Essays on Prospect Theory, Dynamic Contracting and Procurement

by

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Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Economics
in the Graduate School of Duke University
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Abstract

This dissertation collects work concerning the way individuals deal with imperfect information, both related to their knowledge of themselves and of others. The second chapter shows that bounded rationality, in the form of limited knowledge of utility, is an explanation for common stylized facts of prospect theory like loss aversion, *status quo* bias and non-linear probability weighting. Locally limited utility knowledge is considered within a classical demand model framework, suggesting that costs of inefficient search for optimal consumption will produce a value function that obeys the loss aversion axiom of Tversky and Kahneman (1991). Moreover, since this adjustment happens over time, new predictions are made that explain why the *status quo* bias is reinforced over time. This search can also describe the behavior of a consumer facing an uncertain future wealth level. The search cost justifies non-linear forms of probability weighting. The effects that have been observed in experiments will follow as a consequence.

The third chapter looks to understand how firms create and maintain long term relationships with consumers, or how procurement relations evolve over time, by studying a dynamic variant of the classical two-type-buyer contract in mechanism design. It is less trivial and more interesting if the utility determinant (or utility type) is not fixed or completely random, and fair assumptions are that it is either stochastic, or given by a distribution whose parameters are common knowledge. The first approach is that of Battaglini (2005), while the second is pursued in this paper.

With two possible types of buyers, the buyer more likely to have a high utility type will receive the first-best allocations, while the other will receive the first best only if he has the high utility type.

The last chapter analyzes a dynamic procurement setting with promise keeping, where two firms (agents) with private information on their costs contract competitively with a principal. To this end, two models are proposed and the optimal allocations are determined. The agents face liquidity constraints, which induce distortions when high marginal costs are reported. We deduce that the principal uses promised utilities to incentivize the agents, which act as state variables in the recursive maximization problem. High cost types are allocated less than efficient quantities and the inefficiency of the allocation is relieved as the promised utilities increase.

I dedicate my thesis to all the great teachers I had, and to my family for indulging endless whys.

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List of Abbreviations and Symbols

Symbols

Vectors will not be represented by bold letters. Vector components will have indices. Vector multiplication will be denoted by a central dot, or omitted if the context is clear.

$:=$	Denotes a definition of the variable to the left with the expression to the right.
$\mathbb{R}_+$	The positive real numbers, including 0.
$\mathbb{R}_+^n$	The space of vectors with positive or 0 real components.
Δ	Usually denotes a change in a variable.
$\lvert \cdot \rvert$	The absolute value.
$\lVert \cdot \rVert$	Usually the 2-norm. An index will specify a different norm.
$\mathcal{C}^1$	The set of continuously differentiable real functions.
∇	The gradient of a scalar function.

Abbreviations

pdf	Probability density function.
cdf	Probability cumulative density function.

Acknowledgements

I want to thank Professors Curtis Taylor, Philipp Sadowski, Rachel Kranton, Vincent Conitzer and Huseyin Yildirim for the very helpful comments and observations that lead to the creation of this work. I am especially grateful to Duke University for having supported me as a PhD candidate, and for providing a nurturing environment for research. Lastly, my friends and fellow colleagues deserve some appreciation for putting up with my endless need to debate, and for their spirited retorts.

1

Introduction

Experimental evidence will often show that people are not entirely rational, at least according to classical assumptions on what it means to be rational. By irrational, it is usually meant that we observe a person acting in what we believe is not her self-interest, implicitly assuming what her self-interest is. In classical theory, it is not always possible to resolve such contradictions by the usual methods: a better description of the environment, or a more sophisticated description of the preferences of the person observed. This is important because, however we might think of people, we do tend to consider the world as governed by logically consistent rules, and this includes the world of human biology. So then why are biological mechanisms creating a being that behaves irrationally? This question is central to a more refined view of *homo economicus*, and the obvious answer is that perfect brains are hard to grow. I approach behavioral economic questions from the point of view that what we call irrational behavior fits into two categories: (a) classically rational behaviors that are considered irrational only because the environment or preferences are not described correctly, and (b) behaviors based on heuristics that are quasi-rational in the "average" environment, but which can lead to suboptimal choices in the setting observed. Heuristic behaviors can be learned or hardwired, but they are fundamentally related to limitations of reasoning capacities. The test of rationality, with its common meaning, is that every behavior must make sense from an economic maximizing perspective if certain limitations on reasoning abilities are assumed. A very obvious limitation to reasoning is knowledge. In the next chapter, I attempt to bring together a collection of behavioral economic observations by positing such a limitation, in the form of limited knowledge of preferences. The goal is to see if they can be justified as efficient behaviors for a rational but limited individual. With the new theory, one can also look at how uncertainty is analyzed, and then see if the conclusions can be linked to other behavioral observations involving lotteries.

Limited knowledge is an impediment to optimal outcomes also when it pertains to knowing others. Dealing with adverse selection is a typical problem for designing agreements between agents, for example between a buyer and a seller. If it is possible to commit to appropriate transactions and risk is not an issue, the party that will be privately informed can receive a fair price for future information and efficiency can be achieved. Once an agent has private information, efficiency loss is unavoidable, but in repeated interactions this does not need to happen after the first period. Of course, once we limit commitment, introduce restrictions or preferences for transactions – like liquidity constraints and risk aversion, there are further reasons for inefficiency.

The relationships between buyers and sellers are often dynamic in nature, and relevant permanent private information will rarely stay hidden when there is repeated interaction. This has important implications for the pricing of products or the transfer of goods between a principal and an agent, since the inefficiency induced by asymmetric information may be reduced by considering the long term behavior of the agent. Chapter 3 looks at how a principal will design a contract in a situation when there is a permanent partially informative correlate to a typical random period-by-period private information that the agent has. Unlike in repeated interactions where the private information is connected in simpler ways, the dynamic contract cannot be reduced to offering a repetition of the same single period contract. Chapter 4 looks at how two agents with private information can be induced to report truthfully when they are not able to buy out the other party because of a liquidity constraint. As usual, achieving efficiency is limited by the trade-off with the desire to limit information rent, but the repeating nature of the interaction provides an opportunity for reducing the liquidity constraint that induces the inefficiency. This work, which was done together with Prof. Curtis Taylor, extends results from Krishna & Taylor (2011) to more than one privately informed agent.

2

Inefficient Reallocation, Loss Aversion and Prospect Theory

2.1 Introduction and Literature

In many contexts, an individual's choice is influenced by the *status quo*. This means that choices are reference dependent. The seminal paper of Kahneman and Tversky (1979) proposed "prospect theory," and a new direction of research developed, commingling theoretical economics and psychological research. This theory proposes *ad hoc* deviations from expected utility theory, which are used to explain countless experimental observations that are hard to account for in classical terms: all else being equal, gains are discounted more than losses, small changes are discounted more than large changes, speed-ups in a sequence of payments are preferred to delays, improving sequences of returns are preferred over declining sequences (Loewenstein, 1988), gains with small odds are overweighted, and decisions are biased towards certainty. Confirming evidence can be found in, e.g., Bateman et al. (1997), who propose an experimental set-up to contrast the predictions of reference dependent utilities with those of classic Hicksian theory. Many other papers study individual deviations from classic theory, e.g. Thaler (1981), Loewenstein (1988), Loewenstein and Sicherman (1991), etc.

In a more recent treatment, Tversky and Kahneman (1991) present a set of axioms that are used to generalize the deterministic part of old prospect theory, to explain a large set of experimental effects related to reference dependence. Let the choice set $X = \{x, y, z, r, s, ...\}$ be isomorphic to $\mathbb{R}^2_+$, and for $x = (x_1, x_2)$, $x_1, x_2 \geqslant 0$ are the consumption values of goods 1 and 2. If $\geqslant_r$ denotes the preference structure related to r, then:

(A_0) $\forall\, r \in X :\ \geqslant_r$ is complete, transitive and continuous; moreover

$\{x \geqslant_r y \ \wedge\ x \neq y\} \Rightarrow x >_r y\,.$

(A_1) $\forall\, r, s, x, y \in X :$ Let $x_1 \geqslant r_1 > s_1 = y_1,\ y_2 > x_2,$ and $r_2 = s_2$. Then we have

$x =_s y \Rightarrow x >_r y\,.$

(A_2) $\forall\, t, s, x, y \in X :$ Let $x_1 > y_1,\ y_2 > x_2,\ s_2 = t_2,\ y_1 \geqslant s_1 \geqslant t_1 \vee t_1 \geqslant s_1 \geqslant x_1\,.$

Then $x =_s y \Rightarrow y \geqslant_t x\,.$

The indices can be reversed throughout, and the domain is 2–dimensional for simplicity. Assumption A_0 is the standard collection of axioms to insure that there exists a strictly increasing continuous utility representation U_r for the preference structure $\geqslant_r$ given at the reference point r. Classical theory follows if the subscript r is superfluous; i.e., if any reference would generate the same preferences. Assumption A_1 is what recreates *loss aversion.* It implies an asymmetry between gains and losses in terms of the absolute value of the utility change. A_2 is the *diminishing sensitivity* assumption. It says that the absolute value of the change in utility is smaller if the consumption shift happens further from the origin, all else being equal.

It is instructive to consider prospect theory in the one dimensional case, as formulated by Kahneman and Tversky (1979). In the following, consider the reference to be at the origin. The three assumptions for the value function over wealth changes, as expressed by Bowman et al. (1999), are:

(P_0) $\mathcal{V}(x)$ is continuous, strictly increasing, and $\mathcal{V}(0) = 0$.

(P_1) Let $y > x > 0$. Then $\mathcal{V}(y) + \mathcal{V}(-y) < \mathcal{V}(x) + \mathcal{V}(-x)$.

(P_2) $\mathcal{V}(x)$ is strictly concave for $x > 0$ and strictly convex for $x < 0$.

The features of the value function implied by these assumptions can be easily seen in Figure 1, as given by Kahneman and Tversky (1979) or Tversky and Kahneman (1991). Assumption P_1 is the loss aversion assumption, and it implies that a shift from $x > 0$ to $y > 0$ in wealth will produce a lower utility change than a shift from $-x$ to $-y$, all in absolute value. Assumption P_2 is the diminishing sensitivity assumption, and it makes the impact of equal changes of wealth decreasing in the distance to the origin, which is the reference point.

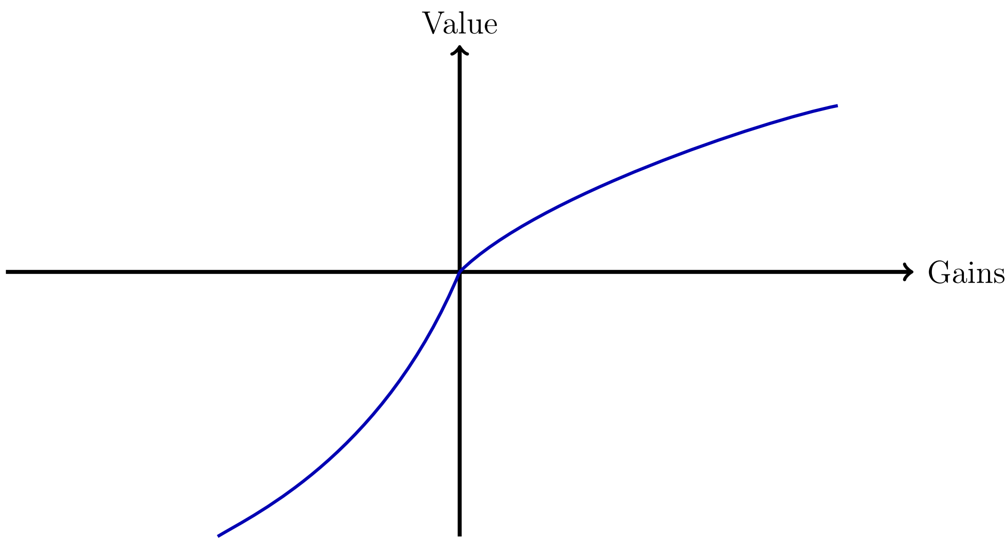

FIGURE 2.1: Value function over wealth changes.

This chapter studies the impact of the inefficiency of adjustment to optimal consumption because of search, in a classical setting and with bounded rationality. Search in the consumption space is necessary since the consumer has incomplete knowledge of his preferences, which will be the assumed limitation. The goal is to model the intuitive notion that it would be hard for a consumer to compare wildly different lifestyles, and in practice he will be more likely to consider small changes to his current consumption behavior. Formally, it is assumed that the consumer has only local knowledge of his utility, and he also knows the rate at which his utility can be improved by marginal changes in his consumption vector. An adjustment in consumption must happen every time there is a wealth shift or, equivalently, a shift in the endowment with a certain good, which is the settings of most experiments. It will be shown that, with relatively weak assumptions, slow or costly search generates loss aversion, as formulated in assumption A_1 in Tversky and Kahneman (1991), or as given by P_1 in Kahneman and Tversky (1979). The main idea is that finding optimal consumption bundles is slow and therefore inefficient, and any shock to the wealth level or to a consumption endowment requires a new optimization of consumption.

Utility losses appear because the allocation is suboptimal during search, and an alternative approach is to make search itself costly, either in terms of lost time, or directly. Both positive and negative changes in wealth or endowment require reallocation, and this means that lost utility will be magnified, while gained utility will be reduced. This effect is the main feature of prospect theory – loss aversion: in absolute values, gains are valued less that losses for equal changes in wealth of endowment. Moreover, the search process can be considered in situations when the consumer faces shocks in wealth or endowment in the form of lotteries. A consumer with a heuristic decision making process that takes search losses into account when planning future allocations will show effects like subcertainty, the overweighting of small probabilities, and subadditivity of the probability weighting function, which are described in Kahneman and Tverky (1979). An interesting observation leading to comparative statics can also be made. The concavity of the instantaneous utility function, a classic assumption leading to risk aversion, can be shown to determine how uncertain gains and losses influence the search process. As we would expect, a more risk averse consumer in the classical sense will search towards lower cost allocations when facing a lottery. We can conclude that a more risk averse consumer is also more pessimistic.

The idea of costly change is not completely new. Samuelson and Zeckhauser (1988) mention that *status quo* bias could be generated by the cost of thinking, as well as by transaction costs and other psychological effects. In this setting, adjustments are costly because thinking about a better consumption decision leads to a utility loss over time and in Section 2.7 a setting in which changing the consumption vector costs time is considered.

One important question for any new model is whether it offers testable implications. Here, costly reallocation of resources happens over time. Having the incurred cost depend on time suggests an experimental set-up in which the *status quo* can be cemented by letting time pass before a new option is presented. The expected effect is that the more time passes with a *status quo*, the more the loss aversion effect will be accentuated, thereby providing an experimentally testable implication of the theory. There is already evidence in support of this interpretation. In Strahilevitz and Loewenstein (1998), the authors look in depth at the effect of the history of ownership on the valuation of an object. They find that, in addition to an instantaneous effect, a longer history of ownership of an object will also increase its valuation, thus the value lost in an exchange will be higher than the value gained in the beginning. Another way of looking at this is to say that increased duration of ownership will make the object more a part of the *status quo*, or more a part of his planned consumption – having an object, or wealth level, for a longer period of time should make it more "comfortable" for the consumer. This is similar to thinking of the *status quo* as a plan, or an expectation for future consumption.

Other attempts have been made to place prospect theory within a framework of classical assumptions. The work of Kőszegi and Rabin (2006, 2009) develops a model for the formation of the reference point, which is set as the probabilistic belief that the person held in the recent past about the possible outcome. This set-up can be used to explain laboratory observations like the endowment effect from the more basic gain-loss bias. Because expectations determine the reference, the theory predicts that traders or merchants will not be affected by the endowment effect, because their beliefs about future outcomes take into account the fast shifts in the consumption vector. The idea is closely related in interpretation to the one in the present chapter. A trader will not search for a new optimal consumption bundle, given a shift in wealth or allocation, and this is because the trader will expect the *status quo* to be temporary.

Another detailed attempt at explaining some parts of prospect theory in classic terms is in the working paper of Rick Harbaugh (2009). The main idea is that winning or losing gambles will signal the skill of the player. Thus, a gamble with some probability for loss, given a perceived default payoff, will signal low skill, while a gamble with a probability of win above the perceived default will signal high skill. Therefore the framing of a trade matters, and loss aversion is the result of avoiding the loss of a gamble. Probability weighting is another consequence, since a high gain with a small probability will also be a strong signal for skill, while losing an almost-sure bet is a strong signal of lack of skill. This approach is superior in explaining framing effects, but is not very useful in settings with little uncertainty.

This chapter has six sections. Section 2.2 lays the theory used to model search, proposes a general search mechanism in continuous time and proves central results. Section 2.3 uses the theory to justify loss aversion by inefficient allocation during search, and justifies the loss aversion axiom A_1. Section 2.4 looks at search when the consumer faces a lottery shock to wealth and shows that risk aversion is related to pessimism in planning. Section 2.5 considers a functional form example to show how observations related to the probability weighting function of prospect theory can be recreated. Section 2.6 develops the search method and useful results, section 2.7 constructs an alternative theory that models costly search as lost utility from lost time, and the last section discusses an experimental method that can be used to test the model for confirmation and to contrast its implications to those of existing theory, and concludes. The proofs are in the Appendix.

2.2 Theory

Consider an individual with a rational, continuous and locally nonsatiated preference relation, in a deterministic continuous time setting. The consumer chooses between consumption bundles $x(t) = (x_1(t), ..., x_n(t))$, based on his utility function $U : \mathbb{R}^n_+ \to \mathbb{R}$, $U(x(t)) = U(x_1(t), ..., x_n(t))$, which obeys the usual assumptions of strict quasiconcavity and continuity. $i \in \{1, .., n\}$ indexes the goods in the consumption bundle, and $t \in (0, \infty)$ denotes the time. The consumer starts with a consumption vector plan $x(0) = (x_1(0), ..., x_n(0))$, and with lifetime wealth $W(0)$. $p = (p_1, ..., p_n)$ is a positive and constant market price vector, known to the consumer, and β is the intertemporal discount factor. The discounting formula is calibrated such that we can write equations with a familiar discount factor $\beta \in (0, 1)$. For that, use equations like:

$$NPV(\text{consumption } x(t) \text{ between times } \tau_1 \text{ and } \tau_2) := \frac{\ln \beta}{\beta - 1} \int_{\tau_1}^{\tau_2} e^{t \ln \beta} p \cdot x(t) dt.$$

$$\left[= \frac{\beta^{\tau_1} - \beta^{\tau_2}}{1 - \beta} p \cdot x \;\; \text{if, say, } x(t) \text{ is constant} \right]$$

Say that $x(0)$ is the optimal consumption vector for the instantaneous utility maximization problem, given the budget $p \cdot x(0)$. An important question is if the consumer, starting with total (lifetime) wealth $W(0)$, is satisfied with continuing to consume $x(0)$ in the future. E.g., if β is a small number, it could be that the consumer would want to use his wealth sooner. Optimal behavior depends on whether β is smaller or larger than $1/(1+r)$, where r is the real interest rate. The life-cycle model (Hall, 1978) argues that, in an economy with many identical consumers, the two values are equal and the consumer uses a constant fraction of his lifetime wealth each period. It is more complicated when we involve economic growth or realistic behavioral models (Shefrin, 1988) and, in either case, consumer heterogeneity would require a more general approach. However, to avoid unnecessary complications, we assume that $\beta = 1/(1+r)$. To support the assumption, consider that the average person, who faces the decisions that we consider on time scales significantly shorter than his lifetime, would be well represented by a stable habitual consumption vector. The consumer will have access to simple banking, in which he can save his wealth for later consumption. In Section 2.4 we'll consider lotteries, so contingent claims cannot be available, at least for the scope of the decisions considered, because our risk averse consumer couldn't distinguish between lotteries and their expected value.

Now, we deviate significantly from classical assumptions by limiting the knowledge of our consumer. The only utility he can know is the utility at the point of his current chosen consumption plan. He can consider changing his consumption $x(0)$, and as soon as he does, he can know his new utility level. The only information the consumer has, besides $x(0)$ and $U(x(0))$, is the rate at which utility can be increased on any of the dimensions, $\nabla U(x(0))$. Therefore, the problem the consumer faces is similar to a simple optimization problem in which we're computationally limited to knowing only the linear approximation of an unknown function. In addition, the utility of 0 consumption, $U(0, ..., 0) =: \underline{U}$, is also always known.

Assume that at the start the consumer has one bundle in mind, $x(0)$, which is optimal. The plan to consume it forever employs all the initial financial resources the consumer has, $W(0) = p \cdot x(0)/(1-\beta)$. The constant consumption decision is optimal. However, if $W(0) \neq p \cdot x(0)/(1-\beta)$, no change means that the consumer will either go bankrupt at some point, or has unused resources.

The consumer's rational behavior is limited by the local nature of the knowledge of his preferences. From his initial choice $x(0)$, he has the opportunity to search for lifetime utility improvements by small changes. The search method utilized is steepest descent in its continuous time limit. Section 2.6 describes it and some of the results in more detail. This means that $x(t)$ will change along $\nabla \mathcal{U}(x)$, where $\mathcal{U}(x)$ is the objective function. Let's assume the consumer starts at the optimum, and there is a $t = 0$ wealth shock such that $W(0) > p \cdot x(0)/(1-\beta)$. He suddenly has access to more wealth than before, so the plan of consuming $x(t) = x(0)$ forever is suboptimal. It then makes sense to look for a new long run optimal allocation that employs more wealth, which is a step in his search process. If the search would converge instantly, we would need to consider only a simple constrained optimization problem:

$$\max_x U(x), \qquad \text{s.t. } \frac{p \cdot x}{1-\beta} \leqslant W. \tag{2.1}$$

If the initial shock in wealth is such that $W(0) < p \cdot x(0)/(1-\beta)$, the initial consumption plan is that the consumer will have $x(0)$ for as long as his wealth $W(0)$ will cover, and then switch to the utility level of no consumption expenditure, $\underline{U}$. As long as the marginal utility gain from spending is higher at $(0, ..., 0)$ than at any $x \neq (0, ..., 0)$, the initial consumption plan is suboptimal, which means that the consumer will want to search for allocations of lower utility in the beginning, and postpone the moment when he runs out of wealth. With instant search, the problem he solves is the following:

$$\max_x U(x) \int_0^{\tau(x)} \beta^t dt + \underline{U} \int_{\tau(x)}^{\infty} \beta^t dt, \quad s.t. \ \frac{1-\beta^{\tau(x)}}{1-\beta} p \cdot x \leqslant W. \tag{2.2}$$

τ is the moment in time when the consumer switches from consumption bundle x to 0. The lifetime budget constraint will bind, so x determines τ, because the consumer will stop x only because he's out of resources. In the continuous time version of the maximization problem, it doesn't matter how low the value $W(0)$ becomes, since there is a small enough duration τ for which the wealth W will be sufficient to consume the initial allocation $x(0)$.

Now let's consider how the search works when the convergence towards the optimum is not instant. The main problem is that, as search proceeds, the wealth level will also change according to the spending on intermediate points, which changes the objective function for the future. This means that, in choosing the next step in the search, the consumer has to also consider the effect of the search step on the future-self's search process. Fortunately, the consumer has only local knowledge of his utility, which greatly simplifies the problem. The next step depends only on the momentary knowledge and wealth level.

A way to formally state the consumer's bounded rationality constraint is to have his prior on $U(x)$ as:

$$U(x) = \begin{cases} U(x(0)) + \nabla U(x(0)) \cdot (x - x(0)), & x \in \mathcal{B}(x(0), \epsilon|\nabla U(x(0))|), \\ \underline{U}, & x \in \mathbb{R}^n_+ / \mathcal{B}(x(0), \epsilon|\nabla U(x(0))|), \end{cases} \tag{2.3}$$

for some small ϵ. If the search step takes time δ_τ, then $\epsilon|\nabla U(x(0))|/\delta_\tau$ is the rate of movement in the search process, assumed constant. Rather than interpreting this as bad *a priori* knowledge, we can think of it as the consumer strongly disliking changes larger than $\epsilon|\nabla U(x(0))|$ in his consumption bundle because, say, he doesn't have any certainty on his utility if he moves too fast and without a clear indication of improvement, $|\nabla U(x(0))|$.

Proposition 1. *Assume that the boundedly rational consumer knows only his current consumption vector, his current utility level and gradient, as well as the utility of no consumption. Given a line search method for the maximum, at any point in time the* *optimal next step is given by the method applied to problems (2.1) or (2.2), where x is the momentary consumption vector and W is the momentary wealth level.*

For proofs, see the Appendix. The entire search process is simply the successions of search steps described in the proposition.

At any point, the search direction depends only on the current consumption x, the direction of the gradient $\nabla U(x)$ and the price vector p.

Proposition 2. *If the lifetime budget constraint doesn't bind and doesn't hold with equality, the search moves in the direction of $\nabla U(x)$. If the constraint binds, the search moves in the direction $\nabla U(x) + [U(x) - \underline{U}] \cdot \frac{-p}{p \cdot x}$.*

When the lifetime budget constraint doesn't bind, search moves in the direction $\nabla U(x)$, towards higher spending allocations. It's interesting to consider whether the direction of search when the lifetime budget constraint binds is towards lower spending per period.

Proposition 3. *Let the utility function $U(x)$ be strictly concave. When the lifetime budget constraint binds, the consumer searches towards allocations with lower spending. If the constraint doesn't bind, but holds with equality, the direction of movement is along the budget constraint.*

An important question is why we need to introduce concavity to generate the result. Intuitively, the corollary proves a smoothing result. When the consumer has less wealth than it would be needed to sustain the level of consumption x forever, consumption smoothing implies that he will look for a cheaper allocation, and we know that smoothing requires assumptions on the intertemporal utility trade-off, or simply that the one-period utility is concave. Therefore, the search steps have to be towards lower allocations only if we have come kind of concavity generating risk aversion. Strict concavity is not be the weakest condition required, but it is sufficient.

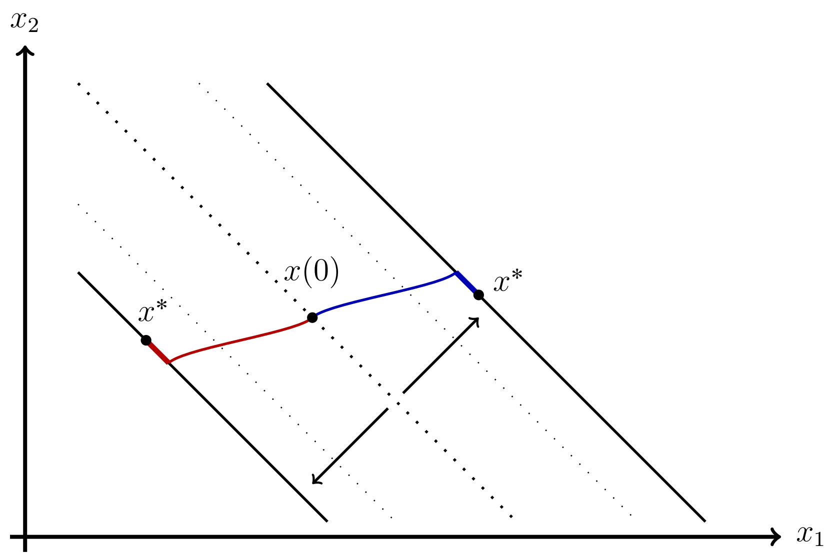

FIGURE 2.2: A search path in 2 dimensions.

For each step in the search algorithm proposed, we have a marginal change in the consumption bundle x, and a change in lifetime wealth remaining. Lifetime wealth is reduced by the instantaneous consumption and increased by interest:

$$W'(t) = (-\ln \beta)\left[-\frac{p \cdot x(t)}{1-\beta} + W(t)\right]. \tag{2.4}$$

The intermediate allocations generate a path $x(t)$ in the allocation space, and the lifetime wealths a function $W(t)$. The last important question left is whether the search method proposed reaches a new optimum, i.e., if $x(t)$ given by the algorithm proposed will converge to a point x^* and $W(t)$ to W^*, where $W^* = p \cdot x^*/(1-\beta)$.

Proposition 4. *Let the search process be a continuous steepest descent algorithm, where the wealth bound is updated after each step. If $U(x)$ is strictly concave, the search for the optimal allocation and wealth converges, and the limit wealth is exactly enough to sustain the limit allocation forever.*

Because the focus is explaining observations in experimental conditions, we can ignore the cases where the the consumer goes bankrupt before he reaches a new optimal allocation. It is also conceivable that wealth is so large that search won't stop and spending per time can increase forever, because of interest. While this can be sensible for some consumers, we ignore it and focus on the cases with convergence. It is also intuitive that these circumstances are likelier to happen if the search is slow.

2.3 Implications of Slow Search

In this section, we show how a loss aversion effect can form from a slow adjustment of the consumption bundle.

2.3.1 Efficiency Loss in Search

First, let's look at the consumer's lifetime utility change from a wealth shock, as the search rate is increased or decreased. For that, we compare with a situation with instant search and one with no search. Consider a sudden change in wealth ΔW at $t = 0$. Assume that the optimum consumption level reached with instant search gives instantaneous utility OU. The lifetime utility gain from $\Delta W > 0$ after instant search is $OU - U(x(0))/(1-\beta)$. If there is no search, or the search is very slow, the gain in lifetime utility is 0. Slow search gives an intermediate lifetime utility gain:

$$\frac{\ln \beta}{\beta - 1} \int_0^{\infty} [U(x(t)) - U(x(0))] \beta^t dt \in \left(0, \frac{OU - U(x(0))}{1-\beta}\right). \tag{2.5}$$

If $\Delta W < 0$, the lifetime utility loss after instant search is $\frac{U(x(0))-OU}{1-\beta}$, while with no search, the utility loss is

$$\begin{aligned} &\frac{U(x(0))}{1-\beta} - \frac{\ln \beta}{\beta - 1} \left[\int_0^{\infty} [U(x(t)) - U(x(0))] \beta^t dt + \int_0^{\infty} \underline{U} \beta^t dt \right] = \\ &= \frac{U(x(0)) - \underline{U}}{1-\beta} \frac{\Delta W}{W}. \end{aligned} \tag{2.6}$$

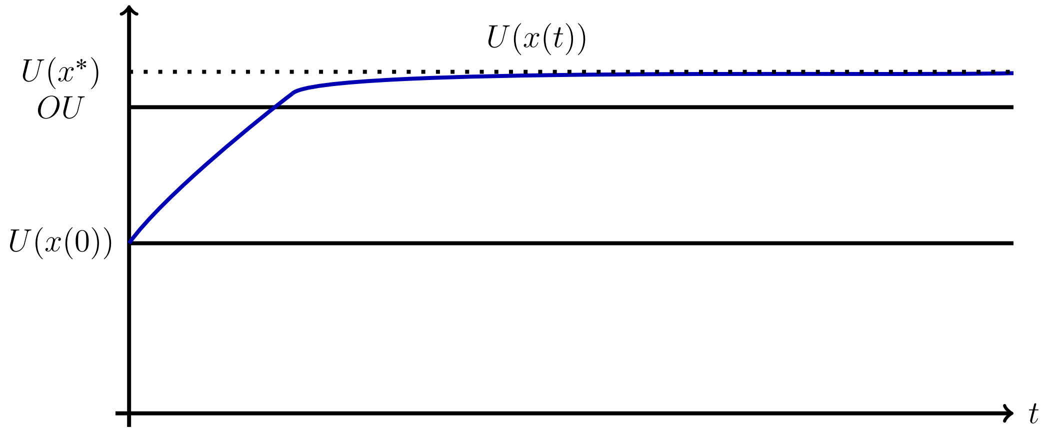

FIGURE 2.3: Instantaneous utility after $\Delta W > 0$.

