbf{u} \cdot \mathbf{n} = 0, \text{ on } \partial \Omega_w \}, \\ \mathbf{W}_\Phi^{1,3}(\Omega) &= \{ \Phi = (\phi_1, \phi_2, \dots, \phi_N)^T \in \mathbf{W}_N^{1,3} \mid -1 \leq \phi_i \leq 1, i = 1, 2, \dots, N, \text{ in } \Omega \}, \\ \mathbf{W}_U^{1,3}(\Omega) &= \{ \mathbf{U} = (\mu_1, \mu_2, \dots, \mu_N)^T \in \mathbf{W}_N^{1,3} \mid \partial_n \mu_i = 0, i = 1, 2, \dots, N, \text{ on } \partial \Omega_w \}, \\ \mathbf{W}_F^{1,3}(\Omega) &= \{ \mathbf{F} = (f_1, f_2, \dots, f_N)^T \in \mathbf{W}_N^{1,3} \mid f_i = 0, i = 1, 2, \dots, N, \text{ on } \partial \Omega_w \}, \\ \mathbf{W}_b &= \mathbf{W}_\Phi^{1,3}(\Omega) \times \mathbf{W}_F^{1,3}(\Omega) \times \mathbf{W}_U^{1,3}(\Omega) \times \mathbf{W}_N^{1,\frac{3}{2}}(\Omega) \times W^{1,\frac{3}{2}}(\Omega) \times \mathbf{W}^{1,3}(\Omega), \end{aligned}$$

and let  $\| \cdot \| = (\int_{\Omega} | \cdot |^2 dx)^{\frac{1}{2}}$  and  $\| \cdot \|_w = (\int_{\partial \Omega_w} | \cdot |^2 ds)^{\frac{1}{2}}$  denote the  $L^2$  norm defined in the domain and on the domain boundary respectively.

**Theorem 3.1.** *If  $(\Phi, F, U, \lambda, P, \mathbf{u}) \in \mathbf{W}_b$  are smooth solutions of the above system (19)–(20), the following energy law holds:*

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_{total} &= \frac{d}{dt} (\mathcal{E}_{kin} + \mathcal{E}_{cell} + \mathcal{E}_w) \\ &= \frac{1}{Re} \left( -2 \|\eta^{1/2} \mathbf{D}_\eta\|^2 - \mathcal{M} \sum_i \|\nabla \mu_i\|^2 - \xi \sum_i \|\epsilon \phi_i \nabla \lambda_i\|^2 - \kappa \sum_i \|\dot{\phi}_i\|_w^2 - \|\mathbf{u}\|_s^{-1/2} \|\mathbf{u}_\tau\|_w^2 \right), \end{aligned} \quad (21)$$

where  $\mathcal{E}_{total} = \mathcal{E}_{kin} + \mathcal{E}_{cell} + \mathcal{E}_w$ ,  $\mathcal{E}_{kin} = \frac{1}{2} \int_{\Omega} |\mathbf{u}|^2 dx$ ,  $\mathcal{E}_{cell} = \frac{\kappa_B}{2Re\epsilon} \sum_i \int_{\Omega} |f_i|^2 dx + \mathcal{M}_s \sum_i \frac{(S(\phi_i) - S(\phi_{i,0}))^2}{2ReS(\phi_{i,0})} + \frac{\alpha}{Re} \int_{\Omega} H dx$  and  $\mathcal{E}_w = \frac{\alpha}{Re} \sum_i \int_{\Omega} f_w(\phi_i) dx$ .

**Proof.** Multiplying the first equation in Eq. (19) with  $\mathbf{u}$  and integration by parts yield

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_{kin} &= \frac{1}{Re} \left\{ - \int_{\Omega} 2\eta |\mathbf{D}_{\eta}|^2 dx + \int_{\partial\Omega_w} (\boldsymbol{\sigma}_{\eta} \cdot \mathbf{n}) \cdot \mathbf{u}_{\tau} ds + \sum_i \int_{\Omega} \mu_i \nabla \phi_i \cdot \mathbf{u} dx \right. \\ &\quad \left. - \sum_i \int_{\Omega} \lambda_i \delta_{\epsilon_i} \mathcal{P}_i : \nabla \mathbf{u} dx + \sum_i \int_{\partial\Omega_w} (\lambda_i \delta_{\epsilon_i} \mathcal{P}_i \cdot \mathbf{n}) \cdot \mathbf{u}_{\tau} ds \right\} \\ &= \frac{1}{Re} \left\{ - \int_{\Omega} 2\eta |\mathbf{D}_{\eta}|^2 dx - \sum_i \int_{\Omega} \lambda_i \delta_{\epsilon_i} \mathcal{P}_i : \nabla \mathbf{u} dx - l_s^{-1} \int_{\partial\Omega_w} |\mathbf{u}_{\tau}|^2 ds \right. \\ &\quad \left. + \sum_i \int_{\partial\Omega_w} L(\phi_i) \partial_{\tau} \phi \cdot \mathbf{u}_{\tau} ds + \sum_i \int_{\Omega} \mu_i \nabla \phi_i \cdot \mathbf{u} dx \right\}, \end{aligned} \tag{22}$$

where the slip boundary condition in Eq. (20) is applied.

Taking the inner product of the third equation in Eq. (19) with  $\frac{\mu_i}{Re}$  and summing up with respect to  $i$  result in

$$\frac{1}{Re} \sum_i \int_{\Omega} \frac{\partial \phi_i}{\partial t} \mu_i dx + \frac{1}{Re} \sum_i \int_{\Omega} \mathbf{u} \cdot \nabla \phi_i \mu_i dx = - \frac{1}{Re} \mathcal{M} \sum_i \int_{\Omega} |\nabla \mu_i|^2 dx, \tag{23}$$

where  $\partial_n \mu_i = 0$  is considered here.

Multiplying the fourth equation in Eq. (19) with  $\frac{1}{Re} \frac{\partial \phi_i}{\partial t}$  and integration by parts give rise to

$$\begin{aligned} &\frac{1}{Re} \sum_i \int_{\Omega} \mu \frac{\partial \phi_i}{\partial t} dx \tag{24} \\ &= \frac{1}{Re} \sum_i \left\{ \kappa_B \int_{\Omega} g_i \frac{\partial \phi_i}{\partial t} dx + \mathcal{M}_s \frac{S(\phi_i) - S(\phi_{i,0})}{S(\phi_{i,0})} \int_{\Omega} f_i \frac{\partial \phi_i}{\partial t} dx + \alpha \int_{\Omega} \frac{\partial H}{\partial \phi_i} \frac{\partial \phi_i}{\partial t} dx + \alpha \int_{\Omega} \frac{f_w(\phi_i)}{\partial \phi_i} \frac{\partial \phi_i}{\partial t} dx \right\} \\ &= \frac{\kappa_B}{Re} \sum_i \int_{\Omega} f_i \frac{\partial}{\partial t} \left( -\Delta \phi_i + \frac{1}{\epsilon^2} (\phi_i^3 - \phi_i) \right) dx - \frac{\kappa_B}{Re} \sum_i \int_{\partial\Omega_w} \partial_n f_i \frac{\partial \phi_i}{\partial t} ds \\ &\quad + \mathcal{M}_s \sum_i \frac{d}{dt} \left( \frac{(S(\phi_i) - S(\phi_{i,0}))^2}{2ReS(\phi_{i,0})} \right) - \mathcal{M}_s \sum_i \left( \frac{S(\phi_i) - S(\phi_{i,0})}{ReS(\phi_{i,0})} \right) \int_{\partial\Omega_w} \epsilon \partial_n \phi_i \frac{\partial \phi_i}{\partial t} ds \\ &\quad + \frac{\alpha}{Re} \sum_i \int_{\Omega} \frac{\partial f_w(\phi_i)}{\partial \phi_i} \frac{\partial \phi_i}{\partial t} dx + \frac{\alpha}{Re} \sum_i \int_{\Omega} \frac{\partial H}{\partial \phi_i} \frac{\partial \phi_i}{\partial t} dx \\ &= \frac{d}{dt} \left( \kappa_B \sum_i \int_{\Omega} \frac{|f_i|^2}{2Re\epsilon} dx \right) + \mathcal{M}_s \frac{d}{dt} \left( \sum_i \frac{(S(\phi_i) - S(\phi_{i,0}))^2}{2ReS(\phi_{i,0})} \right) + \frac{\alpha}{Re} \frac{d}{dt} \sum_i \int_{\Omega} f_w(\phi_i) dx \\ &\quad + \frac{\alpha}{Re} \frac{d}{dt} \int_{\Omega} H dx - \sum_i \int_{\partial\Omega_w} \frac{L(\phi_i)}{Re} \frac{\partial \phi_i}{\partial t} ds \\ &= \frac{d}{dt} (\mathcal{E}_{cell} + \mathcal{E}_w) - \int_{\partial\Omega_w} \frac{L(\phi)}{Re} \frac{\partial \phi}{\partial t} ds, \end{aligned}$$

where the definitions of  $f(\phi)$ ,  $g(\phi)$  and the boundary conditions of  $\phi$  and  $f$  are utilized.

