

COMPLEX ANALYSIS AND A CLASS OF WEINGARTEN SURFACES

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ABSTRACT. An idea of Hopf's for applying complex analysis to the study of constant mean curvature spheres is generalized to cover a wider class of spheres, namely, those satisfying a Weingarten relation of a certain type, namely $H = f(H^2 - K)$ for some smooth function f , where H and K are the mean and Gauss curvatures.

The results are either not new or are minor extensions of known results, but the method, which involves introducing a different conformal structure on the surface than the one induced by the first fundamental form, is different from the one used by Hopf [3] and requires less technical results from the theory of PDE than Hopf's methods.

This is a \TeX ed version of a manuscript dating from early 1984. It was never submitted for publication, though it circulated to some people and has been referred to from time to time in published articles (cf. [5, 6]). It is being provided now for the convenience of those who have asked for a copy. Except for the correction of various grammatical or typographical mistakes and infelicities and the addition of some (clearly marked) comments at the end of the introduction, the text is that of the original.

0. INTRODUCTION

Two of the most satisfying theorems in the differential geometry of surfaces in \mathbb{E}^3 are Hopf's Theorem, asserting that a two-sphere in \mathbb{E}^3 of constant mean curvature is a round 2-sphere, and Liebmann's Theorem, asserting that a 2-sphere in \mathbb{E}^3 of constant Gaussian curvature is a round 2-sphere. The usual proofs of these theorems are by quite different techniques. Liebmann's Theorem is usually proved by assuming that the sphere is not round and then doing local analysis at a point where the difference of the principal curvatures is a maximum (see, for example, O'Neill [4]). The proof of Hopf's Theorem is less direct. It involves treating S^2 as a Riemann surface and constructing a holomorphic quadratic differential on S^2 from the second fundamental form of the immersion.

The original purpose of the investigations that led to this paper was to give a proof of Liebmann's Theorem by Riemann surface theory. To the author's surprise, a much more general theorem developed: If H and K represent the mean and Gaussian curvatures of an immersion $\mathbf{x} : S^2 \rightarrow \mathbb{E}^3$ and they satisfy a Weingarten relation of the form $H = f(H^2 - K)$ where f is *any* smooth function on an

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open interval containing $[0, \infty)$, then $\mathbf{x}(S^2)$ is a round sphere. Note that Hopf's Theorem follows by taking f to be constant and Liebmann's Theorem follows by taking $f(x) = \sqrt{c+x}$ where c (necessarily positive) is the constant Gaussian curvature.

This theorem can also be generalized to immersions into other space forms of dimension three. Moreover, the hypothesis on the form of the Weingarten relation can be weakened considerably. (Note that *some* hypothesis on the form of the Weingarten relation is needed: The ellipsoids of revolution are non-round spherical Weingarten surfaces.) Finally, the differentiability hypotheses can certainly be weakened, but we leave this as an exercise for the interested reader and assume that all given data are smooth for simplicity.

Added October, 2004: The reader may wonder why this manuscript was never published. The reason is that, after it was finished, I realized that the main results were essentially contained in those of Hopf and Alexandrov that are described as Theorem 6.2 in Hopf's book [3]. However, in conversations with others over the intervening years, I have realized that the method introduced in this manuscript, that of considering holomorphic quantities with respect to a Riemann surface structure different from that of the conformal structure induced by the first fundamental form, has certain advantages and simplifications over the proofs and techniques employed by Hopf. Also, in the intervening years, I have had several requests for copies of the old manuscript and some references to it have appeared in the literature. Unfortunately, the old typescript is of poor quality and hard to read. Consequently, I have decided to make this \TeX ed version available.

1. THE MOVING FRAME AND COMPLEX NOTATION FOR SURFACES IN \mathbb{E}^3

We will assume that the reader is familiar with the moving frame notation and the basic definitions of surface theory. This section is mainly to fix notation. We fix an inner product and orientation on \mathbb{R}^3 and denote the resulting oriented Euclidean space by \mathbb{E}^3 .