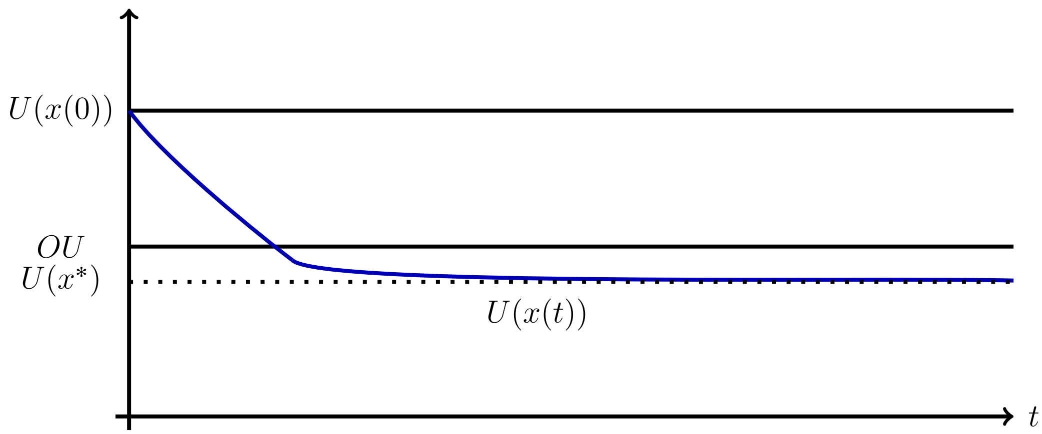

FIGURE 2.4: Instantaneous utility after $\Delta W < 0$.

This time, slow search gives an intermediate lifetime utility loss:

$$\frac{\ln \beta}{\beta - 1} \int_0^\infty [U(x(0)) - U(x(t))]\beta^t dt \in \left(\frac{U(x(0)) - OU}{1-\beta}, \frac{U(x(0)) - \underline{U}}{1-\beta} \cdot \frac{\Delta W}{W} \right). \quad (2.7)$$

Now plot the lifetime utility changes, as a function of ΔW. The ranges obtained show that the change in utility has the properties highlighted in Figure 2.1. Lifetime utility is lost for a wealth shock ΔW because of the inefficiency of the allocation during search. This way the utility of a gain is decreased while the dis-utility of a loss is magnified, creating an effect of loss aversion.

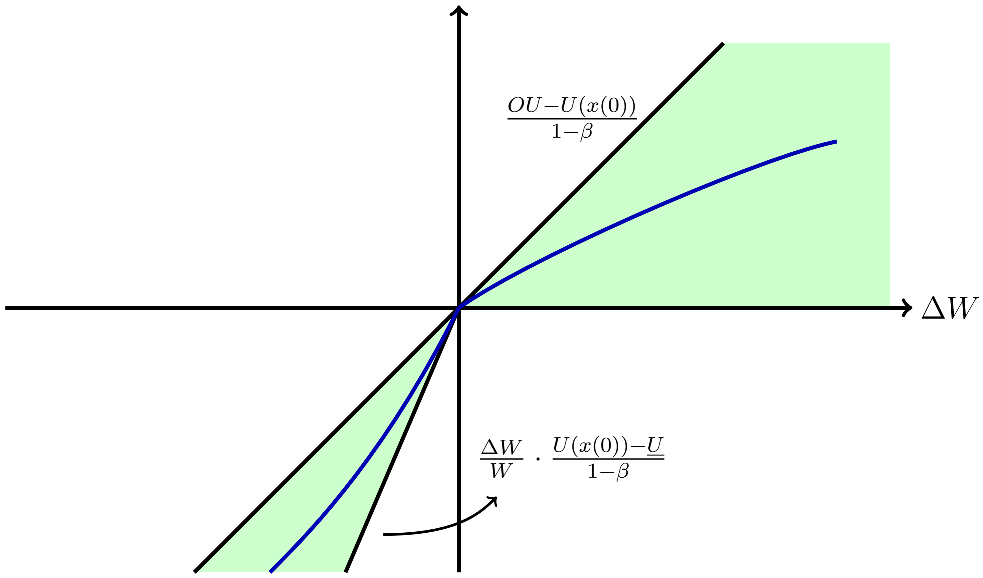

FIGURE 2.5: Lifetime utility change ranges.

In Figure 2.5, both the upper and the lower bounds on the lifetime utility change can be reached, depending on the speed of the search process. We have normalized $\big(OU - U(x(0))\big)/(1-\beta)$ to linearity for exposition. The usual concave shape supports the same conclusion. Observe that we cannot say if the derivative of the lifetime utility function has a discontinuity at 0, which is usually assumed for the value function in prospect theory. If the cost function is assumed *ad hoc*, we only need for a kink that its relative size doesn't go to 0 as the change ΔW goes to 0. Here, we have modeled the cost as lost efficiency of allocations with concave utilities. Small shocks to wealth will lead to quadratically decreasing efficiency losses, since the utility function is locally flat. However, we focus on heuristic decision making, so it is plausible that experiments with small shocks will induce behavior that is optimal for important (large) shocks. It can also be that small decisions introduce direct time costs or other effects, leading again to *status quo* bias and relatively high change costs.

2.3.2 Costly Adjustments

Search for consumption leads to inefficiency, or is costly in some other way. For example, in Section 2.7, a different set-up introduces cost as lost time. It is not necessary to propose any specific search mechanism to make this point. Costly search will generate a reference dependent value function that obeys the loss aversion axiom. To consider the payoff of a new wealth level or endowment vector, the consumer will heuristically estimate the cost of switching to it. For this section, we only need to assume a cost function, $C : \mathbb{R}_+ \to \mathbb{R}_+$, that is differentiable and strictly increasing, with $C(0) = 0$ and $C'(l) > 0$ for $l > 0$.

For any value function $\mathcal{U} : \mathbb{R}^n_+ \to \mathbb{R}$ which describes the consumer's choices absent search, define the reference dependent utility over consumption changes v from the reference r, $V_r(v) := \mathcal{U}(r + v) - \mathcal{U}(r) - C(\|v\|)$, for all possible reference consumptions. $\|\cdot\|$ can be almost any p-norm, e.g., the euclidean distance, or, in light of our previous discussion, the seminorm that gives the money value of the change, $|p \cdot v|$. The following proposition verifies that this reference dependent utility satisfies the loss aversion assumption A_1.

Proposition 5. *Let $C : \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable, with $C(0) = 0$ and $C'(l) > 0$ for all $l > 0$, and let $\mathcal{U}(x) : \mathbb{R}^n_+ \to \mathbb{R}$ of class $\mathcal{C}^1$, strictly quasiconcave. Define $V_r(x - r) := \mathcal{U}(x) - \mathcal{U}(r) - C(\|x - r\|)$, for any $r \in \mathbb{R}^n_+$, where the norm is any d-norm $\|\cdot\|_d$, $1 \leqslant d < \infty$, or the seminorm $|p \cdot (x - r)|$. Then the preference structure given by $V_r(x - r)$ satisfies assumption A_1, for any pair of indices $i \neq j$.*

The *status quo* has increased value because the search cost is sunk cost. Loss aversion also implies *status quo* bias, so costly search will generate the two experimental effects.

2.4 Searching with Lotteries

Slow search has other implications, besides loss aversion, that relate to prospect theory. In this section we consider how the consumer, given his limited knowledge, will evaluate lotteries. We will assume that he is an expected utility maximizer, and that he knows the probabilities of an uncertain event which will change his lifetime wealth level. For simplicity, we assume that the lottery is resolved after his initial search has enough time to converge. This is relevant for many practical situations in which the risk is evaluated, because uncertainties rarely have immediate resolutions. It is intuitive that his heuristic decision making will also apply in decisions that involve lotteries that are resolved instantly.

The next question is if we can describe the path of the allocation $x(t)$ when the consumer faces an initial lottery. Let the consumer start from an optimal initial point $x(0)$, and consider a shock to wealth in the form of a lottery, $(\alpha, W_1(0); 1-\alpha, W_2(0))$, where $W_1(0) < p\cdot x(0)/(1-\beta) < W_2(0)$. $W_j(0)$ is the lifetime wealth if contingency j is realized. As the consumer spends $p\cdot x(t)$ per unit of time, we know that $W_j(t)$ will also change, but the interpretation is the same. Because the initial point is optimal, we must have that $p||\nabla U(x(0))$. Then, using the general result from Proposition 6, $\exists \alpha_c \in (0,1)$ such that

$$\frac{1-\alpha_c}{1-\beta}\nabla U(x(0)) + \frac{\alpha_c W_1}{p\cdot x(0)}\left[\nabla U(x(0)) - (U(x(0)) - \underline{U})\frac{p}{p\cdot x(0)}\right] = 0, \tag{2.8}$$

because we know that $\nabla U(x(0))$ and $\nabla U(x(0)) - (U(x(0)) - \underline{U})p/p\cdot x(0)$ are vectors pointing in opposite directions. In this case, the odds of the gamble are such that there is no need to search for a better allocation, momentarily. However, we know that

$$W_1'(0) = (-\ln\beta)\left[-\frac{p\cdot x(0)}{1-\beta} + W_1(0)\right] < 0. \tag{2.9}$$

This will reduce the effect of the low wealth outcome on the direction of search, so the gradient of the objective changes and we start moving towards higher cost allocations. If the probability of the low wealth event $\alpha < \alpha_c$, the search is going to move towards higher spending allocations, and *vice versa*. The following result gives the search direction at any point for a general lottery function, assuming that all $p \cdot x/(1-\beta) \leqslant W_j$ either bind or don't hold with equality.

Proposition 6. *Let the consumer have current consumption choice x, and say that he faces a lottery $(\alpha_1, W_1; ...; \alpha_l, W_l)$, expressed in terms of the lifetime wealths for each contingency. Let $W_1 \leqslant ... \leqslant W_k < p \cdot x/(1-\beta) < W_{k+1} \leqslant ... \leqslant W_l$, and $\alpha_1 + ... + \alpha_l = 1$. The direction of his search will be given by*

$$\frac{\alpha_{k+1} + ... + \alpha_l}{1-\beta} \nabla U(x) + \frac{\alpha_1 W_1 + ... + \alpha_k W_k}{p \cdot x} \left[\nabla U(x) - (U(x) - \underline{U}) \frac{p}{p \cdot x} \right].$$

Consider the simple lottery again. It cannot be that searching stops at any x^* such that $W_1(t) < W^* := p \cdot x^*/(1-\beta) < W_2(t)$. That is because $W_1(t)$ will be decreasing, and the gradient of the objective function will start pointing towards higher cost allocations. However, without more assumptions on the shape of $U(x)$, we cannot say how the two "forces" will balance out as consumption changes. It is sensible that, if $U(x)$ is sufficiently concave, the bundle $x(t)$ can stay somewhere such that the spending rate will not converge towards any $W_j(t)$. As before, it is also possible that the search will not converge because the values $W_j(t)$ are also changing very fast, but we ignore such unusual behavior assuming that the search is fast enough.

The following result gives us convergence for simple lotteries and $U(x)$ that is not very concave, or equivalently for small enough lottery payoffs. Under the conditions of the proposition, the movement is towards higher cost consumption for small α, or lower cost for large α.

Proposition 7. *If the consumer faces a lottery in the distant future* $(\alpha, W_1; 1-\alpha, W_2)$*, with* $W_1 < p \cdot x(0)/(1-\beta) < W_2$*, and the instantaneous utility function has* $0 \succ \nabla^2 U(x) \geq -MI$*, then for any subdomain* $D \in \mathbb{R}^n_+$*,* $\exists\ 0 < \alpha_l < \alpha_h < 1$ *for* M *small enough such that for* $\alpha \in (0, \alpha_l] \cup [\alpha_h, 1)$ *the search converges,* $x(t) \rightarrow x^*$*, and* $W^* = W_1(t)$ *for* $\alpha \leqslant \alpha_h$*,* $W^* = W_1(t)$ *for* $\alpha \geqslant \alpha_h$*, where* $W^* := p \cdot x^*/(1-\beta)$*, and* W_j *is the lifetime utility for outcome* j*.*

For $\alpha > \alpha_h$, the consumer behaves as if "planing" for the worse outcome, and will reach a point x^* which can be consumed without bankruptcy if the worse outcome happens. For $\alpha < \alpha_l$, the consumer behaves as if he's ignoring the worse outcome, and reaches a point where he consumes at a rate that is sustainable only with the high wealth outcome. We will say that the consumer plans for a certain outcome if his search is dominated by it, leading him to incur the cost of switching his consumption to match the outcome's lifetime wealth. For $\alpha \in (\alpha_l, \alpha_h)$, it is possible that the search will never converge. When the lottery will be resolved, the consumer will have a consumption that is in between what can be afforded under the two wealth outcomes. This behavior is suggestive of hedging. Searching for an optimal allocation leads to utility loss for the initial search, as well as in the future, if the lottery will be resolved opposite to the plan. The behavior suggests a bias towards certainty, since more certain outcomes are less likely to lead to future search costs.

We see here a behavior that suggests the consumer heads towards a discounted consumption level approximated by the expectation of the lottery faced. This is reminiscent of the work of Kőszegi and Rabin (2006), which assume that "the reference point is fully determined by the expectations a person held in the recent past."

We can make another interesting observation leading to comparative statics. Propositions 6, 7 show that planning is influenced by the concavity of the instantaneous utility function. All else being equal, at any point the consumer is more likely to plan for the worse outcome if his instantaneous utility function is more concave, i.e., if $|\nabla U(x) - (U(x) - \underline{U})p/(p \cdot x)|$ is higher. This confirms a classic assumption: a consumer that will be more likely to plan for the worse outcome, i.e., more pessimistic, is a consumer with a more concave utility function, so more risk averse. Also note that, in the limit of no concavity of $U(x)$, $|\nabla U(x) - (U(x) - \underline{U})p/(p \cdot x)| \to 0$ (Proposition 3), and the consumer will act in such a way as to make sure he consumes all possible resources if the high wealth contingency is realized. Low concavity leads to a behavior suggestive of optimism.

Corollary 8. *Let $U^a(x)$, $U^b(x)$ be two instantaneous utility functions for consumers a and b, which have current consumptions x_0. If $\nabla U^a(x_0) = \nabla U^b(x_0)$ and $0 \succ \nabla^2 U^a(x) \succ \nabla^2 U^b(x)$ for all x, then consumer a will increase his spending more than consumer b.*

2.5 Evaluating Lotteries

In this section we will assume that the consumer incurs search costs according to $C(\cdot)$, and that he always plans for the likeliest outcome when facing a lifetime wealth shock in the form of a lottery. We want to see what experimental evidence summarized by prospect theory can be explained this way. To keep things simpler, we'll assume a high intertemporal discount factor β, so that the utility cost of a future consumption adjustment is approximately the same as the cost of a current adjustment. For evaluating specific numerical lotteries, we pick the examples in Kahneman and Tversky (1979), and we use specific examples of utility and cost functions over wealth changes.

Let $V_r(\Delta W) := \mathcal{U}(\Delta W) - C(|\Delta W|)$ be our reference dependent value function for changes ΔW. $\mathcal{U}(\Delta W)$ is the lifetime utility gain from ΔW without any search costs. We make the usual concavity assumption for it. Furthermore, assume $C(|\Delta W|)/|\mathcal{U}(\Delta W)|$ is decreasing when $|\Delta W|$ grows. This means that the relative utility cost of search is smaller for higher wealth changes. In our discussion of search, the loss was created by the inefficient allocation of resources.

First, let's consider the following example for a value function, $V_r(\Delta W)$, where

$$\mathcal{U}(\Delta W) := \frac{\Delta W}{10^5} - \frac{1}{5}\left(\frac{\Delta W}{10^5}\right)^2, \; C(|\Delta W|) := \frac{1}{2}\ln\left(1 + \frac{|\Delta W|}{10^5}\right). \tag{2.10}$$

The utility gain with no search is a quadratic approximation, and the cost function is much more concave than the quadratic, so that search costs become relatively less significant for large shocks in wealth. There is a 1/2 factor in front, so that the cost isn't overwhelming for small $|\Delta W|$.

The Allais paradox, as presented by Kahneman and Tversky (1979), is that in experiments gamble B is preferred to gamble A, and C to D, by a large majority of subjects:

$$\begin{array}{llll} A: & 2,500 \text{ with probability } 0.33, & B: & 2,400 \text{ with certainty,} \\ & 2.400 \text{ with probability } 0.66, & & \\ & 0 \text{ with probability } 0.01, & & \\ & & & \\ C: & 2,500 \text{ with probability } 0.33, & D: & 2,400 \text{ with probability } 0.34, \\ & 0 \text{ with probability } 0.67, & & 0 \text{ with probability } 0.66. \end{array}$$

This means that a number of subjects make decisions that violate the substitution axiom. This result is interpreted to imply subcertainty for the probability weighting function in prospect theory, i.e, $\pi(p) + \pi(1-p) < 1$ for $p \in (0,1)$. However, if we evaluate the lotteries with our value function example:

$$
\begin{aligned}
V_r(\text{"}A\text{"}) &= \mathcal{U}(2400)\cdot 0.66 + \mathcal{U}(2500)\cdot 0.33 - C(2400) - C(100)\cdot 0.33 - C(2400)\cdot 0.01 \\
&= 0.01183, \\
V_r(\text{"}B\text{"}) &= \mathcal{U}(2400) - C(2400) = 0.01202, \\
V_r(\text{"}C\text{"}) &= (\mathcal{U}(2500) - C(2500))\cdot 0.33 = 0.004134, \\
V_r(\text{"}D\text{"}) &= (\mathcal{U}(2400) - C(2400))\cdot 0.34 = 0.004089.
\end{aligned}
$$

Lottery A leads to a lower gain than B. Although the expected value of $\mathcal{U}$ is higher for A, the small possibility of a loss increases search costs. C has a higher gain than D, which comes from the higher expected $\mathcal{U}$ gain, since search costs are similar.

A similar story can justify the subadditivity property in probability waiting (Kahneman and Tversky, 1979). Lottery F is experimentally preferred to E, and G to H:

$$
\begin{aligned}
E:&\quad 3,000 \text{ with probability } 0.002, & F:&\quad 6,000 \text{ with probability } 0.001, \\
&\quad 0 \text{ with probability } 0.998, & &\quad 0 \text{ with probability } 0.999,
\end{aligned}
$$

$$
\begin{aligned}
G:&\quad -3,000 \text{ with probability } 0.002, & H:&\quad -6,000 \text{ with probability } 0.001, \\
&\quad 0 \text{ with probability } 0.998, & &\quad 0 \text{ with probability } 0.999.
\end{aligned}
$$

$$
\begin{aligned}
V_r(\text{"}E\text{"}) &= (\mathcal{U}(3000) - C(3000))\cdot 0.002 = 0.00003008, \\
V_r(\text{"}F\text{"}) &= (\mathcal{U}(6000) - C(6000))\cdot 0.001 = 0.00003015, \\
V_r(\text{"}G\text{"}) &= (\mathcal{U}(-3000) - C(3000))\cdot 0.002 = -0.00008992, \\
V_r(\text{"}H\text{"}) &= (\mathcal{U}(-6000) - C(6000))\cdot 0.001 = -0.00008985.
\end{aligned}
$$

Finally, consider the effect of small probability overweighting, $\pi(p) > p$, for small

p. In the following, I is preferred to J, and L to K in experiments:

$$\begin{array}{llll} I: & 5,000 \text{ with probability } 0.001, & J: & 5 \text{ with certainty}, \\ & 0 \text{ with probability } 0.999, & & \end{array}$$

$$\begin{array}{llll} K: & -5,000 \text{ with probability } 0.001, & L: & -5 \text{ with certainty}. \\ & 0 \text{ with probability } 0.999, & & \end{array}$$

$$V_r(\text{"}I\text{"}) = (\mathcal{U}(5000) - C(5000)) \cdot 0.001 = 0.00002510,$$

$$V_r(\text{"}J\text{"}) = \mathcal{U}(5) - C(5) = 0.00002500,$$

$$V_r(\text{"}K\text{"}) = (\mathcal{U}(-5000) - C(5000)) \cdot 0.001 = -0.00007490,$$

$$V_r(\text{"}L\text{"}) = \mathcal{U}(-5) - C(5) = -0.00007500.$$

When comparing lotteries I and J, the cost of search is much less relative to the change in utility for the high wealth shock, and with our functions this effects overcomes the utility concavity effect. Subproportionality, which is defined as

$$\frac{\pi(pq)}{\pi(p)} \leqslant \frac{\pi(pqr)}{\pi(pr)}, \tag{2.11}$$

for $0 < p, q, r \leqslant 1$, is also partially supported. Our example function has reproduced the experimental observations that justify the shape of the probability weighting function in prospect theory.

2.6 The Search Method

In order to link search to consumption choice, a model of search is proposed in this section. The theory Section 2.7 uses an assumption that can give the time rate of the improvement,

$$\|x(t) - x^*\| = e^{-tr}\|x(0) - x^*\| \Leftrightarrow (\|x(t) - x^*\|)'(t) = -r \cdot \|x(t) - x^*\|. \tag{2.12}$$

In this section, steepest descent is proposed and discussed, but such a result can be achieved by more than one mechanism. For now, let's consider search in discrete time. If $f(x)$ is $\mathcal{C}^1$, bounded from above, the domain is bounded, and the gradient is Lipschitz continuous with some constant N, we know that the steepest descent algorithm will converge for step sizes $0 < \epsilon < 1/N$ (Ruszczyński, 5.3.2, 2006). Maximum lifetime utility is bounded from above in our set-up, since we always assume that lifetime wealth is bounded in every contingency, and by the same argument so is the domain of search. A stronger result can be obtained with more assumptions on $f(x)$.

Proposition 9. *Let $f(x)$ be a twice continuously differentiable function over consumption x, which does not include a measure of time. It is assumed that $-mI \succeq \nabla^2 f(x) \succeq -MI$, $\forall x \in D$, the domain, and $0 < m \leqslant M$. Furthermore D is convex. (This implies that the Hessian is negative definite on the domain D, $\nabla^2 f(x) \preceq 0$.) A unique maximum $x^* \in D$ must exist, which is assumed to be an interior point. The search generated by the steepest descent algorithm, starting from x_0 given, with a step length $\epsilon \in (0, 2/M)$, generates a sequence $\{x_t\} \rightarrow x^*$, and the following results hold:*

$$(a)\ \|x_t - x^*\| \leqslant q^t \|x_0 - x^*\|,$$

$$(b)\ f(x^*) - f(x_t) \leqslant \frac{M}{m} qmax^{2t} [f(x^*) - f(x_0)],$$

where $qmax := \max[|1 - \epsilon m|, |1 - \epsilon M|] < 1$.

When $\|x_{t+1} - x^*\| / \|x_t - x^*\| = const. < 1$, the search is said to converge Q-linearly. For specially constructed f, we can get near the upper bound in the limit, so the upper bound cannot be improved (Cartis et al., 2009). As for a lower bound on the search convergence rate, it is obvious that, given the search process, a step length and a starting point, specific examples can be constructed such that the first steps will coincide with a maximum. But we can try to consider how fast the search method works for generic quadratic examples, which will approximate well any f with $-mI \geq \nabla^2 f(x) \geq -MI$, locally. For quadratics, Akaike (1959) shows that the steepest descent method with directional minimization, which is superior to constant step length, will statistically converge only Q-linearly, unless the first search direction happens to be an eigenvalue of the Hessian. This confirms the observation that in practical use steepest descent is not better than Q-linear. To show that the search speed is limited to Q-linear, we consider steepest descent with constant step length in the best case scenario: where $f(x)$ is quadratic and the initial search direction is an eigenvector of $\nabla^2 f(x)$.

Proposition 10. *Consider a quadratic $f(x)$ with non-degenerate Hessian $-mI \geq \nabla^2 f \geq -MI$, optimized by steepest descent with constant step length $\epsilon \in (0, 2/M)$. Let x_0 be the starting point, and $\nabla f(x_0)$ an eigenvector of $\nabla^2 f(x_0)$. Then the search method will proceed only along the direction of $\nabla f(x_0)$, and the following hold:*

$$(a)\ \|x_t - x^*\| \geqslant qmin^t \|x_0 - x^*\|,$$

$$(b)\ f(x^*) - f(x_t) \geqslant \frac{m}{M} qmin^{2t} [f(x^*) - f(x_0)],$$

where $qmin := \min\{|1 - \epsilon m|, ..., |1 - \epsilon M|\} < 1$.

We can conclude that generally our search process has a $Q - linear$ convergence speed. Now we consider what happens in the continuous time limit to the search process.

Lemma 11. *Consider the steepest descent search with step length inversely proportional to the time interval, and the familiar assumptions on $f(x)$. In the continuous time limit, the upper and lower bounds on the convergence of the objective function and the argument are exponentials, and the search converges Q-linearly to the optimum.*

We have assumed the step length to be inversely proportional to the time of one search iteration. In terms of computational costs, the step length doesn't make a difference. We can, however, justify it with economic intuition, assuming that larger consumption changes take more time. The assumption is needed because we want to consider a continuous time description of individual consumption, even if the search behavior is essentially discrete. Another way to view $x(t)$ is as a continuous approximation of a messy, discrete consumption pattern, in the same way classic period by period aggregation is a discrete approximation. Then the continuous steepest descent is an approximation of a real discrete process.

From Lemma 11, assumption (2.12) holds if we take r to satisfy the upper and lower convergence speed bounds. Since we derive stylized results, we only need that the search process converges Q-linearly in Section 2.7.

2.7 An Alternate Theoretical Set-up

Consider an individual with a rational, continuous and locally nonsatiated preference relation, implying some utility function $\mathcal{U} : \mathbb{R}_+^{n+1} \to \mathbb{R}$, who is making a one-time consumption decision over bundles $(x_1, ..., x_n, t) = (x, t)$, where the last component represents time, or leisure. Furthermore, $\mathcal{U}$ is strictly quasiconcave and of class $\mathcal{C}^1$. Assume that his utility is additively separable in time: $\mathcal{U}(x, t) = U(x) + T(t)$. It must follow that $U \in \mathcal{C}^1$, strictly quasiconcave. The consumption vectors $x = (x_1, ..., x_n)$ are expressed in units of money for simplicity, so the price is normalized to $(1, ..., 1)$. Leisure and every good will optimally be consumed in some amount because:

$$T'(t) \xrightarrow{t \to 0} \infty; \; \forall i : \forall v_1, ..., v_{i-1}, v_{i+1}, ..., v_n : \frac{\partial U(v_1, ..., v_{i-1}, x_i, v_{i+1}, ..., v_n)}{\partial x_i} \xrightarrow{x_i \to 0} \infty \; .$$

So, for simplicity, the solution to the classical utility maximization problem is an interior solution for nonzero wealth. Because total utility is additively separable, the individual solves a classical utility maximization problem if the leisure consumption is fixed:

$$\arg\max_x \; U(x) \quad s.t. \quad \sum_{i=1}^{n} x_i \leqslant B, \quad B \geqslant 0. \tag{2.13}$$

Let $\delta(r,x) \equiv \sum_i (r_i - x_i)$ be the wealth shift measure in units of money from the initial value $B(r) \equiv \sum_i r_i$, given some change in endowment $r \to x$. In general, x will be suboptimal for the wealth value $B + \delta$. Let $\Delta(x, x^*) \equiv \|x - x^*\|$ be the distance in consumption bundle space to the new optimum $x^*(\delta)$ that solves (2.13) for $B + \delta$.

Given any deviation from the reference point, the individual will incur a cost over time at a constant rate while adjusting to the new optimal consumption allocation. To be specific, begin by setting $x(t)$ as the temporary consumption decision at search time t. In the following, the setting is deterministic and the search path $x(t)$ is continuous. Search starts at $t = 0$ and will continue if the payoff from reaching a more efficient allocation offsets the cost incurred. Let $t_s \geqslant 0$ be the stopping time. To simplify notation, say $\Delta(t) = \Delta(x(t), x^*)$, and assume it is differentiable, and:

$$-\frac{\Delta'(t)}{\Delta(t)} \equiv r > 0, \text{ so } \Rightarrow t_s = \int_0^{t_s} \mathrm{d}t = \frac{\log \Delta(0) - \log \Delta(t_s)}{r}. \tag{2.14}$$

The interpretation of this assumption is that reducing the misallocation by a small percentage happens in the same small period of time, i.e., the log-rate at which the distance to optimal allocation is reduced is constant. This assumption will be justified with a mechanism for costly search in section 2.6, but now it is an assumption. We are dealing with a one-time consumption allocation problem and time discounting is ignored; implicitly, the consumption aggregation period is assumed much larger than the search time.

An endowment change $r \to r + v$ gives a change in wealth $\delta(r, r+v) = \sum_i v_i$. For an arbitrary v with $\sum_i v_i = 1$, consider changes of the form $r \to r + \delta v$, $\delta \in$

$(-\sum_i r_i, \infty)$. The following proposition shows that the optimal utility value for the new wealth level is continuous and differentiable in δ, that $U(r + \delta v)$ will deviate as $o(\delta)$ from this optimal utility, and that the optimal demand function is continuous in δ.

Proposition 12. *Let $r \in \mathbb{R}^n_+$, and $\Phi(\delta) \equiv U(x^*(\delta)) = \max_x \{U(x) \mid \sum_i x_i = B(r) + \delta\}$. Let $x^*(\delta)$ and r be the solutions for the classical utility maximization problem (2.13) with budgets $B(r) + \delta$ and $B(r)$. Then $\Phi(\delta)$ is continuous and differentiable. Furthermore, for all v such that $\sum_i v_i = 1$, $\Phi(\delta) - U(r + \delta v) = o(\delta)$, where $\frac{o(\delta)}{|\delta|} \xrightarrow{\delta \to 0} 0$, and $x^*(\delta)$ is continuous in δ.*

So $x^*(\delta)$ is a continuous curve on the domain, that intersects any expenditure boundary once. Without other assumptions on the utility function, we can only establish that $\Delta(r + \delta v, x^*(\delta))$ will be bounded by a positive continuous function passing through the origin:

$$\Delta(r + \delta v, x^*(\delta)) = \|x^*(\delta) - (r + \delta v)\| \leqslant \|x^*(\delta) - r\| + |\delta| \cdot \|v\|. \tag{2.15}$$

This, however, does not say anything about the lower bound of $\Delta(r + \delta v, x^*(\delta))$. We can do more if we employ stronger assumptions on U and a result by Debreu (1972, 1976), which gives conditions for the invertibility of the demand function $x^*(\delta)$. Specifically, the Gaussian curvature of the hypersurface of constant utility $\{x \mid U(x) = U(x^*)\}$ has to be non-zero at x^*, i.e., the level curves have no *flat* parts, in a quadratic sense. Observe that the Hessian condition of Proposition 9 is stronger than and implies that the Gaussian curvature is strictly greater than zero.

Proposition 13. *For all $\delta \in (-B, \infty)$, let $x^*(\delta)$ be the demand induced by the classical utility maximization problem of U with wealth $B + \delta$. If $\kappa(x^*(\delta)) \neq 0$ is the Gaussian curvature of $\{x \mid U(x) = U(x^*(\delta))\}$ at $x^*(\delta)$, then $x^*(\cdot)$ is differentiable at*

δ, and $\frac{\partial x_k^*}{\partial \delta} = -\frac{x_k^*}{B+\delta}\left(\sum_j \epsilon_{kj}\right)$, $\forall k$, where ϵ_{kj} is the price elasticity of demand for good k and price j.

