

A Study of Edge Toric Ideals using Associated Graphs

Yingyi Shen

Mentor: Sonja Mapes

Submitted in partial fulfillment of the requirements
for the degree of Bachelor of Science
in the Trinity College of Arts and Sciences

DUKE UNIVERSITY

2012

Abstract

A Study of Edge Toric Ideals using Associated Graphs

Yingyi Shen

This thesis studies properties of edge toric ideals and resolutions by analyzing the associated graphs of algebraic structures. It mainly focused on proving that the repeated edges in a graph wouldn't change some properties of its underlying algebraic structure. An application of this result is that when we study multi-edge graphs, we can simplify infinite numbers of graphs to a simple one by deleting all the repeated edges.

Acknowledgements

At the time I started this independent research study, I had no idea how rewarding it would be. This thesis is only part of what I gained during this two-year research which greatly enriches my undergraduate experience. I want to thank all of the people who has been helping, supporting and believeing in me along this journey and make such wonderful experience happen.

To Sonja Mapes, thank you for being such a great mentor. Your true passion for mathematics is the greatest inspration for me. You are always encouraging and supportive and you gave me tremedous help in guiding me through this research, providing sound advice and pointing the right direction. I remember all the interesting and exciting discussions we had over these two years and your efforts to explain everything so clearly and your charming conversations make this research such a fun experience for me. I learned not only mathematics from you. Your positive attitudes in life and in work, your diligence and your attention to details are all the great characters that I am learning. I'm very lucky to have you as my mentor.

To David Kraines, thank you for organizing all these different programs and events to enrich the undergraduate math experience at Duke. During my four year's study at Duke, you have always been a great professor and friend to go to for advice. Your introduction to the PRUV program enabled me to start my research. You connected me with Sonja and you organized many events such that students can exchange their ideas with each other. You really build a close math undergraduate community and I learned and

gained a lot from my peers in this community.

To Thomas Witelksi, thank you for being my advisor and helping to finding the right path in completing my study. You are always resourceful and providing me concrete advice in selecting courses and preparing me for my future career. You carefully helped me to go through my academic status and offered great help to help me draft my academic plans. Your trust and help are priceless to me.

To my peers in math department, thank you all for the inspirations that you have given me. It was an honor to sharing these four year's amazing time with each and every one of you. I enjoyed discussing questions with you, no matter in the classroom or on the dinner table. It was always fun to have you around and I will miss you.

To all the staffs in the math department, especially Tyffany C. Kittrell, thank you for all the little but important things you do. Your work does facilitate my study.

To my defense committee, thank you for your time to be on my committee. Your comments on my thesis helped me to improve it. I do appreciate.

To Mr. David J. Zezza, Ms. Diane L. Stewart and Mr. and Mrs. D. Richard Mead, Jr., thank you all for your generosity in providing me the Duke endowments scholarships. Your support is far beyond the financial meanings for me. Knowing that there are people willing to support me and believing in my future achievements always encourages me. I am deeply indebted to your for your help and I will pass your spirit by helping other students.

To my parents, thank you for everything you have done for me. You gave

me my life and you taught me how to live it meaningfully. You always believe in me and you respect my decisions and encourage me to chase my dreams. Having your support and knowing that you will always be there for me make me faithful all the time.

To my parents, Qing Shen and Wei Wang

1 Introduction

The objective of this research project has been to explore the connection between graphs and the toric ideals arising from their incidence matrices. More specifically we sought to understand minimal generating sets for these ideals as well as higher syzygies in terms of the underlying graphs. We began our research in the summer of 2010. We first focused on minimal generating sets, and afterwards we focused on resolutions and syzygies and our goal is to find relations between syzygies and paths in the graph.

This next section will first provide all the preliminaries for the readers. We are going to cover basic definitions, notation and important previous results from other mathematicians. We will start by introducing toric ideals, one of the major subject in this paper. Then we introduce Groebner basis, minimal generators, resolutions and Castelnuovo-Mumford regularity.

After equipped our readers with these concepts, we then present our results in the first part of Section 3. We proved that the quotient ring of any 2-vertex graph will always be normal. Then we introduce our major result: repeating edges or removing repeated edges in a graph will not change the equality or inequality between the toric edge ideal and circuit ideal of that graph. We further use this result to show that the edge toric ideal and circuit ideal of any graph with 3 or less vertices equal each other.

We also read papers by other mathematicians throughout our research. In the second part of section 3, we cite their research as we want to show the audience further information beyond our current research progress. These results from other mathematicians answer some of the questions we have at

the end of summer 2010 and can be regarded as a helpful tool for us to further our understanding in the issue of different generators as well as to progress our research to the next stage about resolutions and syzygies.

The last section is mainly about our results on resolutions and Castelnuovo-Mumford regularity. Once again, we found that the repeated edges in a graph do not play any role in determining the regularity of the associate edge toric ideal. This result, similar to our last result about the equality between edge toric ideal and circuit ideal, can greatly help us to simplify the problem when we study multi-edge graphs since we can simplify infinite numbers of graphs to a simple one by deleting all the repeated edges.

At the end of this paper, we provide the *Macaulay 2* code we used and the reference.

2 Preliminaries

2.1 Toric Ideals

First, we give the definition of a toric ideal. This definition is essential for us to understand the material we are going to discuss in this report.

Definition 2.1. For a subset $A = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ of \mathbb{Z}^d , each vector \mathbf{a}_i is identified with a monomial $\mathbf{t}^{\mathbf{a}_i}$ in the Laurent polynomial ring

$$k[\mathbf{t}^{\pm 1}] := k[t_1, \dots, t_d, t_1^{-1}, \dots, t_d^{-1}].$$

For the semigroup homomorphism

$$\pi : \mathbb{N}^n \rightarrow \mathbb{N}A, \mathbf{u} = (u_1, \dots, u_n) \mapsto u_1 \mathbf{a}_1 + \dots + u_n \mathbf{a}_n.$$

The map π lifts to a homomorphism of semigroup algebras:

$$\overset{\circ}{\pi} : k[\mathbf{x}] \rightarrow k[\mathbf{t}^{\pm 1}], x_i \mapsto \mathbf{t}^{\mathbf{a}_i}.$$

Then, we call the kernel of $\overset{\circ}{\pi}$ the *toric ideal* of A and we denote it as I_A .

[9]

Example 2.2. For example, let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

Then a generating set of toric ideals of A is $\{x_2x_4 - x_3^2, x_1x_4^2 - x_3^3\}$.

We are going to introduce the concept of incidence matrix, which will serve as the A for our edge toric ideals of graphs.

Definition 2.3. Suppose G is a graph. Let V be the set of vertices of the graph where $V = \{v_1, v_2, \dots, v_n\}$, and E be the set of edges of the graph where $E = \{e_1, e_2, \dots, e_m\}$. A_G (or sometimes simply A when it is clear which graph we are referring to, e.g. when there is only one graph) is the *incidence matrix* of G if the j^{th} column of A is used to denote the j^{th} edges e_j where the i^{th} entry in the j^{th} column denotes whether the end point of the e_j is v_i . When one and only one of the end point of the j^{th} edge is the i^{th} vertex V_i , we put 1 into the i, j^{th} entry in the matrix. When both end points of the j^{th} edge are the same i^{th} vertex, we put 2 into the i, j^{th} entry in the matrix. Otherwise, we put 0 into the i, j^{th} entry of the matrix.

Definition 2.4. The *edge toric ideal* of a graph is the toric ideal of the incidence matrix A_G of the graph.

Remark 2.5. We allow the existence of multiple edges and loops in our graphs.

Example 2.6. This example is used to illustrate how we can find the incidence matrix of a graph. See Figure 1.

