MULTITAPER WAVE-SHAPE F-TEST FOR DETECTING NON-SINUSOIDAL OSCILLATIONS

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Abstract. Many practical periodic signals are not sinusoidal and contaminated by complicated noise. The traditional spectral approach is limited in this case due to the energy spreading caused by the non-sinusoidal oscillation. We systematically study the multitaper spectral estimate and generalize the Thomson's F-statistic under the setup physically dependent random process to analyze periodic signals of this kind. The developed statistic is applied to estimate the walking activity from the actinogram signals.

1. Introduction

Oscillatory time series is arguably one of the most commonly encountered data types in almost all scientific disciplines, and it has been long studied for decades. Sometimes, we have prior knowledge that the oscillation always exists, like the electrocardiogram [GGS17] or photoplethysmogram [AS14] recorded from a normal subject. In this case, researchers can focus on handling the potential noise and artifacts and studying the oscillatory behavior of the time series, like the heart rate variability analysis [Mal98] or arrhythmia analysis [Eer+18]. However, in many other situations, we may not know if there exists any oscillation, or when there is an oscillation. Take the popular actinogram as an example. When the inertial measurement unit (IMU) sensor is installed properly and the motion is sampled at high frequency like 10Hz, the walking pattern, like the cadence, of the subject can be studied in the actinogram [WH23], which is encoded in the oscillatory component of the actinogram. However, in the free-living environment, in general we need to determine when the subject is walking from the provided actinogram, before we can study the walking pattern by quantifying its oscillatory behavior. Toward this goal, in general we divide the analysis of an oscillatory time series into two main steps. First, we determine if there exists any oscillation, and sometime we need to determine how many. Second, once we know the existence, we want to know when an oscillation starts and ends. The second step is in general understood as the oscillatory component change point detection problem, which has been recently studied when the oscillation is sinusoidal by taking the time-frequency analysis idea into account [WZ23]. While there are many other challenges, in this paper, we focus on the first step; that is, determining if there exists an oscillatory component in given time series, particularly when the oscillation is not sinusoidal.

A commonly applied method to determine the existence of an oscillatory component is using the power spectrum of the time series, where "dominant peaks" in the spectrum are identified as potential components [BD09]. For example, take the maximum of the power spectrum as the statistic. To handle the inconsistency of the periodogram, we could consider the multitaper estimate and the associated

Thomson F-test [Tho82]. The Thomson F-test allows us to directly determine the presence of an oscillatory component. To our knowledge, while the multitaper method and the Thomson F-test have been widely used (eg. [Mob95; PPVI09], etc.), its theoretical support is limited and often only studied under the Gaussian noise setup, see [PW93; Wal00] among others. We should mention that the Thomson F-test is usually performed over the set of canonical frequencies. However, it has been known that working with the canonical frequencies might not be the best idea (see [Mob95; DW99; GH07; Mey12]) and we may consider a denser grid in the frequency axis for the frequency estimation purpose. Also, it is usually challenging to write down the precise formula of the distribution of the global Thomson F-test, and we need a strategy to approximate the distribution. The above challenges are further complicated when we handle modern time series. See Figure 1 for an illustrative biomedical signal recorded from IMU when a subject is walking as an illustration, where we could conclude the following observations. First, the oscillation is not sinusoidal; that is, the usual sinusoidal assumption in the literature does not hold. Second, the noise might be nonstationary. Third, the amplitude and frequency of the oscillation might change from time to time. Fourth, from the power spectrum it is not clear if there is one or multiple periodic signals. Ultimately, we want to handle these challenges simultaneously, but to simplify the discussion, we will focus on a simplified assumption based on physiological knowledge. In general, we could assume that the amplitude and frequency of the oscillatory component do not change fast locally and that the noise is "almost stationary" locally. This is the model that has been considered in the time-frequency literature [CCW14]. Under this simplified model, we can assume that each oscillatory component has a fixed amplitude and a fixed frequency and the noise is stationary. However, we cannot ignore the fact that the oscillation might be nonsinusoidal. The impact of this nonsinusoidal oscillation is not negligible. Even when the nonsinusoidal oscillation is strong, the energy of each harmonic might be weak. In this situation, the traditional Thomson F-test might fail to detect the existence of the signal. See the discussion in Section 6 for an example.

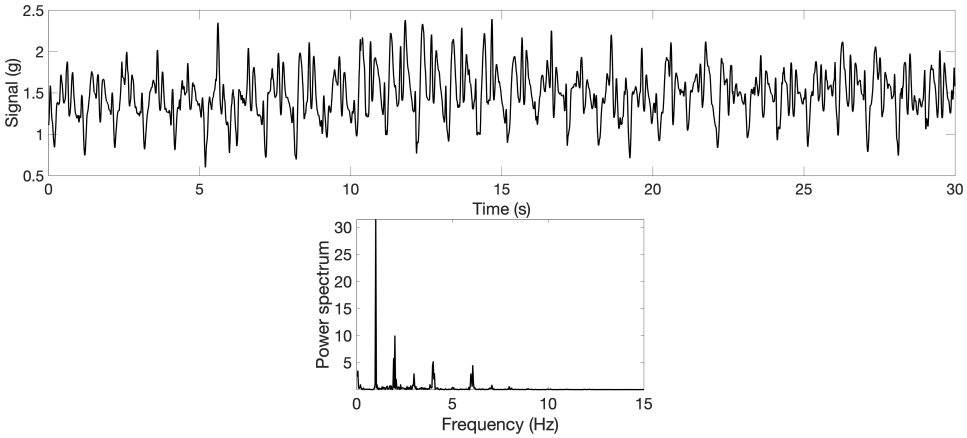

FIGURE 1. Top: An representative actinogram signal during walking. Bottom: the periodogram of the actinogram.

Motivated by the above challenges and its practical needs, in this paper we offer a systematic solution to determine the existence of an oscillatory component when the oscillatory signal might be nonsinusoidal. Under the stationary noise model, particularly the physically dependent model, we provide a series of properties of the multitaper estimate under general taper assumptions, including the consistency and central limit theorem. We also obtain the distribution under the alternative hypothesis when the oscillation is nonsinusoidal. Based on the results, we propose a new statistic, coined as the *wave-shape F-statistic*, that generalizes the traditional Thomson F-statistic. The wave-shape F-statistic is reduced to the traditional Thomson F-statistic if we have the knowledge that the oscillation is sinusoidal. Since the distribution of the wave-shape F-statistic is in general difficult to write down analytically, we propose a linear process bootstrap algorithm to approximate its distribution. Under the alternative hypothesis, the dense periodogram test (DPT) is considered to enhance the frequency estimation, where the DPT approach is motivated by a recent work [WZ23]. The proof depends on a nontrivial application of ideas from previous works (eg. [SW07], [LW10]) that studied properties of other spectral estimates also under the physically dependent noise model, and Gaussiand approximation via the m-dependent approximation along with an application of the martingale CLT. We then show that our statistic can be extended to determine the highest order of harmonics present in the oscillation.

Before jumping to the main article, we summarize some literature review. Central limit theorem (CLT) results for the DFT of time series with and without tapers have been extensively explored in [Ros85], [Lah03] and among others under mixing conditions. In the latter work the author gave characterizations of CLT of the expression $\sum_{j=0}^{N-1} h\left(\frac{j}{N-1}\right) X_j e^{-i2\pi jf}$ for the time series X_j with certain tapers $h(\cdot)$, under the assumption on cumulants. However this does not apply to our problem, as the DPSS taper coming from a solution of an eigenvector problem that cannot be characterized by a fixed function h as N varies. Under the physically dependent noise framework, CLT results were proved for smoothed-periodogram estimates [WS07], for certain *lag-windows* estimates [LW10], and for the periodogram for almost all frequencies under weak second-moment assumptions [PW10]. the asymptotic distribution of the maxima of periodogram estimates [LL09], a CLT result on the lag-window estimate [LW10], and covariance matrix estimate and large deviation bound for quadratic form [XW12]. We refer readers with interest to [Wu11] for a comprehensive review of the topic. We discuss other related results in the next section where we introduce the model and statistics.

The paper is organized as following. In Section 2, we provide the model of the nonsinusoidal oscillatory signal, the stationary noise, multitapers, and the proposed statistics. In Section 3, we study the consistency property of the multitaper estimate under the physically dependent noise model and generic assumptions satisfied by the tapers. Under the same assumptions, in Section 4, we provide the asymptotic distributions of the multitaper estimate when the number of tapers used is fixed and grows as the sampling point increases. In Section 5, the proposed wave-shape F-statistic is studied with the associated bootstrap algorithm, where the DPT is introduced. The introduced statistic is applied to determine the highest order of harmonics present in the wave-shape oscillatory signal. The proposed statistic and bootstrap are validated in the simulated signals and a real actinogram dataset in Section 6.

We summarize the commonly used notations here. The following standard Landau asymptotic symbols apply throughout: $f(N) = O(g(N))$ if there exists $C > 0$ so that $f(N) \leq Cg(N)$ for all N, $f(N) = o(g(N))$ if $f(N)/g(N) \to 0$ when $N \to \infty$, $f(N) = \Theta(g(N))$ if there exists $C_1, C_2 > 0$ so that $C_1 g(N) \leq f(N) \leq C_2 g(N)$, and $f(N) = \Omega(g(N))$ if $f(N)/g(N) \to \infty$ when $N \to \infty$. Denote the centering operator $\mathbb{E}_0[X] = X - \mathbb{E}[X]$.

2. Mathematical model and proposed statistics

2.1. Mathematical model.

Consider a random process

$$Z_n = F_n + G_n + X_n\,, \tag{1}$$

where $n \in \mathbb{Z}$, F_n is a deterministic non-sinusoidal oscillatory signal, G_n is a deterministic trend, and X_n is a mean zero stationary random process that models the noise. Here, G_n is assumed to be sampled from a smooth function that varies slowly, which is usually understood as the *trend* or *baseline wandering* in biomedical signal processing, so that its spectrum is supported in $[-f_a, f_a]$ for some small constant $f_a \geq 0$. We assume that F_n is sampled from a continuous function that models the non-sinusoidal oscillatory signal of interest.

To model a non-sinusoidal oscillatory signal, recall that for a nonzero periodic function $h : \mathbb{R} \to \mathbb{R}$, its *fundamental period* is defined as $\tau := \min\{p | \, p > 0\,, h(t + p) = h(t) \text{ for all } t \in \mathbb{R}\}$. We say that h is τ-periodic, and call $\theta := 1/\tau$ the *fundamental frequency* of h. We could rewrite $h(t)$ as $Bs(t/\tau)$, where $B > 0$ is the amplitude of the oscillation and s is a 1-periodic function with a unit L^2 norm. In the signal processing literature, s is called the *wave-shape function* (WSF) [Wu13]. The Fourier series expansion of h is given by $h(t) = B\sum_{j=0}^{\infty} B_j \cos(2\pi j\theta t + \phi_j)$, where $B_j \geq 0$ and $\phi_j \in [0, 2\pi)$ come from the Fourier coefficients of s, and the convergence depends on the regularity of s. Following the common terminology in signal processing, we call $BB_j \cos(2\pi j\theta t + \phi_j)$ the j-th harmonic of h, where ϕ_j is the associated global phase. When $j = 1$, we specifically call $BB_1 \cos(2\pi\theta t + \phi_1)$ the fundamental component of h. Set $A_j = BB_j$. We make the following assumptions.

Assumption 2.1. *Take a small constant $f_a \geq 0$. For the $1/\theta$-periodic continuous function f defined on $\mathbb{R}$ associated with the model (1), we assume $f(t) = \sum_{j=1}^{L} A_j \cos(2\pi j\theta t + \phi_j)$, where $L \in \mathbb{N}$, $\theta \in (f_a, \frac{1}{2L})$, $A_1 > 0$, $A_L > 0$, $A_j \geq 0$ for $j = 2, \ldots, L-1$, and $\phi_j \in [0, 2\pi)$. Set $F_n = f(n)$ for $n \in \mathbb{Z}$.*

We call L the *harmonic order* of the WSF, and comment that $\theta < \frac{1}{2L}$ is a fundamental assumption ruled by the sampling theorem [Opp99]. In other words, since the time series is sampled at 1Hz, we cannot capture information beyond 0.5Hz. The assumption that $\theta > f_a$ is aiming to distinguish the oscillation from the smooth trend G that is modeled by a function with its power spectrum supported in $[-f_a, f_a]$. In practice, if $G_n = 0$ is a confirmed, we can take $f_a = 1/n$. When $G_n \neq 0$ and f_a is unknown, a common practice is applying the first order differentiation to reduce the impact of G_n. We shall mention that in many real world applications, usually we have some basic knowledge of the oscillation and the trend, and we shall use the knowledge to set f_a or even remove the trend by applying a high pass filter. If $A_j = 0$ for $j > 1$, the $1/\theta$-periodic function is reduced to the harmonic function; that is, the oscillation is sinusoidal.

We model the noise by a mean 0 physically dependent random process [Wu05]

$$X_n := g(\ldots, \varepsilon_{n-1}, \varepsilon_n)\,, \tag{2}$$

where g is a measurable function and ε_n is an i.i.d. random variables. We summarize some useful quantities here. Let ε_0^* denote an independent and identical copy of ε_0. The *physical dependence measure* for $n \geq 0$ is defined as $\theta_{n,p} = \|g(\ldots, \varepsilon_{-1}, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n) - g(\ldots, \varepsilon_{-1}, \varepsilon_0^*, \varepsilon_1, \ldots, \varepsilon_n)\|_p$. Denote the sum of the physical dependence measures as $\Theta_{n,p} = \sum_{i=n}^{\infty} \theta_{i,p}$. Also, introduce two filtrations $\mathcal{F}^n = \sigma(\varepsilon_n, \varepsilon_{n+1}, \ldots)$ and $\mathcal{F}_n = \sigma(\ldots, \varepsilon_{n-1}, \varepsilon_n)$ and the associated projection operators $\mathcal{P}^k := \mathbb{E}[\,\cdot\,|\mathcal{F}^{-k}] - \mathbb{E}[\,\cdot\,|\mathcal{F}^{-k+1}]$ and $\mathcal{P}_k := \mathbb{E}[\,\cdot\,|\mathcal{F}_k] - \mathbb{E}[\,\cdot\,|\mathcal{F}_{k-1}]$. We need some assumptions for the spectral content of X_n.

Assumption 2.2. *Assume* $\mathbb{E}[X_n^2] < \infty$ *and the autocovariance sequence* $r_\tau := \mathbb{E}[X_0 X_\tau]$, *where* $\tau \in \mathbb{Z}$, *satisfies the absolute summable condition* $\sum_{\tau=-\infty}^{\infty} |r_\tau| < \infty$. *Also, assume that* $\min_{f\in[-1/2,1/2]} S_X(f) > 0$, *where* S_X *is the* spectral density function *defined by* $S_X(f) = \sum_{\tau=-\infty}^{\infty} r_\tau e^{-i2\pi\tau f}$, *where* $f \in [-1/2, 1/2]$.

In this paper, we are concerned with studying if there exists an oscillatory component in (1) from a finite observed data, $Z_0, Z_1, \ldots, Z_{N-1}$, where $N \in \mathbb{N}$ and the fundamental frequency lies in a prescribed interval $[f_a, f_b] \subseteq (0, 1/2)$. To this end, consider the null and alternative hypotheses as

$$H_0 : F_n \equiv 0 \quad \text{and} \quad H_1 : F_n \not\equiv 0\,,$$

where the fundamental frequency satisfies $\theta \in [f_a, f_b]$.

2.2. **Multitaper estimate.** The multitaper estimate was introduced in [Tho82] (see also [PW93, Section 7]) to estimate the spectral density function $S(f)$ of a stationary random process from a finite sample points. Choose a set of orthonormal tapers $\mathbf{v}_k = (v_{k,0}, \ldots, v_{k,N-1})^\top \in \mathbb{R}^N$ for $k = 0, \ldots, K-1$, where K is the number of tapers chosen by the user. The multitaper estimate is the average of K tapered spectral estimates given by

$$\hat{S}_X^{mt}(f) := \frac{1}{K} \sum_{k=0}^{K-1} \left| \sum_{j=0}^{N-1} v_{k,j} X_j e^{-i2\pi f j} \right|^2 , \tag{3}$$

where $f \in (0, 1/2)$. The multitaper estimate has been widely applied in various applications, including EEG analysis [XHR99], climate research [Mob95; Tho90], gravitational wave detection [AR99] among many others. The main advantage of the multitaper estimate compared with the widely known tapered estimate is that it further stabilizes spectral estimation by further reducing the variance via taking the average of K tapered spectral estimates that are almost uncorrelated. While many alternative choices of tapers exist, in this paper we follow the original choice in [Tho82] and choose $\mathbf{v}_k$ to be the *discrete prolate spheroidal sequences* (DPSS) tapers. We briefly recall some key properties of DPSS here and delegate details to Appendix A. An important quantity associated with DPSS is the bandwidth function $W : \mathbb{N} \to (0, \frac{1}{2})$, which quantifies the concentration of energy of the taper in the frequency region $[-W, W]$ by the quantity $\lambda_k := \int_{-W}^{W} |V_k(f)|^2 df \in (0, 1)$, where $V_k(f) = \sum_{j=0}^{N-1} v_{k,j} e^{-i2\pi f j}$ is the Fourier transform of $\mathbf{v}_k$. The larger λ_k is, the more concentrated the associated taper is in the frequency domain, and hence less spectral leakage is when it is applied. Theoretically, λ_k associated with the

first $2NW$ DPSS are close to 1 [KRD21a], so in practice we use the first $2NW$ DPSS' in the multitaper estimate [PW93]. We refer readers to Appendix A for this derivation and other properties of the DPSS tapers.

While multitaper estimate has been widely applied, to our knowledge, the statistical properties of the multitaper estimate has only been studied under restrictive assumptions that X_n is a Gaussian stationary random process. Authors in [AR17] studied the bias of the multitaper estimate for the DPSS tapers, under the assumption that the number of tapers taken is $\lfloor 2NW \rfloor$ for fixed W. An improvement by slightly lowering the number of tapers used was suggested in [KRD21a]. Results on the variance and asymptotic distribution of the multitaper estimate for Gaussian processes were obtained in [Tho82], [Wal00], [LR08] and [KRD21b]. Moreover, in all existing literature, only models with a sinusoidal oscillatory component have been considered in the nonnull case, which is limited in practice, particularly when the signal is oscillating with a nontrivial WSF. In this work, we will show that the multitaper estimate is both unbiased and consistent under the model (2) assuming that $\Theta_{0,4} < \infty$, and provide a bootstrapping algorithm with theoretical support for applications.

2.3. **Proposed wave-shape F-statistic.** To perform the hypothesis test for a wave-shape oscillation from the sampled data $Z_0, Z_1, \ldots, Z_{N-1}$, where $N \in \mathbb{N}$, we propose the wave-shape F-statistic, which is based on the Thomson F-statistic first proposed in [Tho82]. Take the first K DPSS tapers with parameters N and $W \in (0, 1/2)$. Take π_N to be a set of frequency bins in $[f_a, f_b] \cap [W, 1-W]$, where f_a and f_b are chosen a priori, and the size of frequency bin is $f_\pi > 0$. Recall that we assume the trend f is a smooth function that "oscillates slowly" so that its spectrum is supported below f_a for some positive constant f_a. Take the first K DPSS tapers, define

$$\hat{\mu}(f) := \frac{\sum_{k=0}^{K-1} V_k(0) \hat{S}_{k,Z}(f)}{\sum_{k=0}^{K-1} |V_k(0)|^2},$$

where $f \in (0, 1/2)$ and $\hat{S}_{k,Z} = \sum_{j=0}^{N-1} v_{k,j} Z_j e^{-i2\pi f j}$ denotes the Fourier transform of Z_n using the k-th DPSS taper. When $f = l\theta$ in Assumption 2.1, $\hat{\mu}(l\theta)$ is an estimate of the Fourier coefficient of the l-th harmonic of the oscillatory signal with $\hat{S}_{k,Z}(l\theta) = \frac{1}{2} A_l (V_k(0) e^{i\phi_l} + V_k(2l\theta) e^{-i\phi_l}) + \hat{S}_{k,X}(l\theta)$. Then, for $f \in (0, 1/2)$, by expanding the "total variation" $\sum_{k=0}^{K-1} |\hat{S}_{k,Z}(f)|^2$, the Thomson's F-statistic is defined as

$$T_Z(f) := \frac{(\sum_{k=0}^{K-1} |V_k(0)|^2) |\hat{\mu}(f)|^2}{\frac{1}{K-1} \sum_{k=0}^{K-1} |\hat{S}_{k,Z}(f) - V_k(0)\hat{\mu}(f)|^2}.$$

Here, the numerator denotes the power of the estimated amplitude at f, and the denominator denotes the power spectrum at f upon removing the estimated oscillation. We shall mention that under the null hypothesis and assuming that X_n is Gaussian, the Thomson's F-statistic asymptotically converges to the F distribution with 2 and $2K-2$ degrees of freedom [Tho82].

With the Thomson's F-statistics, we define the following *wave-shape F-statistic*

$$F_Z := \max_{f \in \pi_N} \sum_{j=1}^{\sigma(f)} T_Z(jf),$$

where we set $\sigma(f) = L$ and $f_b = \frac{1}{2L}$ if the harmonic order L is known. For example, if we know that the oscillation is sinusoidal, we could set $\sigma(f) = 1$. If L is unknown, which is a common situation in practice, we set $\sigma(f) = \hat{L}$, where $\hat{L}$ is the estimated harmonic order, and set $f_b = \frac{1}{2\hat{L}}$.

In practice, under the nonnull setup, both the fundamental frequency θ and its harmonic order in Assumption 2.1 are unknown and need to be estimated. We propose the following algorithm to estimate θ and L simultaneously. Observe that when f is small, the model could afford a high harmonic order a sufficiently large σ, the quantity $\sum_{j=1}^{\sigma(f)} T_Z(jf)$ contains all the non-trivial harmonics when f is near θ/k for $k \in \mathbb{N}$. However, the larger k is in θ/k, the more "noise" spectra will be included in $\sum_{j=1}^{\sigma(f)} T_Z(jf)$. To remedy this, we penalize the number of included harmonics and consider

$$\max_{(f,l),\, f\in\pi_N,\, l\le\sigma(f)} \left(\sum_{j=1}^{l} T_Z(jf) - lN^q \right) \tag{4}$$

for a small $q > 0$. The estimate of fundamental frequency and order of highest harmonic is given by

$$(\hat{\theta}, \hat{L}) = \arg \max_{(f,l),f\in\pi_N,\, l\le\sigma(f)} \left(\sum_{j=1}^{l} T_Z(jf) - lN^q \right) . \tag{5}$$

With $\hat{\theta}$, we could estimate the associated Fourier coefficient by $\hat{s}_l := \hat{\mu}(l\hat{\theta})$, where $l = 1, \ldots, \hat{L}$. We shall mention that when L is known, we could directly estimate the fundamental frequency and the associated Fourier coefficient by

$$\hat{\theta} = \arg \max_{f\in\pi_N} \sum_{j=1}^{L} T_Z(jf) \quad \text{and } \hat{s}_l := \hat{\mu}(l\hat{\theta}) \tag{6}$$

respectively, where $l = 1, \ldots, L$. If there are multiple f achieving the maximum, we pick the smallest one as $\hat{\theta}$.

Remark 2.1. How to choose f_π is a critical issue and deserves a discussion. Instead of looking at $\max_{f\in\pi_N} T_Z(f)$, the original formulation in [Tho82] suggests looking at the individual Thomson F-statistics $T_Z(f)$ over the grid of frequencies $f \in \pi_N$, handling issues of multiple testing, and only those with significance, defined as p-value of $\frac{1}{N}$ or lower, are identified as periodic components. There seems to be no consistent choice of π_N in the literature. [Tho82] suggests that one should choose π_N such that its cardinality is much larger than N, which coincides with the idea of achieving the parametric rate $O_P(n^{-3/2})$ for oscillatory frequency estimation [GH07]. [Mob95] chooses π_N to be the set of Fourier frequencies $\{1/N, 2/N, \ldots, \lfloor N/2 \rfloor/N\}$, which might be too coarse for practical purpose, whereas in other applications the grid itself maybe unspecified and depends on the implementation in the code (see [PPVI09]). Despite the above challenges, the distribution of the global Thomson F-statistic is unknown, so it is unclear what the significance level of $\frac{1}{N}$ means when we perform the multiple tests at the frequencies simultaneously. It is also unclear if the cut-off of $\frac{1}{N}$ has a theoretical justification, and it is well known in the literature that such a procedure may yield undesired false positives (see [Mob95], [DW99], [PPVI09], [Mey12]). If we attempt to choose a sparse grid to avoid the false positives, then such a test performs poorly if the

sinusoidal oscillations have frequencies that lie in between two consecutive frequencies in π_N. For example if we choose π_N to be the set of Fourier frequencies, then the power of detecting sinusoidal oscillations with frequencies close to $\frac{2i+1}{2N}$ would be low due to spectral leakage.

3. Consistency of the multitaper estimate

We establish the consistency of the multitaper estimate under the physically dependent random process. This is a generalization of existing results under more restrictive conditions in [LR08]. The results presented in the following sections will be stated with conditions on the parameters W and K for the DPSS tapers, as well as conditions on the physically dependent measures θ and Θ.

The first theorem gives a bound on the variance of the multitaper estimate when X_n satisfying mild conditions. The proof is based on the m-dependent approximation and Gaussian comparison, and postponed to Appendix B.

Theorem 3.1. *Assume Assumption 2.2 holds. Further assume $X_n \in \mathcal{L}^4$ and $\Theta_{0,4} < \infty$. Then*

$$\mathrm{Var}(\hat{S}_X^{mt}(f)) \leq C(\Theta_{0,4}^3 \Theta_{m,4} + m^3 NW^2 \|X_n\|_4^4 + \Theta_{0,2}^4/K) \tag{7}$$

for some absolute constant $C > 0$ and any $m \in \mathbb{N}$. In particular, if $NW^2 \to 0$, $K \to \infty$ and $K < \lfloor 2NW \rfloor$, we can choose $m = m(N)$ that is strictly monotonic increasing so that $\mathrm{Var}(\hat{S}_X^{mt}(f)) = o(1)$.