If $x^*(\delta)$ is differentiable, then so is $\Delta(r + \delta v, x^*(\delta))$. Therefore, for small δ, $\Delta(r + \delta v, x^*(\delta))$ can be approximated by a linear function, which is nonzero for a.e. v:

$$\Delta(r + \delta v, x^*(\delta)) \approx |\delta| \cdot \left\| \left(\frac{\partial x_1^*}{\partial \delta} - v_1, ..., \frac{\partial x_n^*}{\partial \delta} - v_n \right) \right\|. \tag{2.16}$$

The upper and lower bounds for the linear approximation at small δ values are given by:

$$\begin{aligned} |\delta| \cdot \left| \left\| \left(\frac{\partial x_1^*}{\partial \delta}, ..., \frac{\partial x_n^*}{\partial \delta} \right) \right\| - \| (v_1, ..., v_n) \| \right| &\lessapprox \Delta(r + \delta v, x^*(\delta)) \\ &\lessapprox |\delta| \cdot \left\| \left(\frac{\partial x_1^*}{\partial \delta}, ..., \frac{\partial x_n^*}{\partial \delta} \right) \right\| + |\delta| \cdot \| (v_1, ..., v_n) \|. \end{aligned} \tag{2.17}$$

The slope is positive and bounded if the price elasticities are bounded locally on the demand curve, which is a reasonable assumption far away from extreme allocations. In the following, it will be assumed that $\Delta(r + \delta v, x^*(\delta))$ is monotonic and strictly increasing in δ. The assumption is reasonable if ϵ_{kj} can be taken to change at a small enough rate, which is true if each consumption component x_i is the aggregate of many goods. Alternatively, there can be very many x_i that can have elasticities which change rapidly, but for which $\|(\frac{\partial x_1^*}{\partial \delta} - v_1, ..., \frac{\partial x_n^*}{\partial \delta} - v_n)\|$ still changes at a rate smaller than one, because of the averaging effect.

Effective spending of resources means that the temporary consumption choice $x(t)$ is close to the budget frontier, as optimal allocations should use all the available wealth. The difference to the maximum in the utility value will be $o(x^* - x(t))$, because the budget frontier is tangent to the level curve:

$$U(x) = U(x^*) + \sum_{i=1}^{n} \frac{\partial U}{\partial x_i}(x_i - x_i^*) - o(x^* - x),$$
$$\frac{\partial U}{\partial x_i} = \frac{\partial U}{\partial x_j} \text{ and } \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i^* \Rightarrow U(x) - U(x^*) = -o(x^* - x) \leqslant -o(\|x^* - x\|). \tag{2.18}$$

Define the utility centered around the reference point on the budget frontier:

$$\forall x^* \in \mathbb{R}_+^n : \text{ let } U_x : \left\{ x \middle| \sum_i x_i = \sum_i x_i^* \right\} \to \mathbb{R}\ ,\ U_{x^*}(x - x^*) \equiv U(x) - U(x^*). \tag{2.19}$$

The differential $DU_{x^*}(x^* - x)$ becomes the $\mathbf{0}$ map at $x^* = x$ and it's continuous, and T is strictly increasing in leisure, so:

$$\exists \Delta(t_s) > 0 \text{ s.t. } \left| \frac{\mathrm{d}U_{x^*}(\alpha(x^* - x))}{\mathrm{d}\alpha} \right| \leqslant \left| \frac{\mathrm{d}T(t - \frac{1}{r}\log(\alpha\|x - x^*\|) + \frac{1}{r}\log\Delta(0))}{\mathrm{d}\alpha} \right| =$$
$$\frac{T'(t - \frac{1}{r}\log(\alpha\|x - x^*\|) + \frac{1}{r}\log\Delta(0))}{r\alpha}, \text{ if } \|x - x^*\| \leqslant \Delta(t_s). \tag{2.20}$$

Therefore the search stops at $\Delta(t_s)$. We can find bounds on the final utility in terms of δ and $\Delta(t_s)$:

$$\Phi(\delta) + T(t) \geqslant \mathcal{U}(x(t_s), t - t_s) =$$
$$= \begin{cases} \Phi(\delta) - o(\Delta(t_s)) + T\left(t - \dfrac{\log\frac{\Delta(0)}{\Delta(t_s)}}{r}\right), & \Delta(0) > \Delta(t_s) \\ \Phi(\delta) - o(\Delta(0)) + T(t), & \Delta(0) \leqslant \Delta(t_s). \end{cases} \tag{2.21}$$

Because the budget frontiers intersect the demand function $x^*(\delta)$ only once, $\Phi(\delta)$ can be scaled arbitrarily by any monotonic transformation. For simplicity and convenience, let $\Phi(\delta)$ be linear.$\mathcal{U}(r,t) = \Phi(0) + T(t)$ is the utility at the *status quo*, and $\Phi(\delta) + T(t)$ is the utility in absence of search costs, for a given δ.

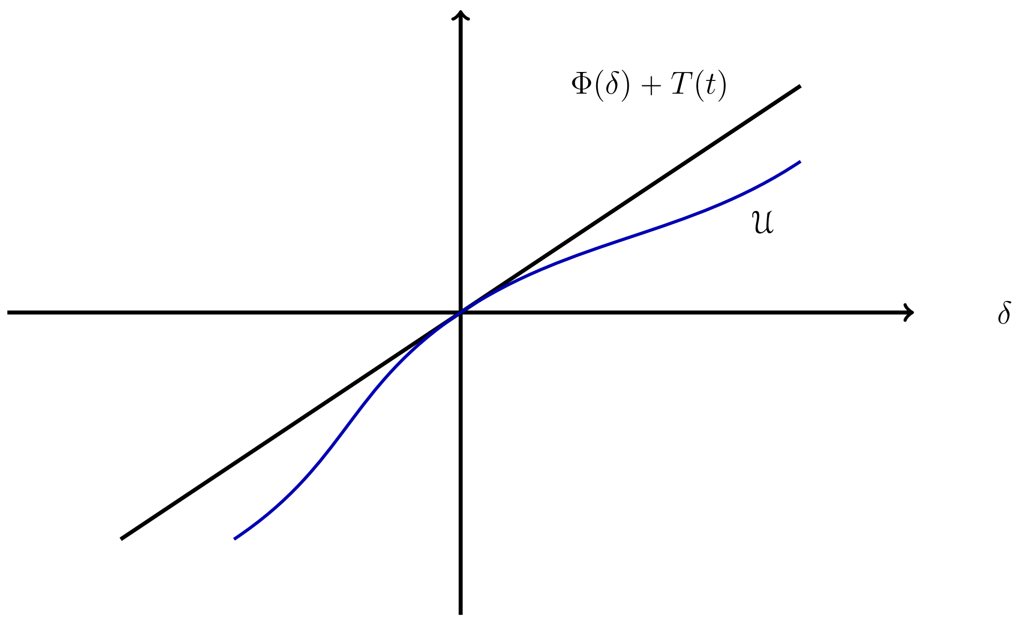

FIGURE 2.6: $\mathcal{U}$ deviates from the linear shape of $\Phi(\delta) + T$.

If δ is small enough, we can simplify things further by making $T(t) \equiv k_1 t$ and $\Delta(0) = \Delta(r + \delta v, x^*(\delta)) \equiv k_2|\delta|$ linear. Consider the effect of the adjustment cost on the slope, for positive and negative δ, in Figure 2.6.

$$\Phi(\delta) + T(t) \geqslant \mathcal{U}(x(t_s), t - t_s) = \begin{cases} \Phi(\delta) - o(\Delta(t_s)) + T(t) - \dfrac{k_1}{r} \log \dfrac{k_2|\delta|}{\Delta(t_s)}, & k_2|\delta| > \Delta(t_s) \\ \Phi(\delta) - o(k_2|\delta|) + T(t), & k_2|\delta| \leqslant \Delta(t_s). \end{cases} \tag{2.22}$$

The absolute value of the slope is decreased for positive δ values and increased for negative ones. This gives the loss aversion effect. Compare the shape of the function with the one in Figure 2.1. The cost function $C(\cdot)$ used before would be given by:

$$C(\|\delta v\|) = \begin{cases} o(\Delta(t_s)) + \dfrac{k_1}{r} \log \dfrac{k_2|\delta|}{\Delta(t_s)}, & k_2|\delta| > \Delta(t_s) \\ o(k_2|\delta|), & k_2|\delta| \leqslant \Delta(t_s). \end{cases} \tag{2.23}$$

This graph requires that we talk about risk aversion. In the original Kahneman and Tversky (1979) treatment, one of the assumptions is that utility of money is concave, to account for risk aversion. The cardinality of the utility function itself is linked to the basic test for risk aversion, which is assumptions (P_1). Here, we deal with choice over a set of goods, with utility arbitrary up to monotonic transformations. To be able to connect it to the one dimensional utility over money only, we need to fixate the cardinal values of U. The natural choice is to set U such that $\Phi(\delta)$ satisfies the usual one-dimensional utility assumptions, including concavity for risk aversion, but here assuming a linear $\Phi(\delta)$ would not change the results.

2.8 Discussion and Conclusion

We have proposed a behavioral model of consumer optimization which leads to inefficient search for optimal allocations. We have modeled the utility losses as an effect of search, once the consumer is faced with a shock to wealth or endowment. This cost can justify loss aversion as observed in experiments. A reference dependent value function that incorporates a cost of search satisfies the loss aversion axiom of Tversky and Kahneman (1991), and the equivalent loss aversion assumption of prospect theory in Kahneman and Tversky (1979). After a lottery shock, the consumer will plan for the likelier outcome, and this will lead to observed features of the probability weighting function in prospect theory. In addition, new predictions can be derived, which can serve to contrast this theoretical set-up from other models. For once, the issue of time in the setting of the reference point is crucial. An experiment that looks at the strength of the deviations from classical Hicksian demand in terms of the historical duration of a *status quo*, and tries to separate this effect from other potentially valid ones, can be used to confirm.

There is important supporting evidence for the effect of time on loss aversion in the experimental work of Strahilevitz and Loewenstein (1998). In four related studies, the authors try to separate the effect of ownership history and conclude that, in addition to an endowment effect component that seems to set in almost instantly, the duration of ownership will also increase the valuation of a given object. In the present framework, the two components of this effect can be explained. The initial effect, which sets in almost immediately, is the reflection of the fact that only a small investment in looking for the optimum at the beginning can have a large effect. Because the rate at which the optimum is approached decreases to zero, we must have an efficiency gain that comes only after a while, as the optimum is approached asymptotically. This gives the later duration-dependent component that the authors observe.

To support the theory, a mechanism that generates costly search is constructed, with some basic assumptions about an individual's search behavior. The search mechanism is proposed to be the heuristic equivalent of steepest descent, which is an intuitive solution in convex maximization problems. The individual is said to behave as if he would be performing such an algorithm, starting from an initial condition – here the *status quo*. This is a bounded rationality setting, where the limitation is that individuals have no *a priori* knowledge of utility values for consumption bundles they do not hold, unless they choose them during search.

The theoretical work of Kőszegi and Rabin (2006, 2009) is further suggesting of the importance of beliefs about future realizations on the choice of *status quo*. If beliefs on the future are important, then it's not a great leap to assume that planning is important. One of the explanations provided by their model – that traders and merchants are not influenced by the endowment effect because they don't hold updated beliefs based on the market traffic – is also paralleled by an explanation within the framework of the current chapter: an agent who doesn't expect to hold on to some bundle of goods will not try to find an optimal level of consumption after every trade. Furthermore, the later explanation has the advantage that it replaces an *ad hoc* assumption – expectation as *status quo* – with an underlining mechanism.

A valuable extension to the current work would be to distinguish which features of a reduced functional form for the consumer's value function, together with search planning, are crucial for recreating subcertainty, the overweighting of small probabilities, and subadditivity, effects which justify the probability weighting function of prospect theory.

3

Dynamic Contracting under Permanent and Transitory Private Information

3.1 Motivation and Related Literature

There are many ways to model repeated interactions between, say, a buyer and a seller, and by the revelation principle the optimal mechanism will involve the seller reporting his private information, as long as adequate incentive compatibility and rationality conditions are satisfied. But, if the private information known before signing the contract has no bearing on future private information, the problem is in a sense trivial. The dynamic contract becomes a pasting of static ones. It is more interesting to consider that there is a link between the private information at different times, and this link can be stochastic, or statistic – given by an underlining condition like a probability distribution, or in general a combination. In Battaglini (2005), we see the simplest form of a stochastic link: the utility type evolves according to a Markov process. Here I considers the simplest statistic link: the utility types are given by a distribution depending on a buyer type. The buyer type could be common knowledge, but this would make the problem trivial again. I will consider chiefly the buyer's type to be the buyer's private information, but the problem becomes even more interesting once we explore the idea of a hidden or partially hidden buyer type, which can be learned by both parties through repeated interactions.

It is important to look at such dynamic contractual relationships, since they are increasingly important for the economy. As information technology advances, tracking customers becomes easier and can lead to better tailored products and offers. Moreover, handling such long term data can improve the customer screening potential of companies which have the resources. A great example is the market for mobile telephony. Mobile phone carriers have the ability to enforce long term contracts, and the service usage can be carefully monitored. Moreover, the service to users can be charged at different rates based on quantity brackets, and can package utility increasing features into the contract – phone quality, text messaging, data plans. The contracts themselves come with initial or overtime financial commitments. The consumer can arguably have knowledge of his potential usage pattern, defining a possible buyer type, and his monthly consumption choice would be equivalent to reporting a utility type by the taxation principle. Consider also a monopolistic supplier selling to a small business or a company dealing with its franchise. The company or franchise hold specific private information about the profitability of the business, as well as about the day to day sales potential, which would determine short term profits. Long term strategy must incorporate the information from repeated interaction.

This chapter presents the interaction between a buyer and a seller, with the buyer has private information about his "utility type" and his "buyer type". The buyer type will determine the utility type statistically, but all buyer types can have one of the same utility types, so there is some overlap which makes buyer types unseparable in a trivial manner. Considering the buyer type as private information, it is clear that the buyer's type will influence the type of contract that is optimally offered only to him, as well as in a setting in which there are more than one buyer type. It turns out that there is separation between buyer types, which for two types happens through basically two contracts, where for reports of a low utility (type) the quantities allocated differ.

An alternative approach to a similar problem in current literature is in Battaglini (2005). He considers a dynamic interaction between a buyer and a seller much like in this chapter, where utility only can change from a utility type determinant, and this type is linked across time by a Markov process. This assumption may describe the setting of a monopolistic supplier or a franchise owner better, at least when it can be said that the profitability of the agent's business is subject to a Markov process because, say, the daily number of returning customers is Markov. In other situations it makes more sense to consider that the profitability of the business is simply determined by unchangeable factors – here the agent type, and randomness.

In a working paper, Pavan et al. (2009) attempt to characterize general dynamic mechanism design problems in finite time, and again for an infinite horizon in Pavan et al. (2010). They consider a general stochastic process for information, with possibly non-time-separable agent payoffs. To obtain more explicit results, they require continuity in the probability measure that determines types, and in general, continuity in the total variation metric for the most general results. This does not translate to the finite type case, where the differential conditions cannot apply. One important observation they make is that the vanishing distortions at the bottom principle (VDB) in Battaglini (2005) is not a robust result. In general, distortions need not be monotonic in type or in time, nor vanish in the long run.

Courti and Li (2000) study a somewhat similar problem with two stages, where the first stage is where the agent, an airline ticket buyer, finds out his type, which restricts somewhat the buyer's possible valuations for the ticket. Considering contracts with partial refunds, the authors find that the informativeness of the signal the buyer gets is what determines the optimal mechanisms, and not the uncertainty that affects all the buyer types.

The next two sections present and solve the model proposed, the following section provides a discussion of the results and their implications, and the last section concludes. The proofs are relegated to the Appendix.

3.2 Describing the Model

Consider a setting with a monopolistic principal, or seller, and an agent, or buyer. The buyer's period by period utility function is given by

$$U(\theta, q) = \theta q - q^2/2 - p(q), \tag{3.1}$$

where q is the buyer's consumption, p is a price he pays for it henceforth determined by q, and θ is a utility, or demand, type determinant which can take the values $\{\theta_h, \theta_l\}$. The buyer has ex-ante private information about the probability of having a demand of type θ_h instead of θ_l in any one period. This probability is determined by the buyer's type, and can take the values $\{\alpha_h, 1-\alpha_h\}$ or $\{\alpha_l, 1-\alpha_l\}$. There is common knowledge that the only two buyer types appear with odds $\{\varphi, 1-\varphi\}$ respectively. Later on, this assumption will be relaxed and generalized. The reservation utility for the buyer is 0, and the seller has the power to commit to a contract. Both buyer and seller are risk neutral and have no liquidity constraints, so they care only about maximizing expected utility. We can restrict attention to direct truthful mechanisms. The game starts in period 0, with the buyer reporting his α value, after which he may have to make a payment, although we may assume *w.l.o.g.* that the payment is deferred. In subsequent rounds, the buyer reports his θ demand determinant, receives an allocation q and makes a new payment – Figure 3.1. Let generic $\theta_h > \theta_l \geqslant 0$, $1 > \alpha_h > \alpha_l > 0$, $\varphi \in (0, 1)$, but strict inequalities are only to

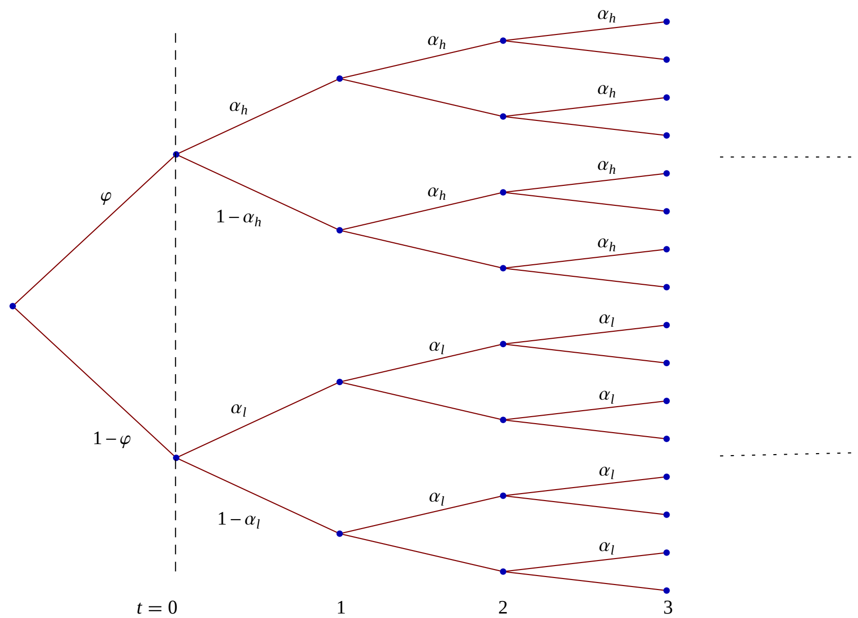

FIGURE 3.1: Moves by nature. The dotted line shows when the contract is signed.

consider the interesting problem. In the following, I will focus on a game with infinite periods, but sometimes I may also consider a finite period game for description or as a stepping stone. Unless explicitly stated, the results will hold for the finite as well as infinite settings. In the finite case I will ignore discounting.

3.3 Solving the Model

It is interesting to consider both the finite and the infinite time periods versions. For the infinite setting, we could possibly use a recursive approach, but simple considerations show that the number of relevant state variables grows linearly in time, and it is not immediate in what way one can summarize the relevant history. Therefore, looking at the optimal direct mechanism for the finite game is easier, and could possibly be extended with a continuity argument to the infinite setting. For that, one typically considers rationality and compatibility incentive constraints, which are then incorporated in a global maximization problem. However, as the number of time periods grows, the number of incentive compatibility conditions will also grow polynomially. It is important then to par down this number to the absolute minimum, before proceeding to any analysis in this direction.

3.3.1 Reducing the number of ICC conditions

The first step in the solving strategy is proving that the set of incentive compatibility conditions can be restricted somewhat. This claim will be proven for a more general buyer utility function, and an arbitrary finite number of utility types l. The utility function is $U_t(\theta, q_t(\cdot))$ in round t, discounted if needed, where θ can take any value from the finite set $\{\theta_1, \ldots, \theta_l\}$, and the allocation $q_t(h_0, h_1, \ldots, h_t)$ is a function of the history of messages the buyer sent. Fix $U_0 \equiv 0$, since we can delay payment. Here α is just an index for one of a finite number of probability distributions over the values θ, and it has full support *w.l.o.g.* We can consider the messages h_t to be determined by a function of the buyer's true type and demand type history, $h^t \equiv \sigma_t(\alpha, \theta^1, \theta^2, \ldots, \theta^t)$, $h^0 \equiv \sigma_0(\alpha)$, where $\sigma \equiv \{\sigma_t(\cdot)\}$ is the player's strategy. Any direct truthful mechanism that implements a menu will have to satisfy individual rationality constraints (IRCs) for each α, since the buyer has ex-ante information about his type before he signs a contract.

$$\forall \alpha : \quad 0 \leqslant \sum_{t=1}^{n} \sum_{\Theta^t} \prod_{i=1}^{t} \lambda_\alpha(\theta^i) U_t(\theta^t, q_t(\alpha, \theta^1, \ldots, \theta^t)), \tag{3.2}$$

where $\Theta^t = (\theta^1, \ldots, \theta^t)$ is one possible history of types for period t, and $\lambda_\alpha(\theta^i)$ is the probability of getting the value θ^i that is given by the distribution indicated by α. For clarification, in our two-types model $\alpha_h \equiv \lambda_{\alpha_h}(\theta_h)$, $1 - \alpha_h \equiv \lambda_{\alpha_h}(\theta_l)$, $\alpha_l \equiv \lambda_{\alpha_l}(\theta_h)$, and $1 - \alpha_l \equiv \lambda_{\alpha_l}(\theta_l)$. The incentive compatibility constraints (ICCs) for a direct truthful mechanism are

$$\begin{aligned}\forall \sigma, \alpha: \quad & \sum_{t=1}^{n} \sum_{\Theta^t} \prod_{i=1}^{t} \lambda_\alpha(\theta^i) U_t(\theta^t, q_t(\alpha, \theta^1, \ldots, \theta^t)) \geqslant \\ & \sum_{t=1}^{n} \sum_{\Theta^t} \prod_{i=1}^{t} \lambda_\alpha(\theta^i) U_t(\theta^t, q_t(h^0, h^1, \ldots, h^t)).\end{aligned} \tag{3.3}$$

Proposition 14. *The set of incentive compatibility conditions (3.3) is satisfied if a stronger set of compatibility conditions ($\mathcal{S}$) hold, whereby in each period the buyer considers only one-time deviations from truth, followed by truthful reporting in the future. That is, the ICCs are implied by the following set of inequalities, for*[1] $k \in \overline{0, n}$, $\forall\ \sigma, \alpha, \Theta^k$:

$$\begin{aligned} & U_k(\theta^k, q_k(h^0, h^1, \ldots, h^{k-1}, \theta^k)) + \\ & \qquad + \sum_{t=k+1}^{n} \sum_{\Theta^t|\Theta^k} \prod_{j=k+1}^{t} \lambda_\alpha(\theta^j) U_t(\theta^t, q_t(h^0, h^1, \ldots, h^{k-1}, \theta^k, \theta^{k+1}, \ldots, \theta^t)) \geqslant \\ \geqslant\ & U_k(\theta^k, q_k(h^0, h^1, \ldots, h^k)) + \\ & \qquad + \sum_{t=k+1}^{n} \sum_{\Theta^t|\Theta^k} \prod_{j=k+1}^{t} \lambda_\alpha(\theta^j) U_t(\theta^t, q_t(h^0, h^1, \ldots, h^k, \theta^{k+1}, \ldots, \theta^t)).\end{aligned}$$

Proofs in this chapter are relegated to the appendix.

Intuitively, the alternative conditions of Proposition 14 allow us to improve, step by step, any non-truthful strategy by considering deviations towards truthful reporting. Starting with an arbitrary strategy σ the last period, it must be that, if it is preferable for the buyer to not lie regardless of the past history, he will improve on σ by reporting truthfully. This must hold after any history, so we immediately have

[1] To be exact, consider that $\theta^0 = \alpha$. At $k = 0$ there are only the incentive compatibility conditions for each type α, which are the same as the ICCs.

that in the last period $\mathcal{S}$ implies truthful reporting. This allows us to move to the previous period, where we again observe, history by history, that truthful reporting is preferred, and so on by backward induction. This is possible since there are compatibility conditions for any α type at each node; i.e., it is optimal for the buyer not to lie, assuming truthful reporting in the future, even if he isn't supposed to get this menu because he misrepresented his α. It is thus possible to improve onto any σ by deviating towards the truthful strategy.

The result holds for $n = \infty$ also, and the proof is by contradiction. Starting again with a possibly non-truthful strategy σ, we can approximate it by assuming that after period k the buyer decides to report truthfully. Only after period k will his utility be changed by the new strategy, and the change is small if k is large. This is true because we will need to incorporate discounting into U_t, and if the period by period utility is bounded, then the partial sum over time that gives the buyer's total utility must converge. With the new strategy it is possible to use conditions in $\mathcal{S}$ as before, to prove by backward induction that true reporting is superior to the approximated σ. Then, in the limit $k \to \infty$, we must have that the true strategy is as good as σ. But this statement is precisely the requirement that truthful strategies satisfy the ICCs, which is what we need to prove.

Proposition 15. *The set of ICCs in (3.3) imply a weaker set of compatibility conditions ($\mathcal{W}$), in which the buyer, who has reported his true type α if in periods after 0th, considers only one-time deviations from truth, followed by truthful reporting in the future. That is, the ICCs imply the following inequalities, for*[2] $k \in \overline{0, n}$, $\forall\ \sigma, \alpha, \Theta^k$:

$$U_k(\theta^k, q_k(\alpha, h^1, \ldots, h^{k-1}, \theta^k)) +$$

$$+ \sum_{t=k+1}^{n} \sum_{\Theta^t|\Theta^k} \prod_{j=k+1}^{t} \lambda_\alpha(\theta^j) U_t(\theta^t, q_t(\alpha, h^1, \ldots, h^{k-1}, \theta^k, \theta^{k+1}, \ldots, \theta^t)) \geqslant$$

[2] At $k = 0$ consider $\theta^0 = \alpha$, and $U_0 = 0$ as defined.

$$\geqslant U_k(\theta^k, q_k(\alpha, h^1, \ldots, h^k)) +$$

$$+ \sum_{t=k+1}^{n} \sum_{\Theta^t|\Theta^k} \prod_{j=k+1}^{t} \lambda_\alpha(\theta^j) U_t(\theta^t, q_t(\alpha, h^1, \ldots, h^k, \theta^{k+1}, \ldots, \theta^t)), \ \textit{and}$$

$$\sum_{t=1}^{n} \sum_{\Theta^t} \prod_{i=1}^{t} \lambda_\alpha(\theta^i) U_t(\theta^t, q_t(\alpha, \theta^1, \ldots, \theta^t)) \geqslant \sum_{t=1}^{n} \sum_{\Theta^t} \prod_{i=1}^{t} \lambda_\alpha(\theta^i) U_t(\theta^t, q_t(h^0, \theta^1, \ldots, \theta^t)).$$

The intuition behind the proof rests on the observation that, if reporting truthfully is always better than not, then it is also better than lying only once, and that the continuation utilities for a person who considers lying once are the same as for a person who may have misrepresented his past utility types – θ, as long as he is the same buyer type α. This is true because future payoffs depend only on reports and the buyer type, which are both assumed the same. So having compatibility conditions in each period checking whether a buyer prefers lying for that period to true reporting is equivalent to considering some of the ICCs.

We can therefore say $\mathcal{W} \subset$ ICCs, and $\mathcal{W} \subset \mathcal{S}$ is by definition. Proposition 14 says that if a menu satisfies $\mathcal{S}$, then it also satisfies the ICCs, so we can think of the set of possible menus under $\mathcal{S}$ as more restrictive than those that satisfy the ICCs. In general, a menu implemented by a mechanism will be a point in the menu parameter space. The sets of conditions $\mathcal{W}, \mathcal{S}$, ICCs will delimit a subset of this space, where the menus inside obey the conditions. Let $\mathcal{D}[\cdot]$ denote the subset. Then we must have $\mathcal{D}[\mathcal{S}] \subset \mathcal{D}[ICCs] \subset \mathcal{D}[W]$ (Figure 3.3). Now consider the maximization problem that determines the optimal mechanism. To find the optimal truthful mechanism we need to restrict the domain of menus to $\mathcal{D}[ICCs]$. A containing domain may generate a false maximum, if it is located outside $\mathcal{D}[ICCs]$, while a subdomain can lead to maxima on its boundaries that could be suboptimal. To solve for the optimal

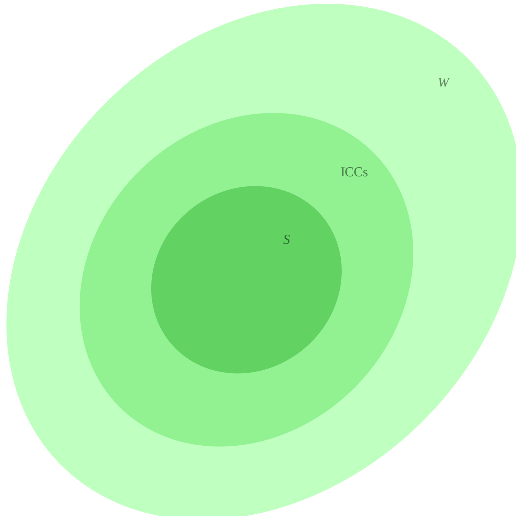

FIGURE 3.2: The restricted domains according to $\mathcal{W}, \mathcal{S}, ICCs$ in the menu parameter space.

mechanism, my strategy will be to find the optimal menu in $\mathcal{D}[\mathcal{W}]$ and show that is is in $\mathcal{D}[\mathcal{S}]$.

3.3.2 Fixing some of the allocations

The next step is to reduce the possible ways in which a menu representing a truthful mechanism can be constructed. From now on, we return to the two-type setting, although the results will hold for an arbitrary number of types, if properly stated. This section shows with a few lemmas that the optimal menu must have first best allocations for reports of θ_h, as well as for θ_l after type α_h was reported. The lemmas work for any $n \leqslant \infty$, and also if we rephrase them with $\mathcal{S}$ instead of $\mathcal{W}$.

Lemma 16. *Consider any period with allocations. Denote with q_h, q_l the quantities and p_h, p_l the prices in the menu offered to the buyer in that period, contingent upon the θ-type report, and let u_h, u_l be the expected continuation utilities for the buyer, given by the buyer type and truthful reporting in the future. Then, if conditions $\mathcal{W}$ hold, it must be that $q_h \geqslant q_l$.*

The result can be understood to mean that, if $q_l > q_h$ and if a buyer with θ_l prefers the low allocation and price to the high ones, then the buyer with θ_h must prefer it too, because his total utility is increased more by an increase in the allocation, and future utility is determined only by today's report, and not by today's true type. But this evidently contradicts a compatibility condition.

Lemma 17. *Consider any period with allocations in the game described. Denote with q_h, q_l the quantities and p_h, p_l the prices in the menu offered to the buyer in that period, and let u_h, u_l be the expected continuation utilities. Assuming that $\mathcal{W}$ hold, it must be that $q_h \geqslant \theta_h$ and $q_l \leqslant \theta_l$ in the optimal menu.*

This lemma shows that the θ_h type must always be allocated at least the first best in an optimal mechanism, because you can always increase a suboptimal allocation, and then charge a higher price to compensate his utility increase, and this change will not make the report θ_h better for any lower utility type. Similarly, allocating more than the first best to the low utility type is suboptimal because, if the allocations is greater, then the seller can decrease it marginally, and compensate by an increase in price, and this compensation would leave the lowest type indifferent, but for all the other types it would be insufficient, therefore leaving any constraints relaxed.

Lemma 18. *Assume that in the setting presented $\mathcal{W}$ hold, and the results of the previous lemmas. Then, for every period, for each history, at most one of the compatibility conditions in $\mathcal{W}$ for an agent α will bind.*

From the previous lemma, we can also see that the high and low utility type allocations must be different, by the conditions in $\mathcal{W}$. If more than one compatibility conditions hold, then it says that the difference in utilities minus the difference in prices for two alternatives has to be 0, and for more than one utility type. This is not possible because utility differences for the same allocations are strictly monotone in θ.

