Pure and Applied Mathematics Quarterly
Volume 19, Number 6, 2627–2662, 2023

The mathematical work of H. Blaine Lawson, Jr.

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In this article, we celebrate the 80^{th} birthday and remarkable career of H. Blaine Lawson, Jr. For more than half a century, Lawson has been a leading figure in mathematics. His work, a masterful combination of differential geometry, topology, algebraic geometry and analysis, has been enormously influential. He has made numerous fundamental contributions to diverse areas involving these subjects. He can be seen as a true "Renaissance man," combining profound mathematical insight with a remarkable talent for expressing his discoveries with elegance and clarity.

Roughly speaking, Lawson has changed the focus of his research every 10 to 15 years, in each instance, illuminating new fields of study with his unique insight and perspective. In the narrative that follows, we will endeavor, albeit with notable omissions, to showcase his most significant achievements. The order of presentation is essentially chronological. We will conclude with a concise overview of his highly influential expository work.

Received October 30, 2023.

1. Minimal variaties, foliations and nonpositive curvature (1970–1979)

By the late 1960s, it was well established that the study of geodesics in a riemannian manifold was a powerful tool to understand the manifold itself. As a natural extension, it became clear that understanding the minimal submanifolds, though more difficult, would be similarly important. The particular case in which the ambient manifold is the round n-sphere is important for a different reason as well. Any singularity of a minimal variety is, up to first order, described by its tangent cone. Hence, to understand such singularities it is necessary to understand minimal cones. On the other hand, a cone is minimal if and only its intersection with the unit sphere is a minimal subvariety of the sphere. Thus, to a significant extent, understanding minimal variety singularities in general, hinges on understanding minimal varieties in the round sphere.

Lawson's first dramatic breakthrough in this area appeared in the remarkable paper [Law70a], published in the *Annals of Mathematics*. At that time, there were only two known examples of closed minimal surfaces in S^3, the totally geodesic S^2 and the Clifford Torus. Lawson proved the existence of infinitely many closed minimal surfaces in S^3. More precisely, he proved the existence of embedded examples of every genus, and immersed examples of every topological type except for the projective plane (for which no minimal immersion exists). His ingenious method was to construct a piece of the example by solving the Plateau problem for a suitable piecewise geodesic boundary curve, and then to extend by Schwarz reflection to get a closed example. This method has proven to be extremely powerful and continues to be used. Here are some examples. It was used to construct interesting new examples in $S^2 \times \mathbb{R}$ (Rosenberg), in the polydisk boundary (Banchoff), and in $\mathbb{R}^3$ (Karcher). More recently, this method was a key ingredient in the proof of existence of "genus-g helicoids" in $\mathbb{R}^3$; see [HTW16]. In [KPS88], Karcher, Pinkall, and Sterling used a method reminiscent of Lawson's to construct counterexamples to the longstanding conjecture that any closed embedded minimal surface in S^3 divides it into regions of equal volume. More recently, Choi and Soret [CS16] used a variant of Lawson's method to construct the first examples of closed embedded minimal surfaces in S^3 with no plane of symmetry.

Another beautiful and fundamental result in [Law70a] is a duality between minimal surfaces in S^3 and constant mean curvature surfaces in $\mathbb{R}^3$. From this and his examples in S^3, Lawson could deduce existence of complete, periodic, embedded constant mean curvature surfaces in $\mathbb{R}^3$ lying between parallel planes. Once again, this construction proved to be a powerful tool. For example, using a gluing method, the analytic aspects of which are highly nontrivial, Kapoules was able to construct 3-ended analogs of the classical Delaunay constant mean curvature surfaces in $\mathbb{R}^3$. However, by taking advantage of Lawson's duality, Grosse-Brauckmann, Kusner, and Sullivan [GBKS03] were able to construct those surfaces in much more elementary way, and to analyze the moduli space of all such examples.

Continuing his tour de force work in the subject, Lawson published a paper in *Inventiones Math.* [Law70b], proving that all closed embedded minimal surfaces in S^3 are standardly embedded i.e. that they divide the complement into two standardly embedded handlebodies. This paper stimulated subsequent related results by Meeks-Yau, Rubinstein, Freedman, and others. In [Law70b], also appeared the famous "Lawson Conjecture", asserting that the Clifford torus is the only minimal *embedded* torus in S^3. (He had already found infinitely many immersed minimal tori.) This conjecture attracted much attention over the ensuing decades, with various partial results, until eventually, more than 40 years later, it was finally proved by Simon Brendle [Bre13].

In the following year, together with Wu-Yi Hsiang, Lawson published a very influential paper, [HL71], in which they vastly increased the number of known examples of closed minimal hypersurfaces in n-spheres. Elsewhere, Lawson showed that for a great many of their examples, the corresponding cone is actually area-minimizing. As Hsiang and Lawson stated, "the nonlinearity of the problem makes even the construction of explicit examples reasonably difficult, and at the same time, makes such examples indispensable guidelines for research." Indeed, their examples have been extremely useful for much further work on minimal varieties and on area-minimizing cones.

In an entirely different direction, in his *Annals* paper [Law71], Lawson proved that for each $k \geq 1$, there exists a codimension 1 $\mathscr{C}^\infty$-foliation of the sphere S^{2k+3}. Even for the case of S^5, *these were the first such examples*, the existence of which had been conjectured by Reeb. Lawson's work on foliations provided the first opportunity for the exercise of his exceptional talent for mathematical exposition. In 1975 he was awarded the Steele Prize for Mathematical Exposition of the AMS for his survey article entitled *Foliations* [Law74].

In yet another direction was the work of Lawson and S.-T. Yau. The famous theorem of Preismann (1942) states that for a compact manifold of strictly negative curvature, any abelian subgroup of the fundamental group is cyclic. The conjecture that this holds for solvable fundamental groups was proved by Yau [Yau71] who, at the time, was a student in Lawson's riemannian geometry course. The two of them went on to prove additional theorems on compact manifolds of nonpositive curvature [LY72]. One such, the Maximal Torus Theorem, states that, if $\pi_1(X)$ contains a subgroup $\mathbb{Z}^k$, then X contains a totally geodesic flat k-torus T^k. Another, the Splitting Theorem, states essentially that, if $\pi_1(X)$ splits as a product, then X splits as a riemannian product. Closely related contemporary work was done by Detlef Gromoll and Joe Wolf [GW71].

During this same period, Lawson published several other influential papers on a variety of different, though not unrelated, topics.

With Yau, [LY74], he showed that a compact smooth manifold that admits a smooth action of a compact connected non-abelian Lie group also admits a riemannian metric whose scalar curvature is everywhere positive. Combined with results of Nigel Hitchin, [Hit74], this severely restricts the possible degree of symmetry of any exotic sphere which does not bound a spin manifold.

In Lawson's joint paper with Jim Simons, "On stable currents and their application to global problems in real and complex geometry", [LS73], a number of beautiful rigidity theorems are proved. They showed in particular that no nontrivial minimal variety in S^n can be stable, and that every stable minimal variety in complex projective space is a positive algebraic cycle. This last result gave a new perspective on the Hodge Conjecture.

