INTERMEDIATE CURVATURE, SPACETIME HARMONIC FUNCTIONS AND THE MONOTONICITY OF THE HAWKING ENERGY

by

Sven Hirsch

Department of Mathematics
Duke University

Date: ______________________________

Approved:

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Hubert Bray, Advisor

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William Allard

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Robert Bryant

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Mark Stern

Dissertation submitted in partial fulfillment of the
requirements for the degree of Doctor of Philosophy
in the Department of Mathematics
in the Graduate School of
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ABSTRACT

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Abstract

First, we introduce m-intermediate curvature $\mathcal{C}_m$ which interpolates between Ricci ($m = 1$) and scalar curvature ($m = n - 1$) and prove in this context a generalized Geroch conjecture. In particular, we show that $M^{n-m} \times \mathbb{T}^m$, $n \leq 7$, does not admit a metric with $\mathcal{C}_m > 0$.

Next, we study initial data sets (M, g, k) which are used in General Relativity to describe isolated gravitational systems. We introduce spacetime harmonic functions, i.e. functions solving the PDE $\Delta u = -\mathrm{tr}_g k |\nabla u|$, to give a new lower bound for the mass of (M, g, k). This lower bound in particular implies the spacetime positive mass theorem including the case of equality.

Finally, we discuss recent progress towards the spacetime Penrose conjecture. We demonstrate how the famous monotonicity formula for the Hawking energy under inverse mean curvature flow can be generalized to initial data sets. This leads to a new notion of spacetime inverse mean curvature flow which is based on double null foliations.

Several of the above results have been obtained in collaboration with Simon Brendle, Florian Johne, Demetre Kazaras, Marcus Khuri and Yiyue Zhang.

Acknowledgements

I would like to express my gratitude towards my advisor Hubert Bray for his constant encouragement, for being so generous with his ideas, energy and time, and for his outstanding guidance.

Also, I am very grateful to Simon Brendle, Marcus Khuri, Martin Li and Pengzi Miao for their excellent mentorship, and for countless hours sharing their knowledge.

Moreover, I would like to thank my colleagues Xiaoxiang Chai, Simone Checcini, Michael Eichmair, Benjamin Hamm, Lan-Hsuan Huang, Gerhard Huisken, Florian Johne, Demetre Kazaras, Thomas Koerber, Martin Lesourd, Michael Lin, Alex Mramor, Alec-Payne, Anna Sakovich, Sarah Schott, Luen-Fai Tam, Tin-Yau Tsang, Shira Viel, James Wheeler, Kai Xu, Yiyue Zhang, and Jonathan Zhu for their friendship and for many interesting discussions.

Furthermore, I am grateful to my friends Arved, Aygul, Carol, Christian, Ilaria, Julian, Karim, Madeline, Misha, Moritz, Oli, Robby and Shri for many fun moments during graduate school.

Finally, and most of all, I would like to thank my family for their endless support.

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Chapter 1

Introduction

1.1 Proof of a generalized Geroch Conjecture

Bonnet-Myers' theorem implies that $M^{n-1} \times \mathbb{S}^1$ does not admit a metric with positive Ricci curvature while the resolution of Geroch's conjecture yields that the torus $\mathbb{T}^n$ does not admit a metric of positive scalar curvature. Together with S. Brendle and F. Johne we showed in [22]:

Theorem A. *Let $n \leq 7$. On $\mathbb{T}^m \times M^{n-m}$ there is no complete metric of positive m-intermediate curvature.*

Here *m-intermediate curvature* is a new notion of curvature we introduced which interpolates between Ricci curvature ($m = 1$) and scalar curvature ($m = n - 1$). Our proof uses stable weighted slicings of order m and delicate extrinsic curvature estimates. We also discuss other recent work about m intermediate curvature [29, 34, 38, 89, 119] by X. Chen, A. Chow, F. Johne, J. Wan, J. Chu, K.-K. Kwong, M.-C. Lee, M. Labbi and K. Xu.

1.2 Spacetime harmonic functions

Besides the initial proofs of the Geroch conjecture due to R. Schoen, S.-T. Yau [106], and M. Gromov, H.B. Lawson [58], there is another argument in dimension 3 due to D. Stern [113] using harmonic maps. This idea has been expanded upon by H. Bray, D. Kazaras,

M. Khuri and D. Stern to give a new proof of the Riemannian positive mass theorem (PMT) [18]. In [64] we showed with D. Kazaras and M. Khuri the spacetime PMT using *spacetime harmonic functions*, and in [72] we proved with Y. Zhang the corresponding rigidity. The main result of the two papers [64, 72] can be stated as follows:

Theorem B. *Let (M, g, k) be a complete, asymptotically flat initial data set satisfying the dominant energy condition $\mu \geq |J|$. Then there exists a spacetime harmonic function u such that*

$$E - |P| \geq \frac{1}{16\pi} \int_{M_{ext}} \left(\frac{|\nabla^2 u + k|\nabla u||^2}{|\nabla u|} + 2\mu|\nabla u| + 2\langle J, \nabla u\rangle \right) d\mu.$$

In particular, we have $E \geq |P|$. Moreover, $E = |P|$, implies $E = |P| = 0$ and that (M, g, k) isometrically embeds into Minkowski space.

Here μ, J are the energy and momentum densities, M_{ext} is the *generalized exterior region* of M, cf. Appendix B.1, and we remark that the case of equality of the spacetime PMT has previously only been known under several additional decay assumptions [10, 64, 77]. In further work joint with H. Bray, D. Kazaras, M. Khuri and Y. Zhang [16, 17, 65, 66] we have found various other applications of spacetime harmonic functions. This includes existence results for black holes, and purely Riemannian statements such as a conjecture of M. Gromov, a generalization of Bonnet-Myers' theorem and band width inequalities. Also, see [3, 27, 28, 114, 115] by A. Alaee, P.-K. Hung, M. Khuri, X. Chai, X. Wan and T.-Y. Tsang for even more examples.

1.3 Monotonicity of the Hawking energy

The success of spacetime harmonic functions raises the question whether other important tools from Riemannian geometry also have "spacetime analogs". One of the central results concerning asymptotically flat manifolds is the Riemannian Penrose inequality which has been proven by G. Huisken and T. Ilmanen [79] for connected horizons, and by H. Bray [13] for arbitrary ones. Husisken-Ilmanen's proof relies on *inverse mean curvature flow* (IMCF) and the monotonicity of the *Hawking energy.* In [63] we showed the following generalization of this monotonicity formula:

Theorem C. *Let (M, g, k) be an asymptotically flat initial data set satisfying the dominant energy condition $\mu \geq |J|$. Let Σ_1, Σ_2 be two surfaces in M with Σ_2 containing enclosing Σ_1. Let (u, v) be a solution to spacetime IMCF*

$$\begin{cases} \Delta u = -\operatorname{tr}_g(k)|\nabla u| + k_{\eta\eta}|\nabla u| + \nabla^2_{\eta\eta}u + \frac{3|\nabla u||\nabla v| + \langle \nabla u, \nabla v\rangle}{u+v} \\ \Delta v = \operatorname{tr}_g(k)|\nabla v| - k_{\eta\eta}|\nabla v| + \nabla^2_{\eta\eta}v + \frac{3|\nabla u||\nabla v| + \langle \nabla u, \nabla v\rangle}{u+v} \end{cases},$$

with appropriate boundary conditions, where $\eta = \frac{\nabla u|\nabla v| + \nabla v|\nabla u|}{|\nabla u|\nabla v| + \nabla v|\nabla u||}$. Then

$$\mathfrak{m}_H(\Sigma_2) \geq \mathfrak{m}_H(\Sigma_1)$$

where $\mathfrak{m}_H$ is the spacetime Hawking energy

$$\mathfrak{m}_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_\Sigma \theta_-\theta_+ dA\right),$$

and where $\theta_\pm = H \pm \operatorname{tr}_\Sigma k$ are the null-expansions.

We will demonstrate that Theorem C implies both the Hawking energy monotonicity in the Riemannian setting, and the integral formula of Theorem B. In particular, the above formula implies the spacetime PMT and the Riemannian Penrose inequality. Compared to other approaches towards spacetime IMCF [15, 19, 51, 80, 98], our approach has the advantage that we both have a monotonicity formula and a PDE with a comparatively simple structure. We will discuss existence results in Chapter 4.5.

1.4 Outline

In chapters 2, 3, and 4 we will prove theorems A, B and C respectively. At the beginning of each section we will include detailed information about the background, literature and related topics. We will focus in our exposition on the central geometric and analytic ideas, and outsource less important technical details to the corresponding appendices A, B and C.

Further papers I wrote during my time in graduate school [14, 16, 17, 65, 66, 67, 68, 69, 70, 71, 73] will not be discussed.

Chapter 2

Proof of a generalized Geroch conjecture

This section is based on joint work with Simon Brendle and Florian Johne [22].

The round metric $g_{\mathbb{S}^2}$ on $\mathbb{S}^2$ has positive Gaussian curvature, and Gauss-Bonnet's theorem shows that there is no such metric on the torus $\mathbb{T}^2$. Similarly, we obtain the following figure in one dimension higher:

Table 2.1: Positive curvature in Dimension 3

Dimension 3	$\mathbb{S}^3$	$\mathbb{S}^2 \times \mathbb{S}^1$	$\mathbb{T}^3$
Existence of a metric with $\mathrm{Ric} > 0$?	✓	✗	✗
Existence of a metric with $R > 0$?	✓	✓	✗

Note that the round metric $g_{\mathbb{S}^3}$ has positive Ricci curvature Ric, and the product metric $g_{\mathbb{S}^2} \oplus g_{\mathbb{S}^1}$ has positive scalar curvature R. The non-existence of metrics with positive Ricci curvature on $\mathbb{S}^2 \times \mathbb{S}^1$ and $\mathbb{T}^3$ follow from Bonnet-Myers' diameter estimate, and the non-existence of metrics of positive scalar curvature on $\mathbb{T}^3$ is the statement of the famous Geroch conjecture.

Theorem 2.0.1 (Bonnet-Myers' theorem). *Let (M^n, g) be Riemannian manifold with* $\mathrm{Ric} \geq (n-1)$. *Then the fundamental group of M^n is finite, and we have diam$(M^n) \leq \pi$.*

Here and in the rest of this manuscript all manifolds are assumed to be complete, orientable and smooth.

Corollary 2.0.2. *There is no metric with* $\mathrm{Ric} > 0$ *on* $M^{n-1} \times \mathbb{S}^1$.

Proof. If $M^{n-1} \times \mathbb{S}^1$ has $\mathrm{Ric} > 0$, so does the cover $M^{n-1} \times \mathbb{R}$ which has infinite diameter.

□

Bonnet-Myers' theorem has first been shown by S. Myers [99] using the second variation formula of geodesics and a corresponding rigidity statement has been established by S.-Y. Cheng [30]. New proofs have been given by C. Croke and B. Kleiner [40] using the distance function and by the author, D. Kazaras, M. Khuri and Y. Zhang using spacetime harmonic functions [65]. We remark that the proof using spacetime harmonic functions also applies to open and incomplete manifolds and includes rigidity.

Theorem 2.0.3 (Geroch conjecture)**.** *There is no metric* g *with* $\mathrm{R} > 0$ *on the torus* $\mathbb{T}^n$.

This has been first established by R. Schoen and S.-T. Yau [106] up to dimension 7 using the minimal surface techniques, and later by M. Gromov and H.B. Lawson [58] in all dimension using the twisted Lichnerowicz formula. A new proof in Dimension 3 has recently been discovered by D. Stern [113] based upon harmonic maps into $\mathbb{S}^1$.

Applying again Bonnet-Myers' theorem and the Geroch conjecture, we obtain the figure below for Dimension 4.

Table 2.2: Positive curvature in Dimension 4

Dimension 4	$\mathbb{S}^4$	$\mathbb{S}^2 \times \mathbb{S}^2$	$\mathbb{S}^3 \times \mathbb{S}^1$	$\mathbb{S}^2 \times \mathbb{T}^2$	$\mathbb{T}^4$
Existence of a metric with $\sec > 0$?	✓	?	✗	✗	✗
Existence of a metric with $\mathrm{Ric} > 0$?	✓	✓	✗	✗	✗
Existence of a metric with $R > 0$?	✓	✓	✓	✓	✗

Note that it is still an open question whether there exists a metric of positive sectional curvature $\sec > 0$ on $\mathbb{S}^2 \times \mathbb{S}^2$. This is known as Hopf's conjecture [54] and remains unsolved for more than half a century.

Observe that neither sectional, Ricci, nor scalar curvature are able to distinguish between the different topological spaces $\mathbb{S}^3 \times \mathbb{S}^1$ and $\mathbb{S}^2 \times \mathbb{T}^2$. This motivated us to introduce a new notion of curvature in [22]:

Definition 2.0.4 (Positive m-intermediate curvature)**.** *Suppose (N^n, g) is a Riemannian manifold. Then (N^n, g) has positive m-intermediate curvature at $p \in M$, if the inequality*

$$\mathcal{C}_m(e_1, \ldots, e_m) := \sum_{p=1}^{m} \sum_{q=p+1}^{n} \sec(e_p, e_q) > 0$$

holds for every orthonormal basis $\{e_1, \ldots, e_n\}$ of the tangent space T_pM. The manifold (N^n, g) has positive m-intermediate curvature, if it has positive m-intermediate curvature for all $p \in M$.

In the tables below we illustrate what it means for $\mathcal{C}_m$, $m = 1, 2, 3$, to be positive in Dimension 4. In general, m-intermediate curvature is obtained by "adding m columns of sectional curvatures":

Table 2.3: The 1-intermediate curvature $\mathcal{C}_1$ is obtained by summing the sectional curvatures highlighted in red. In particular, we have $\mathcal{C}_1(e_1) = \text{Ric}(e_1, e_1)$.

Dimension 4	e_1	e_2	e_3	e_4
e_1	0	$\sec(e_1, e_2)$	$\sec(e_1, e_3)$	$\sec(e_1, e_4)$
e_2	$\sec(e_2, e_1)$	0	$\sec(e_2, e_3)$	$\sec(e_2, e_4)$
e_3	$\sec(e_3, e_1)$	$\sec(e_3, e_2)$	0	$\sec(e_3, e_4)$
e_4	$\sec(e_4, e_1)$	$\sec(e_4, e_2)$	$\sec(e_4, e_3)$	0

Table 2.4: The 2-intermediate curvature $\mathcal{C}_2$ is obtained by summing the sectional curvatures highlighted in blue. In particular, we have $\mathcal{C}_2(e_1, e_2) = \text{BiRic}(e_1, e_2)$. Here the bi-Ricci curvature has been previously introduced by Y. Shen and R. Ye in [110] via $\text{BiRic}(e_1, e_2) = \text{Ric}(e_1, e_1) + \text{Ric}(e_2, e_2) - \sec(e_1, e_2)$.

Dimension 4	e_1	e_2	e_3	e_4
e_1	0	$\sec(e_1, e_2)$	$\sec(e_1, e_3)$	$\sec(e_1, e_4)$
e_2	$\sec(e_2, e_1)$	0	$\sec(e_2, e_3)$	$\sec(e_2, e_4)$
e_3	$\sec(e_3, e_1)$	$\sec(e_3, e_2)$	0	$\sec(e_3, e_4)$
e_4	$\sec(e_4, e_1)$	$\sec(e_4, e_2)$	$\sec(e_4, e_3)$	0

Observe that Theorem A implies that the manifold $\mathbb{S}^2 \times \mathbb{T}^2$ does not admit a metric of positive 2-intermediate curvature, while a straightforward computation shows that the standard metric on $\mathbb{S}^3 \times \mathbb{S}^1$ does have positive 2-intermediate curvature. Thus, $\mathcal{C}_2$ is able to distinguish the topological spaces $\mathbb{S}^2 \times \mathbb{T}^2$ and $\mathbb{S}^3 \times \mathbb{S}^1$, and we can complete the above picture:

Table 2.5: The 3-intermediate curvature $\mathcal{C}_3$ is obtained by summing the sectional curvatures highlighted in purple. In particular, we have in Dimension 4 $\mathcal{C}_3 = \frac{1}{2}\,\mathrm{R}$.

Dimension 4	e_1	e_2	e_3	e_4
e_1	0	$\sec(e_1, e_2)$	$\sec(e_1, e_3)$	$\sec(e_1, e_4)$
e_2	$\sec(e_2, e_1)$	0	$\sec(e_2, e_3)$	$\sec(e_2, e_4)$
e_3	$\sec(e_3, e_1)$	$\sec(e_3, e_2)$	0	$\sec(e_3, e_4)$
e_4	$\sec(e_4, e_1)$	$\sec(e_4, e_2)$	$\sec(e_4, e_3)$	0

Table 2.6: Positive curvature in Dimension 4 revisited

Dimension 4	$\mathbb{S}^4$	$\mathbb{S}^2 \times \mathbb{S}^2$	$\mathbb{S}^3 \times \mathbb{S}^1$	$\mathbb{S}^2 \times \mathbb{T}^2$	$\mathbb{T}^4$
Existence of metric with $\sec > 0$?	✓	?	✗	✗	✗
Existence of metric with $\mathrm{Ric} > 0$?	✓	✓	✗	✗	✗
Existence of metric with $\mathcal{C}_2 > 0$?	✓	✓	✓	✗	✗
Existence of metric with $R > 0$?	✓	✓	✓	✓	✗

Theorem A follows from a slightly more general statement. To state this result we need to introduce the notion of stable weighted slicings:

Definition 2.0.5 (Stable weighted slicing of order m).
Suppose $1 \leq m \leq n-1$ *and let* (N^n, g) *be a Riemannian manifold of dimension* $\dim N = n$. *A stable weighted slicing of order* m *consists of a collection of submanifolds* Σ_k, $0 \leq k \leq m$, *and a collection of positive functions* $\rho_k \in C^\infty(\Sigma_k)$ *satisfying the following conditions:*

- $\Sigma_0 = N$ *and* $\rho_0 = 1$.

- *For each $1 \le k \le m$, Σ_k is a embedded two-sided hypersurface in Σ_{k-1}. Moreover, Σ_k is a stable critical point of the ρ_{k-1}-weighted area*

$$\mathcal{H}^{n-k}_{\rho_{k-1}}(\Sigma) = \int_\Sigma \rho_{k-1}\, d\mu$$

in the class of hypersurfaces $\Sigma \subset \Sigma_{k-1}$.

- *For each $1 \le k \le m$, the function $\frac{\rho_k}{\rho_{k-1}|_{\Sigma_k}} \in C^\infty(\Sigma_k)$ is a first eigenfunction of the stability operator associated with the ρ_{k-1}-weighted area.*

This definition is similar but not identical to the notion of *minimal m-slicings* used by R. Schoen and S.-T. Yau in [107]. See Figure 2.1 for a depiction of a stable weighted slicing.

Now we can state the main result of [22]:

Theorem 2.0.6 (m-intermediate curvature and stable weighted slicings)**.**
Assume that $n(m-2) \le m^2 - 2$. Suppose (N^n, g) is a closed Riemannian manifold with positive m-intermediate curvature. Then N does not admit a stable weighted slicing

$$\Sigma_m \subset \cdots \subset \Sigma_1 \subset \Sigma_0 = N^n$$

of order $m \le n-1$.

The dimensional constraint $n(m-2) \le m^2 - 2$ is always satisfied for $n \le 7$ and arises from several algebraic inequalities originating from stability inequalities. This should be compared with the proof of the codimension-7 regularity for area-minimizing surfaces, cf. [112, Appendix B], which also relies on algebraic inequalities appearing in the stability inequality for area-minimizing surfaces.

Combining Theorem 2.0.6 with the topological existence result below yields Theorem A.

Theorem 2.0.7 (Existence of stable weighted slicings)**.**
Assume $n \leq 7$ and $1 \leq m \leq n-1$. Let N^n be a closed manifold of dimension n, and suppose that there exists a closed manifold M^{n-m} and a map $F : N^n \to M^{n-m} \times \mathbb{T}^m$ with non-zero degree. Then for each Riemannian metric g on N there exists a stable smooth weighted slicing

$$\Sigma_m \subset \Sigma_{m-1} \subset \cdots \subset \Sigma_1 \subset \Sigma_0 = N^n$$

of order m.

Theorem A has been generalized into various directions by several authors: S. Chen [29] extended Theorem A to the non-compact setting, while A. Chow, F. Johne and J. Wan [34] allowed a non-empty boundary to be present. J. Chu, K.-K. Kwong and M.-C. Lee [38] as well as K. Xu [119] addressed the corresponding rigidity, and M. Labbi used Theorem A to compute the Riemann invariants of certain manifolds. Moreover, K. Xu [119] constructed examples demonstrating that the dimensional bound $n(m-2) \leq m^2-2$ is sharp. Finally, we show in an upcoming work a band-width version of Theorem A.

This chapter is organized as follows: In Section 2.1 we establish several elementary properties of $\mathcal{C}_m$ and recall the stability formula for area-minimizing hypersurfaces. Next, we prove in Section 2.2 Theorem A for $m = 1$ and $m = 2$ before proceeding with the general case in Section 2.3. Thereby we highlight the geometric ideas of our argument while several technical aspects are moved to Appendix A.

2.1 Preliminaries

The next two lemma follow from standard computations:

Lemma 2.1.1. *Consider the manifold $\mathbb{S}^{n-m} \times \mathbb{T}^m$ equipped with the standard metric g, i.e. $g = g_{\mathbb{S}^{n-m}} \oplus g_{\mathbb{T}^m}$ where $g_{\mathbb{S}^{n-m}}$ is the round metric on $\mathbb{S}^{n-m}$, and $g_{\mathbb{T}^m}$ is the flat metric* *on $\mathbb{T}^m$. Then $\mathcal{C}_m(g) \geq 0$ and $\mathcal{C}_{m+k}(g) > 0$ for $k \geq 1$.*

Lemma 2.1.2. *Let (N^n, g) be a Riemannian manifold with positive sectional curvature. Then (N^n, g) has positive m-intermediate curvature for all $m \geq 1$. Similarly, if $\mathcal{C}_m > 0$ for some $m \geq 1$, then (N^n, g) has positive scalar curvature.*

Recall that given m orthonormal vectors $\{e_1, \ldots, e_m\}$, m-intermediate curvature is defined as the sum

$$\mathcal{C}_m(e_1, \ldots, e_m) = \sum_{p=1}^{m} \sum_{q=p+1}^{n} \sec(e_p, e_q).$$

In fact, $\mathcal{C}_m$ depends only on the m-plane spanned by the vectors $\{e_1, \ldots, e_m\}$:

Lemma 2.1.3. *m-intermediate curvature is a function on on the Grassmanian, i.e. given two sets of orthonormal vectors $\{e_1, \ldots, e_m\}$ and $\{f_1, \ldots f_m\}$ spanning the same m-dimensional plane in T_pN^n, we have*

$$\mathcal{C}_m(e_1, \ldots, e_m) = \mathcal{C}_m(f_1, \ldots, f_m).$$

Proof. Observe that

$$s_{m,n}(e_1, \ldots, e_m) + 2\,\mathcal{C}_m(e_1, \ldots, e_m) = \mathrm{R}$$

where

$$s_{m,n}(e_1, \ldots, e_m) = \sum_{p=m+1}^{n} \sum_{q=m+1}^{n} \mathrm{Rm}(e_p, e_q, e_p, e_q).$$

Here $s_{m,n}$ is *(m,n)-intermediate scalar curvature* introduced by M.L. Labbi [88] and further studied by M. Burkemper, C. Searle and M. Walsh [23]. Since $s_{m,n}$ depend only on the Grassmanian [23, Section 2], the result follows. □

It is well-known that taking connected sums of two manifolds preserves positive scalar curvature but does not preserve positive Ricci curvature. In our setting we obtain:

Lemma 2.1.4. *Positive m-intermediate curvature is preserved under surgeries of codimension $c \geq n - m + 2$. In other words, if both N^n and M^n admit a metric with $\mathcal{C}_m > 0$, then so does the manifold obtained by surgery of codimension $c \geq n - m + 2$.*

In particular, positive scalar curvature is preserved under codimension 3 surgeries which recovers a classical result of R. Schoen, S.-T. Yau [106] and M. Gromov, H.B. Lawson [59].

Proof. This follows immediately from S. Hoelzel's general surgery result [74, Theorem A]. □

Our arguments employ the first and second variation of a suitably weighted area which we will recall next.

Consider a Riemannian manifold (N^n, g), a smooth positive function $\rho : N^n \to \mathbb{R}$, and an embedded two-sided closed manifold $\Sigma \subset N^n$. For a given smooth function $f \in C^\infty(\Sigma)$ we consider a variation $F : (-\epsilon, \epsilon) \times \Sigma \to N^n$ with $F(0, x) = x$ and $\frac{\partial}{\partial s} F(s, x)\big|_{s=0} = f(x)\, \nu(x)$. In the following, we denote the map $F(s, \cdot)$ by F_s. Moreover, we denote by Σ_s the image of F_s and by ν_s the unit normal vector field to F_s.

By precomposing the maps F_s with suitable tangential diffeomorphisms, we can arrange that the variation is normal in the sense that

$$\frac{\partial}{\partial s} F_s = f_s\, \nu_s,$$

where f_s is a smooth function on Σ_s.

We consider the ρ-weighted area defined by

$$\mathcal{H}^{n-1}_\rho(\Sigma) \coloneqq \int_\Sigma \rho \, d\mu$$

where μ is the area-measure on Σ.

Proposition 2.1.5 (First variation of weighted area)**.**

The first variation of weighted area is given by

$$\frac{d}{ds}\mathcal{H}^{n-1}_{\rho}(\Sigma_s)\Big|_{s=0} = \int_{\Sigma} \rho f \left(H_{\Sigma} + \langle \nabla_N \log \rho, \nu \rangle\right) d\mu.$$

Here ∇_N is the Levi-Civita connection of N^n and H_{Σ} the mean curvature of Σ.

Proof. This is a consequence of the first variation formula for area, and the chain rule. □

Corollary 2.1.6.

Suppose Σ is a critical point of weighted area. Then we have

$$H_{\Sigma} = -\langle \nabla_N \log \rho, \nu \rangle.$$

For a constant weight we recover the minimal surface equation $H_{\Sigma} = 0$.

Proof. This follows immediately from the fundamental lemma of the calculus of variations. □

Proposition 2.1.7 (Second variation formula on critical points)**.**

If Σ is a critical point of the weighted area functional, then the second variation of weighted area is given by

$$\begin{aligned}&\frac{d^2}{ds^2}\mathcal{H}^{n-1}_{\rho}(\Sigma_s)\Big|_{s=0}\\ &= \int_{\Sigma} \rho \left(-f\Delta_{\Sigma} f - \left(|A_{\Sigma}|^2 + \mathrm{Ric}_N(\nu,\nu)\right) f^2 + f^2(\nabla^2_N \log \rho)(\nu,\nu) - f\langle \nabla_{\Sigma} \log \rho, \nabla_{\Sigma} f \rangle\right) d\mu.\end{aligned}$$

Here Δ_{Σ} is the Laplacian operator induced on Σ and A_{Σ} is the scalar-valued second fundamental form of Σ.

Proof. We use normal variations for our computation, and hence the first derivative is given by

$$\frac{d}{ds}\int_{\Sigma_s}\rho\,d\mu_s = \int_{\Sigma_s}\rho f_s\left(H_{\Sigma_s} + \langle\nabla_N\log\rho,\nu_s\rangle\right)\,d\mu_s.$$

We now differentiate both sides of this equation with respect to s, and evaluate the result at $s = 0$. By the variation formulas for hypersurfaces, compare for example with [81], the first order change in the mean curvature is given by

$$\frac{\partial}{\partial s}H_{\Sigma_s}\Big|_{s=0} = -\Delta_\Sigma f - \left(|A_\Sigma|^2 + \mathrm{Ric}_N(\nu,\nu)\right)f,$$

whereas the first order change in the normal vector field is given by

$$\nabla_s\nu_s|_{s=0} = -\nabla_\Sigma f.$$

This implies

$$\begin{aligned}&\frac{\partial}{\partial s}\left(H_{\Sigma_s} + \langle\nabla_M\log\rho,\nu_s\rangle\right)\Big|_{s=0}\\&= -\Delta_\Sigma f - \left(|A_\Sigma|^2 + \mathrm{Ric}_N(\nu,\nu)\right)f + (\nabla_N^2\log\rho)(\nu,\nu)f - \langle\nabla_\Sigma\log\rho,\nabla_\Sigma f\rangle,\end{aligned}$$

hence

$$\begin{aligned}&\frac{d^2}{ds^2}\mathcal{H}^{n-1}_\rho(\Sigma_s)\Big|_{s=0}\\&= \int_\Sigma\rho f\left(-\Delta_\Sigma f - \left(|A_\Sigma|^2 + \mathrm{Ric}_N(\nu,\nu)\right)f + (\nabla_N^2\log\rho)(\nu,\nu)f - \langle\nabla_\Sigma\log\rho,\nabla_\Sigma f\rangle\right)d\mu.\end{aligned}$$

This finishes the proof. □

2.2 Proof of Theorem A for specical cases

2.2.1 Proof of Theorem A for $m = 1$

We show that $N^n = M^{n-1}\times\mathbb{S}^1$ does not admit a metric g with $\mathrm{Ric}(g) > 0$. As discussed in the previous section, this follows immediately from Bonnet-Myers' theorem. Here we give an alternative argument based on area-minimizing hypersurfaces which we will generalize to other values of m.

Minimizing area in a non-trivial homology class of N^n, we obtain a stable minimal surface Σ_1, cf. Appendix A.3. Note that Σ_1 is a stable weighted slicing of order $m = 1$. According to the second variation formula, Proposition 2.1.7, we have for every smooth test function $f \in C^\infty(\Sigma_1)$

$$0 \leq -\int_{\Sigma_1} (f\Delta_\Sigma f + \mathrm{Ric}_N(\nu,\nu)f^2 + |A_{\Sigma_1}|^2 f^2).$$

Choosing $f = 1$ demonstrates that $\mathrm{Ric}_N(g) > 0$ is impossible.

2.2.2 Proof of Theorem A for $m = 2$

Our proof of Theorem A employs stable weighted slicings of order m as depicted in Figure 2.1 below. In the figure, the cube's faces are identified to represent $\mathbb{T}^m$, and each point in the cube represents a copy of M^{n-m}. Minimizing area in a non-trivial homology class leads to a stable minimal surface Σ_1 and the stability inequality for Σ_1 gives rise for a non-trivial eigenfunction u_1. Next, we minimize within Σ_1 the u_1-weighted area to obtain a stable weighted minimal surface Σ_2 with eigenfunction u_2. Iterating this process leads to a stable $u_1 \cdot \dots \cdot u_{m-1}$-weighted minimal surface Σ_m in Σ_{m-1}. Applying the stability inequality on Σ_m rules out the existence of a metric with $\mathcal{C}_m > 0$ on $N^n = M^{n-m} \times \mathbb{T}^m$.

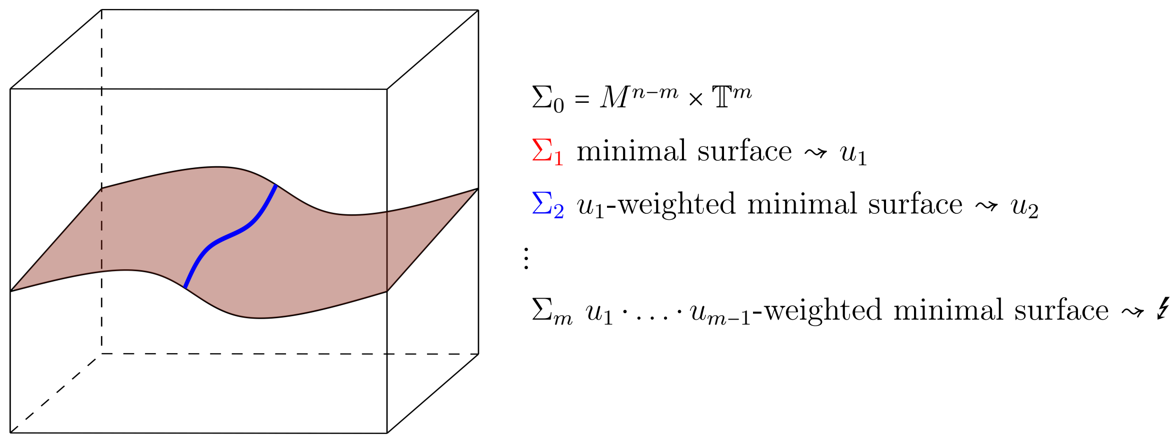


Figure 2.1: Schematic description of our slicing argument

Even though there are no conceptual differences between the case $m = 2$ and the general case, the algebra becomes significantly more involved in the latter setting. Therefore, we include the computation for $m = 2$ separately to highlight the most important analytic ideas.

As in the case $m = 1$ we start with constructing a stable minimal surface Σ_1 in $M^{n-2} \times \mathbb{T}^2$, cf. Appendix A. The hypersurface Σ_1 is smooth since $n \leq 7$. On Σ_1 we have for every smooth test function $f \in C^\infty(\Sigma_1)$

$$0 \leq -\int_{\Sigma_1} (f \Delta_{\Sigma_1} f + \mathrm{Ric}_N(\nu_1, \nu_1) f^2 + |A_{\Sigma_1}|^2 f^2).$$

It is well-known, see for instance [49, Theorem 1], that such an inequality leads to the existence of a first eigenfunction u_1 of the stability operator satisfying

$$\Delta_{\Sigma_1} u_1 = -\lambda_1 u_1 - |A_{\Sigma_1}|^2 u_1 - \mathrm{Ric}_N(\nu_1, \nu_1) u_1$$

for some eigenvalue $\lambda_1 \geq 0$. For the sake of completeness, we include a proof below:

Lemma 2.2.1. *Let Σ be a closed manifold, and let ϕ be a smooth function on Σ. Suppose* *that*

$$0 \leq \int_\Sigma (|\nabla f|^2 + \phi f^2) d\mu$$

for all smooth test functions $f \in C^\infty(\Sigma_1)$. Then there exists a smooth, strictly positive function u solving the equation

$$\Delta u = \phi u + \lambda u$$

for some $\lambda \geq 0$.

Proof. Since $\int_\Sigma (|\nabla f|^2 + \phi f^2) d\mu$ is bounded from below, we can consider the corresponding variational problem

$$\inf \left\{ \int_\Sigma (|\nabla f|^2 + \phi f^2) d\mu \,\middle|\, f \in W^{1,2}(\Sigma),\ \|f\|_{L^2(\Sigma)} = 1 \right\}.$$

Let $\{u_i\}$ be a minimizing sequence with $\|u_i\|_{L^2(\Sigma)} = 1$ and

$$\int_\Sigma \left(|\nabla u_i|^2 + \phi u_i^2 \right) d\mu \to \lambda$$

for some constant $\lambda \geq 0$. Since Σ is compact, ϕ is bounded, and therefore u_i is uniformly bounded in $W^{1,2}(\Sigma)$. This implies that u_i is subsequently converging strongly in $L^2(\Sigma)$ to some function u by Rellich-Kondrachov's theorem, and weakly in $W^{1,2}(\Sigma)$ by Banach-Alaoglu's theorem together with the reflexivity of the space $W^{1,2}(\Sigma)$. Due to the strong convergence in $L^2(\Sigma)$ and the normalization $\|u_i\|_{L^2(\Sigma)} = 1$, we obtain that $\|u\|_{L^2(\Sigma)} = 1$. Due to the weak convergence in $W^{1,2}(\Sigma)$ and the weak lower semicontinuity of $\|\cdot\|_{W^{1,2}(\Sigma)}$, we find

$$\int_\Sigma \left(|\nabla u|^2 + \phi u^2 \right) d\mu = \lambda.$$

Hence u is indeed a minimizer and therefore satisfies the corresponding Euler-Lagrange equation $\Delta u = \phi u + \lambda u$. By standard elliptic theory, u is smooth.

Next, observe that $|u|$ also lies in $W^{1,2}(\Sigma)$ and satisfies $\||u|\|_{L^2(\Sigma)} = 1$. Hence, $|u|$ is a valid competitor of the above variational problem. Since, $\int_\Sigma (|\nabla|u||^2 + f|u|^2) dV = \lambda$, the function $|u|$ is also a minimizer of the variational problem. Thus, $|u|$ solves the corresponding Euler-Lagrange equation and is smooth by elliptic regularity. Therefore, u must have vanishing normal derivative on the boundary of the set $\Sigma_\geq := \{x \in \Sigma : u(x) > 0\}$. Note that $\Sigma_\geq$ is with out loss of generality non-empty since we may replace u by $-u$. Applying Hopf's Lemma to $\Sigma_\geq$ we deduce that $\Sigma_\geq = \Sigma$, i.e. u must be non-negative. Using the strong maximum principle, u is in fact strictly positive. □

Resuming with the proof of Theorem A for $m = 2$, we consider on Σ_1 the weighted area functional

$$\mathcal{H}^{n-2}_{u_1}(\Sigma) = \int_\Sigma u_1 d\mu.$$

Minimizing $\mathcal{H}^{n-2}_u$ in a non-trivial homology class of Σ_1 leads to stable weighted minimal hypersurface $\Sigma_2 \subset \Sigma_1$ of dimension $n-2$. Again, we refer to Appendix A.3 to justify the existence of Σ_2. By the first variation formula, Corollary 2.1.6, we have

$$H_{\Sigma_2} = -\nabla_{\nu_1} u_1$$

where H_{Σ_2} is the mean curvature of $\Sigma_2 \subset \Sigma_1$ with respect to the unit normal ν_2. By the second variation formula, Proposition 2.1.7, we have for every test function $f \in C^\infty(\Sigma_2)$

$$\begin{aligned} 0 \leq &- \int_{\Sigma_2} u_1 \left(f \Delta_{\Sigma_2} f + |A_{\Sigma_2}|^2 f^2 + \mathrm{Ric}_{\Sigma_1}(\nu_2, \nu_2) f^2 \right) d\mu \\ &- \int_{\Sigma_2} u_1 \left(-f^2 (\nabla^2_{\Sigma_1} u_1)(\nu_2, \nu_2) + \langle \nabla_{\Sigma_2} f, \nabla_{\Sigma_2} u_1 \rangle f \right) d\mu \end{aligned}$$

where A_{Σ_2} is the second fundamental form of $\Sigma_2 \subset \Sigma_1$, and Ric_{Σ_1} is the Ricci curvature of Σ_1. Next, we would like to make use of u_1 satisfying the eigenvalue equation $\Delta_{\Sigma_1} u_1 = -\lambda_1 u_1 - |A_{\Sigma_1}|^2 u_1 - \mathrm{Ric}_N(\nu_1, \nu_1) u_1$. Therefore, we insert the text function $f = u_1^{-1}$ which yields

$$0 \le -\int_{\Sigma_2} \left(u_1^{-1}(-u_1^{-1}\Delta_{\Sigma_2}u_1 + |\nabla_{\Sigma_2}\log u_1|^2\right) d\mu$$
$$-\int_{\Sigma_2} \left(|A_{\Sigma_2}|^2 + \mathrm{Ric}_{\Sigma_1}(\nu_2,\nu_2) - u_1^{-1}(\nabla^2_{\Sigma_1}u_1)(\nu_2,\nu_2)\right) d\mu.$$

By choosing $f = u_1^{-1}$ that the factors in front of the $\Delta_{\Sigma_2}u_1$ and $(\nabla^2_{\Sigma_2}u_1)(\nu_2,\nu_2)$ terms coincide. With the help of the first variation formula, this allows us to compute

$$\Delta_{\Sigma_2}u_1 + (\nabla^2_{\Sigma_1}u_1)(\nu_2,\nu_2) = \Delta_{\Sigma_1}u_1 - H_{\Sigma_2}\nabla_{\nu_2}u_1 = \Delta_{\Sigma_1}u_1 + u_1^{-1}(\nabla^{\Sigma_1}_{\nu_2}u_1)^2.$$

Thus, using the eigenvalue equation for u_1 and estimating $|\nabla_{\Sigma_1}u_1|^2 \ge (\nabla_{\nu_2}u_1)^2$, we obtain

$$0 \le -\int_{\Sigma_2} \left(u_1^{-1}(|A_{\Sigma_1}|^2 + \mathrm{Ric}_N(\nu_1,\nu_1) + |\nabla_{\Sigma_2}\log u_1|^2 + |A_{\Sigma_2}|^2 + \mathrm{Ric}_{\Sigma_1}(\nu_2,\nu_2))\right) d\mu$$

Next, we would like to replace $\mathrm{Ric}_{\Sigma_1}(\nu_2,\nu_2)$ with a curvature term of $\Sigma_0 = N^n = M^{n-2} \times \mathbb{T}^2$. To do so, we use the Gauss equations which state

$$\mathrm{Ric}_{\Sigma_1}(\nu_2,\nu_2) = \mathrm{Ric}_N(\nu_2,\nu_2) - \sec_N(\nu_1,\nu_2) - (A_{\Sigma_1}(\nu_2,\nu_2))^2.$$

Hence

$$0 \le -\int_{\Sigma_2} u_1^{-1}(\mathrm{Ric}_N(\nu_1,\nu_1) + \mathrm{Ric}_N(\nu_2,\nu_2) - \sec_N(\nu_1,\nu_2))d\mu = -\int_{\Sigma_2} u_1^{-1}\mathcal{C}_2(\nu_1,\nu_2)d\mu$$

which implies that $\mathcal{C}_2$ cannot be strictly positive on N^n. This finishes the proof of Theorem A for $m = 2$.

We would highlight that m-intermediate curvature is up to some extrinsic curvature terms just the sum of Ricci curvature terms coming from a stable slicing. In other words, m-intermediate curvature is the natural notion of curvature associated to stable weighted slicings.

2.3 The general case

For arbitrary m, we use the topology of the manifold $N^n = M^{n-m} \times \mathbb{T}^m$ to construct a stable weighted slicing $\Sigma_m \subset \Sigma_{m-1} \subset \cdots \subset \Sigma_1 \subset \Sigma_0 = N^n$, cf. Figure 2.1 and Appendix A.3. After a tedious (but otherwise to $m = 2$ similar) computation and some delicate gradient estimates, cf. Appendix A.1, the stability inequality of the $u_1 \cdot \ldots \cdot u_{m-1}$-weighted minimal surface $\Sigma_m \subset \Sigma_{m-1}$ yields:

Lemma 2.3.1 (Stability inequality for weighted slicings)**.** *Let $\Sigma_m \subset \Sigma_{m-1} \subset \cdots \subset \Sigma_1 \subset \Sigma_0 = N^n$ be a stable weighted slicing of order m. Then we have*

$$0- \leq \int_{\Sigma_m} \left(\mathcal{C}_m(\nu_1, \ldots, \nu_m) + \sum_{k=1}^{m} \mathcal{V}_k \right) d\mu$$

where $\mathcal{V}_k$ are extrinsic curvature terms given by

$$\begin{aligned}
\mathcal{V}_1 =& |A_{\Sigma_1}|^2 + \sum_{p=2}^{m} \sum_{q=p+1}^{n} \left(A_{\Sigma_1}(e_p, e_p) A_{\Sigma_1}(e_q, e_q) - A_{\Sigma_1}(e_p, e_q)^2 \right), \\
\mathcal{V}_k =& |A_{\Sigma_k}|^2 - \left(\frac{1}{2} - \frac{1}{2(k-1)} \right) H^2_{\Sigma_k} \\
&+ \sum_{p=k+1}^{m} \sum_{q=p+1}^{n} \left(A_{\Sigma_k}(e_p, e_p) A_{\Sigma_k}(e_q, e_q) - A_{\Sigma_k}(e_p, e_q)^2 \right) \text{ for } 2 \leq k \leq m-1, \\
\mathcal{V}_m =& |A_{\Sigma_m}|^2 - \left(\frac{1}{2} - \frac{1}{2(m-1)} \right) H^2_{\Sigma_m}.
\end{aligned}$$

For $m = 1$ and $m = 2$, we saw in the previous section that $\sum_{k=1}^{m} \mathcal{V}_k \geq 0$ which implied that $N^n = M^{n-m} \times \mathbb{T}^m$ does not admit a metric with positive m-intermediate curvature. Let us analyze the general case next:

Lemma 2.3.2. *Suppose $n(m-2) \leq m^2 - 2$. Then $\sum_{k=1}^{m} \mathcal{V}_k \geq 2$.*

This estimate uses different ideas for the three different cases $k = 1$, $2 \leq k \leq m-1$ and $k = m$. The main philosophy of our argument is that we have good second fundamental form terms competing against bad mean curvature terms. Since for a k-dimensional hypersurface the estimate $|A|^2 \geq \frac{1}{k} H^2$ holds, we should not expect that the good second fundamental form terms can control the bad mean curvature terms in arbitrary large dimensions. Due to the technical nature, the proof of Lemma 2.3.2 will be carried out in Appendix A.2.