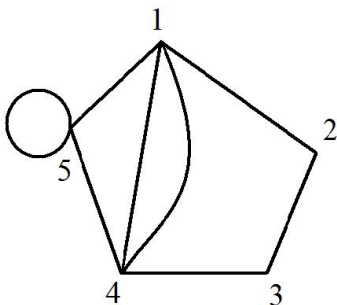


Figure 1: example 2.3

According to Definition 2.3, the incidence matrix of Figure 1 is

$$A_G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2 \end{pmatrix}$$

Also, we can use *Macaulay2* and *4Ti2* to find the toric ideal of this graph:

```
i1 : installPackage "FourTiTwo";
```

```
i2 : load "LLL.m2";
```

```
i3 : R = QQ[a,b,c,d,e,f,g,h]
```

o3 = R

o3 : PolynomialRing

i4 : A =

matrix{{1,0,0,0,1,1,1,0},{1,1,0,0,0,0,0,0},{0,1,1,0,0,0,0,0},
{0,0,1,1,0,1,1,0},{0,0,0,1,1,0,0,2}}

o4 = | 1 0 0 0 1 1 1 0 |
 | 1 1 0 0 0 0 0 0 |
 | 0 1 1 0 0 0 0 0 |
 | 0 0 1 1 0 1 1 0 |
 | 0 0 0 1 1 0 0 2 |

5 8

o4 : Matrix ZZ <--- ZZ

i5 : toricGroebner (A,R)

using temporary file name /tmp/M2-44680-3

4ti2 version 1.3.2, Copyright (C) 2006 4ti2 team.

4ti2 comes with ABSOLUTELY NO WARRANTY.

This is free software, and you are welcome

to redistribute it under certain conditions.

For details, see the file COPYING.

Using 64 bit integers.

4ti2 Total Time: 0.00 secs.

o5 = ideal (- a*c + b*f, b*d*e - a*c*h, - d*e + f*h, - f + g)

o5 : Ideal of R

The output toric ideal is $\langle -ac + bf, bde - acf, -de + fh, -f + g \rangle$.

In our later results, we showed that the repeated edges in a graph do not change some of the properties of the graph. Therefore, we introduce the notation for the columns of a graph with no repeated edges below.

Definition 2.7. Let $A_{G'}$ be the set of columns with no repeated edges of the incidence matrix A_G of graph G , as G' will be the graph with no repeated edges. $A_{G'} = \{\mathbf{a}_1, \dots, \mathbf{a}_m\}$.

Example 2.8. Therefore, for Figure 1,

$$A_{G'} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} \right\}$$

2.2 Groebner Basis

We should note that the ideals in the definition above are in fact finitely generated despite the fact that the definition does not give a description of how to find generators. The following algorithm is one such way.

Before we give the algorithm, we'll first define reduced Groebner bases.

Definition 2.9. Given a term order \succ , every non-zero polynomial $f \in k[\mathbf{x}]$ has a unique initial monomial, denoted $in_\succ(f)$. If I is an ideal in $k[\mathbf{x}]$, then its *initial ideal* is the monomial ideal:

$$in_{\succ}(I) := \langle in_{\succ}(f) : f \in I \rangle$$

A finite subset $G \subset I$ is a *Groebner basis* for I with respect to \succ if $in_{\succ}(I)$ is generated by $\{in_{\succ}(g) : g \in G\}$. A Groebner basis is *reduced* if $\{in_{\succ}(g) : g \in G\}$ is the unique minimal generating set of $in_{\succ}(I)$ and if no non-initial term of any $g \in G$ is divisible by any of $\{in_{\succ}(g) : g \in G\}$.

The reduced Groebner basis is unique for an ideal and a term order, provided one requires the coefficient of $in_{\succ}(g)$ in g to be 1 for each $g \in \mathcal{G}$. [9]

One can compute the Groebner basis for very simple structures by hand. Here below is an example.

Example 2.10. Consider the ideal $\langle x - y, y - z \rangle$ and we are going to build a Groebner basis based on the order $x \succ y \succ z$.

Then, trying to reduce the leading terms in both binomials, we get $y * (x - y) + (-x) * (y - z) = xz - y^2$.

Since the leading term in the result is xz and it can be generated by the leading terms x and y of the original binomials, we therefore don't add it into the Groebner basis and we thus have $\langle x - y, y - z, xz - y^2 \rangle$.

We repeat the process with the new set, $z * (x - y) + (-y) * (y - z) + xz - y^2 = 0$. We can see we have already successfully reduced all the terms. Therefore, no new terms needed to be added. As a result, the Groebner basis of the ideal $\langle x - y, y - z \rangle$ is just $\langle x - y, y - z \rangle$.

Then, with the definition of the reduced Groebner basis, we are able to give the algorithm for finding the ideal.

Algorithm 2.11. *The Conti-Traverso algorithm*

Input: A vector configuration $A \subset \mathbf{Z}^d$ and a term order given by the weight vector \mathbf{w} .

Output: The reduced Groebner basis of I_A with respect to w .

1. Introduce $n + d + 1$ indeterminates $t_0, t_1, \dots, t_d, x_1, \dots, x_n$. Let \succ be any elimination order such that $t_i \succ x_j$ and the x variables are ordered by \mathbf{w} .

2. Compute the reduced Groebner basis $\mathcal{G}_\succ(J)$ of the ideal

$$J = \langle t_0 t_1 \dots t_d - 1, x_j^{t_j^{\mathbf{a}_j^-}} - t_j^{\mathbf{a}_j^+}, j = 1, \dots, n \rangle.$$

3. Output the set $\mathcal{G}_\succ(J) \cap \mathbf{k}[\mathbf{x}]$ which is the reduced Groebner basis of I_A with respect to \mathbf{w} . [6]

2.3 Minimal Generators

One line of questioning that we are interested in is to compare the circuit ideal of a graph with the toric ideal. This is in part because circuits are sometimes easier to understand as we found in [5] a description of circuit ideals in terms of specific patterns in a graph. Thus when these ideals are equal, we have a nice description of our ideal generators.

In order to understand the definition of circuit, we need first introduce the definition of support.

Definition 2.12. The *support* of a monomial $x^{\mathbf{u}^+}$ of $K[x_1, \dots, x_m]$ is $\text{supp}(x^{\mathbf{u}}) := \{i | x_i \text{ divides } x^{\mathbf{u}}\}$ and the support of a binomial $B = x^{\mathbf{u}} - x^{\mathbf{v}}$ is $\text{supp}(B) := \text{supp}(x^{\mathbf{u}}) \cup \text{supp}(x^{\mathbf{v}})$. [7]

With the definition of support, we'll be able to define circuit.

Definition 2.13. An irreducible binomial B belonging to I_A is called a *circuit* of I_A if there is no binomial $B' \in I_A$ such that $\text{supp}(B') \subsetneq \text{supp}(B)$. A binomial $B \in I_A$ is a circuit of I_A if and only if $I_A \cap K[x_i \mid i \in \text{supp}(B)]$ is generated by B . [7]

Definition 2.14. Let \mathcal{C}_A be the set of circuits. The *circuit ideal* of A is the binomial ideal $I_{\mathcal{C}_A} := \langle \mathcal{C}_A \rangle \subset k[\mathbf{x}]$. [1]

The circuit ideal $I_{\mathcal{C}_A}$ is a subideal of I_A [1].

Lastly, we are going to give notations we are going to use for toric ideals and circuit ideals.

Notation 2.15. Let I_{A_G} denote the edge toric ideal corresponding to the graph G and $I_{\mathcal{C}_{A_G}}$ denote the circuit ideal of the graph G .

Then let us look at some theorem and proposition from other papers. Our new findings are based on them.

The following theorem and proposition is from 3.3 and 3.2 of [5] respectively.

Theorem 2.16. *A sub-multigraph H of G is called a circuit of G if H has one of the following forms:*

- (a) H is an even cycle.
- (b) H consists of two odd cycles intersecting in exactly one vertex; a loop is regarded as an odd cycle of length 1.
- (c) H consists of two vertex disjoint odd cycles joined by a path. [5]

Since the definition of normality will be of great importance for us to prove our main theorems, we are going to introduce it here.