A few years later, in [LO77], published in Acta Mathematica, Lawson and Bob Osserman constructed a collection of counterexamples, demonstrating that many of the deep results (existence, uniqueness, regularity) for nonparametric minimal surfaces in codimension 1 utterly fail in higher codimensions. One example was the cone on the graph of the Hopf map which gave a Lipschitz (but not C^1) function $f : \mathbb{R}^4 \to \mathbb{R}^3$ whose graph was minimal. Later, using the coassociative calibration, (see below), Harvey and Lawson showed, that this graph is locally absolutely area-minimizing. In recent years, there has been a substantial amount of work devoted to finding geometrically natural boundary data hypotheses that are sufficient to exclude the pathologies which were uncovered in [LO77].

It was around this time that the first papers in Lawson's remarkable, and ultimately prolific, collaboration with F. Reese Harvey appeared. One of their earliest works was "On boundaries of complex analytic varieties, I" [HL75a] published in the *Annals of Mathematics*. This paper represented a major accomplishment in the field of several complex variables. It vastly generalized the classical Hartog's Theorem and the subsequent work of S. Bochner. Later, it became known simply as the Harvey-Lawson Theorem. It stated, for example, that a compact oriented submanifold of $\mathbb{C}^n$ of dimension $2k-1>1$ bounds a holomorphic chain, i.e. an integral combination of complex k-dimensional subvarieties, if and only if all its tangent planes are maximally complex, i.e. contain a complex subspace of real codimension 1. For a compact oriented curve γ, this necessary and sufficient condition means that the integral of every holomorphic 1-form in $\mathbb{C}^n$ over γ must vanish. In these results, the manifolds are allowed to have singular sets of appropriate dimension, and $\mathbb{C}^n$ can be replaced by any Stein manifold. In Part II, [HL77], analogous results were established with $\mathbb{C}^n$ replaced by $\mathbb{P}^n \setminus \mathbb{P}^m$.

In [HL83], Harvey and Lawson gave an *intrinsic* characterization of Kähler manifolds. Specifically, they proved that, if a compact complex manifold carries no positive (1,1)-currents which are the (1,1)-components of boundaries, then the manifold admits a Kähler metric. This paper has been important in the modern theory of complex surfaces.

Around the same time, Harvey and Lawson wrote a paper on *calibrated foliations*, [HL82a], in which they characterize oriented foliations of a compact manifold X with the following property: There exists a riemannian metric on X such that every foliation current is homologically mass minimizing. In particular, this condition means that every domain in a leaf of the foliation is homologically mass minimizing for its boundary and that all compact leaves are of least mass in their homology classes. As they showed: *The property holds if and only if every d-closed foliation current is non-zero in homology.* In codimension 1, this translates to saying that every compact leaf (oriented by the foliation) is nontrivial in homology.

To be precise, Lawson's collaboration with Reese Harvey began in the Fall semester, 1972, at the Institute for Advanced Study, with [HL75b, HL75a] and [HL77]. In actuality, they had overlapped for two years (1964-66) as graduate students at Stanford. Their collaboration has intensified over the years, with 56 published papers to date, and no end in sight.

2. Gauge theories, calibrated geometries and scalar curvature (1980–1989)

Calibrated geometries Some of Lawson's most fundamental and far-reaching contributions have been connected to his work with Reese Harvey on *Calibrations.* Indeed, their work in this area has exerted an enormous influence on the development of differential geometry ever since the 1970s, to the point where *Calibrations and Calibrated Geometries* has been awarded its own Mathematics Subject Classification, 53C38, by the International Mathematical Union.

The initial motivation was to explore whether Federer's theorem and its proof (see below) could be generalized. Their fundamental discovery was that natural and highly significant generalizations do indeed exist.

The notion of a calibration, i.e., a closed p-form ϕ on a riemannian manifold (M^n, g) with the property that

$$\phi(v_1, \ldots, v_p) \leq \|v_1 \wedge \cdots \wedge v_p\|_g$$

for all tangent vectors $v_1, \ldots, v_p \in T_xM$ and all $x \in M$, is a special case of what was known as a *null Lagrangian* in the calculus of variations, whose classical underpinnings go back to Euler.

The essential fact, now known as the "Fundamental Lemma of Calibrations", is that if $P^p \subset M$ is an oriented submanifold-with-boundary with the property that $P^*\phi$ equals the induced volume form of P (in which case, one says that P is calibrated by ϕ), then $(P, \partial P)$ has minimal g-volume among all cycles in M representing the relative homology class $[(P, \partial P)] \in H_p(M, \partial P)$.

The first important application of this idea in differential geometry was Federer's 1969 observation that Wirtinger's Inequality in Kähler geometry implies that, for a Kähler manifold (M, ω), the form $\phi = \frac{1}{p!}\omega^p$ calibrates the p-dimensional complex subvarieties, hence implying that they are volume minimizing in their homology classes.

In a series of papers beginning in 1977, Harvey and Lawson developed a theory of calibrations that generalized and extended much of the differential geometric theory of Kähler manifolds and their complex subvarieties, discovering in the process whole new realms of geometric applications as well as their connections with theoretical physics. Their 1982 foundational paper, "Calibrated Geometries" [HL82b], both laid out the theory in a broad perspective and developed specific examples that have been, and continue to be, enormously influential.

In addition to setting the theory of Kähler manifolds in a wider context, they consider the specific example of Kähler manifolds M with trivial canonical bundle. By the celebrated theorem of S.-T. Yau, a compact manifold of this kind has a Kähler metric g for which the holomorphic volume form Υ is parallel. In [HL82b], Harvey and Lawson showed that, if one writes $\Upsilon = \phi + i\psi$, where ϕ and ψ are real-valued, then ϕ is itself a calibration for g, calibrating the so-called *special Lagrangian submanifolds* of (M, g). This class of volume-minimizing submanifolds led to a rich new theory, whose importance was soon recognized. Once the central role in string theory of Kähler 3-folds with trivial canonical bundles was realized, the interpretation of duality in string theory led directly to the study of their special Lagrangian submanifolds as the key to understanding the physicists' predictions of duality. This culminated in the famous 'SYZ conjecture' of Strominger, Yau, and Zaslow, which stimulated enormous developments in both differential geometry and theoretical physics, especially after MacLean's 1998 thesis showing that the moduli space of such submanifolds is smooth.

In their exhaustive exploration of calibrated submanifolds, Harvey and Lawson also discovered a connection with the issue of completing the classification of riemannian manifolds with special holonomy. According to a classical result of Marcel Berger, nearly all possible riemannian holonomy groups were known, but as of 1980, there still remained the problem of proving existence in two exceptional cases, G_2 in dimension 7 and Spin(7) in dimension 8. In the G_2 case, it was known that if a riemannian 7-manifold (M^7, g) with holonomy G_2 existed, it would support a parallel 3-form, ϕ, and its dual 4-form, $*\phi$, of a particular algebraic kind. Harvey and Lawson showed that these special forms would necessarily be calibrations with a rich geometry of calibrated submanifolds. For example, the graph of the example of Lawson and Osserman mentioned above is shown to be calibrated by just such a form on $\mathbb{R}^7$. The publication of [HL82b] inspired a shift of focus for the holonomy problem from directly studying the metric g to studying, instead, the calibration ϕ and its dual. This was a crucial insight; it led Bryant in 1984 to an existence proof for metrics with holonomy G_2 in dimension 7, as well as to an analysis of their local generality [Bry85]. Further, it was the key insight that led Dominic Joyce eight years later to his construction of compact manifolds with holonomy G_2.

An entirely parallel story can be told for riemannian 8-manifolds with holonomy Spin(7), which support a parallel 4-form that is a calibration, the so-called *Cayley calibration* [Bry87]. This, too, has led to an enormous body of theory that is still vigorously under development, involving deep problems in differential geometry, as well as in theoretical high-energy physics.