Multiplying the last equations with  $\frac{\lambda_i}{Re}$  and integration by parts and sum up by  $i$  leads to

$$\frac{1}{Re} \sum_i \int_{\Omega} (\lambda_i \delta_{\epsilon_i} \mathcal{P}_i) : \nabla \mathbf{u} dx - \frac{1}{Re} \sum_i \int_{\Omega} \xi \epsilon^2 \phi_i^2 (\nabla \lambda_i)^2 dx = 0. \tag{25}$$

Finally, the energy dissipation law (21) is obtained by combining Eqs. (22), (23), (24) and (25) considering the boundary conditions in (20).  $\square$

In the rest of this section, we propose an energy-law-preserving, second-order accurate in both space and time scheme for solving the model system (19)–(20).

### 3.1. Time-discrete primitive method

The mid-point method is utilized for the temporal discretization of Eqs. (19)–(20). Let  $\Delta t$  denote the time step size,  $(\cdot)^{n+1}$  and  $(\cdot)^n$  denote the values of the variables at times  $(n + 1)\Delta t$  and  $n\Delta t$ , respectively. The semi-discrete in time scheme to solve Eqs. (19)–(20) is as follows:

$$\left\{ \begin{aligned} & \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} + (\mathbf{u}^{n+\frac{1}{2}} \cdot \nabla) \mathbf{u}^{n+\frac{1}{2}} + \frac{1}{Re} \nabla P^{n+\frac{1}{2}} = \frac{1}{Re} \nabla \cdot (\eta^n (\nabla \mathbf{u}^{n+\frac{1}{2}} + (\nabla \mathbf{u}^{n+\frac{1}{2}})^T)) \\ & \quad + \frac{1}{Re} \sum_i \mu_i^{n+\frac{1}{2}} \nabla \phi_i^{n+\frac{1}{2}} + \sum_i \frac{1}{Re} \nabla \cdot \left( \lambda_i^{n+\frac{1}{2}} \mathcal{P}_i^n \delta_{\epsilon_i} \right), \\ & \nabla \cdot \mathbf{u}^{n+\frac{1}{2}} = 0, \\ & \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} + (\mathbf{u}^{n+\frac{1}{2}} \cdot \nabla) \phi_i^{n+\frac{1}{2}} = -\mathcal{M} \Delta \mu_i^{n+\frac{1}{2}}, \\ & \mu_i^{n+\frac{1}{2}} = \kappa_B g(\phi_i^{n+1}, \phi_i^n) + \mathcal{M}_s \frac{(S(\phi_i^{n+\frac{1}{2}}) - S(\phi_{0_i}))}{S(\phi_{0_i})} f(\phi_i^{n+1}, \phi_i^n) \\ & \quad + \alpha \frac{H_i^{n+1} - H_i^n}{\phi_i^{n+1} - \phi_i^n} + \alpha \frac{f_w(\phi_i^{n+1}) - f_w(\phi_i^n)}{\phi_i^{n+1} - \phi_i^n}, \\ & f_i^{n+\frac{1}{2}} = -\epsilon \Delta \phi_i^{n+\frac{1}{2}} + \frac{1}{\epsilon} ((\phi_i^{n+\frac{1}{2}})^2 - 1) \phi_i^{n+\frac{1}{2}}, \\ & \delta_{\epsilon_i} \mathcal{P}_i^n : \nabla \mathbf{u}^{n+\frac{1}{2}} + \xi \epsilon^2 \nabla \cdot ((\phi_i^n)^2 \nabla \lambda_i^{n+\frac{1}{2}}) = 0, \end{aligned} \right. \tag{26}$$

with boundary conditions on  $\partial \Omega_w$ ,

$$\left\{ \begin{aligned} & \kappa \dot{\phi}_i^{n+\frac{1}{2}} = -L_i^{n+\frac{1}{2}}, \\ & L_i^{n+\frac{1}{2}} = \kappa_B \partial_n f_i^{n+\frac{1}{2}} + \mathcal{M}_s \epsilon \frac{S(\phi_i^{n+\frac{1}{2}}) - S_{0_i}}{S_{0_i}} \partial_n \phi_i^{n+\frac{1}{2}}, \\ & -l_s^{-1} \mathbf{u}_{\tau_j}^{n+\frac{1}{2}} = \boldsymbol{\tau}_j \cdot (\eta^n (\nabla \mathbf{u}^{n+\frac{1}{2}} + (\nabla \mathbf{u}^{n+\frac{1}{2}})^T) + \sum_i \lambda_i^{n+\frac{1}{2}} \delta_{\epsilon_i} \mathcal{P}_i^n) \cdot \mathbf{n} \\ & \quad - \sum_i L_i^{n+\frac{1}{2}} \partial_{\tau_j} \phi_i^{n+\frac{1}{2}}, \quad j = 1, 2, \\ & f_i^{n+\frac{1}{2}} = 0, \\ & \partial_n \lambda_i^{n+\frac{1}{2}} = 0, \end{aligned} \right. \tag{27}$$

with  $(\cdot)^{n+\frac{1}{2}} = \frac{(\cdot)^n + (\cdot)^{n+1}}{2}$  and  $\mathcal{P}_i^n = I - \mathbf{n}_m^n \otimes \mathbf{n}_m^n$  with  $\mathbf{n}_m^n = \frac{\nabla \phi_i^n}{|\nabla \phi_i^n|}$  and

$$\left\{ \begin{aligned} & f(\phi_i^{n+1}, \phi_i^n) = -\epsilon \Delta \phi_i^{n+\frac{1}{2}} + \frac{1}{4\epsilon} ((\phi_i^{n+1})^2 + (\phi_i^n)^2 - 2)(\phi_i^{n+1} + \phi_i^n), \\ & g(\phi_i^{n+1}, \phi_i^n) = \left( -\Delta f_i^{n+\frac{1}{2}} + \frac{1}{\epsilon^2} ((\phi_i^{n+1})^2 + (\phi_i^n)^2 + \phi_i^{n+1} \phi_i^n - 1) f_i^{n+\frac{1}{2}} \right), \\ & H_i^n = q_1 (\phi_i^n + 1)^2 \sum_{j \neq i} \left[ (\phi_j^{n+\frac{1}{2}} + 1)^2 \right] - q_2 ((\phi_i^n)^2 - 1)^2 \sum_{j \neq i} \left[ ((\phi_j^{n+\frac{1}{2}})^2 - 1)^2 \right], \\ & H_i^{n+1} = q_1 (\phi_i^{n+1} + 1)^2 \sum_{j \neq i} \left[ (\phi_j^{n+\frac{1}{2}} + 1)^2 \right] - q_2 ((\phi_i^{n+1})^2 - 1)^2 \sum_{j \neq i} \left[ ((\phi_j^{n+\frac{1}{2}})^2 - 1)^2 \right], \\ & f_w(\phi^n) = q_{w_1} (\phi_i^n + 1)^2 (\phi_w + 1)^2 - q_{w_2} ((\phi_i^n)^2 - 1)((\phi_w)^2 - 1). \end{aligned} \right. \tag{28}$$

Thus we have

$$\begin{aligned}
 & \frac{H_i^{n+1} - H_i^n}{\phi_i^{n+1} - \phi_i^n} \\
 &= \frac{1}{\phi_i^{n+1} - \phi_i^n} \left( q_1(\phi_i^{n+1} + 1)^2 \sum_{j \neq i} \left[ (\phi_j^{n+\frac{1}{2}} + 1)^2 \right] - q_2((\phi_i^{n+1})^2 - 1)^2 \sum_{j \neq i} \left[ ((\phi_j^{n+\frac{1}{2}})^2 - 1)^2 \right] \right. \\
 & \quad \left. - q_1(\phi_i^n + 1)^2 \sum_{j \neq i} \left[ (\phi_j^{n+\frac{1}{2}} + 1)^2 \right] + q_2((\phi_i^n)^2 - 1)^2 \sum_{j \neq i} \left[ (\phi_j^{n+\frac{1}{2}})^2 - 1 \right] \right) \\
 &= q_1(\phi_i^{n+1} + \phi_i^n + 2) \sum_{j \neq i} \left[ (\phi_j^{n+\frac{1}{2}} + 1)^2 \right] \\
 & \quad - q_2(\phi_i^{n+1} + \phi_i^n)((\phi_i^{n+1})^2 + (\phi_i^n)^2 - 2) \sum_{j \neq i} \left[ ((\phi_j^{n+\frac{1}{2}})^2 - 1)^2 \right] \tag{29}
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 \frac{f_w(\phi_i^{n+1}) - f_w(\phi_i^n)}{\phi_i^{n+1} - \phi_i^n} &= q_{w1}(\phi_i^{n+1} + \phi_i^n + 2)(\phi_w + 1)^2 \\
 & \quad - q_{w2}(\phi_i^{n+1} + \phi_i^n)((\phi_i^{n+1})^2 + (\phi_i^n)^2 - 2)(\phi_w^2 - 1)^2 \tag{30}
 \end{aligned}$$

We keep using the notations  $\frac{f_w(\phi_i^{n+1}) - f_w(\phi_i^n)}{\phi_i^{n+1} - \phi_i^n}$  and  $\frac{H(\phi_i^{n+1}) - H(\phi_i^n)}{\phi_i^{n+1} - \phi_i^n}$  for convenience in later derivation.

