

ESSAYS IN INTERNATIONAL MACROECONOMICS

by

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Dissertation submitted in partial fulfillment of the
requirements for the degree of Doctor of Philosophy
in the Department of Economics
in the Graduate School of
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ABSTRACT

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Abstract

This dissertation consists of two essays in international macroeconomics. The first essay shows that optimal fiscal and monetary policy is time consistent in a standard small open economy. Further, there exist many maturity structures of public debt capable of rendering the optimal policy time consistent. This result is in sharp contrast with that obtained in the context of closed-economy models. In the closed economy, the time consistency of optimal monetary and fiscal policy imposes severe restrictions on public debt in the form of a unique term structure of public debt that governments can leave to their successors at each point in time. The time consistent result is robust: optimal policy is time consistent when both real and nominal bonds have finite horizons. While in a closed economy, governments must have both nominal and real bonds, and have at least real bonds over an infinite horizon to render optimal policy time consistent.

The second essay uses a dynamic stochastic general equilibrium model to theoretically rationalize the empirical finding that sudden stops have weaker effects on outputs when the small open economy is more open to trade. First, welfare costs of sudden stops are decreasing in trade openness. The reason is that when the economy is more open to trade, the economy will have less volatile capital, which leads to less volatile output. In terms of welfare, when the small open economy is more open to trade, the welfare costs of sudden stops will be smaller. Second, sudden stops may be welfare improving to the small open economy. This is because when the representative household is a net borrower in the international capital market, its consumption will be negatively correlated with country spread. Since utility is a concave function

of consumption, it must be a convex function of country spread. That is, when the country spread is more volatile, the mean utility is higher. The two findings are robust: they hold with one sector economy model, and two sector economy models with homogenous capital and heterogenous capital. In addition, this paper shows that a counter-cyclical tariff rate policy is not welfare-improving.

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Chapter 1

Optimal and Time Consistent Monetary and Fiscal Policy in a Small Open Economy

1.1 Introduction

In an important recent paper, Persson, Persson, and Svensson (2006) (hereafter PPS) show that optimal fiscal and monetary policy is time consistent in the context of closed-economy models. However, the requirement of time consistency imposes an extremely strong restriction on public debt. Namely, there is a unique term structure of public debt capable of rendering optimal fiscal and monetary policy time consistent. This extremely strong restriction, plus the issue of time consistency itself, provokes the following questions: (a) Is the optimal policy time consistent in a small open economy? The question is important because time inconsistent optimal policy will result in an inferior equilibrium with lower overall welfare (Chari (1991)). (b) If so, is the term structure of public debt unique? (c) If so, must the government issue both nominal and real public debt over an infinite horizon? The questions are important because they are of practical relevancy.

Time consistency is defined as follows: From time 0, for any $t \geq 0$, the time $t + 1$ government will choose the time t government's policy continuation. There are two types of time inconsistency: nominal and real. Nominal time inconsistency refers to the government's action to change the outstanding

nominal debt by deviating from the promised path of price levels¹. Real time inconsistency refers to the government's action to deviate from the promised labor income tax rate path. It is usually characterized by changing the discounted present value of real debt by deviating from the promised path of real interest rates through changes in labor income tax rates².

Time inconsistent policy results in an inferior equilibrium, as seen through the following mechanism: The optimal policy requires the use of bonds to smooth out taxes and consumption over time. Whenever the policy is not time consistent, the demand for bonds disappears, and the government cannot use bonds to smooth taxes and consumption over time. It has to rely entirely on distortionary taxes and the economy ends up with lower aggregate welfare. It is thus important for the Ramsey government to render optimal policy time consistent. Note that this argument applies to both closed economies and open economies.

The literature suggests that one way to guarantee real time consistent optimal policy in closed real economies is to choose the term structures of

¹Calvo (1978) shows that in a closed monetary economy, when a lump-sum tax is not available, the government has an ex post incentive to use surprise inflation to lower the real value of the outstanding nominal public debt. The reason for nominal time inconsistency is that the government uses the surprise inflation as a lump-sum tax.

²Lucas and Stokey (1983) (hereafter LS) show that in a closed real economy, the government has an ex post incentive to manipulate the path of real interest rates in order to lower the discounted present value of real public debt. In this case, even though the government cannot find any tax instrument equivalent to lump-sum taxes, it can always take advantage of the inelasticity of outstanding real bonds to minimize the distortion caused by the distortionary taxes. This incentive is reflected in the government's ex post action to compensate for the distortion in the bond market with counterpart distortion in other market(s), usually the labor market. These two compensating distortions are briefly mentioned in PPS (1987). Liu (2006) shows that real time inconsistency exists even with non-zero one-period real bonds because the government can still change the contemporaneous labor income tax rate, even though in this case, the change in the real interest rate path has nothing to do with the present value of the outstanding one-period real bond. Further, Liu (2006) shows that the economic incentives for both nominal and real time inconsistency are unified on the following two arguments: (a) The government wants to minimize the general distortion level caused by distortionary taxes. And (b) the inelasticity of inherited bonds gives the government a room to take time inconsistent action.

debt that the $t = 1$ government will inherit from the $t = 0$ government. This is possible because the term structure of the $t = 1$ government's initial debt holding position does not enter the $t = 0$ government's Ramsey problem (to be defined later). This irrelevance gives the $t = 0$ government freedom to choose the term structure of debt that it can leave to its successor government. The maturity structure of real debt can be pinned down in the exact same way as in LS (1983).

Under a setup with both real and nominal time inconsistency, PPS (2006) incorporate direct costs to the surprise inflation and show that it is possible to construct a unique term structure of nominal debt and real debt in such a way that the optimal policy is time consistent³. Albanesi (2005) introduces heterogenous agents and shows that, if the wealth distribution effect is strong enough, it is possible to find time consistent optimal policy by choosing appropriate distribution of nominal and real bonds.

The present work extends the PPS (2006) framework beyond a closed economy to a small open economy. The main findings are as follows:

(1) In a standard small open economy, the optimal policy is generally time consistent. Specifically, there are many maturity structures of public debt capable of rendering optimal policy time consistent. This is in sharp contrast

³The literature has a debate on whether the optimal policy can be rendered time consistent in this case. LS (1983) argue that if a monetary economy has both cash and credit consumption goods, the consumer optimization requires both nominal and real debt be non-zero in order to smooth out consumption. However, the non zero nominal debt implies nominal time inconsistency as shown in Calvo (1978). LS (1983) conclude that it is impossible to have time consistent optimal policy since any non-zero nominal debt will result in nominal time inconsistency. PPS (1987) study a closed economy with money in the utility function. They suggest that both real and nominal time inconsistency can be removed by choosing an appropriate maturity structure of nominal and real bonds. In particular, they assume that the discounted present value of nominal liabilities in the initial period is zero. Calvo and Obstfeld (1990) (hereafter CO) suggest that the solution in PPS (1987) is actually not an optimum. The CO (1990) finding puts the PPS (1987) conclusion in question. Alvarez, Kehoe, and Neumeyer (2004) (hereafter AKN) show that it is possible to remove both nominal and real time inconsistency when the endogenous restriction on the public debt, which is ignored in PPS (1987), has been taken into consideration.

with what is obtained in the closed economy. In the closed economy, the time consistency of optimal monetary and fiscal policy imposes severe restrictions on public debt in the form of a unique term structure of public debt that governments can leave to their successors at each point in time.

The main reason for the time consistent optimal policy is that the $t = 0$ government always has more policy instruments than the $t = 1$ government has policy choices. The policy choices of the $t = 1$ government include: nominal interest rates and only one labor income tax rate instead of all labor income tax rates. This is true because in the Ramsey problem, other optimal labor income tax rates are pinned down by the exogenously given real interest rates. The policy instruments of the $t = 0$ government include: nominal public and external bonds from $t = 3$ on; and either the present values of real public and external bonds or the nominal public and external bonds at $t = 1$ ⁴; and possibly Lagrange multipliers.

Since the economy is of the infinite horizon, there are far more policy instruments than there are the policy choices of the $t = 1$ government, and it is thus possible to have time consistent optimal policy. With real bonds, the mechanism for the time consistent optimal policy is the following: the $t = 0$ government uses real bonds to control the $t = 1$ government's choice of one labor income tax rate; and uses nominal bonds to control the $t = 1$ government's choices of $t \geq 2$ nominal interest rates. The mechanism of the $t = 0$ government using one instrument to control for one choice of the $t = 1$ government is as follows.

In the small open economy, when the $t = 1$ households inherit a positive

⁴The nominal bonds at $t = 2$ are taken away because the government will choose them in such a way that the next government will smooth consumption the same as policy continuation.

one-period real bond, they feel richer so that they would choose relatively more contemporary consumption. However, since real interest rates are exogenous, the households have to smooth consumption out over time. Other things being equal, the holding position of the one-period real bond will determine the level of the smoothed consumption, so does the contemporary labor income tax rate. If the $t = 0$ government has freedom in choosing the real bond holding position for the $t = 1$ government, it can choose the right amount. As a result, the $t = 1$ government's reoptimized contemporary labor income rate will be the same as policy continuation⁵⁶. In this process, the $t = 0$ government uses real bond holding balance to control the $t = 1$ government's optimal choice of the contemporary labor income tax rate.

When the $t = 1$ government inherits positive multi-period nominal bonds at one particular future period, it has the incentive to increase the nominal interest rate between that period and the period before because this serves as a lump-sum tax. However, the benefit from this surprise inflation will be fully offset by the associated decline of real money balance. This gives the $t = 1$ government the option to choose positive finite nominal interest rates. If the $t = 0$ government has freedom in choosing holding positions of nominal

⁵Liu (2006) shows that in the closed economy, when the $t = 1$ government inherits a positive one-period real bond holding position, it has the incentive to increase the contemporary labor income tax rate comparing to the policy continuation, while setting relative low labor income tax rates for the future. This is because the inelastic real bond holding position provides an indirect and less distortionary source of public financing. This gives rise to the economic incentive of real time inconsistent behavior. The households receive more utility from their unit contemporary consumption than from unit future consumption due to the inelastic and positive one-period real bond. They will consume relatively more, comparing to the policy continuation, which leads to a high contemporary labor income tax rate. The Ramsey government finds it optimal to set a relatively high real interest rate to induce households to save.

⁶In the real small open economy, the level of consumption is determined by discounted value of public surplus, initial public debt, and the present value of utility from working. It can be shown that the sum of the first two components will be time-independent, but the third component is time-dependent if productivity is changing over time. If the government does not have policy instruments, this time-dependent component arises time inconsistent behavior.

bonds, it will choose the right amount so that the reoptimized positive finite nominal rates of the $t = 1$ government will be same as policy continuation. In this process, the $t = 0$ government uses nominal bonds to control the $t = 1$ government's optimal choices of nominal interest rates.

Both processes are possible at the same time because the $t = 0$ government will always have more policy instruments than the policy choices of the $t = 1$ government. More importantly, as I show later, to find the right policy instruments is to solve a system of *linear* equations. The mechanism works because I assume that given an initial asset position, there is one unique equilibrium.

(2) My second finding is: The $t = 0$ government can render optimal policy time consistent without nominal bonds when productivity is constant. In this case, the nominal economy is effectively reduced to a real economy and the government obtains one Lagrange multiplier as an implicit policy instrument. This is in sharp contrast with what is obtained in the context of closed economy models. In closed economies, the government must issue nominal bonds, otherwise the $t = 1$ government will choose a different path of nominal interest rates even if the productivity is assumed to be constant.

(3) Under certain conditions, the $t = 0$ government can render optimal policy time consistent without real bonds. In this case, the $t = 0$ government still has more policy instruments than the $t = 1$ government has policy choices. However, it is required that there are no identification problems caused by the shift from closed economy to a small open economy. I will return later to these identification problems. This is also in sharp contrast to what is obtained in closed economies: without real bonds, the optimal policy in closed economies cannot be time consistent.

(4) Furthermore, the optimal policy in a small open economy may be time

consistent with both public and private bonds issued over a finite horizon. This comes from the fact that the governments in the small open economy also lose their choices over nominal interest rates, on top of the fact that the governments lose their choices of labor income tax rates. However, in the closed economy, it requires that the bonds issued be over an infinite horizon.

There are several policy implications of my findings: (a) Optimal policy will be time consistent when the government issues both real and nominal bonds over the finite horizon; (b) If the government has neutral taste for real versus nominal bonds, it will always issue both nominal and real bonds; (c) If the government dislikes real bonds, it will always issue both nominal and real bonds and set the real debt at an optimally minimum level; and (d) when some explicit policy instruments are unavailable, the government may obtain some implicit policy instruments.

These policy implications are important in the following sense: First, if the necessary condition for time consistent optimal policy is the availability of both real and nominal bonds over infinite horizon, it implies that optimal policy in practice is time inconsistent. The result that optimal policy is time consistent with either real or nominal bonds over finite horizon is of practical importance. Secondly, one interpretation of real bonds is nominal bonds but denominated in US\$, i.e., debt dollarization. From the recent financial crises, debt dollarization exaggerates the adverse effects of crises. The result that the government can have time consistent optimal policy with minimum real bond holdings is important because the propagation effect from debt dollarization is then minimized.

The time consistent results rely on the assumption that for the given combination of monetary and fiscal policy, there is a unique competitive equilibrium.

As a result, as long as the monetary and fiscal policy combination uniquely determines the competitive equilibrium, the optimal policy is time consistent. Note that even though the price level is completely determined in each period, there is still room for the future government to select different optimal policy as opposed to mere policy continuation. However, when the price level is indeterminate, the time consistency problem will be more complicated.

This paper is organized as follows: Section 2 describes the economic setup; Section 3 discusses the Ramsey problem; Section 4 discusses the time consistency of optimal policy; and Section 5 concludes.

1.2 The Small Open Economy

1.2.1 Households

In this model, households are price-takers, and they are given the price of consumption, p_t , the present value in period 0 of goods in period t , $q_{0,t}$, the labor income tax rate, τ_t , and the nominal interest rate i_{t+1} . A representative household chooses the time profile of consumption, $c_t, t \geq 0$, real money balances, $m_{t+1}, t \geq 1$, and labor supply, $h_t, t \geq 0$, to maximize lifetime utility:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, m_t, h_t),$$

subject to its period budget constraints (1.1) and the no-Ponzi game condition (1.2). Real money balances are defined as $m_t = \frac{M_{t-1}}{p_t}$, where M denotes nominal money balance. The choice of beginning-of-period nominal money introduces an inflation cost: when the price level increases, real money balances are reduced, and the household receives less utility from the given level of nominal money balances. PPS (1996) argue that high inflation has large and

well-known social costs. Another rational for the use of the beginning of period money balance is that inflation forces households to economize money thus bringing costs to holding money, see Bailey (1956) and Tower (1971). The parameter β is the subjective discount factor, which weights the consumption bundles over time.

The representative household's period budget constraint is given by:

$$\begin{aligned} & q_{0,t} \left[(1 - \tau_t)w_t h_t + \Pi_t + \frac{M_{t-1}}{p_t} \right] + \sum_{s=t}^{\infty} q_{0,s} \left({}_{t-1}b_s^P + \frac{{}_{t-1}B_s^P}{p_s} \right) \\ & \geq q_{0,t} \left(c_t + \frac{M_t}{p_t} \right) + \sum_{s=t+1}^{\infty} q_{0,s} \left({}_t b_s^P + \frac{{}_t B_s^P}{p_s} \right). \end{aligned} \quad (1.1)$$

The variable $q_{0,t}$ can be regarded as a multiple period discount factor. In this paper, given the exogenous real interest rate, there is a one-to-one relationship between $q_{0,t}$ and β . The same discount factor is applied to both internal and external bonds due to the assumption of perfect capital mobility. The variable $({}_{t-1}b_s^P)$ denotes net claims by the domestic household when entering period t on the amount of goods to be delivered in period s ; $({}_{t-1}B_s^P)$ denotes the net claims on money to be delivered in period s . One can interpret the real bonds as nominal bonds but denominated in US\$. The bonds are real in this small open economy because their purchasing power will not change when the domestic price level changes. The sum $\sum_{s=t}^{\infty} q_{0,s} \left({}_{t-1}b_s^P + \frac{{}_{t-1}B_s^P}{p_s} \right)$ denotes the representative household's initial bond holding position.

Note that both nominal and real bonds appear in the period budget constraint. The reason for this is the following: In a real economy, the government cannot use nominal assets to render optimal policy time consistency, otherwise the price level will go to either infinity or zero. While in a monetary economy, the government must use nominal assets to assure the time consistency of the

optimal policy to avoid zero or infinite price level. When the household in a closed monetary economy receives utility from both consumption and real money balances, the government has to issue both real and nominal assets to smooth both consumption and real money balances. When I extend this framework to small open economies, it at first seems natural to put both nominal and real bonds in the domestic household's budget constraint. However, I will later show that it is not necessary to issue both real and nominal bonds in a small open economy to render optimal policy time consistent.

The no-Ponzi game condition for the representative household is given by:

$$\lim_{j \rightarrow \infty} \left[q_{t,t+j} \frac{M_{t+j}}{p_{t+j}} + \sum_{s=t+j+1}^{\infty} q_{t,s} \left({}_{t+j}b_s^P + \frac{{}_{t+j}B_s^P}{p_s} \right) \right] \geq 0, \quad \forall t \geq 0. \quad (1.2)$$

The no-Ponzi game condition has its usual meaning: the representative household has to keep non-negative financial assets in the limit. This no-Ponzi game condition must hold in each period. In this economy, nominal interest rates are defined as:

$$\frac{1}{1 + i_{t+1}} = \frac{q_{0,t+1}/p_{t+1}}{q_{0,t}/p_t} \leq 1, t \geq 0. \quad (1.3)$$

Combining the period budget constraint and the no-Ponzi game condition, I write the inter-temporal budget constraint of the representative household as equation (1.4) as:

$$\begin{aligned} & \sum_{t=0}^{\infty} q_{0,t} [(1 - \tau_t)w_t h_t + \Pi_t] + \frac{M_{-1}}{p_0} + \sum_{t=0}^{\infty} q_{0,t} \left({}_{-1}b_t^P + \frac{{}_{-1}B_t^P}{p_t} \right) \\ &= \sum_{t=0}^{\infty} q_{0,t} c_t + \sum_{t=1}^{\infty} q_{0,t} m_t i_t. \end{aligned} \quad (1.4)$$

In the Appendix, I show that the time sequences for $\{c_t, m_{t+1}, h_t\}$ satisfying constraints (1.1) and (1.2) are the same as those satisfying the single constraint (1.4). Thus, the representative household maximizes lifetime utility subject to the single constraint (1.4). The optimality conditions for the domestic household are the single inter-temporal budget constraint (1.4) and

$$\beta^t u_{ct} = \lambda q_{0,t}, t \geq 0 \quad (1.5)$$

$$\tau_t = 1 + \frac{1}{w_t} \frac{u_{ht}}{u_{ct}}, t \geq 0 \quad (1.6)$$

$$i_{t+1} = \frac{u_{m_{t+1}}}{u_{c_{t+1}}}, t \geq 0. \quad (1.7)$$

All the optimality conditions have their usual meanings: equation (1.5) says that the marginal utility of consumption should equal the marginal cost of consumption; equation (1.6) shows that the introduction of labor income tax distorts the marginal rate of substitution between consumption and labor; and equation (1.7) states that there is cost to holding money.

1.2.2 Competitive Firms

In each period, competitive firms use decreasing returns-to-scale technology in production:

$$y_t = z_t h_t^\eta, t \geq 0.$$

The technology is non-linear in labor input in order to rule out the possibility of a corner solution. In the closed economy, the choice of zero output is ruled out because it is usually assumed that consumption is positive. But in the open economy, zero output does not imply zero consumption. To rule out the case of zero output and guarantee an interior solution, I assume that at low levels of labor input, marginal product of labor is extremely high. For details,

please see Schmitt-Grohe and Uribe (2003). I assume that output is sold in both domestic and international markets so that the “law of one price” for one tradable good holds in each period, i.e.,

$$p_t = S_t p_t^*, t \geq 0, \quad (1.8)$$

where the variable S_t denotes the nominal exchange rate at time t and the variable p_t^* denotes the world price at time t . Firms maximize profit, which is given by

$$\Pi_t = z_t h_t^\eta - w_t h_t, t \geq 0. \quad (1.9)$$

The optimality condition for labor demand is given by:

$$w_t = z_t \eta h_t^{\eta-1}, t \geq 0. \quad (1.10)$$

1.2.3 The Government

The government finances its expenditures by levying labor income taxes at the rate of τ_t , by printing money and by trading multi-period nominal and real bonds with both domestic households and international investors. The monetary/fiscal regime consists of plans for the policy instruments: money and bonds; and for the policy choices: nominal interest rates and labor income tax rates. Here I assume that lump-sum taxes are not available to the government. The period budget constraint of the government is given by:

$$q_{0,t} \left(g_t + \frac{M_{t-1}}{p_t} \right) + \sum_{s=t}^{\infty} q_{0,s} \left({}_{t-1}b_s^G + \frac{{}_{t-1}B_s^G}{p_s} \right) \leq q_{0,t} \left(\tau_t w_t h_t + \frac{M_t}{p_t} \right) \quad (1.11)$$

$$+ \sum_{s=t+1}^{\infty} q_{0,s} \left({}_t b_s^G + \frac{{}_t B_s^G}{p_s} \right).$$

The variable ${}_{t-1}b_s^G$ denotes total net claims on the amount of goods to be delivered by the government in period s . It follows from the perfect capital mobility assumption that the government applies the same discount factors on both internal and external bonds.

The no-Ponzi game condition for the government is given by:

$$\lim_{j \rightarrow \infty} \left[q_{t,t+j} \frac{M_{t+j}}{p_{t+j}} + \sum_{s=t+j+1}^{\infty} q_{t,s} \left({}_{t+j}b_s^G + \frac{{}_{t+j}B_s^G}{p_s} \right) \right] \leq 0, \forall t \geq 0. \quad (1.12)$$

This condition rules out the possibility that the government borrows infinitely to finance its expenditures. The government's intertemporal budget constraint is given by:

$$\sum_{t=0}^{\infty} q_{0,t} \left(-{}_1b_t^G + \frac{{}_1B_t^G}{p_t} \right) + \frac{M_{-1}}{p_0} = \sum_{t=0}^{\infty} q_{0,t} (\tau_t w_t h_t - g_t) + \sum_{t=1}^{\infty} q_{0,t} i_t r_t \quad (1.13)$$

1.2.4 International Investors

International investors can always borrow and lend at a nominal interest rate of i^* in the international market. Due to assumption of perfect capital mobility, the uncovered interest rate parity condition holds:

$$(1 + i_{t+1}) = \frac{S_{t+1}}{S_t} (1 + i^*), t \geq 0.$$

Combined with the purchasing power parity condition, the following is obtained:

$$1 + i_{t+1} = (1 + i^*) \frac{S_{t+1}}{S_t} = \frac{1 + i^*}{1 + \pi^*} \frac{p_{t+1}}{p_t}, t \geq 0. \quad (1.14)$$

1.2.5 Competitive Equilibrium

Definition A competitive equilibrium is defined as a sequence $\{c_t, m_{t+1}, h_t, w_t, \Pi_t, q_{0,t}\}_{t=0}^{\infty}$, a positive constant λ , an initial price level $p_0 > 0$, and a sequence of government tax policies $\{\tau_t, i_{t+1}\}_{t=0}^{\infty}$, satisfying the conditions of (1.3), (1.4), (1.5), (1.6), (1.7), (1.9), (1.10), (1.13), (1.14), given the initial asset conditions of $\{M_{-1}, (-_1b_t^P), (-_1b_t^G), \text{ and } (-_1B_t^P), (-_1B_t^G), \forall t \geq 0\}$.

Equation (1.3) describes the clearing condition in the bond markets. Equations (1.4) -(1.7) solve the domestic household's utility maximization problem. Equations (1.9) and (1.10) solve the firms' profit maximization problem. Equation (1.13) balances the government's budget constraint. And equation (1.14) solves the international investor's borrowing/lending decision. For con-

venience, I bring the various conditions (1.3) - (1.14) together below:

$$\begin{aligned}
\sum_{t=0}^{\infty} q_{0,t} c_t + \sum_{t=1}^{\infty} q_{0,t} i_t m_t - \frac{M_{-1}}{p_0} &= \sum_{t=0}^{\infty} q_{0,t} \left(-_1 b_t^P + \frac{-_1 B_t^P}{p_t} \right) \\
&+ \sum_{t=0}^{\infty} q_{0,t} [(1 - \tau_t) w_t h_t + \Pi_t] \\
\beta^t u_{ct} &= \lambda q_{0,t}, t \geq 0 \\
\tau_t &= 1 + \frac{1}{w_t} \frac{u_{ht}}{u_{ct}}, t \geq 0 \\
\Pi_t &= z_t h_t^\eta - w_t h_t, t \geq 0 \\
i_{t+1} &= \frac{u_{mt+1}}{u_{ct+1}}, t \geq 0 \\
\frac{1}{1 + i_{t+1}} &= \frac{q_{0,t+1}/p_{t+1}}{q_{0,t}/p_t} \leq 1, t \geq 0 \\
w_t &= z_t \eta h_t^{\eta-1}, t \geq 0 \\
1 + i_{t+1} &= \frac{1 + i^*}{1 + \pi^*} \frac{p_{t+1}}{p_t}, t \geq 0 \\
\sum_{t=0}^{\infty} q_{0,t} (\tau_t w_t h_t - g_t) + \sum_{t=1}^{\infty} q_{0,t} i_t m_t &= \sum_{t=0}^{\infty} q_{0,t} \left(-_1 b_t^G + \frac{-_1 B_t^G}{p_t} \right) + \frac{M_{-1}}{p_0}.
\end{aligned}$$

1.2.6 Characterization of Competitive Equilibrium

To eliminate nonessential dynamics in consumption, I make two assumptions. First, I assume that $\beta \frac{1+i^*}{1+\pi^*} = 1$. Second, I assume that the period utility function is separable in goods, money and labor supply, taking the form of $u(c, m, h) = u(c) + v(m) + g(h)$. These two assumptions do not change this paper's results with respect to time consistent optimal policy, though they simplify the analytical computation. Given these assumptions, the discount factors will grow at the rate of β , and consumption is constant over time,

which can be seen from:

$$q_{0,t} = \left(\frac{1 + \pi^*}{1 + i^*} \right)^t = \beta^t, t \geq 0, \quad (1.15)$$

$$u_{ct} = \lambda \frac{q_{0,t}}{\beta^t} = \lambda \frac{\left(\frac{1 + \pi^*}{1 + i^*} \right)^t}{\beta^t} = \lambda, t \geq 0. \quad (1.16)$$

A third assumption I make is that $u_{mt} \geq 0$. As a result, it is always the case that $i_t \geq 0$. This assumption also does not change the paper's results with respect to time consistent optimal policy.