In another direction, during the same period, together with Marie-Louise Michelsohn, Lawson wrote two impressive *Inventiones Math.* papers [LM84b, LM84a] concerning hypersurfaces with positive mean curvature.

Gauge theories The advent of Yang-Mills theory marked a remarkable moment in mathematics, where the confluence of ideas from differential geometry, topology and physics spawned deep breakthroughs and set whole new paradigms and areas of research.

Starting with their joint work with Jim Simons [BLS79], Jean-Pierre Bourguignon and Lawson made important contributions to the study of non-abelian gauge field theories in two subsequent papers [BL81] and [BL82].

Consider a compact riemannian manifold M, endowed with a principal G-bundle P, where G is a compact Lie group. The Yang-Mills functional, on the space $\mathscr{C}_P$ of connections on P, is defined as

$$\nabla \longmapsto \mathscr{Y}\mathscr{M}(\nabla) = \frac{1}{2}\int_M \|R^\nabla\|^2, \tag{1}$$

where R^∇ is the curvature of $\nabla \in \mathscr{C}_P$ and $\|\cdot\|$ is a suitable norm. The *Yang-Mills* connections are precisely the critical points of this functional, and of particular interest are the local minima.

For the sphere S^4, the absolute minima are precisely the self-dual or anti-self-dual connections. In [BLS79] the authors show that any other critical point cannot be a local minimum. The proof follows from second-variation formulas and the action of conformal vector fields on the curvature tensor. These techniques are used to show that for S^n, with $n \geq 5$, no local minima exist.

In the influential paper [BL81], Bourguignon and Lawson prove a series of results about Yang-Mills fields on locally homogeneous spaces. In particular, they expand and strengthen previous results, and show that any weakly stable Yang-Mills field over S^4, with $G = SU_2, SU_3$ or U_2 is either self-dual or anti-self-dual. This was based on fundamental ideas from Bourguignon. When $G = SU_2$, they show that, on an arbitrary compact orientable riemannian 4-manifold, a weakly stable Yang-Mills field is either self-dual, anti-self-dual or an abelian field.

The authors utilize Weitzenböck formulas on bundles of Lie algebra-valued forms, and prove more general theorems on gap-phenomena, studying explicit $\mathscr{C}^0$-neighborhoods of self-dual fields. They also study topological restrictions coming from Pontrjagin and Euler numbers, exhibit explicit examples over quotients S^n/Γ of spheres, and introduce the notion of self-duality.

In addition to the joint work with Bourguignon on non-abelian gauge theory, Lawson wrote a beautiful exposition on *The theory of gauge fields in four dimensions* [Law85]. See §6.

Scalar curvature One of Lawson's most fundamental contributions is his joint work with Misha Gromov, devoted to the understanding of positive scalar curvature. This subject has a very long history. In the 1960s, Lichnerowicz (and earlier Schrödinger [Sch20]) observed that the square of the Dirac operator D on a riemannian spin manifold M has the form $D^2 = \nabla^*\nabla + \frac{R}{4}$, where R is the scalar curvature and the term $\nabla^*\nabla$ is clearly self-adjoint and non-negative. As an immediate consequence, when the manifold M is complete and $R > \epsilon > 0$, all index invariants associated to D must vanish. In particular, when M is compact, the Atiyah-Singer Theorem immediately implies that $\hat{A}(M) = 0$ if M admits a metric of positive scalar curvature. By using a fancier version of index theory, Hitchin eventually refined this to show that there is a $\mathbb{Z}/2$-valued obstruction to positive scalar curvature on spin manifolds in dimensions 1 or 2 mod 8. The Gauss-Bonnet Theorem also implies that a closed 2-manifold M admits a metric of positive scalar curvature if and only if $M = S^2$ or $\mathbb{RP}^2$, while M admits a metric of negative scalar curvature if and only if $\chi(M) < 0$. However, the apparent symmetry in this result between the cases of positive and negative scalar curvature is broken in higher dimensions.

The big breakthrough came around 1979, through the work of Schoen-Yau [SY79], and Gromov-Lawson [GL80a, GL80b]. The paper [SY79] was an inspiration to Gromov and Lawson, and it was also the first step in the proof of the *positive mass conjecture*. Gromov and Lawson found further obstructions, some using the Atiyah-Singer Index Theorem for Families, and many more from the fundamental group of M and stable minimal hypersurfaces. One of the important concepts of Gromov and Lawson was that of *enlargeability*, the existence of finite coverings with ϵ-contracting maps of positive degree onto a sphere of the same dimension for each $\epsilon > 0$. A key idea was to tensor the spinor bundle with the pull back of a vector bundle on the sphere with non-trivial Chern class. With positive scalar curvature and ϵ sufficiently small, this twisted Dirac operator will again have vanishing index.

Furthermore, a novel technology for creating new manifolds of positive scalar curvature out of old ones was developed, based on surgery in codimension $k \geq 3$. These techniques were developed independently by Schoen-Yau in [SY79], and Gromov-Lawson [GL80a, GL80b]. The surgery results led to an amazing conclusion by Gromov and Lawson: if a closed manifold M^n is simply connected and non-spin, with $n \geq 5$, it always admits a metric of positive scalar curvature. This explains why Lichnerowicz's method fails to find an obstruction to positive scalar curvature in the non-spin case. In the spin case, Gromov and Lawson formulated and came close to proving the conjecture that, in dimensions $n \geq 5$, the Lichnerowicz-Hitchin obstructions are the only obstructions to positive scalar curvature for simply connected spin manifolds. This conjecture was eventually proved by Stephan Stolz [Sto92].

The cases of non-simply connected and of open manifolds proved more difficult. But the technology of Gromov-Lawson, along with the minimal surface method of Schoen-Yau [SY79] was further developed in [GL83]. Here are a few of the results from this paper:

1. The space of positive scalar curvature metrics on S^7 has infinitely many connected components;
2. An enlargeable spin manifold cannot admit a metric of positive scalar curvature;

3. A manifold which admits a hyperbolic metric of finite volume cannot admit a complete metric of positive scalar curvature;
4. In many cases, if M is a closed manifold that does not admit positive scalar curvature, then $M \times \mathbb{R}$ cannot admit a complete metric of positive scalar curvature;
5. An oriented closed 3-manifold can have a metric with positive scalar curvature only if it is a connected sum of 3-manifolds with finite fundamental group and copies of $S^1 \times S^2$. (This can be improved to "if and only if" using the results of Perelman on Ricci flow.)

The paper [GL83] led to the firm belief that the positive scalar curvature problem is closely related to the Novikov Conjecture, an idea that was borne out in the work of J. Rosenberg [Ros83, Ros86a, Ros86b]. It also contained an explicit conjecture about a necessary and sufficient condition for a compact spin manifold to admit positive scalar curvature. While the original form of the Gromov-Lawson Conjecture is not exactly right, it is now known that a modification of it is true "stably", as long as the fundamental group of the manifold in question satisfies the Baum-Connes Conjecture, for instance, if it is amenable. In summary, the work of Gromov and Lawson set the agenda for an entire avenue of research which has remained active for 40 years, and has led to hundreds of research papers.

Finally, a hugely influential contribution of Lawson and Marie-Louise Michelson in this field is the book "Spin Geometry" [LM89], which has become the definitive source on spin structures, Dirac operators, and their geometric applications, including questions related to positive scalar curvature. What makes this book unique is the combination of breadth of topics and clarity of exposition.