**Remark 3.1.** Note that the discrete scheme solving the model system is a fully coupled nonlinear algorithm. This may increase the cost of numerical simulation. Nevertheless, the energy-law-preserving scheme has to involve solving a nonlinear system to the best of the authors' efforts. Recent papers have introduced a number of linear decoupled schemes like IEQ and SAV methods. These methods could be considered in further studies.

**Theorem 3.2.** If  $(\phi_i^n, \mu_i^n, \mathbf{u}^n, P^n)$  are smooth solutions of the above system (26)–(27), the following energy law is satisfied:

$$\begin{aligned}
 \mathcal{E}_{total}^{n+1} - \mathcal{E}_{total}^n &= (\mathcal{E}_{kin}^{n+1} + \sum_i^N [\mathcal{E}_{cell_i}^{n+1} + \mathcal{E}_{i,int}^{n+1} + \mathcal{E}_{w_i}^{n+1}]) - (\mathcal{E}_{kin}^n + \sum_i^N [\mathcal{E}_{cell_i}^n + \mathcal{E}_{i,int}^n + \mathcal{E}_{w_i}^n]) \\
 &= \frac{\Delta t}{Re} \left( -2\|(\eta^n)^{1/2} \mathbf{D}_\eta^{n+\frac{1}{2}}\|^2 - \mathcal{M} \sum_i^N \|\nabla \mu_i^{n+\frac{1}{2}}\|^2 - \xi \sum_i^N \|\epsilon \phi_i^n \nabla \lambda_i^{n+\frac{1}{2}}\|^2 \right. \\
 & \quad \left. - \frac{1}{\kappa} \left\| \sum_i^N L(\phi_i^{n+\frac{1}{2}}) \right\|_w^2 - \|l_s^{-1/2} \mathbf{u}_\tau^{n+\frac{1}{2}}\|_w^2 \right), \tag{31}
 \end{aligned}$$

where  $\mathcal{E}_{total}^n = \mathcal{E}_{kin}^n + \sum_i^N [\mathcal{E}_{cell_i}^n + \mathcal{E}_{i,int}^n + \mathcal{E}_{w_i}^n]$  with  $\mathcal{E}_{kin}^n = \frac{1}{2} \|\mathbf{u}^n\|^2$ ,  $\mathcal{E}_{cell_i}^n = \frac{\kappa_B \|f_i^n\|^2}{2Re\epsilon} + \mathcal{M}_s \frac{(S(\phi_i^n) - S(\phi_{i,0}))^2}{2ReS(\phi_{i,0})} + \frac{\alpha}{Re} H_i^n$  and  $\mathcal{E}_{i,w}^n = \frac{\alpha}{Re} \int_\Omega f_{i,w}^n dx$ .

**Proof.** Multiplying the first equation in system (26) by  $\Delta t \mathbf{u}^{n+\frac{1}{2}}$  gives

$$\begin{aligned} & \int_{\Omega} \frac{1}{2} ((\mathbf{u}^{n+1})^2 - (\mathbf{u}^n)^2) d\mathbf{x} + \int_{\Omega} \Delta t \mathbf{u}^{n+\frac{1}{2}} \cdot ((\mathbf{u}^{n+\frac{1}{2}} \nabla) \cdot \mathbf{u}^{n+\frac{1}{2}}) d\mathbf{x} \\ & - \frac{\Delta t}{Re} \int_{\Omega} \mathbf{P}^{n+\frac{1}{2}} \nabla \cdot \mathbf{u}^{n+\frac{1}{2}} d\mathbf{x} \\ = & - \frac{\Delta t}{Re} \int_{\Omega} \nabla \mathbf{u}^{n+\frac{1}{2}} : \eta^n (\nabla \mathbf{u}^{n+\frac{1}{2}} + (\nabla \mathbf{u}^{n+\frac{1}{2}})^T) d\mathbf{x} + \frac{\Delta t}{Re} \sum_i \int_{\Omega} \mathbf{u}^{n+\frac{1}{2}} \cdot \nabla \phi_i^{n+1} \mu_i^{n+1} d\mathbf{x} \\ & - \frac{\Delta t}{Re} \sum_i \int_{\Omega} \lambda_i \delta_{\epsilon_i} \mathcal{P}_i^n : \nabla \mathbf{u}^{n+\frac{1}{2}} d\mathbf{x} + \frac{\Delta t}{Re} \sum_i \int_{\partial \Omega_w} \lambda_i^{n+\frac{1}{2}} (\delta_{\epsilon_i} \mathcal{P}_i^n \cdot \mathbf{n}) \cdot \mathbf{u}_{\tau}^{n+\frac{1}{2}} ds \\ & + \frac{\Delta t}{Re} \int_{\partial \Omega_w} \mathbf{u}^{n+\frac{1}{2}} \cdot \eta^n ((\nabla \mathbf{u}^{n+\frac{1}{2}} + (\nabla \mathbf{u}^{n+\frac{1}{2}})^T) \cdot \mathbf{n}) ds . \end{aligned} \tag{32}$$

Multiplying the fourth equation in system (26) by  $\frac{\phi_i^{n+1} - \phi_i^n}{Re}$  and integration by parts lead to

$$\begin{aligned} & \frac{1}{Re} \sum_i \int_{\Omega} \mu_i^{n+1/2} (\phi_i^{n+1} - \phi_i^n) d\mathbf{x} \\ = & \frac{\kappa_B}{Re} \sum_i \int_{\Omega} \frac{1}{2\epsilon} ((f_i^{n+1})^2 - (f_i^n)^2) d\mathbf{x} + \frac{\mathcal{M}_s (S(\phi_i^{n+1}) - S_{i,0})^2 - (S(\phi_i^n) - S_{i,0})^2}{2S_{i,0}} \\ & + \frac{\alpha}{Re} \sum_i \int_{\Omega} (H_i^{n+1} - H_i^n) d\mathbf{x} + \frac{\alpha}{Re} \sum_i \int_{\Omega} (f_w(\phi_i^{n+1}) - f_w(\phi_i^n)) d\mathbf{x} \\ & - \frac{\kappa_B}{Re} \sum_i \int_{\partial \Omega_w} \partial_n f_i^{n+\frac{1}{2}} (\phi_i^{n+1} - \phi_i^n) ds \\ & - \frac{\mathcal{M}_s}{Re} \sum_i \int_{\partial \Omega_w} \frac{S(\phi_i^{n+\frac{1}{2}}) - S_{i,0}}{S_{i,0}} \epsilon \partial_n \phi_i^{n+\frac{1}{2}} (\phi_i^{n+1} - \phi_i^n) ds . \end{aligned} \tag{33}$$

Multiplying the third equation in system (26) by  $\frac{\mu_i^{n+\frac{1}{2}} \Delta t}{Re}$  yields

$$\begin{aligned} & \frac{1}{Re} \sum_i \int_{\Omega} \mu_i^{n+\frac{1}{2}} (\phi_i^{n+1} - \phi_i^n) d\mathbf{x} + \frac{\Delta t}{Re} \sum_i \int_{\Omega} \mu_i^{n+\frac{1}{2}} (\mathbf{u}^{n+\frac{1}{2}} \cdot \nabla) \phi_i^{n+\frac{1}{2}} d\mathbf{x} \\ = & - \frac{\mathcal{M} \Delta t}{Re} \sum_i \int_{\Omega} (\nabla \mu_i^{n+\frac{1}{2}})^2 d\mathbf{x} . \end{aligned} \tag{34}$$

Multiplying the last equation in system (26) by  $\frac{\lambda_i^{n+\frac{1}{2}} \Delta t}{Re}$  and integration by parts then sum by  $i$  give

$$\frac{\Delta t}{Re} \sum_i \int_{\Omega} (\lambda_i^{n+\frac{1}{2}} \delta_{\epsilon_i} \mathcal{P}_i^n) : \nabla \mathbf{u}^{n+\frac{1}{2}} d\mathbf{x} - \frac{\Delta t}{Re} \sum_i \int_{\Omega} \xi \epsilon^2 (\phi_i^n)^2 \left| \nabla \lambda_i^{n+\frac{1}{2}} \right|^2 d\mathbf{x} = 0 . \tag{35}$$

The discretized energy dissipation law (31) is obtained by combining Eqs. (32)–(35) and organizing the terms according to the boundary conditions  $L(\phi_i)$  as shown in (27).  $\square$