Let M^2 be a smooth connected oriented surface and let $\mathbf{x} : M \rightarrow \mathbb{E}^3$ be a smooth immersion. An *adapted frame field* on an open set $U \subseteq M$ will be a triple of smooth functions $\mathbf{e}_i : U \rightarrow \mathbb{E}^3$ ($i = 1, 2, 3$) with the property that for all $p \in U$, $(\mathbf{e}_1(p), \mathbf{e}_2(p), \mathbf{e}_3(p))$ is an oriented orthonormal basis of \mathbb{E}^3 and with the property that $\mathbf{e}_3(p)$ is the oriented unit normal to $\mathbf{x}_*(T_p M) \subseteq \mathbb{E}^3$. If $(\mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{e}_3^*)$ is any other adapted frame field on U , then there exists a unique smooth function $\theta : U \rightarrow \mathbb{R}/2\pi\mathbb{Z}$ for which

$$(1) \quad \begin{aligned} \mathbf{e}_1^* &= \cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_2, \\ \mathbf{e}_2^* &= -\sin \theta \mathbf{e}_1 + \cos \theta \mathbf{e}_2, \\ \mathbf{e}_3^* &= \mathbf{e}_3. \end{aligned}$$

We say that $(\mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{e}_3^*)$ is the *rotation of $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ by θ* .

If $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is an adapted frame field on $U \subseteq M$, we define the canonical forms ω_i, ω_{ij} as usual by

$$(2) \quad \omega_i = \mathbf{e}_i \cdot d\mathbf{x} \quad \omega_{ij} = \mathbf{e}_i \cdot d\mathbf{e}_j.$$

As usual, we have the vector-valued 1-form identities

$$(3) \quad d\mathbf{x} = \mathbf{e}_i \omega_i \quad d\mathbf{e}_i = \mathbf{e}_j \omega_{ji},$$

as well as the structure equations

$$(4) \quad d\omega_i = -\omega_{ij} \wedge \omega_j \quad d\omega_{ij} = -\omega_{ik} \wedge \omega_{kj}.$$

Now, by definition, $\mathbf{e}_3 \cdot d\mathbf{x} = \omega_3 = 0$, so, by the structure equations,

$$(5) \quad 0 = -d\omega_3 = \omega_{31} \wedge \omega_1 + \omega_{32} \wedge \omega_2.$$

Since $\omega_1 \wedge \omega_2$ is the oriented area form on U (and hence is not zero), Cartan's Lemma applies to show that there are smooth functions $h_{ij} = h_{ji}$ on U so that

$$(6) \quad \omega_{3i} = h_{ij} \omega_j.$$

The eigenvalues of the matrix (h_{ij}) are the *principal curvatures* of the immersion \mathbf{x} (on the open set U). They are independent of our choice of framing $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. Unfortunately, they are not, in general, smooth functions on a neighborhood of the umbilic locus (the closed subset of U where the eigenvalues are equal) since one must take a square root to compute the eigenvalues. On the other hand, the symmetric functions of the eigenvalues are smooth. The most common symmetric functions taken are

$$(7) \quad H = \frac{1}{2}(h_{11} + h_{22}) \quad K = h_{11}h_{22} - h_{12}^2.$$

These are the mean and Gaussian curvatures, respectively. One easily sees that the locus $H^2 - K = 0$ is the umbilic locus.