3. Algebraic geometry and homotopy theory (1990–2005)

Algebraic cycles are a fundamental tool in the study of algebraic varieties and the source of many fundamental invariants and outstanding conjectures in the field, such as the theory of Chow groups [Ful98], the *Hodge Conjecture* [Lew99] and *Grothendieck's Standard Conjectures* [Kle94].

Lawson's seminal work, *Algebraic Cycles and Homotopy Theory* [Law89], utilizes a blend of geometric measure theory and homotopy theory to study algebraic cycles on complex projective algebraic varieties. This work led to the development of new homology and cohomology theories for complex algebraic varieties that were a precursor of Voevodsky's motivic cohomology.

In general terms, keys ideas used in the duality results of Friedlander and Lawson (described below) inspired the work in [FV00], and are manifest in the proof of the isomorphism between Voevodsky's motivic cohomology and Bloch's higher Chow groups [SV00]. These are central results that played a significant role in the solution of outstanding problems such as the Milnor Conjecture that led to the awarding of the Fields Medal to Voevodsky in 2002.

Let's place Lawson's work in the proper context along with its relation to classical theories and motivic cohomology. Recall that an *algebraic p-cycle* on a variety X is an element of the free abelian group $\mathscr{Z}_p(X)$ generated by the p-dimensional irreducible subvarieties of X. In the language of schemes, $\mathscr{Z}_p(X)$ can be seen as the free abelian group on the p-dimensional points of X.

Lawson's inspiration stemmed from the study of moduli spaces and Chow groups, along with the classical Dold-Thom theorem [DT58], and the work of Almgren [Alm62] representing singular homology as the homotopy groups of spaces of integral currents with a suitable topology.

The classical Chow groups, the main invariants in enumerative geometry and intersection theory, have the form $A_p(X) := \mathscr{Z}_p(X)/\sim$, where $\sim$ is a suitable equivalence relation such as *rational, algebraic or homological equivalence*; see [Ful98]. Roughly speaking, one says that $\sigma \sim_{\mathrm{rat}} \tau$ (respectively, $\sigma \sim_{\mathrm{alg}} \tau$) if one can deform σ into τ through a family of cycles parametrized by rational (respectively, algebraic) curves, and that $\sigma \sim_{\mathrm{hom}} \tau$ if σ and τ represent the same homology class. Naïvely, one may say that the different Chow groups above describe the group $\pi_0(\mathscr{Z}_p(X))$ of connected components for different "topologies" on $\mathscr{Z}_p(X)$.

From the differential-geometric standpoint, an algebraic p-cycle on a complex projective variety X defines a closed integral current $[\![\sigma]\!] \in \mathscr{I}_{2p}(X)$, on the associated analytic space (also denoted by X). The assignment $\sigma \mapsto [\![\sigma]\!]$ embeds $\mathscr{Z}_p(X)$ as a closed subgroup of $\mathscr{I}_{2p}(X)$ in the *flat-norm* topology [Law89]. The resulting topology on $\mathscr{Z}_p(X)$ has several alternative characterizations of a more algebraic nature [LF94].

The first insight on the relevance of this topology is the isomorphism $\pi_0(\mathscr{Z}_p(X)) \cong \mathscr{A}_p(X) = \mathscr{Z}_p(X)/\sim_{\mathrm{alg}}$, where the latter is the Chow group for algebraic equivalence, described above. For $p = 0$, the Dold-Thom theorem [DT58] states that $\pi_j(\mathscr{Z}_0(X)) \cong H_j(X, \mathbb{Z})$, for all $j \geq 0$, thereby recovering singular homology. This leads to the question of which additional information on X is captured by the topology of $\mathscr{Z}_p(X)$, with $p > 0$. This is one of the driving questions behind Lawson's work on algebraic cycles.

The resulting overall picture can be summarized in the diagram below, where the motivic side is related to rational equivalence and $\mathbb{A}^1$-homotopy theory, in the same fashion that Lawson's theory is related to algebraic equivalence and classical homotopy theory. Ultimately, Lawson's work provides a fascinating bridge between the purely algebraic geometric/motivic universe and its differential geometric/topological counterpart, as we summarize next.

$$
\begin{tikzcd}
\text{Alg. Geom./Motivic} \arrow[d, Rightarrow] & \text{Lawson's holom./top.} \arrow[d, Rightarrow] & \text{Diff. Geom./Top.} \arrow[d, Rightarrow] \\
\text{Alg. } K\text{-theory} \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] & \text{Semi-top./Holom } K\text{-theory} \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] & \text{Top. } K\text{-theory} \arrow[d, rightsquigarrow] \\
\text{Motivic cohom/hom} \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] & \text{Lawson hom/Morphic cohom} \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] & \text{Sing. hom/cohom} \arrow[d, rightsquigarrow] \\
\text{Rational Equiv.} \arrow[r, rightsquigarrow] & \text{Algebraic Equiv.} \arrow[r, rightsquigarrow] & \text{Homol. Equiv.}
\end{tikzcd}
$$

Lawson homology for algebraic varieties To describe the main result in [Law89] consider a hyperplane $\mathbb{P}^n \subset \mathbb{P}^{n+1}$ and $x_0 \in \mathbb{P}^{n+1} \setminus \mathbb{P}^n$. For each subvariety $V \subset \mathbb{P}^n$ let $\Sigma\!\!\!\!/\, V \subset \mathbb{P}^{n+1}$ be the *join* $V\#x_0$ consisting of the union of all projective lines from points of V to x_0. Extending this assignment linearly gives the complex suspension homomorphism $\Sigma\!\!\!\!/\, : \mathscr{Z}_p(X) \longrightarrow \mathscr{Z}_{p+1}(\Sigma\!\!\!\!/\, X)$. Lawson's *complex suspension theorem* states that $\Sigma\!\!\!\!/$ is a homotopy equivalence.

Inspired by this result, Eric Friedlander introduced the *Lawson homology groups* of a complex projective variety X in [Fri91], defined as the homotopy groups $L_pH_n(X) := \pi_{n-2p}(\mathscr{Z}_p(X))$, $n \geq 2p$. For general varieties this was done in [LF92]. The bigrading is closely related to Hodge structures and one should think of n as the topological degree and p as the "holomorphic degree" (or weight).

The main properties of Lawson homology are the following.

1. (Functoriality) $L_*H_\bullet(-)$ is a covariant functor for proper maps and contravariant for flat maps [Fri91, LF92] and [LF94].
2. (Localization sequences) Given a closed subvariety $Y \subset X$ denote $U = X \setminus Y$. Then one has a natural long exact sequence [LF92, LF93a]
$$\cdots \to L_pH_n(X) \to L_pH_n(U) \to L_pH_{n-1}(Y) \to L_pH_{n-1}(X) \to \cdots .$$
3. (Cycle maps to ordinary homology) There are natural cycle maps
$$c\colon L_pH_n(X) \to H_n^{\mathrm{BM}}(X;\mathbb{Z}), \quad n \geq 0,$$
into Borel-Moore homology; [LF93b, FM94].

