Complete Mirror Pairs and Their Naïve Stringy Hodge Numbers

by

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Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
in the Graduate School of Duke University
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Abstract

The Batyrev-Borisov construction associates a to dual pair of nef-partitions $\Delta = \Delta_1 + \cdots + \Delta_c$ and $\nabla = \nabla_1 + \cdots + \nabla_c$ a pair of Calabi-Yau complete intersections $(Y_{\Delta_1,\ldots,\Delta_c}, Y_{\nabla_1,\ldots,\nabla_c})$ in Gorenstein Fano toric varieties (X_Δ, X_∇). These Calabi-Yau varieties are singular in general. Batyrev and Nill have developed a generating function E_{st} for the stringy Hodge numbers of Batyrev-Borisov mirror pairs. This function depends solely on the combinatorics of the nef-partitions and, under this framework, Batyrev-Borisov mirror pairs pass the stringy topological mirror symmetry test $h^{p,q}_{\mathrm{st}}(Y_{\Delta_1,\ldots,\Delta_c}) = h^{d-p,q}_{\mathrm{st}}(Y_{\nabla_1,\ldots,\nabla_c})$.

Recently, Aspinwall and Plesser have defined the notion of a complete non-reflexive mirror pair $(\mathscr{A}, \mathscr{B})$ and used this notion to study Calabi-Yau complete intersections in non-Gorenstein toric varieties. Complete mirror pairs generalize the notion of a dual pair of almost reflexive Gorenstein cones $(\sigma, \sigma^\bullet)$ developed by Mavlyutov to propose a generalization of the Batyrev-Borisov mirror construction. The only known example of either of these two notions is the complete intersection of a quintic and a quadric in $\mathbb{P}^5_{211111}$. We construct 2152 distinct examples of complete mirror pairs and 1077 distinct examples of dual pairs of almost reflexive Gorenstein cones. Additionally, we propose a generalization of Batyrev and Nill's stringy E-function, called the naïve stringy E-function $\widetilde{E}_{\mathrm{st}}$, that is well-defined for complete mirror pairs.

For Sarah.

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List of Abbreviations and Symbols

Symbols

X_Σ	The toric variety corresponding to a fan Σ.
$\mathcal{C}_{\Delta_1,\ldots,\Delta_c}$	The Cayley cone of a c-tuple of polytopes $\Delta_1, \ldots, \Delta_c$.

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1

Introduction

Mirror symmetry is an area of string theory that describes a duality in the collection of Calabi-Yau varieties. This duality provides a rich connection between physics and mathematics that has been an intense area of focus for researchers in both fields.

Definition 1.0.1. A *Calabi-Yau variety* is a smooth projective variety Y over $\mathbb{C}$ such that

(CY1) $\dim_{\mathbb{C}} H^p(Y, \mathscr{O}_Y) = 0$ for $0 < p < \dim Y$

(CY2) The canonical bundle K_Y of Y is trivial $K_Y \simeq \mathscr{O}_Y$.

The conditions **(CY1)** and **(CY2)** are called the *Calabi-Yau conditions*.

In [43], Witten associates two 2-dimensional topological field theories to a Calabi-Yau variety Y. The symplectic structure of Y defines a topological field theory called the *A-model* and the complex structure of Y defines a topological field theory called the *B-model*. Mirror symmetry predicts that Calabi-Yau varieties occur in *mirror pairs* $(Y, Y^\vee)$, where the A-model on Y is equivalent to the B-model on $Y^\vee$ and vice versa.

A categorified approach to this story was offered by Kontsevich in [26]. Here, Kontsevich categorifies the A-model on Y with the *derived Fukaya category* $\mathrm{Fuk}(Y)$ and the B-model with the bounded derived category of coherent sheaves $\mathbf{D}^b(Y)$. Under this framework, Konstevich suggests that a mirror pair of Calabi-Yau varieties $(Y, Y^\vee)$ should be equipped with equivalences of categories

$$\mathrm{Fuk}(Y) \simeq \mathbf{D}^b(Y^\vee) \qquad\qquad \mathrm{Fuk}(Y^\vee) \simeq \mathbf{D}^b(Y) \tag{1.1}$$

The conjecture predicting the existence of the equivalences in (1.1) is known as *Homological Mirror Symmetry.*

One consequence of the isomorphisms in (1.1) is *topological mirror symmetry.* The vector spaces underlying the the Frobenius algebras corresponding to the A-model and the B-model are $\mathcal{A}_Y = H^*(Y, \mathbb{C})$ and $\mathcal{B}_Y = \mathrm{HH}^*(Y)$ respectively. These vector spaces are equipped with gradings

$$H^*(Y, \mathbb{C}) = \bigoplus_{p+q=*} H^p(Y, \wedge^q \Omega_Y) \qquad \mathrm{HH}^*(Y) = \bigoplus_{p+q=*} H^p(Y, \wedge^q \mathscr{T}_Y) \tag{1.2}$$

where Ω_Y is the cotangent sheaf of Y and $\mathscr{T}_Y$ is the tangent sheaf. The Calabi-Yau condition implies that $H^p(Y, \wedge^q \mathscr{T}_Y) \simeq H^p(Y, \wedge^{d-p}\Omega_Y)$. So, if $(Y, Y^\vee)$ is a mirror pair of Calabi-Yau varieties, then the isomorphism $\mathcal{A}_Y \simeq \mathcal{B}_{Y^\vee}$ implies

$$H^p(Y, \wedge^q \Omega_Y) \simeq H^p(Y, \wedge^{d-q}\Omega_{Y^\vee}) \tag{1.3}$$

This inspires the following definition.

Definition 1.0.2. Let Y and $Y^\vee$ be d-dimensional algebraic varieties. We say that the pair $(Y, Y^\vee)$ *passes the topological mirror symmetry test* if the Hodge numbers of Y and $Y^\vee$ satisfy the relations

$$h^{p,q}(Y) = h^{d-p,q}(Y^\vee) \tag{1.4}$$

for $0 \le p, q \le d$.

In this dissertation, a *mirror pair of Calabi-Yau varieties* will refer to a pair $(Y, Y^\vee)$ of Calabi-Yau varieties passing the topological mirror symmetry test.

1.1 Toric Geometry and Mirror Symmetry

The first sizeable collection of Calabi-Yau varieties were constructed as complete intersections in products of projective spaces [24, 14]. Later, this collection was expanded to a list of Calabi-Yau threefold hypersurfaces in weighted projective spaces [15]. Eventually, it was noticed by Batyrev [4] that toric geometry offers mathematical tools to explicitly construct and analyze mirror-symmetric Calabi-Yau varieties. The Batyrev construction begins with a *dual pair of reflexive polytopes* (Δ, Δ^*), which correspond to a pair of *Gorenstein Fano toric varieties* (X_Δ, X_{Δ^*}). A *Batyrev mirror pair* is a pair of Calabi-Yau hypersurfaces $Y_\Delta \subset X_\Delta$ and $Y_{\Delta^*} \subset X_{\Delta^*}$.

In [12], Borisov generalizes the Batyrev contruction by considering particular Minkowski sums $\Delta = \Delta_1 + \cdots + \Delta_c$ of reflexive polytopes, called *nef-partitions*. Each nef-partition $\Delta = \Delta_1 + \cdots + \Delta_c$ of a reflexive polytope Δ defines a Calabi-Yau complete intersection $Y_{\Delta_1,\ldots,\Delta_c}$ in the Gorenstein Fano toric variety X_Δ of codimension c. Nef-partitions naturally occur in dual pairs $\Delta = \Delta_1 + \cdots + \Delta_c$ and $\nabla = \nabla_1 + \cdots + \nabla_c$ and thus define pairs of Calabi-Yau complete intersections $(Y_{\Delta_1,\ldots,\Delta_c}, Y_{\nabla_1,\ldots,\nabla_c})$ in the Gorenstein Fano toric varieties (X_Δ, X_∇). The pair $(Y_{\Delta_1,\ldots,\Delta_c}, Y_{\nabla_1,\ldots,\nabla_c})$ is referred to as a *Batyrev-Borisov mirror pair*.

Though the Batyrev-Borisov construction has been one of the most important tools for producing mirror pairs of Calabi-Yau varieties, the method is not without limitations. Another important tool for constructing Calabi-Yau varieties or finding mirrors is the notion of an extremal transition. Suppose Y is a smooth Calabi-Yau threefold admitting a birational contraction $Y \to \overline{Y}$ onto a normal variety $\overline{Y}$. If there exists a complex deformation (or *smoothing*) of $\overline{Y}$ to a smooth Calabi-Yau threefold $\widetilde{Y}$, then the process of going from Y to $\widetilde{Y}$ is called an *extremal transition.* It has long been thought that any two smooth Calabi-Yau threefolds can be connected by extremal transitions. In fact, the Kreuzer-Skarke classification of four-dimensional reflexive polytopes [31] confirms this for the class of Calabi-Yau threefold hypersurfaces in four-dimensional Gorenstein Fano toric varieties.

The class of Calabi-Yau varieties arising from the Batyrev-Borisov construction is not closed under extremal transitions. The ambient spaces ocurring in the Batyrev-Borisov construction are Gorenstein Fano toric varieties, which puts restrictions on the Calabi-Yau varieties for which the tools coming from the Batyrev-Borisov construction can be used. Recently, more attention has been devoted to the study of Calabi-Yau complete intersections in Fano toric varieties that are *not* Gorenstein. In [2], Aspinwall and Plesser study the smooth Calabi-Yau threefold Y given by the intersection of a cubic and a quartic in the weighted projective space $\mathbb{P}^5_{211111}$. The space $\mathbb{P}^5_{211111}$ is a Fano toric variety that is $\mathbb{Q}$-Gorenstein, but not Gorenstein. The Calabi-Yau Y can thus not obviously be realized as part of a Batyrev-Borisov mirror pair. This example is unavoidable if one is to follow extremal transitions from Batyrev-Borisov mirror pairs. In fact, in §4.2.2, we demonstrate this by explicitly constructing an extremal transition between $\mathbb{P}^5_{211111}[3,4]$ and a Calabi-Yau threefold from a Batyrev-Borisov mirror pair.

To address this issue, Aspinwall and Plesser use the gauged linear σ-model to develop the notion of a *complete mirror pair* $(\mathscr{A}, \mathscr{B})$. A *complete mirror pair* consists of the Hilbert bases $(\mathscr{A}, \mathscr{B})$ of a pair of non-reflexive Gorenstin cones $(\sigma, \sigma^\bullet)$. In [2], it is shown that the geometry of the complete intersection $Y = \mathbb{P}^5_{211111}[3,4]$ is encoded by a complete mirror pair $(\mathscr{A}, \mathscr{B})$. Several questions, however, remain. In particular, Aspinwall and Plesser ask whether Y has a mirror $Y^\vee$ that can be realized as a complete intersection in a toric variety. Additionally, they note that it would be interesting to construct other examples of complete mirror pairs $(\mathscr{A}, \mathscr{B})$ that are not reflexive. In this dissertation, we address both of these issues by proposing a toric description for a mirror variety $Y^\vee$ and by giving several thousands of new examples of complete mirror pairs $(\mathscr{A}, \mathscr{B})$ that are not reflexive.

1.2 Stringy Hodge Numbers

It turns out that the Calabi-Yau varieties arising from the Batyrev-Borisov construction often inherit singularities from their ambient spaces. Thus, it is necessary to consider *singular* Calabi-Yau varieties, which are varieties with at worst Gorenstein canonical singularities that satisfy the Calabi-Yau conditions. While studying singular Calabi-Yau varieties, one quickly realizes that the usual Hodge numbers do not behave as expected.

Example 1.2.1 (Example 1.1 from [6])**.** Let $Y \subset \mathbb{P}^4$ be the Fermat quintic threefold, defined by the equation $f(z_0, z_1, z_2, z_3, z_4) = 0$ where

$$f(z_0, z_1, z_2, z_3, z_4) = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 \tag{1.5}$$

Then Y is a Calabi-Yau threefold and physicists predict that a mirror $Y^\vee$ to Y may be obtained by a quotient $Y^\vee = Y/G$ where $G \simeq \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$ is the maximal diagonal subgroup of $\mathrm{PSL}(5, \mathbb{C})$ that acts trivially on f. However, the Hodge diamonds of Y and $Y^\vee$ are

$$\begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix} \qquad \begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 1 & & 1 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix} \tag{1.6}$$

That is, the pair $(Y, Y^\vee)$ fails the topological mirror symmetry test.

The issue here is that $Y^\vee$ is a singular Calabi-Yau. To remedy this situation, one may construct a smooth Calabi-Yau threefold $\widehat{Y}^\vee$ by crepantly resolving $Y^\vee$. Then the pair $(Y, \widehat{Y}^\vee)$ passes the topological mirror symmetry test, regardless of the choice of desingularization $\widehat{Y}^\vee$. $\Diamond$

It is a nontrivial fact [23, Theorem 1.1] in algebraic geometry that if $Y_1 \to Y$ and $Y_2 \to Y$ are two crepant resolutions of a variety Y over $\mathbb{C}$, then the Hodge numbers of Y_1 and Y_2 agree. In light of this, we call the Hodge numbers of the smooth Calabi-Yau $\widehat{Y}^\vee$ from Example 1.2.1 the *stringy Hodge numbers* $h_{\text{st}}^{p,q}(Y^\vee)$ *of* $Y^\vee$. Crepant desingularizations of Calabi-Yau threefolds always exist [39], so we have a well-defined notion of stringy Hodge numbers in dimension three. In dimensions $d \geq 4$, however, the situation is far less clear.[1]

In [10], Batyrev and Nill show that the *Cayley polytope* $\tilde{\Delta} = \Delta_1 * \cdots * \Delta_c$ associated to a nef-partition $\Delta = \Delta_1 + \cdots + \Delta_c$ of a reflexive polytope Δ is a *Gorenstein polytope of index* c. In this paper, the authors construct a polynomial $E_{\text{st}}(\tilde{\Delta}; u, v) \in \mathbb{Q}[u, v]$ called the *stringy* E*-polynomial of* $\tilde{\Delta}$, which takes the form

$$E_{\text{st}}(\tilde{\Delta}; u, v) = \sum_{p,q} (-1)^{p+q} h_{\text{st}}^{p,q}(\tilde{\Delta}) u^p v^q \tag{1.7}$$

for some nonnegative integers $h_{\text{st}}^{p,q}(\tilde{\Delta})$. Since the Calabi-Yau varieties $Y_{\Delta_1,\ldots,\Delta_c}$ arising from the Batyrev-Borisov mirror construction are associated to nef-partitions $\Delta = \Delta_1 + \cdots + \Delta_c$, one may use (1.7) to *define* the stringy Hodge numbers of $Y_{\Delta_1,\ldots,\Delta_c}$ as $h_{\text{st}}^{p,q}(Y_{\Delta_1,\ldots,\Delta_c}) = h_{\text{st}}^{p,q}(\tilde{\Delta})$. Under this framework, Batyrev-Borisov mirror pairs $(Y_{\Delta_1,\ldots,\Delta_c}, Y_{\nabla_1,\ldots,\nabla_c})$ pass the *stringy topological mirror symmetry test*

$$h_{\text{st}}^{p,q}(Y_{\Delta_1,\ldots,\Delta_c}) = h_{\text{st}}^{d-p,q}(Y_{\nabla_1,\ldots,\nabla_c}) \tag{1.8}$$

and the stringy Hodge numbers of a variety agree with the usual Hodge numbers if the variety is smooth [10, Theorem 4.17].

One of the benefits of using (1.7) to define stringy Hodge numbers is that the polynomial E_{st} depends only on the combinatorial data of the reflexive Gorenstein polytope $\tilde{\Delta}$ and its dual $\tilde{\Delta}^*$. We thus have a combinatorial description of the stringy

[1] See Example 1.2 in [6] for an example of a singular Calabi-Yau fourfold that admits no crepant desingularization.

Hodge numbers of complete intersection Calabi-Yau varieties in Gorenstein Fano toric varieties.

The downside is that it is unclear how to generalize the E_{st}-polynomial if our Calabi-Yau of interest Y is a complete intersection in a non-Gorenstein toric variety X_Σ, such as the Calabi-Yau $Y = \mathbb{P}^5_{211111}[3,4]$ studied in [2]. As we will see in §2.5, if Y is smooth and the number of generators of the homogeneous coordinate ring $S(\Sigma)$ of X_Σ is small, then a direct analysis of the tangent sheaf $\mathscr{T}_Y$ is possible. However, if $S(\Sigma)$ has many generators, then direct computations are infeasable. A generalization of the E_{st}-polynomial that applies to Calabi-Yau complete intersections in non-Gorenstein toric varieties is thus desirable.

1.3 Outline

This dissertation is divided into six chapters, including this introduction.

In Chapter 2, we review the basic toric geometry involved in the Batyrev-Borisov mirror construction. Of particular interest is the method developed in §2.5 for computing the Hodge numbers of smooth Calabi-Yau complete intersections in complete simplicial toric varieties. In §2.5.1, we use the spectral sequence from §2.5 to compute the Hodge numbers of the smooth Calabi-Yau complete intersection $Y = \mathbb{P}^5_{211111}[3,4]$. In §2.5.2, we turn our attention to the Calabi-Yau complete intersection $Y = \mathbb{P}^5_{321111}[4,5]$. We show that Y is singular and construct a crepant desingularization $\widehat{Y} \to Y$, where $\widehat{Y}$ is a Calabi-Yau complete intersection in a non-Gorenstein toric variety. Computing the Hodge numbers $h^{p,q}(\widehat{Y})$ using the spectral sequence from §2.5 thus gives the stringy Hodge numbers $h^{p,q}_{\text{st}}(Y)$.

In Chapter 3, we outline the construction of Batyrev-Borisov mirror pairs and their stringy Hodge numbers. In §3.1 and §3.2, we develop the combinatorial notions necessary for defining the *stringy E-polynomial* $E_{\text{st}}(\Delta; u, v)$ of a Gorenstein polytope

Δ. The key ingredient in defining $E_{\text{st}}(\Delta; u, v)$ is the $\widetilde{S}$-polynomial of a lattice polytope, which we describe in §3.2.3. In §3.3, we review the Batyrev-Borisov construction, which associates a pair of Calabi-Yau complete intersections $(Y_{\Delta_1,\ldots,\Delta_c}, Y_{\nabla_1,\ldots,\nabla_c})$ to a dual pair of nef-partitions $\Delta = \Delta_1 + \cdots + \Delta_c$ and $\nabla = \nabla_1 + \cdots + \nabla_c$. In §3.4, we use the E_{st}-polynomial to compute the stringy Hodge numbers of two examples of Batyrev-Borisov mirror pairs. We conclude Chapter 3 with §3.5, where we state a closed-form expression for the stringy Hodge numbers of a bipartite Calabi-Yau complete intersection Y_{Δ_1,Δ_2} corresponding to a length-two nef-partition $\Delta = \Delta_1 + \Delta_2$ such that the divisors D_{Δ_1} and D_{Δ_2} on X_Δ are ample.

In Chapter 4, we turn our attention to Calabi-Yau complete intersections in non-Gorenstein Fano varieties. In §4.1, we summarize Mavlyutov's construction of *dual* $\mathbb{Q}$*-nef-partitions of* $\mathbb{Q}$*-reflexive polytopes*. This construction is based on Mavlyutov's 2011 preprint [34] and generalizes the notation of a dual pair of nef-partitions. In §4.2, we review the notion of a *toric extremal transition* between Calabi-Yau complete intersections. In §4.2.2, we demonstrate that it is necessary to consider Mavlyutov's $\mathbb{Q}$-nef-partitions if one is to follow all extremal transitions. We do so by exhibiting an extremal transition $Y_{\Delta_1,\Delta_2} \dashrightarrow Y_{\tilde{\Delta}_1,\tilde{\Delta}_2}$ where Y_{Δ_1,Δ_2} is a $\mathbb{Q}$-nef Calabi-Yau complete intersection corresponding to a $\mathbb{Q}$-nef-partition $\Delta = \Delta_1 + \Delta_2$ and $Y_{\tilde{\Delta}_1,\tilde{\Delta}_2}$ is a Calabi-Yau complete intersection corresponding to a nef-partition $\tilde{\Delta} = \tilde{\Delta}_1 + \tilde{\Delta}_2$.

To our knowledge, the Calabi-Yau variety $\mathbb{P}^5_{211111}[3, 4]$ is the only known example of a Calabi-Yau complete intersection corresponding to a $\mathbb{Q}$-nef-partition of a non-reflexive polytope. In §4.3, we contribute 1077 more examples to the list of known examples by partially classifying length-two $\mathbb{Q}$-nef-partitions of five-dimensional $\mathbb{Q}$-reflexive polytopes that are not reflexive. By searching for *almost reflexive* polytopes embedded in one of the 833 five-dimensional smooth Fano polytopes, we completely classify all $\mathbb{Q}$-reflexive polytopes Δ whose dual Δ^* embeds into a five-dimensional smooth Fano polytope. By doing so, we obtain the following.

Theorem 4.3.2. *There are exactly* 48 *isomorphism classes of five-dimensional almost reflexive polytopes that are not reflexive contained in a five-dimensional smooth Fano polytope. Equivalently, there are exactly* 48 *isomorphism classes of five-dimensional* $\mathbb{Q}$*-reflexive polytopes that are not reflexive containing a five-dimensional reflexive polytope whose dual is a smooth Fano polytope.*

For each of the 48 $\mathbb{Q}$-reflexive polytopes Δ from Theorem 4.3.2, we compute all length-two dual pairs of of $\mathbb{Q}$-nef-partitions $\Delta = \Delta_1 + \Delta_2$ and $\nabla = \nabla_1 + \nabla_2$ and analyze the geometry of the Calabi-Yau complete intersections Y_{Δ_1,Δ_2}.

Proposition 4.3.5. *There are* 209 *combinatorially distinct length two* $\mathbb{Q}$*-nef-partitions of five-dimensional non-reflexive* $\mathbb{Q}$*-reflexive polytopes containing a polytope whose dual is a smooth Fano polytope. Moreover, each of these* $\mathbb{Q}$*-nef partitions* $\Delta = \Delta_1 + \Delta_2$ *defines a smooth Calabi-Yau threefold* $Y_{\Delta_1,\Delta_2} \subset X_\Delta$.

For each of the 209 dual pairs of $\mathbb{Q}$-nef-partitions from Proposition 4.3.5, we compute the Hodge data of the Calabi-Yau complete intersection Y_{Δ_1,Δ_2} by analyzing the cohomology of the tangent sheaf $\mathscr{T}_{Y_{\Delta_1,\Delta_2}}$ (Table 4.1).

In §4.3.2, we expand this library of examples by searching for almost reflexive subpolytopes in Kreuzer's list of reflexive five-dimensional polytopes found in [27]. In total, we find 1077 combinatorially distinct examples of $\mathbb{Q}$-nef-partitions of 813 $\mathbb{Q}$-reflexive polytopes. For each of our $\mathbb{Q}$-nef-partitions, we compute the Hodge data of Y_{Δ_1,Δ_2} (Table 4.2).

In §4.4, we review Aspinwall and Plesser's *complete mirror pairs*. We show that the notion of a complete mirror pair $(\mathscr{A}, \mathscr{B})$ generalizes the notion of a dual pair of almost reflexive Gorenstein cones $(\sigma, \sigma^\bullet)$ (Theorem 4.4.3). Thus, each of our 1077 combinatorially distinct examples of $\mathbb{Q}$-nef-partitions found in §4.3.2 provides an example of a complete mirror pair $(\mathscr{A}, \mathscr{B})$. We add 2142 more examples of complete mirror pairs to this list in §4.4.2 by considering complete intersections in weighted projective spaces.

In Chapter 5, we seek a combinatorial definition of the stringy Hodge numbers of a complete mirror pair $(\mathscr{A}, \mathscr{B})$. In §5.1, we define the *naïve stringy E-functions* $\widetilde{E}_{\mathrm{st}}(\Delta_{\mathscr{A}}; u, v)$ *and* $\widetilde{E}_{\mathrm{st}}(\Delta_{\mathscr{B}}; u, v)$ of the polytopes $\Delta_{\mathscr{A}} = \mathrm{Conv}(\mathscr{A})$ and $\Delta_{\mathscr{B}} = \mathrm{Conv}(\mathscr{B})$ corresonding to a complete mirror pair $(\mathscr{A}, \mathscr{B})$. We define the *naïve stringy Hodge numbers* $\tilde{h}^{p,q}_{\mathrm{st}}(\Delta_{\mathscr{A}})$ *of* $\Delta_{\mathscr{A}}$ as the coefficients of $\widetilde{E}_{\mathrm{st}}(\Delta_{\mathscr{A}}; u, v)$, providing that $\widetilde{E}_{\mathrm{st}}(\Delta_{\mathscr{A}}; u, v)$ is a polynomial. The naïve stringy Hodge numbers $\tilde{h}^{p,q}_{\mathrm{st}}(\Delta_{\mathscr{B}})$ are defined similarly. When $(\mathscr{A}, \mathscr{B})$ is a reflexive mirror pair, the naïve stringy E-function coincides with the usual stringy E-function.

We formulate three important conjectures concerning our naïve stringy E-function. First, we expect that $\widetilde{E}_{\mathrm{st}}$ obeys the *mirror symmetry law.*

Conjecture 5.1.4. *Let* $(\mathscr{A}, \mathscr{B})$ *be a complete mirror pair with* $\dim_{\mathrm{CY}} \Delta_{\mathscr{A}} = n$. *Then*

$$\widetilde{E}_{\mathrm{st}}(\Delta_{\mathscr{A}}; u, v) = (-u)^n \widetilde{E}_{\mathrm{st}}(\Delta_{\mathscr{B}}; u^{-1}, v) \tag{1.9}$$

That is, $\widetilde{E}_{\mathrm{st}}$ *obeys the mirror symmetry law.*

We also predict that the naïve stringy E-function generates the stringy Hodge numbers of a Calabi-Yau complete intersection arising from Mavlyutov's construction.

Conjecture 5.1.6. *Let* $Y_{\Delta_1,\ldots,\Delta_c} \subset X_\Delta$ *be the* $\mathbb{Q}$*-nef Calabi-Yau complete intersection associated to a* $\mathbb{Q}$*-nef-partition* $\Delta = \Delta_1 + \cdots + \Delta_c$ *and suppose* $\widehat{Y} \to Y_{\Delta_1,\ldots,\Delta_c}$ *is a crepant desingularization of* $Y_{\Delta_1,\ldots,\Delta_c}$ *by a smooth Calabi-Yau variety* $\widehat{Y}$. *Then* $h^{p,q}(\widehat{Y}) = \tilde{h}^{p,q}_{\mathrm{st}}(Y_{\Delta_1,\ldots,\Delta_c})$.

Finally, we suspect that every Mavlyutov mirror pair $(Y_{\Delta_1,\ldots,\Delta_c}, Y_{\nabla_1,\ldots,\nabla_c})$ passes the *naïve* topological mirror symmetry test.

Conjecture 5.1.7. *Let* $Y_{\Delta_1,\ldots,\Delta_c} \subset X_\Delta$ *and* $Y_{\nabla_1,\ldots,\nabla_c} \subset X_\nabla$ *be the* $\mathbb{Q}$*-nef Calabi-Yau complete intersections associated to a dual pair of* $\mathbb{Q}$*-nef-partitions* $\Delta = \Delta_1 + \ldots + \Delta_c$ *and* $\nabla = \nabla_1 + \cdots + \nabla_c$*. Then the naïve stringy Hodge numbers of* $Y_{\Delta_1,\ldots,\Delta_c} \subset X_\Delta$ *and* $Y_{\nabla_1,\ldots,\nabla_c} \subset X_\nabla$ *satisfy*

$$\tilde{h}^{p,q}_{\mathrm{st}}(Y_{\Delta_1,\ldots,\Delta_c}) = \tilde{h}^{d-c-p,q}_{\mathrm{st}}(Y_{\nabla_1,\ldots,\nabla_c}) \tag{1.10}$$

for $0 \leq p, q \leq d - c$*, where* $d = \dim \Delta$*.*

Our conjectures are supported by a good deal of evidence. In §5.2, we consider the 209 dual pairs of $\mathbb{Q}$-nef-partitions $\Delta = \Delta_1 + \Delta_2$ and $\nabla = \nabla_1 + \nabla_2$ found in §4.3.1. By comparing the naïve stringy Hodge numbers with the Hodge data found in §5.2, we find that the naïve stringy Hodge numbers coincide with the usual Hodge numbers for each of the 209 Calabi-Yau threefolds Y_{Δ_1,Δ_2}. That is, Conjecture 5.1.6 for these Calabi-Yau threefolds. Additionally, we find that each of these 209 Mavlyutov mirror pairs $(Y_{\Delta_1,\Delta_2}, Y_{\nabla_1,\nabla_2})$ satisfies Conjecture 5.1.7.

In §5.3, we test our conjectures on the 1077 dual pairs of $\mathbb{Q}$-nef-partitions $\Delta = \Delta_1 + \Delta_2$ and $\nabla = \nabla_1 + \nabla_2$ found in §4.3.2. We find again that Conjecture 5.1.6 holds for the Calabi-Yau threefolds X_{Δ_1,Δ_2} and Conjecture 5.1.7 holds for each Mavlyutov mirror pair $(Y_{\Delta_1,\Delta_2}, Y_{\nabla_1,\nabla_2})$.

In §5.4, we apply the naïve stringy E-function to the 2142 complete mirror pairs $(\mathscr{A}, \mathscr{B})$ produced in §4.4.2 by examining Klemm's list of complete intersections in weighted projective spaces. We find that the Hodge data generated by $\widetilde{E}_{\mathrm{st}}$ agrees with the computations done by Klemm.

In §5.5 we propose a closed-form expression for $\mathbb{Q}$-nef Calabi-Yau threefolds corresponding to $\mathbb{Q}$-nef-partitions $\Delta = \Delta_1 + \Delta_2$ of non-reflexive five-dimensional polytopes when the divisors D_{Δ_1} and D_{Δ_2} are ample. We verify our proposed formula correctly predicts the stringy Hodge numbers of each of the 24 of the 1077 relevant examples from §4.3.2.

We conclude this dissertation with a discussion of possible further directions for this research (Chapter 6).

2

Basic Toric Geometry

The principal objects of interest in this dissertation are Calabi-Yau complete intersections in projective toric varieties. Toric varieties are algebraic varieties that are generalizations of both affine space $\mathbb{A}^d$ and projective space $\mathbb{P}^d$. There is a deep connection between the geometry of toric varieties and the combinatorics of fans. As we will see, this connection often proves useful in answering computational questions about toric varieties and their subvarieties.

In this chapter, we present some basic concepts and results surrounding toric varieties and their subvarieties. In §2.1, we review the construction of a toric variety X_Σ from a combinatorial fan Σ and state some basic results relating the geometry of X_Σ to the combinatorics of Σ. In §2.2, we demonstrate an algorithm for computing the divisor class group $\mathrm{Cl}(X_\Sigma)$ depending only on the combinatorial data in Σ. While the construction of X_Σ from Σ offered in §2.1 is of theoretical interest, it is less useful for studying subvarieties of X_Σ. In Section 2.3, we show how to remidy this by reviewing the quotient construction of X_Σ from Σ, which allows one to define homogeneous coordinates on X_Σ. The toric varieties most important to this dissertation are projective toric varieties corresponding to normal fans of polytopes, so in §2.4 we collect some terminology and results on polytopes and toric varieties. We conclude this chapter by outlining a method involving the derived category for computing Hodge numbers of complete intersections in toric varieties and offer two examples of this method. The examples preceeding §2.5 served as inspiration for this dissertation.

Our principal motivation is to standardize notation and collect results useful for later constructions. Much of the notation and many of the results covered can be found in [17]. A reader familiar with the basic language of toric varieties and fans can focus on the technique developed in Section 2.5 for computing Hodge numbers of complete intersections in toric varieties.

2.1 The Toric Variety of a Fan

There is a well-known equivalence of categories

$$\left\{ \begin{array}{c} \text{category of normal} \\ \text{separated toric varieties} \end{array} \right\} \overset{\simeq}{\longleftrightarrow} \left\{ \begin{array}{c} \text{category} \\ \text{of fans} \end{array} \right\} \tag{2.1}$$

which we will refer to as the *variety-fan equivalence*. In this section, we review the construction and basic utility of this equivalence. Our motivation is mainly to establish notation and terminology. We refer the reader to [17, §3.1, §3.3] for a more detailed exposition.

We begin by describing the category of toric varieties.

Definition 2.1.1 (Definition 3.1.1 and Proposition 1.3.14 in [17])**.** A *toric variety with torus* $T \simeq (\mathbb{C}^*)^d$ is an irreducible variety X containing T as a Zariski open subset such that the action of T on itself extends to an algebraic action of T on X.

Suppose X_1 and X_2 are toric varieties with tori T_1 and T_2 respectively. A *morphism of toric varieties* $\phi : X_1 \to X_2$ is a morphism of varieties $\phi : X_1 \to X_2$ such that $\phi|_{T_1} : T_1 \to T_2$ is a well-defined group homomorphism.

Before introducing the category of fans, let us standardize some notation. Let N be a torsion-free lattice of finite rank d and let M be its dual lattice $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. The *torus* T_N *associated to* N *and* M is

$$T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) \simeq (\mathbb{C}^*)^d \tag{2.2}$$

The isomorphisms in (2.2) allow us to view N as the *lattice of one-parameter subgroups of* T_N and M as the *character lattice of* T_N [17, §1.1]. Put $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and note that the natural pairing $\langle -, - \rangle : M \times N \to \mathbb{Z}$ extends naturally to a pairing $\langle -, - \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$.

Definition 2.1.2 (Definition 3.1.2 and Definition 3.3.1 in [17])**.** A *fan in* $N_{\mathbb{R}}$ is a finite collection Σ of polyhedral cones $\sigma \subset N_{\mathbb{R}}$ such that

(*a*) Every face τ of every cone $\sigma \in \Sigma$ is a cone in Σ.

(*b*) The intersection $\sigma_1 \cap \sigma_2$ of every two cones $\sigma_1, \sigma_2 \in \Sigma$ is a face of both σ_1 and σ_2.

Suppose Σ_1 and Σ_2 are fans in $(N_1)_{\mathbb{R}}$ and $(N_2)_{\mathbb{R}}$ respectively. A *morphism of fans* $\bar{\phi} : \Sigma_1 \to \Sigma_2$ is a $\mathbb{Z}$-linear map $\bar{\phi} : N_1 \to N_2$ such that for every cone $\sigma_1 \in \Sigma_1$ there exists a cone $\sigma_2 \in \Sigma_2$ satisfying $\bar{\phi}_{\mathbb{R}}(\sigma_1) \subset \sigma_2$.

The following list of properties of fans will be useful in the sequel.

Definition 2.1.3. Let Σ be a fan in $N_{\mathbb{R}}$. Denote by $\Sigma(r)$ the collection of cones in Σ with dimension r. The cones in $\Sigma(1)$ are called the *rays of* Σ and each ray ρ of Σ is generated by a unique primitive element $u_\rho \in N$. We say Σ *has no torus factors* if $\mathrm{Span}_{\mathbb{R}}\{u_\rho \mid \rho \in \Sigma(1)\} = N_{\mathbb{R}}$. The *support of* Σ is $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma \subset N_{\mathbb{R}}$.

We now review how the variety-fan equivalence (2.1) assigns a normal separated variety X_Σ to each fan Σ in $N_{\mathbb{R}}$. Each cone $\sigma \in \Sigma$ corresponds to an affine variety

$$U_\sigma = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M]) \tag{2.3}$$

Note that the torus

$$T_N \simeq \mathrm{Spec}(\mathbb{C}[M]) \subset U_\sigma \tag{2.4}$$

acts on each U_σ in (2.3), endowing U_σ with the structure of an *affine toric variety.* The combinatorial structure of Σ describes the data necessary to glue the affine toric varieties in (2.3) together to obtain a variety X_Σ. In fact, each of the tori in (2.4) are identified under this gluing and it can be shown [17, Theorem 3.1.5] that X_Σ is a normal separated toric variety with torus T_N. Conversely, each normal separated toric variety with torus T_N is of the form X_Σ for a fan Σ in $N_\mathbb{R}$ [17, Corollary 3.1.8]. This establishes how the variety-fan equivalence operates on the level of objects.

To see how the variety-fan equivalence operates on the level of morphisms, let $\bar{\phi} : \Sigma_1 \to \Sigma_2$ be a morphism of fans. Each inclusion $\bar{\phi}_\mathbb{R}(\sigma_1) \subset \sigma_2$ induces an affine toric morphism $\phi_{\sigma_1} : U_{\sigma_1} \to U_{\sigma_2}$. These affine toric morphisms patch together [17, Theorem 3.3.4 (a)] to yield a morphism of toric varieties $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$. In particular, $\phi|_{T_{N_1}} = \phi_{\{0\}}$. Conversely, it can be shown [17, Theorem 3.3.4 (b)] that every morphism of toric varieties $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ corresponds to a morphism $\bar{\phi} : \Sigma_1 \to \Sigma_2$ to make (2.1) an equivalence of categories.

The variety-fan equivalence allows one to translate combinatorial properties of a fan Σ into geometric properties of the toric variety X_Σ. To demonstrate this, we introduce some combinatorial terminology.

Definition 2.1.4. Let Σ be a fan in $N_\mathbb{R}$.

(a) A cone $\sigma \in \Sigma$ is *smooth* if its minimal generators form part of a $\mathbb{Z}$-basis of N.

(b) A cone $\sigma \in \Sigma$ is *simplicial* if its minimal generators are linearly independent over $\mathbb{R}$.

We call Σ *smooth* if each of its cones is smooth and we call Σ *simplicial* if each of its cones is simplicial. The fan Σ is *complete* if $|\Sigma| = N_\mathbb{R}$.

These combinatorial notions correspond to geometric notions under the variety-fan equivalence.

Theorem 2.1.5 (Theorem 3.1.19 in [17])**.** *Let Σ be a fan in $N_{\mathbb{R}}$. Then*

(a) *X_Σ is smooth if and only if Σ is smooth.*

(b) *X_Σ is an orbifold if and only if Σ is simplicial.*

(c) *X_Σ is compact in the classical topology if and only if Σ is complete.*

The following example demonstrates many of the concepts developed thus far.