Definition 2.17. The configuration A is *normal* if $\mathbb{N}A = \mathbb{Z}A \cap \mathbb{R}_+A$.

With the definition of normality, we can state the result below from [5].

Proposition 2.18. *Let G be a multigraph and let I_{A_G} be the toric ideal of the edge subring $K[G]$. Then $K[G]$ is normal if and only if I_{A_G} is generated by circuits with a square-free term. [5]*

Therefore, we know that for graphs whose A is normal, the toric ideal is equal to the circuit ideal.

2.4 Resolutions

The definition of free resolutions will help us to understand the concept of Castelnuovo-Mumford regularity in the next section.

Definition 2.19. For any polynomial ring R , denote by $R(a)$ as the polynomial ring R shifted (or "twisted") by a :

$$R(a)_d = R_{a+d}$$

where

$$R_t = \{f \in R \mid \deg f = t\}$$

For a graded R -module M , given homogeneous elements $m_i \in M$ of degree a_i that generate M as an S -module, we may define a map from the graded free module $F_0 = \bigoplus_i R(-a_i)$ onto M by sending the i -th generator to m_i . Let $M_1 \subset F_0$ be the kernel of this map $F_0 \rightarrow M$. By the Hilbert Basis Theorem, M_1 is also a finitely generated module. The elements of M_1 are called *syzygies* of the generators m_i , or simply *syzygies* of M . [3]

Definition 2.20. Choosing finitely many homogeneous syzygies that generate M_1 , we may define a map from a graded free module F_1 to F_0 with image M_1 . Continuing in this way we construct a sequence of maps of graded free modules, called a *graded free resolution* of M :

$$\dots \longrightarrow F_i \xrightarrow{\phi_i} F_{i-1} \longrightarrow \dots \longrightarrow F_1 \xrightarrow{\phi_1} F_0.$$

[3]

Definition 2.21. If \mathbf{F} is a minimal resolution of R/I , then a *subcomplex* of \mathbf{F} is a complex of three module \mathbf{G} , where $\mathbf{G}_i \subset F_i$ for all i with maps d'_i such that $d'_i = d_i|_{\mathbf{G}_i}$ and $d_i'^2 = 0$.

Definition 2.22. If \mathbf{F} is a minimal resolution of R/I and \mathbf{G} is a subcomplex of \mathbf{F} such that all of the nonzero entries in the maps of \mathbf{G} are linear polynomials. Then \mathbf{G} is the *linear strand*.

2.5 Castelnuovo-Mumford Regularity

We began to study the Castelnuovo-Mumford Regularity in the summer of 2010 and we will further explore it in the fall semester. This is related to understanding higher syzygies of the toric ideals. Understanding the generators for the toric ideals was a first step in understanding this invariant. Here are some useful definitions.

Definition 2.23. Suppose that \mathbf{F} is a free complex

$$\mathbf{F} : 0 \rightarrow F_s \rightarrow \dots \rightarrow F_m \rightarrow \dots \rightarrow F_0$$

where $F_i = \bigoplus_j S(-j)^{\beta_{i,j}}$; that is, F_i requires $\beta_{i,j}$ minimal generators of degree j . The Betti table of \mathbf{F} has the form

	0	1	...	s
i	$\beta_{0,i}$	$\beta_{1,i+1}$...	$\beta_{s,i+s}$
i + 1	$\beta_{0,i+1}$	$\beta_{1,i+2}$...	$\beta_{s,i+s+1}$
...
j	$\beta_{0,j}$	$\beta_{1,j+1}$...	$\beta_{s,j+s}$

It consists of a table with $s + 1$ columns, labeled $0, 1, \dots, s$, corresponding to the free modules F_0, \dots, F_s . It has rows labeled with consecutive integers corresponding to degrees. The m -th column specifies the number of generators in each degree of F_m . Thus, for example, the row labels at the left of the diagram correspond to the possible degrees of a generator of F_0 . Then for F_i the row labels at the left of the diagram are the possible degrees $-i$ of the generators of F_i . For clarity we sometimes replace a 0 in the diagram by a “-”.

If \mathbf{F} is the minimal free resolution of a module M , we refer to the Betti diagram of \mathbf{F} as the Betti diagram of M and the $\beta_{m,d}$ of \mathbf{F} are called the graded Betti numbers of M , sometimes written $\beta_{m,d}(M)$

Definition 2.24. We call the index of the last non-zero row of a betti table of a graph as the Castelnuovo-Mumford Regularity of this graph.

2.6 Simplicial Complexes and Simplicial Homology

Our work in spring 2012 focused on simplicial complexes and simplicial homology and we used them to detect syzygies from paths in the graph.

Definition 2.25. A *simplicial complex* on n vertices is a subset of the set of all subsets of $\{1, 2, \dots, n\}$ denoted as,

$$\Delta \subset 2^{[n]} = \{\emptyset, \{1\}, \{2\}, \dots, \sigma, \dots, \{1, 2, \dots, n\}\}$$

where $\sigma \subset \{1, 2, \dots, n\}$ such that Δ satisfies the following property:

if $\sigma \in \Delta$ and $\tau \subset \sigma$ then $\tau \in \Delta$.

Example 2.26. Let $n = 4$ then

$$\Delta_1 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 4\}, \{1, 2, 3\}\}$$

is a simplicial complex.

Given a simplicial complex Δ define a *chain complex* \mathcal{C}_Δ as follows. Let $C_i = \mathbb{R}^{k_i}$ where k_i is the number of size $i + 1$ subsets in Δ for $i \geq 0$. The empty set is the subset with 0 elements, by convention.

We define maps between C_i and C_{i-1} for each i as follows, let $d_k : C_k \rightarrow C_{k-1}$ be defined to be the map defined as $d_k(\sigma) = \sum_{i=1}^{k+1} (-1)^{i-1} \{v_0, \dots, \hat{v}_i, \dots, v_k\}$ where σ is a set in Δ with cardinality $k + 1$, and $\{v_0, \dots, \hat{v}_i, \dots, v_k\}$ is an element in Δ with cardinality k , specifically here we delete the v_i element from σ .

Example 2.27. For $\sigma = \{1, 2, 3\} \in \Delta_1$ we get that

$$d_2(\sigma) = (1) * \{2, 3\} + (-1) * \{1, 3\} + (1) * \{1, 2\}.$$

And as Δ_1 only has one 3 element set in it, this completely determines d_2 . Moreover, this means we can represent d_2 as the 4×1 matrix with entries 1,-1,1,0 (as a column).

Also for this example $d_1(\{1, 2\}) = (1) * \{2\} + (-1) * \{1\}$ and similarly you can compute what d_2 is for the other 2 element sets, and ultimately d_1 should be a 3×4 matrix. Finally, $d_0(\{1\}) = (1) * \emptyset$ and so d_1 is the 1×4 matrix $[1 \ 1 \ 1 \ 1]$.

Definition 2.28. The i -th reduced simplicial homology group is defined as: $\tilde{H}_i(\Delta) = \ker d_i / \text{im } d_{i+1}$ where the d_i 's are the maps for the chain complex of Δ .

Example 2.29. For Δ_1 , we get that $\tilde{H}_i(\Delta_1) = 0$ for all i .

We then connect these definitions to edge toric ideals and their syzygies.

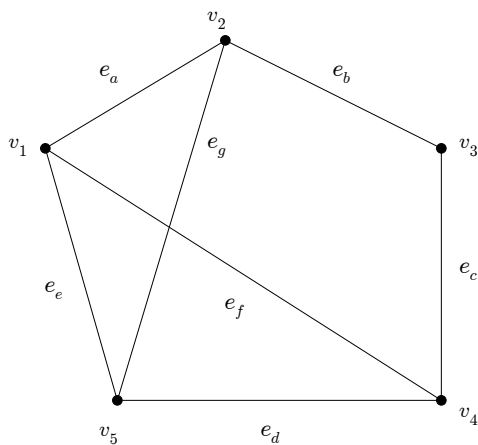
Recall the definition of A -degree, let $C_{\mathbf{b}} = \{\mathbf{x}^{\mathbf{a}} \mid \deg_A(\mathbf{x}^{\mathbf{a}}) = \mathbf{b}\}$.