4. (Cycle maps from Higher Chow Groups) If X has dimension d, one has natural maps
$$CH^p(X,n) \to L_{d-p}pH_{n+2(d-p)}(X),$$
from the Bloch's Higher Chow groups of X to its Lawson homology, extending the natural homomorphism $CH^p(X) \to \mathscr{A}^p(X)$ from the Chow groups of cycles modulo rational equivalence to cycles modulo algebraic equivalence [FG93].
5. (Local-to-Global spectral sequences) There is a spectral sequence
$$E^2_{p,q} = H^{n-p}(X, \mathscr{L}_r\mathscr{H}_{n+q}) \Longrightarrow L_rH_{p+q}(X),$$
where $\mathscr{L}_r\mathscr{H}_k$ denotes the sheaf associated to the presheaf $U \mapsto L_rH_k(U)$ [FG93].
6. (Intersection theory) There is a fully developed intersection theory that gives $L_*H_\bullet(X)$ the structure of a bigraded ring when X is smooth [FG93].
7. (Suspension isomorphism) For a projective variety X one has
$$\not{\Sigma}: \;\; L_pH_n(X) \cong L_{p+1}H_{n+2}(\not{\Sigma}X).$$

Morphic cohomology and duality In a series of joint papers, Friedlander and Lawson developed a corresponding *bivariant* theory of topologized algebraic cocycles $\mathscr{Z}^p(X;Y)$ on a variety X with values in a quasiprojective variety Y; see [FL92, FL97, FL98], along the lines of [FM81]. In [FV00] this was introduced in the motivic context.

Once again, there is a corresponding homotopy equivalence $\mathscr{Z}^p(X;Y) \cong \mathscr{Z}^p(X;\not{\Sigma}Y)$ that allows one to introduce the cohomological counterpart of Lawson homology, called *morphic cohomology groups* in [FL92]. This is similar to defining singular cohomology using Eilenberg-MacLane spaces, and it goes as follows. Consider the affine spaces $Y = \mathbb{A}^m$ and define
$$L^sH^k(X) \;:=\; \pi_{2s-k}\mathscr{Z}^s(X, \mathbb{A}^m),$$
for $2s \geq k$ and m sufficiently large. It is worth mentioning that Lawson's suspension theorem translates into an algebraic *homotopy invariance* property for the cocycles functor $\mathscr{Z}^s(X) \xrightarrow{\simeq} \mathscr{Z}^s(X \times \mathbb{A}^1)$, corresponding to similar properties in Morel-Voevodsky's $\mathbb{A}^1$-homotopy theory [MV99].

Morphic cohomology satisfies all the expected properties, such as *functoriality*; *bigraded ring structure*; *localization exact sequences* and *cycle maps* $\Phi\colon L^sH^k(-) \to H^k(-;\mathbb{Z}), 2s \geq k$, into ordinary cohomology [FL92]. There are also *Chern classes* $\mathbf{c}_k \in L^k H^{2k}(-)$ defined for algebraic vector bundles, that transform under Φ into the standard topological Chern classes.

At a much deeper level lies the duality isomorphism between morphic cohomology and Lawson homology in [FL97], stating that for a smooth projective variety X of dimension n one has a natural isomorphism

$$L^s H^k(X) \xrightarrow{\cong} L_{n-s} H_{2n-k}(X).$$

Under the natural transformations Φ into singular homology and cohomology, these isomorphisms are carried over to the Poincaré duality map $H^k(X;\mathbb{Z}) \xrightarrow{\cong} H_{2n-k}(X;\mathbb{Z})$. Overall, the relationship between Lawson homology and morphic cohomology can be summarized by saying that they satisfy the axioms of a *Poincaré duality theory with supports* for algebraic varieties, or a Bloch-Ogus theory [BO74], as shown in [Fri00].

The key ingredient to prove duality is the *graphing theorem*, from [FL92], that shows the existence of a continuous homomorphism

$$\mathscr{Z}^s(X;Y) := \mathfrak{Mor}(X, \mathscr{Z}^s(Y)) \hookrightarrow \mathscr{Z}^s(X \times Y), \tag{2}$$

that becomes a weak homotopy equivalence when X and Y are non-singular.

At first sight, one may think that duality for non-singular varieties is to be expected. However, in this context the result relies on a deep *moving lemma for families*, proven over arbitrary fields in [FL98]. Contrary to differential topology, where transversality of submanifolds is achieved via smooth homotopies, "moving lemmas" for algebraic cycles were the source of many failed attempts to have a rigorous framework for intersection theory prior to Fulton-MacPherson's approach via deformations to the normal cone [FM77]. An independent corollary of the results in [FL98] is that one can finally provide a rigorous account of the aforementioned constructions of the classical Chow rings and intersection theory via moving cycles.

There are strong results and conjectures relating motivic and Lawson homology. Most notably, Suslin and Voevodsky showed that, for complex quasiprojective varieties, *motivic homology and Lawson homology with finite coefficients coincide.* This led to a conjecture by A. Suslin (still unsolved), which goes as follows. *Let* $\pi\colon (\mathsf{Var}/\mathbb{C})_{analytic} \to (\mathsf{Var}/\mathbb{C})_{Zar}$ *be the change of sites functor. Then, for any complex quasiprojective variety* U,

$$L^q H^n(U) \cong \mathbb{H}^n(U, tr^{\leq q}\mathbf{R}\pi_*\mathbb{Z}).$$

Essentially, Suslin's conjecture is the equivalent of Bloch-Kato's conjecture with $\mathbb{Z}$-coefficients for morphic cohomology, in the case of complex varieties.

Homotopy-theoretic applications Apart from the algebraic geometric impact of Lawson's work on algebraic cycles, there are unexpected applications to homotopy theory. The complex suspension theorem provides an explicit homotopy equivalence $\mathscr{Z}^q(\mathbb{P}^n) \simeq \mathbb{Z} \times \prod_{j=1}^q K(\mathbb{Z}, 2j)$. In [LM88] Lawson and Michelsohn show that the inclusion $Gr^q(\mathbb{P}^n) \to \mathscr{Z}^q(\mathbb{P}^n)_1$, of the Grassmannian of codimension q planes in $\mathbb{P}^n$ into the space of all algebraic cycles of degree 1 in $\mathbb{P}^n$, stabilizes to give a map $\mathbf{c}\colon BU_q \longrightarrow \mathscr{Z}^q(\mathbb{P}^\infty)_1 \cong 1 \times \prod_{j=1}^q K(\mathbb{Z}, 2j)$ that classifies the total Chern class of the universal quotient q-plane bundle over the classifying space BU_q of the unitary group. Using the naturality of the constructions, they show that these maps stabilize to give the classifying map of the total Chern class map in reduced K-theory

$$\mathbf{c}\colon BU \longrightarrow \mathscr{Z}^\infty := 1 \times \prod_{j \geq 1} K(\mathbb{Z}, 2j).$$

The Whitney sum of bundles is represented by a Hopf map $\oplus\colon BU \times BU \to BU$ induced by the direct sum of linear subspaces. At the level of algebraic cycles, this operation extends to the projective *join* of algebraic cycles. Lawson and Michelsohn [LM88] show that the operation $\#\colon \mathscr{Z}^\infty \times \mathscr{Z}^\infty \longrightarrow \mathscr{Z}^\infty$, induced by the join, classifies the cup product. More precisely, if Y is a topological space, consider $\mathfrak{Z}^0(Y) := 1 \times \prod_{j \geq 1} H^{2j}(Y, \mathbb{Z})$ as a group under the cup product. Then the join induces the group operation

$$\cup\colon \mathfrak{Z}^0(Y) \times \mathfrak{Z}^0(Y) \to \mathfrak{Z}^0(Y)$$

and the Whitney formula follows from the fact that the join $\#$ extends the sum $\oplus$ of subspaces.