The spatial discretization using  $C^0$  finite element is straight forward. Let  $\Omega$  be the domain of interest with a Lipschitz-continuous boundary  $\partial \Omega$ . Let  $\mathbf{W}_b^h \subset \mathbf{W}_b$  be a finite element space with respect to the triangulation of the domain  $\Omega$ . The fully discrete scheme of the system is to find

$$(\{\Phi_h\}^{n+1}, \{\mathbf{U}\}_h^{n+1}, \{F_h\}^{n+1}, \{\Lambda_h\}^{n+1}, \{p_h\}^{n+1}, \{\mathbf{u}_h\}^{n+1}) \in \mathbf{W}_b^h,$$

such that for any  $(\psi_{1,h}, \dots, \psi_{N,h}, \chi_{1,h}, \dots, \chi_{N,h}, \zeta_{1,h}, \dots, \zeta_{N,h}, \Theta_{1,h}, \dots, \Theta_{N,h}, q_h, \mathbf{v}_h) \in \mathbf{W}_b^h$ , the following scheme holds.

$$\left\{ \begin{aligned} & \int_{\Omega} \left( \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t} + (\mathbf{u}_h^{n+\frac{1}{2}} \cdot \nabla) \mathbf{u}_h^{n+\frac{1}{2}} + \frac{1}{Re} \nabla P_h^{n+\frac{1}{2}} \right) \cdot \mathbf{v}_h d\mathbf{x} \\ &= - \int_{\Omega} \frac{1}{Re} (\eta_h^n (\nabla \mathbf{u}_h^{n+\frac{1}{2}} + (\nabla \mathbf{u}_h^{n+\frac{1}{2}})^T)) : \nabla \mathbf{v}_h d\mathbf{x} \\ & \quad + \sum_i \int_{\Omega} \frac{1}{Re} \mu_{i,h}^{n+\frac{1}{2}} \nabla \phi_{i,h}^{n+\frac{1}{2}} \cdot \mathbf{v}_h d\mathbf{x} - \sum_i \int_{\Omega} \frac{1}{Re} \lambda_{i,h}^{n+\frac{1}{2}} \mathcal{P}_{i,h}^n \delta_{i,h,\epsilon} : \mathbf{v}_h d\mathbf{x} \\ & \quad + \int_{\partial \Omega_w} \frac{1}{Re} \mathbf{n} \cdot (\eta_h^n (\nabla \mathbf{u}_h^{n+\frac{1}{2}} + (\nabla \mathbf{u}_h^{n+\frac{1}{2}})^T) + \sum_i \lambda_{i,h}^{n+\frac{1}{2}} \mathcal{P}_{i,h}^n \delta_{i,\epsilon}) \cdot \mathbf{v}_h d\mathbf{x} , \\ & \int_{\Omega} (\nabla \cdot \mathbf{u}_h^{n+\frac{1}{2}}) q_h d\mathbf{x} = 0 , \\ & \int_{\Omega} \left( \frac{\phi_{i,h}^{n+1} - \phi_{i,h}^n}{\Delta t} + (\mathbf{u}_h^{n+\frac{1}{2}} \cdot \nabla) \phi_{i,h}^{n+\frac{1}{2}} \right) \psi_{i,h} d\mathbf{x} = - \int_{\Omega} \mathcal{M} \nabla \mu_{i,h}^{n+\frac{1}{2}} \nabla \psi_{i,h} d\mathbf{x} , \\ & \int_{\Omega} \mu_{i,h}^{n+\frac{1}{2}} \chi_{i,h} d\mathbf{x} = \int_{\Omega} \left( \kappa_B \frac{1}{\epsilon^2} ((\phi_{i,h}^{n+1})^2 + (\phi_{i,h}^n)^2 + \phi_{i,h}^{n+1} \phi_{i,h}^n - 1) f_{i,h}^{n+\frac{1}{2}} \right. \\ & \quad \left. + \mathcal{M}_s \frac{(S(\phi_{i,h}^{n+\frac{1}{2}}) - S(\phi_{i,h,0}))}{S(\phi_{i,h,0})} \left( \frac{1}{4\epsilon} ((\phi_{i,h}^{n+1})^2 + (\phi_{i,h}^n)^2 - 2)(\phi_{i,h}^{n+1} + \phi_{i,h}^n) \right) \right) \chi_{i,h} d\mathbf{x} \\ & \quad + \int_{\Omega} (\kappa_B \nabla f_{i,h}^{n+\frac{1}{2}} + \mathcal{M}_s \epsilon \frac{(S(\phi_{i,h}^{n+\frac{1}{2}}) - S(\phi_{i,h,0}))}{S(\phi_{i,h,0})} \nabla \phi_{i,h}^{n+\frac{1}{2}}) \cdot \nabla \chi_{i,h} d\mathbf{x} \\ & \quad + \int_{\Omega} \alpha \frac{f_w(\phi_{i,h}^{n+1}) - f_w(\phi_{i,h}^n)}{\phi_{i,h}^{n+1} - \phi_{i,h}^n} \chi_{i,h} d\mathbf{x} + \int_{\Omega} \alpha \frac{(H_{i,h}^{n+1} - H_{i,h}^n)}{\phi_{i,h}^{n+1} - \phi_{i,h}^n} \chi_{i,h} d\mathbf{x} \\ & \quad - \int_{\partial \Omega_w} (\kappa_B \partial_{\mathbf{n}} f_{i,h}^{n+\frac{1}{2}} + \mathcal{M}_s \epsilon \frac{(S(\phi_{i,h}^{n+\frac{1}{2}}) - S(\phi_{i,h,0}))}{S(\phi_{i,h,0})} \partial_{\mathbf{n}} \phi_{i,h}^{n+\frac{1}{2}}) \chi_{i,h} ds , \\ & \int_{\Omega} f_{i,h}^{n+\frac{1}{2}} \zeta_{i,h} = \int_{\Omega} \epsilon \nabla \phi_{i,h}^{n+\frac{1}{2}} \cdot \nabla \zeta_{i,h} + \int_{\Omega} \frac{1}{\epsilon} ((\phi_{i,h}^{n+\frac{1}{2}})^2 - 1) \phi_{i,h}^{n+\frac{1}{2}} \zeta_{i,h} d\mathbf{x} \\ & \quad - \int_{\partial \Omega_w} \epsilon \partial_{\mathbf{n}} \phi_{i,h}^{n+\frac{1}{2}} \zeta_{i,h} d\mathbf{x} , \\ & \int_{\Omega} \delta_{i,h,\epsilon} \mathcal{P}_{i,h}^n : \nabla \mathbf{u}_{i,h}^{n+\frac{1}{2}} \Theta_{i,h} d\mathbf{x} - \int_{\Omega} \xi \epsilon^2 ((\phi_{i,h}^n)^2 \nabla \lambda_{i,h}^{n+\frac{1}{2}}) \cdot \nabla \Theta_{i,h} d\mathbf{x} \\ & \quad + \int_{\partial \Omega_w} \xi \epsilon^2 ((\phi_{i,h}^n)^2 \partial_{\mathbf{n}} \lambda_{i,h}^{n+\frac{1}{2}}) \Theta_{i,h} d\mathbf{x} = 0 . \end{aligned} \right.$$

Newton’s iteration method is applied to solve the above nonlinear system. The unique solvability of 3.2 can be proved by following the approach introduced in [32].

### 4. Numerical results

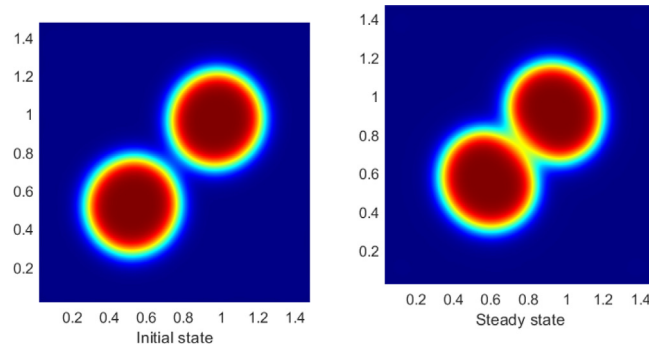
In this section, we first calibrate the model parameters and validate the model by comparing with the experimental data on RBC deformation under different stretching forces. Then cell–wall interaction simulations are used to study the effects of the cell–wall adhesion force and the local insensibility of cell membrane. The energy preservation of the numerical scheme is also illustrated.

Rest of this section is devoted to studying the cell aggregation and RBC motion when passing bifurcated blood vessel.

#### 4.1. Convergence study

The temporal convergence rate of the scheme is tested here. The setting of this convergence study is shown in Fig. 3. We place two cells next to each other initially. Due to the interaction potential  $H$ , motion and deformation of the cells occur, and the system gradually evolves to the steady state in which the two cells no longer move and deform.