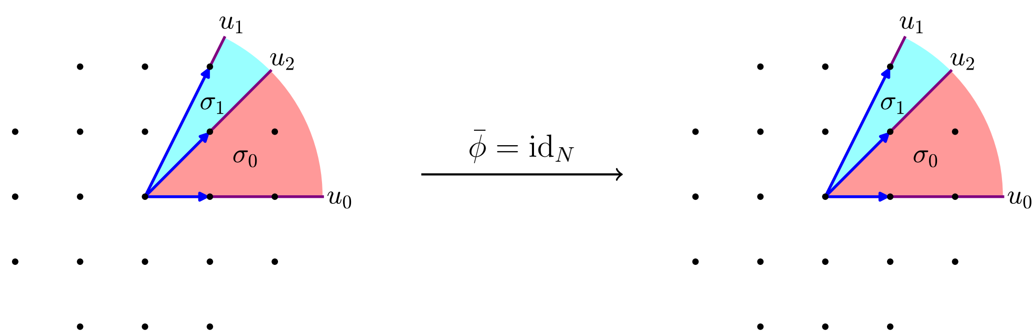


FIGURE 2.1: A morphism of fans $\bar{\phi} : \Sigma_1 \to \Sigma_2$

Example 2.1.6. Let $N = \mathbb{Z}^2$. Figure 2.1 depicts two fans Σ_1 and σ_2 in $N_{\mathbb{R}} \simeq \mathbb{R}^2$ and the morphism of fans $\bar{\phi} : \Sigma_1 \to \Sigma_2$ induced by the identity morphism id_N of N. The fan Σ_1 is generated by the cones $\{\sigma_1, \sigma_2\}$ and the fan Σ_2 is generated by the single cone σ where

$$\begin{array}{lll} u_0 = N(1,\ 0) & \qquad & \sigma = \mathrm{Cone}(u_0,\ u_1) \\ u_1 = N(1,\ 2) & & \sigma_0 = \mathrm{Cone}(u_0,\ u_2) \\ u_2 = N(1,\ 1) & & \sigma_1 = \mathrm{Cone}(u_1,\ u_2) \end{array}$$

Each cone in Σ_1 is smooth so that X_{Σ_1} is a smooth variety by Theorem 2.1.5. The cone σ is simplicial but not smooth and all the proper faces of σ are smooth. Theorem

2.1.5 then implies that X_{Σ_2} is an orbifold. Note that $|\Sigma_1| \neq N_{\mathbb{R}}$ and $|\Sigma_2| \neq N_{\mathbb{R}}$ so neither X_{Σ_1} nor X_{Σ_2} is compact.

The toric variety X_{Σ_2} is affine since Σ_2 is generated by the single cone σ. Note that the semigroup $\sigma^\vee \cap M$ is generated by

$$m_1 = M(0,1) \qquad m_2 = M(2,-1) \qquad m_3 = M(1,0)$$

Equation (2.3) then offers an explicit description of X_{Σ_2} as an orbifold

$$\begin{aligned} X_{\Sigma_2} &= \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M] \\ &= \operatorname{Spec} \mathbb{C}[\chi^{m_1}, \chi^{m_2}, \chi^{m_3}] \\ &\simeq \operatorname{Spec} \mathbb{C}[x,y,z]/\langle xy - z^2 \rangle \\ &\simeq \mathbb{C}^2/\mathbb{Z}_2 \end{aligned} \tag{2.5}$$

By inspecting the affine patches U_{σ_0} and U_{σ_1} of X_{Σ_1}, one obtains an explicit description of X_{Σ_2} as the smooth variety $X_{\Sigma_2} \simeq \mathscr{O}_{\mathbb{P}^1}(2)$. Hence Figure 2.1 depicts the resolution of singularities $\mathscr{O}_{\mathbb{P}^1}(2) \to \mathbb{C}^2/\mathbb{Z}_2$. $\lozenge$

Another important result relating the combinatorics of the fan Σ to the geometry of X_Σ is the *Orbit-Cone Correspondence*. The uninitiated reader is referred to Theorem 3.2.6, Proposition 3.2.7, and Proposition 11.1.2 in [17] for the relevant notation.

Theorem 2.1.7 (The Orbit-Cone Correspondence). *Let Σ be a fan in $N_{\mathbb{R}}$. Then*

(*a*) *There is a one-to-one correspondence*

$$\begin{aligned} \{\text{cones } \sigma \text{ in } \Sigma\} &\longleftrightarrow \{T_N\text{-orbits in } X_\Sigma\} \\ \sigma &\longleftrightarrow O(\sigma) \simeq \operatorname{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*) \end{aligned} \tag{2.6}$$

satisfying $\dim \sigma + \dim O(\sigma) = \dim N_{\mathbb{R}}$.

(*b*) *For each cone $\sigma \in \Sigma$, the affine variety U_σ can be expressed as*

$$U_\sigma = \bigcup_{\tau \preceq \sigma} O(\tau) \tag{2.7}$$

where the union is taken over all faces τ of σ.

(*c*) *A cone τ is a face of $\sigma \in \Sigma$ if and only if $O(\sigma) \subset V(\tau)$, and*

$$V(\tau) = \bigcup_{\tau \preceq \sigma} O(\sigma) \tag{2.8}$$

where $V(\tau) = \overline{O(\tau)}$ denotes the closure in both the classical and Zariski topologies.

(*d*) *For each cone $\sigma \in \Sigma$, the orbit closure $V(\sigma) = \overline{O(\tau)}$ is isomorphic to the toric variety $X_{\mathrm{Star}(\tau)}$.*

(*e*) *The singular locus and the smooth locus of X_Σ are given by*

$$\mathrm{Sing}(X_\Sigma) = \bigcup_{\sigma \text{ not smooth}} V(\sigma) \qquad X_\Sigma \setminus \mathrm{Sing}(X_\Sigma) = \bigcup_{\sigma \text{ smooth}} U_\sigma \tag{2.9}$$

respectively.

In particular, applying Theorem 2.1.7(e) to the toric variety X_{Σ_2} of Example 2.1.6 shows that the singular locus of X_{Σ_2} consists of a single point $V(\sigma)$, as expected from the identification $X_{\Sigma_2} \simeq \mathbb{C}^2/\mathbb{Z}_2$ from (2.5).

2.2 The Divisor Class Group

Let X be an irreducible variety. Recall that a *prime divisor on X* is an irreducible subvariety of codimension one and the *group of Weil divisors on X* is the free abelian group $\mathrm{Div}(X)$ generated by the prime divisors on X. If X is a normal variety, then we define the *divisor class group of X* as

$$\mathrm{Cl}(X) = \mathrm{Div}(X)/\,\mathrm{Div}_0(X) \tag{2.10}$$

where $\mathrm{Div}_0(X)$ is the subgroup of $\mathrm{Div}(X)$ consisting of all principal divisors.

In general, the class group of a variety is difficult to compute. For toric varieties, however, the situation is simplified dramatically.

Let X_Σ be the normal toric variety of a fan Σ in $N_\mathbb{R}$ with $\dim N_\mathbb{R} = d$. Under the Orbit-Cone Correspondence, each ray $\rho \in \Sigma(1)$ corresponds to a codimension one orbit $O(\rho)$ whose closure $V(\rho)$ is a T_N-invariant prime divisor on X_Σ. We write $D_\rho = V(\rho)$ to emphasize that $V(\rho)$ is a divisor. The *group of torus-invariant Weil divisors on* X_Σ is the subgroup $\mathrm{Div}_{T_N}(X_\Sigma)$ of $\mathrm{Div}(X_\Sigma)$ generated by the torus-invariant prime divisors $\{D_\rho \mid \rho \in \Sigma(1)\}$.

One may show [17, Theorem 4.1.3] that the class group $\mathrm{Cl}(X_\Sigma)$ fits into an exact sequence

$$M \longrightarrow \mathrm{Div}_{T_N}(X_\Sigma) \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0 \tag{2.11}$$

where the first map is $m \mapsto \sum_{\rho\in\Sigma(1)} \langle m, u_\rho\rangle D_\rho$. Furthermore, the sequence (2.11) is a short exact sequence

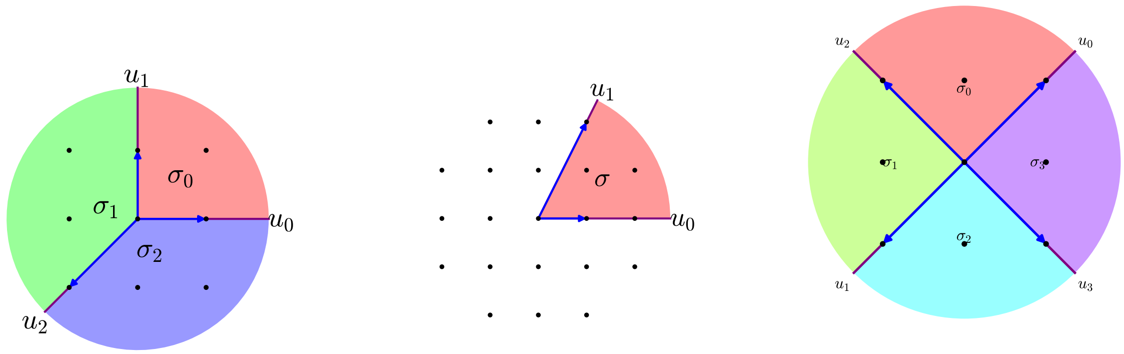
$$0 \longrightarrow M \longrightarrow \mathrm{Div}_{T_N}(X_\Sigma) \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0 \tag{2.12}$$

if and only if Σ has no torus factors.

The exact sequence in (2.12) offers an algorithm for computing $\mathrm{Cl}(X_\Sigma)$. Choose a basis β of N. Then the first nontrivial map in (2.12) can be represented as the matrix A whose rows are $\{[u_\rho]_\beta \mid \rho \in \Sigma(1)\}$. The exactness of (2.12) implies that $\mathrm{Cl}(X_\Sigma)$ is isomorphic to the cokernel of A. Hence

$$\mathrm{Cl}(X_\Sigma) \simeq \mathbb{Z}^{|\Sigma(1)|-d} \oplus \mathbb{Z}/e_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/e_r\mathbb{Z} \tag{2.13}$$

where $e_1, \ldots, e_r$ are the elementary divisors of A. In particular, (2.13) implies that $\mathrm{Cl}(X_\Sigma)$ is finitely generated with rank $|\Sigma(1)| - d$. Figure 2.2 depicts the class groups of three toric varieties computed using (2.13).



(a) $\mathrm{Cl}(\mathbb{P}^2) \simeq \mathbb{Z}$ (b) $\mathrm{Cl}(\mathbb{C}^2/\mathbb{Z}_2) \simeq \mathbb{Z}_2$ (c) $\mathrm{Cl}((\mathbb{P}^1 \times \mathbb{P}^1)/\mathbb{Z}_2) \simeq \mathbb{Z}^2 \oplus \mathbb{Z}_2$

FIGURE 2.2: Computations of various class groups.

2.3 The Quotient Construction and Homogeneous Coordinates

While toric varieties are interesting in their own right, the main objects of interest in this dissertation are *subvarieties* of toric varieties. In §2.1 we saw a coordinate-free construction of the normal toric variety X_Σ from the combinatorial data in Σ. In this section we review the quotient construction of X_Σ. This construction allows us to define coordinates on X_Σ, which will provide a more convenient setting for studying subvarieties.

Let X_Σ be the normal toric variety without torus factors corresponding to a simplicial fan Σ in $N_\mathbb{R}$. To construct X_Σ as a quotient, one begins by associating two important algebraic structures to Σ. The *homogeneous coordinate ring of* Σ is the polynomial algebra

$$S(\Sigma) = \mathbb{C}[z_\rho \,|\, \rho \in \Sigma(1)] \tag{2.14}$$

and the *irrelevant ideal of* Σ is the ideal

$$B(\Sigma) = \langle \textstyle\prod_{\rho \not\subset \sigma} z_\rho \,|\, \sigma \in \Sigma \rangle \tag{2.15}$$

in $S(\Sigma)$. The quotient we are interested in is defined in terms of the geometric counterparts to the algebraic structures in (2.14) and (2.15). The homogeneous coordinate ring (2.14) corresponds to the affine space $\mathbb{C}^{\Sigma(1)} = \mathrm{Spec}(S)$ which contains the Zariski closed subset $\mathbf{V}(B)$ determined by the irrelevant ideal (2.15).

Now, consider the subgroup $G(\Sigma)$ of the torus $(\mathbb{C}^*)^{\Sigma(1)}$ given by

$$G(\Sigma) = \left\{ (t_\rho) \in (\mathbb{C}^*)^{\Sigma(1)} \,\middle|\, u \in M \Rightarrow \prod_{\rho \in \Sigma(1)} t_\rho^{\langle u, u_\rho \rangle} = 1 \right\} \tag{2.16}$$

The vanishing locus $\mathbf{V}(B)$ of the irrelevant ideal is invariant under the diagonal action of G on $\mathbb{C}^{\Sigma(1)}$ [17, Lemma 5.1.1]. The toric variety X_Σ is isomorphic to the geometric quotient

$$X_\Sigma \simeq \frac{\mathbb{C}^{\Sigma(1)} \setminus \mathbf{V}(B)}{G} \tag{2.17}$$

induced by a toric morphism $\pi : \mathbb{C}^{\Sigma(1)} \setminus \mathbf{V}(B) \to X_\Sigma$ [17, Theorem 5.1.11].

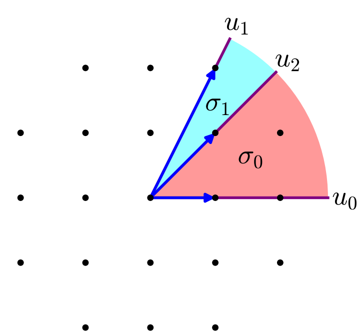


FIGURE 2.3: $X_\Sigma \simeq \mathscr{O}_{\mathbb{P}^1}(2)$

Example 2.3.1. Let Σ be the fan in $N_\mathbb{R} \simeq \mathbb{R}^2$ depicted in Figure 2.3. Then (2.14), (2.15), and (2.16) give

$$S(\Sigma) = \mathbb{C}[z_0, z_1, z_2] \qquad B(\Sigma) = \langle z_0, z_1 \rangle \qquad G(\Sigma) = \{(\lambda, \lambda, \lambda^{-2}) \,|\, \lambda \in \mathbb{C}^*\}$$

Thus (2.17) is given by the quotient

$$X_\Sigma \simeq \frac{\mathbb{C}^3 \setminus \{(0,0,z_2) \,|\, z_2 \in \mathbb{C}\}}{\{(\lambda, \lambda, \lambda^{-2}) \,|\, \lambda \in \mathbb{C}^*\}} \tag{2.18}$$

That is, $X_\Sigma \simeq \mathscr{O}_{\mathbb{P}^2}(2)$, as claimed in Example 2.1.6. $\Diamond$

Following [34, §4], the formula

$$\deg(\prod_{\rho\in\Sigma(1)} z_\rho^{b_\rho}) = [\sum_{\rho\in\Sigma(1)} b_\rho D_\rho] \in \mathrm{Cl}(X_\Sigma) \tag{2.19}$$

endows the homogeneous coordinate ring $S(\Sigma)$ with a $\mathrm{Cl}(X_\Sigma)$-grading

$$S(\Sigma) = \bigoplus_{\beta\in\mathrm{Cl}(X_\Sigma)} S(\Sigma)_\beta \tag{2.20}$$

where $S(\Sigma)_\beta$ is the subalgebra of $S(\Sigma)$ generated by monomials of degree β. The grading (2.20) provides a nice combinatorial description of divisors on X_Σ. There is a one-to-one correspondance

$$\mathrm{Div}_{T_N}(X_\Sigma) \otimes_{\mathbb{Z}} \mathbb{Q} \longleftrightarrow \left\{ \begin{array}{c} \text{polytopes in } M_\mathbb{R} \\ \text{with rational vertices} \end{array} \right\} \tag{2.21}$$

under which a torus-invariant $\mathbb{Q}$-Weil divisor $D = \sum_{\rho\in\Sigma(1)} b_\rho D_\rho$ is assigned to the polytope

$$\Delta_D = \{m \in M_\mathbb{R} \,|\, \rho \in \Sigma(1) \Rightarrow \langle m, u_\rho\rangle \geq -b_\rho\} \tag{2.22}$$

and a polytope Δ is assigned to

$$D_\Delta = \sum_{\rho\in\Sigma(1)} (-\min\langle\Delta, u_\rho\rangle) D_\rho \tag{2.23}$$

Furthermore, for a torus-invariant Weil divisor $D = \sum_{\rho\in\Sigma(1)} b_\rho D_\rho$, there is a one-to-one correspondence

$$\begin{aligned} \{\text{lattice points in } \Delta_D\} &\longleftrightarrow \{\text{monomials in } S(\Sigma)_{[D]}\} \\ m &\longmapsto \prod_{\rho\in\Sigma(1)} z_\rho^{b_\rho + \langle m, u_\rho\rangle} \end{aligned} \tag{2.24}$$

We now have a convenient framework for studying subvarieties of X_Σ. For $\beta = [D] \in \mathrm{Cl}(X_\Sigma)$ there is a natural isomorphism [17, §9.1]

$$H^0(X_\Sigma, \mathscr{O}_{X_\Sigma}) \simeq S(\Sigma)_\beta \tag{2.25}$$

By (2.24), it follows that every hypersurface in X_Σ of degree $\beta = [\sum_{\rho\in\Sigma(1)} b_\rho D_\rho]$ corresponds to a homogeneous polynomial $f \in S(\Sigma)_\beta$ of the form

$$f = \sum_{m\in\Delta_D\cap M} a_m \prod_{\rho\in\Sigma(1)} z_\rho^{b_\rho+\langle m,u_\rho\rangle} \tag{2.26}$$

for $a_m \in \mathbb{C}$. Moreover, the *Toric Ideal-Variety Correspondence* [17, Proposition 5.2.7] provides a one-to-one correspondence

$$\{\text{closed subvarieties of } X_\Sigma\} \longleftrightarrow \left\{ \begin{array}{c} \text{radical homogeneous} \\ \text{ideals } I \subset B(\Sigma) \subset S(\Sigma) \end{array} \right\} \tag{2.27}$$

A *complete intersection in* X_Σ *of codimension* c is a closed subvariety $\mathbf{V}(I) \subset X_\Sigma$ corresponding to a radical homogeneous ideal $I \subset S(\Sigma)$ generated by a regular sequence of homogeneous polynomials $f_1, \ldots, f_c \in S(\Sigma)$ such that $c = \dim X_\Sigma - \dim \mathbf{V}(I)$.

2.4 Polytopes and Toric Varieties

In this section, we collect some definitions and results relating the combinatorics of polytopes to toric geometry. Our primary interest is the correspondence between canonical Fano polytopes and Fano toric varieties, which restricts to a correspondence between reflexive polytopes and Gorenstein Fano toric varieties.

Let N be a torsion-free lattice of finite rank d and let M be its dual lattice $M = \mathrm{Hom}_\mathbb{Z}(N, \mathbb{Z})$. Put $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ and $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$ and note that the natural pairing $\langle -, - \rangle : M \times N \to \mathbb{Z}$ extends naturally to a pairing $\langle -, - \rangle : M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}$.

Definition 2.4.1. A polytope $\Delta \subset M_\mathbb{R}$ *has the IP-property* if $0_M \in \mathrm{int}(\Delta)$.

Note that polytopes with the IP-property are full-dimensional, having a nonempty interior. The *dual polytope* of a polytope $\Delta \subset M_\mathbb{R}$ with the IP-property is the polytope $\Delta^* \subset N_{\mathbb{R}}$ defined by

$$\Delta^* = \{n \in N_{\mathbb{R}} \,|\, m \in \Delta \Rightarrow \langle m, n \rangle \geq -1\} \tag{2.28}$$

Note that $(\Delta^*)^* = \Delta$.

Definition 2.4.2. Let F be a face of a polytope $\Delta \subset M_{\mathbb{R}}$ with the IP-property. The *dual face of* F is the face $F^\vee$ of Δ^* defined by

$$F^\vee = \{n \in \Delta^* \,|\, m \in F \Rightarrow \langle m, n \rangle = -1\} \tag{2.29}$$

The map $F \mapsto F^\vee$ defines an inclusion-reversing one-to-one correspondence

$$\{\text{face lattice of } \Delta\} \longleftrightarrow \{\text{face lattice of } \Delta^*\} \tag{2.30}$$

satisfying $(F^\vee)^\vee = F$ and

$$\dim F + \dim F^\vee = d - 1 \tag{2.31}$$

Note that $\Delta^\vee = \varnothing \neq \Delta^*$. Recall that the *$f$-vector* of a d-dimensional polytope Δ is the vector $f_\Delta = (f_{-1}, f_0, \ldots, f_d)$ where f_i is the number of faces of Δ of dimension i. One consequence of the correspondence (2.30) is that the ith entry of f_Δ is the $(d-1-i)$th entry of f_{Δ^*}. That is, the vectors f_Δ and f_{Δ^*} are reverses of each other.

Two classes of polytopes important to toric geometry are *canonical Fano polytopes* and *reflexive polytopes*.

Definition 2.4.3. A *canonical Fano polytope* is a lattice polytope $\Delta \subset M_{\mathbb{R}}$ such that $\operatorname{int}(\Delta) \cap M = \{0_M\}$.

Definition 2.4.4. A *reflexive polytope* is a lattice polytope $\Delta \subset M_{\mathbb{R}}$ with the IP-property whose dual $\Delta^* \subset N_{\mathbb{R}}$ is also a lattice polytope.

Reflexive polytopes are canonical Fano polytopes and the dual of a reflexive polytope is a reflexive polytope. Thus, if Δ is reflexive, we refer to the pair (Δ, Δ^*) as a *dual pair of reflexive polytopes.* To see how these polytopes relate to toric geometry, we recall the notion of the *normal fan of a polytope.*

Definition 2.4.5. Let Δ be a rational polytope in $M_{\mathbb{R}}$. For each face $F \preceq \Delta$, let σ_F be the cone in $N_{\mathbb{R}}$ given by

$$\sigma_F = \{y \in N_{\mathbb{R}} \mid x \in F \Rightarrow \langle x, y\rangle \leq \min\langle \Delta, y\rangle\} \tag{2.32}$$

The cones in (2.32) define a fan $\Sigma_\Delta = \{\sigma_F \mid F \preceq \Delta\}$ in $N_{\mathbb{R}}$ called the *normal fan of* Δ. We often write X_Δ in place of X_{Σ_Δ} for brevity.

Alternatively, the toric variety X_Δ may be constructed using the Proj functor. Let

$$\mathcal{C}_\Delta = \mathrm{Cone}(\Delta \times 1) \subset M_{\mathbb{R}} \oplus \mathbb{R} \tag{2.33}$$

and put $\mathsf{S}_\Delta = \mathbb{C}[\mathcal{C}_\Delta \cap (M \oplus \mathbb{Z})]$. Then $X_\Delta \simeq \mathrm{Proj}(\mathsf{S}_\Delta)$ and X_Δ is semiprojective [17, Proposition 14.2.12].

Definition 2.4.6. A compact toric variety X is called

- *Fano* if the anticannonical divisor $-K_X$ is ample and $\mathbb{Q}$-Cartier.
- *Gorenstein* if K_X is Cartier.

Proposition 2.4.7 (Theorem 8.3.4 in [17], Proposition 1.4 in [34])**.** *The map* $\Delta \mapsto X_{\Delta^*}$ *defines a one-to-one correspondence between isomorphism classes of canonical Fano polytopes and Fano toric varieties with canonical singularities. This correspondence restricts to a correspondence between isomorphism classes of reflexive polytopes and isomorphism classes of Gorenstein Fano toric varieties.*

Table 2.1: The number of reflexive polytopes in dimensions $2 \leq d \leq 4$

dimension	2	3	4	≥ 5
# of reflexive polytopes	16	4319	473800776	?

Reflexive polytopes play an essential rôle in mirror symmetry and have been studied extensively. Up to lattice isomorphism, there are only a finite number of canonical Fano polytopes in a given dimension [33]. Reflexive polytopes have been classified up to dimension four (Table 2.1). The classifications in dimensions three [30] and four [31] were both done by Kreuzer and Skarke.

2.5 Hodge Numbers of Calabi-Yau Complete Intersections in Toric Varieties

Recall that the *Hodge numbers* of a d-dimensional variety Y are defined as

$$h^{p,q}(Y) = \dim_{\mathbb{C}} H^q(Y, \wedge^p \Omega_Y) \tag{2.34}$$

where Ω_Y is the cotangent sheaf of Y. If Y is Calabi-Yau, then Y the Hodge numbers are given by

$$h^{p,q}(Y) = \dim_{\mathbb{C}} H^q(Y, \wedge^{d-p} \mathscr{T}_Y) \tag{2.35}$$

where $\mathscr{T}_Y$ is the tangent sheaf of Y. In [3] Aspinwall, Melnikov, and Plesser use techniques of the derived category to develop a method for computing the Hodge numbers of a smooth complete intersection in a smooth complete toric variety. In this section we review this method and note that it also applies to smooth complete intersections in *simplicial* complete toric varieties.

Let X_Σ be the d-dimensional toric variety corresponding to a complete simplicial fan Σ in $N_\mathbb{R}$ with no torus factors. The class group $\mathrm{Cl}(X_\Sigma)$ of X_Σ fits into an exact sequence [17, Theorem 4.1.3]

$$0 \longrightarrow M \longrightarrow \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} D_\rho \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0 \tag{2.36}$$

so that $\mathrm{Cl}(X_\Sigma)$ has rank $r = |\Sigma(1)| - d$. Batyrev and Cox [9, Theorem 12.1] show that the cotangent sheaf $\Omega^1_{X_\Sigma}$ of X_Σ fits into an exact sequence

$$0 \longrightarrow \Omega^1_{X_\Sigma} \longrightarrow \bigoplus_{\rho\in\Sigma(1)} \mathscr{O}_{X_\Sigma}(-D_\rho) \longrightarrow \mathscr{O}_{X_\Sigma}^{\oplus r} \longrightarrow 0 \tag{2.37}$$

called *the generalized Euler exact sequence*. Applying $\mathscr{H}om_{\mathscr{O}_{X_\Sigma}}(-, \mathscr{O}_{X_\Sigma})$ to (2.37) then yields an exact sequence

$$0 \longrightarrow \mathscr{O}_{X_\Sigma}^{\oplus r} \longrightarrow \bigoplus_{\rho\in\Sigma(1)} \mathscr{O}_{X_\Sigma}(D_\rho) \longrightarrow \mathscr{T}_{X_\Sigma} \longrightarrow 0 \tag{2.38}$$

where $\mathscr{T}_{X_\Sigma} = \mathscr{H}om_{\mathscr{O}_{X_\Sigma}}(\Omega^1_{X_\Sigma}, \mathscr{O}_{X_\Sigma})$ is the *tangent sheaf* of X_Σ.

Let $Z_1, \ldots, Z_c$ be torus-invariant Weyl divisors on X_Σ. A global section $f_i \in H^0(X_\Sigma, \mathscr{O}_{X_\Sigma}(Z_i))$ defines a hypersurface $Y_i = \mathbf{V}(f_i)$ in X_Σ. Suppose that a generic choice of these hypersurfaces defines a smooth complete intersection $Y = Y_1 \cap \cdots \cap Y_c$ in X_Σ. The adjunction exact sequence for Y [17, Theorem 8.0.18] is

$$0 \longrightarrow \mathscr{T}_Y \longrightarrow \mathscr{T}_{X_\Sigma|Y} \longrightarrow \bigoplus_i \mathscr{O}_Y(Z_i) \longrightarrow 0 \tag{2.39}$$

where we denote the restriction of $\mathscr{O}_{X_\Sigma}(Z_i)$ to Y by $\mathscr{O}_Y(Z_i)$ for brevity. Each of the sheaves in (2.38) is reflexive so [17, Proposition 8.0.1] implies that restricting to Y gives an exact sequence

$$0 \longrightarrow \mathscr{O}_Y^{\oplus r} \longrightarrow \bigoplus_{\rho\in\Sigma(1)} \mathscr{O}_Y(D_\rho) \longrightarrow \mathscr{T}_{X_\Sigma|Y} \longrightarrow 0 \tag{2.40}$$

of $\mathscr{O}_Y$-modules.

Note that (2.39) and (2.40) can be arranged into a diagram of $\mathscr{O}_Y$-modules

$$\begin{array}{ccccccccc} & & p=-1 & & p=0 & & p=1 & & \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathscr{T}_Y & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{O}_Y^{\oplus r} & \longrightarrow & \bigoplus_{\rho\in\Sigma(1)} \mathscr{O}_Y(D_\rho) & \longrightarrow & \bigoplus_i \mathscr{O}_Y(Z_i) & \longrightarrow & 0 \end{array} \tag{2.41}$$

The indices above this diagram suggest that the rows ought to be viewed as complexes of $\mathscr{O}_Y$-modules. In fact, the construction of (2.41) from the exact sequences (2.39) and (2.40) shows that the morphism of complexes defined by the vertical arrows of (2.41) induces isomorphisms on the level of cohomology of complexes. That is, the morphism of complexes in (2.41) is a *quasi-isomorphism*.

At this point, it is clear that the derived category $\mathbf{D}^b(Y)$ will be a useful tool for computing the cohomology of $\mathscr{T}_Y$. The quasi-isomorphism in (2.41) provides an isomorphism in $\mathbf{D}^b(Y)$ between $\mathscr{T}_Y$ and the complex

$$\begin{array}{ccccccccc} & & p=-1 & & p=0 & & p=1 & & \\ 0 & \longrightarrow & \mathscr{O}_Y^{\oplus r} & \longrightarrow & \bigoplus_{\rho\in\Sigma(1)} \mathscr{O}_Y(D_\rho) & \longrightarrow & \bigoplus_i \mathscr{O}_Y(Z_i) & \longrightarrow & 0 \end{array} \tag{2.42}$$

For every complex of $\mathscr{O}_Y$-modules $\mathscr{E}^\bullet$ considered as an object in $\mathbf{D}^b(Y)$ there is a spectral sequence [3] with

$$E_1^{p,q} = H^q(\mathscr{E}^p) \Rightarrow H^{p+q}(\mathscr{E}^\bullet) \tag{2.43}$$

Taking $\mathscr{E}^\bullet$ to be the complex in (2.42) thus gives a spectral sequence that computes the cohomology of $\mathscr{T}_Y$. The E_1 page of this spectral sequence is

$$\begin{array}{c|ccccc}
q & & & & & \\
3 & H^3(\mathscr{O}_Y)^{\oplus r} & \longrightarrow & \bigoplus_{\rho\in\Sigma(1)} H^3\big(\mathscr{O}_Y(D_\rho)\big) & \longrightarrow & \bigoplus_i H^3\big(\mathscr{O}_Y(Z_i)\big) \\
2 & H^2(\mathscr{O}_Y)^{\oplus r} & \longrightarrow & \bigoplus_{\rho\in\Sigma(1)} H^2\big(\mathscr{O}_Y(D_\rho)\big) & \longrightarrow & \bigoplus_i H^2\big(\mathscr{O}_Y(Z_i)\big) \\
1 & H^1(\mathscr{O}_Y)^{\oplus r} & \longrightarrow & \bigoplus_{\rho\in\Sigma(1)} H^1\big(\mathscr{O}_Y(D_\rho)\big) & \longrightarrow & \bigoplus_i H^1\big(\mathscr{O}_Y(Z_i)\big) \\
0 & H^0(\mathscr{O}_Y)^{\oplus r} & \longrightarrow & \bigoplus_{\rho\in\Sigma(1)} H^0\big(\mathscr{O}_Y(D_\rho)\big) & \longrightarrow & \bigoplus_i H^0\big(\mathscr{O}_Y(Z_i)\big) \\
\hline
 & -1 & & 0 & & 1 \quad p
\end{array} \tag{2.44}$$

Since we are primarily interested in Calabi-Yau threefolds we have excluded the terms with $q > 3$. The terms with $p + q = 1$ (blue) represent the contributions to $H^1(\mathscr{T}_Y)$ and the terms with $p + q = 2$ (red) represent the contributions to $H^2(\mathscr{T}_Y)$.

Now, to compute the cohomology groups $H^q(\mathscr{O}_Y(T))$, consider the twisted Koszul exact sequence

$$\begin{array}{l}
0 \longrightarrow \mathscr{O}_{X_\Sigma}(T - \sum_i Z_i) \longrightarrow \cdots \longrightarrow \bigoplus_{i_1<i_2} \mathscr{O}_{X_\Sigma}(T - \sum_j Z_{i_j}) \\
\longrightarrow \bigoplus_{i_1} \mathscr{O}_{X_\Sigma}(T - Z_{i_1}) \longrightarrow \mathscr{O}_{X_\Sigma}(T) \longrightarrow \mathscr{O}_Y(T) \longrightarrow 0
\end{array} \tag{2.45}$$

By introducing 'auxiliary sheaves' $\mathscr{I}_1, \ldots, \mathscr{I}_{c-1}$, one may break this long exact sequence into c short exact sequences

$$
\begin{array}{ccccccccc}
0 \longrightarrow \mathscr{O}_{X_\Sigma}(T - \sum_i Z_i) & \longrightarrow & \bigoplus_{i_1<\cdots<i_{c-1}} \mathscr{O}_{X_\Sigma}(T - \sum_j Z_{i_j}) & \longrightarrow & \mathscr{I}_1 & \longrightarrow 0 \\
0 \longrightarrow \mathscr{I}_1 & \longrightarrow & \bigoplus_{i_1<\cdots<i_{c-2}} \mathscr{O}_{X_\Sigma}(T - \sum_j Z_{i_j}) & \longrightarrow & \mathscr{I}_2 & \longrightarrow 0 \\
 & & \vdots & & & \\
0 \longrightarrow \mathscr{I}_{c-2} & \longrightarrow & \bigoplus_{i_1} \mathscr{O}_{X_\Sigma}(T - Z_{i_1}) & \longrightarrow & \mathscr{I}_{c-1} & \longrightarrow 0 \\
0 \longrightarrow \mathscr{I}_{c-1} & \longrightarrow & \mathscr{O}_{X_\Sigma}(T) & \longrightarrow & \mathscr{O}_Y(T) & \longrightarrow 0
\end{array}
\tag{2.46}
$$

Each of the short exact sequences in (2.46) induces a long exact sequence in cohomology. Computing the cohomology groups $H^q(X_\Sigma, \mathscr{O}_{X_\Sigma}(T))$ can be accomplished with local cohomology using a computer algebra system such as `Macaulay2` or `sage`. Hence each of the terms in (2.44) can be computed from (2.46).

2.5.1 $\mathbb{P}^5_{211111}[3, 4]$

The weighted projective space $\mathbb{P}^5_{211111}$ is a toric variety whose fan Σ is the complete simplicial fan in $N_\mathbb{R}$ given by the data

$$
\begin{array}{lclllll}
u_{\rho_0} & = & N(\ 1, & 0, & 0, & 0, & 0) \\
u_{\rho_1} & = & N(\ 0, & 1, & 0, & 0, & 0) \\
u_{\rho_2} & = & N(\ 0, & 0, & 1, & 0, & 0) \\
u_{\rho_3} & = & N(\ 0, & 0, & 0, & 1, & 0) \\
u_{\rho_4} & = & N(\ 0, & 0, & 0, & 0, & 1) \\
u_{\rho_5} & = & N(-2, & -1, & -1, & -1, & -1)
\end{array}
\qquad
\begin{array}{lcl}
\sigma_0 & = & \mathrm{Cone}(u_{\rho_1},\ u_{\rho_2},\ u_{\rho_3},\ u_{\rho_4},\ u_{\rho_5}) \\
\sigma_1 & = & \mathrm{Cone}(u_{\rho_0},\ u_{\rho_2},\ u_{\rho_3},\ u_{\rho_4},\ u_{\rho_5}) \\
\sigma_2 & = & \mathrm{Cone}(u_{\rho_0},\ u_{\rho_1},\ u_{\rho_3},\ u_{\rho_4},\ u_{\rho_5}) \\
\sigma_3 & = & \mathrm{Cone}(u_{\rho_0},\ u_{\rho_1},\ u_{\rho_2},\ u_{\rho_4},\ u_{\rho_5}) \\
\sigma_4 & = & \mathrm{Cone}(u_{\rho_0},\ u_{\rho_1},\ u_{\rho_2},\ u_{\rho_3},\ u_{\rho_5}) \\
\sigma_5 & = & \mathrm{Cone}(u_{\rho_0},\ u_{\rho_1},\ u_{\rho_2},\ u_{\rho_3},\ u_{\rho_4})
\end{array}
$$

The only singular cone of Σ is the maximal cone σ_0. Hence the orbit-cone correspondence implies that the singular locus of X_Σ consists of a single orbifold point $X_\Sigma(1 : 0 : 0 : 0 : 0 : 0)$ with isotropy $\mathbb{Z}_2$.

Consider the torus-invariant Weyl divisors

$$Z_3 = D_{\rho_0} + D_{\rho_1} \qquad\qquad Z_4 = D_{\rho_2} + D_{\rho_3} + D_{\rho_4} + D_{\rho_5}$$

Our choice of indices is explained by the generic global sections $f_i \in H^0(X_\Sigma, \mathscr{O}_{X_\Sigma}(Z_i))$, which are

$$\begin{aligned} f_3 &= a_0 z_1^3 + a_1 z_2^3 + a_2 z_3^3 + a_3 z_4^3 + a_4 z_5^3 + a_5 z_0 z_1 + a_6 z_0 z_2 + a_7 z_0 z_3 + a_8 z_0 z_4 + a_9 z_0 z_5 \\ f_4 &= b_0 z_1^4 + b_1 z_2^4 + b_2 z_3^4 + b_3 z_4^4 + b_4 z_5^4 + b_5 z_0^2 \end{aligned} \tag{2.47}$$

That is, f_3 is a generic cubic polynomial in $S(\Sigma)$ and f_4 is a generic quartic polynomial in $S(\Sigma)$. Let Y be a complete intersection of a cubic and a quartic in X_Σ. Note that the equations in (2.47) force Y to miss the orbifold point of X_Σ. In fact, Y is a smooth Calabi-Yau threefold.

Our analysis of $\mathscr{T}_Y$ in (2.44) is

$$\begin{array}{c|ccc} q & & & \\ 3 & \mathbb{C} & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \mathbb{C} \longrightarrow & \mathbb{C}^{41} \longrightarrow & \mathbb{C}^{110} \\ \hline & -1 & 0 & 1 \quad p \end{array} \tag{2.48}$$

so that $h^{11}(Y) = 1$ and $h^{21}(Y) = 79$. The Hodge diamond of Y is thus

$$\begin{array}{ccccccc} & & & h^{33} & & & \\ & & h^{32} & & h^{23} & & \\ & h^{31} & & h^{22} & & h^{13} & \\ h^{30} & & h^{21} & & h^{12} & & h^{03} \\ & h^{20} & & h^{11} & & h^{02} & \\ & & h^{10} & & h^{01} & & \\ & & & h^{00} & & & \end{array} = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 79 & & 79 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{2.49}$$

This is the example studied in [2] and served as insipiration for this dissertation. In [2] it is asked whether the Calabi-Yau threefold Y has a mirror $Y^\vee$ that can be realized as a complete intersection in a toric variety. The Batyrev-Borisov method cannot be used to construct such a $Y^\vee$ since $\mathbb{P}^5_{211111}$ is not Gorenstein.

We will see that a generalization of the Batyrev-Borisov construction developed by Mavlyutov [34] offers a candidate for a toric description of a mirror variety $Y^\vee$ and this mirror agrees with the possible mirror described in [2]. In this dissertation we develop a generalization of Batyrev's *stringy* Hodge numbers and, under this framework, show that the proposed mirror from [34] and [2] passes the topological mirror symmetry test.