By bringing in the infinite loop space machinery from [May72, Seg74], Boyer, Lawson, Lima-Filho, Mann and Michelsohn [BLLF$^+$93] used the arguments above to give an affirmative answer to a question, conjectured about 20 years earlier by G. Segal in [Seg75]. This was a surprising "reverse" outcome of Lawson's work, for he was originally utilizing homotopy theoretic tools to study algebraic geometric objects, and at the end his algebraic geometry constructions were used to settle open questions in homotopy theory. In a nutshell, the main result shows that *there is an infinite loop space structure on $\mathscr{Z}^\infty$ induced by the join $\#$ of algebraic cycles such that the inclusion $\mathbf{c}\colon (BU, \oplus) \to (\mathscr{Z}^\infty, \#)$ is a map of infinite loop spaces. In particular, there is a generalized cohomology theory $\mathfrak{Z}^*$ having $\mathfrak{Z}^0(Y)$ as its 0-th group and such that the total Chern class map extends to a map $\mathbf{c}\colon \mathbf{bu}^* \to \mathfrak{Z}^*$ of generalized cohomology theories.*

These constructions are essentially motivic and have been generalized to represent Chern classes from algebraic K-theory to motivic cohomology and other contexts, as in [FW02] and [Wal06].

Equivariant homotopy theory and families of spectra The tools used in [BLLF+93] to construct connective spectra have been further expanded in many directions by Lawson, his students and collaborators. A general account of these constructions is given in [Law03] and further generalized in [LF99]. A brief summary of these developments goes as follows.

1. The *suspension theorem* is generalized to the equivariant context in [LLFM98] with applications to equivariant K-theory and generalized equivariant cohomology [LLFM96].
2. In [LLFM03] and [LLFM05], Lawson et al. study *real* and *quaternionic* algebraic cycles and their relation to the classical groups and characteristic classes for various topological K-theories.
3. In [dS03], P. dos Santos develops Lawson homology for real varieties and develops a treatment of characteristic classes for Atiyah's KR-theory with values in ordinary equivariant cohomology. The quaternionic counterpart of these results appear in [dSLF04].

Euler-Chow series and other developments Lawson's study of moduli spaces of effective cycles goes further back to earlier work with S. S. T. Yau [LSteY87], where they study holomorphic actions of the circle group and calculate, as an example, the Euler characteristic of the Chow varieties $\mathscr{C}_{p,d}(\mathbb{P}^n)$ of effective algebraic cycles of degree d on projective spaces. In view of the work developed in [Law89], this calculation led to the definition of the *Euler-Chow series*

$$E_p(X) := \sum_{\alpha} \chi(\mathscr{C}_{p,\alpha}(X))\ t^{\alpha}$$

of a projective variety X. See [Eli94, ELF98] and [ES02]. In the case $p = 0$, the Euler-Chow series is given simply by MacDonald's formula $E_0(X) = \sum_{d\geq 0} \chi(SP_d(X))t^d = (\frac{1}{1-t})^{\chi(X)}$, where $SP_d(X) = \mathscr{C}_{0,d}(X)$ is the d-fold symmetric product of X.

A motivic version $\mathscr{E}_p(X)$ of the Euler-Chow series, with coefficients in the ring of homological Chow motives $K_0(\mathsf{ChMot})$ of complex varieties, was introduced in [EK09]. This series is closely related to Kapranov's motivic zeta function [Kap00], which can be seen from the case $p = 0$. The Euler characteristic, seen as a motivic measure, takes the motivic Chow series into the Euler-Chow series, generalizing the relationship between the motivic zeta function and MacDonald's formula.

4. Characteristic forms, subvarieties and differential characters (1993–2005)

One of the areas of Lawson's work with Reese Harvey started with their investigation of singular connections and characteristic currents [HL93]. This theory has many applications which directly relate smooth invariants in geometry to singularities of maps. For a very simple example consider a complex k-dimensional vector bundle $E \to X$ with connection on a compact manifold, and a section σ of E transversal to the zero section. Then from their theory of singular connections they construct a form S with integrable coefficients, such that

$$dS = c_k(E) - \operatorname{div}(\sigma). \tag{3}$$

Here $c_k(E)$ is the k^{th} Chern form and $\operatorname{div}(\sigma)$ is the manifold of zeros of σ. Their results covered an immense variety of applications: Thom-Porteous Formulas for the k^{th} degeneracy locus of a map between bundles, invariants for pairs of almost complex structures, or foliations, the formulas of MacPherson for degeneracies of mappings. Many other applications can be found in [HL95, HL00]. This also led to a new and quite useful approach to Morse Theory, with wide-ranging applicability [HL01]. See, for example, the work in [HM06] on Novikov Theory.

Spark complexes As is well known, a homology or cohomology theory on manifolds can be given by a wide variety of complexes, and this is one reason why such theories are so useful. With this perspective, Harvey and Lawson developed a simple and elegant theory of *spark complexes*, which they used to show the abundance of ways one can represent various theories.

A *spark complex* is a triple of complexes

$$\begin{array}{ccccc} & & F & & \\ & \subset & & \supset & \\ E & & & & I \end{array}$$

where $F = F^0 \xrightarrow{d} F^1 \xrightarrow{d} \cdots$ (same for E and I), so that $H^*(E) = H^*(F)$ and $E^k \cap I^k = \{0\}$, for $k > 0$. A *spark* is an element $a \in F^k$ such that

$$da = \phi - r, \qquad \phi \in E^{k+1} \quad \text{and} \quad r \in I^{k+1},$$

(ϕ and r are unique and d-closed). Two sparks a, a' of degree k are *equivalent* if $a - a' = db + s$ for $b \in F^{k-1}$ and $s \in I^k$. Note that $\phi = \phi'$ and $r - r' = ds$.

Defining

$$\hat{\mathbb{H}}^k = \text{the equivalent classes of sparks of degree } k \tag{4}$$

gives two basic homomorphisms $\hat{\mathbb{H}}^k \xrightarrow{d_1} E^{k+1}$ and $\hat{\mathbb{H}}^k \xrightarrow{d_2} H^{k+1}(I)$. A sub-spark complex is an inclusion of spark complexes

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & F' & & \\
E' \arrow[urr, phantom, "\subset"] \arrow[dd, equal] & & \cup & & I' \arrow[ull, phantom, "\supset"] \\
 & & F & & \cup \\
E \arrow[urr, phantom, "\subset"] & & & & I \arrow[ull, phantom, "\supset"]
\end{tikzcd}$$

such that $E = E'$ and $H^*(I) = H^*(I')$. Two spark complexes are *compatible* if they are both sub-spark complexes of a third. A key result states:

(5) Compatibility gives a *canonical* isomorphism of the $\hat{\mathbb{H}}^k$'s.

Differential characters One of the most fruitful applications of Harvey–Lawson's theory of spark complexes lies in their work on differential characters. This comes as a nice and natural consequence of the explicit nature of the $\mathrm{L}^2_{\mathrm{loc}}$-form S in the expression $dS = c_k(E) - \mathrm{div}(\sigma)$, explained in (3).