1.2.7 Intertemporal Budget Constraints

Using the optimality conditions, I can rewrite the government's intertemporal budget constraint containing only the initial price level, p_0 , a constant, λ , and real money and labor allocations $\{m_{t+1}, h_t\}_{t=0}^{\infty}$,

$$\begin{aligned} \frac{\lambda}{p_0} \left[\sum_{t=0}^{\infty} Q_{0,t} ({}_{-1}B_t^G) + M_{-1} \right] &= \sum_{t=0}^{\infty} \beta^t [\lambda(\eta z_t h_t^\eta - g_t - {}_{-1}b_t^G) + u_{ht} h_t] \\ &+ \sum_{t=1}^{\infty} \beta^t u_{mt} m_t, \end{aligned} \quad (1.17)$$

where $Q_{0,t} = \prod_{i=1}^t \left(1 + \frac{u_{mi}}{\lambda} \right)^{-1}$.

Similarly, combining the intertemporal budget constraint of the representative domestic household with that of the government, I obtain the intertemporal budget constraint for the small open economy:

$$\sum_{t=0}^{\infty} \beta^t [c_t + g_t + ({}_{-1}b_t^F) - z_t h_t^\eta] = -\frac{1}{p_0} \sum_{t=0}^{\infty} Q_{0,t} ({}_{-1}B_t^F), \quad (1.18)$$

where ${}_{-1}B_t^F = {}_{-1}B_t^G - {}_{-1}B_t^P$, and ${}_{-1}b_t^F = {}_{-1}b_t^G - {}_{-1}b_t^P$.

1.3 Optimal Policy with Commitment

In Appendix B, I show that sequences for $\{\lambda, m_{t+1}, \text{ and } h_t\}$ satisfying optimality conditions (1.3), (1.4), (1.5), (1.6), (1.7), (1.9), (1.10), (1.13), and (1.14), are the same as those satisfying the optimality conditions (1.17) and (1.18). Thus, when the government can commit to policy, it will choose a constant λ , an initial price level p_0 , and a sequence of $\{m_{t+1}, h_t\}_{t=0}^{\infty}$ to maximize the representative household's lifetime utility:

$$u\left(c(\lambda), \frac{M_{-1}}{p_0}, h_0\right) + \sum_{t=1}^{\infty} \beta^t u(c(\lambda), m_t, h_t), \quad (1.19)$$

subject to (1.17) and (1.18), given the initial money stock, M_{-1} , the initial real and nominal debt, $(-{}_1b_t)_{t=0}^{\infty}$ and $(-{}_1B_t)_{t=0}^{\infty}$.

Let μ_0^G and μ_0^E be the Lagrange multipliers for (1.17), the $t = 0$ government's intertemporal budget constraint, and for (1.18), the economy's intertemporal budget constraint, respectively. Then the Lagrangian associated with the $t = 0$ government is given by:

$$\begin{aligned} \mathbf{L} = & u\left[c(\lambda), \frac{M_{-1}}{p_0}, h_0\right] + \sum_{t=1}^{\infty} \beta^t u[c(\lambda), m_t, h_t] \\ & - \mu_0^G \left\{ \frac{\lambda}{p_0} \left[\sum_{t=0}^{\infty} Q_{0,t} (-{}_1B_t^G) + M_{-1} \right] \right\} \\ & + \mu_0^G \left\{ \sum_{t=0}^{\infty} \beta^t [\lambda(\eta z_t h_t^\eta - g_t - {}_{-1}b_t^G) + u_{ht} h_t] + \sum_{t=1}^{\infty} \beta^t u_{mt} m_t \right\} \\ & + \mu_0^E \left\{ \sum_{t=0}^{\infty} \beta^t [z_t h_t^\eta - c(\lambda) - g_t - {}_{-1}b_s^F] - \frac{1}{p_0} \sum_{t=0}^{\infty} Q_{0,t} (-{}_1B_t^F) \right\}, \end{aligned}$$

and the optimality condition with respect to λ is

$$\begin{aligned}
\sum_{t=0}^{\infty} \beta^t u_{ct} \frac{\partial c}{\partial \lambda} &= \mu_0^E \sum_{t=0}^{\infty} \beta^t \frac{\partial c}{\partial \lambda} - \mu_0^G \left\{ \sum_{t=0}^{\infty} \beta^t (\eta z_t h_t^\eta - g_t - {}_{-1}b_t^G) \right\} \\
&+ \frac{\mu_0^E}{p_0} \sum_{t=1}^{\infty} ({}_{-1}B_t^F) \frac{\partial Q_{0,t}}{\partial \lambda} \\
&+ \frac{\mu_0^G}{p_0} \left\{ \left[\sum_{t=0}^{\infty} Q_{0,t} ({}_{-1}B_t^G) + M_{-1} \right] + \lambda \sum_{t=1}^{\infty} ({}_{-1}B_t^G) \frac{\partial Q_{0,t}}{\partial \lambda} \right\}.
\end{aligned} \tag{1.20}$$

The lefthand side of (1.20) represents the marginal cost in terms of utility due to an increase of λ . Intuitively, when it becomes more expensive to borrow to smooth consumption, the representative household will decrease its consumption in each period. The foregone discounted present value of utility caused by the decrease in consumption is the marginal cost of the change in λ . The righthand side of (1.20) represents the corresponding marginal benefit in terms of utility, which contains four components: the first represents the increased discounted present value utility if the economy's intertemporal resource constraint is relaxed due to the decrease of consumption; the second represents the discounted present value disutility if the government's intertemporal budget constraint is relaxed due to the increase of λ ; the third component is the marginal benefit caused by the change in the discounted present value of outstanding external debt; and the last component comes from the associated change in the discounted present value of outstanding public debt.

The optimality condition with respect to m_t is:

$$\begin{aligned}
u_{mt} = & -\mu_0^G(u_{mmt}m_t + u_{mt}) + \frac{\mu_0^G \lambda}{\beta^t p_0} \sum_{s=t}^{\infty} (-_1B_s^G) \frac{\partial Q_{0,s}}{\partial m_t} \\
& + \frac{\mu_0^E}{\beta^t p_0} \sum_{s=t}^{\infty} (-_1B_s^F) \frac{\partial Q_{0,s}}{\partial m_t}, t \geq 1.
\end{aligned} \tag{1.21}$$

The lefthand side of (1.21) represents the marginal cost in utility if real money balances decrease. The righthand side of (1.21) represents the corresponding marginal benefit in utility, which has three sources: the first source is the relaxing of the government's intertemporal budget constraint; the second is the change in the discounted present value public bonds due to the change in nominal interest rates; and the last source comes from the change in external financing due to the change in nominal interest rates.

The optimality condition with respect to h_t is:

$$-u_{ht} = \mu_0^G (\lambda \eta^2 z_t h_t^{\eta-1} + u_{hht} h_t + u_{ht}) + \mu_0^E \eta z_t h_t^{\eta-1}, t \geq 0. \tag{1.22}$$

Equation (1.22) shows that labor supply is determined by equating the marginal benefit with the marginal cost. This optimality condition has the same components as the corresponding optimality condition in the closed economy. The only difference is that here the Lagrange multiplier in the second component of the right-hand side of (1.22) is the multiplier for the intertemporal budget constraint, while in the closed economy the corresponding multiplier is for the within-period resource constraint.

The optimality condition with respect to p_0 is:

$$u_{m0} M_{-1} = \mu_0^G \lambda \left[\sum_{t=0}^{\infty} Q_{0,t} (-_1B_t^G) + M_{-1} \right] + \mu_0^E \sum_{t=0}^{\infty} Q_{0,t} (-_1B_t^F). \tag{1.23}$$

There is a marginal benefit in utility due to an increase in the price level since inflation reduces the outstanding nominal public debt and the external debt. This marginal benefit is given by the righthand side of (1.23). There is also an associated marginal cost in utility due to an increase in the price level since inflation erodes real money balances. This marginal cost exactly offsets the marginal benefit in equilibrium.

A Ramsey equilibrium is defined as a choice of $(\lambda, p_0, \{h_t\}_{t=0}^\infty, \{m_{t+1}\}_{t=0}^\infty)$ satisfying the optimality conditions (1.17), (1.18), (1.20), (1.21), (1.22), and (1.23), given the initial asset positions, $\{-1B_t^G, -1B_t^F, -1b_t^G, -1b_t^F\}_{t=0}^\infty$ and M_{-1} .

In going from a small closed economy to a small open economy, there arise two potential identification problems. First, there is an identification problem in recovering the Lagrange multipliers. To see this, note that the Lagrange multipliers are the only choice variables that matter in the optimality condition with respect to labor supply and in the optimality condition with respect to price level. The first is true because the period resource constraint does not necessarily hold with equality. The second is true because the discounted values of nominal liabilities are predetermined from equation (1.17) and equation (1.18). Thus, equations (1.22) at $t \geq 1$ may be enough in recovering the two Lagrange multipliers, and equation (1.23) becomes an extra restriction on the choice of Lagrange multipliers. To solve this identification problem associated with the Lagrange multipliers, we need one policy instrument that enters (1.22) and equation (1.23) asymmetrically, and this is where real bonds have an important role.

Second, there may be another potential identification problem. Consider that the optimality conditions with respect to consumption and with respect to $t = 2$ real money balances are functions of the same set of nominal bonds

(this fact will become clear in the next section, Section 4). It is thus possible that there is again an identification problem: the predetermined $t = 2$ nominal interest rate is inconsistent with the other policy continuation. This implies that it may be impossible to recover the term structure of the real and nominal bonds. This second identification problem is not of major concern because it is very unlikely that the two optimality conditions are homogeneous in nominal bonds. We will return to this issue in detail when we discuss time consistent optimal policy without real bonds in Section 4 below.

1.4 Optimal Policy with Discretion

To prove the time consistency of optimal policy, I follow the procedure shown in PPS (2006) and in other papers in the literature. I prove time consistency by showing that the policy continuation of the $t = 0$ government satisfies the optimality conditions of the $t = 1$ government. The policy continuation of the $t = 0$ government refers to the $t = 0$ government's optimal choice of $(\lambda, h_t, m_{t+1}, p_1)$. The optimality conditions of the $t = 1$ government, (1.24) - (1.29), are the one-period updated version of the optimality conditions of the $t = 0$ government, (1.17), (1.18), (1.20), (1.21), (1.22), and (1.23).

In particular, in line with the literature, the time consistency problem becomes whether the $t = 0$ government can find a profile $({}_0B_t^G, {}_0B_t^F, {}_0b_t^G, {}_0b_t^F)$ such that the policy continuation of the $t=0$ government $(\lambda, h_t, m_{t+1}, p_1)$ and the predetermined M_0 satisfy the optimality conditions (1.24) - (1.29). If so, the optimal policy is time consistent. Otherwise, the optimal policy is not time consistent.

To facilitate the discussion, the optimality conditions of the $t = 1$ govern-

ment are rewritten as follows:

$$\sum_{t=1}^{\infty} Q_{1,t} ({}_0B_t^G) + p_1 \sum_{t=1}^{\infty} \beta^t ({}_0b_t^G) = D_{1.24} \quad (1.24)$$

$$\sum_{t=1}^{\infty} Q_{1,t} ({}_0B_t^F) + p_1 \sum_{t=1}^{\infty} \beta^t ({}_0b_t^F) = D_{1.25} \quad (1.25)$$

$$\sum_{t=2}^{\infty} \mu_1^E A_{1.26,t}({}_0B_t^F) + \sum_{t=2}^{\infty} \mu_1^G \lambda A_{1.26,t}({}_0B_t^G) = D_{1.26} \quad (1.26)$$

$$\sum_{s=t}^{\infty} \mu_1^E Q_{1,s}({}_0B_s^F) + \sum_{s=t}^{\infty} \mu_1^G \lambda Q_{1,s}({}_0B_s^G) = D_{1.27,t}, t \geq 2 \quad (1.27)$$

$$\mu_1^G (\lambda \eta^2 z_t h_t^{\eta-1} + u_{hht} h_t + u_{ht}) + \mu_1^E \eta z_t h_t^{\eta-1} = -u_{ht}, t \geq 1 \quad (1.28)$$

$$\mu_1^E \sum_{t=1}^{\infty} Q_{1,t} ({}_0B_t^F) + \mu_1^G \lambda \sum_{t=1}^{\infty} Q_{1,t} ({}_0B_t^G) = u_{m1} M_0 - \mu_1^G \lambda M_0, (1.29)$$

where

$$D_{1.24} = p_1 \sum_{t=1}^{\infty} \beta^{t-1} \left[(\eta z_t h_t^\eta - g_t) + \frac{u_{ht}}{\lambda} h_t \right] + p_1 \sum_{t=2}^{\infty} \beta^{t-1} \frac{u_{mt}}{\lambda} m_t - M_0$$

$$D_{1.25} = p_1 \sum_{t=1}^{\infty} \beta^{t-1} [z_t h_t^\eta - c(\lambda) - g_t]$$

$$A_{1.26,t} = Q_{1,t} \left[\sum_{i=2}^t \frac{u_{mi}}{\lambda(\lambda + u_{mi})} \right]$$

$$D_{1.26} = p_1 \frac{\lambda - \mu_1^E}{1 - \beta} - p_1 \mu_1^G \left[\sum_{t=1}^{\infty} \beta^{t-1} \frac{u_{ht}}{\lambda} h_t + \sum_{t=2}^{\infty} \beta^{t-1} \frac{u_{mt}}{\lambda} m_t \right]$$

$$D_{1.27,t} = \mu_1^G m_t \beta^{t-1} p_1 (\lambda + u_{mt}) + \frac{u_{mt} (1 + \mu_1^G) \beta^{t-1} p_1 (\lambda + u_{mt})}{u_{mmt}}.$$

If the $t = 1$ government follows the $t = 0$ government's policy function, all the A's and D's are functions of the policy continuation. $D_{1.26}$ and $D_{1.27,t}$ are also

functions of the Lagrange multipliers, μ_1^G and μ_1^F . $D_{1.26}$ and $D_{1.27,t}$ appear on the righthand sides of equations (1.26) and (1.27) because the Lagrange multipliers, μ_1^G and μ_1^F , are determined through equations (1.28). Proposition (1) states that in a small open economy the optimal policy is time consistent.

Proposition 1. *In a small open economy with perfect capital mobility and with Svensson timing, if the government issues both nominal and real bonds and if the government is free to choose the term structure of external bonds, the optimal monetary and fiscal policy is time consistent. This result is independent of the productivity process, the government expenditure process, and the initial asset position of the government. Further, there are many maturity structures of bonds that are capable of rendering optimal monetary and fiscal policy time consistent.*

Proof. of Proposition (1) consists of two parts, P1 and P2. The first part (P1) shows by construction that there exist many possible maturity structures (or profiles) of $({}_0B_t^G, {}_0B_t^F, {}_0b_t^G, {}_0b_t^F)$. The second part (P2) shows that the constructed term structure of bonds satisfies the solvency conditions of the $t = 0$ government and of the economy.

P1: I show by construction in six steps (in the text below, S# denotes Step #) that the optimal policy is time consistent. I then show that there exist many term structures of nominal and real bonds that can make the optimal policy time consistent by changing the arbitrary choices assumed in the construction.

S1: From equations (1.28), it is implied that the $t = 1$ government chooses the Lagrange multipliers as:

$$\mu_1^G = \mu_0^G = \mu^G; \mu_1^E = \mu_0^E = \mu^E. \quad (1.30)$$

This choice of Lagrange multipliers comes from the fact that the policy continuation of h_t satisfies the optimality conditions of the $t = 0$ government, as shown below:

$$\mu_0^G (\lambda \eta^2 z_t h_t^{\eta-1} + u_{hht} h_t + u_{ht}) + \mu_0^E \eta z_t h_t^{\eta-1} = -u_{ht}, t \geq 0.$$

It is clear from equations (1.28) that the optimal choice of labor supplies by the $t = 1$ government is sensitive to the Lagrange multipliers. By choosing the same Lagrange multipliers, we can guarantee that the continuation of labor supply will satisfy equations (1.28). However, constant Lagrange multipliers do not rule out the possibility that under certain circumstances, it is possible for the $t = 1$ government to choose different Lagrange multipliers from those of the $t = 0$ government. In that case, the choice of constant Lagrange multipliers is among the $t = 1$ government's choice set. For example, if the productivity is constant, equations (1.28) reduce to a single equation and the $t = 1$ government can have many choices. In another words, the $t = 0$ government gains an implicit policy instrument.

The choice of constant Lagrange multipliers in equation (1.30) is a very strong result derived from this framework. It says that governments want to keep the marginal financing costs constant over time in order to have time consistent optimal policy. In this model, constant Lagrange multipliers imply constant marginal financing costs only if optimal policy is time consistent. This comes from the fact that the product of $\lambda \mu^G$ denotes the marginal public financing cost; while the Lagrange multiplier μ^E denotes the marginal external financing cost.

In the context of this small open economy, the constant marginal financing costs do not imply constant labor income tax rates, however, because labor supplies can change over time. The constant marginal financing costs also do not imply constant nominal interest rates because real money balances can change over time. This choice of constant Lagrange multipliers reflects the fact that the policy continuation of labor supply puts a restriction on the $t = 1$ government's choice of Lagrange multipliers and that governments want to rule out the effect of endogenous responses of labor supply on the time consistency of optimal policy.

In the closed economy, it does not have to consider the possible restriction due to the policy continuation of labor supply because there is a Lagrange multiplier for every period resource constraint, and these Lagrange multipliers can change over time. The changing Lagrange multipliers in the period resource constraint will absorb the effect of endogenous responses in labor supply.

S2: To reveal the term structure of nominal and real bonds, I make the following arbitrary assumptions,

$$({}_0b_t^F) = ({}_0\tilde{b}_t^F), t \geq 2; ({}_0b_t^G) = ({}_0\tilde{b}_t^G), t \geq 1; ({}_0B_t^G) = ({}_0\tilde{B}_t^G), t \geq 3$$

where the variables $({}_0\tilde{b}_t^F)$, $({}_0\tilde{b}_t^G)$, and $({}_0\tilde{B}_t^G)$ denote the values arbitrarily chosen for $({}_0b_t^F)$, $({}_0b_t^G)$, and $({}_0B_t^G)$, respectively. There are two layers of arbitrary assumptions here. First, the values for these bonds are arbitrary. Second, the format of assumption (1.31) itself is arbitrary in the sense that I can interchange the superscript of G by F and vice versa across time.

Note that the arbitrary values for $({}_0B_t^G)$ start at $t = 3$ instead of

$t = 2$. This comes from the fact that there is no period resource constraint, rather an intertemporal resource constraint in this small open economy. Due to that fact, when Lagrange multipliers and $\{(0B_t^G), (0B_t^F)\}_{t=3}^\infty$ are found, equation (1.26) and equation (1.27) at $t = 2$ will become two equations in two unknowns, $(0B_2^G)$ and $(0B_2^F)$. So, there is no extra degree of freedom in arbitrarily choosing $(0B_2^G)$ for the $t = 0$ government, given Lagrange multipliers and $\{(0B_t^G), (0B_t^F)\}_{t=3}^\infty$.

The fact that the arbitrary values for $(0b_t^F)$ and $(0b_t^G)$ include all the values after $t = 2$ reflects that there is no real time inconsistency problem and that the term structure of real bonds is indeterminate. Further, the arbitrary values for $(0b_t^G)$ actually start at $t = 1$. This is due to the equivalence between $(0b_1^G)$ and $(0B_1^G)$ that occurs if the $t = 1$ government follows the policy continuation. This point will become more evident in the following steps.

Assumption (1.31) is of interest because it applies to the case in which the $t = 0$ government has the maximum number of degrees of freedom in choosing the term structure of nominal bonds in order to render time consistent optimal policy. In fact, this assumption says that the $t = 0$ government should pay attention only to one particular multiple-period public bond, and the government is free in choosing all other nominal public bonds.

S3: Given the choice of constant Lagrange multipliers, subtracting equation (1.27) held at $t = S$ from equation (1.27) held at $t = S + 1$ produces the following equation involving $(0B_S^F)$ and $(0B_S^G)$:

$$\mu^E Q_{1,S}(0B_S^F) + \mu^G \lambda Q_{1,S}(0B_S^G) = D_{1.27,S} - D_{1.27,S+1}, S \geq 2 \quad (32)$$

Equation (1.32) is the only generic restriction on the term structure of nominal bonds. It shows that if the government wants to decrease government financing by one unit, it must increase external financing by $\mu^G \lambda / \mu^E$ units. The economic interpretation of equation (1.32) is that once the $t = 1$ government follows the policy continuation, it does not care about the particular source of financing.

Equation (1.32) and assumption (1.31) are sufficient to pin down $({}_0B_S^F)$ and $({}_0B_S^G)$ for any $S \geq 3$. Equation (1.26) and equation (1.27) held at $t = 2$ are two equations in two unknowns, $({}_0B_2^F)$ and $({}_0B_2^G)$, and I can consequently solve for the nominal bond holding positions at $t = 2$.

- S4: After $({}_0B_S^F)$ and $({}_0B_S^G)$ for all $S \geq 2$ are solved, equation (1.24) is an equation in one unknown $({}_0B_1^G)$ given assumption (1.31), and I can solve for it. This construction of $({}_0B_1^G)$ shows that there is one-to-one relation between $({}_0B_1^G)$ and $({}_0b_1^G)$. The $t = 1$ government has a degree of freedom to choose one of these two, and once the value for $({}_0b_1^G)$ is chosen, a corresponding value for $({}_0B_1^G)$ is determined.
- S5: I plug the solutions of $({}_0B_S^F), S \geq 2$ and $({}_0B_S^G), S \geq 1$, into equation (1.29), and I can then solve for $({}_0B_1^F)$. The $t = 0$ government has to choose $({}_0B_1^F)$ in such a way that, under assumption (1.31), the benefit of surprise inflation is completely neutralized by the cost of surprise inflation.
- S6: Finally, I plug all the solved optimal bond holding positions into equation (1.25), and can solve then for $({}_0b_1^F)$. This shows that under assumption (1.31), the choice of $({}_0b_1^F)$ has to satisfy the intertem-

poral budget constraint of the $t = 1$ economy.

Thus, I construct a maturity structure of bonds and a solution to the Lagrange multipliers under arbitrary assumption (1.31). With the constructed term structure of bonds and the solution to the Lagrange multipliers, the optimal policy is time consistent because the policy continuation satisfies the optimality conditions of the $t = 1$ government. Since assumption (1.31) is arbitrary, I can change the values in that assumption to construct different maturity structures of bonds and different solutions for the Lagrange multipliers, which will together make the optimal policy time consistent. The proof of P1 is completed so that when the government issues both nominal and real bonds, the government can choose many term structures of bonds to make the optimal policy time consistent.

P2: In this part, I show that the constructed term structure of bonds is consistent with the $t = 0$ government's intertemporal budget constraint and with the $t = 0$ economy's intertemporal budget constraint.

The proof is presented in Appendix G. ■

□

The main reason time consistent optimal policy arises is that the current period's government always has more policy instruments than the $t = 1$ government has policy choices. In particular, the choices of the $t = 1$ government include all nominal interest rates and only one labor income tax rate as opposed to all labor income tax rates. This is true because in the Ramsey problem, the other optimal labor income tax rates are pinned down by the exogenously

given real interest rates. The policy instruments of the $t = 0$ government are: both nominal public and external bonds from $t = 3$ onward; the present values of real public and external bonds; and, possibly, Lagrange multipliers.

With real bonds, the mechanism for time consistent optimal policy is the following: the $t = 0$ government uses real bonds to control the $t = 1$ government's choice of one labor income tax rate; and uses nominal bonds to influence the $t = 1$ government's choices of $t > 1$ nominal interest rates. When the optimal policy is time consistent, there is an equivalence between the government's nominal liability and the corresponding external nominal liability. This equivalence gives rise to an indeterminacy of the term structure of nominal bonds.

Several assumptions are crucial to obtain the above conclusion. (a) Real interest rates are assumed to be exogenous. This assumption guarantees that there is no real time inconsistency from the change of real interest rates. It then follows that the term structure of real bonds is indeterminate. (b) Svensson timing is assumed: the beginning of period money enters the utility function so that there is a direct cost associated with surprise inflation. This Svensson timing mechanism gives governments freedom in choosing non-zero nominal bonds to smooth household consumption. It is thus the case that governments have far more nominal instruments to choose than the nominal choices they want to influence. And (c) domestic households are assumed to coordinate fully, which affords the government full control in choosing the level of external debt. In Appendix H, I discuss the case when the assumption (c) does not hold.

1.4.1 An Economy without Nominal Bonds

In the closed economy, the government has to issue both nominal and real bonds to render optimal policy time consistent. This subsection explores whether an economy without nominal bonds has time consistent optimal policy. Note that in the discussion about the assumption (c) as shown in the appendix, a restricted labor market plays a role in rendering optimal policy time consistent, especially in the discussion of moving away from the second scenario. This subsection will thus start with the restriction that productivity is constant over time, the special assumption mentioned in the previous discussion.