The condition $K < \lfloor 2NW \rfloor$ is a natural one for the DPSS tapers since it is well known that the power spectra of the top $\lfloor 2NW \rfloor$ tapers are concentrated in the region $[-W, W]$ [Sle78]. $NW^2 \to 0$ ensures that $W \to 0$ when $N \to \infty$.

The following asymptotically unbiased result appears in [KRD21b]. The proof can be done by splitting the bias to the local-bias term and wide-band bias term, following the terminology in [PW93].

Theorem 3.2 (Theorem 2, [KRD21b])**.** *Assume Assumption 2.2 holds. We have*

$$\mathrm{Bias}(\hat{S}_X^{mt}(f)) \leq \frac{1}{K}\left[\left(\max_{\xi\in[f-W,f+W]} S_X(\xi) - \min_{\xi\in[f-W,f+W]} S_X(\xi)\right)(K - \Lambda_N) + \max_{\xi\in[-1/2,1/2]} S_X(\xi)\Lambda_N\right],$$

where $\Lambda_N := \sum_{k=0}^{K-1}(1-\lambda_k)$. If $W \to 0$ and $NW \to \infty$ when $N \to \infty$, we have $\mathrm{Bias}(\hat{S}^{mt}(f)) = o(1)$.

Combining Theorems 3.1 and 3.2, we can bound the mean squared error (MSE) of the multitaper estimate, and establish that the multitaper estimate is consistent. We note that the consistency of lag-window estimates under the physically dependent random process was established in [LW10] under the assumptions that $X_n \in \mathcal{L}^4$ and $\Theta_{0,4} < \infty$.

Corollary 3.3. *Assume Assumption 2.2 holds and suppose $X_n \in \mathcal{L}^4$ and $\Theta_{0,4} < \infty$. Let $NW^2 \to 0$ and $K \to \infty$ when $N \to \infty$ and $K \leq \lfloor 2NW \rfloor - 1$. Then $\mathrm{MSE}(\hat{S}_X^{mt}(f)) = o(1)$.*

Finally, we can strengthen this result to show that the multitaper estimate is uniformly consistent under more restrictive assumptions on W and a faster decay of $\Theta_{N,p}$. Again, under similar conditions for the physically stationary random process the uniform consistency for lag-window estimates was also established in [XW12].

Theorem 3.4. *Assume Assumption 2.2 holds and suppose that $X_n \in \mathcal{L}^p$ for some $p > 4$ and $\Theta_{N,p} = O(N^{-\alpha})$ for $\alpha > 1$ when $N \to \infty$. Take $\beta > 0$ and $\gamma \in (0,1)$ such that $2\beta + \gamma < 1$ and $\beta > \frac{2\gamma}{\alpha p}$ are satisfied. Suppose $W = \Theta(N^{-\gamma})$, $\frac{K}{2NW} \to 1$ and $\frac{2NW-K}{\log N \log NW} \to \infty$ when $N \to \infty$. Then for any $c \in (0, \min\{\frac{1-\gamma-2\beta}{4}, \alpha\beta\})$ and $\delta > 0$, we have*

$$\mathbb{P}\left(\sup_{f\in\mathbb{R}} |\hat{S}_X^{mt}(f) - S_X(f)| \geq \delta N^{-c}\right) \leq CN^{-c'},$$

where the constant $c' > 0$ depends on $c, \alpha, \beta, \gamma, p$, and C depends on $c, c', \delta, \alpha, \beta, \gamma, p, \|X\|_p$.

4. Central limit theorem of multitaper estimate

We establish various CLT results for the multitaper estimate under the physically dependent random process setup. To our knowledge, there are no existing results in the literature discussing the asymptotic distribution of the multitaper estimate. Recall that the Kolmogorov distance between two random variables X and Y is defined as $\rho(X,Y) = \sup_{x\in\mathbb{R}} |\mathbb{P}(X \leq x) - \mathbb{P}(Y \leq x)|$.

The first result shows the asymptotic distribution of the multitaper spectral density estimate at different frequencies when K is fixed. Under much weaker conditions that $X_n \in \mathcal{L}^2$ and $\Theta_{0,2} < \infty$, the joint distribution of periodogram estimates at distinct frequencies was proven in [PW10].

Theorem 4.1. *Assume Assumption 2.2 holds and suppose that $X_n \in \mathcal{L}^4$ and $\Theta_{0,4} < \infty$. Fix $K, s \in \mathbb{N}$. Choose s distinct frequencies $f_1, \ldots, f_s \in (0, 1/2)$ and let $NW^2 \to 0$ and $NW \to \infty$ when $N \to \infty$. Then $\hat{S}_X^{mt}(f_j)$, $j = 1, \ldots, s$, with K tapers asymptotically follows independent $\frac{S(f_j)}{2K}\chi^2_{2K}$, $j = 1, \ldots, s$. If $f_1 = 0$ or $f_s = 1/2$, $\frac{S(f_1)}{2K}\chi^2_{2K}$ is replaced by $\frac{S(f_1)}{K}\chi^2_K$ and similarly $\frac{S(f_s)}{2K}\chi^2_{2K}$ is replaced by $\frac{S(f_s)}{K}\chi^2_K$.*

Finally, we look at the case when $K \to \infty$. Since the multitaper estimate is an asymptotic consistent estimator of S_X when $K \to \infty$, we need to consider a rescaling of the multitaper estimate to obtain an appropriate asymptotic distribution. For $X_n \in \mathcal{L}^p$ for some $p > 4$ and $\Theta_{n,4} = O(\rho^n)$, the joint distribution of the lag-window estimates was shown in [SW07]. The pointwise distribution of lag-window estimates was shown in [LW10] for the weaker conditions $X_n \in \mathcal{L}^4$ and $\Theta_{0,4} < \infty$.

Theorem 4.2. *Assume Assumption 2.2 holds and suppose that $X_n \in \mathcal{L}^p$ for some $p > 4$ and $\Theta_{n,4} = O(\rho^n)$ for some $\rho \in (0,1)$. Let $\frac{K}{\log^2 N \log^2(NW)} \to \infty$, $\frac{K}{2NW} \to 1$ and $\frac{2NW-K}{\log N \log(NW)} \to \infty$. Fix $s \in \mathbb{N}$ and take different frequencies $f_1, \ldots, f_s \in (0, 1/2)$. Then $\sqrt{K}(\hat{S}_X^{mt}(f_j) - \mathbb{E}[\hat{S}_X^{mt}(f_j)])$, $j = 1, \ldots, s$, asymptotically follows independent $N(0, S_X(f_j)^2)$, where we can replace $\mathbb{E}[\hat{S}_X^{mt}(f_j)]$ by $S_X(f_j)$ if $N^2W^3 \to 0$ and $\Theta_{N,2} = o(N^{-1}W(N^2)^{-1/2})$ hold.*

4.1. Asymptotic properties under the alternative hypothesis.

To study the multitaper estimate under the alternative hypothesis, first note that the oscillatory component is presented as a plateaus of width $2W$ in $\hat{S}^{mt}(f)$, which comes from the known fact that the multitaper spectral estimate closely approximates a band-pass filter supported on $[-W, W]$ when $K \to \infty$ [AR17]. Let $\hat{S}_Z^{mt}(f)$ denote the

multitaper spectral estimate for Z_n, which can be expanded into

$$\hat{S}_X^{mt}(f) + \frac{1}{K}\sum_{k=0}^{K-1}\left(\sum_{l=1}^{L}\mathrm{Re}\left(A_l(e^{i\phi_l}V_k(f-l\theta)+e^{-i\phi_l}V_k(f+l\theta))\sum_{j=0}^{N-1}v_{k,j}X_j e^{-i2\pi fj}\right)\right.$$
$$\left. + \frac{1}{4}\left|\sum_{l=1}^{L}A_l\left(e^{i\phi_l}V_k(f-l\theta)+e^{-i\phi_l}V_k(f+l\theta)\right)\right|^2\right). \tag{8}$$

In other words, the spectral density function can be divided into three parts: the pure noise part, the mixed part, and the deterministic component. The pure noise part has already been analyzed from our earlier discussion. The mixed part has mean 0 and behaves like a noise. For the deterministic part, when $f = l\theta$, we would expect $|V_k(f-l\theta)| = |V_k(0)| \to \infty$ as $N \to \infty$, which would dominate.

We now state the CLT result and that the estimates at distinct frequencies are still asymptotically independent, which is similar to Corollary 4.1 when one of the frequencies is a multiple of the fundamental frequency of the wave-shape function.

Theorem 4.3. *Assume Assumptions 2.1 and 2.2 hold and suppose $X_n \in \mathcal{L}^4$ and $\Theta_{0,4} < \infty$. Suppose $N^3W^4 \to 0$ and $NW \to \infty$ when $N \to \infty$ and $K < \lfloor 2NW \rfloor$. Under the alternative hypothesis H_1, for l such that $A_l \neq 0$, we have*

$$\frac{K}{A_l\sqrt{g(l,N)}}\left(\hat{S}_Z^{mt}(l\theta) - \frac{A_l^2 g(l,N)}{4K}\right) \to_d N\left(0, \frac{S(l\theta)}{2}\right),$$

where $g(l,N) := \sum_{k=0}^{K-1}|e^{i\phi_l}V_k(0) + e^{-i\phi_l}V_k(2l\theta)|^2$. Further, let

$$\hat{C}(f) = \begin{cases} \frac{K}{A_l\sqrt{g(l,N)}}\left(\hat{S}_Z^{mt}(l\theta) - \frac{A_l^2 g(l,N)}{4K}\right) & \text{if } A_l \neq 0 \text{ and } f = l\theta, \\ \hat{S}_Z^{mt}(f) & \text{otherwise.} \end{cases}$$

Then for fixed choices of distinct frequencies $f_1, \ldots, f_s$, we have $\hat{C}(f_j)$, $j = 1 \ldots, s$ are also asymptotically independent.

5. Bootstrap the wave-shape F-test

For the hypothesis testing, we need to know the distribution of F_{WS}. However, in general it does not have an explicit formula under the general physically dependent random process setup. To resolve this challenge, we propose to apply the linear process bootstrap procedure as described in [MP10]. The linear process bootstrap depends on realizing a Gaussian stationary process W_n with the covariance structure estimated from the input data. There is a large literature on how to estimate the covariance structure for the stationary process, and we refer readers to [XW12] for the discussion. Here, we propose the following multitaper covariance matrix estimate $\hat{\Sigma}_X^{mt}(f) \in \mathbb{R}^{N\times N}$:

$$[\hat{\Sigma}_X^{mt}]_{ij} = \frac{1}{K}\sum_{k=0}^{K-1}\sum_{l=0}^{N-|i-j|-1} v_{k,l}v_{k,l+|i-j|}X_l X_{l+|i-j|}.$$

The covariance matrix estimator $\hat{\Sigma}_X^{mt}$ is motivated by the dual relationship between the lag-window covariance matrix estimate and lag-window spectral estimate in [XW12]. The performance of $\hat{\Sigma}_X^{mt}$ is stated in the following Proposition.

Proposition 5.1. *Assume Assumptions 2.2 holds. Denote* $\Sigma_X \in \mathbb{R}^{N\times N}$ *to be the covariance matrix associated with* X_n*; that is,* $[\Sigma_X]_{ij} = \mathbb{E}X_iX_j$*. Under the conditions of Theorem 3.4, with probability approaching one* $\hat{\Sigma}_X^{mt}$ *is positive definite and we have*

$$\max_{i,j}\left|[\hat{\Sigma}_X^{mt}]_{ij} - [\Sigma_X]_{ij}\right| = o_P(N^{-c})$$

for some $c > 0$*.*

Note that we can relate the multitaper covariance matrix estimate to the multitaper spectral density estimate by the equations $\hat{S}_X^{mt}(f) = \sum_{\tau=-(N-1)}^{N-1} \hat{r}_\tau^{mt} e^{-i2\pi\tau f}$, where $\hat{r}_{i-j}^{mt} := [\hat{\Sigma}_X^{mt}]_{ij}$. Under the null hypothesis, with $\hat{\Sigma}_X^{mt}$, the linear process bootstrap is carried out in the following steps:

(1) Compute a lower triangular matrix L in the Cholesky decomposition such that $\hat{\Sigma}_X^{mt} = LL^\top$.
(2) Independently sample $\boldsymbol{Q} := [Q_0, \ldots, Q_{N-1}]^\top \sim \mathcal{N}(0, I_N)$ and set $\boldsymbol{W} := [W_0, \ldots, W_{N-1}]^\top = L\boldsymbol{Q}$ for M times.
(3) For each realized $\boldsymbol{W}$, calculate the wave-shape F statistic.
(4) Repeat (1)-(3) for M times, for example, $M = 10,000$, and obtain an approximation of the distribution of F_X.

To apply this algorithm in practice, since we do not know if F_n is zero or not, we need to estimate X_n. We remove the potentially existing F_n by estimating the fundamental frequency and amplitude of F_n by (6), pretending its existence. With $\hat{\theta}$ and $\hat{A}_l$, X_n is estimated by

$$\hat{X}_n := Z_n - \hat{F}_n\,, \tag{9}$$

where $\hat{F}_n := \sum_{l=1}^{L} \hat{s}_l e^{i2\pi l\hat{\theta}n} + \hat{s}_l^* e^{-i2\pi l\hat{\theta}n}$ for $n = 0, 1, \ldots, N-1$. We then run the bootstrap algorithm with $\hat{X}_n$. We shall mention that if F_n is zero, such estimate does not impact the performance, which will be shown in Theorem 5.3.

5.1. **Bootstrap under null hypothesis.** To guarantee the performance of the bootstrap, we carry out the following analyses. First, perform a careful reduction of the wave-shape F statistic to independent pieces via a similar procedure to [LL09]. Second, use a Gaussian approximation argument [CCK17]. Third, apply the Gaussian multiplier bootstrap to the estimation of noise via [MP10]. Below we summarize the result, and the proof can be found in Section 5. First of all, we have the following result about the Gaussian approximation framework in [CCK17] and [WZ23], where one replaces the random process X_n by a Gaussian random process that have the same covariance structure. We postpone the proof to Appendix D.

Theorem 5.2. *Assume Assumption 2.2 holds and suppose that* $X_n \in \mathcal{L}^p$ *for some* $p > 4$ *and* $\Theta_{N,p} = O(N^{-\alpha})$ *for* $\alpha > 1$ *so that* $\frac{\alpha p}{2+\alpha p} > \frac{5p-2}{6p-4}$*. Suppose* $\Lambda_N := \sum_{k=0}^{K-1}(1-\lambda_k) = O(N^{-c_1})$ *for some constant* $c_1 > 0$*, and the conditions of Proposition 3.4 hold for some* $\gamma \in \left(\frac{5p-2}{6p-4}, \frac{\alpha p}{2+\alpha p}\right)$*. Then there exists a constant* $c > 0$ *not depending on* X_n*, such that for a stationary Gaussian process* Y_n *with the same covariance structure as* X_n*, we have*

$$\rho\left(F_X, F_Y\right) = O(N^{-c})$$

when $N \to \infty$*.*

Next, we show that the Kolmogorov distance between $\max \sum T_Y(jf)$ and the bootstrapped statistic goes to zero when $N \to \infty$.

Theorem 5.3. *Grant the conditions in Theorem 5.2. There exists a probability space associated with the bootstrapping, called* $(\Omega, \mathsf{F}, \mathsf{P})$*, where we could construct a sequence of i.i.d. stationary Gaussian random processes* $\bar{W}_n$ *that follows the same covariance matrix as that estimated from* $\hat{X}_n$ *in (9). Then we have*

$$\rho\left(F_Y,\, F_{\bar{W}}|\hat{X}\right) = O(N^{-c})$$

when $N \to \infty$.

By combining Theorems 5.2 and 5.3, we show that the propose bootstrapping algorithm allows us to recover the distribution of F_X since asymptotically when $N \to \infty$ we have

$$\rho\left(F_X,\, F_{\bar{W}}|\hat{X}\right) = O(N^{-c}).$$

5.2. **Bootstrap under alternative hypothesis.** We now turn to the case of the alternative hypothesis. It is important that the estimated fundamental frequency $\hat{\theta}$ is close to the true fundamental frequency θ, so that $\bar{X}_n$ is sufficiently close to a realization of X_n. To achieve this, we impose conditions on the frequency grid. Note that the frequency grid should be sufficiently dense so that we can obtain a sufficiently accurate estimate of the fundamental frequency. The following proposition guarantees that with probability approaching one, we could obtain an accurate estimate of θ.

Proposition 5.4. *Grants the assumptions in Theorem 5.2. Set* $f_\pi = N^{-1-\delta}$ *for* $\delta \geq \frac{1}{2}$*. Assume* γ *is sufficiently close to* 1*. Define* $\hat{\theta} := \arg\max_{f \in \pi_N} \sum_{j=1}^{\sigma(f)} T_Z(jf)$*. Then, if the highest non-trivial harmonic* L *is known and we set* $\sigma(f) \equiv L$*, there exists* $\omega(N) = o(N^{-1})$ *as* $N \to \infty$*, where under the alternative hypothesis we have* $\mathbb{P}(|\hat{\theta} - \theta| \leq \omega(N)) \to 1$.

The following theorem shows that with an accurate oscillatory component recovery, guarantees that with probability approaching one, we could obtain an accurate estimate of θ.

Theorem 5.5. *Under the alternative hypothesis, and assumptions of Proposition 5.4, there exists a probability space associated with the bootstrapping, called* $(\Omega, \mathsf{F}, \mathsf{P})$*, where we could construct a sequence of i.i.d. stationary Gaussian random processes* $\bar{W}_n$ *that follows the same covariance matrix as that estimated from the realized estimated noise* $\hat{X}_n$ *in (9), where the power of the test evaluating* F_Z *against the bootstrapped distribution* $F_{\bar{W}}|\hat{X}$ *converges to* 1.

5.3. **Parameter tuning: the order of the highest harmonics.** As per before, we can bootstrap this statistic using a Gaussian approximation, and state the results for both the null and alternative hypothesis.

Theorem 5.6. *Grant the conditions of Theorem 5.2. Suppose that* $\bar{W}_n$ *is a Gaussian stationary process obtained from the bootstrap of a realization of the stationary process* $\hat{X}_n$ *in (9). Then we have*

$$\rho\left(\max_{\substack{(f,l),f\in\pi_N,\\ l\leq\sigma(f)}}\left(\sum_{j=1}^{l} T_X(jf) - lN^q\right),\ \max_{\substack{(f,l),f\in\pi_N,\\ l\leq\sigma(f)}}\left(\sum_{j=1}^{l} T_{\bar{W}}(jf) - lN^q\right)\Big|\bar{X}\right) = O(N^{-c})\,.$$

Under the alternative hypothesis, and assumptions of Proposition 5.4, the power of the test evaluating $\max_{(f,l),f\in\pi_N,l\leq\sigma(f)}\left(\sum_{j=1}^{l}T_Z(jf)-lN^q\right)$ *against the bootstrapped distribution* $\max_{(f,l),\in\pi_N,l\leq\sigma(f)}\left(\sum_{j=1}^{l}T_{\bar{W}}(jf)-lN^q\right)|\bar{X}$ *converges to* 1, *where* $\bar{W}_n$ *is a Gaussian stationary process obtained from the realized estimation of the noise* $\hat{X}_n$ *in* (9).

As a result we also have

Proposition 5.7. *Grant the assumption in Theorem 5.6 for the alternative hypothesis. Then we have* $\mathbb{P}(|\hat{\theta}-\theta|\leq\omega(N),\hat{L}=L)\to 1$.

6. Numerical results

In the simulations, we consider both Gaussian and non-Gaussian processes with innovations having a heavy tail, e.g. the Lomax distribution, to show that our methods are robust to different types of background noise.

6.1. **Simulated signal.** We first assume that L is known. Consider the AR(2) process, where $\epsilon_n\sim\sqrt{\frac{27}{5}}(\text{Lomax}(5,1)-\frac{1}{4})$ are i.i.d. and $X_n=0.75X_{n-1}-0.5X_{n-2}+\epsilon_n$, as the noise. Under the null hypothesis, we compare the true distribution with the bootstrap distribution with $M=10,000$ pseudo-observed Gaussian noises. The chosen parameters are $\sigma(f)=L=3$, $N=500$, $NW=4$, $K=7$, and $f_\pi=\lfloor N^{1.5}\ln N\rfloor^{-1}$. See Figure 2 for the result, where the Kolmogorov distance ρ is 0.106 ± 0.011 across bootstrap distributions.

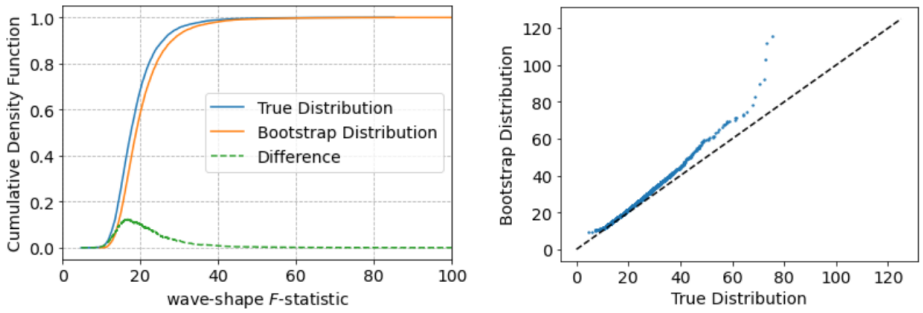

Figure 2. Comparison between wave-shape F-statistic and its bootstrap. Left: Kolmogorov-Smirnov plot, Right: QQ plot.

Next, we assume L is unknown, and evaluate how accurate we can simultaneously estimate the fundamental frequency θ and the harmonic order L by the proposed approach in Section 5.3. Use the same AR(2) noise as the above. The oscillatory signal is generated as follows. For $1\leq i\leq L-1$, sample independently $A_i'\sim\text{Ber}(0.2)\cdot\text{Unif}[0.2,1]$, and $A_L'\sim\text{Unif}[0.2,1]$. We only consider those realizations so that $\gcd(\{j|A_j'\neq 0\})=1$. Then, independently sample $\phi_i\sim\text{Unif}[0,2\pi]$ for all $1\leq i\leq L$. Finally, set $A_i=\sqrt{2}A_i'/\sqrt{\sum_{j=1}^{L}A_j'^2}$. Note that the signal-to-noise ratio is 1. Choose $\sigma(f)=20$, $W=N^{-3/4}$, $K=\lfloor 2NW-(\ln N)^2/20\rfloor$, and $q=1/5$ for the penalty term. We consider the recovery of the fundamental frequency θ and the

highest order of multiples L to be accurate if $\hat{L} = L$ and $|\hat{\theta} - \theta| < \frac{1}{2N}$. We report the ratio of accurate estimates among 1000 realizations in Table 1. Clearly, θ and L can be recovered successfully even for large L, as long as N is sufficiently large.

	$L = 3$	5	6	10
$N = 100$	0.476	0.439	0.428	0.484
250	0.713	0.682	0.71	0.679
500	0.828	0.855	0.855	0.794
1000	0.893	0.918	0.913	0.864
2000	0.934	0.955	0.96	0.921

TABLE 1. The accuracy of the recovery of $(\hat{\theta}, \hat{L})$ for different values of L and N

Finally, we discuss the importance of considering the signal as a wave-shape function instead of a sum of sinusoidal oscillations, and how the wave-shape F-test has higher power compared to the traditional testing methods. Consider the same AR(2) process X_n and $F_n = \frac{A}{2}\cos(2\pi n \cdot 0.2 + 2) + \frac{A}{3}\cos(2\pi n \cdot 0.4 + 3)$, and so $L = 2$. Recall that if we set $\sigma(f) = 1$ we recover the traditional Thomson F-test over a dense grid. For the parameters hypothesis test, we set $N = 500$, $NW = 4$, $K = 7$, $f_\pi = \lfloor N^{1.5} \ln N \rfloor^{-1}$, $f_a = W$, $f_b = 0.25$, and perform the hypothesis test at the 0.05 significance level over 200 trials. We also require that $|\hat{\theta} - \theta| < \frac{1}{2N}$. As shown in Figure 3, the test has higher power when we choose the correct number of harmonics $\sigma(f) = 2$. Intuitively, we expect $T(jf)$, $j \in \mathbb{N}$ to be asymptotically independent of each other, and hence the power of the statistic increases upon considering it as a wave-shape function.

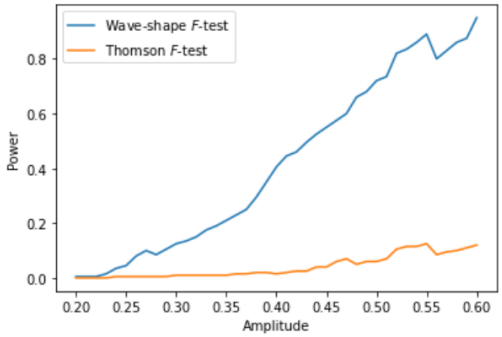

FIGURE 3. Comparison between the power of two statistical tests: Wave-shape F-test ($\sigma(f) = 2$) and Thomson F-test ($\sigma(f) = 1$).

6.2. **Real world application – Accelerometer data.** We now apply our statistic can be applied in the real world situation of detecting whether a subject is walking on flat ground, using data obtained from wearable devices. The database is from the Indiana University Walking and Driving Study (IUWDS) collected in 2015 on 13 males and 19 females in a partially controlled environment. All subjects

are physically healthy with age ranging from 21 to 51 years of age. The study was approved by the Institutional Review Board (IRB) of the Indiana University, and all participants provided written informed consent to participate. Here, the signal is the magnitude of the signal recorded from the tri-axial ActiGraph GT3X+ accelerometer attached on the left hip. The sampling rate is 100Hz.