A more general formulation of utility that would preserve the result has $\mathcal{U}_\theta(\theta, q) > 0$, and $\mathcal{U}_{\theta,q}(\theta, q) > 0$ (known as the *Spence-Mirrlees single-crossing condition*).

Lemma 19. *Consider any period with allocations, and assume the compatibility condition in $\mathcal{W}$ for the buyer with θ_h-type utility does not bind. Then $q_l \geqslant \theta_l$. Similarly, if the θ_l-type condition does not bind, then $q_h \leqslant \theta_h$.*

Decreasing the high utility allocations marginally towards the first best and then compensating the price will leave the high utility type indifferent, but it will improve the high report option for the other type. However, if the old compatibility constraint was slack, a marginal increase in the high type allocation will not violate the it. Similarly, you can argue that increasing the low type allocation towards the first best and then compensating the low utility type with a higher price will also be an improvement, but only if the high type's compatibility constraint is slack.

3.3.3 Solving the maximization problem

From now on assume $\mathcal{W}$ hold and that in every period with allocations the θ_l-type incentive compatibility condition does not bind, while for the case with $\sigma_0 = \alpha_h$ the θ_h-type compatibility condition will also not have to bind, i.e., it is slack. With the previous lemmas, the assumption gives us that, in every period after the 0th, $q_h = \theta_h$, and that after $\sigma_0 = \alpha_h$ we must have in every subsequent period $q_l = \theta_l$. Moreover, at period 0, we assume that only the rationality constraint for the α_l-type and the compatibility constraint for the α_h-type bind. The proof requires that we verify the assumptions after we find the solution.

With the assumptions above, the allocations after reporting α_h are fixed. The seller needs to optimize the low demand type allocations if the buyer reported α_l in period 0. Since we have assumed that only the α_l-type compatibility condition binds in period 0, we can maximize the seller's value, keeping track of the fact that

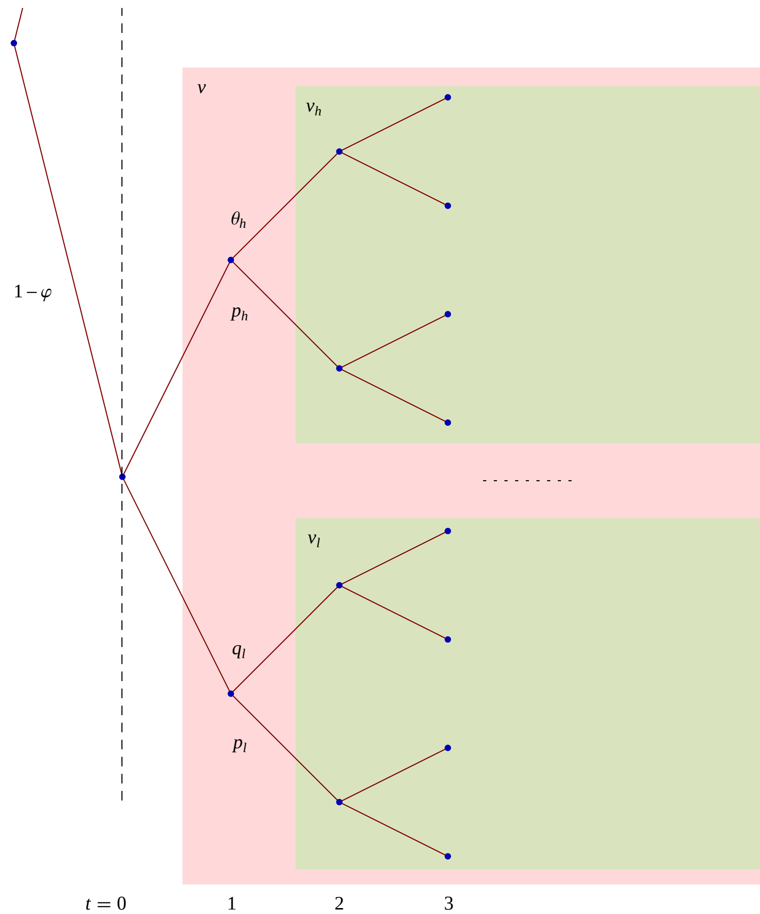

FIGURE 3.3: The simplified maximization problem.

the information rent earned by a α_h buyer pretending to have the α_l-type must be subtracted with the appropriate weight.

Lemma 20. *The simplified seller's maximization problem is given by the following recursive form:*

$$P(v) = \max_{\{p_h, p_l, v_h, v_l, q_l\}} (1-\varphi)[\alpha_l p_h + (1-\alpha_l)p_l] + \alpha_l \delta P(v_h) + (1-\alpha_l)\delta P(v_l) + \varphi[-\alpha_h(\theta_h^2/2 - p_h) - (1-\alpha_h)(\theta_l q_l - q_l^2/2 - p_l)] \tag{3.4}$$

$$
\begin{aligned}
s.t.\quad (i)\quad & \theta_h^2/2 - p_h + \delta v_h = \theta_h q_l - q_l^2/2 - p_l + \delta v_l, \\
(ii)\quad & v = \alpha_l(\theta_h^2/2 - p_h) + (1-\alpha_l)(\theta_l q_l - q_l^2/2 - p_l) + \alpha_l \delta v_h + (1-\alpha_l)\delta v_l, \\
(iii)\quad & q_l \geqslant 0.
\end{aligned}
$$

Here v is the utility value for the buyer at the beginning of period 1, v_h, v_l are his continuation utilities if he reports θ_h and θ_l respectively. Since we assumed that the α_l-type rationality constraint in period 0 holds with equality, we must have that $v = 0$ in the first period. Because there is an indeterminacy in the price allocation, we can also fix $v_h = v_l = 0$ to find explicit prices, but after we have determined the quantity allocations from the maximization problem.

Proposition 21. *Assuming that* (iii) *doesn't bind, the allocation after reporting* θ_l *in the maximization problem (3.4) is always given by*

$$q_l = \theta_l - (\theta_h - \theta_l)\frac{\varphi}{1-\varphi} \cdot \frac{\alpha_h - \alpha_l}{1-\alpha_l}$$

and with the normalization $v_h = v_l = 0$*,*

$$
\begin{aligned}
p_h =& \frac{\theta_h^2}{2} - (\theta_h - \theta_l)(1-\alpha_l)q_l, \\
p_l =& (\alpha_l(\theta_h - \theta_l) + \theta_l)q_l - \frac{q_l^2}{2}.
\end{aligned}
$$

(iii) *holds with equality when* $\frac{\theta_l(1-\alpha_l)}{\theta_h(\alpha_h-\alpha_l)+\theta_l(1-\alpha_h)} \leqslant \varphi \leqslant 1$*, and in this case* $q_l = 0$*,* $p_h = \frac{\theta_h^2}{2}$ *and* $p_l = 0$ *in every period.*

The next result gives the allocations and prices offered if the buyer reports α_h in period 0.

Proposition 22. *The allocations after reporting α_h are $q_h = \theta_h$ and $q_l = \theta_l$, and*

$$p_h = \frac{\theta_h^2}{2} - \theta_h\theta_l + \alpha_h\theta_h\theta_l + \theta_l^2 - \alpha_h\theta_l^2$$

$$p_l = \alpha_h\theta_h\theta_l + \frac{\theta_l^2}{2} - \alpha_h\theta_l^2,$$

as well as a transfer of $\frac{(\alpha_h-\alpha_l)(\theta_h-\theta_l)q_l}{1-\delta}$ from the seller to cover the information rent for the α_h type buyer.

It is left is to check that all the assumptions on which individual rationality or compatibility conditions in $\mathcal{W}$ will bind have been correct. Because of the symmetry of the allocations after reporting α, it is obvious that all the compatibility conditions in $\mathcal{S}$ are also satisfied. It was assumed that the θ_h-type condition holds with equality, for the buyer who has reported his true α, which means that the θ_l-type compatibility condition will also hold. If the buyer has not reported his α-type truthfully, then the compatibility conditions starting from period 1 will be essentially unchanged, because of the symmetry of the allocations, which means that $v_h = v_l$.

The last thing we need to check are the conditions in period 0. The IRCs are satisfied by construction. If an α_h-type buyer reports α_l, then the information rent he earns is given by:

$$v^h = (\alpha_h - \alpha_l)(\theta_h - \theta_l)q_l + \delta\alpha_h v^h + \delta(1-\alpha_h)v^h, \tag{3.5}$$

which solves to

$$\frac{(\alpha_h - \alpha_l)(\theta_h - \theta_l)q_l}{1-\delta}. \tag{3.6}$$

This is the transfer a α_h-type buyer will get in period 0 in our solution, and the expected utility starting from period 1 will be 0 by construction. Similarly, we can find the expected utility of an α_l-type buyer who reports α_h:

$$\frac{(\alpha_h - \alpha_l)(\theta_h - \theta_l)(q_l - \theta_l)}{1-\delta} < 0. \tag{3.7}$$

Therefore, the lower type wouldn't report α_h, so he gets a continuation utility of 0. Because we have found a solution satisfying the stronger collection of compatibility conditions in $\mathcal{S}$, it is the correct solution for the set of $ICCs$.

3.4 Discussion

From the solution in Proposition 21, it can be seen intuitively that the distortion of θ_l allocations is induced by the need to decrease rents for the α_h type buyer. A marginal increase Δq in all allocations for θ_l reports of a α_l buyer, at any period, would lead to a proportional increase in rents $\varphi(\alpha_h - \alpha_l)(\theta_h - \theta_l)\Delta q$, and the extra revenue for the seller is $(1-\varphi)(1-\alpha_l)(\theta_l - q_l)\Delta q$, in the first order approximation. At equilibrium, the two must be equal, and this gives the result.

Now we want to discuss some comparative statics. We vary the different parameters, to see if the results coincide with those of simpler problems. If $\varphi = 0$, then there is only an α_l type, who must sign a contract before he knows his first θ. As expected, our solution becomes $q_l = \theta_l, q_h = \theta_h$, which is the classical dynamic contract when the buyer has no private information. If $\varphi \to 1$, this leads to $q_l \to 0$, and to 0 rent for the high type buyer, because the α_h buyer becomes increasingly important, so his rent is weighted progressively more. Again, in the limit we find the classical dynamic contract with one buyer type and no private information. The same holds true when $\alpha_h = \alpha_l$, and it is because we don't have to distinctive buyer types anymore.

It is interesting to contrast the result with those of Battaglini (2005), who considers that there is only one buyer type, but his utility determinant θ evolves according to a Markov process, from a known prior distribution, and assumes that high types will be correlated. He showed that the optimal dynamic contract for such a setting converges to first best allocations as time passes. Furthermore, only reporting a sequence of θ_l would lead to suboptimal allocations; once the buyer has reported

θ_h, it would be followed only by first best allocations. In this setting, allocation distortions are permanent for a low buyer type, which leads to consistent losses of social welfare. This comes in part from the fact that a Markov process becomes progressively less connected to past information, so the buyer's private information – which usually leads to the allocation distortions – is in a sense weaker. However, it is also arguable that the two settings are fundamentally different, because in Battaglini (2005) the contract is signed after the buyer knows his first θ value. We can also modify our problem to consider such a situation (pending work), and the result is that there is a difference only in that first period, when if θ_l is reported then $q = \theta_l - (\theta_h - \theta_l)Pr[\theta_h]/Pr[\theta_l]$, after which allocations revert to the ones here. It is also interesting to note that, if we consider only two periods of the game mentioned, then we must have the same solution as in Battaglini (2005), since the information process is trivially Markov, and this checks out.

Consider now the problem faced by a mobile phone carrier. He must distinguish between two types of buyers, one that is either relatively poor, or that has a lifestyle that doesn't require many phone calls, and one who is likely to consume many minutes. By the taxation principle, the contracts offered to them are the implementation of a truthful mechanism. The result presented here suggests that the optimal choice of contracts would have basically two, for each buyer type. Let's say that one contract has sufficient minutes included, and one has a flat pay-per-use. When the buyer has a high utility draw, it means that his need for the call is great, so he will take the call no matter what his phone plan is. If his utility draw is low, then the flat rate buyer will decide, maybe, to not make the call, or talk less; but the buyer of the plan with minutes need not bother. It is also obvious that the buyer who expects to have many calls will prefer the better plan, and he will have to pay less per minute. The difference can be considered his information rent, because he had the option to choose the flat rate plan in the beginning. This description is consistent with typical consumer behavior. One can also think of contracts that may or may not include additional features, like messages or data plans, where the utility draw is over the usefulness of the feature. People who believe they will often need such features are more likely to buy a contract that will allow them their use more often.

3.5 Conclusion

This chapter deals with long term contracts between a principal and an agent, and considers that the agent has private information that is not completely random or fixed, but comes from the same prior distribution, which is his private information. The optimal contracts for two types of buyers with two types of utilities have some of the usual properties from the static problem, like no distortions for high utility draws, but are radically different from the static or the infinite games with random or persistent information. There is consistent distortion for the low type buyer, when he reports a low utility, and the distortion decreases social welfare. The results also provide some insight into the pricing strategies of monopolistic suppliers of firms or franchises, and of mobile phone carriers.

4

Notes on a two-firms procurement model

4.1 Motivation

Repeated principal-agent interactions in which the agent has private information, but is also liquidity constrained, are very relevant to a modern economy. We can think of many agreements that are reasonably described in such a way: franchise agreements, government procurement contracts, start-ups with venture capital, etc. This chapter presents joint work with Prof. Curtis Taylor[1], which extends a model from Krishna & Taylor (Working Paper, 2011). In that paper, a principal contracts one agent to operate a production technology, for example in a procurement contract, over an infinite number of discrete periods. The agent is realistically liquidity constrained, and has private knowledge of a sequence of marginal costs for his production technology. The results show that the principal uses his power of commitment to incentivize the agent with increases in promised utility, because output distortions induced by the liquidity constraint are mitigated by the increased stake the agent has in the partnership. Once the promised utility to the agent reaches a certain value, the principal

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can be bought out and thereafter production is efficient. Here we consider the same dynamic procurement setting with promise keeping, but this time we consider two agents acting competitively. Some results will parallel old ones, but this time we deal with competition each period between the two firms. For example, high cost types are allocated less than efficient quantities, "up to a point" the principal will only incentivize through promised utility, and the promised utilities to the two firms act as state variables in the recursive formulation of the principal's maximization problem.

In the following sections we develop two different theoretical set-ups. In section 4.2, we consider two firms which can have a private cost type with only two possible values, and also a public cost component from a continuum distribution, and we analyze the interaction in one period. Sections 4.3 and 4.4 considers the recursive problem over an infinity of periods. We will start with a simple one period game, and then analyze the dynamic interaction for an infinity of periods. The second set-up considers two firms which can have a marginal cost extracted from a continuum distribution. We will study the interaction between the firms in one period in section 4.5, and then extend the analysis to a two period game in section 4.6, but the results can be applied to solving numerically a similar game with an arbitrary finite number of periods. The last section concludes.

4.2 Interaction for one period

For now, consider a static two-firms (agents) procurement model with promise keeping. Both the principal and the firms (agents) are risk neutral, and the firms are *ex post* liquidity constrained. The principal wishes to procure a certain good by employing the firms, and for quantity q he gets a benefit of $R(q)$, where $R : \mathbb{R}_+ \to \mathbb{R}_+$ is $\mathcal{C}^1$, strictly increasing, concave, with a strictly decreasing first derivative, and

$\lim_{q\to 0} R'(q) = \infty, \lim_{q\to\infty} R'(q) = 0$.

The two risk neutral firms, indexed by $\{a, b\}$, can supply the good in any quantity. The marginal cost of the firms, denoted by θ^i, is a private draw from $\{\theta_l, \theta_h\}$, where $0 \leqslant \theta_l < \theta_h$. The draw probabilities are $Pr[\theta^i = \theta_l] = f \in (0, 1)$, so $Pr[\theta^i = \theta_h] = 1 - f$. Moreover, after the participation decision is made, there is a public draw of an extra marginal cost $e^i \in [\underline{e}, \overline{e}] \subset \mathbb{R}_+$ for each firm, given by the distribution *pdf* $g(e, \tilde{e})$ with support $[\underline{e}, \overline{e}]^2$. Define $\Delta = \theta_h - \theta_l$ and assume that $\Delta \geqslant \overline{e} - \underline{e}$, i.e., that $\theta_l + \overline{e} \leqslant \theta_h + \underline{e}$.

The principal wants to maximize his utility by designing a mechanism that assigns production to the firms and allocates transfers, all contingent on the reports of the firms and the public draws e^i. The firms receive an *ex post* payoff of $u^i_{jk}(e, \tilde{e}) := m^i_{jk}(e, \tilde{e}) - (\theta_j + e)q^i_{jk}(e, \tilde{e})$ in the case of truthful reporting, where the superindex $i \in \{a, b\}$ represents the firm, the first subindex $j \in \{l, h\}$ represents the report of the firm for its marginal cost type, the second subindex $k \in \{l, h\}$ is the report of the other firm on its type, while the first and second arguments are the firms' own draw of extra marginal cost, and the competitions' draw. $m^i_{jk}(e, \tilde{e})$ are monetary transfers to the firms, and since firms will be facing *ex post* liquidity constraints $u^i_{jk}(e, \tilde{e}) \geqslant 0$, we must have that $m^i_{jk}(e, \tilde{e}) \geqslant 0$.

Define the interim expected payoff after a report of j for firm i, conditional on the extra marginal cost draws $(e, \tilde{e})$:

$$U^i_j(e, \tilde{e}) := f u^i_{jl}(e, \tilde{e}) + (1 - f)u^i_{jh}(e, \tilde{e}),$$

and from this the expected promised utility for firm i:

$$v^i := \iint\limits_{[\underline{e}, \overline{e}]^2} [f U^i_l(e, \tilde{e}) + (1 - f)U^i_h(e, \tilde{e})]g(e, \tilde{e}) de d\tilde{e}.$$

The expected output of type j for firm i conditional on $(e, \tilde{e})$ is:

$$Q_j^i(e, \tilde{e}) := f q_{jl}^i(e, \tilde{e}) + (1 - f) q_{jh}^i(e, \tilde{e}).$$

Ex post liquidity constraints, $u_{jk}^i(e, \tilde{e}) \geqslant 0$, will also imply weaker interim liquidity constraints, $U_j^i(e, \tilde{e}) \geqslant 0$. For implementing a mechanism with truthful reporting, individual rationality will require that the low cost type firm doesn't report a high cost type, and we call the condition an incentive constraint. The binding (interim) upward adjacent incentive constraint is thus:

$$\iint\limits_{[\underline{e}, \bar{e}]^2} U_l^i(e, \tilde{e}) g(e, \tilde{e}) de d\tilde{e} = \iint\limits_{[\underline{e}, \bar{e}]^2} U_h^i(e, \tilde{e}) g(e, \tilde{e}) de d\tilde{e} + \Delta \iint\limits_{[\underline{e}, \bar{e}]^2} Q_h^i(e, \tilde{e}) g(e, \tilde{e}) de d\tilde{e}. \quad (4.1)$$

Using the expected promised utility formula, we can deduce:

$$\begin{aligned} \iint\limits_{[\underline{e}, \bar{e}]^2} U_h^i(e, \tilde{e}) g(e, \tilde{e}) de d\tilde{e} &= v^i - \Delta f \iint\limits_{[\underline{e}, \bar{e}]^2} Q_h^i(e, \tilde{e}) g(e, \tilde{e}) de d\tilde{e}, \\ \iint\limits_{[\underline{e}, \bar{e}]^2} U_l^i(e, \tilde{e}) g(e, \tilde{e}) de d\tilde{e} &= v^i + \Delta(1 - f) \iint\limits_{[\underline{e}, \bar{e}]^2} Q_h^i(e, \tilde{e}) g(e, \tilde{e}) de d\tilde{e}. \end{aligned} \quad (4.2)$$

For now, let us consider only the interim liquidity constraints, and it will remain to be shown that the *ex post* liquidity constraints are also satisfied. Because of positive allocations, it is easy to see that only the high type constraint needs to be considered:

$$\iint\limits_{[\underline{e}, \bar{e}]^2} U_h^i(e, \tilde{e}) g(e, \tilde{e}) de d\tilde{e} \geqslant 0 \Leftrightarrow v^i - \Delta f \iint\limits_{[\underline{e}, \bar{e}]^2} Q_h^i(e, \tilde{e}) g(e, \tilde{e}) de d\tilde{e} \geqslant 0. \quad (4.3)$$

Let's express now the principal's maximization problem:

$$\max_{\{q_{jk}^i\}} \iint\limits_{[\underline{e}, \bar{e}]^2} \left\{ \sum_{j,k} Pr[\theta_j] Pr[\theta_k] \left[R(q_{jk}^a + q_{kj}^b) - (\theta_j + e) q_{jk}^a - (\theta_k + \tilde{e}) q_{kj}^b \right] \right\} g(e, \tilde{e}) de d\tilde{e} - v^a - v^b \quad (4.4)$$

s.t. $\forall i$:

$$v^i \geqslant \Delta f^2 \iint\limits_{[\underline{e},\bar{e}]^2} q^i_{hl}(e,\tilde{e})g(e,\tilde{e})ded\tilde{e} + \Delta f(1-f) \iint\limits_{[\underline{e},\bar{e}]^2} q^i_{hh}(e,\tilde{e})g(e,\tilde{e})ded\tilde{e},$$

$\forall i, j, k, \forall (e, \tilde{e})$:

$$q^i_{jk}(e,\tilde{e}) \geqslant 0.$$

The first set of constraints are (4.3) above, with substituted expected output. The second set are non-negativity constraints on allocations.

Proposition 23. *The Lagrangian corresponding to the principal's maximization problem (4.4) is the following:*

$$\begin{aligned} S := \iint\limits_{[\underline{e},\bar{e}]^2} \Bigg\{ & \sum_{j,k} Pr[\theta_j] Pr[\theta_k] \left[R\left(q^a_{jk}(e,\tilde{e}) + q^b_{kj}(\tilde{e},e)\right) - (\theta_j + e) q^a_{jk}(e,\tilde{e}) - (\theta_k + \tilde{e}) q^b_{kj}(\tilde{e},e) \right] \\ & - \lambda^a \Delta \left[f^2 q^a_{hl}(e,\tilde{e}) + f(1-f) q^a_{hh}(e,\tilde{e}) \right] - \lambda^b \Delta \left[f^2 q^b_{hl}(\tilde{e},e) + f(1-f) q^b_{hh}(\tilde{e},e) \right] \\ & + \sum_{i,j,k} \mu^i_{jk}(e,\tilde{e}) q^i_{jk}(e,\tilde{e}) \Bigg\} g(e,\tilde{e}) ded\tilde{e} - v^a - v^b + \lambda^a v^a + \lambda^b v^b. \end{aligned} \tag{4.5}$$

Proof. Denote the Lagrangian multipliers for the liquidity constraints by λ^i, and consider multipliers $\mu^i_{jk}(e,\tilde{e})$ for each allocation values $q^i_{jk}(e,\tilde{e})$. Using (4.4) and the high type liquidity constraint (4.3), it follows from standard theory that the liquidity constraints can be incorporated into a Lagrangian function, and the non-negativity constraints can also be included. For details, see J.L. Troutman - "Variational Calculus and Optimal Control", 7.4. □

To characterize the solution, first consider the following pair of first order conditions (FOC) for the allocations q^i_{ll}, for a certain extra marginal cost draw $(e,\tilde{e})$:

$$\begin{aligned} f^2 \left[R'\left(q^a_{ll}(e,\tilde{e}) + q^b_{ll}(\tilde{e},e)\right) - (\theta_l + e) \right] + \mu^a_{ll}(e,\tilde{e}) = 0, \\ f^2 \left[R'\left(q^a_{ll}(e,\tilde{e}) + q^b_{ll}(\tilde{e},e)\right) - (\theta_l + \tilde{e}) \right] + \mu^b_{ll}(\tilde{e},e) = 0. \end{aligned} \tag{4.6}$$

Since $\lim_{q\to 0} R'(q) = \infty$, it must be that at least one of the allocations q^i_{ll} is not null. Because $\mu^i_{jk} \geqslant 0$, the only possible solution is:

$$\begin{cases} q^a_{ll} + q^b_{ll} = R'^{-1}\left(\theta_l + e\right), \text{ if } e = \tilde{e}, \\ q^b_{ll} = 0, q^a_{ll} = R'^{-1}\left(\theta_l + e\right), \text{ if } e < \tilde{e}, \\ q^a_{ll} = 0, q^b_{ll} = R'^{-1}\left(\theta_l + \tilde{e}\right), \text{ if } e > \tilde{e}. \end{cases} \tag{4.7}$$

Now consider another pair of FOC for the allocations q^a_{hl}, q^b_{lh}, and any $(e, \tilde{e})$:

$$\begin{aligned} &f(1-f)\left[R'\left(q^a_{hl}(e,\tilde{e}) + q^b_{lh}(\tilde{e},e)\right) - (\theta_h + e)\right] - \lambda^a \Delta f^2 + \mu^a_{hl}(e,\tilde{e}) = 0, \\ &f(1-f)\left[R'\left(q^a_{hl}(e,\tilde{e}) + q^b_{lh}(\tilde{e},e)\right) - (\theta_l + \tilde{e})\right] + \mu^b_{lh}(\tilde{e},e) = 0. \end{aligned} \tag{4.8}$$

Again, we need a nonzero allocation among the two, and positive multipliers, so as long as $\Delta > \overline{e} - \underline{e}$, the only way to satisfy the FOC is to have:

$$\begin{cases} q^a_{hl} = 0, \\ q^b_{lh} = R'^{-1}\left(\theta_l + \tilde{e}\right). \end{cases} \tag{4.9}$$

There is obviously a symmetric pair of equations where firms are switched. Finally, consider the pair of FOC:

$$\begin{aligned} &(1-f)^2\left[R'\left(q^a_{hh}(e,\tilde{e}) + q^b_{hh}(\tilde{e},e)\right) - (\theta_h + e)\right] - \lambda^a \Delta f(1-f) + \mu^a_{hh}(e,\tilde{e}) = 0, \\ &(1-f)^2\left[R'\left(q^a_{hh}(e,\tilde{e}) + q^b_{hh}(\tilde{e},e)\right) - (\theta_h + \tilde{e})\right] - \lambda^b \Delta f(1-f) + \mu^b_{hh}(\tilde{e},e) = 0. \end{aligned} \tag{4.10}$$

Considering again all the cases for whether μ^i_{hh} are positive or not, we deduce that:

$$\begin{cases} q^a_{hh} + q^b_{hh} = R'^{-1}\left(\theta_h + e + \lambda^a \Delta \frac{f}{1-f}\right), \text{ if } e + \lambda^a \Delta \frac{f}{1-f} = \tilde{e} + \lambda^b \Delta \frac{f}{1-f}, \\ q^b_{hh} = 0, q^a_{hh} = R'^{-1}\left(\theta_h + e + \lambda^a \Delta \frac{f}{1-f}\right), \text{ if } e + \lambda^a \Delta \frac{f}{1-f} < \tilde{e} + \lambda^b \Delta \frac{f}{1-f}, \\ q^a_{hh} = 0, q^b_{hh} = R'^{-1}\left(\theta_h + \tilde{e} + \lambda^b \Delta \frac{f}{1-f}\right), \text{ if } e + \lambda^a \Delta \frac{f}{1-f} > \tilde{e} + \lambda^b \Delta \frac{f}{1-f}. \end{cases} \tag{4.11}$$

Based on the values of $e, \tilde{e}, \lambda^i$, we see whether the principal allocates to firm a or b. Let's define the domain $\mathcal{D}(\lambda^a, \lambda^b) := \left\{(e, \tilde{e}) | \tilde{e} + \lambda^b > e + \lambda^a\right\}$, in which the principal allocates to firm a, when both firms report a high cost type. Because $q^i_{hl} = 0$, the values of λ^i can be obtained from the implicit equations:

$$\begin{aligned} v^a &= \Delta f(1-f) \iint\limits_{\mathcal{D}(\lambda^a, \lambda^b)} g(e, \tilde{e}) q^a_{hh}(e, \tilde{e}) de d\tilde{e}, \\ v^b &= \Delta f(1-f) \iint\limits_{[\underline{e}, \overline{e}]^2 \setminus \mathcal{D}(\lambda^a, \lambda^b)} g(e, \tilde{e}) q^b_{hh}(e, \tilde{e}) de d\tilde{e}. \end{aligned} \tag{4.12}$$

It is very easy to see that the area of $\mathcal{D}(\lambda^a, \lambda^b)$ is decreasing in λ^a and increasing in λ^b, and that $R'^{-1}\left(\theta_h + e + \frac{\lambda^i \Delta f}{1-f}\right)$ is decreasing in λ^i so, using (4.11), it follows immediately that v^i is decreasing in λ^i and increasing in $\lambda^{j \neq i}$.

If the principal can choose the optimal values v^i, he will choose them such that $\lambda^i = 1$, as we can see from FOC for v^i, which are:

$$\begin{aligned} -1 + \lambda^a &= 0, \\ -1 + \lambda^b &= 0. \end{aligned} \tag{4.13}$$

This means that optimal value for both v^i is:

$$\overline{v^a} = \overline{v^b} = \frac{1}{2} \Delta f(1-f) \iint\limits_{[\underline{e}, \overline{e}]^2} g(e, \tilde{e}) R'^{-1}\left(\theta_h + \frac{\Delta f}{1-f} + \min\{e, \tilde{e}\}\right) de d\tilde{e}. \tag{4.14}$$

To finish proving that the derivation above describes the solution, we must argue why the *ex post* liquidity constraints hold, since we have assumed interim liquidity constraints.

Lemma 24. *The ex post liquidity constraints for each firm hold, if the allocations are given by (4.7),(4.9) and (4.11).*

Proof. Now, since:

$$v^i \geqslant \Delta f \iint\limits_{[\underline{e},\overline{e}]^2} Q_h^i(e,\tilde{e})g(e,\tilde{e})ded\tilde{e} \geqslant 0,$$

set $m_{hk}^i(e,\tilde{e}) = (\theta_h + e)q_{hk}^i(e,\tilde{e})$, $m_{lk}^i(e,\tilde{e}) = (\theta_l + e)q_{lk}^i(e,\tilde{e}) + v^i/f$. We see then that $u_{jk}^i(e,\tilde{e}) \geqslant 0$ always, as required. $\square$

From now on, let's assume that v^i are given, and define the following value:

$$v^* := \frac{1}{2}\Delta f(1-f) \iint\limits_{[\underline{e},\overline{e}]^2} R'^{-1}\left(\theta_h + \min\{e,\tilde{e}\}\right) g(e,\tilde{e})ded\tilde{e}.$$

We want to know for what values v^b does the liquidity constraint for firm a become binding. For that, we must consider $\mathcal{D} := \mathcal{D}(0,\lambda^b)$. The area of $\mathcal{D}$ will be increasing in λ_b, and the allocations for b when both firms report high types are decreasing in λ^b. Therefore:

$$v^b = \Delta f(1-f) \iint\limits_{[\underline{e},\overline{e}]^2 \setminus \mathcal{D}} g(e,\tilde{e})R'^{-1}\left(\theta_h + \tilde{e} + \frac{\lambda^b \Delta f}{1-f}\right) ded\tilde{e}, \tag{4.15}$$

so v^b is strictly decreasing in λ^b. Define v^{a*} to be the value of v^a at which the liquidity constraint for firm a becomes non-binding, i.e., $\lambda_a = 0$:

$$v^{a*} := \Delta f(1-f) \iint\limits_{\mathcal{D}} g(e,\tilde{e})R'^{-1}\left(\theta_h + e\right) ded\tilde{e}.$$

$v^{a*}(\lambda^b)$ is strictly increasing in λ^b. As we increase v^b, λ^b must decrease, so the domain $\mathcal{D}$ will also decrease. Therefore v^{a*} is strictly decreasing in $v^b(\lambda^b)$, as long as $0 < R' < \infty$ for every possible allocation of the mechanism. We have a proper map from v^b to v^{a*}, and that map is also continuous, as a composition of continuous maps. Furthermore:

$$v^{a*}(0) = \Delta f(1-f) \int_{\underline{e}}^{\overline{e}} g(e)R'^{-1}(\theta_h + e)de. \tag{4.16}$$

Observe that, because $R'^{-1}(\theta_h + \min\{e, \tilde{e}\}) \geqslant R'^{-1}(\theta_h + e)$, $v^* \leqslant v^{a*}(v^b) < 2v^*$. $v^{a*}(v^b)$ will decrease from $v^{a*}(0)$ to $v^{a*}(v^*) = v^*$. From this point on, the liquidity constraint for the allocations of b will not bind anymore, so when the promised utility v^b is increased further, it translates into transfers from the principal to b, so we can conclude that v^{a*} won't be affected anymore and will stay constant at v^*. Figure (4.1) represents $v^{a*}(v^b)$.