2.5.2 $\mathbb{P}^5_{321111}[4,5]$

The weighted projective space $\mathbb{P}^5_{321111}$ is a complete simplicial toric variety X_Σ whose singular locus consists of an orbifold point $X_\Sigma(1:0:0:0:0:0)$ with isotropy $\mathbb{Z}_3$ and an orbifold point $X_\Sigma(0:1:0:0:0:0)$ with isotropy $\mathbb{Z}_2$. The equations of a generic quartic f_4 and a generic quintic f_5 in $S(\Sigma)$ are

$$\begin{aligned} f_4 &= z_2^4 + z_3^4 + z_4^4 + z_5^4 + z_1^2 + z_0 z_2 + z_0 z_3 + z_0 z_4 + z_0 z_5 \\ f_5 &= z_2^5 + z_3^5 + z_4^5 + z_5^5 + z_1^2 z_2 + z_0 z_2^2 + z_1^2 z_3 \\ &\quad + z_0 z_3^2 + z_1^2 z_4 + z_0 z_4^2 + z_1^2 z_5 + z_0 z_5^2 + z_0 z_1 \end{aligned} \tag{2.50}$$

where we have suppressed the monomial coefficients. Let Y be a complete intersection of a quartic and a quintic in X_Σ. Then Y is a Calabi-Yau threefold and our analysis of $\mathscr{T}_Y$ in (2.44) is

$$
\begin{array}{c|ccc}
q & & & \\
3 & \mathbb{C} & 0 & 0 \\
2 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & \mathbb{C} & \longrightarrow \mathbb{C}^{53} & \longrightarrow \mathbb{C}^{135} \\
\hline
 & -1 & 0 & 1 \quad p
\end{array}
\tag{2.51}
$$

The Hodge diamond of Y is thus

$$
\begin{array}{ccccccc}
 & & & h^{33} & & & \\
 & & h^{32} & & h^{23} & & \\
 & h^{31} & & h^{22} & & h^{13} & \\
h^{30} & & h^{21} & & h^{12} & & h^{03} \\
 & h^{20} & & h^{11} & & h^{02} & \\
 & & h^{10} & & h^{01} & & \\
 & & & h^{00} & & &
\end{array}
=
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
1 & & 84 & & 84 & & 1 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\tag{2.52}
$$

Note, however, that the equations in (2.50) force Y to pass through the $\mathbb{Z}_3$-singularity of X_Σ. In fact, Y is a *singular* Calabi-Yau threefold whose singular locus consists of this orbifold point.

This situation can be remedied as follows. Let $\widehat{\Sigma}$ be the fan obtained from Σ by inserting the ray ρ_6 generated by $u_{\rho_6} = N(-1,0,0,0,0)$ and subdividing. Generic sections of the torus-invariant Weyl divisors

$$D_{\rho_0} + D_{\rho_1} \qquad\qquad D_{\rho_2} + D_{\rho_3} + D_{\rho_4} + D_{\rho_5} + D_{\rho_6}$$

define a smooth Calabi-Yau complete intersection $\widehat{Y}$ in $X_{\widehat{\Sigma}}$. This gives a discrepant toric morphism $X_{\widehat{\Sigma}} \to X_\Sigma$ that restricts to a crepant desingularization $\widehat{Y} \to Y$. The spectral sequence (2.44) computing the cohomology of $\mathscr{T}_{\widehat{Y}}$ is

$$\begin{array}{c|ccccc}
q & & & & & \\
3 & \mathbb{C}^2 & & 0 & & 0 \\
2 & 0 & & 0 & & 0 \\
1 & 0 & & 0 & & 0 \\
0 & \mathbb{C}^2 & \longrightarrow & \mathbb{C}^{53} & \longrightarrow & \mathbb{C}^{135} \\
\hline
 & -1 & & 0 & & 1 \quad p
\end{array} \tag{2.53}$$

The Hodge diamond of $\widehat{Y}$ is thus

$$\begin{array}{ccccccc}
 & & & h^{33} & & & \\
 & & h^{32} & & h^{23} & & \\
 & h^{31} & & h^{22} & & h^{13} & \\
h^{30} & & h^{21} & & h^{12} & & h^{03} \\
 & h^{20} & & h^{11} & & h^{02} & \\
 & & h^{10} & & h^{01} & & \\
 & & & h^{00} & & &
\end{array}
=
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 2 & & 0 & \\
1 & & 84 & & 84 & & 1 \\
 & 0 & & 2 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array} \tag{2.54}$$

A general result [5] states that if $Y_1 \to Y$ and $Y_2 \to Y$ are crepant desingularizations of a Calabi-Yau threefold X by smooth Calabi-Yau threefolds Y_1 and Y_2, then the Hodge numbers of Y_1 and Y_2 coincide. In light of this result, we call (2.54) the *stringy* Hodge diamond of Y.

Note that both of the ambient spaces $\mathbb{P}^5_{321111}$ and $X_{\widehat{\Sigma}}$ are $\mathbb{Q}$-Gorenstein but not Gorenstein. Again, the Batyrev-Borisov mirror construction fails to match both Y and $\widehat{Y}$ to mirror Calabi-Yau threefolds. We will show that Mavlyutov's construction does apply to $\widehat{Y}$ and Mavlyutov's proposed mirror $\widehat{Y}^\vee$ satisfies the topological mirror symmetry test under our framework.

It is also worth noting that the example $\mathbb{P}^5_{321111}[4,5]$ was studied by Kreuzer and Riegler [32]. Here the authors attempt to construct a desingularization $\widehat{Y} \to Y$ where $\widehat{Y}$ is a complete intersection in a Gorenstein Fano toric variety. Their search involves a rather intense computation. Such a desingularization $\widehat{Y}$ could theoretically be found by locating a certain reflexive subpolytope of a polytope Δ with 575 lattice points. By omitting up to 30 lattice points, Kreuzer and Riegler find 21 reflexive subpolytopes of Δ, none of which give the desired $\widehat{Y}$. From this they conclude "a more detailed analysis of the geometry would be required to check if a toric description exists." Our toric complete intersection $\widehat{Y} \subset X_{\widehat{\Sigma}}$ was found by excluding a single lattice point from Δ.

3

The Batyrev-Borisov Mirror Construction

In this dissertation, we seek to generalize a method for producing mirror pairs of Calabi-Yau varieties called the *Batyrev-Borisov mirror construction.* This construction relates the combinatorics of a *dual pair of nef-partitions*

$$\Delta = \Delta_1 + \cdots + \Delta_c \qquad\qquad \nabla = \nabla_1 + \cdots + \nabla_c \tag{3.1}$$

of two reflexive polytopes Δ and ∇ to the geometry of a *stringy mirror pair* of Calabi-Yau varieties $(Y_{\Delta_1,\ldots,\Delta_c}, Y_{\nabla_1,\ldots,\nabla_c})$ in the Gorenstein Fano toric varieties (X_Δ, X_∇).

In this chapter, we outline the Batyrev-Borisov construction of a Batyrev-Borisov mirror pair $(Y_{\Delta_1,\ldots,\Delta_c}, Y_{\nabla_1,\ldots,\nabla_c})$ and present the combinatorial definition of the stringy Hodge numbers $h^{p,q}_{\mathrm{st}}$ of these varieties. In Section 3.1, we review the notions of *Gorenstein polytopes* and *Gorenstein cones*, both of which play a pivotal rôle in defining stringy Hodge numbers. Gorenstein cones will be a key focus of interest throughout the rest of this dissertation.

In §3.2, we present the definition of the *stringy E-polynomial* $E_{\mathrm{st}}(\Delta; u, v)$ of a Gorenstein polytope Δ, which computes the stringy Hodge numbers $h^{p,q}_{\mathrm{st}}(\Delta)$ of Δ. This polynomial depends on both the structure of Δ as a lattice polytope and as an

Eulerian poset. In particular, we need the notions of the *h^*-polynomial* of Δ (§3.2.1) and the *g-polynomial* of the poset of faces of Δ (Section 3.2.2), which combine to give the *$\widetilde{S}$-polynomial* of Δ (§3.2.3). Finally, we give the definition of and basic properties of the E_{st}-polynomial in §3.2.4.

In §3.3, we show how the stringy E-polynomial of a Gorenstein polytope can be used to define the stringy Hodge numbers of a Batyrev-Borisov mirror pair.

We conclude this chapter with two examples where we compute the stringy Hodge numbers of a Batyrev-Borisov mirror pair.

3.1 Gorenstein Polytopes and Gorenstein Cones

The duality of reflexive polytopes can be stated in the more general setting of *reflexive Gorenstein cones*. Here, we develop language necessary to make this generalization precise. Note that we are interested in non-reflexive Gorenstein cones as well as reflexive ones. The main references are [10] and [34].

Let (N, M) and $(\bar{N}, \bar{M})$ be dual pairs of torsion-free lattices of finite ranks d and $\bar{d}$ respectively. Recall from Definition 2.4.4 that a *reflexive polytope* is a lattice polytope Δ containing the origin in its interior whose dual polytope Δ^* (2.28) is also a lattice polytope. Let us now generalize this notion.

Definition 3.1.1. Let Δ be a polytope in $M_{\mathbb{R}}$. Then Δ is a *Gorenstein polytope of index c* if $\operatorname{int}(c\Delta) \cap M = \{m\}$ for some nonnegative integer c and $c\Delta - m$ is reflexive.

Note that Gorenstein polytopes are full-dimensional, having nonempty interior. In fact, we immediately see that reflexive polytopes are precisely those Gorenstein polytopes of index one with $\operatorname{int}(\Delta) \cap M = \{0_M\}$. It is less obvious, however, how one should define the dual of a Gorenstein polytope. Naïvely, one may define the dual of a Gorenstein polytope as the dual of the reflexive polytope $c\Delta - m$. This naïve definition turns out to be geometrically uninteresting, so we must search for duals of

Gorenstein polytopes elsewhere. To do so, we need the notion of a *Gorenstein cone.*

For $\mu \in \bar{N}$ and r a nonnegative integer, let $H_\mu^{(r)}$ be the affine hyperplane in $\bar{M}_\mathbb{R}$ given by

$$H_\mu^{(r)} = \{x \in \bar{M}_\mathbb{R} \mid \langle x, \mu \rangle = r\} \tag{3.2}$$

For $\nu \in \bar{M}$ we define $H_\nu^{(r)}$ similarly.

Definition 3.1.2. A *Gorenstein cone with degree vector* h_σ is a cone σ in $\bar{M}_\mathbb{R}$ generated by finitely many lattice points in $H_{h_\sigma}^{(1)} \cap \bar{M}$ for some $h_\sigma \in \bar{N}$.

The degree vector h_σ of a Gorenstein polytope σ is unique. The *rth slice* of σ is the polytope

$$\sigma_{(r)} = \sigma \cap H_{h_\sigma}^{(r)} \tag{3.3}$$

considered as a polytope in the affine hyperplane $H_{h_\sigma}^{(r)}$. The *support of* σ is the polytope $\sigma_{(1)}$.

Now, recall that the *dual cone* of a cone σ in $\bar{M}_\mathbb{R}$ is the cone $\sigma^\vee$ in $\bar{N}_\mathbb{R}$ defined by

$$\sigma^\vee = \{y \in \bar{N}_\mathbb{R} \mid x \in \sigma \Rightarrow \langle y, x \rangle \geq 0\} \tag{3.4}$$

Note that $(\sigma^\vee)^\vee = \sigma$.

Definition 3.1.3. Let τ be a face of a cone σ in $\bar{M}_\mathbb{R}$. The *dual face of* τ is the face τ^* of $\sigma^\vee$ defined by

$$\tau^* = \{y \in \sigma^\vee \mid x \in \tau \Rightarrow \langle y, x \rangle = 0\} \tag{3.5}$$

The map $\tau \mapsto \tau^*$ defines an inclusion-reversing one-to-one correspondence

$$\{\text{face lattice of } \sigma\} \longleftrightarrow \{\text{face lattice of } \sigma^\vee\} \tag{3.6}$$

satisfying $(\tau^*)^* = \tau$ and

$$\dim \tau + \dim \tau^* = \bar{d} \tag{3.7}$$

Note that $\sigma^* = \{0_{\bar{N}}\} \neq \sigma^\vee$.

Remark 3.1.4. The notation we have adopted to denote the dual faces of faces of polytopes and cones departs slightly from that in much of the literature. To summarize, we have defined two one-to-one correspondences

$$\left\{\begin{array}{c}\text{IP-polytopes}\\ \text{in } M_{\mathbb{R}}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{IP-polytopes}\\ \text{in } N_{\mathbb{R}}\end{array}\right\} \qquad \left\{\text{cones in } \bar{M}_{\mathbb{R}}\right\} \longleftrightarrow \left\{\text{cones } \bar{N}_{\mathbb{R}}\right\}$$
$$\Delta \longmapsto \Delta^* \qquad\qquad \sigma \longmapsto \sigma^\vee$$

satisfying $(\Delta^*)^* = \Delta$ and $(\sigma^\vee)^\vee = \sigma$. For a dual pair of IP-polytopes (Δ, Δ^*) and a dual pair of cones $(\sigma, \sigma^\vee)$ we have defined one-to-one correspondences

$$\left\{\begin{array}{c}\text{face lattice}\\ \text{of } \Delta \subset M_{\mathbb{R}}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{face lattice}\\ \text{of } \Delta^* \subset N_{\mathbb{R}}\end{array}\right\} \qquad \left\{\begin{array}{c}\text{face lattice}\\ \text{of } \sigma \subset \bar{M}_{\mathbb{R}}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{face lattice}\\ \text{of } \sigma^\vee \subset \bar{N}_{\mathbb{R}}\end{array}\right\}$$
$$F \longmapsto F^\vee \qquad\qquad \tau \longmapsto \tau^*$$

satisfying $(F^\vee)^\vee = F$, $(\tau^*)^* = \tau$, and

$$\dim F + \dim F^\vee = d - 1 \qquad\qquad \dim \tau + \dim \tau^* = \bar{d} \tag{3.8}$$

In particular, $\Delta^\vee = \varnothing \neq \Delta^*$ and $\sigma^* = \{0_{\bar{N}_{\mathbb{R}}}\} \neq \sigma^\vee$. It is common in the literature to denote a face F of Δ by F^*, which leads to the apparent contradiction $\Delta^* = \varnothing$ when Δ is viewed as a face of itself. ◊

Recall that a lattice polytope Δ with the IP-property may have a non-lattice polytope Δ^* as its dual. It is also true that a Gorenstein cone σ may have a non-Gorenstein dual $\sigma^\vee$.

Definition 3.1.5. Let σ be a Gorenstein cone in $\bar{M}_{\mathbb{R}}$. Then σ is *reflexive* if $\sigma^\vee$ is a Gorenstein cone in $\bar{N}_{\mathbb{R}}$.

The relation $(\sigma^\vee)^\vee = \sigma$ implies that reflexive Gorenstein cones occur in dual pairs $(\sigma, \sigma^\vee)$. The *index* of a reflexive Gorenstein cone σ is the positive integer $c = \langle h_{\sigma^\vee}, h_\sigma \rangle$.

The following proposition demonstrates the relationship between reflexive Gorenstein cones and reflexive polytopes.

Proposition 3.1.6 (Proposition 2.11 in [8])**.** *Let σ be a Gorenstein cone in $\bar{M}_{\mathbb{R}}$. Then the following are equivalent.*

(*a*) *σ is reflexive of index c.*

(*b*) *The support polytope $\sigma_{(1)}$ is a Gorenstein polytope of index c when considered as a polytope in the affine hyperplane $H^{(1)}_{h_\sigma}$.*

(*c*) *The polytope $\sigma_{(c)} - h_{\sigma^\vee}$ is a reflexive polytope when considered as a polytope in $(\bar{M} \cap h_\sigma^\perp)_{\mathbb{R}}$.*

Proposition 3.1.6 allows us to generalize the duality of reflexive polytopes. The *Gorenstein cone associated* to a d-dimensional polytope $\Delta \subset M_{\mathbb{R}}$ is $\mathcal{C}_\Delta = \mathrm{Cone}(\Delta \times 1) \subset M_{\mathbb{R}} \oplus \mathbb{R}$.

Definition 3.1.7. Let $\Delta \subset M_{\mathbb{R}}$ be a Gorenstein polytope of index c. The *dual Gorenstein polytope* Δ^* is the support polytope of $\mathcal{C}_\Delta^\vee$.

The duality $\tau \mapsto \tau^*$ between the face lattices of $\mathcal{C}_\Delta$ and $\mathcal{C}_\Delta^\vee$ induces a duality $F \mapsto F^\vee$ between the face lattices of Δ and Δ^* satisfying $\dim F + \dim F^\vee = d - 1$.

Example 3.1.8. Fix $M \simeq \mathbb{Z}^7$ and let σ_1 and σ_2 be the cones generated by the two lists of primitive M-vectors

```
M(0, 1,  0,  0,  0,  0,  0),     M(0, 1,  0,  0,  0,  0,  0),
M(0, 1,  0,  0,  0,  1,  0),     M(0, 1,  0,  0,  0,  0,  1),
M(0, 1,  0,  0,  1,  0,  0),     M(0, 1,  0,  0,  0,  1,  0),
M(0, 1,  0,  1,  0,  0,  0),     M(0, 1,  0,  0,  1,  0,  0),
M(0, 1,  1,  0,  0,  0,  0),     M(0, 1,  0,  1,  0,  0,  0),    (3.9)
M(1, 0, -1, -1, -1, -1, -1),     M(1, 0, -2, -1, -1, -1, -1),
M(1, 0,  0,  0,  0,  0,  0),     M(1, 0,  0,  0,  0,  0,  0),
M(1, 0,  0,  0,  0,  0,  1)      M(1, 0,  1,  0,  0,  0,  0)
in 7-d lattice M                 in 7-d lattice M
```

respectively. Each of the lists in (3.9) consist of points m satisfying $\langle m, \mu \rangle = 1$ where

$$\mu = N(1, 1, 0, 0, 0, 0, 0) \tag{3.10}$$

It follows that both σ_1 and σ_2 are Gorenstein cones with degree vectors $h_{\sigma_1} = h_{\sigma_2} = \mu$.

The dual cones $\sigma_1^\vee$ and $\sigma_2^\vee$ are generated by the lists of primitive N-vectors

```
N(1, 0,  0,  0,  0,  0,  1),    N(2, 0,  1,  0,  0,  0,  0),
N(0, 1, -1, -1, -1, -1,  4),    N(0, 1,  2, -1, -1, -1, -1),
N(0, 1, -1, -1, -1, -1,  0),    N(0, 1,  0, -1, -1, -1, -1),
N(0, 1,  3, -1, -1, -1,  0),    N(0, 1,  0,  3, -1, -1, -1),
N(0, 1, -1,  3, -1, -1,  0),    N(0, 1,  0, -1,  3, -1, -1),
N(0, 1, -1, -1, -1,  3,  0),    N(0, 1,  0, -1, -1, -1,  3),
N(0, 1, -1, -1,  3, -1,  0),    N(0, 1,  0, -1, -1,  3, -1),   (3.11)
N(1, 0,  0,  0,  2,  0, -1),    N(1, 0, -1,  0,  0,  3,  0),
N(1, 0,  0,  0,  0,  2, -1),    N(1, 0, -1,  0,  0,  0,  3),
N(1, 0,  0,  2,  0,  0, -1),    N(1, 0, -1,  0,  3,  0,  0),
N(1, 0,  2,  0,  0,  0, -1),    N(1, 0, -1,  3,  0,  0,  0),
N(1, 0,  0,  0,  0,  0, -1)     N(1, 0, -1,  0,  0,  0,  0)
in 7-d lattice N                in 7-d lattice N
```

respectively. Note that the generators n of $\sigma_1^\vee$ satisfy $\langle n, h_{\sigma_1^\vee} \rangle = 1$ where

$$h_{\sigma_1^\vee} = M(1, 1, 0, 0, 0, 0, 0) \tag{3.12}$$

However, the generators of $\sigma_2^\vee$ are not contained in a primitive affine hyperplane. Hence $(\sigma_1, \sigma_1^\vee)$ is a dual pair of reflexive Gorenstein cones with index $\langle h_{\sigma_1^\vee}, h_{\sigma_1} \rangle = 2$ while the Gorenstein cone σ_2 is not reflexive. In particular, note that the f-vectors of the support polytopes of σ_1 and $\sigma_1^\vee$ are

$$f_1 = (1,\ 8,\ 27,\ 50,\ 55,\ 36,\ 12,\ 1) \qquad f_2 = (1,\ 12,\ 36,\ 55,\ 50,\ 27,\ 8,\ 1) \tag{3.13}$$

respectively. Note that the entries of f_2 are the entries of f_1 in reverse order. This is a consequence of the duality between the face lattice of the Gorenstein polytopes corresponding to σ_1 and $\sigma_1^\vee$. ♢

3.2 The Stringy Hodge Numbers of a Gorenstein Polytope

The Batyrev-Borisov construction produces a *dual pair of Calabi-Yau varieties* $(Y, Y^\vee)$. In this section we review the machinery necessary for making these notions precise.

3.2.1 Basic Ehrhart Theory

Let M be a lattice with $\dim M_{\mathbb{R}} = d$ and let Δ be a lattice polytope in $M_{\mathbb{R}}$. The *Ehrhart series of* Δ is the power series $\mathrm{Ehr}_{\Delta}(t) \in \mathbb{Z}[\![t]\!]$ defined by

$$\mathrm{Ehr}_{\Delta}(t) = \sum_{k=0}^{\infty} |k\Delta \cap M| t^k \tag{3.14}$$

In [19], Ehrhart shows that the Ehrhart series of Δ is a rational function

$$\mathrm{Ehr}_{\Delta}(t) = \frac{h^*_{\Delta}(t)}{(1-t)^{1+\dim \Delta}} \tag{3.15}$$

where $h^*_{\Delta}(t) \in \mathbb{Z}[t]$ is a polynomial with nonnegative coefficients satisfying $\deg h^*_{\Delta}(t) \leq \dim \Delta$.

The polynomial $h^*_{\Delta}(t)$ encodes several interesting invariants of Δ [36, §2.2]. We define the *degree and codegree of* Δ by the formulae

$$\deg \Delta = \deg h^*_{\Delta}(t) \qquad\qquad \operatorname{codeg} \Delta = 1 + \dim \Delta - \deg \Delta \tag{3.16}$$

The codegree of Δ is the smallest nonnegative integer k for which $k\Delta$ has a lattice point in its relative interior. The leading coefficient of $h^*_{\Delta}(t)$ is the number of lattice points in the relative interior of $\operatorname{codeg}(\Delta) \cdot \Delta$. Note that the index of a Gorenstein polytope is its codegree. By convention, we define $h^*_{\varnothing}(t) = 1$ so $\deg \varnothing = \operatorname{codeg} \varnothing = 0$.

A remarkable theorem of Hibi characterizes Gorenstein polytopes in terms of the h^*-polynomial. Recall that a polynomial $f(t)$ is *palindromic* if $f(t) = t^{\deg f} f(t^{-1})$.

Theorem 3.2.1 (Theorem 2.1 in [22])**.** *A lattice polytope Δ is Gorenstein if and only if $h^*_{\Delta}(t)$ is palindromic.*

This gives an alternative way to test whether a given Gorenstein cone is reflexive.

Example 3.2.2. Consider the Gorenstein cones σ_1 and σ_2 in $M_{\mathbb{R}} \simeq \mathbb{R}^7$ from Example 3.1.8. Let Δ_1 and Δ_2 be the support polytopes of σ_1 and σ_2 respectively. Then

$$h^*_{\Delta_1}(t) = t^5 + t^4 + t^3 + t^2 + t + 1 \qquad h^*_{\Delta_2}(t) = t^5 + t^4 + 2t^3 + t^2 + t + 1$$

Theorem 3.2.1 then implies that Δ_1 is Gorenstein while Δ_2 is not. By Proposition 3.1.6, σ_1 is reflexive while σ_2 is not reflexive. $\Diamond$

3.2.2 Combinatorial Polynomials of Eulerian Posets

The stringy E-polynomial of a Gorenstein polytope Δ relies on the structure of Δ as an *Eulerian poset*. In this section we develop the notions required to define an Eulerian poset $\mathcal{P}$ and review some of the basic polynomial invariants of $\mathcal{P}$.

Following [11], a *ranked poset* is a poset $\mathcal{P}$ equipped with a map $\rho : \mathcal{P} \to \mathbb{Z}$ such that every maximal chain $x_0 < x_2 < \cdots < x_k$ in $\mathcal{P}$ satisfies $k = \rho(x_k) - \rho(x_0)$. A *graded poset* is a ranked poset $\mathcal{P}$ possessing a minimal element $\hat{0}$ satisfying $\rho(\hat{0}) = 0$ and a maximal element $\hat{1}$. The *rank* of a graded poset $\mathcal{P}$ is $\rho(\mathcal{P}) = \rho(\hat{1})$. A graded poset $\mathcal{P}$ of rank d satisfies the *Euler-Poincaré relation* if

$$f_0 - f_1 + \cdots + (-1)^d f_d = 0 \tag{3.17}$$

where f_r is the number of elements in $\mathcal{P}$ of rank r.

Each pair of elements x and y in a graded poset $\mathcal{P}$ defines an *interval*

$$[x, y] = \{z \in \mathcal{P} : x \leq z \leq y\} \tag{3.18}$$

Note that the map $z \mapsto \rho(z) - \rho(x)$ endows each interval $[x, y]$ in a graded poset $\mathcal{P}$ with the structure of a graded poset of rank $\rho(y) - \rho(x)$. A graded poset $\mathcal{P}$ is *Eulerian* if each interval in $\mathcal{P}$ satisfies the Euler-Poincaré relation (3.17).

In [42, p. 122], Stanley proves that the map $\rho(F) = \dim F + 1$ endows the face lattice of a lattice polytope Δ with the structure of a Eulerian poset.

The polynomial invariants we are interested in are most easily defined in terms of the *truncation operator* $\tau_r : \mathbb{Z}[t] \to \mathbb{Z}[t]$, which is given by

$$\tau_r \left(\sum_i a_i t^i \right) = \sum_{i<r} a_i t^i \tag{3.19}$$

For $f(t) = \sum_i a_i t^i \in \mathbb{Z}[t]$ we write $f(t) \geq 0$ if each coefficient a_i in f satisfies $a_i \geq 0$.

Definition 3.2.3. Let $\mathcal{P}$ be an Eulerian poset of rank d. The *g-polynomial of $\mathcal{P}$ and the h-polynomial of $\mathcal{P}$* are the polynomials $g_{\mathcal{P}}(t), h_{\mathcal{P}}(t) \in \mathbb{Z}[t]$ defined by $g_{\mathcal{P}}(t) = h_{\mathcal{P}}(t) = 1$ if $d = 0$ and by the intertwining recursions

$$g_{\mathcal{P}}(t) = \tau_{d/2}\big((1-t)h_{\mathcal{P}}(t)\big) \qquad h_{\mathcal{P}}(t) = \sum_{\hat{0} < x \leq \hat{1}} (t-1)^{\rho(x)-1} g_{[x,\hat{1}]}(t) \tag{3.20}$$

if $d > 0$.

A poset is *boolean* if it is isomorphic to the face lattice of a simplex.

Proposition 3.2.4. *The g-polynomial has the following properties.*

1. *If $\mathcal{P}$ is the poset of faces of a lattice polytope, then $g_{\mathcal{P}}(t)$ has nonnegative coefficients [41].*

2. *A poset $\mathcal{P}$ is boolean if and only if $g_{\mathcal{P}}(t) = 1$.*

3. $g_{\mathcal{P}\times\mathcal{Q}}(t) = g_{\mathcal{P}}(t) \cdot g_{\mathcal{Q}}(t)$

3.2.3 The $\widetilde{S}$-Polynomial of a Lattice Polytope

In [13], Borisov and Mavlyutov combine the combinatorial data encoded in the g-polynomial of a lattice polytope Δ with the enumeration of lattice points from $h^*_\Delta(t)$ to obtain a rather sophisticated polynomial invariant of Δ.

Definition 3.2.5. Let Δ be a lattice polytope in $M_{\mathbb{R}}$. The *$\widetilde{S}$-polynomial of* Δ is the polynomial $\widetilde{S}_\Delta(t) \in \mathbb{Z}[t]$ defined by

$$\widetilde{S}_\Delta(t) = \sum_{\varnothing \preceq F \preceq \Delta} (-1)^{\dim\Delta - \dim F} h^*_F(t) \cdot g_{[F,\Delta]}(t) \tag{3.21}$$

where the sum is taken over all faces F of Δ.

Note that $\widetilde{S}_{\varnothing}(t) = 1$ while $\widetilde{S}_{\Delta}(t) = 0$ if $\dim \Delta = 0$. The $\widetilde{S}$-polynomial enjoys a number of nice properties.

Proposition 3.2.6. *Let Δ be a lattice polytope. Then*

(a) $\widetilde{S}_{\Delta}(0) = 0$ *if* $\dim \Delta \geq 0$.

(b) $\widetilde{S}_{\Delta}(t) \geq 0$ *[13, Proposition 5.5]*

(c) $\widetilde{S}_{\Delta}(t) = t^{1+\dim \Delta}\widetilde{S}_{\Delta}(t^{-1})$ *[13, Remark 5.4]*

(d) *If Δ is a lattice pyramid, then $\widetilde{S}_{\Delta}(t) = 0$ [10, Lemma 4.5].*

(e) $\widetilde{S}_{\Delta}(t) \leq h^*_{\Delta}(t)$ *[36, Corollary 2.16 (1)]*

Example 3.2.7. Let Δ be the Gorenstein polytope corresponding to the reflexive Gorenstein cone σ_1 from Example 3.1.8. The vertices of Δ are precisely the points listed in the first column of (3.9). Of the 190 faces of Δ, only $F = \varnothing$ and $F = \Delta$ satisfy $\widetilde{S}_F(t) \neq 0$. For these two faces, we have

$$\widetilde{S}_{\varnothing}(t) = 1 \qquad\qquad \widetilde{S}_{\Delta}(t) = t^5 + t^4 + t^3 + t^2 \tag{3.22}$$

The vertices of the dual Gorenstein polytope Δ^* are the points listed in the first column of (3.11). The faces dual to the faces $F = \varnothing$ and $F = \Delta$ of Δ are $\varnothing^\vee = \Delta^*$ and $\Delta^\vee = \varnothing$. Computing the $\widetilde{S}$-polynomials gives

$$\widetilde{S}_{\Delta^*}(t) = t^5 + 89\,t^4 + 89\,t^3 + t^2 \qquad\qquad \widetilde{S}_{\varnothing}(t) = 1 \tag{3.23}$$

The lattice structure of Δ^* is significantly more complicated than that of Δ. Of the 190 faces of Δ^*, 131 have a nonvanishing $\widetilde{S}$-polynomial. $\Diamond$

The phenomena exhibited in Example 3.2.7 is not rare. In fact, the main result of [36] states that for a dual pair of Gorenstein polytopes (Δ, Δ^*), a face $F \preceq \Delta$ and its dual $F^\vee \preceq \Delta^*$ satisfy

$$c \leq \operatorname{codeg} F + \operatorname{codeg} F^\vee \tag{3.24}$$

where c is the index of Δ. Heuristically, this means that F and $F^\vee$ cannot both be 'complicated'.

3.2.4 The E_{st}-Polynomial of a Gorenstein Polytope

We now have enough technology to define the stringy E-function of a Gorenstein polytope.

Definition 3.2.8. Let Δ be a d-dimensional Gorenstein polytope of index c. The *stringy E-function of* Δ is the function $E_{\text{st}}(\Delta; u, v) \in \mathbb{Q}(u, v)$ defined by

$$E_{\text{st}}(\Delta; u, v) = (uv)^{-c} \sum_{\varnothing \preceq F \preceq \Delta} (-u)^{1+\dim F} \widetilde{S}_F(u^{-1}v) \cdot \widetilde{S}_{F^\vee}(uv) \tag{3.25}$$

where the sum is taken over all faces F of Δ and $F^\vee$ is the face of Δ^* dual to F. We call the integer

$$\dim_{\text{CY}} \Delta = d + 1 - 2\,c \tag{3.26}$$

the *Calabi-Yau dimension of* Δ.

Note that a dual pair of Gorenstein polytopes (Δ, Δ^*) satisfies

$$\dim_{\text{CY}} \Delta = \dim_{\text{CY}} \Delta^* \tag{3.27}$$

since $\dim \Delta = \dim \Delta^*$ and each of Δ and Δ^* has the same index. The properties of the $\widetilde{S}$-polynomial in Proposition 3.2.6 endow the E_{st}-function with several nice properties.

Proposition 3.2.9. *Let Δ be a Gorenstein polytope with* $\dim_{\text{CY}} \Delta = n$*. Then*

(a) $E_{\text{st}}(\Delta; u, v) = E_{\text{st}}(\Delta; v, u)$

(b) $(uv)^n E_{\text{st}}(\Delta; u^{-1}, v^{-1}) = E_{\text{st}}(\Delta; u, v)$

(c) $E_{\text{st}}(\Delta; u, v) = (-u)^n E_{\text{st}}(\Delta^*; u^{-1}, v)$

(*d*) $\dim_{\text{CY}} \Delta = \deg \Delta - \operatorname{codeg} \Delta$

(*f*) $E_{\text{st}}(\Delta; u, v)$ *is a polynomial and* $E_{\text{st}}(\Delta; u, v) = 0$ *if* $\dim_{\text{CY}} \Delta < 0$.

Proposition 3.2.9 (*b*) is known as *Poincaré duality* and Proposition 3.2.9 (c) is known as the *mirror symmetry law.* Proposition 3.2.9 (*f*) was proved by Nill and Schepers in 2010 [36, Corollary 3.7].

Remark 3.2.10. Suppose Δ is a Gorenstein polytope with $\dim_{\text{CY}} \Delta = n$. Then Proposition 3.2.6 and Proposition 3.2.9 imply that

$$E_{\text{st}}(\Delta; u, v) = \sum_{p,q=0}^{n} (-1)^{p+q} h_{\text{st}}^{p,q}(\Delta) u^p v^q \tag{3.28}$$

where $h_{\text{st}}^{p,q}(\Delta)$ are nonnegative integers, which we call the *stringy Hodge numbers of* Δ. ◊

Let us now compute the stringy Hodge numbers of a few Gorenstein polytopes.

Example 3.2.11. Let Δ be the Gorenstein polytope corresponding to the reflexive Gorenstein cone σ_1 from Example 3.1.8. The vertices of Δ are precisely the points listed in the first column of (3.9). The computations in (3.22) and (3.23) immediately imply that

$$E_{\text{st}}(\Delta; u, v) = u^3v^3 + 89\, u^2v^2 - u^3 - u^2v - uv^2 - v^3 + 89\, uv + 1 \tag{3.29}$$

The stringy Hodge numbers of Δ are thus

$$\begin{matrix} & & & h_{\text{st}}^{33} & & & \\ & & h_{\text{st}}^{32} & & h_{\text{st}}^{23} & & \\ & h_{\text{st}}^{31} & & h_{\text{st}}^{22} & & h_{\text{st}}^{13} & \\ h_{\text{st}}^{30} & & h_{\text{st}}^{21} & & h_{\text{st}}^{12} & & h_{\text{st}}^{03} \\ & h_{\text{st}}^{20} & & h_{\text{st}}^{11} & & h_{\text{st}}^{02} & \\ & & h_{\text{st}}^{10} & & h_{\text{st}}^{01} & & \\ & & & h_{\text{st}}^{00} & & & \end{matrix} \quad = \quad \begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 89 & & 0 & \\ 1 & & 1 & & 1 & & 1 \\ & 0 & & 89 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix} \tag{3.30}$$

The mirror symmetry law implies that the dual Gorenstein polytope Δ^*, whose vertices are listed in the first column of (3.11) has $E_{\rm st}$-function given by

$$E_{\rm st}(\Delta^*; u, v) = u^3v^3 + u^2v^2 - u^3 - 89\,u^2v - 89\,uv^2 - v^3 + uv + 1 \qquad (3.31)$$

Of course, this implies that the stringy Hodge numbers of Δ^* are given by the transpose of the diamond in (3.30).

Note that $\dim_{\rm CY} \Delta = 6 + 1 - 2 \cdot 2 = 3$. ♢

The next example demonstrates that a Gorenstein polytope Δ and its dual Δ^* may have the same stringy Hodge numbers.

Example 3.2.12. Fix $M \simeq \mathbb{Z}^7$ and let (Δ, Δ^*) be the dual pair of Gorenstein polytopes of index $c = 2$ whose vertices are given by the lists

```
M(0, 1,  0, -1, -2, -1, -1),   N(0, 1,  0,  0,  0,  0,  0),
M(0, 1,  0,  0,  0,  0,  1),   N(0, 1,  0,  0,  0,  0,  1),
M(0, 1,  0,  1,  1,  1, -1),   N(0, 1,  1,  0,  0, -2, -1),
M(0, 1,  0,  1,  2,  1, -1),   N(0, 1,  3,  0,  4, -6, -1),
M(1, 0, -2, -2,  0, -1,  0),   N(1, 0, -2, -1, -1,  3,  0),
M(1, 0,  0, -2, -3, -2,  0),   N(1, 0, -1, -1,  0,  1,  0),     (3.32)
M(1, 0,  0, -1, -1, -1,  2),   N(1, 0, -1,  1, -1,  1,  0),
M(1, 0,  0,  0,  1,  0,  0),   N(1, 0,  0, -1, -1,  3,  0),
M(1, 0,  0,  1,  0,  0,  0),   N(1, 0,  0,  1,  0, -1,  0),
M(1, 0,  2,  0,  0,  1,  0)    N(1, 0,  1, -1,  0,  1,  0)
in 7-d lattice M               in 7-d lattice N
```

respectively. Then

$$E_{\rm st}(\Delta; u, v) = u^3v^3 + 11\,u^2v^2 - u^3 - 11\,u^2v - 11\,uv^2 - v^3 + 11\,uv + 1 \qquad (3.33)$$

and the stringy Hodge numbers of Δ are thus

$$\begin{matrix} & & & h_{\rm st}^{33} & & & \\ & & h_{\rm st}^{32} & & h_{\rm st}^{23} & & \\ & h_{\rm st}^{31} & & h_{\rm st}^{22} & & h_{\rm st}^{13} & \\ h_{\rm st}^{30} & & h_{\rm st}^{21} & & h_{\rm st}^{12} & & h_{\rm st}^{03} \\ & h_{\rm st}^{20} & & h_{\rm st}^{11} & & h_{\rm st}^{02} & \\ & & h_{\rm st}^{10} & & h_{\rm st}^{01} & & \\ & & & h_{\rm st}^{00} & & & \end{matrix} \quad = \quad \begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 11 & & 0 & \\ 1 & & 11 & & 11 & & 1 \\ & 0 & & 11 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix} \tag{3.34}$$

The mirror symmetry law then implies that $h_{\rm st}^{p,q}(\Delta) = h_{\rm st}^{p,q}(\Delta^*)$.