The theory of differential characters has its genesis in the work of Jeff Cheeger, presented first in the *Convegno di Geometria, Rome, 1971* [Che73], and Cheeger-Simons [CS73] where they developed, on the base-space level, invariants associated to the Chern-Simons invariants on total spaces of principal bundles [CS74]. These new objects provide a lift of the Weil homomorphism at the level of the ring structure. The theory was further developed in [Sim74], and carried forward by J. Simons, S.-S. Chern and J. Cheeger; see [CS74]. It must be said that the work of Cheeger-Simons was only published in final form 14 years later, in [CS85]. From their inception, differential characters were shown to have many applications to geometry (e.g., proving non-existence of conformal immersions), and later on they became very important in theoretical physics.

Utilizing spark complexes, Harvey, Lawson and John Zweck [HLZ03] exhibited a myriad of ways to represent differential characters, much as homology and cohomology classes can be represented. Some representations coming from interesting bi-complexes, some closely aligned to the original definition.

For example, they can be represented by a family of currents T (i.e., forms with distribution coefficients) such that

$$dT = \varphi - R$$

where φ is a smooth form and R is a manifold-like object (such as a submanifold, or a submanifold with singularities, a rectifiable current, a singular cycle, etc.). Under the spark complex formalism, the groups $\mathbb{H}^k$ obtained from these T's, as in (4), give exactly the space of differential characters. In fact, as a consequence of (5) one can vary the choice of currents and manifold-like objects greatly, and the quotient is always the same. In particular, the representation of differential characters by forms with singularities, that was introduced in [Che73], is one of the various forms that spark complexes encapsulate the theory. (Incidentally, the fact that differential characters have an analytic interpretation was noted in the foundational paper of Cheeger and Simons [CS73].)

Harvey, Lawson and Zweck showed in [HLZ03] that differential characters on a compact oriented manifold X obey a *Poincaré-Pontryagin Duality*. If $\mathbb{H}^k(X)$ is the group of characters of degree k and if X has dimension n, then

$$\mathbb{H}^{n-k-1}(X) \times \mathbb{H}^k(X) \to \mathbb{R}, \qquad \text{given by } \; (a, b) \mapsto (a * b)([X])$$

is non-degenerate and gives an isomorphism $\mathbb{H}^{n-k-1}(X) \to \operatorname{Hom}(\mathbb{H}^k(X), \mathbb{R})$ onto the differential homomorphisms.

Note that we have not discussed a product in $\hat{\mathbb{H}}^*$. However, there is a product defined by Jeff Cheeger [Che73] on differential characters, whose original definition involved the fact that the limit of cup product under subdivision is a wedge product for a (suitable) singular theory based on cubes, and whose motivation was the aforementioned lifting of the Weil homomorphism. In [HLZ03] a different way of defining this product, at the level of complexes, was given. The two products agree, and carry over by the canonical isomorphisms above to the spark classes of any other compatible complex.

The general theory of spark complexes goes far beyond differential characters. In [HL08] the theory gave a new $\overline{\partial}$-character theory that is an expansion of Deligne cohomology.

In a different direction, Harvey and Lawson gave a complete projective version of the classical notion of the algebraic hull of a set $X \subset \mathbb{C}^n$, and also a projective analogue of the classical Gelfand Transform [HL06]. This led to a conjecture that, if $\gamma \subset \mathbb{P}^n$ is a compact real analytic curve, then $\hat{\gamma}_{\mathbb{P}} \setminus \gamma$ is a complex curve of $\mathbb{P}^n \setminus \gamma$ with γ as boundary (where $\hat{\gamma}_{\mathbb{P}}$ is the projective hull of γ). This is a natural analogue of a classical result of John Wermer [Wer58], and the conjecture is still outstanding despite many partial results.

5. Geometric PDEs and plurisubharmonicity (2005–present)

Harvey and Lawson have continued to develop fundamental new ideas in calibration theory, extending their original work into an extremely fertile generalization of the notion of plurisubharmonicity from the theory of complex manifolds to calibrated manifolds in general, developing a natural geometric generalization of potential theory in calibrated geometries. The key idea is to introduce the notions of φ-plurisubharmonicity, and $(X;\varphi)$-convexity on a calibrated manifold $(X;\varphi)$. The foundational paper is: *An introduction to potential theory in calibrated geometry* [HL09c]. A duality theory between positive currents and plurisubharmonic functions is developed in [HL09b]. The potential theories they have found have all the fundamental properties of the classical case. In particular there is now a notion of φ-harmonic functions on $(X;\varphi)$, and one can solve the Dirichlet problem for these functions on any domain whose boundary is "φ-convex". These and other applications of their ideas in calibration theory have already led to some remarkable advances. In particular, we now know many more examples, far beyond the Kähler case, in which we can completely classify the homologically minimizing cycles in certain riemannian manifolds, including many symmetric spaces for which the local geometry of minimizing cycles is quite rich. (See [GMM95] for example.)

Later, Harvey and Lawson expanded the notion of φ-subharmonicity beyond calibrated manifolds to a general geometric context [HL11b]. This was quite surprising, since one could use any closed subset of the Grassmannian of p-planes (see below). An attractive case is when G is the whole Grassmannian. Here the potential theory is related to p-convexity. They discussed p-subharmonicity in real and complex manifolds. Moreover, they defined a new operator of Monge-Ampère type, given by the product of all p-fold sums of the eigenvalues of D^2u (or $\operatorname{Hess} u$ on a riemannian manifold). The Dirichlet Problem was solved for this operator on p-convex domains.

A very important special case takes place on symplectic manifolds with a Gromov metric. Here a new operator MA_{Lag} of Monge-Ampère type was discovered [HL18b]. One way it can be understood is in terms of representations of the spinor groups. The homogeneous and inhomogeneous Dirichlet Problem for MA_{Lag} have been solved, in the viscosity sense, in great generality [HL13b, HL19].

The results above have now been extended by Harvey and Lawson to the case of fully non-linear, possibly degenerate, elliptic differential equations. They discovered a hidden symmetry in their general theory which gave new insights and, for example, led to the right boundary conditions for the Dirichlet problem. Their approach was to consider *subequations*, that is, constraint sets for subsolutions. For equations of the form $\mathbf{F}(D^2u) = 0$, where the condition for subsolutions is $\mathbf{F}(D^2u) \geq 0$, this is simply a closed set $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ (symmetric $n \times n$-matrices) with the *positivity property* that

$$F + \mathcal{P} \ \subset \ F \qquad \text{where} \ \ \mathcal{P} = \{A : A \geq 0\}.$$

Subsolutions u of class C^2 are functions which satisfy $D^2u \in F$. *Supersolutions* v are defined similarly by requiring $-v$ to be a subsolution of the *Dirichet dual equation*

$$\widetilde{F} \equiv -(\sim \mathrm{Int}\, F).$$

Note the duality property that

$$\widetilde{\widetilde{F}} = F.$$

Note that the operator $\mathbf{F}$ has vanished from the picture! The set F gives a potential theory of functions which can be extended to the upper semicontinuous case using subaffine functions. (This extension is equivalent to the viscosity extension.) A *solution* u is a subsolution such that $-u$ is a supersolution.

The independence of the operator is very useful for calibrated geometries, since the F is easy to define, but many times a nice algebraic operator is elusive or non-existent. In fact, the independence allows one to study all *geometric subequations*. Here one starts with a closed set G in the Grassmannian $G(p, \mathbb{R}^n)$ of p-planes in $\mathbb{R}^n$ and defines

$$F(G) = \{A \in \mathrm{Sym}^2(\mathbb{R}^n) : \mathrm{tr}(A\big|_P) \geq 0, \forall P \in G\}.$$

If G is the set of lines in $\mathbb{R}^n$, or complex lines in $\mathbb{C}^n$ or quaternionic lines in $\mathbb{H}^n$, one gets the Monge-Ampère subequations. Other relevant examples were discussed above.