To facilitate this discussion, it is convenient to rewrite the optimality conditions corresponding to (1.24) - (1.29) as:

$$0 = \sum_{t=1}^{\infty} \beta^{t-1} [c(\lambda) + g_t - h^\eta] + \sum_{t=1}^{\infty} \beta^{t-1} ({}_0b_t^F) \quad (1.33)$$

$$0 = \sum_{t=1}^{\infty} \beta^{t-1} [\lambda(\eta h^\eta - g_t) + u_h h] + \sum_{t=2}^{\infty} \beta^{t-1} u_{mt} m_t - \sum_{t=1}^{\infty} \beta^{t-1} \lambda ({}_0b_t^G) - \frac{\lambda M_0}{p_1} \quad (1.34)$$

$$\frac{\lambda - \mu_1^E}{1 - \beta} = \mu_1^G \sum_{t=1}^{\infty} \beta^{t-1} (\eta h^\eta - g_t) - \mu_1^G \sum_{t=1}^{\infty} \beta^{t-1} ({}_0b_t^G) - \frac{\mu_1^G M_0}{p_1} \quad (1.35)$$

$$-u_h = \mu_1^G (\lambda \eta^2 h^{\eta-1} + u_{hh} h + u_h) + \mu_1^E \eta z_t h_t^{\eta-1}, t \geq 1 \quad (1.36)$$

$$u_{mt} (1 + \mu_1^G) = -\mu_1^G m_t u_{mmt}, t \geq 2. \quad (1.37)$$

$$u_{m1} = \mu_1^G \lambda. \quad (1.38)$$

From equation (1.37), the real balances from $t = 2$ onward are constant.

Comparing this equation to the corresponding optimality condition for the $t = 0$ government, it is obvious that the choice of μ_1^G is given by $\mu_1^G = \mu_0^G$. The constant μ^G will imply constant μ^E if considering equation (1.36). Using this result and comparing equation (1.38) with the corresponding optimality condition for the $t = 0$ government, it is obvious that $m_0 = m_1$, implying that real money balances are constant over the full horizon. The outcome of constant real money balances comes from the fact that the Ramsey problem becomes a completely real-economy problem, as characterized by equations (1.33) - (1.38). This is achieved because the only role played by p_0 is to adjust the nominal money balance M_{-1} such that the real money balances are constant.

Given the constant Lagrange multipliers, the $t = 0$ government chooses the real bond holdings as:

$$\sum_{t=1}^{\infty} \beta^{t-1}({}_0b_t^F) = g_0 + \sum_{t=0}^{\infty} \beta^t({}_{-1}b_t^F)$$

$$\sum_{t=1}^{\infty} \beta^{t-1}({}_0b_t^G) = g_0 + \sum_{t=0}^{\infty} \beta^t({}_{-1}b_t^G).$$

It can be shown by simple algebra that equation (1.33), (1.34), and (1.35) are the same as those of the $t = 0$ government. It is clear that the $t = 1$ government solves the identical Ramsey problem as the $t = 0$ government does, which implies time consistent optimal policy under the assumption of constant labor productivity.

Next, I show that if I relax the assumption of constant productivity, then the optimal policy is time inconsistent. This is due to the fact that in order to have time consistent optimal policy, labor supply must be constant to isolate the endogenous effect of labor supply on consumption. If productivity is not

constant, labor supply will change over time and the governments will not have enough policy instruments.

When nominal bonds are not available, on the one hand, the $t = 1$ government cannot use nominal interest rates as policy choices, and the only choice is the labor income tax rate at $t = 1$. On the other hand, the $t = 0$ government does not have any explicit policy instruments because the liquidity constraints take away the degree of freedom in choosing real bonds. However, the government may have an implicit policy instrument, one of the Lagrange multipliers. When this implicit policy instrument is available, as in the case when productivity is constant, optimal policy is time consistent. Even though these implicit policy instruments are ignored in the discussion of closed economies, they are the reason why the government in the closed economy does not have to worry about changes in the labor market in rendering optimal policy time consistent.

1.4.2 An Economy without Real Bonds

In this subsection, I assume that only nominal bonds and money are available. This is an interesting scenario because: in some open economies, the bonds issued are mainly nominal bonds; and in the closed economy, real bonds are needed to assure time consistency of optimal policy.

Equations of Labor Supply Do Not Pin Down Lagrange Multipliers

Since the labor market plays a role in rendering optimal policy time consistent, this subsection starts with the assumption that productivity is constant, i.e., equations (1.43) are not sufficient to pin down the Lagrange multipliers. This gives the $t = 0$ government an implicit policy instrument. As before, I show

time consistency by constructing a term structure and a solution to Lagrange multipliers. The optimality conditions faced by the $t = 1$ government are the updated version of equations (1.24) - (1.29):

$$0 = \sum_{t=1}^{\infty} \beta^{t-1} [c(\lambda) + g_t - h^\eta] + \frac{1}{p_1} \sum_{t=1}^{\infty} Q_{1,t} ({}_0B_t^F) \quad (1.39)$$

$$0 = \sum_{t=1}^{\infty} \beta^{t-1} [\lambda(\eta h^\eta - g_t) + u_h h] + \sum_{t=2}^{\infty} \beta^{t-1} u_{mt} m_t \quad (1.40)$$

$$\begin{aligned} & -\frac{\lambda}{p_1} \left[\sum_{t=1}^{\infty} Q_{1,t} ({}_0B_t^G) + M_0 \right] \\ \frac{\lambda - \mu_1^E}{1 - \beta} &= \mu_1^G \sum_{t=1}^{\infty} \beta^{t-1} (\eta h^\eta - g_t) - \frac{\mu_1^G}{p_1} \left[\sum_{t=1}^{\infty} Q_{1,t} ({}_0B_t^G) + M_0 \right] \\ & + \frac{\mu_1^E}{p_1} \sum_{t=2}^{\infty} Q_{1,t} \left[\sum_{i=2}^t \frac{u_{mi}}{\lambda(\lambda + u_{mi})} \right] ({}_0B_t^F) \\ & - \frac{\mu_1^G \lambda}{p_1} \sum_{t=2}^{\infty} Q_{1,t} \left[\sum_{i=2}^t \frac{u_{mi}}{\lambda(\lambda + u_{mi})} \right] ({}_0B_t^G) \end{aligned} \quad (1.41)$$

$$\begin{aligned} u_{mt} (1 + \mu_1^G) &= \left[\frac{\mu_1^E \sum_{s=t}^{\infty} ({}_0B_s^F) Q_{1,s}}{\beta^{t-1} p_1 (\lambda + u_{mt})} + \frac{\mu_1^G \lambda \sum_{s=t}^{\infty} ({}_0B_s^G) Q_{1,s}}{\beta^{t-1} p_1 (\lambda + u_{mt})} \right] u_{mmt} \\ & - \mu_1^G m_t u_{mmt}, t \geq 2 \end{aligned} \quad (1.42)$$

$$-u_h = \mu_1^G (\lambda \eta^2 h^{\eta-1} + u_{hh} h + u_h) + \mu_1^E \eta h^{\eta-1} \quad (1.43)$$

$$u_{m1} M_0 = \mu_1^G \lambda \left[\sum_{t=1}^{\infty} Q_{1,t} ({}_0B_t^G) + M_0 \right] + \mu_1^E \sum_{t=1}^{\infty} Q_{1,t} ({}_0B_t^F). \quad (1.44)$$

From the intertemporal budget constraint of the $t = 1$ government, (1.40), the discounted present value of nominal debt is given by

$$\begin{aligned} \left[\sum_{t=1}^{\infty} Q_{1,t} ({}_0B_t^G) + M_0 \right] &= p_1 \sum_{t=1}^{\infty} \beta^{t-1} \left[(\eta h^\eta - g_t) + \frac{u_h}{\lambda} h \right] \\ &+ p_1 \sum_{t=2}^{\infty} \beta^{t-1} \frac{u_{mt}}{\lambda} m_t. \end{aligned} \quad (1.45)$$

From the intertemporal budget constraint of the small open economy, (1.39), the discounted present value of external nominal bonds is given by

$$\sum_{t=1}^{\infty} Q_{1,t} ({}_0B_t^F) = p_1 \sum_{t=1}^{\infty} \beta^{t-1} [h^\eta - c(\lambda) - g_t]. \quad (1.46)$$

Plugging (1.45) and (1.46) into (1.44), I get the following equation involving μ_1^G and μ_1^E

$$\begin{aligned} u_{m1} M_0 &= \mu_1^E p_1 \sum_{t=1}^{\infty} \beta^{t-1} [h^\eta - c(\lambda) - g_t] \\ &+ \mu_1^G \lambda \left\{ p_1 \sum_{t=1}^{\infty} \beta^{t-1} \left[(\eta h^\eta - g_t) + \frac{u_h}{\lambda} h \right] + \sum_{t=2}^{\infty} \beta^{t-1} \frac{u_{mt}}{\lambda} m_t \right\}. \end{aligned} \quad (1.47)$$

Equation (1.47) and (1.43) are two linear equations in two unknowns, and I solve for μ_1^G and μ_1^E . The next is to see whether it is possible to find $({}_0B_t^G)$ and $({}_0B_t^F)$ that satisfy equation (1.41), which holds in period $t = 1$, equation (1.42), which holds in each period $t \geq 2$, equation (1.39), and equation (1.40). To simplify the discussion, I rewrite the rest of the optimality conditions as

follows:

$$\sum_{t=2}^{\infty} \mu_1^E A_{1.41,t}({}_0B_t^F) + \sum_{t=2}^{\infty} \mu_1^G \lambda A_{1.41,t}({}_0B_t^G) = D_{1.41} \quad (1.48)$$

$$\sum_{s=t}^{\infty} \mu_1^E A_{1.42,s}({}_0B_s^F) + \sum_{s=t}^{\infty} \mu_1^G \lambda A_{1.42,s}({}_0B_s^G) = D_{1.42,t}, t \geq 2 \quad (1.49)$$

$$\sum_{t=1}^{\infty} A_{1.39,t}({}_0B_t^G) = D_{1.39} \quad (1.50)$$

$$\sum_{t=1}^{\infty} A_{1.40,t}({}_0B_t^F) = D_{1.40}, \quad (1.51)$$

where

$$A_{1.41,t} = Q_{1,t} \left[\sum_{i=2}^t \frac{u_{mi}}{\lambda(\lambda + u_{mi})} \right]$$

$$D_{1.41} = p_1 \frac{\mu_1^E - \lambda}{1 - \beta} - p_1 \mu_1^G \left[\sum_{t=1}^{\infty} \beta^{t-1} \frac{u_h}{\lambda} h + \sum_{t=2}^{\infty} \beta^{t-1} \frac{u_{mt}}{\lambda} m_t \right]$$

$$A_{1.42,s} = Q_{1,s}$$

$$D_{1.42,t} = \mu_1^G m_t \beta^{t-1} p_1 (\lambda + u_{mt}) + \frac{u_{mt} (1 + \mu_1^G) \beta^{t-1} p_1 (\lambda + u_{mt})}{u_{mmt}}$$

$$A_{1.39,t} = Q_{1,t} = A_{1.40,t}$$

$$D_{1.39} = p_1 \sum_{t=1}^{\infty} \beta^{t-1} \left[(\eta h^\eta - g_t) + \frac{u_h}{\lambda} h \right] + p_1 \sum_{t=2}^{\infty} \beta^{t-1} \frac{u_{mt}}{\lambda} m_t - M_0$$

$$D_{1.40} = p_1 \sum_{t=1}^{\infty} \beta^{t-1} [h^\eta - c(\lambda) - g_t].$$

As before, if the $t = 1$ government follows the $t = 0$ government's policy function, all A's and D's are functions of the policy continuation. In addition, $D_{1.41}$ and $D_{1.42,t}$ are also functions of the Lagrange multipliers, μ_1^G and μ_1^F . I place $D_{1.41}$ and $D_{1.42,t}$ on the right-hand sides of equation (1.48) and (1.49)

because the Lagrange multipliers, μ_1^G and μ_1^F , are determined through equation (1.47) and (1.43).

Since both equation (1.48) and equation (1.49) at time $t = 2$ are functions of $\{(0B_t^G), (0B_t^F)\}_{t=2}^\infty$, there is a potential identification problem: it is possible that there is no solution of $(0B_t^G)$ and $(0B_t^F)$ if the following condition is satisfied

$$\frac{A_{1.41,t}}{A_{1.42,t}} \equiv k \neq \frac{D_{1.41}}{D_{1.42,2}}, \forall t, \quad (1.52)$$

where k is a constant. When condition (1.52) is satisfied, there is no solution, i.e., the $t = 0$ government cannot find $\{(0B_t^G), (0B_t^F)\}_{t=2}^T$ to make the Ramsey policy time consistent. The economic argument underlying the scenario is as follows: every nominal bond holding of $\{(0B_t^G), (0B_t^F)\}_{t=2}^T$ will have an effect on the smoothed consumption, and on the real money balance in the period $t = 2$ through the change in nominal interest rates. If the two effects from any nominal bond holding are in the same proportion, i.e., $\frac{A_{1.41,t}}{A_{1.42,t}} \equiv k, \forall t$, then the net effects due to other non-bond channels on the consumption should be in the same proportion to the net effects due to other channels on the real money balance holding in the period $t = 2$. In another words, $k = \frac{D_{1.41}}{D_{1.42,2}}$. Whenever condition (1.52) holds, i.e., $\frac{A_{1.41,t}}{A_{1.42,t}} \equiv k \neq \frac{D_{1.41}}{D_{1.42,2}}$, no $\{(0B_t^G), (0B_t^F)\}_{t=2}^T$ can be found to satisfy equation (1.48) and (1.49) at the same time. It is straightforward to check whether condition (1.52) holds or not because all the variables in (1.52) are functions of the Ramsey policy continuation.

The first part of condition (1.52), $\frac{A_{1.41,t}}{A_{1.42,t}} \equiv k$, is possible whenever $u_{mt} = 0, \forall t \geq 3$. Given the setup of the problem, it is possible to have a particular initial asset position for the $t = 0$ government to choose certain optimal choices

that make condition (1.52) hold. The economic interpretation for this result is that the initial asset condition of the $t = 0$ government matters. In the closed nominal economy without real bonds, the initial asset position of the $t = 0$ government does not matter because the real interest rates are of governments' choices. However, this identification problem is not a big concern because it only happens in very few cases.

Now suppose the condition (1.52) does not hold so that the potential identification problem associated with the $t = 2$ nominal interest rate does not exist. The difference between equation (1.49) at time S and time $S + 1$ will give the following relation between $({}_0B_S^G)$ and $({}_0B_S^F)$:

$$\mu_1^E A_{1.42,S}({}_0B_S^F) + \mu_1^G \lambda A_{1.42,S}({}_0B_S^G) = D_{1.42,S} - D_{1.42,S+1}, S \geq 3.$$

In addition, I can impose the following extra restrictions:

$${}_0B_S^G = {}_0\tilde{B}_S^G, S \geq 3. \tag{1.53}$$

With these two equations held in each period since $S = 3$, I can pin down $({}_0B_S^G)$ and $({}_0B_S^F)$ for all $S \geq 3$. I then use equation (1.48) and equation (1.49) to solve for $({}_0B_2^G)$ and $({}_0B_2^F)$; and use equation (1.50) and equation (1.51) to solve for $({}_0B_1^G)$ and $({}_0B_1^F)$. Thus I show there exists a unique term structure of nominal bonds that assures the time consistency of optimal policy with arbitrary assumptions. It is then clear that by changing the values of the arbitrary assumptions, I can find many term structures of nominal bonds that are consistent with the optimal and time consistent policy.

To make the result in this case more straightforward, one way is to compare the number of policy instruments the $t = 0$ government has and the number of constraints the $t = 0$ government has, as I have shown before. However, when real bonds are not available, the government needs one more instrument

to control for the choice of the particular labor income tax rate. In this case, the implicit policy instrument (the Lagrange multipliers) takes that role and functions the same way as real bonds do; and thus optimal policy is again time consistent.

Equations of Labor Supply Pin Down Lagrange Multipliers

Suppose equations (1.43) for $t \geq 1$ pin down the Lagrange multipliers, so that Lagrange multipliers for the $t = 1$ government should be the same at the $t = 0$ government as shown in the previous discussion. In this case, equation (1.44) imposes an extra restriction on these Lagrange multipliers because $[\sum_{t=1}^{\infty} Q_{1,t}({}_0B_t^G) + M_0]$ and $\sum_{t=1}^{\infty} Q_{1,t}({}_0B_t^F)$ in equation (1.44) are predetermined if the $t = 1$ government follows the policy continuation. This is another potential identification problem associated with Lagrange multipliers. To overcome this identification problem, condition (1.54) must be satisfied. To see this, given the budget constraints of the $t = 0$ government and the $t = 0$ economy, and given the budget constraints of the $t = 1$ government and the $t = 1$ economy, I can obtain the optimality condition with respect to p_0 :

$$u_{m0}m_0 = \frac{\mu^G \lambda}{p_0} \left[\sum_{t=0}^{\infty} Q_{0,t}(-{}_1B_t^G) + M_{-1} \right] + \frac{\mu^E}{p_0} \sum_{t=0}^{\infty} Q_{0,t}(-{}_1B_t^F),$$

and the optimality condition with respect to p_1

$$u_{m1}m_1 = \frac{\mu^G \lambda}{p_1} \left[\sum_{t=1}^{\infty} Q_{1,t}({}_0B_t^G) + M_0 \right] + \frac{\mu^E}{p_1} \sum_{t=1}^{\infty} Q_{1,t}({}_0B_t^F),$$

in order to obtain

$$\begin{aligned} u_{m0}m_0 - u_{m1}m_1 &= \mu^G \lambda \left[(\eta z_0 h_0^\eta - g_0) + \frac{u_{h0}}{\lambda} h_0 + \frac{u_{m1}}{\lambda} m_1 \right] \\ &\quad + \mu^E [z_0 h_0^\eta - c(\lambda) - g_0]. \end{aligned} \tag{1.54}$$

Condition (1.54) tests whether the optimality equation of the $t = 1$ government can be satisfied by the policy continuation. Condition (1.54) says that given the government chooses constant marginal domestic and international financing costs, (i.e., the constant Lagrange multipliers,) the government must assure that difference of costs due to changing prices in $t = 0$ and $t = 0$, $u_{m0}m_0 - u_{m1}m_1$, are equal to the marginal benefit due to the relax of period budgets. However, all the variables besides the Lagrangian multipliers in condition (1.54) are predetermined variables, so it is not necessarily true that condition (1.54) holds. This implies that the optimal policy is not time consistent. Proposition (2) summarizes:

Proposition 2. *In a nominal small open economy without real bonds, with perfect capital mobility, and with Svensson timing, when equations of labor supply are not sufficient to pin down Lagrange multipliers, the optimal policy is time consistent if condition (1.52) does not hold for each government. Condition (1.52) may hold under a particular initial asset position. When equations of labor supply are sufficient to pin down the Lagrange multipliers, the optimal policy is not time consistent unless condition (1.54) holds for each government. Condition (1.54) may hold under a particular initial asset position.*

In the following subsection, I show how time consistency of optimal policy is assured with a special example.

Assume the Friedman Rule is Optimal

In the related literature, optimal monetary is usually characterized by the Friedman rule, which says that it is optimal to set nominal interest rates at zero. Thus, it is interesting to explore whether the optimality of the Friedman rule can be a part of the optimal monetary policy in this small open economy.

In addition, when the Friedman rule is optimal, the $t = 1$ government does not have freedom in choosing nominal interest rates. As a result, it is also important to know how the findings with respect to time consistent optimal policy in this small open economy will change when the Friedman rule is assumed to be optimal under commitment. To initiate the discussion, I assume that the Friedman rule is optimal. As before I start with the assumption of constant productivity and rewrite the corresponding optimality conditions as:

$$0 = \sum_{t=1}^{\infty} \beta^{t-1} [c(\lambda) + g_t - h^\eta] + \frac{1}{p_1} \sum_{t=1}^{\infty} ({}_0B_t^F) \quad (1.55)$$

$$0 = \sum_{t=1}^{\infty} \beta^{t-1} [\lambda(\eta h^\eta - g_t) + u_h h] - \frac{\lambda}{p_1} \left[\sum_{t=1}^{\infty} ({}_0B_t^G) + M_0 \right] \quad (1.56)$$

$$\frac{\lambda - \mu_1^E}{1 - \beta} = \mu_1^G \sum_{t=1}^{\infty} \beta^{t-1} (\eta h^\eta - g_t) - \frac{\mu_1^G}{p_1} \left[\sum_{t=1}^{\infty} ({}_0B_t^G) + M_0 \right] \quad (1.57)$$

$$0 = \mu_1^G m_t u_{mmt} \quad (1.58)$$

$$-u_h = \mu_1^G (\lambda \eta^2 h^{\eta-1} + u_{hh} h + u_h) + \mu_1^E \eta h^{\eta-1} \quad (1.59)$$

$$0 = \mu_1^G \lambda \left[\sum_{t=1}^{\infty} ({}_0B_t^G) + M_0 \right] + \mu_1^E \sum_{t=1}^{\infty} ({}_0B_t^F). \quad (1.60)$$

From the above optimality conditions, it is clear that only the stock of outstanding nominal liabilities matters for equilibrium. Nominal liabilities appear in equation (1.57), (1.60), (1.55), and (1.56). When the government expenditure is constant, it is straightforward to show that the following choices of nominal liabilities will make the optimal policy time consistent:

$$\sum_{t=1}^{\infty} ({}_0B_t) + M_0 = p_1^1 \sum_{t=1}^{\infty} \beta^{t-1} (\lambda \eta h^\eta + u_h h - \lambda g)$$

$$\sum_{t=1}^{\infty} ({}_0B_t^F) = p_1^1 \sum_{t=1}^{\infty} \beta^{t-1} (h^\eta - c(\lambda) - g).$$

where the price level is characterized by:

$$p_1^0 = \beta p_0 \tag{1.61}$$

which is exactly what has been shown in the Section 3. These nominal liability positions will make the policy continuation of the $t = 0$ government satisfy equations (1.57), (1.60), (1.55), and (1.56). In this case, everything repeats itself. Optimal policy is time consistent.

However, when government expenditure is not constant, equation (1.57) puts an extra restriction on the choices of μ_1^G , i.e., it has an identification problem. This implies that when government expenditures are changing over time, the Friedman rule cannot be simultaneously time consistent and optimal.

When the equations of labor supply are sufficient to pin down the Lagrange multipliers, it is subject to the same identification problem considered in the previous subsection. I summarize this in the following proposition:

Proposition 3. *In a nominal small open economy without real bonds, with perfect capital mobility, and with Svenssen timing, the optimal policy can be characterized by the optimality of the Friedman rule and is time consistent only if government expenditures are constant and if equations of labor supply are not sufficient to pin down the Lagrange multipliers,.*

1.4.3 Perfect Substitution between Implicit Instruments and Explicit Instruments

As shown in the previous sections, optimal policy may be time consistent, in the economies either without real bonds or without nominal bonds, depending on the availability of implicit policy instrument. Proposition (4) summarizes this result:

Proposition 4. *The implicit policy instrument is a perfect substitute of real bonds and of nominal bonds. When implicit policy instruments are available, the governments can issue either nominal bonds or real bonds to render optimal policy time consistent.*

1.4.4 The Time Horizon of Bonds

This subsection address the question under what conditions the time horizon of bonds will be finite. The question is interesting because given the general solution, any term structure of bonds that guarantees the time consistent optimal policy will potentially be over the infinite horizon. However this result cannot have a practical value for central bankers because it is impossible to design and organize a term structure of bonds over the infinite horizon. Here I claim that the time horizon of the bonds to guarantee time consistent optimal policy can be finite.

In the nominal small open economy with both real bonds and nominal bonds, the horizon for government bonds can be finite, which can be seen from assumption (1.31). In the nominal small open economy without real bonds, either domestic or international bond positions can be over finite periods. This can be seen from assumption (1.53). The key reason for the finite time horizon of either type of bonds is that the $t = 0$ government has far more policy instruments than the $t = 1$ government has policy choices. However, whether both public and private bond holding positions are over finite periods is of particular interest. Now turning to the example economy in the section 1.4.2 and assuming that condition (1.54) is satisfied for each period, so it is true that optimal policy is time consistent. Then the question of interest is whether the horizon for the nominal bonds is finite or infinite if the $t = 0$

initial nominal bonds are over finite periods?