This example is computationally more interesting than the computation in Example 3.2.11. The f-vectors of both Δ and Δ^* are

$$(1,\ 10,\ 36,\ 66,\ 66,\ 36,\ 10,\ 1) \tag{3.35}$$

Of the 226 faces of Δ, 40 have a nonvanishing $\widetilde{S}$-polynomial and the same is true of Δ^*. ◊

3.3 Stringy Hodge Numbers of Batyrev-Borisov Mirror Pairs

Let (Δ, Δ^*) be a dual pair of reflexive polytopes, which corresponds to a pair (X_Δ, X_{Δ^*}) of Gorenstein Fano toric varieties by Proposition 2.4.7. The canonical divisor K_{X_Σ} of any toric variety X_Σ is given by [17, Theorem 8.2.3]

$$K_{X_\Sigma} = -\sum_{\rho\in\Sigma(1)} D_\rho \tag{3.36}$$

From (2.22), it follows that (Δ, Δ^*) are the polytopes corresponding to the anticanonical divisors $(-K_{X_\Delta}, -K_{X_{\Delta^*}})$. Generic global sections of $D_\Delta = -K_{X_\Delta}$ and $D_{\Delta^*} = -K_{\Delta^*}$ thus define ample anticanonical hypersurfaces $Y_\Delta \subset X_\Delta$ and $Y_{\Delta^*} \subset$

X_{Δ^*}. A *Batyrev mirror pair* is a generic choice (Y_Δ, Y_{Δ^*}) of these hypersurfaces, which are Calabi-Yau by the adjunction formula. In [4], Batyrev obtains a mirror pair of Calabi-Yau varieties $(\widehat{Y}_\Delta, \widehat{Y}_{\Delta^*})$ by taking maximal projective crepant partial resolutions $\widehat{Y}_\Delta \to Y_\Delta$ and $\widehat{Y}_{\Delta^*} \to Y_{\Delta^*}$ induced by toric blow ups.

In this section, we review a generalization of this construction developed by Borisov [12] applying to complete intersections.

3.3.1 Nef-Partitions of Reflexive Polytopes

Let (N, M) be a dual pair of torsion-free lattices of rank d and let $\langle -, - \rangle : M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}$ be the natural pairing.

Definition 3.3.1. A *nef-partition of length* c of a reflexive polytope $\Delta \subset M_\mathbb{R}$ is a Minkowski sum

$$\Delta = \Delta_1 + \cdots + \Delta_c \tag{3.37}$$

where each Δ_i is a lattice polytope in $M_\mathbb{R}$ with $0_M \in \Delta_i$.

Note that the each of summands Δ_i in (3.37) need not be full-dimensional and thus may have empty interior. A nef-partition $\Delta = \Delta_1 + \cdots + \Delta_c$ is *proper* if each summand Δ_i satisfies $\dim \Delta_i > 0$.

By (2.23) and [17, §4.3 and Theorem 6.3.12], each summand Δ_i in (3.37) corresponds to a numerically effective divisor D_{Δ_i} on the Gorenstein Fano toric variety X_Δ. This explains the 'nef' part of the term 'nef-partition'. We will explain the 'partition' part shortly. First, we observe that nef-partitions occur in dual pairs.

Proposition 3.3.2 ([12])**.** *Let $\Delta = \Delta_1 + \cdots + \Delta_c$ be a nef-partition of a reflexive polytope Δ in $M_\mathbb{R}$. For $1 \le j \le c$ put*

$$\nabla_j = \{y \in N_\mathbb{R} \,|\, x \in \Delta_i \Rightarrow \langle x, y \rangle \ge -\delta_{ij}, 1 \le i \le c\} \tag{3.38}$$

and let $\nabla = \nabla_1 + \cdots + \nabla_c$. Then (*a*) $\nabla = \nabla_1 + \cdots + \nabla_c$ *is a nef-partition of length* c.

(*b*) *The polytope* Δ_i *satisfies*

$$\Delta_i = \{x \in M_{\mathbb{R}} \,|\, y \in \nabla_j \Rightarrow \langle x, y\rangle \geq -\delta_{ij}, 1 \leq j \leq c\} \tag{3.39}$$

for each $1 \leq i \leq c$.

(*c*) *The dual polytopes* Δ^* *and* ∇^* *satisfy*

$$\Delta^* = \mathrm{Conv}(\nabla_1, \ldots, \nabla_c) \qquad\qquad \nabla^* = \mathrm{Conv}(\Delta_1, \ldots, \Delta_c) \tag{3.40}$$

(*d*) *For each* $1 \leq i \leq c$, $\Delta_i = \{0_M\}$ *if and only if* $\nabla_i = \{0_N\}$. *Consequently, the nef-partition* $\Delta = \Delta_1 + \cdots + \Delta_c$ *is proper if and only if the nef-partition* $\nabla = \nabla_1 + \cdots + \nabla_c$ *is proper.*

(*e*) *For each nonzero vertex* $n \in \nabla_i$ *and each nonzero vertex* $m \in \Delta_i$, *we have*

$$\min_{x\in\Delta_j}\langle x, n\rangle = -\delta_{ij} \qquad\qquad \min_{y\in\nabla_j}\langle m, y\rangle = -\delta_{ij} \tag{3.41}$$

for $1 \leq j \leq c$.

(*f*) *Generic global sections* $f_i \in H^0(X_\Delta, \mathscr{O}_{X_\nabla}(D_{\Delta_i}))$ *and* $g_i \in H^0(X_\nabla, \mathscr{O}_{X_\nabla}(D_{\nabla_i}))$ *define (possibly singular) Calabi-Yau complete intersections* $Y_{\Delta_1,\ldots,\Delta_c} \subset X_\Delta$ *and* $Y_{\nabla_1,\ldots,\nabla_c} \subset X_\nabla$ *each of codimension* c.

We refer to the nef-partitions $\Delta = \Delta_1 + \cdots + \Delta_c$ and $\nabla = \nabla_1 + \cdots + \nabla_c$ as a *dual pair of nef partitions.* The pair $(Y_{\Delta_1,\ldots,\Delta_c}, Y_{\nabla_1,\ldots,\nabla_c})$ of Calabi-Yau complete intersections in (X_Δ, X_∇) from Proposition (3.3.2) (*f*) are called a *stringy Batyrev-Borisov mirror pair.*

The 'partition' part of the term 'nef-partition' can now be explained. Proposition 3.3.2 implies that each nef divisor D_{Δ_i} is of the form

$$D_{\Delta_i} = \sum_{\rho\in\Sigma_\Delta(1)} -\min\langle \Delta_i, u_\rho\rangle D_\rho = \sum_{u_\rho\in\nabla_i} D_\rho \tag{3.42}$$

By (3.36), the support of the anticannonical divisor on X_Δ is

$$\mathrm{Supp}(-K_{X_\Delta}) = \bigcup_{\rho\in\Sigma_\Delta(1)} D_\rho \tag{3.43}$$

which can be expressed as the disjoint union

$$\mathrm{Supp}(-K_{X_\Delta}) = \bigcup_{u_\rho\in\nabla_1} D_\rho \sqcup\cdots\sqcup \bigcup_{u_\rho\in\nabla_c} D_\rho \tag{3.44}$$

Thus, the data of a nef-partition determines a partition of the set of torus-invariant prime divisors $\{D_\rho\,|\,\rho\in\Sigma_\Delta(1)\}$ on X_Δ.

3.3.2 Gorenstein Polytopes, Nef-Partitions, and Special Simplices

In this section, we demonstrate how a dual pair of nef-partitions gives rise to a dual pair of reflexive Gorenstein cones.

Recall that the *affine hull* of a subset S of a vector space V is the smallest affine subset $\mathrm{aff}(S)$ of V containing S.

Definition 3.3.3. Let $\Delta_1,\dots,\Delta_c$ be lattice polytopes in $M_\mathbb{R}$ with $\dim M_\mathbb{R} = d$. Consider the lattice $\bar{M} = M\oplus\mathbb{Z}^c$ and let $\{e_1,\dots,e_c\}$ be the standard basis of $\mathbb{Z}^c$. The *Cayley polytope of length c* associated to the c-tuple of polytopes $\Delta_1,\dots,\Delta_c$ is the polytope

$$\Delta_1 * \cdots * \Delta_c = \mathrm{Conv}(\Delta_1\times e_1,\dots,\Delta_c\times e_c) \tag{3.45}$$

considered as a polytope in the affine subspace

$$\big((M\oplus\mathrm{aff}(e_1,\dots,e_c))\cap\bar{M}\big)_\mathbb{R} \tag{3.46}$$

The cone in $\bar{M}_\mathbb{R}$ given by

$$\mathcal{C}_{\Delta_1,\dots,\Delta_c} = \mathrm{Cone}(\Delta_1 * \cdots * \Delta_c) \tag{3.47}$$

is called the *Cayley cone* of the c-tuple of polytopes is $\Delta_1,\dots,\Delta_c$.

If $\text{aff}(\Delta_1, \dots, \Delta_c) = M_{\mathbb{R}}$, then the Cayley cone $\mathcal{C}_{\Delta_1,\dots,\Delta_c}$ is a Gorenstein cone of dimension $d + r$ whose support polytope is the Cayley polytope $\Delta_1 * \cdots * \Delta_c$.

Proposition 3.3.4 (Definition 3.11 and Proposition 3.13 in [10])**.** *Let $\Delta = \Delta_1 + \cdots + \Delta_c \subset M_{\mathbb{R}}$ and $\nabla = \nabla_1 + \cdots + \nabla_c \subset N_{\mathbb{R}}$ be a dual pair of nef-partitions where* $\dim M_{\mathbb{R}} = \dim N_{\mathbb{R}} = d$. *Then the Cayley cones $\mathcal{C}_{\Delta_1,\dots,\Delta_c}$ and $\mathcal{C}_{\nabla_1,\dots,\nabla_c}$ form a dual pair of reflexive Gorenstein cones of index c. In particular, the Cayley polytopes $\Delta_1 * \cdots * \Delta_c$ and $\nabla_1 * \cdots * \nabla_c$ form a dual pair of Gorenstein polytopes of index c.*

A convenient way to categorize Gorenstein polytopes that arise as Cayley polytopes of nef-partitions uses the concept of a *special* $(c-1)$*-simplex.*

Definition 3.3.5. Let Δ be a lattice polytope in $M_{\mathbb{R}}$. A *special* $(c-1)$*-simplex of* Δ is a simplex S spanned by c affinely independent lattice points in Δ such that every facet of Δ contains precisely $c-1$ vertices of S.

Proposition 3.3.6 (Corollary 3.7 in [10])**.** *A Gorenstein polytope P of index c is the Cayley polytope of a nef-partition of length c if and only if both P and P^* contain special $(c-1)$-simplices.*

3.3.3 The E_{st}-Function of a Stringy Batyrev-Borisov Mirror Pair

We are now ready to define the *stringy Hodge numbers* of a stringy Batyrev-Borisov mirror pair $(Y_{\Delta_1,\dots,\Delta_c}, Y_{\nabla_1,\dots,\nabla_c})$.

Let (N, M) be a dual pair of torsion-free lattices of rank d and let

$$\Delta = \Delta_1 + \cdots + \Delta_c \subset M_{\mathbb{R}} \qquad \nabla = \nabla_1 + \cdots + \nabla_c \subset N_{\mathbb{R}} \tag{3.48}$$

be a dual pair of nef-partitions. Let $\tilde{\Delta}$ and $\tilde{\nabla}$ be the Cayley polytopes

$$\tilde{\Delta} = \Delta_1 * \cdots * \Delta_c \qquad \tilde{\nabla} = \nabla_1 * \cdots * \nabla_c \tag{3.49}$$

defined in (3.45). By Proposition 3.3.4, the pair $(\tilde{\Delta}, \tilde{\nabla})$ is a dual pair of Gorenstein polytopes of index c and dimension $d + c - 1$. Let $(Y_{\Delta_1,\dots,\Delta_c}, Y_{\nabla_1,\dots,\nabla_c})$ be the stringy

Batyrev-Borisov mirror pair of Calabi-Yau complete intersections in the Gorenstein Fano toric varieties (X_Δ, X_∇) from Proposition (3.3.2) (f). Note that

$$\begin{aligned} \dim_{\mathrm{CY}} \tilde{\Delta} &= \dim \tilde{\Delta} + 1 - 2\,c \\ &= (d + c - 1) + 1 - 2\,c \\ &= d - c \\ &= \dim Y_{\Delta_1,\ldots,\Delta_c} \end{aligned} \tag{3.50}$$

and similarly $\dim_{\mathrm{CY}} \tilde{\nabla} = \dim Y_{\nabla_1,\ldots,\nabla_c}$.

Definition 3.3.7. The *stringy Hodge numbers of* $Y_{\Delta_1,\ldots,\Delta_c}$ *and* $Y_{\nabla_1,\ldots,\nabla_c}$ are the stringy Hodge numbers of the associated Cayley polytopes $h^{p,q}_{\mathrm{st}}(Y_{\Delta_1,\ldots,\Delta_c}) = h^{p,q}_{\mathrm{st}}(\tilde{\Delta})$ and $h^{p,q}_{\mathrm{st}}(Y_{\nabla_1,\ldots,\nabla_c}) = h^{p,q}_{\mathrm{st}}(\tilde{\nabla})$.

This defines the stringy Hodge numbers $h^{p,q}_{\mathrm{st}}(Y)$ for any $Y \simeq Y_{\Delta_1,\ldots,\Delta_c}$ for a nef-partition $\Delta = \Delta_1 + \cdots + \Delta_c$. A pair of varieties $(Y, Y^\vee)$ with dimension d is a *stringy topological mirror pair* if their stringy Hodge numbers satisfy

$$h^{p,q}_{\mathrm{st}}(Y) = h^{d-p,q}_{\mathrm{st}}(Y^\vee) \tag{3.51}$$

The mirror symmetry law of Proposition 3.2.9 (c) implies that every stringy Batyrev-Borisov mirror pair $(Y_{\Delta_1,\ldots,\Delta_c}, Y_{\nabla_1,\ldots,\nabla_c})$ is a stringy topological pair.

Note that our definition of the stringy Hodge numbers depends only on the combinatorics of the dual pair of cayley polytopes $\Delta_1 * \cdots * \Delta_c$ and $\nabla_1 * \cdots * \nabla_c$. It is a remarkable fact that this definition coincides with the usual Hodge numbers when the variety of interest is smooth.

Theorem 3.3.8 (Theorem 7.2 in [7]). *Let* $\Delta = \Delta_1 + \cdots + \Delta_c$ *be a nef-partition and let* $Y_{\Delta_1,\ldots,\Delta_c}$ *be the corresponding Calabi-Yau complete intersection of codimension* c *in the Gorenstein Fano toric variety* X_Δ. *Suppose* $\phi : \widehat{Y} \to Y_{\Delta_1,\ldots,\Delta_c}$ *is a crepant resolution. Then* $h^{p,q}(\widehat{Y}) = h^{p,q}_{\mathrm{st}}(Y_{\Delta_1,\ldots,\Delta_c})$. *Moreover, if* $\widehat{Y}$ *is smooth, then the usual Hodge numbers of* Y *and the stringy Hodge numbers of* Y *agree.*

The proof of Theorem 3.3.8 is originally due to Batyrev, who used a motivic integration over the spaces of arcs to relate the E_{st}-functions of different resolutions.

3.4 Examples

In this section, we demonstrate the Batyrev-Borisov mirror construction of a stringy mirror pair $(Y_{\Delta_1,\ldots,\Delta_c}, Y_{\nabla_1,\ldots,\nabla_c})$ corresponding to a dual pair of nef-partitions $\Delta = \Delta_1 + \cdots + \Delta_c$ and $\nabla = \nabla_1 + \cdots + \nabla_c$.

3.4.1 A Smooth Calabi-Yau in $\mathbb{P}^2 \times \mathbb{P}^3$

Fix $M \simeq \mathbb{Z}^5$ and let $\Delta, \Delta_1, \Delta_2 \subset M_{\mathbb{R}}$ be the polytopes whose vertices are

```
M(-1, -1,  3, -1, -1),   M( 2, -1, -1, 1, 0),
M(-1, -1, -1,  3, -1),   M(-1,  2, -1, 1, 0),
M( 2, -1,  3, -1, -1),   M( 2, -1,  0, 0, 0),
M( 2, -1, -1,  3, -1),   M(-1,  2,  0, 0, 0),
M(-1,  2,  3, -1, -1),   M(-1,  2, -1, 0, 1),   M(0, 0, 0,  2, -1),
M(-1,  2, -1,  3, -1),   M( 2, -1, -1, 0, 1),   M(0, 0, 3, -1, -1),
M(-1, -1, -1, -1,  3),   M( 2, -1, -1, 0, 0),   M(0, 0, 0, -1,  2),
M( 2, -1, -1, -1,  3),   M(-1, -1, -1, 0, 0),   M(0, 0, 0, -1, -1)
M(-1,  2, -1, -1,  3),   M(-1, -1, -1, 1, 0),   in 5-d lattice M
M(-1, -1, -1, -1, -1),   M(-1,  2, -1, 0, 0),
M( 2, -1, -1, -1, -1),   M(-1, -1,  0, 0, 0),
M(-1,  2, -1, -1, -1)    M(-1, -1, -1, 0, 1)
in 5-d lattice M         in 5-d lattice M
```

(3.52)

respectively. Then $\Delta = \Delta_1 + \Delta_2$ is a nef-partition and the dual nef-partition $\nabla = \nabla_1 + \nabla_2$ is defined by the polytopes $\nabla, \nabla_1, \nabla_2 \subset N_{\mathbb{R}}$ whose vertices are given by

```
N( 1,  0,  0,  1,  0),
N( 1,  0,  0,  0,  1),
N( 1,  0, -1, -1, -1),
N( 1,  0,  0,  0,  0),
N( 0,  1,  0,  1,  0),
N( 0,  1,  0,  0,  1),
N( 0,  1, -1, -1, -1),
N( 0,  1,  0,  0,  0),
N(-1, -1,  0,  1,  0),
N(-1, -1,  0,  0,  1),
N(-1, -1, -1, -1, -1),
N(-1, -1,  0,  0,  0),
N( 0,  0,  1,  1,  0),
N( 0,  0,  1,  0,  1),
N( 0,  0,  0, -1, -1),
N( 0,  0,  1,  0,  0)
in 5-d lattice N
```

```
N( 1,  0, 0, 0, 0),
N( 0,  1, 0, 0, 0),
N(-1, -1, 0, 0, 0),
N( 0,  0, 1, 0, 0)
in 5-d lattice N
```

```
N(0, 0,  0,  1,  0),
N(0, 0,  0,  0,  1),
N(0, 0, -1, -1, -1),
N(0, 0,  0,  0,  0)
in 5-d lattice N
```

(3.53)

respectively. The dual polytopes Δ^* and ∇^* have vertices

```
N( 1,  0,  0,  0,  0),
N( 0,  1,  0,  0,  0),
N(-1, -1,  0,  0,  0),
N( 0,  0,  1,  0,  0),
N( 0,  0,  0,  1,  0),
N( 0,  0,  0,  0,  1),
N( 0,  0, -1, -1, -1)
in 5-d lattice N
```

```
M( 0,  0,  0,  2, -1),
M( 2, -1, -1,  1,  0),
M(-1,  2, -1,  1,  0),
M( 0,  0,  3, -1, -1),
M( 2, -1,  0,  0,  0),
M(-1,  2,  0,  0,  0),
M(-1,  2, -1,  0,  1),
M( 0,  0,  0, -1,  2),
M( 2, -1, -1,  0,  1),
M( 0,  0,  0, -1, -1),
M( 2, -1, -1,  0,  0),
M(-1, -1, -1,  0,  0),
M(-1, -1, -1,  1,  0),
M(-1,  2, -1,  0,  0),
M(-1, -1,  0,  0,  0),
M(-1, -1, -1,  0,  1)
in 5-d lattice M
```

(3.54)

respectively. From (3.54) we see that $X_\Delta = \mathbb{P}^2 \times \mathbb{P}^3$. The Calabi-Yau complete intersection $Y_{\Delta_1,\Delta_2} \subset X_\Delta = \mathbb{P}^2 \times \mathbb{P}^3$ has configuration matrix

$$Y_{\Delta_1,\Delta_2} = \left[\begin{array}{c||cc} \mathbb{P}^2 & 3 & 0 \\ \mathbb{P}^3 & 1 & 3 \end{array} \right] \tag{3.55}$$

That is, a generic global section of $\mathscr{O}_{X_\Delta}(D_{\Delta_1})$ has multidegree $(3,1)$ in $\mathbb{P}^2 \times \mathbb{P}^3$ and a generic global section of $\mathscr{O}_{X_\Delta}(D_{\Delta_2})$ has multidegree $(0,3)$ in $\mathbb{P}^2 \times \mathbb{P}^3$.

A generic complete intersection Y_{Δ_1,Δ_2} described by the configuration matrix (3.55) is a smooth Calabi-Yau threefold in $\mathbb{P}^2 \times \mathbb{P}^3$. The spectral sequence (2.44) computing the cohomology of $\mathscr{T}_{Y_{\Delta_1,\Delta_2}}$ has E_2-page

$$\begin{array}{c|ccc} q & & & \\ 3 & \mathbb{C}^2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & \mathbb{C}^6 \\ 0 & 0 & 0 & \mathbb{C}^{35} \\ \hline & -1 & 0 & 1 \; p \end{array} \tag{3.56}$$

which gives $h^{11}(Y_{\Delta_1,\Delta_2}) = 8$ and $h^{21}(Y_{\Delta_1,\Delta_2}) = 35$.

The toric variety X_∇ can be shown to have a singular locus of dimension three and the Calabi-Yau complete intersection $Y_{\nabla_1,\nabla_2} \subset X_\nabla$ intersects this singular locus. A direct analysis of $\mathscr{T}_{Y_{\nabla_1,\nabla_2}}$ from (2.44) is impossible. We may, however, compute the *stringy* Hodge numbers of Y_{∇_1,∇_2}. The Cayley polytope $\nabla_1 * \nabla_2$ is the Gorenstein polytope of index two with vertices

```
ZZ^2+N(0, 1,  0,  0,  0,  0,  0),
ZZ^2+N(0, 1,  0,  0,  0,  0,  1),
ZZ^2+N(0, 1,  0,  0,  0,  1,  0),
ZZ^2+N(0, 1,  0,  0,  1,  0,  0),
ZZ^2+N(1, 0, -1, -1,  0,  0,  0),
ZZ^2+N(1, 0,  0,  0, -1, -1, -1),
ZZ^2+N(1, 0,  1,  0,  0,  0,  0),
ZZ^2+N(1, 0,  0,  1,  0,  0,  0)
in 7-d lattice ZZ^2+N
```

(3.57)

The stringy E-function of $\nabla_1 * \nabla_2$ is

$$E_{\mathrm{st}}(\nabla_1 * \nabla_2; u, v) = u^3 v^3 + 35\, u^2 v^2 - u^3 - 8\, u^2 v - 8\, u v^2 - v^3 + 35\, uv + 1 \quad (3.58)$$

Hence $h^{11}_{\mathrm{st}}(Y_{\nabla_1,\nabla_2}) = 35$ and $h^{21}_{\mathrm{st}}(Y_{\nabla_1,\nabla_2}) = 8$, confirming that $(Y_{\Delta_1,\Delta_2}, Y_{\nabla_1,\nabla_2})$ is indeed a stringy mirror pair.

3.4.2 A Singular Calabi-Yau Threefold

Fix $M \simeq \mathbb{Z}^5$ and let $\Delta, \Delta_1, \Delta_2 \subset M_{\mathbb{R}}$ be the polytopes whose vertices are

```
M(-1, -1, -1,  9, -1),
M(-1, -1,  4, -1, -1),   M(0, 0, -1,  3, -1),
M( 2, -1, -1,  0, -1),   M(0, 0,  1, -1, -1),
M( 1, -1,  1, -1, -1),   M(1, 0, -1,  0, -1),   M(-1, -1, 0, 6,  0),
M(-1,  2, -1,  0, -1),   M(0, 0, -1, -1, -1),   M(-1, -1, 3, 0,  0),
M(-1,  1,  1, -1, -1),   M(1, 0, -1, -1, -1),   M( 1, -1, 0, 0,  0),
M(-1, -1, -1, -1, -1),   M(1, 0, -1, -1,  1),   M(-1, -1, 0, 0,  0),
M( 2, -1, -1, -1, -1),   M(0, 0, -1, -1,  7),   M(-1, -1, 0, 0, 12),
M(-1,  2, -1, -1, -1),   M(0, 1, -1,  0, -1),   M(-1,  1, 0, 0,  0)
M(-1, -1, -1, -1, 19),   M(0, 1, -1, -1,  1),   in 5-d lattice M
M( 2, -1, -1, -1,  1),   M(0, 1, -1, -1, -1)
M(-1,  2, -1, -1,  1)    in 5-d lattice M
in 5-d lattice M
```

(3.59)

respectively. Then $\Delta = \Delta_1 + \Delta_2$ is a nef-partition and the dual nef-partition $\nabla = \nabla_1 + \nabla_2$ is defined by the polytopes $\nabla, \nabla_1, \nabla_2 \subset N_{\mathbb{R}}$ whose vertices are given by

```
N( 1,  0,  1,  0,  0),
N( 0,  1,  1,  0,  0),
N( 0,  0,  1,  0,  0),
N(-1, -2, -1,  0,  0),
N(-2, -1, -1,  0,  0),
N(-2, -2, -1,  0,  0),
N( 1,  0,  0,  1,  0),
N( 0,  1,  0,  1,  0),
N( 0,  0,  0,  1,  0),
N( 1,  0,  0,  0,  1),
N( 0,  1,  0,  0,  1),
N( 0,  0,  0,  0,  1),
N(-5, -6, -4, -2, -1),
N(-6, -5, -4, -2, -1),
N(-6, -6, -4, -2, -1),
N( 1,  0,  0,  0,  0),
N( 0,  1,  0,  0,  0)
in 5-d lattice N
```

```
N( 0,  0,  1,  0,  0),
N(-2, -2, -1,  0,  0),
N( 0,  0,  0,  1,  0),
N( 0,  0,  0,  0,  1),
N(-6, -6, -4, -2, -1),
N( 0,  0,  0,  0,  0)
in 5-d lattice N
```

```
N(1, 0, 0, 0, 0),
N(0, 1, 0, 0, 0),
N(0, 0, 0, 0, 0)
in 5-d lattice N
```

(3.60)

respectively. The dual polytopes Δ^* and ∇^* have vertices

```
N( 1,  0,  0,  0,  0),
N( 0,  1,  0,  0,  0),
N( 0,  0,  1,  0,  0),
N(-2, -2, -1,  0,  0),
N( 0,  0,  0,  1,  0),
N( 0,  0,  0,  0,  1),
N(-6, -6, -4, -2, -1)
in 5-d lattice N
```

```
M( 0,  0, -1,  3, -1),
M( 0,  0,  1, -1, -1),
M( 1,  0, -1,  0, -1),
M(-1, -1,  0,  6,  0),
M(-1, -1,  3,  0,  0),
M( 1, -1,  0,  0,  0),
M( 0,  0, -1, -1, -1),
M(-1, -1,  0,  0,  0),
M( 1,  0, -1, -1, -1),
M( 1,  0, -1, -1,  1),
M( 0,  0, -1, -1,  7),
M( 0,  1, -1,  0, -1),
M( 0,  1, -1, -1,  1),
M(-1, -1,  0,  0, 12),
M( 0,  1, -1, -1, -1),
M(-1,  1,  0,  0,  0)
in 5-d lattice M
```

(3.61)

respectively. The data defining the quotient construction of X_Δ is

$$S(\Sigma_\Delta) = \mathbb{C}[z_0, \ldots, z_6] \qquad B(\Sigma_\Delta) = \langle z_0 z_1 z_3, z_2 z_4 z_5 z_6 \rangle \qquad G(\Sigma_\Delta) \simeq (\mathbb{C}^*)^2 \tag{3.62}$$

where the weights of the action $G(\Sigma_\Delta) \simeq (\mathbb{C}^*)^2 \curvearrowright \mathbb{C}^7 \simeq \operatorname{Spec} S(\Sigma_\Delta)$ are given by

$$\begin{bmatrix} 2 & 2 & 1 & 1 & 0 & 0 & 0 \\ 6 & 6 & 4 & 0 & 2 & 1 & 1 \end{bmatrix} \tag{3.63}$$

The fan Σ_Δ is simplicial and has two singular cones of dimension two, ten singular cones of dimension three, seventeen singular cones of dimension four, and ten singular cones of dimension five. The toric variety X_{Σ_Δ} is thus an orbifold with three-dimensional singular locus.

The equations defining the Calabi-Yau threefold $Y_{\Delta_1,\Delta_2} \subset X_\Delta$ are

$$\begin{aligned} f_{\Delta_1} &= z_3^4 z_5^{12} + z_3^4 z_6^{12} + z_3^4 z_4^6 + z_2^3 z_3 + z_0^2 + z_1^2 \\ f_{\Delta_2} &= z_3^2 z_5^8 + z_3^2 z_6^8 + z_3^2 z_4^4 + z_0 z_5^2 + z_1 z_5^2 + z_0 z_6^2 + z_1 z_6^2 + z_2^2 + z_0 z_4 + z_1 z_4 \end{aligned} \tag{3.64}$$

The spectral sequence (2.44) computing the cohomology of $\mathscr{T}_{Y_{\Delta_1,\Delta_2}}$ is

$$\begin{array}{c|ccc} q & & & \\ 3 & \mathbb{C}^2 & 0 & 0 \\ 2 & 0 & \mathbb{C} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \mathbb{C}^2 \longrightarrow & \mathbb{C}^{63} \longrightarrow & \mathbb{C}^{168} \\ \hline & -1 & 0 & 1 \quad p \end{array} \tag{3.65}$$

which predicts $h^{11}(Y_{\Delta_1,\Delta_2}) = 3$ and $h^{21}(Y_{\Delta_1,\Delta_2}) = 97$.

However, the stringy E-polynomial of $\Delta_1 * \Delta_2$ is

$$E_{\mathrm{st}}(\Delta_1 * \Delta_2; u, v) = u^3 v^3 + 6\, u^2 v^2 - u^3 - 98\, u^2 v - 98\, u v^2 - v^3 + 6\, uv + 1 \tag{3.66}$$

so the stringy Hodge numbers of Y_{Δ_1,Δ_2} are $h^{11}_{\text{st}}(Y_{\Delta_1,\Delta_2}) = 6$ and $h^{21}_{\text{st}}(Y_{\Delta_1,\Delta_2}) = 98$.

The descrepancy between the stringy Hodge numbers of Y_{Δ_1,Δ_2} and the usual Hodge numbers of Y_{Δ_1,Δ_2} is can be remedied as follows. The equations (3.64) force Y_{Δ_1,Δ_2} to pass through the singular locus of X_Δ. The non-vertex lattice points on the boundary of Δ^* are

```
N(-3, -3, -2, -1, 0),
N(-1, -1,  0,  0, 0)
in 5-d lattice N
```
(3.67)

Including the cones generated by the points (3.67) in Σ_Δ and subdividing yields a fan $\widehat{\Sigma}_\Delta$ and a crepant partial resolution $X_{\widehat{\Sigma}_\Delta} \to X_\Delta$. This partial resolution restricts to a crepant resolution $\widehat{Y} \to Y_{\Delta_1,\Delta_2}$ where $\widehat{Y}$ is a smooth Calabi-Yau complete intersection in $X_{\widehat{\Sigma}}$. The spectral sequence (2.44) computing the cohomology of $\mathscr{T}_{\widehat{Y}}$ is

$$\begin{array}{c|ccc}
q & & & \\
3 & \mathbb{C}^4 & 0 & 0 \\
2 & 0 & \mathbb{C}^2 & 0 \\
1 & 0 & \mathbb{C} & 0 \\
0 & \mathbb{C}^4 \longrightarrow & \mathbb{C}^{65} \longrightarrow & \mathbb{C}^{168} \\
\hline
 & -1 & 0 & 1 \quad p
\end{array} \tag{3.68}$$

which gives $h^{11}(\widehat{Y}) = h^{11}_{\text{st}}(Y) = 6$ and $h^{21}(\widehat{Y}) = h^{21}_{\text{st}}(Y) = 98$, as expected.

3.5 Closed Form Expressions for Stringy Hodge Numbers of Codimension Two Threefolds

The formula for the E_{st}-polynomial of the Gorenstein polytope $\Delta_1 * \cdots * \Delta_c$ corresponding to a nef-partition $\Delta = \Delta_1 + \cdots + \Delta_c$ depends on the $\widetilde{S}$-polynomials of the faces of both $\Delta_1 * \cdots * \Delta_c$ and its dual $\nabla_1 * \cdots * \nabla_c$. If the divisors D_{Δ_i} are ample and $Y_{\Delta_1,\dots,\Delta_c}$ has dimension three, then it is possible to compute $h^{11}_{\mathrm{st}}(Y_{\Delta_1,\dots,\Delta_c})$ and $h^{21}_{\mathrm{st}}(Y_{\Delta_1,\dots,\Delta_c})$ without computing the dual nef-partition $\nabla = \nabla_1 + \cdots + \nabla_c$. We state this result in the case where $\dim \Delta = 5$ and $c = 2$.

For a lattice polytope $\theta \subset M_{\mathbb{R}}$, let $\ell(\theta) = |\theta \cap N|$ and let $\ell^*(\theta) = |\mathrm{relint}(\theta) \cap N|$.

Theorem 3.5.1 (Corollary 5.2 in [18]). *Let $\Delta = \Delta_1 + \Delta_2$ be a nef-partition of a five-dimensional reflexive polytope $\Delta \subset M_{\mathbb{R}}$ such that the divisors D_{Δ_1} and D_{Δ_2} on X_Δ are ample. Then*

$$\begin{aligned} h^{11}_{\mathrm{st}}(Y_{\Delta_1,\Delta_2}) &= \ell(\Delta^*) - 6 - \sum_{\dim\theta=0} \ell^*(\theta^\vee) - \sum_{\dim\theta=1} \ell^*(\theta^\vee) \\ &\qquad + \sum_{\dim\theta=2} \ell^*(\theta^\vee)\cdot\big[\ell^*(\theta) - \ell^*(\theta_1) - \ell^*(\theta_2)\big] \\ h^{21}_{\mathrm{st}}(Y_{\Delta_1,\Delta_2}) &= \big[\ell^*(2\Delta_1+\Delta_2) - \ell^*(2\Delta_1) + \ell^*(\Delta_1+2\Delta_2) - \ell^*(2\Delta_2)\big] - 7 \\ &\qquad - \sum_{\dim\theta=4} \big[\ell^*(\theta) - \ell^*(\theta_1) - \ell^*(\theta_2)\big] \\ &\qquad\quad + \sum_{\dim\theta=3} \ell^*(\theta^\vee)\cdot\big[\ell^*(\theta) - \ell^*(\theta_1) - \ell^*(\theta_2)\big] \end{aligned} \tag{3.69}$$

where the sums are taken over the faces of Δ of the indicated dimensions, $\theta^\vee$ is the face of Δ^* dual to θ, and $\theta = \theta_1 + \theta_2$ is the decomposition in to Minkowski sum with θ_i being a face of Δ_i.

Computing $h^{11}_{\mathrm{st}}(Y_{\Delta_1,\Delta_2})$ and $h^{21}_{\mathrm{st}}(Y_{\Delta_1,\Delta_2})$ using (3.69) is generally faster than using the E_{st}-polynomial.

4

Non-Reflexive Mirror Symmetry

The Batyrev-Borisov construction produces a stringy mirror pair of Calabi-Yau complete intersections $(Y_{\Delta_1,\ldots,\Delta_c}, Y_{\nabla_1,\ldots,\nabla_c})$ in Gorenstein Fano toric varieties (X_Δ, X_∇). In Sections 2.5.1 and 2.5.2, however, we saw two Calabi-Yau complete intersections in *non-Gorenstein* toric varieties. In this Chapter, we present methods and motivations for constructing Calabi-Yau varieties that cannot be constructed Batyrev-Borisov method from Chapter 3.

We begin in §4.1 with the notions of $\mathbb{Q}$-*reflexive* and *almost reflexive* polytopes. These polytopes generalize the notion of a reflexive polytope and are developed in Mavlyutov's preprint [34]. In §4.1.2, we show that the notion of a nef-partition of a reflexive polytope can be generalized to the notion of a $\mathbb{Q}$-*nef-partition of a* $\mathbb{Q}$-*reflexive polytope*. This allows one to construct Calabi-Yau complete intersections in Fano toric varieties. We also demonstrate how $\mathbb{Q}$-nef-partitions occur in dual pairs. It is conjectured that the Calabi-Yau varieties coming from a dual pair of $\mathbb{Q}$-nef-partitions are stringy mirror pairs.

In §4.2, we motivate our interest in non-reflexive polytopes by reviewing the notion of an *extremal transition*. It has long been conjectured that any two smooth

Calabi-Yau threefolds are connected by an extremal-transition. In §4.2.1, we show how the extremal transitions coming from an inclusion of reflexive polytopes generalizes to the setting of $\mathbb{Q}$-nef-partitions. In §4.2.2, we offer an example of an extremal transition from a reflexive model to a $\mathbb{Q}$-reflexive one.

Finally, in §4.4, we review the notion of a *general mirror pair* developed by Aspinwall and Plesser [2]. We prove that an 'almost reflexive' Gorenstein cone of Mavlyutov corresponds to a general mirror pair in Theorem 4.4.3, though in Example 4.4.4 we demonstrate that Aspinwall and Plesser's general mirror pairs are indeed more general than Mavlyutov's almost reflexive Gorenstein cones.

4.1 Mavlyutov's Almost Reflexive Mirror Pairs

Recall that a dual pair of nef-partitions $\Delta = \Delta_1 + \cdots + \Delta_c$ and $\nabla = \nabla_1 + \cdots + \nabla_c$ define a mirror pair of Calabi-Yau complete intersections $Y_{\Delta_1,\ldots,\Delta_c} \subset X_\Delta$ and $Y_{\nabla_1,\ldots,\nabla_c} \subset X_\nabla$ where X_Δ and X_∇ are Gorenstein Fano toric varieties. In [34], Mavlyutov introduces a combinatorial duality that generalizes the duality of nef-partitions of reflexive polytopes. This gives a method for constructing Calabi-Yau complete intersections in *non-Gorenstein* toric varieties. It is conjectured that the Calabi-Yau complete intersections arising from these generalized nef-partitions are stringy mirror pairs.

In this section, we present Mavlyutov's construction from [34]. We offer examples throughout our exposition.

4.1.1 $\mathbb{Q}$-Reflexive Polytopes and Almost Reflexive Polytopes

Let (N, M) be a pair of dual torsion-free lattices of rank d. We begin by introducing three operators. Recall from Definition 2.4.1 that a polytope $\Delta \subset M_\mathbb{R}$ *has the IP-property* if $0_M \in \operatorname{int} \Delta$.