An adapted frame field $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is said to be *principal* if the matrix (h_{ij}) is diagonal, i.e., $h_{12} = 0$. Let us say that $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is *positive principal* if $h_{12} = 0$ and $h_{11} > h_{22}$. At any given non-umbilic point $p \in U$, there will exist exactly *two* positive principal adapted frames, each being the rotation of the other by an angle of π . Suppose that $p_0 \in U$ is an isolated umbilic point. We define the *umbilic index* $\iota_{\mathbf{x}}(p_0)$ at p_0 as follows: Let γ be a counterclockwise loop around p_0 that does not encircle any other umbilic points. Let $\iota_{\mathbf{x}}(p_0)$ be the multiple of 2π by which a positive principal frame rotates (counterclockwise) as it is transported around γ . Note that it is possible for $\iota_{\mathbf{x}}(p_0)$ to be a *half integer* (see Spivak [7, Chapter 4, Addendum 2]). We have the classical result:

Theorem 0 (Hopf). *Let M be compact and let $\mathbf{x} : M \rightarrow \mathbb{E}^3$ be an immersion for which the umbilic locus \mathcal{U} is finite. Then*

$$(8) \quad \chi(M) = \sum_{p \in \mathcal{U}} \iota_{\mathbf{x}}(p).$$

Finally, in order to simplify our computations in the next section, we introduce the *complex notation* for an adapted frame field $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ on $U \subset M$. We define the complex quantities

$$(9) \quad \begin{aligned} \mathbf{e} &= \frac{1}{2}(\mathbf{e}_1 - i\mathbf{e}_2) & \omega &= \omega_1 + i\omega_2 \\ \pi &= \omega_{31} - i\omega_{32} & \rho &= \omega_{21} \\ z &= \frac{1}{2}(h_{11} - h_{22}) - i h_{12} & H &= \frac{1}{2}(h_{11} + h_{22}). \end{aligned}$$

If $(\mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{e}_3^*)$ is the rotation of $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ by θ , we easily compute

$$(10) \quad \begin{aligned} \mathbf{e}^* &= e^{i\theta} \mathbf{e} & \omega^* &= e^{-i\theta} \omega \\ \pi^* &= e^{i\theta} \pi & \rho^* &= \rho + d\theta \\ z^* &= e^{2i\theta} z & H^* &= H. \end{aligned}$$

In general, we say that a quantity α computed with respect to a frame field $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ has *spin* k if $\alpha^* = e^{ik\theta}\alpha$. The quantities of spin zero are obviously independent of the choice of frame field and hence are globally well defined on M .

Note that $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is a (positive) principal adapted framing iff z is a (positive) real function on U . In fact, the umbilic locus is defined by $z = 0$ in this notation, while we have the important identity

$$(11) \quad \iota_{\mathbf{x}}(p_0) = -\frac{1}{2} \deg(z/|z|)$$

when p_0 is an isolated umbilic point, $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is a smooth adapted frame field on a neighborhood U of p_0 , and $\deg(z/|z|)$ is the degree of the smooth mapping $z/|z| : \gamma \rightarrow S^1$ where γ is a small loop that encircles p_0 counterclockwise (and no other umbilics).

We shall also need the following structure equations (as well as the fact that $\omega \wedge \bar{\omega} \neq 0$):

$$(12) \quad \begin{aligned} d\omega &= -i\rho \wedge \omega \\ d\pi &= i\rho \wedge \pi \\ \pi &= z\omega + H\bar{\omega} \end{aligned}$$

We leave these as an exercise in complex notation for the reader. Note that these equations are just the Codazzi equations. We shall not need the Gauss equation

$$(13) \quad d\rho = \frac{i}{2} \pi \wedge \bar{\pi}$$

at all. This will be useful in §3 when we consider generalizations to other spaces of constant curvature.

2. A CLASS OF WEINGARTEN EQUATIONS

In this section, we prove our main theorem. Let $\mathbf{x} : M^2 \rightarrow \mathbb{E}^3$ be a smooth immersion of a smooth oriented surface into \mathbb{E}^3 . Let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be an adapted frame field on $U \subseteq M$. If we substitute the equation $\pi = z\omega + H\bar{\omega}$ into $d\pi = i\rho \wedge \pi$ and expand, we get

$$(14) \quad (dz - 2iz\rho) \wedge \omega + dH \wedge \bar{\omega} = 0.$$

Since $\omega \wedge \bar{\omega} \neq 0$, it follows that there exist smooth functions on U , say, u and v , so that

$$(15) \quad \begin{aligned} dz - 2iz\rho &= v\omega + u\bar{\omega} \\ dH &= u\omega + \bar{u}\bar{\omega}. \end{aligned}$$