To discuss this question, I start with one simple case by making the following assumptions: (a) The $t = 0$ government inherits nominal bonds over two or three periods where $\mu^E(-_1B_t^F) + \mu^G\lambda(-_1B_t^G) = 0, t = 1, 2$; this is to guarantee that the real money balances are constant from $t = 1$ onward and that optimal policy is characterized by $m_0 = m_1$; (b) the $t = 0$ government leaves the $t = 1$ government with nominal bond positions over three periods and $\mu^E({}_0B_t^F) + \mu^G\lambda({}_0B_t^G) = 0, t = 2, 3$; this is to guarantee that the policy continuation of real money balances satisfies the optimality conditions of the $t = 1$ government; and (c) the $t = 0$ government gives the $t = 1$ government nominal bonds over three periods. These assumption imply that the optimality

conditions of the $t = 0$ government without real bonds are given by

$$0 = \sum_{t=0}^{\infty} \beta^t [c(\lambda) + g_t - h^\eta] - \frac{1}{p_0} \sum_{t=0}^2 Q_{0,t} (-_1B_t^F) \quad (1.62)$$

$$0 = \sum_{t=0}^{\infty} \beta^t [\lambda(\eta h^\eta - g_t) + u_h h] + \sum_{t=1}^{\infty} \beta^t u_m m - \frac{\lambda}{p_0} \left[\sum_{t=0}^2 Q_{0,t} (-_1B_t^G) + M_{-1} \right] \quad (1.63)$$

$$\frac{\lambda - \mu^E}{1 - \beta} = \mu^G \sum_{t=0}^{\infty} \beta^t (\eta h^\eta - g_t) + \frac{\mu^E}{p_0} \sum_{t=1}^2 Q_{0,t} \left[\sum_{i=1}^t \frac{u_{mi}}{\lambda(\lambda + u_{mi})} \right] (-_1B_t^F) - \frac{\mu^G}{p_0} \left[\sum_{t=0}^2 Q_{0,t} (-_1B_t^G) + M_{-1} \right] \quad (1.64)$$

$$- \frac{\mu^G \lambda}{p_0} \sum_{t=1}^2 Q_{0,t} \left[\sum_{i=1}^t \frac{u_{mi}}{\lambda(\lambda + u_{mi})} \right] (-_1B_t^G)$$

$$u_{m1} (1 + \mu^G) = \left[\frac{\mu^E \sum_{s=1}^2 (-_1B_s^F) Q_{0,s}}{\beta^t p_0 (\lambda + u_{m1})} + \frac{\mu^G \lambda \sum_{s=1}^2 (-_1B_s^G) Q_{0,s}}{\beta^t p_0 (\lambda + u_{m1})} \right] u_{mm1} - \mu^G m_1 u_{mm1} \quad (1.65)$$

$$u_{m2} (1 + \mu^G) = \left[\frac{\mu^E (-_1B_2^F) Q_{0,2}}{\beta p_0 (\lambda + u_{m2})} + \frac{\mu^G \lambda (-_1B_2^G) Q_{0,2}}{\beta p_0 (\lambda + u_{m2})} - \mu^G m_2 \right] u_{mm2}, \quad (1.66)$$

$$u_{mt} (1 + \mu^G) = -\mu^G m_t u_{mmt}, t \geq 3 \quad (1.67)$$

$$u_{ht} = \mu^G (\lambda \eta^2 z_t h_t^{\eta-1} + u_{hht} h_t + u_{ht}) + \mu^E \eta h_t^{\eta-1}, t \geq 0 \quad (1.68)$$

$$\frac{u_{m0} M_{-1}}{p_0} = \frac{\mu^G \lambda}{p_0} \left[\sum_{t=0}^2 Q_{0,t} (-_1B_t^G) + M_{-1} \right] - \mu^E \sum_{t=0}^2 Q_{0,t} (-_1B_t^F). \quad (1.69)$$

And the optimality conditions of the $t = 1$ government are given by:

$$0 = \sum_{t=1}^{\infty} \beta^{t-1} [c(\lambda) + g_t - h^\eta] + \frac{1}{p_1} \sum_{t=1}^3 Q_{1,t} ({}_0B_t^F) \quad (1.70)$$

$$0 = \sum_{t=1}^{\infty} \beta^{t-1} [\lambda(\eta h^\eta - g_t) + u_h h] + \sum_{t=2}^{\infty} \beta^{t-1} u_m m \quad (1.71)$$

$$-\frac{\lambda}{p_1} \left[\sum_{t=1}^3 Q_{1,t} ({}_0B_t^G) + M_0 \right]$$

$$\begin{aligned} \frac{\lambda - \mu^E}{1 - \beta} &= \mu^G \sum_{t=1}^{\infty} \beta^{t-1} (\eta h^\eta - g_t) - \frac{\mu^G}{p_1} \left[\sum_{t=1}^3 Q_{1,t} ({}_0B_t^G) + M_0 \right] \\ &+ \frac{\mu^E}{p_1} \sum_{t=2}^3 Q_{1,t} \left[\sum_{i=2}^t \frac{u_{mi}}{\lambda(\lambda + u_{mi})} \right] ({}_0B_t^F) \quad (1.72) \end{aligned}$$

$$-\frac{\mu^G \lambda}{p_1} \sum_{t=2}^3 Q_{1,t} \left[\sum_{i=2}^t \frac{u_{mi}}{\lambda(\lambda + u_{mi})} \right] ({}_0B_t^G)$$

$$\begin{aligned} u_{m2} (1 + \mu^G) &= \left[\frac{\mu^E \sum_{s=2}^3 ({}_0B_s^F) Q_{1,s}}{\beta^{t-1} p_1 (\lambda + u_{m2})} + \frac{\mu^G \lambda \sum_{s=2}^3 ({}_0B_s^G) Q_{1,s}}{\beta^{t-1} p_1 (\lambda + u_{m2})} \right] u_{mm2} \\ &- \mu^G m_2 u_{mm2} \quad (1.73) \end{aligned}$$

$$\begin{aligned} u_{m3} (1 + \mu^G) &= \left[\frac{\mu^E ({}_0B_3^F) Q_{1,3}}{\beta^2 p_1 (\lambda + u_{m3})} + \frac{\mu^G \lambda ({}_0B_3^G) Q_{1,3}}{\beta^2 p_1 (\lambda + u_{m3})} \right] u_{mm3} \\ &- \mu^G m_3 u_{mm3} \quad (1.74) \end{aligned}$$

$$u_{mt} (1 + \mu^G) = -\mu^G m_t u_{mmt}, t \geq 4 \quad (1.75)$$

$$u_{ht} = \mu^G (\lambda \eta^2 h_t^{\eta-1} + u_{hht} h_t + u_{ht}) + \mu^E \eta h_t^{\eta-1}, t \geq 1 \quad (1.76)$$

$$\frac{u_{m1} M_0}{p_1} = \frac{\mu^G \lambda}{p_1} \left[\sum_{t=1}^3 Q_{1,t} ({}_0B_t^G) + M_0 \right] + \frac{\mu^E}{p_1} \sum_{t=1}^3 Q_{1,t} ({}_0B_t^F) \quad (1.77)$$

I drop the t subscript of real money balance in equation (1.71) because due to equation (1.67) and assumption (a), all the real balances from $t = 1$ are

constant.

When the government expenditure is constant, I use the following conditions to determine the maturity structure of bonds that the $t = 0$ government chooses to leave for the $t = 1$ government:

$$\begin{aligned}
0 &= \frac{\mu^E}{p_1} \sum_{t=2}^3 Q_{1,t} \left[\sum_{i=2}^t \frac{u_{mi}}{\lambda(\lambda + u_{mi})} \right] ({}_0B_t^F) \\
&\quad - \frac{\mu^G \lambda}{p_0} \sum_{t=1}^2 Q_{0,t} \left[\sum_{i=1}^t \frac{u_{mi}}{\lambda(\lambda + u_{mi})} \right] ({}_0B_t^G) \\
&\quad - \frac{\mu^E}{p_0} \sum_{t=1}^2 Q_{0,t} \left[\sum_{i=1}^t \frac{u_{mi}}{\lambda(\lambda + u_{mi})} \right] ({}_{-1}B_t^F) \\
&\quad - \frac{\mu^G \lambda}{p_0} \sum_{t=1}^2 Q_{0,t} \left[\sum_{i=1}^t \frac{u_{mi}}{\lambda(\lambda + u_{mi})} \right] ({}_{-1}B_t^G) \\
0 &= \frac{1}{p_1} \sum_{t=1}^3 Q_{1,t} ({}_0B_t^F) + \sum_{t=1}^{\infty} \beta^{t-1} [c(\lambda) + g_t - h^\eta] \\
0 &= \frac{\lambda}{p_1} \left[\sum_{t=1}^3 Q_{1,t} ({}_0B_t^G) + M_0 \right] - \sum_{t=1}^{\infty} \beta^{t-1} [\lambda(\eta h^\eta - g_t) + u_h h] \\
&\quad + \sum_{t=2}^{\infty} \beta^{t-1} u_m m \\
0 &= \mu^E ({}_0B_t^F) + \mu^G \lambda ({}_0B_t^G), t = 2, 3.
\end{aligned}$$

The system has five equations and six unknowns. As a result, I have many choices of maturity structures of bonds. Given the assumption that $m_0 = m_1$, it is straightforward to show that equation (1.70) - equation (1.77) are exact replicates of equation (1.62) - equation (1.69) by construction. This implies that the economy replicates itself period by period. If I assume that condition (1.54) is satisfied at $t = 1$, the equations corresponding to equation (1.77) of

all the future governments will satisfy the condition (1.54). Thus, the optimal policy is time consistent with only three period nominal bonds and without real bonds.

However, when the government expenditure in the above example is volatile over time, it is then not sufficient to assume that $m_0 = m_1$ and condition (1.54) holds for the $t = 1$ government. I then have to make the following stronger assumption: condition (1.54) must hold for all the $t \geq 1$ governments, i.e., the following optimality conditions must be satisfied by the policy continuation of the $t = 0$ government:

$$\begin{aligned} u_{mt}m_t - u_{m_{t+1}}m_{t+1} &= \mu^G \lambda \left[(\eta z_t h_t^\eta - g_t) + \frac{u_{ht}}{\lambda} h_t + u_{m_{t+1}}m_{t+1} \right] \\ &+ \mu^E [z_t h_t^\eta - c(\lambda) - g_t]. \end{aligned} \quad (1.78)$$

I then conclude that in a small open economy, the government bonds can always be over finite horizon. When government can issue both nominal and real bonds, it is straightforward to show that the governments can have optimal policy time consistent because the governments actually lose their choices of nominal interest rates as well as their choices of labor income tax rates. To see this, I list two competitive optimality conditions in closed and small open economy, respectively,

$$\frac{u_{mt}}{i_t} = \beta(1+r) \frac{u_{m_{t+1}}}{i_{t+1}} \quad (1.79)$$

$$\frac{u_{mt}}{i_t} = \beta(1+r_t) \frac{u_{m_{t+1}}}{i_{t+1}} \quad (1.80)$$

Compare equation (1.79), the intertemporal optimality condition for money demand in the small open economy, and equation (1.80), the intertemporal optimality condition for the money demand in the closed economy, it is clear that: in the closed economy, the governments can choose both i_t and i_{t+1} and

leave r_t to adjust. However, in the small open economy, since r is exogenous, the governments do not have freedom in choosing both i_t and i_{t+1} . Since the governments lose their choices of nominal interest rates as well, the maturity structure of both public and external debt can be over finite horizon.

1.4.5 Policy Implications

In a nominal closed economy, in order to have time consistent optimal policy, the government has to issue both nominal and real bonds, and to have real bonds over an infinite horizon. This is a very strong restriction on the governments' options. However, in a small open economy, governments have more options: they have at least more term structures of public debt rather than a unique maturity structure; and they can render optimal policy time consistent even with incomplete set of bonds.

These results have important policy implications. (a) Optimal policy is generally time consistent when the government issues both real and nominal bonds over the finite horizon, and many term structures of bonds are capable of rendering optimal policy time consistent. The result has practical merit and it implies that governments have the ability to handle extra policy issues which have not been modeled in this paper. (b) If the government is neutral between real and nominal bonds, it will always issue both nominal and real bonds. This justifies the use of both nominal and real bonds at the same time. (c) If the government dislikes real bonds, it will always issue both nominal and real bonds and set the real debt at an optimally minimum level. This result is important for the following reason. Since real bonds are interpreted as nominal bonds but denominated in US\$, (c) says that for a government that dislikes real bonds, i.e., dislike debt dollarization, it can choose among

these term structures to minimize the real public debt position. (d) When some explicit policy instruments are unavailable, the government may obtain some implicit policy instruments through non-financial market(s). And (e) the concern about the time consistency of optimal policy sheds light on optimal public debt management. The current literature about public debt management is mainly about to minimize the financial cost given the liquidity concern of the government and the term structure is not an issue at all.

1.5 Conclusion

This paper is my first step in exploring time consistency in a small open economy. The paper shows that in a nominal small open economy, optimal policy is generally time consistent and there are many maturity structures of public debt capable of rendering optimal policy time consistent. This finding is in sharp contrast with that obtained in the context of closed-economy models. In closed economies, the optimal policy is time consistent but only one unique maturity structure of public debt supporting time consistent optimal policy. This paper also shows that Ramsey governments in a small open economy can have time consistent optimal policy with implicit policy instruments and over finite horizon. In conclusion, the governments in the small open economy lose their choices of labor income tax rates, and this gives rise to the multiple maturity structures of debt. The governments also lose their choices of nominal interest rates, and this gives rise to the finite maturity structure of debt. In contrast, the Ramsey government in a *closed* economy needs to issue both nominal and real bonds, and to have real bonds over an infinite horizon to have time consistent optimal policy.

There are several extensions of this paper. One is about public debt man-

agement. The second extension will be to discuss time consistency issue in dynamic economy with incomplete asset market, which is under construction. The third one would be to extend the small open economy to a large open economy. The fourth one would be to study how the results change if we have different capital market or if we have different fiscal policy regime.

Chapter 2

Trade Openness and the Costs of Sudden Stops

2.1 Introduction

One empirical finding about sudden stops is that economies more open to trade will adjust their outputs less when hit by sudden stops, see Calvo et. al (2004)¹. Calvo et. al (2005) show that on the impact of 1998 Russia shock, Chile experienced less severe output adjustments than Argentina did because Chile was more open to trade (56%) than Argentina (19%) for the period of 91-97. These empirical evidences challenge the conventional wisdom to close the economy in order to reduce the adverse effect of exogenous shocks. The empirical finding raises a question: are the welfare costs² of sudden stops³ decreasing in trade openness⁴?

To answer this question, we solve a dynamic stochastic general equilibrium model up to second order approximation to analyze the relation between the welfare costs of sudden stops and the trade openness. The benchmark model is

¹Calvo et. al (2004) work with a sample of 32 countries, 15 emerging economies and 17 developed economies in the period 1990-2001.

²Welfare cost is defined as a lump sum consumption by which the representative household is willing to sacrifice to live in an economy with less volatile economy rather than to live in an otherwise equivalent economy but more volatile.

³A sudden stop economy is an economy with higher country spread volatility. There are other definitions of sudden stops: Chari et al (2005) define sudden stops as exogenous capital inflow reversal. Mendoza (2002) defines sudden stops as events that are characterized by three features: (1) recessions in output and decreases in private consumption; (2) collapses in asset prices; and (3) reversals in capital inflows and current accounts.

⁴Trade openness is defined as the ratio of $\frac{exports+imports}{GDP}$.

an otherwise standard small open economy model⁵ but with the introduction of imported intermediate inputs. In this paper, we do two-step experiment. First, we adjust trade openness by changing the tariff rate on the imported intermediate input. This adjustment serves to have economies with different trade openness. In this paper, the non-stochastic steady state trade openness is monotonically decreasing in the tariff rate. Secondly, we keep trade openness constant and adjust the unconditional distribution of the exogenous driven forces. The second adjustment serves to introduce sudden stops. For the given trade openness, the welfare cost of sudden stops is derived by comparing the welfare of two economies with the same trade openness. The only difference between these two economies is that one economy is an economy with possibility of sudden stops and the other is without the possibility of sudden stops.⁶

The simulation results show that less open economy tends to be more volatile (in terms of magnitudes of changes). In terms of welfare, the costs of sudden stops are decreasing in trade openness. The intuition is that the more open economy will have less volatile capital stock. It is thus true that the

⁵SGU (2003) have a brief discussion of the history of this type of economies.

⁶The welfare cost comparison can be done for theoretical and computational reasons. The computational algorithm by SGU (2004b) helps solve these rich economies with many state variables in the small neighborhood around non-stochastic steady state up to second order approximation. The disadvantage of this algorithm is that it assumes up to second order differentiability of policy functions so it could not handle the economies in which the policy functions have kink(s). That's another reason why sudden stops are defined different from that in Mendoza (2002). If we follow Mendoza (2002), sudden stops will be the results of binding borrowing constraints and all choice variables' policy functions will show kinks. At these kinks, the policy functions are not differentiable and the algorithm by SGU (2004b) is not applicable. Theoretically, all economies share one feature: the mean of trade openness is unique; and the unconditional distribution of trade openness is unique; so that the computational algorithm by SGU (2004b) is applicable. The theoretical feature of all model economies comes from two common building blocks: (1) the transitional distribution of the exogenous state variables is uniquely given; and (2) the invariant limiting distribution of state variables are unique. The block (2) is true because the preference used in this paper is the stationary cardinal utility (SCU), which uses the Green et. al (1988) (GHH) momentum utility function and contains endogenous subjective discount factors. With some regularity conditions, the SCU guarantees that the state variables have unique invariant limiting distributions.

output will be less volatile. The second finding is that sudden stops may be good to this small open economy. When the representative household is a net borrower in the international capital market, its consumption is going to be negatively correlated with country spread. Since utility function is a concave function of consumption, it must be a convex function of country spread. That is, when country spread is more volatile, the representative household is happier.

To check the robustness of these findings, we extend the experiment into two sector economies with homogeneous and heterogeneous capital. The two sector economies⁷ with homogeneous capital are very close those studied by Rebelo and Vegh (1995) and Mendoza and Uribe (2000). To introduce the heterogeneity of capital is to mimic the reality that capital in general cannot be reallocated among sectors without any costs. The numerical results show that the two findings are indeed robust: they not only hold in one sector economy model with and without working capital constraint, but also hold in two sector economy models with homogeneous capital and heterogeneous capital.

Since Calvo et. al (2004), Mendoza (2002) and this paper are different in bringing up sudden stops, it may be helpful to briefly comment on the welfare implications of different models. In Calvo et. al (2004), trade openness will not have any significant welfare implication if there is no liability-dollarization. On the contrary, the paper assumes zero-liability-dollarization and explores the generic relationship between the welfare cost of extra cycles caused by sudden stops and trade openness. The effects of sudden stops are different

⁷The two sector economies are rich in the sense that these models contain more state variables than those similar models have been widely studied in the literature. With our knowledge, these rich economies have not been studied to identify the welfare consequences of sudden stops up to second order approximation. One feature makes these rich economies less attractive is that these rich economies usually have many state variables and the curse of dimension makes them hard to solve.

in one significant way between Mendoza (2002) and this paper: In Mendoza (2002), sudden stops are endogenous results of the binding constraints only on one side. Whenever a sudden stop is realized, there are dramatic changes in macroeconomic aggregates. But in the long run, the volatility of macroeconomic aggregates increases only a little if there is possibility of sudden stops. In our model, the short-run abrupt adjustments have been smoothed out.

Given that the increase of tariff rate can be regarded as a permanent narrowing of the connection between the economy and the rest of the world, this paper naturally extends the discussion to evaluate the welfare effect of counter-cyclical tariff rate policy, which can be regarded as a widening of the connection between the economy and the rest of the world in bad times, but a narrowing in good times. When the tariff rates are counter-cyclical and constitute a stationary process around a fixed mean, it does not necessarily improve the welfare of the economy - in the economies studied in this paper, the reduction in welfare due to the increase of tariff rate in the good times outweighs the gain in welfare due to the decrease of tariff rate in the bad times.

The rest of the paper is organized as follows: Section 2 solves the competitive equilibrium problem of the one sector model. Section 3 and 4 solve the same problem for the two sector model with homogenous and heterogenous capital, respectively. Section 5 analyzes the welfare role of counter-cyclical tariff rate policy. Section 6 concludes.

2.2 One Sector Economy

2.2.1 The Benchmark Economy

The one sector economy⁸ has three types of agents: The domestic households maximize their discounted expected lifetime utility. Firms (100% owned by domestic households) produce the final goods by hiring labor, renting capital, and buying imported intermediate input from households. The government collects tariff from the imports and rebate the tariff income back the domestic households. One real friction in the model is that there exist investment adjustment costs. The economy is driven by a joint process of the total productivity factor, the world interest rate and the country spread.

The Representative Household

The representative household chooses hours and consumption to maximize his lifetime utility given his constraints. One feature about households is that they endogenize but not internalize their subjective discount rates of time preference. The representative household owns the firms so that he receives profit from the firms in each period. The representative household also rents capital, and provides labor and intermediate input to the firms so that he receives capital rent, labor income and intermediate input income from the firms as well. The representative household has two means to smooth his consumption. One way is to purchase one-period real foreign international bonds (bonds are tradable good here). The return on the bond is the product of the world interest rate and the country spread. The two components of the return are non-state-contingent, i.e. the return is predetermined, so is the return on the bond. This non-state-contingent return implies that the representative

⁸The economy is based on Model 1a in SGU (2003).

household could not completely smooth consumption by the purchase of the one-period bond. Another way is through his investment decision. Mathematically, the representative household's problem is to maximize his expected life time utility:

$$\max E_0 \sum_{t=0}^{\infty} \theta_t U(c_t, h_t), \quad (2.1)$$

$$\theta_{t+1} = \beta(\tilde{c}_t, \tilde{h}_t)\theta_t, \quad t \geq 0,$$

$$\theta_0 = 1,$$

subject to period budget constraint

$$\begin{aligned} d_t + r_t k_t + w_t h_t + r_t^m m_t + \Pi_t + \Gamma_t &\geq R_{t-1} d_{t-1} + c_t + i_t + (1 + \tau) m_t \\ &+ \Phi(k_{t+1} - k_t), \end{aligned} \quad (2.2)$$

and the law of motion of capital is

$$k_{t+1} = (1 - \delta)k_t + i_t, \quad (2.3)$$

where E_0 denotes the mathematical expectation operator conditional on information available at time 0. The variables θ_t , c_t , h_t , i_t , d_t , and m_t denote the subjective discount factor, consumption, hours, investment, foreign debt position, and imported intermediate inputs. The price of m_t , p_t^m , is normalized to unity. By assuming the relative price of m_t to be unity, we close the door through which the dynamics of terms of trade can affect the economy we study here. The variables r_t and w_t denote the capital return and wage rate. The variable τ denotes the tariff rate levied on the imports. The variables \tilde{c}_t and \tilde{h}_t denote the cross-sectional averages of consumption and labor supply, respectively, which the individual household takes as given. The variable Γ_t denotes the government transfer. The expression $\Phi(k_{t+1} - k_t)$ denotes investment adjustment costs.

The interest rate faced by the small open economy, R_t , is defined as the product of the world interest rate, R_t^{us} , and the country spread⁹, CR_t . The relationship is governed by $R_t = R_t^{us} CR_t$. The law of motion of interest rate is taken from Neumeyer and Perri (2001), which is:

$$\begin{pmatrix} \hat{R}_t^{us} \\ \hat{C}R_t \end{pmatrix} = \begin{pmatrix} 0.73 & 0.04 \\ 0.70 & 0.58 \end{pmatrix} \begin{pmatrix} \hat{R}_{t-1}^{us} \\ \hat{C}R_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{t,R^{us}} \\ \varepsilon_{t,CR} \end{pmatrix}. \quad (2.4)$$

The innovations in the process of interest rate are characterized by $\sigma_{\varepsilon_{R^{us}}} = 0.42\%$, $\sigma_{\varepsilon_{CR}} = 1.96\%$, and $\rho_{\varepsilon_R, \varepsilon_{CR}} = 0.30$. The exogenous process of country spreads is arbitrary.

The momentum utility function and the subjective discount factor take the following functional forms:

$$U(c, h) = \frac{(c - \omega^{-1}h^\omega)^{1-\gamma} - 1}{1-\gamma},$$

$$\beta(c, h) = \left(1 + c - \frac{h^\omega}{\omega}\right)^{-\beta_1}.$$

This preference is called as stationary cardinal utility (SCU). As long as $\beta_1 < \gamma$, (see Mendoza (1991)), this preference guarantees that there is unique limiting distribution of state variables, and the SCU is suitable for dynamic programming, and consumption good in every period is a normal good.

The endogenous subjective discount factor is decreasing in past consumption (the composite of final good consumption and disutility of labor supply). So, whenever the representative household changes his current consumption, it will not only change the marginal utility of current consumption, but also change the impatience level to the future consumption. If the representative

⁹Country spread measures the extra return required to compensate for taking the country risk.

household increases his current consumption, not only will his marginal utility from current consumption decrease, which is a standard result, but also will he relatively less value his future consumption. The endogenous subjective discount factors is one way to modify the standard real business cycle models to assure stationary behavior of the model economy (See SGU (2003) and Mulraine (2004) for a detailed discussion)¹⁰.

The GHH momentum utility function has a nice property that this utility function will rule out the wealth effect on the labor supply decision and the labor supply is only determined by the wage rate. This can be seen from the first order condition of the representative household $h_t^{\omega-1} = w_t$, where the wage rate w_t is taken as given by the representative household.

The representative household is subject to the non-Ponzi-game constraint of the form

$$\lim_{j \rightarrow \infty} E_t \frac{d_{t+j+1}}{\prod_{s=0}^j R_{t+s}} \geq 0. \quad (2.5)$$

The condition rules out the possibility that the representative household borrow to finance its consumption without limit.

¹⁰Mendoza (1991) points out that compared with the model with constant subjective discount factor, the model with endogenous (internalized) subjective discount factors will produce the same responses to technology shocks except two significant changes: the consumption-GNP correlation decreases and volatility of investment increases. SGU (2003) shows that the model with endogenous (but not internalized) subjective discount factors will produce almost identical responses to technology shocks as the model with endogenous (internalized) subjective discount factors.

The first order conditions associated with the representative household are

$$\lambda_t = \left(c_t - \frac{h_t^\omega}{\omega} \right)^{-\gamma}, \quad (2.6)$$

$$h_t^{\omega-1} = w_t, \quad (2.7)$$

$$\lambda_t = \left(1 + c_t - \frac{h_t^\omega}{\omega} \right)^{-\beta_1} R_t E_t \lambda_{t+1}, \quad (2.8)$$

$$1 + \tau = r_t^m, \quad (2.9)$$

$$q_t = 1, \quad (2.10)$$

$$\begin{aligned} \lambda_t (1 + \phi(k_{t+1} - k_t)) &= \left(1 + c_t - \frac{h_t^\omega}{\omega} \right)^{-\beta_1} \times \\ &E_t \lambda_{t+1} [1 - \delta + \phi(k_{t+2} - k_{t+1}) + r_{t+1}]. \end{aligned} \quad (2.11)$$

plus the non-Ponzi game condition and period budget constraint hold in equality; and the law of motion of capital. All the first order conditions have their usual explanations.

It is straightforward to show that in this economy, the small open economy is more open to trade when the tariff rate is lower, as it can be seen in the Figure 2.1.

The Firms

There are large number of final good production firms. All firms are identical and using constant return to scale production technology, which is given by:

$$y_t = z_t k_t^{\alpha_k} h_t^{\alpha_h} m_t^{\alpha_m},$$

where the variables y_t , z_t , k_t , h_t , and m_t denote output of final good, total productivity factor, capital, labor and intermediate input, respectively. In

these economies, the total productivity factor follows the stationary process¹¹

$$\ln(z_{t+1}) = \rho \ln(z_t) + \varepsilon_{t+1}^z, \varepsilon_{t+1}^z \sim NIID(0, \sigma_z^2). \quad (2.12)$$

Since firms do not make the investment decision, firm's problem is reduced to a static problem to maximize its period profit by choosing k_t , h_t , and m_t given z_t , r_t , w_t , and r_t^m ,

$$\max \Pi_t = z_t k_t^{\alpha_k} h_t^{\alpha_h} m_t^{\alpha_m} - r_t k_t - w_t h_t - r_t^m m_t.$$

The first order conditions for firms are standard:

$$r_t = \alpha_k z_t k_t^{\alpha_k - 1} h_t^{\alpha_h} m_t^{\alpha_m}, \quad (2.13)$$

$$w_t = \alpha_h z_t k_t^{\alpha_k} h_t^{\alpha_h - 1} m_t^{\alpha_m}, \quad (2.14)$$

$$r_t^m = \alpha_m z_t k_t^{\alpha_k} h_t^{\alpha_h} m_t^{\alpha_m - 1}. \quad (2.15)$$

These optimality conditions have their usual meaning.