Definition 4.1.1. For a polytope $\Delta \subset M_\mathbb{R}$ we define $[\Delta]$ as the (possibly empty) polytope $[\Delta] = \mathrm{Conv}(\Delta \cap M) \subset M_{\mathbb{R}}$. An IP-polytope Δ is *IP-confined* if $[\Delta]$ is a lattice polytope. If Δ is IP-confined, then we define $\Delta^\circ \subset N_{\mathbb{R}}$ and $\Delta^\bullet \subset N_{\mathbb{R}}$ as

$$\Delta^\circ = [\Delta]^* = \big(\mathrm{Conv}(\Delta \cap M)\big)^* \qquad \Delta^\bullet = [\Delta^*] = \mathrm{Conv}(\Delta^* \cap N) \tag{4.1}$$

If $\Delta \subset N_{\mathbb{R}}$, then $[\Delta] \subset N_{\mathbb{R}}$, $\Delta^\circ \subset M_{\mathbb{R}}$, and $\Delta^\bullet \subset M_{\mathbb{R}}$ are defined similarly.

Note that if Δ is a lattice polytope, then $\Delta^\circ = \Delta^*$ and if Δ^* is a lattice polytope, then $\Delta^\bullet = \Delta^*$. In particular, if Δ is reflexive, then $\Delta^\circ = \Delta^\bullet = \Delta^*$.

We are particularly interested in polytopes left invariant under repeated applications of the operators $(-)^\circ$ and $(-)^\bullet$.

Definition 4.1.2. An IP-confined polytope Δ in $M_{\mathbb{R}}$ is $\mathbb{Q}$-*reflexive* if

$$\Delta^* = \mathrm{Conv}\Big(\big(\mathrm{Conv}(\Delta \cap M)\big)^* \cap N\Big) \tag{4.2}$$

or, equivalently, if $\Delta = (\Delta^\circ)^\circ$.

The following proposition lists several of the basic properties of $\mathbb{Q}$-reflexive polytopes.

Proposition 4.1.3. *Let $\Delta \subset M_{\mathbb{R}}$ be an IP-confined polytope.*

(*a*) *If Δ is reflexive, then Δ is $\mathbb{Q}$-reflexive.*

(*b*) *If Δ is $\mathbb{Q}$-reflexive, then $\Delta^* \subset N_{\mathbb{R}}$ is a lattice polytope.*

(*c*) *If Δ is $\mathbb{Q}$-reflexive, then Δ has rational vertices.*

(*d*) *The map $\Delta \mapsto \Delta^\circ$ is an involution on the collection of $\mathbb{Q}$-reflexive polytopes.*

If $\Delta \subset M_{\mathbb{R}}$ is $\mathbb{Q}$-reflexive, then Proposition 4.1.3 (d) implies that $\Delta^\circ \subset N_{\mathbb{R}}$ is also $\mathbb{Q}$-reflexive. Thus we refer to the pair (Δ, Δ°) as a *dual pair of* $\mathbb{Q}$-*reflexive polytopes.*

Definition 4.1.4. An IP-confined polytope $\Delta \subset N_{\mathbb{R}}$ is *almost reflexive* if

$$\Delta = \operatorname{Conv}\big(\big(\Delta^* \cap M\big)^* \cap N\big) \tag{4.3}$$

or, equivalently, if $\Delta = (\Delta^\bullet)^\bullet$.

Almost reflexive polytopes enjoy properties similar to those of $\mathbb{Q}$-reflexive polytopes.

Proposition 4.1.5. *Let $\Delta \subset N_{\mathbb{R}}$ be an IP-confined polytope.*

(*a*) *If Δ is reflexive, then Δ is almost reflexive.*

(*b*) *If Δ is almost reflexive, then Δ is a lattice polytope.*

(*c*) *The map $\Delta \mapsto \Delta^\bullet$ is an involution on the collection of almost reflexive polytopes.*

If $\Delta \subset N_{\mathbb{R}}$ is almost reflexive, then Proposition 4.1.5 (*c*) implies that $\Delta^\bullet \subset M_{\mathbb{R}}$ is also an almost reflexive polytope. Thus we refer to the pair $(\Delta, \Delta^\bullet)$ as a *dual pair of almost reflexive polytopes.*

The relationship between $\mathbb{Q}$-reflexive polytopes and almost reflexive polytopes is as follows.

Proposition 4.1.6. *A polytope $\Delta \subset M_{\mathbb{R}}$ is $\mathbb{Q}$-reflexive if and only if its dual $\Delta^* \subset N_{\mathbb{R}}$ is almost reflexive.*

Recall from Definition 2.4.3 that a *canonical Fano polytope* is a lattice polytope $\Delta \subset M_{\mathbb{R}}$ such that $\operatorname{int}(\Delta) \cap M = \{0_M\}$. Canonical Fano polytopes are precisely those polytopes that correspond to Fano toric varieties under the correspondence $\Delta \leftrightarrow X_{\Delta^*}$ (Proposition 2.4.7).

Proposition 4.1.7 (Corollary 1.17 in [34])**.** *Almost reflexive polytopes are canonical Fano polytopes. Hence, if $\Delta \subset N_{\mathbb{R}}$ is almost reflexive, then X_{Δ^*} is a Fano toric variety.*

Remark 4.1.8. It should be noted that we have adopted the nomenclature and notation from Mavlyutov's 2011 preprint [34]. The relation (4.3) defining an almost reflexive polytope, however, was studied by Kreuzer in 2008 [28]. Here, Kreuzer uses the term 'IP-closed' in place of 'almost reflexive'. ◊

We have seen that every reflexive polytope is an almost reflexive polytope. By a theorem of Skarke, the converse is true, if $d \leq 4$.

Theorem 4.1.9 (Theorem 3.1 in [40]). *Suppose* $\dim N_{\mathbb{R}} \leq 4$. *Then every almost reflexive polytope in* $N_{\mathbb{R}}$ *is reflexive.*

The following examples show Theorm 4.1.9 does not hold in dimension five.

Example 4.1.10. Let Δ_1 be the polytope whose vertices are given by

$\Delta_1^v =$

```
M_QQ( -1, -1, -1, -1, -1),
M_QQ(5/2, -1, -1, -1, -1),
M_QQ( -1,  6, -1, -1, -1),
M_QQ( -1, -1, -1, -1,  6),
M_QQ( -1, -1,  6, -1, -1),
M_QQ( -1, -1, -1,  6, -1)
in 5-d vector space M_QQ
```

$\Delta_1^{*v} =$

```
N( 1,  0,  0,  0,  0),
N( 0,  1,  0,  0,  0),
N( 0,  0,  1,  0,  0),
N( 0,  0,  0,  1,  0),
N( 0,  0,  0,  0,  1),
N(-2, -1, -1, -1, -1)
in 5-d lattice N
```

(4.4)

and let Δ_2 be the polytope whose vertices are given by

$\Delta_2^v =$

```
M_QQ(-1/2,-1/2,  -1,  -1,  -1),
M_QQ(   3,   3,  -1,  -1,  -1),
M_QQ(   3,  -1,   3,  -1,  -1),
M_QQ(   3,  -1,  -1,   3,  -1),
M_QQ(   0,  -1,   3,  -1,   2),
M_QQ(  -1,   0,  -1,  -1,  -1),
M_QQ(  -1,   0,  -1,  -1,   0),
M_QQ(   0,  -1,   3,  -1,  -1),
M_QQ(   0,  -1,   0,  -1,  -1),
M_QQ(   0,  -1,  -1,   3,   2),
M_QQ(   0,  -1,  -1,   3,  -1),
M_QQ(   0,  -1,  -1,   0,  -1),
M_QQ(  -1,   0,  -1,   2,  -1),
M_QQ(  -1,   3,  -1,  -1,   3),
M_QQ(  -1,   3,  -1,  -1,  -1),
M_QQ(  -1,   0,  -1,   2,   3),
M_QQ(  -1,   0,   2,  -1,   3),
M_QQ(  -1,   0,   2,  -1,  -1)
in 5-d vector space M_QQ
```

$\Delta_2^{*v} =$

```
N( 1,  0,  0,  0,  0),
N( 0,  1,  0,  0,  0),
N( 0,  0,  1,  0,  0),
N( 0,  0,  0,  1,  0),
N( 0,  0,  0,  0,  1),
N(-1,  1,  1,  1, -1),
N( 1,  1,  0,  0,  0),
N( 0, -1, -1, -1,  0)
in 5-d lattice N
```

(4.5)

Both Δ_1 and Δ_2 are $\mathbb{Q}$-reflexive. Since each has a non-lattice point as a vertex, each is not reflexive. The polytopes Δ_1^* and Δ_2^* are therefore almost reflexive but not reflexive.

Note that $X_{\Delta_1} = \mathbb{P}^5_{211111}$. The toric variety X_{Δ_2} is singular with exactly one orbifold point. $\Diamond$

The examples in Example 4.1.10 show that Theorem 4.1.9 does not hold in dimension five. That is, in dimension five, almost reflexive polytopes are not necessarily reflexive.

4.1.2 $\mathbb{Q}$-Nef Partitions of $\mathbb{Q}$-Reflexive Polytopes

In this section, we show how to generalize the notion of a nef-partition of a reflexive polytope to the setting of $\mathbb{Q}$-reflexive polytopes.

Definition 4.1.11. A $\mathbb{Q}$-*nef-partition of length* c of a $\mathbb{Q}$-reflexive polytope $\Delta \subset M_{\mathbb{R}}$ is a Minkowski sum

$$\Delta = \Delta_1 + \cdots + \Delta_c \tag{4.6}$$

satisfying

$$[\Delta] = [\Delta_1] + \cdots + [\Delta_c] \tag{4.7}$$

where each Δ_i is a polytope in $M_{\mathbb{R}}$ such that $0_M \in \Delta_i$.

Of course, if Δ is a reflexive polytope, then the equations (4.6) and (4.7) are equivalent. Hence a $\mathbb{Q}$-nef-partition of a reflexive polytope is a nef-partition.

Note that each of the summands Δ_i in (4.6) need not be full-dimensional and thus may have empty interior. A $\mathbb{Q}$-nef-partition is *proper* if each summand Δ_i satisfies $\dim \Delta_i > 0$.

It turns out that $\mathbb{Q}$-nef-partitions enjoy many properties similar to nef-partitions. In particular, $\mathbb{Q}$-nef-partitions occur in dual pairs.

Proposition 4.1.12 ([34]). *Let* $\Delta = \Delta_1 + \cdots + \Delta_c$ *be a* $\mathbb{Q}$*-nef-partition of a* $\mathbb{Q}$*-reflexive polytope* $\Delta \subset M_{\mathbb{R}}$*. For* $1 \leq j \leq c$ *put*

$$\nabla_j = \{y \in N_{\mathbb{R}} \,|\, x \in [\Delta_i] \Rightarrow \langle x, y\rangle \geq -\delta_{ij}, 1 \leq j \leq c\} \tag{4.8}$$

and let $\nabla = \nabla_1 + \cdots + \nabla_c \subset N_{\mathbb{R}}$*. Then*

(*a*) $\nabla = \nabla_1 + \cdots + \nabla_c$ *is a* $\mathbb{Q}$*-nef-partition of length* c*.*

(*b*) *The polytope* Δ_i *satisfies*

$$\Delta_i = \{x \in M_{\mathbb{R}} \,|\, y \in [\nabla_j] \Rightarrow \langle x, y\rangle \geq -\delta_{ij}, 1 \leq j \leq c\} \tag{4.9}$$

for each $1 \leq i \leq c$*.*

(*c*) *The dual polytopes* Δ^* *and* ∇^* *satisfy*

$$\Delta^* = \mathrm{Conv}([\nabla_1], \ldots, [\nabla_c]) \qquad \nabla^* = \mathrm{Conv}([\Delta_1], \ldots, [\Delta_c]) \tag{4.10}$$

(*d*) *The dual* $\mathbb{Q}$*-reflexive polytopes* Δ° *and* ∇° *satisfy*

$$\Delta^\circ = \operatorname{Conv}(\nabla_1, \ldots, \nabla_c) \qquad\qquad \nabla^\circ = \operatorname{Conv}(\Delta_1, \ldots, \Delta_c) \tag{4.11}$$

(*e*) *For each* $1 \le i \le c$*,* $\Delta_i = \{0_M\}$ *if and only if* $\nabla_i = \{0_N\}$*. Consequently, the* $\mathbb{Q}$*-nef-partition* $\Delta = \Delta_1 + \cdots + \Delta_c$ *is proper if and only if the* $\mathbb{Q}$*-nef-partition* $\nabla = \nabla_1 + \cdots + \nabla_c$ *is proper.*

(*f*) *For each nonzero vertex* $n \in [\nabla_i]$ *and each nonzero vertex* $m \in [\Delta_i]$*, we have*

$$\min_{x \in \Delta_j} \langle x, n \rangle = -\delta_{ij} \qquad\qquad \min_{y \in \nabla_j} \langle m, y \rangle = -\delta_{ij} \tag{4.12}$$

for $1 \le j \le c$*.*

(*g*) *The Cayley cones* $\mathcal{C}_{\Delta_1, \ldots, \Delta_c}$ *and* $\mathcal{C}_{\nabla_1, \ldots, \nabla_c}$ *satisfy*

$$\mathcal{C}^\vee_{\Delta_1, \ldots, \Delta_c} = \mathcal{C}_{[\nabla_1], \ldots, [\nabla_c]} \qquad\qquad \mathcal{C}^\vee_{\nabla_1, \ldots, \nabla_c} = \mathcal{C}_{[\Delta_1], \ldots, [\Delta_c]} \tag{4.13}$$

We refer to the $\mathbb{Q}$-nef-partitions $\Delta = \Delta_1 + \cdots + \Delta_c$ and $\nabla = \nabla_1 + \cdots + \nabla_c$ as a *dual pair of* $\mathbb{Q}$*-nef-partitions*. Proposition 4.1.12 (*f*) implies that the divisor D_{Δ_i} on the Fano toric variety X_Δ is given by

$$D_{\Delta_i} = \sum_{\rho \in \Sigma_\Delta(1)} -\min\langle \Delta_i, u_\rho \rangle D_\rho = \sum_{u_\rho \in \nabla_i} D_\rho \tag{4.14}$$

and similarly $D_{\nabla_i} = \sum_{u_\rho \in \Delta_i} D_\rho$.

Definition 4.1.13. A $\mathbb{Q}$-*nef* divisor on a complete variety X is a $\mathbb{Q}$-Cartier divisor $D \in \operatorname{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $D \cdot C \ge 0$ for all irreducible curves $C \subset X$.

Proposition 4.1.14 (Corollary 4.4 in [34]). *Let* $\Delta \subset M_{\mathbb{R}}$ *be a canonical Fano polytope and suppose* $\Delta = \Delta_1 + \cdots + \Delta_c$ *is a Minkowski sum decomposition of* Δ *by rational polytopes. Then each of the divisors* D_{Δ_i} *on* X_Δ *is* $\mathbb{Q}$*-nef.*

Thus, for a $\mathbb{Q}$-nef-partition $\Delta = \Delta_1 + \cdots + \Delta_c$, the divisors D_{Δ_i} on X_Δ are $\mathbb{Q}$-nef.

Proposition 4.1.15 (§5 in [34]). *Let $\Delta = \Delta_1 + \cdots + \Delta_c$ be a proper $\mathbb{Q}$-nef-partition where* $\dim \Delta = d < c$. *Then a generic global section of $\mathscr{O}_{X_\Delta}(D_{\Delta_i})$ is of the form*

$$f_{\Delta_i} = \left(\sum_{m \in \Delta_i \cap M} a_{i,m} \prod_{u_\rho \in \Sigma_\Delta(1)} z_\rho^{\langle m, u_\rho \rangle} \right) \prod_{u_\rho \in \nabla_i} z_\rho \tag{4.15}$$

and a generic choice of these global sections defines a (possibly singular) $\mathbb{Q}$-nef Calabi-Yau complete intersection $Y_{\Delta_1,\ldots,\Delta_c}$ of codimension c in the Fano toric variety X_Δ.

We call the pair of $\mathbb{Q}$-nef Calabi-Yau varieties $(Y_{\Delta_1,\ldots,\Delta_c}, Y_{\nabla_1,\ldots,\nabla_c})$ corresponding to a dual pair of $\mathbb{Q}$-nef-partitions an *almost-reflexive stringy mirror pair.* Mavlyutov concludes his preprint [34] by conjecturing that almost-reflexive stringy mirror pairs pass the stringy topological mirror symmetry test

$$h_{\text{st}}^{p,q}(Y_{\Delta_1,\ldots,\Delta_c}) = h_{\text{st}}^{d-c-p,q}(Y_{\nabla_1,\ldots,\nabla_c}) \tag{4.16}$$

for $0 \leq p, q \leq d - c$.

One notable question, however, remains: How should one *define* the stringy Hodge numbers of an almost reflexive stringy mirror pair? In this dissertation, we propose a definition of the stringy Hodge numbers of almost reflexive stringy mirror pairs, under which Mavlyutov's conjecture holds for every example known to us.

Example 4.1.16. Fix $M \simeq \mathbb{Z}^5$. Let Δ be the $\mathbb{Q}$-reflexive polytope from (4.4) in Example 4.1.10 and recall that $X_\Delta = \mathbb{P}^5_{211111}$. Let $\Delta_1, \Delta_2 \subset M_\mathbb{R}$ be the polytopes whose vertices are

```
M_QQ(1/2, -1,  0,  0,  0),      M(2, 0, -1, -1, -1),
M_QQ( -1,  2,  0,  0,  0),      M(0, 4, -1, -1, -1),
M_QQ( -1, -1,  0,  0,  0),      M(0, 0, -1, -1, -1),
M_QQ( -1, -1,  3,  0,  0),      M(0, 0,  3, -1, -1),      (4.17)
M_QQ( -1, -1,  0,  0,  3),      M(0, 0, -1, -1,  3),
M_QQ( -1, -1,  0,  3,  0)       M(0, 0, -1,  3, -1)
in 5-d vector space M_QQ        in 5-d lattice M
```

respectively. Then $\Delta = \Delta_1 + \Delta_2$ is a $\mathbb{Q}$-nef partition. The lattice polytopes $[\Delta], [\Delta_1], [\Delta_2] \subset M_{\mathbb{R}}$ have vertices

```
M(-1, -1, -1, -1, -1),
M( 2,  0, -1, -1, -1),
M( 2, -1,  0, -1, -1),
M( 2, -1, -1,  0, -1),
M( 2, -1, -1, -1,  0),
M( 2, -1, -1, -1, -1),
M(-1,  6, -1, -1, -1),
M(-1, -1, -1, -1,  6),
M(-1, -1,  6, -1, -1),
M(-1, -1, -1,  6, -1)
in 5-d lattice M
```

```
M(-1, -1, 0, 0, 0),
M( 0,  0, 0, 0, 0),
M( 0, -1, 1, 0, 0),
M(-1, -1, 0, 0, 3),
M( 0, -1, 0, 1, 0),
M( 0, -1, 0, 0, 1),
M( 0, -1, 0, 0, 0),
M(-1,  2, 0, 0, 0),
M(-1, -1, 3, 0, 0),
M(-1, -1, 0, 3, 0)
in 5-d lattice M
```

```
M(2, 0, -1, -1, -1),
M(0, 4, -1, -1, -1),
M(0, 0, -1, -1, -1),
M(0, 0,  3, -1, -1),
M(0, 0, -1, -1,  3),
M(0, 0, -1,  3, -1)
in 5-d lattice M
```

(4.18)

respectively. Generic global sections f_{Δ_i} of $\mathscr{O}_{X_\Delta}(D_{\Delta_i})$ are of the form

$$\begin{aligned} f_{\Delta_1} &= z_1^3 + z_2^3 + z_3^3 + z_4^3 + z_5^3 + z_0 z_1 + z_0 z_2 + z_0 z_3 + z_0 z_4 + z_0 z_5 \\ f_{\Delta_2} &= z_1^4 + z_2^4 + z_3^4 + z_4^4 + z_5^4 + z_0^2 \end{aligned} \tag{4.19}$$

That is, the $\mathbb{Q}$-nef Calabi-Yau complete intersection Y_{Δ_1,Δ_2} is a generic complete intersection of a cubic and a quartic in $\mathbb{P}^5_{211111}$. This is the example we studied in Section 2.5.1, where we computed $h^{11}(Y_{\Delta_1,\Delta_2}) = 1$ and $h^{21}(Y_{\Delta_1,\Delta_2}) = 79$.

We may also compute the dual $\mathbb{Q}$-nef partition. Let $\nabla, \nabla_1, \nabla_2 \subset N_{\mathbb{R}}$ be the polytopes with vertices

```
N_QQ(  -2,  -1,  -1,  -1,  -1),
N_QQ(  -2,   0,  -1,  -1,  -1),
N_QQ(  -1,  -1,  -1,  -1,  -1),
N_QQ(   1,   0,   1,   0,   0),
N_QQ(-1/2,   1,   0,   0,   0),
N_QQ(   1,   0,   0,   1,   0),
N_QQ(   0,   0,   0,   0,   1),
N_QQ(   0,   0,   0,   1,   0),
N_QQ(   0,   0,   1,   0,   0),
N_QQ(   0,   1,   0,   0,   0),
N_QQ(   0,   1,   0,   0,   1),
N_QQ(   0,   1,   0,   1,   0),
N_QQ(   0,   1,   1,   0,   0),
N_QQ(   1,   0,   0,   0,   0),
N_QQ(   1,   0,   0,   0,   1)
in 5-d vector space N_QQ
```

```
N(0, 0, 0, 0, 0),
N(0, 1, 0, 0, 0),
N(1, 0, 0, 0, 0)
in 5-d lattice N
```

```
N_QQ(  -2,  -1,  -1,  -1,  -1),
N_QQ(   0,   0,   0,   0,   1),
N_QQ(   0,   0,   0,   1,   0),
N_QQ(   0,   0,   1,   0,   0),
N_QQ(   0,   0,   0,   0,   0),
N_QQ(-1/2,   0,   0,   0,   0)
in 5-d vector space N_QQ
```

(4.20)

respectively. Then $\nabla = \nabla_1 + \nabla_2$ is a $\mathbb{Q}$-nef partition. The lattice polytopes $[\nabla], [\nabla_1], [\nabla_2] \subset N_{\mathbb{R}}$ have vertices

```
N(-2, -1, -1, -1, -1),
N(-2,  0, -1, -1, -1),
N(-1, -1, -1, -1, -1),
N( 1,  0,  1,  0,  0),
N( 0,  0,  0,  0,  1),
N( 0,  0,  0,  1,  0),
N( 0,  0,  1,  0,  0),
N( 0,  1,  0,  0,  0),
N( 0,  1,  0,  0,  1),
N( 0,  1,  0,  1,  0),
N( 0,  1,  1,  0,  0),
N( 1,  0,  0,  0,  0),
N( 1,  0,  0,  0,  1),
N( 1,  0,  0,  1,  0)
in 5-d lattice N
```

```
N(0, 0, 0, 0, 0),
N(0, 1, 0, 0, 0),
N(1, 0, 0, 0, 0)
in 5-d lattice N
```

```
N(-2, -1, -1, -1, -1),
N( 0,  0,  0,  0,  0),
N( 0,  0,  0,  0,  1),
N( 0,  0,  0,  1,  0),
N( 0,  0,  1,  0,  0)
in 5-d lattice N
```

(4.21)

respectively and the almost reflexive polytope ∇^* has vertices

```
M(-1, -1,  0,  0,  0),
M( 0,  0, -1,  3, -1),
M( 0,  0,  3, -1, -1),
M(-1, -1,  0,  0,  3),
M( 0,  0, -1, -1,  3),
M( 2,  0, -1, -1, -1),
M( 0,  0, -1, -1, -1),
M( 0, -1,  1,  0,  0),
M( 0, -1,  0,  1,  0),
M(-1, -1,  0,  3,  0),
M( 0, -1,  0,  0,  1),
M( 0, -1,  0,  0,  0),
M(-1,  2,  0,  0,  0),
M( 0,  4, -1, -1, -1),
M(-1, -1,  3,  0,  0)
in 5-d lattice M
```

(4.22)

Generic global sections f_{∇_i} of $\mathscr{O}_{X_\nabla}(D_{\nabla_i})$ are of the form

$$\begin{aligned} f_{\nabla_1} &= z_1z_3z_6z_7z_8z_9z_{11}z_{13}z_{14} + z_0^4z_{14}^3 + z_7z_8z_{11}z_{12}^2z_{13} \\ f_{\nabla_2} &= z_1^3z_4^4z_8 + z_7z_9^3z_{10}^4 + z_5^4z_6^3z_{11} + z_2^4z_3^3z_{13} + z_0z_2z_4z_5z_{10}z_{12} \end{aligned} \tag{4.23}$$

The Fano toric variety X_∇ has a three-dimensional singular locus and the $\mathbb{Q}$-nef Calabi-Yau complete intersection Y_{∇_1,∇_2} intersects this singular locus. Thus an analysis of the tangent sheaf $\mathscr{T}_{Y_{\nabla_1,\nabla_2}}$ as in (2.44) is impossible.

The canonical Fano polytope ∇^* has 125 lattice points. Inserting the rays generated by each of the 124 lattice points on the boundary of ∇^* in the fan Σ_∇ and subdividing yields a toric variety $X_{\widehat{\Sigma}_\nabla}$ whose singular locus consists of exactly one orbifold point. The partial crepant resolution $X_{\widehat{\Sigma}_\nabla} \to X_\nabla$ restricts to a smooth crepant resolution $\widehat{Y} \to Y_{\nabla_1,\nabla_2}$ where $\widehat{Y}$ is a smooth Calabi-Yau complete intersection in $X_{\widehat{\Sigma}_\nabla}$. An analysis of the tangent sheaf $\mathscr{T}_{\widehat{Y}}$ as in (2.44) is theoretically possible, but with 124 variables in the ring $S(\widehat{\Sigma}_\nabla)$, the local cohomology computations are hopeless. Thus an alternate description of the stringy Hodge numbers $h^{p,q}_{\text{st}}(Y_{\nabla_1,\nabla_2})$ is desirable. $\Diamond$

4.1.3 Almost Reflexive Gorenstein Polytopes and Almost Reflexive Gorenstein Cones

In Proposition 3.3.4, we saw that for a dual pair of nef-partitions $\Delta = \Delta_1 + \cdots + \Delta_c$ and $\nabla = \nabla_1 + \cdots + \nabla_c$, the Cayley cones $\mathcal{C}_{\Delta,\dots,\Delta_c}$ and $\mathcal{C}_{\nabla_1,\dots,\nabla_c}$ form a dual pair of reflexive Gorenstein cones of index c. In this section, we generalize this construction to the setting of a dual pair of $\mathbb{Q}$-nef-partitions.

Let (N, M) and $(\bar{N}, \bar{M})$ be dual pairs of torsion-free lattices of finite ranks d and $\bar{d}$ respectively.

Definition 4.1.17. Let Δ be a polytope in $M_\mathbb{R}$. Then Δ is an *almost reflexive Gorenstein polytope of index* c if $\operatorname{int}(c\Delta) \cap M = \{m\}$ for some nonnegative integer c and $c\Delta - m$ is almost reflexive.

Since reflexive polytopes are almost reflexive, we immediately see that reflexive Gorenstein polytopes are almost reflexive Gorenstein polytopes.

Definition 4.1.18. A Gorenstein cone $\sigma \subset \bar{M}_{\mathbb{R}}$ is *almost reflexive of index* c if the support polytope $\sigma_{(1)}$ is an almost reflexive Gorenstein polytope of index c when considered as a polytope in the affine hyperplane $H^{(1)}_{h_\sigma}$.

For an almost reflexive Gorenstein cone of index c, we denote the unique interior point of $c\sigma_{(1)}$ by $h_{\sigma^\vee}$. If σ is a reflexive Gorenstein cone, then this notation is consistent with the notation used for the degree vector of the dual Gorenstein cone $\sigma^\vee$.

Definition 4.1.19. For an almost reflexive Gorenstein cone $\sigma \subset \bar{M}_{\mathbb{R}}$, let

$$\sigma^\vee_{(r)} = \sigma^\vee \cap H^{(r)}_{h_{\sigma^\vee}} = \{y \in \sigma^\vee \mid \langle h_{\sigma^\vee}, y\rangle = r\} \subset \bar{N}_{\mathbb{R}} \tag{4.24}$$

and let

$$\sigma^\bullet = \operatorname{Cone}\big(\operatorname{Conv}(\sigma^\vee \cap H^{(1)}_{h_{\sigma^\vee}} \cap \bar{N})\big) \subset \bar{N}_{\mathbb{R}} \tag{4.25}$$

If $\sigma \subset \bar{N}_{\mathbb{R}}$ is an almost reflexive Gorenstein cone, then $\sigma^\vee_{(r)} \subset \bar{M}_{\mathbb{R}}$ and $\sigma^\bullet \subset \bar{M}_{\mathbb{R}}$ are defined similarly.

The operator $(-)^\bullet$ extends the duality of reflexive Gorenstein cones to almost reflexive Gorenstein cones.

Proposition 4.1.20 (§3 in [34])**.** *Let $\sigma \subset \bar{M}_{\mathbb{R}}$ be an almost reflexive Gorenstein cone. Then $\sigma^\bullet \subset \bar{N}_{\mathbb{R}}$ is almost reflexive and $(\sigma^\bullet)^\bullet = \sigma$.*

Thus almost reflexive Gorenstein cones occur in dual pairs $(\sigma, \sigma^\bullet)$.

Proposition 4.1.21 (Proposition 3.13 in [34])**.** *Let $\Delta = \Delta_1 + \cdots + \Delta_c \subset M_{\mathbb{R}}$ and $\nabla = \nabla_1 + \cdots + \nabla_c \subset N_{\mathbb{R}}$ be a dual pair of $\mathbb{Q}$-nef-partitions. Then $\mathcal{C}_{[\Delta_1],\ldots,[\Delta_c]}$ and $\mathcal{C}_{[\nabla_1],\ldots,[\nabla_c]}$ are almost reflexive Gorenstein cones of index c and*

$$\mathcal{C}^\bullet_{[\Delta_1],\ldots,[\Delta_c]} = \mathcal{C}_{[\nabla_1],\ldots,[\nabla_c]} \tag{4.26}$$

That is, $\mathcal{C}_{[\Delta_1],\ldots,[\Delta_c]}$ and $\mathcal{C}_{[\nabla_1],\ldots,[\nabla_c]}$ form a dual pair of almost reflexive Gorenstin cones of index c.

Example 4.1.22. Fix $M \simeq \mathbb{Z}^5$ and let $\Delta = \Delta_1 + \cdots + \Delta_c$ and $\nabla = \nabla_1 + \cdots + \nabla_c$ be the dual pair of $\mathbb{Q}$-nef-partitions from Example 4.1.16. Then the Cayley cones $\mathcal{C}_{[\Delta_1],\ldots,[\Delta_c]}$ and $\mathcal{C}_{[\nabla_1],\ldots,[\nabla_c]}$ are generated by the rays

```
ZZ^2+M(0, 1,  0,  0, -1, -1, -1),
ZZ^2+M(1, 0,  0,  0,  0,  0,  0),
ZZ^2+M(1, 0,  0, -1,  1,  0,  0),
ZZ^2+M(1, 0,  0, -1,  0,  1,  0),
ZZ^2+M(0, 1,  0,  0, -1, -1,  3),
ZZ^2+M(1, 0,  0, -1,  0,  0,  1),
ZZ^2+M(1, 0,  0, -1,  0,  0,  0),
ZZ^2+M(1, 0, -1,  2,  0,  0,  0),
ZZ^2+M(1, 0, -1, -1,  3,  0,  0),
ZZ^2+M(1, 0, -1, -1,  0,  3,  0),
ZZ^2+M(1, 0, -1, -1,  0,  0,  3),
ZZ^2+M(1, 0, -1, -1,  0,  0,  0),
ZZ^2+M(0, 1,  2,  0, -1, -1, -1),
ZZ^2+M(0, 1,  0,  4, -1, -1, -1),
ZZ^2+M(0, 1,  0,  0, -1,  3, -1),
ZZ^2+M(0, 1,  0,  0,  3, -1, -1)
in 7-d lattice ZZ^2+M
```

```
ZZ^2+N(0, 1, -2, -1, -1, -1, -1),
ZZ^2+N(0, 1,  0,  0,  0,  0,  0),
ZZ^2+N(0, 1,  0,  0,  0,  0,  1),
ZZ^2+N(0, 1,  0,  0,  0,  1,  0),
ZZ^2+N(0, 1,  0,  0,  1,  0,  0),
ZZ^2+N(1, 0,  0,  0,  0,  0,  0),
ZZ^2+N(1, 0,  0,  1,  0,  0,  0),
ZZ^2+N(1, 0,  1,  0,  0,  0,  0)
in 7-d lattice ZZ^2+N
```

(4.27)

respectively. The cones $\mathcal{C}_{[\Delta_1],\ldots,[\Delta_c]}$ and $\mathcal{C}_{[\nabla_1],\ldots,[\nabla_c]}$ form a dual pair of almost reflexive Gorenstein cones by Proposition 4.1.21. ◊

It is worth noting that some useful features of the duality $(-)^\vee$ between reflexive Gorenstein cones are lost when extending to the duality $(-)^\bullet$ between almost reflexive Gorenstein cones. For example, the f-vectors of the Cayley polytopes $[\Delta_1] * [\Delta_2]$ and $[\nabla_1] * [\nabla_2]$ from Example 4.1.22 are

$$f_{[\Delta_1]*[\Delta_2]} = (1,\ 16,\ 50,\ 75,\ 65,\ 33,\ 9,\ 1) \quad f_{[\nabla_1]*[\nabla_2]} = (1,\ 8,\ 27,\ 50,\ 55,\ 36,\ 12,\ 1) \tag{4.28}$$

Unlike the reflexive situation, the two vectors in (4.28) are not reverses of each other.

4.2 Extremal Transitions

Let Y be a smooth Calabi-Yau threefold and $\phi : Y \to \overline{Y}$ be a *birational contraction* onto a normal variety $\overline{Y}$. If there exists a complex deformation (or *smoothing*) of $\overline{Y}$ to a smooth Calabi-Yau threefold $\widetilde{Y}$, then the process of going from Y to $\widetilde{Y}$ is called an *extremal transition.* We denote an extremal transition symbolically by a diagram

$$Y \xrightarrow{\phi} \overline{Y} \leftrightsquigarrow \widetilde{Y}, \qquad Y \overset{T}{\dashrightarrow} \widetilde{Y} \tag{4.29}$$

It has long been thought that any two smooth Calabi-Yau threefolds can be connected by extremal transitions. Kreuzer and Skarke have verified this claim for threefolds appearing as hypersurfaces in Gorenstein Fano toric varieties by showing that any pair of the $473, 800, 776$ four-dimensional reflexive polytopes can be connected by a chain of inclusions. It has also been conjectured by Morrision [35] that if X and Y are smooth Calabi-Yau threefolds connected by an extremal transition $X \dashrightarrow Y$, then their mirrors are connected by an extremal transition $Y^\vee \dashrightarrow X^\vee$.

In this section, we demonstrate that, by following extremal transitions, one is forced to consider Calabi-Yau complete intersections in non-Gorenstein toric varieties.

4.2.1 Toric Extremal Transitions

Following [16, Example 6.2.4.2], let $\Delta, \tilde{\Delta} \subset M_{\mathbb{R}}$ be $\mathbb{Q}$-reflexive polytopes. Suppose

$$\Delta = \Delta_1 + \cdots + \Delta_c \qquad\qquad \tilde{\Delta} = \tilde{\Delta}_1 + \cdots + \tilde{\Delta}_c \tag{4.30}$$

are two proper $\mathbb{Q}$-nef-partitions with $\Delta_i \subset \tilde{\Delta}_i$. Then the $\mathbb{Q}$-reflexive polytopes Δ and $\tilde{\Delta}$ satisfy $\Delta \subset \tilde{\Delta}$. Taking duals, we see that the almost reflexive polytopes Δ^* and $\tilde{\Delta}^*$ satisfy $\tilde{\Delta}^* \subset \Delta^*$, so any subdivison $\widehat{\Sigma}_{\tilde{\Delta}}$ of $\Sigma_{\tilde{\Delta}}$ can be refined to a subdivision $\widehat{\Sigma}_{\Delta}$ of Σ_{Δ}. Such subdivisions yield a morphism $\pi : X_{\widehat{\Sigma}_{\Delta}} \to X_{\widehat{\Sigma}_{\tilde{\Delta}}}$. The Calabi-Yau complete intersections $\widehat{Y}_{\Delta_1,\dots,\Delta_c} \subset X_{\widehat{\Sigma}_\Delta}$ and $\widehat{Y}_{\tilde{\Delta}_1,\dots,\tilde{\Delta}_c} \subset X_{\widehat{\Sigma}_{\tilde{\Delta}}}$ are defined by the equations

$$f_{\Delta_i} = \left(\sum_{m \in \Delta_i \cap M} a_{i,m} \prod_{u_\rho \in \widehat{\Sigma}_\Delta(1)} z_\rho^{\langle m, u_\rho \rangle} \right) \prod_{u_\rho \in \nabla_i} z_\rho \tag{4.31}$$

and

$$f_{\tilde{\Delta}_i} = \left(\sum_{m \in \tilde{\Delta}_i \cap M} b_{i,m} \prod_{u_\rho \in \widehat{\Sigma}_{\tilde{\Delta}}(1)} z_\rho^{\langle m, u_\rho \rangle} \right) \prod_{u_\rho \in \tilde{\nabla}_i} z_\rho \tag{4.32}$$

respectively.

Put $Y = \widehat{Y}_{\Delta_1,\dots,\Delta_c}$ and $\widetilde{Y} = \widehat{Y}_{\tilde{\Delta}_1,\dots,\tilde{\Delta}_c}$. Then $\phi = \pi|_{\widehat{Y}_{\Delta_1,\dots,\Delta_c}}$ gives a birational contraction $\phi : Y \to \overline{Y}$ where $\overline{Y} \subset X_{\widehat{\Sigma}_{\tilde{\Delta}}}$ is defined by the equations in (4.32), except that $b_{i,m} = 0$ for $m \in (\tilde{\Delta}_i \setminus \Delta_i) \cap M$. The extremal transition is completed by smoothing $\overline{Y}$ to $\widetilde{Y}$ by allowing arbitrary values for each of the $b_{i,m}$ in (4.32). The process of going from $Y = \widehat{Y}_{\Delta_1,\dots,\Delta_c}$ to $\widetilde{Y} = \widehat{Y}_{\tilde{\Delta}_1,\dots,\tilde{\Delta}_c}$ is called a *toric extremal transition.*

Now, by Proposition 4.1.12, the containment $\Delta_i \subset \tilde{\Delta}_i$ implies that $\tilde{\nabla}_i \subset \nabla_i$, where

$$\nabla = \nabla_i + \cdots + \nabla_c \qquad\qquad \tilde{\nabla} = \tilde{\nabla}_i + \cdots + \tilde{\nabla}_c \tag{4.33}$$

are the $\mathbb{Q}$-nef-partitions dual to the $\mathbb{Q}$-nef-partitions in (4.30). Thus we may repeat the above construction to obtain an extremal transition $\widehat{Y}_{\tilde{\nabla}_1,\dots,\tilde{\nabla}_c} \dashrightarrow \widehat{Y}_{\nabla_1,\dots,\nabla_c}$. Since we expect $(\widehat{Y}_{\Delta_1,\dots,\Delta_c})^\vee = \widehat{Y}_{\nabla_1,\dots,\nabla_c}$ and $(\widehat{Y}_{\tilde{\Delta}_1,\dots,\tilde{\Delta}_c})^\vee = \widehat{Y}_{\nabla_1,\dots,\nabla_c}$, this gives Morrison's expected 'mirror transition' [35].