Lemma 2.3.3 (Algebraic Lemma)**.** *Let* $n \leq 7$. *Then* $n(m-2) \leq m^2 - 2$.

We remark that for $m = 1, m = 2, m = n-2, m = n-1$ the dimension n can be arbitrarily large. On the other side for $m = 3$, we obtain $n(3-2) \leq 3^2 - 2$, i.e. $n \leq 7$. Similarly, we also obtain $n \leq 7$ for $m = 4$ and $m = 5$. This dimensional constraint is sharp as demonstrated by K. Xu [119] who constructed counterexamples to Theorem A in dimensions $n(m-2) > m^2 - 2$. Moreover, we observe that the dimensional constraint is symmetric in the sense that $n(m-2) \leq m^2 - 2$ is equivalent to $n(\tilde{m} - 2) \leq \tilde{m}^2 - 2$ for $\tilde{m} = n - m$.

Chapter 3

Spacetime harmonic functions

This chapter is based upon joint work with Demetre Kazaras, Marcus Khuri [64] (the mass formula), and Yiyue Zhang [72] (the case of equality).

In Dimension 3 there is a surprisingly elegant proof of the Geroch conjecture due to D. Stern [113]. While R. Schoen and S.-T. Yau [106] exploit the existence of non-trivial homology classes in $H_2(\mathbb{T}^3;\mathbb{Z})$ by constructing area-minimizing surfaces, D. Stern considers the dual problem of minimizing energy of maps $\mathbb{T}^3$ to $\mathbb{S}^1$. This leads to the existence of a harmonic map $u:\mathbb{T}^3\to\mathbb{S}^1$ where we assume for simplicity that $\nabla u\neq 0$[1]. The harmonicity of u implies the Bochner identity

$$\frac{1}{2}\Delta|\nabla u|^2=\operatorname{Ric}(\nabla u,\nabla u)+|\nabla^2 u|^2.$$

Now the crucial observation is that the Ricci curvature term can be expressed as $\operatorname{Ric}(\nu,\nu)|\nabla u|^2$ where $\nu=\frac{\nabla u}{|\nabla u|}$ is the unit normal to the level-sets Σ_t of u. Hence, we can use the contracted Gauss equations $2\operatorname{Ric}(\nu,\nu)=R-2K+H^2-|A|^2$ where K is the Gaussian curvature, H the mean curvature and A the second fundamental form of the level sets $\Sigma_t=\{u=t\}$. This trick has been previously been known in the physics literature where it found several applications due to J. Jezierski, J. Kijowski and P. Waluk [84, 85]. Expressing the second fundamental form A as $A|\nabla u|=\nabla^2 u|_{T\Sigma\otimes T\Sigma}$, and inte-

[1]For a rigorous proof see Section 3.3.

grating Bochner's formula and the Gauss equations yields

$$0 = 2\int_{\mathbb{T}^3} \Delta|\nabla u| = \int_{\mathbb{T}^3}\left(\frac{|\nabla^2 u|^2}{|\nabla u|} + R|\nabla u|\right)d\mu - 2\int_{\mathbb{T}^3} K|\nabla u|d\mu. \tag{3.1}$$

Since $\mathbb{T}^3$ has no spherical classes, the maximum principle implies that the level sets $\Sigma_t = \{u = t\}$ cannot be spherical. Hence, the coarea formula yields

$$-2\int_{\mathbb{T}^3} K|\nabla u| = -2\int_{\mathbb{S}^1}\int_{\Sigma_t} K dA \geq 0.$$

Therefore, $\mathbb{T}^3$ cannot admit a metric of positive scalar curvature.

Besides harmonic maps, level-sets also arise naturally in the study of geometric flows. For instance G. Huisken and T. Ilmanen proved in [79] the Riemannian Penrose inequality for a single black hole via a level-set formulation of inverse mean curvature flow.

In this section we introduce several new differential equation and demonstrate how we can use the level-sets of their solutions to obtain powerful applications in both mathematical relativity and Riemannian geometry. In particular, we use *spacetime harmonic functions* to prove Theorem B which implies the spacetime positive mass theorem.

3.1 Initial data sets

Asymptotically flat initial data sets (M^3, g, k) naturally arise in General Relativity where they are used to model isolated gravitational systems such as stars, galaxies and black holes. Here (M^3, g) is a complete non-compact Riemannian manifold where g approaches the Euclidean metric δ at infinity and k is a symmetric two-tensor approaching zero at infinity.

More precisely, we say a triple (M^3, g, k) is *an asymptotically flat initial data set* of order $\tau \in (\frac{1}{2}, 1]$ if (M^3, g) contains a compact set $\mathcal{C} \subset M$ such that we can write

$M \setminus \mathcal{C} = \cup_{\ell=1}^{\ell_0} M_{end}^{\ell}$ where the ends M_{end}^{ℓ} are pairwise disjoint and diffeomorphic to the complement of a ball $\mathbb{R}^3 \setminus B_1$. Moreover, we require that there exists a coordinate system in each end satisfying

$$\begin{aligned} |\partial^l (g_{ij} - \delta_{ij})(x)| =& O(|x|^{-\tau - l}), \quad l = 0, 1, 2, \\ |\partial^l k_{ij}(x)| =& O(|x|^{-\tau-1-l}), \quad l = 0, 1. \end{aligned} \tag{3.2}$$

Here $O(\cdot)$ is the standard Landau notation.

In Einstein's theory of gravity, Lorentzian manifolds $(\bar{M}^4, \bar{g})$ are used to model spacetimes, and initial data sets (M^3, g, k) above arise as spacelike slices inside $(\bar{M}^4, \bar{g})$. Here g is metric induced from $\bar{g}$, and k the second fundamental form. According to general relativity, matter curves spacetime and the curvature of spacetime determines the motions of matter. Mathematically, this corresponds to $(\bar{M}^4, \bar{g})$ solving the famous *Einstein equations* $\bar{\mathrm{Ric}} - \frac{1}{2}\bar{\mathrm{R}}\bar{g} = 8\pi T$ where T is the stress energy tensor. Since *geometry equals physics*, making physically reasonable assumptions on T leads to geometric assumptions on $(\bar{M}^4, \bar{g})$ which translate to geometric assumptions on (M^3, g, k) via the Gauss-Codazzi equations. In particular, we will assume that our spacetimes satisfy the *dominant energy condition* which physically amounts to us not being able to observe non-negative mass densities. Mathematically, this translates on each initial data set to the condition

$$\mu \geq |J|,$$

where μ is the energy density, and J the momentum density J defined by

$$\mu = \frac{1}{2}(\mathrm{R} + \mathrm{tr}_g\, k^2 - |k|^2), \qquad J = \mathrm{div}_g(k - \mathrm{tr}_g\, kg).$$

Moreover, we define the ADM energy E and linear momentum P by

$$\begin{aligned} E =& \lim_{r\to\infty} \frac{1}{16\pi} \int_{\mathbb{S}_r} \sum_i (g_{ij,i} - g_{ii,j})\, \upsilon^j \, d\mu, \\ P_i =& \lim_{r\to\infty} \frac{1}{8\pi} \int_{\mathbb{S}_r} (k_{ij} - \mathrm{tr}_g\, k g_{ij})\, \upsilon^j \, d\mu \end{aligned} \tag{3.3}$$

where υ is the outer unit normal to the sphere $\mathbb{S}_r$ and $d\mu$ is its area element. Furthermore, we set $\mathfrak{m} = \sqrt{E^2 - |P|^2}$. In order to ensure that E and P are well-defined in equation (3.3), we impose additionally $\mu, J \in L^1(M)$.

A fundamental results about initial data sets is the positive mass theorem (PMT):

Theorem 3.1.1. *Suppose* (M, g, k) *is a complete asymptotically flat initial data set satisfying the dominant energy condition (DEC)* $\mu \geq |J|$. *Then* $E \geq |P|$.

This result has been first established by R. Schoen and S.-T. Yau in [109] using the Jang equation and by E. Witten in [118] using spinors. Further proofs have been given in [46, 47, 64], and the important special case $k = 0$ has been treated in [2, 18, 70, 79, 92, 96, 108]. We refer to [64] for a more detailed historical overview and to the monograph [90] for an in-depth discussion of mathematical relativity.

In this chapter we will analyze how so-called *spacetime harmonic functions*, i.e. functions solving the PDE $\Delta u = -\operatorname{tr}_g k|\nabla u|$ can be used to study initial data sets (M, g, k). This will not just allow us to obtain a more elementary proof of the spacetime PMT, but it also allows us to classify the initial data sets satisfying the identity $E = |P|$.

3.2 Further applications

Before proceeding with an in-depth study of the spacetime positive mass theorem, we first discuss several other applications of spacetime harmonic functions. Surprisingly, there is a whole zoo of applications including purely geometric statements and physically motivated results.

3.2.1 Comparison geometry

Given a Riemannian manifold (M,g) we can artificially construct an initial data set (M,g,k) by setting $k = fg$ for a suitably chosen function f. Thus, we can use spacetime harmonic functions in a purely Riemannian context.

Similarly, other techniques from GR can be applied to geometric problems: In case $k = fg$, MOTS (marginally outer trapped surfaces) are called *μ-bubbles*, and the Dirac operator is called *Callias operator*. Both μ-bubbles and the Callias operator led to many important geometric results, cf. [31, 55, 91, 103, 121] and [25, 26, 120].

In a joint work with D. Kazaras, M. Khuri and Y. Zhang [65] we used spacetime harmonic functions to obtain a new proof of *Bonnet-Myers' diameter estimate*, cf. Theorem 2.0.1, including Cheng's rigidity [30]. Compared to Myers' original argument using geodesics, and to Croke-Kleiner's [40] argument using the distance function, our proof has the advantage that it also works for open and incomplete manifolds. In the same paper, we also showed the following result classifying lens spaces:

Theorem 3.2.1. *Let (M^3,g) be a closed Riemannian manifold with 2-Ricci curvature at least 4. If $\Sigma^2 \subset M^3$ is a connected embedded closed surface of positive genus, then*

$$Inj_n(\Sigma^2) \le \frac{\pi}{4}. \tag{3.4}$$

If additionally $\mathrm{Ric} \ge 2g$ and equality occurs in (3.4), then the universal cover of (M^3,g) is isometric to the round sphere and Σ^2 lifts to the Clifford torus. Moreover, in this case (M^3,g) is isometric to a round sphere or a round lens space.

Here Inf_n is the normal injectivity radius, and we say that the 2-Ricci curvature is at least 4 if the sum of any two eigenvalues of the operator Ric is greater or equal than 4.

Compared to other techniques spacetime harmonic functions excel at describing complicated rigidity phenomena which we later exploit for the spacetime PMT. Here is another example from [65]:

Theorem 3.2.2. *Let* $(M^3, \partial_\pm M^3, g)$ *be a 3-dimensional Riemannian band with no spherical classes in* $H_2(M^3;\mathbb{Z})$. *Consider the sign reversed minimal outward mean curvature* $H_0 = -\min_{\partial M^3} H$. *If* (M^3, g) *has 2-Ricci curvature at least 4, then* $H_0 > 0$ *and the width of the band satisfies*

$$w := dist(\partial_- M^3, \partial_+ M^3) \le \arctan(H_0/2). \tag{3.5}$$

If additionally $\mathrm{Ric} \ge 2g$ *and equality is achieved in* (3.5), *then the universal cover of* (M^3, g) *is isometric to* $([-\frac{w}{2}, \frac{w}{2}] \times \mathbb{R}^2, g_\Upsilon)$ *where*

$$g_\Upsilon = d\rho^2 + \phi_\Upsilon^2(\rho)dx^2 + \psi_\Upsilon^2(\rho)dy^2, \qquad \rho \in \left[-\frac{w}{2}, \frac{w}{2}\right], \quad (x,y) \in \mathbb{R}^2,$$

and

$$\begin{cases} \phi_\Upsilon(\rho) = 2^{\frac{1-\Upsilon}{2}} \cos^{1-\Upsilon}(\rho + \frac{\pi}{4}) \cos^{\frac{\Upsilon}{2}}(2\rho) \\ \psi_\Upsilon(\rho) = 2^{\frac{1-\Upsilon}{2}} \sin^{1-\Upsilon}(\rho + \frac{\pi}{4}) \cos^{\frac{\Upsilon}{2}}(2\rho) \end{cases}$$

for some $\Upsilon \in [0,1]$.

There are counterexamples for the case of equality in case the additional assumption $\mathrm{Ric} \ge 2$ is not satisfied [65].

Llarull's theorem [93] states that if $g \ge g_{\mathbb{S}^n}$ and $\mathrm{R}(g) \ge \mathrm{R}(g_{\mathbb{S}^n}) = n(n-1)$, one must have $g = g_{\mathbb{S}^n}$. Conjecturally, this extremal character of the round metric on $\mathbb{S}^n$ is even more robust: Gromov has suggested [55, Conjecture D] that the open and incomplete manifold formed by removing finitely many points from the round sphere enjoys the same property, also see [56, Section 3.9]. We confirm this statement in the next result, for dimension 3 in the special case of a pair of antipodal points:

Theorem 3.2.3. *Let g be a Riemannian metric on $S^3 \setminus \{N, S\}$ where N, S are the north and south pole. If $g \geq g_{S^3}$, then there is a point $x \in \mathbb{S}^3 \setminus \{N, S\}$ where the scalar curvature satisfies $\mathrm{R}(x) \leq 6$. If additionally $\mathrm{R} \geq 6$, then g agrees with the round metric $g_{\mathbb{S}^3}$.*

The proof of both this result and Theorems 3.2.2, 3.2.1 above, are also based on spacetime harmonic functions.

3.2.2 Existence of black holes

The idea to construct a new symmetric two-tensor and to study the associated spacetime harmonic functions also works for initial data sets and leads to black hole existence results. Besides spacetime harmonic functions, the techniques described in Chapter 2 also lead to such as existence results. More precisely, in [66] we showed together with D. Kazaras, M. Khuri and Y. Zhang:

Theorem 3.2.4. *Let $3 \leq n \leq 7$, and suppose that (M^n, g, k) is an asymptotically flat n-dimensional initial data set. Assume that there is an n-cube within M^n on which*

$$\mu - |J| \geq \frac{2n\pi^2}{n+1} \sum_{i=1}^{n} \frac{1}{\ell_i^2},$$

where ℓ_i is the distance between the ith pair of opposite faces of the cube. Then the data contains a closed properly embedded smooth apparent horizon.

We remark that in view of Penrose's incompleteness theorem [100], an apparent horizon is contained within the black hole region under physically reasonable assumptions. Thus, the above statement demonstrates that if sufficiently much matter accumulates in a fixed region, gravitational collapse must occur and a black hole must have formed.

However, we note that the Theorem 3.2.4 does not contain information about the location of the apparent horizon; in particular the horizon could be both inside or outside the cube.

3.3 The integral formula

Next, we move on to the foundation of the proof of Theorem B. More precisely, we compute in this section the central divergence formula for the case where ∇u is non-vanishing. This allows us to highlight the most important geometric and analytic aspects of the computation, while the general case is outsourced to Appendix B.3. This integral formula does not just crucial ingredient of the proof of Theorem B, but also for several of the results from previous subsection.

Theorem 3.3.1. *Let u be a smooth solution of the spacetime Laplace equation $\Delta u = -\operatorname{tr}_g k|\nabla u|$ with $\nabla u \neq 0$. Then*

$$\operatorname{div}(\nabla|\nabla u| + k(\nabla u, \cdot)) + K|\nabla u| = \frac{|\bar{\nabla}^2 u|^2}{2|\nabla u|} + \mu|\nabla u| + \langle J, \nabla u\rangle \tag{3.6}$$

where $\bar{\nabla}^2 u = \nabla^2 u + k|\nabla u|$, and where K is the Gaussian curvature of the level-sets Σ_t of u.

Before proceeding with the proof, let us comment more on the structure of Equation (3.6). Note that the right hand side will always be non-negative in case the dominant energy condition $\mu \geq |J|$ is satisfied. This will allows us to deduce geometric and physical consequences. Moreover, understanding the *spacetime Hessian* term $\bar{\nabla}^2 u$ will be the key for proving rigidity statements. After integration we can also control the left hand side by applying the divergence theorem to the first term, and the coarea formula together with Gauss-Bonnet's theorem to the second term. This leads to terms involving the

Euler characteristic of the level-sets and boundary integral terms depending on the null expansions $\theta_\pm = H \pm \operatorname{tr}_{g_\Sigma} k$. We will discuss in depth how to control the Euler characteristic of the level-set in Appendix B.1 and B.2.

Proof. We begin with recalling the standard Bochner identity which states

$$\frac{1}{2}\Delta|\nabla u|^2 = |\nabla^2 u|^2 + \mathrm{Ric}(\nabla u, \nabla u) + \langle \nabla u, \nabla \Delta u\rangle.$$

Exploiting that ∇u is non-vanishing, we can rewrite this identity as

$$\Delta|\nabla u| = \frac{1}{|\nabla u|}(|\nabla^2 u|^2 - |\nabla|\nabla u||^2 + \mathrm{Ric}(\nabla u, \nabla u) + \langle \nabla u, \nabla \Delta u\rangle). \tag{3.7}$$

Let us denote with $\nu = \frac{\nabla u}{|\nabla u|}$ the unit normal to the level-sets of u, and with A, H the second fundamental form and the mean curvature. We have

$$A = \frac{1}{|\nabla u|}\nabla^2 u, \qquad H = \frac{1}{|\nabla u|}(\Delta u - \nabla_{\nu\nu} u).$$

This leads to

$$|\nabla u|^2 (H^2 - |A|^2) = 2|\nabla|\nabla u||^2 - |\nabla^2 u|^2 + (\Delta u)^2 - 2\Delta u \nabla_{\nu\nu} u. \tag{3.8}$$

Combining the contracted Gauss equations $\mathrm{Ric}(\nu,\nu) = \frac{1}{2}(\mathrm{R} + H^2 - |A|^2 - 2K)$ with equations (3.7) and (3.8), we obtain

$$\Delta|\nabla u| = \frac{1}{2|\nabla u|}(|\nabla^2 u|^2 + |\nabla u|^2(\mathrm{R} - 2K) + 2\langle \nabla u, \nabla\Delta u\rangle + (\Delta u)^2 - 2\Delta u \nabla_{\nu\nu} u).$$

Next, we introduce the notation $\bar\nabla^2 u = \nabla^2 u + k|\nabla u|$ which yields

$$\begin{aligned}\Delta|\nabla u| =& \frac{1}{|\nabla u|}(|\bar\nabla^2 u|^2 + |\nabla u|^2(\mathrm{R} - 2K) + 2\langle \nabla u, \nabla\Delta u\rangle + (\Delta u)^2 - 2\Delta u \nabla_{\nu\nu} u)\\ &+ \frac{1}{2|\nabla u|}(-2\nabla_{ij} u k_{ij}|\nabla u| - |k|^2|\nabla u|^2).\end{aligned}$$

Inserting $\Delta u = -\operatorname{tr}_g k|\nabla u|$ leads to

$$\begin{aligned}\Delta|\nabla u| =& \frac{1}{2|\nabla u|}(|\bar{\nabla}^2 u|^2 + |\nabla u|^2(\mathrm{R} - 2K) + 2\langle \nabla u, \nabla(\operatorname{tr}_g k|\nabla u|)\rangle + \operatorname{tr}_g k^2|\nabla u|^2) \\ &+ \frac{1}{2|\nabla u|}(-2\operatorname{tr}_g k|\nabla u|\nabla_{\nu\nu}u - 2\nabla_{ij}uk_{ij}|\nabla u| - |k|^2|\nabla u|^2).\end{aligned}$$

Observe that

$$\langle \nabla u, \nabla|\nabla u|\rangle = |\nabla u|\nabla_{\nu\nu}u.$$

Hence the above term simplifies to

$$\Delta|\nabla u| = \frac{1}{2|\nabla u|}(|\bar{\nabla}^2 u|^2 + 2|\nabla u|^2(\mu - K) + 2|\nabla u|\langle \nabla u, \nabla \operatorname{tr}_g k\rangle - 2\nabla_{ij}uk_{ij}|\nabla u|).$$

Finally, we use the identity

$$-\nabla_{ij}uk_{ij} = -\operatorname{div}(k(\nabla u, \cdot)) + \nabla_i u\nabla_j k_{ij}$$

to finish the proof. □

3.3.1 Technical difficulties

We would like to point out several technical challenges which are present for spacetime harmonic functions but not for harmonic functions:

- The spacetime Laplace equation $\Delta u = -\operatorname{tr}_g k|\nabla u|$ is non-linear which complicates the existence theory. This is especially an issue *asymptotically hyperbolic manifolds* [17]where $\operatorname{tr}_g k$ does not decay to zero at ∞.
- In general, the solution u will not be smooth which leads to some subtleties. For instance, Sard's theorem needs C^3-regularity.
- To be able to exploit Gauss-Bonnet's theorem we need to control the topology of the level-sets of u. For the Riemannian positive mass theorem one can pass to the

exterior region [79, Lemma 4.1] and then solve the spacetime Laplace equation with Neumann boundary data. However, both of these proof steps break down in the spacetime setting and we refer to Appendix B.1 and B.2 for a detailed discussion.

However,the non-linearity is still mild and the spacetime Laplace equation is significantly better behaved than comparable PDEs such as *Jang's equation*. We will exploit this in Section 4.

3.4 Why spacetime harmonic functions?

In this section we discuss where spacetime harmonic functions come from and explain several special properties.

3.4.1 Spacetime harmonic functions in Minkowski space

On any manifold, the perhaps most natural differential operator is the Laplacian. Now suppose (M, g, k) is contained in Minkowski space $(\mathbb{R}^{3,1}, \bar{g})$. Within Minkowski space, harmonic functions are given by $u = ax + by + cz + dt$ where $a, b, c, d \in \mathbb{R}$. Restricting u onto (M, g, k) leads to the equation

$$\Delta u = -\bar{\nabla}^2_{NN} u - \operatorname{tr}_g k N(u) = -\operatorname{tr}_g k N(u).$$

where N is a unit normal of $(M, g, k) \subset \mathbb{R}^{3,1}$, and k the second fundamental form with respect to N. However, $N(u)$ does not solely depend on the initial data (M, g, k). In case u is an optical function, i.e. $\bar{\nabla} u$ is null, we have the additional identity $N(u) = |\nabla u|$. Hence, we recover the spacetime Laplace equation $\Delta u = -\operatorname{tr}_g k|\nabla u|$.

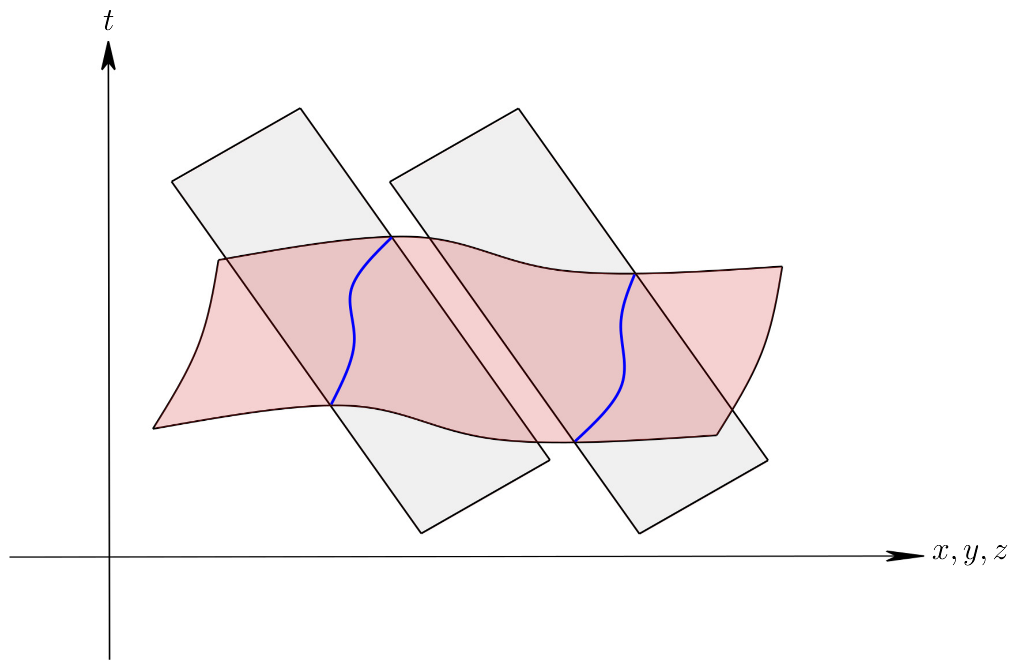


Figure 3.1: Given an initial data set (M, g, k) contained in Minkowski space, the level-sets Σ_t of a spacetime harmonic functions are given by intersections with null-planes. We will use a similar idea to construct *spacetime IMCF* in Chapter 4.

3.4.2 Comparison with other techniques

Perhaps surprisingly, spacetime harmonic functions are closely related to many other techniques used to study manifolds with non-negative scalar curvature (or more generally, initial data sets satisfying the dominant energy condition).

The idea to apply Gauss-Bonnet's theorem in combination with the contracted Gauss-equations lies at the heart of the minimal surface technique, though in this case these theorems are only applied to a single surface instead of a whole family of surfaces.

Moreover, there is a connection between spacetime harmonic functions and inverse mean curvature flow. In fact, Theorem C generalizes the integral formula from the previous section and the famous Hawking mass monotonicity formula for IMCF. We will discuss the precise relationship in more detail in Section 4.

Furthermore, *Witten spinors*[2] give in a natural way rise to a vector field. If this vector field is integrable, the corresponding function will be spacetime harmonic. We refer to [16, Section 5] for details.

Finally, instead of choosing the harmonic function in Minkowski space to be null in the previous subsection, we could have also chosen it be timelike, i.e. $u = t$. In this case $N(u) = \sqrt{1 + |\nabla u|^2}$, and we obtain the PDE $\Delta u = -\operatorname{tr}_g k|\nabla u|$ which has been referred to in the literature as generalized *Jang's equation*. Again, we refer to [16, Section 5] for details.

3.5 The positive mass theorem

Given the integral formula

$$2\operatorname{div}(\nabla|\nabla u| + k(\nabla u, \cdot)) + 2K|\nabla u| = \frac{|\bar{\nabla}^2 u|^2}{|\nabla u|} + 2\mu|\nabla u| + 2\langle J, \nabla u\rangle. \tag{3.9}$$

we are now almost in the position to prove Theorem B. The remaining subtleties are to control the level-set topology which is outsourced to Appendix B.1 and B.2, and to establish an existence theory for spacetime harmonic functions which is the subject of the next subsection. Assuming both these results are established, we will now demonstrate how formula (4.6) leads to the spacetime PMT.

[2]The spinors used by Witten [118] in his proof of the spacetime PMT. Roughly speaking, they are just the usual Dirac spinors with respect to a "spacetime" tangent bundle.

3.5.1 The asymptotically flat case

Now let (M^3, g, k) be an asymptotically flat manifold satisfying the dominant energy condition $\mu \geq |J|$. Let u be the spacetime harmonic functions whose gradient is asymptotic to the unit vector $\vec{x}$ at ∞. The existence of u is guaranteed by Theorem 3.6.2 established in the next subsection. For simplicity we assume $M^3 = \mathbb{R}^3$ and that ∇u is non-vanishing everywhere - both general cases are discussed in detail in Appendix B.1 and B.3.

Integrating the formula (4.6) we obtain with the help of Gauss-Bonnet's theorem and the divergence theorem the equation

$$E + P_{\vec{x}} = \frac{1}{16\pi} \int_{M^3} \left(\frac{|\bar{\nabla}^2 u|^2}{|\nabla u|} + 2\mu|\nabla u| + 2\langle J, \nabla u \rangle \right) d\mu.$$

The computation of the boundary term is slightly tedious and therefore outsourced to Appendix B.4.

Without loss of generality we may assume that $P \neq 0$. Choosing u to be asymptotic to $\vec{x} = -\frac{P}{|P|}$, the mass formula of Theorem B follows. It remains to establish the case of equality.

3.5.2 The case of equality

We assume that $E = |P|$, i.e. (M, g, k) has vanishing mass $\mathfrak{m} = \sqrt{E^2 - |P|^2}$. Our goal is to show that (M, g) embeds isometrically into Minkowski space with second fundamental form k. This has already been established previously by R. Beig, P. Chrusciel [10], L.-H. Huang, D. Lee [77], and together with D. Kazaras and M. Khuri in [64] under additionally imposed regularity and decay assumptions for g, k, μ, J.

Our proof will proceed in four steps:

- We show that the pair (g, k) satisfy most of the *Gauss and Codazzi equations* using the mass formula of Theorem B.
- The remaining Gauss and Codazzi terms have the form $\mathcal{A}_{\alpha\beta} := \nabla_3 k_{\alpha\beta} - \nabla_\alpha k_{\beta 3}$ where e_α, e_β are tangent to the level-sets of u. We show that $\nabla^\Sigma_{\alpha\beta} F = |\nabla u|^{-2} \mathcal{A}_{\alpha\beta}$ for some function F where $\nabla^\Sigma_{\alpha\beta}$ is the level-set Hessian.
- We have $\Delta^\Sigma F = |\nabla u|^{-2}\mu \geq 0$. In combination with the asymptotics of g, k, *Liouville's theorem* yields that the Hessian of F vanishes.
- Since all the Gauss and Codazzi equations are satisfied, the Lorentzian version of the *fundamental theorem of hypersurfaces* implies that (M, g) isometrically embeds in Minkowski space with second fundamental form k.

Let u be the spacetime harmonic function whose gradient is asymptotic to $-\frac{P}{|P|}$. From mass formula we obtain

$$0 = E - |P| = \frac{1}{16\pi} \int_{M^3} \left(\frac{|\bar{\nabla}^2 u|^2}{|\nabla u|} + 2\mu |\nabla u| + 2\langle J, \nabla u \rangle \right) d\mu.$$

Hence $\bar{\nabla}^2 u = 0$ and $\mu|\nabla u| + \langle J, \nabla u\rangle = 0$.[3] This yields 6 equations for u and 3 equations for J.[4] Having so much information at our disposal makes the level-set technique a powerful tool to study rigidity questions, and we would like to highlight that there is no spinor or minimal surface proof of the case of equality.

Conceptually, in the case of equality, the minimal surface technique only gives information of the ambient manifold in a neighborhood of a hypersurface. Spinors do

[3] In [64] we have addressed the case of equality under the additional assumption that $E = |P| = 0$. In this case we have not just one, but three functions whose spacetime Hessian $\bar{\nabla}^2$ vanishes.

[4] The normal component of J (with respect to the level-sets of u) equals $-\mu$ while the tangential components vanish since $\mu \geq |J|$ by assumption.

yield information about the entire manifold but for level-sets we have an additional tool available:

Lemma 3.5.1. *Let* (M, g, k) *be an initial data set with* $E = |P|$*, and let* u *be the spacetime harmonic functions asymptotic to* $-\frac{P}{|P|}$*. Then the level-sets* Σ_t *of* u *are flat, i.e. their Gaussian curvature* K *is vanishing.*

Proof. Since $\mu|\nabla u| = -\langle J, \nabla u\rangle$, Bochner's identity in combination with the contracted Gauss equation yield

$$\Delta|\nabla u| = \frac{1}{|\nabla u|}(-K|\nabla u| + |k|^2|\nabla u|^2 - \langle \operatorname{div} k, \nabla u\rangle|\nabla u|).$$

On the other side, we have by the spacetime Hessian equation $\nabla^2 u = -k|\nabla u|$

$$\Delta|\nabla u| = |k|^2|\nabla u| - \langle \operatorname{div} k, \nabla u\rangle$$

which finishes the proof. □

Next, let us introduce the notation $e_3 = \nu = \frac{\nabla u}{|\nabla u|}$. For a fixed level set Σ, we can express the level set metric by $dx_1^2 + dx_2^2$ which is possible since Σ is flat. We let $e_1 = \partial_{x_1}$, $e_2 = \partial_{x_2}$, and then we extend e_1, e_2 to the entire manifold such that $\{e_1, e_2, e_3\}$ forms an orthonormal frame. We use Greek letter α, β, γ to denote tangential vectors e_1, e_2, and Roman letters i, j, k, l to denote arbitrary vectors e_1, e_2, e_3, and as usual we employ Einstein's summation convention.

We define $\bar{R}_{ijkl} = R_{ijkl} + k_{il}k_{jk} - k_{ik}k_{jl}$ and say that (M, g, k) satisfies the Gauss and Codazzi equations if $\bar{R}_{ijkl} = 0$ and $\nabla_i k_{jk} - \nabla_j k_{ik} = 0$ for all i, j, k, l. Here we use the notation $R^l_{ijk} e_l = [\nabla_i, \nabla_j]e_k - \nabla_{[e_i, e_j]}e_k$ as well as $R_{ijkl} = \langle [\nabla_i, \nabla_j]e_k - \nabla_{[e_i, e_j]}e_k, e_l\rangle$. The reason why we are interested in the Gauss and Codazzi equations is the Lorentzian version of the fundamental theorem of hypersurfaces, also see [8, Corollary 7.3].

Proposition 3.5.2 (Fundamental theorem of hypersurfaces)**.** *Suppose* (M, g, k) *satisfies the Gauss and Codazzi equations, and assume that* M *is diffeomorphic to* $\mathbb{R}^3$*. Then* (M, g, k) *arises as a subset of Minkowski spacetime.*

We provide a proof of the fundamental theorem in Appendix B.5. In the next two lemma we demonstrate that the majority of the Gauss and Codazzi equations are already satisfied.

Lemma 3.5.3. *We have*

$$
\begin{aligned}
0 =& \nabla_1 k_{23} - \nabla_2 k_{13}, \\
0 =& \nabla_\alpha k_{\beta\beta} - \nabla_\beta k_{\alpha\beta}, \\
0 =& \nabla_\alpha k_{33} - \nabla_3 k_{\alpha 3}.
\end{aligned}
$$

Proof. The first identity follows from

$$\nabla_1 k_{23} - \nabla_2 k_{13} = - \nabla_1 \frac{\nabla^2_{23} u}{|\nabla u|} + \nabla_2 \frac{\nabla^2_{13} u}{|\nabla u|} = R_{2133} = 0.$$

Observe that $\mu |\nabla u| = -\langle J, \nabla u \rangle$ together with the DEC $\mu \geq |J|$ yields $J_\alpha = 0$. This implies

$$\nabla_\beta k_{\alpha\beta} - \nabla_\alpha k_{\beta\beta} + \nabla_3 k_{\alpha 3} - \nabla_\alpha k_{33} = 0.$$

Thus, we have

$$\nabla_3 k_{\alpha 3} - \nabla_\alpha k_{33} = -\nabla_3 \frac{\nabla^2_{\alpha 3} u}{|\nabla u|} + \nabla_\alpha \frac{\nabla^2_{33} u}{|\nabla u|} = R_{\alpha 333} = 0$$

which implies the last two identities. □

Lemma 3.5.4. *We have*

$$
\begin{aligned}
\bar{R}_{1212} =& 0, \\
\bar{R}_{\alpha\beta 3\alpha} =& 0, \\
\bar{R}_{\alpha 33\beta} =& \mathcal{A}_{\alpha\beta}.
\end{aligned}
$$

where $\mathcal{A}_{\alpha\beta} := \nabla_3 k_{\alpha\beta} - \nabla_\alpha k_{\beta 3}$.

Proof. Using the Gauss equations we obtain

$$R_{1212} = 2K + h_{11}h_{22} - h_{12}^2.$$

Thus, the first identity follows from $K = 0$ and $h = -k|_{T\Sigma}$. Next, we compute

$$\begin{aligned} R_{\alpha\beta 3\alpha} =& |\nabla u|^{-1}(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\nabla_\alpha u \\ =& -|\nabla u|^{-1}\nabla_\alpha(k_{\alpha\beta}|\nabla u|) + |\nabla u|^{-1}\nabla_\beta(k_{\alpha\alpha}|\nabla u|) \\ =& -\nabla_\alpha k_{\alpha\beta} + \nabla_\beta k_{\alpha\beta} + k_{\alpha 3}k_{\alpha\beta} - k_{\beta 3}k_{\alpha\alpha}. \end{aligned}$$

Using the spacetime Hessian equation $\nabla^2 u = -k|\nabla u|$, we obtain

$$\begin{aligned} \bar{R}_{\alpha\beta\alpha 3} &= R_{\alpha\beta\alpha 3} + k_{\alpha 3}k_{\beta\alpha} - k_{\alpha\alpha}k_{\beta 3} \\ &= \nabla_\alpha k_{\alpha\beta} - \nabla_\beta k_{\alpha\alpha} = 0, \end{aligned}$$

where the last equality follows from the previous lemma. Finally, the third identity follows in the same spirit as the second one. □

Next, we show that $\mathcal{A}_{\alpha\beta}$ is vanishing. This will be achieved by PDE methods in combination with the asymptotics of g, k.

Lemma 3.5.5. *On each level set, there exists a twice differentiable function F such that*

$$\nabla^\Sigma_{\alpha\beta} F = |\nabla u|^{-2}\mathcal{A}_{\alpha\beta}.$$

Here $\nabla^\Sigma_{\alpha\beta}$ denotes the leve-set Hessian.

For the proof of this lemma we need to additionally assume that $g \in C^3(M)$ and $k \in C^2(M)$. However, we provide an alternative approach to the spacetime PMT rigidity in Appendix B.6 which is based on a Killing development and the study of certain pp-wave spacetimes. This approach does not require such additional regularity of g and k and therefore establishes the case of equality of Theorem B in full generality.

Proof. We first show that $\partial_2(|\nabla u|^{-2}\mathcal{A}_{11}) = \partial_1(|\nabla u|^{-2}\mathcal{A}_{12})$ and $\partial_1(|\nabla u|^{-2}\mathcal{A}_{22}) = \partial_2(|\nabla u|^{-2}\mathcal{A}_{12})$. Since the level sets are flat, we can choose $\{e_1, e_2\}$ such that $\langle \nabla_{e_\alpha} e_\beta, e_\gamma \rangle = 0$. Because $\langle \nabla_{e_\alpha} e_3, e_\beta \rangle = -k_{\alpha\beta}$ and applying Lemma 3.5.3, we obtain

$$\begin{aligned}\partial_2\mathcal{A}_{11} =&\partial_2(\nabla_3 k_{11} - \nabla_1 k_{13})\\ =&\nabla_2(\nabla_3 k_{11} - \nabla_1 k_{13}) - k_2^\alpha \nabla_\alpha k_{11} + 2k_{21}\nabla_3 k_{31}\\ &- k_{21}\nabla_3 k_{13} - k_{21}\nabla_1 k_{33} + k_2^\alpha \nabla_1 k_{1\alpha}\\ =&\nabla_2(\nabla_3 k_{11} - \nabla_1 k_{13}).\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\partial_2\mathcal{A}_{11} - \partial_1\mathcal{A}_{12}\\ =&\nabla_2(\nabla_3 k_{11} - \nabla_1 k_{13}) - \nabla_1(\nabla_3 k_{12} - \nabla_2 k_{13})\\ =&\nabla_3\nabla_2 k_{11} - 2R_{231i}k_{1i} - \nabla_2\nabla_1 k_{13} - (\nabla_3\nabla_1 k_{12} - R^i_{131i}k_{i2} - R_{132i}k_{1i})\\ &+ (\nabla_2\nabla_1 k_{13} - R_{121i}k_{i3} - R_{123i}k_{1i})\\ =&\nabla_3\nabla_2 k_{11} - \nabla_3\nabla_1 k_{12} - R_{2312}k_{12} - R_{2313}k_{13} + R_{1312}k_{22}\\ &+ R_{1313}k_{23} - R_{1212}k_{23} - R_{1213}k_{33}\end{aligned}$$

Applying Lemma 3.5.4 to replace the curvature terms in the last two lanes, we obtain

$$\begin{aligned}&\partial_2\mathcal{A}_{11} - \partial_1\mathcal{A}_{12}\\ =&\nabla_3\nabla_2 k_{11} - \nabla_3\nabla_1 k_{12} - (k_{12}k_{23} - k_{22}k_{13})k_{12} - (k_{12}k_{33} - k_{23}k_{13} - \nabla_3 k_{12} + \nabla_1 k_{23})k_{13}\\ &+ (k_{11}k_{23} - k_{12}k_{13})k_{22} + (k_{11}k_{33} - k_{13}^2 - \nabla_3 k_{11} + \nabla_1 k_{13})k_{23}\\ &- (k_{11}k_{22} - k_{12}^2)k_{23} - (k_{11}k_{23} - k_{13}k_{12})k_{33}\\ =&\nabla_3\nabla_2 k_{11} - \nabla_3\nabla_1 k_{12} - (-\nabla_3 k_{12} + \nabla_1 k_{23})k_{13} + (-\nabla_3 k_{11} + \nabla_1 k_{13})k_{23},\end{aligned} \tag{3.10}$$

Due to the spacetime Hessian equation $\nabla^2 u = -k|\nabla u|$, we have $\langle \nabla_3 e_\alpha, e_3 \rangle = -\langle \nabla_3 e_3, e_\alpha \rangle = k_{\alpha 3}$. Combining this identity with Lemma 3.5.3, we deduce

$$\begin{aligned}
&\nabla_3\nabla_2 k_{11} - \nabla_3\nabla_1 k_{12} \\
=&\partial_3(\nabla_2 k_{11}) - \nabla_{\nabla_3 e_2} k_{11} - 2\nabla_2 k(\nabla_3 e_1, e_1) \\
&- \partial_3(\nabla_1 k_{12}) + \nabla_{\nabla_3 e_1} k_{12} + \nabla_1 k(\nabla_3 e_1, e_2) + \nabla_1 k(e_1, \nabla_3 e_2) \\
=& - \langle e_1, \nabla_3 e_2 \rangle \nabla_1 k_{11} - k_{23}\nabla_3 k_{11} - 2\langle \nabla_3 e_1, e_2\rangle \nabla_2 k_{21} - 2k_{13}\nabla_2 k_{31} + \langle e_2, \nabla_3 e_1 \rangle \nabla_2 k_{12} \\
&+ k_{13}\nabla_3 k_{12} + \langle e_2, \nabla_3 e_1 \rangle \nabla_1 k_{22} + k_{13}\nabla_1 k_{32} + \langle e_1, \nabla_3 e_2 \rangle \nabla_1 k_{11} + k_{23}\nabla_1 k_{13} \\
=&k_{23}(\nabla_1 k_{13} - \nabla_3 k_{11}) - k_{13}(\nabla_1 k_{32} - \nabla_3 k_{12}).
\end{aligned} \tag{3.11}$$

Here we also used that $\partial_3(\nabla_2 k_1 1 - \nabla_1 k_{12}) = 0$ by Lemma 3.5.3. Combing Equation (3.10) and (3.11) yields

$$\partial_2 \mathcal{A}_{11} - \partial_1 \mathcal{A}_{12} = 2\mathcal{A}_{12}k_{13} - 2\mathcal{A}_{11}k_{23}.$$

Moreover, we have $\partial_\alpha |\nabla u| = -k_{\alpha 3}|\nabla u|$ which implies

$$\begin{aligned}
&\partial_2(|\nabla u|^{-2}\mathcal{A}_{11}) - \partial_1(|\nabla u|^{-2}\mathcal{A}_{12}) \\
=&|\nabla u|^{-2}(\partial_2\mathcal{A}_{11} - \partial_1\mathcal{A}_{12}) + \mathcal{A}_{11}\partial_2|\nabla u|^{-2} - \mathcal{A}_{12}\partial_1|\nabla u|^{-2} \\
=&|\nabla u|^{-2}(2\mathcal{A}_{12}k_{13} - 2\mathcal{A}_{11}k_{23}) + 2\mathcal{A}_{11}|\nabla u|^{-2}k_{23} - 2\mathcal{A}_{12}|\nabla u|^{-2}k_{13} \\
=&0.
\end{aligned}$$

Therefore, $|\nabla u|^{-2}\mathcal{A}_{11}dx_1 + |\nabla u|^{-2}\mathcal{A}_{12}dx_2$ is closed, where dx_1 and dx_2 are the dual 1-forms of e_1 and e_2. Since the topology of a level set is trivial, there exists on each level set a function which we suggestively denote by F_1 such that $dF_1 = |\nabla u|^{-2}\mathcal{A}_{11}dx_1 + |\nabla u|^{-2}\mathcal{A}_{12}dx_2$. Replacing the roles of e_1 and e_2, there exists another function F_2 such that $dF_2 = |\nabla u|^{-2}\mathcal{A}_{12}dx_1 + |\nabla u|^{-2}\mathcal{A}_{22}dx_2$. Next, we compute

$$\begin{aligned} d(F_1 dx_1 + F_2 dx_2) =& \frac{\partial F_1}{\partial x_2} dx_2 \wedge dx_1 + \frac{\partial F_2}{\partial x_1} dx_1 \wedge dx_2 \\ =& (|\nabla u|^{-2}\mathcal{A}_{12} - |\nabla u|^{-2}\mathcal{A}_{12}) dx_2 \wedge dx_1 = 0. \end{aligned}$$

Thus there exists a function F with $dF = F_1 dx_1 + F_2 dx_2$. □

Lemma 3.5.6. *On each level set, F is a linear function with respect to x_1 and x_2, i.e. $\nabla^2_\Sigma F = 0$.*

Proof. First observe that F is superharmonic on each level set, i.e.

$$\Delta^\Sigma F \geq 0$$

which follows immediately from

$$\Delta^\Sigma F = |\nabla u|^{-2}(\mathcal{A}_{11} + \mathcal{A}_{22}) = -|\nabla u|^{-2} J_3 = |\nabla u|^{-2}\mu \geq 0.$$

Since $\partial^l k_{ij} = O(|x|^{-\tau-l-1})$, $l = 0, 1$, for some $\tau > \frac{1}{2}$, and $|\nabla u| = 1 + O(|x|^{-\tau})$, we obtain

$$F_{\alpha\beta} = \nabla^\Sigma_{\alpha\beta} F = |\nabla u|^{-2}(\nabla_3 k_{\alpha\beta} - \nabla_\alpha k_{\beta 3}) = O(|x|^{-\tau-2}).$$

Integrating $\nabla^2_\Sigma F$ twice over the level set Σ, we see that $F = L + B$, where L is a linear function with respect to $\{x_1, x_2\}$, and B is a bounded function. Combining this with our previous observation yields $\Delta^\Sigma B = \Delta^\Sigma F \geq 0$. Thus, B is constant in view of Liouville's theorem. □

Proof of the rigidity part of Theorem B. Since $\nabla^2_\Sigma F = 0$, (M, g, k) satisfies the Gauss and Codazzi equations which completes the proof in view of the Proposition 3.5.2. □

3.5.3 The PMT in the hyperbolic and other settings

Finally, let us comment on the adjustments needed for the asymptotically hyperbolic case [17]. In Theorem B the initial data set was assumed to be asymptotically flat which means that g approaches the Euclidean metric and infinity, and k goes to zero. Also Physically motivated are*asymptotically hyperboloidal initial data sets.* In this case g approaches the hyperbolic metric and k approaches g at infinity. The model case is the unit sphere $\{(x,y,z,t) \mid x^2+y^2+z^2-t^2=-1\}$ in Minkowski space. The induced metric g is the hyperbolic metric

$$g = \frac{1}{1+r^2}dr^2 + r^2 g_{\mathbb{S}^2}$$

where $r = \sqrt{x^2+y^2+z^2}$, and $g_{\mathbb{S}^2}$ is the round metric on $\mathbb{S}^2$. Moreover, the second fundamental form k equals g, and $\mathrm{R}(g) = -6$. In general, one assumes again the dominant energy condition $\mu \geq |J|$, and at ∞ one can associate a mass to (M,g,k) which is well-defined. There is also a positive mass theorem in this setting with a similar statement as in Theorem B. This has been first established for special asymptotics by X. Wang using spinors [117] and later by P. Chrusciel, M. Herzlich and E. Delay [42, 41]. There are are also proofs by A. Sakovich using the Jang equation, M. Anderson, M. Cai and G. Galloway using isoperimetric surfaces [4] and the rigidity in case of X. Wang's asymptotics has been addressed by L.H. Huang, H.C. Jang and D. Martin in [76]. The most general result concerning the hyperbolic PMT (in Dimension 3) is obtained in a joint paper with H. Bray, D. Kazaras, M. Khuri and Y. Zhang [17].