In the following data obtained from a participant, the activities were marked as one of the following: walking on flat ground, ascending stairs, descending stairs, driving and clapping. Our goal is to determine when a subject is walking on flat ground. Since a significant feature of the signal when a subject is walking is a non-sinusoidal oscillation, our algorithm would be proper for the purpose. We divide the signal into segments of length $N = 1000$, with an overlap of 875. The large overlap is helpful in determining a more precise start and ending time of when the subject is walking.

Note that the highest order of harmonics in the accelerometer data when the subject is walking on flat ground is unknown, and hence needs to be determined via the scheme in Section 5.3. For segments of the signal corresponding to when the subject is walking on flat ground, we perform the wave-shape F-statistic test with unknown highest order, where we choose the DPSS tapers with parameters $NW = 4$ and $K = 7$, and set $f_a = 0.4$Hz, $f_b = 4$Hz, $\sigma = 50$ with penalty term as 4. The average of $\hat{L}$ across samples from all 32 subjects is 15.96, which is rounded to 16 and used as the highest order of harmonics L.

Now, we perform the wave-shape F-statistic test with known highest order of 16 to determine the segments where the subject is walking on flat ground. We mark those with a p-value of < 0.05 as when the subject is walking. The distribution of the wave-shape F-statistic is obtained by performing 1000 resamplings of the reconstructed noise under the LPB framework, and this is performed once for every 10 consecutive segments due to the limitations of computation power. Finally, we apply a median filter with length 5, which reduces the number of false positives and negatives because we expect the subjects to perform each action for a continuous time greater than 2 seconds.

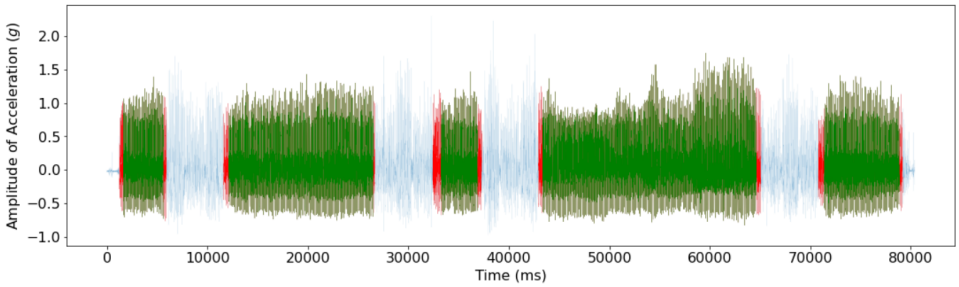

FIGURE 4. Example of a data from a participant. Green indicates data points that were correctly labelled as walking on flat ground, whereas red indicates false positive or false negative.

Over all the available data from 32 participants, the wave-shape F-test achieved a sensitivity of 0.97, specificity of 0.96, and F_1 score of 0.93, and so the statistic worked well in detecting when the subject was walking over each segment. An example can be seen in the labelling of one of the participant's data in Figure 4.

Moreover, we were able to determine precisely the fundamental frequency of the wave-shape function, which is around 0.95Hz and matches the time it takes for a subject to complete two steps. This makes sense as the sensor is placed on the left hip, and so the data obtained from a left step and right step are anti-symmetric. Should we perform this with the traditional Thomson F-test without considering that the harmonics are all due to a single source, we obtain inconsistent values of the most significant frequencies that can be multiples of 0.95Hz.

It is important to note that we do not claim that the noise in the accelerometer data is a stationary process, and we also do not claim that the walking pace is of a fixed frequency and the variation can be seen from computing the fundamental frequency via the wave-shape F-statistic. Despite these limitations our statistic still demonstrated reasonable effectiveness. In future extensions of our work, we will consider generalizing our statistic for locally stationary noise, and also using models of periodic components with changing amplitudes and frequencies.

References

[AR17] L. D. Abreu and J. L. Romero. "MSE Estimates for Multitaper Spectral Estimation and Off-Grid Compressive Sensing". In: *IEEE Transactions on Information Theory* 63.12 (2017), pp. 7770–7776.

[AR99] Bruce Allen and Joseph D Romano. "Detecting a stochastic background of gravitational radiation: Signal processing strategies and sensitivities". In: *Physical Review D* 59.10 (1999), p. 102001.

[AS14] Aymen A Alian and Kirk H Shelley. "Photoplethysmography". In: *Best Practice & Research Clinical Anaesthesiology* 28.4 (2014), pp. 395–406.

[BD09] Peter J Brockwell and Richard A Davis. *Time series: theory and methods*. Springer science & business media, 2009.

[CCK17] Victor Chernozhukov, Denis Chetverikov, and Kengo Kato. "Central limit theorems and bootstrap in high dimensions". In: *The Annals of Probability* 45.4 (2017), pp. 2309–2352.

[CCW14] Yu-Chun Chen, Ming-Yen Cheng, and Hau-Tieng Wu. "Non-parametric and adaptive modelling of dynamic periodicity and trend with heteroscedastic and dependent errors". In: *Journal of the Royal Statistical Society: Series B: Statistical Methodology* (2014), pp. 651–682.

[DJV85] Ph Delsarte, AJEM Janssen, and LB Vries. "Discrete prolate spheroidal wave functions and interpolation". In: *SIAM Journal on Applied Mathematics* 45.4 (1985), pp. 641–650.

[DW99] David GT Denison and Andrew T Walden. "The search for solar gravity-mode oscillations: an analysis using ulysses magnetic field data". In: *The Astrophysical Journal* 514.2 (1999), p. 972.

[Eer+18] Linda M Eerikäinen et al. "Comparison between electrocardiogram-and photoplethysmogram-derived features for atrial fibrillation detection in free-living conditions". In: *Physiological measurement* 39.8 (2018), p. 084001.

[Fre75] David A Freedman. "On tail probabilities for martingales". In: *the Annals of Probability* (1975), pp. 100–118.

[GGS17] Ary L Goldberger, Zachary D Goldberger, and Alexei Shvilkin. *Clinical electrocardiography: a simplified approach e-book*. Elsevier Health Sciences, 2017.

[GH07] Marc G Genton and Peter Hall. "Statistical inference for evolving periodic functions". In: *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 69.4 (2007), pp. 643–657.

[HH81] Peter Hall and CC Heyde. "Rates of convergence in the martingale central limit theorem". In: *The Annals of Probability* (1981), pp. 395–404.

[KRD19] Santhosh Karnik, Justin Romberg, and Mark A Davenport. "Fast multitaper spectral estimation". In: *2019 13th International conference on Sampling Theory and Applications (SampTA)*. IEEE. 2019, pp. 1–4.

[KRD21a] Santhosh Karnik, Justin Romberg, and Mark A Davenport. "Improved bounds for the eigenvalues of prolate spheroidal wave functions and discrete prolate spheroidal sequences". In: *Applied and Computational Harmonic Analysis* 55 (2021), pp. 97–128.

[KRD21b] Santhosh Karnik, Justin Romberg, and Mark A Davenport. "Thomson's Multitaper Method Revisited". In: *arXiv preprint arXiv:2103.11586* (2021).

[Lah03] SN Lahiri. "A necessary and sufficient condition for asymptotic independence of discrete Fourier transforms under short-and long-range dependence". In: *Annals of Statistics* 31.2 (2003), pp. 613–641.

[LL09] Zhengyan Lin and Weidong Liu. "On maxima of periodograms of stationary processes". In: *The Annals of Statistics* (2009), pp. 2676–2695.

[LR08] KS Lii and M Rosenblatt. "Prolate spheroidal spectral estimates". In: *Statistics & probability letters* 78.11 (2008), pp. 1339–1348.

[LW10] Weidong Liu and Wei Biao Wu. "Asymptotics of spectral density estimates". In: *Econometric Theory* (2010), pp. 1218–1245.

[Mal98] Marek Malik. "Heart rate variability". In: *Current opinion in cardiology* 13.1 (1998), pp. 36–44.

[Mey12] Stephen R Meyers. "Seeing red in cyclic stratigraphy: Spectral noise estimation for astrochronology". In: *Paleoceanography* 27.3 (2012).

[Mob95] Anders Moberg. "Multi-taper spectral analysis of the Stockholm air temperature record: A significant 22.8-year cyclic pulse observed". In: *Journal of Coastal Research* (1995), pp. 39–44.

[MP10] Timothy L McMurry and Dimitris N Politis. "Banded and tapered estimates for autocovariance matrices and the linear process bootstrap". In: *Journal of Time Series Analysis* 31.6 (2010), pp. 471–482.

[Nag79] Sergey V Nagaev. "Large deviations of sums of independent random variables". In: *The Annals of Probability* (1979), pp. 745–789.

[Opp99] Alan V Oppenheim. *Discrete-time signal processing*. Pearson Education India, 1999.

[PPVI09] German Andrés Prieto, RL Parker, and FL Vernon Iii. "A Fortran 90 library for multitaper spectrum analysis". In: *Computers & Geosciences* 35.8 (2009), pp. 1701–1710.

[PW10] Magda Peligrad and Wei Biao Wu. "Central limit theorem for Fourier transforms of stationary processes". In: *The Annals of Probability* 38.5 (2010), pp. 2009–2022.

[PW93] Donald B Percival and Andrew T Walden. *Spectral analysis for physical applications*. Cambridge University Press, 1993.

[Ros85] Murray Rosenblatt. *Stationary sequences and random fields*. Springer Science & Business Media, 1985.

[Sle78] David Slepian. "Prolate spheroidal wave functions, Fourier analysis, and uncertainty—V: The discrete case". In: *Bell System Technical Journal* 57.5 (1978), pp. 1371–1430.

[SW07] Xiaofeng Shao and Wei Biao Wu. "Asymptotic spectral theory for nonlinear time series". In: *The Annals of Statistics* 35.4 (2007), pp. 1773–1801.

[Tho82] D. J. Thomson. "Spectrum estimation and harmonic analysis". In: *Proceedings of the IEEE* 70.9 (1982), pp. 1055–1096.

[Tho90] David J Thomson. "Time series analysis of Holocene climate data". In: *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences* 330.1615 (1990), pp. 601–616.

[Wal00] Andrew T Walden. "A unified view of multitaper multivariate spectral estimation". In: *Biometrika* 87.4 (2000), pp. 767–788.

[WH23] Hau-Tieng Wu and Jaroslaw Harezlak. "Application of de-shape synchrosqueezing to estimate gait cadence from a single-sensor accelerometer placed in different body locations". In: *Physiological Measurement* (2023).

[WS07] Wei Biao Wu and Xiaofeng Shao. "A limit theorem for quadratic forms and its applications". In: *Econometric Theory* (2007), pp. 930–951.

[Wu05] Wei Biao Wu. "Nonlinear system theory: Another look at dependence". In: *Proceedings of the National Academy of Sciences* 102.40 (2005), pp. 14150–14154.

[Wu11] Wei Biao Wu. "Asymptotic theory for stationary processes". In: *Statistics and its Interface* 4.2 (2011), pp. 207–226.

[Wu13] Hau-tieng Wu. "Instantaneous frequency and wave shape functions (I)". In: *Applied and Computational Harmonic Analysis* 35.2 (2013), pp. 181–199.

[WVN67] Michael B Woodroofe and John W Van Ness. "The maximum deviation of sample spectral densities". In: *The Annals of Mathematical Statistics* (1967), pp. 1558–1569.

[WZ23] Hau-Tieng Wu and Zhou Zhou. "Frequency detection and change point estimation for time series of complex oscillation". In: *Journal of the American Statistical Association* (2023).

[XHR99] Yan Xu, Simon Haykin, and Ronald J Racine. "Multiple window time-frequency distribution and coherence of EEG using Slepian sequences and Hermite functions". In: *IEEE Transactions on biomedical engineering* 46.7 (1999), pp. 861–866.

[XW12] Han Xiao and Wei Biao Wu. "Covariance matrix estimation for stationary time series". In: *The Annals of Statistics* 40.1 (2012), pp. 466–493.

Appendix A. A quick summary of DPSS tapers and their properties

We quickly summarize DPSS tapers that are used in the multitaper estimate and their properties for the sake of self-containedness. The DPSS tapers form the key ingredient in the original formulation of the multitaper estimate. We define it following [PW93, Section 3.9]. Further properties of the DPSS tapers and the corresponding discrete prolate spheroidal wave functions (DPSWF) can be found in [Sle78; PW93].

Recall that for a discrete sequence of real numbers $g := \{g_j\}_{j\in\mathbb{Z}}$ such that $\sum_{j=-\infty}^{\infty} g_j^2 = 1$, its Fourier transform is defined as $G(f) = \sum_{-\infty}^{\infty} g_j e^{-i2\pi jf}$. By Parseval's Theorem we have $\int_{-\frac{1}{2}}^{\frac{1}{2}} |G(f)|^2 df = 1$. It is well-known that a simultaneously band- and time-limited signal cannot exist due to the Heisenberg's uncertainty principle. A natural question to ask is if we could find a function that is as close to such simultaneously band- and time-limited signal as possible. This can be quantified by two types of concentration. First, among all band-limited g such that $G(f) = 0$ for $|f| > W$, where $W \in (0, 1/2)$, find one that maximizes $\sum_{j=0}^{N-1} g_j^2$ for a given $N \in \mathbb{N}$; that is, we need solve the eigenvalue problem of the following integral equation:

$$\int_{-W}^{W} \frac{\sin(N\pi(f'-f))}{\sin(\pi(f'-f))} G(f)df = \lambda G(f') \text{ for } |f'| \le W\,. \tag{10}$$

Second, among all time-limited g such that $g_j = 0$ for $j \in \mathbb{Z}\setminus\{0,\dots,N-1\}$, where $N \in \mathbb{N}$, find one that maximizes $\int_{-W}^{W} |G(f)|^2 df$ for a given $W \in (0, 1/2)$; that is, solve the following eigenvalue problem

$$A(g_0,\dots,g_{N-1})^\top = \lambda(g_0,\dots,g_{N-1})^\top\,, \tag{11}$$

where $A \in \mathbb{R}^{N\times N}$ is defined as

$$[A]_{i+1,j+1} := \frac{\sin(2\pi W(i-j))}{\pi(i-j)} \text{ for } 0 \le i,j \le N-1\,. \tag{12}$$

As shown in [Sle78], there are N eigenfunctions to (10), also called the discrete prolate spheroidal wave functions (DPSWF), denoted by $U_0(f;W,N),\dots,U_{N-1}(f;W,N)$. Note that $U_0(f;W,N),\dots,U_{N-1}(f;W,N)$ are defined in the frequency domain. The eigenvalues are ordered by $1 > \lambda_0(W,N) > \ldots > \lambda_{N-1}(W,N) > 0$. We extend $U_k(f;W,N)$ to $[-\frac{1}{2},\frac{1}{2}]$ using (10) and normalize them such that $\int_{-\frac{1}{2}}^{\frac{1}{2}} |U_k(f;W,N)|^2 = 1$, $U_k(f;W,N) \ge 0$, and $\frac{dU_k(0;W,N)}{df} \ge 0$. The DPSWFs also satisfy $\int_{-W}^{W} |U_k(f;W,N)|^2 = \lambda_k$. Below, when there is no danger of confusion, we omit the notation dependence on W and N.

It turns out that the kth eigenvector of (11), which we call the kth order DPSS taper $\mathbf{v}_k \in \mathbb{R}^N$, can be obtained directly from $U_k(f)$ via

$$\mathbf{v}_{k,j} = \frac{1}{\lambda_k}\int_{-W}^{W} U_k(f) e^{i2\pi f(j-(N-1)/2)} df \tag{13}$$

for $j = 0,1,\dots,N-1$. The kth eigenvalue is also λ_k. We also have for $f \in [-1/2, 1/2]$

$$U_k(f) = \sum_{j=0}^{N-1} \mathbf{v}_{k,j} e^{-i2\pi f(j-(N-1)/2)}. \tag{14}$$

Finally, define the Fourier transform of $\mathbf{v}_k$ to be

$$V_k(f) = \sum_{j=0}^{N-1} \mathbf{v}_{k,j} e^{-i2\pi fj} = (-1)^k e^{i2\pi f(N-1)} U_k(f)\,. \tag{15}$$

where $f \in [-1/2, 1/2]$.

We state some further results on the DPSS tapers.

Lemma A.1. *For the DPSS tapers, We have* $\sum_{k=0}^{N-1} \lambda_k = 2NW$.

Proof. This follows immediately by taking the trace of the matrix A in (11). □

Lemma A.2 (Section 3.3, [KRD21a])**.** *For the DPSS tapers, if* $k \le \lfloor 2NW \rfloor - 1$, *we have*

$$\lambda_k \ge 1 - 10 \exp\left[- \frac{\lfloor 2NW \rfloor - k - 7}{\frac{2}{\pi^2} \log(100NW + 25)} \right].$$

Consequently if $K \le \lfloor 2NW \rfloor - 1$,

$$\sum_{k=0}^{K-1} (1 - \lambda_k) \le \frac{20}{\pi^2} \log(100NW + 25) \exp\left[- \frac{\lfloor 2NW \rfloor - K - 7}{\frac{2}{\pi^2} \log(100NW + 25)} \right].$$

We can calculate a bound on $\sup_{k,j} |\mathbf{v}_{k,j}|$ for the individual taper entries.

Proposition A.3. *For the DPSS tapers, if* $NW \to \infty$ *and* $K \le \lfloor 2NW \rfloor - 1$, *when* N *is sufficiently large, we have*

$$\sup_{k,j} |\mathbf{v}_{k,j}| \le 2\sqrt{W}.$$

for $0 \le k \le K-1$ *and* $0 \le j \le N-1$.

Proof. From the condition and applying the Cauchy-Schwartz Inequality we have

$$\begin{aligned} |\mathbf{v}_{k,j}| &= \left| \frac{1}{\lambda_k} \int_{-W}^{W} V_k(f) e^{i2\pi fj} df \right| \\ &\le \frac{1}{\lambda_k} \left(2W \cdot \int_{-W}^{W} |V_k(f)|^2 df \right)^{1/2} \le \sqrt{\frac{2W}{\lambda_k}} \le 2\sqrt{W}\,, \end{aligned}$$

where the last bound comes from Lemma A.2. □

Appendix B. Technical lemmas

Define the m-dependence process associated with X_n as $\tilde{X}_n(m) = \mathbb{E}[X_n | \varepsilon_{n-m}, \ldots, \varepsilon_n]$. We summarize a couple of inequalities that describes the properties of physically dependent random process and dthe relationship between various dependence measures of X_n and $\tilde{X}_n(m)$, and the associated autocovariance sequences of these processes.

Lemma B.1 (Lemma 8, [XW12])**.** *If* $X_n \in \mathcal{L}^2$ *and* $\Theta_{0,2} < \infty$, *we have* $|r_\tau| \le \sum_{j=0}^{\infty} \theta_{j,2}\theta_{j+|\tau|,2}$, *and hence* $\sum_{\tau \in \mathbb{Z}} |r_\tau| \le 2\Theta_{0,2}^2$.

Lemma B.2 (Lemma 8, [XW12])**.** *Take* $m \in \mathbb{N}$. *Suppose* $\tilde{X}_n(m)$ *is the* m*-dependence process associated with* X_n, $\tilde{r}_\tau(m)$ *is the autocovariance sequence of* $\tilde{X}_n(m)$ *and* $\tilde{\theta}_{i,p}(m)$ *denotes the physical dependence measure on* $\tilde{X}_n(m)$. *If* $X_n \in \mathcal{L}^2$ *and* $\Theta_{0,2} < \infty$, *we have the following bounds:*

(1) $\|\mathcal{P}^i X_0\|_p \le \theta_{i,p}$ *and* $\tilde{\theta}_{i,p}(m) \le \theta_{i,p}$.
(2) $\sum_{\tau\in\mathbb{Z}} |\tilde{r}_\tau(m)| \le 2\Theta_{0,2}^2$.

By this lemma, the spectral density function of $\tilde{X}_n(m)$ can be defined, which is denoted as $S_m(f)$.

Lemma B.3 (Lemma 1, [LW10])**.** *Suppose* $X_n \in \mathcal{L}^p$ *for some* $p \ge 2$, *and* $\Theta_{0,p} < \infty$. *Let* $w_0, w_1, \ldots, w_{N-1}$ *be an arbitrary sequence of real numbers. Then*

$$\sup_{f\in\mathbb{R}} \mathbb{E}\left[\left|\sum_{j=0}^{N-1}(X_j - \tilde{X}_j(m))w_j e^{-i2\pi fj}\right|^p\right] \le C_p\left(\sum_{j=0}^{N-1} w_j^2\right)^{p/2} \Theta_{m+1,p}^p,$$

where $C_p > 0$ *is a constant only depending on* p. *Additionally, we have*

(1) $\mathbb{E}\left[\left|\sum_{j=0}^{N-1} X_j w_j e^{-i2\pi fj}\right|^p\right] \le C_p\left(\sum_{j=0}^{N-1} w_j^2\right)^{p/2} \Theta_{0,p}^p$;
(2) $\mathbb{E}\left[\left|\sum_{j=0}^{N-1} \tilde{X}_j(m) w_j e^{-i2\pi fj}\right|^p\right] \le C_p\left(\sum_{j=0}^{N-1} w_j^2\right)^{p/2} \Theta_{0,p}^p$.

We have the following convergence of spectral density function for $\tilde{X}_n(m)$.

Proposition B.4. *Suppose* $X_n \in \mathcal{L}^2$ *and* $\Theta_{0,2} < \infty$. *Then* $S_m(f) \xrightarrow{m\to\infty} S(f)$ *uniformly in* f.

Proof. It suffices to show that $\sum_{\tau\in\mathbb{Z}} |r_\tau - \tilde{r}_\tau(m)| \xrightarrow{m\to\infty} 0$. Assume $\tau \ge 0$, then

$$\begin{aligned}
&|r_\tau - \tilde{r}_\tau(m)| \\
&= |\mathbb{E}[X_\tau(X_0 - \tilde{X}_0(m)) + \tilde{X}_0(m)(X_\tau - \tilde{X}_\tau(m))]| \\
&= \left|\mathbb{E}\left[\left(\sum_{k=-\tau}^{\infty}\mathcal{P}^k X_\tau\right)\left(\sum_{k=0}^{\infty}\mathcal{P}^k X_0 - \sum_{k=0}^{m}\mathcal{P}^k X_0\right)\right]\right. \\
&\quad \left. + \mathbb{E}\left[\left(\sum_{k=0}^{\infty}\mathcal{P}^k \tilde{X}_0(m)\right)\left(\sum_{k=-\tau}^{\infty}\mathcal{P}^k X_\tau - \sum_{k=-\tau}^{m-\tau}\mathcal{P}^k X_\tau\right)\right]\right| \\
&\le \left|\mathbb{E}\left[\sum_{k=m+1}^{\infty}(\mathcal{P}^k X_\tau)(\mathcal{P}^k X_0)\right]\right| + \left|\mathbb{E}\left[\sum_{k=m+1-\tau}^{\infty}(\mathcal{P}^k \tilde{X}_0(m))(\mathcal{P}^k X_\tau)\right]\right| \\
&\le \sum_{k=m+1}^{\infty}\|\mathcal{P}^k X_\tau\|_2\|\mathcal{P}^k X_0\|_2 + \sum_{k=m+1-\tau}^{\infty}\|\mathcal{P}^k \tilde{X}_0(m)\|_2\|\mathcal{P}^k X_\tau\|_2 \\
&\le \sum_{k=m+1}^{\infty}\theta_{k+\tau}\theta_k + \sum_{k=\max(0,m+1-\tau)}^{m}\tilde{\theta}_k(m)\theta_{k+\tau} \le \sum_{k=\max(0,m+1-\tau)}^{\infty}\theta_{k+\tau}\theta_k.
\end{aligned}$$

where the last two inequalities follow from Lemma B.2. For $\tau < 0$ we have the same bound from above with τ replaced by $-\tau$. Hence,

$$\sum_{\tau\in\mathbb{Z}}|r_\tau - \tilde{r}_\tau(m)| \le 2\sum_{\tau\ge 0}\sum_{k=\max(0,m+1-\tau)}^{\infty}\theta_{k+\tau}\theta_k \le 2\Theta_{0,2}\Theta_{m+1,2} \xrightarrow{m\to\infty} 0\,,$$

as desired. □

We will now show a couple of results that are slight modifications of similar results in [LL09]. The goal is to decompose the components in the statistic into many independent parts.

Lemma B.5. *Suppose that $X_n \in \mathcal{L}^2$ and $\Theta_{N,2} = O(N^{-\alpha})$ and $m = \lfloor N^\beta \rfloor$ for some $\alpha, \beta > 0$. Then for any fixed $0 < \alpha' < \alpha\beta$, we have*

$$\mathbb{P}\left(\max_{f\in\pi_N}\left|\sum_{l=0}^{N-1} w_l(X_l - \tilde{X}_l(m))e^{-i2\pi lf}\right| \geq \delta N^{-\alpha'}\right) \leq C(\sup_k |w_k|^2 N)N^{\alpha'-\alpha\beta}.$$

for any $\delta > 0$ and some constant C (depending only on $\delta, \alpha', \alpha, \beta, \|X\|_2$) uniformly over any arbitrary sequence w_j, $j = 0, \dots N-1$ of real numbers such that $\sum_{j=0}^{N-1} w_j^2 = 1$.