Moreover, we also compute

$$(16) \quad u^* = e^{i\theta}u \quad v^* = e^{3i\theta}v.$$

Now let us suppose that \mathbf{x} satisfies a Weingarten equation of the form $H = f(H^2 - K)$ where f is a smooth function on the domain $(-\epsilon, \infty) \subset \mathbb{R}$ where $\epsilon > 0$ is arbitrary. Since $H^2 - K = z\bar{z}$ by definition, our relation is written in the form $H = f(z\bar{z})$. If we differentiate this relation, we get

$$(17) \quad u\omega + \bar{u}\bar{\omega} = dH = f'(z\bar{z})(\bar{z}dz + z d\bar{z}) = f'(z\bar{z})(\bar{z}(v\omega + u\bar{\omega}) + z(\bar{v}\bar{\omega} + \bar{u}\omega)).$$

Comparing coefficients of ω , we get the crucial relation

$$(18) \quad u = f'(z\bar{z})(\bar{z}v + z\bar{u}).$$

We are also going to need two smooth functions F and G defined on \mathbb{R} with the following three properties for all $x \geq 0$:

$$(19) \quad \begin{aligned} (F(x))^2 - x(G(x))^2 &= 1, \\ 2F'(x) &= f'(x)G(x), \\ 2xG'(x) &= f'(x)F(x) - G(x). \end{aligned}$$

We construct these functions as follows: Consider the smooth function ϕ defined by

$$(20) \quad \phi(r) = \int_0^r f'(s^2) ds.$$

Obviously, $\phi(-r) = -\phi(r)$. Using the substitution $s = rt$, we see that

$$(21) \quad \phi(r) = r \int_0^1 f'(r^2 t^2) dt,$$

so that $\phi(r) = r\bar{\phi}(r)$ where $\bar{\phi}$ is also smooth. Using this, it is easy to see that there exist smooth functions F and G satisfying

$$(22) \quad F(r^2) = \cosh \phi(r) \quad G(r^2) = \frac{\sinh \phi(r)}{r}.$$

This uniquely specifies F and G for all $x \geq 0$. One easily verifies that they have the three desired properties. Now consider the 1-form on U

$$(23) \quad \sigma = F(z\bar{z})\omega + G(z\bar{z})\bar{z}\bar{\omega}.$$

We easily compute that $\sigma^* = e^{-i\theta}\sigma$ and that

$$(24) \quad \begin{aligned} \frac{i}{2}\sigma \wedge \bar{\sigma} &= \frac{i}{2}(F(z\bar{z})\omega + G(z\bar{z})\bar{z}\bar{\omega}) \wedge (F(z\bar{z})\bar{\omega} + G(z\bar{z})z\omega) \\ &= \frac{i}{2}(F(z\bar{z})^2 - z\bar{z}G(z\bar{z})^2)\omega \wedge \bar{\omega} \\ &= \frac{i}{2}\omega \wedge \bar{\omega} = \omega_1 \wedge \omega_2 > 0 \end{aligned}$$

by the first property of F and G . If we write $\sigma = \sigma_1 + i\sigma_2$, then it follows that σ_1 and σ_2 are independent on U and that the quadratic form

$$(25) \quad ds^2 = \sigma \circ \bar{\sigma} = \sigma_1^2 + \sigma_2^2 = \sigma^* \circ \bar{\sigma}^*$$

is smooth, positive definite and globally well defined on M .

It follows from the theorem of Korn and Lichtenstein on isothermal coordinates (see Courant-Hilbert [1, Chapter VII, §8]) that there is a unique complex structure on M compatible with the metric ds^2 and the orientation $\frac{i}{2}\sigma \wedge \bar{\sigma} > 0$. We endow M with this unique complex structure. Note that if $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is any adapted frame field on $U \subseteq M$, then σ is of type $(1, 0)$ on U (by definition of the complex structure).