The Government

The government is assumed to levy tariff from the imports of intermediate inputs and to transfer the tariff income back the households. The government balances its period budget each period:

$$\tau m_t = \Gamma_t. \quad (2.16)$$

Since the imposition of tariff distorts the economy, it will reduce the welfare of the representative household. However, this is not the answer to the question this paper tries to answer: conditioning on the different levels of trade openness, what is the welfare cost of sudden stops?

¹¹The structure parameters, ρ and σ_z , are calibrated to match two observed second moments in the data.

Equilibrium Conditions

In equilibrium, all markets, capital market, labor market, and intermediate input market, are clearing. And the aggregates equal to their corresponding parts of the representative household's because households are assumed to be identical:

$$\tilde{c}_t = c_t; \tilde{h}_t = h_t. \quad (2.17)$$

Competitive Equilibrium

The competitive equilibrium is defined in a standard way as a sequence of real allocations $\{c_t, k_{t+1}, d_t, i_t, h_t, m_t, \Gamma_t\}_{t=0}^{\infty}$ and prices $\{r_t, w_t, r_t^m\}_{t=0}^{\infty}$, given d_{-1}, k_0 and the law of motion of interest rate, (2.4), and the law of motion of the total productivity factor (2.12), satisfying the conditions (2.2) with equality, (2.3), (2.5)-(2.11), and (2.13)-(2.17).

2.2.2 Numerical Solution and Results

There is no analytical solution to this dynamic stochastic general equilibrium problem. To solve the problem, we first calibrate the structure parameters; secondly, we solve the non-stochastic steady state of the problem; and we finally apply the algorithm developed in SGU (2004b) to get the numerical solution of the dynamic stochastic general equilibrium up to second order approximation. For details, see SGU (2004b). We put a brief description of the process of the numerical solution method in the appendix.

Calibration

Notice that Neumeyer and Perri (2001) obtain the law of motion of interest rate by using quarterly data, to be consistent, we calibrate all the parameters

to match the quarterly data assumption. Once the deep structure parameters are calibrated, their values will be kept constant. The calibration is as follows.

The risk aversion coefficient, γ , is set at 2. There is no consensus on the value of γ . For example, Mendoza and Uribe (2000) sets it at 5 while Mendoza (1991) sets it at 2. The empirical estimate of γ ranges between 1.25 and 10 (see Reinhart and Vegh (1994)). The capital depreciation rate, δ , is set at 0.025, which has been commonly used in the literature. The exponent of labor supply in utility, ω , is set at 1.455. This is taken from Mendoza (1991) to mimic the percentage variability of hours. The share of labor income in value added, s_h , and the share of capital income in value added, s_k , come from Neumayer (2001) and are set at 0.62 and 0.38, respectively. The non-stochastic steady state interest rate, R , is set at 1.0275, a value from Uribe and Yue (2004) and consistent with the average 11% annual real interest rate faced by a small open economy in the international capital market. And the non-stochastic steady state world interest rate R^{us} is set at 1.01625, a value from Mendoza and Uribe (2000). The steady state value for trade openness, TO is taken from Edwards (2002b) and set at 0.65. And for the benchmark economy, the non-stochastic steady state ratio of trade balance to GDP, s_{tb} is set at 0.02.

The parameter α_m , denoting the intermediate input shares in output, is chosen to make sure that in the steady state, the trade openness is 65%. The parameters, α_k and α_h , are determined by two conditions: first, in the value added, capital income share is $s_k = 0.37$ and labor income share is $s_h = 0.62$; second, the production is homogeneous of degree one, so we have $\alpha_k + \alpha_h = 1 - \alpha_m$. The steady state marginal return to capital, μ_k , is calculated from the deterministic steady state optimal condition $\mu_k = R - 1 + \delta$. The

share of investment in value added, s_i , is calculated by the following equation

$$s_i = \frac{i}{y - m} = \frac{\delta \mu_k k}{\mu_k (y - m)} = \frac{\delta s_k}{\mu_k}.$$

The share of consumption is derived by using the accounting identity in the steady state, $s_c = 1 - s_i - s_{tb}$. From the setup of the problem, the determination of the steady state values of c and h are independent of β_1 . Thus, after we calculate the steady state values of c and h , the parameter β_1 can be calibrated by the deterministic steady state optimal condition:

$$1 = \left(1 + c - \frac{h^\omega}{\omega}\right)^{-\beta_1} R.$$

The implied value for β_1 is 0.28123, which is less than $\gamma = 2$. In the analysis of welfare costs, we fix the parameter β_1 , so the change of the long-term tariff rate will change the non-stochastic steady state trade balance to GDP share.

The last three parameters are the serial correlation of productivity shock, ρ ; the standard deviation of innovation to the technology shock, σ_z , and the parameter of adjustment cost function, ϕ . To calibrate these parameters, we choose values for them, simulate the model, and repeat this process until the simulated volatilities of output and investment, and the first order autocorrelation coefficient between output and investment, match the observed values in the data as close as possible.

The values for the second type parameters are summarized as follows:

Symbol	Definition	Value
γ	risk averse coefficient	2
δ	capital depreciation rate	0.025
ω	exponent of labor supply in utility	1.455
R^{us}	steady state of world interest rate	1.01625
R	steady state of interest rate	1.0275
TO	steady state of trade openness	0.65
s_h	share of labor income in value added	0.62
s_k	share of capital income in value added	0.37
s_{tb}	share of trade balance in value added	0.02
α_k	capital elasticity in tradable sector	0.2799
α_h	labor elasticity in tradable sector	0.4566
α_M	imported inputs elasticity	0.2635
β_1	subjective discount parameter	0.28123
μ_k	marginal return to capital	0.0525
s_i	investment share in value added	0.1753
s_c	consumption share in value added	0.8047
ρ	serial correlation of productivity shocks	0.659
σ_z	standard deviation of productivity shocks	0.01143
ϕ	investment adjustment cost coefficient	5.377

The non-stochastic steady state is derived in the Technique Appendix. The non-stochastic steady states of some variables are plotted against tariff rates, as shown in Figure 2.1. In particular, the trade openness is decreasing in tariff rate.

Impulse Response to a Positive Country Spread Shock

Figure 2.2 shows the dynamics of the main variables in the benchmark one-sector model to a positive country spread shock. When there is a country spread shock, the representative household is willing to borrow less, thus a Chari et. al (2005) sudden stop emerges. Since here we use SCU, so that there is no wealth effect on labor supply. As it is shown in Figure 2.2, a positive country spread shock does not have any effect on the the labor and the contemporary output is not be affected when a shock is realized. In this economy, a sudden stop is not associated with output drop. Mathematically,

this can also be seen from the optimality conditions (2.7), (2.9), (2.14) and (2.15). With some manipulations, we get the following expressions for h_t and m_t :

$$\begin{aligned} h_t &= A_1 (z_t k_t^{\alpha_k})^{\frac{1}{\omega - \alpha_h - \omega \alpha_m}}, \\ m_t &= B_1 (z_t k_t^{\alpha_k})^{\frac{\omega}{\omega - \alpha_h - \omega \alpha_m}}, \end{aligned}$$

where A_1 and B_1 are functions of structure parameters:

$$\begin{aligned} A_1 &= \left[(\alpha_h)^{1 - \alpha_m} \left(\frac{\alpha_m}{1 + \tau} \right)^{\alpha_m} \right]^{\frac{1}{\omega - \alpha_h - \omega \alpha_m}}, \\ B_1 &= \frac{1}{\alpha_h} \frac{\alpha_m}{1 + \tau} \left[(\alpha_h)^{1 - \alpha_m} \left(\frac{\alpha_m}{1 + \tau} \right)^{\alpha_m} \right]^{\frac{\omega}{\omega - \alpha_h - \omega \alpha_m}}. \end{aligned}$$

Since k_t is predetermined and z_t is independent of the realization of R_t , the output and labor will not change on the impact of the positive country spread shock.

Consumption drops because of the negative welfare effect. Investment drops a lot because the opportunity cost of investing is high. Trade balance is thus improved. The current account is improved at the same time because the representative household is borrowing much less due to the high borrowing costs.

Second Moments

The modified benchmark one sector model, which is a model with 10% tariff rate and low country spread volatility, replicates the dynamics of Argentina economy, as is shown in Table 1 which lists some second moments associated with the four economies we study here. For example, in the benchmark economy, the standard deviation of output is 4.59%, very close to the corresponding

number in the actual data, which is 4.59%. The similar results apply to the first order autocorrelation coefficient of output and the volatility of investment.

Table 1: Some Second Moments in the One Sector Model¹²

	σ_y	$\rho(y, y_{-1})$	σ_I
Actual Data	4.59	0.79	13.30
Free Trade - Low Country Spread Volatility	4.43	0.78	11.47
Free Trade - High Country Spread Volatility	4.77	0.81	14.30
Tariff = 10% - Low Country Spread Volatility	4.59	0.79	13.30
Tariff = 10% - High Country Spread Volatility	4.98	0.82	16.60

The most noticeable feature of Table 1 is that, for a given exogenous process of driven forces, economies less open to trade tend to have more volatile macro-economic aggregates. When the economy is driven by the joint process governed by (2.4) and (2.12), the standard deviation of output in the free trade economy is 4.43%, which is less than that in the corresponding economy with 10% tariff rate, 4.59%. This result is in line with Calvo et. al (2004, 2005) and Edwards (2004a, 2004b) but at a more deep generic relationship. In the next section, we will discuss the reason why this small open economy is less volatile when it is more open.

Volatility of Capital

The high volatility of output in a less opened economy is mainly because the capital is more volatile. Given our model specification, the return to capital is inversely related to the tariff rate, i.e., inversely related to trade openness. To see this relationship, we do some algebra and get the following equation:

$$r_t = C_1 k_t^{\frac{-(\omega-1)\alpha_h}{D_1}} \left(\frac{\alpha_m}{1+\tau} \right)^{\frac{(\alpha_h+\omega\alpha_m)\alpha_m}{D_1}+1}$$

¹²This is without working capital constraint.

where

$$C_1 = \left(\frac{\alpha_k}{\alpha_h} \right) z^{\frac{\omega}{D_1}} \alpha_h^{\frac{(1-\alpha_m)(\alpha_h+\omega\alpha_m)}{D_1}}$$

$$D_1 = (\omega - 1)\alpha_h + \omega\alpha_k$$

Given that $\omega > 1$, it is true that $C_1 > 0$ and $D_1 > 0$. It is thus clear from the above equation that the negative relationship holds: for a given state of capital stock and productivity, the return to capital is inversely related to the trade openness.

To see how this relationship leads to more volatile capital stock in a less open economy, we combine (2.8) and (2.11), assume no adjustment costs and constant productivity, and get the following simple condition:

$$R_t = \frac{E_t \lambda_{t+1} (1 - \delta + r_{t+1})}{E_t \lambda_{t+1}} = 1 - \delta + C_1 k_{t+1}^{\frac{-(\omega-1)\alpha_h}{D_1}} \left(\frac{\alpha_m}{1 + \tau} \right)^{\frac{(\alpha_h + \omega\alpha_m)\alpha_m}{D_1} + 1}$$

The last equality comes from the fact that k_{t+1} is known at the time t ; since we assume that productivity is constant, r_{t+1} is known at the time t .

When there is a positive country spread shock, i.e., R_t goes up, the representative household will accumulate less capital for the next period, $\Delta k_{t+1} < 0$. However, the representative household in the less open economy will accumulate even less capital:

$$\Delta k_{t+1}(\text{high tariff}) < \Delta k_{t+1}(\text{low tariff}) < 0$$

When there is a negative country spread shock, i.e., R_t goes down, the representative household will accumulate more capital for the next period, $\Delta k_{t+1} > 0$. However, the representative household in the less open economy will accumulate even more capital:

$$\Delta k_{t+1}(\text{high tariff}) > \Delta k_{t+1}(\text{low tariff}) > 0$$

Since for any country spread shock, the less open economy has a bigger adjustment in capital accumulation, as a result, capital is more volatile.

When we introduce adjustment costs and productivity shock, the numerical results show the same pattern: in the less open economy, the standard deviation of capital is 3.84%; while in the more open economy, the standard deviation of capital is 4.23%. The same pattern is also confirmed by the impulse response of capital to country spread shocks, see Figure 2.2. The line represents the case of high trade openness (low tariff rate). The dotted line represents the case of low trade openness (high tariff rate).

2.2.3 Welfare Costs

The negative relationship between volatility and openness shown in the above table implies that welfare costs (or benefits) of sudden stops may be higher (or less) if economies are less open to trade. The next question is how does the representative household like the volatile economy and how does the representative household's preference over volatile economy change with trade openness? In this section, we will derive the welfare cost measure, and report and explain our numerical results.

Unconditional Welfare Cost Measure

The key question is to evaluate welfare cost of sudden stops. Here is how we derive the corresponding welfare cost. In general, the conditional lifetime welfare of the representative household, V_0 , is defined as

$$V_0(x_0, \sigma) = U(C_0(x_0, \sigma), H_0(x_0, \sigma)) + \beta(\tilde{C}_0(x_0, \sigma), \tilde{H}_0(x_0, \sigma))E_0V_1(x_1(x_0, \sigma), \sigma).$$

The unconditional lifetime welfare of household, EV , can be written as

$$EV(\sigma) = EU(C(x, \sigma), H(x, \sigma)) + E\beta(\tilde{C}(x, \sigma), \tilde{H}(x, \sigma))V(x'(x, \sigma), \sigma).$$

The parameter σ controls the size of volatility of the economy; the variable x represents the set of state variables. The E denotes the unconditional expectation operator. Other variables have their usual meanings. Define the unconditional welfare associated with the reference shock process r as

$$EV^r(\sigma) = EU(C^r(x, \sigma), H^r(x, \sigma)) + E\beta(\tilde{C}^r(x, \sigma), \tilde{H}^r(x, \sigma))V^r(x'(x, \sigma), \sigma).$$

where the variables c^r and h^r denote the unconditional expectations for consumption and hours under the exogenous shock process r . Similarly, the unconditional welfare associated with the alternative exogenous shock process a is defined as

$$EV^a(\sigma) = EU(C^a(x, \sigma), H^a(x, \sigma)) + E\beta(\tilde{C}^a(x, \sigma), \tilde{H}^a(x, \sigma))V^a(x'(x, \sigma), \sigma).$$

The reference economy represents the economy in which the standard deviation of country spread is 0.0049. The alternative economy represents the economy in which the standard deviation of country spread is 0.0074. The welfare cost ratio, λ , is implicitly defined by

$$\begin{aligned} EV^a(\sigma) &= EU((1 - \lambda)C^r(x, \sigma), H^r(x, \sigma)) \\ &\quad + E\beta((1 - \lambda)\tilde{C}^r(x, \sigma), \tilde{H}^r(x, \sigma))V^a(x'(x, \sigma), \sigma). \end{aligned}$$

The ratio λ represents the constant ratio of consumption across states. Following the same argument in SGU (2004): the welfare cost ratio, λ , is a function of σ only.

$$\lambda = \lambda(\sigma). \tag{2.18}$$

Approximate λ up to the second order with respect to σ around its non-stochastic steady state

$$\lambda(\sigma) = \lambda(0) + \lambda_\sigma(0)\sigma + \frac{1}{2}\lambda_{\sigma\sigma}(0)\sigma^2 = \frac{1}{2}\lambda_{\sigma\sigma}(0)\sigma^2. \tag{2.19}$$

The result in the equation (2.19) follows from the same argument in SGU (2004), λ vanishes as $\sigma \rightarrow 0$, i.e. $\lambda(0) = 0$. As we shown in the Technique Appendix, the ratio of welfare cost to the non-stochastic steady state final consumption of sudden stops, $\bar{\lambda}_{\sigma\sigma}$, is

$$\bar{\lambda}_{\sigma\sigma} = \frac{\frac{2(C_2 - A_2 - B_2)}{\sigma^2} - E_2 - F_2}{\bar{U}_C \bar{C}^r + \bar{\beta}_C \bar{C}^r \bar{V}'^a},$$

where

$$A_2 = \sum_{i=1}^{n_x} (\bar{S}_{x_i}^r + \bar{T}_{x_i}^r) E(x_i^r - \bar{x}_i) - \sum_{i=1}^{n_x} (\bar{S}_{x_i}^a + \bar{T}_{x_i}^a) E(x_i^a - \bar{x}_i),$$

$$B_2 = \frac{1}{2} \sum_{i=1}^{n_x} \sum_{j=1}^{n_x} (\bar{S}_{x_i x_j}^r + \bar{T}_{x_i x_j}^r) E(x_i^r - \bar{x}_i)(x_j^r - \bar{x}_j) - \\ \frac{1}{2} \sum_{i=1}^{n_x} \sum_{j=1}^{n_x} (\bar{S}_{x_i x_j}^a + \bar{T}_{x_i x_j}^a) E(x_i^a - \bar{x}_i)(x_j^a - \bar{x}_j),$$

$$C_2 = EV^r(\sigma) - EV^a(\sigma),$$

$$E_2 = \bar{\beta}_C \bar{C}^r \bar{c}_{\sigma\sigma}^r \bar{V}'^r + \bar{\beta}_H \bar{H}^r \bar{h}_{\sigma\sigma}^r \bar{V}'^r + \bar{\beta} \sum_{l=1}^{n_x} \bar{V}'_{x_l}{}^r \bar{x}_{l,\sigma\sigma}{}^r + \bar{\beta} \bar{V}'_{\sigma\sigma}{}^r \\ + \bar{\beta} \sum_{l=1}^{n_x} \sum_{n=1}^{n_x} \bar{V}'_{x_l, x_n}{}^r E \varepsilon_n{}^r \varepsilon_l{}^r - \bar{\beta} \sum_{l=1}^{n_x} \sum_{n=1}^{n_x} \bar{V}'_{x_l, x_n}{}^r E \varepsilon_n{}^a \varepsilon_l{}^a \\ - \bar{\beta}_C \bar{C}^r \bar{c}_{\sigma\sigma}^a \bar{V}'^a - \bar{\beta}_H \bar{H}^r \bar{h}_{\sigma\sigma}^a \bar{V}'^a - \bar{\beta} \sum_{l=1}^{n_x} \bar{V}'_{x_l}{}^a \bar{x}_{l,\sigma\sigma}{}^a - \bar{\beta} \bar{V}'_{\sigma\sigma}{}^a,$$

$$F_2 = \bar{U}_C \bar{C}^r \bar{c}_{\sigma\sigma}^r + \bar{U}_H \bar{H}^r \bar{h}_{\sigma\sigma}^r - \bar{U}_C \bar{C}^r \bar{c}_{\sigma\sigma}^a - \bar{U}_H \bar{H}^r \bar{h}_{\sigma\sigma}^a.$$

Even though the formula is complicated, the economic meaning of the above welfare cost measure is straightforward: When the volatility is increasing, this is going to change the unconditional welfare of the representative household, measured by C_2 . The change of the unconditional welfare can be decomposed

into three parts: (1) the first part is associated with the changes of unconditional mean of state variables and the variance and covariance among state variables, measured by $A_2 + B_2$; (2) the second part is associated with the changes of some choice variables, such as consumption, leisure, etc., measured by $1/2(E_2 + F_2)\sigma^2$; and (3) the last part is associated with the representative household's willingness to pay to avoid extra volatility.

Unconditional Welfare Cost

The welfare costs are shown in the following table:

Table 2: Welfare Costs of Sudden Stops in One Sector Open Economy

	Free Trade	Restricted Trade (=10%)
Unconditional	-0.0024	-0.0008

There are two findings: (1) the unconditional welfare cost of sudden stops is less when the economy is more open to trade. In the more open economy, the unconditional welfare cost of sudden stops is -0.0024 unit of consumption; while in the less open economy, the unconditional welfare cost is only -0.0008 unit of consumption. This finding comes directly from the negative relationship between volatility and trade openness; and it is independent of the borrowing/lending position. Here the risk averse household is a net borrower at the non-stochastic steady state. Later, we show that when the risk averse household is a net lender, the welfare costs of sudden stops are higher when economies are less open to trade. (2) Sudden stops can actually be good to the risk averse household: we get results of negative welfare costs, which imply welfare benefits. This finding is quite counter-intuitive because it is widely believed that the more volatile economy should mean a lower expected welfare

for the risk averse representative household. Next, we explain why we have second finding.

Why Prefer Uncertainty? - Technology Shock

When the economy is driven by three joint shocks, we have counter-intuitive result: risk averse household likes uncertainty. To understand this result, it is necessary to disentangle the effects of each individual driven force. First, let's examine the effects of technology shocks.

In the literature, it has been shown that risk averse household may like volatile economy if the economy is driven by productivity shock and/or exchange rate shock. In the case of exchange rate shock, Obstfeld and Rogoff (1999) give a close form solution to an open economy driven by productivity and money shocks; and illustrate the possibility of positive welfare effects of uncertainty through their effects of economic activities levels. Bacchetta and Wincoop (2000) show that depending on the economic structure, high exchange rate volatility may lead to high welfare of the risk averse households. In the case of productivity shock, the intuition is: even though the utility function is concave in consumption, after plugging in the optimal condition, the utility function may be a convex function of productivity. In that sense, the risk averse household will like volatile economy more than a corresponding less volatile economy. However, the mechanism why volatile productivity leads to higher utility is different, as we show below.

To illustrate this possible positive effect due to the productivity shock, we modify the model by assuming two periods. It can be shown that

$$\left(c_2 - \frac{h_2^\omega}{\omega} \right)^{1-\gamma} = \left[(1 - \delta) k_1 - \frac{\phi}{2} k_1^2 + G_3 z_2^{D_3 \omega} - R_1 d_1 \right]^{1-\gamma},$$

where

$$\begin{aligned}
G_3 &= \left[A_3^{\alpha_h} B_3^{\alpha_m} - (1 - \tau) B_3 - \frac{A_3^\omega}{\omega} \right] (k_1^{\alpha_k})^{D_3 \omega}, \\
A_3 &= \left[(\alpha_h)^{1 - \alpha_m} \left(\frac{\alpha_m}{1 + \tau} \right) \right]^{\frac{1}{\omega - \alpha_h - \omega \alpha_m}}, \\
B_3 &= \frac{1}{\alpha_h} \frac{\alpha_m}{1 + \tau} A_3^\omega, \\
D_3 &= \frac{1}{\omega - \alpha_h - \omega \alpha_m}.
\end{aligned}$$

Depending on the signs of G_3 and $1 - \gamma$, it is possible that the uncertainty will push up or down the unconditional period utility with respect to the non-stochastic steady state welfare. Suppose the rest terms are summed to be zero, i.e., $(1 - \delta) k_1 - \frac{\phi}{2} k_1^2 - R_1 d_1 = 0$. This can be true because k_1 and d_1 are the initially given. then we have:

$$E \left(c_2 - \frac{h_2^\omega}{\omega} \right)^{1 - \gamma} = G_3^{1 - \gamma} E z_2^{(1 - \gamma) D_3 \omega} = G_3^{1 - \gamma} e^{\frac{1}{2} [(1 - \gamma) D_3 \omega]^2 \sigma_z^2},$$

The last equality comes from the assumption that z_2 is lognormal. Given our calibration, G_3 is positive. As a result, the above equation shows that there is a positive relationship between unconditional expected period utility and the uncertainty through some channel: for a certain level of debt position, productivity uncertainty increases the expected utility of the risk averse households.

Why Prefer Uncertainty? - Country Spread Shock - Two Islands

When we shut down the two other shocks, technology shocks and world interest rates shocks, and let the economy be driven by the exogenous country spread shock only, the simulation results show similar pattern as follows:

Welfare Costs: Free Trade v.s. Restricted Trade

	Free Trade	Restricted Trade (=10%)
Unconditional	-0.0016	-0.0005

The above results shows that when the economy is driven by country spread shock only, the representative household may still like uncertainty. This is a less understood fact: the literature has not provide an answer why the risk averse household likes uncertainty if the economy is driven by exogenous country spread process only. To explain this result, we use the following two island example.

Suppose a household in period 0 has to choose one of two islands to live for the next two periods of his life. The two islands have the same endowment flow: 0 in period 1, and y_2 in the period 2. This endowment distribution makes sure that the household will be a borrower in period 1. The only difference between two islands is the interest rate of bonds in period 1. For Island 1, the interest rate is known in period 0 and is a constant R^* . For Island 2, the interest rate is random variable in period 0. But in period 0, the household knows that it will be either $R^*(1 + \varepsilon_1)$ with a probability of π or $R^*(1 - \varepsilon_2)$ with a probability of $1 - \pi$. The restriction on ε_1 and ε_2 is $\pi\varepsilon_1 = (1 - \pi)\varepsilon_2$. This restriction guarantees that the expected interest rate is exactly R^* . After the household has decided which island to go, he enters period 1, in which the household will solve a perfect foresight problem because the interest rate is known. The household maximizes its utility function given by

$$V = \log(c_1) + \log(c_2).$$

His period budget constraint is

$$\begin{aligned}c_1 + b_1 &= 0, \\c_2 &= R_1 b_1 + y_2.\end{aligned}$$

It can be shown that the solution is

$$\begin{aligned}c_1 &= \frac{1}{R_1} \frac{y_2}{2}, \\c_2 &= \frac{y_2}{2}.\end{aligned}$$

And the conditional optimal utility is then:

$$V = 2 \log\left(\frac{y_2}{2}\right) - \log(R_1).$$

The expected utility is given by

$$EV = 2 \log\left(\frac{y_2}{2}\right) - E \log(R_1). \quad (2.20)$$

Thus the expected utility if the household chooses Island 1 is

$$EV_1 = 2E \log\left(\frac{y_2}{2}\right) - E \log(R_1) = 2 \log\left(\frac{y_2}{2}\right) - \log(R^*) = V_{ss}. \quad (2.21)$$

The expected utility if the household chooses Island 2 is

$$EV_2 = V_{ss} - [\pi \log(1 + \varepsilon_1) + (1 - \pi) \log(1 - \varepsilon_2)].$$

Since $\pi \log(1 + \varepsilon_1) + (1 - \pi) \log(1 - \varepsilon_2)$ is concave, its negative correspondence must be convex, which means the household enjoys uncertainty. The positive relation between volatility and welfare is plotted in Figure 2.3. Then the choice of the borrower in period 0 is obvious: he will choose Island 2 even though Island 2 has a more volatile economy. It can be shown that with the increase

of ε_1 , the interest rate becomes more volatile and the expected utility to stay in Island 2 is higher. In another words, when the representative household is a net borrower in the international capital market, its consumption will be negatively correlated with country spread. Since utility is a concave function of consumption, it must be a convex function of country spread. That is, when the country spread is more volatile, the mean utility is higher.