The parameter values used for this simulation are chosen as follows:  $Re = 2 \times 10^{-5}$ ,  $\mathcal{M} = 5$ ,  $\kappa_B = 2 \times 10^{-2}$ ,  $\epsilon = 4 \times 10^{-2}$ ,  $\mathcal{M}_s = 0$ ,  $\alpha = 50$ ,  $q_1 = 2$ ,  $q_2 = 1$ . The non-slip boundary condition is used for convenience. We take

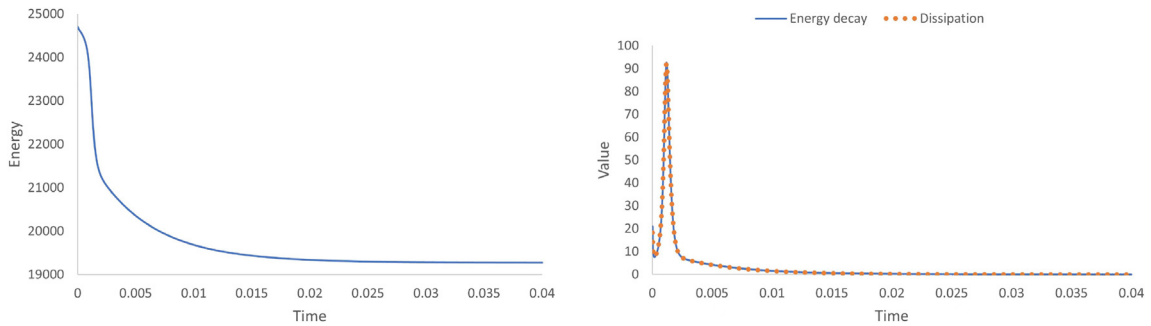


**Fig. 3.** The initial and steady states of the temporal convergence test. Two cells are placed side by side with each other initially. As time evolves, these two cells adhere to form an aggregate, and show deformation at the steady state.

**Table 1**

$L^2$  norm of the error and convergence rate for velocity  $\mathbf{u} = (u_x, u_y)$ , phase-field function  $\phi$ , at time  $t = 3.24 \times 10^{-4}$  with both intercellular and extracellular fluid viscosities being 1.

Time step $\Delta t (\times 10^{-6})$	Error( $u_x$ )	Convergence rate( $u_x$ )	Error( $u_y$ )	Convergence rate( $u_y$ )	Error( $\phi$ )	Convergence rate( $\phi$ )
10.125	5.81e-4		5.82e-4		1.75e-3	
6.75	2.68e-4	1.90	2.68e-4	1.90	8.33e-4	1.83
4.5	1.16e-4	2.07	1.16e-4	2.07	3.76e-4	1.97
3	3.93e-5	2.67	3.93e-5	2.68	1.33e-4	2.58

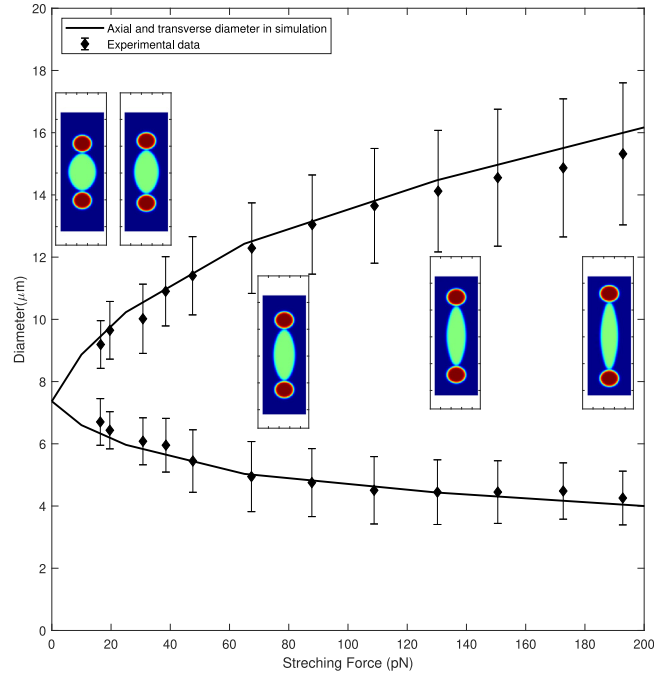


**Fig. 4.** Energy dissipation law. Left: The evolution of discrete energy. Right: Scheme energy-law-preserving. Blue solid line: the energy change between each time step  $\mathcal{E}_{total}^{n+1} - \mathcal{E}_{total}^n$ ; Red dots: the dissipation RHS of Eq. (31) in each time step. The curves fit each other very well, which means that the energy-law-preserving law holds well. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

the solution computed using  $\Delta t = 2 \times 10^{-6}$  as the reference solution. The convergence result is shown in Table 1. Clearly, the discrete scheme is second-order accurate in time.

In Fig. 4, the results demonstrate the energy-law-preserving property of the scheme. The left panel provides evidence of energy decay over time. In the right panel, the blue solid line represents the energy change between each time step  $\mathcal{E}_{total}^{n+1} - \mathcal{E}_{total}^n$ , while the red dots correspond to the discrete dissipation (RHS of Eq. (31)). The plot confirms that the change of the discrete energy agrees with the energy law in Eq. (31), indicating that the proposed scheme is energy-law-preserving.

**Remark 4.1.** In this case, we set the coefficient  $\mathcal{M}_s$  to 0 in order to eliminate the possible effect on the convergence order from the  $\mathcal{M}_s$  related global term. In addition, this term is added for the purpose of keeping the total surface area constant. In the rest of the paper  $\mathcal{M}_s$  is set to a nonzero value to preserve the surface area of the cell.



**Fig. 5.** Nonlinear elastic deformation of red blood cells. The curve represents the relationship between diameter and stretching force by simulation. The diamonds on the curve represent the experimental data. In the experimental schematic diagram, the central phase represents the cell, and the surrounding regions on both sides represent the optical tweezers. The force applied to the membrane remains constant while the tweezers move. Equilibrium is reached when the membrane no longer extends under a specific stretching force. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

**Remark 4.2.** Here we only present the results of the temporal convergence test. The spatial convergence rate is also the second order, which is the same as that in [32]. The same  $C^0$  finite element discretization is used in both this paper and [32].

#### 4.2. Benchmark: Red blood cell deformation under stretching force

Laboratory experiments have tested the non-linear elasticity and deformation of RBC [67] in which optical tweezers are used to provide stretching force to the cells. We set up a numerical simulation mimicking RBC deformation in the experiment. Here we take the top view of the cell, i.e., the circle shape, for 2D simulation. See the inset of Fig. 5. In order to model the interaction between two optical tweezers and the RBC,  $\phi_2$  in Eq. (9) is replaced by new phases  $\phi_{tw_1}$ ,  $\phi_{tw_2}$  to represent the two optical tweezers, respectively. The interaction energy thus becomes

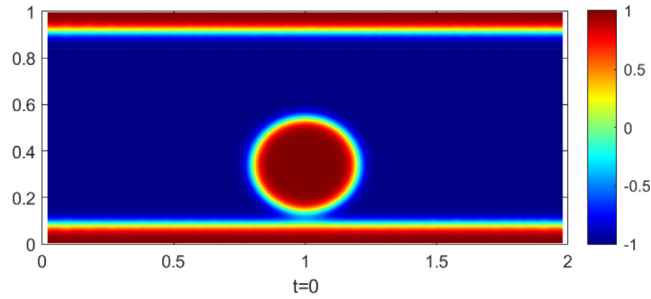
$$H_{tw_i} = Q_{tw_1}(\phi_1 + 1)^2(\phi_{tw_i} + 1)^2 - Q_{tw_2}(\phi_1^2 - 1)^2(\phi_{tw_i}^2 - 1)^2, \quad i = 1, 2. \quad (36)$$

The force of the optical tweezers applied on the RBC is calculated by the following equation:

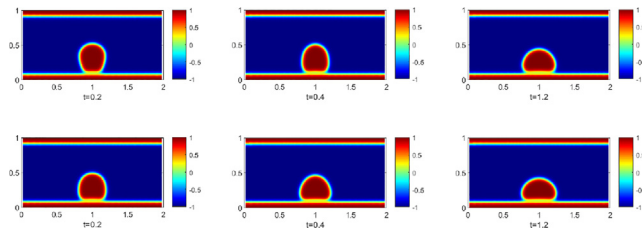
$$\mathbf{F} = \int_{\Omega} \frac{\partial H_{tw_i}}{\partial \phi} \nabla \phi d\mathbf{x}. \quad (37)$$

Values of other parameters used here are as follows:  $Re = 2 \times 10^{-4}$ ,  $\mathcal{M} = 0.25$ ,  $\kappa_B = 2 \times 10^{-3}$ ,  $k = 2 \times 10^{-12}$ ,  $l_s = 5 \times 10^{-3}$ ,  $\mathcal{M}_s = 2$ .