Definition 2.30. $\Delta_{gcd}(\mathbf{b})$ is defined to be the simplicial complex with vertices the elements of $C_{\mathbf{b}}$ and faces all the subsets $T \subset C_{\mathbf{b}}$ such that $\gcd\{x^{\mathbf{a}} \mid x^{\mathbf{a}} \in T\} \neq 1$

The key formula comes from Corollary 3.3 in [10] and is the following:

$$\beta_{i,\mathbf{b}}(R/I_G) = \dim \tilde{H}_i(\Delta_{gcd}(\mathbf{b}))$$

We conclude with a good example.



Example 2.31. For the graph G above since there are 5 vertices the \mathcal{A} -degrees \mathbf{b} are in \mathbb{Z}^5 (because there are 5 rows in the incidence matrix). Moreover, we can compute the generators and syzygies for this example using *Macaulay2*.

Here the variable a corresponds to edge e_a and so on. The ideal that we get is $\langle ac - bf, ad - fg, bd - cg \rangle$, and the minimal resolution given by *Macaulay2* is the following:

```
i4 : (res I).dd
      1                3
o4 = 0 : R <----- R : 1
      | ac-bf ad-fg bd-cg|

      3                2
1 : R <----- R : 2
      {2} | g  -d |
      {2} | -b  c |
      {2} | a  -f |

      2
2 : R <----- 0 : 3
      0

o4 : ChainComplexMap
```

From here using Theorem 4.14 in [7] we can see that the minimal generators are indispensable and correspond to \mathcal{A} -degrees:

1. $\mathbf{b} = (1, 1, 0, 1, 1)$
2. $\mathbf{b} = (1, 1, 1, 1, 0)$
3. $\mathbf{b} = (0, 1, 1, 1, 1)$

Moreover computing the A -degrees of the first syzygies we get:

1. $\mathbf{b} = (1, 2, 1, 1, 1)$
2. $\mathbf{b} = (1, 1, 1, 2, 1)$

3 Minimal Generators

Below are the findings we made.

Theorem 3.1. *If G is a graph with 2 vertices, then R/I_{A_G} is always normal.*

This result is particularly interesting to us since we are going to use it to prove Theorem 3.4 later: according to Proposition 2.18, Theorem 3.1 can lead us to the result that for all 2-vertex graphs, the circuit ideals equals the toric ideas. Here below is the proof for Theorem 3.1.

Proof. First, for the incidence matrix of any 2-vertex graph G , there are only 3 possibilities for the columns of the matrix A :

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{3.1}$$

We know that R/I_{A_G} is normal if

$$\mathbb{N}A = \mathbb{Z}A \cap \mathbb{R}_+A.$$

[5]

We therefore conclude that the normality of R/I_{A_G} only depends on which vectors appears in A .

The only combinations possibilities for A with A having two vertices are

$$\begin{aligned} & \{\{2, 0\}, \{1, 1\}, \{0, 2\}\}, \\ & \{\{2, 0\}, \{1, 1\}\}, \\ & \{\{2, 0\}, \{0, 2\}\}, \\ & \{\{1, 1\}, \{0, 2\}\}, \\ & \{\{2, 0\}\}, \\ & \{\{1, 1\}\} \\ & \text{and } \{\{0, 2\}\}. \end{aligned}$$

Therefore all we need to do is to test whether

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad (3.2)$$

are all normal.

If they are all normal, then any matrices with these combinations of columns and repeated edges will be normal.

As we tested using *Macaulay 2*, they are all normal.

Since these combinations of columns with possible repeated edges cover all 2-vertex graphs, R/I_A is normal for all 2-vertex graphs. \square

This result is particularly interesting to us since we will use this result to prove Theorem 3.4. This result will help us to rule out the possibility of the existence of a 2-vertex graph whose toric ideal is not equal to its circuit ideal.

In order to prove Theorem 3.4. We need to first prove another result. Let us give a definition first.

Definition 3.2. For $\mathbf{u} = (u_1, \dots, u_m) \in \mathbb{N}^m$, we define the A -degree of the monomial $\mathbf{x}^{\mathbf{u}} := x_1^{u_1} \dots x_m^{u_m}$ to be $deg_A(\mathbf{x}^{\mathbf{u}}) := u_1 \mathbf{a}_1 + \dots + u_m \mathbf{a}_m \in \mathbb{N}A$. [7]

Definition 3.3. The monomials of R that do not lie in the initial ideal $in_{\succ}(I)$ are called the *standard monomials* of $in_{\succ}(I)$. [6]

Theorem 3.4. *If for every $\mathbf{b} \in \mathbb{N}A$ and generic weight vector $\omega \in \mathbb{R}^n$, $in_{\omega}(I_{C_A})$ has a unique standard monomial of A -degree \mathbf{b} , then $I_A = I_{C_A}$ [1].*

With the theorem above, we then get the result below.

Theorem 3.5. *Suppose for a graph $G = \{V, E\}$, where $V = \{v_1, v_2, \dots, v_n\}$ and $E = \{e_1, e_2, \dots, e_m\}$, $I_{A_G} = I_{C_{A_G}}$. If we add an edge $e_{m+1} = e_m$ to the graph and call the new graph G' , then $I_{A_{G'}} = I_{C_{A_{G'}}$. Similarly, if for the graph G , $I_{A_G} \neq I_{C_{A_G}}$, then for the graph G' with repeated edges, $I_{A_{G'}} \neq I_{C_{A_{G'}}$.*

Proof. Suppose

$$A_G = \left(\mathbf{B} \mid \mathbf{a} \right),$$

where A_G is the incidence matrix of the graph G , \mathbf{B} is the matrix composed of from the 1st column to the $m - 1$ th column of A_G and \mathbf{a} is the m th column of A_G .

Then, we will have the incidence matrix of G' as

$$A_{G'} = \left(\mathbf{B} \mid \mathbf{a} \mid \mathbf{a} \right).$$

If $I_{A_G} = I_{C_{A_G}}$, we want to show the same is true for G' , i.e., $I_{A_{G'}} = I_{C_{A_{G'}}}$.
 If $I_{A_G} \neq I_{C_{A_G}}$, we want to show $I_{A_{G'}} \neq I_{C_{A_{G'}}}$.

Fixing \mathbf{b} (\mathbf{b} can be anything in $\mathbb{N}A$), we show this by considering how standard monomials of $A(G')$ -degree \mathbf{b} are related to the standard monomials of $A(G)$ -degree \mathbf{b} .

Let $\mathbf{u} = \{u_1, u_2, \dots, u_m\}$ be a solution of $A_G \mathbf{x} = \mathbf{b} \in \mathbb{N}(A(G))$.

Then we know that

$$\mathbf{u}' = \{u_1, u_2, \dots, u_{m-1}, u_m, 0\}$$

and

$$\mathbf{u}'' = \{u_1, u_2, \dots, u_{m-1}, 0, u_m\}$$

are both solutions of $A_{G'} \mathbf{x} = \mathbf{b} \in \mathbb{N}A$.

Case 1: If $u_m = 0$;

Then the two solutions are the same.

And in order to get the second independent solution, we can construct $\mathbf{u}_n = \{u_1, u_2, \dots, u_{m-1}, 1, -1\}$ and we find it is a solution.

However there is a negative element in u_n , making u_n generate a binomial instead of a monomial.

As a result, the standard monomial of $A_{G'}$ -degree \mathbf{b} will be the same as the standard monomial of A_G -degree \mathbf{b} .