The more relevant case is the case where the next period output is a decreasing function of today's interest rate. Suppose

$$Y_{t+1} = R_t^{-\frac{1}{\gamma}}, \text{ where } \gamma > 0,$$

In this case, the choice of consumption is

$$c_1 = \frac{R_1^{-1-\frac{1}{\gamma}}}{2},$$

$$c_2 = \frac{R_1^{-\frac{1}{\gamma}}}{2}.$$

The expected utility of the household is

$$EV = - \left(1 + \frac{2}{\gamma}\right) E \log R_1 - 2 \log 2. \quad (2.22)$$

Compared to (2.20), it is clear that endogenizing output will re-enforce the positive effect of interest rate volatility on the expected welfare.

Borrowing and Lending Position

Since the unconditional welfare may increase in both productivity volatility and country spread volatility, it is not strange here to find that sudden stops are actually good to the representative household in the above case. Then the

question is what is the most important factor in determining whether sudden stops are good or bad?

In the two island example, the important conveyed information is that the borrower/lender position may be important in determining whether the representative household enjoys the volatile economy or not. To check the hypothesis that when the representative is a net lender in the international capital market, then it will choose certainty, I modify the calibration process by changing the non-stochastic state trade balance ratio to -30% and the trade openness level to 25% ¹³. In the new economy, the representative household is a heavy lender at the non-stochastic steady state. The simulation results are listed below:

Welfare Costs: Free Trade v.s. Restricted Trade

	Free Trade	Restricted Trade (=10%)
Unconditional	0.0001	0.0003

This experiment confirms the hypothesis: the risk averse household dislikes uncertainty when he is a net lender. In addition, we still have the result: less open to trade will bring higher cost; even though the experiment has its own weakness because the calibrated economy is not even close to a typical true real economy. we extend the experiments to include more wider range of domains of parameters. The results show that the borrowing and lending position is a very important factor in determining whether sudden stops are good or bad to the representative household.

¹³In this case, we change the value of β_1 from 0.28123 to 0.1171 to accommodate the new borrowing/lending position. The welfare is not directly comparable to other cases discussed so far.

2.2.4 Working Capital Constraint

So far, output does not drop when there is a positive country spread shock. To have the model to produce output drop, it is necessary to introduce new mechanism. Here we extend the model to including working capital constraint in the exactly same way as in Uribe and Yue (2004). The working capital constraint takes the following form

$$WK_t \geq \varphi w_t h_t,$$

where the variable WK_t denotes the amount of working capital. The representative firm's debt position evolves as

$$d_t^f = R_{t-1}d_{t-1}^f - y_t + w_t h_t + \mu_t k_t + (1 + \tau m_t) + \pi_t - WK_{t-1} + WK_t.$$

Define the net liability of the representative firm as $a_t = R_t d_t^f - WK_t$, we can rewrite the budget constraint of the representative firm as

$$\frac{a_t}{R_t} = a_{t-1} - y_t + w_t h_t + \mu_t k_t + (1 + \tau m_t) + \pi_t + \left(\frac{R_t - 1}{R_t} \right) WK_t. \quad (2.23)$$

Since the representative firm is owned by the representative household, the objective function of firms is defined by

$$\max E_0 \sum_{t=0}^{\infty} \theta_t \frac{\lambda_t}{\lambda_0} \pi_t,$$

where λ_t denotes the marginal wealth utility of the representative household. The representative firm is also subject to the following no-ponzi-game constraint

$$\lim_{j \rightarrow \infty} E_t \frac{a_{t+j}}{\prod_{s=0}^j R_{t+s}} \leq 0. \quad (2.24)$$

The introduction of working capital constraint will only change the optimal condition for labor demand. Instead of the equation (2.14), labor demand is determined by the following equation

$$w_t \left[1 + \varphi \left(\frac{R_t - 1}{R_t} \right) \right] = \alpha_h z_t k_t^{\alpha_k} h_t^{\alpha_h - 1} m_t^{\alpha_m}. \quad (2.25)$$

Since any process a_t satisfies (2.23) and (2.24) will be optimal for the representative firm, here we follow Uribe and Yue (2004) and set $a_t = 0$.

The only parameter needs to be calibrated is the parameter φ . we follow Uribe and Yue (2004) and set it at 1.2, which means the representative firm needs to save money to be able to pay at least 1.2 quarter wage bills.

Figure 2.4 shows the impulse responses to a positive country spread shock in this one sector economy. It is easy to see that a positive country spread shock will cause output drop when working capital constraint is introduced. This is because given a rise of country spread, the labor demand from firms' side decreases thus the output drop. The model can also produce current account reversal and output drop for a given negative total productivity factor shock at the same time. When the model is hit by a negative productivity shock, output drops because the total productivity factor drops and the labor input decreases. Current account reverses because of the dramatic decrease in investment.

Table 3 below lists some second moments after the working capital constraint is introduced. First of all, the second moments show similar pattern as in the case where there is no working capital constraint: less open to trade means more volatile macroeconomic aggregates. Second, after the working capital is introduced, economies become more volatile than the corresponding economies without working capital constraint.

Table 3: Some Second Moments in the One Sector Model with Working Capital Constraint

	σ_y	$\rho(y, y_{-1})$	σ_I
Actual Data	4.59	0.79	13.30
Free Trade - Low Country Spread Volatility	4.80	0.80	13.33
Free Trade - High Country Spread Volatility	5.28	0.83	16.62
Tariff = 10% - Low Country Spread Volatility	5.00	0.82	15.44
Tariff = 10% - High Country Spread Volatility	5.54	0.85	19.28

The welfare costs of sudden stops are listed in Table 4.

Table 4: Welfare Costs in the Presence of Working Capital Constraint

	Free Trade	Restricted Trade (=10%)
Unconditional	-0.0016	-0.0005

Compared to the previous case, some benefits of sudden stops are eroded by the distortion caused by the working capital constraint. The two findings still hold when we introduce working capital constraint: sudden stops become less harmful when the economy is more open to trade; and sudden stops may be good to this small open economy if it is a net borrower in the international capital market. Thus, the fact that the findings are generally true implies that commercial policies which promote trade openness should be strictly preferred to other commercial policies, given that others are equal.

2.3 Two Sector Economy with Homogenous Capital

In this section, non-tradable good is introduced. It has been accepted that the introduction of non-tradable good is important for the understanding of the

dynamics of open economy hitting by different shocks since the non-tradable sector is a large sector the economy, almost around 50%. The key element of this two sector model is the supply elasticity of non-tradable good. In the simple setup, non-tradable good is inelastically supplied. The demand change can only be balanced through the change of the relative price of non-tradable good. Introducing capital accumulation into the non-tradable good production sector serves to change the completely inelastic supply to elastic supply of non-tradable good. It is expected that the dynamics of the two sector economy with homogenous capital should be alike the dynamics of the corresponding one sector economy. In Section 4 when capital is sector specific, the supply of non-tradable good will become less elastic. In that case, it is expected that the dynamics of the two sector economy with heterogenous capital will be much more different from the dynamics of the corresponding one sector economy.

2.3.1 Model Modification

The introduction of non-tradable good modifies the model in Section 2 as follows. First, there are productions of non-tradable good and final good. The production of non-tradable takes capital and labor as inputs:

$$Y_t^N = z_t^N (k_t^N)^{\alpha_k^N} (h_t^N)^{\alpha_h^N}, \quad (2.26)$$

where the superscript N denotes non-tradable and all the variables have their usual meanings. The exclusion of m from the production of non-tradable good is in line with the usual empirical definition of non-tradable good sector. In the literature, the non-tradable good sector usually includes: construction, utilities, retailing, restaurant and hotels, financial services and real estate, and social and personal services. For all these industries, the total trade is less than 5% in the gross output at the current prices. And the $\log(z_t^N)$ follows

an AR(1) process with the serial correlation coefficient, ρ_{z^N} , and the standard deviation of the innovation of the AR(1) process, σ_{z^N} .

The production of final good takes tradable good and non-tradable good as inputs:

$$Y_t^F = \left[\kappa (c_t^T)^{-\eta} + (1 - \kappa) (c_t^N)^{-\eta} \right]^{-\frac{1}{\eta}} = V(c_t^T, c_t^N), \quad (2.27)$$

where the superscript F denotes "final"; the parameter η governs the intra-temporal substitution elasticity between c_T and c_N , which is $\frac{1}{1+\eta}$; and the parameter κ is the CES weighting factor. All the final good is used as final consumption, investment, and investment adjustment costs, $Y_t^F = c_t + i_t + \Phi(k_{t+1} - k_t)$.

By assuming the productions of both non-tradable good and final good are perfectly competitive, the relative prices (in units of tradable good) of non-tradable good (i.e., real exchange rate) and final good are determined by:

$$p_t^N = \frac{V_N}{V_T}, \quad (2.28)$$

$$1 = p_t V_T. \quad (2.29)$$

Since capital and labor are homogenous and can be reallocated across sectors without any costs, the marginal productivity of capital should be equalized across sectors. So is the marginal productivity of labor.

The representative still maximize his lifetime utility. By introducing the non-tradable good, the marginal utility of consuming one more unit of tradable good is $U_c V_T$. Compared with the model in Section 2, the first order conditions

become

$$\begin{aligned}
0 &= \lambda_t p_t [1 + \Phi_{k_{t+1}}(k_{t+1} - k_t)] \\
&\quad - \beta(\tilde{c}_t, \tilde{h}_t) E_t \lambda_{t+1} \{p_{t+1} [1 - \delta - \Phi_{k_{t+1}}(\hat{k}_{t+2} - k_{t+1})] + r_{t+1}\}, \\
\frac{U_h}{U_c V_T} &= -w_t, \\
\lambda_t &= U_c V_T.
\end{aligned}$$

And there are three extra equilibrium conditions:

$$\begin{aligned}
c_t^N &= y_t^N, \\
k_t &= k_t^T + k_t^N, \\
h_t &= h_t^T + h_t^N.
\end{aligned}$$

And The full description of the representative household's problem is found in the appendix.

2.3.2 Calibration Updating

Consider the data availability, we calibrate the model parameters to match the Mexico (instead of Argentine) empirical regularities of business cycles found in Uribe (1997) and Mendoza (2002). The parameter ω is taken from Mendoza (2002) and set at 2.12. The parameter η is taken from Ostry and Reinhart (1992) and set at 0.316. The parameter κ is set at 0.38 to make sure that in the non-stochastic steady state, the non-tradable consumption to GDP ratio is 56%. The labor income share in the non-tradable sector is set at 0.72 and the labor income share in the value-added in the tradable sector is set at 0.55. The imported inputs elasticity α_M is set at 0.2584. The values for ρ_{z^T} , ρ_{z^N} , σ_{z^T} , σ_{z^N} , and ϕ are calibrated to make the simulated volatilities of tradable output, non-tradable output, investment, and the simulated first order serial

correlation coefficients of tradable output and non-tradable output to be as close as possible to their corresponding parts found in the data as shown in Mendoza (2002). In summary,

Symbol	Definition	Value
ω	exponent of labor supply in utility	2.12
s_h^T	share of labor income in tradable value added	0.55
s_h^N	share of labor income in non-tradable value added	0.72
α_M	imported inputs elasticity	0.2584
α_{k^T}	capital elasticity in tradable sector	0.3337
α_{h^T}	labor elasticity in tradable sector	0.4079
α_{k^N}	capital elasticity in non-tradable sector	0.2800
α_{h^N}	labor elasticity in non-tradable sector	0.7200
η	about the substitution elasticity between (c_T, c_N)	0.316
κ	CES weighing factor	0.38
ϕ	investment adjustment cost coefficient	6.15
ρ_T	serial correlation of tradable productivity shocks	0.4
ρ_N	serial correlation of non-tradable productivity shocks	0.45
$\sigma_{z,T}$	standard deviation of tradable productivity shocks	0.00135
$\sigma_{z,N}$	standard deviation of tradable productivity shocks	0.0006
β_1	subjective discount parameter	0.21479

2.3.3 Simulation Results

Table 5 below lists some second moments of the economies in this setting. From the table, the same results show up: the less open economy tends to be more volatile.

Table 5: Some Second Moments in the Two Sector Model with Homogenous Capital¹⁴¹⁵

¹⁴ \star denotes the case of low country spread volatility.

¹⁵ $\star\star$ denotes the case of high country spread volatility.

	σ_y^T	$\rho(y^T, y^T-1)$	σ_y^N	$\rho(y^N, y^N-1)$	σ_I	σ_p^N
Actual Data	3.37	0.55	2.74	0.66	11.08	11.93
Free Trade*	4.03	0.85	3.20	0.82	10.79	0.48
Free Trade**	4.95	0.83	4.01	0.80	13.50	0.57
Tariff = 10%*	4.39	0.87	3.17	0.82	11.30	0.48
Tariff = 10%**	5.39	0.85	3.97	0.79	14.15	0.58

One thing needs to mention is that the two sector model could not replicate the low first order autocorrelations of tradable and nontradable goods; nor could it replicate the high volatile relative price of nontradable goods.

Figure 2.5 shows the impulse responses to a positive country spread shock. For the same reason, final consumption and investment drop; and trade balance and current account improve. The reduced demand for final good drives down the relative price of final good, p^F . As a result, real exchange rate, p^N , goes down (appreciation), since in our model the two price indexes are positively correlated. The appreciation makes it more profitable to produce tradable good instead of nontradable good, thus more capital will be allocated to the tradable sector.

Table 6 lists the welfare costs of sudden stops. They have a very similar pattern as shown in Table 2: sudden stops are good and the welfare benefits increase in trade openness.

Table 6: Welfare Costs of Sudden Stops

	Free Trade	Restricted Trade (=10%)
Unconditional	-0.0012	-0.0010

2.4 Two Sector Economy with Heterogenous Capital

In this section, we relax the homogenous capital assumption so that the modified model will be more realistic: it is costly to reallocate capital among sectors. There are many ways to introduce the reallocation costs. Here we assume that capital accumulation in each sector is subject to investment adjustment cost. When the capital is sector specific, the return to capital may not be equal across sectors. Each sector has its own optimal capital decision equation:

$$\begin{aligned}
0 &= \lambda_t p_t^T [1 + \Phi_{k_{t+1}^T} (k_{t+1}^T - k_t^T)] \\
&\quad - \beta(\tilde{c}_t, \tilde{h}_t) E_t \lambda_{t+1} \{ p_{t+1}^T [1 - \delta + \Phi_{k_{t+1}^T} (k_{t+2}^T - k_{t+1}^T)] \\
&\quad - z_{t+1}^T F_k(k_{t+1}^T, h_{t+1}^T, m_{t+1}) \}, \tag{2.30}
\end{aligned}$$

$$\begin{aligned}
0 &= \lambda_t p_t^N [1 + \Phi_{k_{t+1}^N} (k_{t+1}^N - k_t^N)] \\
&\quad - \beta(\tilde{c}_t, \tilde{h}_t) E_t \lambda_{t+1} \{ p_{t+1}^N [1 - \delta + \Phi_{k_{t+1}^N} (k_{t+2}^N - k_{t+1}^N)] \\
&\quad - z_{t+1}^N F_k(k_{t+1}^N, h_{t+1}^N) \}. \tag{2.31}
\end{aligned}$$

The table below lists some second moments of the economies in this setting. From the table, the same results show up: the less open economy tends to be more volatile. Compare to the homogenous capital case, now the relative price becomes much more volatile since capital is costly to change. Still, it's standard deviation, around 2.5%, is far below the observed result in data, which is 11.93%. However, when it is costly to adjust capital across sectors, the first order autocorrelations of tradable and nontradable goods are even higher than those found in the data.

Table 7: Some Second Moments in the Two Sector Model with Heterogenous

	σ_y^T	$\rho(y^T, y^T - 1)$	σ_y^N	$\rho(y^N, y^N - 1)$	σ_I	σ_p^N
Actual Data	3.37	0.55	2.74	0.66	11.08	11.93
Free Trade*	3.40	0.91	2.61	0.87	12.27	2.40
Free Trade**	4.12	0.89	3.24	0.85	15.33	3.03
Tariff* = 10%	3.71	0.92	2.56	0.87	12.76	2.41
Tariff** = 10%	4.51	0.90	3.17	0.85	15.94	3.05

The welfare costs of sudden stops here show similar pattern as those in the above cases: sudden stops are good and the welfare benefits increase in trade openness.

Table 8: Welfare Costs

	Free Trade	Restricted Trade (=10%)
Unconditional	-0.0013	-0.0010

2.5 Counter-Cyclical Tariff Rate Policy

It is natural for the government to have counter-cyclical tariff rate policy with the intention to stabilize the economy. The empirical regularity of counter-cyclical tariff rate policy has been established. The most famous example is the Smoot-Hawley Tariff Act passed in June of 1930 to increase tariffs to 50%. In the early 1980s, the tariff rate in Chile rose in face of the debt crisis. After the December 1994 Peso crisis, the general tariff in Mexico rose from 8.7% in 1994 to a peak of 12.5% in 1995 (source: Haltiwanger et. al (2004)). The theoretical explanation often goes as the political factors will affect decision

¹⁶_{*} denotes the case of low country spread volatility.

¹⁷_{**} denotes the case of high country spread volatility.

makers in such a way that the counter-cyclical trade policy is the final outcome. One reference for this argument is Bagwell and Staiger (1995).

One question is whether the counter-cyclical policy will really achieve the goal of stabilizing the economy? To answer this question, this section extends the two sector model in Section 4 by introducing counter-cyclical tariff rate policy. In particular, the tariff rate is no longer constant, but follows the rule:

$$\hat{\tau}_t = -\psi * \hat{z}_t^T, \quad (2.32)$$

where the coefficient ψ takes two values 0.5 and 1.5, which corresponding to "passive" and "aggressive" stances of the government.

To evaluate the counter-cyclical tariff rate policy against the constant tariff rate policy, it is necessary to use second order approximation. With just first order approximation of the policy function, policies or economies with the same non-stochastic steady state lifetime welfare will give the same first order approximation of stochastic steady state lifetime welfare for the two reasons: the first order condition with respect to the parameter that controls the volatility of the cycles is zero because this is one of the optimality conditions; and the first order approximation of the argument in the utility function is always the non-stochastic steady state value. Using the following expression, it will be clear why the first order approximation cannot rank the policies or economies with the same non-stochastic steady state values:

$$E(u) = u^* + u_c(E(c) - c^*) + u_\sigma\sigma,$$

where $*$ denotes the non-stochastic steady state value, the variable c denotes the argument in the utility function, and the variable σ controls the volatility of the cycles. The first order condition requires that $u_\sigma = 0$ and the first order approximation makes it true that $E(c) = c^*$. It then comes true that

$E(u) = u^*$. So, we cannot differentiate policies or economies with the same u^* . But with second order approximation, it is clear that at least $E(c) \neq c^*$ and $E(u) \neq u^*$. As a result, the second order approximation will allow us to evaluate different policies or economies with the same non-stochastic steady state lifetime welfare.

The welfare costs of sudden stops given the aggressive stance of the government are listed below. It can be seen that unconditionally, the aggressive stance will greatly reduce the benefits brought by sudden stops: the reduction in welfare due to the increase of tariff rate in the good times outweighs the gain in welfare due to the decrease of tariff rate in the bad times.

	Fixed Tariff	Changing Tariff
Unconditional	-0.0013	-0.0001

2.6 Conclusion

The paper has shown that the welfare costs of sudden stops decrease when a small open economy becomes more open to trade. One explanation for this result is that an exogenous shock will make the economy more volatile if it is less open. It confirms our guess that commercial policy making the economy more open to trade is strictly preferable for a very wide range of parameter calibration.

The second simulation result that sudden stops can be good seems counter-intuitive because: given the usual risk averse household assumption, it is naturally expected that the household in a more volatile economy, in terms of the major macro-economic aggregates, should expect less utility. But such a claim has been recently attacked by many authors, for example, Obstfeld and Rogoff

(1998, 1999). In Obstfeld and Rogoff (1999, 1998), the volatility of shocks will change the level of variables thus having either positive or negative impact on the welfare of the household. Another example is a production economy only with productivity shock. In this economy, even though the household is risk averse, the production function can be convex in productivity thus it becomes possible that the household likes the volatile economies. This paper makes two contributions in this respect: (1) All the previous work studies static problem and this project is regarded as an extension of the welfare-volatility analysis into a dynamic setting; and (2) We explain why volatile country spread may increase the utility of the small open economy.

The paper makes unreasonable assumptions. One particular assumption is the exogenous country spread process. To get more accurate picture of welfare costs of sudden stop, it is necessarily to endogenize country spread.

Figure 2.1: Steady State Plotted against Tariff Rate

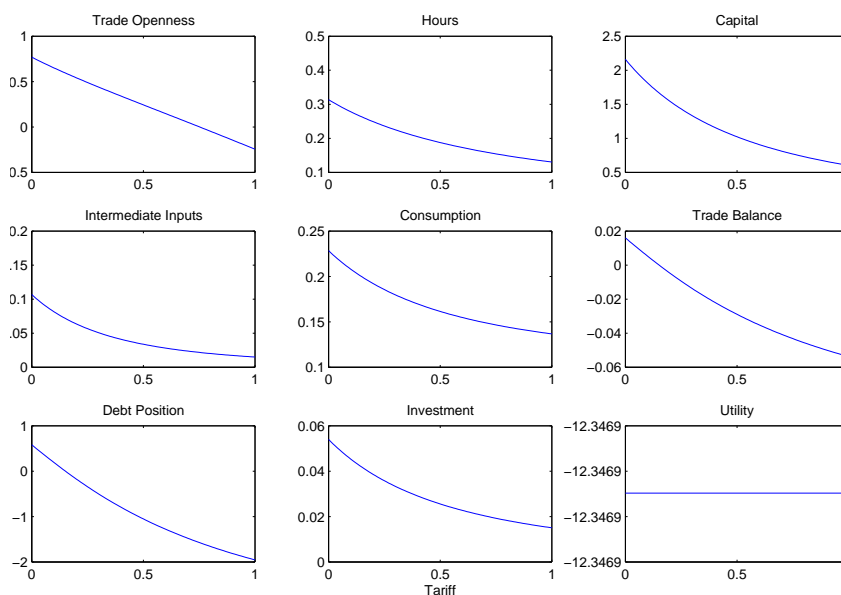


Figure 2.2: Impulse Responses to a Positive Country Spread Shock (One Sector Economy without Working Capital Constraint; Line corresponds to 0% tariff rate; Dotted Line corresponds to 10% tariff rate.)

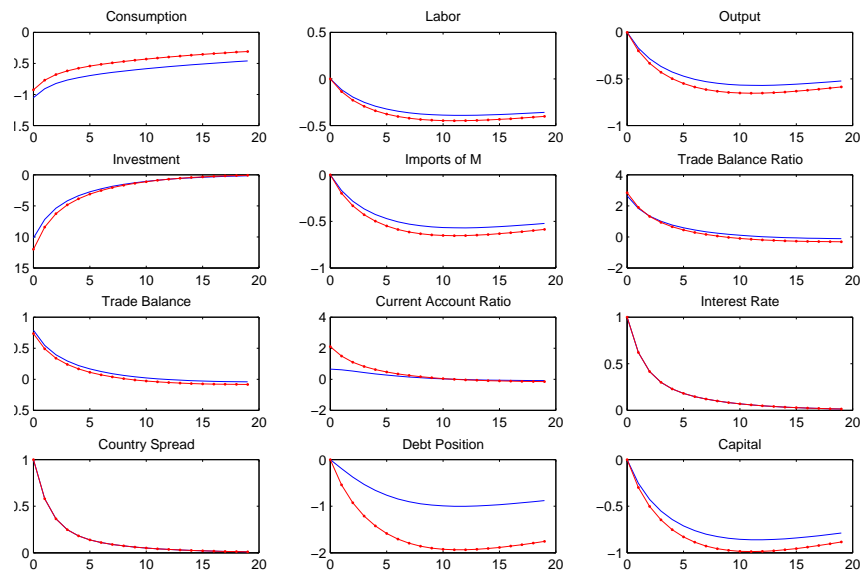


Figure 2.3: Unconditional utility plotted against country spread (R)

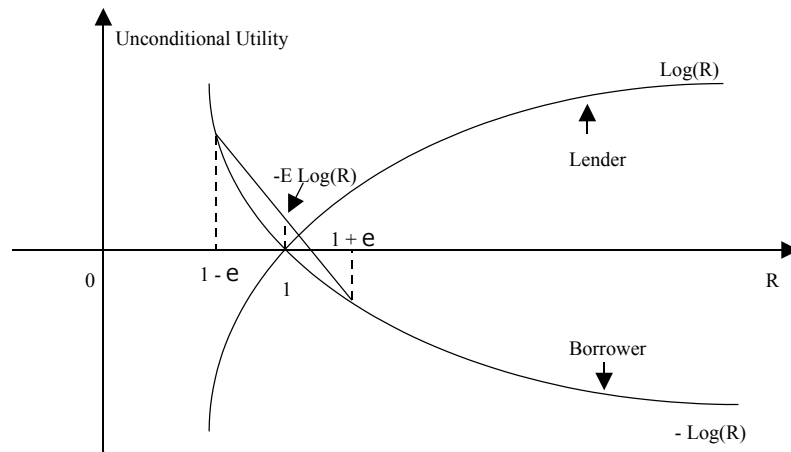


Figure 2.4: Impulse Responses to a Positive Country Spread Shock (One Sector Economy with Working Capital Constraint; Line corresponds to 0% tariff rate; Dotted Line corresponds to 10% tariff rate.)