4.2.2 Transitioning Between Reflexive and Non-Reflexive Mirror Pairs

We now demonstrate that by following extremal transitions, one is forced to consider $\mathbb{Q}$-nef-partitions. Fix $M \simeq \mathbb{Z}^5$ and let $\Delta = \Delta_1 + \Delta_2$ be the $\mathbb{Q}$-nef-partition from Example 4.1.16. Recall that $X_\Delta = \mathbb{P}^5_{211111}$ and Y_{Δ_1,Δ_2} is a generic complete intersection of a cubic and a quartic in $\mathbb{P}^5_{211111}$, which we have shown to have Hodge numbers $h^{1,1}(Y_{\Delta_1,\Delta_2}) = 1$ and $h^{2,1}(Y_{\Delta_1,\Delta_2}) = 79$.

Now, let

$$\tilde{\Delta}_1 = \mathrm{Conv}(\Delta_1 \cap M \setminus \{M(2,0,-1,-1,-1)\}) \tag{4.34}$$

and let $\tilde{\Delta}_2 = \Delta_2$. Then $\tilde{\Delta} = \Delta_1 + \Delta_2$ is a nef-partition and the dual polytope $\tilde{\Delta}^*$ has vertices

```
N( 1,  0,  0,  0,  0),
N(-1,  0,  0,  0,  0),
N( 0,  1,  0,  0,  0),
N( 0,  0,  1,  0,  0),
N( 0,  0,  0,  1,  0),
N( 0,  0,  0,  0,  1),
N( 2, -1, -1, -1, -1)
in 5-d lattice N
```
(4.35)

The toric variety $X_{\tilde{\Delta}}$ is a resolution of $\mathbb{P}^5_{211111}$ and one easily shows that $Y_{\tilde{\Delta}_1,\tilde{\Delta}_2}$ is a smooth Calabi-Yau threefold with $h^{1,1}(Y_{\tilde{\Delta}_1,\tilde{\Delta}_2}) = 2$ and $h^{2,1}(Y_{\tilde{\Delta}_1,\tilde{\Delta}_2}) = 78$.

This gives an extremal transition $Y_{\Delta_1,\Delta_2} \dashrightarrow Y_{\tilde{\Delta}_1,\tilde{\Delta}_2}$ where Y_{Δ_1,Δ_2} is a Calabi-Yau threefold arising from a $\mathbb{Q}$-nef-partition while $Y_{\tilde{\Delta}_1,\tilde{\Delta}_2}$ is a Calabi-Yau threefold arising from a nef-partition. By Section 4.2.1 above, we obtain another extremal transition $\widehat{Y}_{\tilde{\nabla}_1,\tilde{\nabla}_2} \dashrightarrow \widehat{Y}_{\nabla_1,\nabla_2}$ where $\widehat{Y}_{\tilde{\nabla}_1,\tilde{\nabla}_2}$ is a resolution of a Calabi-Yau threefold arising from a nef-partition and $\widehat{Y}_{\nabla_1,\nabla_2}$ is a resolution of a Calabi-Yau threefold arising from a $\mathbb{Q}$-nef-partition.

4.3 Partial Classifications of Five Dimensional Almost Reflexive Polytopes

Now that We have seen that it is necessary to consider $\mathbb{Q}$-nef-partitions if one is to follow extremal transitions, it is interesting to classify the Calabi-Yau complete intersections arising from them. By Theorem 4.1.9, there are no examples of $\mathbb{Q}$-nef-partitions of non-reflexive polytopes in dimension $d \leq 4$. In dimension $d \geq 5$, however, there is much work to be done. In Example 4.1.16, we saw a $\mathbb{Q}$-nef-partition $\Delta = \Delta_1 + \Delta_2$ corresponding to the Calabi-Yau threefold $Y_{\Delta_1,\Delta_2} = \mathbb{P}^5_{211111}[3,4]$ where Δ was not reflexive. Thus, $\mathbb{Q}$-nef-partitions of non-reflexive polytopes exist but, to our knowledge, the example $\mathbb{P}^5_{211111}[3,4]$ is the only example known in the literature.

In this section, we partially classify the $\mathbb{Q}$-reflexive polytopes in dimension five. Recall that a polytope $\Delta \subset M_{\mathbb{R}}$ is $\mathbb{Q}$-reflexive if and only if its dual Δ^* is almost reflexive. Our strategy for generating $\mathbb{Q}$-reflexive polytopes is thus to search for almost reflexive subpolytopes of known reflexive polytopes. We conduct our search within two well-known lists of reflexive polytopes: the complete list of 866 five-dimensional smooth Fano polytopes[1] and Kreuzer's partial list[2] of 7661 reflexive polytopes admitting nef-partitions with Picard numbers ≤ 6. By utilizing the a normal form for lattice polytopes described in [21, §3.4], we find 48 distinct isomorphism classes of almost reflexve polytopes embedded in a smooth Fano polytope and we find 765 distinct isomorphism classes of almost reflexive polytopes embedded in a reflexive polytope on Kreuzer's list. In total, we produce 775 new examples of five-dimensional $\mathbb{Q}$-reflexive polytopes that are not reflexive.

For each of our almost reflexive polytopes Δ^*, we compute all dual pairs of $\mathbb{Q}$-nef-partitions $\Delta = \Delta_1 + \Delta_2$ and $\nabla = \nabla_1 + \nabla_2$ of length two. This classifies all possible $\mathbb{Q}$-nef Calabi-Yau complete intersection threefolds in the Fano toric varieties X_Δ and X_∇. In total, we produce 1077 distinct isomorphism classes of dual pairs of $\mathbb{Q}$-nef-partitions of non-reflexive polytopes. This provides us with a library of examples to explore. To our knowledge, these are the only known $\mathbb{Q}$-nef-partitions of nonreflexive polytopes. The data resulting from our computations can be found in the data supplements to this dissertation at [20].

[1] The classification of all 866 five-dimensional smooth Fano polytopes was done by Øbro in [37] and the list of these polytopes can be found at [29].

[2] This list can be found at [27]

It should be noted that, since $\mathbb{Q}$-reflexive polytopes are generalizations of reflexive polytopes, a complete classification of $\mathbb{Q}$-reflexive polytopes in dimension $d \geq 5$ is currently incalculable. In [28] Kreuzer estimates there to be $6.5 \cdot 10^{14}$ reflexive polytopes in dimension five. The number of $\mathbb{Q}$-reflexive polytopes in dimension five is likely far greater.

4.3.1 Almost Reflexive Polytopes Embedded in a Smooth Fano Polytope

Let (M, N) be a dual pair of torsion-free lattices of rank d. A lattice polytope $\Delta^* \subset N_{\mathbb{R}}$ is a *smooth Fano polytope* if $0_N \in \text{int}(\Delta)$ and the vertices of every facet of Δ^* form a $\mathbb{Z}$-basis of N. Equivalently, Δ^* is a smooth Fano polytope if X_Δ is a smooth Fano toric variety.

Now, fix $d = 5$. In [37], it is shown that there are exactly 866 isomorphism classes of smooth Fano polytopes in dimension five. Each smooth Fano polytope $\Delta \subset N_{\mathbb{R}}$ is reflexive and terminal, meaning that $\partial\Delta \cap N$ consists only of vertex points. The set $\{6, 7, \ldots, 14\}$ consists of the number of possible vertices of a smooth Fano polytope $\Delta \subset N_{\mathbb{R}}$ and, consequently, the set $\{7, 8, \ldots, 15\}$ consists of the number of possible lattice points in Δ.

Consider a reflexive polytope $\tilde{\Delta} \subset M_{\mathbb{R}}$ whose dual $\tilde{\Delta}^* \subset N_{\mathbb{R}}$ is a smooth Fano polytope. Let $\tilde{\Delta} = \tilde{\Delta}_1 + \tilde{\Delta}_2$ be a nef-partition of $\tilde{\Delta}$, which corresponds to a Calabi-Yau complete intersection threefold $Y_{\tilde{\Delta}_1, \tilde{\Delta}_2}$ in the five-dimensional smooth Fano toric variety $X_{\tilde{\Delta}}$. By §4.2.1 above, a toric extremal transition $Y_{\tilde{\Delta}_1, \tilde{\Delta}_2} \dashrightarrow Y_{\Delta_1, \Delta_2}$ from the Calabi-Yau threefold $Y_{\tilde{\Delta}_1, \tilde{\Delta}_2}$ to a $\mathbb{Q}$-nef Calabi-Yau complete intersection Y_{Δ_1, Δ_2} corresponding to a $\mathbb{Q}$-nef-partition $\Delta = \Delta_1 + \Delta_2$ can only be obtained if $\tilde{\Delta} \subset \Delta$. Thus, to classify all toric extremal transitions $Y_{\tilde{\Delta}_1, \tilde{\Delta}_2} \dashrightarrow Y_{\Delta_1, \Delta_2}$, we must classify all $\mathbb{Q}$-reflexive polytopes Δ containing $\tilde{\Delta}$. By taking duals, this is equivalent to classifying all almost reflexive polytopes Δ^* contained in $\tilde{\Delta}^*$. Since the maximum possible number of lattice points in $\tilde{\Delta}^*$ is 15, this is a computationally reasonable task. The following simple algorithm produces our desired almost reflexive polytopes.

Algorithm 4.3.1. Let $\tilde{\Delta}^* \subset N_{\mathbb{R}}$ be a five-dimensional smooth Fano polytope. For each proper subset $S \subset \{\text{vertices of } \tilde{\Delta}^*\}$ with $|S| \geq 6$, let $\Delta^* = \text{Conv}(S)$ and check that Δ^* is almost reflexive but not reflexive.

Performing Algorithm 4.3.1 on the list of 833 smooth Fano polytopes results in checking if 440914 polytopes are almost reflexive but not reflexive. These computations yield 660 nondistinct almost reflexive polytopes and computing their normal forms [21, §3.4] reduces this number to 48 distinct isomorphism classes of almost reflexive polytopes.

Theorem 4.3.2. *There are exactly* 48 *isomorphism classes of five-dimensional almost reflexive polytopes that are not reflexive contained in a five-dimensional smooth Fano polytope. Equivalently, there are exactly* 48 *isomorphism classes of five-dimensional* $\mathbb{Q}$*-reflexive polytopes that are not reflexive containing a five-dimensional reflexive polytope whose dual is a smooth Fano polytope.*

Remark 4.3.3. The computation involved in obtaining these 48 almost reflexive polytopes yields a few interesting results.

1. Of the 866 five-dimensional smooth Fano polytopes, 92 contain an almost reflexive polytope that is not reflexive.

2. The set

$$\{0, 2, 4, 6, 8, 12, 14, 16, 18, 20, 24, 28, 58\} \tag{4.36}$$

consists of the number of subpolytopes of a given five-dimensional smooth Fano polytope that are almost reflexive but not reflexive.

3. The set

$$\{0, 1, 2, 3, 4, 6, 7, 9, 10, 11, 19\} \tag{4.37}$$

consists of the number of isomorphism classes of subpolytopes of a given five-dimensional smooth Fano polytope that are almost reflexive but not reflexive.

The code written to perform this computation was written in `sage`. In particular, the algorithm to find the normal forms of the 660 almost reflexive polytopes utilized the access to the lattice polytope software `PALP` built into `sage`. ◊

We are interested in the $\mathbb{Q}$-nef partitions $\Delta = \Delta_1 + \Delta_2$ of those $\mathbb{Q}$-reflexive polytopes $\Delta \subset M_{\mathbb{R}}$ whose duals $\Delta^* \subset N_{\mathbb{R}}$ are the 48 almost reflexive polytopes from Theorem 4.3.2. These $\mathbb{Q}$-nef partitions define $\mathbb{Q}$-nef Calabi-Yau threefolds Y_{Δ_1,Δ_2} in X_{Δ}.

We may use the Orbit-Cone Correspondence to infer the following.

Proposition 4.3.4. *Let $\Delta^* \subset N_{\mathbb{R}}$ be a five-dimensional almost reflexive polytope that embeds into a smooth Fano polytope. Then X_{Δ} is an orbifold with either a zero-dimensional singular locus or a one-dimensional singular locus.*

Suppose $\Delta^* \subset N_{\mathbb{R}}$ is a five-dimensional almost reflexive polytope that embeds into a smooth Fano polytope. To compute the length two $\mathbb{Q}$-nef-partitions of Δ, we need only find all nontrivial partitions

$$\Sigma_{\Delta}(1) = \{\text{vertices of } \Delta^*\} = S_1 \sqcup S_2 \tag{4.38}$$

and check if $\Delta_1 + \Delta_2$ is a $\mathbb{Q}$-nef-partition of Δ, where $\Delta_i = \sum_{u_\rho \in S_i} D_\rho$. Performing this computation yields 209 combinatorially distinct $\mathbb{Q}$-nef-partitions.

Proposition 4.3.5. *There are* 209 *combinatorially distinct length two $\mathbb{Q}$-nef-partitions of five-dimensional non-reflexive $\mathbb{Q}$-reflexive polytopes containing a polytope whose dual is a smooth Fano polytope. Moreover, each of these $\mathbb{Q}$-nef partitions $\Delta = \Delta_1 + \Delta_2$ defines a smooth Calabi-Yau threefold $Y_{\Delta_1,\Delta_2} \subset X_{\Delta}$.*

Since each of the Calabi-Yau threefolds Y_{Δ_1,Δ_2} in Proposition 4.3.5 is smooth, we may use the spectral sequence from (2.44) to compute the cohomology of the tangent sheaf $\mathscr{T}_{Y_{\Delta_1,\Delta_2}}$. Completing these computations for each of the 209 $\mathbb{Q}$-nef-partitions in Proposition 4.3.5 gives the Hodge numbers in Table 4.1. Checking these Hodge numbers against Kreuzer's list of Hodge numbers of Calabi-Yau hypersurfaces in four-dimensional Gorenstein Fano toric varieties gives 22 new pairs of hodge numbers.

Table 4.1: The Hodge numbers of the smooth Calabi-Yau threefolds $Y_{\Delta_1,\Delta_2} \subset X_\Delta$ where $\Delta = \Delta_1 + \Delta_2$ is a $\mathbb{Q}$-nef partition of a $\mathbb{Q}$-reflexive polytope whose dual Δ^* embeds into a smooth Fano polytope. The highlighted pairs are not the Hodge numbers of a Calabi-Yau threefold hypersurface in a Gorenstein Fano toric variety.

h^{11}	h^{21}	h^{11}	h^{21}	h^{11}	h^{21}	h^{11}	h^{21}	h^{11}	h^{21}	h^{11}	h^{21}	h^{11}	h^{21}
1	**79**	**2**	**56**	**3**	**41**	3	66	4	36	4	58	5	35
1	101	**2**	**58**	3	43	3	67	4	40	4	59	5	45
1	103	**2**	**62**	3	45	3	69	**4**	**41**	4	60	5	51
		2	**64**	**3**	**47**	**3**	**70**	4	42	4	61	7	37
		2	**66**	**3**	**49**	3	71	4	44	4	62	8	36
		2	**68**	3	51	3	72	4	46	4	66		
		2	**72**	**3**	**53**	3	73	4	48	4	68		
		2	**76**	**3**	**54**	3	75	4	49	4	70		
		2	**78**	**3**	**55**	3	76	4	50				
		2	84	3	57	3	77	**4**	**51**				
		2	86	3	59	3	79	**4**	**53**				
		2	90	**3**	**60**	3	81	4	54				
		2	92	**3**	**61**	3	83	4	55				
		2	95	3	63	3	85	4	56				
		2	122	3	65	3	99	4	57				

While each of the Calabi-Yau threefolds Y_{Δ_1,Δ_2} in Proposition 4.3.5 is smooth, their proposed mirrors $Y_{\nabla_1,\nabla_2} \subset X_\nabla$ are not. We may, however, obtain a crepant resolution $X_{\widehat{\Sigma}_\nabla} \to X_\nabla$ by including each of the lattice points in $\partial\nabla$ in the fan Σ_∇ and subdividing. Each of these crepant resolutions $X_{\widehat{\Sigma}_\nabla} \to X_\nabla$ restricts to a crepant desingularization $\widehat{Y}_{\nabla_1,\nabla_2} \to Y_{\nabla_1,\nabla_2}$ where $\widehat{Y}_{\nabla_1,\nabla_2}$ is a smooth Calabi-Yau complete intersection in $X_{\widehat{\Sigma}_\nabla}$.

It is thus theoretically possible to compute the *stringy* Hodge numbers of Y_{∇_1,∇_2}

by using the spectral sequence (2.44) to compute the cohomology of $\mathscr{T}_{\widehat{Y}_{\nabla_1,\nabla_2}}$. Of course, we expect $h^{p,q}_{\text{st}}(Y_{\nabla_1,\nabla_2}) = h^{3-p,q}(Y_{\Delta_1,\Delta_2})$. However, of the 209 dual $\mathbb{Q}$-nef-partitions $\nabla = \nabla_1 + \nabla_2$, the $\mathbb{Q}$-nef-partition with the fewest lattice points in $\partial\nabla$ has $|\partial\nabla \cap N| = 44$. With this many coordinates in the homogeneous coordinate ring, the analysis of $\mathscr{T}_{\widehat{Y}_{\nabla_1,\nabla_2}}$ is computationally unfeasable. Again, as in Example 4.1.16, an alternate description of the stringy Hodge numbers $h^{p,q}_{\text{st}}(Y_{\nabla_1,\nabla_2})$ is desirable.

Example 4.3.6. Let $\tilde{\Delta}^*$ be the smooth Fano polytope with vertices

```
M( 1,  0,  0,  0,  0),
M( 0,  1,  0,  0,  0),
M( 0, -1,  0,  0,  0),
M(-1,  0,  0,  0,  0),
M(-1,  1,  0,  0,  0),
M( 0,  0,  1,  0,  0),
M( 0,  1, -1,  0,  0),
M( 0,  0,  0,  1,  0),
M( 0,  0,  0,  0,  1),
M( 1,  0, -1,  0, -1),
M( 1, -1,  1, -1,  0)
in 5-d lattice M
```
(4.39)

The Hodge numbers of the nef-partitions of Δ are

h^{11}	5	6	6	6	6	6	6
h^{21}	51	32	34	36	41	44	52

(4.40)

There are 58 subpolytopes $\Delta^* \subset \tilde{\Delta}^*$ that are almost reflexive but not reflexive. These 58 subpolytopes represent 19 isomorphism classes of lattice polytopes. The Hodge numbers of the $\mathbb{Q}$-nef-partitions of the corresponding $\mathbb{Q}$-reflexive polytopes Δ containing $\tilde{\Delta}$ are given by

h^{11}	h^{21}	h^{11}	h^{21}	h^{11}	h^{21}	h^{11}	h^{21}	h^{11}	h^{21}	h^{11}	h^{21}	h^{11}	h^{21}
1	101	2	56	3	45	3	67	4	36	4	60	5	35
1	103	2	62	3	47	3	69	4	41	4	62	5	45
1	79	2	66	3	51	3	70	4	42	4	68	5	51
		2	68	3	53	3	71	4	46	4	70	7	37
		2	72	3	54	3	72	4	48			8	36
		2	76	3	55	3	75	4	49				
		2	78	3	57	3	77	4	50				
		2	84	3	59	3	79	4	53				
		2	86	3	60	3	81	4	54				
		2	90	3	61			4	55				
		2	92	3	66			4	59				

The indices of these 19 reflexive polytopes in the data supplement at [20] are

$$\{5, 6, 7, 8, 10, 11, 14, 15, 16, 17, 18, 19, 24, 26, 30, 36, 42, 44, 46\} \qquad (4.41)$$

See §5.3 for more information. ◊

4.3.2 Almost Reflexive Polytopes Embedded in a Reflexive Polytope on Kreuzer's List

In §4.3.1, we found 209 pairs of dual $\mathbb{Q}$-nef-partitions of non-reflexive polytopes by searching for almost reflexive subpolytopes of five-dimensional smooth Fano polytopes. One strategy for expanding our search for examples of $\mathbb{Q}$-nef-partitions is to look for five-dimensional almost reflexive subpolytopes that embed into a given reflexive polytope. Since the number of five-dimensional reflexive polytopes is likely innumerable, this strategy is somewhat artifical. Nonetheless, we perform this task by searching through Kreuzer's partial list of 7661 five-dimensional reflexive polytopes reflexive polytopes admitting nef-partitions with $h^{11} \leq 6$, which can be found at [27].

In total, we find 764 isomorphism classes of almost reflexive polytopes that embed into a polytope on Kreuzer's list. The duals of these almost reflexive polytopes are

$\mathbb{Q}$-reflexive polytopes that admit 868 combinatorially distinct $\mathbb{Q}$-nef-partitions. Of these 868 $\mathbb{Q}$-nef-partitions $\Delta = \Delta_1 + \Delta_2$, 734 define smooth Calabi-Yau threefolds $Y_{\Delta_1,\Delta_2} \subset X_\Delta$.

The other 134 $\mathbb{Q}$-nef-partitions define singular Calabi-Yaus $Y_{\Delta_1,\Delta_2} \subset X_\Delta$. Inserting the rays generated by the lattice points in $\partial\Delta^* \cap N$ in the fan Σ_Δ and subdividing yields a crepant resolution $X_{\widehat{\Sigma}_\Delta} \to X_\Delta$ which induces a crepant desingularization $\widehat{Y}_{\Delta_1,\Delta_2} \to Y_{\Delta_1,\Delta_2}$. Thus, we may compute the stringy Hodge numbers of each of the 868 Calabi-Yau threefolds $Y_{\Delta_1,\Delta_2} \subset X_\Delta$ corresponding to a $\mathbb{Q}$-nef-partition $\Delta = \Delta_1 + \Delta_2$ where Δ^* embeds in a reflexive polytope on Kreuzer's list. These stringy Hodge numbers are shown in Table 4.2.

Table 4.2: The stringy Hodge numbers of the Calabi-Yau threefolds $Y_{\Delta_1,\Delta_2} \subset X_\Delta$ where $\Delta = \Delta_1 + \Delta_2$ is a $\mathbb{Q}$-nef partition of a $\mathbb{Q}$-reflexive polytope whose dual Δ^* embeds into a reflexive polytope on Kreuzer's list of reflexive polytopes admitting nef-partitions with 'small' Picard number. The highlighted pairs are not the Hodge numbers of a Calabi-Yau threefold hypersurface in a Gorenstein Fano toric variety.

h^{11}_{st}	h^{21}_{st}	h^{11}_{st}	h^{21}_{st}	h^{11}_{st}	h^{21}_{st}	h^{11}_{st}	h^{21}_{st}	h^{11}_{st}	h^{21}_{st}	h^{11}_{st}	h^{21}_{st}	h^{11}_{st}	h^{21}_{st}	h^{11}_{st}	h^{21}_{st}	h^{11}_{st}	h^{21}_{st}
1	79	2	30	2	92	3	23	3	58	3	79	4	38	5	39	6	54
1	101	2	44	2	95	3	29	3	59	3	81	4	40	5	41	6	78
1	103	2	50	2	106	3	31	3	60	3	83	4	42	5	45	6	90
1	145	2	56	2	116	3	33	3	61	3	84	4	46	5	47	7	37
		2	58	2	120	3	37	3	63	3	85	4	50	5	53	8	56
		2	60	2	122	3	39	3	65	3	87	4	52	5	55		
		2	62	2	128	3	41	3	66	3	89	4	54	5	56		
		2	64	2	132	3	43	3	67	3	91	4	56	5	59		
		2	66	2	144	3	45	3	69	3	93	4	58	5	62		
		2	68			3	47	3	70	3	95	4	60	5	65		
		2	72			3	48	3	71	3	99	4	64	5	71		
		2	74			3	49	3	72	3	103	4	66	5	77		
		2	76			3	51	3	73	3	105	4	70	5	80		
		2	78			3	53	3	75	3	111	4	82	5	83		
		2	84			3	54	3	76	3	123	4	91	5	89		
		2	86			3	55	3	77			4	94	5	101		
		2	90			3	57	3	78			4	112				

The data defining these 868 $\mathbb{Q}$-nef-partitions can be found in the data supplement to this dissertation at [20].

4.4 Aspinwall and Plesser's General Mirror Pairs

A dual pair of Gorenstein cones is an example of the data required to define a *gauged linear σ-model*. In this section we review this notion, following [1, 2].

Let N be a torsion-free lattice of rank d and let M be its dual lattice $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. Let $\mathscr{A} \subset N$ and $\mathscr{B} \subset M$ be finite collections of points such that

- $\mathrm{Span}_{\mathbb{Z}}\, \mathscr{A} = N$
- $\mathscr{A} = N \cap \mathrm{Conv}(\mathscr{A})$
- There exists a $\mu \in M$ such that $\langle \mu, \alpha \rangle = 1$ for all $\alpha \in \mathscr{A}$.

- $\mathrm{Span}_{\mathbb{Z}}\, \mathscr{B} = M$
- $\mathscr{B} = M \cap \mathrm{Conv}(\mathscr{B})$
- There exists a $\nu \in N$ such that $\langle \beta, \nu \rangle = 1$ for all $\beta \in \mathscr{B}$.

and $\mathrm{Cone}(\mathrm{Conv}(\mathscr{B})) \subset \mathrm{Cone}(\mathrm{Conv}(\mathscr{A}))^\vee$. This is equivalent to requiring that $\mathscr{A}$ and $\mathscr{B}$ are Hilbert bases for two Gorenstein cones $\sigma_1 \subset N$ and $\sigma_2 \subset M$ where $\mu = h_{\sigma_1}$, $\nu = h_{\sigma_2}$, and $\sigma_2 \subset \sigma_1^\vee$. The *index* of the pair $(\mathscr{A}, \mathscr{B})$ is the nonnegative integer $\langle \mu, \nu \rangle$.

Insert the points of $\mathscr{A}$ into the columns of a matrix A to obtain an exact sequence

$$0 \longrightarrow L \longrightarrow \mathbb{Z}^{\mathscr{A}} \xrightarrow{\;A\;} N \longrightarrow 0 \tag{4.42}$$

where L is the so-called *lattice of relations* of rank $r = |\mathscr{A}| - d$. Applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ to (4.42) yields an exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{\mathscr{A}} \xrightarrow{\;Q\;} L^\vee \longrightarrow 0 \tag{4.43}$$

where Q is the *matrix of charges* of the points in $\mathscr{A}$. Let

$$S(\mathscr{A}) = \mathbb{C}[x_\alpha \,|\, \alpha \in \mathscr{A}] \tag{4.44}$$

We will write S in place of $S(\mathscr{A})$ when the context is clear. The formula

$$x_\alpha \mapsto \lambda_1^{Q_{1\alpha}} \lambda_2^{Q_{2\alpha}} \cdots \lambda_r^{Q_{r\alpha}} x_\alpha \tag{4.45}$$

defines a $(\mathbb{C}^*)^r$ torus action on S. Let S_0 be the $(\mathbb{C}^*)^r$-invariant subalgebra of S. Then the algebra S decomposes into a sum of S_0-modules

$$S = \bigoplus_{\mathbf{u}\in L^\vee} S_{\mathbf{u}} \tag{4.46}$$

We will denote a shift in grading by $S(\mathbf{u})_{\mathbf{v}} = S_{\mathbf{u}+\mathbf{v}}$.

A choice of a regular triangulation Σ of the pointset $\mathscr{A}$ is called a *phase*. The *Cox ideal* of a phase Σ is the ideal of S generated by the monomials

$$\prod_\sigma \{x_\alpha \,|\, \alpha \in \mathscr{A} \setminus \{\text{vertices of } \sigma\}\} \tag{4.47}$$

where the product is taken over the maximal faces of Σ.

For a phase Σ, define the stack

$$Z_\Sigma = \mathbb{P}\mathrm{roj}_\Sigma S = \left[\frac{\operatorname{Spec} S - \mathbf{V}(B_\Sigma)}{(\mathbb{C}^*)^r}\right] \tag{4.48}$$

and the *superpotential*

$$W_X = \sum_{\beta\in\mathscr{B}} a_\beta \prod_{\alpha\in\mathscr{A}} x_\alpha^{\langle\alpha,\beta\rangle} \tag{4.49}$$

where $a_\beta \in \mathbb{C}$. Note that the condition $\mathrm{Cone}(\mathrm{Conv}(\mathscr{B})) \subset \mathrm{Cone}(\mathrm{Conv}(\mathscr{A}))^\vee$ ensures that W_X is a polynomial. In fact, W_X is a homogeneous polynomial in S of multi-degree zero. Finally, we define the *Jacobian ideal of* W_X by

$$\mathrm{Jac}(W_X) = \left\langle \frac{\partial W}{\partial x_\alpha} \,\middle|\, \alpha \in \mathscr{A} \right\rangle \tag{4.50}$$

and the corresponding stack

$$X_\Sigma = \mathbb{P}\mathrm{roj}_\Sigma \frac{S}{\mathrm{Jac}(W_X)} \tag{4.51}$$

Of course, X_Σ depends on the choice of coefficients a_β in the superpotential W_X (4.49).

Now, one may exchange the rôles of $\mathscr{A}$ and $\mathscr{B}$ to obtain a *mirror superpotential*

$$W_Y = \sum_{\alpha \in \mathscr{A}} b_\alpha \prod_{\beta \in \mathscr{B}} y_\beta^{\langle \alpha, \beta \rangle} \tag{4.52}$$

for $b_\alpha \in \mathbb{C}$. Fixing a regular triangulation Υ of $\mathscr{B}$ and choices of coefficients $b_\alpha \in \mathbb{C}$ gives a corresponding Y_Υ associated to $\mathrm{Jac}(W_Y)$.

When $\mathrm{Cone}(\mathrm{Conv}(\mathscr{B})) = \mathrm{Cone}(\mathrm{Conv}(\mathscr{A}))^\vee$, the statement of mirror symmetry for the gauged linear sigma model [43] is that X_Σ and Y_Υ form a mirror pair. For our purposes, we will use this as the *definition* of a mirror pair.

Aspinwall and Plesser relax the condition

$$\mathrm{Cone}(\mathrm{Conv}(\mathscr{B})) = \mathrm{Cone}(\mathrm{Conv}(\mathscr{A}))^\vee \tag{4.53}$$

to

$$\mathrm{Cone}(\mathrm{Conv}(\mathscr{B})) \subset \mathrm{Cone}(\mathrm{Conv}(\mathscr{A}))^\vee \tag{4.54}$$

in [2]. Here, the authors use the term *general mirror pair*.

We will often abuse language and refer to the pair of pointsets $(\mathscr{A}, \mathscr{B})$ as a *general mirror pair*.

4.4.1 General Mirror Pairs and Almost Reflexive Gorenstein Cones

In [2], two important properties of general mirror pairs are discussed.

Definition 4.4.1 (Definition 2 in [2])**.** The pair of pointsets $(\mathscr{A}, \mathscr{B})$ is *reflexive* if

$$\mathrm{Cone}(\mathrm{Conv}(\mathscr{B})) = \mathrm{Cone}(\mathrm{Conv}(\mathscr{A}))^\vee \tag{4.55}$$

The term 'reflexive' is used here to indicate that the cones in (4.55) form a dual pair of reflexive Gorenstein cones.

Definition 4.4.2 (Definition 9 in [2])**.** The pair of pointsets $(\mathscr{A}, \mathscr{B})$ is *$\mathscr{B}$-complete* if

$$\mathrm{Conv}\big(\mathrm{Cone}(\mathrm{Conv}\,\mathscr{A})^\vee \cap M \cap H_\nu^{(1)}\big) = \mathrm{Conv}(\mathscr{B}) \tag{4.56}$$

The pair of pointsets $(\mathscr{A},\mathscr{B})$ is $\mathscr{A}$-*complete* if

$$\operatorname{Conv}\bigl(\operatorname{Cone}(\operatorname{Conv}\mathscr{B})^\vee \cap N \cap H_\mu^{(1)}\bigr) = \operatorname{Conv}(\mathscr{A}) \tag{4.57}$$

The pair of pointsets $(\mathscr{A},\mathscr{B})$ is *complete* if it is both $\mathscr{A}$-complete and $\mathscr{B}$-complete.

The notion of a complete mirror pair $(\mathscr{A},\mathscr{B})$ is closely related to the notion of a dual pair of almost reflexive gorenstein cones $(\sigma,\sigma^\bullet)$ from Definition 4.1.19. Recall that the *Hilbert basis* of a strictly convex cone $\sigma \subset N_\mathbb{R}$ is the set of all irreducible elements in the semigroup $\sigma \cap N$.

Theorem 4.4.3. *Let $\sigma \subset N_\mathbb{R}$ be an almost reflexive Gorenstein cone so that the pair $(\sigma,\sigma^\bullet)$ is a dual pair of almost reflexive Gorenstein cones. Then the pair of Hilbert bases $(\mathscr{A},\mathscr{B})$ of the cones $(\sigma,\sigma^\bullet)$ is a complete mirror pair. Moreover, $(\mathscr{A},\mathscr{B})$ is a reflexive mirror pair if and only if $\sigma^\bullet = \sigma^\vee$.*

Proof. Recall from Definition 4.1.19 that

$$\sigma^\bullet = \operatorname{Cone}\bigl(\operatorname{Conv}(\sigma^\vee \cap H^{(1)}_{h_{\sigma^\vee}})\bigr) \subset M_\mathbb{R} \tag{4.58}$$

Since $\mathscr{B}$ is the Hilbert basis of $\sigma^\bullet$ we then have

$$\begin{aligned}\operatorname{Conv}(\mathscr{B}) &= \operatorname{Conv}\bigl(\sigma^\vee \cap H^{(1)}_{h_{\sigma^\vee}} \cap M\bigr) \\ &= \operatorname{Conv}\bigl(\operatorname{Cone}(\operatorname{Conv}\mathscr{A})^\vee \cap H_\nu^{(1)} \cap M\bigr)\end{aligned} \tag{4.59}$$

where $\nu = h_{\sigma^\vee}$. Thus $(\mathscr{A},\mathscr{B})$ is $\mathscr{B}$-complete. Since $\mathscr{A}$ is the Hilbert basis of σ we have

$$\begin{aligned}\operatorname{Conv}(\mathscr{A}) &= \operatorname{Conv}\bigl((\sigma^\bullet)^\vee \cap H^{(1)}_{h_\sigma} \cap N\bigr) \\ &= \operatorname{Conv}\bigl(\operatorname{Cone}(\operatorname{Conv}\mathscr{B})^\vee \cap H_\mu^{(1)} \cap N\bigr)\end{aligned} \tag{4.60}$$

where $\mu = h_\sigma$. Hence $(\mathscr{A},\mathscr{B})$ is complete.

That $(\mathscr{A},\mathscr{B})$ is a reflexive mirror pair if and only if $\sigma^\bullet = \sigma^\vee$ follows from $\operatorname{Cone}(\operatorname{Conv}\mathscr{B}) = \sigma^\bullet$ and $\operatorname{Cone}(\operatorname{Conv}\mathscr{A}) = \sigma$. ☐

Theorem 4.4.3 allows us to generate complete mirror pairs $(\mathscr{A}, \mathscr{B})$ by constructing almost reflexive Gorenstein cones. It is interesting to note that the notion of a complete mirror pair is, in fact, more general than the notion of an almost reflexive Gorenstein cone.

Example 4.4.4. Let σ_1 and σ_2 be the Gorenstein cones generated by the lists

```
N(0, 1, -1, -1,  0,  0,  0),
N(0, 1,  0,  0,  1,  0,  0),
N(0, 1,  0,  1, -1,  0,  0),
N(0, 1,  0,  1,  0,  0,  0),
N(1, 0,  0,  0,  0,  0,  0),
N(1, 0,  0,  0,  0,  0,  1),
N(1, 0,  0,  0,  0,  1,  0),
N(1, 0,  2,  0,  0, -1, -1)
in 7-d lattice N
```

```
M(0,  1,  0, -1, -1,  0,  0),
M(0,  1,  0, -1,  0,  0,  0),
M(2, -1, -3,  2,  1, -2, -2),
M(1,  0,  0,  0,  0,  2, -1),
M(1,  0,  0,  0,  0, -1,  2),
M(0,  1,  0,  1, -1,  0,  0),
M(1,  0,  0,  0,  0, -1, -1),
M(1,  0, -1,  0,  0,  0, -1),
M(0,  1,  0,  1,  2,  0,  0),
M(1,  0, -1,  0,  0, -1,  0),
M(1,  0, -1,  0,  0, -1, -1),
M(0,  1,  2, -1,  0,  4,  0),
M(0,  1,  2, -1,  0,  0,  4),
M(0,  1,  2, -1,  0,  0,  0),
M(0,  1,  2, -1, -1,  4,  0),
M(0,  1,  2, -1, -1,  0,  4),
M(0,  1,  2, -1, -1,  0,  0)
in 7-d lattice M
```

(4.61)

respectively. Then σ_1 and σ_2 are not almost reflexive Gorenstein cones. However, the Hilbert bases $(\mathscr{A}, \mathscr{B})$ of the Gorenstein cones (σ_1, σ_2) form a complete mirror pair. $\diamondsuit$

4.4.2 Complete Mirror Pairs of Weighted Projective CICY Threefolds of Codimension Two

We have seen that the pair of Hilbert bases $(\mathscr{A}, \mathscr{B})$ of the Cayley cones of a dual pair of $\mathbb{Q}$-nef-partitions $\Delta = \Delta_1 + \cdots + \Delta_c$ and $\nabla = \nabla_1 + \cdots + \nabla_c$ define a complete mirror pair. We can also produce pairs of complete mirror pairs by studying Calabi-Yau complete intersections in weighted projective spaces. A list of 4239 transversal configurations for codimension two CalabiYau threefolds in weighted projective spaces was produced by Klemm at [25]. These Calabi-Yau threefolds are *quasismooth* but not necessarily smooth. The data set includes the Hodge numbers of desingularizations of each threefold.