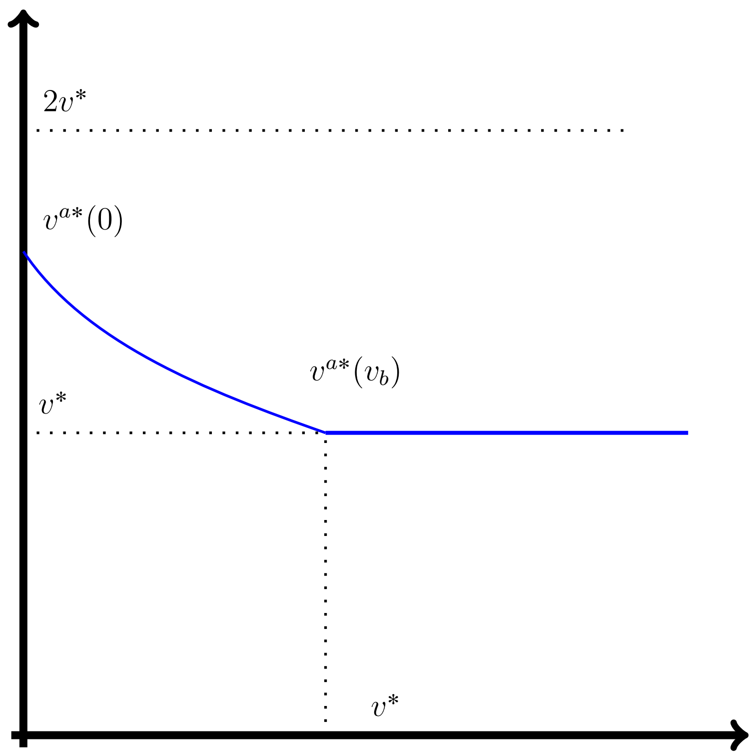

FIGURE 4.1: The parametrization of the ex ante value function for which the liquidity constraint becomes binding.

4.3 Interaction for an infinity of periods

We try to repeat the analysis in a more realistic setting in which firms interact repeatedly. $0 < \delta < 1$ is the intertemporal discount factor. To characterize the solution for the dynamic problem with repeated interaction, we use a recursive formulation.

For any one period, the setting is the same and all the variables defined before have the same meaning. In addition, we define $P(v^a, v^b)$ as the principal's value function, determined by the value functions of the individual firms. The principal's problem becomes:

$$P(v^a, v^b) = \max_{\{q^i_{jk}, w^i_{jk}\}} \iint\limits_{[\underline{e},\bar{e}]} \Big\{ \sum_{j,k} Pr[\theta_j] Pr[\theta_k] \big[R(q^a_{jk} + q^b_{kj}) - (\theta_j + e^a) q^a_{jk} - (\theta_k + e^b) q^b_{kj}$$

$$+\delta(P(w^a_{jk}, w^b_{kj}) + w^a_{jk} + w^b_{kj})\big]\Big\} g(e^a, e^b) de^a de^b - v^a - v^b, \tag{4.17}$$

$$\text{s.t. } \forall i, j, k, (e^a, e^b):$$

$$v^i \geqslant \iint [\Delta f^2 q^i_{hl} + \Delta f(1-f) q^i_{hh} + \delta f w^i_{hl} + \delta(1-f) w^i_{hh}] g(e^a, e^b) de^a de^b,$$

$$v^i \geqslant \iint [-\Delta f(1-f) q^i_{hl} - \Delta(1-f)^2 q^i_{hh} + \delta f w^i_{ll} + \delta(1-f) w^i_{lh}] g(e^a, e^b) de^a de^b,$$

$$q^i_{jk}(e^a, e^b) \geqslant 0.$$

Let λ^i_j be the liquidity constraint for firm i reporting j, let $\mu^i_{jk}(e^a, e^b)$ be the non-negativity constraint for $q^i_{jk}(e^a, e^b)$, and $\alpha^i := \Delta \frac{f}{1-f} \lambda^i_h - \Delta \lambda^i_l$. The Lagrangean formulation is:

$$S := \iint\limits_{[\underline{e},\bar{e}]^2} \Big\{ \sum_{j,k} Pr[\theta_j] Pr[\theta_k] \big[R\left(q^a_{jk}(e^a, e^b) + q^b_{kj}(e^a, e^b)\right) - (\theta_j + e) q^a_{jk}(e^a, e^b) -$$

$$-(\theta_k + e^b) q^b_{kj}(e^a, e^b)\big] +$$

$$+ \sum_i \big(\lambda^i_h \left[-\Delta f^2 q^i_{hl}(e^a, e^b) - \Delta f(1-f) q^i_{hh}(e^a, e^b) - \delta f w^i_{hl}(e^a, e^b) - \delta(1-f) w^i_{hh}(e^a, e^b) \right]$$

$$+ \lambda^i_l \left[\Delta f(1-f) q^i_{hl}(e^a, e^b) + \Delta(1-f)^2 q^i_{hh}(e^a, e^b) - \delta f w^i_{ll}(e^a, e^b) - \delta(1-f) w^i_{lh}(e^a, e^b) \right] \big)$$

$$+ \sum_{i,j,k} \mu^i_{jk}(e^a, e^b) q^i_{jk}(e^a, e^b) \Big\} g(e^a, e^b) de^a de^b - v^a - v^b + (\lambda^a_h + \lambda^b_l) v^a + (\lambda^a_h + \lambda^b_l) v^b. \tag{4.18}$$

The FOC that describe the allocations are:

$$f^2\left[R'\left(q^a_{ll}(e^a,e^b)+q^b_{ll}(e^a,e^b)\right)-(\theta_l+e^a)\right]+\mu^a_{ll}(e^a,e^b)=0,$$
$$f^2\left[R'\left(q^a_{ll}(e^a,e^b)+q^b_{ll}(e^a,e^b)\right)-(\theta_l+e^b)\right]+\mu^b_{ll}(e^a,e^b)=0. \tag{4.19}$$

$$f(1-f)\left[R'\left(q^a_{hl}(e^a,e^b)+q^b_{lh}(e^a,e^b)\right)-(\theta_h+e^a)\right]-f(1-f)\alpha^a+\mu^a_{hl}(e^a,e^b)=0,$$
$$f(1-f)\left[R'\left(q^a_{hl}(e^a,e^b)+q^b_{lh}(e^a,e^b)\right)-(\theta_l+e^b)\right]+\mu^b_{lh}(e^a,e^b)=0. \tag{4.20}$$

$$f(1-f)\left[R'\left(q^a_{lh}(e^a,e^b)+q^b_{hl}(e^a,e^b)\right)-(\theta_l+e^a)\right]+\mu^a_{lh}(e^a,e^b)=0,$$
$$f(1-f)\left[R'\left(q^a_{lh}(e^a,e^b)+q^b_{hl}(e^a,e^b)\right)-(\theta_h+e^b)\right]-f(1-f)\alpha^b+\mu^b_{hl}(e^a,e^b)=0. \tag{4.21}$$

$$(1-f)^2\left[R'\left(q^a_{hh}(e^a,e^b)+q^b_{hh}(e^a,e^b)\right)-(\theta_h+e^a)\right]-(1-f)^2\alpha^a+\mu^a_{hh}(e^a,e^b)=0,$$
$$(1-f)^2\left[R'\left(q^a_{hh}(e^a,e^b)+q^b_{hh}(e^a,e^b)\right)-(\theta_h+e^b)\right]-(1-f)^2\alpha^b+\mu^b_{hh}(e^a,e^b)=0. \tag{4.22}$$

For now, let's assume that $\alpha^i \geqslant 0$, which will be proven later. Then we can duplicate of analysis of the single period setting to describe the optimal allocations.

Proposition 25. *If $\alpha^i \geqslant 0$, then contingent on the public marginal cost components (e^a, e^b), the optimal allocations in the principal's problem (4.17) are given by:*

$$\begin{cases}
q^a_{ll}+q^b_{ll}=R'^{-1}\left(\theta_l+e^a\right), \text{ if } e^a=e^b,\\
q^b_{ll}=0, q^a_{ll}=R'^{-1}\left(\theta_l+e^a\right), \text{ if } e^a<e^b,\\
q^a_{ll}=0, q^b_{ll}=R'^{-1}(\theta_l+e^b), \text{ if } e^a>e^b,\\
q^i_{hl}=0 \text{ and } q^i_{lh}=R'^{-1}(\theta_l+e^i),\\
q^a_{hh}+q^b_{hh}=R'^{-1}\left(\theta_h+e^a+\alpha^a\right), \text{ if } e^a+\alpha^a=e^b+\alpha^b,\\
q^b_{hh}=0, q^a_{hh}=R'^{-1}\left(\theta_h+e^a+\alpha^a\right), \text{ if } e^a+\alpha^a<e^b+\alpha^b,\\
q^a_{hh}=0, q^b_{hh}=R'^{-1}(\theta_h+e^b+\alpha^b), \text{ if } e^a+\alpha^a>e^b+\alpha^b.
\end{cases} \tag{4.23}$$

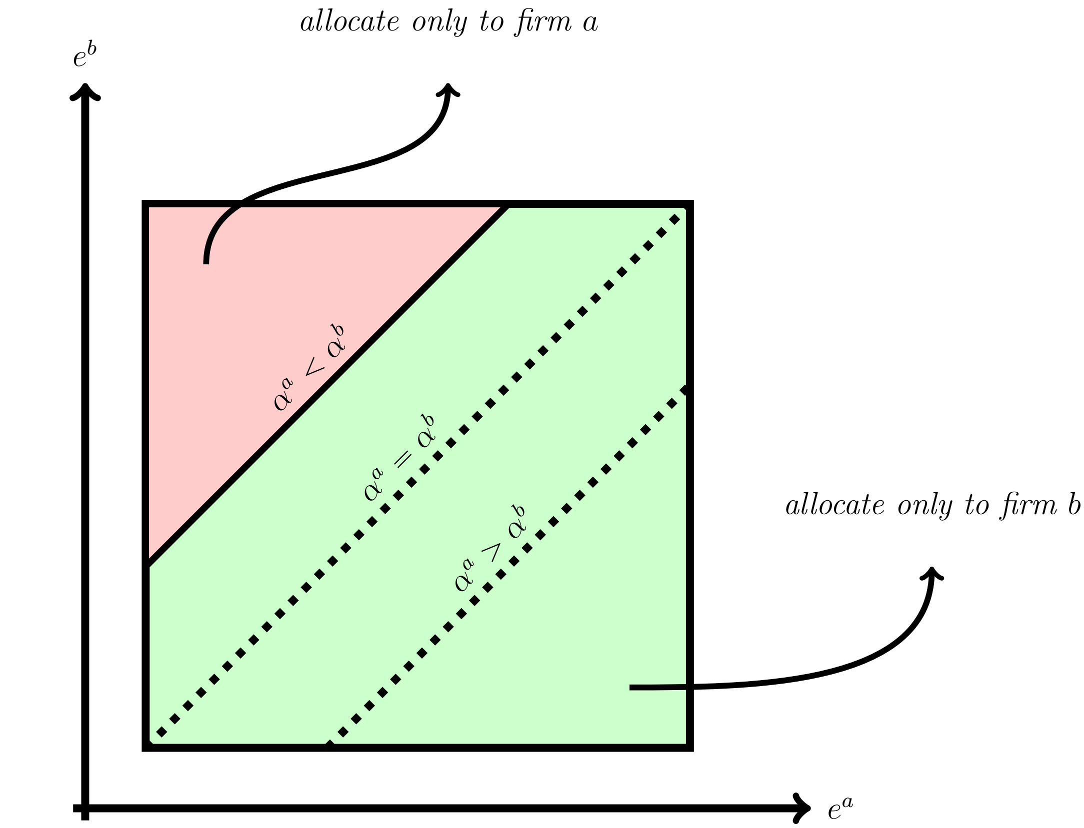

FIGURE 4.2: Allocations when both firms report θ_h, for different parameters (e^a, e^b). In the red region, the contract is allocated to firm a, in the green to b.

Proof. Considering the pairs of FOC for allocations derived from (4.18). The proof is a repetition of the analysis used to derive (4.7), (4.9), and (4.11). □

From (4.18) we can also extract the FOC governing the evolution of promised utility are:

$$\begin{aligned} &\delta f^2[P_a(w^a_{ll}, w^b_{ll}) + 1] - \lambda^a_l \delta f = 0, \\ &\delta f^2[P_b(w^a_{ll}, w^b_{ll}) + 1] - \lambda^b_l \delta f = 0, \end{aligned} \tag{4.24}$$

$$\begin{aligned} &\delta f(1-f)[P_a(w^a_{hl}, w^b_{lh}) + 1] - \lambda^a_h \delta f = 0, \\ &\delta f(1-f)[P_b(w^a_{hl}, w^b_{lh}) + 1] - \lambda^b_l \delta (1-f) = 0, \end{aligned} \tag{4.25}$$

$$\delta f(1-f)[P_a(w^a_{lh}, w^b_{hl}) + 1] - \lambda^a_l \delta(1-f) = 0,$$
$$\delta f(1-f)[P_b(w^a_{lh}, w^b_{hl}) + 1] - \lambda^b_h \delta f = 0, \tag{4.26}$$

$$\delta(1-f)^2[P_a(w^a_{hh}, w^b_{hh}) + 1] - \lambda^a_h \delta(1-f) = 0,$$
$$\delta(1-f)^2[P_b(w^a_{hh}, w^b_{hh}) + 1] - \lambda^b_h \delta(1-f) = 0, \tag{4.27}$$

$$P_a(v^a, v^b) = -1 + \lambda^a_h + \lambda^a_l,$$
$$P_b(v^a, v^b) = -1 + \lambda^b_h + \lambda^b_l. \tag{4.28}$$

$$\tag{4.29}$$

We see that, for $i \in \{a, b\}$:

$$\frac{\lambda^i_h}{1-f} = P_i(w^i_{hl}, w^j_{lh}) + 1 = P_i(w^i_{hh}, w^j_{hh}) + 1,$$
$$\frac{\lambda^i_l}{f} = P_i(w^i_{lh}, w^j_{hl}) + 1 = P_i(w^i_{ll}, w^j_{ll}) + 1. \tag{4.30}$$

From the FOC, we get:

$$\begin{aligned}
&P_a(w^a_{hl}, w^b_{lh}) = P_a(w^a_{hh}, w^b_{hh}),\ P_a(w^a_{ll}, w^b_{ll}) = P_a(w^a_{lh}, w^b_{hl}),\\
&P_b(w^a_{lh}, w^b_{hl}) = P_b(w^a_{hh}, w^b_{hh}),\ P_b(w^a_{ll}, w^b_{ll}) = P_b(w^a_{hl}, w^b_{lh}),\\
&P_a(v^a, v^b) = fP_a(w^a_{ll}, w^b_{ll}) + (1-f)P_a(w^a_{hl}, w^b_{lh})\\
&P_b(v^a, v^b) = fP_b(w^a_{ll}, w^b_{ll}) + (1-f)P_b(w^a_{lh}, w^b_{hl})
\end{aligned} \tag{4.31}$$

We can also see that the future firm values w^i_{jk} don't depend on (e^a, e^b). Since it will be shown that that $\alpha^i \geqslant 0$, we have P_i evolving as a martingale, and so we can also infer its convergence. Define $\mathcal{D}^a := \{(e^a, e^b) | e^b + \alpha^b > e^a + \alpha^a\}$, and $\mathcal{D}^b := [\underline{e}, \overline{e}]^2 \backslash \mathcal{D}^a$. Then we can write:

$$v^a = \Delta f(1-f) \iint\limits_{\mathcal{D}^a} R'^{-1}(\theta_h + e^a + \alpha^a) g(e^a, e^b) de^a de^b + \delta f w^a_{hl} + \delta(1-f) w^a_{hh},$$

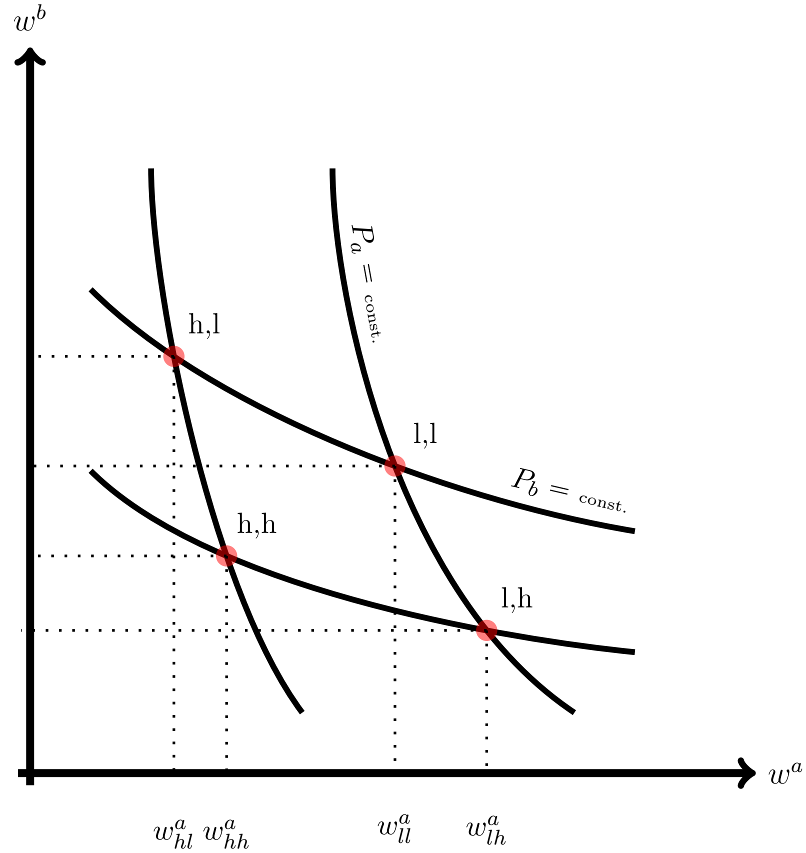

FIGURE 4.3: The evolution of the firms value function.

$$
\begin{aligned}
v^a &= -\Delta(1-f)^2 \iint_{\mathcal{D}^a} R'^{-1}(\theta_h + e^a + \alpha^a) g(e^a, e^b) de^a de^b + \delta f w^a_{ll} + \delta(1-f) w^a_{lh}, \\
v^b &= \Delta f(1-f) \iint_{\mathcal{D}^b} R'^{-1}(\theta_h + e^b + \alpha^b) g(e^a, e^b) de^a de^b + \delta f w^b_{hl} + \delta(1-f) w^b_{hh}, \\
v^b &= -\Delta(1-f)^2 \iint_{\mathcal{D}^b} R'^{-1}(\theta_h + e^b + \alpha^b) g(e^a, e^b) de^a de^b + \delta f w^b_{ll} + \delta(1-f) w^b_{lh}.
\end{aligned}
\tag{4.32}
$$

In principle, we can solve the liquidity constraints with equality for λ^i_j, then find the allocations, and then solve for w^i_{jk} from the liquidity constraints and the promise keeping conditions. Figure 4.3 represents a sketch for the evolution of the firms' future promised utility.

After we find the variables α^i, w^i_{jk}, we can write the following expression, which should verify either numerically or as an expression:

$$P(v^a, v^b) = \iint\limits_{[\underline{e},\overline{e}]} \Big\{ f^2 \left[R(R'^{-1}(\theta_l + \min\{e^a, e^b\})) - (\theta_l + \min\{e^a, e^b\}) R'^{-1}(\theta_l + \min\{e^a, e^b\}) \right.$$

$$\left. + \delta(P(w^a_{ll}, w^b_{ll}) + w^a_{ll} + w^b_{ll}) \right]$$

$$+ f(1-f) \left[R(R'^{-1}(\theta_l + e^a)) - (\theta_l + e^a) R'^{-1}(\theta_l + e^a) + \delta(P(w^a_{lh}, w^b_{hl}) + w^a_{lh} + w^b_{hl}) \right]$$

$$+ (1-f) f \left[R(R'^{-1}(\theta_l + e^b)) - (\theta_l + e^b) R'^{-1}(\theta_l + e^b) + \delta(P(w^a_{hl}, w^b_{lh}) + w^a_{hl} + w^b_{lh}) \right]$$

$$+ (1-f)^2 \left[R(R'^{-1}(\theta_h + \min\{e^a + \alpha^a, e^b + \alpha^b\})) - (\theta_h + \min\{e^a + \alpha^a, e^b + \alpha^b\}) \cdot \right.$$

$$\left. \cdot R'^{-1}(\theta_l + \min\{e^a + \alpha^a, e^b + \alpha^b\}) + \delta(P(w^a_{hh}, w^b_{hh}) + w^a_{hh} + w^b_{hh}) \right] \Big\} g(e^a, e^b) de^a de^b - v^a - v^b. \tag{4.33}$$

4.4 Proof that α^i is non-negative

In this section, we detail a proof that, in the infinite horizon model, the compound liquidity constraint of firm i, $\alpha^i := \Delta \frac{f}{1-f} \lambda^i_h - \Delta \lambda^i_l$, is non-negative. This was assumed in the previous section. For now, let's take for granted that there exists a globally concave principal value function $P(v^a, v^b)$ over the promised utilities to the firms a and b.

We know that P is globally concave if and only if its Hessian is negative semidefinite at every point in the domain. For the special case $P : \mathbb{R}^2 \to \mathbb{R}^2$, the Sylvester criterion gives:

$$P_{a,a} \leqslant 0, \quad \begin{vmatrix} P_{a,a} & P_{a,b} \\ P_{b,a} & P_{b,b} \end{vmatrix} = P_{a,a} P_{b,b} - P^2_{a,b} \geqslant 0. \tag{4.34}$$

This already gives $P_{b,b} \leqslant 0$. In the following, we will use the FOC for the infinite horizon recursive problem, expressed in equations (4.19) through (4.31).

If both liquidity constraints bind, we can deduce that $f w^i_{ll} + (1-f) w^i_{lh} \geqslant f w^i_{hl} + (1-f) w^i_{hh}$, with equality only when the information rent allocated is 0. From obvious efficiency considerations, it is intuitive that if one firm reports a high type and another a low type, we will always allocate to only the low type firm, since whatever the realizations of $e, \tilde{e}$ are, the low type firm will still have a lower marginal cost. This leads us to $\alpha^i \geqslant 0$, so we must have that for $i \in \{a, b\}$:

$$\frac{\lambda_l^i}{f} \leqslant \frac{\lambda_h^i}{1-f} \Leftrightarrow P_i(w_{lh}^i, w_{hl}^j) = P_i(w_{ll}^i, w_{ll}^j) \leqslant P_i(w_{hl}^i, w_{lh}^j) = P_i(w_{hh}^i, w_{hh}^j). \tag{4.35}$$

Proposition 26. *In (4.23), the compound liquidity constraint of firm i, $\alpha^i \geqslant 0$.*

Proof. The proof is by contradiction. *W.l.o.g.* say $\alpha^a < 0$, so that $P_a(w_{lh}^a, w_{hl}^b) = P_a(w_{ll}^a, w_{ll}^b) > P_a(w_{hl}^a, w_{lh}^b) = P_a(w_{hh}^a, w_{hh}^b)$.

The proof is an application of the implicit function theorem. However, the fact that we don't necessarily have strict concavity doesn't allow us to simply posit the implicit function, and we have to take a small detour into correspondences. For intuition, simply ignore the correspondence sections and jump right to the end.

Consider the set $\mathcal{S} := \{(v^a, v^b) | P_b(v^a, v^b) = P_b(w_{ll}^a, w_{ll}^b)\}$. Both points (w_{ll}^a, w_{ll}^b) and (w_{hl}^a, w_{lh}^b) are in this set, which is the graph of a correspondence $\Gamma(v^a)$. By the implicit function theorem, this correspondence is locally the graph of a function $v^b(v^a)$ if and only if $P_{b,b}(v^a, v^b(v^a)) \neq 0$.

We have that $P(v^a, v^b)$ is $\mathcal{C}^2$, so $P_b(v^a, v^b)$ is $\mathcal{C}^1$ and the set $\mathcal{S}$ is closed. So, by the closed graph theorem we have that $\Gamma(v^a)$ is upper hemicontinuous[2].

Lemma 27. *$\Gamma(v^a)$ is a closed interval for all v^a.*

Proof. $P_{b,b} \leqslant 0$, which means that $P_b(v_0^a, v^b)$ is continuous and non-increasing in v^b, so the set $\Gamma(v_0^a)$ is a closed interval. □

[2] Although it seems almost intuitive that Γ is also lower hemicontinuous, there will be portions of $\mathcal{S}$ that will be lower hemicontinuous, and portions that may not be, so we can't use the Michael selection theorem.

Lemma 28. *Define the correspondence* $\Pi(v^a) = \Gamma(v^a)$ *if* $\Gamma(v^a) \neq \varnothing$, *else* $\Pi(v^a) = \{0\}$. *Then the graph of* Π *is simply connected.*

Proof. This follows from the continuity of P_b and its monotonicity in v^b, as before. Imagine that there are disconnected pieces in the graph of Π. Then have a curve starting from bellow one of the pieces and ending above the other. Because of monotonicity, the initial value of P_b on the curve was lower and at the end it is higher. But by continuity it must intersect Γ's graph. Π is just a construction to deal with the fact that we live on the restricted domain $\mathbb{R}^2_+$. $\square$

With this claim, we know that there exists a piecewise differentiable curve in the graph of Π that connects (w^a_{ll}, w^b_{ll}) and (w^a_{hl}, w^b_{lh}), which we can construct from a single value selection of Π by adding vertical segments were needed. (We will obviously select a continuous function of v^a in the hemicontinuous sections of Π, by the Michael selection theorem, and one more very technical argument has to be made that there are only countably many discontinuities necessary.)

By the implicit function theorem, wherever the correspondence Γ is just one value, it is the graph of $v^b(v^a)$, therefore the piecewise differentiable curve selected is also the same graph. Moreover, $v^{b\prime}(v^a) = -\frac{P_{a,b}}{P_{b,b}}$, where $P_{b,b} \neq 0$ at those points. If the curve is not the graph of $v^b(v^a)$, then it must be that that $P_{b,b} = 0 \Rightarrow P_{a,b} = 0$ at those points, with the help of global concavity. For simplicity, extend $v^b(v^a)$ on the hemicontinuous portions of Π by the single valued selection, and parametrize the curve on the non-vertical parts with the help of $v^b(v^a)$. The parametrization of the vertical parts is irrelevant, because it will drop out in the following integral. With this in mind, it is without loss of generality to write:

$$\begin{aligned} P_a(w^a_{hl}, w^b_{lh}) = P_a(w^a_{ll}, w^b_{ll}) + \int_{w^a_{ll}}^{w^a_{hl}} \{\mathbb{1}_{[P_{b,b}=0]} P_{a,a}(v^a, v^b) + \\ + \mathbb{1}_{[P_{b,b}\neq 0]}[P_{a,a}(v^a, v^b) + P_{a,b}(v^a, v^b) v^{b\prime}(v^a)]\} dv^a. \end{aligned} \tag{4.36}$$

The integral on the vertical segments is 0 and was omitted, since we know that only the partial $P_{a,b}$ would matter, but that is 0 since $P_{b,b} = 0$ on the vertical segments. With the assumption $P_a(w^a_{hl}, w^b_{lh}) < P_a(w^a_{ll}, w^b_{ll})$, and after manipulating the expression, we get that:

$$\int_{w^a_{ll}}^{w^a_{hl}} \left\{ \mathbb{1}_{[P_{b,b}=0]} P_{a,a}(v^a, v^b) + \mathbb{1}_{[P_{b,b}\neq 0]} \frac{[P_{a,a}(v^a, v^b) P_{b,b}(v^a, v^b) + P_{a,b}(v^a, v^b)^2]}{P_{b,b}(v^a, v^b)} \right\} dv^a < 0. \tag{4.37}$$

The integrand is non-positive from concavity, so we must have that the integral is on a strictly positive interval, i.e., $w^a_{hl} > w^a_{ll}$. The proof that $w^a_{hh} > w^a_{lh}$ is the same. This leads us to $fw^i_{ll} + (1-f)w^i_{lh} < fw^i_{hl} + (1-f)w^i_{hh}$, which is a contradiction. □

4.5 Continuous types in one period

Having only two marginal cost types for firms is usually an unrealistic simplification, so it is important to check the robustness of the results when there are a multitude of types. In the discrete-private-type setting considered above we also introduced a publicly observed marginal cost addition e, which has two purposes. First, it makes the set-up more realistic, since not all cost information on a firm will be private, and second, it simplifies the mathematical analysis. The critical feature is the continuity of the distribution of $(e, \tilde{e})$, which ensures that the solution contract doesn't involve mixing. It is therefore natural to consider a slightly modified setting, in which the private costs are drawn from a continuous distribution, and we omit any public component to cost. As before, we start with a static two-firms procurement model with promise keeping, where the principal and the firms are risk neutral. The assumptions on the benefit function $R(q)$ are the same. Again, the two risk neutral firms, indexed by $\{a, b\}$, can supply the good in any quantity. The marginal cost of the firms, denoted by θ^i, is a private draw from $[\underline{\theta}, \overline{\theta}]$, where $\underline{\theta} \geqslant 0$. The draw is described by the distribution *cdf* $F(\theta)$, where the *pdf* $f(\theta)$ has full support over the domain. Furthermore, let $\frac{F(\theta)}{f(\theta)}$ be non-decreasing. This assumptions is not unjustified. Consider the distribution $f(\cdot)$ mirror reversed, and call it $h(\cdot)$. If $h(\cdot)$ has a non-decreasing hazard rate, then our assumption must also be true. The hazard rate assumption is quite commonly made to simplify to simplify problems with continuous type distributions. In our case, $h(\cdot)$ needs to have the non-decreasing hazard rate property instead of $f(\cdot)$ because the problem is set up in such a way that the lower θ gets the higher rents.