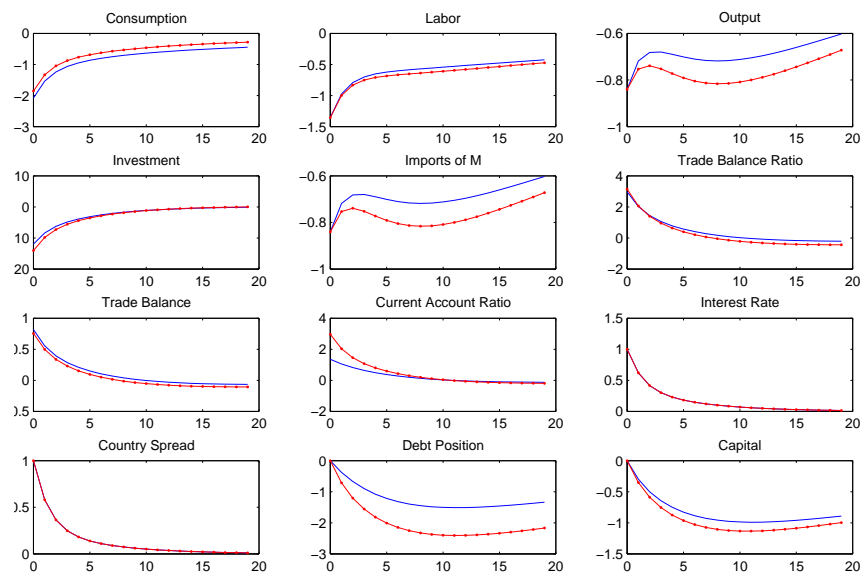
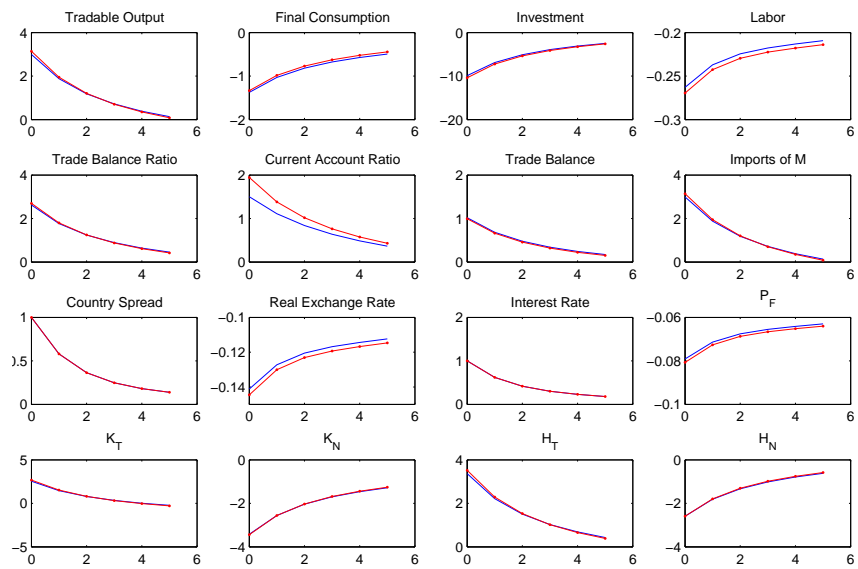


Figure 2.5: Impulse Responses to a Positive Country Spread Shock (Two Sector Economy with Homogenous Capital; Line corresponds to 0% tariff rate; Dotted Line corresponds to 10% tariff rate.)



Appendix A

Appendix for Chapter 1

A.1 Equivalence of constraints (1.1) and (1.2) with the single constraint (1.4)

It is straightforward to show that sequences for $\{c_t, m_{t+1}, h_t\}$ satisfying the constraints (1.1) and (1.2) with equality satisfy the single constraint (1.4). Here I show the converse that sequences for $\{c_t, m_{t+1}, h_t\}$ satisfying the single constraint (1.4) also satisfy constraints (1.1) and (1.2). Rewrite constraint (1.4) at t in the following way

$$\begin{aligned}
 D_t &= \sum_{s=t+1}^{\infty} q_{t,s} \{c_s [(1 - \tau_s)w_s h_s + \Pi_s]\} + \sum_{s=t+2}^{\infty} q_{t,s} i_s m_s \\
 &\quad + c_t - [(1 - \tau_t)w_t h_t + \Pi_t] + q_{t,t+1} i_{t+1} m_{t+1} \\
 &= \sum_{s=t+2}^{\infty} \{q_{t,s} c_s - [(1 - \tau_s)w_s h_s + \Pi_s]\} + \sum_{s=t+2}^{\infty} q_{t,s} i_s m_s \\
 &\quad + \sum_{s=t}^{t+1} q_{t,s} \{c_t - [(1 - \tau_t)w_t h_t + \Pi_t]\} + \sum_{s=t+1}^{t+2} q_{t,s} i_s m_s
 \end{aligned}$$

where $D_t = \frac{M_{t-1}}{P_t} + \sum_{s=t}^{\infty} q_{t,s} \left({}_{t-1}b_s^G + \frac{{}_{t-1}B_s^G}{p_s} \right)$. Recursively substituting the above expressions and taking the limit, I have

$$D_t = D_t + \lim_{j \rightarrow \infty} q_{t,t+j} D_{t+j} \Rightarrow \lim_{j \rightarrow \infty} q_{t,t+j} D_{t+j} = 0$$

Thus constraints (1.2) are satisfied. Taking the difference between the intertemporal budget constraints at time t and time $t + 1$, and multiplying by $q_{t,t+1}$, it can be shown that constraints (1.1) are satisfied. ■

A.2 Primary form of competitive equilibrium.

Sequences for $\{\lambda, m_{t+1}, \text{and } h_t\}$ satisfying optimality conditions (1.4), (1.5), (1.6), (1.7), (1.9), (1.10), (1.13), (1.14), and (1.3), are the same as those satisfying optimality conditions of (1.17) and (1.18).

It is straightforward to show that the sequences for $\{\lambda, m_{t+1}, \text{and } h_t\}$ satisfying optimality conditions (1.3), (1.4), (1.5), (1.6), (1.7), (1.9), (1.10), (1.13), (1.14), (1.15), and (1.17), are consistent with those satisfying the optimality conditions (1.17) and (1.18).

Here I show the converse that the sequences for $\{\lambda, m_{t+1}, \text{and } h_t\}$ satisfying optimality conditions (1.17) and (1.18) are consistent with those satisfying the optimality conditions (1.3), (1.4), (1.5), (1.6), (1.7), (1.9), (1.10), (1.13), (1.14), (1.15), and (1.17). Following the similar steps in SGU (2003), it can be shown that the claim is true. ■

A.3 Intertemporal budget constraint of the government

The period budget constraint of the government is

$$\begin{aligned} & q_{0,t} \left(g_t + \frac{M_{t-1}}{p_t} \right) + \sum_{s=t}^{\infty} q_{0,s} \left({}_{t-1}b_s^G + \frac{{}_{t-1}B_s^G}{p_s} \right) \\ & \leq q_{0,t} \left(\tau_t w_t h_t + \frac{M_t}{p_t} \right) + \sum_{s=t+1}^{\infty} q_{0,s} \left({}_t b_s^G + \frac{{}_t B_s^G}{p_s} \right) \end{aligned}$$

Rearranging and summing over the infinite horizon:

$$\begin{aligned} & \sum_{t=0}^{\infty} \sum_{s=t}^{\infty} q_{0,s} \left({}_{t-1}b_s^G + \frac{{}_{t-1}B_s^G}{p_s} \right) - \sum_{t=0}^{\infty} \sum_{s=t+1}^{\infty} q_{0,s} \left({}_t b_s^G + \frac{{}_t B_s^G}{p_s} \right) \\ & \leq \sum_{t=0}^{\infty} q_{0,t} (\tau_t w_t h_t - g_t) + \sum_{t=0}^{\infty} q_{0,t} \left(\frac{M_t}{p_t} - \frac{M_{t-1}}{p_t} \right) \end{aligned}$$

Simplifying the left-hand side:

$$\begin{aligned} & \sum_{t=0}^{\infty} \sum_{s=t}^{\infty} q_{0,s} \left({}_{t-1}b_s^G + \frac{{}_{t-1}B_s^G}{p_s} \right) - \sum_{t=0}^{\infty} \sum_{s=t+1}^{\infty} q_{0,s} \left({}_t b_s^G + \frac{{}_t B_s^G}{p_s} \right) \\ & = q_{0,0} \sum_{s=0}^{\infty} \left({}_{-1}b_s^G + \frac{{}_{-1}B_s^G}{p_s} \right) + \sum_{t=1}^{\infty} \sum_{s=t}^{\infty} q_{0,s} \left({}_{t-1}b_s^G + \frac{{}_{t-1}B_s^G}{p_s} \right) \\ & \quad - \sum_{t=0}^{\infty} \sum_{s=t+1}^{\infty} q_{0,s} \left({}_t b_s^G + \frac{{}_t B_s^G}{p_s} \right) \\ & = \sum_{s=0}^{\infty} q_{0,s} \left({}_{-1}b_s^G + \frac{{}_{-1}B_s^G}{p_s} \right) - \lim_{T \rightarrow \infty} \sum_{s=T}^{\infty} q_{0,s} \left({}_T b_s^G + \frac{{}_T B_s^G}{p_s} \right) \end{aligned}$$

Simplifying the second term in the right-hand side:

$$\begin{aligned} \sum_{t=0}^{\infty} q_{0,t} \left(\frac{M_t}{p_t} - \frac{M_{t-1}}{p_t} \right) & = \sum_{t=0}^{\infty} q_{0,t} \frac{M_t}{p_t} - \sum_{t=0}^{\infty} q_{0,t} \frac{M_{t-1}}{p_t} \\ & = \sum_{t=0}^{\infty} q_{0,t+1} \frac{(1+i_{t+1})M_t}{p_{t+1}} - \sum_{t=1}^{\infty} q_{0,t} \frac{M_{t-1}}{p_t} - \frac{M_{-1}}{p_0} \\ & = \sum_{t=1}^{\infty} q_{0,t} \frac{(1+i_t)M_{t-1}}{p_t} - \sum_{t=1}^{\infty} q_{0,t} \frac{M_{t-1}}{p_t} - \frac{M_{-1}}{p_0} \\ & = \sum_{t=1}^{\infty} q_{0,t} i_t \frac{M_{t-1}}{p_t} + \lim_{t \rightarrow \infty} \frac{q_{0,t} M_t}{p_t} - \frac{M_{-1}}{p_0} \end{aligned}$$

Thus the intertemporal budget constraint of the government can be written as

$$\begin{aligned}
& \sum_{s=0}^{\infty} q_{0,s} \left(-{}_1b_s^G + \frac{{}_1B_s^G}{p_s} \right) - \lim_{t \rightarrow \infty} \sum_{s=t+1}^{\infty} q_{0,s} \left({}_t b_s^G + \frac{{}_t B_s^G}{p_s} \right) \\
& \leq \sum_{t=0}^{\infty} q_{0,t} (\tau_t w_t h_t - g_t) + \sum_{t=1}^{\infty} q_{0,t} (1 + i_t) \frac{M_{t-1}}{p_t} + \lim_{t \rightarrow \infty} \frac{q_{0,t} M_t}{p_t} - \frac{M_{-1}}{p_0} \\
& \quad \sum_{s=0}^{\infty} q_{0,s} \left(-{}_1b_s^G + \frac{{}_1B_s^G}{p_s} \right) + \frac{M_{-1}}{p_0} \\
& \leq \sum_{t=0}^{\infty} q_{0,t} (\tau_t w_t h_t - g_t) + \sum_{t=1}^{\infty} q_{0,t} i_t \frac{M_{t-1}}{p_t} + \\
& \quad \lim_{t \rightarrow \infty} \left\{ \frac{q_{0,t} M_t}{p_t} + \sum_{s=t+1}^{\infty} q_{0,s} \left({}_t b_s^G + \frac{{}_t B_s^G}{p_s} \right) \right\}
\end{aligned}$$

By using the no-Ponzi game condition, it can further simplify the inter-temporary budget constraint as

$$\sum_{s=0}^{\infty} q_{0,s} \left(-{}_1b_s^G + \frac{{}_1B_s^G}{p_s} \right) + \frac{M_{0-1}}{p_0} \leq \sum_{t=0}^{\infty} q_{0,t} (\tau_t w_t h_t - g_t) + \sum_{t=1}^{\infty} q_{0,t} i_t m_t$$

A.4 Implementability constraint of the Ramsey government (Hereafter RG)

Use the optimal condition of domestic household, $i_t = \frac{u_{mt}}{u_{ct}}$ and $u_{ct} = \lambda$, rewrite the intertemporal budget constraint of the government as

$$\begin{aligned}
& \sum_{s=0}^{\infty} q_{0,s} \left(-{}_1b_s^G + \frac{{}_{-1}B_s^G}{p_s} \right) + \frac{M_{-1}}{p_0} \\
&= \sum_{t=0}^{\infty} q_{0,t} (\tau_t w_t h_t - g_t) + \sum_{t=1}^{\infty} q_{0,t} i_t m_t \\
&= \sum_{t=0}^{\infty} q_{0,t} \left[\left(1 + \frac{1}{w_t} \frac{u_{ht}}{u_{ct}} \right) w_t h_t - g_t \right] + \sum_{t=1}^{\infty} q_{0,t} i_t m_t \\
&= \sum_{t=0}^{\infty} \beta^t \left[(\eta z_t h_t^\eta - g_t) + \frac{u_{ht}}{\lambda} h_t \right] + \sum_{t=1}^{\infty} q_{0,t} \frac{u_{mt}}{\lambda} m_t
\end{aligned}$$

Rearrange and we get:

$$\begin{aligned}
\sum_{s=0}^{\infty} q_{0,s} \left(\frac{{}_{-1}B_s^G}{p_s} \right) + \frac{M_{-1}}{p_0} &= \sum_{t=0}^{\infty} \beta^t \left[(\eta z_t h_t^\eta - g_t - {}_{-1}b_s^G) + \frac{u_{ht}}{\lambda} h_t \right] \\
&+ \sum_{t=1}^{\infty} q_{0,t} \frac{u_{mt}}{\lambda} m_t
\end{aligned}$$

A.5 Simplification of Condition (1.21)

Since $Q_{0,t}^{-1} = \prod_{i=1}^t \left(1 + \frac{u_{mi}}{\lambda} \right)$, it is true that:

$$\frac{\partial Q_{0,s}}{\partial m_t} = Q_{0,s} \frac{-1}{1 + \frac{u_{mt}}{\lambda}} \frac{u_{mmt}}{\lambda} = -Q_{0,s} \frac{u_{mmt}}{\lambda + u_{mt}}$$

Rewrite condition (1.21) as

$$u_{mt} = \left[\frac{\mu_0^G \lambda}{\beta^t p_0} \sum_{s=t}^{\infty} (-1 B_s^G) Q_{0,s} + \frac{\mu_0^E}{\beta^t p_0} \sum_{s=t}^{\infty} (-1 B_s^F) Q_{0,s} \right] \frac{-u_{mmt}}{\lambda + u_{mt}} - \mu_0^G (u_{mmt} m_t + u_{mt}) - \psi_t u_{mmt}$$

Or rewrite this as

$$u_{mt} (1 + \mu_0^G) = \left[\frac{\mu_0^E \sum_{s=t}^{\infty} (-1 B_s^F) Q_{0,s}}{\beta^t p_0 (\lambda + u_{mt})} + \frac{\mu_0^G \lambda \sum_{s=t}^{\infty} (-1 B_s^G) Q_{0,s}}{\beta^t p_0 (\lambda + u_{mt})} \right] u_{mmt} - [\mu_0^G m_t + \psi_t] u_{mmt} \quad (\text{A.1})$$

Relaxing the non-negativity constraint, and I obtain the modified (1.21), i.e., (1.42).

A.6 Time Consistency when Both Real and Nominal Bonds Are Available

The Lagrange associated with the $t = 0$ government is given by:

$$\begin{aligned} \mathbf{L} = & u \left[c(\lambda), \frac{M_{-1}}{p_0}, h_0 \right] + \sum_{t=1}^{\infty} \beta^t u [c(\lambda), m_t, h_t] \\ & - \frac{\mu_0^G \lambda}{p_0} \left[\sum_{t=0}^{\infty} Q_{0,t} (-1 B_t^G) + M_{-1} \right] + \sum_{t=0}^{\infty} \beta^t \psi_t u_{mt} \\ & + \mu_0^G \left\{ \sum_{t=0}^{\infty} \beta^t [\lambda (\eta z_t h_t^\eta - g_t - {}_{-1} b_t^G) + u_{ht} h_t] + \sum_{t=1}^{\infty} \beta^t u_{mt} m_t \right\} \\ & + \mu_0^E \left\{ \sum_{t=0}^{\infty} \beta^t [z_t h_t^\eta - c(\lambda) - g_t - {}_{-1} b_t^F] - \frac{1}{p_0} \sum_{t=0}^{\infty} Q_{0,t} (-1 B_t^F) \right\} \end{aligned}$$

The optimality conditions $t = 0$ for the $t = 0$ government are:

$$\begin{aligned}
\frac{\lambda - \mu_0^E}{1 - \beta} &= \frac{\mu_0^E}{p_0} \sum_{t=0}^{\infty} (-_1B_t^F) \frac{\partial Q_{0,t}}{\partial \lambda} + \frac{\mu_0^G}{p_0} \left[\sum_{t=0}^{\infty} Q_{0,t} (-_1B_t^G) + M_{-1} \right] \\
&\quad + \frac{\mu_0^G \lambda}{p_0} \sum_{t=0}^{\infty} (-_1B_t^G) \frac{\partial Q_{0,t}}{\partial \lambda} - \mu_0^G \sum_{t=0}^{\infty} \beta^t (\eta z_t h_t^\eta - g_t -_{-1}b_t^G) \\
0 &= u_{m0} \left(-\frac{M_{-1}}{p_0^2} \right) + \frac{\mu_0^G \lambda}{p_0^2} \left[\sum_{t=0}^{\infty} Q_{0,t} (-_1B_t^G) + M_{-1} \right] \\
&\quad + \frac{\mu_0^E}{p_0^2} \sum_{t=0}^{\infty} Q_{0,t} (-_1B_t^F) \\
-u_{ht} &= \mu_0^G (\lambda \eta^2 z_t h_t^{\eta-1} + u_{hht} h_t + u_{ht}) + \mu_0^E \eta z_t h_t^{\eta-1} \\
u_{mt} (1 + \mu_0^G) &= \left[\frac{\mu_0^E \sum_{s=t}^{\infty} (-_1B_s^F) Q_{0,s}}{\beta^t p_0 (\lambda + u_{mt})} + \frac{\mu_0^G \lambda \sum_{s=t}^{\infty} (-_1B_s^G) Q_{0,s}}{\beta^t p_0 (\lambda + u_{mt})} \right] u_{mmt} \\
&\quad - \mu_0^G m_t u_{mmt} \\
0 &= \sum_{t=0}^{\infty} \beta^t [c(\lambda) + g_t +_{-1}b_t^F - z_t h_t^\eta] + \frac{1}{p_0} \sum_{t=1}^{\infty} Q_{0,t} (-_1B_t^F) \\
0 &= \sum_{t=0}^{\infty} \beta^t [\lambda (\eta z_t h_t^\eta - g_t -_{-1}b_t^G) + u_{ht} h_t] \\
&\quad + \sum_{t=1}^{\infty} \beta^t u_{mt} m_t - \frac{\lambda}{p_0} \left[\sum_{t=0}^{\infty} Q_{0,t} (-_1B_t^G) + M_{-1} \right]
\end{aligned}$$

The optimality conditions for the $t = 1$ government are:

$$\begin{aligned}
0 &= \frac{1}{p_1} \sum_{t=1}^{\infty} Q_{1,t} ({}_0B_t^F) + \sum_{t=1}^{\infty} \beta_0^t b_t^F - \sum_{t=1}^{\infty} \beta^{t-1} [h^\eta - c(\lambda) - g_t] \\
0 &= \sum_{t=1}^{\infty} \beta^{t-1} [\lambda(\eta h^\eta - g_t) + u_h h] + \sum_{t=2}^{\infty} \beta^t u_{mt} m_t - \frac{\lambda}{p_1} M_0 \\
&\quad - \frac{\lambda}{p_1} \sum_{t=1}^{\infty} Q_{1,t} ({}_0B_t^G) - \sum_{t=1}^{\infty} \beta_0^t b_t^G \\
\frac{\lambda - \mu_1^E}{1 - \beta} &= \frac{\mu_1^E}{p_1} \sum_{t=2}^{\infty} ({}_0B_t^F) \frac{\partial Q_{1,t}}{\partial \lambda} + \frac{\mu_1^G}{p_1} \left[\sum_{t=1}^{\infty} Q_{1,t} ({}_0B_t^G) + M_0 \right] \\
&\quad + \frac{\mu_1^G \lambda}{p_1} \sum_{t=2}^{\infty} ({}_0B_t^G) \frac{\partial Q_{1,t}}{\partial \lambda} + \mu_1^G \sum_{t=1}^{\infty} \beta^{t-1} ({}_0b_t)^G \\
&\quad - \mu_1^G \sum_{t=1}^{\infty} \beta^{t-1} (\eta h^\eta - g_t) \\
u_{mt} (1 + \mu_1^G) &= \left[\frac{\mu_1^E \sum_{s=t}^{\infty} ({}_0B_s^F) Q_{1,s}}{\beta^{t-1} p_1 (\lambda + u_{mt})} + \frac{\mu_1^G \lambda \sum_{s=t}^{\infty} ({}_0B_s^G) Q_{1,s}}{\beta^{t-1} p_1 (\lambda + u_{mt})} \right] u_{mmt} \\
&\quad - \mu_1^G m_t u_{mmt} \\
u_h &= \mu_1^G (\lambda \eta^2 h^{\eta-1} + u_{hh} h + u_h) + \mu_1^E \eta h^{\eta-1} \\
u_{m1} M_0 &= \mu_1^G \lambda \left[\sum_{t=1}^{\infty} Q_{1,t} ({}_0B_t^G) + M_0 \right] + \mu_1^E \sum_{t=1}^{\infty} Q_{1,t} ({}_0B_t^F)
\end{aligned}$$

A.7 Liquidity Constraint of the $t = 0$ Government

Here I use some simplified notations, such as

$$b_0^G = \sum_{s=0}^{\infty} \beta^s (-_1b_s^G); b_1^G = \sum_{s=1}^{\infty} \beta^{s-1} ({}_0b_s^G);$$

$$b_0^F = \sum_{s=0}^{\infty} \beta^s (-_1b_s^F); b_1^F = \sum_{s=1}^{\infty} \beta^{s-1} ({}_0b_s^F).$$

The intertemporal budget constraint of the $t = 0$ government is given by

$$\sum_{s=0}^{\infty} Q_{0,s} \left(\frac{-_1B_s^G}{p_0} \right) + \frac{M_{-1}}{p_0} + b_0^G = \sum_{t=0}^{\infty} \beta^t \left[(\eta z_t h_t^\eta - g_t) + \frac{u_{ht}}{\lambda} h_t \right]$$

$$+ \sum_{t=1}^{\infty} \beta^t \frac{u_{mt}}{\lambda} m_t$$

The intertemporal budget constraint of the $t = 1$ government is given by

$$\sum_{s=1}^{\infty} Q_{1,s} \left(\frac{-_1B_s^G}{p_1} \right) + \frac{M_0}{p_1} + b_1^G = \sum_{t=1}^{\infty} \beta^{t-1} \left[(\eta z_t h_t^\eta - g_t) + \frac{u_{ht}}{\lambda} h_t \right]$$

$$+ \sum_{t=2}^{\infty} \beta^{t-1} \frac{u_{mt}}{\lambda} m_t$$

Here I want to show that given that the policy continuation satisfies the $t = 1$ government's budget constraint, the constructed term structure of bonds is consistent with the intertemporal budget constraint of the $t = 0$ government.

Let's start with the $t = 0$ government's period budget constraint,

$$\sum_{s=0}^{\infty} Q_{0,s} \left(\frac{-_1B_s^G}{p_0} \right) + \frac{M_{-1}}{p_0} + b_0^G = \left[(\eta z_0 h_0^\eta - g_0) + \frac{u_{h0}}{\lambda} h_0 \right]$$

$$+ \sum_{s=1}^{\infty} Q_{0,s} \left(\frac{{}_0B_s^G}{p_0} \right) + \frac{M_0}{p_0} + \beta b_1^G$$

Substituting into the $t = 0$ government's intertemporal budget constraint:

$$\begin{aligned} & \sum_{t=1}^{\infty} \beta^t \left[(\eta z_t h_t^\eta - g_t) + \frac{u_{ht}}{\lambda} h_t \right] + \sum_{t=2}^{\infty} \beta^t \frac{u_{mt}}{\lambda} m_t \\ &= \sum_{s=1}^{\infty} Q_{0,s} \left(\frac{{}_0B_s^G}{p_0} \right) + \frac{M_0}{p_0} - \beta \frac{u_{m1}}{\lambda} m_1 + \beta b_1^G. \end{aligned}$$

Now given that

$$\begin{aligned} \frac{1}{1+i_1} &= \frac{q_1/p_1}{q_0/p_0} = \frac{\beta p_0}{p_1} \\ Q_{0,t} &= Q_{1,t}/(1+i_1) \end{aligned}$$

Then I have:

$$\begin{aligned} & \sum_{t=1}^{\infty} \beta^{t-1} \left[(\eta z_t h_t^\eta - g_t) + \frac{u_{ht}}{\lambda} h_t \right] + \sum_{t=2}^{\infty} \beta^{t-1} \frac{u_{mt}}{\lambda} m_t \\ &= \sum_{s=1}^{\infty} Q_{0,s} \left(\frac{{}_0B_s^G}{\beta p_0} \right) + \frac{M_0}{\beta p_0} - \frac{u_{m1}}{\lambda} m_1 + b_1^G \\ &= \sum_{s=1}^{\infty} Q_{1,s} \left(\frac{{}_0B_s^G}{p_1} \right) + (1+i_1)m_1 - i_1 m_1 + b_1^G \\ &= \sum_{s=1}^{\infty} Q_{1,s} \left(\frac{{}_0B_s^G}{p_1} \right) + \frac{M_0}{p_1} + b_1^G \end{aligned}$$

This is the $t = 1$ government's intertemporal budget constraint. I simplify this budget constraint as follows

$$\begin{aligned} \sum_{s=1}^{\infty} Q_{0,s} \left(\frac{{}_{-1}B_s^G}{p_0} \right) + \frac{\beta M_0}{p_1} + b_1^G &= \sum_{t=1}^{\infty} \beta^t \left[(\eta z_t h_t^\eta - g_t) + \frac{u_{ht}}{\lambda} h_t \right] \\ &+ \sum_{t=2}^{\infty} \beta^t \frac{u_{mt}}{\lambda} m_t \end{aligned}$$

Similarly, it can be shown that the constructed term structure of bonds is consistent with the intertemporal budget constraint of the $t = 0$ economy.