The curves of axial and transverse diameter versus stretching forces are shown in Fig. 5 together with the experimental data (diamonds with bars) from [67]. The results show that our model fits the experimental data very well.



**Fig. 6.** The initial condition for the test case of cell–wall attraction.



**Fig. 7.** The top three pictures show the deformation of a cell with local inextensibility of its membrane; while the bottom three pictures show the deformation of a cell without the membrane local inextensibility at different times, respectively.

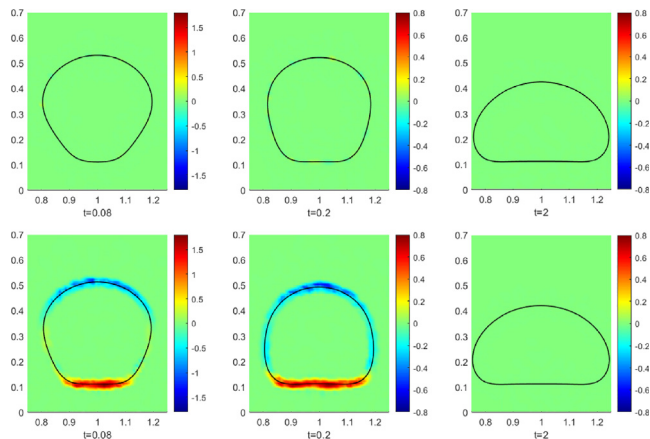
### 4.3. Cell–wall attraction

Cell–wall interaction under blood flow conditions plays important roles during blood clotting [12] and cancer cell invasion [18]. Simulation results presented in this section are used to investigate the effects of cell membrane local inextensibility and adhesion force on the cell–wall interaction. We first consider the effect of cell membrane local inextensibility using a 2D setting that a round cell is initially placed at a location with a point-wise contact with the wall boundary as shown in Fig. 6. The parameter values of this simulation are listed as follows:  $Re = 2 \times 10^{-4}$ ,  $\mathcal{M} = 5 \times 10^{-4}$ ,  $\kappa_B = 2 \times 10^{-2}$ ,  $\varepsilon = 2 \times 10^{-3}$ ,  $\mathcal{M}_s = 10^2$ ,  $k = 4 \times 10^{-11}$ ,  $l_s = 5 \times 10^{-6}$ ,  $\alpha = 1000$ ,  $q_{w_1} = 2$ ,  $q_{w_2} = 1$ .

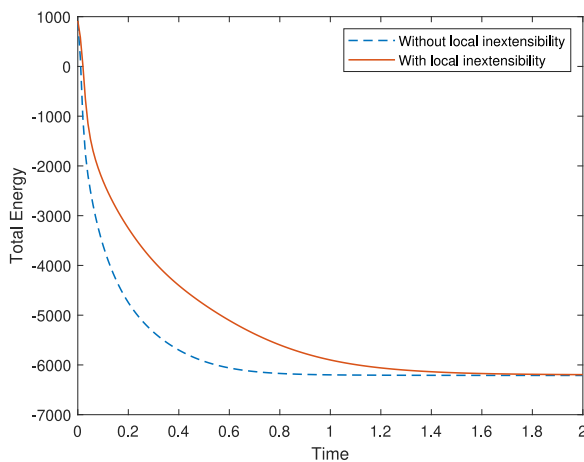
Because of the cell–wall adhesion, the cell gradually develops a line that contacts with the wall. Snapshots of the cells modeled with and without local inextensibility of their membranes at different times are shown in Fig. 7. We can clearly see that these two cells exhibit different dynamics in terms of changes in their shapes during cell–wall adhesion. As shown in Fig. 8, the local inextensibility constraint impedes the deformation of the cell. For the cell modeled without the local inextensibility constraint, the cell membrane attached to the wall is allowed to be stretched (red color in Fig. 8) as well as compressed (blue color in Fig. 8) to achieve the equilibrium faster. In contrast, the membrane of the cell modeled without local inextensibility displays almost no extensile or contractile phenomenon during the adhesion, which is consistent with experimental findings that lipid membrane is almost inextensible. Fig. 9 shows that in both cases, the numerically computed energy of the two systems monotonously decays to the same value. It is also worth noting that the numerical energy of the system in which the cell is modeled without local inextensibility decays faster initially.

The effect of the strength of the adhesion force on the equilibrium profiles of the cells is illustrated in Fig. 10. As expected, when the strength increases from 400 to 2500, the length of the cell–wall contact increases.

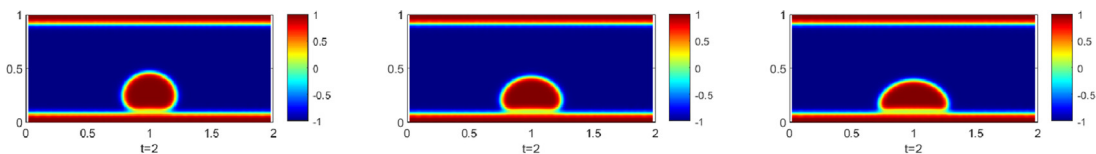
Competition between cell–wall adhesion and flow shear is studied in Fig. 11. Many important subjects of study in biology involve this competition. E.g., free flowing platelets can adhere to inner surface of blood vessels when endothelial cell (EC) is altered or extracellular matrix substrates are exposed. This is a critical initial step in hemostasis and thrombosis. Recruitment of free flowing leukocytes to sites of inflammation is a key step in the body’s innate immune response. This process is initiated by selectin-mediated leukocytes tethering and rolling along the EC surface, followed by integrin-dependent firm adhesion, prior to leukocytes extravasation into the tissue space [68]. Here we simulate two cells interacting with blood vessel wall under the shear flow condition. One cell is with strong adhesion force, and the other is with weak adhesion force with the vessel wall. Our simulations show



**Fig. 8.** The top pictures show the values of surface divergence  $\mathcal{P}_i : \nabla \mathbf{u}$  for the cell with membrane local inextensibility. The bottom pictures show the surface divergence of the cell without membrane local inextensibility.



**Fig. 9.** Total energy of the two system versus time.

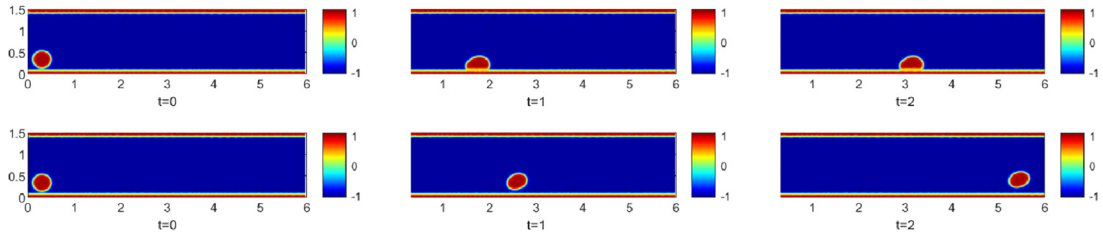


**Fig. 10.** Equilibrium of the cells at different strengths of the adhesion force. (Left:  $\alpha = 400$ ; Middle:  $\alpha = 1000$ ; Right:  $\alpha = 2500$ ).

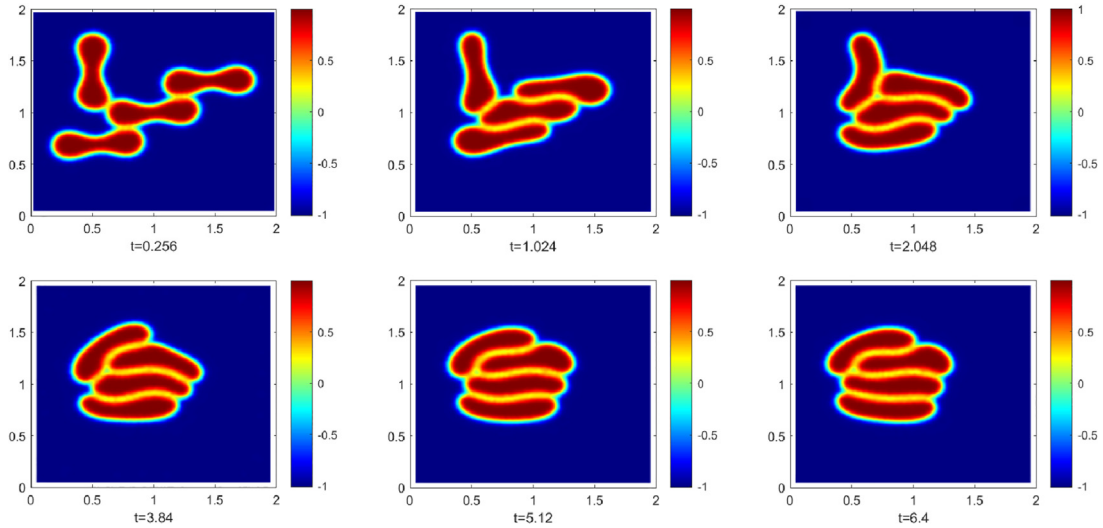
that the cell with strong adhesion force is captured by the wall; while the cell with weak adhesion force is washed away by the shear flow. These are consistent with experimental observations. Nevertheless, note that our model is deterministic and uses a coarse-grained approach for modeling adhesion. This limits the capability of the model for studying stochastic effects of those selectin- or integrin-mediated adhesion.