This time, the firms receive an *ex post* payoff of $u^i(\theta,\tilde{\theta}) := m^i(\theta,\tilde{\theta}) - \theta q^i(\theta,\tilde{\theta})$ in the case of truthful reporting, where the superindex $i \in \{a,b\}$ represents the firm, while the first and second arguments are the firms' own draw for marginal cost, and the competitions' draw. $m^i(\theta,\tilde{\theta})$ are monetary transfers to the firms, and since firms will be facing *ex post* liquidity constraints $u^i(\theta,\tilde{\theta}) \geqslant 0$, we must have that $m^i(\theta,\tilde{\theta}) \geqslant 0$. Define the interim expected payoff after a report of θ for firm i:

$$U^i(\theta) := \int_{\underline{\theta}}^{\overline{\theta}} u^i(\theta,\tilde{\theta}) f(\tilde{\theta}) d\tilde{\theta},$$

and from this the expected promised utility for firm i:

$$v^i := \iint\limits_{[\underline{\theta},\overline{\theta}]^2} u^i(\theta,\tilde{\theta}) f(\theta) f(\tilde{\theta}) d\theta d\tilde{\theta} = \int_{\underline{\theta}}^{\overline{\theta}} U^i(\theta) f(\theta) d\theta.$$

The expected output for firm i with marginal cost θ is:

$$Q^i(\theta) := \int_{\underline{\theta}}^{\overline{\theta}} q^i(\theta,\tilde{\theta}) f(\tilde{\theta}) d\tilde{\theta}.$$

An interim monotonicity constraint can be expressed as $\theta_h > \theta_l \Rightarrow Q^i(\theta_h) \leqslant Q^i(\theta_l)$. An interim incentive constraint is:

$$U^i(\theta) \geqslant \int_{\underline{\theta}}^{\overline{\theta}} [m^i(\theta',\tilde{\theta}) - \theta q^i(\theta',\tilde{\theta})] f(\tilde{\theta}) d\tilde{\theta}, \text{ for all } \theta'. \tag{4.38}$$

Rewriting $U^i(\theta)$, the above is

$$\Leftrightarrow \int_{\underline{\theta}}^{\overline{\theta}} [m^i(\theta,\tilde{\theta}) - \theta q^i(\theta,\tilde{\theta}) - m^i(\theta',\tilde{\theta}) + \theta q^i(\theta',\tilde{\theta})] f(\tilde{\theta}) d\tilde{\theta} \geqslant 0, \text{ for all } \theta'$$

$$\Leftrightarrow U^i(\theta) \geqslant U^i(\theta') + (\theta' - \theta) Q^i(\theta'), \text{ for all } \theta'.$$

By the envelope theorem, this implies $U^{i\prime}(\theta) = -Q^i(\theta)$.

$$\Rightarrow U^i(\theta) = U^i(\overline{\theta}) + \int_{\theta}^{\overline{\theta}} Q^i(\tilde{\theta}) d\tilde{\theta}$$

$$\Rightarrow v^i = U^i(\overline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} Q^i(\theta) F(\theta) d\theta \tag{4.39}$$

$$\Rightarrow U^i(\theta) = v^i - \int_{\underline{\theta}}^{\overline{\theta}} Q^i(\theta) F(\theta) d\theta + \int_{\theta}^{\overline{\theta}} Q^i(\tilde{\theta}) d\tilde{\theta}.$$

We assume the interim liquidity constraints, $U^i(\theta) \geqslant 0$ for all θ, but in the end we must make sure that the ex-post liquidity constraints $u^i(\theta,\tilde{\theta}) \geqslant 0$ hold.Because $Q^i(\theta) \geqslant 0$, we can see from the previous equations that the necessary and sufficient condition is $U^i(\overline{\theta}) \geqslant 0$, which is equivalent with $v^i \geqslant \int_{\underline{\theta}}^{\overline{\theta}} Q^i(\theta) F(\theta) d\theta$.

The principal's maximization problem can be stated as:

$$\max_{\{q^i(\cdot,\cdot)\}} \iint\limits_{[\underline{\theta},\overline{\theta}]^2} \left[R\left(q^a(\theta^a,\theta^b) + q^b(\theta^b,\theta^a)\right) - \theta^a q^a(\theta^a,\theta^b) - \theta^b q^b(\theta^b,\theta^a) \right] f(\theta^a) f(\theta^b) d\theta^a d\theta^b - v^a - v^b, \tag{4.40}$$

$$\text{s.t. } \forall i,\theta,\tilde{\theta}: \; v^i \geqslant \int_{\underline{\theta}}^{\overline{\theta}} Q^i(\theta) F(\theta) d\theta, \quad q^i(\theta,\tilde{\theta}) \geqslant 0.$$

Proposition 29. *The Lagrangian corresponding to the principal's maximization problem for one period, (4.40), is the following:*

$$S := \iint\limits_{[\underline{\theta},\overline{\theta}]^2} \left[R\left(q^a(\theta^a,\theta^b)+q^b(\theta^b,\theta^a)\right)-\theta^a q^a(\theta^a,\theta^b)-\theta^b q^b(\theta^b,\theta^a)\right] f(\theta^a)f(\theta^b)d\theta^a d\theta^b - v^a - v^b$$

$$+ \lambda^a \left[v^a - \iint\limits_{[\underline{\theta},\overline{\theta}]^2} q^a(\theta^a,\theta^b)F(\theta^a)f(\theta^b)d\theta^a d\theta^b \right] + \lambda^b \left[v^b - \iint\limits_{[\underline{\theta},\overline{\theta}]^2} q^b(\theta^b,\theta^a)f(\theta^a)F(\theta^b)d\theta^a d\theta^b \right]$$

$$+ \iint\limits_{[\underline{\theta},\overline{\theta}]^2} \left[\mu^a(\theta^a,\theta^b)q^a(\theta^a,\theta^b)+\mu^b(\theta^b,\theta^a)q^b(\theta^b,\theta^a)\right] f(\theta^a,\theta^b)d\theta^a d\theta^b. \tag{4.41}$$

Proof. Denote the Lagrangian multipliers for the liquidity constraints by λ^i, and consider multipliers $\mu^i(\theta,\tilde{\theta})$ for each allocation value $q^i(\theta,\tilde{\theta})$. It follows from standard theory that the liquidity constraints can be incorporated into a Lagrangian function, and the non-negativity constraints can also be included as follows (for details, see J.L. Troutman - "Variational Calculus and Optimal Control", 7.4.). □

The FOC and envelope conditions for generic draws of (θ^a,θ^b) will be

$$\begin{aligned}
&[q^a]: \quad f(\theta^a)f(\theta^b)\left[R'\left(q^a(\theta^a,\theta^b)+q^b(\theta^b,\theta^a)\right)-\theta^a+\mu^a(\theta^a,\theta^b)\right]-\lambda^a F(\theta^a)f(\theta^b)=0,\\
&[q^b]: \quad f(\theta^a)f(\theta^b)\left[R'\left(q^a(\theta^a,\theta^b)+q^b(\theta^b,\theta^a)\right)-\theta^b+\mu^b(\theta^b,\theta^a)\right]-\lambda^b f(\theta^a)F(\theta^b)=0,\\
&[\text{Env.1}]: \quad P_a(v^a,v^b)=-1+\lambda^a,\\
&[\text{Env.2}]: \quad P_b(v^a,v^b)=-1+\lambda^b.
\end{aligned} \tag{4.42}$$

To characterize the solution, consider the pair of first order conditions for q^a, q^b, for a certain marginal cost draw (θ^a,θ^b). We must have that $\mu^a(\theta^a,\theta^b)q^a(\theta^a,\theta^b) = \mu^b(\theta^b,\theta^a)q^b(\theta^b,\theta^a) = 0$, so it's not possible that $\mu^a \neq 0 \neq \mu^b$ at the same time, because both allocations q^a, q^b would be 0, and the marginal benefit of increasing either would be infinite. For $\mu^a = \mu^b = 0$,

$$\Rightarrow \begin{cases} R'\left(q^a + q^b\right) = \theta^a + \lambda^a \frac{F(\theta^a)}{f(\theta^a)} \\ R'\left(q^a + q^b\right) = \theta^b + \lambda^b \frac{F(\theta^b)}{f(\theta^b)}, \end{cases} \Rightarrow \theta^a + \lambda^a \frac{F(\theta^a)}{f(\theta^a)} = \theta^b + \lambda^b \frac{F(\theta^b)}{f(\theta^b)}. \tag{4.43}$$

For $\mu^a > 0, \mu^b = 0$, we must have that $q^a = 0$, so $q^b > 0$. Moreover,

$$R'(q^b) = \theta^b + \lambda^b \frac{F(\theta^b)}{f(\theta^b)} \text{ and } \theta^a + \lambda^a \frac{F(\theta^a)}{f(\theta^a)} > \theta^b + \lambda^b \frac{F(\theta^b)}{f(\theta^b)}. \tag{4.44}$$

Vice-versa, for $\mu^a = 0, \mu^b > 0$,

$$R'(q^a) = \theta^a + \lambda^a \frac{F(\theta^a)}{f(\theta^a)} \text{ and } \theta^a + \lambda^a \frac{F(\theta^a)}{f(\theta^a)} < \theta^b + \lambda^b \frac{F(\theta^b)}{f(\theta^b)}. \tag{4.45}$$

The equations (4.43) through (4.45) implicitly determine the allocations, if we know the expression of $R(\cdot)$. We can interpret them considering that the square domain $[\underline{\theta}, \overline{\theta}]^2$ is partitioned into two regions. There is a draw of (θ^a, θ^b), which is a point in this domain. If it falls into the region where firm a has a smaller value for $\theta + \lambda \frac{F(\theta)}{f(\theta)}$, then firm a will get the entire allocation, and firm b nothing. Similarly for the other region.

Now let's describe the boundary between the two regions. This is important because, to completely solve the problem, one needs to integrate the allocations over the entire domain, to be able to employ the liquidity constraint equation. Define $l^a(\theta^b)$ to be the upper limit value of θ^a at which the principal still allocates to firm a, or $\underline{\theta}$ if the upper limit does not exist. Then we must have that

$$l^a + \lambda^a \frac{F(l^a)}{f(l^a)} := \theta^b + \lambda^b \frac{F(\theta^b)}{f(\theta^b)}. \tag{4.46}$$

Because the function $\cdot + \lambda \frac{F(\cdot)}{f(\cdot)}$ is strictly increasing, it is therefore invertible so $l^a(\cdot)$ exists, and we must have that l^a is strictly increasing in θ^b. $l^b(\theta^a)$ is similarly defined,

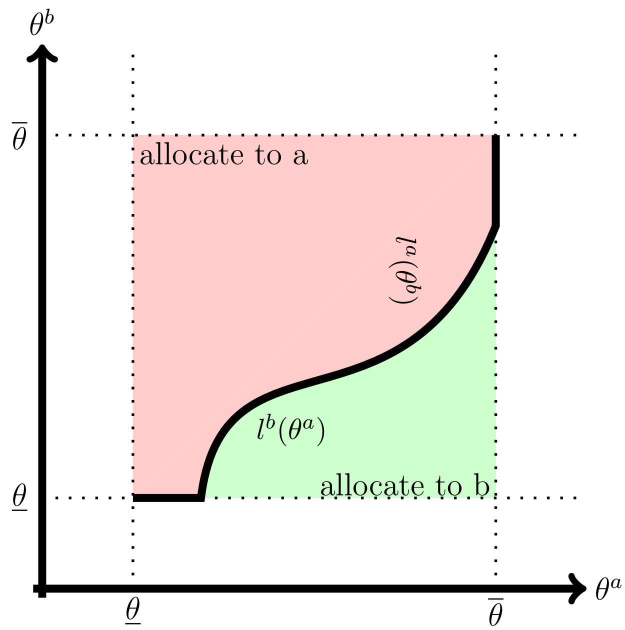

FIGURE 4.4: The red parameter region is where the contract is allocated to a, the green for b.

and it must be that $l^a(l^b(\theta)) = \theta$. Visually, the graph of $l^a(\theta^b)$ divides the domain $[\underline{\theta}, \overline{\theta}]^2$ into two connected open sets for each firm, and for all the pairs (θ^a, θ^b) in the interior of one set, the principal allocates to only one firm. For a representation, see Figure 4.4. It's also easy to see that l^a is strictly increasing in λ^b and strictly decreasing in λ^a.

With this, we can write

$$Q^i(\theta^i) = \int_{l^j(\theta^i)}^{\overline{\theta}} R'^{-1}\left(\theta^i + \lambda^i \frac{F(\theta^i)}{f(\theta^i)}\right) f(\theta^j) d\theta^j \tag{4.47}$$

$$\begin{aligned} v^i = \int_{\underline{\theta}}^{\overline{\theta}} Q^i(\theta^i) F(\theta^i) d\theta^i = \int_{\underline{\theta}}^{\overline{\theta}} \int_{l^j(\theta^i)}^{\overline{\theta}} R'^{-1}\left(\theta^i + \lambda^i \frac{F(\theta^i)}{f(\theta^i)}\right) F(\theta^i) f(\theta^j) d\theta^i d\theta^j = \\ = \int_{\underline{\theta}}^{\overline{\theta}} R'^{-1}\left(\theta^i + \lambda^i \frac{F(\theta^i)}{f(\theta^i)}\right) F(\theta^i) \left(1 - F(l^j(\theta^i))\right) d\theta^i. \end{aligned} \tag{4.48}$$

With the assumptions that $\frac{F(\theta)}{f(\theta)}$ is non-decreasing, we can check the monotonicity of $Q^i(\theta^i)$. First, let $\theta_h > \theta_l$ be two marginal cost values. Because R'^{-1} is strictly decreasing and $\cdot + \lambda \frac{F(\cdot)}{f(\cdot)}$ strictly increasing, $R'^{-1}\left(\theta_h + \lambda \frac{F(\theta_h)}{f(\theta_h)}\right) < R'^{-1}\left(\theta_h + \lambda \frac{F(\theta_h)}{f(\theta_h)}\right)$. Also, $l^i(\theta_h) > l^i(\theta_l)$. So, with a smaller integrand, integrated on a smaller domain, it follows that $Q^i(\theta_h) \leqslant Q^i(\theta_l)$.

To completely solve the problem, we would need to find λ^i as functions of v^i. It's not clear to me how this can be done without more assumptions on $f(\cdot), R(\cdot)$, but other nonspecific results may not need anything.

4.6 Continuous types in more than one period

Now we consider the considerably more complicated problem of repeated interaction between the principal and the firms. We will determine equations for allocations considering two periods, but the same formulas can be used to solve recursively when there are an arbitrary but finite number of periods. For two periods, add time subindices 1,2. In the beginning v_1^a, v_1^b are given, and v_2^a, v_2^b are determined endogenously. Define

$$V_1^i(\theta) := U_1^i(\theta) + \int_{\underline{\theta}}^{\overline{\theta}} \delta v_2^i(\theta, \tilde{\theta}) f(\tilde{\theta}) d\tilde{\theta}, \quad W^i(\theta) := \int_{\underline{\theta}}^{\overline{\theta}} v_2^i(\theta, \tilde{\theta}) f(\tilde{\theta}) d\tilde{\theta}.$$

As in the previous section, it's easy to see that

$$V_1^{i\prime}(\theta) = -Q_1^i(\theta) \ \Rightarrow V_1^i(\theta) = V_1^i(\overline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} Q_1^i(\tilde{\theta}) d\tilde{\theta},$$

and

$$v^i = \int_{\underline{\theta}}^{\overline{\theta}} V_1^i(\theta) f(\theta) d\theta = V_1^i(\overline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} Q_1^i(\theta) F(\theta) d\theta. \tag{4.49}$$

Replace $V_1^i(\overline{\theta})$ from the first equation into the second, and obtain that

$$U_1^i(\theta) = v_1^i - \int_{\underline{\theta}}^{\overline{\theta}} Q_1^i(\tilde{\theta})F(\tilde{\theta})d\tilde{\theta} + \int_{\theta}^{\overline{\theta}} Q_1^i(\tilde{\theta})d\tilde{\theta} - \delta W^i(\theta).$$

The interim liquidity constraint is $U_1^i(\theta) \geqslant 0, \forall\theta$. Then, the necessary and sufficient conditions to satisfy the liquidity constraints are

$$v_1^i - \delta W^i(\theta) - \int_{\underline{\theta}}^{\overline{\theta}} Q_1^i(\tilde{\theta})F(\tilde{\theta})d\tilde{\theta} + \int_{\theta}^{\overline{\theta}} Q_1^i(\tilde{\theta})d\tilde{\theta} \geqslant 0. \tag{4.50}$$

Let $P_2(v_2^a, v_2^b)$ be the principal's value function in the second (last) period, expressed as a function of the valuations of the two firms, and let $P_1(v_1^a, v_1^b)$ be the value function in period 1, which depends on the value function in period 2. The principal's recursive two-period maximization problem can be stated as:

$$\begin{multline} P_1(v_1^a, v_1^b) = \max_{\{q_1^i, v_2^i(\theta^a,\theta^b)\}} \iint\limits_{[\underline{\theta},\overline{\theta}]^2} \left[R\left(q_1^a(\theta^a,\theta^b) + q_1^b(\theta^b,\theta^a)\right) - \theta^a q_1^a(\theta^a,\theta^b) - \theta^b q_1^b(\theta^b,\theta^a) + \right. \\ \left. + \delta P_2(v_2^a(\theta^a,\theta^b), v_2^b(\theta^a,\theta^b)) + \delta v_2^a(\theta^a,\theta^b) + \delta v_2^b(\theta^a,\theta^b)\right] f(\theta^a)f(\theta^b)d\theta^a d\theta^b - v_1^a - v_1^b, \end{multline} \tag{4.51}$$

$$\text{s.t. } \forall i, \theta, \tilde{\theta}: \; v_1^i - \delta W^i(\theta) - \int_{\underline{\theta}}^{\overline{\theta}} Q_1^i(\tilde{\theta})F(\tilde{\theta})d\tilde{\theta} + \int_{\theta}^{\overline{\theta}} Q_1^i(\tilde{\theta})d\tilde{\theta} \geqslant 0, \quad q_1^i(\theta,\tilde{\theta}) \geqslant 0.$$

$$\begin{multline} P_2(v_2^a, v_2^b) = \max_{\{q_2^i\}} \iint\limits_{[\underline{\theta},\overline{\theta}]^2} \left[R\left(q_2^a(\theta^a,\theta^b) + q_2^b(\theta^b,\theta^a)\right) - \theta^a q_2^a(\theta^a,\theta^b) - \theta^b q_2^b(\theta^b,\theta^a)\right] \cdot \\ \cdot f(\theta^a)f(\theta^b)d\theta^a d\theta^b - v_2^a - v_2^b, \end{multline} \tag{4.52}$$

$$\text{s.t. } \forall i, \theta, \tilde{\theta}: \; v_2^i \geqslant \int_{\underline{\theta}}^{\overline{\theta}} Q_2^i(\theta)F(\theta)d\theta \quad q_2^i(\theta,\tilde{\theta}) \geqslant 0.$$

The second (or last) period behavior is known from the analysis of the one period game. For the first period, we need to solve a maximization problem with an integral constraint. Getting a workable Lagrangian is convoluted, but the idea is to end up with function Lagrange multipliers, $\lambda^i(\theta^i)$, inside the double integral.

Proposition 30. *The Lagrangian corresponding to the principal's maximization problem for two periods, (4.51) and (4.52), can be written as follows:*

$$
\begin{aligned}
S_1 := \iint\limits_{[\underline{\theta},\bar{\theta}]^2} & \left[R\left(q_1^a(\theta^a,\theta^b) + q_1^b(\theta^b,\theta^a)\right) - \theta^a q_1^a(\theta^a,\theta^b) - \theta^b q_1^b(\theta^b,\theta^a)\right. \\
& \left. +\delta P_2(v_2^a(\theta^a,\theta^b), v_2^b(\theta^a,\theta^b)) + \delta v_2^a(\theta^a,\theta^b) + \delta v_2^b(\theta^a,\theta^b)\right] f(\theta^a)f(\theta^b)d\theta^a d\theta^b - v_1^a - v_1^b \\
& + \iint\limits_{[\underline{\theta},\bar{\theta}]^2} \left[\mu^a(\theta^a,\theta^b)q_1^a(\theta^a,\theta^b) + \mu^b(\theta^b,\theta^a)q_1^b(\theta^b,\theta^a)\right] f(\theta^a,\theta^b)d\theta^a d\theta^b \\
& + v_1^a \int_{\underline{\theta}}^{\bar{\theta}} \lambda^a(\theta)f(\theta)d\theta + \iint\limits_{[\underline{\theta},\bar{\theta}]^2} \left[-q_1^a(\theta^a,\theta^b)\frac{F(\theta^a)}{f(\theta^a)} \cdot \int_{\underline{\theta}}^{\bar{\theta}} \lambda^a(\theta)f(\theta)d\theta\right. \\
& \left. + \frac{q_1^a(\theta^a,\theta^b)}{f(\theta^a)} \int_{\underline{\theta}}^{\theta^a} \lambda^a(\theta)f(\theta)d\theta - \delta\lambda^a(\theta^a)v_2^a(\theta^a,\theta^b)\right] f(\theta^a)f(\theta^b)d\theta^a d\theta^b \\
& + v_1^b \int_{\underline{\theta}}^{\bar{\theta}} \lambda^b(\theta)f(\theta)d\theta + \iint\limits_{[\underline{\theta},\bar{\theta}]^2} \left[-q_1^b(\theta^b,\theta^a)\frac{F(\theta^b)}{f(\theta^b)} \cdot \int_{\underline{\theta}}^{\bar{\theta}} \lambda^b(\theta)f(\theta)d\theta\right. \\
& \left. + \frac{q_1^b(\theta^b,\theta^a)}{f(\theta^b)} \int_{\underline{\theta}}^{\theta^b} \lambda^b(\theta)f(\theta)d\theta - \delta\lambda^b(\theta^b)v_2^b(\theta^b,\theta^a)\right] f(\theta^a)f(\theta^b)d\theta^a d\theta^b.
\end{aligned}
\tag{4.53}
$$

Proof. First start by writing the constraints according to the theory into a generic Lagrangian formulation:

$$
\begin{aligned}
S_1 := \iint\limits_{[\underline{\theta},\bar{\theta}]^2} & \left[R\left(q_1^a(\theta^a,\theta^b) + q_1^b(\theta^b,\theta^a)\right) - \theta^a q_1^a(\theta^a,\theta^b) - \theta^b q_1^b(\theta^b,\theta^a)\right. \\
& \left. +\delta P_2(v_2^a(\theta^a,\theta^b), v_2^b(\theta^a,\theta^b)) + \delta v_2^a(\theta^a,\theta^b) + \delta v_2^b(\theta^a,\theta^b)\right] f(\theta^a)f(\theta^b)d\theta^a d\theta^b - v_1^a - v_1^b
\end{aligned}
$$

$$
+ \iint\limits_{[\underline{\theta},\overline{\theta}]^2} \left[\mu^a(\theta^a,\theta^b)q_1^a(\theta^a,\theta^b) + \mu^b(\theta^b,\theta^a)q_1^b(\theta^b,\theta^a)\right] f(\theta^a,\theta^b)d\theta^a d\theta^b
$$

$$
+ \int_{\underline{\theta}}^{\overline{\theta}} \lambda^a(\theta^a) \left[v_1^a - \iint\limits_{[\underline{\theta},\overline{\theta}]^2} q^a(\theta^a,\theta^b)F(\theta^a)f(\theta^b)d\theta^a d\theta^b + \int_{\theta^a}^{\overline{\theta}} Q_1^a(\tilde{\theta})d\tilde{\theta} - \right.
$$

$$
\left. -\delta \int_{\underline{\theta}}^{\overline{\theta}} v_2^a(\theta^a,\tilde{\theta})f(\tilde{\theta})d\tilde{\theta} \right] f(\theta^a)d\theta^a
$$

$$
+ \int_{\underline{\theta}}^{\overline{\theta}} \lambda^b(\theta^b) \left[v_1^b - \iint\limits_{[\underline{\theta},\overline{\theta}]^2} q^b(\theta^b,\theta^a)f(\theta^a)F(\theta^b)d\theta^a d\theta^b + \int_{\theta^b}^{\overline{\theta}} Q_1^b(\tilde{\theta})d\tilde{\theta} - \right.
$$

$$
\left. -\delta \int_{\underline{\theta}}^{\overline{\theta}} v_2^b(\theta^b,\tilde{\theta})f(\tilde{\theta})d\tilde{\theta} \right] f(\theta^b)d\theta^b.
$$

After some reworking and changing integral positions, we can get the expression required. □

The FOC and envelope conditions for generic draws of (θ^a,θ^b) will be

$$
\begin{aligned}
&[q_1^a]: \quad R'\left(q_1^a(\theta^a,\theta^b) + q_1^b(\theta^b,\theta^a)\right) - \theta^a + \mu^a(\theta^a,\theta^b) - \\
&\qquad - \frac{F(\theta^a)}{f(\theta^a)} \int_{\underline{\theta}}^{\overline{\theta}} \lambda^a(\theta)f(\theta)d\theta + \frac{1}{f(\theta^a)} \int_{\underline{\theta}}^{\theta^a} \lambda^a(\theta)f(\theta)d\theta = 0, \\
&[q_1^b]: \quad R'\left(q_1^a(\theta^a,\theta^b) + q_1^b(\theta^b,\theta^a)\right) - \theta^b + \mu^b(\theta^b,\theta^a) - \\
&\qquad - \frac{F(\theta^b)}{f(\theta^b)} \int_{\underline{\theta}}^{\overline{\theta}} \lambda^b(\theta)f(\theta)d\theta + \frac{1}{f(\theta^b)} \int_{\underline{\theta}}^{\theta^b} \lambda^b(\theta)f(\theta)d\theta = 0, \\
&[v_2^a]: \quad P_{2,a}(v_2^a(\theta^a,\theta^b), v_2^b(\theta^a,\theta^b)) = -1 + \lambda^a(\theta^a), \\
&[v_2^b]: \quad P_{2,b}(v_2^b(\theta^b,\theta^a), v_2^a(\theta^b,\theta^a)) = -1 + \lambda^b(\theta^b), \\
&[\text{Env.1}]: \quad P_{1,a}(v_1^a,v_1^b) = -1 + \int_{\underline{\theta}}^{\overline{\theta}} \lambda^a(\theta)f(\theta)d\theta, \\
&[\text{Env.2}]: \quad P_{1,b}(v_1^a,v_1^b) = -1 + \int_{\underline{\theta}}^{\overline{\theta}} \lambda^b(\theta)f(\theta)d\theta.
\end{aligned}
\tag{4.54}
$$

It's not hard to notice that these conditions are very similar to the FOC in Krishna & Taylor (2011), except for having integrals instead of sums. This parallel lends credence to the result.

From the FOC (4.54), and with arguments very similar to the ones used for the one period problem, we can derive the allocations. For firm a, the allocations in the first round is as follows:

$$q_1^a(\theta^a, \theta^b) =$$

$$= \begin{cases} R'^{-1}\left(\theta^a + \frac{1-F(\theta^a)}{f(\theta^a)}\mathbb{E}_\theta[\lambda^a(\theta)]\right), \text{ if } \theta^a + \frac{1-F(\theta^a)}{f(\theta^a)}\mathbb{E}_\theta[\lambda^a(\theta)] \leqslant \theta^b + \frac{1-F(\theta^b)}{f(\theta^b)}\mathbb{E}_\theta[\lambda^b(\theta)], \\ 0, \text{ else.} \end{cases} \tag{4.55}$$

The allocation for firm b is given by the same equation, if we swap indices. Once more, we have a situation where the square domain $[\underline{\theta}, \overline{\theta}]^2$ is partitioned by the principal into two regions, as in Figure 4.4. There is a draw of (θ^a, θ^b), which falls into this domain, in one of the regions. In the region where firm a has a smaller marginal cost, it will get the entire allocation, possibly distorted by the liquidity constraints. In the other region, firm b will get the entire allocation. The formulas derived so far for the first period do not require any assumptions on the value function of the second period, so they apply to recursive settings with any number of periods.

4.7 Conclusion

This chapter explores the question of how a principal might formulate an optimal contract when interacting repeatedly with two liquidity constrained agents that have private information on their marginal costs. We presented two models to that end, and we reached allocation solutions that are similar. In both models, the quantity is allocated to the firm with the lowest marginal cost. Moreover, in both cases there are no distortions on the first best allocations when the firms report the lowest costs, and individual rationality is ensured by reducing the quantity supplied by firms reporting a higher marginal cost. In the discrete time model, reporting the high cost type leads to an allocation that is lower than the first best while in the continuous type model the allocations are progressively reduced. These distortions are a consequence of the liquidity constrains that the firms face, and the degree to which these constraints distort the allocations is dependent on the promised utility to the firms. Higher promised utilities lead to lower distortions. Like in Krishna & Taylor (2011), there is a critical region in which the liquidity constraints do not bind and agents receive only promised utility, a region in which the firms have enough promised utility so that the liquidity constraints will not bind and they can buy out the principal, but now there are two more regions in which only one firm has enough equity. Navigating the boundary of these regions is an open question and the goal of improving on these results.

The evolution of the promised utilities in the continuous type model is not describable explicitly, both because of the large space of parameters and because we have left the distribution of types unspecified. The allocations themselves depend on the values of the liquidity constraints, the specific type distribution of types and the function that describes the production technology, so in principle we can use backward induction to solve the implicit relations once we parametrize the model. In the discrete type model, we can obtain one more result that describes the value function. As in the previous work with one agent, we can get a martingale property for the derivatives of the principal's value function. Here of course it applies to the partial derivatives, and we can also conclude its convergence, but because we have two dimensions there are now a multitude of accumulation points. Once again, the broad intuitive description appears to be the same, and we have that both firms build sweat equity towards a point of non-binding liquidity constraints. Classifying the accumulation points according to the probability that they are reached is the next step in improving the results, and this would help with describing the evolution of the firms' promised utility and the long term structure of the contract.

Appendix A

Proofs

Proposition 1. *Assume that the boundedly rational consumer knows only his current consumption vector, his current utility level and gradient, as well as the utility of no consumption. Given a line search method for the maximum, at any point in time the optimal next step is given by the method applied to problems (2.1) or (2.2), where x is the momentary consumption vector and W is the momentary wealth level.*

Proof. Consider the search process at a moment in time. Let the current consumption vector be $x(0)$ and current wealth $W(0)$. Besides $U(x(0))$, the consumer knows the linear approximation $U(x) \approx L(x) := U(x(0)) + \nabla U(x(0)) \cdot (x - x(0))$. The consumer's problem is to find the best move from $x(0)$ that will maximize his lifetime utility, which means he should potentially consider the effect of his next step on the search process in the future. However, even as he anticipates future knowledge and future search, he doesn't have access to the utility levels of choices outside of $\mathcal{B}(x(0), \epsilon)$, so, in the absence of other information, his best move is the same as if he would be doing a one step search. Therefore, he will maximize the objective function derived from problems (2.1):

$$\max_x \mathcal{U}(x) = \max_x \{L(x) + \lambda [(1-\beta)W^* - p \cdot x]\},$$

or (2.2):

$$\max_x \mathcal{U}(x) = \max_x \left\{L(x) - (L(x) - \underline{U})\left(1 + \frac{W}{p \cdot x}(\beta - 1)\right)\right\},$$

s.t. $x \in \mathcal{B}(x(0), \epsilon|\nabla U(x(0))|)$. The best point to move to is $x(0) + \epsilon\nabla\mathcal{U}(x(0))$, in the direction of steepest descent. This maximization will be one step in the search algorithm. As $\epsilon, \delta_\tau \to 0$, we get a continuous movement along the line of steepest descent. □

Proposition 2. *If the lifetime budget constraint doesn't bind and doesn't hold with equality, the search moves in the direction of $\nabla U(x)$. If the constraint binds, the search moves in the direction $\nabla U(x) + [U(x) - \underline{U}] \cdot \frac{-p}{p \cdot x}$.*

Proof. The maximization problem is

$$\max_x \; U(x) \int_0^{\tau(x)} e^{t \ln \beta} dt + \underline{U} \int_{\tau(x)}^{\infty} e^{t \ln \beta} dt,$$

$$s.t. \frac{1 - \beta^{\tau(x)}}{1 - \beta} p \cdot x \leqslant W.$$

If the budget constraint doesn't bind, we have that $\tau(x) = \infty$, and the local maximization problem is trivial because we are away from the boundary. That is, in the Lagrangian we can ignore the constraint. Then the objective maximizing direction is given by $\nabla U(x)$. When the lifetime budget constrain binds, we have that

$$1 - \frac{W}{p \cdot x}(1 - \beta) = \beta^{\tau(x)},$$

so we can rewrite the maximization problem as

$$\max_x \mathfrak{U}(x) := -\left(U(x) - \underline{U}\right)\left[1 - \frac{W}{p \cdot x}(1-\beta)\right] + U(x) = \left(U(x) - \underline{U}\right)\frac{W(1-\beta)}{p \cdot x} + \underline{U}.$$

This gives us

$$\nabla \mathfrak{U}(x) = \frac{W}{p \cdot x}(1-\beta)\left[\nabla U + \left(U(x) - \underline{U}\right) \cdot \frac{-p}{p \cdot x}\right].$$

□

Proposition 3. *Let the utility function $U(x)$ be strictly concave. When the lifetime budget constraint binds, the consumer searches towards allocations with lower spending. If the constraint doesn't bind, but holds with equality, the direction of movement is along the budget constraint.*

Proof. By the multivariate mean value theorem, we have that $U(x) - \underline{U} = U(x) - U(0) = \nabla U((1-c)x) \cdot x$, for some value $c \in (0,1)$. Therefore, we can write the projection on p of the expression that shows the direction of search like

$$\frac{p}{|p|} \cdot \left[\nabla U(x) + \left(U(x) - \underline{U}\right) \cdot \frac{-p}{p \cdot x}\right] = \nabla U \cdot \frac{p}{|p|} - \left(U(x) - \underline{U}\right)\frac{|p|}{p \cdot x} =$$

$$= \nabla U(x) \cdot \frac{p}{|p|} - \nabla U((1-c)x) \cdot x \frac{|p|}{p \cdot x}.$$

Multiply by $\frac{p \cdot x}{|p|}$ to get

$$\left(\nabla U(x) \cdot \frac{p}{|p|}\right)\left(x \cdot \frac{p}{|p|}\right) - \nabla U((1-c)x) \cdot x \leqslant \nabla U(x) \cdot x - \nabla U((1-c)x) \cdot x,$$

because the first term is positive. $\nabla U(x) \cdot x - \nabla U((1-c)x) \cdot x$ is negative since U is concave, therefore the projection is negative, which is what we required.