Proof. We adapt the proof of Lemma 3.4 in [LL09]. Choose any $0 < \alpha' < \alpha\beta$ and define the term $\sum_{s=\max(0,-l-1+m)}^{N-1} w_s \mathcal{P}^{l+1}(X_s)e^{-i2\pi sf} =: R_l(f)$. The quantity at hand is given by $\sum_{l=m+1-N}^{\infty} R_l(f)$. We also define random variables related to its truncation

$$\begin{aligned}
&\tilde{R}_l(f) = R_l(f) I_{|R_l(f)|\leq N^{-\alpha\beta}},\\
&\bar{R}_l(f) = \tilde{R}_l(f) - \mathbb{E}[\tilde{R}_l(f)|\mathcal{F}_{-l,\infty}], \quad \hat{R}_l(f) = R_l(f) - \bar{R}_l(f).
\end{aligned}$$

Now we compute the tail probability for $\hat{R}_l(f)$. We have for any fixed $\delta > 0$ that there exists C_δ such that

$$\begin{aligned}
&\mathbb{P}\left(\max_{f\in\mathbb{R}}\left|\sum_{l=m+2-N}^{\infty} \hat{R}_l(f)\right| \geq \delta N^{-\alpha'}\right)\\
&\leq C_\delta N^{\alpha'} \sum_{l=m+2-N}^{\infty} \mathbb{E}[\max_{f\in\mathbb{R}} |\hat{R}_l(f)|]\\
&\leq 2C_\delta N^{\alpha'} \sum_{l=m+2-N}^{\infty} \mathbb{E}[\max_{f\in\mathbb{R}} |\hat{R}_l(f)|]^2 N^{\alpha\beta}\\
&\leq 2C_\delta N^{\alpha'+\alpha\beta} \sum_{l=m+2-N}^{\infty} \left(\sum_{k=\max(0,-l+m)}^{N-1} |w_k|\theta_{k+l,2}\right)^2\\
&\leq 2C_\delta N^{\alpha'+\alpha\beta} \sup_k |w_k|^2 N\Theta_{m+1,2}^2 \lesssim C_\delta(\sup_k |w_k|^2 N)N^{\alpha'-\alpha\beta}
\end{aligned}$$

Hence we can replace $R_l(f)$ with $\bar{R}_l(f)$. Note that $|\bar{R}_l(f)|$ is bounded by $2N^{-\alpha\beta}$. Since the latter forms a martingale difference sequence, Freedman's inequality [Fre75] tells us that

$$\begin{aligned}
\mathbb{P}\left(\max_{f\in\pi_N}\left|\sum_{l=m+2-N}^{\infty} \bar{R}_l(f)\right| \geq \delta N^{-\alpha'}\right) &\leq 2|\pi_N| \cdot \exp\left(-\frac{\delta^2 N^{\alpha\beta-\alpha'}}{(4+2\delta)}\right)\\
+ \mathbb{P}\left(\max_{f\in\mathbb{R}} \sum_{l=m+2-N}^{\infty} \mathbb{E}[\left|\bar{R}_l(f)\right|^2 |\mathcal{F}_{-l,\infty}] \geq N^{-(\alpha'+\alpha\beta)}\right).
\end{aligned}$$

On the RHS, the first quantity is clearly $o(N^{\alpha'-\alpha\beta})$. For the second quantity, we can proceed identically as before to obtain

$$\mathbb{P}\left(\max_{f\in\mathbb{R}}\sum_{l=m+2-N}^{\infty}\mathbb{E}[\left|\bar{R}_l(f)\right|^2|\mathcal{F}_{-l,\infty}]\geq N^{-(\alpha'+\alpha\beta)}\right)$$
$$\lesssim C_\delta N^{\alpha'-\alpha\beta}.$$

This concludes the proof. □

Hence it suffices to show the result for the m-dependent process $\tilde{X}_n(m)$. The next step is introduce the blocking methods by removing small sections of size m. The result is that the remaining blocks are independent which will be crucial for the Gaussian approximation. Consider the following blocks of indices for $0<m<M$:

$A_j=[(j-1)(M+2m),(j-1)(M+2m)+M-1]$, these are the main blocks of size M

$B_j=[(j-1)(M+2m)+M,j(M+2m)-1]$, these are the small blocks of size m.

Here the indices run from $1\leq j\leq n$ for $n(N)=\lceil N/(M+2m)\rceil=O(N/M)$. Note that we truncate the indices in A_n up to $N-1$. We also define a new taper based on the weighted sum of the tapers, given by $w_j=\frac{\sum_{k=0}^{K-1}V_k(0)v_{k,j}}{\sqrt{\sum_{k=0}^{K-1}V_k(0)^2}}$, and we have $a_j(f):=\sum_{l\in A_j}w_l\tilde{X}_l(m)e^{-i2\pi lf}$ and $b_j(f):=\sum_{l\in B_j}w_l\tilde{X}_l(m)e^{-i2\pi lf}$. Note that $|w_l|\leq\sqrt{KW}$ upon an application of Cauchy-Schwarz inequality. We will choose $0<\beta<\beta'$ such that $m:=[N^\beta],M:=[N^{\beta'}]$.

Before we continue, we first summarize the following Fuk-Nagaev type inequality, which is a large deviation result on sum of independent random variables that is critical to the proof of the next result.

Lemma B.6 (Corollary 1.6, [Nag79])**.** *Let $X_1,\ldots,X_n$ be independent random variables such that $\mathbb{E}[X_i]=0$, and $S_n=\sum_{i=1}^n X_i$. Define $F_i(x)=\mathbb{P}(X_i<x)$ and $A_t=\sum_{i=1}^n\int_{u\geq 0}u^t dF_i(u)$. If $A_t<\infty$ and $1\leq t\leq 2$, then for $y^t\geq 4A_t$ and $x>y$, we have*

$$\mathbb{P}(S_n\geq x)\leq\sum_{i=1}^n\mathbb{P}(X_i>y)+(e^2A_t/xy^{t-1})^{x/2y}.$$

Lemma B.7. *Suppose that for some $p>2$, $\gamma\in\left(\frac{5p-2}{6p-4},1\right)$ holds, $X_n\in\mathcal{L}^p$ and $\Theta_{0,p}<\infty$. Then there exists constant c depending only on γ,p,β',β, $\delta>0$, such that we have*

$$\mathbb{P}\left(\max_{f\in\pi_N}\left|\sum_{j=1}^n b_j(f)\right|\geq\delta N^{-c}\right)\leq CN^{\tau p+(1-\beta')+\beta' p+(1-2\gamma)p/2}$$

for any $\delta>0$, $\tau>0$ satisfying $\tau p+(1-\beta')+\beta' p+(1-2\gamma)p/2\leq 0$ and $\tau\leq(\beta'-\beta)/2+\gamma-1$, and constant C depending only on $c,\delta,\gamma,p,\|X\|_p,\beta',\beta,\tau$.

Proof. Choose $\tau>0$ such that $\tau p+(1-\beta')+\beta' p+(1-2\gamma)p/2\leq 0$ and $\tau\leq(\beta'-\beta)/2+\gamma-1$, and also choose any $\tau'\in(0,\tau)$. Consider the truncation:

$$\bar{b}_j(f)=b_j(f)I_{|b_j(f)|\leq N^{-\tau}}-\mathbb{E}[b_j(f)I_{|b_j(f)|\leq N^{-\tau}}].$$

Then by triangle inequality it suffices to show the result for $\max_{f\in\pi_N}|\sum_{j=1}^n\bar{b}_j(f)|$ and $\max_{f\in\pi_N}|\sum_{j=1}^n b_j(f)-\bar{b}_j(f)|$ respectively for some c. For the first term, we

can apply Lemma B.3 and the Fuk-Nagaev inequality in Lemma B.6, to obtain that for Q sufficiently large, there exists a constant $C_{Q,\delta}$ such that

$$\begin{aligned}
&\mathbb{P}\left(\max_{f\in\pi_N}|\sum_{j=1}^{n}\bar{b}_j(f)|\geq\delta N^{-\tau'}\right)\\
&\leq C_{Q,\delta}\sum_{f\in\pi_N}\left(\left(\sum_{j=1}^{n}\mathbb{E}[\bar{b}_j(f)^2]N^{2\tau'}\right)^Q+2\sum_{j=1}^{n}\mathbb{P}(|\bar{b}_j(f)|\geq 2N^{-\tau'})\right)\\
&= C_{Q,\delta}\sum_{f\in\pi_N}\left(\sum_{j=1}^{n}\mathbb{E}[\bar{b}_j(f)^2]N^{2\tau'}\right)^Q\lesssim N(nKWm^2(\log N))^Q\\
&\leq N(N^{2+\beta-\beta'-2\gamma+2\tau'})^Q
\end{aligned}$$

Since $2+\beta-\beta'-2\gamma+2\tau'<0$, when Q is sufficiently large this term is $\leq-\tau'$. For the other term, we can apply Markov's inequality

$$\begin{aligned}
&\mathbb{P}\left(\max_{f\in\pi_N}|\sum_{j=1}^{n}b_j(f)-\bar{b}_j(f)|\geq\delta N^{-\tau'}\right)\\
&\leq\delta N^{\tau'}\mathbb{E}[\max_{f\in\pi_N}|\sum_{j=1}^{n}b_j(f)-\bar{b}_j(f)|]\\
&\leq 2\delta N^{\tau'}\sum_{j=1}^{n}\max_{f\in\mathbb{R}}\mathbb{E}[|b_j(f)|I_{|b_j(f)|\geq N^{-\tau}}]\\
&\leq 2\delta N^{\tau'}\sum_{j=1}^{n}\max_{f\in\mathbb{R}}\mathbb{E}[|b_j(f)|]\frac{\mathbb{E}[|b_j(f)|^{p-1}]}{N^{-\tau(p-1)}}\\
&\leq 2\delta N^{\tau'+\tau(p-1)}\sum_{j=1}^{n}\max_{f\in\mathbb{R}}\mathbb{E}[|b_j(f)|^p]\\
&\leq 2\delta N^{\tau'+\tau(p-1)}\sum_{j=1}^{n}C_p(\sup_{l\in B_j}w_l^p)m^p\\
&= o(N^{\tau p+(1-\beta')+\beta p+(1-2\gamma)p/2})
\end{aligned}$$

This proves the result for $c=\tau'$. □

Remark B.8. Under the same conditions and defining $\bar{a}_j(f)$ analogously, we have

$$\begin{aligned}
&\mathbb{P}(\max_{f\in\pi_N}|\sum_{j=1}^{n}a_j(f)-\bar{a}_j(f)|\geq\delta N^{-\tau'})\\
&\leq 2\delta N^{\tau'+\tau(p-1)}\sum_{j=1}^{n}C_p(\sup_{l\in A_j}w_l^p)M^p\\
&= o(N^{\tau p+(1-\beta')+\beta' p+(1-2\gamma)p/2}).
\end{aligned}$$

We can now reduce to proving the result for $\max_{f\in\mathbb{R}}|\sum_{j=1}^{n}\bar{a}_j(f)|^2/S(f)$.

We show a third moment bound for blocks, in the same spirit as Lemma 5.2 in [LL09].

Lemma B.9. *Suppose $X_n \in \mathcal{L}^p$ for $p > 2$, such that $\gamma \in \left(\frac{5p-2}{6p-4}, 1\right)$ holds. Using the terminology defined in the previous lemma, we have*

$$\sum_{j=1}^{n} \mathbb{E}[|\bar{a}_j(f)|^3] = O(N^{-\tau}).$$

Proof. Making use of Lemma B.3, if $2 < p \leq 3$, we have

$$\begin{aligned}
\sum_{j=1}^{n} \mathbb{E}[|\bar{a}_j(f)|^3] &\lesssim N^{(3-p)\tau} \sum_{j=1}^{n} \mathbb{E}[|\bar{a}_j(f)|^p] \\
&\leq C_X N^{(3-p)\tau} \sum_{j=1}^{n} \left(\sum_{l \in A_j} w_l^2 \right)^{p/2} \\
&\leq C_X N^{(3-p)\tau} n M^{p/2} (KW)^{p/2} \\
&\leq C_X N^{(3-p)\tau + (1-\beta') + \beta' p/2 + (1-2\gamma)p/2}.
\end{aligned}$$

We have $(3-p)\tau + (1-\beta') + \beta' p/2 + (1-2\gamma)p/2 \leq (3-2p)\tau - \beta' p/2 < -\tau$. Finally if $p \geq 3$ we can reduce it to the $p = 3$ case. □

Let V be the function that denote the spectral window of w_j and $\Lambda_N = \sum_{k=0}^{K-1}(1-\lambda_k)$.

Lemma B.10. *The following are true*

(1) We have for $f \in [0, \frac{1}{2}]$ that

$$\left| \mathbb{E}\left[\sum_{j=0}^{N-1} w_j X_j e^{-i2\pi jf} \sum_{j=0}^{N-1} w_j X_j e^{i2\pi jf} - S(f) \right] \right| \leq \max_{\alpha \in [f-W, f+W]} |S(\alpha) - S(f)| + 2\Lambda_N \|S\|_\infty.$$

(2) If $|f - f'| \geq 2W$, then

$$\left| \mathbb{E}\left[\sum_{j=0}^{N-1} w_j X_j e^{-i2\pi jf} \sum_{j=0}^{N-1} w_j X_j e^{i2\pi jf'} \right] \right| \leq (2\sqrt{\Lambda_N} + \Lambda_N)\|S\|_\infty.$$

(3) For all $f \neq f'$ where $f, f' \in [W, 1/2 - W]$,

$$\left| \mathbb{E}\left[\sum_{j=0}^{N-1} w_j X_j e^{-i2\pi jf} \sum_{j=0}^{N-1} w_j X_j e^{-i2\pi jf'} \right] \right| \leq (2\sqrt{\Lambda_N} + \Lambda_N)\|S\|_\infty.$$

Proof. We will first find the concentration of energy of $V(f)$. We have the broadband energy

$$\begin{aligned}\int_{f\in[-1/2,1/2]\setminus[-W,W]} V(f)^2 df &= \int_{f\in[-1/2,1/2]\setminus[-W,W]} \frac{(\sum_{k=0}^{K-1} V_k(0)V_k(f))^2}{\sum_{k=0}^{K-1} V_k(0)^2} df \\ &\leq \sum_{k=0}^{K-1} \int_{f\in[-1/2,1/2]\setminus[-W,W]} V_k(f)^2 df \\ &= \sum_{k=0}^{K-1} (1-\lambda_k) = \Lambda_N.\end{aligned}$$

Hence we have

$$\begin{aligned}&\left|\mathbb{E}\left[\sum_{j=0}^{N-1} w_j X_j e^{-i2\pi jf} \sum_{j=0}^{N-1} w_j X_j e^{-i2\pi jf'}\right]\right| \\ =&\left|\int_{-1/2}^{1/2} V(f-\alpha)V(\alpha+f')S(\alpha)d\alpha\right| \\ \leq&\left|\int_{|f-\alpha|\leq W} V(f-\alpha)V(\alpha+f')S(\alpha)d\alpha\right| \\ &+\left|\int_{|f'+\alpha|\leq W} V(f-\alpha)V(\alpha+f')S(\alpha)d\alpha\right| \\ &+\left|\int_{\alpha\in[-1/2,1/2],|f-\alpha|>W,|f'+\alpha|>W} V(f-\alpha)V(\alpha+f')S(\alpha)d\alpha\right| \\ \leq&(2\sqrt{\Lambda_N}+\Lambda_N)\|S\|_\infty.\end{aligned}$$

If $|f-f'|\geq 2W$, a similar computation as above yields

$$|\mathbb{E}\left[\sum_{j=0}^{N-1} w_j X_j e^{-i2\pi jf} \sum_{j=0}^{N-1} w_j X_j e^{i2\pi jf'}\right]| \leq (2\sqrt{\Lambda_N}+\Lambda_N)\|S\|_\infty.$$

If $f=f'$, we have

$$\begin{aligned}&\left|\mathbb{E}\left[\sum_{j=0}^{N-1} w_j X_j e^{-i2\pi jf} \sum_{j=0}^{N-1} w_j X_j e^{i2\pi jf} - S(f)\right]\right| \\ =&\left|\int_{-1/2}^{1/2} V(f-\alpha)V(\alpha-f)(S(\alpha)-S(f))d\alpha 0\right| \\ \leq&\left|\int_{|f-\alpha|\leq W} V(f-\alpha)V(\alpha-f)(S(\alpha)-S(f))d\alpha\right| \\ &+\left|\int_{\alpha\in[-1/2,1/2],|f-\alpha|>W} V(f-\alpha)V(\alpha-f)(S(\alpha)-S(f))d\alpha\right| \\ \leq&\max_{\alpha\in[f-W,f+W]}|S(\alpha)-S(f)|+2\Lambda_N\|S\|_\infty.\end{aligned}$$

□

Note that in the application below we will require $\Lambda_N = O(N^{-c})$ for some c. This can be achieved for the DPSS tapers when we take $g(N,W) = \Omega(\log N \log(NW))$.

Now assume that $S(f) > 0$. Let $(\bar{a}_{j,1}(f), \bar{a}_{j,2}(f))$ be the real and imaginary parts of $\bar{a}_j(f)$ divided by $\sqrt{S(f)}$. We have the following error for the covariance of terms.

Lemma B.11. *Suppose that $X_n \in \mathcal{L}^p$, for $p > 4$, $\Theta_{N,p} = O(N^{-\alpha})$ and $\gamma \in \left(\frac{5p-2}{6p-4}, 1\right)$ holds, then there exists a constant $c > 0$ such that the following bound is uniform for any two frequencies $f, f' \in \pi_N$, and $j', k' = 1$ or 2:*

1. *If $j' = k' = 1$, then*

$$\left| \mathrm{Cov}\left(\sum_{j=1}^{n} \bar{a}_{j,1}(f), \sum_{k=1}^{n} \bar{a}_{k,1}(f') \right) - \frac{\mathbb{E}\left[\sum_{j=0}^{N-1} w_j X_j \cos(2\pi j f) \sum_{j=0}^{N-1} w_j X_j \cos(2\pi j f') \right]}{(S(f)S(f'))^{1/2}} \right| = O(N^{-c}).$$

2. *If $j' = k' = 2$, then*

$$\left| \mathrm{Cov}\left(\sum_{j=1}^{n} \bar{a}_{j,1}(f), \sum_{k=1}^{n} \bar{a}_{k,1}(f') \right) - \frac{\mathbb{E}\left[\sum_{j=0}^{N-1} w_j X_j \sin(2\pi j f) \sum_{j=0}^{N-1} w_j X_j \sin(2\pi j f') \right]}{(S(f)S(f'))^{1/2}} \right| = O(N^{-c}).$$

3. *If $j' = 1, k' = 2$, then*

$$\left| \mathrm{Cov}\left(\sum_{j=1}^{n} \bar{a}_{j,1}(f), \sum_{k=1}^{n} \bar{a}_{k,1}(f') \right) - \frac{\mathbb{E}\left[\sum_{j=0}^{N-1} w_j X_j \cos(2\pi j f) \sum_{j=0}^{N-1} w_j X_j \sin(2\pi j f') \right]}{(S(f)S(f'))^{1/2}} \right| = O(N^{-c}).$$

Proof. This follows immediately from the reduction from X_j to $\bar{a}_{j,1}$. More explicitly, we have

$$\begin{aligned} \mathbb{E}[|\sum_{j=1}^{n} a_j(f) - \bar{a}_j(f)|^2] &\leq \sum_{j=1}^{n} \mathbb{E}[|a_j(f)|^2] \frac{\mathbb{E}[|a_j(f)|^{p-2}]}{N^{-\tau'(p-2)}}. \\ &\lesssim N^{(1-\beta')+\tau'(p-2)+\beta' p+(1-2\gamma)p/2} = N^{-2\tau'}. \end{aligned}$$

As well as

$$\mathbb{E}[|\sum_{j=1}^{n} b_j(f)|^2] \lesssim N^{(1-\beta')+\tau'(p-2)+\beta p+(1-2\gamma)p/2} = N^{-2\tau'}.$$

Lemma B.3 also tells us that $\mathbb{E}[|\sum_{j=0}^{N-1} w_j(X_j - \tilde{X}_j(m))e^{-i2\pi j f}|^2] \lesssim \Theta_{m,2}^2 = O(N^{-2\alpha\beta})$. This implies the desired result. □

We will now consider the Gaussian approximation outlined in [CCK17]. In the next few results, we will frequently use a *polytope approximation* for the region $\{\sum_{m=1}^{P_n(j)} |\frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_{ij}^m|^2 \leq x\}$ for some random variables Z_{ij}^m. To achieve this, consider the standard d-dimensional spehrical coordinates defined by $\theta = (\theta_1, \ldots, \theta_d)^T$. We define the subset of unit vectors $V_{n,d} \subset \mathbb{R}^d$ to be those of the form $\{\theta : \theta_i = \pi l_i/n$ for $l_i = 1, \ldots, 2n$ for all $1 \leq i \leq d\}$. These unit vectors then define a set of hyperplanes which for the polytope approximation.

Proposition B.12 (Modified version of Proposition 3.1 of [CCK17])**.** *Fix a natural number c, and let $P_n : \{1, \ldots, h(n)\} \to \{0, 1, \ldots, c\}$. Let $Z_{ij}^m = Z_{ij,1}^m + iZ_{ij,2}^m$ be a complex centered random variable for $i = 1, \ldots, n$, $j = 1, \ldots, h$ and $m = 1, \ldots, P_n(j)$ and let $Z_{ij} = [Z_{ij,1}^1, Z_{ij,2}^1, \ldots, Z_{ij,1}^{P_n(j)}, Z_{ij,2}^{P_n(j)}]^T$. Suppose we can choose*

a sequence of constants $B_n \geq 1$ that may grow to infinity, such the random variables satisfy the following conditions:

(1) $\mathbb{E}[\exp(|Z_{ij,l}^m|/B_n)] \leq 2$ *for* $i = 1, \ldots, n$, $j = 1, 2, \ldots, h$, $l = 1, 2$ *and* $m = 1, \ldots, P_n(i)$.

(2) For $k = 1, 2$, *we have* $n^{-1} \sum_{i=1}^n \mathbb{E}[|Z_{ij,l}^m|^{2+k}] \leq B_n^k$ *for all* $j = 1, 2, \ldots, h$, $l = 1, 2$, *and* $m = 1, \ldots, P_n(i)$.

(3) $n^{-1} \sum_{i=1}^n \mathbb{E}[\left(v^T Z_{ij}\right)^2] \geq b$ *for all* $j = 1, 2, \ldots, h$, *and* $v \in V_{n,2P_n(j)}$.

Suppose that Y is a set of centered complex Gaussian random vectors indexed in the same way as Z, with the same covariance matrix. Then we have

$$\sup_{x \leq n^2} |\mathbb{P}(\max_{1 \leq j \leq h} \sum_{m=1}^{P_n(j)} |\frac{1}{\sqrt{n}} \sum_{i=1}^n Z_{ij}^m|^2 \leq x) - \mathbb{P}(\max_{1 \leq j \leq h} \sum_{m=1}^{P_n(j)} |\frac{1}{\sqrt{n}} \sum_{i=1}^n Y_{ij}^m|^2 \leq x)| \leq C_{b,c} \left(\frac{B_n^2 \log^7(hn)}{n} \right)^{1/6},$$

where $C_{b,c}$ is a constant that only depends on b and c.

Proof. We will first perform a *polytope approximation* for the subset $A_{j,x} := \{\sum_{m=1}^{P_n(j)} |\frac{1}{\sqrt{n}} \sum_{i=1}^n Z_{ij}^m|^2 \leq x\}$. Then

$$\bigcap_{v \in V_{n,2P_n(j)}} \{\sum_{i=1}^n \frac{v^T Z_{ij}}{\sqrt{n}} \leq x \cos(\pi c/n)\} \subset A_{j,x} \subset \bigcap_{v \in V_{n,2P_n(j)}} \{\sum_{i=1}^n \frac{v^T Z_{ij}}{\sqrt{n}} \leq x\}.$$

Since $\cos(\pi c/n) > 1 - \frac{16c^2}{n^2}$ and $x \leq n$, the region $\{\max_{1 \leq j \leq h} \sum_{m=1}^{P_n(j)} |\frac{1}{\sqrt{n}} \sum_{i=1}^n Z_{ij}^m|^2 \leq x\} = \cap_{j=1}^h A_{j,x}$ has a polytope approximation of precision $\frac{16c^2}{n}$. In addition, the number of hyperplanes needed to define the polytope approximation is $\leq h(2n)^{2c}$, and so condition (C) of [CCK17] is satisfied for $a = 4c$ and $d = 2c + 1$ for $x \leq n$ and n sufficiently large.

We will now show that condition (M.1'), (M.2') and (E.1') of Proposition 3.1 of [CCK17] are satisfied, for B_n in the conditions replaced by $B_n' = C_1 B_n$ for some constant C_1 depending on c. (M.1') follows directly from Condition 3. (M.2') follows from Condition 2 and the bound

$$|v^T Z_{ij}|^{2+k} \lesssim \sum_{m=1}^{P_n(j)} \left(|Z_{ij,1}^m|^{2+k} + |Z_{ij,2}^m|^{2+k} \right). \tag{16}$$

(E.1') follows from Condition 1 and the bound

$$\begin{aligned} &\exp(|v^T Z_{ij}|/C_1 B_n) \\ &\leq \frac{1}{(2c+1)} \sum_{m=1}^{P_n(j)} \left(\exp(\frac{(2c+1)|Z_{ij,1}^m|}{C_1 B_n}) + \exp(\frac{(2c+1)|Z_{ij,2}^m|}{C_1 B_n}) \right) \leq 2 \end{aligned} \tag{17}$$

for sufficiently large C_1 where here we used the convexity of exp. Applying Proposition 3.1 of [CCK17], we have the desired result. □

Remark B.13. This step is one of the main obstruction that forces us to limit π_N to frequencies in the range $[W, \frac{1}{2} - W]$. For a frequency f outside of this range, Condition 3 no longer holds as the variance of the imaginary components of the Fourier transform at f will tend to 0 as $f \to 0$, and so the lower bound of the variance term can no longer hold uniformly for all frequencies.

We also show an anti-concentration result:

Lemma B.14. *Fix a natural number c, and let $P_n : \{1, \dots, h(n)\} \to \{0, 1, \dots, c\}$. Let $Z_{ij}^m = Z_{ij,1}^m + iZ_{ij,2}^m$ be a complex centered Gaussian random variable for $i = 1, \dots, n$, $j = 1, \dots, h$ and $m = 1, \dots, P_n(j)$ and let $Z_{ij} = [Z_{ij,1}^1, Z_{ij,2}^1, \dots, Z_{ij,1}^{P_n(j)}, Z_{ij,2}^{P_n(j)}]^T$. Define $c' = 2\log(2^c n^{c+1} h)^{-\delta}$. Then for any $\delta > 0$,*

$$\sup_{\sqrt{x}\cos(\pi\sqrt{c}/2n) > c'} \left| \mathbb{P}\left(\max_{1\le j\le h} \sum_{m=1}^{P_n(j)} \left| \sum_{i=1}^{n} Z_{ij}^m/\sqrt{n} \right|^2 \le x + a \right) - \mathbb{P}\left(\max_{1\le j\le h} \sum_{m=1}^{P_n(j)} \sum_{i=1}^{n} \left| Z_{ij}^m/\sqrt{n} \right|^2 \le x \right) \right|$$
$$\le C\left((a^{1/2} + 4x^{1/2}/n^2)(\log 2^c n^{c+1} h)^{1+\delta} + (\log 2^c n^{c+1} h)^{-1/2} (2^c n^{c+1} h)^{-1} \right),$$

where C is an absolute constant.