The same motivation for studying the spacetime Laplace equation via the intersection of null planes, cf. Figure 3.1, also holds true in this case. Moreover, the main integral formula

$$\mathrm{div}(\nabla|\nabla u| + k(\nabla u,\cdot)) + K|\nabla u| = \frac{|\bar{\nabla}^2 u|^2}{2|\nabla u|} + \mu|\nabla u| + \langle J, \nabla u\rangle.$$

can still be applied. However, there are several major technical challenges to overcome which causes the proof to be significantly longer and more complicated than the one in the asymptotically flat setting [64, 72].

- While the underlying PDE $\Delta u = -\operatorname{tr}_g k|\nabla u|$ are identical, $\operatorname{tr}_g k$ approaches $tr_g g = 3$ at infinity in the hyperbolic setting (compared to 0 in the asymptotically flat setting). This makes the non-linearity significantly more severe.
- In the asymptotically flat setting, $|\nabla u|$ goes to 1 at ∞, but in the hyperbolic setting $|\nabla u|$ goes to zero in some and to ∞ in other directions at ∞. This lack of spherical symmetry at ∞ leads to technical difficulties.
- To obtain the mass at ∞ the construction of an *interpolation region* is required. We refer to [16, 17] for details.

We have also proved the PMT with charge for manifolds with electrical field E using the charged Laplace equation $\Delta u = \langle \nabla u, E \rangle$ [16], and for manifolds with boundary [71] and refer to the respective papers for details.

3.6 Existence and regularity

Let (M, g, k) be an asymptotically flat initial data set of order τ with (possibly empty) smooth boundary ∂M. The purpose of this section is to establish the appropriate existence, uniqueness, and asymptotic properties of spacetime harmonic functions. The proof proceeds in four steps:

- First, we solve the *linear* equation $\Delta v = -\operatorname{tr}_g k$.
- Second, we solve on compact domains B_r the spacetime Laplace equation $\Delta u^r = -\operatorname{tr}_g k|\nabla u^r|$ with boundary data v via fixedpoint methods.

- Next, we use barriers to obtain uniform estimates for u^r.
- Finally, we use the uniform estimates to pass to a limit $u^r \to u$.

For simplicity of discussion, it will be assumed here that M possesses a single end, although the final result stated at the end of the section will be given in full generality. Let $a_i x^i$ be a linear function of the asymptotic coordinates in the end M_{end}, with $\sum_i a_i^2 = 1$, and let $h \in C^\infty(\partial M)$. By slightly generalizing [9, Theorem 3.1] we may solve the asymptotically linear Dirichlet problem

$$\Delta v = -\operatorname{tr}_g k \qquad \text{on} \qquad M, \tag{3.12}$$

$$v = 0 \quad \text{on} \quad \partial M, \qquad\qquad v = a_i x^i + O_2(r^{1-\tau}) \quad \text{as} \quad r \to \infty, \tag{3.13}$$

where $r = |x|$, q is as in (3.2), and O_2 indicates in the usual way additional fall-off for each derivative taken up to order 2. Consider now the corresponding problem for the spacetime harmonic function equation

$$\Delta u + \operatorname{tr}_g k |\nabla u| = 0 \qquad \text{on} \qquad M, \tag{3.14}$$

$$u = h \quad \text{on} \quad \partial M, \qquad\qquad u = v + O_2(r^{-\beta}) \quad \text{as} \quad r \to \infty, \tag{3.15}$$

where $\beta \in (0,1)$. As mentioned before, the strategy will be to first solve for u on a sequence of compact domains exhausting M, use a barrier in the asymptotic end to obtain uniform estimates, and then find a subsequence that converges to the desired solution.

3.6.1 Solutions on compact exhausting domains

Let $S_r \subset M_{end}$ be a coordinate sphere in the asymptotic end, and let M_r denote the compact component of $M \setminus S_r$ having boundary $\partial M_r = \partial M \cup S_r$. Consider now the preliminary Dirichlet problem

$$\Delta u^r + \operatorname{tr}_g k|\nabla u^r| = 0 \qquad \text{on} \qquad M_r, \tag{3.16}$$

$$u^r = h \quad \text{on} \quad \partial M, \qquad\qquad u^r = v \quad \text{on} \quad S_r. \tag{3.17}$$

For this boundary value problem we will use the Leray-Schauder fixed point theorem [53, Theorem 11.3].

Theorem 3.6.1. *Let $\mathcal{B}$ be a Banach space and $\mathcal{F} : \mathcal{B}\times[0,1] \to \mathcal{B}$ a compact mapping with $\mathcal{F}(b,0) = 0$ for all $b \in \mathcal{B}$. If there is a constant c, such that any solution $(b,\sigma) \in \mathcal{B}\times[0,1]$ of $b = \mathcal{F}(b,\sigma)$ satisfies the a priori inequality $\| b \| \le c$, then there is a fixed point at $\sigma = 1$. That is, there exists $b_1 \in \mathcal{B}$ with $b_1 = \mathcal{F}(b_1, 1)$.*

To set up the fixed point method write $u^r = \tilde{v} + w^r$ and $f = \Delta\tilde{v} + \operatorname{tr}_g k|\nabla\tilde{v}|$, where $\tilde{v} = v + v_0$ with $v_0 \in C^\infty(M)$ a fixed function satisfying $v_0 = h$ on ∂M and $v_0 \equiv 0$ on M_{end}. Then boundary value problem (3.16), (3.17) becomes

$$\begin{aligned}\Delta w^r &= -\operatorname{tr}_g k\,(|\nabla u^r| - |\nabla\tilde{v}|) - f \\ &= -\operatorname{tr}_g k\left(\frac{\nabla(w^r + 2\tilde{v})}{|\nabla(w^r+\tilde{v})| + |\nabla\tilde{v}|}\right)\cdot\nabla w^r - f \quad \text{on} \quad M_r,\end{aligned} \tag{3.18}$$

$$w^r = 0 \quad \text{on} \quad \partial M_r. \tag{3.19}$$

Let $C_0^{2,\alpha}(M_r)$ denote the space of $C^{2,\alpha}(M_r)$ functions which vanish on the boundary, and observe that $\Delta^{-1} : C_0^{2,\alpha}(M_r) \to C^{0,\alpha}(M_r)$ is an isomorphism. Now set

$$\begin{aligned}\mathcal{F}(w,\sigma) =&\sigma\Delta^{-1}\left[-\operatorname{tr}_g k\left(\frac{\nabla(w+2\tilde{v})}{|\nabla(w+\tilde{v})| + |\nabla\tilde{v}|}\right)\cdot\nabla w - f\right] \\ =:&\sigma\Delta^{-1}F(w), \qquad w \in C_0^{1,\alpha}(M_r).\end{aligned} \tag{3.20}$$

Observe that $F(w) \in C^{0,\alpha}(M_r)$ and hence $\mathcal{F}(w,\sigma) \in C_0^{2,\alpha}(M_r)$. We choose $\mathcal{B} = C_0^{1,\alpha}(M_r)$ and note that the composition

$$C_0^{1,\alpha}(M_r) \xrightarrow{F} C^{0,\alpha}(M_r) \xrightarrow{\Delta^{-1}} C_0^{2,\alpha}(M_r) \xrightarrow{\iota} C_0^{1,\alpha}(M_r),$$

yields a compact map $\mathcal{F} : \mathcal{B} \times [0,1] \to \mathcal{B}$ since the first two pieces F and Δ^{-1} are bounded while the inclusion ι is compact. Furthermore, finding a fixed point $w^r = \mathcal{F}(w^r, 1)$ is equivalent to solving (3.18), (3.19) in $C_0^{2,\alpha}(M_r)$ by elliptic regularity. Then $u^r = \tilde{v} + w^r$ is the desired solution of (3.16), (3.17).

It remains to establish the a priori estimate $|w_\sigma|_{C^{1,\alpha}(M_r)} \leq c$, independent of σ, for a fixed point $w_\sigma = \mathcal{F}(w_\sigma, \sigma)$. Such a fixed point satisfies (3.18), (3.19) with $\mathrm{tr}_g\, k$ and f replaced by $\sigma\, \mathrm{tr}_g\, k$ and σf. This may be viewed as a linear equation with coefficients that depend on the solution. However, since the coefficients remain uniformly bounded independent of the solution, L^p estimates for linear elliptic equations may be applied to obtain

$$\| w_\sigma \|_{W^{2,p}(M_r)} \leq C \left(\| f \|_{L^p(M_r)} + \| w_\sigma \|_{L^p(M_r)} \right), \tag{3.21}$$

where $W^{l,p}$ denotes the Sobolev space with l weak derivatives in L^p, $p > 1$. Moreover, since the coefficient of the zeroth order term in (3.18) vanishes, the maximum principle is valid and leads to a C^0 bound for w_σ which in turn gives a bound for $\| w_\sigma \|_{L^p(M_r)}$. Hence we obtain the a priori estimate

$$\| w_\sigma \|_{W^{2,p}(M_r)} \leq C, \tag{3.22}$$

independent of σ where C may change its value from line to line. According to the Sobolev embedding $W^{2,p}(M_r) \hookrightarrow C^{1,\alpha}(M_r)$ for p sufficiently large, we obtain

$$|w_\sigma|_{C^{1,\alpha}(M_r)} \leq C,$$

independent of σ. The Leray-Schauder theorem may now be applied to obtain a fixed point at $\sigma = 1$.

3.6.2 Barriers

Rotationally symmetric asymptotic barrier functions will be constructed to obtain uniform bounds on the solutions w^r of (3.18), (3.19) independent of r. To this end, in the asymptotically flat region set

$$\overline{w}(r) = \lambda r^{-\beta}, \qquad \overline{w}'(r) = -\lambda\beta r^{-1-\beta}, \qquad \overline{w}''(r) = \lambda\beta(1+\beta)r^{-2-\beta},$$

for $\beta \in (0,1)$ and where $\lambda > 0$ is a constant to be chosen sufficiently large. Using the level sets of r, the metric may be expressed as

$$g = |\partial_r|^2 dr^2 + g_r = |\nabla r|^{-2} dr^2 + g_{\mathbb{S}^2_r},$$

where $g_{\mathbb{S}^2_r}$ is the induced metric on the coordinate spheres $\mathbb{S}^2_r$. If υ denotes the unit outer normal to the coordinate spheres then

$$\upsilon = |\partial_r|^{-1}\partial_r = |\nabla r|\partial_r, \qquad |\nabla r|^2 = g^{ij}\partial_i r \partial_j r = \frac{g^{ij}x^i x^j}{r^2} = 1 + O_2(r^{-q}),$$

and the Laplacian becomes

$$\Delta\overline{w} = \nabla^2_\upsilon \overline{w} + H_{S_r}\upsilon(\overline{w}),$$

where H_{S_r} denotes mean curvature. Observe that

$$\nabla^2_\upsilon \overline{w} = \upsilon^i \upsilon^j \nabla_{ij}\overline{w} = |\nabla r|^2 \nabla_{rr}\overline{w} = |\nabla r|^2 \left(\overline{w}'' - \Gamma^r_{rr}\overline{w}'\right),$$

and

$$\Gamma^r_{rr} = \frac{1}{2} g^{rr}\partial_r g_{rr} = -\partial_r \log|\nabla r|,$$

so that

$$\begin{aligned}\Delta\overline{w} =& |\nabla r|^2 \overline{w}'' + |\nabla r|\left(H_{S_r} + \partial_r|\nabla r|\right)\overline{w}' \\ =& \left(1 + O(r^{-q})\right)\left(\overline{w}'' + \frac{2}{r}\overline{w}' + O(r^{-q-1})\overline{w}'\right) \\ =& -\lambda\beta(1-\beta)r^{-2-\beta}\left(1 + O(r^{-q})\right).\end{aligned}$$

Furthermore

$$
\begin{aligned}
|\mathcal{K}(w^r)\cdot\nabla\overline{w}| :=& \left|\operatorname{tr}_g k\left(\frac{\nabla(w^r+2\tilde{v})}{|\nabla(w^r+\tilde{v})|+|\nabla\tilde{v}|}\right)\cdot\nabla\overline{w}\right| \\
\leq& Cr^{-q-1}|\overline{w}'| = C\lambda\beta r^{-q-2-\beta}.
\end{aligned}
$$

It follows that

$$L\overline{w} := \Delta\overline{w} + \mathcal{K}(w^r)\cdot\nabla\overline{w} = -\lambda\beta(1-\beta)r^{-2-\beta}\left(1+O(r^{-q})\right).$$

Next, comes the estimate which justifies the use of v as Dirichlet boundary data for u^r (over the naive use of the boundary condition $u^r = x$ on ∂M_r). Consider now the asymptotics for f. According to (3.12), (3.13) we have

$$|f| = |\Delta v + \operatorname{tr}_g k|\nabla v|| = |\operatorname{tr}_g k||1-|\nabla v|| \leq C_1 r^{-2q-1} = C_1 r^{-2-\beta},$$

by setting $\beta = 2q-1 > 0$. Therefore, given a large radius r_0, it holds that

$$L\overline{w} \leq -f \qquad \text{for} \qquad r > r_0 \tag{3.23}$$

if λ is sufficiently large. Hence, $\overline{w}$ is a super-solution of (3.18) on $M_r \setminus M_{r_0}$.

In order to obtain a global barrier let $\tilde{w}^{r_0}$ solve (3.18), (3.19) on M_{r_0} with $\mathcal{K}(\tilde{w}^{r_0})$ replaced by $\mathcal{K}(w^r)$, noting that this is a linear boundary value problem. Next define

$$\hat{w}_\lambda = \begin{cases} \tilde{w} := \tilde{w}^{r_0} + \lambda r_0^{-\beta} & \text{on} \quad M_{r_0}, \\ \overline{w} & \text{on} \quad M_r \setminus M_{r_0}. \end{cases}$$

This function is smooth everywhere, except at S_{r_0} where it is continuous, and is a super-solution for (3.18) on M_{r_0} and $M_r \setminus M_{r_0}$ separately. Furthermore we have

$$\partial_r\tilde{w} > \partial_r\overline{w} \qquad \text{at} \qquad S_{r_0},$$

if λ is sufficiently large (independent of r), and this allows for an application of the weak maximum principle. To see this, let $\phi \in C_c^\infty(M_r)$ be a nonnegative test function and observe that

$$\begin{aligned}
0 &= -\int_{M_{r_0}} \phi L(\tilde{w} - w^r) dV \\
&= \int_{M_{r_0}} \left(\nabla\phi \cdot \nabla(\tilde{w} - w^r) - \phi\mathcal{K}(w^r) \cdot \nabla(\tilde{w} - w^r)\right) dV \\
&\quad - \int_{\mathbb{S}^2_{r_0}} \phi \partial_r(\tilde{w} - w^r) dA,
\end{aligned}$$

and

$$\begin{aligned}
0 &\leq -\int_{M_r \setminus M_{r_0}} \phi L(\overline{w} - w^r) dV \\
&= \int_{M_r \setminus M_{r_0}} \left(\nabla\phi \cdot \nabla(\overline{w} - w^r) - \phi\mathcal{K}(w^r) \cdot \nabla(\overline{w} - w^r)\right) dV \\
&\quad + \int_{\mathbb{S}^2_{r_0}} \phi \partial_r(\overline{w} - w^r) dA,
\end{aligned}$$

so that upon adding these two inequalities

$$\begin{aligned}
&\int_{M_r} \left(\nabla\phi \cdot \nabla(\hat{w}_\lambda - w^r) - \phi\mathcal{K}(w^r) \cdot \nabla(\hat{w}_\lambda - w^r)\right) dV \\
\geq &\int_{\mathbb{S}^2_{r_0}} \phi(\partial_r \tilde{w} - \partial_r \overline{w}) dA \geq 0.
\end{aligned}$$

Hence, according to [53, Theorem 8.1] the weak maximum principle yields

$$\inf_{M_r}(\hat{w}_\lambda - w^r) \geq \inf_{\partial M_r}(\hat{w}_\lambda - w^r) \geq 0.$$

A similar argument with $\hat{w}_{-\lambda}$ yields a lower bound, and therefore

$$\hat{w}_{-\lambda} < w^r < \hat{w}_\lambda \quad \text{on} \quad M_r.$$

Consequently we obtain a global C^0 estimate for w^r independent of r.

3.6.3 The global existence result

Here we will show that w^r subconverges on compact subsets as $r \to \infty$ to a $C^{2,\alpha}$ solution of (3.18) on all of M. In the previous subsection a uniform C^0 estimate was achieved. Consider (3.18) as a linear equation with coefficients depending on w^r but which are uniformly bounded, and apply the local L^p estimates to find

$$\| w^r \|_{W^{2,p}(\Omega)} \leq C \left(\| f \|_{L^p(\Omega')} + \| w^r \|_{L^p(\Omega')} \right),$$

where $\Omega \subset\subset \Omega'$ are any fixed compact subsets of M_r and C is independent of r. The uniform C^0 bound implies a uniform L^p bound in Ω', and therefore

$$\| w^r \|_{W^{2,p}(\Omega)} \leq C'.$$

By Sobolev embedding this yields a uniform $C^{1,\alpha}(\Omega)$ bound, so that in particular the right-hand side of (3.18) is controlled in $C^{0,\alpha}(\Omega)$. Now applying the local Schauder estimates we obtain the desired $C^{2,\alpha}$ estimate on a subset of Ω, for any $\alpha \in (0,1)$. It follows then by a diagonal argument that there is a subsequence w^{r_i} converging in $C^{2,\alpha}$ on any compact subset of M to a smooth function w which solves

$$\Delta w = -\operatorname{tr}_g k \left(\frac{\nabla(w + 2\tilde{v})}{|\nabla(w + \tilde{v})| + |\nabla \tilde{v}|} \right) \cdot \nabla w - f \quad \text{on} \quad M, \tag{3.24}$$

$$w = 0 \quad \text{on} \quad \partial M, \qquad -\overline{w} < w < \overline{w} \quad \text{on} \quad M \setminus M_{r_0}. \tag{3.25}$$

Finally, by setting $u = \tilde{v} + w$ we obtain the desired solution of (3.14), (3.15).

As mentioned at the start of the section, this global existence result extends in a straightforward manner to the case of multiple asymptotically flat ends M^{ℓ}_{end}, $\ell = 1, \ldots, \ell_0$. For this situation let $a^{\ell}_i x^i$ be a linear function of the asymptotic coordinates in the end M^{ℓ}_{end}, with $\sum_i (a^{\ell}_i)^2 = 1$, and let $h \in C^{\infty}(\partial M)$. Then the background function satisfies

$$\Delta v = -\operatorname{tr}_g k \qquad \text{on} \qquad M, \tag{3.26}$$

$$v = 0 \quad \text{on} \quad \partial M,$$

$$v = a_i^\ell x^i + O_2(r^{1-q}) \quad \text{as} \quad r \to \infty \quad \text{in} \quad M_{end}^\ell, \quad \ell = 1, \dots, \ell_0. \tag{3.27}$$

Theorem 3.6.2. *Suppose that (M, g, k) is a smooth asymptotically flat initial data set with (possibly empty) boundary ∂M, and $h \in C^\infty(\partial M)$. Let v be a solution of* (3.26) *and* (3.27)*. Then for each $\alpha \in (0,1)$ there exists a solution $u \in C^{2,\alpha}(M)$ of the spacetime harmonic function equation*

$$\Delta u + \mathrm{tr}_g\, k |\nabla u| = 0 \qquad \textit{on} \qquad M, \tag{3.28}$$

such that

$$u = h \quad \textit{on} \quad \partial M, \qquad\qquad u = v + O_2(r^{1-2q}) \quad \textit{as} \quad r \to \infty. \tag{3.29}$$

The solution u is unique among those which satisfy (3.29)*.*

Proof. The existence portion was proven in the discussion above, while the uniqueness follows from the maximum principle in the same manner as the barrier argument at the end of Section 3.6.2. Lastly, the decay of derivatives in the asymptotic ends may be established analogously to [109, Proposition 3]. □

Chapter 4

Monotonicity of the Hawking energy

Harmonic functions are not the only tool to study asymptotically flat Riemannian manifolds (M, g). The perhaps other most important tools are minimal surfaces, spinors, H. Bray's conformal flow and inverse mean curvature flow (IMCF). However, in general an isolated gravitational system is modeled by an initial data set (M, g, k) where k is non-vanishing. This raises the question what the corresponding spacetime (i.e. $k \neq 0$) versions of the above techniques are. We have already seen in the previous section the success of spacetime harmonic functions to study initial data sets, and spinors generalize without major adjustments to the spacetime case [118]. There are also 'spacetime' minimal surfaces called MOTS and MITS (marginally outer and inner trapped surfaces) which have found have been used in [47] to prove the spacetime PMT, also see the survey [6].

IMCF and the conformal flow have been used to proof the Riemannian ($k = 0$) *Penrose inequality* which is an important geometric statement related to *Cosmic Censorship* which will be explained in more detail in Section 4.1 below. Interestingly, more elementary techniques such using harmonic functions, spinors or minimal surfaces are insufficient to prove the Riemannian Penrose inequality. There has been a long history of attempts to generalize IMCF and Bray's conformal flow to the spacetime setting [19, 15, 51, 80, 98], but all of them have been so far without success of proving the spacetime ($k \neq 0$) Penrose conjecture which is open since 1973 [101].

The goal of this section is to introduce a new notion of spacetime IMCF which is based on double null foliations.

4.1 The Penrose conjecture

As mentioned in the previous chapter, General Relativity (GR) is concerned with the study of Lorentzian manifolds $(\overline{M}^4, \overline{g})$ satisfying the Einstein equations $\overline{\mathrm{Ric}} - \frac{1}{2}\overline{g} = 8\pi T$ where $\overline{\mathrm{Ric}}$ is the Ricci curvature, $\overline{R}$ is the scalar curvature of $\overline{g}$, and T is the stress-energy-momentum tensor. For instance, Minkowski space $\mathbb{R}^{3,1}$ with metric

$$\bar{g} = -dt^2 + dx^2 + dy^2 + dz^2,$$

and the Schwarzschild spacetime with metric

$$\bar{g} = -(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})^{-1}dr^2 + r^2 g_{\mathbb{S}^2}$$

are examples of such Lorentzian manifolds with $T = 0$. As shown by Y. Choquet-Bruhat [33], the Einstein equations can also be understood as a Cauchy problem for a system of hyperbolic PDE. We refer to the books of D. Lee [90] and R. Wald [116] for a detailed introduction to this topic.

An interesting feature of GR is the existence of singularities[1] which can arise even in elementary examples such as the Schwarzschild spacetime above. In Schwarzschild the singularity is hidden behind the event horizon and it is believed that this is also generically the case[2] which is known as the *Cosmic Censorship Conjecture*.

However, to understand whether a singularity is located within the black hole region, the entire time evolution of the spacetime has to be known. This is a notoriously difficult task, and even in the simplest cases such as for perturbations of slices of Minkowski space, this requires a substantial amount of analysis [37].

[1]The precise definition of a singularity is a subtle issue and we refer to [116] for details.

[2]The additional assumption of genericity is necessary as Christodoulou demonstrated in [35, 36].

To understand Cosmic Censorship, R. Penrose proposed in 1973 a test which relates the conjecture to a more feasible geometric problem which is depicted in Figure 4.1 below.

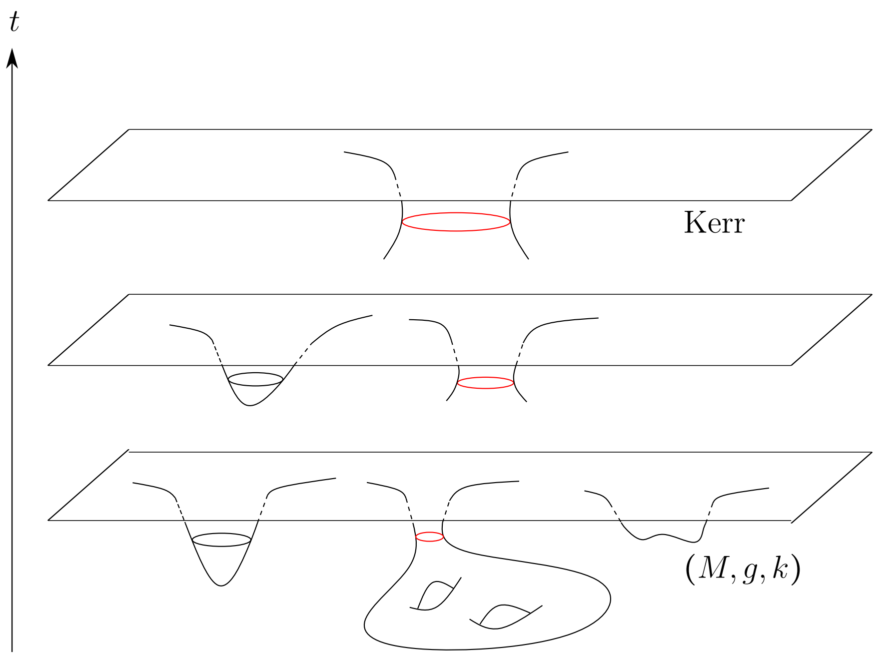


Figure 4.1: Penrose's heuristic argument

Given an initial data set (M, g, k), it is believed that going forward in time, on all matter either falls into the black hole or radiates away to ∞. Moreover, the final state of (M, g, k) is described by a static solution to the Einstein equations which is a Kerr solution, i.e. a rotating black hole. This is known as the *Final State conjecture*. For any slice in Kerr, it is known that $|\Sigma| \leq 16\pi\mathfrak{m}^2$ where $\mathfrak{m}$ is the ADM mass of the slice and Σ is the surface which is formed by the intersection of the black hole's event horizon. In fact, for Schwarzschild (i.e. the Kerr solution with zero angular momentum) we have $|\Sigma| = 16\pi\mathfrak{m}^2$. Next, we would like to trace back this inequality in time. Since matter is radiating away to ∞, the mass decreases, and Hawking's area theorem [60, 116] states that the area of Σ is increasing. Hence, we have $\mathfrak{m} \geq \sqrt{\frac{|\Sigma|}{16\pi}}$ on (M, g, k). To locate Σ within (M, g, k) the Cosmic Censorship is used: The incompleteness theorem of R. Penrose states that each apparent horizon (i.e. a surface with $\theta_+ = 0$) leads to a singularity in the future. Hence, $|\Sigma|$ can be estimated by the minimal area enclosure of an apparent horizon (highlighted in red).

Hence, we are led with the help of Cosmic Censorship to the following geometric conjecture which is known as *Penrose conjecture*.

Conjecture 4.1.1. *Let (M, g, k) be an initial data set satisfying the DEC. Let Σ_0 be a MOTS in (M, g, k), and let Σ be the minimal area enclosure of Σ_0. Then the mass $\mathfrak{m} = \sqrt{E^2 - |P|^2}$ of (M, g, k) is bounded from below by*

$$\mathfrak{m} \geq \sqrt{\frac{|\Sigma|}{16\pi}}.$$

Moreover, we have equality if and only if (M, g, k) is a slice in Schwarzschild spacetime.

Here a marginally outer trapped surfaces (MOTS) is a surface Σ satisfying $\theta_+ = 0$ and models an apparent horizons. Moreover, we recall the definitions of ADM energy and momentum of (M, g, k)

$$E = \lim_{r\to\infty} \frac{1}{16\pi} \int_{S_r} \sum_i (g_{ij,i} - g_{ii,j})\, \upsilon^j dA, \qquad P_i = \lim_{r\to\infty} \frac{1}{8\pi} \int_{S_r} (k_{ij} - (\mathrm{tr}_g\, k) g_{ij})\, \upsilon^j dA.$$

A counter example to the Penrose conjecture would pose a serious challenge to the Cosmic Censorship Conjecture which is considered to be the weakest link the above heuristic argument. Besides its physical significance, the Penrose conjecture also presents a strengthening of the famous positive mass theorem [2, 18, 46, 47, 64, 79, 92, 96, 108, 109]. By time-reversal, i.e. by replacing k with $-k$, one also expects Conjecture 4.1.1 hold also for marginally inner trapped surfaces (MITS), i.e surfaces satisfying $\theta_- = 0$.

The conjecture has been established in the case $k = 0$ by G. Huisken and T. Ilmanen [79] (for connected horizons), and by H. Bray [13] (for arbitrary horizons). H. Bray's proof employs the conformal flow and has also been generalized up to dimension 7 by H. Bray and D. Lee in [21] and to the electrostatic setting by M. Khuri, G. Weinstein and S. Yamada in [86].

In the general case $k \neq 0$ the conjecture is wild open outside spherical symmetry [19, 20, 61, 62, 82, 94] and H. Roesch' result on certain null cones [104]. In the pioneering work [19] H. Bray and M. Khuri proposed a method to couple IMCF and Jang's equation to solve the conjecture. This leads to a complicated system of PDE which (if it can be solved) implies the Penrose conjecture for initial data sets which are asymptotic to the Riemannian Schwarzschild manifold. In fact, this system would even imply the Penrose conjecture for *generalized horizons*, i.e. surfaces satisfying $\theta_+\theta_- = 0$. Thus, there have to arise some complications in the existence theory in view of A. Carrasco and M. Mars' counter example [24]. For more information we refer to the survey [95] by M. Mars and the references therein.

We remark that in the statement of the Penrose inequality it is necessary to consider the minimal area enclosure Σ instead of the MOTS Σ_0. It is easy to construct counterexamples to $m \geq \sqrt{\frac{|\Sigma_0|}{16\pi}}$, see for instance Figure 1 in [79], and even the assumption of Σ_0 being an outermost MOTS is insufficient as demonstrated by I. Ben-Dov in [11].

One difficulty most approaches towards the Penrose conjecture face, is to solve certain PDE. For instance, P. Jang and R. Wald already showed in [83] that R. Geroch's monotonicity formula [52] implies the Riemannian Penrose inequality if an existence theory for IMCF can be established. This has also been observed in the spacetime setting for *Inverse Mean Curvature Vector Flow* by J. Frauendiener [51]. In the Riemannian case this has been resolved in [1, 79, 97], but the spacetime case this is still completely open. Similarly, the pioneering approach by H. Bray and M. Khuri [19] did not yield a proof of the Penrose conjecture due to the difficulties of solving the underlying PDE systems outside spherical symmetry.

Our notion of spacetime IMCF has a comparatively simple PDE and we will discuss existence results in Section 4.5.

4.2 The Riemannian Penrose inequality

To motivate our definition of spacetime IMCF, we begin with recalling the two proofs of the Riemannian Penrose inequality, i.e. Conjecture 4.1.1 for $k = 0$.

Theorem 4.2.1 (Riemannian Penrose inequality)**.** *Let* (M^3, g) *be an asymptotically flat manifold with non-negative scalar curvature* $\mathrm{R} \geq 0$ *and outermost minimal surface* Σ*. Then*

$$\mathfrak{m} \geq \sqrt{\frac{|\Sigma|}{16\pi}}.$$

As mentioned in the previous section, this theorem has been proven by G. Huisken and T. Ilmanen [79] in case Σ is connected using IMCF and by H. Bray [13] in the general case using the conformal flow.

4.2.1 Bray's conformal flow

Starting with an asymptotically flat initial data set (M^3, g), the Einstein equations are expected to deform the initial data set to a slice in Schwarzschild (or more generally, Kerr) according to the final state conjecture. In this process, the area of the horizon Σ increases (Hawking's area theorem) and and the mass decreases. However, the mass and area monotonicities aren't established fully rigorously, and our understanding of the

Einstein equations is insufficient (by a large amount) to verify the final state conjecture. H. Bray discovered a new flow, the so-called *conformal flow*, which possesses similar properties but has a much simpler existence theory. The area of the outermost minimal surface stays constant while the mass is non-decreasing[3], and (M^3, g) is converging to the $(t = 0)$-slice of Schwarzschild. Here, the deformation of (M^3, g) is achieved by conformal deformations of the metric g. We refer to H. Bray's pioneering work [13] for more details.

4.2.2 Huisken-Ilmanen's weak inverse mean curvature flow

Given a surface Σ, the Hawking mass $\mathfrak{m}_h(\Sigma)$ is defined via

$$\mathfrak{m}_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_\Sigma H^2 d\mu\right)$$

where H is the mean curvature of Σ. Since the outermost minimal surface has zero mean curvature, we have

$$\mathfrak{m}_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}}$$

Now letting $\Sigma = \Sigma_0$ evolve by IMCF, i.e. the (∞-pointing) normal speed is given by $\frac{1}{H}$, one obtains a family of surfaces $\{\Sigma_t\}$ foliating (M^3, g). More precisely, Huisken-Ilmanen introduced a weak notion of IMCF which allows the possibility jumps. An easy computation which will be carried out below and which goes back to R. Geroch, P. Jang and R. Wald in [52, 83] yields that $\mathfrak{m}_H(\Sigma_t)$ is monotonically non-decreasing along the flow in case $\mathrm{R}(g) \geq 0$ and Σ is connected. Moreover, $\mathfrak{m}_H(\Sigma_t) \to \mathfrak{m}$ at ∞, and the result follows.

While there has not been much progress made on generalizing the conformal flow to the $k \neq 0$ setting, there have been several proposals for IMCF in this case. This includes

[3] This follows from H. Bray's famous *mass-capacity inequality*.

the uniformly area expanding flow by H. Bray, S. Hayard, M. Mars and W. Simon [15], H. Bray's and M. Khuri's Jang IMCF [19], J. Frauendiener inverse mean curvature vector flow [51], and K. Moore's [98] and G. Huisken's and M. Wolf's null IMCF [80]. However, the existence theory for the flows in [19, 15, 51] appears to be out of reach, while for the flows in [98, 80] there appears to be no analogue to the Hawking mass monotonicity. In this manuscript we suggest a new approach to generalize IMCF to the spacetime setting and introduce systems of PDE which model double null foliations. This leads simultaneously to a generalization of the Hawking mass monotonicity, cf. Theorem C, and to existence results outside spherical symmetry, cf. Theorem 4.5.1.

4.3 Monotonicity formula vs integral formula

As before, let (M^3, g) be an asymptotically flat complete manifold with non-negative scalar curvature R, and let Σ_0 be the outermost minimal surface. We begin with computing the aforementioned evolution of the Hawking mass:

$$\begin{aligned}16\pi\partial_t\mathfrak{m}_H(\Sigma_t) =&16\pi\partial_t\left[\sqrt{\frac{|\Sigma|}{16\pi}}\left(1-\frac{1}{16\pi}\int_\Sigma H^2 dA\right)\right]\\ =&8\pi\mathfrak{m}_H(\Sigma_t)-\sqrt{\frac{|\Sigma|}{16\pi}}\int_{\Sigma_t}\left[H^2-2H\left(\Delta^\Sigma\frac{1}{H}+|A|^2\frac{1}{H}+\mathrm{Ric}(\nu,\nu)\frac{1}{H}\right)\right]dA.\end{aligned}$$

Here ν is the ∞-pointing unit normal to Σ_t and A is its second fundamental form. Using the contracted Gauss equations and integrating by parts yield

$$16\pi\partial_t\mathfrak{m}_H(\Sigma_t) =8\pi\mathfrak{m}_H(\Sigma_t)-\sqrt{\frac{|\Sigma|}{16\pi}}\int_{\Sigma_t}\left(-\frac{|\nabla H|^2}{H^2}-|A|^2-\mathrm{R}+2K\right)dA$$

where K is the Gaussian curvature of Σ_t. Using Gauss-Bonnet's theorem, we obtain

$$16\pi\partial_t\mathfrak{m}_H(\Sigma_t) = \int_{\Sigma_t}\left(\frac{|\nabla H|^2}{H^2}+|A|^2-\frac{1}{2}H^2+R\right)d\mu\geq 0.$$

Besides this *monotonicity formula*, there is also an equivalent *integral formula* which we will discuss next.

An important ingredient in G. Huisken and T. Ilmanen's proof of the Riemannian Penrose inequality is to recognize that there is a level-set formulation of IMCF for which one can find weak solutions. More precisely, by defining the function U via $\Sigma_t = \partial\{x \in M : U(x) < t\}$, we see that U satisfies the degenerate elliptic equation

$$\operatorname{div}\left(\frac{\nabla U}{|\nabla U|}\right) = |\nabla U|$$

where we note that the term on the left hand side is the mean curvature of the level-sets Σ_t. Reparametrizing $u = e^{\frac{1}{2}U}$, we obtain the *homogeneous* equation

$$\Delta u = \nabla^2_{\nu\nu} u + 2\frac{|\nabla u|^2}{u} \tag{4.1}$$

where ν is the outer normal to the level sets Σ_t. In this context, we can rephrase the Hawking mass *monotonicity formula* $\mathfrak{m}_H(\Sigma_t) - \mathfrak{m}_H(\Sigma_0) \geq 0$, $t \geq 0$, as *integral formula*

$$\mathfrak{m}_H(\Sigma_t) - \mathfrak{m}_H(\Sigma_0) = \frac{1}{16\pi}\int_{\Omega_t}\left(R|\nabla u| + \frac{|\mathcal{H}^2 u|^2 - (\mathcal{H}^2_{\nu\nu} u)^2}{|\nabla u|}\right) dV. \tag{4.2}$$

Here Ω_t is the region bounded by Σ_0 and Σ_t, and $\mathcal{H}$ is a symmetric 2-tensor defined by

$$\mathcal{H}_{ij} u = \nabla_{ij} u - \frac{|\nabla u|^2}{u} g_{ij} + \frac{\nabla_i u \nabla_j u}{u}. \tag{4.3}$$

The RHS of equation (4.2) is non-negative in case $R \geq 0$.

To formulate such a monotonicity formula in the spacetime setting, we will take a more general point of view. In case we do not integrate the integrand on the RHS of equation (4.2) over a domain Ω, we obtain

$$\mathrm{R}\,|\nabla u| + \frac{|\mathcal{H}^2 u|^2 - (\mathcal{H}^2_{\nu\nu} u)^2}{|\nabla u|} - 2K_u|\nabla u| = 2\operatorname{div}\left(\nabla|\nabla u| + \frac{|\nabla u|}{u}\nabla u - \Delta u\frac{\nabla u}{|\nabla u|}\right) \tag{4.4}$$

where K_u is the Gaussian curvature of $\Sigma_t = \{u(x) = t\}$. The above version of the Hawking mass monotonicity generalizes to the spacetime setting.

4.4 The Hawking mass monotonicity formula

Suppose now that (M^3, g, k) is an initial data set, and recall the definitions of the energy density μ and the momentum density J

$$2\mu = \mathrm{R} + (\mathrm{tr}_g(k))^2 - |k|^2, \qquad J = \mathrm{div}(k - \mathrm{tr}_g(k)g).$$

We say (M, g, k) satisfies the dominant energy condition (DEC), in case $\mu \geq |J|$ everywhere on M. Our main result generalizes the Hawking mass integral formula (4.4) to initial data sets:

Theorem 4.4.1. *Let $a \in [0,1]$ and suppose $u, v \in C^{2,\alpha}(M)$ are positive solutions of the system*

$$\begin{aligned}\Delta u &= -\,\mathrm{tr}_g(k)|\nabla u| + ak_{\eta\eta}|\nabla u| + a\nabla^2_{\eta\eta}u + \frac{3|\nabla u||\nabla v| + \langle \nabla u, \nabla v\rangle}{u+v},\\ \Delta v &= \mathrm{tr}_g(k)|\nabla v| - ak_{\eta\eta}|\nabla v| + a\nabla^2_{\eta\eta}v + \frac{3|\nabla u||\nabla v| + \langle \nabla u, \nabla v\rangle}{u+v}\end{aligned} \tag{4.5}$$

with $|\nabla u|, |\nabla v| \neq 0$, where $\eta = \frac{\nabla u|\nabla v| + \nabla v|\nabla u|}{|\nabla u|\nabla v| + \nabla v|\nabla u||}$. Then

$$\begin{aligned}\mathrm{div}\, Y =& \frac{|\mathcal{H}^2_+ u|^2 - (a(\mathcal{H}^2_+)_{\eta\eta}u)^2}{|\nabla u|} + \frac{|\mathcal{H}^2_- v|^2 - (a(\mathcal{H}^2_-)_{\eta\eta}v)^2}{|\nabla v|}\\ &+ 2\mu(|\nabla u| + |\nabla v|) + 2\langle J, \nabla u - \nabla v\rangle\\ &- 2K_u|\nabla u| - 2K_v|\nabla v|\end{aligned} \tag{4.6}$$

where K_u, K_v are the Gaussian curvatures of the level sets of u, v,

$$\begin{aligned}Y =& 2\nabla(|\nabla u| + |\nabla v|) + 2k(\nabla(u-v), \cdot) + 4(|\nabla u|\nabla v + |\nabla v|\nabla u)\frac{1}{u+v}\\ &- 2\Delta u\frac{\nabla u}{|\nabla u|} - 2\Delta v\frac{\nabla v}{|\nabla v|} - 2\,\mathrm{tr}_g(k)\nabla(u-v)\end{aligned}$$

and

$$\begin{aligned}(\mathcal{H}^2_+)_{ij}u &= \nabla_{ij}u + k_{ij}|\nabla u| - 2\frac{\nabla_\eta u\nabla_\eta v}{u+v}g_{ij} + \frac{\nabla_i u\nabla_j v + \nabla_j u\nabla_i v}{u+v},\\ (\mathcal{H}^2_-)_{ij}v &= \nabla_{ij}v - k_{ij}|\nabla v| - 2\frac{\nabla_\eta u\nabla_\eta v}{u+v}g_{ij} + \frac{\nabla_i u\nabla_j v + \nabla_j u\nabla_i v}{u+v}.\end{aligned}$$

Observe that the first line in (4.6) is always non-negative, and the second line (4.6) is non-negative in case the DEC is satisfied. In Section 4.6.2 we see that the above formula implies upon integrating:

Corollary 4.4.2. *Let (M, g) be an annulus satisfying the dominant energy condition with spherical boundary components $\partial_- M$ and $\partial_+ M$. Suppose u, v are constant on both $\partial_- M$ and $\partial_+ M$ and that $u|_{\partial_- M} < u|_{\partial_+ M}$, $v|_{\partial_- M} < v|_{\partial_+ M}$. Then we have under the same assumptions as in Theorem 4.4.1*

$$\begin{aligned}(u+v)|_{\partial_+ M}&\left[1 - \frac{1}{8\pi}\int_{\partial_+ M}\left(2\theta_+\frac{|\nabla u|}{u+v} + 2\theta_-\frac{|\nabla v|}{u+v} - 8\frac{|\nabla u||\nabla v|}{(u+v)^2}\right)dA\right]\\ \geq (u+v)|_{\partial_- M}&\left[1 - \frac{1}{8\pi}\int_{\partial_- M}\left(2\theta_+\frac{|\nabla u|}{u+v} + 2\theta_-\frac{|\nabla v|}{u+v} - 8\frac{|\nabla u||\nabla v|}{(u+v)^2}\right)dA\right]\end{aligned}$$

where $\theta_\pm = H \pm (\mathrm{tr}_g(k) - k_{\eta\eta})$ are the null expansions. In the case $a = 1$ we furthermore have $|\nabla u| = \frac{1}{4}\theta_-(u+v)$ and $|\nabla v| = \frac{1}{4}\theta_+(u+v)$ on $\partial_\pm M$ which implies the generalized Hawking mass monotonicity

$$(u+v)|_{\partial_+ M}\left(1 - \frac{1}{16\pi}\int_{\partial_+ M}\theta_-\theta_+ dA\right) \geq (u+v)|_{\partial_- M}\left(1 - \frac{1}{16\pi}\int_{\partial_- M}\theta_-\theta_+ dA\right). \tag{4.7}$$

In particular, prescribing the boundary data $(u+v) = \sqrt{\frac{|\Sigma_1|}{16\pi}}$, $(u+v) = \sqrt{\frac{|\Sigma_2|}{16\pi}}$ on Σ_1, Σ_2 respectively, Theorem C follows.