Therefore, if initially we have only one standard monomial of A_G -degree \mathbf{b} , then we still have one standard monomial of $A_{G'}$ -degree \mathbf{b} . If initially we have multiple standard monomials of A_G -degree \mathbf{b} , then we still have multiple standard monomials of $A_{G'}$ -degree \mathbf{b} . As a result, the equality or

non-equality between the circuit ideal and the toric ideal is not going to change.

Case 2: If $u_m \neq 0$;

If $I_{A_G} = I_{C_{A_G}}$, then as stated in the theorem 2.1, we know that for every $\mathbf{b} \in \mathbb{N}A$ and generic weight vector $\omega \in \mathbb{R}^n$, $in_\omega(I_{C_{A_{G'}}})$ has a unique standard monomial of A -degree \mathbf{b} .

Therefore in order to prove the same statement is true for G' , all we need to prove is that with the variables x_m and x_{m+1} , only one of them is left as a standard monomial of $R/I_{C_{A_{G'}}}$.

Recall definition 1.4, we need to prove one of x_m and x_{m+1} must lie in the initial ideal $in_{\succ}(I_{C_{A_{G'}}})$.

We know that with the new repeated edge e_{m+1} in G' , there is one more circuit ideal generator $x_m - x_{m+1}$.

Then either $x_m \succ x_{m+1}$ or $x_{m+1} \succ x_m$.

Thus, only one of x_m and x_{m+1} will be a standard monomial for any generic ordering.

Though we have one more element x_{m+1} for G' , either this monomial or x_m will be eliminated from the standard monomials.

Therefore, the uniqueness or non-uniqueness of the standard monomial of $A(G')$ -degree \mathbf{b} will be the same as that of A_G -degree \mathbf{b} .

If initially $I_{A_G} = I_{C_{A_G}}$, as a result, for every $\mathbf{b} \in \mathbb{N}A_{G'}$ and generic weight vector $\omega \in \mathbb{R}^n$, $in_\omega(I_{C_{A_{G'}}})$ has a unique standard monomial of $A_{G'}$ -degree \mathbf{b} .

If initially $I_{A_G} \neq I_{C_{A_G}}$, similarly, $I_{A_{G'}} \neq I_{C_{A_{G'}}}$.

If there are r repeated edge,

then

$$A_{G''} = \left(\mathbf{B} \mid \mathbf{a} \mid \mathbf{a} \mid \dots \mid \mathbf{a} \right).$$

As a result, we will have $r-1$ new circuit ideal generators $x_m - x_{m+1}, x_m - x_{m+2}, \dots, x_m - x_{m+r-1}$.

Therefore, we will pick up a strict order for $x_m, x_{m+1}, \dots, x_{m+r-1}$.

On the other hand, we have r solutions for $A_{G''}\mathbf{x} = \mathbf{b}$ arising from $\mathbf{u} = \{u_1, u_2, \dots, u_m\}$:

$$\mathbf{u}_1 = \{u_1, \dots, u_{m-1}, u_m, 0, 0, \dots, 0\}$$

$$\mathbf{u}_2 = \{u_1, \dots, u_{m-1}, 0, u_m, 0, \dots, 0\}$$

...

$$\mathbf{u}_r = \{u_1, \dots, u_{m-1}, 0, 0, 0, \dots, u_m\}.$$

When $u_m = 0$, similarly to the discussion for one repeated edge, the equality or non-equality between the circuit ideal and the toric ideal won't change.

When $u_m \neq 0$, for some generic ordering we will set $(r-1)$ of the variables from the set $S = \{x_m, x_{m+1}, \dots, x_{m+r-1}\}$ as elements in $in_\omega I_{C_{A_{G''}}}$. Thus we only have one variable from S as a standard monomial.

Therefore, if initially there is only one standard monomial of A_G -degree \mathbf{b} , then exactly one standard monomial of $A_{G''}$ -degree \mathbf{b} will be left.

Similarly, if initially there are multiple standard monomials of A_G -degree \mathbf{b} , then the same number of multiple standard monomials of $A_{G''}$ -degree \mathbf{b} will be left.

As a result, $I_{A_{G''}} = I_{C_{A_{G''}}}$ if $I_{A_G} = I_{C_{A_G}}$. Also, $I_{A_{G''}} \neq I_{C_{A_{G''}}}$ if $I_{A_G} \neq I_{C_{A_G}}$.

□

Theorem 3.6. *In order to have $I_{A_G} \neq I_{C_{A_G}}$ for the graph G , G has to have at least 4 vertices (i.e., for $G = \{V, E\}$, if $\|V\| \leq 3$, then $I_{A_G} = I_{C_{A_G}}$).*

Proof. First, we prove for 1-, 2-, and 3-vertex graphs, $I_{A_G} = I_{C_{A_G}}$ must hold.

For a 1-vertex graph, the only kind of edge is the loop. We can have no loop, a single loop, or multiple loops.

In all the three situations, the graph is normal since the only possible non-zero column for incidence matrix is 2 and we know

$$\begin{pmatrix} 2 \end{pmatrix}.$$

is normal.

As we proved in Theorem 3.1 above, 2-vertex graphs are all normal as well.

Therefore, $I_{A_G} = I_{C_{A_G}}$ always hold for 1- and 2-vertex graphs.

For 3-vertex graphs, similarly, we can consider only graphs without multiple edges by Theorem 3.5.

There are two types of non-normal graphs by exhaustively checking graphs without multiple edges using *Macaulay2*:

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 1 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

Since all other non-normal graphs are just these two graphs with repeated edges, with Theorem 3.5, we know that their equality for I_{A_G} and $I_{C_{A_G}}$ are the same as that of the two matrices above.

With *Macaulay 2* and *FourTiTwo*, we got the result that for the two above matrices, $I_{A_G} = I_{C_{A_G}}$.

Therefore, $I_{A_G} = I_{C_{A_G}}$ holds for all non-normal 3-vertex graphs.

As a result, for all 3-vertex graphs, $I_{A_G} = I_{C_{A_G}}$.

So far, we have proved that for all 1-, 2-, and 3-vertex graphs, $I_{A_G} = I_{C_{A_G}}$.

Next we are going to find an example of 4-vertex graph for which $I_{A_G} \neq I_{C_{A_G}}$.

An example is,

$$\begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Therefore there is at least one 4-vertex graph for which $I_{A_G} \neq I_{C_{A_G}}$.

As a result, the bound for $I_{A_G} \neq I_{C_{A_G}}$ is sharp.

As a result, in order for a graph G to have $I_{A_G} \neq I_{C_{A_G}}$, G must have at least 4 vertices. □

3.1 Results from other Mathematicians

In this section, we are going to give the results we found in the paper *Minimal Generators of Toric Ideals of Graphs* by *Enrique Reyes*, *Christos Tatakis*, and *Apostolos Thoma*. We are going to accompany the definitions with examples to help the readers better understand these terms. Please note that everything in this subsection is credited to the authors of that paper. Also, this paper in particular answers the question we were considering in the summer of 2010. Incidentally this paper had come out just a few months earlier.

With their results we were able to move on to understand higher syzygies which was our ultimate goal.

We are especially interested in this paper because it provides a good way of understanding how algebraically characterize a structure by looking at its associated graphs.

They first defined what is *indispensable*, *irreducible*, *primitive* and *fundamental* binomials.

Please note that in this subsection specifically, we are going to give A a new definition in the context.

Notation 3.7. Let $A = \mathbf{a}_1, \dots, \mathbf{a}_m \subset \mathbb{N}^n$ be a vector corresponding in \mathbb{Q}^n and $\mathbb{N}A := l_1\mathbf{a}_1 + \dots + l_m\mathbf{a}_m \mid l_i \in \mathbb{N}$ the corresponding affine semigroup.

Remark 3.8. The minimal binomials, up to scalar multiple, are finitely many. Their number is computed in terms of combinatorial invariants of a simplicial complex associated to the toric ideal.

Definition 3.9. A binomial $B \in A$ is called *indispensable* if there exists a nonzero constant multiple of it to every minimal system of generators of I_A .