Proof. This result is implied by Corollary B.1 and the proof of Proposition B.2 of [WZ23], which we will state the proof here explicitly. We recall the notations and the *polytope approximation* as in Proposition B.12 which gives

$$\mathbb{P}\left(\max_{\substack{1\le j\le h \\ v\in V_{n,2P_n(j)}}} \sum_{i=1}^{n} \frac{v^T Z_{ij}}{\sqrt{n}} \le \sqrt{x}\cos(\pi c/n) \right) \le \mathbb{P}\left(\max_{1\le j\le h} \sum_{m=1}^{P_n(j)} |\sum_{i=1}^{n} Z_{ijl}^m/\sqrt{n}|^2 \le x \right)$$
$$\le \mathbb{P}\left(\max_{\substack{1\le j\le h \\ v\in V_{n,2P_n(j)}}} \sum_{i=1}^{n} \frac{v^T Z_{ij}}{\sqrt{n}} \le \sqrt{x} \right).$$

We have $\cos(\pi c/2n) > 1 - \frac{16c^2}{n^2}$. By Corollary B.1 of [WZ23], if $\sqrt{x}\cos(\pi c/n) > c'$ we conclude that

$$\mathbb{P}\left(\max_{\substack{1\le j\le h \\ v\in V_{n,2P_n(j)}}} \sum_{i=1}^{n} \frac{v^T Z_{ij}}{\sqrt{n}} \le \sqrt{x+a} \right) - \mathbb{P}\left(\max_{\substack{1\le j\le h \\ v\in V_{n,2P_n(j)}}} \sum_{i=1}^{n} \frac{v^T Z_{ij}}{\sqrt{n}} \le \sqrt{x}\cos(\pi c/n) \right)$$
$$\le C\left((a^{1/2} + 4x^{1/2}c^2/n^2)(\log 2^c n^{c+1} h)^{1+\delta} + (\log 2^c n^{c+1} h)^{-1/2} (2^c n^{c+1} h)^{-1} \right)$$

where C is an absolute constant, concluding the proof. □

The final piece is the Gaussian comparison result, which is a generalization of a similar result from [WZ23] to the sum of a variable number of terms.

Proposition B.15. *Fix a natural number c, and let $P : \{1, \dots, h\} \to \{0, 1, \dots, c\}$. Let $Y_j^m = Y_{j,1}^m + iY_{j,2}^m$ be a complex centered Gaussian random variable for $j = 1, \dots, h$ and $m = 1, \dots, P_n(j)$ and let $Y_j = [Y_{j,1}^1, Y_{j,2}^1, \dots, Y_{j,1}^{P_n(j)}, Y_{j,2}^{P_n(j)}]^T$ and let Y be all the random variables Y_j^m. Define W and its corresponding random variables in a similar fashion. Let $\Delta = \max|\mathrm{Cov}((Y,Y)) - \mathrm{Cov}((W,W))|$. Suppose that $c' > 0$ is a constant such that $c' > \mathbb{E}[(Y_{j,l}^m)^2], \mathbb{E}[(W_{j,l}^m)^2] > 0$ for all $j = 1, \dots, h, m = 1, \dots, P(j)$, $l = 1, 2$. Then we have*

$$\sup_{x \ge d^2} \left| \mathbb{P}\left(\max_{1\le j\le h} \sum_{m=1}^{P(j)} |Y_j^m|^2 \le x \right) - \mathbb{P}\left(\max_{1\le j\le h} \sum_{m=1}^{P(j)} |W_j^m|^2 \le x \right) \right| \tag{18}$$
$$\lesssim \Delta^{1/3} \log^{1+2\delta}(2^c h^{c+1}) + h^{-1} \log^{1+\delta}(2^c h^{c+1}),$$

where $d = C(\Delta^{1/3} \log^{\delta/3}(2^c h^{c+1}) + \log^{-\delta}(2^c h^{c+1}))$ for some constant C depending only on c, c'.

Proof. The proof follows from a similar idea used in Proposition C.3 in [WZ23]. Using a *polytope approximation*, we have

$$\mathbb{P}(\max_{\substack{1\le j\le h\\ v\in V_{h,2P_n(j)}}}\sum_{i=1}^n v^TY_{ij}\le\sqrt{x}\cos(\pi c/h))\le\mathbb{P}(\max_{1\le j\le h}\sum_{m=1}^{P(j)}|Y_j^m|^2\le x)$$
$$\le\mathbb{P}(\max_{\substack{1\le j\le h\\ v\in V_{h,2P_n(j)}}}\sum_{i=1}^n v^TY_{ij}\le\sqrt{x})$$

Now, for a sufficiently large d, we can apply Theorem C.1 of [WZ23] to obtain

$$\sup_{x\ge d^2}\left|\mathbb{P}(\max_{\substack{1\le j\le h\\ v\in V_{h,2P_n(j)}}}\sum_{i=1}^n v^TY_{ij}\le\sqrt{x}\cos(\pi c/h))-\mathbb{P}(\max_{\substack{1\le j\le h\\ v\in V_{h,2P_n(j)}}}\sum_{i=1}^n v^TW_{ij}\le\sqrt{x}\cos(\pi c/h))\right|$$
$$\lesssim\Delta^{1/3}\log^{1+2\delta}(2^ch^{c+1})+(2^ch^{c+1})^{-1}\log(2^ch^{c+1})^{-1/2}$$

Similarly

$$\sup_{x\ge d^2}\left|\mathbb{P}(\max_{\substack{1\le j\le h\\ v\in V_{h,2P_n(j)}}}\sum_{i=1}^n v^TY_{ij}\le\sqrt{x})-\mathbb{P}(\max_{\substack{1\le j\le h\\ v\in V_{h,2P_n(j)}}}\sum_{i=1}^n v^TW_{ij}\le\sqrt{x})\right|$$
$$\lesssim\Delta^{1/3}\log^{1+2\delta}(2^ch^{c+1})+(2^ch^{c+1})^{-1}\log(2^ch^{c+1})^{-1/2}.$$

Now for $x\le h^2$, we have $|\sqrt{x}-\sqrt{x}\cos(\pi c/h)|\lesssim ch^{-1}$, thus by Lemma B.14, we obtain

$$\sup_{h^2\ge x\ge d^2}\left|\mathbb{P}(\max_{\substack{1\le j\le h\\ v\in V_{h,2P_n(j)}}}\sum_{i=1}^n v^TY_{ij}\le\sqrt{x})-\mathbb{P}(\max_{\substack{1\le j\le h\\ v\in V_{h,2P_n(j)}}}\sum_{i=1}^n v^TY_{ij}\le\sqrt{x}\cos(\pi c/h))\right|$$
$$\lesssim c^{1/2}h^{-1/2}\log^{1+\delta}(2^ch^{c+1})+h^{-2}\log^{-1/2}(2^ch^{c+1}).$$

A similar anti-concentration bound also holds for W, and so the result then follows for the case $h^2\ge x\ge d^2$. Finally if $x>h^2$, we have

$$\mathbb{P}(\max_{1\le j\le h}\sum_{m=1}^{P(j)}|Y_j^m|^2\ge x) \tag{19}$$
$$\le\sum_{m=1}^{P(j)}\mathbb{P}(\max_{1\le j\le h,1\le l\le 2}|Y_{j,l}^m|\ge\sqrt{x/c})\le 4ch\mathbb{P}(N(0,c')\ge\sqrt{x/c}/2)$$
$$\lesssim ch\cdot\sqrt{cc'/x}\exp(-x/(4cc'))\lesssim c^{3/2}c'^{1/2}h^{-1}\exp(-h^2/(2c^2c')),$$

and the result follows. □

We also include a result on the maximum deviation of multitaper estimates for a Gaussian stationary process, which is necessary for the final stage of the Gaussian approximation. This is modified from Theorem 5 of [KRD21b] for a general set of tapers.

Lemma B.16. *Suppose that X_n is a Gaussian stationary process. Then*

$$\mathbb{P}(\hat{S}^{mt}(f)\ge\beta\mathbb{E}[\hat{S}^{mt}(f)])\le\beta^{-1}\exp(-\kappa_f(\beta-1-\ln\beta))\text{ for }\beta>1.$$
$$\mathbb{P}(\hat{S}^{mt}(f)\le\beta\mathbb{E}[\hat{S}^{mt}(f)])\le\exp(-\kappa_f(\beta-1-\ln\beta))\text{ for }\beta\in(0,1),$$

where $\kappa_f = \frac{(\sum_{k=0}^{K-1}\lambda_k)\max_{\omega\in[f-W,f+W]}S(\omega) - N(2W\max_{\omega\in[f-W,f+W]}S(\omega) - \int_{f-W}^{f+W}S(\omega)d\omega)}{\|S\|_\infty}$.

Proof. Write the matrix of tapers

$$V = \begin{bmatrix} v_0^T & v_1^T & \dots & v_{K-1}^T \end{bmatrix} \in \mathbb{R}^{N\times N}$$

and

$$E_f = \operatorname{diag}\begin{bmatrix} 1 & e^{i2\pi f} & \dots & e^{i2\pi(N-1)f} \end{bmatrix} \in \mathbb{C}^{N\times N}.$$

By Lemma 4 and the first part of the proof of Theorem 5 of [KRD21b], it suffices to show that $\|V^*E_f^*\Sigma E_f V\|_2 \le \|S\|_\infty$. We have

$$\begin{aligned}
&V^*E_f^*\Sigma E_f V \\
&= V^*\left(\int_{-1/2}^{1/2} S(f+f')[1 \quad e^{i2\pi f'} \dots e^{i2\pi(N-1)f'}][1 \quad e^{-i2\pi f'} \dots e^{-i2\pi(N-1)f'}]^T df'\right)V \\
&\preceq V^*\left(\int_{-1/2}^{1/2} \|S\|_\infty[1 \quad e^{i2\pi f'} \dots e^{i2\pi(N-1)f'}][1 \quad e^{-i2\pi f'} \dots e^{-i2\pi(N-1)f'}]^T df'\right)V \\
&= \|S\|_\infty I_K.
\end{aligned}$$

This gives the desired result. □

This section provides the lemmas used and also additional proofs for results stated in Section 3. First, we quote Theorem 2 of [LR08] which bounds the covariance of the multitaper spectral estimates at any two frequencies.

Theorem B.17 (Theorem 2, [LR08])**.** *Suppose that* $Y_n \in \mathcal{L}^2$ *is a Gaussian process with the spectral density function* $S(f)$*. Then for* $f, f' \in [0, 1/2]$*, we have*

$$\operatorname{Cov}\left(\frac{1}{K}\sum_{k=0}^{K-1}\left|\sum_{j=0}^{N-1} v_{k,j}Y_j e^{-i2\pi fj}\right|^2, \frac{1}{K}\sum_{k=0}^{K-1}\left|\sum_{j=0}^{N-1} v_{k,j}Y_j e^{-i2\pi f'j}\right|^2\right) \le \frac{2\|S\|_\infty^2}{K}.$$

We show that the covariance between two tapered estimates at frequencies f and f' respectively changes little when we switch to a m-dependent process.

Lemma B.18. *Suppose that* $X_n \in \mathcal{L}^4$ *and* $\Theta_{0,4} < \infty$*. Take the normalized tapers* $\mathbf{v}_r$ *for* $r = 0, 1, \dots, K-1$*. Then there exists a constant* $C > 0$ *such that*

$$\begin{aligned}
&\left|\operatorname{Cov}\left(\left|\sum_{j=0}^{N-1} v_{r,j}X_j e^{-2\pi fj}\right|^2, \left|\sum_{j=0}^{N-1} v_{s,j}X_j e^{-2\pi f'j}\right|^2\right)\right. \\
&\left. - \operatorname{Cov}\left(\left|\sum_{j=0}^{N-1} v_{r,j}\tilde{X}_j(m) e^{-2\pi fj}\right|^2, \left|\sum_{j=0}^{N-1} v_{s,j}\tilde{X}_j(m) e^{-2\pi f'j}\right|^2\right)\right| \le C\Theta_{0,4}^3\Theta_{m,4}
\end{aligned}$$

for any $0 \le r, s \le K-1$.

Proof. For each $r = 0, 1, \dots, K-1$, rewrite

$$\sum_{j=0}^{N-1} v_{r,j}X_j e^{-2\pi fj} = \sum_{j=0}^{N-1} v_{r,j}\tilde{X}_j(m)e^{-2\pi fj} + \sum_{j=0}^{N-1} v_{r,j}(X_j - \tilde{X}_j(m))e^{-2\pi fj} =: X(r,m) + Z(r,m)\,,$$

which is controlled by $\mathbb{E}[|X(r,m)|^p] \le C_p \Theta_{0,p}^p$ and $\mathbb{E}[|Z(r,m)|^p] \le C_p \Theta_{m,p}^p$ by Lemma B.3. Now rewrite the difference in covariance as

$$\begin{aligned}
&\mathrm{Cov}(|X(r,m)+Z(r,m)|^2, |X(s,m)+Z(s,m)|^2) - \mathrm{Cov}(|X(r,m)|^2, |X(s,m)|^2) \\
=& \mathrm{Cov}(|Z(r,m)|^2, |Z(s,m)|^2) + 2\mathrm{Re}(\mathrm{Cov}(X(r,m)Z(r,m), |Z(s,m)|^2)) \\
&+ 2\mathrm{Re}(\mathrm{Cov}(X(s,m)Z(s,m), |Z(r,m)|^2)) + 2\mathrm{Re}(\mathrm{Cov}(|X(r,m)|^2, |Z(s,m)|^2)) \\
&+ 2\mathrm{Re}(\mathrm{Cov}(|X(s,m)|^2, |Z(r,m)|^2)) + 2\mathrm{Re}(\mathrm{Cov}(X(r,m)Z(r,m), X(s,m)Z(s,m))) \\
&+ 2\mathrm{Re}(\mathrm{Cov}(X(r,m)Z(r,m), |X(s,m)|^2)) + 2\mathrm{Re}(\mathrm{Cov}(X(s,m)Z(s,m), |X(r,m)|^2)) .
\end{aligned}$$

These terms can be bounded in a similar manner using Hölder's inequality. We only show the control for the last term, and the other terms are controlled in the same way.

$$\begin{aligned}
&|\mathrm{Re}(\mathrm{Cov}(X(s,m)Z(s,m), |X(r,m)|^2))| \\
\le& |\mathrm{Re}(\mathbb{E}[X(s,m)Z(s,m)|X(r,m)|^2] - \mathbb{E}[X(s,m)Z(s,m)]\mathbb{E}[|X(r,m)|^2])| \\
\le& \mathbb{E}[|X(s,m)| \cdot |Z(s,m)| \cdot |X(r,m)|^2] + \mathbb{E}[|X(s,m)| \cdot |Z(s,m)|]\mathbb{E}[|X(r,m)|^2] \\
\le& \left(\mathbb{E}[|X(s,m)|^4]\mathbb{E}[|Z(s,m)|^4]\mathbb{E}[|X(r,m)|^4]^2\right)^{1/4} + \left(\mathbb{E}[|X(s,m)|^2]\mathbb{E}[|Z(s,m)|^2]\right)^{1/2} \mathbb{E}[|X(r,m)|^2] \\
\le& 2\Theta_{0,4}^3 \Theta_{m,4} .
\end{aligned}$$

The other terms in the sum can be similarly bounded by constant multiples of $\Theta_{0,4}^{4-k}\Theta_{m,4}^k$ for $1 \le k \le 4$, which are all controlled by $\Theta_{0,4}^3\Theta_{m,4}$. This completes the proof. □

Proof of Theorem 3.1. Denote $X(r,m) := \sum_{j=0}^{N-1} v_{r,j}\tilde{X}_j(m)e^{-2\pi f j}$. By a direct expansion,

$$\begin{aligned}
&\mathbb{E}[|X(r,m)|^2|X(s,m)|^2] \\
=:& \sum_{i\le j\le k\le l} R_{i,j,k,l}\mathbb{E}[\tilde{X}_i(m)\tilde{X}_j(m)\tilde{X}_k(m)\tilde{X}_l(m)] \\
=& \sum_{\substack{i\le j\le k\le l \\ |j-i|,|k-j|,|l-k|\le m}} R_{i,j,k,l}\mathbb{E}[\tilde{X}_i(m)\tilde{X}_j(m)\tilde{X}_k(m)\tilde{X}_l(m)] \\
&+ \sum_{\substack{i\le j\le k\le l \\ |k-j|>m}} R_{i,j,k,l}\mathbb{E}[\tilde{X}_i(m)\tilde{X}_j(m)]\mathbb{E}[\tilde{X}_k(m)\tilde{X}_l(m)] ,
\end{aligned}$$

where $R_{i,j,k,l}$ is a sum of at most 24 terms of the form $e^{i\alpha}v_{r,i}v_{r,j}v_{s,k}v_{s,l}$ and thus applying Proposition A.3 we have $|R_{i,j,k,l}| \le 24W^2$. Also, introduce a Gaussian process $\tilde{Y}_n(m)$ with the same autocovariance sequence as that of $\tilde{X}_n(m)$, and define $Y(r,m) = \sum_{j=0}^{N-1} v_{r,j}\tilde{Y}_j(m)e^{-2\pi f j}$. We can thus write

$$\begin{aligned}
&\mathbb{E}[|X(r,m)|^2|X(s,m)|^2] - \mathbb{E}[|Y(r,m)|^2|Y(s,m)|^2] \\
=& \sum_{(i,j,k,l)\in I} R_{i,j,k,l}(\mathbb{E}[\tilde{X}_i(m)\tilde{X}_j(m)\tilde{X}_k(m)\tilde{X}_l(m)] - \mathbb{E}[\tilde{Y}_i(m)\tilde{Y}_j(m)\tilde{Y}_k(m)\tilde{Y}_l(m)])
\end{aligned}$$

where $I = \{0 \le i \le j \le k \le l \le N-1, |j-i|, |k-j|, |l-k| \le m\}$. From a direct counting argument there are at most $N(m+1)^3$ terms in the above summation. By applying Hölder's Inequality and the fact that $\|\tilde{Y}_i(m)\|_4^4 = 3\|\tilde{Y}_i(m)\|_2^4 =$

$3\|\tilde{X}_i(m)\|_2^4$, we have for a constant $C_1 > 0$ so that

$$|\mathrm{Cov}(|X(r,m)|^2, |X(s,m)|^2) - \mathrm{Cov}(|Y(r,m)|^2, |Y(s,m)|^2)|$$

(20)

$$\leq 24W^2 \sum_{(i,j,k,l)\in I} |\mathbb{E}[\tilde{X}_i(m)\tilde{X}_j(m)\tilde{X}_k(m)\tilde{X}_l(m)]| + |\mathbb{E}[\tilde{Y}_i(m)\tilde{Y}_j(m)\tilde{Y}_k(m)\tilde{Y}_l(m)]|$$

(21)

$$\leq C_1(m+1)^3 NW^2 \|X_n\|_4^4.$$

Let $\hat{S}_m^{mt}(f)$ (3) denote the estimate of $S_m(f)$, which is the spectral density of both $\tilde{Y}_n(m)$ and $\tilde{X}_n(m)$. Applying Lemma B.18, we have for a constant $C_2 > 0$ so that

$$\mathrm{Var}(\hat{S}^{mt}(f)) \leq C_2(\Theta_{0,4}^3\Theta_{m,4} + m^3NW^2\mathbb{E}[X_n^4]) + \mathrm{Var}(\hat{S}_m^{mt}(f)) \tag{22}$$

By Lemma B.2, the absolute value of autocovariance sequence $\tilde{r}_\tau$ for $\tilde{Y}_n(m)$ is bounded by $\sum_{i=0}^{\infty}\theta_{i,2}\theta_{i+|\tau|,2}$. Thus, $S_m(f) \leq 2\Theta_{0,2}^2$. By Theorem B.17, we have $\mathrm{Var}(\hat{S}_G^{mt}(f)) \leq 2[\sup S(f)]^2/K$, and hence

$$\mathrm{Var}(\hat{S}_m^{mt}(f)) \leq 2\Theta_{0,2}^4/K\,.$$

By putting all together, we obtain (7). If we choose m to be a function of N such that $m(N) \to \infty$ and $m(N) = o(W^{-2/3}N^{-1/3})$, as $N \to \infty$, we have $\mathrm{Var}(\hat{S}^{mt}(f)) = o(1)$. □

Appendix C. Proof of theorems in Section 4

To prove Theorem 4.1, we introduce the m-dependent approximation of X_n by $\tilde{X}_n(m) = \mathbb{E}[X_n|\varepsilon_{n-m}, \ldots, \varepsilon_n]$. Note that $\tilde{X}_n(m)$ is also a physically dependent stationary process. We provide a Berry-Esseen type bound for the sum of Fourier transforms of the stationary process with multiple tapers, across multiple frequencies. Before stating the next theorem, we summarize the following CLT result on martingale differences which is critical to our proof.

Lemma C.1 (Special case of Theorem 1, [HH81]). *Let $\{S_i = \sum_{j=1}^{i} X_j, \mathcal{F}_j, 1 \leq i \leq n\}$ be a zero-mean, square integrable martingale. Define*

$$L = \sum_{i=1}^{n} \mathbb{E}[|X_i|^4] + \mathbb{E}[|(\sum_{j=1}^{n} \mathbb{E}[X_j^2|\mathcal{F}_{j-1}])^2 - \eta^2|^2]$$

where η is a constant. Suppose that for some constants c, c', we have $c \leq \eta \leq c'$, then there exists a constant A depending only on c and c', such that for all m and $L \leq 1$, we have

$$|\mathbb{P}(S_n \leq x) - \Phi(\eta^{-1}x)| \leq AL^{1/5}(1 + |x|^{16/5})^{-1}.$$

Theorem C.2. *Take distinct frequencies $f_1, \ldots, f_s \in [0, 1/2]$, and $\alpha_{l,k}, \beta_{l,k} \in \mathbb{R}$, where $l = 1, \ldots, s$ and $k = 0, \ldots, K-1$. Define*

$$T^l(N) := \sum_{k=0}^{K-1} \left(\alpha_{l,k}\mathrm{Re}\left(\sum_{j=0}^{N-1} v_{k,j}X_j e^{-i2\pi f_l j} \right) + \beta_{l,k}\mathrm{Im}\left(\sum_{j=0}^{N-1} v_{k,j}X_j e^{-i2\pi f_l j} \right) \right)$$

and $T(N) := \sum_{l=1}^{s} T^l(N)$. Define $T_m^l(N)$ and $T_m(N)$ in the same way but for the m-dependent approximation stationary process $\tilde{X}_n(m)$. Let $c \leq \|T_m(N)\|_2 \leq c'$. If

$X_n \in \mathcal{L}^4$ and $\Theta_{0,4} < \infty$, then there exists a constant $C > 0$ depending only on c and c', such that

$$
\begin{aligned}
(23) \qquad & \rho\left(T(N), G\right) \\
& \leq C\Bigg[\left((\sum_{l,k} |\alpha_{l,k}| + |\beta_{l,k}|)^4 m^6 (N+m) (\sup_{k,j} |v_{k,j}|)^4 \|\tilde{X}_n(m)\|_4^4 \right)^{1/5} \\
& \qquad + \left(\sum_{l,k} (\alpha_{l,k}^2 + \beta_{l,k}^2) \right)^{1/3} \Theta_{m+1,2}^{2/3} \|T(N)\|_2^{-2/3} \\
& \qquad + \left| \|T(N)\|_2^{-1} - \|T_m(N)\|_2^{-1} \right| \max\{\|T_m(N)\|_2, \|T(N)\|_2\} \Bigg],
\end{aligned}
$$

where $G \sim N(0,\, \|T(N)\|_2^2)$.