#### 4.4. Cell aggregation

Aggregation of RBCs is observed in experiment [3]. [69,70] report that there is RBC hyperaggregatability phenomenon observed in Type 2 diabetes mellitus (T2DM) patients due to fibrinogen-dependent aggregation dynamics [2]. The hyperaggregatability enhances the formation of rouleau [3,8], and subsequently leads to ischemic



**Fig. 11.** The top three picture shows the motion of the cell in strong adhesion case. The bottom three pictures shows the cell motion in weak adhesion case.



**Fig. 12.** Aggregating of four red blood cells. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

tissue [71]. In this section, we set up a simulation with four RBCs initially touching with each other at a small area of cell membranes. The parameter values are chosen to mimic real RBC and keep the adhesion force in the same range of tens to thousands of pN as reported in [2,9].

More specifically, the parameter values are:  $Re = 2 \times 10^{-5}$ ,  $\mathcal{M} = 5 \times 10^{-4}$ ,  $\kappa_B = 4 \times 10^{-2}$ ,  $k = 4 \times 10^{-12}$ ,  $l_s = 5 \times 10^{-3}$ ,  $\mathcal{M}_s = 10^3$ ,  $\alpha = 3 \times 10^3$ ,  $q_1 = 1$ ,  $q_2 = 0.5$ .

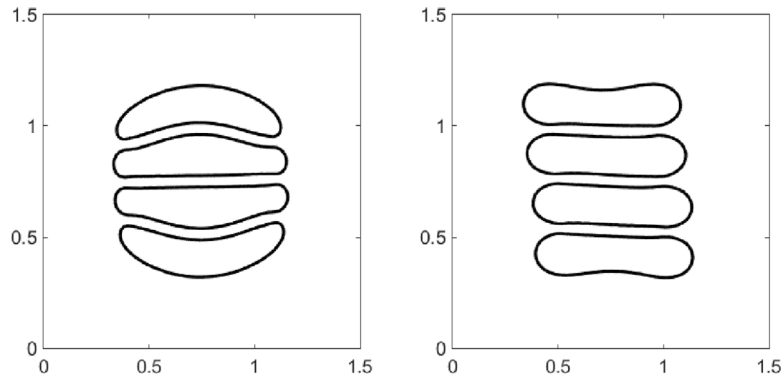
The evolution of this RBC system with time is shown in Fig. 12. From the result we can see that RBCs creep together with respect to time under the attractive force, and form rouleaux. This is consistent with the experimental result shown in [3,8].

The deformation of the RBCs at equilibrium is related to the value of the attractive force [8,19,24]. Fig. 13 shows clusters of RBCs with moderate and strong attractive (or aggregation) forces. The parameter values for simulations in Fig. 13 are as follows:  $Re = 2 \times 10^{-4}$ ,  $\mathcal{M} = 5 \times 10^{-4}$ ,  $\kappa_B = 2 \times 10^{-2}$ ,  $k = 2 \times 10^{-11}$ ,  $l_s = 5 \times 10^{-3}$ ,  $\mathcal{M}_s = 10$ .

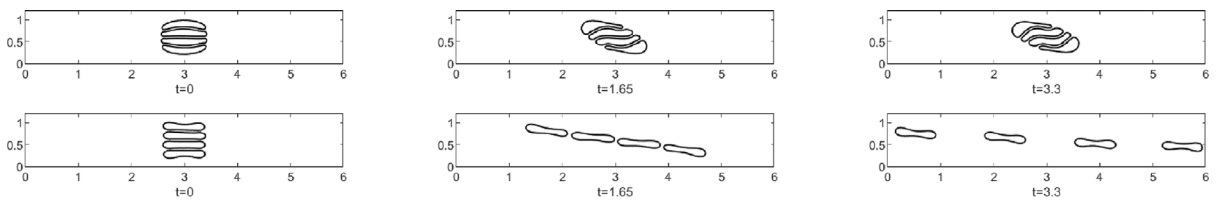
From Fig. 13, we can see that under strong aggregation force, obvious terminal hemispherical caps is shown and the offset between each adjacent cell is smaller as well. This result is consistent with the experimental observation [3].

In the following test, we place the RBCs in a Couette flow with flow shear rate being equal to  $20s^{-1}$  with the same dimension as in [21]. Motions of the cells are shown in Fig. 14. Under shear flow, the rouleaux with strong adhesion force still aggregate together; while the one with weak adhesion force is broken up by the shear flow. This is consistent with the results reported in [21].

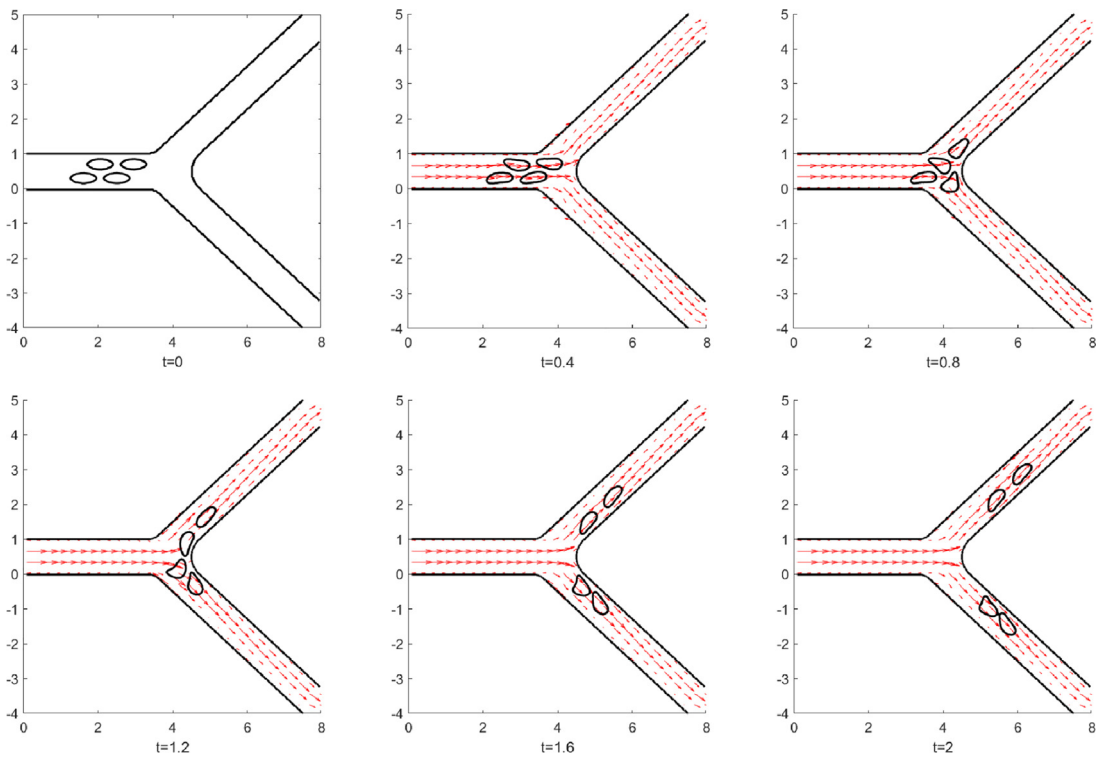
Now we simulate the motion of RBCs in branched vessels. Four RBCs are initially placed in the Y-shaped vessel. The width of the main channel (in which the inflow boundary condition is specified) of the Y-shaped vessel and the bottom branch of the vessel is set to be  $1 \times 10^{-7}$  meters. The width of the top branch of the vessel is  $0.7 \times 10^{-7}$