For $k = 0$, system (4.5) decouples if u, v have the same boundary data, and we recover several important monotonicity formulas: For $k = 0$ and $a = 1$, the function $u = v$ is rescaled IMCF (as in equation (4.1)), and we obtain the Hawking mass monotonicity formula (4.4). For $k = 0$ and $0 \leq a < 1$, the function $u = v$ solves the rescaled[4] p-Laplacian equation

$$\Delta u = a\nabla^2_{\nu\nu}u + 2\frac{|\nabla u|^2}{u}$$

[4] i.e. $U = u^{-\frac{1+a}{1-a}}$ is $(2-a)$-harmonic

with monotonicity formula

$$\mathrm{R}\,|\nabla u| + \frac{|\mathcal{H}^2 u|^2 - (a\mathcal{H}^2_{\nu\nu} u)^2}{|\nabla u|} - 2K_u|\nabla u| = 2\operatorname{div}\left(\nabla|\nabla u| + \frac{|\nabla u|}{u}\nabla u - \Delta u\frac{\nabla u}{|\nabla u|}\right).$$

This formula has been first discovered by V. Agostiniani, L. Mazzieri and F. Oronzio in [2] which enabled them to give a new proof of the Riemannian Positive Mass Theorem [2], and, together with C. Mantegazza, the Riemannian Penrose inequality [1]. However, even in the special case $k = 0$, the above formula has some new contents since we can prescribe different boundary conditions for u and v, such that $u \neq v$ and the system does not decouple.

Another special case is given by $v = 0$. Then u is a spacetime harmonic function. i.e. u solves the PDE $\Delta u = -\operatorname{tr}_g(k)|\nabla u|$, and we recover the main integral formula of [64], Proposition 3.2. Moreover, we will see in Theorem 4.4.4 that (4.6) recovers the monotonicity formula of the spacetime Hawking energy [61]

$$\mathfrak{m}_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_\Sigma \theta_+\theta_- dA\right).$$

under IMCF in spherical symmetry which implies the Penrose inequality in this setting.

4.4.1 Origins of spacetime IMCF

As discussed in Section 3, to give a new proof of the spacetime PMT, D. Kazaras, M. Khuri and the author introduced in [64] spacetime harmonic functions. In case (M, g, k) arises as subset of Minkowski space $\mathbb{R}^{3,1}$, the spacetime harmonic function u can be obtained by restricting a null coordinate function of Minkowski space such as $x + t$, to (M, g, k). Hence, in the case of equality of the spacetime PMT the level-sets Σ_t of u can be obtained by intersecting null planes with the initial data set $(M, g, k) \subset \mathbb{R}^{3,1}$ as visualized in Figure 3.1.

A similar situation occurs for any $a \in [0,1]$ for the system

$$\Delta u = -\operatorname{tr}_g(k)|\nabla u| + ak_{\eta\eta}|\nabla u| + a\nabla^2_{\eta\eta}u + \frac{3|\nabla u||\nabla v| + \langle \nabla u, \nabla v\rangle}{u+v},$$
$$\Delta v = \operatorname{tr}_g(k)|\nabla v| - ak_{\eta\eta}|\nabla v| + a\nabla^2_{\eta\eta}v + \frac{3|\nabla u||\nabla v| + \langle \nabla u, \nabla v\rangle}{u+v}.$$

However, instead of leading to a single null foliation, the level sets Σ_u, Σ_v of u, v lead to a *double null foliation.* More precisely, we have the following:

Theorem 4.4.3. *Let $u = r+t$ and $v = r-t$ where r, t are the radial and the time coordinate functions of Minkowski space $\mathbb{R}^{3,1}$. Then the restrictions of u, v to any initial data set $(M, g, k) \subset \mathbb{R}^{3,1}$ solve system (4.5) for any $a \in [0,1]$. In fact, we have $(\mathcal{H}^2_+)_{ij}u = 0$ and $(\mathcal{H}^2_-)_{ij}v = 0$, and have equality in Corollary 4.4.2.*

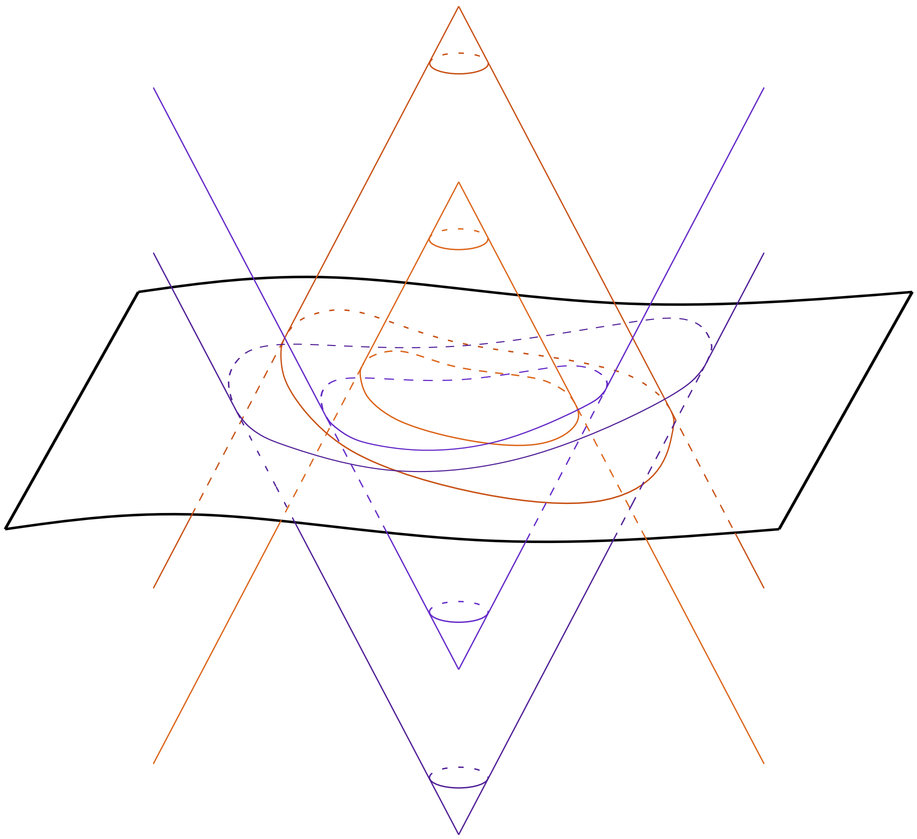
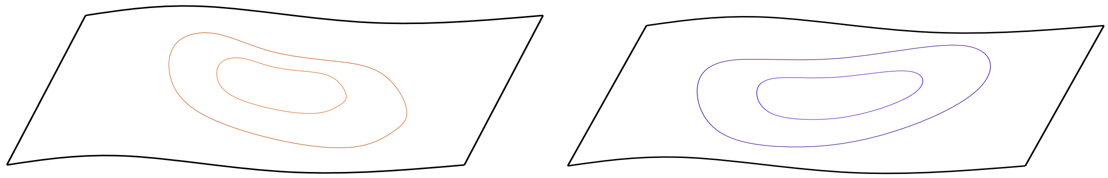


Figure 4.2: The double null foliation (Σ_u,Σ_v) for the initial data set $(\mathbf{M,g,k}) \subset \mathbb{R}^{3,1}$ is obtained by intersecting past and future directed lightcones in $\mathbb{R}^{3,1}$ with $(\mathbf{M,g,k})$. We would like to highlight that the individual null foliations Σ_u and Σ_v differ. This implies that an integral formula as in Theorem 4.4.1 is a more general concept than a monotonicity formula such as the one for the Hawking mass under IMCF.

Furthermore, we can interpret system (4.5) for $a = 1$ as *coupled inverse null mean curvature flow* and for $a = 0$ as *coupled spacetime harmonic functions*. Given an initial data set (M, g, k) and a surface $\Sigma \subset M$, we can define the future and past null expansions



$(\mathbf{M,g,k}) \subset \mathbb{R}^{3,1}$ with Σ_u Foliation $\qquad$ $(\mathbf{M,g,k}) \subset \mathbb{R}^{3,1}$ with Σ_v Foliation

Figure 4.3: There are no monotone quantities associated with the level sets Σ_u and Σ_v individually.

θ_+ and θ_- by

$$\theta_+ = H + \mathrm{tr}_g(k) - k_{\nu\nu}, \qquad \theta_- = H - \mathrm{tr}_g(k) + k_{\nu\nu}$$

where ν is the outer normal to Σ. A generalization of IMCF to initial data sets is given by flows with speeds $\frac{1}{\theta_+}$ and $\frac{1}{\theta_-}$ in the outward normal direction. These so called *inverse null mean curvature flows* have been studied K. Moore in [98] where an existence theory under the assumptions $\mathrm{tr}_g(k) \geq 0$ has been developed, also see [80]. Inverse null mean curvature flows A, B have like regular IMCF level-set formulations which after rescaling $\alpha = e^{\frac{1}{2}A}$, $\beta = e^{\frac{1}{2}B}$ become

$$\Delta\alpha = -\mathrm{tr}_g(k)|\nabla\alpha| + \nabla^2_{\nu\nu}\alpha + k_{\nu\nu}|\nabla\alpha| + 2\frac{|\nabla\alpha|^2}{\alpha}$$

for the speed $\frac{1}{\theta_+}$, and

$$\Delta\beta = -\mathrm{tr}_g(k)|\nabla\beta| + \nabla^2_{\nu\nu}\beta + k_{\nu\nu}|\nabla\beta| + 2\frac{|\nabla\beta|^2}{\beta}$$

for the speed $\frac{1}{\theta_-}$. We emphasize the similarities of these above equations with our system (4.5) for $a = 1$. More rigorously, we observe in Section 4.6.2 that the solutions (u, v) to our system (4.5) are in spherical symmetry rescalings of $\frac{1}{\theta_-}$ and $\frac{1}{\theta_+}$ flows. The rescaling factor is given by the usual IMCF.

4.4.2 Applications to the Penrose conjecture

Given that system (4.5) with $a = 1$ and integral formuala (4.6) generalizes IMCF including the Hawking mass monotonicity formula, it is natural to ask whether there are applications towards the Penrose conjecture.

Theorem 4.4.4. *Let (M, g, k) be a spherically symmetric initial data set satisfying the DEC, and let $a = 1$. Then system (4.5) can be solved, and the integral formula (4.6) reduces to the monotonicity formula of the spacetime Hawking energy*

$$\mathfrak{m}_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_\Sigma \theta_+\theta_- dA\right).$$

It is well-known that the monotonicity of spacetime Hawking energy on spherically symmetric initial data sets satisfying the DEC leads to the Penrose inequality, see for instance [61]. Therefore, Theorem 4.4.4 (and thus Theorem 4.4.1) implies the Penrose inequality in spherical symmetry. In the next section we discuss existence results outside of spherical symmetry.

4.5 Existence results

Our systems (4.5) have the advantage that there are no second-order coupling terms, and there is a simple expression for $\Delta(v - u)$. This allows us to obtain an existence theory for system (4.5) with $a = 0$ in full generality without having to assume any symmetry:

Theorem 4.5.1. *Let (M, g, k) be a compact 3-dimensional Riemannian manifold equipped with symmetric 2-tensor k. Suppose that the boundary of M has two connected components $\partial_- M$ and $\partial_+ M$. Then we can solve system (4.5) for $a = 0$, i.e. there exist* *functions* $u, v \in C^{2,\alpha}(M)$ *solving*

$$\Delta u = -\operatorname{tr}_g(k)|\nabla u| + \frac{3|\nabla u||\nabla v| + \langle \nabla u, \nabla v\rangle}{u+v},$$
$$\Delta v = \operatorname{tr}_g(k)|\nabla v| + \frac{3|\nabla u||\nabla v| + \langle \nabla u, \nabla v\rangle}{u+v}$$

on M*, with Dirichlet boundary data* $u = c_\pm$*,* $v = d_\pm$ *on* $\partial_\pm M$ *for positive constants* $c_\pm, d_\pm$*.*

To solve the system

$$\begin{aligned}\Delta u &= -\operatorname{tr}_g(k)|\nabla u| + \frac{3|\nabla u||\nabla v| + \langle \nabla u, \nabla v\rangle}{u+v},\\ \Delta v &= \operatorname{tr}_g(k)|\nabla v| + \frac{3|\nabla u||\nabla v| + \langle \nabla u, \nabla v\rangle}{u+v}\end{aligned} \tag{4.8}$$

on M with $u = c_\pm$ on $\partial_\pm M$ and $v = d_\pm$ on $\partial_\pm M$, we will first obtain uniform estimates for the system

$$\begin{aligned}\Delta u_{\sigma,\varepsilon} &= -\sigma\operatorname{tr}_g(k)|\nabla u_{\sigma,\varepsilon}| + \frac{3|\nabla u_{\sigma,\varepsilon}||\nabla v_{\sigma,\varepsilon}| + \langle \nabla u_{\sigma,\varepsilon}, \nabla v_{\sigma,\varepsilon}\rangle}{|u_{\sigma,\varepsilon}+v_{\sigma,\varepsilon}| + \varepsilon},\\ \Delta v_{\sigma,\varepsilon} &= \sigma\operatorname{tr}_g(k)|\nabla v_{\sigma,\varepsilon}| + \frac{3|\nabla u_{\sigma,\varepsilon}||\nabla v_{\sigma,\varepsilon}| + \langle \nabla u_{\sigma,\varepsilon}, \nabla v_{\sigma,\varepsilon}\rangle}{|u_{\sigma,\varepsilon}+v_{\sigma,\varepsilon}| + \varepsilon}.\end{aligned} \tag{4.9}$$

Here $\sigma \in [0,1]$, $\varepsilon > 0$, and we consider the boundary data $u_{\sigma,\varepsilon} = c_\pm$ on $\partial_\pm M$ and $v_{\sigma,\varepsilon} = \sigma d_\pm + (1-\sigma)c_\pm - \varepsilon$ on $\partial_\pm M$. We assume ε to be sufficiently small such that $\sigma d_\pm + (1-\sigma)c_\pm - \varepsilon > 0$ for all $\sigma \in [0,1]$. Without loss of generality we assume that $c_- < c_+$ and $d_- < d_+$.

Lemma 4.5.2. *Suppose* $u_{\sigma,\varepsilon} \in C^{2,\alpha}(M)$ *and* $v_{\sigma,\varepsilon} \in C^{2,\alpha}(M)$ *solve the system* (4.9)*. Then we have*

$$c_- \le u_{\sigma,\varepsilon} \le c_+, \qquad \sigma d_- + (1-\sigma)c_- - \varepsilon \le v_{\sigma,\varepsilon} \le \sigma d_+ + (1-\sigma)c_+ - \varepsilon$$

on M*.*

Proof. This follows immediately from the maximum principle. □

Lemma 4.5.3. *Suppose* $u_{\sigma,\varepsilon}$ *and* $v_{\sigma,\varepsilon}$ *solve system* (4.9)*. Then there exists a constant* C *independent of* $\sigma, u_\varepsilon, v_\varepsilon, \varepsilon$ *such that*

$$\|u_{\sigma,\varepsilon}\|_{W^{2,p}(M)} + \|v_{\sigma,\varepsilon}\|_{W^{2,p}(M)} \leq C.$$

Proof. To prove this proposition, it will be helpful to rewrite the above system in terms of

$$w_{\sigma,\varepsilon} = v_{\sigma,\varepsilon} - u_{\sigma,\varepsilon}, \qquad h_{\sigma,\varepsilon} = \frac{1}{u_{\sigma,\varepsilon} + v_{\sigma,\varepsilon} + \epsilon}.$$

We compute for $w_{\sigma,\varepsilon}$

$$\Delta w_{\sigma,\varepsilon} = \sigma \operatorname{tr}_g(k)(|\nabla u_{\sigma,\varepsilon}| + |\nabla v_{\sigma,\varepsilon}|),$$

and for $h_{\sigma,\varepsilon}$

$$\begin{aligned}\frac{1}{2}h_{\sigma,\varepsilon}^{-2}\Delta h_{\sigma,\varepsilon} =& -\frac{1}{2}\Delta(u_{\sigma,\varepsilon} + v_{\sigma,\varepsilon} + \epsilon) + \frac{|\nabla(u_{\sigma,\varepsilon} + v_{\sigma,\varepsilon} + \epsilon)|^2}{u_{\sigma,\varepsilon} + v_{\sigma,\varepsilon} + \epsilon} \\ =& \frac{1}{u_{\sigma,\varepsilon} + v_{\sigma,\varepsilon} + \epsilon}(3|\nabla u_{\sigma,\varepsilon}|^2 - 3|\nabla u_{\sigma,\varepsilon}||\nabla u_{\sigma,\varepsilon} + w_{\sigma,\varepsilon}| + 3\langle \nabla u_{\sigma,\varepsilon}, \nabla w_{\sigma,\varepsilon}\rangle + |\nabla w_{\sigma,\varepsilon}|^2) \\ &+ \sigma\frac{\operatorname{tr}_g(k)}{2}|\nabla u_{\sigma,\varepsilon}| - \sigma\frac{\operatorname{tr}_g(k)}{2}|\nabla u_{\sigma,\varepsilon} + w_{\sigma,\varepsilon}|.\end{aligned}$$

Using the identity

$$|\nabla(u_{\sigma,\varepsilon} + w_{\sigma,\varepsilon})| - |\nabla u_{\sigma,\varepsilon}| = \frac{1}{|\nabla(u_{\sigma,\varepsilon} + w_{\sigma,\varepsilon})| + |\nabla u_{\sigma,\varepsilon}|}(|\nabla w_{\sigma,\varepsilon}|^2 + 2\langle \nabla u_{\sigma,\varepsilon}, \nabla w_{\sigma,\varepsilon}\rangle),$$

we obtain

$$\begin{aligned}
&\frac{1}{2}h_{\sigma,\varepsilon}^{-2}\Delta h_{\sigma,\varepsilon}\\
=&-\frac{1}{u_{\sigma,\varepsilon}+v_{\sigma,\varepsilon}+\epsilon}\frac{3|\nabla u_{\sigma,\varepsilon}|}{|\nabla(u_{\sigma,\varepsilon}+v_{\sigma,\varepsilon})|+|\nabla u_{\sigma,\varepsilon}|}(|\nabla w_{\sigma,\varepsilon}|^2+2\langle\nabla u_{\sigma,\varepsilon},\nabla w_{\sigma,\varepsilon}\rangle)\\
&+\frac{1}{u_{\sigma,\varepsilon}+v_{\sigma,\varepsilon}+\epsilon}\left(3\langle\nabla u_{\sigma,\varepsilon},\nabla w_{\sigma,\varepsilon}\rangle+|\nabla w_{\sigma,\varepsilon}|^2\right)\\
&-\sigma\frac{\mathrm{tr}_g(k)}{2}\frac{1}{|\nabla(u_{\sigma,\varepsilon}+w_{\sigma,\varepsilon})|+|\nabla u_{\sigma,\varepsilon}|}(|\nabla w_{\sigma,\varepsilon}|^2+2\langle\nabla u_{\sigma,\varepsilon},\nabla w_{\sigma,\varepsilon}\rangle)\\
=&\frac{1}{u_{\sigma,\varepsilon}+v_{\sigma,\varepsilon}+\epsilon}\left(-3|\nabla u_{\sigma,\varepsilon}|\frac{1}{|\nabla(u_{\sigma,\varepsilon}+w_{\sigma,\varepsilon})|+|\nabla u_{\sigma,\varepsilon}|}|\nabla w_{\sigma,\varepsilon}|^2+|\nabla w_{\sigma,\varepsilon}|^2\right)\\
&+\frac{1}{u_{\sigma,\varepsilon}+v_{\sigma,\varepsilon}+\epsilon}3\frac{\langle\nabla u_{\sigma,\varepsilon},\nabla w_{\sigma,\varepsilon}\rangle}{(|\nabla(u_{\sigma,\varepsilon}+w_{\sigma,\varepsilon})|+|\nabla u_{\sigma,\varepsilon}|)^2}(|\nabla w_{\sigma,\varepsilon}|^2+2\langle\nabla u_{\sigma,\varepsilon},\nabla w_{\sigma,\varepsilon}\rangle)\\
&-\sigma\frac{\mathrm{tr}_g(k)}{2}\frac{1}{|\nabla(u_{\sigma,\varepsilon}+w_{\sigma,\varepsilon})|+|\nabla u_{\sigma,\varepsilon}|}(|\nabla w_{\sigma,\varepsilon}|^2+2\langle\nabla u_{\sigma,\varepsilon},\nabla w_{\sigma,\varepsilon}\rangle).
\end{aligned}$$

Having established our identities for $\Delta w_{\sigma,\varepsilon}$ and $\Delta h_{\sigma,\varepsilon}$ we proceed with estimating the above terms. We have

$$\begin{aligned}
|\Delta w_{\sigma,\varepsilon}|\leq&C(|\nabla(u_{\sigma,\varepsilon}+v_{\sigma,\varepsilon})|+|\nabla(u_{\sigma,\varepsilon}-v_{\sigma,\varepsilon})|\\
\leq&C(|\nabla h_{\sigma,\varepsilon}|h_{\sigma,\varepsilon}^{-2}+|\nabla w_{\sigma,\varepsilon}|)
\end{aligned}\tag{4.10}$$

where C is depending on $M,k,c_\pm,d_\pm$ whose value may change from line to line. Moreover,

$$|\Delta h_{\sigma,\varepsilon}|\leq Ch_{\sigma,\varepsilon}^3|\nabla w_{\sigma,\varepsilon}|^2+Ch_{\sigma,\varepsilon}^2|\nabla w_{\sigma,\varepsilon}|\tag{4.11}$$

where we used

$$\frac{|\nabla w_{\sigma,\varepsilon}|}{|\nabla(u_{\sigma,\varepsilon}+w_{\sigma,\varepsilon})|+|\nabla u_{\sigma,\varepsilon}|}\leq 1.$$

The $W^{2,p}$ estimate for solutions of elliptic equations states

$$\|w_{\sigma,\varepsilon}\|_{W^{2,p}(M)}\leq C(\|w_{\sigma,\varepsilon}\|_{L^p(M)}+\|h_{\sigma,\varepsilon}\|_{W^{1,p}(M)})\leq C+C\|h_{\sigma,\varepsilon}\|_{W^{1,p}(M)}$$

and

$$\|h_{\sigma,\varepsilon}\|_{W^{2,p}(M)}\leq C(\|h_{\sigma,\varepsilon}\|_{L^p(M)})+\|w_{\sigma,\varepsilon}\|^2_{W^{1,2p}(M)})\leq C+C\|w_{\sigma,\varepsilon}\|^2_{W^{1,2p}(M)}.$$

By the Gagliardo-Nirenberg interpolation inequality, we have

$$\|\nabla w_{\sigma,\varepsilon}\|^2_{L^{2p}(M)} \le C\|\nabla^2 w_{\sigma,\varepsilon}\|_{L^p(M)}\|w_{\sigma,\varepsilon}\|_{L^\infty(M)}$$

and

$$\|\nabla h_{\sigma,\varepsilon}\|_{L^p(M)} \le C\|\nabla^2 h_{\sigma,\varepsilon}\|^{\delta}_{L^p(M)}\|h_{\sigma,\varepsilon}\|^{1-\delta}_{L^\infty(M)}$$

where

$$\delta = \frac{p-3}{2p-3} < 1.$$

Hence, we are lead to

$$\begin{aligned}\|h_{\sigma,\varepsilon}\|_{W^{2,p}(M)} \le& C + C\|w_{\sigma,\varepsilon}\|_{W^{2,p}(M)} \le C + C\|h_{\sigma,\varepsilon}\|_{W^{1,p}(M)}\\ \le& C + C\|h_{\sigma,\varepsilon}\|^{\alpha}_{W^{2,p}(M)} \le +\frac{1}{2}\|h_{\sigma,\varepsilon}\|_{W^{2,p}(M)}.\end{aligned}$$

Thus, we have $\|h_{\sigma,\varepsilon}\|_{W^{2,p}(M)} \le C$ which implies $\|w_{\sigma,\varepsilon}\|_{W^{2,p}(M)} \le C$. Reconstructing $u_{\sigma,\varepsilon}, v_{\sigma,\varepsilon}$ from $h_{\sigma,\varepsilon}, w_{\sigma,\varepsilon}$, we also obtain $\|u_{\sigma,\varepsilon}\|_{W^{2,p}(M)} + \|v_{\sigma,\varepsilon}\|_{W^{2,p}(M)} \le C$ which finishes the proof.. □

We can use the Sobolev inequality and Schauder estimates to improve the above estimate to $C^{2,\alpha}$. More precisely, we obtain:

Lemma 4.5.4. *Suppose $u_{\sigma,\varepsilon}, v_{\sigma,\varepsilon}$ solve system* (4.9). *Then there exists a constant C independent of $\sigma, u_{\sigma,\varepsilon}, v_{\sigma,\varepsilon}, \varepsilon$ such that*

$$\|u_{\sigma,\varepsilon}\|_{C^{2,\alpha}(M)} + \|v_{\sigma,\varepsilon}\|_{C^{2,\alpha}(M)} \le C.$$

Having obtained uniform estimates for system (4.9) we will use Leray-Schauder's fixed point theorem below to obtain solutions of (4.9) for $\sigma = 1$. Passing to a limit $\varepsilon \to 0$ then gives a solution to (4.8) establishing Theorem 4.5.1.

Proof of Theorem 4.5.1. Let $\phi_\pm^{\sigma,\varepsilon}$ be two functions on M with $\phi_\pm^{\sigma,\varepsilon} = c_\pm$, $\phi_+^{\sigma,\varepsilon} = \sigma d_\pm + (1-\sigma)c_\pm - \varepsilon$ on $\partial_\pm M$. We denote with $C_0^{2,\alpha}(M)$ the set $C^{2,\alpha}$-functions on M which vanish on $\partial_\pm M$. Observe that $C_0^{2,\alpha}(M)$ is a Banach space. We define a family of maps $\mathcal{F}_{\sigma,\epsilon} : C_0^{2,\alpha}(M) \oplus C_0^{2,\alpha}(M) \to C_0^{2,\alpha}(M) \oplus C_0^{2,\alpha}(M)$ via

$$\mathcal{F}_{\sigma,\varepsilon}(u,v) = \left[\Delta_0^{-1}\left(\mathcal{G}_-^{\sigma,\varepsilon}(u,v)\right), \Delta_0^{-1}\left(\mathcal{G}_+^{\sigma,\varepsilon}(u,v)\right)\right]$$

where

$$\begin{aligned}\mathcal{G}_\pm^{\sigma,\varepsilon}(u,v) = &\pm \sigma \operatorname{tr}_g(k)|\nabla(u+\phi_-)| \\ &+ \frac{3|\nabla(u+\phi_-)||\nabla(v+\phi_+)| + \langle \nabla(u+\phi_-), \nabla(v+\phi_+)\rangle}{|u+\phi_- + v+\phi_+| + \varepsilon} - \Delta\phi_\pm^{\sigma,\varepsilon}\end{aligned}$$

and where Δ_0^{-1} maps a function f to the solution ψ of $\Delta\psi = f$ on M with vanishing Dirichlet boundary data. By standard elliptic theory, $\mathcal{F}_{\sigma,\varepsilon}$ is indeed a map into $C_0^{2,\alpha}(M) \oplus C_0^{2,\alpha}(M)$. Moreover, $\mathcal{F}_{\sigma,\varepsilon}$ is a compact operator since the image of a bounded sequence $\{(u_i, v_i)\}$ has a convergent subsequence. Observe that if $\mathcal{F}_{\sigma,\varepsilon}(u,v) = (u,v)$, then $(u+\phi_-, v+\phi_+)$ solve system (4.9). Hence, we can use our uniform estimates, Lemma 4.5.4, and Leray-Schauder's fixed point theorem, see for instance Theorem 11.6 in [53], to deduce that there exists a solution of $\mathcal{F}_{1,\varepsilon}(u_{1,\varepsilon}, v_{1,\varepsilon}) = (u_{1,\varepsilon}, v_{1,\varepsilon})$ if there exists a solution of $\mathcal{F}_{0,\varepsilon}(u_{0,\varepsilon}, v_{0,\varepsilon}) = (u_{0,\varepsilon}, v_{0,\varepsilon})$. Let $U_{0,\varepsilon}$ be the harmonic function with $U_{0,\varepsilon} = \frac{1}{c_\pm}$ on $\partial_\pm M$. Then $u_{0,\varepsilon} = \frac{1}{U_{0,\varepsilon}}$ satisfies

$$\Delta u_{0,\varepsilon} = 2\frac{|\nabla u_{0,\varepsilon}|^2}{u_{0,\varepsilon}}$$

with $u_{0,\varepsilon} = c_\pm$ on $\partial_\pm M$. Next, let $v_{0,\varepsilon} = u_{0,\varepsilon} - \varepsilon$. Note that $v_{0,\varepsilon} = c_\pm - \varepsilon$ on $\partial_\pm M$ and $v_{0,\varepsilon} > 0$ on M. Then

$$\begin{aligned}\Delta u_{0,\varepsilon} =& \frac{3|\nabla u_{0,\varepsilon}||\nabla v_{0,\varepsilon}| + \langle \nabla u_{0,\varepsilon}, \nabla v_{0,\varepsilon}\rangle}{|u_{0,\varepsilon} + v_{0,\varepsilon}| + \varepsilon}, \\ \Delta v_{0,\varepsilon} =& \frac{3|\nabla u_{0,\varepsilon}||\nabla v_{0,\varepsilon}| + \langle \nabla u_{0,\varepsilon}, \nabla v_{0,\varepsilon}\rangle}{|u_{0,\varepsilon} + v_{0,\varepsilon}| + \varepsilon}.\end{aligned}$$

Thus, we may find solutions $(u_{1,\varepsilon}, v_{1,\varepsilon})$ to system (4.9) for $\sigma = 1$. Since $u_{1,\varepsilon}, v_{1,\varepsilon}$ are uniformly bounded away from zero, and we have uniform $C^{2,\alpha}$-estimates for $(u_{1,\varepsilon}, v_{1,\varepsilon})$ in terms of ε, we can take the limit $\varepsilon \to 0$ to obtain solutions (u, v) to system (4.8). □

A crucial ingredient of the above proof is that the system (4.5) takes a simpler form for $w = u - v$ and $h = \frac{1}{u+v}$ as in (4.10) and (4.11). We remark that for the p-harmonic system, i.e. system (4.5) for $a \in (0, 1)$, can be rewritten in a very similar form to (4.10) and (4.11) though we have to re-define $h = (u+v)^{-\frac{1+a}{1-a}}$. Note that the radial function $r^{-\frac{1+a}{1-a}}$ is p-harmonic in $\mathbb{R}^3$ for $p = 2 - a$. We also expect that the solutions of system (4.5) for $a = 0$ established in this section can be used to give a new proof for the spacetime PMT.

4.6 Further discussions

In this section we better understand spacetime IMCF within Minkowski space and Schwarzschild, and also give an example of another PDE system incorporating electrical fields.

4.6.1 Minkowski space

In this section we show that for any initial data set (M, g, k) contained in Minkowski space $(\mathbb{R}^{3,1}, \bar{g})$ the functions $u = r + t$ and $v = r - t$ solve system (4.5) and satisfy $\mathcal{H}^2_+ u = 0$, $\mathcal{H}^2_- v = 0$ where

$$(\mathcal{H}^2_+)_{ij} u = \nabla_{ij} u + k_{ij}|\nabla u| - 2\frac{\nabla_\eta u \nabla_\eta v}{u+v} g_{ij} + \frac{\nabla_i u \nabla_j v + \nabla_j u \nabla_i v}{u+v},$$
$$(\mathcal{H}^2_-)_{ij} v = \nabla_{ij} v - k_{ij}|\nabla v| - 2\frac{\nabla_\eta u \nabla_\eta v}{u+v} g_{ij} + \frac{\nabla_i u \nabla_j v + \nabla_j u \nabla_i v}{u+v}.$$

Moreover, we analyze what the existence of functions satisfying the conditions $\mathcal{H}^2_+ u = 0$, $\mathcal{H}^2_- v = 0$ implies for general initial data sets.

Proof of Theorem 4.4.3. We begin with observing that $\bar\nabla^2(v-u) = 0$ and $\bar\nabla^2(uv) = 2\bar g$ where $\bar g$ is the metric of Minkowski space. This implies

$$\begin{aligned}2\bar g =& v\bar\nabla^2 u + u\bar\nabla^2 v + \bar\nabla u \otimes \bar\nabla v + \bar\nabla v \otimes \bar\nabla u\\ =& (u+v)\bar\nabla^2 u + \bar\nabla u \otimes \bar\nabla v + \bar\nabla v \otimes \bar\nabla u.\end{aligned}$$

Similarly,

$$2\bar g = (u+v)\bar\nabla^2 v + \bar\nabla u \otimes \bar\nabla v v + \bar\nabla v \otimes \bar\nabla u.$$

Restricting the above two equalities onto $T^*M \otimes T^*M$ we obtain

$$\begin{aligned}2g =& (u+v)\bar\nabla^2|_{T^*M\otimes T^*M} u + \nabla u \otimes \nabla v + \nabla v \otimes \nabla u,\\ 2g =& (u+v)\bar\nabla^2|_{T^*M\otimes T^*M} v + \nabla u \otimes \nabla v + \nabla v \otimes \nabla u.\end{aligned}$$

Next, let us denote with N the future pointing unit normal of $M \subset \mathbb{R}^{3,1}$. It is well-known that

$$\begin{aligned}\bar\nabla^2|_{T^*M\otimes T^*M} u =& \nabla^2 u + kN(u),\\ \bar\nabla^2|_{T^*M\otimes T^*M} v =& \nabla^2 v + kN(v).\end{aligned}$$

Since $\bar\nabla u$ and $\bar\nabla v$ are null, we have

$$\begin{aligned}\bar\nabla^2|_{T^*M\otimes T^*M} u =& \nabla^2 u + k|\nabla u|,\\ \bar\nabla^2|_{T^*M\otimes T^*M} v =& \nabla^2 v - k|\nabla u|.\end{aligned}$$

Combining everything yields

$$\begin{aligned}2g =& (u+v)(\nabla^2 u + k|\nabla u|) + \nabla u \otimes \nabla v + \nabla v \otimes \nabla u,\\ 2g =& (u+v)(\nabla^2 v - k|\nabla u|) + \nabla u \otimes \nabla v + \nabla v \otimes \nabla u.\end{aligned}$$

Observe that Lemma C.1.2 implies

$$|\nabla u||\nabla v| + \langle \nabla u, \nabla v\rangle = 2\nabla_\eta u \nabla_\eta v. \tag{4.12}$$

Therefore, it suffices to show that

$$2 = |\nabla u||\nabla v| + \langle \nabla u, \nabla v\rangle$$

for any initial data set $(M, g, k) \subset \mathbb{R}^{3,1}$. To do so, we observe that

$$2 = \bar{g}(\bar{\nabla} u, \bar{\nabla} v) = \langle \nabla u, \nabla v\rangle + \bar{g}(N(u)N, N(v)N).$$

Since

$$\bar{g}(N(u)N, N(v)N) = -N(u)N(v) = |\nabla u||\nabla v|,$$

the result follows. □

We remark that in Minkowski space we can also solve for any $c > 0$ the boosted system

$$\begin{aligned} 0 &= \nabla_{ij} u + k_{ij}|\nabla u| - 2\frac{\nabla_\eta u \nabla_\eta v}{\frac{1}{c^2}u + v} g_{ij} + \frac{\nabla_i u \nabla_j v + \nabla_j u \nabla_i v}{\frac{1}{c^2}u + v}, \\ 0 &= \nabla_{ij} v - k_{ij}|\nabla v| - 2\frac{\nabla_\eta u \nabla_\eta v}{u + c^2 v} g_{ij} + \frac{\nabla_i u \nabla_j v + \nabla_j u \nabla_i v}{u + c^2 v}. \end{aligned}$$

where $u = c(r + t)$ and $v = \frac{1}{c}(r - t)$.

The vector field η appears in both the integral formulas (4.6) and (4.24). Next, we describe η for IDS (M, g, k) in Minkowski space. In the following proposition we equip $(\mathbb{R}^{3,1}, \bar{g})$ with spherical coordinates $(\partial_r, \partial_t, \partial_\phi, \partial_\theta)$.

Proposition 4.6.1. *Let $(M, g, k) \subset (\mathbb{R}^{3,1}, \bar{g})$ be an IDS contained in Minkowski space. Then $\bar{g}(\eta, \partial_\theta) = 0$ and $\bar{g}(\eta, \partial_\phi) = 0$.*

Proof. We have

$$\partial_r + \partial_t = \bar{\nabla} u = \nabla u + N(u)N$$

and

$$\partial_r - \partial_t = \nabla v + N(v)N.$$

Therefore,

$$\eta |\nabla u|\nabla v + |\nabla v|\nabla u| = |\nabla v|\nabla u + |\nabla u|\nabla v = \partial_r(|\nabla u| + |\nabla v|) + \partial_t(|\nabla u| - |\nabla v|),$$

and the result follows. □

4.6.2 Schwarzschild

We begin with the proof of Corollary 4.1.1 before proceeding with the Penrose inequality in spherical symmetry 4.4.4 and studying arbitrary slices of Schwarzschild.

Proof of Corollary 4.4.2. Recall from Theorem 4.4.1 that

$$\begin{aligned}\operatorname{div} Y =& \frac{|\mathcal{H}^2_+ u|^2 - ((\mathcal{H}^2_+)_{\eta\eta} u)^2}{|\nabla u|} + \frac{|\mathcal{H}^2_- v|^2 - ((\mathcal{H}^2_-)_{\eta\eta} v)^2}{|\nabla v|} \\ &+ 2\mu(|\nabla u| + |\nabla v|) + 2\langle J, \nabla u - \nabla v\rangle \\ &- 2K_u|\nabla u| - 2K_v|\nabla v| \\ \geq& - 2K_u|\nabla u| - 2K_v|\nabla v|.\end{aligned} \tag{4.13}$$

where we used that (M, g, k) satisfies the DEC, and where

$$\begin{aligned}Y =& 2\nabla(|\nabla u| + |\nabla v|) + 2k(\nabla(u-v), \cdot) + 4(|\nabla u|\nabla v + |\nabla v|\nabla u)\frac{1}{u+v} \\ &- 2\Delta u\frac{\nabla u}{|\nabla u|} - 2\Delta v\frac{\nabla v}{|\nabla v|} - 2\operatorname{tr}_g(k)\nabla(u-v).\end{aligned}$$

Next, recall that each boundary component $\partial_\pm M$ is a level set for both u and v which can be interpreted as $\partial_- M$ and $\partial_+ M$ being *unboosted* with respect to each other. Hence $\nu := \eta = \nu_u = \nu_v$ on $\partial_\pm M$ and

$$H|\nabla u| + \nabla_{\nu\nu} u = \Delta u = -\operatorname{tr}_k |\nabla u| + \nabla_{\nu\nu} u + k_{\nu\nu}|\nabla u| + 4\frac{|\nabla u||\nabla v|}{u+v},$$
$$H|\nabla v| + \nabla_{\nu\nu} v = \Delta v = \operatorname{tr}_k |\nabla v| + \nabla_{\nu\nu} v - k_{\nu\nu}|\nabla v| + 4\frac{|\nabla u||\nabla v|}{u+v}$$

on $\partial_\pm M$. Since $|\nabla u|, |\nabla v|$ are non-zero, this implies

$$|\nabla v| = \frac{1}{4}\theta_+(u+v), \qquad |\nabla u| = \frac{1}{4}\theta_-(u+v). \tag{4.14}$$

Combining these equations with the identity $\Delta u = \nabla_{\nu\nu} u + H\nabla_\nu u$, we obtain

$$\begin{aligned} Y_\nu =& 2\nabla_{\nu\nu}(u+v) + 2k_{\nu\nu}(|\nabla u| - |\nabla v|) + \frac{8|\nabla u||\nabla v|}{u+v} \\ & - 2\Delta(u+v) - 2\operatorname{tr}_g(k)(|\nabla u| - |\nabla v|) \\ =& -2H(|\nabla u| + |\nabla v|) + 2k_{\nu\nu}(|\nabla u| - |\nabla v|) \\ & + \frac{8|\nabla u||\nabla v|}{u+v} - 2\operatorname{tr}_g(k)(|\nabla u| - |\nabla v|) \\ =& -2\theta_+|\nabla u| - 2\theta_-|\nabla v| + \frac{8|\nabla u||\nabla v|}{u+v} \\ =& -\frac{1}{2}\theta_+\theta_-(u+v). \end{aligned}$$

Combining this with equation (4.13) yields after integration

$$-\frac{(u+v)|_{\partial_+ M}}{16\pi}\int_{\partial_+ M}\theta_+\theta_- dA \geq -\frac{(u+v)|_{\partial_- M}}{16\pi}\int_{\partial_- M}\theta_+\theta_- dA + \frac{1}{4\pi}\int_M (K_u|\nabla u| + K_v|\nabla v|)dV$$

Next, we use twice the coarea formula and Gauss-Bonnet's theorem to obtain

$$\int_M (K_u|\nabla u| + K_v|\nabla v|)dV = \int_{u|_{\partial_- M}}^{u|_{\partial_+ M}} 4\pi dt + \int_{v|_{\partial_- M}}^{v|_{\partial_+ M}} 4\pi dt = 4\pi(u+v)|_{\partial_- M} - 4\pi(u+v)|_{\partial_+ M}.$$

Hence, we have

$$(u+v)|_{\partial_+ M}\left(1 - \frac{1}{16\pi}\int_{\partial_+ M}\theta_+\theta_- dA\right) \geq (u+v)|_{\partial_- M}\left(1 - \frac{1}{16\pi}\int_{\partial_- M}\theta_+\theta_- dA\right) \tag{4.15}$$

which finishes the proof. □

In order to prove Theorem 4.4.4 which implies the Penrose inequality in spherical symmetry, we first establish the following Lemma.