Proof. We start with the m-dependent approximation and deduce a CLT convergence rate. Define the quantity

$$
D_{l,k,t} := \mathcal{P}_t \left(\sum_{j=-\infty}^{\infty} v_{k,j} \tilde{X}_j(m) e^{-i2\pi f_l j} \right),
$$

where $v_{k,j} = 0$ if $j \notin \{0, \ldots, N-1\}$. A closer look at $D_{l,k,t}$ shows that it is a sum of at most $m+1$ terms, is m-dependent, and forms a martingale difference sequence. We can write

$$
T_m(N) = \sum_{t=-m}^{N-1} \sum_{l=1}^{s} \sum_{k=0}^{K-1} (\alpha_{l,k} \cdot \mathrm{Re}(D_{l,k,t}) + \beta_{l,k} \cdot \mathrm{Im}(D_{l,k,t})).
$$

Then by setting $\eta = \|T_m(N)\|_2$ in Lemma C.1, we have

$$
\left| \mathbb{P}\left(T_m(N) \leq x\right) - \Phi\left(\frac{x}{\|T_m(N)\|_2} \right) \right| \leq \frac{A(L_1(N) + L_2(N))^{1/5}}{1 + |x|^{16/5}},
$$

where $A > 0$ is a constant depending on c, c' and

$$
L_1(N) := \sum_{t=-m}^{N-1} \mathbb{E}\left[\left(\sum_{l=1}^{s} \sum_{k=0}^{K-1} \alpha_{l,k} \cdot \mathrm{Re}(D_{l,k,t}) + \beta_{l,k} \cdot \mathrm{Im}(D_{l,k,t}) \right)^4 \right],
$$

$$
L_2(N) := \mathbb{E}\left[\left(\sum_{t=-m}^{N-1} \mathbb{E}\left[\left(\sum_{l=1}^{s} \sum_{k=0}^{K-1} \alpha_{l,k} \cdot \mathrm{Re}(D_{l,k,t}) + \beta_{l,k} \cdot \mathrm{Im}(D_{l,k,t}) \right)^2 \middle| F_{t-1} \right] - \mathbb{E}[T_m(N)^2] \right)^2 \right].
$$

From $\mathbb{E}[|D_{l,k,t}|^4] \leq \mathbb{E}[|\sum_{j=t}^{t+m} v_{k,j} \mathcal{P}_t \tilde{X}_j(m) e^{-i2\pi f_l j}|^4] \leq (m+1)^4 (\sup_{k,j} |v_{k,j}|)^4 \mathbb{E}[\tilde{X}_n^4]$, we have

$$
\begin{aligned}
L_1(N) &\leq \sum_{t=-m}^{N-1} \mathbb{E}\left[\left(\sum_{l=1}^{s} \sum_{k=0}^{K-1} (|\alpha_{l,k}| + |\beta_{l,k}|) \cdot |D_{l,k,t}| \right)^4 \right] \\
&\lesssim (\sum_{l,k} |\alpha_{l,k}| + |\beta_{l,k}|)^4 m^4 \sup_{k,j} |v_{k,j}|^4 \mathbb{E}[\tilde{X}_n^4] (N+m)
\end{aligned}
$$

Next, we control $L_2(N)$. First we rewrite $\mathbb{E}[T_m(N)^2]$ using the orthogonality of projection and obtain

$$\mathbb{E}[T_m(N)^2] = \sum_{t=-m}^{N-1} \mathbb{E}[(\sum_{l=1}^{s}\sum_{k=0}^{k-1} \alpha_{l,k}\cdot \mathrm{Re}(D_{l,k,t}) + \beta_{l,k}\cdot \mathrm{Im}(D_{l,k,t}))^2],$$

Further using the orthogonality of projection, we obtain the bound

$$\begin{aligned}
L_2(N) &= \Big\| \sum_{t=-m}^{N-1} \mathbb{E}\Big[\Big(\sum_{l=1}^{s}\sum_{k=0}^{k-1} \alpha_{l,k}\cdot \mathrm{Re}(D_{l,k,t}) + \beta_{l,k}\cdot \mathrm{Im}(D_{l,k,t})\Big)^2 \Big| F_{t-1}\Big] \\
&\quad - \mathbb{E}\Big[\Big(\sum_{l=1}^{s}\sum_{k=0}^{k-1} \alpha_{l,k}\cdot \mathrm{Re}(D_{l,k,t}) + \beta_{l,k}\cdot \mathrm{Im}(D_{l,k,t})\Big)^2\Big]\Big\|_2^2 \\
&= \Big\| \sum_{t=-m}^{N-1}\sum_{j=1}^{m} \mathcal{P}_{t-j}\Big(\sum_{l=1}^{s}\sum_{k=0}^{k-1} \alpha_{l,k}\cdot \mathrm{Re}(D_{l,k,t}) + \beta_{l,k}\cdot \mathrm{Im}(D_{l,k,t})\Big)^2\Big\|_2^2 \\
&= \sum_{t=-m}^{N-1} \Big\| \sum_{j=1}^{m} \mathcal{P}_{t-j}\Big(\sum_{l=1}^{s}\sum_{k=0}^{k-1} \alpha_{l,k}\cdot \mathrm{Re}(D_{l,k,t}) + \beta_{l,k}\cdot \mathrm{Im}(D_{l,k,t})\Big)^2\Big\|_2^2 \\
&\lesssim \sum_{t=-m}^{N-1} \Big\| \sum_{j=1}^{m} \Big(\sum_{l=1}^{s}\sum_{k=0}^{k-1} |\alpha_{l,k}| + |\beta_{l,k}|\Big)^2\Big\|_2^2 \max_{l,k,t} |D_{l,k,t}|^4 \\
&\le (\sum_{l=1}^{s}\sum_{k=0}^{k-1} |\alpha_{l,k}| + |\beta_{l,k}|\Big)^4 (N+m) m^6 \sup_{k,j} |v_{k,j}|^4 \mathbb{E}[\tilde{X}_n^4].
\end{aligned}$$

To tie the m-approximated stationary process back to the theorem, we will use the anti-concentration properties of the normal distribution. We first control the Kolmogorov distance between $N(0, \|T_m(N)\|_2^2)$ and $N(0, \|T(N)\|_2^2)$ by

$$\begin{aligned}
&\rho\left(N(0,\|T_m(N)\|_2^2), N(0,\|T(N)\|_2^2)\right) = \sup_{x\in\mathbb{R}} \frac{2}{\sqrt{\pi}} \left| \int_{\frac{x}{\sqrt{2}\|T_m(N)\|_2}}^{\frac{x}{\sqrt{2}\|T(N)\|_2}} e^{-t^2} dt \right| \\
&\le \frac{2}{\sqrt{\pi}} \sup_{x\in\mathbb{R}} \left| \int_{\frac{x}{\sqrt{2}\|T_m(N)\|_2}}^{\frac{x}{\sqrt{2}\|T(N)\|_2}} \exp\left(-\min\left(\frac{x^2}{2\|T_m(N)\|_2^2}, \frac{x^2}{2\|T(N)\|_2^2}\right)\right) dt \right| \\
&\le \frac{\sqrt{2}}{\sqrt{\pi}} \left| \frac{1}{\|T(N)\|_2} - \frac{1}{\|T_m(N)\|_2} \right| \sup_{x\in\mathbb{R}} |x| \exp\left(-\min\left(\frac{x^2}{2\|T_m(N)\|_2^2}, \frac{x^2}{2\|T(N)\|_2^2}\right)\right) \\
&\lesssim \frac{1}{\sqrt{\pi}} \left| \frac{1}{\|T(N)\|_2} - \frac{1}{\|T_m(N)\|_2} \right| \max(\|T_m(N)\|_2, \|T(N)\|_2).
\end{aligned}$$

By Chebyshev's inequality, we have

$$\begin{aligned}
\mathbb{P}(T(N) \le x) &\le \mathbb{P}(T_m(N) \le x+\varepsilon) + \mathbb{P}(|T(N) - T_m(N)| > \varepsilon) \\
&\le \mathbb{P}(T_m(N) \le x+\varepsilon) + \frac{\mathbb{E}[|T(N) - T_m(N)|^2]}{\varepsilon^2}.
\end{aligned}$$

Using Lemma B.3, we find that

$$\frac{\mathbb{E}[|T(N) - T_m(N)|^2]}{\varepsilon^2} \le C_2 \sum_{k,l} (\alpha_{k,l}^2 + \beta_{k,l}^2) \Theta_{m+1,2}^2 / \varepsilon^2$$

for some constant $C_2 > 0$. Note the trivial uniform bound

$$\left|\Phi\left(\frac{x+\varepsilon}{\|T(N)\|_2}\right) - \Phi\left(\frac{x}{\|T(N)\|_2}\right)\right| \le (2\pi)^{-1/2}\frac{\varepsilon}{\|T(N)\|_2}\,,$$

where Φ is the cumulative distribution function of the standard normal distribution and $\varepsilon > 0$. Set $\varepsilon = \left(\sum_{k,l}(\alpha_{k,l}^2+\beta_{k,l}^2)\right)^{1/3}\Theta_{m+1,2}^{2/3}\|T(N)\|_2^{1/3}$. By putting everything together, we have

$$\begin{aligned}
&\rho(T(N), N(0,\|T(N)\|_2^2)) = \sup_{x\in\mathbb{R}}|\mathbb{P}(T(N)\le x) - \Phi(x/\|T(N)\|_2)| \\
\le &\sup_{x\in\mathbb{R}}\left(|\mathbb{P}(T(N)\le x) - \mathbb{P}(T_m(N)\le x+\varepsilon)| + \left|\mathbb{P}(T_m(N)\le x+\varepsilon) - \Phi\left(\frac{x+\varepsilon}{\|T_m(N)\|_2}\right)\right|\right. \\
&\left. + \left|\Phi\left(\frac{x+\varepsilon}{\|T_m(N)\|_2}\right) - \Phi\left(\frac{x+\varepsilon}{\|T(N)\|_2}\right)\right| + \left|\Phi\left(\frac{x+\varepsilon}{\|T(N)\|_2}\right) - \Phi\left(\frac{x}{\|T(N)\|_2}\right)\right|\right) \\
\le &C\Bigg[\Big((\sum_{l,k}|\alpha_{l,k}|+|\beta_{l,k}|)^4 m^6(N+m)(\sup_{k,j}|v_{k,j}|)^4\mathbb{E}[\tilde{X}_n^4]\Big)^{1/5} \\
&\quad + \Big(\sum_{k,l}(\alpha_{k,l}^2+\beta_{k,l}^2)\Big)^{1/3}\Theta_{m+1,2}^{2/3}\|T(N)\|_2^{-2/3} \\
&\quad + \left|\|T(N)\|_2^{-1} - \|T_m(N)\|_2^{-1}\right|\max\{\|T_m(N)\|_2,\,\|T(N)\|_2\}\Bigg]\,,
\end{aligned}$$

where $C > 0$ is a constant. This concludes the proof. □

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Adapt notation in Theorem C.2. Rewrite

$$\sum_{j=0}^{N-1} v_{k,j}X_j e^{-i2\pi f_l j} =: E_{l,k} + iF_{l,k}\,,$$

where $l = 1,\dots,s$ and $k = 0,\dots,K-1$. We claim that the limiting multivariate distribution of the $2Ks$ random variables, $E_{l,k}$ and $F_{l,k}$, is the multivariate Gaussian distribution with a diagonal matrix, where the entry for $E_{l,k}$ and $F_{l,k}$ is $S(f_l)/2$. Suppose the claim is true, then by the continuous mapping theorem the proof is done. Hence it suffices to prove the claim. By the Cramér-Wold device, it suffices to show that for any coefficients $\alpha_{l,k}, \beta_{l,k} \in \mathbb{R}$,

$$\sum_{l=1}^{s}\sum_{k=0}^{K-1}(\alpha_{l,k}E_{l,k} + \beta_{l,k}F_{l,k}) \to_d N\left(0, \sum_{l=1}^{s}\sum_{k=0}^{K-1}(\alpha_{l,k}^2+\beta_{l,k}^2)S(f_l)/2\right). \tag{24}$$

To this end, we apply Theorem C.2. First, we control the right hand side of (23). By $NW^2 \to 0$ and Proposition A.3, we have $\sup_{k,j}|v_{k,j}(N)| = o(N^{-1/4})$. For any monotonically increasing sequence $m(N) \to \infty$ as $N \to \infty$, we have

$$\|T_m(N) - T(N)\|_2 \le \left(\sum_{l=1}^{s}\sum_{k=0}^{K-1}(\alpha_{l,k}^2+\beta_{l,k}^2)\right)^{1/2}\left\|\sum_{j=0}^{N-1}(X_j - \tilde{X}_j(m))v_{k,j}e^{-i2\pi fj}\right\|_2 \to 0$$

by Lemma B.3. Also, we can choose $m(N) \to \infty$, which might increase slowly, so that the right hand side of Theorem C.2 is $o(1)$. Thus, it suffices to show that

$$\mathbb{E}\left[\left(\sum_{l=1}^{s}\sum_{k=0}^{K-1}(\alpha_{l,k}E_{l,k}+\beta_{l,k}F_{l,k})\right)^2\right] \xrightarrow{N\to\infty} \sum_{l=1}^{s}\sum_{k=0}^{K-1}(\alpha_{l,k}^2+\beta_{l,k}^2)S(f_l)/2\,. \tag{25}$$

The left hand side can be rewritten as

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{l=1}^{s}\sum_{k=0}^{K-1}(\alpha_{l,k}E_{l,k}+\beta_{l,k}F_{l,k})\right)^2\right]\\
=&\mathbb{E}\left[\left(\sum_{l=1}^{s}\sum_{k=0}^{K-1}\frac{\alpha_{l,k}-i\beta_{l,k}}{2}\sum_{j=0}^{N-1}v_{k,j}X_j e^{-i2\pi f_l j}+\frac{\alpha_{l,k}+i\beta_{l,k}}{2}\sum_{j=0}^{N-1}v_{k,j}X_j)e^{i2\pi f_l j}\right)^2\right].
\end{aligned}$$

We directly expand the latter and compute the expectation for each term. For any k, k' and f, f', we have

$$\begin{aligned}
&\left|\mathbb{E}\left[\left(\sum_{j=0}^{N-1}v_{k,j}X_j e^{-i2\pi f_l j}\right)\left(\sum_{j=0}^{N-1}v_{k',j}X_j e^{-i2\pi f_{l'} j}\right)\right]\right|\\
=&\left|\int_{-1/2}^{1/2}V_k(f_l-\alpha)V_{k'}(\alpha+f_{l'})S(\alpha)d\alpha\right|\\
=&\left|\int_{I}V_k(f_l-\alpha)V_{k'}(\alpha+f_{l'})S(\alpha)d\alpha\right|\\
&+\left|\int_{[-1/2,1/2]\setminus I}V_k(f_l-\alpha)V_{k'}(\alpha+f_{l'})S(\alpha)d\alpha\right|\\
\le&(\sqrt{1-\lambda_k}+\sqrt{1-\lambda_{k'}}+\sqrt{(1-\lambda_k)(1-\lambda_{k'})})\|S\|_\infty\,,
\end{aligned}$$

where $I := \{\alpha : |f_l - \alpha| \le W \text{ or } |f_{l'} + \alpha| \le W\}$. Since $f_1, \ldots, f_s$ are distinct, for sufficiently large N, we have $W(N) < |f_l - f_{l'}|/2$ for $l \ne l'$. Thus, we could carry a similar argument and get

$$\begin{aligned}
&\left|\mathbb{E}\left[\left(\sum_{j=0}^{N-1}v_{k,j}X_j e^{-i2\pi f_l j}\right)\left(\sum_{j=0}^{N-1}v_{k',j}X_j e^{i2\pi f_{l'} j}\right)\right]\right|\\
\le&(\sqrt{1-\lambda_k}+\sqrt{1-\lambda_{k'}}+\sqrt{(1-\lambda_k)(1-\lambda_{k'})})\|S\|_\infty\,.
\end{aligned}$$

We also have

$$\begin{aligned}
&\left|\mathbb{E}\left[\left(\sum_{j=0}^{N-1} v_{k,j}X_j e^{-i2\pi f_l j}\right)\left(\sum_{j=0}^{N-1} v_{k,j}X_j e^{i2\pi f_l j}\right)\right] - S(f_l)\right| \\
&= \left|\int_{-1/2}^{1/2} V_k(f_l-\alpha)V_k(\alpha-f_l)(S(\alpha)-S(f_l))d\alpha\right| \\
&= \left|\int_{I_1} V_k(f_l-\alpha)V_k(\alpha-f_l)(S(\alpha)-S(f_l))d\alpha\right| \\
&\quad + \left|\int_{[-1/2,1/2]\setminus I_1} V_k(f_l-\alpha)V_k(\alpha+f_l)(S(\alpha)-S(f_l))d\alpha\right| \\
&\leq \max_{\alpha\in[f_l-W,f_l+W]} |S(\alpha)-S(f_l)| + 2(1-\lambda_k)\|S\|_\infty
\end{aligned}$$

where $I_1 := \{\alpha : |f_l - \alpha| \leq W\}$. Finally, for $k \neq k'$,

$$\begin{aligned}
&\left|\mathbb{E}\left[\left(\sum_{j=0}^{N-1} v_{k,j}X_j e^{-i2\pi f_l j}\right)\left(\sum_{j=0}^{N-1} v_{k',j}X_j e^{i2\pi f_l j}\right)\right]\right| \\
&= \left|\int_{-1/2}^{1/2} V_k(f_l-\alpha)V_{k'}(\alpha-f_l)S(\alpha)d\alpha\right| \\
&= \left|\int_{I_1} V_k(f_l-\alpha)V_{k'}(\alpha-f_l)(S(\alpha)-S_m(f_l))d\alpha\right| \\
&\quad + \left|\int_{[-1/2,1/2]\setminus I_1} V_k(f_l-\alpha)V_{k'}(\alpha-f_l)S(\alpha)d\alpha\right| \\
&\leq \max_{\alpha\in[f_l-W,f_l+W]} |S(\alpha)-S(f)| + \sqrt{(1-\lambda_k)(1-\lambda_{k'})}\|S\|_\infty\,,
\end{aligned}$$

where in the second equality we use the fact that $\int_{-W}^{W} V_k(f_l-\alpha)V_{k'}(\alpha-f_l)d\alpha = 0$. By the assumptions, $\lambda_k \to 1$ and $S(f)$ is continuous, and hence (25) is proven.

Finally, in the case when some $f_j = 0$ or $1/2$, the term $\frac{S(f_j)}{2K}\chi^2_{2K}$ is replaced by $\frac{S(f_j)}{K}\chi^2_K$. This follows by noting that $F_{j,k} = 0$ and rewriting the RHS of (24) as $N\left(0, \sum_{l=1}^{s}\sum_{k=0,k\neq j}^{K-1}(\alpha_{l,k}^2+\beta_{l,k}^2)S(f_l)/2 + \sum_{l=1}^{s}\alpha_{l,k}^2 S(f_j)\right)$. □

Before we move on to the proofs of the next two theorems, we introduce the following lag-window estimate with kernel function $\kappa(x) = \frac{\sin \pi x}{\pi x}$ and bandwidth function $B(N) = \frac{1}{2}W(N)^{-1}$:

$$\hat{S}^{eig}(f) = \sum_{0\leq j,j'\leq N-1} \frac{\sin(2\pi W(j-j'))}{2NW\pi(j-j')} X_j X_{j'} e^{i2\pi(j-j')f}.$$

Note that κ also gives the terms in the matrix in (12), which can be used to define the DPSS tapers. In fact, the multitaper estimate $\hat{S}^{mt}(f)$ is in fact closely related to the lag-window estimate $\hat{S}^{eig}(f)$ as described in [KRD19], which can be a topic for further discussion and the unification of the two distinct approaches in spectral analysis. This relationship can be quantified by a bound on the difference between the two terms which is shown in the proof of Theorem 4.2.

Define the κ-lag-window estimate with bandwidth function $B(N)$ to be

$$\hat{S}^{\kappa}(f) = \frac{1}{N} \sum_{0 \le j,j' \le N-1} \kappa\left(\frac{j-j'}{B(N)}\right) X_j X_{j'} e^{i2\pi(j-j')f}.$$

Theorem C.3 (Corollary 3.1 of [SW07]). *Suppose that $X_n \in \mathcal{L}^p$ for $p > 4$ and $\Theta_{0,4} < \infty$. Let κ be a function that is symmetric, Lipschitz continuous function supported on $[-1,1]$. Then for any s distinct frequencies $f_1, f_2, \ldots, f_s$, we have that $\sqrt{\frac{N}{B(N)}}\mathbb{E}_0[\hat{S}^{\kappa}(f_j)]$ for $j = 1, \ldots, s$, is asymptotically join independent $N\left(0, \omega(f_j)\left(\int_{-1}^{1}\kappa^2(x)dx\right)S(f_j)^2\right), j = 1, \ldots, s$. Here $\omega(f) = 1$ if $f \neq 0, \frac{1}{2}$ and $\omega(f) = 2$ otherwise.*

Proof of Theorem 4.2. We first consider the case where $s = 1$ and $f_1 = f$. We note that by Lemma A.2, $\frac{2NW-K}{\log N \log NW} \to \infty$ implies that $\epsilon_N := 1 - \lambda_K \le c/N$ for a fixed c. Theorem 2 of [KRD21a] tells us that for sufficiently large N, the number of k such that $c/N \le \lambda_k \le 1 - c/N$ is bounded by $O(\log(NW)\log N)$. We also have the following bounds from [KRD19]:

$$\left|\sqrt{K}\hat{S}^{mt}(f) - \sqrt{K}\tilde{S}^{mt}(f)\right| \lesssim \epsilon_N K^{-1/2} \sum_{j=0}^{N-1} X_j^2.$$
$$\mathbb{E}\left|\sqrt{K}\tilde{S}^{mt}(f) - \frac{2NW}{\sqrt{K}}\hat{S}^{eig}(f)\right| \lesssim O(\log(NW)\log N)\|S\|_{\infty}/\sqrt{K} = o(1).$$

where $\tilde{S}^{mt}(f)$ is as defined in [KRD19]. Hence $\sqrt{K}\mathbb{E}\left|\hat{S}^{mt}(f) - \frac{2NW}{\sqrt{K}}\hat{S}^{eig}(f)\right| = o(1)$ and an application of Markov's inequality gives $\sqrt{K}\left|\hat{S}^{mt}(f) - \frac{2NW}{\sqrt{K}}\hat{S}^{eig}(f)\right| = o_P(1)$. It remains to show that $\mathbb{E}_0[\frac{2NW}{\sqrt{K}}\hat{S}^{eig}(f)] \to_d N(0, S(f)^2)$, Since κ satisfies the conditions of Theorem C.3, and we have

$$\frac{2NW}{\sqrt{K}}\mathbb{E}_0[\hat{S}^{eig}(f)] \to_d N(0, \frac{K}{2NW}S(f)^2).$$

Combining with $\frac{K}{2NW} \to 1$, we have the result for $s = 1$.

For $s > 1$, we describe the modifications needed to the above argument, as other details follow similarly. By the Cramer-Wold device it suffices to show that $\sqrt{K}\mathbb{E}_0[\sum_{j=1}^{s} a_j \hat{S}^{mt}(f_j)] \to_d N(0, \sum_{j=1}^{s} a_j^2 S(f_j)^2)$, which also follows from Theorem C.3.

Suppose $N^2W^3 \to 0$ and $\Theta_{N,2} = o(N^{-1}W(N^2)^{-1/2})$ hold. This reduces to showing that the bias of the multitaper estimate decays sufficiently fast, at the rate $\mathbb{E}[\hat{S}^{mt}(f)] - S(f) = o(K^{-1/2})$. From Theorem 3.2 and $\Lambda_N \to 0$, it suffices to show

that $\sqrt{K}\sup_{\{|f'-f|\le W\}}|S(f')-S(f)|\to 0$. We have for any $|f'-f|\le W$ that

$$\begin{aligned}\sqrt{K}|S(f')-S(f)| &\le \sqrt{K}\sum_{\tau\in\mathbb{Z}}|r_\tau|\cdot|e^{-i2\pi f'\tau}-e^{-i2\pi f\tau}|\\ &\le 2\sqrt{K}\left(\sum_{0\le\tau<N^{1/2}}|r_\tau|\tau W+2\sum_{\tau\ge N^{1/2}}|r_\tau|\right)\\ &\lesssim \sqrt{K}N^{1/2}W\Theta_{0,2}^2+\sqrt{K}\sum_{\tau\ge N^{1/2}}|r_\tau|\\ &\le \Theta_{0,2}(NW^{3/2}\Theta_{0,2}+N^{1/2}W^{1/2}\Theta_{N^{1/2},2}).\end{aligned}$$

By Lemma B.1, we deduce that $\sum_{\tau\ge N^{1/2}}|r_\tau|\le\Theta_{0,2}\Theta_{N^{1/2},2}$, and so the last term is asymptotically 0 as desired. □

We now introduce a few results that are critical to the proof of Theorem 3.4. The first result gives a bound on the supremum over all frequencies of the multitaper estimate by the max over a grid.

Lemma C.4 (Corollary 2.1, [WVN67]). *Let $p(f)=\sum_{v=-k}^{k}\alpha_v e^{i2\pi vf}$ be a trigonometric polynomial. Let $f_j=j/rN$, $|j|\le rN/2$, then*

$$\max_{|f|\le\pi}p(f)\le\max_{|j|\le rN/2}|p(f_j)/(1-3\pi r^{-1})|.$$

The next two results concern technical bounds on general lag-window spectral estimates.

Lemma C.5 (Lemma 9, [XW12]). *Let $B(N)$ be a bandwidth function satisfying $B(N)\to\infty$ and $B(N)=O(N^\gamma)$ for some $\gamma\in(0,1)$. Let $a_{s,t}$ be coefficients such that $\sup_{s,t}|a_{s,t}|=1$ and $a_{s,t}=0$ when $|s-t|>B(N)$. Assume that $X_n\in\mathcal{L}^p$ for some $p\ge 4$, $\mathbb{E}[X_n]=0$ and $\Theta_{0,p}<\infty$. Then*

$$\begin{aligned}&\left\|\mathbb{E}_0\left[\sum_{s,t}a_{s,t}(X_sX_t-\tilde{X}_s(m)\tilde{X}_t(m))\right]\right\|_{p/2}\\ &\le 4\Theta_{m,p}^2+11(p-2)\Theta_{0,p}\sqrt{NB(N)}\Theta_{m,p}+6(p-2)\sqrt{NB(N)}\Theta_{\lfloor m/2\rfloor,p}\Theta_{m,p}.\end{aligned}$$

Note that in the original result a stronger bound using $\Delta_{n,p}$ (as defined in [XW12]) was stated. We simplified the bound here using $\Delta_{n,p}\le\Theta_{n,p}$.

The next lemma occurs in the proof of Theorem 4, [XW12].

Lemma C.6 (From proof of Theorem 4, [XW12]). *Suppose that κ satisfies*

$$\kappa(0)=1,|\kappa(x)|\le 1 \text{ and } \kappa(x)=0 \text{ for } |x|>1.$$

Suppose further that $B(N)\to\infty$ and $B(N)=O(N^\gamma)$ where $\gamma\in(0,1)$. Let $X_n\in\mathcal{L}^p$ for $p>4$, $\mathbb{E}[X_n]=0$, $\Theta_{N,p}=O(N^{-\alpha})$, and that

$$\gamma<\alpha p/2 \text{ and } (1-2\alpha)\gamma<(p-4)/p.$$

Then

$$\begin{aligned}&\mathbb{P}\left(\sup_{f\in\mathbb{R}}|\mathbb{E}_0[\hat{S}^\kappa(f)]|\ge\delta\sqrt{N^{-1}B(N)\log B(N)}\right)\\ &\le C\left(\log N[N^{\gamma-\alpha\beta p/2}+N^{1-p/4}+N^{(p/4-\alpha\beta p/2)\gamma-(p/4-1)}]+B(N)^{-1}\right),\end{aligned}$$

where β is any constant satisfying

$$\beta \in (0,1), \gamma - \alpha\beta p/2 < 0, (p/4 - \alpha\beta p/2)\gamma - (p/4 - 1) < 0,$$

and C depends only on $\delta, \beta, p, \|X_n\|_p$.