Now let's consider the case when the lifetime budget holds with equality, $p \cdot x/(1-\beta) = W$, but doesn't bind. We'll prove that the best marginal change direction has to be perpendicular on p. *W.l.o.g.* consider a small change in x by $a + b$, where $a \cdot p = 0$, $b \cdot p = |b||p| > 0$. If b increases spending per unit of time, then lifetime utility can be increased by a change of consumption of $-b$, by $\nabla \mathcal{U}(x) \cdot (-b)$, since we have proved that $\nabla \mathcal{U}(x)$ is directed towards allocations of lower spending when the spending rate exceeds W. This means that a change of a is better than a change of $a+b$. Similarly, if b decreases spending per unit of time, then we can increase lifetime utility by moving back by $\nabla U(x) \cdot (-b)$, because $\nabla U(x)$ points towards allocations of higher spending.

□

Proposition 4. *Let the search process be a continuous steepest descent algorithm, where the wealth bound is updated after each step. If $U(x)$ is strictly concave, the search for the optimal allocation and wealth converges, and the limit wealth is exactly enough to sustain the limit allocation forever.*

Proof.

$$W'(t) = (-\ln \beta) \left[-\frac{p \cdot x(t)}{1-\beta} + W(t) \right].$$

If the lifetime wealth constraint doesn't bind and doesn't hold with equality, the direction of movement is $\nabla U(x)$. Because preferences are assumed non-satiated, as the algorithm proceeds we have that consumption per unit of time, $p \cdot x(t)$, is increased. Moreover, as spending increases, current lifetime utility is also increased. As mentioned, we ignore the case of unbounded growth of consumption, when the new initial wealth is so large that it exceeds the net present cost of the path of consumption growing infinitely. In other words,

$$\frac{p \cdot x(t)}{1-\beta} \to W(t),$$

and

$$\frac{p \cdot x(t)}{1-\beta} = W(t) \Rightarrow W'(t) = 0,$$

so the wealth level converges, $W(t) \to W^* > 0$. Search when the lifetime wealth constraint holds with equality is discussed last.

Now assume that the lifetime wealth constraint binds, and begin by incorporating the constraint into (2.2). We then have the following generic objective function that we consider for a step in the algorithm:

$$\max_x \left\{ -(U(x) - \underline{U}) \left[1 - \frac{W}{p \cdot x}(1-\beta) \right] + U(x) \right\}.$$

In the steepest descent algorithm, a step in the search should weakly increase the value of the objective function. (See Theorems 5.1, 5.3 in Ruszczyński (2006).) This function is the discounted lifetime utility of the consumer, where he stops searching and consumes x as long as he can afford it. Therefore, each step in the search increases lifetime utility. Moreover, lifetime wealth is decreased after each step, since the consumer spends more than he can afford in the long run. From equation (2.4):

$$\frac{1}{1-\beta} p \cdot x(t) \geqslant W \Rightarrow \frac{\ln \beta}{\beta - 1} p \cdot x(t) \geqslant W(-\ln \beta) \Rightarrow W'(t) \leqslant 0.$$

Since lifetime wealth is bounded from below by 0 and $W'(t)$ is monotonic, $W(t) \to W^*$. As mentioned, we ignore the case where the consumer searches so slowly that he goes bankrupt before he reaches a new optimum, i.e., $W^* > 0$.

When the lifetime budget constraint holds with equality, the consumer solves

$$\max_x \mathfrak{U}(x) := \max_x \{U(x) + \lambda [(1-\beta)W^* - p \cdot x]\},$$

by Proposition 3. With $U(x)$ strictly concave, steepest descent will find the unique maximum, x^*, and $W^* = p \cdot x^*/(1-\beta)$. □

Proposition 5. *Let $C : \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable, with $C(0) = 0$ and $C'(l) > 0$ for all $l > 0$, and let $\mathcal{U}(x) : \mathbb{R}^n_+ \to \mathbb{R}$ of class $\mathcal{C}^1$, strictly quasiconcave. Define $V_r(x - r) := \mathcal{U}(x) - \mathcal{U}(r) - C(\|x - r\|)$, for any $r \in \mathbb{R}^n_+$, where the norm is any d-norm $\|\cdot\|_d$, $1 \leqslant d < \infty$, or the seminorm $|p \cdot (x - r)|$. Then the preference structure given by $V_r(x - r)$ satisfies assumption A_1, for any pair of indices $i \neq j$.*

Proof. Let $r, s, x, y \in \mathbb{R}^n_+$, and $i = 1, j = 2$, *w.l.o.g.* In addition, $x_1 \geqslant r_1 > s_1 = y_1$, $y_2 > x_2$, $r_2 = s_2$, and $x =_s y$. Define $a \equiv y_2 - s_2$, $b \equiv x_2 - s_2$, $c \equiv r_1 - s_1 > 0$ and $d \equiv x_1 - r_1 \geqslant 0$. Then $x =_s y$ means:

$$V_s(x - s) = V_s(c + d, b, 0, ..., 0) = \mathcal{U}(x) - \mathcal{U}(s) - C\big(\|(c + d, b, 0, ..., 0)\|\big) =$$

$$V_s(y - s) = V_s(0, a, 0, ..., 0) = \mathcal{U}(y) - \mathcal{U}(s) - C\big(\|(0, a, 0, ..., 0)\|\big).$$

With respect to the reference r, the value functions are:

$$V_r(x - r) = V_r(d, b, 0, ..., 0) = \mathcal{U}(x) - \mathcal{U}(r) - C\big(\|(d, b, 0, ..., 0)\|\big), \text{ and}$$

$$V_r(y - r) = V_r(-c, a, 0, ..., 0) = \mathcal{U}(y) - \mathcal{U}(r) - C\big(\|(-c, a, 0, ..., 0)\|\big),$$

$$\Rightarrow V_r(x - r) - V_r(y - r) = \mathcal{U}(x) - \mathcal{U}(y) - C\big(\|(d, b, 0, ..., 0)\|\big) + C\big(\|(-c, a, 0, ..., 0)\|\big).$$

So, solving for $\mathcal{U}(x) - \mathcal{U}(y)$ in the first equation:

$$\begin{aligned} V_r(x - r) - V_r(y - r) = &+ C\big(\|(c + d, b, 0, ..., 0)\|\big) + C\big(\|(-c, a, 0, ..., 0)\|\big) - \\ &- C\big(\|(d, b, 0, ..., 0)\|\big) \qquad - C\big(\|(0, a, 0, ..., 0)\|\big) \\ &> 0. \end{aligned}$$

The last inequality follows from the fact that, for $p < \infty$, $c > 0$, C strictly increasing, and:

$$\|(0, a, 0, ..., 0)\| < \|(-c, a, 0, ..., 0)\|,$$

$$\|(d, b, 0, ..., 0)\| < \|(c + d, b, 0, ..., 0)\|.$$

Therefore $x >_r y$, which was required. □

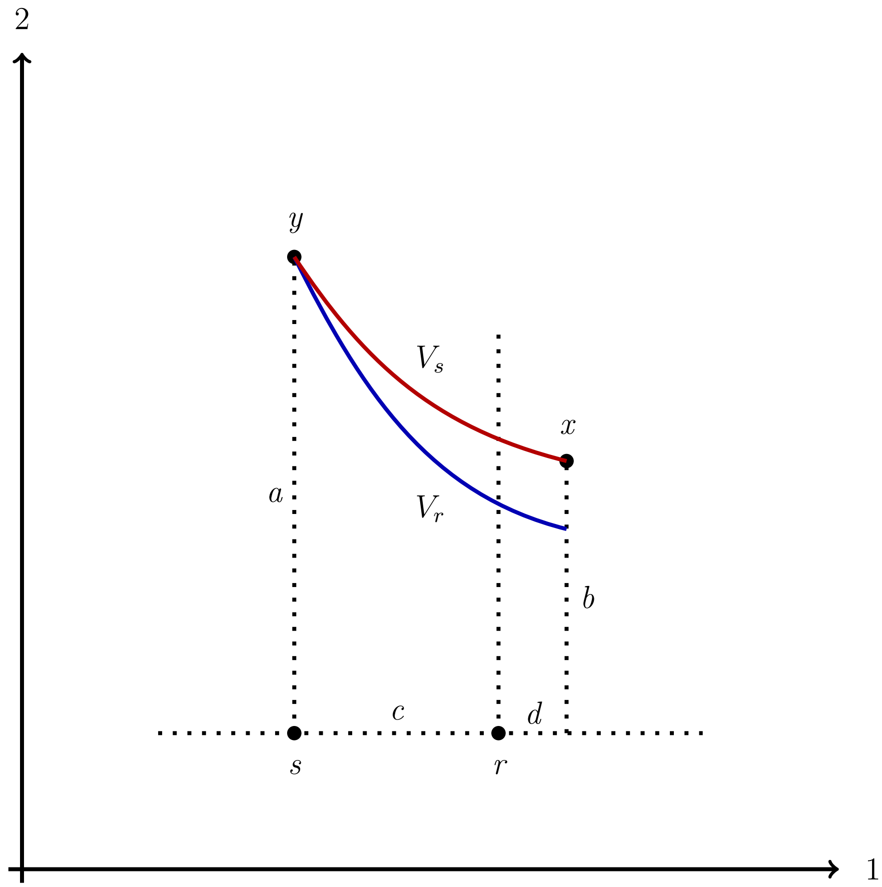

FIGURE A.1: Reference dependence in the multidimensional setting.

Proposition 6. *Let the consumer have current consumption choice x, and say that he faces a lottery $(\alpha_1, W_1; ...; \alpha_l, W_l)$, expressed in terms of the lifetime wealths for each contingency. Let $W_1 \leqslant ... \leqslant W_k < \frac{p \cdot x}{1-\beta} < W_{k+1} \leqslant ... \leqslant W_l$, and $\alpha_1 + ... + \alpha_l = 1$. The direction of his search will be given by*

$$\frac{\alpha_{k+1} + ... + \alpha_l}{1-\beta} \nabla U(x) + \frac{\alpha_1 W_1 + ... + \alpha_k W_k}{p \cdot x} \left[\nabla U(x) - (U(x) - \underline{U}) \frac{p}{p \cdot x} \right].$$

Proof. We have argued that the consumer, having restricted knowledge, behaves at any point as if he's maximizing his lifetime utility given his current consumption plan. For this, write the maximization problem for the consumer:

$$\max_x \frac{\ln \beta}{\beta - 1} \left\{ \sum_j \alpha_j \left[U(x) \int_0^{\tau_j(x)} \beta^t dt + \underline{U} \int_{\tau_j(x)}^{\infty} \beta^t dt \right] \right\}$$

$$\text{s.t. } \forall j \in \{1, ..., k\} : \frac{1 - \beta^{\tau_j(x)}}{1 - \beta} p \cdot x = W_j, \text{ and } \tau_{k+1} = ... = \tau_n = \infty.$$

It's easy to solve the constraints for $\beta^{\tau_j(x)}$ and introduce them into the objective function, and we get

$$\max_x \left\{ \sum_{j=1}^{k} \frac{\alpha_j W_j}{p \cdot x} \left[U(x) - (U(x) - \underline{U}) \left(1 + \frac{W_j}{p \cdot x}(\beta - 1) \right) \right] + U(x) \sum_{j=k+1}^{l} \alpha_j \right\}.$$

The gradient of this new objective function is the formula given in the proposition.

□

Proposition 7. *If the consumer faces a lottery in the distant future* $(\alpha, W_1; 1 - \alpha, W_2)$*, with* $W_1 < p \cdot x(0)/(1 - \beta) < W_2$*, and the instantaneous utility function has* $0 > \nabla^2 U(x) \geq -MI$*, then for any subdomain* $D \in \mathbb{R}^n_+$*,* $\exists\ 0 < \alpha_l < \alpha_h < 1$ *for* M *small enough such that for* $\alpha \in (0, \alpha_l] \cup [\alpha_h, 1)$ *the search converges,* $x(t) \to x^*$*, and* $W^* = W_1(t)$ *for* $\alpha \leqslant \alpha_h$*,* $W^* = W_1(t)$ *for* $\alpha \geqslant \alpha_h$*, where* $W^* := p \cdot x^*/(1 - \beta)$*, and* W_j *is the lifetime utility for outcome* j*.*

Proof. First, let's consider $W_1 < p \cdot x/(1 - \beta) < W_2$. At any x, the direction of search is given by

$$\frac{1 - \alpha}{1 - \beta} \nabla U(x) + \frac{\alpha W_1}{p \cdot x} \left[\nabla U(x) - (U(x) - \underline{U}) \frac{p}{p \cdot x} \right]. \tag{A.1}$$

Let's rewrite the expression for component i, by dividing by the positive scalar $\frac{\alpha(1-\beta)W_1}{p \cdot x}$ and regrouping:

$$U_i(x) \left(1 + \frac{1 - \alpha}{\alpha} \frac{p \cdot x}{(1 - \beta) W_1} \right) - (U(x) - \underline{U}) \frac{p_i}{p \cdot x}.$$

Now consider a small change in x_j by δ_j, happening in the time δ_τ, and write the marginal change in the component i of the search direction vector:

$$U_{ij}(x)\delta_j\left(1+\frac{1-\alpha}{\alpha}\frac{p\cdot x}{(1-\beta)W_1}\right)+U_i(x)\frac{1-\alpha}{\alpha}\frac{p_j\delta_j}{(1-\beta)W_1}+$$

$$+U_i(x)\frac{1-\alpha}{\alpha}\frac{p\cdot x}{(1-\beta)W_1^2}\left(-W_1'\delta_\tau\right)-U_i(x)\frac{p_j\delta_j}{p\cdot x}-(U(x)-\underline{U})\frac{-p_ip_j\delta_j}{(p\cdot x)^2}.$$

For small enough maximum curvatures M, the negative first term is dominated by the other. Observe that the 2nd, 3rd and 5th terms are strictly positive ($W_1' < 0!$), and the 4th is negative. We know that $(1-\beta)W_1 < p\cdot x$, so for any $\alpha < 1/2$ we have that the second term is bigger in absolute value than the 4th. Given a small M, $\exists\alpha_l^{ij} > 0$ such that the expression is positive on D for x, and define $\alpha_l := \min_{i,j}\{\alpha_l^{ij}\} > 0$. We can conclude that, for any $0 < \alpha \leqslant \alpha_l$, a marginal movement towards higher spending will not lead to a reversal of direction.

Similarly, we can discuss high α values. Group the last two terms together by factoring $-p_j\delta_j/p\cdot x$, and observe that the expression is negative, by Proposition 3. Terms 2 and 3 can be grouped together into a positive expression as well, with a factor $U_i(x)(1-\alpha)/\alpha$ in front. $U_i(x) > 0$, so $\exists\alpha_h^{ij} < 1$ such that the expression is negative for $\alpha > \alpha_h^{ij}$ and $x \in D$. This means that, for $\alpha > \alpha_h := \max_{i,j}\{\alpha_h^{ij}\} < 1$, a marginal movement towards lower spending will not lead to a reversal of direction.

The above deals with the case when the budget constraint for the low wealth contingency binds, but the other one doesn't bind or hold with equality. As usual, we exclude the cases when the search is not fast enough and $W_1 \to 0$, or when the consumer increases his consumption forever: $W_1, W_2 \to \infty$ and $p\cdot x \to \infty$. The search moves either toward lower or higher spending, until the spending level $p\cdot x(t)$ reaches either $W_1(t)$ or $W_2(t)$ respectively. At that point, the respective contingent wealths converge, and maximization problem changes.

Now, let's assume that $W_1(t) \to W^* > 0$, so the current consumption level $x(t)$ is such that $p \cdot x(t)/(1-\beta) = W^*$. Then the claim is that the search will move perpendicular to p, similarly to the situation in Proposition 3. The argument is similar and tedious; from (A.1), note that moving away from the plane $p \cdot x(t)$ means that utility is lost proportional to

$$\frac{\alpha}{1-\beta}\left[\nabla U(x) - (U(x) - \underline{U})\frac{p}{p \cdot x}\right],$$

and gained proportional to $(1-\alpha)\nabla U(x)/(1-\beta)$. If that were positive, we wouldn't have moved towards the lower spending. So the movement has to be along the plane. If $W_2(t) \to W^*$, then a similar argument shows that the search will then move along the plane $p \cdot x(t)$, because

$$\frac{\alpha W_1}{p \cdot x}\left[\nabla U(x) - (U(x) - \underline{U})\frac{p}{p \cdot x}\right]$$

is decreasing from W_1. In this stage, the search is a simple optimization problem, and we get $x(t) \to x^*$ and $p \cdot x^* = W^*(1-\beta)$. $\square$

Corollary 8. *Let $U^a(x)$, $U^b(x)$ be two instantaneous utility functions for consumers a and b, which have current consumptions x_0. If $\nabla U^a(x_0) = \nabla U^b(x_0)$ and $0 \succ \nabla^2 U^a(x) \succ \nabla^2 U^b(x)$ for all x, then consumer a will increase his spending more than consumer b.*

Proof.

$$U^j(x_0) - U^j(0) = \int_0^1 \nabla U^j(\theta x_0) \cdot x_0 d\theta =$$

$$= \int_0^1 \left[\nabla U^j(x_0) - \int_0^1 \nabla^2 U^j(x_0 - \rho(1-\theta)x_0) \cdot x_0 d\rho\right] \cdot x_0 d\theta =$$

$$= \int_0^1 \nabla U^j(x_0) \cdot x_0 d\theta - \int_0^1\int_0^1 x_0^T \cdot \nabla^2 U^j(x_0 - \rho(1-\theta)x_0) \cdot x_0 d\rho d\theta.$$

We know that, for strictly concave utilities, $\nabla^2 U^j(x)$ are negative definite, so we can conclude that $U^a(x_0) - \underline{U}^a < U^b(x_0) - \underline{U}^b$. From (A.1), we see that the gradient of the objective of b minus that of a is $p/(p \cdot x)(U^a(x_0) - U^b(x_0) - \underline{U}^a + \underline{U}^b) < 0$, which is directed towards lower cost consumptions.

□

Proposition 9. *Let $f(x)$ be a twice continuously differentiable function over consumption x, which does not include a measure of time. It is assumed that $-mI \succeq \nabla^2 f(x) \succeq -MI$, $\forall x \in D$, the domain, and $0 < m \leqslant M$. Furthermore D is convex. (This implies that the Hessian is negative definite on the domain D, $\nabla^2 f(x) \preceq 0$.) A unique maximum $x^* \in D$ must exist, which is assumed to be an interior point. The search generated by the steepest descent algorithm, starting from x_0 given, with a step length $\epsilon \in (0, 2/M)$, generates a sequence $\{x_t\} \to x^*$, and the following results hold:*

$$(a)\ \|x_t - x^*\| \leqslant q^t \|x_0 - x^*\|,$$

$$(b)\ f(x^*) - f(x_t) \leqslant \frac{M}{m} qmax^{2t}[f(x^*) - f(x_0)],$$

where $qmax := \max[|1 - \epsilon m|, |1 - \epsilon M|] < 1$.

Proof. Apply, for example, Theorem 5.5 in Ruszczyński (2006) to obtain (a). The Hessian condition $-mI \succeq \nabla^2 f(x) \succeq -MI$, and the fact that $\nabla f(x^*) = 0$ give the following inequalities:

$$\frac{m}{2}\|x - x^*\|^2 \leqslant f(x^*) - f(x) \leqslant \frac{M}{2}\|x - x^*\|^2,\ \forall x \in D.$$

Using (a), this implies:

$$f(x^*) - f(x_t) \leqslant \frac{M}{2}\|x_t - x^*\|^2 \leqslant \frac{M}{2} q^{2t} \|x_0 - x^*\|^2 \leqslant \frac{M}{m} q^{2t}[f(x^*) - f(x_0)].$$

□

Proposition 10. *Consider a quadratic $f(x)$ with non-degenerate Hessian $-mI \geq \nabla^2 f \geq -MI$, optimized by steepest descent with constant step length $\epsilon \in (0, 2/M)$. Let x_0 be the starting point, and $\nabla f(x_0)$ an eigenvector of $\nabla^2 f(x_0)$. Then the search method will proceed only along the direction of $\nabla f(x_0)$, and the following hold:*

$$(a)\ \|x_t - x^*\| \geqslant qmin^t \|x_0 - x^*\|,$$

$$(b)\ f(x^*) - f(x_t) \geqslant \frac{m}{M} qmin^{2t} [f(x^*) - f(x_0)],$$

where $qmin := \min\{|1 - \epsilon m|, ..., |1 - \epsilon M|\} < 1$.

Proof. For steepest descent with constant step length, $x^{t+1} = x^t + \epsilon \cdot \nabla f(x^t)$, so

$$\begin{aligned}\nabla f(x_{t+1}) = \nabla f(x_t) + (x_{t+1} - x_t) \cdot \nabla f(x_t) &= \nabla f(x_t) + \epsilon \nabla^2 f(x_t) \nabla f(x_t) = \\ &= \nabla f(x_t)(1 - \epsilon K),\end{aligned}$$

where $K \in [m, M]$ is the eigenvalue of $\nabla f(x^t)$. After each step, the gradient has the same direction, which means that all search steps are on the same direction. Therefore, we can analyze the search in one dimension. The section of f on one dimension will be a 1-dimensional quadratic function, and we'll parametrize the points with $y \in \mathbb{R}$. Let y^* be the maximum, and y_0 the starting point. Steepest descent gives:

$$y_{t+1} - y_t = \epsilon K(y_t - y^*) \Rightarrow y_n = (1 - \epsilon K)^t y_0 + \epsilon K \frac{(1 - \epsilon K)^t - 1}{1 - \epsilon K - 1}$$

$$\Rightarrow y_t - y^* = (1 - \epsilon K)^n (y_0 - y^*) \Rightarrow \|x_t - x^*\| = (1 - \epsilon K)^t \|x_0 - x^*\|.$$

The condition for convergence is $\epsilon \in (0, 2/K)$. From this, the fastest speed of convergence is reached for the direction with the eigenvalue that minimizes $|1 - \epsilon K|$, so $qmin := \min\{|1 - \epsilon m|, ..., |1 - \epsilon M|\}$. As in Proposition 9,

$$\frac{m}{2} \|x - x^*\|^2 \leqslant f(x^*) - f(x) \leqslant \frac{M}{2} \|x - x^*\|^2,\ \forall x \in D.$$

This means that

$$f(x^*) - f(x_t) \geqslant \frac{m}{2}\|x_t - x^*\|^2 \geqslant \frac{m}{2} qmin^{2t}\|x_0 - x^*\|^2 \geqslant \frac{m}{M} qmin^{2t}[f(x^*) - f(x_0)].$$

□

Lemma 11. *Consider the steepest descent search with step length inversely proportional to the time interval, and the familiar assumptions on $f(x)$. In the continuous time limit, the upper and lower bounds on the convergence of the objective function and the argument are exponentials, and the search converges Q-linearly to the optimum.*

Proof. Let s be the size of the time period, $s\epsilon$ the step length in the search, and T a total finite amount of time. The number of time periods is $t := T/s$. Define $q := 1 - s\epsilon K$, where K is such that q^t can describe either the upper or lower bounds on the search speed. Now consider the bonds in the limit when the search period goes to 0:

$$\lim_{s\to 0} q^t = \lim_{s\to 0}(1 - s\epsilon K)^{T/s} = (e^{\epsilon K})^{-T}.$$

We know that $(1 - \frac{1}{n})^n$, where $n := \frac{1}{s\epsilon K}$, is monotonically increasing in n, so as s decreases, the upper bond on $\|x_t - x^*\|$ after a finite period of time T is reduced. Also, for arbitrarily small time periods s, the search process will converge to the same limit, and it is Q-linear. The same arguments apply to $[f(x^*) - f(x_t)]$. □

Proposition 12. *Let $r \in \mathbb{R}^n_+$, and $\Phi(\delta) \equiv U(x^*(\delta)) = \max_x \{U(x) \mid \sum_i x_i = B(r) + \delta\}$. Let $x^*(\delta)$ and r be the solutions for the classical utility maximization problem (2.13) with budgets $B(r) + \delta$ and $B(r)$. Then $\Phi(\delta)$ is continuous and differentiable. Furthermore, for all v such that $\sum_i v_i = 1$, $\Phi(\delta) - U(r + \delta v) = o(\delta)$, where $\frac{o(\delta)}{|\delta|} \xrightarrow{\delta\to 0} 0$, and $x^*(\delta)$ is continuous in δ.*

Proof. By Proposition 2.13(b) of Kreps (1990), $\Phi(\delta)$ is continuous. $U(x)$ is differentiable so, by the envelope theorem, $\Phi(\delta)$ is differentiable also. Moreover, from the envelope theorem:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\delta} \equiv \lambda = \left.\frac{\partial U}{\partial x_i}\right|_{x=x^*}, \ \forall i, \text{ and } \frac{\mathrm{d}U(r+\delta v)}{\mathrm{d}\delta} = \sum_i v_i \left.\frac{\partial U}{\partial x_i}\right|_{x=r+\delta v}.$$

Therefore:

$$\left.\frac{\mathrm{d}(\Phi(\delta) - U(r+\delta v))}{\mathrm{d}\delta}\right|_{\delta=0} = \left.\frac{\partial U}{\partial x_i}\right|_{x=r} - \sum_i v_i \left.\frac{\partial U}{\partial x_i}\right|_{x=r} = \lambda(1 - \sum_i v_i) = 0.$$

Applying Taylor's Theorem gives $\Phi(\delta) - U(r+\delta v) = o(\delta)$. To show that $x^*(\delta)$ is continuous, let $\delta_n \to \delta$, with $x^*(\delta_n) \nrightarrow x^*(\delta)$. But $\Phi(\delta_n) \to \Phi(\delta)$, since Φ is continuous. Therefore, $U(x^*(\delta)) = \Phi(\delta) = \lim_n \Phi(\delta_n) = \lim_n U(x^*(\delta_n)) = U(\lim_n x^*(\delta_n))$, where the last equality follows from the continuity of U. Moreover:

$$\sum_{i=1}^n x_i^*(\delta_n) - (B+\delta_n) = 0 \Rightarrow \sum_{i=1}^n \lim_n x_i^*(\delta_n) - (B+\delta) = 0$$

$$\Rightarrow \lim_n x^*(\delta_n) \in \arg\max_x \left\{ U(x) \;\middle|\; \sum_{i=1}^n x_i - (B+\delta) = 0 \right\},$$

because $\lim_n x^*(\delta_n)$ solves the utility maximization problem. Since U is strictly quasiconcave, the solution to the maximization problem must be unique, so $\lim_n x^*(\delta_n) = x^*(\delta)$, so $x^*(\delta)$ is continuous in δ. □

Proposition 13. *For all $\delta \in (-B, \infty)$, let $x^*(\delta)$ be the demand induced by the classical utility maximization problem of U with wealth $B+\delta$. If $\kappa(x^*(\delta)) \neq 0$ is the Gaussian curvature of $\{x \mid U(x) = U(x^*(\delta))\}$ at $x^*(\delta)$, then $x^*(\cdot)$ is differentiable at δ, and $\frac{\partial x_k^*}{\partial \delta} = -\frac{x_k^*}{B+\delta}\left(\sum_j \epsilon_{kj}\right)$, $\forall k$, where ϵ_{kj} is the price elasticity of demand for good k and price j.*

Proof. Define the general demand function $f(p, W)$ given some arbitrary nonzero price vector p and wealth W. By Debreu (1972, 1976), the Jacobian of f at x is given by:

$$J(f) = \frac{\|\nabla U\|}{\frac{\partial U}{\partial x_n}\kappa(x)}, \text{ if } \kappa(x) \neq 0.$$

Therefore $x^*(\delta) \equiv f((1, ..., 1), B + \delta)$ is differentiable as a function of δ. Because the demand function is homogeneous of degree zero, we can write:

$$\sum_{i=1}^{n} \frac{\partial x_k^*}{\partial p_i} p_i + \frac{\partial x_k^*}{\partial \delta}(B + \delta) = 0 \Rightarrow \frac{\partial x_k^*}{\partial \delta} = -\frac{x_k^*}{B + \delta}\left(\sum_j \epsilon_{kj}\right), \; \forall k.$$

$\square$

Proposition 14. *The set of incentive compatibility conditions (3.3) is satisfied if a stronger set of compatibility conditions ($\mathcal{S}$) hold, whereby in each period the buyer considers only one-time deviations from truth, followed by truthful reporting in the future. That is, the ICCs are implied by the following set of inequalities, for*[1] $k \in \overline{0, n}$, $\forall\ \sigma, \alpha, \Theta^k$:

$$U_k(\theta^k, q_k(h^0, h^1, \ldots, h^{k-1}, \theta^k)) +$$

$$+ \sum_{t=k+1}^{n} \sum_{\Theta^t|\Theta^k} \prod_{j=k+1}^{t} \lambda_\alpha(\theta^j) U_t(\theta^t, q_t(h^0, h^1, \ldots, h^{k-1}, \theta^k, \theta^{k+1}, \ldots, \theta^t)) \geqslant$$

$$\geqslant U_k(\theta^k, q_k(h^0, h^1, \ldots, h^k)) +$$

$$+ \sum_{t=k+1}^{n} \sum_{\Theta^t|\Theta^k} \prod_{j=k+1}^{t} \lambda_\alpha(\theta^j) U_t(\theta^t, q_t(h^0, h^1, \ldots, h^k, \theta^{k+1}, \ldots, \theta^t)).$$

Proof. For now let $n < \infty$. The proof is by induction. First consider a finite setting with n periods. Let σ be any strategy, and consider a deviation from this strategy to

[1] To be exact, consider that $\theta^0 = \alpha$. At $k = 0$ there are only the incentive compatibility conditions for each type α, which are the same as the ICCs.

truth telling, only in the last period n, on only one of the possible histories. Assume the following inequality holds. $\forall\ \sigma, \alpha, \Theta^n$:

$$U_n(\theta^n, q_n(h^0, h^1, \dots, h^{n-1}, \theta^n)) \geqslant U_n(\theta^n, q_n(h^0, h^1, \dots, h^{n-1}, h^n)).$$