Lemma 4.6.2. *Let (M, g, k) be a spherically symmetric initial data set and let $\Sigma_0 \subset M$ be the outermost horizon. Let s be a smooth solution of rescaled IMCF starting from Σ_0, i.e. $\Delta s = \nabla_{\nu\nu} s + 2\frac{|\nabla s|^2}{s}$ with $s(\Sigma_0) = \sqrt{\frac{|\Sigma_0|}{16\pi}}$. Outside Σ_0 we define the spherically symmetric function $w = w(r)$ via*

$$w(r) = \int_0^r \frac{1}{2}(\mathrm{tr}_g(k) - k_{\nu\nu}) s d\rho$$

where r is the distance to Σ_0. Then u, v, implicitly defined by

$$u + v = s, \qquad v - u = w,$$

solve system (4.5) *for $a = 1$. Moreover, we have*

$$|\nabla v| = \frac{1}{4}\theta_+ s, \qquad |\nabla u| = \frac{1}{4}\theta_- s. \tag{4.16}$$

Here ν is the unit normal to the spherically symmetric surfaces and we note that $\nu = \nu_u = \nu_v = \eta$.

Proof. Since s solves rescaled IMCF, and using $\Delta s = \nabla_{\nu\nu} s + H\nabla_\nu s$, we deduce that $|\nabla s| = \frac{1}{2}Hs$. Moreover, we have $|\nabla w| = \frac{1}{2}|\,\mathrm{tr}_g(k) - k_{\nu\nu}|s$. Since Σ_0 is the outermost horizon and is therefore not enclosed by any MITS or MOTS, we also obtain that $\theta_+, \theta_- > 0$ for all spherically symmetric surfaces outside Σ_0 . This implies $H > |\,\mathrm{tr}_g(k) - k_{\nu\nu}|$ for all spherically symmetric surfaces outside Σ_0, and since $u = \frac{1}{2}(s - w)$ and $v = \frac{1}{2}(s + w)$ we obtain

$$|\nabla v| = \frac{1}{4}\theta_+ s, \qquad |\nabla u| = \frac{1}{4}\theta_- s.$$

Note that this in particular implies $\nabla u, \nabla v \neq 0$ outside Σ_0 as well as $\nabla_r u, \nabla_r v > 0$. Multiplying the above identities by $|\nabla u|, |\nabla v|$, we obtain in the same fashion as in the computation of equation (4.14) that (u, v) solve system (4.5) with $a = 1$. This finishes the proof. □

Observe that (4.16) implies that the level sets of u move by rescaled $\frac{1}{\theta_-}$ flow, and the level sets of v move by rescaled $\frac{1}{\theta_+}$ flow. The rescaling factor is in both cases given by $\frac{1}{4}s$ where s is rescaled IMCF.

The above lemma immediately yields

Corollary 4.6.3. *We can solve the system* (4.5) *for* $a = 1$ *in spherical symmetry.*

We remark that although system (4.5) is in many ways the most complicated for $a = 1$ due to its degenerate elliptic character, the existence theory for $a = 1$ is substantially simpler than for $a \in [0, 1)$ in spherical symmetry. This contrasts the Riemannian (i.e. $k = 0$ case) where the existence theory for harmonic functions is elementary compared to the sophisticated existence theory for IMCF [79, 97]. The reason for this reverse behavior stems from the fact that the system decouples for $a = 1$ in spherical symmetry as demonstrated in Lemma 4.6.2. However, the system appears not to decouple in spherical symmetry for $a \neq 1$, and the function $u + v$ is not the rescaling of a p-harmonic function.

Proof of Theorem 4.4.4. Let (M, g, k) be a spherically symmetric initial data set satisfying the DEC, and let u, v be solutions to system (4.5) for $a = 1$ outside the horizon Σ_0 as described in Lemma 4.6.2. As in the proof of Corollary 4.4.2 above, we obtain

$$(u+v)|_{\Sigma_2}\left(1 - \frac{1}{16\pi}\int_{\Sigma_2} \theta_+\theta_- dA\right) \geq (u+v)|_{\Sigma_1}\left(1 - \frac{1}{16\pi}\int_{\Sigma_1} \theta_+\theta_- dA\right) \tag{4.17}$$

for any spherically symmetric surface Σ_2 enclosing Σ_1 enclosing Σ_1. Since $u+v = s$ solves rescaled IMCF $\Delta s = \nabla_{\nu\nu} s + 2\frac{|\nabla s|^2}{s}$ with $s(\Sigma_0) = \sqrt{\frac{|\Sigma_0|}{16\pi}}$, and because IMCF is uniformly area expanding, we obtain

$$\sqrt{\frac{|\Sigma_2|}{16\pi}}\left(1-\frac{1}{16\pi}\int_{\Sigma_2}\theta_+\theta_- dA\right)\geq\sqrt{\frac{|\Sigma_1|}{16\pi}}\left(1-\frac{1}{16\pi}\int_{\Sigma_1}\theta_+\theta_- dA\right)$$

Hence, the spacetime Hawking energy

$$\mathfrak{m}_H(\Sigma_t)=\sqrt{\frac{|\Sigma_t|}{16\pi}}\left(1-\frac{1}{16\pi}\int_{\Sigma_t}\theta_-\theta_+ dA\right) \tag{4.18}$$

is monotonically increasing for spherically symmetric initial data sets satisfying the DEC. □

In Minkowski space we can obtain for any initial data set $(M,g,k)\subset\mathbb{R}^{3,1}$ solutions to system (4.5) by restricting the optical functions $u=r+t$ and $v=r-t$ to M. Moreover $(\mathcal{H}^2_+)_{ij}u=0$ and $(\mathcal{H}^2_-)_{ij}v=0$. For Schwarzschild the situation is similar, though the underlying objects are null vector fields rather than null functions:

Proposition 4.6.4. *Let $(\bar{M},\bar{g})$ be the Schwarzschild spacetime of mass $\mathfrak{m}\geq 0$ in static coordinates, i.e.*

$$\bar{g}=-\phi dt^2+\phi^{-1}dr^2+r^2g_{S^2}$$

where $\phi=(1-\frac{2\mathfrak{m}}{r})$ On $(\bar{M},\bar{g})$ we define the null vector fields $X=\phi\nabla(r^*+t)$ and $Y=\phi\nabla(r^*-t)$ where $r^*=r+2\mathfrak{m}\ln(\frac{r}{2m}-1)$ is the tortoise coordinate. Then on each spherically symmetric initial data set (M,g,k) in $(\bar{M},\bar{g})$ the vector fields $X|_{TM}$ and $Y|_{TM}$ are integrable, i.e. there are functions u,v on M with $\nabla u=X|_{TM}$ and $\nabla v=Y|_{TM}$. These functions u,v solve the system (4.5) for $a=1$ and we have $|\mathcal{H}^2_+u|^2-((\mathcal{H}^2_+)_{\eta\eta}u)^2=0$ as well as $|\mathcal{H}^2_-v|^2-((\mathcal{H}^2_-)_{\eta\eta}v)^2=0$. Moreover, $\nabla_\eta X_\eta=\nabla_\eta Y_\eta=\frac{m}{r^2}$.

We would like to remark that $X+Y=2\nabla r$ and $X-Y=2T$ where T is the time-like Killing vector field $T=(1-\frac{2\mathfrak{m}}{r})\nabla t$.

Proof. Recall that we have in view of equation (4.12)

$$(\mathcal{H}^2_+)_{ij}u = \nabla_{ij}u + k_{ij}|\nabla u| - \frac{|\nabla u||\nabla v| + \langle \nabla u, \nabla v\rangle}{u+v}g_{ij} + \frac{\nabla_i u \nabla_j v + \nabla_j u \nabla_i v}{u+v},$$
$$(\mathcal{H}^2_-)_{ij}v = \nabla_{ij}v - k_{ij}|\nabla v| - \frac{|\nabla u||\nabla v| + \langle \nabla u, \nabla v\rangle}{u+v}g_{ij} + \frac{\nabla_i u \nabla_j v + \nabla_j u \nabla_i v}{u+v}.$$

Next, observe that we have $\nabla_i X_j = \nabla_j X_i$ unless $(i,j) = (r,t),(t,r)$. Thus, X, Y are integrable on each spherically symmetric IDS. Moreover, $\bar{\nabla} X = \nabla X + k\langle N, X\rangle_{\bar{g}} = \nabla X + |X|k$ and $\bar{\nabla} Y = \nabla Y - k|\nabla Y|$ since X, Y are null vectors. This implies

$$|X||Y| + \langle X, Y\rangle = \bar{g}(N(u)N, N(v)N) + \langle X, Y\rangle = \bar{g}(X,Y) = 2\phi.$$

To prove the above proposition, it thus suffices to show on $(\bar{M}, \bar{g})$

$$\bar{\nabla}_\alpha X_\beta = \frac{\phi}{r}\bar{g}_{\alpha\beta} - \frac{X_\alpha Y_\beta + X_\beta Y_\alpha}{2r}, \qquad \bar{\nabla}_\alpha Y_\beta = \frac{\phi}{r}\bar{g}_{\alpha\beta} - \frac{X_\alpha Y_\beta + X_\beta Y_\alpha}{2r} \tag{4.19}$$

for all α, β apart form $(\alpha,\beta) = (r,r),(r,t),(t,r),(t,t)$. We merely perform the computation for $\bar{\nabla}_\alpha X_\beta$ since the ones for $\bar{\nabla}_\alpha Y_\beta$ are analogous.

Denoting with $A = \partial_\phi, \partial_\theta$ the standard spherical coordinates, we compute for $\alpha \neq A$

$$\bar{\nabla}_A X_\alpha = \bar{\nabla}_\alpha X_A = 0$$

and

$$\frac{\phi}{r}\bar{g}_{A\alpha} - \frac{X_\alpha Y_A + X_A Y_\alpha}{2r} = 0.$$

Moreover, we have

$$\bar{\nabla}_A X_A = \bar{\Gamma}^r_{AA} X_r = \frac{\phi}{r}\bar{g}_{AA} - \frac{X_\alpha Y_A + X_A Y_\alpha}{2r}.$$

Next, observe that η has only components in ∇r and ∇t direction, i.e. $\eta = a\nabla_r + b\nabla_t$. We calculate

$$\nabla^2_{rr} r = \phi\frac{\mathfrak{m}}{r^2}, \qquad \nabla^2_{tt} r = -\phi^{-1}\frac{\mathfrak{m}}{r^2}.$$

This leads to

$$\nabla_\eta X_\eta = \nabla^2_{\eta\eta} r = \frac{m}{r^2}\bar{g}(\eta,\eta) = \frac{m}{r^2}$$

which finishes the proof. □

We would like to remark that in case $\mathfrak{m} \neq 0$, the Killing vector field $T = \frac{1}{2}(X - Y)$ does not satisfy $\nabla_r T_t = 0$, $\nabla_t T_r = 0$. Hence, equation (4.19) is not satisfied for $(\alpha,\beta) = (r,t)$, and X, Y are not integrable on the entire spacetime $(\bar{M},\bar{g})$. Thus, in contrast to Minkowski space, there are no globally defined functions u, v such that when they are restricted to an IDS, they solve system (4.5) for $a = 1$. However, there are other such globally defined null functions $u = r^* + t$ and $v = r^* - t$ which do satisfy another nice set of equations. More precisely, we have:

Proposition 4.6.5. *Consider the functions $u = r^* + t$ and $v = r^* - t$ in the Schwarzschild spacetime $(\bar{M},\bar{g})$. Then on any IDS $(M,g,k) \subseteq (\bar{M},\bar{g})$ (not necessarily spherically symmetric) the restrictions of the functions u, v onto M satisfy*

$$\begin{aligned}\theta_+|\nabla u| =&\Delta u + \operatorname{tr}_g k|\nabla u| - \nabla^2_{\nu_u\nu_u} u - k_{\nu_u\nu_u}|\nabla u| = \phi\frac{|\nabla u||\nabla v| + \langle\nabla u,\nabla v\rangle}{r},\\ \theta_-|\nabla v| =&\Delta v - \operatorname{tr}_g k|\nabla v| - \nabla^2_{\nu_v\nu_v} v + k_{\nu_v\nu_v}|\nabla v| = \phi\frac{|\nabla u||\nabla v| + \langle\nabla u,\nabla v\rangle}{r}.\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}\Delta u =&(k_{\eta\eta} - \operatorname{tr}_g k)|\nabla u| + \nabla^2_{\eta\eta} u - \frac{\mathfrak{m}}{r^2}(|\nabla u|^2 - (\nabla_\eta u)^2) + \phi\frac{3|\nabla u||\nabla v| + \langle\nabla u,\nabla v\rangle}{2r},\\ \Delta v =&(\operatorname{tr}_g k - k_{\eta\eta})|\nabla v| + \nabla^2_{\eta\eta} v - \frac{\mathfrak{m}}{r^2}(|\nabla v|^2 - (\nabla_\eta v)^2) + \phi\frac{3|\nabla u||\nabla v| + \langle\nabla u,\nabla v\rangle}{2r}.\end{aligned} \tag{4.21}$$

as well as

$$\begin{aligned}\nabla^2 u =& - k|\nabla u| + \frac{g}{2r}\phi(|\nabla u||\nabla v| + \langle\nabla u,\nabla v\rangle)\\ &- \frac{\mathfrak{m}}{r^2}\nabla u\otimes\nabla u - \frac{\phi}{2r}(\nabla u\otimes\nabla v + \nabla v\otimes\nabla u),\\ \nabla^2 v =& - k|\nabla v| + \frac{g}{2r}\phi(|\nabla u||\nabla v| + \langle\nabla u,\nabla v\rangle)\\ &- \frac{\mathfrak{m}}{r^2}\nabla v\otimes\nabla v - \frac{\phi}{2r}(\nabla u\otimes\nabla v + \nabla v\otimes\nabla u).\end{aligned} \tag{4.22}$$

Proof. We only show the computation for u since the one for v is analogous. We have

$$\bar{\nabla}^2_{\alpha\beta} t = \bar{\Gamma}^t_{\alpha\beta} = \frac{1}{2}\bar{g}^{tt}(\nabla_\alpha g_{\beta t} + \nabla_\beta g_{\alpha t})$$

which implies

$$\bar{\nabla}^2_{rt} t = \bar{\nabla}^2_{tr} t = -\frac{\mathfrak{m}}{r^2}\phi^{-1}$$

and $\bar{\nabla}^2 t = 0$ otherwise. Next, we compute

$$\begin{aligned}
\bar{\nabla}^2_{rr} r^* &= \partial^2_{rr} r^* + \bar{\Gamma}^r_{rr}\partial_r r^* = -\frac{\mathfrak{m}}{r^2}\phi^{-2}, \\
\bar{\nabla}^2_{AA} r^* &= \partial^2_{AA} r^* + \bar{\Gamma}^r_{AA}\partial_r r^* = \frac{1}{r}, \\
\bar{\nabla}^2_{tt} r^* &= \Gamma^r_{tt}\partial_r r^* = -\frac{\mathfrak{m}}{r^2}
\end{aligned}$$

where $A \in \{\theta, \phi\}$. Moreover, $\nabla^2_{\theta\phi} r^* = 0$, $\nabla^2_{At} r^* = 0$ and $\bar{\nabla}^2_{tr} r^* = 0$. Hence,

$$\bar{\nabla}^2 u = \frac{\bar{g}}{r} - \frac{\phi}{2r}(\bar{\nabla} u \otimes \bar{\nabla} v + \bar{\nabla} v \otimes \bar{\nabla} u) - \frac{\mathfrak{m}}{r^2}\bar{\nabla} u \otimes \bar{\nabla} u$$

which implies on any initial data set (M, g, k)

$$\nabla^2 u = -k|\nabla u| + \frac{g}{r} - \frac{\phi}{2r}(\nabla u \otimes \nabla v + \nabla v \otimes \nabla u) - \frac{\mathfrak{m}}{r^2}\nabla u \otimes \nabla u.$$

Next, we observe that on each initial data set

$$|\nabla u||\nabla v| + \langle \nabla u, \nabla v\rangle = \bar{g}(N(u)N, N(v)N) + \langle \nabla u, \nabla v\rangle = 2\phi^{-1}.$$

Therefore, we are lead to the equation

$$\nabla^2 u = -k|\nabla u| + \frac{g}{2r}\phi(|\nabla u||\nabla v| + \langle \nabla u, \nabla v\rangle) - \frac{\phi}{2r}(\nabla u \otimes \nabla v + \nabla v \otimes \nabla u) - \frac{\mathfrak{m}}{r^2}\nabla u \otimes \nabla u.$$

Taking the trace, we obtain

$$\Delta u = -\operatorname{tr}_g k|\nabla u| + \phi\frac{3|\nabla u||\nabla v| + \langle \nabla u, \nabla v\rangle}{2r} - \frac{\mathfrak{m}}{r^2}|\nabla u|^2.$$

Moreover,

$$\nabla^2_{\nu_u\nu_u} u = -k_{\nu_u\nu_u}|\nabla u| - \frac{\mathfrak{m}}{r^2}|\nabla u|^2 + \phi\frac{|\nabla u||\nabla v| - \langle \nabla u, \nabla v\rangle}{2r}$$

and

$$\nabla^2_{\eta\eta} u = -k_{\eta\eta}|\nabla u| - \frac{\mathfrak{m}}{r^2}(\nabla_\eta u)^2.$$

Thus, the result follows. □

Note that by relating u and v to r via the identity $u + v = 2r^*$, systems (4.20) and (4.21) can also be studied for an arbitrary initial data set which does not arise as slice in Schwarzschild. Moreover, system (4.21) reduces to system (4.5) with $a = 1$ in case $m = 0$.

Finally, we would like to point out the importance of equations such as (4.22) lies in the fact that they can be used to characterize slices in certain spacetimes. See for instance [72, 64] for slices in Minkowski space and Proposition 2 in J. Krohn's paper [87] for slices of Schwarzschild.

4.6.3 The charged setting

Finally, we would like to give another example where the double null foliation concept is useful. In [64] the spacetime PMT has been proven via spacetime harmonic functions, and in [16] the PMT with charge has been proven via *charged harmonic* functions, i.e. functions solving $\Delta u = \langle E, \nabla u\rangle$ where E is the electrical field. Given an initial data set (M, g, k) equipped with an electrical field E, we need to combine both approaches and use double null foliations:

Theorem 4.6.6. *Let E be a divergence-free vector field on (M,g,k). Suppose u,v solve the system*

$$\begin{aligned}\Delta u &= \xi E_\eta - \mathrm{tr}_g(k)|\nabla u| \\ \Delta v &= \xi E_\eta + \mathrm{tr}_g(k)|\nabla v|\end{aligned} \tag{4.23}$$

with $|\nabla u|, |\nabla v| \neq 0$, where $\xi = \sqrt{|\nabla v||\nabla u|}$ and $\eta = \frac{\nabla u|\nabla v| + \nabla v|\nabla u|}{|\nabla u|\nabla v| + \nabla v|\nabla u||}$. Then we have

$$\begin{aligned}\mathrm{div}(Z) =& \frac{1}{2|\nabla u|}\left(|\mathcal{E}_+^2 u|^2 + |\nabla u|^2(2\mu - 2K_u - 2|E|^2) + 2|\nabla u|\langle J, \nabla u\rangle\right) \\ &+ \frac{1}{2|\nabla v|}\left(|\mathcal{E}_-^2 v|^2 + |\nabla v|^2(2\mu - 2K_v - 2|E|^2) - 2|\nabla v|\langle J, \nabla v\rangle\right).\end{aligned} \tag{4.24}$$

where K_u, K_v are the Gaussian curvatures of the level-sets of u,v,

$$\begin{aligned}Z =& \nabla|\nabla u| - \Delta u\frac{\nabla u}{|\nabla u|} + \nabla|\nabla v| - \Delta v\frac{\nabla v}{|\nabla v|} + 2\xi^{-1}(|\nabla u||\nabla v| + \langle\nabla u, \nabla v\rangle)E \\ &- \mathrm{tr}_g(k)\nabla u + \mathrm{tr}_g(k)\nabla v + k(\nabla u, \cdot) - k(\nabla v, \cdot),\end{aligned}$$

and

$$\begin{aligned}(\mathcal{E}_+^2)_{ij}u =& \nabla_{ij}^2 u + \xi\eta_i E_j + \xi\eta_j E_i - \xi E_\eta g_{ij} + k_{ij}|\nabla u|, \\ (\mathcal{E}_-^2)_{ij}v =& \nabla_{ij}^2 v + \xi\eta_i E_j + \xi\eta_j E_i - \xi E_\eta g_{ij} - k_{ij}|\nabla v|.\end{aligned}$$

We remark the important role the vector field η plays in both integral formulas. Observe that the above formula recovers Proposition 3.2 of [64] in case $E = 0$ which has been the main ingredient to prove the spacetime PMT, and equation (8.7) of [16] in case $k = 0$ which has been the main ingredient to prove PMT with charge.

Chapter 5

Conclusion

We have successfully proven Theorems A, B and C. It remains an open question to establish the spacetime Penrose inequality in full generality.

Appendix A

Technical aspects of Theorem A

This appendix is based on joint work with Simon Brendle and Florian Johne [22].

We discuss the technical aspects of our proof of the generalized Geroch conjecture which we omitted in the main text. We begin with establishing the stability inequality for stable weighted slicings, Lemma 2.3.1.

A.1 The stability inequality

Let (N^n, g) be a closed Riemannian manifold of dimension $\dim N = n$. Throughout this section, we assume that we are given an stable weighted slicing of order m. Our goal is to show that metric g cannot have positive m-intermediate curvature.

By the first variation formula for weighted area, Corollary 2.1.6, the mean curvature H_{Σ_k} of the slice Σ_k in the manifold Σ_{k+1} satisfies for $1 \leq k \leq m$ the relation

$$H_{\Sigma_k} = -\langle \nabla_{\Sigma_{k-1}} \log \rho_{k-1}, \nu_k \rangle$$

where we set $\rho_k = u_1 \cdot \dots \cdot u_k$. By the second variation formula for weighted area (compare Proposition 2.1.7) we obtain for $1 \leq k \leq m$ the inequality

$$\begin{aligned} 0 \leq &\int_{\Sigma^k} \rho_{k-1} \left(-\psi \Delta_{\Sigma_k} \psi - \psi \langle \nabla_{\Sigma_k} \log \rho_{k-1}, \nabla_{\Sigma_k} \psi \rangle\right) d\mu \\ &- \int_{\Sigma_k} \rho_{k-1} \left(|A_{\Sigma_k}|^2 + \mathrm{Ric}_{\Sigma_{k-1}}(\nu_k, \nu_k) - (\nabla^2_{\Sigma_{k-1}} \log \rho_{k-1})(\nu_k, \nu_k) \right) \psi^2 \, d\mu \end{aligned}$$

for all $\psi \in C^\infty(\Sigma_k)$. By Definition 2.0.5 we may write $\rho_k = \rho_{k-1} v_k$, where $v_k > 0$ is the first eigenfunction of the stability operator for the weighted area functional. The function v_k satisfies

$$\begin{aligned}\lambda_k v_k =& -\Delta_{\Sigma_k} v_k - \langle \nabla_{\Sigma_k} \log \rho_{k-1}, \nabla_{\Sigma_k} v_k \rangle - \left(|A_{\Sigma_k}|^2 + \mathrm{Ric}_{\Sigma_{k-1}}(\nu_k, \nu_k) \right) v_k \\ &+ (\nabla^2_{\Sigma_{k-1}} \log \rho_{k-1})(\nu_k, \nu_k) v_k,\end{aligned}$$

where $\lambda_k \geq 0$ denotes the first eigenvalue of the stability operator.

By setting $w_k = \log v_k$ we record the following equation:

$$\begin{aligned}\lambda_k =& -\Delta_{\Sigma_k} w_k - \langle \nabla_{\Sigma_k} \log \rho_{k-1}, \nabla_{\Sigma_k} w_k \rangle - \left(|A_{\Sigma_k}|^2 + \mathrm{Ric}_{\Sigma_{k-1}}(\nu_k, \nu_k) \right) \\ &+ (\nabla^2_{\Sigma_{k-1}} \log \rho_{k-1})(\nu_k, \nu_k) - |\nabla_{\Sigma_k} w_k|^2.\end{aligned} \tag{A.1}$$

We next record two lemmata connecting the second derivatives on consecutive slices.

Lemma A.1.1 (First slicing identity)**.**
We have for $1 \leq k \leq m$ the identity

$$\Delta_{\Sigma_k} \log \rho_{k-1} + (\nabla^2_{\Sigma_{k-1}} \log \rho_{k-1})(\nu_k, \nu_k) = \Delta_{\Sigma_{k-1}} \log \rho_{k-1} + H^2_{\Sigma_k}.$$

Proof. The above formula follows by applying the formula relating the Laplace operator on a submanifold to the Laplace operator on the ambient space

$$\Delta_{\Sigma_k} f + (\nabla^2_{\Sigma_{k-1}} f)(\nu_k, \nu_k) = \Delta_{\Sigma_{k-1}} f - H_{\Sigma_k} \langle \nabla_{\Sigma_{k-1}} f, \nu_k \rangle.$$

to the function $f = \log \rho_{k-1}$. The gradient term on the right-hand side is rewritten by using the first variation formula for weighted area

$$H_{\Sigma_k} = -\langle \nabla_{\Sigma_{k-1}} \log \rho_{k-1}, \nu_k \rangle.$$

from Corollary 2.1.6. □

Lemma A.1.2 (Second slicing identity)**.**

We have for $1 \le k \le m-1$ *the identity*

$$\begin{aligned}\Delta_{\Sigma_k} \log \rho_k =& \Delta_{\Sigma_k} \log \rho_{k-1} + (\nabla^2_{\Sigma_{k-1}} \log \rho_{k-1})(\nu_k, \nu_k) \\ &- \left(\lambda_k + |A_{\Sigma_k}|^2 + \mathrm{Ric}_{\Sigma_{k-1}}(\nu_k, \nu_k) + \langle \nabla_{\Sigma_k} \log \rho_k, \nabla_{\Sigma_k} w_k \rangle\right).\end{aligned}$$

Proof. This follows from the identity $\log \rho_k = w_k + \log \rho_{k-1}$ together with the equation (A.1). □

Lemma A.1.3 (Stability inequality on the bottom slice)**.**

On the bottom slice Σ_m *we have the inequality*

$$\int_{\Sigma_m} \rho_{m-1}^{-1} \left(\Delta_{\Sigma_{m-1}} \log \rho_{m-1} + H^2_{\Sigma_m}\right) d\mu \ge \int_{\Sigma_m} \rho_{m-1}^{-1} \left(|A_{\Sigma_m}|^2 + \mathrm{Ric}_{\Sigma_{m-1}}(\nu_m, \nu_m)\right) d\mu.$$

Proof. By the second variation of weighted area (compare Proposition 2.1.7) the stability inequality on the bottom slice Σ_m gives

$$\begin{aligned}0 \le& \int_{\Sigma_m} \rho_{m-1} \left(-f \Delta_{\Sigma_m} f - f \langle \nabla_{\Sigma_m} \log \rho_{m-1}, \nabla_{\Sigma_m} f \rangle\right) d\mu \\ &- \int_{\Sigma_m} \rho_{m-1} \left(|A_{\Sigma_m}|^2 + \mathrm{Ric}_{\Sigma_{m-1}}(\nu_m, \nu_m) - (\nabla^2_{\Sigma_{m-1}} \log \rho_{m-1})(\nu_m, \nu_m)\right) f^2 \, d\mu\end{aligned}$$

for all test functions $f \in C^\infty(\Sigma_m)$. Since the weight ρ_{m-1} is positive, we may use the direction $f = \rho_{m-1}^{-1}$ in the stability inequality, and observe

$$\begin{aligned}-\Delta_{\Sigma_m} f = -\Delta_{\Sigma_m} \rho_{m-1}^{-1} &= \rho_{m-1}^{-1} \Delta_{\Sigma_m} \log \rho_{m-1} - \rho_{m-1}^{-3} |\nabla_{\Sigma_m} \rho_{m-1}|^2, \\ -\langle \nabla_{\Sigma_m} \log \rho_{m-1}, \nabla_{\Sigma_m} f \rangle = -\langle \nabla_{\Sigma_m} \log \rho_{m-1}, \nabla_{\Sigma_m} \rho_{m-1}^{-1} \rangle &= +\rho_{m-1}^{3} |\nabla_{\Sigma_m} \rho_{m-1}|^2.\end{aligned}$$

The gradient terms in the previous formulae cancel, and we obtain by rearrangement

$$\begin{aligned}&\int_{\Sigma_m} \rho_{m-1}^{-1} \left(\Delta_{\Sigma_m} \log \rho_{m-1} + (\nabla^2_{\Sigma_{m-1}} \log \rho_{m-1})(\nu_m, \nu_m)\right) d\mu \\ \geq &\int_{\Sigma_m} \rho_{m-1}^{-1} \left(|A_{\Sigma_m}|^2 + \mathrm{Ric}_{\Sigma_{m-1}}(\nu_m, \nu_m)\right) d\mu.\end{aligned}$$

Finally, we use the first slicing equality from Lemma A.1.1 to replace

$$\Delta_{\Sigma_m} \log \rho_{m-1} + (\nabla^2_{\Sigma_{m-1}} \log \rho_{m-1})(\nu_m, \nu_m) = \Delta_{\Sigma_{m-1}} \log \rho_{m-1} + H^2_{\Sigma_m}$$

which finishes the proof. □

Lemma 2.3.1 will follow now by carefully computing and estimating all terms appearing in Lemma A.1.3.

Lemma A.1.4.

We have the inequality

$$\int_{\Sigma_m} \rho_{m-1}^{-1} \left(\Lambda + \mathcal{R} + \mathcal{G} + \mathcal{E}\right) d\mu \leq 0,$$

where the eigenvalue term Λ*, the intrinsic curvature term* $\mathcal{R}$*, the extrinsic curvature term* $\mathcal{E}$*, and the gradient term* $\mathcal{G}$ *are given by*

$$\begin{aligned}&\Lambda = \sum_{k=1}^{m-1} \lambda_k,\ \mathcal{R} = \sum_{k=1}^{m} \mathrm{Ric}_{\Sigma_{k-1}}(\nu_k, \nu_k),\ \mathcal{G} = \sum_{k=1}^{m-1} \langle \nabla_{\Sigma_k} \log \rho_k, \nabla_{\Sigma_k} w_k \rangle, \\ &\text{and } \mathcal{E} = \sum_{k=1}^{m} |A_{\Sigma_k}|^2 - \sum_{k=2}^{m} H^2_{\Sigma_k}.\end{aligned}$$

Proof. If we combine the first slicing equality, Lemma A.1.1, and the second slicing equality, Lemma A.1.2, we obtain for $1 \leq k \leq m-1$ the identity

$$\Delta_{\Sigma_k} \log \rho_k = \Delta_{\Sigma_{k-1}} \log \rho_{k-1} + H^2_{\Sigma_{k-1}} - \left(\lambda_k + |A_{\Sigma_k}|^2 + \mathrm{Ric}_{\Sigma_{k-1}}(\nu_k, \nu_k) + \langle \nabla_{\Sigma_k} \log \rho_k, \nabla_{\Sigma_k} w_k \rangle\right).$$

Summation of the above formula over k from 1 to $m-1$ yields

$$\begin{aligned}\Delta_{\Sigma_{m-1}} \log \rho_{m-1} =& \Delta_{\Sigma_0} \log \rho_0 + \sum_{k=1}^{m-1} H_{\Sigma_k}^2 \\ & - \sum_{k=1}^{m-1} \left(\lambda_k + |A_{\Sigma_k}|^2 + \mathrm{Ric}_{\Sigma_{k-1}}(\nu_k, \nu_k) + \langle \nabla_{\Sigma_k} \log \rho_k, \nabla_{\Sigma_k} w_k \rangle \right).\end{aligned}$$

We plug this equation into the stability inequality, Lemma A.1.3. Moreover, we observe that the weight ρ_0 is constant, the mean curvature of the top slice H_{Σ_1} vanishes, and that the stability inequality contains the mean curvature term $H_{\Sigma_m}^2$ the extrinsic curvature term $|A_{\Sigma_m}|^2$ and the curvature term $\mathrm{Ric}_{\Sigma_{m-1}}(\nu_m, \nu_m)$. Then the lemma follows by grouping the terms suitably. □

The eigenvalue term Λ is non-negative, since it is the sum of the non-negative eigenvalues. We will estimate the other terms below.

The first step is to estimate the gradient terms:

Lemma A.1.5 (Estimate of gradient terms)**.**
We have the estimate

$$\mathcal{G} \geq \sum_{k=2}^{m} \left(\frac{1}{2} + \frac{1}{2(k-1)} \right) H_{\Sigma_k}^2 .$$

Proof. We define for $k \geq 1$ the non-negative real numbers α_k by

$$\alpha_k = \frac{k-1}{2k}.$$

By direct computation one verifies the identity

$$1 - \alpha_{k-1} = \frac{1}{4\alpha_k}$$

for $k \geq 2$. Using the identity $H_{\Sigma_{k+1}} = -\langle \nabla_{\Sigma_k} \log \rho_k, \nu_{k+1} \rangle$, we obtain

$$\begin{aligned}
&\langle \nabla_{\Sigma_k} \log \rho_k, \nabla_{\Sigma_k} w_k \rangle \\
=& \langle \nabla_{\Sigma_k} \log \rho_k, \nabla_{\Sigma_k} (\log \rho_k - \log \rho_{k-1}) \rangle \\
=& (1-\alpha_k) |\nabla_{\Sigma_k} \log \rho_k|^2 - \frac{1}{4\alpha_k} |\nabla_{\Sigma_k} \log \rho_{k-1}|^2 \\
&+ \alpha_k \left| \nabla_{\Sigma_k} \log \rho_k - \frac{1}{2\alpha_k} \nabla_{\Sigma_k} \log \rho_{k-1} \right|^2 \\
=& (1-\alpha_k)\, H^2_{\Sigma_{k+1}} + (1-\alpha_k)\, |\nabla_{\Sigma_{k+1}} \log \rho_k|^2 - (1-\alpha_{k-1})\, |\nabla_{\Sigma_k} \log \rho_{k-1}|^2 \\
&+ \alpha_k \left| \nabla_{\Sigma_k} \log \rho_k - \frac{1}{2\alpha_k} \nabla_{\Sigma_k} \log \rho_{k-1} \right|^2
\end{aligned}$$

for $2 \leq k \leq m-1$. Summation over k from 2 to $m-1$ yields the formula

$$\sum_{k=2}^{m-1} \langle \nabla_{\Sigma_k} \log \rho_k, \nabla_{\Sigma_k} w_k \rangle \geq \sum_{k=2}^{m-1} (1-\alpha_k)\, H^2_{\Sigma_{k+1}} + (1-\alpha_{m-1})\, |\nabla_{\Sigma_m} \log \rho_{m-1}|^2 - |\nabla_{\Sigma_2} \log \rho_1|^2.$$

Moreover, the identity $H_{\Sigma_2} = -\langle \nabla_{\Sigma_1} \log \rho_1, \nu_2 \rangle$ implies

$$\langle \nabla_{\Sigma_1} \log \rho_1, \nabla_{\Sigma_1} w_1 \rangle = |\nabla_{\Sigma_1} \log \rho_1|^2 = H^2_{\Sigma_2} + |\nabla_{\Sigma_2} \log \rho_1|^2.$$

Adding the two inequalities gives

$$\sum_{k=1}^{m-1} \langle \nabla_{\Sigma_k} \log \rho_k, \nabla_{\Sigma_k} w_k \rangle \geq \sum_{k=1}^{m-1} (1-\alpha_k)\, H^2_{\Sigma_{k+1}} + (1-\alpha_{m-1})\, |\nabla_{\Sigma_m} \log \rho_{m-1}|^2$$

which finishes the proof. □

In the next step we rewrite the intrinsic curvature terms with the help of the Gauss equations:

Lemma A.1.6 (Iterated Gauss equations)**.**

The curvature term $\mathcal{R}$ is given by

$$\mathcal{R} = \mathcal{C}_m(e_1, \dots, e_m) + \sum_{k=1}^{m-1} \sum_{p=k+1}^{m} \sum_{q=p+1}^{n} \left(A_{\Sigma_k}(e_p, e_p) A_{\Sigma_p}(e_q, e_q) - A_{\Sigma_k}(e_p, e_q)^2 \right),$$

where $\mathcal{C}_m$ denotes the m-intermediate curvature of the Riemannian manifold (N^n, g).

Proof. Fix a point $x \in \Sigma_m$ and consider an orthonormal basis $\{e_1, \dots, e_n\}$ of $T_x N$ with $e_j = \nu_j$ for $1 \le j \le m$ as above. We observe by the definition of the Ricci curvature on the slice Σ_{k-1}, and by the Gauss equations the formula

$$\begin{aligned}\mathrm{Ric}_{\Sigma_{p-1}}(\nu_p, \nu_p) = \mathrm{Ric}_{\Sigma_{p-1}}(e_p, e_p) &= \sum_{q=p+1}^{n} \mathrm{Rm}_{\Sigma_{p-1}}(e_p, e_q, e_p, e_q) \\ &= \sum_{q=p+1}^{n} \mathrm{Rm}_N(e_p, e_q, e_p, e_q) + \sum_{q=p+1}^{n} \sum_{k=1}^{p-1} \left(A_{\Sigma_k}(e_p, e_p) A_{\Sigma_k}(e_q, e_q) - A_{\Sigma_k}(e_p, e_q)^2 \right).\end{aligned}$$

Summation over p from 1 to m then implies

$$\begin{aligned}\mathcal{R} &= \sum_{p=1}^{m} \mathrm{Ric}_{\Sigma_{p-1}}(\nu_p, \nu_p) \\ &= \sum_{p=1}^{m} \sum_{q=p+1}^{n} \mathrm{Rm}_N(e_p, e_q, e_p, e_q) + \sum_{p=1}^{m} \sum_{q=p+1}^{n} \sum_{k=1}^{p-1} \left(A_{\Sigma_k}(e_p, e_p) A_{\Sigma_k}(e_q, e_q) - A_{\Sigma_k}(e_p, e_q)^2 \right) \\ &= \mathcal{C}_m(e_1, \dots, e_m) + \sum_{p=1}^{m} \sum_{q=p+1}^{n} \sum_{k=1}^{p-1} \left(A_{\Sigma_k}(e_p, e_p) A_{\Sigma_k}(e_q, e_q) - A_{\Sigma_k}(e_p, e_q)^2 \right).\end{aligned}$$

If we interchange the order of summation, the assertion follows. □

Combining the lemmata A.1.6, A.1.4, A.1.5, and reorganizing terms, Lemma 2.3.1 follows.

A.2 Extrinsic curvature estimates

In this section we prove Lemma 2.3.2. We use different estimates for the top slice, the intermediate slices and the bottom slice.

Lemma A.2.1 (Extrinsic curvature terms on top slice).

We have the estimate

$$\mathcal{V}_1 \ge \frac{m^2 - 2 - n(m-2)}{2(n-m)(m-1)} \left(\sum_{p=2}^{m} A_{\Sigma_1}(e_p, e_p) \right)^2 .$$

Proof. To estimate the term $\mathcal{V}_1$, we begin by discarding the off-diagonal terms of the second fundamental form h_{Σ_1}:

$$\begin{aligned}\mathcal{V}_1 =& |A_{\Sigma_1}|^2 + \sum_{p=2}^{m}\sum_{q=p+1}^{n}\left(A_{\Sigma_1}(e_p,e_p)A_{\Sigma_1}(e_q,e_q) - A_{\Sigma_1}(e_p,e_q)^2\right)\\ \geq& \sum_{p=2}^{n} A_{\Sigma_1}(e_p,e_p)^2 + \sum_{p=2}^{m}\sum_{q=p+1}^{n} A_{\Sigma_1}(e_p,e_p)A_{\Sigma_1}(e_q,e_q).\end{aligned}$$

The terms on the right hand side can be rewritten as follows:

$$\mathcal{V}_1 \geq \frac{1}{2}\sum_{p=2}^{m} A_{\Sigma_1}(e_p,e_p)^2 + \sum_{q=m+1}^{n} A_{\Sigma_1}(e_q,e_q)^2 + \sum_{p=2}^{m} A_{\Sigma_1}(e_p,e_p)\, H_{\Sigma_1} - \frac{1}{2}\left(\sum_{p=2}^{m} A_{\Sigma_1}(e_p,e_p)\right)^2.$$

Recall that $H_{\Sigma_1} = 0$. By the Cauchy–Schwarz inequality,

$$\sum_{p=2}^{m} A_{\Sigma_1}(e_p,e_p)^2 \geq \frac{1}{m-1}\left(\sum_{p=2}^{m} A_{\Sigma_1}(e_p,e_p)\right)^2$$

and

$$\sum_{q=m+1}^{n} A_{\Sigma_1}(e_q,e_q)^2 \geq \frac{1}{n-m}\left(\sum_{q=m+1}^{n} A_{\Sigma_1}(e_q,e_q)\right)^2 = \frac{1}{n-m}\left(\sum_{p=2}^{m} A_{\Sigma_1}(e_p,e_p)\right)^2,$$

where in the last step we have used the fact that $H_{\Sigma_1} = 0$. Putting these facts together, the assertion follows. □

Lemma A.2.2 (Extrinsic curvature terms on intermediate slices)**.**
We have for $2 \leq k \leq m-1$ *the estimate*

$$\mathcal{V}_k \geq \frac{m^2 - 2 - n(m-2)}{2(m-1)(n-m)}\left(\sum_{q=m+1}^{n} A_{\Sigma_k}(e_q,e_q)\right)^2.$$

Proof. To estimate the term $\mathcal{V}_k$, we start by discarding the off-diagonal terms:

$$\begin{aligned}\mathcal{V}_k =& |h_{\Sigma_k}|^2 - \left(\frac{1}{2} - \frac{1}{2(k-1)}\right)H_{\Sigma_k}^2 + \sum_{p=k+1}^{m}\sum_{q=p+1}^{n}\left(A_{\Sigma_k}(e_p,e_p)A_{\Sigma_k}(e_q,e_q) - A_{\Sigma_k}(e_p,e_q)^2\right)\\ \geq& \sum_{p=k+1}^{n} A_{\Sigma_k}(e_p,e_p)^2 - \left(\frac{1}{2} - \frac{1}{2(k-1)}\right)H_{\Sigma_k}^2 + \sum_{p=k+1}^{m}\sum_{q=p+1}^{n} A_{\Sigma_k}(e_p,e_p)A_{\Sigma_k}(e_q,e_q).\end{aligned}$$

The terms on the right hand side can be rewritten as follows:

$$\begin{aligned}\mathcal{V}_k \geq & \frac{1}{2} \sum_{p=k+1}^{m} A_{\Sigma_k}(e_p, e_p)^2 + \sum_{q=m+1}^{n} A_{\Sigma_k}(e_q, e_q)^2 \\ & + \frac{1}{2(k-1)} \left(\sum_{p=k+1}^{m} A_{\Sigma_k}(e_p, e_p) \right)^2 - \left(\frac{1}{2} - \frac{1}{2(k-1)} \right) \left(\sum_{q=m+1}^{n} A_{\Sigma_k}(e_q, e_q) \right)^2 \\ & + \frac{1}{k-1} \left(\sum_{p=k+1}^{m} A_{\Sigma_k}(e_p, e_p) \right) \left(\sum_{q=m+1}^{n} A_{\Sigma_k}(e_q, e_q) \right).\end{aligned}$$

The Cauchy–Schwarz inequality gives

$$\sum_{p=k+1}^{m} A_{\Sigma_k}(e_p, e_p)^2 \geq \frac{1}{m-k} \left(\sum_{p=k+1}^{m} A_{\Sigma_k}(e_p, e_p) \right)^2$$

and

$$\sum_{q=m+1}^{n} A_{\Sigma_k}(e_q, e_q)^2 \geq \frac{1}{n-m} \left(\sum_{q=m+1}^{n} A_{\Sigma_k}(e_q, e_q) \right)^2.$$

Moreover, Young's inequality implies

$$\begin{aligned}\left(\sum_{p=k+1}^{m} A_{\Sigma_k}(e_p, e_p) \right) \left(\sum_{q=m+1}^{n} A_{\Sigma_k}(e_q, e_q) \right) \geq & -\frac{m-1}{2(m-k)} \left(\sum_{p=k+1}^{m} A_{\Sigma_k}(e_p, e_p) \right)^2 \\ & - \frac{m-k}{2(m-1)} \left(\sum_{q=m+1}^{n} A_{\Sigma_k}(e_q, e_q) \right)^2.\end{aligned}$$

Putting these facts together, the assertion follows. □

Lemma A.2.3 (Extrinsic curvature terms on bottom slice)**.**

We have the estimate

$$\mathcal{V}_m \geq \frac{m^2 - 2 - n(m-2)}{2(n-m)(m-1)} H_{\Sigma_m}^2. \tag{A.2}$$

Proof. We observe by the the trace estimate for symmetric two-tensors the inequality

$$\begin{aligned}\mathcal{V}_m = |A_{\Sigma_m}|^2 - \left(\frac{1}{2} - \frac{1}{2(m-1)} \right) H_{\Sigma_m}^2 & \geq \left(\frac{1}{n-m} - \left(\frac{1}{2} - \frac{1}{2(m-1)} \right) \right) H_{\Sigma_m}^2 \\ & = \frac{m^2 - 2 - n(m-2)}{2(n-m)(m-1)} H_{\Sigma_m}^2.\end{aligned}$$

□

Clearly, Lemma 2.3.2 follows.