Proof of Theorem 3.4. By the assumption, $K = \Theta(N^{1-\gamma})$. From Lemma A.2 we have $\Lambda_N \to 0$. In addition, we can choose a sufficiently large $\beta < (1-\gamma)/2$ such that $\gamma < \alpha\beta p/2$. Choose $m := \lfloor N^\beta \rfloor$.

First, we can bound the autocovariance sequence

$$\begin{aligned} \sum_{\tau\in\mathbb{Z}} |\tau||r_\tau| &\le 2\sum_{j=0}^{\infty}\theta_{j,2}(\sum_{k=1}^{\infty} k\theta_{j+k,2}) \\ &= 2\sum_{j=0}^{\infty}\theta_{j,2}(\sum_{k=j+1}^{\infty}\Theta_{k,2}) \lesssim \sum_{j=0}^{\infty}\theta_{j,2} = \Theta_{0,2} < \infty \end{aligned} \tag{26}$$

where the first inequality comes from Lemma B.1 and the second inequality comes from the fast decay of $\Theta_{N,p}$ so that $\sum_{k=j+1}^{\infty}\Theta_{k,2} < \infty$. Hence, $S(f)$ is differentiable and $\max_{\xi\in[f-W,f+W]} S(\xi) - \min_{\xi\in[f-W,f+W]} S(\xi) = O(W)$. Applying Theorem 3.2 we have

$$|\mathbb{E}\hat{S}^{mt}(f) - S(f)| = O(W + K^{-1}) = O(N^{\gamma-1})$$

Hence we have shown that we can replace $S(f)$ in the result by $\mathbb{E}[\hat{S}^{mt}(f)]$. It remains to show that $\sup_{f\in\mathbb{R}}|\mathbb{E}_0[\hat{S}^{mt}(f)]| = o_P(N^{-c})$ for some $c > 0$. Applying Lemma C.4, we deduce that

$$\sup_{f\in\mathbb{R}}|\mathbb{E}_0[\hat{S}^{mt}(f)]| \le (1 - 3\pi/N)^{-1}\max_{f=j/N^2}|\mathbb{E}_0[\hat{S}^{mt}(f)]|.$$

Now we reduce the problem to an m-dependent version. To do so, we bound the difference between the tapered spectral estimate for X_n and $\tilde{X}_n(m)$. Start with

$$\begin{aligned} &\max_{f=j/N^2}\left|\mathbb{E}_0\left[|\sum_{j=0}^{N-1}v_{k,j}X_j e^{-i2\pi fj}|^2 - |\sum_{j=0}^{N-1}v_{k,j}\tilde{X}_j(m)e^{-i2\pi fj}|^2\right]\right| \\ \le &\max_{f=j/N^2}\left||\sum_{j=0}^{N-1}v_{k,j}X_j e^{-i2\pi fj}|^2 - |\sum_{j=0}^{N-1}v_{k,j}\tilde{X}_j(m)e^{-i2\pi fj}|^2\right| \\ &+ \max_{f=j/N^2}\left|\mathbb{E}\left[|\sum_{j=0}^{N-1}v_{k,j}X_j e^{-i2\pi fj}|^2 - |\sum_{j=0}^{N-1}v_{k,j}\tilde{X}_j(m)e^{-i2\pi fj}|^2\right]\right|. \end{aligned}$$

Now we can rewrite both terms on the RHS by

$$\begin{aligned} &|\sum_{j=0}^{N-1}v_{k,j}X_j e^{-i2\pi fj}|^2 - |\sum_{j=0}^{N-1}v_{k,j}\tilde{X}_j(m)e^{-i2\pi fj}|^2 \\ &= (\sum_{j=0}^{N-1}v_{k,j}X_j e^{-i2\pi fj})\overline{(\sum_{j=0}^{N-1}v_{k,j}(X_j - \tilde{X}_j(m))e^{-i2\pi fj})} \\ &\quad - (\sum_{j=0}^{N-1}v_{k,j}\tilde{X}_j(m)e^{-i2\pi fj})\overline{(\sum_{j=0}^{N-1}v_{k,j}(X_j - \tilde{X}_j(m))e^{-i2\pi fj})}. \end{aligned} \tag{27}$$

Applying Lemma B.5 we have for any $0 < \alpha' < \alpha\beta$ that

$$\mathbb{P}\left(\max_{f=j/N^2}\left|\left|\sum_{j=0}^{N-1} v_{k,j}X_j e^{-i2\pi fj}\right|^2 - \left|\sum_{j=0}^{N-1} v_{k,j}\tilde{X}_j(m)e^{-i2\pi fj}\right|^2\right| \geq \delta N^{-\alpha}\right) \lesssim N^{(\alpha'-\alpha)\beta}.$$

Applying Lemma B.3 gives

$$\max_{f=j/N^2}\left|\mathbb{E}\left[\left|\sum_{j=0}^{N-1} v_{k,j}X_j e^{-i2\pi fj}\right|^2 - \left|\sum_{j=0}^{N-1} v_{k,j}\tilde{X}_j(m)e^{-i2\pi fj}\right|^2\right]\right| \lesssim N^{-\alpha\beta}.$$

For the quantities below, the subscript m implies that the random variable is calculated using the m-dependent approximation of X_n. Thus up to this step, we have reduced the problem to showing that

$$\mathbb{P}\left(\sup_{f\in\mathbb{R}}|\mathbb{E}_0\hat{S}_m^{mt}(f)| \geq \delta N^{-c}\right) \leq CN^{-c'},$$

Now we will approximate the multitaper estimate with the lag-window estimate $\hat{S}^{eig}(f)$. As shown in the proof of Theorem 4.2, we have

$$\sup_{f\in\mathbb{R}}\mathbb{E}[\left|\mathbb{E}_0[\hat{S}_m^{mt}(f)] - \frac{2NW}{K}\mathbb{E}_0[\hat{S}_m^{eig}(f)]\right|] = o(K^{-1/2}),$$

and so by Markov's inequality

$$\mathbb{P}\left(\sup_{f\in\mathbb{R}}\left|\mathbb{E}_0[\hat{S}_m^{mt}(f)] - \frac{2NW}{K}\mathbb{E}_0[\hat{S}_m^{eig}(f)]\right| \geq \delta N^{-c}\right) \lesssim N^{c-(1-\gamma)/2}$$

for any $c < (1-\gamma)/2$. Based on the lag-window spectral estimate $\hat{S}_m^{eig}(f) = \sum_{0\leq j,j'\leq N-1}\frac{\sin(2\pi W(j-j'))}{2NW\pi(j-j')}\tilde{X}_j\tilde{X}_{j'}$, we define the truncated lag-window spectral estimate

$$\tilde{S}_m^{eig}(f) := \sum_{0\leq j,j'\leq N-1, |j-j'|W<2}\frac{\sin(2\pi W(j-j'))}{2NW\pi(j-j')}\tilde{X}_j\tilde{X}_{j'}.$$

Since $2m < 1/W$ for sufficiently large N, we have

(28)

$$\begin{aligned}
&\sup_{f\in\mathbb{R}}\|\frac{2NW}{K}\mathbb{E}_0[\hat{S}_m^{eig}(f)] - \frac{2NW}{K}\mathbb{E}_0[\tilde{S}_m^{eig}(f)]\|_2^2\\
=&\frac{2NW}{K}\mathbb{E}[\sum_{\substack{|j-j'|>2/W,\\|k-k'|>2/W}}\frac{\sin(2\pi W(j-j'))}{2NW\pi(j-j')}\frac{\sin(2\pi W(k-k'))}{2NW\pi(k-k')}\tilde{X}_j\tilde{X}_{j'}\tilde{X}_k\tilde{X}_{k'}]\\
=&\frac{2NW}{K}\sum_{\substack{|j-j'|>2/W,|k-k'|>2/W,\\|k-j|\leq m\vee|k-j'|\leq m,\\|k'-j|\leq m\vee|k'-j'|\leq m}}\frac{\sin(2\pi W(j-j'))}{2NW\pi(j-j')}\frac{\sin(2\pi W(k-k'))}{2NW\pi(k-k')}\mathbb{E}[\tilde{X}_j\tilde{X}_{j'}\tilde{X}_k\tilde{X}_{k'}]\\
\lesssim&\sum_{|j-j'|>2/W} m^2N^{-2}W^{-2}(j-j')^{-1}(j-j'-2m)^{-1}\mathbb{E}[\tilde{X}_n^4]\\
=&O(m^2W^{-1}N^{-1}).
\end{aligned}$$

Hence by Markov's inequality we have

$$\mathbb{P}\left(\sup_{f\in\mathbb{R}}|\frac{2NW}{K}\mathbb{E}_0[\hat{S}^{eig}_m(f)]-\frac{2NW}{K}\mathbb{E}_0[\tilde{S}^{eig}_m(f)]|\geq\delta N^{-c}\right)\lesssim N^{2c}mW^{-1/2}N^{-1/2}\lesssim N^{2c+\beta-(1-\gamma)/2}.$$

Note that $\tilde{S}^{eig}_m(f)$ is a trigonometric polynomial of order W^{-1} and the coefficients of $\tilde{S}^{eig}_m(f)$ are bounded by $\frac{1}{N}$. Thus Lemma C.4 and Lemma C.5 together with Markov's inequality and union bound gives us that for $\tau_N=\sqrt{2\log(2/W)/NW)}=\Theta(\frac{(\log N)^{1/2}}{K^{1/2}})$,

$$\begin{aligned}
&\mathbb{P}(\max_{f\in\mathbb{R}}|\frac{2NW}{K}\mathbb{E}_0[N\tilde{S}^{eig}_m(f)]-\frac{2NW}{K}\mathbb{E}_0[N\tilde{S}^{eig}(f)]|\leq N\tau_N)\\
&\lesssim W^{-1}(\Theta^2_{m,p}+(p-2)\sqrt{N/W}\Theta_{0,p}\Theta_{m,p}\\
&\quad+(p-2)\sqrt{N/W}\Theta_{\lfloor m/2\rfloor,p}\Theta_{m,p})^{p/2}(N\tau_N)^{-p/2}\\
&\lesssim W^{-1}(\sqrt{W^{-1}N^{-1}}\Theta_{m,p}/\tau_N)^{p/2}=W^{-1}(\Theta_{m,p}/(-\log W)^{1/2})^{p/2}\\
&\lesssim N^{\gamma-\alpha\beta p/2}/(\log N)^{p/4}.
\end{aligned}$$

Finally from Lemma C.6 we deduce that $\mathbb{P}(\max_{f\in\mathbb{R}}|\frac{2NW}{K}\mathbb{E}_0[\tilde{S}^{eig}(f)]|\leq\delta\tau_N)\lesssim N^{-c}$ for some $c>0$. This implies the desired conclusion. □

Proof of Theorem 4.3. We know that the 0th-order DPSS taper is positive everywhere and the kth-order DPSS taper changes sign k times [DJV85]. So from Proposition A.3 and $\sum_{j=0}^{N-1}(v_{0,j})^2=1$ we deduce that $V_0(0)\geq\frac{c}{\sqrt{W}}$. For $f>W$, we have

$$\begin{aligned}
|V_k(f)|&=\left|\lambda_k^{-1}\int_{-W}^{W}\frac{\sin(N\pi(f-f'))}{\sin(\pi(f-f'))}V_k(f')df'\right|\\
&\leq\lambda_k^{-1}\left(\int_{-W}^{W}\left|\frac{\sin(N\pi(f-f'))}{\sin(\pi(f-f'))}\right|^2df'\right)^{1/2}\left(\int_{-W}^{W}|V_k(f')|^2df'\right)^{1/2}\lesssim\frac{W^{1/2}}{f-W}.
\end{aligned}\tag{29}$$

and so we have

$$\max_{l\geq1,0\leq k\leq K-1}|V_k(l\theta)|=O(W^{1/2})\tag{30}$$

. From here we can also deduce that $g(l,N)=\Omega(W^{-1})$. Now note that

$$\mathbb{E}\left[\frac{1}{A_l\sqrt{g(l,N)}}\sum_{k=0}^{K-1}\left|\sum_{j=0}^{N-1}v_{k,j}X_je^{-i2\pi l\theta j}\right|^2\right]\leq\frac{K}{A_l\sqrt{g(l,N)}}\|S\|_\infty=O(NW^{3/2})=o(1).$$

Hence $\frac{1}{A_l\sqrt{g(l,N)}}\sum_{k=0}^{K-1}\left|\sum_{j=0}^{N-1}v_{k,j}X_je^{-i2\pi l\theta j}\right|^2=o_P(1)$. It suffices to show that

$$\frac{1}{A_l\sqrt{g(l,N)}}\sum_{k=0}^{K-1}\mathrm{Re}\left(\sum_{l'=1}^{L}A_{l'}(e^{i\phi_{l'}}V_k((l-l')\theta)+e^{-i\phi_{l'}}V_k((l+l')\theta))\sum_{j=0}^{N-1}v_{k,j}X_je^{-i2\pi l\theta j}\right)\to_d N\left(0,\frac{S(l\theta)}{2}\right).$$

Using (30) and $g(l,N)\to\infty$, we have

$$\left|\frac{1}{A_l\sqrt{g(l,N)}}\sum_{k=0}^{K-1}\mathrm{Re}\left(\sum_{l'=1,l'\neq l}^{L}A_{l'}(e^{i\phi_{l'}}V_k((l-l')\theta)+e^{-i\phi_{l'}}V_k((l+l')\theta))\sum_{j=0}^{N-1}v_{k,j}X_je^{-i2\pi l\theta j}\right)\right|$$
$$\leq\frac{1}{|A_l|\sqrt{g(l,N)}}\sum_{k=0}^{K-1}\sum_{l'=1,l'\neq l}^{L}2|A_{l'}|\cdot W^{1/2}\cdot\left|\sum_{j=0}^{N-1}v_{k,j}X_je^{-i2\pi l\theta j}\right|$$
$$\leq\frac{W^{1/2}L\max_{l'\in\{1,\dots,L\}}|A_{l'}|}{|A_l|\sqrt{g(l,N)}}\sum_{k=0}^{K-1}\left|\sum_{j=0}^{N-1}v_{k,j}X_je^{-i2\pi l\theta j}\right|$$
$$\leq\frac{WL\max_{l'\in\{1,\dots,L\}}|A_{l'}|}{|A_l|}K\hat{S}^{mt}(l\theta)^{1/2}$$

By Theorem 3.1 and Markov's inequality, we have $\mathbb{P}(\hat{S}^{mt}(l\theta)^{1/2}>\delta)=O(\delta^{-1})$ uniformly over all N. Since $KW=O(NW^2)=o(1)$, we have $\frac{WL\max_{l'\in\{1,\dots,L\}}|A_{l'}|}{|A_l|}K\hat{S}^{mt}(l\theta)^{1/2}=o_P(1)$ and it remains to show

$$\frac{1}{\sqrt{g(l,N)}}\sum_{k=0}^{K-1}\mathrm{Re}\left((e^{i\phi_l}V_k(0)+e^{-i\phi_l}V_k(2l\theta))\sum_{j=0}^{N-1}v_{k,j}X_je^{-i2\pi l\theta j}\right)\to_d N\left(0,\frac{S(l\theta)}{2}\right).$$

Define $\alpha_k,\beta_k\in\mathbb{R}$ such that $\alpha_k+i\beta_k=e^{i\phi_l}V_k(0)+e^{-i\phi_l}V_k(2l\theta)$, then the LHS distribution can be rewritten as

$$\frac{\sum_{k=0}^{K-1}\alpha_k\mathrm{Re}\left(\sum_{j=0}^{N-1}v_{k,j}X_je^{-i2\pi l\theta j}\right)-\beta_k\mathrm{Im}\left(\sum_{j=0}^{N-1}v_{k,j}X_je^{-i2\pi l\theta j}\right)}{\sqrt{\sum_{k=0}^{K-1}\alpha_k^2+\beta_k^2}}. \tag{31}$$

We now apply Theorem C.2. Given the normalization factors we have chosen, the variance of (31) is asymptotically $\frac{S(l\theta)}{2}$ by the computations in Theorem 4.1. Of the three terms in the RHS of (23), the last two are asymptotically 0 whenever $m(N)\to\infty$, and it remains to compute the asymptotic value of the first term. Using $(\sum|\alpha_k|+|\beta_k|)^2\leq 2K(\sum\alpha_k^2+\beta_k^2)$, we have for $m(N)$ increasing at a sufficiently slow rate that

$$\frac{(\sum|\alpha_k|+|\beta_k|)^4}{(\sum\alpha_k^2+\beta_k^2)^2}m^6(N+m)W^2\lesssim K^2m^6(N+m)W^2=O(m^6W^4N^3)=o(1).$$

Hence the CLT result holds.

For $f\neq l\theta$, we can deduce from (29) that

$$\|\hat{S}_Z^{mt}(f)-\hat{S}^{mt}(f)\|_2=o(1).$$

The proof of the last part then follows via the Cramer-Wold argument (as applied in Theorem 4.1) and Theorem C.2. □

Appendix D. Proofs in Section 5

Before proceeding to the proof of the Gaussian approximation, we first return to the Thomson F-statistic and show that in the denominator, the deviation resulting from removing the periodic component has asymptotically no effect when we assume the null hypothesis.

Proposition D.1. *Suppose that $X_n \in \mathcal{L}^p$, for $p > 4$, $\Theta_{N,p} = O(N^{-\alpha})$, and $\gamma \in \left(\frac{5p-2}{6p-4}, 1\right)$ holds. Then for any $\delta > 0$, we have for some $c > 0$ independent of X_n and $C > 0$ only depending on $\|X\|_p$ that*

$$\mathbb{P}\left(\max_{f\in\mathbb{R}}\left|\frac{\sum_{k=0}^{K-1}|\sum_{j=0}^{N-1} v_{k,j}X_j e^{-i2\pi fj} - V_k(0)\hat{A}|^2}{K} - \hat{S}^{mt}(f)\right| > N^{\delta}K^{-1}\right) \leq CN^{-c}$$

Proof. We will first rewrite the left term, which gives $\frac{\sum_{k=0}^{K-1}|\sum_{j=0}^{N-1} v_{k,j}X_j e^{-i2\pi fj} - V_k(0)\hat{A}|^2}{K} = \hat{S}^{mt}(f) - \frac{|\sum_{k=0}^{K-1} V_k(0)(\sum_{j=0}^{N-1} v_{k,j}X_j e^{-i2\pi fj})|^2}{K(\sum_{k=0}^{K-1} V_k(0)^2)}$. By Lemma C.4 we can reduce the problem to $f \in \{\frac{i}{N^2}\}$. We have

$$\begin{aligned}\max_{f=\frac{i}{N^2}} \frac{|\sum_{k=0}^{K-1} V_k(0)(\sum_{j=0}^{N-1} v_{k,j}X_j e^{-i2\pi fj})|^2}{K(\sum_{k=0}^{K-1} V_k(0)^2)} &= \max_{f=\frac{i}{N^2}} \frac{|\sum_{j=0}^{N-1} w_j X_j e^{-i2\pi jf}|^2}{K} \\ &= \max_{f=\frac{i}{N^2}} \frac{|\sum_{j=1}^{n} \bar{a}_j(f)|^2}{K} + O(N^{-c'}/K).\end{aligned}$$

Using the Bernstein's inequality applied in a similar way as (32) the proposition follows. □

Proof of Theorem 5.2. Note that $\max_{f\in\pi_N}\sigma(f) \leq \frac{1}{2f_a}$. Start with $\mathbb{P}(\max_{f\in\pi_N}\sum_{l=1}^{\sigma(f)} T_X(lf) \leq x)$. By Proposition 3.4 and D.1 we have for some constants $\alpha_0, c_1 > 0$ that

$$\mathbb{P}(\max_{f\in\pi_N}\sum_{l=1}^{\sigma(f)} T_X(lf) \leq x) \leq \mathbb{P}\left(\max_{f\in\pi_N}\sum_{l=1}^{\sigma(f)} \frac{|\sum_{j=1}^{N-1} w_j X_j e^{-i2\pi jlf}|^2}{S(lf)} \leq x(1+N^{-c_1})\right) + O(N^{-\alpha_0}).$$

By Lemma B.5, Lemma B.7 and Remark B.8, we have for $\alpha_1 > 0$ that

$$\begin{aligned}&\mathbb{P}\left(\max_{f\in\pi_N}\sum_{l=1}^{\sigma(f)} \frac{|\sum_{j=1}^{N-1} w_j X_j e^{-i2\pi jlf}|^2}{S(lf)} \leq x(1+N^{-c_1})\right) \\ \leq &\mathbb{P}\left(\max_{f\in\pi_N}\sum_{l=1}^{\sigma(f)} \frac{|\sum_{j=1}^{n} \bar{a}_j(lf)|^2}{S(lf)} \leq x(1+N^{-c_1}) + N^{-\tau'}\right) + O(N^{-\alpha_1}).\end{aligned}$$

Define the following real random variables for $1 \leq l \leq \sigma(f)$:

$$(U^l_{ij,1}, U^l_{ij,2}) = \frac{\sqrt{n}}{\sqrt{S(lf)}}(\Re(\bar{a}_j(lf)), \Im(\bar{a}_j(lf))).$$

We will now prove that they satisfy the conditions of Proposition B.12 (recall that $n = N^{1-\beta'}$). Choose $B_n = c_2 n^{\frac{1}{2}-\frac{\tau}{1-\beta'}}$ for some constant c_2 to be determined. Then $|(U_{i,l})_j| \leq n^{1/2}N^{-\tau} = n^{\frac{1}{2}-\frac{\tau}{1-\beta'}}$. Thus Condition 1 of Proposition B.12 is satisfied for $c_2 > 2$.

Lemma B.9 then gives $n^{-1}\sum_{j=1}^n \mathbb{E}[|U^m_{ij,l}|^3] = O(B_n)$, and $n^{-1}\sum_{j=1}^n \mathbb{E}[|U^m_{ij,l}|^4] \lesssim B_n n^{-1}\sum_{j=1}^n \mathbb{E}[|U^m_{ij,l}|^3] = O(B_n^2)$. Hence Condition 2 of Proposition B.12 is also satisfied for sufficiently large c_2.