Definition 3.10. A binomial $B \in A$ is called *irreducible* if B cannot be expressed as a product of $B_1 \in A$ and $B_2 \in A$.

Definition 3.11. An irreducible binomial $x^{\mathbf{u}^+} - x^{\mathbf{u}^-}$ in I_A is called *primitive* if there exists no other binomial $x^{\mathbf{v}^+} - x^{\mathbf{v}^-} \in I_A$ such that $x^{\mathbf{v}^+}$ divides $x^{\mathbf{u}^+}$ and $x^{\mathbf{v}^-}$ divides $x^{\mathbf{u}^-}$.

Definition 3.12. A binomial $B \in I_A$ is called *fundamental* if there exists a combinatorial pure subring $K[\mathbb{N}A_F]$ such that $K[x_i \mid \mathbf{a}_i \in A_F] \cap I_A = I_{A_F} = \langle B \rangle$.

It is known from this paper: The fundamental binomials are indispensable. The indispensable binomials are always minimal. The minimal binomials are always primitive. The fundamental binomials are circuits and the circuits are primitive.

In the following discussion, G will be a finite simple connected graph on the vertex set $V(G) = \{v_1, \dots, v_n\}$. Let $E(G) = \{e_1, \dots, e_m\}$ be the set of edges of G .

Definition 3.13. A *walk* connecting $v_1 \in V(G)$ and $v_{q+1} \in V(G)$ is a finite sequence of the form

$$w = (\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \dots, \{v_{i_q}, v_{i_{q+1}}\})$$

with each $e_{i_j} = \{v_{i_j}, v_{i_{j+1}}\} \in E(G)$.

Length of the walk w is called the number of q of edges of the walk. An *even* (respective *odd*) walk is a walk of *even* (respectively *odd*) length.

A walk $w = (\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \dots, \{v_{i_q}, v_{i_{q+1}}\})$ is called *closed* if $v_{i_{q+1}} = v_1$.

A *cycle* is a closed walk $w = (\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \dots, \{v_{i_q}, v_{i_1}\})$ with $v_{i_k} \geq v_{i_j}$ for every $1 \leq k > j \leq q$.

Theorem 3.14. Let G be a finite connected graph. If $B \in I_{AG}$ is primitive, then we have $B = B_w$ where w is one of the following even closed walks:

- (1) w is an even cycle of G
- (2) $w = (c_1, c_2)$, where c_1 and c_2 are odd cycles of G having exactly one common vertex
- (3) $w = (c_1, w_1, c_2, w_2)$, where c_1 and c_2 are odd cycles of G having exactly one common vertex and where w_1 and w_2 are walks of G both of which

combine a vertex v_1 of c_1 and a vertex v_2 of c_2 .

Definition 3.15. A *cut edge* (respectively *cut vertex*) is an edge (respectively vertex) of the graph whose removal increases the number of connected components of the remaining subgraph.

Definition 3.16. A graph is called *biconnected* if it is connected and does not contain a cut vertex.

Definition 3.17. A *block* is maximal biconnected subgraph of a given graph G .

Definition 3.18. *Sink* of a block C is a common vertex of two odd or two even edges of the walk w which belong to the block B .

Definition 3.19. Let \mathbf{w} be the underlying graph of the walk w .

Theorem 3.20. Let G be a graph and w be an even closed walk of G . The walk w is primitive if and only if

- (1) every block of \mathbf{w} is a cycle or a cut edge
- (2) every multiple edge of the walk w is a double edge of the walk and a cut edge of \mathbf{w}
- (3) every cut vertex of \mathbf{w} belongs to exactly two blocks and it is a sink of both

Corollary 3.21. Let G be a graph and W be a subgraph of G . The subgraph W is the graph \mathbf{w} of a primitive walk w if and only if

- (1) every block of W is a cycle or a cut edge

(2) every cut vertex of W belongs to exactly two blocks and separates the graph in two parts, the total number of edges of the cyclic blocks in each part of odd.

Definition 3.22. An edge f of the graph G is called a *chord* of the walk w if the vertices of the edge f belong to $V(\mathbf{w})$ and $f \notin E(\mathbf{w})$.

Definition 3.23. A chord $f = v_1, v_2$ is called *bridge* of a primitive walk w if there exist two different blocks B_1, B_2 of \mathbf{w} such that $v_1 \in B_1$ and $v_2 \in B_2$. A chord is called *even* (respectively *odd*) if it is not a bridge and breaks the walk in two even (respectively odd) walks.

Definition 3.24. Let $w = (\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \dots, \{v_{i_q}, v_{i_1}\})$ be a primitive walk. Let $f = \{v_{i_s}, v_{i_j}\}$ and $f' = \{v_{i_{s'}}, v_{i_{j'}}\}$ be two odd chords with $1 \leq s < j \leq 2q$ and $1 \leq s' < j' \leq 2q$. We say that f and f' *cross effectively* in w if $s' - s$ is odd and neither $s < s' < j < j'$ nor $s' < s < j' < j$.

Definition 3.25. We call an F_4 of the walk w a cycle (e, f, e', f') of length four which consists of two edges e, e' of the walk w both odd or both even, and two odd chords f and f' which cross effectively in w .

Definition 3.26. Let w be a primitive walk and f, f' be two odd chords. We say that f, f' *cross strongly effectively* in w if they cross effectively and they do not form an F_4 in w .

Proposition 3.27. *Let w be a primitive walk. If B_w is a minimal binomial then all the chords of w are odd and there are not two of them which cross strongly effectively.*

Theorem 3.28. *Let w be an even closed walk. B_w is a minimal binomial if and only if w is strongly primitive, all the chords of w are odd, there are not two of them which cross strongly effectively and no odd chord crosses an F_4 of the walk w .*

Theorem 3.29. *Let w be an even closed walk. B_w is an indispensable binomial if and only if w is strongly primitive, all the chords of w are odd, there are not two of them which cross effectively.*

Theorem 3.30. *If w is an even closed walk, then the binomial B_w is fundamental if and only if w is a circuit and has no chords except in the case that it is a cycle with no even chords and at most one odd chord.*

Example 3.31. An example given in the paper is below.

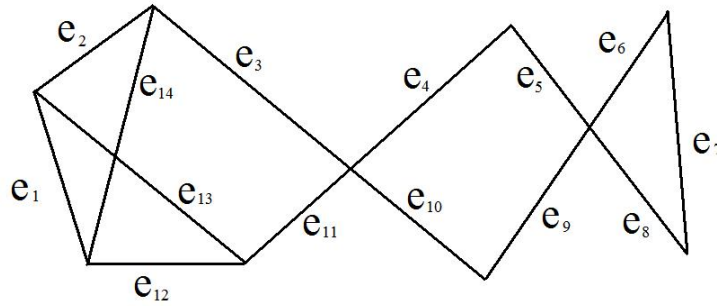


Figure 2: Example

$B_1 = e_2e_{12} - e_{13}e_{14}$ is a fundamental binomial.

$B_2 = e_{14}e_2e_4e_6e_8e_{10} - e_1e_3^2e_5e_7e_9$ is an indispensable binomial and it's not fundamental.

$B_3 = e_3e_1e_{11}e_5e_9e_7 - e_2e_{12}e_4e_{10}e_6e_8$ is a minimal binomial and it's not indispensable.

$B_4 = e_{14}e_1e_{11}^2e_9^2e_7 - e_2e_{12}^2e_{10}^2e_6e_8$ is a primitive binomial and it's not minimal.

B_1 and B_4 are circuits and B_2 and B_3 are not circuits.

4 Higher Syzygies and Regularity

During the fall semester of 2010, we mainly focused on studying the Castelnuovo-Mumford regularity of edge toric ideals. Our main research focused on determining if it was sufficient to study graphs with no repeated edges.

We tested some examples.

Example 4.1. First, let's see this example. In the graph on the left, there isn't any repeated edge. Then, we add a repeated edge, *edge 8*, and generate the graph on the right.