We now consider the quadratic form $Q = z\sigma^2$ of type $(2, 0)$ on U . We compute

$$(26) \quad Q^* = z^*(\sigma^*)^2 = e^{2i\theta}z(e^{-i\theta}\sigma)^2 = z\sigma^2 = Q,$$

so Q has spin zero and hence is well defined globally on M . The following proposition is the heart of our results:

Proposition 1. *Q is a holomorphic quadratic form on M .*

Proof. This will be a pure computation. Let $U \subseteq M$ be an open set on which there exists a local holomorphic coordinate $\zeta : U \rightarrow \mathbb{C}$ (clearly M is covered by such open sets). It is easy to see that there is a unique adapted frame field $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ on U so that $\sigma = \lambda d\zeta$ where $\lambda > 0$ is a positive real-valued smooth function on U . Then

$$(27) \quad Q|_U = (z\lambda^2) (d\zeta)^2.$$

It suffices to show that $\partial(z\lambda^2)/\partial\bar{\zeta} \equiv 0$ on U . This is equivalent to

$$(28) \quad d(z\lambda^2) \wedge d\zeta = 0.$$

Now we expand this to

$$(29) \quad d(z\lambda^2) \wedge d\zeta = \lambda(dz \wedge \sigma + 2z d\sigma) = 0.$$

By the structure equations derived so far, we expand this last term (writing F, F' , etc., instead of $F(z\bar{z}), F'(z\bar{z})$, etc.):

$$\begin{aligned} dz \wedge \sigma + 2z d\sigma &= (2iz\rho + v\omega + u\bar{\omega}) \wedge (F\omega + G\bar{z}\bar{\omega}) \\ &\quad + 2z[F'(\bar{z}u + z\bar{v})\bar{\omega} \wedge \omega - F i\rho \wedge \omega] \\ &\quad + 2z[\bar{z}G'(\bar{z}v + z\bar{u})\omega \wedge \bar{\omega} \\ &\quad \quad + G\bar{z}(i\rho \wedge \bar{\omega}) + G(-2i\bar{z}\rho + \bar{u}\omega) \wedge \bar{\omega}] \end{aligned}$$

(note that all the terms containing ρ cancel)

$$\begin{aligned} &= [-uF + \bar{z}vG - zf'G(\bar{z}u + z\bar{v}) \\ &\quad \quad + (f'F - G)(\bar{z}v + z\bar{u}) + 2z\bar{u}G] \omega \wedge \bar{\omega} \end{aligned}$$

(using $u = f'(z\bar{z})(\bar{z}v + z\bar{u})$, this becomes)

$$\begin{aligned} &= [-uF + \bar{z}vG - zG\bar{u} + uF - G(\bar{z}v + z\bar{u}) + 2z\bar{u}G] \omega \wedge \bar{\omega} \\ &= 0 \quad (\text{as desired}). \end{aligned}$$

□

Proposition 2. *Either $\mathbf{x} : M \rightarrow \mathbb{E}^3$ is totally umbilic or else the umbilic locus consists entirely of isolated points of strictly negative index.*

Proof. Because M is connected and Q is holomorphic on M , either $Q \equiv 0$ or else Q has only isolated zeroes. If $Q \equiv 0$ then $z\sigma^2 \equiv 0$ on each $U \subseteq M$ with an adapted frame field $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. Since $\sigma^2 \neq 0$ on U , it follows that $z \equiv 0$, so that every point of U is umbilic.