A.3 Existence of minimal slicings

In this section we prove existence of stable weighted slicings of order m. The argument uses the mapping degree and is essentially contained in Theorem 4.5 of [107]. Alternatively, one could also use an argument based on homology, compare with Theorem 4.6 in [107].

Proof of Theorem 2.0.7. Suppose $F : N^n \to \mathbb{T}^m \times M^{n-m}$ has degree $d \neq 0$. The projection of F onto the factors yields maps $f_0 : N \to M$ and maps $f_1, \ldots, f_m : N \to \mathbb{S}^1$. Let Θ be a top-dimensional form of the manifold M normalized such that $\int_M \Theta = 1$, and let θ be a one-form on the circle $\mathbb{S}^1$ with $\int_{\mathbb{S}^1} \theta = 1$. We define the pull-back forms $\Omega := f_0^* \Theta$ and $\omega_j := f_j^* \theta$. By the normalization condition we deduce that $\int_N \omega_1 \wedge \cdots \wedge \omega_m \wedge \Omega = d$.

We claim that one can construct the slices Σ_k and the weights ρ_k, such that $\int_{\Sigma_k} \omega_{k+1} \wedge \cdots \wedge \omega_m \wedge \Omega = d$ holds. We prove the claim by induction. The base case $k = 0$ holds by the previous observation and by setting $\Sigma_0 := N$ and $\rho_0 := 1$. For the induction step we suppose that we have constructed the slice Σ_{k-1} and the weight ρ_{k-1}, such that $\int_{\Sigma_{k-1}} \omega_k \wedge \cdots \wedge \omega_m \wedge \Omega = d$.

We define a class $\mathcal{A}_k$ by

$$\mathcal{A}_k = \left\{ \Sigma \text{ is an } (n-k) - \text{integer rectifiable current in } \Sigma_k \text{ with } \int_\Sigma \omega_{k+1} \wedge \cdots \wedge \omega_m \wedge \Omega = d \right\}.$$

The first step is to show that the above class is non-empty. Suppose $p_k \in \mathbb{S}^1$ is a regular value of the map $f_k|_{\Sigma_{k-1}} : \Sigma_{k-1} \to \mathbb{S}^1$. Then the the preimage $\tilde{\Sigma}_k = \{x \in \Sigma_{k-1} : f_k(x) = p_k\}$ is a smooth and embedded hypersurface in Σ_{k-1}. A priori $\tilde{\Sigma}_k$ might be empty. On the complement $\mathbb{S}^1 \backslash \{p_k\}$ the one-form θ is exact. In other words, there exists a function $\psi_k : \mathbb{S}^1 \backslash \{p_k\} \to \mathbb{R}$, such that $d\psi_k = \theta$. Moreover, due to the normalization condition $\int_{\mathbb{S}^1} \theta = 1$, the function ψ_k jumps by 1 at p_k.

We define a function $\varphi_k : \Sigma_{k-1} \backslash \tilde{\Sigma}_k \to \mathbb{R}$ by setting $\varphi_k := \psi_k \circ f_k$. Since the pull-back commutes with the differential we deduce $d\varphi_k = f_k^* (d\psi_k) = f_k^* \theta = \omega_k$ on $\Sigma_{k-1} \backslash \tilde{\Sigma}_k$.

The above observation (and the closedness of the forms $\omega_k, \ldots, \omega_m, \Omega$) implies

$$d(\varphi_k \, \omega_{k+1} \wedge \cdots \wedge \omega_m \wedge \Omega) = \omega_k \wedge \omega_{k+1} \wedge \cdots \wedge \omega_m \wedge \Omega.$$

Let us integrate the above relation over $\Sigma_{k-1} \smallsetminus \tilde{\Sigma}_k$. By Stokes theorem, the integral of the left hand side yields two boundary integrals over $\tilde{\Sigma}_k$. Since φ_k jumps by 1 at $\tilde{\Sigma}_k$, we obtain

$$\begin{aligned} \int_{\tilde{\Sigma}_k} \omega_{k+1} \wedge \cdots \wedge \omega_m \wedge \Omega &= \int_{\Sigma_{k-1} \smallsetminus \tilde{\Sigma}_k} d(\varphi_k \, \omega_{k+1} \wedge \cdots \wedge \omega_m \wedge \Omega) \\ &= \int_{\Sigma_{k-1} \smallsetminus \tilde{\Sigma}_k} \omega_k \wedge \omega_{k+1} \wedge \cdots \wedge \omega_m \wedge \Omega \\ &= d. \end{aligned}$$

In particular, $\tilde{\Sigma}_k$ is non-empty and belongs to the class $\mathcal{A}_k$. This shows that the class $\mathcal{A}_k$ is non-empty.

We consider the variational problem

$$\sigma_k = \inf \left\{ \mathbb{M}_{\rho_{k-1}, n-k}(\Sigma) : \Sigma \in \mathcal{A}_k \right\},$$

where $\mathbb{M}_{\rho_{k-1}, n-k}$ denotes the ρ_{k-1}-weighted mass functional on $(n-k)$-integer rectifiable currents. By the compactness theory for integer rectifiable currents, compare for example Theorem 7.5.3 in [?], we deduce that there exists an $(n-k)$-integer rectifiable current Σ_{k+1} with mass $\mathbb{M}_{\rho_{k-1}, n-k}(\Sigma_k) = \sigma_k$.

By the regularity theory for integer rectifiable currents, compare for example Theorem 7.5.8 in [?] or the survey [43], and the dimension bound $n \le 7$ we deduce that Σ_k

is a smooth, two-sided and embedded hypersurface. Moreover, the smooth surface Σ_k is stable with respect to variations of the weighted area, and therefore we can find a positive first eigenfunction v_k of the weighted stability operator. Defining the weight ρ_k by the formula $\rho_k = \rho_{k-1} \cdot v_k$ completes the induction step. $\square$

Appendix B

Technical aspects of Theorem B

This appendix is based upon joint work with D. Kazaras and M. Khuri [64] and with Yiyue Zhang HirschZhang.

We address several technical difficulties arising in the proof of Theorem B. We would like to highlight that the majority of these difficulties are caused by the vanishing of the gradient of the spacetime harmonic function.

B.1 The generalized exterior region

Recall the central formula

$$\operatorname{div}(\nabla|\nabla u| + k(\nabla u, \cdot)) + K|\nabla u| = \frac{|\bar{\nabla}^2 u|^2}{2|\nabla u|} + \mu|\nabla u| + \langle J, \nabla u\rangle.$$

Upon integration and by use of Gauss-Bonnet's theorem, this formula leads to the spacetime PMT assuming the level-sets of u are not spherical. In the main text we assumed that M^3 is topologically $\mathbb{R}^3$. In this case spherical level-sets must bound a region. However, in view of the strong maximum principle, this is impossible.

In general this does not need to be the case. For instance, for $\mathbb{R}^3 \# \mathbb{S}^2 \times \mathbb{S}^1$ spherical level-sets can be present. To solve this issue we will construct a so-called *generalized exterior region*, i.e. an asymptotically flat manifold (M_{ext}, g, k) whose end coincides with (M, g, k), whose second homology is trivial and whose boundary consist entirely of MITS and MOTS (i.e. surfaces where $\theta_\pm = H \pm \operatorname{tr}_\Sigma k = 0$). On these MOTS and MITS boundaries we then carefully choose appropriate boundary conditions for the spacetime harmonic function u. This will be the subject of this and the next section.

In Lemma 4.1 of [79] G. Huisken and T. Ilmanen established the existence of an exterior region for asymptotically flat Riemannian 3-manifolds, showing that for each asymptotic end there is such a region which is diffeomorphic to the complement of a finite union of balls in $\mathbb{R}^3$. They accomplished this by removing all compact minimal surfaces, including immersed ones, to identify the trapped region and remove it. As pointed out by Lee in [90, page 140], the weaker topological simplification $H_2(M_{ext}, \partial M_{ext}; \mathbb{Z}) = 0$ may still be achieved by only removing embedded compact minimal surfaces. His proof relies on the classical result that within each nontrivial 2-dimensional homology class there exists an area minimizing minimal surface representative. Due to the lack of a variational characterization, such a result is not currently known for MOTS. Nevertheless, the conclusion of Lee's observation still remains valid in spirit with the role of minimal surfaces replaced by that of MOTS and MITS.

Proposition B.1.1. *Let (M, g, k) be a smooth asymptotically flat initial data set satisfying the dominant energy condition. Then for each end M_{end}, there exists a new initial data set $(M_{ext}, g_{ext}, k_{ext})$ having a single end which is isometric (as initial data) to (M_{end}, g, k). Furthermore, M_{ext} is orientable, satisfies $H_2(M_{ext}, \partial M_{ext}; \mathbb{Z}) = 0$, and has a boundary ∂M_{ext} consisting entirely of MOTS and MITS.*

Proof. There are two primary steps. The first is to identify appropriate (possibly immersed) MOTS and MITS to remove from M in order to obtain a subset $M' \supset M_{end}$, whose compactification admits a positive scalar curvature metric. The second step entails reducing the first Betti number of M' to zero via an iterative process which involves passing to finite sheeted covers. The proof of the first step is based on a reorganization of the arguments used for [5, Theorem 1.2], and thus only an outline of the main ideas will be given here. The second step will be described in detail. In what follows, we assume without loss of generality that M is orientable by passing to the orientable double cover if necessary.

According to [47, Theorem 22] there is a sequence of perturbed initial data (M, g_i, k_i) with $g_i \to g$ in $W^{3,p}_{-q}(M)$ and $k_i \to k$ in $W^{2,p}_{-q-1}(M)$ as $i \to \infty$ for $p > 3$, such that a strict dominant energy condition is satisfied $\mu_i > |J_i|_{g_i}$. To this end, solve the Jang equation [46, Proposition 7] for (M, g_i, k_i) with standard asymptotic decay in each end. Note that the assumed decay on $\mathrm{Tr}_g k$ is not in general sufficient to guarantee bounded solutions of Jang's equation near infinity. However, as pointed out in [5, Remarks 2.2 and 3.1], this technicality can be avoided by an appropriate deformation of the initial data in the asymptotic ends. The solution of Jang's equation gives rise to a hypersurface in $\mathbb{R} \times M$ which is a vertical graph over an open subset of M containing the asymptotic ends; $\Omega_i \subset M$ will denote the component of this open set that contains the designated end M_{end}. The components of the boundary $\partial\Omega_i$ are spherical MOTS or MITS that satisfy a uniform C-almost minimization property [5, Remark 2.3], [44]. Note that the spherical topology is due to the strict dominant energy condition and stability property of the Jang graph. Observe that due to the strict dominant energy condition, the proof of [5, Theorem 1.2] shows that a conformal change of metric may be introduced, after preliminary deformations along the asymptotically cylindrical ends as well as in the asymptotically flat ends, to arrive at a positive scalar curvature (PSC) metric on the manifold obtained by compactifying the asymptotically flat ends of Ω_i, which also has a Riemannian product structure near each boundary component.

Next, by the compactness theory of [44, 45], the sequence $\partial\Omega_i$ subconverges in the $C^{2,\alpha}$ local graph sense to a set $\mathcal{S}$ which is a finite collection of MOTS $\{\mathcal{S}^+_a\}_{a=1}^{a_0}$ and MITS $\{\mathcal{S}^-_b\}_{b=1}^{b_0}$ in (M, g, k). Moreover, each of these MOTS and MITS arises from a sequence of connected closed properly embedded MOTS $\mathcal{S}_{ai}^{+} \subset \partial\Omega_i$ or MITS $\mathcal{S}_{bi}^{-} \subset \partial\Omega_i$ with respect to (g_i, k_i). We claim that $\mathcal{S}$ is a smooth submanifold. If a MOTS $\mathcal{S}_a^+$ or a MITS $\mathcal{S}_b^-$ remains disjoint from the other MOTS and MITS of $\mathcal{S}$, then this component is a smooth submanifold. If $\mathcal{S}_a^+$ or $\mathcal{S}_b^-$ has nontrivial intersection and does not coincide with another member of the MOTS and MITS comprising $\mathcal{S}$, this violates the C-almost minimization property of $\partial\Omega_i$ for large i. Thus the MOTS and MITS in $\mathcal{S}$ are pairwise disjoint, and hence are smooth submanifolds.

To conclude the first step, remove the surface $\mathcal{S}$ from M and take the metric completion of the component containing the designated end M_{end} to obtain an initial data set (M', g, k). Note that this contains (M_{end}, g, k), has boundary components consisting entirely of smooth MOTS and MITS, and the topology of M' agrees with that of Ω_i for large i. Because Ω_i admits a PSC metric having Riemannian product structure near each boundary component, we may apply the prime decomposition theorem along with a result of Gromov-Lawson [58] and the resolution of the Poincaré conjecture to deduce that manifold M' has PSC topology. That is, M' is diffeomorphic to a finite connected sum of spherical spaces, $S^1 \times S^2$'s, and $\mathbb{R}^3$'s representing the ends, all with a finite number of 3-balls removed which indicate the horizons. Thus, to conclude the first step of the proof, we have produced an asymptotically flat initial data set (M', g, k) having PSC topology, with boundary $\partial M'$ consisting of MOTS and MITS components, and is such that one of the ends coincides with (M_{end}, g, k).

In the second step of the proof the first Betti number of M' will be reduced to zero with an iterative procedure. Since $H_2(M', \partial M'; \mathbb{Z})$ is Poincaré dual to $H^1(M'; \mathbb{Z})$, which is itself isomorphic to the torsion-free subgroup of $H_1(M'; \mathbb{Z})$, this procedure will result in the desired conclusion of vanishing second homology relative to the boundary. As observed above, M' can be expressed as the compliment of finitely many disjoint balls in $\#^l(S^1 \times S^2)\#N$ where N is a rational homology sphere. Since N has vanishing first Betti number, $b_1(M')$ is equal to the number of its handle $S^1 \times S^2$ summands. We proceed by constructing a particular double cover of M'. Let $\Sigma' \subset \mathring{M}'$ be the image of an embedding of S^2 in one of the $S^1 \times S^2$ summands of M' which is homologous $\{\text{pt}\} \times S^2 \subset S^1 \times S^2$. Define W to be the metric completion of $M' \setminus \Sigma'$ and notice that its boundary can be decomposed as

$$\partial W = \partial M' \cup \Sigma'_1 \cup \Sigma'_2,$$

where Σ'_1 and Σ'_2 are copies of Σ'. Next, consider the manifold

$$\overline{M} = W_1 \sqcup W_2 / \sim,$$

where W_1 and W_2 are copies of W and the relation $\sim$ identifies $\Sigma'_1 \subset W_1$ with $\Sigma'_2 \subset W_2$ and $\Sigma'_2 \subset W_1$ with $\Sigma'_1 \subset W_2$. The manifold $\overline{M}$ is a two-fold cover of M, classified by the mod 2 reduction of the cohomology class Poincaré dual to $[\Sigma]$, and the pullback of the data (g, k) to $\overline{M}$ will be denoted by $(\overline{g}, \overline{k})$. Furthermore, observe that $\overline{M}$ is diffeomorphic to the complement of finitely many disjoint balls in

$$\left(\#^{l-1}(S^1 \times S^2)\#N\right)\#(S^1 \times S^2)\#\left(\#^{l-1}(S^1 \times S^2)\#N\right),$$

so that

$$b_1(\overline{M}) = 2b_1(M') - 1. \tag{B.1}$$

Consider the two ends of $\overline{M}$ that are isometric to M_{end}, and choose one for reference and denote it by $\mathcal{E}$. The boundary of the double cover may be decomposed as $\partial\overline{M} = \partial_+\overline{M} \cup \partial_-\overline{M}$, where $\theta_\pm = 0$ on $\partial_\pm\overline{M}$ and the null expansions are computed with respect to the unit normal pointing inside $\overline{M}$. Now let $\mathcal{D} \subset \overline{M}$ be the bounded component that remains after removing sufficiently large coordinate spheres in each of the asymptotic ends of $\overline{M}$. The boundary may be decomposed into two types of surfaces $\partial\mathcal{D} = \partial_{out}\mathcal{D} \cup$

$\partial_{in}\mathcal{D}$, in which $\theta_+ \geq 0$ on $\partial_{out}\mathcal{D}$ with respect to the normal pointing out of $\mathcal{D}$, and $\theta_+ \leq 0$ on $\partial_{in}\mathcal{D}$ with respect to the normal pointing into $\mathcal{D}$. Note that MOTS boundary components belong to $\partial_{in}\mathcal{D}$, while MITS components belong to $\partial_{out}\mathcal{D}$. Moreover the coordinate sphere boundary in $\mathcal{E}$ satisfies the strict inequality $\theta_+ > 0$ and belongs to $\partial_{out}\mathcal{D}$, while the coordinate sphere boundaries lying in the remaining ends satisfy the strict inequality $\theta_+ < 0$ and belong to $\partial_{in}\mathcal{D}$. It follows that we may apply the MOTS existence result [48, Theorem 4.2], or rather a slight generalization of it to allow for nonstrict inequalities (see [7, Section 5] or [45, Remark 4.1]), to obtain an outermost (with respect to $\mathcal{E}$) MOTS $\Sigma \subset \mathcal{D}$ that separates $\partial_{out}\mathcal{D}$ from $\partial_{in}\mathcal{D}$. Furthermore, this surface separates $\overline{M}$ into two disjoint regions $\overline{M} \setminus \Sigma = \overline{M}_{out} \cup \overline{M}_{in}$, where $\overline{M}_{out}$ is the component containing the reference end $\mathcal{E}$, see Figure B.1.

In the remainder of the argument, we will first consider the case in which $(\overline{M}, \overline{g}, \overline{k})$ satisfies a strict dominant energy condition, and will subsequently explain the alterations required for the general case. By the strict dominant energy condition, stability of outermost MOTS, and orientability of $\overline{M}$, it follows that Σ consists of finitely many disjoint embedded spheres. Now consider the Mayer-Vietoris sequence associated with the decomposition $\overline{M} = \overline{M}_{out} \cup \overline{M}_{in}$, that is

$$\cdots \longrightarrow H_1(\Sigma;\mathbb{R}) \longrightarrow H_1(\overline{M}_{out};\mathbb{R}) \oplus H_1(\overline{M}_{in};\mathbb{R}) \longrightarrow H_1(\overline{M};\mathbb{R}) \longrightarrow \cdots.$$

Since $H_1(\Sigma;\mathbb{R}) = 0$ we find that that $b_1(\overline{M}_{out}) + b_1(\overline{M}_{in}) \leq b_1(\overline{M})$. Taking (B.1) into consideration shows that either $\overline{M}_{out}$ or $\overline{M}_{in}$ must have first Betti number strictly less than $b_1(M')$; label the component of this manifold that contains an isometric copy of M_{end}, by $\overline{M}'$. Notice that each component of the boundary of $\overline{M}'$ is either a MOTS or a MITS. Moreover, as Σ is spherical, the sets $\overline{M}_{out}$ and $\overline{M}_{in}$ give rise to a connected sum decomposition of $\overline{M}$. It follows that both $\overline{M}_{out}$ and $\overline{M}_{in}$ are diffeomorphic to the compliment of finitely many disjoint balls in the connected sum of $S^1 \times S^2$'s and a rational homology sphere. Furthermore, we may assume that $\overline{M}'$ has a single end, since if necessary attention may be restricted to the region outside the outermost MOTS to isolate the isometric copy of M_{end}. This same procedure can be applied to $\overline{M}'$ to once again reduce the first Betti number by at least one. Continuing in this manner yields the desired initial data $(M_{ext}, g_{ext}, k_{ext})$.

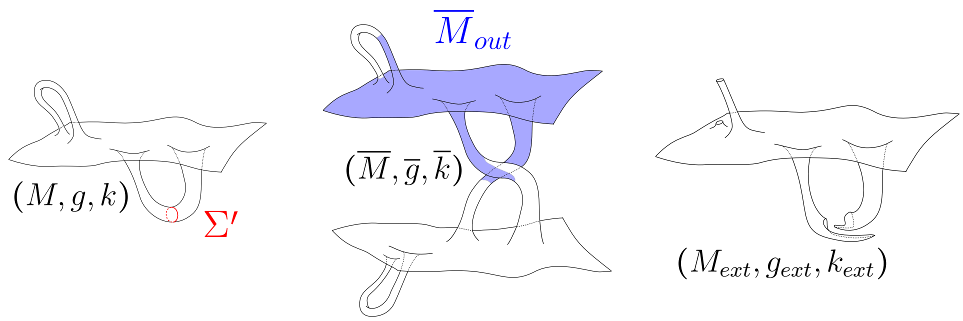


Figure B.1: A schematic description of the stages in the second step in the proof of Proposition B.1.1.

To finish, we describe the modifications necessary to accomplish the construction in the above paragraph in the general case when $(\overline{M}, \overline{g}, \overline{k})$ satisfies the dominant energy condition, but not strictly so. In this case, apply the approximating argument from the first step to obtain a sequence $(\overline{g}_i, \overline{k}_i)$ on $\overline{M}$ satisfying the strict dominant energy condition and which converges to $(\overline{g}, \overline{k})$. Note that a minor refinement of [47, Theorem 22] is required for this due to the presence of boundary components, see [5, footnote - page 869]. The outermost MOTS Σ_i that induces a separation $\overline{M} = \overline{M}^i_{out} \cup \overline{M}^i_{in}$, admits the C-almost minimization property and consists of spherical MOTS and MITS. By the arguments of the previous paragraph, the first Betti number of either $\overline{M}^i_{out}$ or $\overline{M}^i_{in}$ is strictly less than $b_1(M')$. As described in the first step of the proof, Σ_i subconverges to a limiting MOTS/MITS surface $\overline{\mathcal{S}}$ in $\overline{M}$, and we may consider the metric completion $\widehat{M}$ of $\overline{M} \setminus \overline{\mathcal{S}}$. The two components of $\widehat{M}$ containing the isometric copies of M_{end}, have the same topology as components of $\overline{M}^i_{out}$ or $\overline{M}^i_{in}$ for sufficiently large i. It follows that one of them, $\widehat{M}'$, satisfies $b_1(\widehat{M}') < b_1(M')$. As above it may be assumed that the component $\widehat{M}'$ possesses one end modeling $\mathcal{E}$. Moreover its boundary consists of MOTS and MITS, and it is diffeomorphic to the compliment of finitely many disjoint balls in the connected sum of $S^1 \times S^2$'s and a rational homology sphere. Thus the iteration may be continued to obtain the desired conclusion. □

B.2 Controlling the level-set topology

As seen in Figure B.1, we have now created a manifold with vanishing second homology, but we might have created a MITS / MOTS boundary. In the Riemannian case (i.e. the $k = 0$ setting) two strategies have been pursued in [18]: First, it is possible to fill-in the boundary components to create a manifold without boundary. Second, it is possible to use Neumann boundary conditions for the spacetime harmonic funtions for which one can verify that they don't contribute negatively to the integral formula. However, both approaches are not available in the spacetime ($k \neq 0$) setting.

In order to apply the main integral inequality Proposition B.3.2 below successfully, it is important to ensure that the Euler characteristic of regular level sets for the spacetime harmonic function does not exceed 1. In this section, we show that it is possible to choose the spacetime harmonic function, by carefully selecting its Dirichlet data, to achieve this goal for the level sets. Since this will be employed for generalized exterior regions, here we consider asymptotically flat initial data (M, g, k) with a single asymptotic end, although the boundary may have several components $\partial M = \cup_{i=1}^{n} \partial_i M$. Let v solve (3.26), (3.27) and consider the Dirichlet problem

$$\Delta u_{\mathbf{c}} + \mathrm{tr}_g\, k |\nabla u_{\mathbf{c}}| = 0 \qquad \text{on} \qquad M, \tag{B.2}$$

$$u_{\mathbf{c}} = c_i \quad \text{on} \quad \partial_i M, \quad i = 1, \ldots, n, \qquad u_{\mathbf{c}} = v + O_2(r^{1-2q}) \quad \text{as} \quad r \to \infty, \tag{B.3}$$

where $\mathbf{c} = (c_1, \ldots, c_n)$ are constants. The following is a technical preliminary result that indicates how to choose the constants $\mathbf{c}$ in order to achieve the main topological conclusions of Theorem B.2.2 concerning level sets, as well as to aid with the computation of boundary terms in the integral inequality Proposition B.3.2.

Lemma B.2.1. *Let $u_{\mathbf{c}}$ be the solution of* (B.2), (B.3) *given by Theorem 3.6.2.*

1. *Let $(-1)^{\varsigma_i}$, $\varsigma_i \in \{0, 1\}$ be a choice of sign associated with each boundary component $i = 1, \ldots, n$. There exists a set of constants $\mathbf{c}$ such that for each boundary component there is a point $y_i \in \partial_i M$ with $|\nabla u_{\mathbf{c}}(y_i)| = 0$, and in addition $(-1)^{\varsigma_i} \partial_\upsilon u_{\mathbf{c}} \geq 0$ on $\partial_i M$, where υ is the unit normal to ∂M pointing outside M.*

2. *If $v \neq 0$, then within each boundary component $\partial_i M$, the set of points at which $|\nabla u_{\mathbf{c}}| = 0$ is nowhere dense.*

Proof. A priori estimates established in the previous section show that $u_{\mathbf{c}}$ is continuously differentiable in $\mathbf{c}$. Set $u_i := \partial_{c_i} u_{\mathbf{c}}$ and observe that these functions satisfy

$$\Delta u_i + \operatorname{tr}_g k \frac{\nabla u_{\mathbf{c}}}{|\nabla u_{\mathbf{c}}|} \cdot \nabla u_i = 0 \qquad \text{on} \qquad M, \tag{B.4}$$

$$u_i = \delta_{ij} \quad \text{on} \quad \partial_j M, \;\; j = 1, \ldots, n, \qquad\qquad u_i = O(r^{1-2q}) \quad \text{as} \quad r \to \infty. \tag{B.5}$$

Clearly the set of functions $\{u_1, \ldots, u_n\}$ is linearly independent. Pick $y_i \in \partial_i M$, $i = 1, \ldots, n$ and set $\mathbf{y} = (y_1, \ldots, y_n)$. We claim that the Jacobian matrix

$$U(\mathbf{c}, \mathbf{y}) = \begin{bmatrix} \partial_\upsilon u_1(y_1) & \partial_\upsilon u_2(y_1) & \ldots & \partial_\upsilon u_n(y_1) \\ \partial_\upsilon u_1(y_2) & \partial_\upsilon u_2(y_2) & \ldots & \partial_\upsilon u_n(y_2) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_\upsilon u_1(y_n) & \partial_\upsilon u_2(y_n) & \ldots & \partial_\upsilon u_n(y_n) \end{bmatrix}$$

is invertible, where υ is the unit outer normal to ∂M. Suppose by way of contradiction that it is not invertible. Then there exist constants b_i, $i = 1, \ldots, n$, not all zero, such that $u = \sum_{i=1}^n b_i u_i$ satisfies $\partial_\upsilon u(y_j) = 0$, $j = 1, \ldots, n$. Note that the function u satisfies

$$\Delta u + \operatorname{tr}_g k \frac{\nabla u_{\mathbf{c}}}{|\nabla u_{\mathbf{c}}|} \cdot \nabla u = 0 \quad \text{on} \quad M,$$

$$u = b_j \quad \text{on} \quad \partial_j M, \;\; j = 1, \ldots, n, \qquad\qquad u = O(r^{1-2q}) \quad \text{as} \quad r \to \infty.$$

Since b_i are not all zero, we have that $u \neq 0$. On the other hand, by the maximum principle either the global max or min must be achieved on $\partial_{i_0} M$ for some i_0. By the Hopf lemma, we then have $\partial_\upsilon u(y_{i_0}) \neq 0$. However this contradicts the basic property of u described above. It follows that U is invertible.

We now show that $U(\mathbf{c}, \mathbf{y})$ stays uniformly bounded and away from being singular. To see this, suppose that for a sequence $\{(\mathbf{c}_l, \mathbf{y}_l)\}_{l=1}^\infty$ either $\| \, U(\mathbf{c}_l, \mathbf{y}_l) \, \| \to \infty$ or $\det U(\mathbf{c}_l, \mathbf{y}_l) \to 0$. Observe that the solutions $\partial_{c_i} u_{\mathbf{c}_l}$ of (B.4), (B.5) are uniformly controlled in $W^{2,p}_{loc}(M)$ by the L^p estimates, since the first order coefficients remain uniformly bounded. It follows that there is subsequential convergence in $C^{1,\alpha}(M)$ to a solution $\partial_{c_i} u_\infty$. Consequently, using that the sequence $\{\mathbf{y}_l\} \subset \Pi_{i=1}^n \partial_i M$ lies in a compact set, we find that there is subconvergence $U(\mathbf{c}_l, \mathbf{y}_l) \to U(\infty)$. However, the arguments of the previous paragraph show that $U(\infty)$ is invertible, and this contradiction yields the desired conclusion. In particular, $U^{-1}(\mathbf{c}, \mathbf{y})$ is uniformly bounded.

Consider the map $\mathfrak{U} : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\mathfrak{U}(c_1, \ldots, c_n) = (\partial_\upsilon u_{\mathbf{c}}(y_1(\mathbf{c})), \ldots, \partial_\upsilon u_{\mathbf{c}}(y_n(\mathbf{c}))),$$

where $y_i(\mathbf{c}) \in \partial_i M$ is a point at which $\partial_\upsilon u_{\mathbf{c}}$ achieves its: minimum over this component when $\varsigma_i = 0$, or maximum over this component when $\varsigma_i = 1$. Observe that $\mathfrak{U}$ is continuous. Moreover, it will be shown that this function is differentiable in certain directions, and the matrix U will play a role similar to a Jacobian for $\mathfrak{U}$. Set $\mathbf{p}_0 = (\partial_\upsilon u_0(y_1(0)), \ldots, \partial_\upsilon u_0(y_n(0)))$, and let $\mathbf{p}(t) \subset \mathbb{R}^n$ be a smooth curve emanating from $\mathbf{p}_0 = \mathbf{p}(0)$ and ending at $\mathbf{p}(1) = 0$. We claim that there is a smooth curve $\mathbf{c}(t)$, $t \in [0,1]$, emanating from $\mathbf{c}(0) = 0$, such that $\mathfrak{U}(\mathbf{c}(t)) = \mathbf{p}(t)$. To find this solve the ODE initial value problem

$$\mathbf{c}'(t) = U^{-1}(\mathbf{c}(t), \mathbf{y}(\mathbf{c}(t)))\mathbf{p}'(t), \qquad \mathbf{c}(0) = 0. \tag{B.6}$$

Observe that global existence holds since $U^{-1}(\mathbf{c}, \mathbf{y}(\mathbf{c}))$ is uniformly bounded independent of $\mathbf{c}$.

We will now show that $\mathfrak{U}(\mathbf{c}(t))$ is differentiable. Let $y_i(\mathbf{c})$ be a minimum point for

$\partial_\upsilon u_{\mathbf{c}}$ on $\partial_i M$, and $0 \le s < t \le 1$; a similar argument holds or a maximum point. Then

$$\begin{aligned}
&\partial_\upsilon u_{\mathbf{c}(t)}(y_i(\mathbf{c}(t))) - \partial_\upsilon u_{\mathbf{c}(s)}(y_i(\mathbf{c}(s)))\\
=&\left[\partial_\upsilon u_{\mathbf{c}(t)}(y_i(\mathbf{c}(t))) - \partial_\upsilon u_{\mathbf{c}(s)}(y_i(\mathbf{c}(t)))\right]\\
&+\left[\partial_\upsilon u_{\mathbf{c}(s)}(y_i(\mathbf{c}(t))) - \partial_\upsilon u_{\mathbf{c}(s)}(y_i(\mathbf{c}(s)))\right]\\
\ge&\partial_\upsilon u_{\mathbf{c}(t)}(y_i(\mathbf{c}(t))) - \partial_\upsilon u_{\mathbf{c}(s)}(y_i(\mathbf{c}(t)))\\
=&\sum_j \partial_\upsilon \partial_{c_j} u_{\mathbf{c}(t)}(y_i(\mathbf{c}(t))c_j'(t)(t-s) + o(t-s)\\
=&p_i'(t)(t-s) + o(t-s),
\end{aligned}$$

and

$$\begin{aligned}
&\partial_\upsilon u_{\mathbf{c}(t)}(y_i(\mathbf{c}(t))) - \partial_\upsilon u_{\mathbf{c}(s)}(y_i(\mathbf{c}(s)))\\
=&\left[\partial_\upsilon u_{\mathbf{c}(t)}(y_i(\mathbf{c}(t))) - \partial_\upsilon u_{\mathbf{c}(t)}(y_i(\mathbf{c}(s)))\right]\\
&+\left[\partial_\upsilon u_{\mathbf{c}(t)}(y_i(\mathbf{c}(s))) - \partial_\upsilon u_{\mathbf{c}(s)}(y_i(\mathbf{c}(s)))\right]\\
\le&\partial_\upsilon u_{\mathbf{c}(t)}(y_i(\mathbf{c}(s))) - \partial_\upsilon u_{\mathbf{c}(s)}(y_i(\mathbf{c}(s)))\\
=&\sum_j \partial_\upsilon \partial_{c_j} u_{\mathbf{c}(s)}(y_i(\mathbf{c}(s))c_j'(s)(t-s) + o(t-s)\\
=&p_i'(s)(t-s) + o(t-s),
\end{aligned}$$

where we have used Taylor's theorem and (B.6) with the notation $\mathbf{p}(t) = (p_1(t), \dots, p_n(t))$. Dividing both sides of these equations by $t-s$ and letting $t \to s$ shows that $\mathfrak{U}(\mathbf{c}(t))$ is differentiable, and

$$\frac{d}{dt}\mathfrak{U}(\mathbf{c}(t)) = \mathbf{p}'(t).$$

Integrating this equation then gives the desired relation. We now have $\mathfrak{U}(\mathbf{c}(1)) = 0$, so that $\mathbf{c}(1)$ is the claimed set of constants such that

$$\partial_\upsilon u_{\mathbf{c}(1)}(y_i(\mathbf{c}(1))) = 0, \qquad i = 1, \dots, n.$$

This completes the proof of (1).

Consider now part (2). Suppose that the set within $\partial_i M$ on which $|\nabla u_{\mathbf{c}}| = 0$ has a nonempty interior. Then since equation (B.2) may be viewed as a linear equations with bounded coefficients, the unique continuation result [**?**, Theorem 1.7] applies to show that $u_{\mathbf{c}} \equiv const$. This contradicts the assumption that $v \neq 0$. Since the set on which $|\nabla u_{\mathbf{c}}| = 0$ is also closed, it follows that it is nowhere dense. □

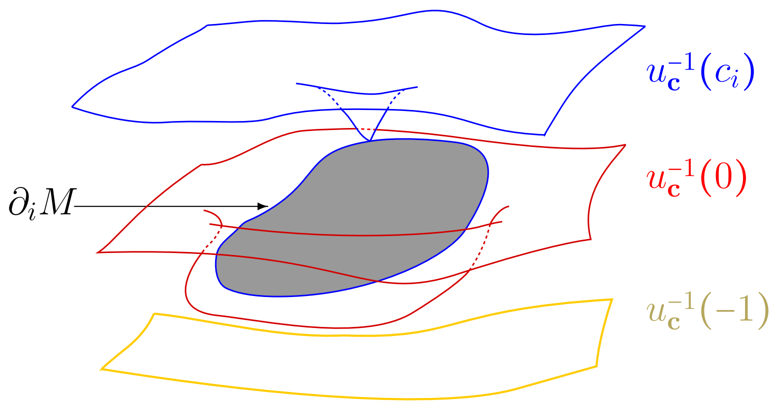


Figure B.2: Possible level sets of the function $u_{\mathbf{c}}$ constructed in Lemma B.2.1.

We are now in a position to establish the main topological result concerning regular level sets of the spacetime harmonic function $u_{\mathbf{c}}$ arising from Lemma B.2.1. This will later be employed in generalized exterior regions which have a simplified topology, although we do not use here the MOTS and MITS condition on the boundary of such regions.

Theorem B.2.2. *Let (M, g, k) be a smooth asymptotically flat initial data set having a single asymptotic end, and satisfying $H_2(M, \partial M; \mathbb{Z}) = 0$. Let $u_{\mathbf{c}}$ be the solution of* (B.2)*,* (B.3) *with $\mathbf{c}$ given by Lemma B.2.1. Then all regular level sets of $u_{\mathbf{c}}$ are connected and noncompact with a single end modeled on $\mathbb{R}^2 \setminus B_1$. In particular, if Σ_t is a regular level set then its Euler characteristic satisfies $\chi(\Sigma_t) \leq 1$.*

Proof. Let $\Sigma_t = u_{\mathbf{c}}^{-1}(t)$ be a regular level set, and suppose that there is a compact connected component $\Sigma'_t \subset \Sigma_t$. Note that Σ'_t is a 2-sided properly embedded submanifold. Since $H_2(M, \partial M; \mathbb{Z}) = 0$ the boundary cycles $\partial_i M$, $i = 1, \ldots, n$ generate $H_2(M)$. Thus, either Σ'_t is homologous to zero or it is homologous to a sum of boundary cycles. In the former case Σ'_t bounds a compact region of M, and since the spacetime harmonic function equation admits a maximum principle the solution $u_{\mathbf{c}} \equiv t$ in this region. This, however, contradicts the assumption that t is a regular value. So now consider the later case in which $[\Sigma'_t]$ can be represented as the sum of boundary classes $\sum_{i \in I} [\partial_i M]$, for some index set I. Let $D \subset M$ denote the compact region bounded by $\Sigma'_t \cup (\cup_{i \in I} \partial_i M) = \partial D$. Since the maximum and minimum of $u_{\mathbf{c}}$ on D are achieved on the boundary, it follows that either the max or min is achieved on $\partial_{i_0} M$, for some $i_0 \in I$. In particular, this max or min is achieved at $y_{i_0} \in \partial_{i_0} M$. Next observe that the Hopf lemma applies to the spacetime harmonic function equation, since the nonlinear first order part may be expressed as a linear term with bounded coefficients, and therefore $\partial_\upsilon u_{\mathbf{c}}(y_{i_0}) \neq 0$. However this contradicts the fact that y_{i_0} is a critical point for $u_{\mathbf{c}}$, as stated in Lemma

B.2.1. We conclude that all components of Σ_t are noncompact. Moreover Σ_t is a closed subset of M, since it is properly embedded. Therefore if any component of Σ_t stays within M_r (see Section 3.6.1), it must be compact which is a contradiction. It follows that each component must extend beyond S_r for all r.

The asymptotics of $u_{\mathbf{c}} \sim a_i x^i$ in the end M_{end} imply that for all sufficiently large r the level set Σ_t stays within a slab $\{x \in M \setminus M_r \mid t - C < a_i x^i < t + C\}$, for some constant C. Indeed, by the implicit function theorem Σ_t may be presented uniquely in this region as a graph over the plane $t = a_i x^i$. Hence, Σ_t is connected and has a single end modeled on $\mathbb{R}^2 \setminus B_1$. □

B.3 The integral formula with vanishing gradient

We have already established the main formula for spacetime harmonic functions u in Theorem 3.3.1 assuming that ∇u is not vanishing. However, this may not be the case in general, and we establish in this setting a version of Theorem 3.3.1 which allows ∇u to vanish.

Before stating the primary integral formula for spacetime harmonic functions, we give a technical lemma based on a refined version of Kato's inequality. This will be used in the proof of the main result of this section. Note that the natural regularity for spacetime harmonic functions is $C^{2,\alpha}(M)$, $0 < \alpha < 1$. By Rademacher's theorem $|\nabla u|$ is then differentiable almost everywhere, and from the equation the same holds for Δu. Thus, the inequality of the next result holds away from a set of measure zero.

Lemma B.3.1. *Let u be a spacetime harmonic function for the initial data set (M, g, k). Then there exists a constant $C > 0$ depending only on $\mathcal{K}$ and its first derivatives such* *that*

$$|\nabla^2 u|^2 - |\nabla|\nabla u||^2 + \langle \nabla u, \nabla \Delta u\rangle \geq -C|\nabla u|^2. \tag{B.7}$$

Proof. By using the spacetime Laplace equation $\Delta u = -\operatorname{tr}_g k|\nabla u|$, we have

$$\langle \nabla u, \nabla \Delta u\rangle \geq -\operatorname{tr}_g k\langle \nabla u, \nabla|\nabla u|\rangle - C_0|\nabla u|^2 \geq -\frac{1}{4}|\nabla|\nabla u||^2 - C_1|\nabla u|^2. \tag{B.8}$$

Moreover, a refined version of the Kato inequality produces

$$|\nabla^2 u|^2 \geq \frac{5}{4}|\nabla|\nabla u||^2 - C_2|\nabla u|^2. \tag{B.9}$$

Note that as discussed above, these inequalities hold almost everywhere. Combining (B.8) and (B.9) yields the desired result.

It remains to establish (B.9). To this end denote $u_i = \partial_i u$ and set

$$X_i = \frac{1}{2}\partial_i|\nabla u|^2 - \frac{1}{3}(\Delta u)u_i, \qquad W_{ij} = X_{(i}u_{j)} - \frac{1}{3}\langle X, \nabla u\rangle g_{ij},$$

where parentheses are used to indicate symmetrization of indices. Observe that

$$\begin{aligned}
|W|^2 =& X^i u^j \left(X_{(i}u_{j)} - \frac{1}{3}\langle X, \nabla u\rangle g_{ij}\right) \\
=& \frac{1}{2}|X|^2|\nabla u|^2 + \frac{1}{6}\langle X, \nabla u\rangle^2 \\
\leq& \frac{2}{3}|X|^2|\nabla u|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
\frac{1}{2}X^i\partial_i|\nabla u|^2 =& X^i u^j \nabla_{ij} u \\
=& X^i u^j \left(\nabla_{ij}u - \frac{1}{3}(\Delta u)g_{ij}\right) + \frac{1}{3}(\Delta u)\langle X, \nabla u\rangle \\
=& W^{ij}\left(\nabla_{ij}u - \frac{1}{3}(\Delta u)g_{ij}\right) + \frac{1}{3}(\Delta u)\langle X, \nabla u\rangle \\
\leq& |W|\sqrt{|\nabla^2 u|^2 - \frac{1}{3}(\Delta u)^2} + \frac{1}{3}(\Delta u)\langle X, \nabla u\rangle \\
\leq& \sqrt{\frac{2}{3}}|X||\nabla u|\sqrt{|\nabla^2 u|^2 - \frac{1}{3}(\Delta u)^2} + \frac{1}{3}(\Delta u)\langle X, \nabla u\rangle.
\end{aligned}$$

It follows that

$$|X| \leq \sqrt{\frac{2}{3}}|\nabla u|\sqrt{|\nabla^2 u|^2 - \frac{1}{3}(\Delta u)^2}.$$

Squaring both sides, utilizing the spacetime harmonic function equation, and applying Young's inequality then gives

$$\begin{aligned}|\nabla u|^2|\nabla^2 u|^2 \geq& \frac{1}{3}(\Delta u)^2|\nabla u|^2 + \frac{3}{2}|X|^2\\ =& \frac{1}{2}(\Delta u)^2|\nabla u|^2 + \frac{3}{2}|\nabla|\nabla u||^2|\nabla u|^2 - (\Delta u)|\nabla u|\langle\nabla u, \nabla|\nabla u|\rangle\\ \geq& \frac{5}{4}|\nabla|\nabla u||^2|\nabla u|^2 - C_2|\nabla u|^4.\end{aligned}$$

This gives inequality (B.9), if $|\nabla u| \neq 0$. At points where $|\nabla u| = 0$ and $|\nabla u|$ is differentiable, we have that $|\nabla|\nabla u|| = 0$ since the nonnegative function $|\nabla u|$ achieves its minimum value. Inequality (B.9) thus holds trivially at such points. The remaining points, where $|\nabla u| = 0$ and $|\nabla u|$ is not differentiable, form a set of measure zero. □

We are now in a position to establish the main integral formula for spacetime harmonic functions. This may be viewed as a generalization of [18, Proposition 4.2], see also [113].

Proposition B.3.2. *Let (Ω, g, k) be a 3-dimensional oriented compact initial data set with smooth boundary $\partial\Omega$, having outward unit normal υ. Let $u : \Omega \to \mathbb{R}$ be a spacetime harmonic function, and denote the open subset of $\partial\Omega$ on which $|\nabla u| \neq 0$ by $\partial_{\neq 0}\Omega$. If $\overline{u}$ and $\underline{u}$ denote the maximum and minimum values of u and Σ_t are t-level sets, then*

$$\int_{\partial_{\neq 0}\Omega} (\partial_\upsilon|\nabla u| + k(\nabla u, \upsilon))\, dA \geq \int_{\underline{u}}^{\overline{u}} \int_{\Sigma_t} \left(\frac{1}{2}\frac{|\bar{\nabla}^2 u|^2}{|\nabla u|^2} + \mu + J(\nu) - K\right) dAdt,$$

where $\nu = \frac{\nabla u}{|\nabla u|}$ and K is the level set Gauss curvature.