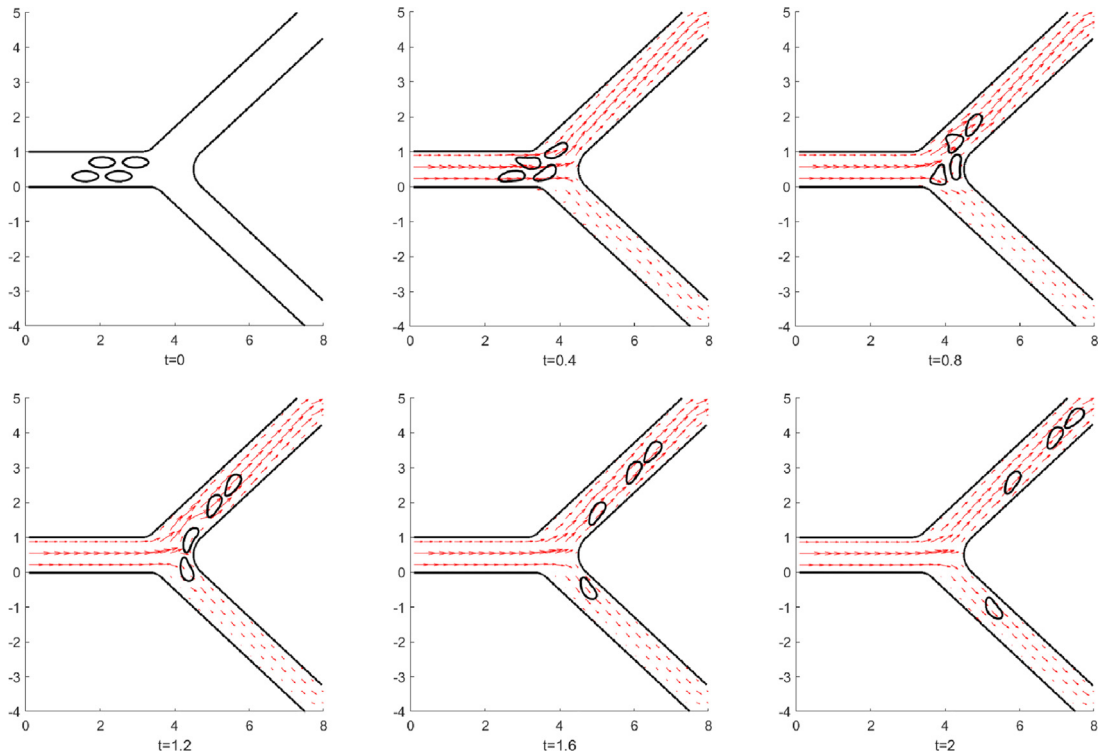
**Fig. 13.** Left: Strong aggregate  $\alpha = 15 \times 10^3, q_1 = 1, q_2 = 1$ . Right: moderate aggregate  $\alpha = 100, q_1 = 1, q_2 = 1$ .



**Fig. 14.** The top three figures shows the motion of the cells with strong aggregation. The bottom three figures shows the motion under moderate aggregation.



**Fig. 15.** Cells are set in a cluster initially under a moderate aggregating force with  $\alpha = 25, q_1 = 1, q_2 = 1$ . The main channel width is 1 and the width of the two branches is 0.7. RBCs divide equally at the vessel bifurcation. The velocity field is indicated by the vector field.



**Fig. 16.** Cells are set in a cluster initially under a moderate aggregating force with  $\alpha = 25$ ,  $q_1 = 1$ ,  $q_2 = 1$ . The top branch is set to be 1 with the other stays 0.7. One of the cells is going into the branched vessel. The velocity field is shown as well.

meters, which is close to the size of a red blood cell. A pressure drop boundary condition is used to introduce a shear flow in the vessel with a velocity around  $5 \times 10^{-4}$  m/s, which is close to the blood flow in capillaries. Other model parameter values in this simulation are as follows:  $Re = 2 \times 10^{-4}$ ,  $\mathcal{M} = 5 \times 10^{-4}$ ,  $\kappa_B = 4 \times 10^{-2}$ ,  $k = 4 \times 10^{-11}$ ,  $l_s = 2 = 5 \times 10^{-6}$ ,  $\mathcal{M}_s = 20$ .

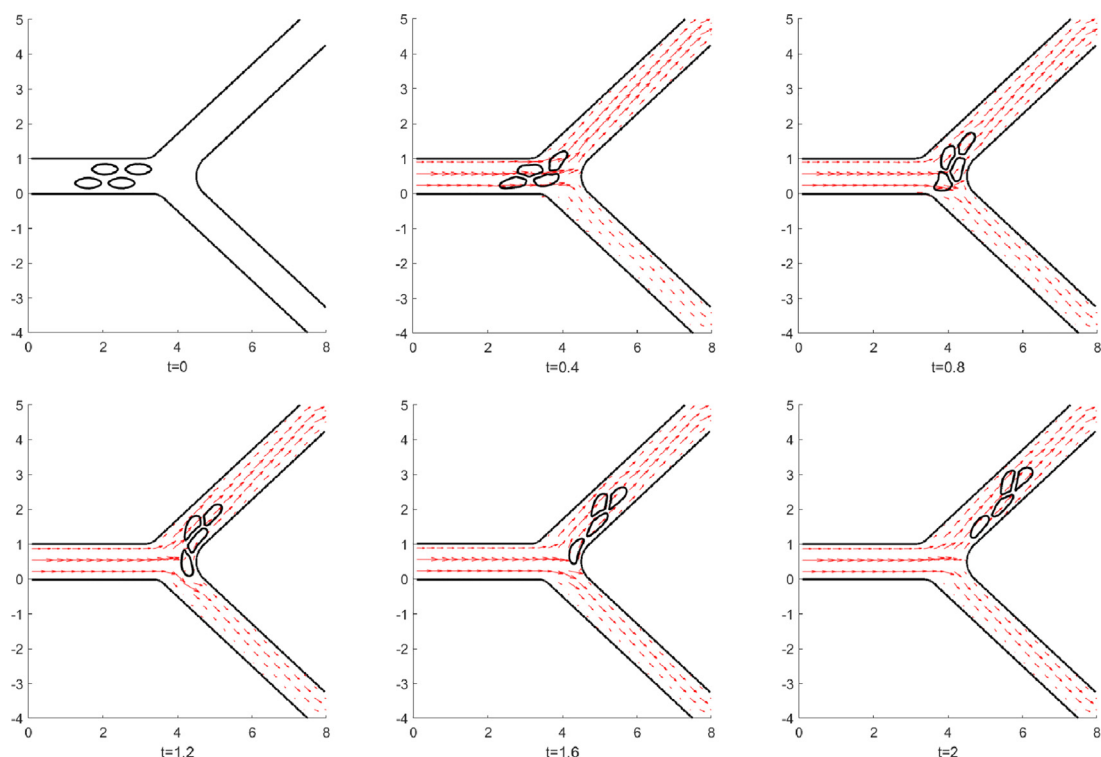
The motion of the RBCs and the velocity field of the flow are shown in Figs. 15, 16 and 17, respectively. We first simulate the motion of the RBC group with a moderate aggregation force when they pass a Y-shaped channel with the same width of both branches as a baseline. The cells divide equally at the vessel bifurcation location. Then one of the channels is widened and a non-equal deviation is observed (3 move upwards, 1 moves downward) due to the lower resistance in the upper branch. Then, under the same geometric setup, a strong aggregation force is applied to the cells. In this case, all RBCs go into the wider channel. The simulation result explains the experimental result in [3], in which a great number of RBCs are observed to be absent in the branched vessel under a strong aggregate case compared with moderate aggregate.

**Remark 4.3.** In Figs. 15, 16 and 17, the position and deformation of the cells and the distribution of the velocity field are both critical information that needs to be visualized. Thus we draw the cells with contour curves and the velocity field is marked with red arrows.

## 5. Conclusions

In this paper, we have presented a thermodynamically consistent phase-field model for simulating cell deformation and aggregation, incorporating a new multi-dimensional Lennard-Jones type interaction energy that accounts for both repulsion and adhesion between cells and walls. The model's mechanical properties, including bending, surface area conservation, and local inextensibility, are modeled using different energy functionals.

We have proposed an efficient numerical scheme using  $C^0$  finite element discretization in space and mid-point temporal discretization, which preserves energy unconditionally. The model and its parameters were calibrated and



**Fig. 17.** Cells are set in a cluster initially under a strong aggregating force with  $\alpha = 1.5 \times 10^3$ ,  $q_1 = 1$ ,  $q_2 = 1$ . The set up is the same as Fig. 16. None of the cells is going into the branched vessel. The velocity field is shown as well.

validated using experimental data on cell deformation under different stretch forces, and we have investigated the effects of adhesion strength on cell–wall and cell–cell interaction. The model was also used to study the motion of red blood cells near vessel bifurcations, which confirmed the role of hyperaggregability in inducing ischemic tissue in Type 2 diabetes mellitus patients.

It is important to note that while the Lennard-Jones potential provides a useful approximation for intermolecular interactions, cellular interactions are highly complex and involve a wide range of additional factors, such as specific cell adhesion molecules, signaling pathways, and mechanical properties. The Lennard-Jones type potential serves as a simplified representation to capture some aspects of cell interactions but may not fully capture the intricacies of biological systems. Future work will focus on extending the model to account for reaction and mass transportation on the membrane [72,73], as well as studying the effects of T2DM on cell deformability and oxygen transportation. The model could also be combined with viscoelastic models [74–76] for vessels to simulate free-flowing red blood cells in narrow deformable vessels using the wall phase-field label  $\phi_w$ . Overall, this work offers a promising framework for simulating cell–cell and cell–wall interactions in complex flow conditions, with potential applications in biomedical engineering and pathology.

### Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Shixin Xu reports financial support was provided by Duke Kunshan University.

### Data availability

No data was used for the research described in the article

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