Now suppose $Q \not\equiv 0$. Then the zeroes of Q are isolated and are clearly the umbilic points of the immersion \mathbf{x} . Suppose that p_0 is an umbilic of the immersion. Then there exists an integer $k > 0$ and a holomorphic local coordinate $\zeta : U \rightarrow \mathbb{C}$ with $p_0 \in U$ and $\zeta(p_0) = 0$ so that

$$(30) \quad Q|_U = \zeta^k (d\zeta)^2.$$

(The proof is an elementary exercise in analytic function theory.) We choose the frame field on U for which $\sigma = \lambda d\zeta$ with λ real and positive. Then on $U \setminus \{p_0\}$, we have

$$(31) \quad \frac{z}{|z|} = \frac{\zeta^k/\lambda^2}{|\zeta^k/\lambda^2|} = \frac{\zeta^k}{|\zeta^k|}.$$

Let γ be the counterclockwise loop $|\zeta| = \delta > 0$ where δ is very small. Obviously, the degree of the mapping $\zeta^k/|\zeta^k| : \gamma \rightarrow S^1$ is k . Thus, $\deg(z/|z|) = k$. By our identity from §1:

$$(32) \quad \iota_{\mathbf{x}}(p_0) = -k/2 < 0.$$

□

We will now prove our main theorem.

Theorem 1. *Let $\mathbf{x} : S^2 \rightarrow \mathbb{E}^3$ be a smooth immersion that satisfies a Weingarten equation of the form $H = f(H^2 - K)$ where f is a smooth function on some interval $(-\epsilon, \infty)$ where $\epsilon > 0$. Then $\mathbf{x}(S^2)$ is a round 2-sphere in \mathbb{E}^3 .*

Proof. If $\mathbf{x} : S^2 \rightarrow \mathbb{E}^3$ is totally umbilic, we are done, so suppose otherwise. Then, by Proposition 2, the umbilics of \mathbf{x} form a finite set $\mathcal{U} \subset S^2$ and each umbilic has negative index. However

$$(33) \quad \sum_{p \in \mathcal{U}} \iota_{\mathbf{x}}(p) = \chi(S^2) = 2 > 0$$

by Hopf's Theorem, which is a contradiction. □

Corollary 1 (Hopf). *If $\mathbf{x} : S^2 \rightarrow \mathbb{E}^3$ is an immersion with constant mean curvature, then $\mathbf{x}(S^2)$ is a round sphere.*

Proof. Merely take $f \equiv \text{const}$. □

Corollary 2. *Suppose that M^2 is a compact oriented surface and that $\mathbf{x} : M \rightarrow \mathbb{E}^3$ is a smooth immersion satisfying a Weingarten equation of the form $H = f(H^2 - K)$ where f is a smooth function on the interval $(-\epsilon, \infty)$ ($\epsilon > 0$) and also satisfies $f(x)^2 \geq x$ for all $x \geq 0$. Then M is a 2-sphere and $\mathbf{x}(M) \subseteq \mathbb{E}^3$ is a round 2-sphere.*

Proof. Since $K = H^2 - (H^2 - K) = (f(H^2 - K))^2 - (H^2 - K) \geq 0$, it follows that the induced metric on M has non-negative curvature. Since M is compact, we must have $K(p) > 0$ for some $p \in M$. But then, by Gauss-Bonnet, $\chi(M) > 0$, so $M = S^2$. Now Theorem 1 applies. □

Corollary 3 (Liebmann). *Suppose M is compact and oriented and that $\mathbf{x} : M \rightarrow \mathbb{E}^3$ has constant positive Gaussian curvature $K_0 > 0$. Then $\mathbf{x}(M)$ is a round 2-sphere.*

Proof. Apply Corollary 2 with $f(x) = \sqrt{K_0 + x}$. □

Of course, we also get some information about more complicated surfaces:

Theorem 2. *Let $\mathbf{x} : T^2 \rightarrow \mathbb{E}^3$ be a smooth immersion of the torus T^2 that satisfies a Weingarten equation of the form $H = f(H^2 - K)$ where f is smooth on some interval $(-\epsilon, \infty)$ with $\epsilon > 0$. Then \mathbf{x} is free of umbilics and there is a global positive principal frame field on T^2 .*