Proof. Recall Bochner's identity

$$\frac{1}{2}\Delta|\nabla u|^2 = |\nabla^2 u|^2 + \mathrm{Ric}(\nabla u, \nabla u) + \langle\nabla u, \nabla\Delta u\rangle.$$

For $\varepsilon > 0$ set $\varphi_\varepsilon = \sqrt{|\nabla u|^2 + \varepsilon}$, and use Bochner's identity to find

$$\begin{aligned}\Delta\varphi_\varepsilon =& \frac{\Delta|\nabla u|^2}{2\varphi_\varepsilon} - \frac{|\nabla|\nabla u|^2|^2}{4\varphi_\varepsilon^3} \\ \geq& \frac{1}{\varphi_\varepsilon}\left(|\nabla^2 u|^2 - |\nabla|\nabla u||^2 + \operatorname{Ric}(\nabla u, \nabla u) + \langle \nabla u, \nabla \Delta u\rangle\right).\end{aligned} \tag{B.10}$$

On a regular level set Σ, the unit normal is $\nu = \frac{\nabla u}{|\nabla u|}$ and the second fundamental form is given by $A = \frac{\nabla^2_\Sigma u}{|\nabla u|}$, where $\nabla^2_\Sigma u$ represents the Hessian of u restricted to $T\Sigma \otimes T\Sigma$. We then have

$$|A|^2 = |\nabla u|^{-2}\left(|\nabla^2 u|^2 - 2|\nabla|\nabla u||^2 + [\nabla^2 u(\nu,\nu)]^2\right),$$

and the mean curvature satisfies

$$|\nabla u| H = \Delta u - \nabla^2_\nu u. \tag{B.11}$$

Furthermore by taking two traces of the Gauss equations

$$2\operatorname{Ric}(\nu,\nu) = \mathrm{R} - 2K - |A|^2 + H^2, \tag{B.12}$$

where R is the scalar curvature of g. Combining these formulas with (B.10) produces

$$\begin{aligned}\Delta\varphi_\varepsilon \geq& \frac{1}{\varphi_\varepsilon}\left(|\nabla^2 u|^2 - |\nabla|\nabla u||^2\right) \\ &+ \frac{1}{\varphi_\varepsilon}\left(\langle\nabla u, \nabla\Delta u\rangle + \frac{|\nabla u|^2}{2}\left(\mathrm{R} - 2K + H^2 - |A|^2\right)\right) \\ =& \frac{1}{2\varphi_\varepsilon}\left(|\nabla^2 u|^2 + (R_g - R_\Sigma)|\nabla u|^2\right) \\ &+ \frac{1}{2\varphi_\varepsilon}\left(2\langle\nabla u, \nabla\Delta u\rangle + (\Delta u)^2 - 2(\Delta u)\nabla^2_\nu u\right).\end{aligned} \tag{B.13}$$

Let us now replace the Hessian with the spacetime Hessian via the relation $\bar{\nabla}^2 u = \nabla^2 u + k|\nabla u|$, and utilize the spacetime harmonic function equation $\Delta u = -\operatorname{tr}_g k|\nabla u|$ to find

$$\begin{aligned}\Delta\varphi_\varepsilon \geq& \frac{1}{2\varphi_\varepsilon}\Big(|\bar{\nabla}^2 u|^2 - 2\langle k, \nabla^2 u\rangle|\nabla u| - |k|_g^2|\nabla u|^2 + (\mathrm{R} - 2K)|\nabla u|^2 \\ &- 2\langle\nabla u, \nabla \operatorname{tr}_g k\rangle|\nabla u| - 2\operatorname{tr}_g k\langle\nabla u, \nabla|\nabla u|\rangle + \operatorname{tr}_g k|\nabla u|^2 + 2\operatorname{tr}_g k|\nabla u|\nabla^2_\nu u\Big).\end{aligned}$$

Moreover noting that

$$\langle \nabla u, \nabla |\nabla u| \rangle = \frac{1}{2} \langle \nu, \nabla |\nabla u|^2 \rangle = u^i \nabla_{i\nu} u = |\nabla u| \nabla^2_{\nu} u,$$
$$2\mu = \mathrm{R} + \operatorname{tr}_g k^2 - |k|^2_g,$$

gives rise to the following inequality on a regular level set

$$\Delta \varphi_\varepsilon \geq \frac{1}{2\varphi_\varepsilon} \left(|\bar{\nabla}^2 u|^2 + (2\mu - 2K)|\nabla u|^2 - 2\langle k, \nabla^2 u \rangle |\nabla u| - 2\langle \nabla u, \nabla \operatorname{tr}_g k \rangle |\nabla u| \right). \tag{B.14}$$

Consider an open set $\mathcal{A} \subset [\underline{u}, \overline{u}]$ containing the critical values of u, and let $\mathcal{B} \subset [\underline{u}, \overline{u}]$ denote the complementary closed set. Then integrate by parts to obtain

$$\int_{\partial\Omega} \partial_\upsilon \varphi_\varepsilon dA = \int_\Omega \Delta \varphi_\varepsilon dV = \int_{u^{-1}(\mathcal{A})} \Delta \varphi_\varepsilon dV + \int_{u^{-1}(\mathcal{B})} \Delta \varphi_\varepsilon dV.$$

According to Lemma B.3.1 and (B.10) there is a positive constant C_0, depending only on $\mathrm{Ric}(g)$ along with $\operatorname{tr}_g k$ and its first derivatives, such that

$$\Delta \varphi_\varepsilon \geq -C_0 |\nabla u|. \tag{B.15}$$

An application of the coarea formula to $u : u^{-1}(\mathcal{A}) \to \mathcal{A}$ then produces

$$-\int_{u^{-1}(\mathcal{A})} \Delta \varphi_\varepsilon dV \leq C_0 \int_{u^{-1}(\mathcal{A})} |\nabla u| dV = C_0 \int_{t \in \mathcal{A}} \mathcal{H}^2(\Sigma_t) dt, \tag{B.16}$$

where $\mathcal{H}^2(\Sigma_t)$ is the 2-dimensional Hausdorff measure of the t-level set Σ_t. Next, apply the coarea formula to $u : u^{-1}(\mathcal{B}) \to \mathcal{B}$ together with (B.14) to find

$$\begin{aligned}
&\int_{u^{-1}(\mathcal{B})} \Delta \varphi_\varepsilon dV \\
\geq& \frac{1}{2} \int_{t \in \mathcal{B}} \int_{\Sigma_t} \frac{|\nabla u|}{\varphi_\varepsilon} \left[\frac{|\bar{\nabla}^2 u|^2}{|\nabla u|^2} + 2\mu - 2K \right] dA dt. \\
\geq& \frac{1}{2} \int_{t \in \mathcal{B}} \int_{\Sigma_t} \frac{|\nabla u|}{\varphi_\varepsilon} \left[-\frac{2}{|\nabla u|} (\langle k, \nabla^2 u \rangle + \langle \nabla u, \nabla \operatorname{tr}_g k \rangle) \right] dA dt.
\end{aligned}$$

Combining all this together produces

$$\begin{aligned}
&\int_{\partial\Omega} \partial_\upsilon \varphi_\varepsilon dA + C_0 \int_{t\in\mathcal{A}} \mathcal{H}^2(\Sigma_t)dt \\
\geq &\frac{1}{2}\int_{t\in\mathcal{B}}\int_{\Sigma_t} \frac{|\nabla u|}{\varphi_\varepsilon}\left(\frac{|\bar{\nabla}^2 u|^2}{|\nabla u|^2} + 2\mu - 2K\right) dAdt \\
&- \int_{t\in\mathcal{B}}\int_{\Sigma_t} \varphi_\varepsilon^{-1}\left(\langle k, \nabla^2 u\rangle + \langle \nabla u, \nabla \operatorname{tr}_g k\rangle\right) dAdt.
\end{aligned} \tag{B.17}$$

On the set $u^{-1}(\mathcal{B})$, we have that $|\nabla u|$ is uniformly bounded from below. In addition, on $\partial_{\neq 0}\Omega$ it holds that

$$\partial_\upsilon \varphi_\varepsilon = \frac{|\nabla u|}{\varphi_\varepsilon}\partial_\upsilon |\nabla u| \to \partial_\upsilon|\nabla u| \quad \text{as} \quad \varepsilon \to 0.$$

Therefore, the limit $\varepsilon \to 0$ may be taken in (B.17), resulting in the same bulk expression except that φ_ε is replaced by $|\nabla u|$, and with the boundary integral taken over the restricted set. Furthermore, by Sard's theorem (see Remark B.3.3 below) the measure $|\mathcal{A}|$ of $\mathcal{A}$ may be taken to be arbitrarily small. Since the map $t \mapsto \mathcal{H}^2(\Sigma_t)$ is integrable over $[\underline{u}, \overline{u}]$ in light of the coarea formula, by then taking $|\mathcal{A}| \to 0$ we obtain

$$\begin{aligned}
\int_{\partial_{\neq 0}\Omega} \partial_\upsilon |\nabla u| dA \geq &\frac{1}{2}\int_{\underline{u}}^{\overline{u}}\int_{\Sigma_t}\left(\frac{|\bar{\nabla}^2 u|^2}{|\nabla u|^2} + 2\mu - 2K\right) dAdt \\
&- \int_\Omega \left(\langle k, \nabla^2 u\rangle + \langle \nabla u, \nabla \operatorname{tr}_g k\rangle\right) dV.
\end{aligned}$$

Lastly integrating parts

$$-\int_\Omega \langle k, \nabla^2 u\rangle dV = -\int_\Omega k^{ij}\nabla_{ij} u dV = \int_\Omega u^i \nabla^j k_{ij} - \int_{\partial\Omega} k(\nabla u, \upsilon) dA,$$

and recalling that $J = \operatorname{div}_g(k - \mathcal{K}g)$, yields the desired result. □

Remark B.3.3. *The classical statement of Sard's theorem in the current context requires $u \in C^3$, while spacetime harmonic functions typically only satisfy $u \in C^{2,\alpha}$, $0 < \alpha < 1$. Nevertheless, Sard's theorem still applies. To see this, observe that since $|\nabla u|$ is Lipschitz and hence in L^p_{loc} for all p, elliptic regularity yields $u \in W^{2,p}_{loc}$. It follows from Kato's inequality that $|\nabla u| \in W^{1,p}_{loc}$, and therefore $u \in W^{3,p}_{loc}$. According to [50, Theorem 5] the conclusion of Sard's theorem holds for such functions.*

B.4 Computations at infinity

Let (M, g, k) be a complete asymptotically flat initial data set for the Einstein equations, having generalized exterior region M_{ext} associated with a particular end M_{end} and given by Proposition B.1.1; for convenience we will continue denoting the metric and extrinsic curvature on M_{ext} by (g, k). Suppose that $x = (x^1, x^2, x^3)$ are *spacetime harmonic coordinates* on M_{ext}. This means that each function x^l satisfies (B.2), (B.3) and is given by Theorem 3.6.2 and Lemma B.2.1 (1), with asymptotics $x^l \sim \tilde{x}^l$ for some given asymptotically flat coordinate system $\tilde{x} = (\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ on M_{end}. More precisely, by the first part of Lemma B.2.1 we may choose the sign of the normal derivative at each boundary component $\partial_i M_{ext}$, $i = 1, \ldots, n$ so that:

$$\begin{aligned} &\partial_\upsilon x^l \leq 0 \text{ on } \partial_i M_{ext} \text{ if } \theta_+ (\partial_i M_{ext}) = 0, \\ &\partial_\upsilon x^l \geq 0 \text{ on } \partial_i M_{ext} \text{ if } \theta_- (\partial_i M_{ext}) = 0, \;\; l = 1, 2, 3. \end{aligned} \tag{B.18}$$

Note that although x^l are referred to as spacetime harmonic coordinates and are defined on all of M_{ext}, they are only guaranteed to form a coordinate system in M_{end}. Observe that due to the asymptotic expansion in Theorem 3.6.2, the ADM energy and linear momentum computed in spacetime harmonic coordinates will agree with the computation in any other valid asymptoticaly flat coordinate system [9].

For $L > 0$ sufficiently large define the cylindrical boundaries

$$\begin{aligned} &D_L^{\pm} = \{x \in M_{end} \mid (x^2)^2 + (x^3)^2 \leq L^2, \; x^1 = \pm L\}, \\ &T_L = \{x \in M_{end} \mid (x^2)^2 + (x^3)^2 = L^2, \; |x^1| \leq L\}, \\ &C_L = D_L^+ \cup T_L \cup D_L^-, \end{aligned}$$

and denote by Ω_L the bounded component of $M_{ext} \setminus C_L$, so that $\partial \Omega_L = C_L \sqcup \partial M_{ext}$. Let $u = x^1$ be the spacetime harmonic function described above, and set $\Sigma_t = u^{-1}(t)$ as well as $\Sigma_t^L = \Sigma_t \cap \Omega_L$. If t is a regular value of u, then $\partial \Sigma_t^L$ lies entirely within C_L, due to the fact that u has critical points on each component $\partial_i M_{ext}$, $i = 1, \ldots, n$. Note also that the regular level sets Σ_t^L meet T_L transversely, and by Theorem B.2.2, Σ_t^L has only one component so that $\chi(\Sigma_t^L) \leq 1$. Therefore we may apply Proposition B.3.2 together with the Gauss-Bonnet theorem to obtain

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_L}\left(\frac{|\bar{\nabla}^2 u|^2}{|\nabla u|} + 2(\mu - |J|_g)|\nabla u|\right) dV \\
\leq &\int_{-L}^{L}\left(2\pi\chi(\Sigma_t) - \int_{\Sigma_t^L \cap T_L} \kappa_{t,L}\right) dt + \int_{\partial_{\neq 0}\Omega_L} (\partial_\upsilon|\nabla u| + k(\nabla u, \upsilon))\, dA \\
\leq &4\pi L - \int_{-L}^{L}\left(\int_{\Sigma_t^L \cap T_L} \kappa_{t,L}\right) dt + \int_{C_L} (\partial_\upsilon|\nabla u| + k(\nabla u, \upsilon))\, dA \\
&+ \int_{\partial_{\neq 0} M_{ext}} (\partial_\upsilon|\nabla u| + k(\nabla u, \upsilon))\, dA,
\end{aligned} \tag{B.19}$$

where $\kappa_{t,L}$ is the geodesic curvature of $\Sigma_t^L \cap T_L$ interpreted as the boundary curve in Σ_t, $\partial_{\neq 0} M_{ext}$ denotes the subset of ∂M_{ext} where $|\nabla u| \neq 0$, and we have used that $|\nabla u| > 0$ on C_L.

In what follows we will compute first the outer boundary integral along C_L in the asymptotic end, from which the ADM energy and linear momentum will arise. The inner boundary integral along $\partial_{\neq 0} M_{ext}$ will then be computed and shown to vanish, due to the fact that ∂M_{ext} consists of MOTS and MITS. Below, the notation $\int_{D_L^\pm} \pm f$ will be used to represent $\int_{D_L^+} f - \int_{D_L^-} f$.

B.4.1 Computation of the outer boundary integral

In this section we show that the boundary term at ∞ of the main integral formula B.3.2 indeed yields the mass which follows the argument in the Riemannian case.

In [18, Lemma 6.1], a computation was carried out in harmonic coordinates. Each step of the proof applies here without change using spacetime harmonic coordinates, except for equation [18, (6.9)] where harmonicity was used. By replacing the harmonic function equation with the spacetime harmonic function equation, in this calculation, we find that

$$\begin{aligned}
&\int_{C_L} \partial_\upsilon |\nabla u| dA \\
=&\int_{D_L^\pm} \pm \left(\frac{1}{2} \sum_j (g_{1j,j} - g_{jj,1}) - \mathrm{tr}_g\, k \right) dA \\
&+ \frac{1}{2L} \int_{T_L} \left[x^2 (g_{21,1} - g_{11,2}) + x^3 (g_{31,1} - g_{11,3}) \right] dA + O(L^{1-2q}).
\end{aligned}$$

Similarly, [18, Lemma 6.2] may be carried over without change to the current setting to yield

$$\begin{aligned}
&\int_{-L}^{L} \left(\int_{\Sigma_t \cap T_L} \kappa_{t,L} \right) dt \\
=&4\pi L + \frac{1}{2L} \int_{T_L} \left[x^2 (g_{33,2} - g_{23,3}) + x^3 (g_{22,3} - g_{32,2}) \right] dA \\
&+ O(L^{1-2q} + L^{-q}).
\end{aligned}$$

Next, observe that the outward normal υ to C_L takes the form

$$\upsilon = \begin{cases} \pm \partial_1 + O(|x|^{-q}) & \text{on } D_L^\pm, \\ \frac{x^2}{L} \partial_2 + \frac{x^3}{L} \partial_3 + O(|x|^{-q}) & \text{on } T_L. \end{cases}$$

Furthermore

$$\nabla u = g^{i1} \partial_i = \partial_1 + O(|x|^{-q}).$$

It follows that

$$k(\nabla u, \upsilon) = \pm k_{11} + O(|x|^{-1-2q}) \quad \text{on} \quad D_L^\pm,$$

and

$$k(\nabla u, \upsilon) = \frac{x^2}{L} k_{12} + \frac{x^3}{L} k_{13} + O(|x|^{-1-2q}).$$

Finally, combining these computations produces

$$
\begin{aligned}
&4\pi L - \int_{-L}^{L}\left(\int_{\Sigma_t^L\cap T_L}\kappa_{t,L}\right)dt + \int_{C_L}\left(\partial_\upsilon|\nabla u| + k(\nabla u,\upsilon)\right)dA\\
=&\frac{1}{2}\int_{D_L^\pm}\pm\sum_j(g_{1j,j}-g_{jj,1})dA\\
&+\frac{1}{2}\int_{T_L}\left[\frac{x^2}{L}(g_{21,1}-g_{11,2})+\frac{x^3}{L}(g_{31,1}-g_{11,3})\right]dA\\
&+\frac{1}{2}\int_{T_L}\left[\frac{x^2}{L}(g_{23,3}-g_{33,2})+\frac{x^3}{L}(g_{32,2}-g_{22,3})\right]dA\\
&+\int_{D_L^\pm}\pm(k_{11}-\mathcal{K})\,dA+\int_{T_L}\left(\frac{x^2}{L}k_{12}+\frac{x^3}{L}k_{13}\right)dA+O(L^{1-2q}+L^{-q})\\
=&\frac{1}{2}\int_{C_L}\sum_j(g_{ij,j}-g_{jj,i})\upsilon^i dA\\
&+\int_{C_L}(k_{1i}-(\mathrm{Tr}_g k)g_{1i})\,\upsilon^i dA+O(L^{1-2q}+L^{-q}).
\end{aligned}
\tag{B.20}
$$

B.4.2 Computation of the inner boundary integral

Here we show that the inner boundary integral over $\partial_{\neq 0}M_{ext}$ vanishes, due to boundary behavior of the spacetime harmonic function combined with the fact that each boundary component is either a MOTS or MITS. Moreover, if the boundary components consist of weakly trapped surfaces then the inner boundary integral is nonpositive, which is an advantageous sign with respect to positivity of the ADM energy. Let υ denote the unit normal to a boundary component $\partial_i M_{ext}$, which points outside of M_{ext}. Then because u is constant on $\partial_i M_{ext}$, the spacetime harmonic function equation and gradient may be rewritten on this surface as

$$\nabla^2_{\upsilon\upsilon}u = H\upsilon(u) - \mathrm{tr}_g\, k|\nabla u|, \qquad \nabla u = \upsilon(u)\upsilon.$$

Note that here, the mean curvature H is computed with respect to $-\upsilon$. Observe that

$$|\nabla u|\partial_\upsilon|\nabla u| = \frac{1}{2}\partial_\upsilon|\nabla u|^2 = \frac{1}{2}\partial_\upsilon\left(g^{ij}u_i u_j\right) = u^j\nabla_{j\upsilon}u = \upsilon(u)\nabla^2_\upsilon u,$$

and hence

$$\partial_\upsilon |\nabla u| = \frac{\upsilon(u)}{|\nabla u|}\nabla^2_\upsilon u = \frac{\upsilon(u)}{|\nabla u|}\left(H\upsilon(u) - \mathrm{tr}_g\, k|\nabla u|\right) = H|\upsilon(u)| - \mathrm{tr}_g\, k\upsilon(u).$$

Furthermore since

$$\mathrm{tr}_g\, k = k(\upsilon,\upsilon) + \mathrm{Tr}_{\partial M_{ext}} k, \qquad\qquad k(\nabla u,\upsilon) = k(\upsilon,\upsilon)\upsilon(u), \tag{B.21}$$

it follows that the inner boundary integral becomes

$$\begin{aligned}\int_{\partial_{\neq 0} M_{ext}} \left(\partial_\upsilon|\nabla u| + k(\nabla u,\upsilon)\right) dA &= \int_{\partial M_{ext}} \left[H|\upsilon(u)| - \left(\mathrm{Tr}_{\partial M_{ext}} k\right)\upsilon(u)\right] dA \\ &= \sum_{i=1}^{n}\int_{\partial_i M_{ext}} \theta_\pm |\upsilon(u)| dA,\end{aligned} \tag{B.22}$$

where we have used (B.18) in the last step. The notation $\theta_\pm$ above indicates that the integrand contains θ_+ for a MOTS component and θ_- for a MITS component. We conclude that the inner boundary integral vanishes. Similarly, if the boundary of the generalized exterior region consists of weakly trapped surfaces then this boundary integral is nonpositive.

B.4.3 Proof of Theorem B (the inequality)

By combining (B.19), (B.20), (B.22), and taking the limit as $L \to \infty$ we obtain

$$E + P_1 \geq \frac{1}{16\pi}\int_{M_{ext}} \left(\frac{|\bar{\nabla}^2 u|^2}{|\nabla u|} + 2(\mu - |J|_g)|\nabla u|\right) dV, \tag{B.23}$$

since $q > \frac{1}{2}$. Furthermore, it may be assumed without loss of generality that the ADM linear momentum satisfies $P_1 = -|P|$, by applying an appropriate rotation of the asymptotically flat coordinates $\tilde{x}$. This yields the desired inequalities.

B.5 Classifying hypersurfaces in Minkowski space

In this section we address the Lorentzian version of fundamental theorem of hypersurfaces which is needed to prove the case of equality of the spacetime PMT.

Proof of Proposition 3.5.2. We follow the proof of [102, page 100]. Let U be a compact subset of M. We construct the metric $\bar{g} = -dt^2 + g_t$ on $(-\varepsilon, \varepsilon) \times U$ by prescribing

$$\begin{aligned} \partial_t g_t(\partial_i, \partial_j) =& 2\bar{\nabla}^2_{ij} t, \\ \bar{g}|_{t=0} =& g, \\ \partial_t(\bar{\nabla}^2_{ij} t) - (\bar{\nabla}^2 t)^2_{ij} =& 0, \\ \bar{\nabla}^2_{ij} t|_{t=0} =& k_{ij} \end{aligned} \tag{B.24}$$

where $(\bar{\nabla}^2 t)^2_{ij} = \bar{g}^{kl}(\bar{\nabla}^2_{ik} t)(\bar{\nabla}^2_{jl} t)$. We will use Roman letters $\{i, j, k, l\}$ to denote indices tangential to M. By standard ODE existence theory there exists a small $\varepsilon > 0$ such that we can solve the above equation for $t \in (-\varepsilon, \varepsilon)$. Next, we take a cover $\{U_i\}$ of M. According to the asymptotics of (M, g, k), there exists a uniform $\varepsilon > 0$ for each U_i. Therefore, we can patch together above's construction and (M, g) can be embedded in $((-\varepsilon, \varepsilon) \times M, \bar{g})$ with the second fundamental form k.

To verify the flatness of $\bar{g}$ we proceed exactly as in [102]. It suffices to verify that the curvatures $\bar{R}_{tijt}$, $\bar{R}_{ijkl}$ and $\bar{R}_{tijk}$ are vanishing. Observe that $\langle \bar{\nabla} t, \bar{\nabla} t \rangle = -1$ implies

$\bar{\nabla}_i \bar{\nabla}_t t = 0$. Combining this with the third line of equation (B.24) yields

$$\begin{aligned} 0 =& \partial_t(\bar{\nabla}^2_{ij} t) - (\bar{\nabla}^2 t)^2_{ij} \\ =& \bar{\nabla}_t \bar{\nabla}_i \bar{\nabla}_j t + (\bar{\nabla}^2 t)^2_{ij} \\ =& \bar{\nabla}_i \bar{\nabla}_t \bar{\nabla}_j t - \bar{R}_{tijt} + (\bar{\nabla}^2 t)^2_{ij} \\ =& \partial_i(\bar{\nabla}_t \bar{\nabla}_j t) - \bar{\nabla}^2 t(\partial_t, \bar{\nabla}_i \partial_j) - \bar{\nabla}^2 t(\partial_j, \bar{\nabla}_i \partial_t) - \bar{R}_{tijt} + (\bar{\nabla}^2 t)^2_{ij} \\ =& - \bar{R}_{tijt}. \end{aligned}$$

Since $\bar{R}_{tijt} = 0$, $\bar{\nabla}_t \partial_t = 0$ and $\bar{\Gamma}^t_{ti} = 0$, we obtain

$$\begin{aligned} \partial_t(\bar{R}_{tijk}) =& (\bar{\nabla}_t \bar{R})_{tijk} + \bar{R}_{tljk}\bar{\Gamma}^l_{ti} + \bar{R}_{tilk}\bar{\Gamma}^l_{tj} + \bar{R}_{tljk}\bar{\Gamma}^l_{tk} \\ =& (\bar{\nabla}_j \bar{R})_{titk} + (\bar{\nabla}_k \bar{R})_{tijt} + \bar{R}_{tljk}\bar{\Gamma}^l_{ti} + \bar{R}_{tilk}\bar{\Gamma}^l_{tj} + \bar{R}_{tljk}\bar{\Gamma}^l_{tk} \\ =& \partial_j(\bar{R}_{titk}) - \bar{R}_{litk}\bar{\Gamma}^l_{jt} - \bar{R}_{tilk}\bar{\Gamma}^l_{jt} + \partial_k(\bar{R}_{tijt}) - \bar{R}_{lijt}\bar{\Gamma}^l_{kt} - \bar{R}_{tijl}\bar{\Gamma}^l_{kt} \\ & + \bar{R}_{tljk}\bar{\Gamma}^l_{ti} + \bar{R}_{tilk}\bar{\Gamma}^l_{tj} + \bar{R}_{tljk}\bar{\Gamma}^l_{tk} \\ =& - \bar{R}_{litk}\bar{\Gamma}^l_{jt} - \bar{R}_{tilk}\bar{\Gamma}^l_{jt} - \bar{R}_{lijt}\bar{\Gamma}^l_{kt} - \bar{R}_{tijl}\bar{\Gamma}^l_{kt} + \bar{R}_{tljk}\bar{\Gamma}^l_{ti} + \bar{R}_{tilk}\bar{\Gamma}^l_{tj} + \bar{R}_{tljk}\bar{\Gamma}^l_{tk}. \end{aligned}$$

According to the Codazzi equation, $\bar{R}_{tijk}|_{t=0} = 0$, and thus $\bar{R}_{tijk} = 0$. Next, we compute

$$\begin{aligned} \partial_t(\bar{R}_{ijkl}) =& (\bar{\nabla}_t \bar{R})_{ijkl} + \bar{R}_{sjkl}\bar{\Gamma}^s_{ti} + \bar{R}_{iskl}\bar{\Gamma}^s_{tj} + \bar{R}_{ijsl}\bar{\Gamma}^s_{tk} + \bar{R}_{ijks}\bar{\Gamma}^s_{tl} \\ =& (\nabla_k \bar{R})_{ijtl} + (\nabla_l \bar{R})_{ijkt} + \bar{R}_{sjkl}\bar{\Gamma}^s_{ti} + \bar{R}_{iskl}\bar{\Gamma}^s_{tj} + \bar{R}_{ijsl}\bar{\Gamma}^s_{tk} + \bar{R}_{ijks}\bar{\Gamma}^s_{tl} \\ =& - \bar{R}_{ijsl}\bar{\Gamma}^s_{kt} - \bar{R}_{ijks}\bar{\Gamma}^s_{lt} + \bar{R}_{sjkl}\bar{\Gamma}^s_{ti} + \bar{R}_{iskl}\bar{\Gamma}^s_{tj} + \bar{R}_{ijsl}\bar{\Gamma}^s_{tk} + \bar{R}_{ijks}\bar{\Gamma}^s_{tl} \\ =& \bar{R}_{sjkl}\bar{\Gamma}^s_{ti} + \bar{R}_{iskl}\bar{\Gamma}^s_{tj}. \end{aligned}$$

According to the Gauss equations, $\bar{R}_{ijkl}|_{t=0} = 0$, and thus $\bar{R}_{ijkl} = 0$. Therefore, $\bar{M}$ is flat which implies together with $M \cong \mathbb{R}^3$ that $\bar{M}$ is a subset of Minkowski spacetime. $\square$

B.6 The Killing development

Another way to prove rigidity for the spacetime PMT, is to construct a spacetime using spacetime harmonic function, and demonstrating that this spacetime is Minkowski space. For this purpose, we define on $\tilde{M}^4 = \mathbb{R} \times M^3$ the Lorentzian metric

$$\tilde{g} = 2d\tau du + g$$

where τ is the flat coordinate on the $\mathbb{R}$-factor. This so-called *Killing Development* is motivated by [10, 64], though we note that the Killing Development in [10, 64] was obtained from three, rather than a single vector field. Since $M^3 \cong \mathbb{R}^3$, we have $\tilde{M}^4 \cong \mathbb{R}^4$, and thus it suffices to show that $\tilde{g}$ is flat. The flatness of $\tilde{g}$ follows essentially from the Gauss and Codazzi equations computed in Section 3.5.2. We present here another approach which has the advantage that it does not require the additional regularity assumptions $g \in C^3(M^3)$ and $k \in C^2(M^3)$ used in Lemma 3.5.5, and therefore establishes the rigidity of Theorem B in full generality.

We first claim that we can write

$$g = (|\nabla u|^{-2} + a^2 + b^2)du^2 + 2adudx_1 + 2bdudx_2 + dx_1^2 + dx_2^2, \tag{B.25}$$

for some functions $a, b \in C^2(M^3)$. This essentially follows from the flatness of the level-sets of u, but let us elaborate more on this construction:

To write g in the above form, we need to define globally defined coordinates x_1, x_2. To do so, we begin with introducing global polar coordinates. Given some point $p_0 \in M^3$, let $\Gamma : (-\infty, +\infty) \to M^3$ be the integral curve through p_0 with respect to the vector field ∇u. We define the function $\rho(p) = d(p, \Gamma \cap \Sigma_{u(p)})$ where d denotes the distance within the level set $\Sigma_{u(p)}$. Since $u \in C^3(M)$ and $|\nabla u| \neq 0$, the second fundamental form of $\Sigma_{u(p)}$ is C^1. On each level set Σ_t of u, we can write the metric g_{Σ_t} as $d\rho^2 + \rho^2 d\theta^2$. We would like g to have globally such a form, i.e., we need to define an angle function $\theta(p) \in [0, 2\pi)$ for any $p \in M^3 \backslash \Gamma$. To uniquely determine $\theta(p)$, we fix another point $p_1 \in M^3$ not contained in the image $\mathrm{im}(\Gamma)$. Let $\Gamma_1 : (-\infty, \infty) \to M^3$ be the integral curve through p_1 with respect to the vector field ∇u. Since $|\nabla u| \neq 0$, we have $\mathrm{im}(\Gamma) \cap \mathrm{im}(\Gamma_1) = \varnothing$. We set $\theta(\Gamma_1) = 0$. Thus, the Lorentzian metric $\tilde{g}$ can be written in the form

$$\tilde{g} = 2d\tau du + (|\nabla u|^{-2} + a_0^2 + \rho^{-2} b_0^2) du^2 + 2a_0 du d\rho + 2b_0 du d\theta + d\rho^2 + \rho^2 d\theta^2$$

for some functions $a_0, b_0 \in C^2(M^3 \backslash \Gamma)$, where the C^2 regularity follows from the second fundamental form being C^1. Finally, we change coordinates via $x_1 = \rho \cos\theta$, $x_2 = \rho \sin\theta$ and set

$$a = a_0 \cos\theta - b_0 \rho^{-1} \sin\theta, \quad b = a_0 \sin\theta + b_0 \rho^{-1} \cos\theta$$

to obtain

$$\tilde{g} = 2d\tau du + (|\nabla u|^2 + a^2 + b^2) du^2 + 2a du dx_1 + 2b du dx_2 + dx_1^2 + dx_2^2$$

as desired.

In (τ, u, x_1, x_2) coordinates, the inverse metric $\tilde{g}^{-1}$ is given by

$$\tilde{g}^{-1} = \begin{bmatrix} -|\nabla u|^{-2} & 1 & -a & -b \\ 1 & 0 & 0 & 0 \\ -a & 0 & 1 & 0 \\ -b & 0 & 0 & 1 \end{bmatrix}.$$

Therefore, we have

$$\tilde{\nabla} u = \tilde{g}^{ui} \partial_i = \partial_\tau.$$

Moreover, the null vector $\tilde{\nabla} u = \partial_\tau$ is covariantly constant, i.e., $\tilde{\nabla}^2 u = 0$. Thus, $(\tilde{M}^4, \tilde{g})$ is a pp-wave. See [12] for a more detailed discussion of such spacetimes. Therefore, we have on (M^3, g, k)

$$0 = \tilde{\nabla}^2_{ij} u|_{TM^3} = \nabla^2_{ij} u + \mathrm{II}_{ij} \hat{N}(u) = (A_{ij} - k_{ij})|\nabla u|$$

where $N = |\nabla u|(-|\nabla u|^{-2}\partial_\tau + \partial_u - a\partial_1 - b\partial_2)$ is a time-like unit normal vector. Thus, the second fundamental form A of $(M^3, g) \subset (\tilde{M}^4, \tilde{g})$ is given by k.

The vector fields $\{\partial_1, \partial_2, \partial_u, \partial_\tau\}$ form a frame of $T\tilde{M}^4$ and $\{\nabla u, \partial_1, \partial_2\}$ form an orthogonal frame of TM^3. Using Mathematica, we obtain that the only non-vanishing Ricci curvature terms of $\tilde{g}$ are given by

$$\begin{aligned}
\widetilde{\mathrm{Ric}}(\partial_u, \partial_1) &= \frac{1}{2}(-a_{x_2x_2} + b_{x_1x_2}), \\
\widetilde{\mathrm{Ric}}(\partial_u, \partial_2) &= \frac{1}{2}(a_{x_1x_2} - b_{x_1x_1}), \\
\widetilde{\mathrm{Ric}}(\partial_u, \partial_u) &= \frac{1}{2}(a_{x_2} - b_{x_1})^2 - \frac{1}{2}\Delta_{\mathbb{R}^2}(|\nabla u|^{-2} + a^2 + b^2) + a_{ux_1} + b_{ux_2}.
\end{aligned} \tag{B.26}$$

Taking the trace of $\widetilde{\mathrm{Ric}}$, we have $\tilde{R} = 0$, then $\mu = \widetilde{\mathrm{Ric}}(N, N)$ and $J = \widetilde{\mathrm{Ric}}(N, \cdot)$. The identity $\langle J, \partial_1 \rangle = \langle J, \partial_2 \rangle = 0$ yields $\widetilde{\mathrm{Ric}}(N, \partial_1) = \widetilde{\mathrm{Ric}}(N, \partial_2) = 0$. Combining this with $\mu \geq 0$, we obtain $\widetilde{\mathrm{Ric}}(\partial_u, \partial_u) \geq 0$. The equation $\widetilde{\mathrm{Ric}}(N, \partial_1) = \widetilde{\mathrm{Ric}}(N, \partial_2) = 0$ also implies

$$a_{x_2x_2} = b_{x_1x_2} \quad \text{and} \quad a_{x_1x_2} = b_{x_1x_1}.$$

Thus, $\psi := a_{x_2} - b_{x_1}$ only depends on u. Hence, there exists a function l such that $a = x_2\psi(u) + l_{x_1}$ and $b = -x_1\psi(u) + l_{x_2}$. Inserting this into the third line of Equation (B.26), we obtain

$$\Delta_{\mathbb{R}^2}\left(\frac{1}{2}|\nabla u|^{-2} + \frac{1}{2}l_{x_1}^2 + \frac{1}{2}l_{x_2}^2 + l_{x_1}x_2\psi(u) - l_{x_2}x_1\psi(u) - l_u\right) \leq 0. \tag{B.27}$$

Next, we define

$$F(u, x_1, x_2) := \frac{1}{2}|\nabla u|^{-2} + \frac{1}{2}l_{x_1}^2 + \frac{1}{2}l_{x_2}^2 + l_{x_1}x_2\psi(u) - l_{x_2}x_1\psi(u) - l_u.$$

Another computation and the fact that $\tilde{\nabla} u$ is covariantly constant, yield that the only non-vanishing Riemann curvature terms of $\tilde{g}$ in the frame $\{\partial_1, \partial_2, \partial_\tau, \nabla u\}$ are given by

$$\begin{aligned}
\tilde{R}(\nabla u, \partial_1, \partial_1, \nabla u) =& R(\nabla u, \partial_1, \partial_1, \nabla u) + k(\nabla u, \nabla u)k(\partial_1, \partial_1) - k^2(\nabla u, \partial_1) \\
=& -|\nabla u|^4 F_{x_1x_1}, \\
\tilde{R}(\nabla u, \partial_2, \partial_2, \nabla u) =& R(\nabla u, \partial_1, \partial_1, \nabla u) + k(\nabla u, \nabla u)k(\partial_2, \partial_2) - k^2(\nabla u, \partial_2) \\
=& -|\nabla u|^4 F_{x_2x_2}, \\
\tilde{R}(\nabla u, \partial_1, \partial_2, \nabla u) =& -|\nabla u|^4 F_{x_1x_2}.
\end{aligned}$$

According to Theorem 4.2 in [64], we have $|\nabla u| = 1 + O_1(|x|^{-\tau})$. Combining this with the asymptotics for g and k in (3.2), we obtain $F_{x_ix_j} = O(|x|^{-\tau-2})$, where $i, j = 1, 2$. Therefore, we can follow the proof of Lemma 3.5.6 to conclude that F is a linear function with respect to x_1, x_2. Thus, $\tilde{g}$ is flat which finishes the proof.

Appendix C

Technical aspects of Theorem C

In this section we compute the integral formulas (4.6) and (4.24). They in particular generalize the Hawking mass monotonicity formula for IMCF [52, 79, 83], the spacetime Hawking energy monotonicity in spherical symmetry [61], the integral formula for harmonic and p-harmonic functions [1, 2, 18], for spacetime harmonic functions [64, 17] and for charged harmonic functions [16]. We remark that the aforementioned formulas led to proofs of the Riemannian Penrose inequality [1, 79], the spacetime Penrose inequality in spherical symmetry [61], the Riemannian [2, 18], spacetime [64] and hyperbolic PMT [17], as well as the PMT with charge [16] and corners [71].

C.1 Spacetime IMCF and p-harmonic functions

We denote with $\nu_u = \frac{\nabla u}{|\nabla u|}$ and $\nu_v = \frac{\nabla v}{|\nabla v|}$ the unit normals to the level sets of u and v. Throughout this section we assume that both ν_u and ν_v are well-defined, i.e. $|\nabla u|, |\nabla v| \neq 0$. We expect that the cases where $\nabla u, \nabla v$ are allowed to vanish can be treated in a similar fashion as in [1, 64, 113].

Next, we define $\eta = \frac{\nu_u + \nu_v}{|\nu_u + \nu_v|}$ in case $\nu_u \neq -\nu_v$, and $\eta = 0$ in case $\nu_u = -\nu_v$. Similarly, we define $f = \frac{\nu_u - \nu_v}{|\nu_u - \nu_v|}$ in case $\nu_u \neq \nu_v$ and in case $\nu_u = \nu_v$ we set $f = 0$ (which is the case in spherical symmetry). It is convenient to compute formula (4.6) in this frame. We remark that $\nu_u \neq -\nu_v$ for any initial data set contained in Minkowski space.

We start with collecting several elementary properties about η and f:

Lemma C.1.1. *We have*

$$\nabla_\eta u \nabla_f v = -\nabla_f u \nabla_\eta v.$$

In particular, for any symmetric 2-tensor A^{ij}

$$A^{ij}(\nabla_i u \nabla_j v + \nabla_j u \nabla_i v) = 2A^{\eta\eta}\nabla_\eta u \nabla_\eta v + 2A^{ff}\nabla_f u \nabla_f v.$$

Proof. To prove the first identity we can assume without loss of generality that $\nu_u \neq \nu_v$ and $\nu_u \neq -\nu_v$. We compute

$$\nabla_\eta u \nabla_f v = \frac{1}{|\nu_u + \nu_v||\nu_u - \nu_v||\nabla u||\nabla v|}(|\nabla u||\nabla v| + \langle \nabla u, \nabla v\rangle)(-|\nabla u||\nabla v| + \langle \nabla u, \nabla v\rangle)$$

and

$$\nabla_\eta v \nabla_f u = \frac{1}{|\nu_u + \nu_v||\nu_u - \nu_v||\nabla u||\nabla v|}(|\nabla u||\nabla v| + \langle \nabla u, \nabla v\rangle)(|\nabla u||\nabla v| - \langle \nabla u, \nabla v\rangle).$$

The second identity directly follows from the first one. □

Lemma C.1.2. *We have*

$$\langle \nabla u, \nabla v\rangle = \nabla_\eta u \nabla_\eta v + \nabla_f u \nabla_f v$$

and

$$|\nabla u||\nabla v| = \nabla_\eta u \nabla_\eta v - \nabla_f u \nabla_f v.$$

Proof. The first identity is trivial, so it suffices to show the second one. Observe that $|\nabla u|^2 = (\nabla_\eta u)^2 + (\nabla_f u)^2$ which holds also in case $\nu_u = \nu_v$ or $\nu_u = -\nu_v$. We compute using

Lemma C.1.1

$$
\begin{aligned}
&|\nabla u|^2|\nabla v|^2\\
=&((\nabla_\eta u)^2+(\nabla_f u)^2)((\nabla_\eta v)^2+(\nabla_f v)^2)\\
=&(\nabla_\eta u\nabla_\eta v)^2+(\nabla_f u\nabla_f v)^2+(\nabla_\eta u)^2(\nabla_f v)^2+(\nabla_\eta v)^2(\nabla_f u)^2\\
=&(\nabla_\eta u\nabla_\eta v)^2+(\nabla_f u\nabla_f v)^2-2\nabla_\eta u\nabla_\eta v\nabla_f u\nabla_f v\\
=&(\nabla_\eta u\nabla_\eta v-\nabla_f u\nabla_f v)^2.
\end{aligned}
$$

Taking the square root on both sides yields

$$|\nabla u||\nabla v| = |\nabla_\eta u\nabla_\eta v - \nabla_f u\nabla_f v|.$$

We clearly have $\nabla_\eta u\nabla_\eta v - \nabla_f u\nabla_f v \geq 0$ in case $\nu_u = \nu_v$ or $\nu_u = -\nu_v$. In case $\nu_u \neq \nu_v$ and $\nu_u = -\nu_v$ we have

$$
\begin{aligned}
&|\nabla u|^2|\nabla v|^2(\nabla_\eta u\nabla_\eta v-\nabla_f u\nabla_f v)\\
=&\frac{1}{|\nu_u+\nu_v|^2}(|\nabla u||\nabla v|+\langle\nabla u,\nabla v\rangle)^2+\frac{1}{|\nu_u-\nu_v|^2}(|\nabla u||\nabla v|-\langle\nabla u,\nabla v\rangle)^2\\
\geq&0
\end{aligned}
$$

which finishes the proof. □

Lemma C.1.3. *We have*

$$|\nu_u+\nu_v| = 2\langle\nu_u,\eta\rangle = 2\langle\nu_v,\eta\rangle$$

as well as

$$|\nu_u-\nu_v| = 2\langle\nu_u,f\rangle = -2\langle\nu_v,f\rangle.$$

Proof. Recall that $\nu_u = \frac{\nabla u}{|\nabla u|}$ and $\nu_v = \frac{\nabla v}{|\nabla v|}$. We compute

$$|\nu_u+\nu_v|^2 = \left|\frac{\nabla u|\nabla v|+\nabla v|\nabla u|}{|\nabla u||\nabla v|}\right|^2 = 2\frac{(|\nabla u||\nabla v|+\langle\nabla u,\nabla v\rangle)}{|\nabla u||\nabla v|}$$

and

$$|\nu_u + \nu_v|\langle \nu_u, \eta\rangle = \langle \nu_u, \nu_u + \nu_v\rangle = \frac{1}{|\nabla u||\nabla v|}\left(|\nabla u||\nabla v| + \langle \nabla u, \nabla v\rangle\right)$$

which implies $|\nu_u + \nu_v| = 2\langle \nu_u, \eta\rangle$. In case $\nu_u = -\nu_v$, we clearly have $\langle \nu_u, \eta\rangle = \langle \nu_v, \eta\rangle$. Moreover, observe in case $\nu_u \neq -\nu_v$

$$\langle \nu_u, \eta\rangle = \frac{1 + \langle \nu_u, \nu_v\rangle}{|\nu_u + \nu_v|} = \langle \nu_v, \eta\rangle$$

which implies the first identity. Replacing v by $-v$, the second identity follows. □

Recall that

$$(\mathcal{H}^2_+)_{ij} u = \nabla_{ij} u + k_{ij}|\nabla u| - 2\frac{\nabla_\eta u \nabla_\eta v}{u+v} g_{ij} + \frac{\nabla_i u \nabla_j v + \nabla_j u \nabla_i v}{u+v},$$
$$(\mathcal{H}^2_-)_{ij} v = \nabla_{ij} v - k_{ij}|\nabla v| - 2\frac{\nabla_\eta u \nabla_\eta v}{u+v} g_{ij} + \frac{\nabla_i u \nabla_j v + \nabla_j u \nabla_i v}{u+v}.$$

The proof of Theorem 4.4.1 is implied by the following two proposition regarding $\mathcal{H}^2_\pm$:

Proposition C.1.4. *Let $a \in [0,1]$ and suppose u, v solve system (4.5). Then we have*

$$\begin{aligned}
&\frac{|\mathcal{H}^2_+ u|^2}{|\nabla u|} - \frac{(a(\mathcal{H}^2_+)_{\eta\eta} u)^2}{|\nabla u|} \\
=&R|\nabla u| - 2K_u|\nabla u| + |k|^2|\nabla u| - \mathrm{tr}_g(k)^2|\nabla u| + 2|\nabla u|\nabla_{\nu_u}\mathrm{tr}_g(k) - 2|\nabla u|\nabla_i k_{i\nu_u} \\
&+ \mathrm{div}\left(-2\nabla|\nabla u| + 2\Delta u\frac{\nabla u}{|\nabla u|} + 2k(\nabla u, \cdot) - 2(\nabla u\,\mathrm{tr}_g(k)) + 4|\nabla u|\frac{\nabla v}{u+v}\right) \\
&- 4a\nabla_{\eta\eta} v\langle \nu_u, f\rangle^2\frac{|\nabla u|}{u+v} + 4a\nabla_{\eta\eta} u\langle \nu_u, f\rangle^2\frac{|\nabla v|}{u+v} + 4(|\nabla u| - |\nabla v|)\frac{|\nabla u||\nabla v|}{(u+v)^2} \\
&+ 2k_{ij}\frac{\nabla_i u\nabla_j v + \nabla_j u \nabla_i v}{u+v} - 4|\nabla u||\nabla v|\frac{\mathrm{tr}_g(k)}{u+v} - 8|\nabla u||\nabla v|\frac{k_{\eta\eta}}{u+v}.
\end{aligned} \tag{C.1}$$

Here R is the scalar curvature of g, and K_u, K_v are the Gaussian curvatures of the level sets of u, v.