Finally from Lemma B.10 we have for any $v \in V_{n,2\sigma(f)}$ that

$$\begin{aligned}
&\sum_{j=1}^{n}\mathbb{E}[(\sum_{l=1}^{\sigma(f)} v_{2l-1}\bar{a}_{j,1}(lf)/\sqrt{S(lf)} + v_{2l}\bar{a}_{j,2}(lf)/\sqrt{S(lf)})^2] \\
&\quad = \mathbb{E}[(\sum_{j=1}^{n}\sum_{l=1}^{\sigma(f)} v_{2l-1}\bar{a}_{j,1}(lf)/\sqrt{S(lf)} + v_{2l}\bar{a}_{j,2}(lf)/\sqrt{S(lf)})^2] \\
&\quad = \mathbb{E}[(\sum_{l=1}^{\sigma(f)} v_{2l-1}/\sqrt{S(lf)}\sum_{j=0}^{N-1} w_j X_j \cos(2\pi jlf) \\
&\quad - v_{2l}/\sqrt{S(lf)}\sum_{j=0}^{N-1} w_j X_j \sin(2\pi jlf))^2] + O(c_\sigma^2 N^{-c}) \\
&\quad = 1 - O(\|S'\|_\infty W + \|S\|_\infty \cdot \sqrt{\Lambda_N}) + O(c_\sigma^2 N^{-c}).
\end{aligned}$$

Since $\Lambda_N = O(N^{-c_3})$ for some $c_3 > 0$, for sufficiently large N we can choose $b = 1/2$, and Condition 3 of Proposition B.12 is satisfied. Hence for $x \le N^{2-2\beta'}/2$ (recall that $n = N^{1-\beta'}$), and C a constant depending only on $\max\sigma$, we apply Proposition B.12 to get

$$\begin{aligned}
&\mathbb{P}\left(\max_{f\in\pi_N}\sum_{l=1}^{\sigma(f)}\frac{|\sum_{j=1}^{n}\bar{a}_j(lf)|^2}{S(lf)} \le x(1+N^{-c_1}) + N^{-\tau'}\right) \\
&\le \mathbb{P}\left(\max_{f\in\pi_N}\sum_{l=1}^{\sigma(f)}\frac{|\sum_{j=1}^{n}\bar{a}_j^G(lf)|^2}{S(lf)} \le x(1+N^{-c_1}) + N^{-\tau'}\right) + C\left(\frac{B_n^2\log^7(|\pi_N|n)}{n}\right)^{1/6}.
\end{aligned}$$

Choose a constant $c' < c_1$ and $d = \min(\tau', c_1 - c', 2(1-\beta') - c') > 0$. For $x \in ((4\log|\pi_N|hn)^{-2} + 2N^{-d}, N^{c'})$ and N is sufficiently large, We can then apply Lemma B.14 for $h = |\pi_N|$ and $\delta = 1$ to obtain for some absolute constant C that

$$\begin{aligned}
&\mathbb{P}\left(\max_{f\in\pi_N}\sum_{l=1}^{\sigma(f)}|\sum_{j=1}^{n}\bar{a}_j^G(lf)|^2/S(lf) \le x(1+N^{-c_1}) + N^{-\tau'}\right) \\
&\qquad \le \mathbb{P}\left(\max_{f\in\pi_N}\sum_{l=1}^{\sigma(f)}|\sum_{j=1}^{n}\bar{a}_j^G(lf)|^2/S(lf) \le x(1-N^{-c_1})\right) \\
&\qquad\quad + C(\log(|\pi_N|n)^2 N^{-d} + (\log|\pi_N|n)^{-1/2}(|\pi_N|n)^{-1}).
\end{aligned}$$

We will now consider the difference in the covariance structure of $\sum_{j=1}^{n}\bar{a}_j^G(lf)$ and $\sum_{j=1}^{n} w_j Y_j e^{-i2\pi lfj}$ across f and l. Define

$$\Delta = \text{Diff in covariance structure} = \text{Diff in covariance structure for original.}$$

The proof of Lemma B.11 immediately implies that $\Delta = N^{-c''}$ for some constant c''. Applying the Gaussian comparison in Proposition B.15, for $\delta = 1$ and $h = |\pi_N|$ we obtain that if $x \ge C_4(\Delta^{1/3}\log^{1/3}(2^c|\pi_N|^{c+1}) + \log^{-1}(2^c|\pi_N|^{c+1}))^2$ for some

constant C_4 depending only on $\max \sigma$ and C_4' depending on c, that

$$\begin{aligned}
&\mathbb{P}\left(\max_{f\in\pi_N}\sum_{l=1}^{\sigma(f)}\frac{|\sum_{j=1}^n \bar{a}_j^G(lf)|^2}{S(lf)} \le x(1-N^{-c_1})\right)\\
&\le \mathbb{P}\left(\max_{f\in\pi_N}\sum_{l=1}^{\sigma(f)}\frac{|\sum_{j=1}^n w_j Y_j e^{-i2\pi lfj}|^2}{S(lf)} \le x(1-N^{-c_1})\right)\\
&+ C_4'(N^{-c''/3}\log^3(2^c|\pi_N|^{c+1}) + |\pi_N|^{-1}\log^3(2^c|\pi_N|^{c+1}).
\end{aligned}$$

For the last step, it suffices to show that $\sup_{f\in\mathbb{R}}|\hat{S}_G^{mt}(f) - S(f)| = o_P(N^{-c})$ for some $c > 0$, where the superscript or subscript G appearing denote that the quantity is defined in terms of Y. By the analysis in Proposition 3.4 it suffices to show that $\sup_{f=j/N^2}|\mathbb{E}_0[\hat{S}_G^{mt}(f)]| = o_P(N^{-c})$. Now the conditions imply that $\kappa_f \ge \frac{K}{2}$ in Lemma B.16 and $\mathbb{E}[\hat{S}^{mt}(f)] \ge \min_{f\in\mathbb{R}} S(f)/2$ for sufficiently large N. Hence we have that there is a fixed constant D such that for any $f \in \mathbb{R}$ and d,

$$\mathbb{P}(|\mathbb{E}_0[\hat{S}_G^{mt}(f)]| \ge dN^{-c}) \le 2\exp(-Kd^2N^{-2c}/D)$$

and so $\mathbb{P}(\sup_{f=j/N^2}|\mathbb{E}_0[\hat{S}_G^{mt}(f)]| \ge dN^{-c}) = 2N^2\exp(-Kd^2N^{-2c}/D)$, thus showing that

$$\mathbb{P}(\max_{f\in\pi_N}\sum_{l=1}^{\sigma(f)}T_X(lf)\le x) \le \mathbb{P}(\max_{f\in\pi_N}\sum_{l=1}^{\sigma(f)}T_Y(lf)\le x) + O(N^{-c}).$$

uniformly for $x \in \left(\max(4\log(|\pi_N|n)^{-2} + 2N^{-d}, C_4(N^{-c''/3}\log(|\pi_N|^2)^{1/3} + \log(|\pi_N|^2)^{-1})^2), N^{c'}\right)$ for some constants $c', c > 0$. The other direction $\mathbb{P}(\max_{f\in\pi_N}T_Y(f) \le x) \le \mathbb{P}(\max_{f\in\pi_N}T_X(f) \le x) + O(N^{-c})$. can be shown similarly, and so the proof is completed for this case.

For the case $x \le \max(4\log(|\pi_N|n)^{-2}+2N^{-d}, C_4(N^{-c''/3}\log(|\pi_N|^2)^{1/3}+\log(|\pi_N|^2)^{-1})^2)$, recall that we have that the variance of $\Re(\sum_{j=1}^n \bar{a}_j^G(f))$ converges uniformly to $S(f)/2$ for any $f \in \pi_N$. Hence for some constant c_4, we have

$$\begin{aligned}
&\mathbb{P}(\max_{f\in\pi_N}\sum_{l=1}^{\sigma(f)}T_X(lf)\le x)\\
&\le \mathbb{P}\left(\max_{f\in\pi_N}\sum_{l=1}^{\sigma(f)}\frac{|\sum_{j=1}^n \bar{a}_j^G(lf)|^2}{S(lf)} \le x(1+N^{-c}) + N^{-\tau'}\right) + O\left(N^{-\alpha_0} + N^{-\alpha_1} + \left(\frac{B_n^2\log^7(|\pi_N|n)}{n}\right)^{1/6}\right)\\
&\le \mathbb{P}\left(\max_{f\in\pi_N}\frac{\Re(\sum_{j=1}^n \bar{a}_j^G(f))^2}{S(f)} \le c_4(\log N)^{-2}\right) + O\left(N^{-\alpha_0} + N^{-\alpha_1} + \left(\frac{B_n^2\log^7(|\pi_N|n)}{n}\right)^{1/6}\right).
\end{aligned}$$

Lemma B.10 and B.11 implies that we can choose a set π_N' of $\log N$ frequencies in π_N that are at least $2W$ apart, such that for any $f_1, f_2 \in \pi_N'$ we have $\mathrm{Cov}\left(\frac{\sum_{j=1}^n \bar{a}_j^G(f_1)}{S(f_1)^{1/2}}, \frac{\sum_{j=1}^n \bar{a}_j^G(f_2)}{S(f_2)^{1/2}}\right) = O(N^{-\beta_0})$ for some $\beta_0 > 0$. Hence we can perform a whitening transformation of the Gaussian random variables $\left\{\frac{\sum_{j=1}^n \bar{a}_j^G(f)}{S(f)^{1/2}}\right\}_{f\in\pi_N'}$ to

get independent Gaussian random variables $\{a'(f)\}_{f\in\pi'_N}$ each with variance $> 1/2$. Hence

$$\mathbb{P}\left(\max_{f\in\pi_N}\frac{\Re(\sum_{j=1}^n \bar{a}_j^G(f))^2}{S(f)}\le c_4(\log N)^{-2}\right)\le \mathbb{P}\left(\max_{f\in\pi'_N} a'(f)\le c_4'(\log N)^{-1}\right)$$
$$\le (c_4'/\log N)^{\log N} = O(N^{-1}).$$

Likewise, we can check that $\mathbb{P}(\max_{f\in\pi_N}\sum_{l=1}^{\sigma(f)} T_Y(lf)\le x) = O(N^{-1})$, and so this case is settled.

Finally for the case $x\ge N^{c'}$ and N sufficiently large, we have

$$\mathbb{P}(\max_{f\in\pi_N}\sum_{l=1}^{\sigma(f)} T_X(lf)\le x)\ge \mathbb{P}\left(\max_{f\in\pi_N}\sum_{l=1}^{\sigma(f)}\frac{|\sum_{j=1}^{N-1} w_j X_j e^{-i2\pi jlf}|^2}{S(lf)}\le x/(1+N^{-c_1})\right) - O(N^{-a_0})$$
$$\ge \mathbb{P}\left(\max_{f\in\pi_N}\sum_{l=1}^{\sigma(f)}\frac{|\sum_{j=1}^{n}\bar{a}_j(lf)|^2}{S(lf)}\le x(1-N^{-c_1})-N^{-\tau'}\right) - O(N^{-\alpha_0}+N^{-\alpha_1})$$
$$\ge \mathbb{P}\left(\max_{f\in\pi_N}\sum_{l=1}^{\sigma(f)}\frac{|\sum_{j=1}^{n}\bar{a}_j(lf)|^2}{S(lf)}\le N^{c'}/2\right) - O(N^{-\alpha_0}+N^{-\alpha_1})$$

Recall that $\mathbb{E}[\sum_{j=1}^n |\bar{a}_j(f)|^2] = \mathbb{E}\left[\left|\sum_{j=0}^{N-1} w_j X_j e^{-i2\pi jf}\right|^2\right] + O(N^{-c}) \lesssim \sup_{\omega\in\mathbb{R}} S(\omega)$. Applying Bernstein's inequality, we get that for constant C depending on $\sup_{\omega\in\mathbb{R}} S(\omega)$ and σ that

(32)
$$\mathbb{P}\left(\max_{f\in\pi_N}\sum_{l=1}^{\sigma(f)}\frac{|\sum_{j=1}^{n}\bar{a}_j(lf)|^2}{S(lf)} > N^{c'}/2\right)\le |\pi_N|\cdot\exp\left(-\frac{CN^{c'}}{\min S(\omega)+N^{c'/2-\tau}}\right).$$

Thus $1-\mathbb{P}(\max_{f\in\pi_N}\sum_{l=1}^{\sigma(f)} T_X(lf)\le x) = O(N^{-c})$ for some $c>0$. Similarly for $\{Y_n\}$, we also have

$$\mathbb{P}(\max_{f\in\pi_N}\sum_{l=1}^{\sigma(f)} T_Y(lf)\le x)\ge \mathbb{P}\left(\max_{f\in\pi_N}\sum_{l=1}^{\sigma(f)}\frac{|\sum_{j=1}^{n}\bar{a}_j^G(lf)|^2}{S(lf)}\le N^{c'}/2\right) - O(N^{-\alpha_0}+N^{-\alpha_1})$$

Now for sufficiently large N we have that $\mathbb{E}[(\sum_{j=1}^n \bar{a}_{j,1}^G(f))^2], \mathbb{E}[(\sum_{j=1}^n \bar{a}_{j,2}^G(f))^2] \ge S(f)/4$ for all $f\in[W,1/2-W]$. Hence

(33)
$$\mathbb{P}\left(\max_{f\in\pi_N}\sum_{l=1}^{\sigma(f)}\frac{|\sum_{j=1}^{n}\bar{a}_j^G(lf)|^2}{S(lf)} > N^{c'}/2\right)$$
$$\le \mathbb{P}\left(\max_{f\in\pi_N}\sum_{l=1}^{\sigma(f)}\frac{(\sum_{j=1}^{n}\bar{a}_{j,1}^G(lf))^2}{S(lf)} > N^{c'}/4\right) + \mathbb{P}\left(\max_{f\in\pi_N}\sum_{l=1}^{\sigma(f)}\frac{(\sum_{j=1}^{n}\bar{a}_{j,2}^G(lf))^2}{S(lf)} > N^{c'}/4\right)$$
$$\lesssim |\pi_N|\cdot\sum_{l=1}^{\sigma(f)}\exp\left(-\frac{N^{c'}S(lf)/4}{\sigma(f)S(lf)/2}\right).$$

This concludes the $x \geq N^{c'}$ case and the proof. □

Proof of Corollary 5.3. From Proposition 4.2 and 3.4 we can fix a constant c such that $\mathbb{P}(\sup_{f\in\mathbb{R}} |\hat{S}^{mt}(f) - S(f)| > N^{-c}) \leq CN^{-\alpha_0}$. A slight modification of the proof of Proposition D.1 shows that $\mathbb{P}(\max_{f\in\mathbb{R}} |\hat{S}^{mt}(f) - \hat{S}^{mt}_{\bar{X}}(f)| > N^{-c}) \leq CN^{-\alpha_0}$.

Now assuming that $\sup_{f\in\mathbb{R}} |\hat{S}^{mt}_{\bar{X}}(f) - S(f)| \leq N^{-c}$, we can apply the final parts of the proof of Theorem 5.2 to our problem. The change to Δ to include the difference in covariance due to bootstrap can be bounded using the following inequality:

$$\max_{f,f'\in\mathbb{R}} \left| \mathbb{E}\left[\left(\sum_{j=0}^{N-1} w_j X_j e^{-i2\pi jf} \right) \left(\sum_{j=0}^{N-1} w_j X_j e^{-i2\pi jf'} \right) \right] - \mathbb{E}\left[\left(\sum_{j=0}^{N-1} w_j \bar{W}_j e^{-i2\pi jf} \right) \left(\sum_{j=0}^{N-1} w_j \bar{W}_j e^{-i2\pi jf'} \right) \right] \right|$$

$$= \max_{f,f'\in\mathbb{R}} \left| \int_{-1/2}^{1/2} V(f-\alpha)V(\alpha+f')(S(\alpha) - \hat{S}^{mt}_{\bar{X}}(\alpha))d\alpha \right|$$

$$\leq \max_{f,f'\in\mathbb{R}} N^{-c} \int_{-1/2}^{1/2} |V(f-\alpha)| \cdot |V(\alpha+f')| d\alpha = O(N^{-c}).$$

Hence the proof in Theorem 5.2 follows through upon replacing the stationary process Y_t with $\bar{W}_t$. □

D.1. **Covariance estimates of process.**

Proposition D.2. *Under the conditions of Proposition 3.4, we have*

$$\max_{\tau=-(N-1),\ldots,N-1} |\hat{r}^{mt}_\tau - \mathbb{E}[\hat{r}^{mt}_\tau]| = o_P(N^{-c}).$$

Proof. By Proposition 3.4, for any $\delta > 0$ we have with probability going to 1 that $\max_{f\in\mathbb{R}} |\mathbb{E}_0[\hat{S}^{mt}(f)| \leq \delta N^{-c}$ and

$$\max_{\tau=-(N-1),\ldots,N-1} |\mathbb{E}_0[\hat{r}^{mt}_\tau]| \leq \max_{\tau=-(N-1),\ldots,N-1} \int_{-1/2}^{1/2} |\mathbb{E}_0[\hat{S}^{mt}(f)]| \cdot |e^{2\pi\tau f}| df \leq \delta N^{-c}.$$

□

Proposition D.3. *Under the conditions of Proposition 3.4, the multitaper covariance matrix estimate $\hat{\Sigma}^{mt}$ is positive definite with probability approaching one, i.e.*

$$\mathbb{P}(\hat{\Sigma}^{mt} \text{ is positive definite}) \xrightarrow{N\to\infty} 1.$$

Proof. Define the matrix

$$L^{mt}_k = \begin{bmatrix} v_{k,0}X_0 & \ldots & v_{k,N-2}X_{N-2} & v_{k,N-1}X_{N-1} & 0 & \ldots & 0 & 0 \\ v_{k,1}X_1 & \ldots & v_{k,N-1}X_{N-1} & 0 & 0 & \ldots & 0 & v_{k,0}X_0 \\ & & & \vdots & & & & \\ v_{k,N-1}X_{N-1} & \ldots & 0 & 0 & v_{k,0}X_0 & \ldots & v_{k,N-3}X_{N-3} & v_{k,N-2}X_{N-2}. \end{bmatrix}$$

Then $\hat{\Sigma}^{mt}$ can be written as a sum of positive semidefinite matrices

$$\hat{\Sigma}^{mt} = \frac{1}{K} \sum_{k=0}^{K-1} L^{mt}_k (L^{mt}_k)^T$$

Now suppose that $\hat{\Sigma}^{mt}$ is not strictly positive definite, then we can deduce that $v_{k,j}X_j = 0$ for all $0 \leq k \leq K-1, 0 \leq j \leq N-1$. Then $\hat{S}^{mt}(f) = 0$ for all f. Since

$\sup_{f\in\mathbb{R}}|\hat{S}^{mt}(f)-S(f)|=o_P(N^{-c})$, this happens with probability approaching 0 and the proof is complete. □

D.2. **Alternative hypothesis for wave-shape F-test.** Our main idea here is to show that $\max_{f\in\pi_N}\sum_{j=1}^{\sigma(f)}T_X(jf)$ is a consistent statistic when evaluated against the bootstrap distribution $\max_{f\in\pi_N}\sum_{j=1}^{\sigma(f)}T_W(jf)|\bar{X}$, and also to show that we can accurately estimate the fundamental frequency, i.e. $|\hat{\theta}-\theta|$ is sufficiently small.

We first describe the terms in the statistic under the alternative hypothesis. The numerator of $T(f)$ is given by

$$\left|\sum_{j=0}^{N-1}w_j(X_j+\sum_{l=1}^{L}A_l\cos(2\pi l\theta j+\phi_l))e^{-i2\pi fj}\right|^2$$
$$=\left|\sum_{j=0}^{N-1}w_jX_je^{-i2\pi fj}+\sum_{l=1}^{L}\left(\frac{A_le^{-i\phi_l}}{2}V(f-l\theta)+\frac{A_le^{-i\phi_l}}{2}V(f+l\theta)\right)\right|^2.$$

For the denominator term, noting that $V(0)=\sqrt{\sum_{l=0}^{K-1}V_l(0)^2}$ we have

$$\frac{\sum_{k=0}^{K-1}|\sum_{j=0}^{N-1}v_{k,j}Z_je^{-i2\pi fj}-V_k(0)\hat{A}(f)|^2}{K-1}$$
$$=\frac{1}{K-1}\sum_{k=0}^{K-1}\left|\sum_{j=0}^{N-1}v_{k,j}X_je^{-i2\pi fj}-\frac{V_k(0)\sum_{j=0}^{N-1}w_jX_je^{-i2\pi fj}}{V(0)}\right.$$
$$\left.+\sum_{l=1}^{L}\left(\frac{A_le^{i\phi_l}}{2}\left(V_k(f-l\theta)-\frac{V_k(0)V(f-l\theta)}{V(0)}\right)+\frac{A_le^{-i\phi_l}}{2}\left(V_k(f+l\theta)-\frac{V_k(0)V(f+l\theta)}{V(0)}\right)\right)\right|^2.$$

Before we move on to the proof of the main theorems, we first consider a technical regarding the behaviour of DPSWF.

Lemma D.4. *Let $W=\Theta(N^{-\gamma})$ for $\gamma\in(0,1)$ and $\frac{K}{2NW}\to 1$ as before. Choose π_N where $f_\pi=O(N^{-1-\delta})$ for $\delta\geq 1/2$. Take $\epsilon\in(0,\delta)$. Then for γ sufficiently close to 1, some $\epsilon>0$ and $\omega(N)=O(N^{-1-\epsilon})$, the DPSS tapers satisfy*

- *For $f\in[-f_\pi,f_\pi]$, we have for some $c_{D.4,1}>1/2-\epsilon$ that $|V(f)|=\Omega(N^{c_{D.4,1}})$ and $|V_k(f)-V_k(0)|=O(1)$ for $k=0,\dots,K-1$.*
- *For $f\in[-\omega(N),\omega(N)]$, we have $|1-\frac{|V(f)|}{|V(0)|}|<N^{-c_{D.4,2}}$ for some $c_{D.4,2}>0$.*
- *For f such that $|f|>\omega(N)$, we have either $\frac{|V(f)|}{|V(0)|}<N^{-c_{D.4,3}}$ or $|V_1(f)|=\Omega(N^{c_{D.4,4}})$ for some $c_{D.4,3}>1-\gamma$ and $c_{D.4,4}>(1-\gamma)/2$.*

Proof. These results can be deduced from the asymptotic approximation for the DPSWF found in [Sle78], although rigorous proofs for the results cannot be found in the literature to our best knowledge. □

Proof of Proposition 5.4. First, with probability approaching 1, we have $\max_{f\in\mathbb{R}}|\hat{S}^{mt}(f)-S(f)|=o(N^{-c})$ as shown in Proposition 3.4, which we now assume for the rest of the proof.

Note that the conditions imply that Lemma D.4 holds, which terminology we use in the remainder of the proof. We split $f\in\pi_N$ into two regions $F_1=\{f\in$

$\pi_N : |f-\theta| < \omega(N)\}, F_2 = \{f \in \pi_N : \omega(N) < |f-\theta|\}$. Also, we define

$$A_k(f) = \sum_{j=0}^{N-1} w_j X_j e^{-i2\pi fj} + \sum_{l=1}^{L} \left(\frac{A_l e^{-i\phi_l}}{2} V(f-l\theta) + \frac{A_l e^{-i\phi_l}}{2} V(f+l\theta) \right),$$

$$B_k(f) = \sum_{j=0}^{N-1} v_{k,j} X_j e^{-i2\pi fj} - \frac{V_k(0) \sum_{j=0}^{N-1} w_j X_j e^{-i2\pi fj}}{V(0)},$$

$$C_k(f) = \sum_{l=1}^{L} \left(\frac{A_l e^{i\phi_l}}{2} \left(V_k(f-l\theta) - \frac{V_k(0)V(f-l\theta)}{V(0)} \right) + \frac{A_l e^{-i\phi_l}}{2} \left(V_k(f+l\theta) - \frac{V_k(0)V(f+l\theta)}{V(0)} \right) \right),$$

which are stochastic and deterministic components of the denominator of $T(f)$ respectively. An application of Bernstein's inequality similar to (32) also shows that

$$\max_{f \in \pi_N} | \sum_{j=0}^{N-1} w_j X_j e^{-i2\pi fj} | = o_P(N^{c'})$$

for any $c' > 0$.

Suppose that $\arg\min_{f \in F_1} |f-\theta| = \hat{\theta}_1$. Now we show that $\max_{\eta \in F_2} \sum_{j=1}^{\sigma(\eta)} T_Z(j\eta) \leq \sum_{j=1}^{\sigma(\hat{\theta}_1)} T_Z(j\hat{\theta}_1)$ with probability approaching 1. For the case $|V_1(f) - \frac{V(f)}{V(0)} V_1(0)| = \Omega(N^{c_{D.4,4}})$, the denominator is $\Omega(N^{2c_{D.4,4}-(1-\gamma)})$ with probability approaching 1, and the desired conclusion follows. Otherwise, we have for some $c'' > 0$ that $|A_k(\hat{\theta}_1)|^2 > N^{c''} |A_k(f)|^2$ for all $f \in F_2$. Note that for any fixed $C > 0$ we have over all f and N that $\max_{k=0,\ldots,K-1} \mathbb{P}(|B_k(f) + C_k(f)|^2 \leq C) \leq 1 - \epsilon'$ for some $\epsilon' > 0$. Hence with probability approaching 1, we have for sufficiently large N that over all such $f \in F_2$ that $\sum_{k=0}^{K-1} |B_k(f) + C_k(f)|^2 > 1 \geq N^{-c''} \sum_{k=0}^{K-1} |B_k(\hat{\theta}_1) + C_k(\hat{\theta}_1)|^2$. Hence indeed we have with probability approaching 1 that $\max_{\eta \in F_2} \sum_{j=1}^{\sigma(\eta)} T_Z(j\eta) \leq \sum_{j=1}^{\sigma(\hat{\theta}_1)} T_Z(j\hat{\theta}_1)$. □

Proof of Theorem 5.5. The spectral density function of the Gaussian bootstrap process W is given by $\hat{S}^{mt}_{\hat{X}}(f)$. Since

$$|V_k(f+f') - V_k(f)| \leq ((\sum_{j=0}^{N-1} (v_{k,j})^2)(\sum_{j=0}^{N-1} |e^{i2\pi f'j} - 1|^2))^{1/2} \leq (\sum_{j=0}^{N-1} 2 - e^{-i2\pi f'j} - e^{i2\pi f'j})^{1/2} \lesssim (N^3 f'^2)^{1/2},$$

for $|f'| < \omega(N)$ this quantity is $O(N^{1/2-\epsilon})$. We can then deduce that $\hat{S}^{mt}_{\hat{X}}(f) = O(N^{1-2\epsilon})$, which implies that the variance of the Gaussian random variable $\sum_{j=0}^{N-1} w_j W_j e^{-i2\pi fj}$ is $O(N^{1-2\epsilon})$ for all f. Hence with probability 1, we have $\mathbb{P}(\max \sum_{j=1}^{\sigma(f)} T_Y(jf) \geq N^{2c_{D.4,1}}) \to 0$ as desired.

□

Proof of Theorem 5.6 . Applying Bernstein's inequality, we obtain that

$$\mathbb{P}(\max_{f \in \pi_N} T_X(f) > N^{q/2}) = o(N^{-c}), \mathbb{P}(\max_{f \in \pi_N} T_{\bar{W}}(f) | \bar{X} > N^{q/2}) = o(N^{-c}),$$

and so we can reduce the theorem to the case where $\sigma(f) \equiv 1$. Then the proof for the null hypothesis follows from that of Theorem 5.3.

The alternative hypothesis follows as well from the proof of Theorem 5.5. □

Proof of Proposition 5.7. The proof is an extension to the proof of Proposition 5.4. Recall that for sufficiently small $q < c_{D.4,1}$, we have $T_Z(f) > N^q$ for $|f - j\theta| < f_\pi, A_j \neq 0$ and sufficiently large N. Bernstein's inequality and the assumptions on the tapers also imply that we have

$$\mathbb{P}\left(\max_{f\in\pi_N, |f-j\theta|\geq f_a/4 \forall A_j\neq 0} T_Z(f) \leq N^{q/2}\right) \to 1.$$

Let $\hat{\theta}_1 = \arg\min_{f\in\pi_N} |f - \theta|$, and suppose that the argmax is achieved by $\hat{\theta}$ and $\hat{L}$ such that $|\hat{\theta} - \theta| > \omega(N)$. Then we must have that $|\{j \in \mathbb{N} | A_j \neq 0, \min_{1\leq l\leq \hat{L}} |l\hat{\theta} - j\theta| > \omega(N)\}| = 0$ and $|\hat{L}\hat{\theta} - L\theta| < \omega(N)$, and so $\hat{\theta} < \frac{\theta}{2} + \omega(N)$. Hence $|\{1 \leq j \leq \hat{L} | \min_{\{1\leq l\leq L | A_l\neq 0\}} |j\hat{\theta} - l\theta| \geq f_a/4\}| > |\{1 \leq j \leq L | \min_{\{1\leq l\leq L | A_l\neq 0\}} |j\hat{\theta}_1 - l\theta| \geq f_a/4\}|$. This implies that we have $T_Z(f) > N^{q/2}$ for some $f \in \pi_N$, $|f - j\theta| \geq f_a/4$ and $A_j \neq 0$, which happens with probability approaching 0.

Thus we have $|\hat{\theta} - \theta| \leq \omega(N)$ with probability approaching 1, and a similar analysis also shows that $\hat{L} = L$ with probability approaching 1, as desired. □