This list was studied extensively by Kreuzer and Riegler in [32]. Here, the authors attempt to resolve each quasismooth Calabi-Yau threefold by a complete intersection in a Gorenstin Fano toric variety. Their strategy is to try to relate the Newton polytopes of the defining polynomial equations of degrees (d_1, d_2) to a nef partition $\Delta = \Delta_1 + \Delta_2$ of a reflexive polyhedron Δ. By using this method, Kreuzer and Riegler are able to replicate Klemm's Hodge number computations in "many, but not all" cases.

We have found that the notion of a complete mirror pair is a more useful tool for studying the Calabi-Yau threefolds on Klemm's list. Consider the following algorithm.

Algorithm 4.4.5. Let $Y = \mathbb{P}^5_{w_0,\ldots,w_5}[d_1, d_2]$ be a quasismooth Calabi-Yau threefold on Klemm's list and let Σ be the fan in $N_\mathbb{R}$ of $\mathbb{P}^5_{w_0,\ldots,w_5}$.

1. Let $\Sigma(1) = S_{d_1} \sqcup S_{d_2}$ be a partition of the rays of the fan Σ such that generic sections of the torus-invariant Weyl divisors

$$Z_{d_1} = \sum_{u_\rho \in S_{d_1}} D_\rho \qquad\qquad Z_{d_2} = \sum_{u_\rho \in S_{d_2}} D_\rho \tag{4.62}$$

   have homogeneous degrees d_1 and d_2 in $S(\Sigma)$ respectively.

2. Let $\Delta_1 = \Delta_{Z_{d_1}}$ and $\Delta_2 = \Delta_{Z_{d_2}}$ be the polytopes in $M_\mathbb{R}$ assigned to the divisors Z_{d_1} and Z_{d_2} in the correspondence described in (2.22). Let $\Delta \subset M_\mathbb{R}$ be given by the Minkowski sum $\Delta = \Delta_1 + \Delta_2$.

3. Let $\nabla_1, \nabla_2 \subset N_\mathbb{R}$ be given by

$$\nabla_j = \{y \in N_\mathbb{R} \,|\, x \in \Delta_i \Rightarrow \langle x, y\rangle \geq -\delta_{ij}, 1 \leq i \leq c\} \tag{4.63}$$

and let $\nabla = \nabla_1 + \nabla_2$.

4. Let σ be the Cayley cone $\mathcal{C}_{\nabla_1,\nabla_2}$ (3.47) of the polytopes ∇_1 and ∇_2.

5. If $2\sigma_{(1)}$ contains a unique interior point $h_{\sigma^\vee}$, then let $\sigma^\bullet$ be given by the formula (4.25).

6. Let $(\mathscr{A}, \mathscr{B})$ be the Hilbert bases of the pair of cones $(\sigma, \sigma^\bullet)$.

7. Return $(\mathscr{A}, \mathscr{B})$.

Note that the Minkowski sums $\Delta = \Delta_1 + \Delta_2$ and $\nabla = \nabla_1 + \nabla_2$ from Algorithm 4.4.5 need not be nef-partitions or even $\mathbb{Q}$-nef-partitions. We suspect the Minkowski sums for which $\nabla = \nabla_1 + \nabla_2$ are nef-partitions are precisely those successfully studied by Kreuzer and Riegler in [32].

We have run Algorithm 4.4.5 on 3906 of the 4239 configurations on Klemm's list. Each configuration successfully yields a complete mirror pair: 1754 are reflexive mirror pairs and 2152 are complete but not reflexive mirror pairs.

Note that if the Minkowski sum $\nabla = \nabla_1 + \nabla_2$ is a nef-partition, then $(\sigma, \sigma^\bullet)$ is a dual pair of reflexive Gorenstein cones and the stringy E-polynomial may be used to compute the Hodge numbers. However, if $\nabla = \nabla_1 + \nabla_2$ is not a nef-partition, then an alternative description of the Hodge numbers is necessary.

The data from our computations can be found at [20].

5

Stringy Hodge Numbers of Non-Reflexive Mirror Pairs

In Chapter 4, we saw two generalizations of the Batyrev-Borisov mirror construction. We showed in Theorem 4.4.3 that the Hilbert bases $(\mathscr{A}, \mathscr{B})$ of a dual pair of almost reflexive Gorenstein cones $(\sigma, \sigma^\bullet)$ is a complete mirror pair, in the sense of Aspinwall and Plesser [2]. We have also seen that the Cayley cones associated to a dual pair of $\mathbb{Q}$-nef-partitions

$$\Delta = \Delta_1 + \cdots + \Delta_c \qquad\qquad \nabla = \nabla_1 + \cdots + \nabla_c \tag{5.1}$$

define a dual pair almost reflexive Gorenstein cones and that this pair is reflexive if and only if the $\mathbb{Q}$-nef-partitions are nef-partitions of reflexive polytopes. The $\mathbb{Q}$-nef-partitions in (5.1) define a pair of Calabi-Yau complete intersections $(Y_{\Delta_1,\ldots,\Delta_c}, Y_{\nabla_1,\ldots,\nabla_c})$ in the Fano toric varieties (X_Δ, X_∇). If the $\mathbb{Q}$-nef-partitions in (5.1) are nef-partitions of reflexive polytopes, then the stringy Hodge numbers $h^{p,q}_{\mathrm{st}}(Y_{\Delta_1,\ldots,\Delta_c})$ and $h^{p,q}_{\mathrm{st}}(Y_{\nabla_1,\ldots,\nabla_c})$ are defined by E_{st}-polynomial of the Cayley polytopes $\Delta_1 * \cdots * \Delta_c$ and $\nabla_1 * \cdots * \nabla_c$ respectively. Moreover, Theorem 3.3.8 implies that the pair of Calabi-Yau complete intersections $(Y_{\Delta_1,\ldots,\Delta_c}, Y_{\nabla_1,\ldots,\nabla_c})$ passes the stringy topological mirror symmetry in test in this case.

Note, however, that if the polytopes Δ and ∇ in (5.1) are *not* reflexive, then the E_{st}-polynomials of the Cayley polytopes $\Delta_1 * \cdots * \Delta_c$ and $\nabla_1 * \cdots * \nabla_c$ are not well-defined. Indeed, in Example 4.1.22 we saw an example of a dual pair of $\mathbb{Q}$-nef-partitions (5.1) with $c = 2$ where the f-vectors of the Cayley polytopes were

$$f_{[\Delta_1]*[\Delta_2]} = (1,\ 16,\ 50,\ 75,\ 65,\ 33,\ 9,\ 1) \quad f_{[\nabla_1]*[\nabla_2]} = (1,\ 8,\ 27,\ 50,\ 55,\ 36,\ 12,\ 1) \tag{5.2}$$

Unlike the reflexive situation, the two vectors in (5.2) are not reverses of each other. That is, there is no obvious duality between the face lattices of the Cayley polytopes corresponding to the dual pair of $\mathbb{Q}$-nef-partitions in (5.1) if the polytopes Δ and ∇ are not reflexive.

In this chapter, we offer a definition of the stringy E-polynomial, which we refer to as the *naïve stringy E-function* $\widetilde{E}_{\text{st}}$, of a complete mirror pair $(\mathscr{A}, \mathscr{B})$. Since complete mirror pairs are generalizations of almost reflexive Gorenstein cones, our stringy E-function is well-defined for Cayley polytopes corresponding to the $\mathbb{Q}$-nef-partitions in (5.1), even if the polytopes Δ and ∇ are not reflexive. We conjecture that $\widetilde{E}_{\text{st}}$ is a generating function for the stringy Hodge numbers $h^{p,q}_{\text{st}}(Y_{\Delta_1,\ldots,\Delta_c})$ and $h^{p,q}_{\text{st}}(Y_{\nabla_1,\ldots,\nabla_c})$ of the $\mathbb{Q}$-nef complete intersections $(Y_{\Delta_1,\ldots,\Delta_c}, Y_{\nabla_1,\ldots,\nabla_c})$ in the non-Gorenstein Fano toric varieties (X_Δ, X_∇). Moreover, we conjecture that $\widetilde{E}_{\text{st}}$ passes the mirror symmetry law. Our latter conjecture is a generalization of Mavlyutov's conjecture in [34, Conjecture 5.1].

Our conjectures involving the generalized stringy E-function $\widetilde{E}_{\text{st}}$ are supported by a good deal of computational evidence. In Section 5.2, we verify that $\widetilde{E}_{\text{st}}$ generates the stringy hodge numbers of Y_{Δ_1,Δ_2} if $\Delta = \Delta_1 + \Delta_2$ is a $\mathbb{Q}$-reflexive polytope whose dual Δ^* is embedded in a five-dimensional smooth Fano polytope. In §5.3, we verify that our conjectures hold when Δ^* is embedded in one of the 7660 reflexive polytopes on Kreuzer's list. Finally, in Section 5.4, we demonstrate that the naïve stringy E-function correctly predicts the Hodge data of each of the 2152 complete non-reflexive mirror pairs corresponding to Calabi-Yau complete intersections in weighted projective spaces we found in §4.4.2.

5.1 The Naïve Stringy E-Function

Let (N, M) be a dual pair of torsion-free lattices of rank d and let $(\mathscr{A}, \mathscr{B})$ be a complete mirror pair of index c (Definition 4.4.2) where $\mathscr{A} \subset N$ and $\mathscr{B} \subset M$. Let

$$\Delta_{\mathscr{A}} = \mathrm{Conv}(\mathscr{A}) \qquad \sigma_{\mathscr{A}} = \mathrm{Cone}(\mathrm{Conv}(\mathscr{A})) \tag{5.3}$$

$$\Delta_{\mathscr{B}} = \mathrm{Conv}(\mathscr{B}) \qquad \sigma_{\mathscr{B}} = \mathrm{Cone}(\mathrm{Conv}(\mathscr{B})) \tag{5.4}$$

Note that $\sigma_{\mathscr{B}} = \sigma_{\mathscr{A}}^{\vee}$ if $(\mathscr{A}, \mathscr{B})$ is reflexive while $\sigma_{\mathscr{B}} = \sigma_{\mathscr{A}}^{\bullet} \subsetneq \sigma_{\mathscr{A}}^{\vee}$ if $(\mathscr{A}, \mathscr{B})$ is not reflexive.

Definition 5.1.1. Let F be a face of $\Delta_{\mathscr{A}}$. The *naïve dual of* F is

$$F^{\ddot{\vee}} = \mathrm{Cone}(F)^* \cap \mathrm{Conv}(\mathscr{B}) \subset \mathrm{Conv}(\mathscr{B}) \tag{5.5}$$

where $\mathrm{Cone}(F)^*$ is the face of $\sigma_{\mathscr{A}}^{\vee}$ dual to $\mathrm{Cone}(F)$. The naïve dual of a face $\Delta_{\mathscr{B}}$ is defined similarly.

By construction, the naïve dual $F^{\ddot{\vee}}$ of a face F of $\Delta_{\mathscr{A}}$ is a face of $\Delta_{\mathscr{B}}$. If the pair $(\mathscr{A}, \mathscr{B})$ is reflexive, then $(\Delta_{\mathscr{A}}, \Delta_{\mathscr{B}})$ is a dual pair of Gorenstein polytopes and Definition 5.1.1 coincides with the usual duality of the face lattice of dual Gorenstein polytopes. However, if $(\mathscr{A}, \mathscr{B})$ is not reflexive, then the face lattices of $\Delta_{\mathscr{A}}$ and $\Delta_{\mathscr{B}}$ are not dual to one another and $(F^{\ddot{\vee}})^{\ddot{\vee}} \neq F$ in general.

We now have enough technology to generalize Batyrev and Nill's stringy E-function.

Definition 5.1.2. The *naïve stringy E-function of* $\Delta_{\mathscr{A}}$ is the function $\widetilde{E}_{\mathrm{st}}(\Delta_{\mathscr{A}}; u, v) \in \mathbb{Q}(u, v)$ defined by

$$\widetilde{E}_{\mathrm{st}}(\Delta_{\mathscr{A}}; u, v) = (uv)^{-c} \sum_{\varnothing \preceq F \preceq \Delta_{\mathscr{A}}} (-u)^{1+\dim F} \widetilde{S}_F(u^{-1}v) \cdot \widetilde{S}_{F^{\ddot{\vee}}}(uv) \tag{5.6}$$

where the sum is taken over all faces F of $\Delta_{\mathscr{A}}$ and $F^{\ddot{\vee}}$ is the face of $\Delta_{\mathscr{B}}$ naïvely dual to F, as defined in Definition 5.1.1. We call the integer

$$\dim_{\mathrm{CY}} \Delta_{\mathscr{A}} = \dim \Delta_{\mathscr{A}} + 1 - 2\,c \tag{5.7}$$

the *Calabi-Yau dimension* of $\Delta_{\mathscr{A}}$. The function $\widetilde{E}_{\mathrm{st}}(\Delta_{\mathscr{B}}; u, v)$ is defined similarly.

We use the notation $\widetilde{E}_{\mathrm{st}}$ to indicate that the polytope $\Delta_{\mathscr{A}}$ is not necessarily Gorenstein. Note, however, that if $(\mathscr{A}, \mathscr{B})$ is reflexive, then $\widetilde{E}_{\mathrm{st}}(\Delta_{\mathscr{A}}; u, v) = E_{\mathrm{st}}(\Delta_{\mathscr{A}}; u, v)$. That is, $\widetilde{E}_{\mathrm{st}}$ indeed generalizes Batyrev and Nill's stringy E-function E_{st} to the setting of complete mirror pairs $(\mathscr{A}, \mathscr{B})$.

Since the face lattices of $\Delta_{\mathscr{A}}$ and $\Delta_{\mathscr{B}}$ are not necessarily dual to one another, it is not immediately clear if $\widetilde{E}_{\mathrm{st}}$ satisfies the mirror-symmetry law. The following proposition, however, follows immediately from the properties of the $\widetilde{S}$-polynomial.

Proposition 5.1.3. *Let $(\mathscr{A}, \mathscr{B})$ be a complete mirror pair with* $\dim_{\mathrm{CY}} \Delta_{\mathscr{A}} = n$. *Then*

(a) $\widetilde{E}_{\mathrm{st}}(\Delta_{\mathscr{A}}; u, v) = \widetilde{E}_{\mathrm{st}}(\Delta_{\mathscr{A}}; v, u)$

(b) $(uv)^n \widetilde{E}_{\mathrm{st}}(\Delta_{\mathscr{A}}; u^{-1}, v^{-1}) = \widetilde{E}_{\mathrm{st}}(\Delta_{\mathscr{A}}; u, v)$

We conjecture that the generalized stringy E-function obeys the mirror symmetry law.

Conjecture 5.1.4. *Let $(\mathscr{A}, \mathscr{B})$ be a complete mirror pair with* $\dim_{\mathrm{CY}} \Delta_{\mathscr{A}} = n$. *Then*

$$\widetilde{E}_{\mathrm{st}}(\Delta_{\mathscr{A}}; u, v) = (-u)^n \widetilde{E}_{\mathrm{st}}(\Delta_{\mathscr{B}}; u^{-1}, v) \tag{5.8}$$

That is, $\widetilde{E}_{\mathrm{st}}$ obeys the mirror symmetry law.

5.1.1 Stringy Hodge Numbers of a Mavlyutov Stringy Mirror Pair

In the previous section, we defined $\widetilde{E}_{\rm st}(\Delta_{\mathscr{A}}; u, v)$ and $\widetilde{E}_{\rm st}(\Delta_{\mathscr{B}}; u, v)$ for a complete mirror pair $(\mathscr{A}, \mathscr{B})$. Let $\Delta = \Delta_1 + \cdots + \Delta_c$ and $\nabla = \nabla_1 + \cdots + \nabla_c$ be a dual pair of $\mathbb{Q}$-nef-partitions. Recall that the Cayley cones $(\mathcal{C}_{[\Delta_1],\ldots,[\Delta_c]}, \mathcal{C}_{[\nabla_1],\ldots,[\nabla_c]})$ form a dual pair of almost reflexive Gorenstein cones $(\sigma, \sigma^\bullet)$ of index c. By Theorem 4.4.3, the Hilbert bases $(\mathscr{A}, \mathscr{B})$ of the dual pair of almost reflexive Gorenstein cones $(\sigma, \sigma^\bullet)$ form a complete mirror pair.

Definition 5.1.5. Let $Y_{\Delta_1,\ldots,\Delta_c} \subset X_\Delta$ be the $\mathbb{Q}$-nef Calabi-Yau complete intersection corresponding to a $\mathbb{Q}$-nef-partition $\Delta = \Delta_1 + \cdots + \Delta_c$ and let $\mathscr{A}$ be the Hilbert basis of the Cayley cone $\mathcal{C}_{[\Delta_1],\ldots,[\Delta_c]}$. Consider the naïve stringy E-function $\widetilde{E}_{\rm st}(\Delta_{\mathscr{A}}; u, v)$ of the polytope $\Delta_{\mathscr{A}} = \mathrm{Conv}(\mathscr{A})$ and write

$$\widetilde{E}_{\rm st}(\Delta_{\mathscr{A}}; u, v) = \sum_{p,q} (-1)^{p+q} \tilde{h}^{p,q}_{\rm st}(\Delta_{\mathscr{A}}) \tag{5.9}$$

The *naïve stringy Hodge numbers of* $Y_{\Delta_1,\ldots,\Delta_c}$ are the numbers $\tilde{h}^{p,q}_{\rm st}(Y_{\Delta_1,\ldots,\Delta_c}) = \tilde{h}^{p,q}_{\rm st}(\Delta_{\mathscr{A}})$.

Definition 5.1.5 gives a possible combinatorial interpretation of the stringy Hodge numbers of the $\mathbb{Q}$-nef Calabi-Yau complete intersections $(Y_{\Delta_1,\ldots,\Delta_c}, Y_{\nabla_1,\ldots,\nabla_c})$ in the Fano toric varieties (X_Δ, X_∇) corresponding to a dual pair of $\mathbb{Q}$-nef-partitions $\Delta = \Delta_1 + \cdots + \Delta_c$ and $\nabla = \nabla_1 + \cdots + \nabla_c$. We suspect that our naïve stringy Hodge numbers coincide with the usual Hodge numbers when our variety is smooth.

Conjecture 5.1.6. *Let $Y_{\Delta_1,\ldots,\Delta_c} \subset X_\Delta$ be the $\mathbb{Q}$-nef Calabi-Yau complete intersection associated to a $\mathbb{Q}$-nef-partition $\Delta = \Delta_1 + \cdots + \Delta_c$ and suppose $\widehat{Y} \to Y_{\Delta_1,\ldots,\Delta_c}$ is a crepant desingularization of $Y_{\Delta_1,\ldots,\Delta_c}$ by a smooth Calabi-Yau variety $\widehat{Y}$. Then $h^{p,q}(\widehat{Y}) = \tilde{h}^{p,q}_{\rm st}(Y_{\Delta_1,\ldots,\Delta_c})$.*

Under this framework, we expect that the pair $(Y_{\Delta_1,\ldots,\Delta_c}, Y_{\nabla_1,\ldots,\nabla_c})$ passes the stringy topological mirror symmetry test.

Conjecture 5.1.7. *Let $Y_{\Delta_1,\ldots,\Delta_c} \subset X_\Delta$ and $Y_{\nabla_1,\ldots,\nabla_c} \subset X_\nabla$ be the $\mathbb{Q}$-nef Calabi-Yau complete intersections associated to a dual pair of $\mathbb{Q}$-nef-partitions $\Delta = \Delta_1 + \ldots + \Delta_c$ and $\nabla = \nabla_1 + \cdots + \nabla_c$. Then the naïve stringy Hodge numbers of $Y_{\Delta_1,\ldots,\Delta_c} \subset X_\Delta$ and $Y_{\nabla_1,\ldots,\nabla_c} \subset X_\nabla$ satisfy*

$$\tilde{h}^{p,q}_{\mathrm{st}}(Y_{\Delta_1,\ldots,\Delta_c}) = \tilde{h}^{d-c-p,q}_{\mathrm{st}}(Y_{\nabla_1,\ldots,\nabla_c}) \tag{5.10}$$

for $0 \le p, q \le d - c$.

5.1.2 A Stringy Approach to $\mathbb{P}^5_{211111}[3,4]$

Fix $M \simeq \mathbb{Z}^5$ and let $\Delta = \Delta_1 + \Delta_2$ and $\nabla = \nabla_1 + \nabla_2$ be the $\mathbb{Q}$-nef-partitions defined in Example 4.1.16. Recall that $X_\Delta = \mathbb{P}^5_{211111}$ and that the $\mathbb{Q}$-nef Calabi-Yau complete intersection Y_{Δ_1,Δ_2} is a generic complete intersection of a cubic and a quartic in $\mathbb{P}_{211111}$. In Section 2.5.1, we computed $h^{11}(Y_{\Delta_1,\Delta_2}) = 1$ and $h^{21}(Y_{\Delta_1,\Delta_2}) = 79$.

We also saw in Example 4.1.16 that the $\mathbb{Q}$-nef Calabi-Yau complete intersection Y_{∇_1,∇_2} is singular. A crepant desingularization $\widehat{Y}_{\nabla_1,\nabla_2} \to Y_{\nabla_1,\nabla_2}$ may be obtained, but the analysis of $\mathscr{T}_{\widehat{Y}_{\nabla_1,\nabla_2}}$ as in (2.44) is hopeless.

We may, however, compute the *naïve* stringy hodge numbers of Y_{Δ_1,Δ_2} and Y_{∇_1,∇_2}. Let $(\sigma, \sigma^\bullet)$ be the dual pair of almost reflexive Gorenstein cones given by the Cayley cones $\sigma = \mathcal{C}_{[\Delta_1],[\Delta_c]}$ and $\sigma^\bullet = \mathcal{C}_{[\nabla_1],[\nabla_2]}$. We saw in Example 4.1.22 that σ and $\sigma^\bullet$ are generated by the rays

```
ZZ^2+M(0, 1,  0,  0, -1, -1, -1),
ZZ^2+M(1, 0,  0,  0,  0,  0,  0),
ZZ^2+M(1, 0,  0, -1,  1,  0,  0),
ZZ^2+M(1, 0,  0, -1,  0,  1,  0),
ZZ^2+M(0, 1,  0,  0, -1, -1,  3),
ZZ^2+M(1, 0,  0, -1,  0,  0,  1),
ZZ^2+M(1, 0,  0, -1,  0,  0,  0),
ZZ^2+M(1, 0, -1,  2,  0,  0,  0),
ZZ^2+M(1, 0, -1, -1,  3,  0,  0),
ZZ^2+M(1, 0, -1, -1,  0,  3,  0),
ZZ^2+M(1, 0, -1, -1,  0,  0,  3),
ZZ^2+M(1, 0, -1, -1,  0,  0,  0),
ZZ^2+M(0, 1,  2,  0, -1, -1, -1),
ZZ^2+M(0, 1,  0,  4, -1, -1, -1),
ZZ^2+M(0, 1,  0,  0, -1,  3, -1),
ZZ^2+M(0, 1,  0,  0,  3, -1, -1)
in 7-d lattice ZZ^2+M
```

```
ZZ^2+N(0, 1, -2, -1, -1, -1, -1),
ZZ^2+N(0, 1,  0,  0,  0,  0,  0),
ZZ^2+N(0, 1,  0,  0,  0,  0,  1),
ZZ^2+N(0, 1,  0,  0,  0,  1,  0),
ZZ^2+N(0, 1,  0,  0,  1,  0,  0),
ZZ^2+N(1, 0,  0,  0,  0,  0,  0),
ZZ^2+N(1, 0,  0,  1,  0,  0,  0),
ZZ^2+N(1, 0,  1,  0,  0,  0,  0)
in 7-d lattice ZZ^2+N
```

(5.11)

respectively. Let $(\mathscr{A},\mathscr{B})$ be the Hilbert bases of $(\sigma,\sigma^\bullet)$, which are given by the lattice points in the polytopes spanned by the two lists in (5.11).

The only pairs of faces $(F,F^{\ddot\vee})$ where F is a face of $\Delta_{\mathscr{A}}$ and $F^{\ddot\vee}$ is the corresponding naïve dual face of $\Delta_{\mathscr{B}}$ are $(F,F^{\ddot\vee})=(\varnothing,\Delta_{\mathscr{B}})$ and $(F,F^{\ddot\vee})=(\Delta_{\mathscr{A}},\varnothing)$. The corresponding $\widetilde{S}$-polynomials are

$$S_\varnothing(t)=1 \qquad S_{\varnothing^{\ddot\vee}}=t^5+t^4+t^3+t^2 \tag{5.12}$$

$$S_{\Delta_{\mathscr{A}}}=t^5+79\,t^4+79\,t^3+t^2 \qquad S_{\Delta_{\mathscr{A}}^{\ddot\vee}}=1 \tag{5.13}$$

By definition $\widetilde{E}_{\rm st}(Y_{\Delta_1,\Delta_2};u,v)=\widetilde{E}_{\rm st}(\Delta_{\mathscr{A}};u,v)$. Thus (5.12) and (5.13) imply

$$\widetilde{E}_{\rm st}(Y_{\Delta_1,\Delta_2};u,v)=u^3v^3+u^2v^2-u^3-79\,u^2v-79\,uv^2-v^3+uv+1 \tag{5.14}$$

Hence

$$\tilde{h}^{11}_{\rm st}(Y_{\Delta_1,\Delta_2};u,v)=h^{11}(Y_{\Delta_1,\Delta_2};u,v)=1 \tag{5.15}$$

and

$$\tilde{h}^{21}_{\rm st}(Y_{\Delta_1,\Delta_2};u,v)=h^{21}(Y_{\Delta_1,\Delta_2};u,v)=79 \tag{5.16}$$

as expected from Conjecture 5.1.6.

We can now compute the naïve stringy Hodge numbers of Y_{∇_1,∇_2}. The only pairs of faces $(F, F^{\ddot\vee})$ where F is a face of $\Delta_{\mathscr{B}}$ and $F^{\ddot\vee}$ is the corresponding naïve dual face of $\Delta_{\mathscr{A}}$ are $(F, F^{\ddot\vee}) = (\varnothing, \Delta_{\mathscr{A}})$ and $(F, F^{\ddot\vee}) = (\Delta_{\mathscr{B}}, \varnothing)$. The corresponding $\widetilde{S}$-polynomials are

$$S_{\varnothing}(t) = 1 \qquad S_{\varnothing^{\ddot\vee}}(t) = t^5 + 79\,t^4 + 79\,t^3 + t^2 \tag{5.17}$$

$$S_{\Delta_{\mathscr{B}}}(t) = t^5 + t^4 + t^3 + t^2 \qquad S_{\Delta_{\mathscr{B}}^{\ddot\vee}}(t) = 1 \tag{5.18}$$

By definition $\widetilde{E}_{\rm st}(Y_{\nabla_1,\nabla_2}; u, v) = \widetilde{E}_{\rm st}(\Delta_{\mathscr{B}}; u, v)$. Thus (5.17) and (5.18) imply

$$\widetilde{E}_{\rm st}(Y_{\nabla_1,\nabla_2}; u, v) = u^3v^3 + 79\,u^2v^2 - u^3 - u^2v - uv^2 - v^3 + 79\,uv + 1 \tag{5.19}$$

Hence $\tilde{h}^{11}_{\rm st}(Y_{\nabla_1,\Delta_2}; u, v) = 79$ and $\tilde{h}^{11}_{\rm st}(Y_{\nabla_1,\nabla_2}; u, v) = 1$, as expected from Conjecture 5.1.7.

5.1.3 A Stringy Approach to $\mathbb{P}^5_{321111}[4,5]$

Fix $M \simeq \mathbb{Z}^5$ and let $\Delta \subset M_{\mathbb{R}}$ and $\nabla \subset N_{\mathbb{R}}$ be the polytopes whose vertices are given by

```
M_QQ(  1,1/2, -1, -1, -1),
M_QQ(  1, -1,  2, -1, -1),
M_QQ(  1, -1, -1,  2, -1),
M_QQ(  1, -1, -1, -1,  2),
M_QQ( -1,7/2, -1, -1, -1),
M_QQ( -1, -1, -1, -1, -1),
M_QQ(  1, -1, -1, -1, -1),
M_QQ( -1, -1,  8, -1, -1),
M_QQ( -1, -1, -1, -1,  8),
M_QQ( -1, -1, -1,  8, -1)
in 5-d vector space M_QQ
```

```
N_QQ(  -3,  -2,  -1,  -1,  -1),
N_QQ(  -3,  -1,  -1,  -1,  -1),
N_QQ(  -2,  -2,  -1,  -1,  -1),
N_QQ(  -1,-1/2,   0,   0,   0),
N_QQ(   1,   0,   1,   0,   0),
N_QQ(   1,   0,   0,   1,   0),
N_QQ(  -1,   1,   0,   0,   0),
N_QQ(   1,   0,   0,   0,   1),
N_QQ(   1,   0,   0,   0,   0),
N_QQ( 1/2,-1/2,   0,   0,   0),
N_QQ(   0,   0,   0,   0,   1),
N_QQ(   0,   0,   0,   1,   0),
N_QQ(   0,   0,   1,   0,   0),
N_QQ(   0,   1,   0,   0,   0),
N_QQ(   0,   1,   0,   0,   1),
N_QQ(   0,   1,   0,   1,   0),
N_QQ(   0,   1,   1,   0,   0)
in 5-d vector space N_QQ
```

(5.20)

respectively. Then Δ and ∇ are $\mathbb{Q}$-reflexive polytopes that are not reflexive. The dual polytopes $\Delta^* \subset N_\mathbb{R}$ and $\nabla^* \subset M_\mathbb{R}$ have vertices

```
N( 1,  0,  0,  0,  0),
N( 0,  1,  0,  0,  0),
N( 0,  0,  1,  0,  0),
N( 0,  0,  0,  1,  0),
N( 0,  0,  0,  0,  1),
N(-3, -2, -1, -1, -1),
N(-1,  0,  0,  0,  0)
in 5-d lattice N
```

```
M(-1, -1,  0,  0,  0),
M(-1, -1,  0,  0,  5),
M(-1, -1,  0,  5,  0),
M(-1, -1,  5,  0,  0),
M(-1,  1,  0,  0,  0),
M(-1,  1,  0,  0,  1),
M(-1,  1,  0,  1,  0),
M(-1,  1,  1,  0,  0),
M( 0, -1,  0,  0,  0),
M( 0, -1,  0,  0,  2),
M( 0, -1,  0,  2,  0),
M( 0, -1,  2,  0,  0),
M( 0,  0, -1, -1, -1),
M( 0,  0, -1, -1,  3),
M( 0,  0, -1,  3, -1),
M( 0,  0,  3, -1, -1),
M( 0,  2, -1, -1, -1),
M( 1,  0, -1, -1, -1),
M( 1,  0, -1, -1,  0),
M( 1,  0, -1,  0, -1),
M( 1,  0,  0, -1, -1)
in 5-d lattice M
```

(5.21)

respectively. Note that X_Δ is a partial resolution of the weighted projective space $\mathbb{P}_{321111}$.

Now, let $\Delta_1, \Delta_2 \subset M_\mathbb{R}$ be the polytopes with respective vertices

```
M_QQ(  0, -1,  0,  0,  2),
M_QQ(  0,  0,  0,  0,  0),
M_QQ(  0, -1,  2,  0,  0),
M_QQ(  0, -1,  0,  0,  0),
M_QQ(  0, -1,  0,  2,  0),
M_QQ( -1, -1,  0,  5,  0),
M_QQ( -1, -1,  0,  0,  0),
M_QQ( -1, -1,  5,  0,  0),
M_QQ( -1,3/2,  0,  0,  0),
M_QQ( -1, -1,  0,  0,  5)
in 5-d vector space M_QQ
```

```
M_QQ(  1,  0, -1, -1,  0),
M_QQ(  1,1/2, -1, -1, -1),
M_QQ(  1,  0,  0, -1, -1),
M_QQ(  1,  0, -1, -1, -1),
M_QQ(  1,  0, -1,  0, -1),
M_QQ(  0,  0, -1,  3, -1),
M_QQ(  0,  0, -1, -1, -1),
M_QQ(  0,  0,  3, -1, -1),
M_QQ(  0,  2, -1, -1, -1),
M_QQ(  0,  0, -1, -1,  3)
in 5-d vector space M_QQ
```

(5.22)

and let $\nabla_1, \nabla_2 \subset N_{\mathbb{R}}$ be the polytopes with respective vertices

```
N_QQ(0,0,0,0,0),
N_QQ(0,1,0,0,0),
N_QQ(1,0,0,0,0)
in 5-d vector space N_QQ
```

```
N_QQ(  -3,  -2,  -1,  -1,  -1),
N_QQ(  -1,   0,   0,   0,   0),
N_QQ(   0,   0,   0,   0,   1),
N_QQ(   0,   0,   0,   1,   0),
N_QQ(-1/2,-1/2,   0,   0,   0),
N_QQ(   0,   0,   0,   0,   0),
N_QQ(   0,   0,   1,   0,   0),
N_QQ(  -1,-1/2,   0,   0,   0)
in 5-d vector space N_QQ
```

(5.23)

Then $\Delta = \Delta_1 + \Delta_2$ and $\nabla = \nabla_1 + \nabla_2$ form a dual pair of $\mathbb{Q}$-nef-partitions. The Cayley cones $\mathcal{C}_{[\Delta_1],[\Delta_2]}$ and $\mathcal{C}_{[\nabla_1],[\nabla_2]}$ are generated by the points

```
ZZ^2+M(0, 1,  0,  0, -1, -1, -1),
ZZ^2+M(1, 0,  0,  0,  0,  0,  0),
ZZ^2+M(1, 0,  0, -1,  2,  0,  0),
ZZ^2+M(1, 0,  0, -1,  0,  2,  0),
ZZ^2+M(0, 1,  0,  0, -1, -1,  3),
ZZ^2+M(1, 0,  0, -1,  0,  0,  2),
ZZ^2+M(1, 0,  0, -1,  0,  0,  0),
ZZ^2+M(1, 0, -1,  1,  1,  0,  0),
ZZ^2+M(1, 0, -1,  1,  0,  1,  0),
ZZ^2+M(1, 0, -1,  1,  0,  0,  1),
ZZ^2+M(1, 0, -1,  1,  0,  0,  0),
ZZ^2+M(1, 0, -1, -1,  5,  0,  0),
ZZ^2+M(1, 0, -1, -1,  0,  5,  0),
ZZ^2+M(1, 0, -1, -1,  0,  0,  5),
ZZ^2+M(0, 1,  0,  0, -1,  3, -1),
ZZ^2+M(1, 0, -1, -1,  0,  0,  0),
ZZ^2+M(0, 1,  1,  0,  0, -1, -1),
ZZ^2+M(0, 1,  1,  0, -1,  0, -1),
ZZ^2+M(0, 1,  1,  0, -1, -1,  0),
ZZ^2+M(0, 1,  1,  0, -1, -1, -1),
ZZ^2+M(0, 1,  0,  2, -1, -1, -1),
ZZ^2+M(0, 1,  0,  0,  3, -1, -1)
in 7-d lattice ZZ^2+M
```

```
ZZ^2+N(0, 1, -3, -2, -1, -1, -1),
ZZ^2+N(0, 1, -1,  0,  0,  0,  0),
ZZ^2+N(0, 1,  0,  0,  0,  0,  0),
ZZ^2+N(0, 1,  0,  0,  0,  0,  1),
ZZ^2+N(0, 1,  0,  0,  0,  1,  0),
ZZ^2+N(0, 1,  0,  0,  1,  0,  0),
ZZ^2+N(1, 0,  0,  0,  0,  0,  0),
ZZ^2+N(1, 0,  0,  1,  0,  0,  0),
ZZ^2+N(1, 0,  1,  0,  0,  0,  0)
in 7-d lattice ZZ^2+N
```

(5.24)

respectively. Let $(\mathscr{A}, \mathscr{B})$ be the Hilbert bases of these cones.

We may now compute the naïve stringy Hodge numbers of Y_{Δ_1,Δ_2} and Y_{∇_1,∇_2}.

To compute $\tilde{h}^{p,q}_{\rm st}(Y_{\Delta_1,\Delta_2})$, note that he only pairs of faces $(F, F^{\ddot{\vee}})$ where F is a face of $\Delta_{\mathscr{A}}$ and $F^{\ddot{\vee}}$ is the corresponding naïve dual face of $\Delta_{\mathscr{B}}$ are $(F, F^{\ddot{\vee}}) = (\varnothing, \Delta_{\mathscr{B}})$ and $(F, F^{\ddot{\vee}}) = (\Delta_{\mathscr{A}}, \varnothing)$. The corresponding $\widetilde{S}$-polynomials are

$$S_{\varnothing}(t) = 1 \qquad S_{\varnothing^{\ddot{\vee}}} = t^5 + 2\,t^4 + 2\,t^3 + t^2 \tag{5.25}$$

$$S_{\Delta_{\mathscr{A}}} = t^5 + 84\,t^4 + 84\,t^3 + t^2 \qquad S_{\Delta^{\ddot{\vee}}_{\mathscr{A}}} = 1 \tag{5.26}$$

By definition $\widetilde{E}_{\rm st}(Y_{\Delta_1,\Delta_2}; u, v) = \widetilde{E}_{\rm st}(\Delta_{\mathscr{A}}; u, v)$. Thus (5.25) and (5.26) imply

$$\widetilde{E}_{\rm st}(Y_{\Delta_1,\Delta_2}; u, v) = u^3v^3 + 2\,u^2v^2 - u^3 - 84\,u^2v - 84\,uv^2 - v^3 + 2\,uv + 1 \tag{5.27}$$

Hence

$$\tilde{h}^{11}_{\rm st}(Y_{\Delta_1,\Delta_2}; u, v) = h^{11}(Y_{\Delta_1,\Delta_2}; u, v) = 2 \tag{5.28}$$

and

$$\tilde{h}^{21}_{\rm st}(Y_{\Delta_1,\Delta_2}; u, v) = h^{21}(Y_{\Delta_1,\Delta_2}; u, v) = 84 \tag{5.29}$$

as expected from Conjecture 5.1.6.

To compute $\tilde{h}^{p,q}_{\rm st}(Y_{\nabla_1,\nabla_2})$ note that the only pairs of faces $(F, F^{\ddot{\vee}})$ where F is a face of $\Delta_{\mathscr{B}}$ and $F^{\ddot{\vee}}$ is the corresponding naïve dual face of $\Delta_{\mathscr{A}}$ are $(F, F^{\ddot{\vee}}) = (\varnothing, \Delta_{\mathscr{A}})$ and $(F, F^{\ddot{\vee}}) = (\Delta_{\mathscr{B}}, \varnothing)$. The corresponding $\widetilde{S}$-polynomials are

$$S_{\varnothing}(t) = 1 \qquad S_{\varnothing^{\ddot{\vee}}}(t) = t^5 + 84\,t^4 + 84\,t^3 + t^2 \tag{5.30}$$

$$S_{\Delta_{\mathscr{B}}}(t) = t^5 + 2\,t^4 + 2\,t^3 + t^2 \qquad S_{\Delta^{\ddot{\vee}}_{\mathscr{B}}}(t) = 1 \tag{5.31}$$

By definition $\widetilde{E}_{\rm st}(Y_{\nabla_1,\nabla_2}; u, v) = \widetilde{E}_{\rm st}(\Delta_{\mathscr{B}}; u, v)$. Thus (5.30) and (5.31) imply

$$\widetilde{E}_{\rm st}(Y_{\nabla_1,\nabla_2}; u, v) = u^3v^3 + 84\,u^2v^2 - u^3 - 2\,u^2v - 2\,uv^2 - v^3 + 84\,uv + 1 \tag{5.32}$$

Hence $\tilde{h}^{11}_{\rm st}(Y_{\nabla_1,\Delta_2}; u, v) = 84$ and $\tilde{h}^{11}_{\rm st}(Y_{\nabla_1,\nabla_2}; u, v) = 2$, as expected from Conjecture 5.1.7.