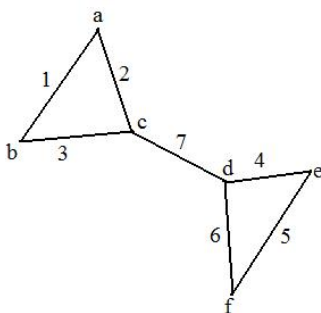


Figure 3: example 3.1

Here, we used *Macaulay2* to test the regularities of toric ideals and circuit ideals of these two graphs respectively.

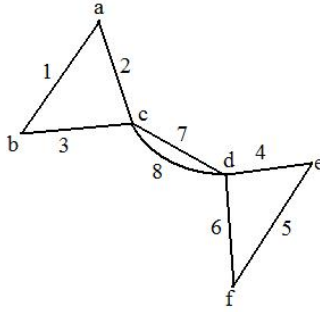


Figure 4: example 3.1

```

R = QQ[a,b,c,d,e,f,g,h,i];
A = matrix gTm (6,{{1,2},{2,3},{1,3},{4,5},{5,6},{4,6},{3,4}});
regularity toricGroebner (A,R)
o1 = 4
B = matrix gTm (6,{{1,2},{2,3},{1,3},{4,5},{5,6},{4,6},{3,4},{3,4}});
regularity toricGroebner (B,R)
o3 = 4

```

We see that the regularity of I_A is the same as the regularity of I_B .

Example 4.2. In this example, we test the situation where there are more repeated edges. Here we used the second graph from the example above. Then we add another repeated edge to this graph to generate the third graph below.

We then use *Macaulay2* to test the regularity of the edge toric ideal.

```

C = matrix gTm (6,{{1,2},{2,3},{1,3},{4,5},{5,6},{4,6},{3,4},
{3,4},{3,4}});

```

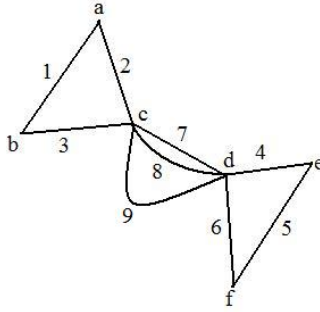


Figure 5: example 3.2

regularity toricGroebner (C,R)

o5 = 4

We can see the same result still holds even we have more repeated edges.

Suppose we have a graph G . We found that when we repeat an edge in a graph, or if we remove a repeated edge in a graph to generate a new graph G' . The regularity of the edge toric ideal doesn't change.

We tried to prove this with a proposition from [4].

Proposition 4.3. *If M is a finitely generated graded S -module and x is a linear form of S that is a nonzerodivisor on M , then $\text{reg}M = \text{reg}(M/xM)$. More generally, if x is a linear form whose annihilator $(0 :_M x)$ in M has finite length, then*

$$\text{reg}M = \max(\text{reg}(0 :_M x), \text{reg}M/xM).$$

Then we give our own theorem.

Theorem 4.4. *Suppose a graph $G = \{V, E\}$, where $V = \{v_1, v_2, \dots, v_n\}$ and $E = \{e_1, e_2, \dots, e_m\}$, has regularity $\text{reg}I_G$. If we add an edge $e_{m+1} = e_m$ to*

the graph and call the new graph G' , then this new graph G' has regularity $\text{reg}I_{G'} = \text{reg}I_G$. By induction, we know if we remove a repeated edge from a graph F to construct a new graph F' , then the regularity of the new graph $\text{reg}I_{F'}$ equals the regularity of the original graph $\text{reg}I_F$. I.e., $\text{reg}I_{F'} = \text{reg}I_F$.

Proof. We are going to prove Theorem 4.4 by Proposition 4.3.

First, we let $R/I_G = M$.

Then we use x to denote the binomial of the two repeated edges.

Since x is a linear form of the module, all we need to show is to prove that x is a nonzerodivisor on M .

That is equivalent to prove that

$$xg \in I_G \text{ if and only if } g \in I_G$$

Since I_G is a prime ideal, the statement above is true and we therefore showed that all the conditions in proposition 1.2 are satisfied.

As a result, according to proposition 1.2, we know that $\text{reg}M = \text{reg}(M/xM)$, which is equivalent to $\text{reg}I_G = \text{reg}I_{G'}$. \square

We also raised the question below that we are hoping to answer in the future:

Question 1. Can we determine which graphs G have edge toric ideals whose resolution has a linear strand?

In particular, we can focus our attention to graphs without repeated edges. To further understand this question, we return to the ideas of trying to detect syzygies from paths in the graph. This is the focus of our work in spring 2012. We then have the following questions.

Question 2. Given the A -degrees of the generators or indispensable generators of the ideal, how do we determine the A -degrees of the higher syzygies or indispensable syzygies? Specifically, how do we find collections of edges in the graph for which the corresponding $\Delta_{gcd}(\mathbf{b})$ has nonzero homology in some degree?

We now revisit Example 2.31 in order to give an idea of how we might try to answer Question 2.

Example 4.5. *This is a continuation of Example 2.31*

If we call $F_1 = ac - bf$, $F_2 = ad - fg$, and $F_3 = bd - cg$ we can see the syzygies in terms of the even and odd components of the corresponding closed walks in the following way.

Let A be the closed walk corresponding to F_1 and denote A_1 as the set of edges $\{e_a, e_c\}$ and $A_2 = \{e_b, e_f\}$. Likewise, let $B = B_1 \cup B_2$ where $B_1 = \{e_a, e_d\}$ and $B_2 = \{e_f, e_g\}$ and $C = C_1 \cup C_2$ where $C_1 = \{e_b, e_d\}$ and $C_2 = \{e_c, e_g\}$.

Now define a new operation: $S * T$ of two sets to be $(S \cup T) - (S \cap T)$.

If we compute all possible $A_i * B_j$ for $i, j = 1, 2$ and compare the resulting sets to C_1 and C_2 we get the following:

$$A_1 * B_1 = \{e_c, e_d\}; \text{ and } e_d \in C_1, e_c \in C_2$$

$$A_1 * B_2 = \{e_a, e_c, e_f, e_g\}$$

$$A_2 * B_1 = \{e_b, e_f, e_a, e_d\}$$

$$A_2 * B_2 = \{e_b, e_g\}; \text{ and } e_b \in C_1, e_g \in C_2.$$

So somehow here the “interesting” pairs are A_1, B_1 and A_2, B_2 .

Do the same for $A_i * C_j$ with B_1, B_2 , and $B_i * C_j$ with A_1, A_2 for $i, j = 1, 2$.

$$A_1 * C_1 = \{e_a, e_c, e_b, e_d\}$$

$$A_1 * C_2 = \{e_a, e_g\}; \text{ and } e_a \in B_1, e_g \in B_2$$

$$A_2 * C_1 = \{e_f, e_d\}; \text{ and } e_d \in B_1, e_f \in B_2$$

$$A_2 * C_2 = \{e_b, e_f, e_c, e_g\}$$

$$B_1 * C_1 = \{e_a, e_b\}; \text{ and } e_a \in A_1, e_b \in A_2$$

$$B_1 * C_2 = \{e_a, e_d, e_c, e_g\}$$

$$B_2 * C_1 = \{e_f, e_g, e_b, e_d\}$$

$$B_2 * C_2 = \{e_f, e_c\}; \text{ and } e_c \in A_1, e_f \in A_2.$$

So we add to the “interesting” pairs, A_1, C_2 ; A_2, C_1 ; B_1, C_1 ; and B_2, C_2 .

Now we want to look at which pairs of “interesting” pairs include a subset from each of A, B and C . Note that a pair of “interesting” pairs by definition includes 4 subsets total each coming from A, B and C so we will have a fourth such subset which can come from one of A, B , or C and we want to require that we get only one instance of having the subset 1 and subset 2 of either A, B , or C .