It is just a special case of (3.3), and of ($\mathcal{S}$) when $k = n$. Now, for a generic σ, it implies that $\forall\ \alpha$:

$$\sum_{t=1}^{n} \sum_{\Theta^t} \prod_{i=1}^{t} \lambda_\alpha(\theta^i) U_t(\theta^t, q_t(h^0, h^1, \dots, h^t)) \leqslant$$

$$\sum_{t=1}^{n-1} \sum_{\Theta^t} \prod_{i=1}^{t} \lambda_\alpha(\theta^i) U_t(\theta^t, q_t(h^0, h^1, \dots, h^t)) + \sum_{\Theta^n} \prod_{i=1}^{n} \lambda_\alpha(\theta^i) U_n(\theta^n, q_n(h^0, h^1, \dots, h^{n-1}, \theta^n)),$$

so it makes sense to always report truthfully in the last period. This is the first step in the induction process. In general, consider that the following holds for an arbitrary $0 \leqslant k \leqslant n$, i.e. that it is better to report truthfully from period $k+1$ on. $\forall\ \sigma, \alpha$:

$$\sum_{t=1}^{n} \sum_{\Theta^t} \prod_{i=1}^{t} \lambda_\alpha(\theta^i) U_t(\theta^t, q_t(h^0, h^1, \dots, h^t)) \leqslant$$

$$\sum_{t=1}^{k} \sum_{\Theta^t} \prod_{i=1}^{t} \lambda_\alpha(\theta^i) U_t(\theta^t, q_t(h^0, h^1, \dots, h^t)) +$$

$$+ \sum_{t=k+1}^{n} \sum_{\Theta^t} \prod_{i=1}^{t} \lambda_\alpha(\theta^i) U_t(\theta^t, q_t(h^0, h^1, \dots, h^k, \theta^{k+1}, \dots, \theta^t)).$$

Assume that $\mathcal{S}$ hold. $\mathcal{S}$ say that, assuming future truthful reporting, the buyer finds the total discounted utility from reporting his true type today is larger than for any other report. Notice that this has to hold for all α, and for any history prior to period k. Using $\mathcal{S}$, rework the previous expression. $\forall\ \sigma, \alpha$:

$$\sum_{t=1}^{n}\sum_{\Theta^t}\prod_{i=1}^{t}\lambda_\alpha(\theta^i)U_t(\theta^t, q_t(h^0, h^1, \dots, h^t)) \leqslant$$

$$\sum_{t=1}^{k-1}\sum_{\Theta^t}\prod_{i=1}^{t}\lambda_\alpha(\theta^i)U_t(\theta^t, q_t(h^0, h^1, \dots, h^t)) + \sum_{\Theta^k}\prod_{i=1}^{k}\lambda_\alpha(\theta^i)U_k(\theta^k, q_k(h^0, h^1, \dots, h^k)) +$$

$$\sum_{t=k+1}^{n}\sum_{\Theta^t}\prod_{i=1}^{t}\lambda_\alpha(\theta^i)U_t(\theta^t, q_t(h^0, h^1, \dots, h^k, \theta^{k+1}, \dots, \theta^t)) =$$

$$= \sum_{t=1}^{k-1}\sum_{\Theta^t}\prod_{i=1}^{t}\lambda_\alpha(\theta^i)U_t(\theta^t, q_t(h^0, h^1, \dots, h^t)) + \sum_{\Theta^k}\prod_{i=1}^{k}\lambda_\alpha(\theta^i)\left[U_k(\theta^k, q_k(h^0, h^1, \dots, h^k)) +\right.$$

$$\left.\sum_{t=k+1}^{n}\sum_{\Theta^t|\Theta^k}\prod_{j=k+1}^{t}\lambda_\alpha(\theta^j)\ U_t(\theta^t, q_t(h^0, h^1, \dots, h^k, \theta^{k+1}, \dots, \theta^t))\right] \leqslant$$

$$\leqslant \sum_{t=1}^{k-1}\sum_{\Theta^t}\prod_{i=1}^{t}\lambda_\alpha(\theta^i)U_t(\theta^t, q_t(h^0, h^1, \dots, h^t)) + \sum_{\Theta^k}\prod_{i=1}^{k}\lambda_\alpha(\theta^i)\left[U_k(\theta^k, q_k(h^0, h^1, \dots, h^{k-1}, \theta^k)) +\right.$$

$$\left.\sum_{t=k+1}^{n}\sum_{\Theta^t|\Theta^k}\prod_{j=k+1}^{t}\lambda_\alpha(\theta^j)\ U_t(\theta^t, q_t(h^0, h^1, \dots, h^{k-1}, \theta^k, \dots, \theta^t))\right] =$$

$$\sum_{t=1}^{k-1}\sum_{\Theta^t}\prod_{i=1}^{t}\lambda_\alpha(\theta^i)U_t(\theta^t, q_t(h^0, h^1, \dots, h^t)) +$$

$$+ \sum_{t=k}^{n}\sum_{\Theta^t}\prod_{i=1}^{t}\lambda_\alpha(\theta^i)U_t(\theta^t, q_t(h^0, h^1, \dots, h^{k-1}, \theta^k, \dots, \theta^t)).$$

This result is the induction step, which takes us to $k = 1$. The last step is applying the 0-period ICC, which is also in S. Therefore, we have used expressions $\mathcal{S}$ to obtain the ICCs.

Now consider $n = \infty$. The partial sum from k to ∞, of the series that gives the buyer's utility is

$$\sum_{t=k}^{\infty}\sum_{\Theta^t}\prod_{i=1}^{t}\lambda_\alpha(\theta^i)U_t(\theta^t, q_t(h^0, h^1, \dots, h^t)),$$

which is bounded and must go to 0 as $k \to \infty$, if the function U_t is exponentially bounded because of discounting. For an infinite direct mechanism that implements a menu, say that the infinite set of constraints $\mathcal{S}$ will hold. Let σ represent the optimal strategy, mapping any buyer type and sequence of utility types to optimal reports. From this, devise an approximate strategy that is a "truncated" version of σ, in which the buyer reports according with the strategy σ up to period k, and then he reports truthfully. Then it must be that the payoff from the new strategy is an increasingly good approximation as k increases, since only the partial sum from k on is changed. Fix k, and apply the compatibility conditions in $\mathcal{S}$ from period k backwards to prove, again by induction, that the completely truthful strategy is weakly better than the "truncated" σ, which approximates in payoff the original. As k goes to infinity, we obtain that truthful reporting is weakly optimal in the context of a menu satisfying $\mathcal{S}$, which simply means that the regular ICCs have to hold too. $\square$

Proposition 15. *The set of ICCs in (3.3) imply a weaker set of compatibility conditions ($\mathcal{W}$), in which the buyer, who has reported his true type α if in periods after 0th, considers only one-time deviations from truth, followed by truthful reporting in the future. That is, the ICCs imply the following inequalities, for*[2] $k \in \overline{0, n}$, $\forall\ \sigma, \alpha, \Theta^k$:

$$U_k(\theta^k, q_k(\alpha, h^1, \ldots, h^{k-1}, \theta^k)) +$$

$$+ \sum_{t=k+1}^{n} \sum_{\Theta^t|\Theta^k} \prod_{j=k+1}^{t} \lambda_\alpha(\theta^j) U_t(\theta^t, q_t(\alpha, h^1, \ldots, h^{k-1}, \theta^k, \theta^{k+1}, \ldots, \theta^t)) \geqslant$$

$$\geqslant U_k(\theta^k, q_k(\alpha, h^1, \ldots, h^k)) +$$

$$+ \sum_{t=k+1}^{n} \sum_{\Theta^t|\Theta^k} \prod_{j=k+1}^{t} \lambda_\alpha(\theta^j) U_t(\theta^t, q_t(\alpha, h^1, \ldots, h^k, \theta^{k+1}, \ldots, \theta^t)), \text{ and}$$

$$\sum_{t=1}^{n} \sum_{\Theta^t} \prod_{i=1}^{t} \lambda_\alpha(\theta^i) U_t(\theta^t, q_t(\alpha, \theta^1, \ldots, \theta^t)) \geqslant \sum_{t=1}^{n} \sum_{\Theta^t} \prod_{i=1}^{t} \lambda_\alpha(\theta^i) U_t(\theta^t, q_t(h^0, \theta^1, \ldots, \theta^t)).$$

[2] At $k = 0$ consider $\theta^0 = \alpha$, and $U_0 = 0$ as defined.

Proof. The idea is to observe that the compatibility conditions expressed in $\mathcal{W}$ can be seen as just a subset of the ICCs. For the last inequality above, it is obvious. In general, consider the ICC for two strategies, the truthful strategy and a deviation to the strategy that reports truthfully except after a history $(\alpha, h^1, \ldots, h^{k-1})$, when $\sigma_k(\alpha, h^1, h^2, \ldots, h^{k-1}, \theta^k) = h^k$ and, subsequently, again truthful reporting after any continuation history. Define $H^k \equiv (\alpha, h^1, \ldots, h^{k-1}, \theta^k)$. You can simplify the ICC by eliminating the payoffs for the histories $\Theta^n - \Theta^n|H^k$ and their continuations, which must be equal, so you get:

$$\prod_{i=1}^{k} \lambda_\alpha(\theta^i) U_k(\theta^k, q_k(\alpha, h^1, \ldots, h^{k-1}, \theta^k)) +$$

$$+ \prod_{i=1}^{k} \lambda_\alpha(\theta^i) \sum_{t=k+1}^{n} \sum_{\Theta^t|H^k} \prod_{j=k+1}^{t} \lambda_\alpha(\theta^j) U_t(\theta^t, q_t(\alpha, h^1, \ldots, h^{k-1}, \theta^k, \theta^{k+1}, \ldots, \theta^t)) \geqslant$$

$$\geqslant \prod_{i=1}^{k} \lambda_\alpha(\theta^i) U_k(\theta^k, q_k(\alpha, h^1, \ldots, h^k)) +$$

$$+ \prod_{i=1}^{k} \lambda_\alpha(\theta^i) \sum_{t=k+1}^{n} \sum_{\Theta^t|H^k} \prod_{j=k+1}^{t} \lambda_\alpha(\theta^j) U_t(\theta^t, q_t(\alpha, h^1, \ldots, h^k, \theta^{k+1}, \ldots, \theta^t))$$

Simplify the common factor to get the generic compatibility condition of $\mathcal{W}$. In this construction, we haven't assumed $n < \infty$, so the result holds for the infinite setting. □

Lemma 16. *Consider any period with allocations. Denote with q_h, q_l the quantities and p_h, p_l the prices in the menu offered to the buyer in that period, contingent upon the θ-type report, and let u_h, u_l be the expected continuation utilities for the buyer, given by the buyer type and truthful reporting in the future. Then, if conditions $\mathcal{W}$ hold, it must be that $q_h \geqslant q_l$.*

Proof. Assume the $\mathcal{W}$ compatibility conditions in the period hold. By them it is *w.l.o.g.* that in the future the player will report truthfully. Then there are only two continuation values that are needed to describe a player's incentives, say u_h and u_l, and:

$$\theta_h q_h - q_h^2/2 - p_h + u_h \geqslant \theta_h q_l - q_l^2/2 - p_l + u_l$$
$$\theta_l q_l - q_l^2/2 - p_l + u_l \geqslant \theta_l q_h - q_h^2/2 - p_h + u_h.$$

We can rewrite the equations to:

$$\theta_h(q_h - q_l) \geqslant p_h - p_l + q_h^2/2 - q_l^2/2 + u_l - u_h$$
$$p_h - p_l + q_h^2/2 - q_l^2/2 + u_l - u_h \geqslant \theta_l(q_h - q_l),$$

from which you can deduce that $(\theta_h - \theta_l)(q_h - q_l) \geqslant 0$. Since $\theta_h > \theta_l$, we must have that $q_h \geqslant q_l$. This lemma remains true if we replace $\mathcal{W}$ with $\mathcal{S}$, because we would have similar inequalities for each pair u_l^α, u_h^α, leading to the same conclusion. □

Lemma 17. *Consider any period with allocations in the game described. Denote with q_h, q_l the quantities and p_h, p_l the prices in the menu offered to the buyer in that period, and let u_h, u_l be the expected continuation utilities. Assuming that $\mathcal{W}$ hold, it must be that $q_h \geqslant \theta_h$ and $q_l \leqslant \theta_l$ in the optimal menu.*

Proof. Prove by contradiction, i.e., assume that $q_h < \theta_h$. Then $\exists \epsilon > 0$ s.t. $q_h + \epsilon < \theta_h$. Now consider the following changes in the menu offered: $q_h \to q_h + \epsilon$, and $p_h \to p_h + (\theta_h - q_h)\epsilon - \epsilon^2/2$. Observe that both q_h and p_h are strictly increased.

The new expected utility of the buyer, when he truthfully reports θ_h, is:

$$\theta_h(q_h + \epsilon) - (q_h + \epsilon)^2/2 - (p_h + (\theta_h - q_h)\epsilon - \epsilon^2/2) + u_h = \theta_h q_h - q_h^2/2 - p_h + u_h.$$

When the buyer reports θ_l, everything stays the same. Now verify that the new incentive compatibility conditions will also be satisfied:

$$\theta_h(q_h + \epsilon) - (q_h + \epsilon)^2/2 - (p_h + (\theta_h - q_h)\epsilon - \epsilon^2/2) + u_h \geqslant \theta_h q_l - q_l^2/2 - p_l + u_l$$
$$\theta_l q_l - q_l^2/2 - p_l + u_l \geqslant \theta_l(q_h + \epsilon) - (q_h + \epsilon)^2/2 - (p_h + (\theta_h - q_h)\epsilon - \epsilon^2/2) + u_h$$

$$\Leftrightarrow$$

$$\theta_h q_h - q_h^2/2 - p_h + u_h \geqslant \theta_h q_l - q_l^2/2 - p_l + u_l$$
$$\theta_l q_l - q_l^2/2 - p_l + u_l \geqslant \theta_l q_h - q_h^2/2 - p_h - (\theta_h - \theta_l)\epsilon + u_h.$$

Because $(\theta_h - \theta_l)\epsilon > 0$, the old incentive compatibility conditions imply that these hold. Any other constraints, either incentive compatibility constraints at other nodes or individual rationality constraints, will not be changed, because the expected utility for the buyer does not change. Therefore we have found that there is a way to strictly improve the seller's expected return, without changing the expected payoff of the buyer. So this allocation is not optimal, therefore $q_h \geqslant \theta_h$. For the statement $q_l \leqslant \theta_l$, prove by contradiction by assuming that $q_l \geqslant \theta_l$. Then $\exists \epsilon > 0$ s.t. $q_l - \epsilon > \theta_l$. Now change the menu offered: $q_l \rightarrow q_l - \epsilon$, and $p_l \rightarrow p_l + (q_l - \theta_l)\epsilon - \epsilon^2/2$. Observe that q_l is strictly decreased while p_l is strictly increased. The new expected utility for the buyer that truthfully reports θ_l does not change:

$$\theta_l(q_l - \epsilon) - (q_l - \epsilon)^2/2 - (p_l + (q_l - \theta_l)\epsilon - \epsilon^2/2) + u_l = \theta_l q_l - q_l^2/2 - p_l + u_l.$$

Neither does the utility of the buyer that truthfully reports θ_h. The new incentive compatibility constraints are:

$$\theta_h q_h - q_h^2/2 - p_h + u_h \geqslant \theta_h(q_l - \epsilon) - (q_l - \epsilon)^2/2 - (p_l + (q_l - \theta_l)\epsilon - \epsilon^2/2) + u_l$$
$$\theta_l(q_l - \epsilon) - (q_l - \epsilon)^2/2 - (p_l + (q_l - \theta_l)\epsilon - \epsilon^2/2) + u_l \geqslant \theta_l q_h - q_h^2/2 - p_h + u_h$$

$$\Leftrightarrow$$

$$\theta_h q_h - q_h^2/2 - p_h + u_h \geqslant \theta_h q_l - q_l^2/2 - p_l - (\theta_h - \theta_l)\epsilon + u_l$$
$$\theta_l q_l - q_l^2/2 - p_l + u_l \geqslant \theta_l q_h - q_h^2/2 - p_h + u_h.$$

These new conditions must hold if the old ones did. Also, no other compatibility or rationality conditions will be changed. So this change improves the seller's return, therefore it must be that $q_l \geqslant \theta_l$ was suboptimal. $\square$

Lemma 18. *Assume that in the setting presented $\mathcal{W}$ hold, and the results of the previous lemmas. Then, for every period, for each history, at most one of the compatibility conditions in $\mathcal{W}$ for an agent α will bind.*

Proof. Write again the ICCs:

$$\theta_h q_h - q_h^2/2 - p_h + u_h \geqslant \theta_h q_l - q_l^2/2 - p_l + u_l$$

$$\theta_l q_l - q_l^2/2 - p_l + u_l \geqslant \theta_l q_h - q_h^2/2 - p_h + u_h.$$

Assume that both conditions above hold with equality. Then

$$\theta_h(q_h - q_l) = q_h^2/2 + p_h - u_h - q_l^2/2 - p_l + u_l$$

$$\theta_l(q_h - q_l) = q_h^2/2 + p_h - u_h - q_l^2/2 - p_l + u_l,$$

so $\theta_h(q_h - q_l) = \theta_l(q_h - q_l)$. Because $q_h \geqslant \theta_h > \theta_l \geqslant q_l$, we have a contradiction. □

Lemma 19. *Consider any period with allocations, and assume the compatibility condition in $\mathcal{W}$ for the buyer with θ_h-type utility does not bind. Then $q_l \geqslant \theta_l$. Similarly, if the θ_l-type condition does not bind, then $q_h \leqslant \theta_h$.*

Proof. Say that $q_l < \theta_l$. Then $\exists \epsilon > 0$ s.t. $q_l + \epsilon < \theta_l$. Consider the menu transformation $q_l \to q_l + \epsilon, p_l \to p_l + (\theta_l - q_l)\epsilon - \epsilon^2/2$. Then, the new incentive compatibility conditions will be:

$$\theta_h q_h - q_h^2/2 - p_h + u_h \geqslant \theta_h(q_l + \epsilon) - (q_l + \epsilon)^2/2 - p_l + (\theta_l - q_l)\epsilon - \epsilon^2/2 + u_l$$

$$\theta_l(q_l + \epsilon) - (q_l + \epsilon)^2/2 - p_l + (\theta_l - q_l)\epsilon - \epsilon^2/2 + u_l \geqslant \theta_l q_h - q_h^2/2 - p_h + u_h.$$

$$\Leftrightarrow$$

$$\theta_h q_h - q_h^2/2 - p_h + u_h \geqslant \theta_h q_l - q_l^2/2 - p_l + u_l + (\theta_h - \theta_l)\epsilon$$

$$\theta_l q_l - q_l^2/2 - p_l + u_l \geqslant \theta_l q_h - q_h^2/2 - p_h + u_h.$$

As long as ϵ is small enough, the first incentive compatibility condition will hold for the new menu. Moreover, the expected revenue of the seller is strictly increased without affecting the buyer's utility. For the second part of the statement, consider now $q_h \to q_h - \epsilon, p_h \to p_h + (q_h - \theta_h)\epsilon - \epsilon^2/2$. After reworking, the new incentive compatibility conditions will be:

$$\begin{cases} \theta_h q_h - q_h^2/2 - p_h + u_h \geqslant \theta_h q_l - q_l^2/2 - p_l + u_l \\ \theta_l q_l - q_l^2/2 - p_l + u_l \geqslant \theta_l q_h - q_h^2/2 - p_h + u_h + (\theta_h - \theta_l)\epsilon. \end{cases}$$

If the θ_l compatibility condition will not bind, then for a small enough ϵ the above conditions will hold. As before, the seller's expected revenue is increased, but the buyer's payoffs are left the same. □

Lemma 20. *The simplified seller's maximization problem is given by the following recursive form:*

$$\begin{aligned} P(v) = \max_{\{p_h, p_l, v_h, v_l, q_l\}} \ & (1-\varphi)[\alpha_l p_h + (1-\alpha_l)p_l] + \alpha_l \delta P(v_h) + (1-\alpha_l)\delta P(v_l) + \\ & \varphi[-\alpha_h(\theta_h^2/2 - p_h) - (1-\alpha_h)(\theta_l q_l - q_l^2/2 - p_l)] \end{aligned} \tag{A.2}$$

$$\begin{aligned} s.t. \quad (i) \quad & \theta_h^2/2 - p_h + \delta v_h = \theta_h q_l - q_l^2/2 - p_l + \delta v_l, \\ (ii) \quad & v = \alpha_l(\theta_h^2/2 - p_h) + (1-\alpha_l)(\theta_l q_l - q_l^2/2 - p_l) + \alpha_l \delta v_h + (1-\alpha_l)\delta v_l, \\ (iii) \quad & q_l \geqslant 0. \end{aligned}$$

Proof. The statement of the maximization problem is implied by the lemmas 16 through 19. □

Proposition 21. *Assuming that (iii) doesn't bind, the allocation after reporting θ_l in the maximization problem (A.2) is always given by*

$$q_l = \theta_l - (\theta_h - \theta_l)\frac{\varphi}{1-\varphi} \cdot \frac{\alpha_h - \alpha_l}{1-\alpha_l}$$

and with the normalization $v_h = v_l = 0$,

$$p_h = \frac{\theta_h^2}{2} - (\theta_h - \theta_l)(1-\alpha_l)q_l,$$

$$p_l = (\alpha_l(\theta_h - \theta_l) + \theta_l)q_l - \frac{q_l^2}{2}.$$

(iii) holds with equality when $\frac{\theta_l(1-\alpha_l)}{\theta_h(\alpha_h-\alpha_l)+\theta_l(1-\alpha_h)} \leqslant \varphi \leqslant 1$, and in this case $q_l = 0$, $p_h = \frac{\theta_h^2}{2}$ and $p_l = 0$ in every period.

Proof. First assume that (iii) doesn't bind. Solve (i) and (ii) for p_h and p_l:

$$p_h = -v + \delta v_h + \frac{\theta_h^2}{2} - q_l(1-\alpha_l)(\theta_h - \theta_l),$$

$$p_l = -\frac{q_l^2}{2} - v + v_l\delta + q_l(\alpha_l(\theta_h - \theta_l) + \theta_l).$$

Replace these expressions into the maximization problem, and get the first order conditions. The first order condition determining q_l will be:

$$\frac{(1-\alpha_l)(-1+\varphi)q_l}{-1+\varphi} + \frac{\theta_l(1+\alpha_h\varphi - \varphi - \alpha_l) - (\alpha_h - \alpha_l)\theta_h\varphi}{-1+\varphi} = 0.$$

This solves for the q_l expression. Observe that the allocation is determined for every period after period 0. Because we ignored the non-negativity constraint (iii), it is necessary to see if q_l is positive. It turns out that a necessary and sufficient condition for that is:

$$\varphi \leqslant \frac{\theta_l(1-\alpha_l)}{\theta_h(\alpha_h - \alpha_l) + \theta_l(1-\alpha_h)}.$$

When this condition doesn't hold, then $q_l = 0$. To find the prices, it is necessary to fix the indeterminacy by making some assumptions, because the seller can commit on the contract offered, therefore changing the all the prices at period t by ϵ and compensating at period $t+k$ by $\delta^k\epsilon$ leaves the buyer with equivalent choices. Moreover, there is a one-to-one trade-off for the buyer and the truthful seller between v_h

or v_l and p_h, p_l. This suggests that neither the absolute, nor the relative values of the continuation utilities v_h, v_l are fixed, and the easiest way to deal with them is to assume $v_h = v_l = 0$. Of course, we have to check that the compatibility constraints of the buyer that reports non-truthfully will also hold, but this assumption makes all the four compatibility constraints for each node hold, because of (i). Using the assumptions, we get the prices p_h, p_l in every period. □

Proposition 22. *The allocations after reporting α_h are $q_h = \theta_h$ and $q_l = \theta_l$, and*

$$p_h = \frac{\theta_h^2}{2} - \theta_h\theta_l + \alpha_h\theta_h\theta_l + \theta_l^2 - \alpha_h\theta_l^2$$

$$p_l = \alpha_h\theta_h\theta_l + \frac{\theta_l^2}{2} - \alpha_h\theta_l^2,$$

as well as a transfer of $\frac{(\alpha_h-\alpha_l)(\theta_h-\theta_l)q_l}{1-\delta}$ from the seller to cover the information rent for the α_h type buyer.

Proof. Solve the following equations, representing the binding θ_h-type incentive compatibility and the buyer value identity respectively, after assuming again $v = v_h = v_l = 0$:

$$\alpha_h\left(\frac{\theta_h^2}{2} - p_h\right) + (1-\alpha_h)\left(\frac{\theta_l^2}{2} - p_l\right) = 0,$$

$$\frac{\theta_h^2}{2} - p_h = \theta_h\theta_l - \frac{\theta_l^2}{2} - p_l,$$

and this gives the results. We have assumed that the θ_h-type compatibility condition must not bind, so that we can get $q_l = \theta_l$, yet we have obtained the above solution making the exact opposite assumption. But that is fine, since the compatibility condition is slack. To see that, change the solution above with a small value κ:

$$p_h = \frac{\theta_h^2}{2} - \theta_h\theta_l + \alpha_h\theta_h\theta_l + \theta_l^2 - \alpha_h\theta_l^2 - (1-\alpha_h)\kappa$$

$$p_l = \alpha_h\theta_h\theta_l + \frac{\theta_l^2}{2} - \alpha_h\theta_l^2 + \alpha_h\kappa.$$

It is easy to check that, because the high and low demand compatibility conditions cannot both bind at the same time, a κ small enough always exists that will leave every condition satisfied with strict inequality. This allows us to deduce that $q_l = \theta_l$ was a good assumption. $\square$

Bibliography

Athey, S. and Segal, I., “An Efficient Dynamic Mechanism,” *Draft*, February 2007.

Andreoni, J. and Sprenger, C., “Uncertainty Equivalents: Testing the Limits of the Independence Axiom,” *Working Paper*, April, 2010.

Akaike, H., “On a Successive Transformation of Probability Distribution and its Application to the Analysis of the Optimum Gradient Method,” *Annals of the Institute of Statistical Mathematics,*, July, 1959, Volume 11, Issue 1, pp. 1 - 16.

Bateman, I., Munro, A., Rhodes, B., Starmer, C. and Sugden, R., “A Test of the Theory of Reference-Dependent Preferences,” *The Quarterly Journal of Economics*, 1997, 2, 479–505.

Battaglini, M., “Long-Term Contracting with Markovian Consumers,” *The American Economic Review*, June 2005, Vol. 85, No.3, 637–658.

Bergeman, D. and Valimaki, J., “The Dynamic Pivot Mechanism,” *Econometrica*, March 2010, Vol. 78, No. 2 , 771-789.

Bolton, P. and Dewatripont, M., “Contract theory,” Cambridge, MA: MIT Press, 2005.

Bowman, D., Minehart, D., and Rabin, M., “Loss Aversion in a Consumption–Savings Model,” *Journal of Economic Behavior and Organization*, 1999, 38, 155–178.

Cartis, C., Gould, N. I. M., and Toint, Ph. L., “On the complexity of steepest descent, Newton’s and regularized Newton’s methods for nonconvex unconstrained optimization,” *http://www.maths.ed.ac.uk/ ccartis/papers/cgt36.pdf*, 15 October 2009.

Courty, P. and Li, H., “Sequential Screening,” *Review of Economic Studies*, 2000, 67, 697–717.

Farber, H.,“Reference-Dependent Preferences and Labor Supply: The Case of New York City Taxi Drivers,” *American Economic Review*, 2008, 98:3, 1069–1082.

Frederik, S., Loewenstein, G., and O’Donoghue, T., “Time Discounting and Time Preference: A Critical Review,” *Journal of Economic Literature*, 2002, XL, 351–401.

Goldstein, A. A., “On Steepest Descent,” *SIAM Journal on Control and Optimization*, 1965, A, 3, 1, 1147–151.

Hall, R. E., “Stochastic Implications of the Life Cycle-Permanent Income Hypothesis: Theory and Evidence,” *Journal of Political Economy*, 1978, 86, 971–987.

Harbaugh, R., “Prospect Theory or Skill Signaling?,” *Working Paper*, 2009.

Johnson, E. J. and Goldstein, D., “Do Defaults Save Lives?,” *Science*, 2003, 302, 1338–1339.

Kahneman, D. and Tversky, A., “Prospect Theory: A Analysis of Decision under Risk,” *Econometrica*, 1979, 46, 263–292.

Kőszegi, B. and Rabin, M., “A Model of Reference-Dependent Preferences,” *The Quarterly Journal of Economics*, 2006, 4, 1133–1165.

Kőszegi, B. and Rabin, M., “Reference-Dependent Consumption Plans,” *American Economic Review*, 2009, 99:3, 909–936.

Krishna, V. and Taylor, C. R., “Stairway to Heaven or Highway to Hell: Liquidity, Sweat Equity, and the Uncertain Path to Ownership,” *Duke University, Department of Economics Working Papers*, 2011.

Loewenstein, G., “Frames of Mind in Intertemporal Choice,” *Management Science*, 1988, 34, 200–14.

Loewenstein, G. and Sicherman, N., “Do Workers Prefer Increasing Wage Profiles?,” *Journal of Labour Economics*, 1991, 9:1, 67–84.

McKenzie, C. R. M., Liersch, M. J. and Finkelstein, S. R., “Recommendations Implicit in Policy Defaults,” *Psychological Science*, 2006, 17–5, 414–421.

Miravete, E., “Choosing the Wrong Calling Plan? Ignorance and Learning,” *The American Economic Review*, 2003, 93(1), 297–310.

Munro, A. and Sugden, R., “On the Theory of Reference-Dependent Preferences,” *Journal of Economic Behavior & Organization*, 2003, 50, 407–428.

Ok, E. A., “Real Analysis with Economic Applications,” *Princeton University Press*, 2007.

Pavan, A., Segal, I., Rhodes, B., Starmer, C. and Sugden, R., “A Test of the Theory of Reference-Dependent Preferences,” *The Quarterly Journal of Economics*, 1997, 2, 479–505.

Pavan, A., Segal, I. and Toikka, J., “Dynamic Mechanism Design: Incentive Compatibility, Profit Maximization and Information Disclosure,” *Working Paper*, August 1st, 2009.

Pavan, A., Segal, I. and Toikka, J., “Infinite-Horizon Mechanism Design: The Independent Shock Approach,” *Working Paper*, July 16th, 2010.

Ruszczyński, A., “Nonlinear optimization,” *Princeton University Press*, 2006.

Samuelson, W. and Zeckhauser, R., “Status Quo Bias in Decision Making,” *Journal of Risk and Uncertainty*, 1988, 1, 7–59.

Shea, J., “Union Contracts and the Life-Cycle/Permanent-Income Hypothesis,” *The The American Economic Review*, 1995, 85, 186–200.

Shefrin, H. and Thaler, R., “The Behavioral Life-Cycle Hypothesis,” *Economic Inquiry*, 1988, 24, 609–643.

Stigler, G., “The Economics of Information,” *The Journal of Political Economy*, 1961, 69, 213-225

Strahilevitz, M. A. and Loewenstein, G., “The Effect of Ownership History on the Valuation of Objects,” *Journal of Consumer Research*, 1998, 25, 276–289.

Thaler, R., “Some Empirical Evidence on Dynamic Inconsistency,” *Economic Letters*, 1981, 8, 201–07.

Tversky, A. and Kahneman, D., “Loss Aversion in Riskless Choice: A Reference Dependent Model,” *The Quarterly Journal of Economics*, 1991, 4, 1039–1061.

Tversky, A. and Simonson, I., “Context-Dependent Preferences,” *Management Science*, 1993, 39, 1179–1189.

Yuan, Y., "A Short Note on the Q-linear Convergence of the Steepest Descent Method," *ftp://lsec.cc.ac.cn/pub/yyx/papers/p0703.pdf*.

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