Proof. The form Q constructed above cannot vanish identically on T^2 since T^2 obviously has no totally umbilic immersion into \mathbb{E}^3 . Since $\chi(T^2) = 0$, it follows that Q has no zeroes at all. It is well known that, as a Riemann surface, T^2 must be isomorphic to \mathbb{C}/Λ where $\Lambda \subseteq \mathbb{C}$ is a rank two discrete lattice (see Griffiths-Harris [2]). Moreover, a linear coordinate ζ can be chosen on \mathbb{C} so that $d\zeta$ is well defined and holomorphic on \mathbb{C}/Λ (Λ is the lattice of periods of $d\zeta$) and so

that $Q = (d\zeta)^2$. This $d\zeta$ is unique up to multiplication by ± 1 . We then choose the unique frame field for which $\sigma = \lambda d\zeta$ with λ real and positive. Since $Q = (d\zeta)^2$, it follows that $z = \lambda^{-2} > 0$, so this frame field is positive and principal. \square

We close this section with a couple of remarks.

The first remark is that *some* hypothesis about the Weingarten relation

$$R(H, H^2 - K) = 0$$

must be made in order to deduce results about the umbilic locus corresponding to Proposition 2. For example, any surface of revolution is always a Weingarten surface and the ellipsoids of revolution give examples of non-round spherical Weingarten surfaces. Of course, the corresponding Weingarten relation cannot be solved smoothly for H in terms of $H^2 - K$. On the other hand, we could considerably weaken our hypothesis and still have the conclusion of Theorem 1. For example, suppose $\mathbf{x} : S^2 \rightarrow \mathbb{E}^3$ is a smooth immersion such that, on a neighborhood of each umbilic point $p \in S^2$, \mathbf{x} satisfies a Weingarten relation of the form $H = f_p(H^2 - K)$ where f_p is a smooth function on some interval $(-\epsilon, \infty)$ where $\epsilon > 0$. Here, f_p can depend on p . Then we can still conclude that \mathbf{x} is totally umbilic as follows: Applying Proposition 2 to \mathbf{x} restricted to such a neighborhood of p , we see that either p is an isolated umbilic of strictly negative index or else p has an open neighborhood consisting entirely of umbilics. Obviously, the non-isolated umbilics will then form an open and closed set. Thus, if \mathbf{x} were not totally umbilic, the umbilic locus would consist of isolated umbilics of negative index. Since this latter is impossible by Hopf's Theorem, we are done.

Perhaps the main interest in such an improvement of Theorem 1 comes from studying Weingarten relations that satisfy the solvability hypothesis only locally. For example, the relation $H^2 + (H^2 - K)^2 = 1$ does not satisfy the hypothesis of Theorem 1, but at the points where $H^2 - K = 0$ (i.e., the umbilic locus), we can solve for H smoothly, locally as $H = \sqrt{1 - (H^2 - K)^2}$ or as $H = -\sqrt{1 - (H^2 - K)^2}$. From our above argument, it follows that an immersion $\mathbf{x} : S^2 \rightarrow \mathbb{E}^3$ satisfying $H^2 + (H^2 - K)^2 = 1$ must be totally umbilic.

Our second remark concerns the nature of the equation $H = f(H^2 - K)$ as a second order partial differential equation for the immersion $\mathbf{x} : M^2 \rightarrow \mathbb{E}^3$. If we suppose that \mathbf{x} satisfies $H = f(H^2 - K)$ and define the function

$$(34) \quad A = 4(H^2 - K)(f'(H^2 - K))^2 \geq 0$$

on M , then it can be shown that the linearization of the above equation is elliptic on the regions where $A < 1$ and hyperbolic on the regions where $A > 1$. (The linearization is computed with respect to *normal* variations to avoid the degeneracies of reparametrization.) In particular, the equation $H = f(H^2 - K)$ has an elliptic linearization near the umbilic locus, since A vanishes on the umbilic locus. Perhaps this accounts for the simple behavior of the umbilics.