So here we can take the following pairs of “interesting” pairs:

1. A_1, B_1 and A_2, C_1
2. A_1, B_1 and B_2, C_2
3. B_2, C_2 and A_2, C_1
4. A_2, B_2 and A_1, C_2

5. A_2, B_2 and B_1, C_1

6. A_1, C_2 and B_1, C_1

Now remembering that A goes with F_1 , B goes with F_2 , and C goes with F_3 examine the overlaps between the pairs listed above.

Observe that for number 4 above, $(A_2 * B_2) \cap (A_1 * C_2) = \{e_g\}$ and $(A_2 * B_2) * (A_1 * C_2) = \{e_b, e_a\}$. This seems to correspond exactly to the syzygy $gF_1 - bF_2 + aF_3$.

Testing this for the other pairs of pairs in the above list we see the following:

1. $(A_1 * B_1) * (A_2 * C_1) = \{e_c, e_d, e_f\} - \{e_d\}$, which seems to correspond to $-dF_1 + cF_2 - fF_3$

2. $(A_1 * B_1) * (B_2 * C_2)$, ends up roughly the same as for 1

3. $(B_2, *C_2) * (A_2 * C_1)$, ends up being roughly the same as for 1

4. done above

5. $(A_2 * B_2) * (B_1 * C_1)$, ends up being roughly the same as for 4

6. $(A_1 * C_2) * (B_1 * C_1)$, ends up being roughly the same as for 4

Question 3. Given the minimal generators of an ideal, does some sort of analysis similar to above detect all of the first syzygies?

Question 4. What is the connection between the above analysis and figuring out the A -degree of the corresponding syzygy?

We hope to answer these question in the future.

5 Appendix

Here is the *Macaulay2* and *4ti2* code we wrote.

```
-- function computes the toric Ideal using the Hosten-Sturmfels  
Algorithm, this code can be found along with a description of  
the algorithm in reference
```

```
[5]
```

```
iG = (A,w) -> (  
  n := #(A_0);  
  R = QQ[x_1..x_n, Degrees => transpose A, MonomialSize =>1 6,  
  Weights => w];  
  B := transpose LLL syz matrix A;  
  J := ideal apply (entries B, b -> toBinomial(b,R));  
  scan (gens ring J, f -> J = saturate (J,f));  
  gens gb J  
)
```

```
-- function used to convert edges list of a graph to the  
incidence matrix list
```

```
gTm = (n, L) -> (  
  entries transpose matrix apply(L, edge -> apply(n, i->  
  # select(edge, a-> a == i+1)))  
)
```

```
-- function used to find the toric ideal of a graph
```

```

gTi = (n, L) -> (
    G := iG (gTm (n, L), apply(#L,1 -> 1));
    ideal flatten entries G
)

-- function used to check the normality
normal = (n, L) -> (
    J := gTi (n,L);
    K := R/J;
    isNormal K
)

-- function used to find examples of graphs whose toric
ideal is not equal to its circuit ideal
testRandomExamples = (n, trials) -> (
    nots := {};
    M = 2*n;
    R = QQ[x_1..x_M];
    for i from 1 to trials do (
        vertices = n;
        m := random (2, M);
        edges := apply (m, i-> {random (1,vertices),
            random(1,vertices)});
        A := matrix gTm (vertices,edges);
        if toricGroebner (A,R) != toBinomial

```

```

    (toricCircuits A, R) then notes = append(notes,A);
  );
  notes )

-- function used to generate an N-gon
Ngon = (n) -> (
  L := apply (n-1, i -> {i+1, i+2});
  append (L, {1,n})
)

-- function used to add a cycle to a vertex to the
existing graph
-- input:
-- A = the edge list of existing graph
-- v1 = the vertex the cycle will be attached onto
-- m = the number of edges of the attached cycle
addCycle = (A, v1, m) -> (
  -- if m != 1 then (
    a := max unique apply (#(flatten A), i->
      (flatten A)_i);
    L := apply ((m-2), i->{a+i+1,a+i+2});
    L1 := append (L, {v1, a+1});
    edges := flatten {A, append (L1, {v1, a+m-1})}
  -- )
  -- else edges := append (A, {v1,v1});

```

```

-- edges
)

-- function used to generate all the chords for
some Ngon
Chords = (n) -> (
  toList( set subsets(apply(n, i-> 1+i),2)
  - set Ngon(n))
)

-- take a graph with a ngon in it, and add a set
of chords
addAllChords = (n) -> (
  allPossChords := subsets Chords(n);
  apply(allPossChords, 1-> flatten {Ngon(n), 1})
)

-- testreg
-- G := the graph of interest
-- L := the list of #edges of attached cycles

-- first try to find the number of edges and
define R
nedge = (G, L) ->(
  x := max unique apply (#(flatten G_0 ),

```

```

i->(flatten G_0)_i);
y := x - sum L + #L ;
m := y*(y-1)//2 + sum L
)

-- after defining R, calculate the regularity
testReg = (G, L) -> (
  p := max unique apply (#(flatten G_0 ),
  i->(flatten G_0)_i);
  q := p - sum L + #L ;
  m := q*(q-1)//2 + sum L ;
  R := QQ[x_1..x_m];
  unique apply (G, g -> (regularity toricGroebner
  (matrix gTm (p, g), R)))
)

-- regularity of the graph w/o chords
wocReg = (G, L) -> (
  p := max unique apply (#(flatten G_0 ),
  i->(flatten G_0)_i);
  q := p - sum L + #L ;
  m := q*(q-1)//2 + sum L ;
  R := QQ[x_1..x_m];
  regularity toricGroebner (matrix gTm (p, G_0), R)
)

```

```
-- function to test the equality of the regularity
gTm = (n, L) -> (
  entries transpose matrix apply (L, edge ->
  apply (n, i -> # select (edge, a -> a == i+1)))
)
```

References

- [1] TRISTRAM BOGART, ANDERS N. JENSEN, AND REKHA R. THOMAS, *The Circuit Ideal of a Vector Configuration*. J. Algebra 309 (2007), no.2, 518-542.
- [2] REINHARD DIESTEL, *Graph Theory*. Graduate Texts in Mathematics, 173. Springer-Verlag, Berlin, 2005.
- [3] DAVID EISENBUD, *The Geometry of Syzygies: A Second Course in Algebraic Geometry and Commutative Algebra* Graduate Texts in Mathematics, 229. Springer-Verlag, New York, 2005.
- [4] DAVID EISENBUD, *Commutative Algebra: with a View Toward Algebraic Geometry (Graduate Texts in Mathematics)*. Springer (March 30, 1995).
- [5] JOSE MARTINEZ-BERNAL AND RAFAEL H. VILLARREAL, *Toric Ideals generated by circuits* arXiv:0812.0195v2.
- [6] DIANE MACLAGAN, REKHA R. THOMAS, SARA FARIDI, LEAH GOLD, A.V. JAYANTHAN, AMIT KHETAN, AND TONY PUTHENPURAKAL,

Computational Algebra and Combinatorics of Toric Ideals. Notes from workshop at Harish Chandra Research Institute in December 2003

- [7] ENRIQUE REYES, CHRISTOS TATAKIS, AND APOSTOLOS THOMA, *Minimal Generators of Toric Ideals of Graphs*. preprint
- [8] ARON SIMIS, WOLMER V. VASCONCELOS, AND RAFAEL H. VILLARREAL, *The Integral Closure of Subrings Associated to Graphs*. *Journal of Algebra* 199, 281-289 (1998).
- [9] BERND STURMFELS, *Grobner bases and convex polytopes*. University Lecture Series, 8. American Mathematical Society, Providence, RI, 1996.
- [10] HARA CHARALAMBOUS AND APOSTOLOS THOMA, *On simple \mathcal{A} -multigraded minimal resolutions*. Exploratory workshop on combinatorial commutative algebra and computer algebra, Mangalia, Romania, May 29–31, 2008, Providence, RI, Amer. Math. Soc., *Contemp. Math.*, 502 (2009), 33-44