Proposition C.1.5. *Let $a \in [0,1]$ and suppose u, v solve system* (4.5)*. Then we have*

$$
\begin{aligned}
&\frac{|\mathcal{H}_-^2 v|^2}{|\nabla v|} - \frac{(a(\mathcal{H}_-^2)_{\eta\eta} v)^2}{|\nabla v|} \\
=& R|\nabla v| - 2K_v|\nabla v| + |k|^2|\nabla v| - \operatorname{tr}_g(k)^2|\nabla v| - 2|\nabla v|\nabla_{\nu_v}\operatorname{tr}_g(k) + 2|\nabla v|\nabla_i k_{i\nu_v} \\
&+ \operatorname{div}\left(-2\nabla|\nabla v| + 2\Delta v\frac{\nabla v}{|\nabla v|} - 2k(\nabla v, \cdot) + 2(\nabla v \operatorname{tr}_g(k)) + 4|\nabla v|\frac{\nabla u}{u+v}\right) \qquad \text{(C.2)} \\
&- 4a\nabla_{\eta\eta}u\langle\nu_u, f\rangle^2\frac{|\nabla u|}{u+v} + 4a\nabla_{\eta\eta}v\langle\nu_u, f\rangle^2\frac{|\nabla v|}{u+v} + 4(|\nabla v| - |\nabla u|)\frac{|\nabla u||\nabla v|}{(u+v)^2} \\
&- 2k_{ij}\frac{\nabla_i u\nabla_j v + \nabla_j u\nabla_i v}{u+v} + 4|\nabla u||\nabla v|\frac{\operatorname{tr}_g(k)}{u+v} + 8|\nabla u||\nabla v|\frac{k_{\eta\eta}}{u+v}.
\end{aligned}
$$

Proof of Theorem 4.4.1. This follows immediately from adding equation (C.1) to equation (C.2). Observe how the last two lines of both (C.1) and (C.2) cancel. □

To prove Proposition C.1.4 and Proposition C.1.5 we will make use of several auxiliary lemma:

Lemma C.1.6. *Let $a \in [0,1]$ and suppose u, v solve system* (4.5)*. Then we have*

$$
\begin{aligned}
&|\mathcal{H}_+^2 u|^2 + (\Delta u)^2 \\
=& |\nabla^2 u|^2 + |k|^2|\nabla u|^2 + 8\frac{(\nabla_\eta u\nabla_\eta v)^2}{(u+v)^2} \\
&+ 2\nabla_{ij}uk_{ij}|\nabla u| - 4\operatorname{tr}_g(k)|\nabla u|\frac{\nabla_\eta u\nabla_\eta v}{u+v} + 2k_{ij}|\nabla u|\frac{\nabla_i u\nabla_j v + \nabla_j u\nabla_i v}{u+v} \\
&+ (4+4a)\nabla_{\eta\eta}u\frac{\nabla_\eta u\nabla_\eta v}{u+v} + 4\nabla_{ff}u\frac{\nabla_f u\nabla_f v}{u+v} \\
&+ (\operatorname{tr}_g(k) - ak_{\eta\eta})^2|\nabla u|^2 + a^2(\nabla_{\eta\eta}u)^2 - 2a\nabla_{\eta\eta}u(\operatorname{tr}_g(k) - ak_{\nu\nu})|\nabla u| \\
&- 4\Delta u\frac{\nabla_f u\nabla_f v}{u+v} \\
&- 4\frac{\nabla_\eta u\nabla_\eta v}{u+v}(\operatorname{tr}_g(k) - ak_{\nu\nu})|\nabla u|.
\end{aligned}
$$

Proof. Using Lemma C.1.1 several times we obtain

$$
\begin{aligned}
&(\nabla_i u\nabla_j v + \nabla_j u\nabla_i v)(\nabla_i u\nabla_j v + \nabla_j u\nabla_i v) \\
=& 4(\nabla_\eta u\nabla_\eta v)^2 + 4(\nabla_f u\nabla_f v)^2
\end{aligned}
$$

and

$$\nabla_{ij}u(\nabla_i u\nabla_j v + \nabla_j u\nabla_i v) = 2\nabla_{\eta\eta}u\nabla_\eta u\nabla_\eta v + 2\nabla_{ff}u\nabla_f\nabla_f v$$

as well as

$$g_{ij}(\nabla_i u\nabla_j v + \nabla_j u\nabla_i v) = 2\nabla_\eta u\nabla_\eta v + 2\nabla_f u\nabla_f v.$$

This allows us to compute

$$\begin{aligned}|\mathcal{H}_+^2 u|^2 =& |\nabla^2 u|^2 + |k|^2|\nabla u|^2 + 12\frac{(\nabla_\eta u\nabla_\eta v)^2}{(u+v)^2} + 4\frac{(\nabla_\eta u\nabla_\eta v)^2 + (\nabla_f u\nabla_f v)^2}{(u+v)^2}\\ &+ 2\nabla_{ij}uk_{ij}|\nabla u| - 4\operatorname{tr}_g(k)|\nabla u|\frac{\nabla_\eta u\nabla_\eta v}{u+v} + 2k_{ij}|\nabla u|\frac{\nabla_i u\nabla_j v + \nabla_j u\nabla_i v}{u+v}\\ &- 4\Delta u\frac{\nabla_\eta u\nabla_\eta v}{u+v} - 8\nabla_\eta u\nabla_\eta v\frac{\nabla_\eta u\nabla_\eta v + \nabla_f u\nabla_f v}{(u+v)^2}\\ &+ 4\nabla_{\eta\eta}u\frac{\nabla_\eta u\nabla_\eta v}{u+v} + 4\nabla_{ff}u\frac{\nabla_f u\nabla_f v}{u+v}.\end{aligned}$$

Grouping together terms, we obtain

$$\begin{aligned}|\mathcal{H}_+^2 u|^2 =& |\nabla^2 u|^2 + |k|^2|\nabla u|^2 + 8\frac{(\nabla_\eta u\nabla_\eta v)^2}{(u+v)^2} + 4\frac{(\nabla_f u\nabla_f v)^2}{(u+v)^2}\\ &+ 2\nabla_{ij}uk_{ij}|\nabla u| - 4\operatorname{tr}_g(k)|\nabla u|\frac{\nabla_\eta u\nabla_\eta v}{u+v} + 2k_{ij}|\nabla u|\frac{\nabla_i u\nabla_j v + \nabla_j u\nabla_i v}{u+v}\\ &- 4\Delta u\frac{\nabla_\eta u\nabla_\eta v}{u+v} - 8\nabla_\eta u\nabla_\eta v\frac{\nabla_f u\nabla_f v}{(u+v)^2}\\ &+ 4\nabla_{\eta\eta}u\frac{\nabla_\eta u\nabla_\eta v}{u+v} + 4\nabla_{ff}u\frac{\nabla_f u\nabla_f v}{u+v}.\end{aligned} \tag{C.3}$$

Next, we recall that the PDE (4.5) for u states

$$\Delta u = -\operatorname{tr}_g(k)|\nabla u| + ak_{\nu\nu}|\nabla u| + a\nabla_{\eta\eta}^2 u + \frac{3|\nabla u||\nabla v| + \langle\nabla u, \nabla v\rangle}{u+v}.$$

Moreover, we note that Lemma C.1.2 implies

$$3|\nabla u||\nabla v| + \langle\nabla u, \nabla v\rangle = 4\nabla_\eta u\nabla_\eta v - 2\nabla_f u\nabla_f v.$$

Thus, we are able to calculate

$$\begin{aligned}
&(\Delta u)^2 - 4\Delta u \frac{\nabla_\eta u \nabla_\eta v}{u+v} \\
=&\Delta u \left(a\nabla_{\eta\eta} u - (\mathrm{tr}_g(k) - k_{\eta\eta})|\nabla u| - 2\frac{\nabla_f u \nabla_f v}{u+v} \right) \\
=&(\mathrm{tr}_g(k) - a k_{\eta\eta})^2 |\nabla u|^2 + a^2 (\nabla_{\eta\eta} u)^2 - 2a\nabla_{\eta\eta} u (\mathrm{tr}_g(k) - a k_{\nu\nu})|\nabla u| \\
&- 2\Delta u \frac{\nabla_f u \nabla_f v}{u+v} + 4\frac{\nabla_\eta u \nabla_\eta v}{u+v}(a\nabla_{\eta\eta} u - (\mathrm{tr}_g(k) - k_{\eta\eta})|\nabla u|) \\
&- 2\frac{\nabla_f u \nabla_f v}{u+v}(a\nabla_{\eta\eta} u - (\mathrm{tr}_g(k) - k_{\eta\eta})|\nabla u|) \\
=&(\mathrm{tr}_g(k) - a k_{\eta\eta})^2 |\nabla u|^2 + a^2 (\nabla_{\eta\eta} u)^2 - 2a\nabla_{\eta\eta} u (\mathrm{tr}_g(k) - a k_{\nu\nu})|\nabla u| \\
&- 4\Delta u \frac{\nabla_f u \nabla_f v}{u+v} - 4\frac{(\nabla_f u \nabla_f v)^2}{(u+v)^2} \\
&- 4\frac{\nabla_\eta u \nabla_\eta v}{u+v}(\mathrm{tr}_g(k) - a k_{\nu\nu})|\nabla u| + 4a\nabla_{\eta\eta} u \frac{\nabla_\eta u \nabla_\eta v}{u+v} \\
&+ 8\nabla_\eta u \nabla_\eta v \frac{\nabla_f u \nabla_f v}{(u+v)^2}.
\end{aligned}$$

Combining the above identity with equation (C.3), we obtain

$$\begin{aligned}
&|\mathcal{H}_+^2 u|^2 + (\Delta u)^2 \\
=&|\nabla^2 u|^2 + |k|^2 |\nabla u|^2 + 8\frac{(\nabla_\eta u \nabla_\eta v)^2}{(u+v)^2} \\
&+ 2\nabla_{ij} u k_{ij} |\nabla u| - 4\,\mathrm{tr}_g(k)|\nabla u| \frac{\nabla_\eta u \nabla_\eta v}{u+v} + 2k_{ij}|\nabla u| \frac{\nabla_i u \nabla_j v + \nabla_j u \nabla_i v}{u+v} \\
&+ (4+4a)\nabla_{\eta\eta} u \frac{\nabla_\eta u \nabla_\eta v}{u+v} + 4\nabla_{ff} u \frac{\nabla_f u \nabla_f v}{u+v} \\
&+ (\mathrm{tr}_g(k) - a k_{\eta\eta})^2 |\nabla u|^2 + a^2 (\nabla_{\eta\eta} u)^2 - 2a\nabla_{\eta\eta} u (\mathrm{tr}_g(k) - a k_{\nu\nu})|\nabla u| \\
&- 4\Delta u \frac{\nabla_f u \nabla_f v}{u+v} \\
&- 4\frac{\nabla_\eta u \nabla_\eta v}{u+v}(\mathrm{tr}_g(k) - a k_{\nu\nu})|\nabla u|
\end{aligned}$$

which finishes the proof. □

Lemma C.1.7. *Let $a \in [0,1]$ and suppose u, v solve system (4.5). Then we have*

$$\begin{aligned}
&\frac{4}{|\nabla u|}(1+a)\nabla_{\eta\eta}u\frac{\nabla_\eta u \nabla_\eta v}{u+v}\\
=&4\langle \nu_u, f\rangle^2 \nabla_{ff}u\frac{|\nabla v|}{u+v}\\
&-4a\nabla_{\eta\eta}u\langle \nu_u, f\rangle^2\frac{|\nabla u|}{u+v} + 4a\nabla_{\eta\eta}(u-v)\frac{|\nabla u|}{u+v}\\
&+4\operatorname{div}\left(|\nabla u|\frac{\nabla v}{u+v}\right) + 4|\nabla u||\nabla v|\frac{|\nabla v| - 3|\nabla u|}{(u+v)^2}\\
&-4|\nabla u||\nabla v|\frac{\operatorname{tr}_g(k) - ak_{\eta\eta}}{u+v}
\end{aligned}$$

Proof. Recall that

$$\Delta v = \operatorname{tr}_g(k)|\nabla v| - ak_{\nu\nu}|\nabla v| + a\nabla^2_{\eta\eta}v + \frac{3|\nabla u||\nabla v| + \langle \nabla u, \nabla v\rangle}{u+v}.$$

Using this equation we compute

$$\begin{aligned}
&4\nabla_{\nu_v\nu_u}u\frac{|\nabla v|}{u+v}\\
=&4\left\langle \nabla|\nabla u|, \frac{\nabla v}{u+v}\right\rangle\\
=&4\operatorname{div}\left(|\nabla u|\frac{\nabla v}{u+v}\right) - 4|\nabla u|\frac{\Delta v}{u+v} + 4|\nabla u|\frac{|\nabla v|^2 + \langle \nabla u, \nabla v\rangle}{(u+v)^2} \qquad \text{(C.4)}\\
=&4\operatorname{div}\left(|\nabla u|\frac{\nabla v}{u+v}\right) - 4a|\nabla u|\frac{\nabla_{\eta\eta}v}{u+v} + 4|\nabla u||\nabla v|\frac{|\nabla v| - 3|\nabla u|}{(u+v)^2}\\
&-4|\nabla u||\nabla v|\frac{\operatorname{tr}_g(k) - ak_{\eta\eta}}{u+v}.
\end{aligned}$$

Next, observe by Lemma C.1.3

$$
\begin{aligned}
&\langle \nu_u, \eta\rangle^2 \nabla_{\eta\eta} u \\
=&\langle \nu_u, \eta\rangle^2 \frac{1}{|\nu_u + \nu_v|^2} \nabla_{(\nu_u+\nu_v)(\nu_u+\nu_v)} u \\
=&\frac{1}{4}(2\nabla_{\nu_u\nu_v} u + \nabla_{\nu_u\nu_u} u + \nabla_{\nu_v\nu_v} u) \\
=&\frac{1}{4}(2\nabla_{\nu_u\nu_v} u + \nabla_{\nu_u(f+\nu_v)} u + \nabla_{\nu_v(\nu_u-f)} u) \\
=&\nabla_{\nu_u\nu_v} u + \frac{1}{4}\nabla_{(\nu_u-\nu_v)(\nu_u-\nu_v)} u \\
=&\nabla_{\nu_u\nu_v} u + \langle \nu_u, f\rangle^2 \nabla_{ff} u.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\frac{4}{|\nabla u|}(1+a)\nabla_{\eta\eta} u \frac{\nabla_\eta u \nabla_\eta v}{u+v} \\
=&\frac{4}{|\nabla u|}\nabla_{\eta\eta} u \frac{\nabla_\eta u \nabla_\eta v}{u+v} + \frac{4a}{|\nabla u|}\nabla_{\eta\eta} u \frac{\nabla_\eta u \nabla_\eta v}{u+v} \\
=&4\nabla_{\nu_v\nu_u} u \frac{|\nabla v|}{u+v} + 4\langle \nu_u, f\rangle^2 \nabla_{ff} u \frac{|\nabla v|}{u+v} \\
&+ 4a\nabla_{\eta\eta} u \frac{|\nabla u|}{u+v} - 4a\nabla_{\eta\eta} u \langle \nu_u, f\rangle^2 \frac{|\nabla u|}{u+v}.
\end{aligned}
$$

Combining this with equation (C.4) yields

$$
\begin{aligned}
&\frac{4}{|\nabla u|}(1+a)\nabla_{\eta\eta} u \frac{\nabla_\eta u \nabla_\eta v}{u+v} \\
=&4\langle \nu_u, f\rangle^2 \nabla_{ff} u \frac{|\nabla v|}{u+v} \\
&- 4a\nabla_{\eta\eta} u \langle \nu_u, f\rangle^2 \frac{|\nabla u|}{u+v} + 4a\nabla_{\eta\eta}(u-v)\frac{|\nabla u|}{u+v} \\
&+ 4\operatorname{div}\left(|\nabla u| \frac{\nabla v}{u+v}\right) + 4|\nabla u||\nabla v| \frac{|\nabla v| - 3|\nabla u|}{(u+v)^2} \\
&- 4|\nabla u||\nabla v| \frac{\operatorname{tr}_g(k) - a k_{\eta\eta}}{u+v}
\end{aligned}
$$

as desired. □

Lemma C.1.8. *Let $a \in [0,1]$ and suppose u, v solve system* (4.5). *Then we have*

$$\begin{aligned}-2a\operatorname{tr}_g(k)\nabla_{\eta\eta}u = &-2\operatorname{div}(\nabla u\operatorname{tr}_g(k)) + 2(-\operatorname{tr}_g(k)|\nabla u| + ak_{\eta\eta}|\nabla u|)\operatorname{tr}_g(k)\\ &+ 2\operatorname{tr}_g(k)\frac{4\nabla_\eta u\nabla_\eta v - 2\nabla_f u\nabla_f v}{u+v} + 2|\nabla u|\nabla_{\nu_u}\operatorname{tr}_g(k).\end{aligned}$$

Proof. Using the PDE for u (4.5), we compute

$$\begin{aligned}&-2a\operatorname{tr}_g(k)\nabla_{\nu_u\nu_u}u\\ =&-2a\left\langle\nabla|\nabla u|, \frac{\nabla u\operatorname{tr}_g(k)}{|\nabla u|}\right\rangle\\ =&-2a\operatorname{div}(\nabla u\operatorname{tr}_g(k)) + 2a(-\operatorname{tr}_g(k)|\nabla u| + ak_{\eta\eta}|\nabla u| + a\nabla_{\eta\eta}u)\operatorname{tr}_g(k)\\ &+2a\operatorname{tr}_g(k)\frac{4\nabla_\eta u\nabla_\eta v - 2\nabla_f u\nabla_f v}{u+v}\\ &-2a\nabla_{\nu_u\nu_u}u\operatorname{tr}_g(k) + 2a|\nabla u|\nabla_{\nu_u}\operatorname{tr}_g(k).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}-2a\operatorname{tr}_g(k)\nabla_{\eta\eta}u = &-2\operatorname{div}(\nabla u\operatorname{tr}_g(k)) + 2(-\operatorname{tr}_g(k)|\nabla u| + ak_{\eta\eta}|\nabla u|)\operatorname{tr}_g(k)\\ &+ 2\operatorname{tr}_g(k)\frac{4\nabla_\eta u\nabla_\eta v - 2\nabla_f u\nabla_f v}{u+v} + 2|\nabla u|\nabla_{\nu_u}\operatorname{tr}_g(k)\end{aligned}$$

which finishes the proof. □

Lemma C.1.9. *For any twice-differentiable function u we have*

$$\operatorname{div}\left(\nabla|\nabla u| - \Delta u\frac{\nabla u}{|\nabla u|}\right) = \frac{1}{2|\nabla u|}(|\nabla^2 u|^2 + |\nabla u|^2(R - 2K_u) - (\Delta u)^2).$$

Proof. This formula has already been established in equation (4.8) of [16], also see the article of D. Stern [113]. We nonetheless include a proof to make this manuscript more self-contained. We compute using Bochner's identity and the Gauss equations

$$\begin{aligned}2\Delta|\nabla u| =&2|\nabla u|^{-1}(\operatorname{Ric}(\nabla u,\nabla u) + |\nabla^2 u|^2 + \langle\nabla\Delta u,\nabla u\rangle - |\nabla|\nabla u||^2)\\ =&2|\nabla u|^{-1}(|\nabla^2 u|^2 + \langle\nabla\Delta u,\nabla u\rangle - |\nabla|\nabla u||^2) + (R-2K_u)|\nabla u|\\ &+|\nabla^2 u|^2 + |\nabla u|^{-1}((\Delta u - \nabla_{\nu\nu}u)^2 - |\nabla^2 u|^2 + 2|\nabla|\nabla u||^2 - (\nabla_{\nu\nu}u)^2)\end{aligned}$$

Rewriting the term $2|\nabla u|^{-1}\langle \nabla \Delta u, \nabla u\rangle$, the result follows. $\square$

Now we have all the auxiliary ingredients to proceed with the proof of Proposition C.1.4. The proof of Proposition C.1.5 is identical so we will omit it.

Proof of Proposition C.1.4. Combining Lemma C.1.6 and Lemma C.1.7, we obtain

$$
\begin{aligned}
&|\mathcal{H}^2_+ u|^2 + (\Delta u)^2 \\
=&|\nabla^2 u|^2 + |k|^2|\nabla u|^2 + 8\frac{(\nabla_\eta u \nabla_\eta v)^2}{(u+v)^2} \\
&+ 2\nabla_{ij}u k_{ij}|\nabla u| - 4\,\mathrm{tr}_g(k)|\nabla u|\frac{\nabla_\eta u\nabla_\eta v}{u+v} + 2k_{ij}|\nabla u|\frac{\nabla_i u\nabla_j v + \nabla_j u \nabla_i v}{u+v} \\
&+ (\mathrm{tr}_g(k) - ak_{\eta\eta})^2|\nabla u|^2 + a^2(\nabla_{\eta\eta}u)^2 - 2a\nabla_{\eta\eta}u(\mathrm{tr}_g(k) - ak_{\nu\nu})|\nabla u| \\
&- 4\Delta u\frac{\nabla_f u \nabla_f v}{u+v} \\
&- 4\frac{\nabla_\eta u\nabla_\eta v}{u+v}(\mathrm{tr}_g(k) - ak_{\nu\nu})|\nabla u| \\
&- 4a\nabla_{\eta\eta}v\langle \nu_u, f\rangle^2\frac{|\nabla u|^2}{u+v} \\
&+ 4|\nabla u|\,\mathrm{div}\left(|\nabla u|\frac{\nabla v}{u+v}\right) - 8|\nabla u||\nabla v|\frac{|\nabla u||\nabla v|}{(u+v)^2} \\
&+ 4|\nabla u|(|\nabla u| - |\nabla v|)\frac{|\nabla u||\nabla v|}{(u+v)^2} - 4|\nabla u|^2|\nabla v|\frac{\mathrm{tr}_g(k) - ak_{\eta\eta}}{u+v}.
\end{aligned}
$$

Observe how the $\nabla_{ff}u$ terms cancel. Next, we calculate using Lemma C.1.2

$$
\begin{aligned}
&8\frac{(\nabla_\eta u\nabla_\eta v)^2}{(u+v)^2} - 4\Delta u\frac{\nabla_f u\nabla_f v}{u+v} - 8|\nabla u||\nabla v|\frac{|\nabla u||\nabla v|}{(u+v)^2} \\
=& - 4\Delta u\frac{\nabla_f u\nabla_f v}{u+v} - 8\frac{(\nabla_f u\nabla_f v)^2}{(u+v)^2} + 16\frac{\nabla_f u\nabla_f v\nabla_\eta u\nabla_\eta v}{(u+v)^2} \\
=& - 4a\nabla_{\eta\eta}u\frac{\nabla_f u\nabla_f v}{u+v} + 4(\mathrm{tr}_g(k) - ak_{\nu\nu})|\nabla u|\frac{\nabla_f u\nabla_f v}{u+v}.
\end{aligned}
$$

Moreover,

$$((\mathcal{H}^2_+)_{\eta\eta}u)^2 =(\nabla_{\eta\eta}u + k_{\eta\eta}u|\nabla u|)^2 = (\nabla_{\eta\eta}u)^2 + 2k_{\eta\eta}|\nabla u|\nabla_{\eta\eta}u + k^2_{\eta\eta}|\nabla u|^2.$$

Hence, we obtain

$$
\begin{aligned}
&|\mathcal{H}^2_+ u|^2 + (\Delta u)^2 - (a(\mathcal{H}^2_+)_{\eta\eta} u)^2 \\
=&|\nabla^2 u|^2 + |k|^2 |\nabla u|^2 \\
&+ 2\nabla_{ij} u k_{ij} |\nabla u| - 4 \operatorname{tr}_g(k) |\nabla u| \frac{\nabla_\eta u \nabla_\eta v}{u+v} + 2k_{ij}|\nabla u| \frac{\nabla_i u \nabla_j v + \nabla_j u \nabla_i v}{u+v} \\
&+ (\operatorname{tr}_g(k)^2 - 2a \operatorname{tr}_g(k) k_{\eta\eta}) |\nabla u|^2 - 2a \nabla_{\eta\eta} u \operatorname{tr}_g(k) |\nabla u| \\
&- 4 \frac{\nabla_\eta u \nabla_\eta v}{u+v} (\operatorname{tr}_g(k) - a k_{\nu\nu}) |\nabla u| \\
&- 4a \nabla_{\eta\eta} v \langle \nu_u, f\rangle^2 \frac{|\nabla u|^2}{u+v} - 4a \nabla_{\eta\eta} u \frac{\nabla_f u \nabla_f v}{u+v} \\
&+ 4|\nabla u| \operatorname{div}\left(|\nabla u| \frac{\nabla v}{u+v} \right) \\
&+ 4|\nabla u| (|\nabla u| - |\nabla v|) \frac{|\nabla u||\nabla v|}{(u+v)^2} - 4|\nabla u|^2 |\nabla v| \frac{\operatorname{tr}_g(k) - a k_{\eta\eta}}{u+v} \\
&+ 4(\operatorname{tr}_g(k) - a k_{\nu\nu}) |\nabla u| \frac{\nabla_f u \nabla_f v}{u+v}.
\end{aligned}
$$

Next, we use the divergence identity

$$
\nabla_{ij} u k_{ij} = \operatorname{div} k(\nabla u, \cdot) - |\nabla u| \nabla_i k_{i\nu_u}
$$

and Lemma C.1.8 which results in

$$
\begin{aligned}
&|\mathcal{H}_+^2 u|^2 + (\Delta u)^2 - (a(\mathcal{H}_+^2)_{\eta\eta} u)^2 \\
=&|\nabla^2 u|^2 + |k|^2 |\nabla u|^2 \\
&+ 2|\nabla u| \operatorname{div} k(\nabla u, \cdot) - 2|\nabla u|^2 \nabla_i k_{i\nu_u} + 2k_{ij}|\nabla u| \frac{\nabla_i u \nabla_j v + \nabla_j u \nabla_i v}{u+v} \\
&- \operatorname{tr}_g(k)^2 |\nabla u|^2 \\
&- 4\frac{\nabla_\eta u \nabla_\eta v}{u+v}(-ak_{\nu\nu})|\nabla u| \\
&- 4a\nabla_{\eta\eta} v \langle \nu_u, f\rangle^2 \frac{|\nabla u|^2}{u+v} - 4a\nabla_{\eta\eta} u \frac{\nabla_f u \nabla_f v}{u+v} \\
&+ 4|\nabla u| \operatorname{div}\left(|\nabla u| \frac{\nabla v}{u+v}\right) \\
&+ 4|\nabla u|(|\nabla u| - |\nabla v|)\frac{|\nabla u||\nabla v|}{(u+v)^2} - 4|\nabla u|^2 |\nabla v| \frac{\operatorname{tr}_g(k) - ak_{\eta\eta}}{u+v} \\
&- 4ak_{\nu\nu}|\nabla u| \frac{\nabla_f u \nabla_f v}{u+v} \\
&- 2|\nabla u| \operatorname{div}(\nabla u \operatorname{tr}_g(k)) \\
&+ 2|\nabla u|^2 \nabla_{\nu_u} \operatorname{tr}_g(k).
\end{aligned}
$$

By collecting terms which are homogeneous of degree 1 in k (though note they will cancel anyways with the corresponding terms from Proposition C.1.5), this simplifies further to

$$
\begin{aligned}
&|\mathcal{H}_+^2 u|^2 + (\Delta u)^2 - (a(\mathcal{H}_+^2)_{\eta\eta} u)^2 \\
=&|\nabla^2 u|^2 + |k|^2 |\nabla u|^2 - \operatorname{tr}_g(k)^2 |\nabla u|^2 + 2|\nabla u|^2 \nabla_{\nu_u} \operatorname{tr}_g(k) - 2|\nabla u|^2 \nabla_i k_{i\nu_u} \\
&+ 2|\nabla u| \operatorname{div} k(\nabla u, \cdot) - 2|\nabla u| \operatorname{div}(\nabla u \operatorname{tr}_g(k)) + 4|\nabla u| \operatorname{div}\left(|\nabla u| \frac{\nabla v}{u+v}\right) \\
&- 4a\nabla_{\eta\eta} v \langle \nu_u, f\rangle^2 \frac{|\nabla u|^2}{u+v} - 4a\nabla_{\eta\eta} u \frac{\nabla_f u \nabla_f v}{u+v} + 4|\nabla u|(|\nabla u| - |\nabla v|)\frac{|\nabla u||\nabla v|}{(u+v)^2} \\
&+ 2k_{ij}|\nabla u| \frac{\nabla_i u \nabla_j v + \nabla_j u \nabla_i v}{u+v} - 4|\nabla u|^2 |\nabla v| \frac{\operatorname{tr}_g(k)}{u+v} - 8a|\nabla u|^2 |\nabla v| \frac{k_{\eta\eta}}{u+v}
\end{aligned}
$$

Next, we use Lemma (C.1.9) to obtain

$$\begin{aligned}
&\frac{|\mathcal{H}_+^2 u|^2}{|\nabla u|} - \frac{(a(\mathcal{H}_+^2)_{\eta\eta} u)^2}{|\nabla u|} \\
=&R|\nabla u| - 2K_u|\nabla u| + |k|^2|\nabla u| - \operatorname{tr}_g(k)^2|\nabla u| + 2|\nabla u|\nabla_{\nu_u}\operatorname{tr}_g(k) - 2|\nabla u|\nabla_i k_{i\nu_u} \\
&+ \operatorname{div}\left(-2\nabla|\nabla u| + 2\Delta u\frac{\nabla u}{|\nabla u|} + 2k(\nabla u, \cdot) - 2(\nabla u \operatorname{tr}_g(k)) + 4|\nabla u|\frac{\nabla v}{u+v}\right) \\
&- 4a\nabla_{\eta\eta}v\langle \nu_u, f\rangle^2\frac{|\nabla u|}{u+v} + 4a\nabla_{\eta\eta}u\langle \nu_u, f\rangle^2\frac{|\nabla v|}{u+v} + 4(|\nabla u| - |\nabla v|)\frac{|\nabla u||\nabla v|}{(u+v)^2} \\
&+ 2k_{ij}\frac{\nabla_i u\nabla_j v + \nabla_j u\nabla_i v}{u+v} - 4|\nabla u||\nabla v|\frac{\operatorname{tr}_g(k)}{u+v} - 8|\nabla u||\nabla v|\frac{k_{\eta\eta}}{u+v}
\end{aligned}$$

which finishes the proof. □

C.2 Spacetime charged harmonic functions

Again, we set $\eta = \frac{\nu_u+\nu_v}{|\nu_u+\nu_v|}$ in case $\nu_u \neq -\nu_v$, and $\eta = 0$ in case $\nu_u = -\nu_v$. Note that the integral formula (4.24) in Theorem 4.6.6 reduces to the integral formula for spacetime harmonic functions in case $\eta = 0$, cf. Proposition 3.2 in [64]. Therefore, we assume without loss of generality that $\nu_u \neq -\nu_v$ in the proof below.

Proof of Theorem 4.6.6. Recall that the charged spacetime Hessians are given by

$$\begin{aligned}
(\mathcal{E}_+^2)_{ij}u =&\nabla_{ij}^2 u + \xi\eta_i E_j + \xi\eta_j E_i - \xi E_\eta g_{ij} + k_{ij}|\nabla u|, \\
(\mathcal{E}_-^2)_{ij}v =&\nabla_{ij}^2 v + \xi\eta_i E_j + \xi\eta_j E_i - \xi E_\eta g_{ij} - k_{ij}|\nabla v|
\end{aligned}$$

where $\xi = \sqrt{|\nabla u||\nabla v|}$. We compute

$$
\begin{aligned}
|\mathcal{E}_+^2 u|^2 =& |\nabla^2 u|^2 + 2\xi^2 |E|^2 + 3\xi^2 E_\eta^2 \\
& + 4\xi \nabla_{ij} u \eta_i E_j - 2a\Delta u E_\eta + 2a^2 E_\eta^2 - 4a^2 E_\eta^2 \\
& + 2(\xi \eta_i E_j + \xi \eta_j E_i - \xi E_\eta g_{ij}) k_{ij} |\nabla u| \\
& + |k|^2 |\nabla u|^2 + 2\nabla_{ij}^2 u k_{ij} |\nabla u| \\
=& |\nabla^2 u|^2 + 2\xi^2 |E|^2 \\
& + 4\xi \nabla_{ij} u \eta_i E_j - \xi^2 E_\eta^2 \\
& + 2(\xi \eta_i E_j + \xi \eta_j E_i - \xi E_\eta g_{ij}) k_{ij} |\nabla u| \\
& + |k|^2 |\nabla u|^2 + 2\nabla_{ij}^2 u k_{ij} |\nabla u| + 2\operatorname{tr}_g(k) |\nabla u| \xi E_\eta.
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
|\mathcal{E}_-^2 v|^2 =& |\nabla^2 v|^2 + 2\xi^2 |E|^2 \\
& + 4\xi \nabla_{ij} v \eta_i E_j - \xi^2 E_\eta^2 \\
& - 2(\xi \eta_i E_j + \xi \eta_j E_i - \xi E_\eta g_{ij}) k_{ij} |\nabla v| \\
& + |k|^2 |\nabla u|^2 - 2\nabla_{ij}^2 u k_{ij} |\nabla u| - 2\operatorname{tr}_g(k) |\nabla u| \xi E_\eta.
\end{aligned}
$$

Using Lemma C.1.9, we obtain

$$
\begin{aligned}
& \operatorname{div}\left(\nabla|\nabla u| - \Delta u \frac{\nabla u}{|\nabla u|} + \nabla|\nabla v| - \Delta v \frac{\nabla v}{|\nabla v|} \right) \\
=& \frac{1}{2|\nabla u|} (|\mathcal{E}_+^2 u|^2 - 2\xi^2 |E|^2 + \xi^2 E_\eta^2 - 4\xi \nabla_{ij}^2 u \eta_i E_j) \\
& + \frac{1}{2|\nabla u|} (|\nabla u|^2 (R_M - 2K_u) - (\xi E_\eta - \operatorname{tr}_g(k)|\nabla u|)^2 - |k|^2 |\nabla u|^2 - 2\nabla_{ij}^2 u k_{ij} |\nabla u|) \\
& + \frac{1}{2|\nabla v|} (|\mathcal{E}_-^2 v|^2 - 2\xi^2 |E|^2 + \xi^2 E_\eta^2 - 4\xi \nabla_{ij}^2 v \eta_i E_j) \\
& + \frac{1}{2|\nabla v|} (|\nabla v|^2 (R_M - 2K_v) - (\xi E_\eta + \operatorname{tr}_g(k)|\nabla v|)^2 - |k|^2 |\nabla v|^2 + 2\nabla_{ij}^2 v k_{ij} |\nabla v|).
\end{aligned}
$$

Simplifying yields

$$
\begin{aligned}
&\operatorname{div}\left(\nabla|\nabla u| - \Delta u\frac{\nabla u}{|\nabla u|} + \nabla|\nabla v| - \Delta v\frac{\nabla v}{|\nabla v|}\right)\\
=&\frac{1}{2|\nabla u|}(|\mathcal{E}_+^2 u|^2 - 2\xi^2|E|^2 - 4\xi\nabla_{ij}^2 u\eta_i E_j)\\
&+\frac{1}{2|\nabla u|}(|\nabla u|^2(R_M - 2K_u) + 2\operatorname{tr}_g(k)\Delta u|\nabla u| + (\operatorname{tr}_g(k)^2 - |k|^2)|\nabla u|^2 - 2\nabla_{ij}^2 u k_{ij}|\nabla u|)\\
&+\frac{1}{2|\nabla v|}(|\mathcal{E}_-^2 v|^2 - 2\xi^2|E|^2 - 4\xi\nabla_{ij}^2 v\eta_i E_j)\\
&+\frac{1}{2|\nabla v|}(|\nabla v|^2(R_M - 2K_v) - 2\operatorname{tr}_g(k)\Delta v|\nabla v| + (\operatorname{tr}_g(k)^2 - |k|^2)|\nabla v|^2 + 2\nabla_{ij}^2 v k_{ij}|\nabla v|).
\end{aligned}
$$

Next, we compute

$$
\frac{1}{|\nabla u|}\xi\nabla_{ij}^2 u\eta_i E_j = \operatorname{div}\left(\sqrt{\frac{|\nabla v|}{|\nabla u|}}\nabla_i u\eta_i E\right) - \nabla_j\sqrt{\frac{|\nabla v|}{|\nabla u|}}E_j\nabla_\eta u - \sqrt{\frac{|\nabla v|}{|\nabla u|}}\nabla_i u\nabla_j\eta_i E_j
$$

and similarly

$$
\frac{1}{|\nabla v|}\xi\nabla_{ij}^2 v\eta_i E_j = \operatorname{div}\left(\sqrt{\frac{|\nabla u|}{|\nabla v|}}\nabla_i v\eta_i E\right) - \nabla_j\sqrt{\frac{|\nabla u|}{|\nabla v|}}E_j\nabla_\eta v - \sqrt{\frac{|\nabla u|}{|\nabla v|}}\nabla_i v\nabla_j\eta_i E_j.
$$

Observe that

$$
\begin{aligned}
&2\nabla_j\sqrt{\frac{|\nabla v|}{|\nabla u|}}E_j\nabla_\eta u + 2\nabla_j\sqrt{\frac{|\nabla u|}{|\nabla v|}}E_j\nabla_\eta v\\
=&\xi^{-1}\nabla_j|\nabla v|E_j\nabla_\eta u - \xi^{-1}|\nabla v||\nabla u|^{-1}\nabla_j|\nabla u|\nabla_\eta u E_j\\
&+\xi^{-1}\nabla_j|\nabla u|E_j\nabla_\eta v - \xi^{-1}|\nabla u||\nabla v|^{-1}\nabla_j|\nabla v|\nabla_\eta v E_j = 0
\end{aligned}
$$

where we used Lemma C.1.3 in combination with $|\nabla v|\nabla_\eta u = |\nabla v||\nabla u|\langle\nu_u, \eta\rangle$. Moreover,

$$
\begin{aligned}
&\sqrt{\frac{|\nabla v|}{|\nabla u|}}\nabla_i u\nabla_j\eta_i E_j + \sqrt{\frac{|\nabla u|}{|\nabla v|}}\nabla_i v\nabla_j\eta_i E_j\\
=&\xi^{-1}\langle\nabla_E\eta, \nabla u|\nabla v| + \nabla v|\nabla u|\rangle\\
=&\xi^{-1}\|\nabla u|\nabla v + \nabla u|\nabla v\|\langle\nabla_E\eta, \eta\rangle = 0
\end{aligned}
$$

where we used that $\langle \eta, \eta \rangle = 1$ and $\langle \nabla_E \eta, \eta \rangle = 0$. Hence, we obtain

$$\begin{aligned}
&\operatorname{div}\left(\nabla|\nabla u| - \Delta u \frac{\nabla u}{|\nabla u|} + \nabla|\nabla v| - \Delta v \frac{\nabla v}{|\nabla v|} + 2\sqrt{\frac{|\nabla u|}{|\nabla v|}} \nabla_\eta v E + 2\sqrt{\frac{|\nabla v|}{|\nabla u|}} \nabla_\eta u E \right) \\
=& \frac{1}{2|\nabla u|} (|\mathcal{E}_+^2 u|^2 - 2a\xi^2 |E|^2) \\
&+ \frac{1}{2|\nabla u|} (|\nabla u|^2 (R_M - 2K_u) + 2\operatorname{tr}_g(k) \Delta u |\nabla u| + (\operatorname{tr}_g(k)^2 - |k|^2)|\nabla u|^2 - 2\nabla_{ij}^2 u k_{ij} |\nabla u|) \\
&+ \frac{1}{2|\nabla v|} (|\mathcal{E}_-^2 v|^2 - 2\xi^2 |E|^2) \\
&+ \frac{1}{2|\nabla v|} (|\nabla v|^2 (R_M - 2K_v) - 2\operatorname{tr}_g(k) \Delta v |\nabla v| + (\operatorname{tr}_g(k)^2 - |k|^2)|\nabla v|^2 + 2\nabla_{ij}^2 v k_{ij} |\nabla v|).
\end{aligned}$$

Integrating by parts, we find

$$\begin{aligned}
\operatorname{div}(Z) =& \frac{1}{2|\nabla u|} (|\mathcal{E}_+^2 u|^2 - 2|\nabla v||\nabla u||E|^2) \\
&+ \frac{1}{2|\nabla u|} (|\nabla u|^2 (2\mu - 2K_u) + 2|\nabla u| \langle J, \nabla u \rangle) \\
&+ \frac{1}{2|\nabla v|} (|\mathcal{E}_-^2 v|^2 - 2|\nabla u||\nabla v||E|^2) \\
&+ \frac{1}{2|\nabla v|} (|\nabla v|^2 (2\mu - 2K_v) - 2|\nabla v| \langle J, \nabla v \rangle).
\end{aligned}$$

where

$$\begin{aligned}
Z =& \nabla|\nabla u| - \Delta u \frac{\nabla u}{|\nabla u|} + \nabla|\nabla v| - \Delta v \frac{\nabla v}{|\nabla v|} + 2\xi^{-1}(|\nabla u||\nabla v| + \langle \nabla u, \nabla v \rangle) E \\
&- \operatorname{tr}_g(k) \nabla u + \operatorname{tr}_g(k) \nabla v + k(\nabla u, \cdot) - k(\nabla v, \cdot).
\end{aligned}$$

This finishes the proof. □

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