What seems remarkable to this author is that the "elliptic" conclusion of Proposition 1 continues to hold even in the hyperbolic region, where $A > 1$. This phenomenon of a hyperbolic equation implying an elliptic one is surely unusual and probably deserves further study.

3. WEINGARTEN SURFACES IN SPACES OF CONSTANT CURVATURE

We now consider the case of Weingarten immersions $\mathbf{x} : M^2 \rightarrow N^3$ where N^3 is a space of constant sectional curvature R . For simplicity, we assume that M^2 and N^3 are oriented. An adapted frame field on $U \subseteq M^2$ will now be given by a triple of smooth functions $\mathbf{e}_i : U \rightarrow TN^3$ with the property that, for all $p \in U$, $(\mathbf{e}_1(p), \mathbf{e}_2(p), \mathbf{e}_3(p))$ is an oriented orthonormal basis of $T_{\mathbf{x}(p)}N^3$ and with the property that $\mathbf{e}_3(p)$ is the oriented unit normal to $\mathbf{x}_*(T_pM)$.

The forms ω_i, ω_{ij} are defined by the equations

$$(35) \quad d\mathbf{x} = \mathbf{e}_i \omega_i \quad \nabla \mathbf{e}_i = \mathbf{e}_j \omega_{ji}$$

where ∇ is the Levi-Civita connection. The structure equations are now (see Spivak [7]):

$$(36) \quad d\omega_i = -\omega_{ij} \wedge \omega_j \quad d\omega_{ij} = -\omega_{ik} \wedge \omega_{kj} + R\omega_i \wedge \omega_j.$$

Again, we have $\omega_3 = 0$ and consequently $\omega_{3i} = h_{ij} \omega_j$ ($h_{ij} = h_{ji}$). The formulae for the mean and Gaussian curvatures become

$$(37) \quad H = \frac{1}{2}(h_{11} + h_{22}) \quad K = h_{11}h_{22} - h_{12}^2 + R.$$

As far as the complex notation goes, we define $\omega, \pi, \rho,$ and z exactly as before. We then verify that the structure equations are

$$(38) \quad \begin{aligned} d\omega &= -i\rho \wedge \omega \\ d\pi &= i\rho \wedge \pi \quad (\pi = z\omega + H\bar{\omega}) \\ d\rho &= \frac{i}{2}(\pi \wedge \bar{\pi} - R\omega \wedge \bar{\omega}). \end{aligned}$$

Note that $z\bar{z} = H^2 - K + R$ and that the first two structure equations are unchanged. Since we did not use the formula for $d\rho$ (i.e., the Gauss equation) in the proof of Proposition 1, §2, it follows that Proposition 1 remains valid for immersions $\mathbf{x} : M^2 \rightarrow N^3$ that satisfy an equation of the form $H = f(H^2 - K + R)$ where f is a smooth function on an interval $(-\epsilon, \infty)$ ($\epsilon > 0$).

This leads directly to the following theorem (we omit the proof):

Theorem 3. *Let M^2 be connected and let $\mathbf{x} : M^2 \rightarrow N^3$ be a smooth immersion where N^3 has constant sectional curvature R . Suppose that every umbilic point $p \in M$ has an open neighborhood on which \mathbf{x} satisfies a Weingarten equation of the form $H = f_p(H^2 - K + R)$ where f_p is a smooth function on a neighborhood of $0 \in \mathbb{R}$. Then either X is a totally umbilic immersion or else each umbilic point is isolated and of strictly negative index. In particular, if $M = S^2$ or $\mathbb{R}P^2$, then \mathbf{x} is totally umbilic.*

Remark 1. *This theorem includes many of the classical results about Weingarten surfaces in spaces of constant curvature. For example, one deduces immediately from Theorem 3 Hopf's result that a sphere of constant mean curvature in a space form is totally umbilic and a generalization of Liebmann's result that a sphere of constant curvature $K_0 \neq R$ in N^3 is a round sphere.*

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