5.2 Almost Reflexive Polytopes Embedded in a Smooth Fano Polytope

We now wish to test the naïve stringy E-function on more examples of complete mirror pairs. Recall from §4.3.1 that there are 209 distinct dual pairs of $\mathbb{Q}$-nef-partitions $\Delta = \Delta_1 + \Delta_2$ and $\nabla = \nabla_1 + \nabla_2$ where Δ is a five-dimensional $\mathbb{Q}$-reflexive polytope that is not reflexive such that Δ^* embeds in a five-dimensional smooth Fano polytope. The Hodge data for the Calabi-Yau complete intersections Y_{Δ_1,Δ_2} can be found in Table 4.1. By computing the naïve stringy Hodge numbers of Y_{Δ_1,Δ_2} and Y_{∇_1,∇_2}, we are able to replicate this Hodge data.

Theorem 5.2.1. *Let $\Delta = \Delta_1 + \Delta_2$ and $\nabla = \nabla_1 + \nabla_2$ be a dual pair of $\mathbb{Q}$-nef-partitions where Δ is a five-dimensional $\mathbb{Q}$-reflexive polytope that is not reflexive such that Δ^* embeds in a five-dimensional smooth Fano polytope. Then the naïve stringy Hodge numbers $\tilde{h}^{p,q}_{\text{st}}(Y_{\Delta_1,\Delta_2})$ and the Hodge numbers $h^{p,q}(Y_{\Delta_1,\Delta_2})$ of Y_{Δ_1,Δ_2} agree. Moreover, the pair $(Y_{\Delta_1,\Delta_2}, Y_{\nabla_1,\nabla_2})$ passes the naïve stringy mirror symmetry test.*

The data for the computations involved in proving Theorem 5.2.1 is as follows. The first column of this table represents the index of the almost reflexive polytope Δ^* in the data supplement to this dissertation found at [20].

Δ^*	D_{Δ_1}	D_{Δ_2}	h^{11}_{st}	h^{21}_{st}	$-\chi$	$\#\Delta^* \cap N$	$\#\Delta^{*v}$	$\#\nabla^* \cap M$	$\#\nabla^{*v}$
0	$\{0,1,4,5\}$	$\{2,3\}$	1	79	156	7	6	129	13
1	$\{0,1,2,3,4,5\}$	$\{6\}$	2	95	186	8	7	169	23
1	$\{0,3,6\}$	$\{1,2,4,5\}$	3	77	148	8	7	125	17
2	$\{0,3,4,5,6,7,8\}$	$\{1,2\}$	4	68	128	10	9	100	20
2	$\{0,2,3,4,5,6,7\}$	$\{8,1\}$	3	61	116	10	9	86	20
2	$\{0,1,2,3,4,5,6\}$	$\{8,7\}$	4	61	114	10	9	102	32
2	$\{0,3,4,5,6,7\}$	$\{8,1,2\}$	4	50	92	10	9	67	26
2	$\{0,2,3,4,5,6\}$	$\{8,1,7\}$	4	51	94	10	9	68	26
2	$\{0,3,4,5,6\}$	$\{8,1,2,7\}$	4	44	80	10	9	58	32
2	$\{0,1,6,8,7\}$	$\{2,3,4,5\}$	4	40	72	10	9	51	20
3	$\{0,1,2,4,5,6,7\}$	$\{3\}$	2	92	180	9	8	193	20
3	$\{0,1,2,3,5,6,7\}$	$\{4\}$	3	79	152	9	8	138	26
3	$\{0,4\}$	$\{1,2,3,5,6,7\}$	3	73	140	9	8	119	26
3	$\{0,1,2,5,6,7\}$	$\{3,4\}$	3	61	116	9	8	92	32
3	$\{0,3,4\}$	$\{1,2,5,6,7\}$	3	55	104	9	8	73	20

Δ^*	D_{Δ_1}	D_{Δ_2}	$h^{11}_{\rm st}$	$h^{21}_{\rm st}$	$-\chi$	$\#\Delta^*\cap N$	$\#\Delta^{*v}$	$\#\nabla^*\cap M$	$\#\nabla^{*v}$
4	$\{0,2,3,4,5,6\}$	$\{1\}$	1	103	204	8	7	178	21
4	$\{0,1,3,4,5,6\}$	$\{2\}$	2	122	240	8	7	260	17
4	$\{0,3,4,5,6\}$	$\{1,2\}$	2	84	164	8	7	140	21
5	$\{0,7\}$	$\{1,2,3,4,5,6,8\}$	4	54	100	10	9	94	30
5	$\{0,1,2,3,4,5,6\}$	$\{8,7\}$	4	55	102	10	9	85	26
5	$\{0,8,7\}$	$\{1,2,3,4,5,6\}$	4	49	90	10	9	65	26
5	$\{0,3,4,5,6\}$	$\{8,1,2,7\}$	4	36	64	10	9	45	22
5	$\{0,2,4,5,6\}$	$\{8,1,3,7\}$	4	41	74	10	9	52	34
6	$\{0,6\}$	$\{1,2,3,4,5,7\}$	2	84	164	9	8	130	17
6	$\{0,2,3,4,5,6\}$	$\{1,7\}$	2	84	164	9	8	130	17
6	$\{0,1,2,3,4,5\}$	$\{6,7\}$	3	67	128	9	8	104	25
6	$\{0,6,7\}$	$\{1,2,3,4,5\}$	3	61	116	9	8	84	21
6	$\{0,2,3,4,5\}$	$\{1,6,7\}$	3	61	116	9	8	84	21
6	$\{0,1,6,7\}$	$\{2,3,4,5\}$	3	55	104	9	8	64	17
7	$\{0,2,3,4,5,6\}$	$\{1\}$	1	103	204	8	7	214	17
7	$\{0,1,3,4,5,6\}$	$\{2\}$	2	90	176	8	7	160	21
7	$\{0,2\}$	$\{1,3,4,5,6\}$	2	84	164	8	7	140	21
7	$\{0,3,4,5,6\}$	$\{1,2\}$	2	68	132	8	7	104	25
7	$\{0,1,2\}$	$\{3,4,5,6\}$	2	62	120	8	7	84	17
8	$\{0,1,2,4,5,6,7,8\}$	$\{3\}$	3	69	132	10	9	126	29
8	$\{0,1,2,3,4,5,6,7\}$	$\{8\}$	4	68	128	10	9	124	33
8	$\{0,1,2,4,5,6,7\}$	$\{8,3\}$	4	50	92	10	9	72	41
8	$\{0,1,2,8\}$	$\{3,4,5,6,7\}$	4	50	92	10	9	68	25
8	$\{0,1,2,3,8\}$	$\{4,5,6,7\}$	8	36	56	10	9	52	17
9	$\{0\}$	$\{1,2,3,4,5,6\}$	2	86	168	8	7	150	16
9	$\{0,1\}$	$\{2,3,4,5,6\}$	2	72	140	8	7	115	22
9	$\{0,3,4\}$	$\{1,2,5,6\}$	2	58	112	8	7	84	22
10	$\{0\}$	$\{1,2,3,4,5\}$	1	101	200	7	6	181	15
10	$\{0,2,3,4,5\}$	$\{1\}$	1	103	204	7	6	300	10
10	$\{0,1\}$	$\{2,3,4,5\}$	1	79	156	7	6	125	15
11	$\{0,2,3,4,5,6,7\}$	$\{1\}$	2	86	168	9	8	170	23
11	$\{0,1,2,3,4,5,6\}$	$\{7\}$	3	79	152	9	8	142	29
11	$\{0,2,3,4,5,6\}$	$\{1,7\}$	3	59	112	9	8	88	35
11	$\{0,2,7\}$	$\{1,3,4,5,6\}$	3	67	128	9	8	104	25
11	$\{0,1,2,7\}$	$\{3,4,5,6\}$	3	47	88	9	8	68	19
12	$\{0\}$	$\{1,2,3,4,5,6,7\}$	1	103	204	9	8	180	14
12	$\{0,1,3,4,5,6,7\}$	$\{2\}$	3	83	160	9	8	147	21
12	$\{0,2\}$	$\{1,3,4,5,6,7\}$	3	61	116	9	8	91	22
12	$\{0,3,4,5,6,7\}$	$\{1,2\}$	2	84	164	9	8	137	18
12	$\{0,1,2\}$	$\{3,4,5,6,7\}$	2	62	120	9	8	81	17
13	$\{0,1\}$	$\{2,3,4,5,6,7\}$	3	69	132	9	8	103	16
13	$\{0,7\}$	$\{1,2,3,4,5,6\}$	3	61	116	9	8	90	22
13	$\{0,1,7\}$	$\{2,3,4,5,6\}$	3	51	96	9	8	70	28
13	$\{0,3,6,7\}$	$\{1,2,4,5\}$	3	41	76	9	8	54	22
14	$\{0\}$	$\{1,2,3,4,5,6,7\}$	3	69	132	9	8	140	22
14	$\{0,1,2,3,4,5,6\}$	$\{7\}$	3	72	138	9	8	129	26
14	$\{0,7\}$	$\{1,2,3,4,5,6\}$	3	54	102	9	8	77	34
14	$\{0,3,4,5,6\}$	$\{1,2,7\}$	3	51	96	9	8	69	22
14	$\{0,1,2,7\}$	$\{3,4,5,6\}$	7	37	60	9	8	53	14
15	$\{0,1,2,3,4,5,6,7\}$	$\{8\}$	3	71	136	10	9	121	23
15	$\{0,2\}$	$\{1,3,4,5,6,7,8\}$	4	70	132	10	9	134	22
15	$\{0,1,2,3,4,5,7\}$	$\{8,6\}$	4	62	116	10	9	101	35
15	$\{0,8,2\}$	$\{1,3,4,5,6,7\}$	4	46	84	10	9	65	19
15	$\{0,8,2,6\}$	$\{1,3,4,5,7\}$	4	48	88	10	9	68	35
16	$\{0,2,3,4,5,6\}$	$\{1\}$	2	86	168	8	7	215	16
16	$\{0,1,2,3,4,5\}$	$\{6\}$	2	86	168	8	7	154	24
16	$\{0,6\}$	$\{1,2,3,4,5\}$	2	76	148	8	7	120	18
16	$\{0,2,3,4,5\}$	$\{1,6\}$	2	66	128	8	7	100	24
16	$\{0,1,6\}$	$\{2,3,4,5\}$	2	56	108	8	7	84	18
17	$\{0,1,3,4,5,6,8\}$	$\{2,7\}$	3	69	132	10	9	105	23
17	$\{0,1,2,3,4,5,6\}$	$\{8,7\}$	4	59	110	10	9	92	31

Δ^*	D_{Δ_1}	D_{Δ_2}	h^{11}_{st}	h^{21}_{st}	$-\chi$	$\#\Delta^* \cap N$	$\#\Delta^{*v}$	$\#\nabla^* \cap M$	$\#\nabla^{*v}$
17	{0,1,3,4,5,6}	{8,2,7}	4	53	98	10	9	72	29
17	{0,1,8,7}	{2,3,4,5,6}	4	48	88	10	9	64	25
18	{0,1,2,3,4,5,6}	{7}	3	75	144	9	8	132	25
18	{0,2,3,4,5,6}	{1,7}	3	69	132	9	8	112	29
18	{0,2,7}	{1,3,4,5,6}	3	55	104	9	8	80	31
19	{0,2,3,4,5,6,7}	{1}	3	77	148	9	8	175	20
19	{0,1,2,3,4,5,6}	{7}	2	86	168	9	8	146	19
19	{0,2,3,4,5,6}	{1,7}	3	59	112	9	8	85	19
19	{0,1,2,4,5,6}	{3,7}	3	70	134	9	8	113	29
19	{0,2,4,5,6}	{1,3,7}	3	55	104	9	8	79	29
20	{0,1,2,3,4,5,7}	{6}	3	69	132	9	8	122	22
20	{0,2,3,4,5,7}	{1,6}	3	59	112	9	8	88	23
20	{0,1,6}	{2,3,4,5,7}	3	49	92	9	8	68	25
21	{0,1,2,4,5,6,7}	{3}	2	86	168	9	8	146	20
21	{0,1,2,3,5,6,7}	{4}	3	83	160	9	8	165	26
21	{0,2,4,5,6,7}	{1,3}	3	71	136	9	8	112	26
21	{0,1,2,5,6,7}	{3,4}	3	60	114	9	8	88	32
21	{0,3,4,5}	{1,2,6,7}	3	57	108	9	8	81	20
22	{0,2,3,4,5,6,7}	{1}	1	103	204	9	8	158	17
22	{0,3,4,5,6,7}	{1,2}	2	68	132	9	8	94	21
22	{0,1,3,4,5,6}	{2,7}	3	83	160	9	8	140	21
22	{0,3,4,5,6}	{1,2,7}	3	61	116	9	8	84	21
23	{0,1,2,3,4,5,6}	{7}	3	71	136	9	8	126	26
23	{0,4}	{1,2,3,5,6,7}	3	75	144	9	8	140	20
23	{0,2,3,4,5,6}	{1,7}	3	61	116	9	8	92	26
23	{0,4,7}	{1,2,3,5,6}	3	53	100	9	8	76	32
23	{0,1,4,7}	{2,3,5,6}	3	43	80	9	8	60	20
24	{0,2,3,4,5,6}	{1}	2	92	180	8	7	225	14
24	{0,1,2,3,4,5}	{6}	2	86	168	8	7	153	23
24	{0,2,3,4,5}	{1,6}	2	66	128	8	7	101	23
25	{0}	{1,2,3,4,5,6,7}	3	85	164	9	8	185	20
25	{0,4}	{1,2,3,5,6,7}	3	63	120	9	8	92	23
25	{0,1,2,5,6,7}	{3,4}	3	76	146	9	8	129	29
25	{0,3,4}	{1,2,5,6,7}	3	57	108	9	8	83	29
26	{0,1,2,3,4,5,6}	{7}	3	72	138	9	8	134	18
26	{0,7}	{1,2,3,4,5,6}	3	66	126	9	8	114	26
26	{0,3,4,5,6}	{1,2,7}	3	51	96	9	8	70	26
27	{0,2,3,4,5,6,7}	{1}	3	99	192	9	8	190	17
27	{0,1,3,4,5,6,7}	{2}	1	103	204	9	8	165	18
27	{0,3,4,5,6,7}	{1,2}	3	69	132	9	8	106	18
28	{0,2,3,4,5,6,7}	{1}	3	75	144	9	8	180	20
28	{0,1,2,3,4,5,6}	{7}	2	86	168	9	8	150	21
28	{0,2,3,4,5,6}	{1,7}	3	61	116	9	8	90	21
28	{0,1,2,3,5,6}	{4,7}	3	75	144	9	8	119	22
28	{0,4,7}	{1,2,3,5,6}	3	65	124	9	8	100	22
28	{0,2,3,5,6}	{1,4,7}	3	55	104	9	8	80	22
28	{0,1,4,7}	{2,3,5,6}	3	51	96	9	8	74	22
29	{0}	{1,2,3,4,5,6,7}	3	75	144	9	8	170	22
29	{0,1,2,3,4,6,7}	{5}	2	86	168	9	8	148	18
29	{0,5}	{1,2,3,4,6,7}	3	60	114	9	8	88	24
29	{0,1,2,3,4,6}	{5,7}	3	72	138	9	8	124	24
29	{0,5,7}	{1,2,3,4,6}	3	55	104	9	8	80	24
30	{0,1,2,3,4,5,6,7}	{8}	4	60	112	10	9	104	25
30	{0,1,3,4,5,6,7}	{8,2}	4	54	100	10	9	84	33
30	{0,1,3,8}	{2,4,5,6,7}	4	42	76	10	9	56	33
31	{0}	{1,2,3,4,5,6}	1	103	204	8	7	159	10
31	{0,1}	{2,3,4,5,6}	2	68	132	8	7	95	18
32	{0}	{1,2,3,4,5,6}	1	103	204	8	7	192	10
32	{0,1,3,4,5,6}	{2}	2	84	164	8	7	149	18
32	{0,2}	{1,3,4,5,6}	2	62	120	8	7	93	18
33	{0}	{1,2,3,4,5,6,7}	2	84	164	9	8	164	16

Δ^*	D_{Δ_1}	D_{Δ_2}	$h^{11}_{\rm st}$	$h^{21}_{\rm st}$	$-\chi$	$\#\Delta^* \cap N$	$\#\Delta^{*v}$	$\#\nabla^* \cap M$	$\#\nabla^{*v}$
33	$\{0,1,2,3,4,5,7\}$	$\{6\}$	2	84	164	9	8	143	18
33	$\{0,6\}$	$\{1,2,3,4,5,7\}$	3	55	104	9	8	77	15
33	$\{0,2,3,4,5,7\}$	$\{1,6\}$	3	69	132	9	8	113	24
33	$\{0,1,6\}$	$\{2,3,4,5,7\}$	3	53	100	9	8	78	24
34	$\{0,2,3,4,5,6,7\}$	$\{1\}$	3	73	140	9	8	151	20
34	$\{0,1,2,3,4,5,6\}$	$\{7\}$	2	86	168	9	8	144	17
34	$\{0,2,3,4,5,6\}$	$\{1,7\}$	3	59	112	9	8	82	23
34	$\{0,1,2,4,5,6\}$	$\{3,7\}$	3	65	124	9	8	101	29
34	$\{0,2,4,5,6\}$	$\{1,3,7\}$	3	51	96	9	8	70	29
35	$\{0,4\}$	$\{1,2,3,5,6,7,8\}$	3	73	140	10	9	110	20
35	$\{0,1,2,4,5,6,7\}$	$\{8,3\}$	3	77	148	10	9	119	20
35	$\{0,1,2,3,5,6,7\}$	$\{8,4\}$	4	60	112	10	9	92	32
35	$\{0,8,4\}$	$\{1,2,3,5,6,7\}$	4	54	100	10	9	73	26
35	$\{0,1,2,5,6,7\}$	$\{8,3,4\}$	4	56	104	10	9	76	26
35	$\{0,2,5,6\}$	$\{8,1,3,4,7\}$	4	46	84	10	9	61	32
35	$\{0,8,3,4\}$	$\{1,2,5,6,7\}$	4	50	92	10	9	57	20
36	$\{0,2,3,4,5,6\}$	$\{1\}$	2	84	164	8	7	216	14
36	$\{0,1,2,3,4,5\}$	$\{6\}$	1	101	200	8	7	171	15
36	$\{0,6\}$	$\{1,2,3,4,5\}$	2	78	152	8	7	125	19
36	$\{0,2,3,4,5\}$	$\{1,6\}$	2	72	140	8	7	105	15
36	$\{0,1,6\}$	$\{2,3,4,5\}$	2	62	120	8	7	90	19
37	$\{0\}$	$\{1,2,3,4,5\}$	1	103	204	7	6	179	14
38	$\{0,2,3,4,5,6,7,8\}$	$\{1\}$	2	84	164	10	9	162	20
38	$\{0,1,3,4,5,6,7,8\}$	$\{2\}$	3	75	144	10	9	120	21
38	$\{0,3,4,5,6,7,8\}$	$\{1,2\}$	4	58	108	10	9	79	22
38	$\{0,1,4,5,6,7,8\}$	$\{2,3\}$	4	58	108	10	9	86	26
38	$\{0,2,3\}$	$\{1,4,5,6,7,8\}$	4	54	100	10	9	76	24
38	$\{0,4,5,6,7,8\}$	$\{1,2,3\}$	4	48	88	10	9	66	25
38	$\{0,1,2,3\}$	$\{8,4,5,6,7\}$	4	44	80	10	9	56	20
39	$\{0,2,3,4,5,6,7\}$	$\{1\}$	1	103	204	9	8	211	14
39	$\{0,1,3,4,5,6,7\}$	$\{2\}$	3	75	144	9	8	126	21
39	$\{0,2\}$	$\{1,3,4,5,6,7\}$	3	69	132	9	8	106	18
39	$\{0,3,4,5,6,7\}$	$\{1,2\}$	3	61	116	9	8	91	22
39	$\{0,1,2\}$	$\{3,4,5,6,7\}$	3	55	104	9	8	71	17
40	$\{0\}$	$\{1,2,3,4,5,6,7\}$	2	92	180	9	8	184	20
40	$\{0,1,2,3,4,6,7\}$	$\{5\}$	3	81	156	9	8	144	26
40	$\{0,5\}$	$\{1,2,3,4,6,7\}$	3	61	116	9	8	92	32
40	$\{0,1,3,4,6,7\}$	$\{2,5\}$	3	77	148	9	8	128	26
40	$\{0,2,5\}$	$\{1,3,4,6,7\}$	3	57	108	9	8	76	20
41	$\{0,1,2,3,4,5,6,7\}$	$\{8\}$	3	71	136	10	9	119	21
41	$\{0,4\}$	$\{1,2,3,5,6,7,8\}$	4	66	124	10	9	110	20
41	$\{0,1,2,3,4,5,7\}$	$\{8,6\}$	4	57	106	10	9	89	35
41	$\{0,8,4\}$	$\{1,2,3,5,6,7\}$	4	46	84	10	9	62	23
41	$\{0,8,4,6\}$	$\{1,2,3,5,7\}$	4	44	80	10	9	59	33
42	$\{0,1,2,3,4,5,6\}$	$\{7\}$	3	71	136	9	8	125	27
42	$\{0,2\}$	$\{1,3,4,5,6,7\}$	3	81	156	9	8	150	14
42	$\{0,2,7\}$	$\{1,3,4,5,6\}$	3	53	100	9	8	77	27
43	$\{0,2,3,4,5,6,7\}$	$\{1\}$	2	84	164	9	8	165	23
43	$\{0,1,2,3,4,5,6\}$	$\{7\}$	2	90	176	9	8	154	21
43	$\{0,2,3,4,5,6\}$	$\{1,7\}$	3	61	116	9	8	88	25
43	$\{0,1,3,4,5,6\}$	$\{2,7\}$	3	71	136	9	8	114	24
43	$\{0,2,7\}$	$\{1,3,4,5,6\}$	3	67	128	9	8	104	27
43	$\{0,3,4,5,6\}$	$\{1,2,7\}$	3	55	104	9	8	79	28
43	$\{0,1,2,7\}$	$\{3,4,5,6\}$	3	51	96	9	8	69	20
44	$\{0,1,2,4,5,6,7,9\}$	$\{8,3\}$	4	54	100	11	10	80	29
44	$\{0,1,2,3,4,5,6,7\}$	$\{8,9\}$	5	51	92	11	10	80	33
44	$\{0,1,2,4,5,6,7\}$	$\{8,9,3\}$	5	45	80	11	10	60	33
44	$\{0,1,2,8,9\}$	$\{3,4,5,6,7\}$	5	35	60	11	10	44	25
45	$\{0,1,2,3,5,6,7\}$	$\{4\}$	2	84	164	9	8	142	17
45	$\{0,1,2,3,4,6,7\}$	$\{5\}$	2	84	164	9	8	142	17
45	$\{0,1,2,3,6,7\}$	$\{4,5\}$	3	55	104	9	8	76	18

Δ^*	D_{Δ_1}	D_{Δ_2}	$h^{11}_{\rm st}$	$h^{21}_{\rm st}$	$-\chi$	$\#\Delta^* \cap N$	$\#\Delta^{*v}$	$\#\nabla^* \cap M$	$\#\nabla^{*v}$
45	{0, 2, 5, 7}	{1, 3, 4, 6}	3	51	96	9	8	71	24
46	{0, 2, 3, 4, 5, 6, 7}	{1}	3	69	132	9	8	150	22
46	{0, 1, 2, 3, 4, 5, 6}	{7}	2	86	168	9	8	147	18
46	{0, 2, 3, 4, 5, 6}	{1, 7}	3	60	114	9	8	84	24
46	{0, 1, 2, 3, 5, 6}	{4, 7}	3	66	126	9	8	104	26
46	{0, 4, 7}	{1, 2, 3, 5, 6}	3	57	108	9	8	80	24
46	{0, 2, 3, 5, 6}	{1, 4, 7}	3	51	96	9	8	70	26
46	{0, 1, 4, 7}	{2, 3, 5, 6}	3	45	84	9	8	62	24
47	{0}	{1, 2, 3, 4, 5, 6}	2	86	168	8	7	188	16
47	{0, 1, 2, 4, 5, 6}	{3}	2	86	168	8	7	151	19
47	{0, 3}	{1, 2, 4, 5, 6}	2	64	124	8	7	95	25

5.3 Almost Reflexive Polytopes Embedded in a Reflexive Polytope on Kreuzer's List

Recall from §4.3.2 the 868 distinct pairs of dual $\mathbb{Q}$-nef-partitions $\Delta = \Delta_1 + \Delta_2$ and $\nabla = \nabla_1 + \nabla_2$ where Δ is a five-dimensional $\mathbb{Q}$-reflexive polytope that is not reflexive such that Δ^* embeds in a reflexive polytope on Kreuzer's list. The stringy Hodge data for the Calabi-Yau complete intersections Y_{Δ_1,Δ_2} can be found in Table 4.2. By computing the naïve stringy Hodge numbers of Y_{Δ_1,Δ_2} and Y_{∇_1,∇_2}, we are able to replicate this stringy Hodge data.

Theorem 5.3.1. *Let $\Delta = \Delta_1 + \Delta_2$ and $\nabla = \nabla_1 + \nabla_2$ be a dual pair of $\mathbb{Q}$-nef-partitions where Δ is a five-dimensional $\mathbb{Q}$-reflexive polytope that is not reflexive such that Δ^* embeds in a reflexive polytope on Kreuzer's list. Then the naïve stringy Hodge numbers $\tilde{h}^{p,q}_{\rm st}(Y_{\Delta_1,\Delta_2})$ and the Hodge numbers $h^{p,q}(Y_{\Delta_1,\Delta_2})$ of Y_{Δ_1,Δ_2} agree. Moreover, the pair $(Y_{\Delta_1,\Delta_2}, Y_{\nabla_1,\nabla_2})$ passes the naïve stringy mirror symmetry test.*

The data defining each of these 868 dual $\mathbb{Q}$-nef-partitions can be found in the data supplement to this dissertation found at [20].

5.4 Stringy Hodge Numbers of Weighted Projective Complete Intersections

Recall from §4.4.2 the complete mirror pairs $(\mathscr{A}, \mathscr{B})$ obtained by applying Algorithm 4.4.5 to 3906 of the 4239 transversal configurations of codimension two Calabi-Yau threefolds in weighted projective spaces on Klemm's list. By computing the naïve stringy Hodge numbers of $\Delta_{\mathscr{A}}$ and $\Delta_{\mathscr{B}}$, we are able to replicate the Hodge computations done by Klemm.

The data for these computations can be found in the file `wpcicydata.txt` in the data supplement to this dissertation found at [20]. For example, the lines

```
20 7 'A 11 1 1 1 2 3 3 d=6 5 h=3 77 complete'
 0  1  0 -1 -1  0  0
 1  0  1  0  0 -1 -1
 0  1  0 -1 -1  5  0
 1  0 -1  2  0 -1 -1
 1  0 -1  0  3 -1 -1
 0  1  0 -1 -1  0  5
 1  0 -1  0  0  5 -1
 1  0 -1  0  0 -1  5
 1  0 -1  0  0 -1 -1
 0  1  1 -1  0  0  0
 0  1  1 -1 -1  2  0
 0  1  1 -1 -1  0  2
 0  1  1 -1 -1  0  0
 0  1  0  0  0  0  0
 0  1  0  0 -1  2  0
 0  1  0  0 -1  0  2
 0  1  0  0 -1  0  0
 0  1  0 -1  1  1  0
 0  1  0 -1  1  0  1
 0  1  0 -1  1  0  0
9 7 'B 11 1 1 1 2 3 3 d=6 5 h=3 77 complete'
 0  1  0  0  0  0  0
```

```
0  1  0  0  1  0  0
0  1  0  1  0  0  0
1  0 -3 -3 -2 -1 -1
1  0 -1 -1  0  0  0
1  0  0  0  0  0  0
1  0  0  0  0  0  1
1  0  0  0  0  1  0
1  0  1  0  0  0  0
```

from the file `wpcicydata.txt` indicate that applying Algorithm 4.4.5 to the quasismooth weighted complete intersection $\mathbb{P}^5_{1,1,1,2,3,3}[6,5]$ yields a complete non-reflexive mirror pair $(\mathscr{A}, \mathscr{B})$ whose points are given by the indicated rows. The Hodge numbers computed by Klemm of the resolved $\mathbb{P}^5_{1,1,1,2,3,3}[6,5]$ are $h^{11} = 3$ and $h^{21} = 77$. Our naïve stringy Hodge numbers are $\tilde{h}^{11}_{\text{st}}(\Delta_{\mathscr{A}}) = \tilde{h}^{21}_{\text{st}}(\Delta_{\mathscr{B}}) = 3$ and $\tilde{h}^{21}_{\text{st}}(\Delta_{\mathscr{A}}) = \tilde{h}^{11}_{\text{st}}(\Delta_{\mathscr{B}}) = 77$.

5.5 A Closed-Form Expression for $\mathbb{Q}$-Nef Partitions of Length Two

We now propose a closed-form expression for the the stringy Hodge numbers of a $\mathbb{Q}$-nef Calabi-Yau bipartite complete intersection Y_{Δ_1,Δ_2} corresponding to a $\mathbb{Q}$-nef-partition $\Delta = \Delta_1+\Delta_2$ such that the divisors D_{Δ_1} and D_{Δ_2} analgous to the expression in Theorem 3.5.1.

Definition 5.5.1. Let $\Delta = \Delta_1 + \Delta_2$ be a $\mathbb{Q}$-nef partition of a five-dimensional $\mathbb{Q}$-reflexive polytope Δ such that the divisors D_{Δ_1} and D_{Δ_2} on X_Δ are ample. The *closed-form naïve Stringy Hodge numbers* of Y_{Δ_1,Δ_2} are

$$\begin{aligned}
\tilde{h}^{11}_{\text{cl-st}}(Y_{\Delta_1,\Delta_2}) &= \ell(\Delta^*) - 6 - \sum_{\dim\theta=0} \ell^*(\theta^\vee) - \sum_{\dim\theta=1} \ell^*(\theta^\vee) \\
&\qquad + \sum_{\dim\theta=2} \ell^*(\theta^\vee)\cdot\big[\ell^*(\theta) - \ell^*(\theta_1) - \ell^*(\theta_2)\big] \\
\tilde{h}^{21}_{\text{cl-st}}(Y_{\Delta_1,\Delta_2}) &= \big[\ell^*(2\Delta_1+\Delta_2) - \ell^*(2\Delta_1) + \ell^*(\Delta_1+2\Delta_2) - \ell^*(2\Delta_2)\big] - 7 \qquad (5.33) \\
&\qquad - \sum_{\dim\theta=4} \big[\ell^*(\theta) - \ell^*(\theta_1) - \ell^*(\theta_2)\big] \\
&\qquad\quad + \sum_{\dim\theta=3} \ell^*(\theta^\vee)\cdot\big[\ell^*(\theta) - \ell^*(\theta_1) - \ell^*(\theta_2)\big]
\end{aligned}$$

where the sums are taken over the faces of Δ of the indicated dimensions, $\theta^\vee$ is the face of Δ^* dual to θ, and $\theta = \theta_1 + \theta_2$ is the decomposition in to Minkowski sum with θ_i being a face of Δ_i.

We conjecture that these closed-form expressions yield the correct stringy Hodge numbers of Y_{Δ_1,Δ_2}. The data below shows the $\mathbb{Q}$-nef-partitions $\Delta = \Delta_1 + \Delta_2$ from §4.3.2 where D_{Δ_1} and D_{Δ_2} are ample. In each of these examples, the closed-form expressions $\tilde{h}^{p,q}_{\text{cl-st}}(Y_{\Delta_1,\Delta_2})$ coincide with the stringy Hodge numbers $h^{p,q}_{\text{st}}(Y_{\Delta_1,\Delta_2})$.

Δ^*	D_{Δ_1}	D_{Δ_2}	h^{11}_{st}	h^{21}_{st}	$-\chi$	$\#\Delta^*\cap N$	$\#\Delta^{*v}$	$\#\nabla^*\cap M$	$\#\nabla^{*v}$
14	$\{0\}$	$\{1,2,3,4,5\}$	5	89	168	8	6	158	12
28	$\{0\}$	$\{1,2,3,4,5\}$	1	145	288	8	6	267	13
54	$\{0\}$	$\{1,2,3,4,5\}$	3	45	84	9	6	69	12
152	$\{0,3\}$	$\{1,2,4,5,6\}$	2	64	124	8	7	95	25
240	$\{0,1,4,5\}$	$\{2,3\}$	1	79	156	7	6	129	13
247	$\{0\}$	$\{1,2,3,4,5\}$	5	89	168	8	6	203	10
261	$\{0,1,2,3\}$	$\{4,5\}$	2	30	56	8	6	42	10
291	$\{0\}$	$\{1,2,3,4,5\}$	1	101	200	7	6	181	15
291	$\{0,2,3,4,5\}$	$\{1\}$	1	103	204	7	6	300	10
291	$\{0,1\}$	$\{2,3,4,5\}$	1	79	156	7	6	125	15
293	$\{0,1,2,4,5\}$	$\{3,6\}$	2	58	112	8	7	86	22
367	$\{0,3\}$	$\{1,2,4,5,6\}$	3	73	140	9	7	120	27
390	$\{0,1\}$	$\{2,3,4,5,6\}$	2	72	140	8	7	115	22
437	$\{0,2\}$	$\{1,3,4,5,6\}$	2	84	164	8	7	140	21
437	$\{0,3,4,5,6\}$	$\{1,2\}$	2	68	132	8	7	104	25
459	$\{0\}$	$\{1,2,3,4,5\}$	2	74	144	8	6	124	13
507	$\{0,4\}$	$\{1,2,3,5,6\}$	3	57	108	9	7	84	24
575	$\{0,3,4,5,6\}$	$\{1,2\}$	2	84	164	8	7	140	21
598	$\{0,1\}$	$\{2,3,4,5,6\}$	3	57	108	9	7	84	27
629	$\{0\}$	$\{1,2,3,4,5\}$	2	74	144	8	6	138	11

Δ^*	D_{Δ_1}	D_{Δ_2}	$h^{11}_{\rm st}$	$h^{21}_{\rm st}$	$-\chi$	$\#\Delta^* \cap N$	$\#\Delta^{*v}$	$\#\nabla^* \cap M$	$\#\nabla^{*v}$
649	$\{0\}$	$\{1,2,3,4,5\}$	3	69	132	9	6	147	11
707	$\{0\}$	$\{1,2,3,4,5\}$	1	103	204	7	6	179	14
758	$\{0\}$	$\{1,2,3,4,5\}$	3	45	84	9	6	77	10
764	$\{0,2,3,4,5\}$	$\{1,6\}$	2	66	128	8	7	100	24

6

Future Work

Obviously, it would be nice to find a proof of, or a counterexample to, one of Conjectures 5.1.4, 5.1.6, or 5.1.7. The computational evidence we have in support of these conjectures seems to suggest that they are, indeed, true. If the conjectures are false, it would be interesting to know if one could adjust our definition of $\widetilde{E}_{\text{st}}$ so that the conjectures hold.

There are also several combinatorial questions that remain. Since $\mathbb{Q}$-reflexive and almost reflexive polytopes generalize relfexive polytopes, nearly any question about reflexive polytopes may be asked. A few of these questions are:

1. Can the h^*-polynomial detect whether or not a polytope Δ is the support polytope of an almost reflexive Gorenstein cone?

2. Let $(\mathscr{A}, \mathscr{B})$ be a complete mirror pair of index c and let F be a face of $\Delta_{\mathscr{A}}$. Is it true that F and its naïve dual $F^{\ddot{\vee}}$ satisfy $c \leq \operatorname{codeg} F + \operatorname{codeg} F^{\ddot{\vee}}$? A positive answer to this question could possibly be used to prove that $\widetilde{E}_{\text{st}}$ obeys the mirror-symmetry law.

3. If F and $F^{\ddot{\vee}}$ satisfy $\dim F + \dim F^{\ddot{\vee}} \neq d-1$, then must one of $\widetilde{S}_F(t)$ and $\widetilde{S}_{F^{\ddot{\vee}}}(t)$

be zero?

4. Is there a combinatorial way to detect when a pair of cones (σ_1, σ_2) are of the form $\sigma_1 = \mathrm{Cone}(\mathrm{Conv}(\mathscr{A}))$ and $\sigma_2 = \mathrm{Cone}(\mathrm{Conv}(\mathscr{B}))$ for a complete mirror pair $(\mathscr{A}, \mathscr{B})$?

5. Do maximal projective crepant partial resolutions $\widehat{Y}_{\Delta_1,\ldots,\Delta_c}$ always exist?

It would also be interesting to search for dual pairs of $\mathbb{Q}$-nef-partitions of length larger than two. One possible strategy to accomplish this would be to search for almost reflexive subpolytopes of smooth Fano polytopes in dimension higher than five. A complete classification of smooth Fano polytopes in dimensions $3 \leq d \leq 9$ can be found at [38]. We suspect that improvements to our code would need to be made to perform such a search effectively.

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Biography

Brian David Fitzpatrick was born on February 16, 1988 in Rocky Mount, North Carolina. In 2010, he earned a BA in mathematics from Trinity University in San Antonio, Texas. While at Trinity, he wrote an honors thesis *On Minimal Surfaces and Their Representations* under the advisement of Dr. Eduardo Cabral Balreira and was the recipient of the *Junior Mathematics Award* in 2009. He spent the Summer of 2008 studying mathematics at the University of California, Berkeley. In the Spring and the Fall of 2009, he attended the *Budapest Semesters in Mathematics* program in Budapest, Hungary. In the summer of 2009, he participated in the *Summer in Analysis and Geometry* program at Princeton University in Princeton, New Jersey.

In Fall 2010, Brian entered the Duke Mathematics Department as a PhD candidate. He earned a MA in mathematics in 2013 and a PhD in 2017 under the supervision of Dr. Paul Aspinwall. During his time at Duke, Brian served as Instructor of Record of 15 undergraduate mathematics courses. He was the recipient of the LP and Barbara Smith award for teaching excellence in 2013 and 2015.

In Fall 2017, Brian will join the faculty of the Mathematics Department of Duke University as a Lecturer.