The independence of F also means that F gives a unified approach to the wide class of operators which have the same solution, namely $D^2u \in \partial F$. This gives a much richer approach to the equation. For example, this family of operators has much to say about the potential theory defined by F. Conversely, the F-potential theory gives insights to all the associated equations. See [CHLP23] for a full elaboration of this and much more.

This work was originally done in Euclidean space [HL09a]. Then Harvey and Lawson established an important generalization to manifolds [HL11a]. One of the key ideas was their new concept of *jet equivalence of equations and subequations*. This allowed one to solve the Dirichlet problem for *almost* *all natural operators on riemannian manifolds*. It also lead to analytic results on almost complex manifolds. This is because for natural operators given, say, by symmetric functions of the eigenvalues of the riemannian $\mathrm{Hess}(u)$, or from $i\partial\overline{\partial}u$, the operator is locally jet equivalent to a constant coefficient operator!

In [HL11a] they introduced a weak notion of *comparison* which had two important properties:

1. It is invariant under jet equivalence;
2. If one can establish comparison locally, then comparison holds globally.

Thus, in the presence of a global strictly F-subharmonic function, local weak comparison implies global comparison, which immediately implies the main results. These and many other results are explained in the survey paper [HL13b].

This work led to further interesting results. They wrote two important papers on tangents to viscosity subsolutions in $\mathbb{R}^n$ [HL18a, HL17]. They introduced a new algebraically defined and easily computable invariant of a subequation F, called the *Riesz characteristic* p_F, which governs much of the behavior of subsolutions. The name comes from the fact that, when $p \equiv p_F$ is finite, the classical p^{th} *Riesz kernel* $K_p(|x|)$, where

$$K_p(t) = \begin{cases} t^{2-p} & \text{if } 1 \leq p < 2 \\ \log t & \text{if } p = 2 \\ -\frac{1}{t^{p-2}} & \text{if } 2 < p < \infty. \end{cases}$$

is a solution of the non-linear equation determined by F.

When p is finite, there is an associated tangential p-flow on F-subharmonic functions u at each point x_0, given for $x_0 = 0$ by

$$u_r(x) = \begin{cases} r^{p-2}u(rx) & \text{if } p \neq 2, \quad \text{and} \\ u(rx) - M(u,r) & \text{if } p = 2, \end{cases}$$

where

$$M(u,r) \equiv \sup_{|x|\leq r} u.$$

The *tangents* to u at $0 \in \mathbb{R}^n$ are defined to be the set, $T_0(u)$, of cluster points of the flow above. When F is convex, these cluster points are taken in $L^1_{\text{loc}}(\mathbb{R}^n)$. When $1 \leq p_F < 2$ (but F not necessarily convex), they can be taken in the local β-Hölder norm for $\beta < 2 - p$. In either case, $U \in T_0(u)$ if and only if there exists a sequence $r_j \downarrow 0$ such that $u_{r_j} \to U$ (in the appropriate space). They prove that tangents are always F-subsolutions.

A basic result is the Existence of Tangents: *If F is convex or if $p_F < 2$, then tangents always exist.* Another such result is the existence of an upper semi-continuous density function. In [HL18a] two fundamental theorems establishing the strong Uniqueness of Tangents are proved. Namely, *Every tangent is a Riesz kernel.* This holds for all $O(n)$-invariant convex cone equations and their complex and quaternionic analogues, with the single exception of the homogeneous Monge-Ampère equations, over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, where the uniqueness fails. It also holds for a large class of geometrically defined subequations which includes those coming from calibrations. They also establish a discreteness result and a Hölder continuity theorem for subsolutions, when the Riesz characteristic p satisfies $1 \leq p < 2$.

Their notion of tangents was inspired by the results of Hörmander and Kiselman in complex analysis. In fact this is true of much of the Harvey-Lawson work in this area, going back for example to the Poincaré-Lelong formula, which was motivation for the results described in Section 4.

Two foundational theorems of Harvey and Lawson provide important underpinnings for nonlinear potential theory, and in particular, for all the geometric cases.

First, they established a "Restriction Theorem" for viscosity solutions to fully nonlinear pde's on manifolds [HL14]. This fact is often very valuable to have in hand. In the geometric case, it says that a function u is G-plurisubharmonic in the viscosity sense on a riemannian manifold $X \iff$ the restriction of u to every minimal G-submanifold $M \subset X$ (one with $T_xM \in G$ for all x) is Δ_M-subharmonic.

Second, in [HL13a] Harvey and Lawson proved that for fully nonlinear pde's, when it is possible to define subharmonics via distribution theory (which requires the use of linearizations), this definition agrees with the viscosity definition of a subsolution. The proof requires some technicalities. Surprisingly, this result was not part of classical viscosity theory, even for a linear operator!

They were also able to solve the Dirichlet problem with prescribed asymptotic singularities. This is to say, suppose we have a differential equation given by a subequation F of Riesz characteristic p on a manifold X, and we are given points $p_1, \ldots, p_N$ in a domain $\Omega \subset X$. Suppose we assign functions, like the p^{th}-Riesz kernel, at each of these points. Then, given some assumptions on Ω, one can solve the Dirichlet problem with a solution which is asymptotic to the given singularity at each p_k [HL16].

They also solved the inhomogeneous Dirichlet problem for fully nonlinear operators on manifolds where the right hand side f is allowed to take on all acceptable values [HL19]. In many examples, *acceptable* means $f \geq 0$, not just $f > 0$.

With the fundamental tools mentioned above, Harvey and Lawson were able to solve the Dirichlet Problem for the complex Monge-Ampère equation on *almost complex* manifolds [HL15], and to solve a conjecture of Nefton Pali. Together with Szymon Pliś, they used their obstacle version of the Dirichlet Problem to establish smooth approximations of plurisubharmonic functions on almost complex manifolds [HLP16].

6. Expository work

Apart from his many ground-breaking mathematical achievments, Lawson's expository works have been hugely important and influential. For his survey article entitled "Foliations" [Law74], he was awarded the Steele Prize for Mathematical Exposition of the American Mathematical Society 1975.

Lawson's 1980 book "Lectures on minimal submanifolds, I" [Law77] gives an excellent and accessible introduction to the theory.

Lawson was invited to give the principal series of lectures for the CBMS conference held at Santa Barbara in August 1983. The write up was published by the AMS in 1985 [Law85]. The lectures summarize the differential geometric work of Donaldson and others which, when combined with results of Freedman, led to a spectacular advance in the theory of 4-manifolds and a significant new connection between differential geometry and topology. The lectures are aimed both at specialist and nonspecialists, including topologists who wish to see a thorough treatment of the differential-geometric aspects of the theory.

Upon its publication in 1989 by Princeton University Press, "Spin geometry", by Lawson and Marie-Louise Michelson [LM89] became an instant classic and the standard reference work for the theory of Clifford algebras, spinors, index theory for Dirac operators and related subjects. Presently, this remarkable book has 1,343 citations on MathSciNet and 3,924 citations on Google Scholar.

In summary, the scientific output of H. Blaine Lawson, Jr., that we have recounted in this overview is phenomenal. Surely, the flow of new ideas will continue in the years to come.

Acknowledgements

We would like to express our sincere gratitude to Jean-Pierre Bourguignon for his meticulous review and editing of this article, and for his insightful suggestions and corrections.

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