A.8 What will happen if households do not cooperate?

The assumption (c) in the section 4 is strong and debatable for the following reasons: Domestic households make individual choice of the term structure of bonds. Note since these individual choices are not based on optimization behavior, there is no guarantee that after aggregation, the realized term structures of $({}_0B_S^P)$ and $({}_0b_S^P)$ are the same as those implied in Proposition 2. The only way to guarantee optimal choice is for domestic households to have complete coordination. However, since there is no clear mechanism in the model for household coordination, it is very unlikely that the required complete coordination obtains. Once the $t = 0$ government does not have control over the aggregated $({}_0B_S^P)$ and $({}_0b_S^P)$, it loses control over $({}_0B_S^F)$ and $({}_0b_S^F)$ as well. As a result, the $t = 0$ government cannot freely choose the level of external debt and only has control over $({}_0B_S^G)$ and $({}_0b_S^G)$.

If households do not coordinate at all, in order to discuss time consistent optimal policy, I replace assumption (1.31) by the following restriction faced by the $t = 0$ government:

$$({}_0B_S^F) = ({}_0\tilde{B}_S^F); ({}_0b_S^F) = ({}_0\tilde{b}_S^F); S \geq 1. \quad (\text{A.2})$$

The main difference between assumption (1.31) and restriction (A.2) is on the $t = 0$ government's choice of one particular multi-period nominal public bond, for which I choose $({}_0B_2^G)$. Given assumption (1.31), the $t = 0$ government has zero (or one, depending on the labor market) degree of freedom in choosing $({}_0B_2^G)$, and it can accordingly pin down one $({}_0b_2^G)$. With restriction (A.2), the $t = 0$ government may not be able to find a choice of $({}_0B_2^G)$ consistent with the time consistent optimal policy, not even to mention its corresponding

degree of freedom.

The reason for the lack of time consistent optimal policy in this case is as follows. From step S1 of Proposition 1, I can choose constant Lagrange multipliers. With constant Lagrange multipliers and with restriction (A.2), I obtain the following equation for $({}_0B_S^G)$ by subtracting equation (1.27) held at $t = S$ from equation (1.27) held at $t = S + 1$:

$$\mu^E A_{1.27,S}(\tilde{B}_S^F) + \mu^G \lambda A_{1.27,S}({}_0B_S^G) = D_{1.27,S} - D_{1.27,S+1}, S \geq 2. \quad (\text{A.3})$$

It is clear that equations (A.3) determine $({}_0B_S^G)$ for $S \geq 2$. This is problematic because given these chosen nominal bonds and the chosen constant Lagrange multipliers, both sides of equation (1.26) are predetermined, which implies it is not necessarily the case that equation (1.26) holds with the policy continuation. As a result, the $t = 0$ government cannot find a term structure of nominal bonds that renders optimal policy time consistent. Proposition 5 summarizes these points:

Proposition 5. *In a small open economy with perfect capital mobility and with Svensson timing, if the government issues both nominal and real bonds, if the government does not have the freedom to choose the term structure of external bonds, and if constant Lagrange multipliers are needed, then the optimal policy is not time consistent. This result is independent of the government expenditure process and the initial asset position of the government.*

Proposition 1 and Proposition 5 give two extreme and opposite scenarios. In the first scenario, the government has full power to influence household choices of bonds, the optimal policy is time consistent, and there are many term structures of public debt capable of rendering optimal policy time consistent. In the second scenario, the government does not have any power to influence

households' choices of bonds, and the optimal policy is not time consistent. Then, if this unpreferred second scenario is in practice more likely, the question of interest is whether the government can move away from the second scenario and still ultimately arrive at the first scenario. The surprising answer is yes. The intuition is the following: the time inconsistency in the second scenario comes from two assumptions: first, the $t = 0$ government does not have control over $({}_0B_S^F)$. Second, it is necessary to have constant Lagrange multipliers to isolate the effect of the endogenous response in labor supply on optimal policy. When one of these two assumptions is relaxed, the government will be equipped with sufficiently more policy instrument(s) to render optimal policy time consistent.

Accordingly, there are two ways to move away from the second scenario. The first way is to have a particular kind of labor market, for example, a labor market with constant labor productivity. In this case, it would not be necessary to have constant Lagrange multipliers, i.e., equations (1.28) alone are no longer sufficient to determine the Lagrange multipliers, and the $t = 0$ government acquires one more policy instrument. To see this, note that equations (A.3) yield $({}_0B_S^G)$ for $S \geq 2$ as functions of μ_1^G and μ_1^E . Plugging $({}_0B_S^G)$ back into equation (1.26) yields one equation in two unknowns, the Lagrange multipliers. This updated version of (1.26) and equation (1.28) are two linear equations in two unknowns, μ_1^G and μ_1^E , which can be solved to determine the unique term structure of nominal public bonds capable of rendering optimal policy time consistent. In this case, the government gains one policy instrument, namely, one of the two Lagrange multipliers, due to the special feature of the labor market (in this case, constant productivity). As a result, even though individual households make their own arbitrary choices of term structure, the

government still is free to choose the maturity structure of nominal public debt in such a way to affectuate time consistent optimal policy. However, the government in this case can at best do just as good as the government in the corresponding closed economy can do. The government in the closed economy can always find a unique term structure of nominal public debt to render optimal policy time consistent, independent of the features of the labor market. In contrast, the government in a small open economy can render optimal policy time consistent only with some particular assumption about the conditions of the labor market.

The second way to move away from the unpreferred second scenario to the preferred first scenario is as follows. Consider that even with constant Lagrange multipliers, the $t = 0$ government in a small open economy can do better by introducing asymmetric taxes on households' purchases of nominal bonds. The intuition is that, with asymmetric taxes, households give up their degrees of freedom in choosing the term structure of nominal bonds, which gives the government control over one or all of $\{ {}_0B_S^P \}_{S=2}^{\infty}$. This reverts the small open economy back to time consistent optimal monetary and fiscal policy. In particular, when the government has control over one particular multi-period nominal bond, $({}_0B_t^P)$ for $t \geq 2$, a unique term structure of bonds is capable of rendering optimal policy time consistent; and when the government has control over two multi-period nominal bonds, the government has one degree of freedom in choosing the term structure of bonds to render optimal policy time consistent. I label these asymmetric taxes as auxiliary policy instruments. Below I present an example that illustrates the use of asymmetric taxes to bring an economy away from the second scenario and thus towards time consistency of optimal policy.

Suppose there are N households and that the term structure of nominal bonds for household j , where $j \in (1, 2, 3, \dots, N)$, is given by $\{ {}_0B_S^j \}_{S=1}^\infty$. The $t = 0$ government randomly chooses one household k and collects asymmetric tax τ_2^j on household purchases of each unit of $({}_0B_2^j)$. The tax is asymmetric in the following sense

$$\begin{aligned}\tau_2^j &= x > 0, \text{ if } j \neq k \\ \tau_2^j &= 0, \text{ if } j = k.\end{aligned}$$

Any household $j \neq k$ will set $({}_0B_2^j) = 0$. This is the case because the household only cares about the discounted present value of its bond holding positions, and when it sets $({}_0B_2^j) \neq 0$, it has to pay a tax $x|{}_0B_2^j| > 0$, which lowers the present value of its asset positions and thus its utility. Other than $({}_0B_2^j)$, the household is free to choose any term structure it likes as long as its liquidity condition is satisfied. Household k is also free to choose $({}_0B_S^k)$ for $S \geq 3$ (note that $({}_0B_1^k)$ is chosen to satisfy the household's liquidity condition). Household k 's choice of $({}_0B_2^k)$ will become clear below.

The arbitrary behavior of households leaves the $t = 0$ government the following restriction:

$$({}_0B_S^P) = ({}_0\tilde{B}_S^P); S \geq 3.$$

This restriction occurs because $({}_0B_1^k)$ and $({}_0B_2^k)$ have not yet been determined. With this restriction and with equations (A.3) held at $S \geq 3$, I get the government's choice of $({}_0B_S^G)$ for $S \geq 3$. Given the two constant Lagrange multipliers, equation (A.3) for $S = 2$ and equation (1.26), I have two equations in two unknowns $({}_0B_2^k)$ (note that ${}_0B_2^k = {}_0B_2^P$) and $({}_0B_2^G)$. I solve the simultaneous equations and obtain a value of $({}_0B_2^{k*})$ that is called for by the time consistent optimal policy in the model.

Returning now to household k , this household still has one degree of freedom in choosing $({}_0B_1^k)$ and $({}_0B_2^k)$. If household k chooses ${}_0B_2^k \neq {}_0B_2^{k*}$, then optimal policy is time inconsistent and the household, along with other households, will achieve sub-optimal utility. If, instead, the household chooses ${}_0B_2^k = {}_0B_2^{k*}$, then optimal policy is time consistent and the household, along with other households, obtains the optimal level of utility. As a result, household k gives up its degree of freedom by choosing $({}_0B_2^{k*})$. The $({}_0B_1^k)$ is chosen accordingly. It is straightforward to show that both the liquidity constraints of all households and the liquidity constraint of the aggregate economy are satisfied.

The above example shows how the government can use an asymmetric tax on the purchases of two-period nominal bonds in order to yield a time consistency outcome. This works because in equilibrium, the asymmetric tax τ_2^j will never be used and thus have no impact on the Ramsey equilibrium of this economy. The government can use two or more asymmetric taxes on multi-period nominal bonds to gain control over more $({}_0B_S^P)$'s. This way, the government not only renders optimal policy time consistent, but also gains degrees of freedom in arranging the term structure of public debt, which gives the government greater ability to render optimal policy time consistent than its counterpart in the closed economy. In the closed economy, there is no role for these asymmetric taxes because households have to choose what the government supplies.

Appendix B

Appendix for Chapter 2

B.1 One Sector Model

B.1.1 Lagrange and Optimality Conditions

The Lagrange for the household is

$$\begin{aligned} \mathcal{L} = E_0 \sum_{t=0}^{\infty} \theta_t \{ & U(c_t, h_t) + \lambda_t [z_t F(k_t, h_t, m_t) + \Gamma_t + d_t - R_{t-1}d_{t-1} - c_t - i_t \\ & - (1 + \tau)m_t - \Phi(k_{t+1} - k_t)] + \lambda_t q_t [(1 - \delta)k_t + i_t - k_{t+1}] \}. \end{aligned}$$

The first order conditions are:

$$\begin{aligned} \lambda_t &= \left(c_t - \frac{h_t^\omega}{\omega} \right)^{-\gamma}, \\ h_t^{\omega-1} &= \alpha_h z_t k_t^{\alpha_k} h_t^{\alpha_h-1} m_t^{\alpha_m}, \\ \lambda_t &= \left(1 + \tilde{c}_t - \frac{\tilde{h}_t^\omega}{\omega} \right)^{-\beta_1} R_t E_t \lambda_{t+1}, \\ 1 + \tau &= (\alpha_m) z_t k_t^{\alpha_k} h_t^{\alpha_h} m_t^{\alpha_m-1}, \\ q_t &= 1, \\ \lambda_t (1 + \phi(k_{t+1} - k_t)) &= \left(1 + \tilde{c}_t - \frac{\tilde{h}_t^\omega}{\omega} \right)^{-\beta_1} E_t \lambda_{t+1} \\ &\quad [1 - \delta + \phi(k_{t+2} - k_{t+1}) + z_{t+1} \alpha_k k_{t+1}^{\alpha_k-1} h_{t+1}^{\alpha_h} m_{t+1}^{\alpha_m}]. \end{aligned}$$

The first order conditions for firms are standard:

$$\begin{aligned} r_t &= \alpha_k z_t k_t^{\alpha_k - 1} h_t^{\alpha_h} m_t^{\alpha_m}, \\ w_t &= \alpha_h z_t k_t^{\alpha_k} h_t^{\alpha_h - 1} m_t^{\alpha_m}, \\ r_t^m &= \alpha_m z_t k_t^{\alpha_k} h_t^{\alpha_h} m_t^{\alpha_m - 1}. \end{aligned}$$

The equilibrium conditions are:

$$\begin{aligned} d_t &= R_{t-1}d_{t-1} + c_t + i_t + (1 + \tau)m_t + \frac{\phi}{2}(k_{t+1} - k_t)^2 - z_t k_t^{\alpha_k} h_t^{\alpha_h} m_t^{\alpha_m}, \\ k_{t+1} &= (1 - \delta)k_t + i_t, \\ c_t &= \tilde{c}_t, \\ h_t &= \tilde{h}_t. \end{aligned}$$

We can rearrange and get the following

$$\begin{aligned} h_t &= A_1 (z_t k_t^{\alpha_k})^{\frac{1}{\omega - \alpha_h - \omega \alpha_m}}, \\ m_t &= B_1 (z_t k_t^{\alpha_k})^{\frac{\omega}{\omega - \alpha_h - \omega \alpha_m}}, \end{aligned}$$

where

$$\begin{aligned} A_1 &= \left[(\alpha_h)^{1 - \alpha_m} \left(\frac{\alpha_m}{1 + \tau} \right)^{\alpha_m} \right]^{\frac{1}{\omega - \alpha_h - \omega \alpha_m}}, \\ B_1 &= \frac{1}{\alpha_h} \frac{\alpha_m}{1 + \tau} \left[(\alpha_h)^{1 - \alpha_m} \left(\frac{\alpha_m}{1 + \tau} \right)^{\alpha_m} \right]^{\frac{\omega}{\omega - \alpha_h - \omega \alpha_m}}. \end{aligned}$$

So, the optimal choices of h_t and m_t depend on the states of z_t and k_t only. After get the solution for h_t and m_t , we can further simplify the system.

$$\begin{aligned}
\left(c_t - \frac{h_t^\omega}{\omega}\right)^{-\gamma} &= \left(1 + c_t - \frac{h_t^\omega}{\omega}\right)^{-\beta_1} R_t E_t \left(c_{t+1} - \frac{h_{t+1}^\omega}{\omega}\right)^{-\gamma}, \\
0 &= \left(c_t - \frac{h_t^\omega}{\omega}\right)^{-\gamma} [1 + \phi(k_{t+1} - k_t)] \\
&\quad - \left(1 + c_t - \frac{h_t^\omega}{\omega}\right)^{-\beta_1} R_t E_t \left(c_{t+1} - \frac{h_{t+1}^\omega}{\omega}\right)^{-\gamma} \\
&\quad [1 - \delta + \phi(k_{t+2} - k_{t+1}) + z_{t+1} \alpha_k k_{t+1}^{\alpha_k - 1} h_{t+1}^{\alpha_h} m_{t+1}^{\alpha_m}], \\
d_t &= R_{t-1} d_{t-1} + c_t + k_{t+1} - (1 - \delta)k_t + (1 + \tau)m_t \\
&\quad + \frac{\phi}{2}(k_{t+1} - k_t)^2 - z_t k_t^{\alpha_k} h_t^{\alpha_h} m_t^{\alpha_m}.
\end{aligned}$$

B.1.2 Functional Form and Non-stochastic Steady State

The functional form for the optimal conditions are:

$$\begin{aligned}
F(k, h, m) &= k^{\alpha_k} h^{\alpha_h} m^{\alpha_m}, \\
1 &= \alpha_k + \alpha_h + \alpha_m, \\
F(k^T, h^T, m) &= k^{\alpha_{k^T}} h^{\alpha_{h^T}} m^{\alpha_m}, \\
F(k^N, h^N) &= k^{\alpha_{k^N}} h^{\alpha_{h^N}}, \\
\alpha_k &\geq 0, \alpha_h \geq 0, \alpha_k + \alpha_h \leq 1, \\
U(c, h) &= \frac{[c - \omega^{-1} h^\omega]^{1-\gamma} - 1}{1 - \gamma}, \\
\Phi(x) &= \frac{\phi}{2} x^2, \\
\beta(c, h) &= \left(1 + c - \frac{h^\omega}{\omega}\right)^{-\beta_1}.
\end{aligned}$$

In the non-stochastic steady state, the optimality conditions are reduced to

$$\begin{aligned}
d &= Rd + c + i + m - k^{\alpha_k} h^{\alpha_h} m^{1-\alpha_k-\alpha_h}, \\
i &= \delta k, \\
\lambda &= (c - \omega^{-1} h^\omega)^{-\gamma}, \\
h^{\omega-1} &= \alpha_h k^{\alpha_k} h^{\alpha_h-1} m^{1-\alpha_k-\alpha_h}, \\
1 &= \left(1 + c - \frac{h^\omega}{\omega}\right)^{-\beta_1} R, \\
1 + \tau &= (1 - \alpha_k - \alpha_h) k^{\alpha_k} h^{\alpha_h} m^{-\alpha_k-\alpha_h}, \\
q &= 1, \\
R &= [1 - \delta + \alpha_k k^{\alpha_k-1} h^{\alpha_h} m^{1-\alpha_k-\alpha_h}].
\end{aligned}$$

Rearrange and we get three equations of k , h , and m only:

$$\begin{aligned}
\frac{1}{\alpha_h} &= k^{\alpha_k} h^{\alpha_h-\omega} m^{1-\alpha_k-\alpha_h}, \\
\frac{1 + \tau}{(1 - \alpha_k - \alpha_h)} &= k^{\alpha_k} h^{\alpha_h} m^{-\alpha_k-\alpha_h}, \\
\frac{R + \delta - 1}{\alpha_k} &= k^{\alpha_k-1} h^{\alpha_h} m^{1-\alpha_k-\alpha_h}.
\end{aligned}$$

We can solve for h as:

$$h = \left[\frac{(R + \delta - 1)^{\alpha_k} (1 + \tau)^{1-\alpha_k-\alpha_h}}{\alpha_k^{\alpha_k} \alpha_h^{\alpha_h} (1 - \alpha_k - \alpha_h)^{1-\alpha_k-\alpha_h}} \right]^{\frac{1}{\alpha_h(1-\omega)}}.$$

Notice without m , the solution for h is:

$$h = \left[\frac{(R + \delta - 1)^{\alpha_k}}{\alpha_k^{\alpha_k} \alpha_h^{\alpha_h}} \right]^{\frac{1}{\alpha_h(1-\omega)}}.$$

Consider $\alpha_k + \alpha_h = 1$ when there is no m , the solution is the same as the one in SGU (2003), which is:

$$h = \left[\left(\frac{1}{1 - \alpha_k} \right) \left(\frac{(R + \delta - 1)}{\alpha_k} \right)^{\frac{\alpha_k}{1 - \alpha_k}} \right]^{\frac{1}{1 - \omega}}.$$

Thus, the output is

$$\begin{aligned} Y &= k^{\alpha_k} h^{\alpha_h} m^{\alpha_m} \\ &= \left[\frac{1}{\alpha_h} \frac{\alpha_k}{R + \delta - 1} h^\omega \right]^{\alpha_k} h^{\alpha_h} \left[\frac{1}{\alpha_h} \frac{\alpha_m}{1 + \tau} h^\omega \right]^{\alpha_m} \\ &= \left(\frac{\alpha_h}{1} \right)^{\alpha_h} \left(\frac{\alpha_k}{R + \delta - 1} \right)^{\alpha_k} \left(\frac{\alpha_m}{1 + \tau} \right)^{\alpha_m} \times \\ &\quad \left[\left(\frac{1}{\alpha_h} \right)^{\alpha_h} \left(\frac{R + \delta - 1}{\alpha_k} \right)^{\alpha_k} \left(\frac{1 + \tau}{\alpha_m} \right)^{\alpha_m} \right]^{1 + \frac{\omega}{\alpha_h(1 - \omega)}} \\ &= \left[\left(\frac{1}{\alpha_h} \right)^{\alpha_h} \left(\frac{R + \delta - 1}{\alpha_k} \right)^{\alpha_k} \left(\frac{1 + \tau}{\alpha_m} \right)^{\alpha_m} \right]^{\frac{\omega}{\alpha_h(1 - \omega)}}. \end{aligned}$$

B.2 Two Sectors with Homogenous Capital - Tradable Good

In this part, we show how to solve the model when the investment good is measured in the units of tradable good.

B.2.1 Lagrange and Optimality Conditions

The Lagrangian for the household's problem is:

$$\begin{aligned}
\mathcal{L} = & E_0 \sum_{t=0}^{\infty} \theta_t \{ U(c_t, h_t) + \lambda_t [z_t^T F(k_t^T, h_t^T, m_t) - (1 + \tau)m_t + p_t^N z_t^N F(k_t^N, h_t^N) \\
& + \Gamma_t + d_t - R_{t-1}d_{t-1} - c_t^T - p_t^N c_t^N - i_t - \Phi(k_{t+1} - k_t)] \\
& + \lambda_t q_t [(1 - \delta)k_t + i_t - k_{t+1}] + \lambda_t r_t (k_t - k_t^T - k_t^N) \\
& + \lambda_t w_t (h_t - h_t^T - h_t^N) \}.
\end{aligned}$$

The first order conditions are:

$$\begin{aligned}
U_c V_T &= \lambda_t, \\
p_t^N &= \frac{V_N}{V_T}, \\
\lambda_t &= \beta(\tilde{c}_t, \tilde{h}_t) R_t E_t \lambda_{t+1}, \\
r_t &= z_t^T F_k(k_t^T, h_t^T, m_t), \\
r_t &= p_t^N z_t^N F_k(k_t^N, h_t^N), \\
w_t &= z_t^T F_h(k_t^T, h_t^T, m_t), \\
w_t &= p_t^N z_t^N F_h(k_t^N, h_t^N), \\
1 + \tau &= z_t^T F_M(k_t^T, h_t^T, m_t), \\
q_t &= 1, \\
\lambda_t [q_t + \Phi_{k_{t+1}}(k_{t+1} - k_t)] &= \beta(\tilde{c}_t, \tilde{h}_t) E_t \lambda_{t+1} \\
& \quad [(1 - \delta) + r_{t+1} - \Phi_{k_{t+1}}(k_{t+2} - k_{t+1})], \\
U_h &= -\lambda_t w_t.
\end{aligned}$$

In equilibrium, we have the period budget constraint holding with equality and

$$\begin{aligned}
d_t &= R_{t-1}d_{t-1} + c_t^T + p_t^N c_t^N + i_t + (1 + \tau)m_t + \Phi(k_{t+1} - k_t), \\
&\quad - z_t^T F(k_t^T, h_t^T, m_t) - p_t^N z_t^N F(k_t^N, h_t^N) - \Gamma_t, \\
c_t^N &= z_t^N F_h(k_t^N, h_t^N), \\
\tilde{c}_t &= c_t, \\
\tilde{h}_t &= h_t, \\
k_{t+1} &= i_t + (1 - \delta)k_t, \\
k_t &= k_t^T + k_t^N, \\
h_t &= h_t^T + h_t^N, \\
\Gamma_t &= \tau m_t.
\end{aligned}$$

B.2.2 Non-Stochastic Steady State

Use the functional forms, and rewrite the equilibrium conditions in steady state

$$\lambda = U' \left[\kappa (c^T)^{-\eta} + (1 - \kappa) (c^N)^{-\eta} \right]^{-\frac{1}{\eta}-1} \kappa (c^T)^{\eta-1}, \quad (\text{B.1})$$

$$p^N = \frac{1 - \kappa}{\kappa} \left(\frac{c^T}{c^N} \right)^{1+\eta}, \quad (\text{B.2})$$

$$R = \left\{ 1 + \left[\kappa (c^T)^{-\eta} + (1 - \kappa) (c^N)^{-\eta} \right]^{-\frac{1}{\eta}} - \frac{h^\omega}{\omega} \right\}^{\beta_1}, \quad (\text{B.3})$$

$$r = z^T \alpha_{k^T} (k^T)^{\alpha_{k^T}-1} (h^T)^{\alpha_{h^T}} m^{\alpha_m}, \quad (\text{B.4})$$

$$r = p^N z^N \alpha_{k^N} (k^N)^{\alpha_{k^N}-1} (h^N)^{\alpha_{h^N}}, \quad (\text{B.5})$$

$$w = z^T \alpha_{h^T} (k^T)^{\alpha_{k^T}} (h^T)^{\alpha_{h^T}-1} m^{\alpha_m}, \quad (\text{B.6})$$

$$w = p^N z^N \alpha_{h^N} (k^N)^{\alpha_{k^N}} (h^N)^{\alpha_{h^N}-1}, \quad (\text{B.7})$$

$$1 + \tau = z^T \alpha_m (k^T)^{\alpha_{k^T}} (h^T)^{\alpha_{h^T}} m^{\alpha_m-1}, \quad (\text{B.8})$$

$$q = 1, \quad (\text{B.9})$$

$$r = R + \delta - 1, \quad (\text{B.10})$$

$$h^{\omega-1} = \frac{z^T \alpha_{h^T} (k^T)^{\alpha_{k^T}} (h^T)^{\alpha_{h^T}-1} m^{\alpha_m}}{\left[\kappa (c^T)^{-\eta} + (1 - \kappa) (c^N)^{-\eta} \right]^{\frac{1}{\eta}+1} \kappa^{-1} (c^T)^{\eta+1}}, \quad (\text{B.11})$$

$$h = h^T + h^N, \quad (\text{B.12})$$

$$k = k^T + k^N, \quad (\text{B.13})$$

$$i = \delta k \quad (\text{B.14})$$

$$d = R d + c^T + i + m - F(k^T, h^T, m), \quad (\text{B.15})$$

$$\Gamma = \tau m, \quad (\text{B.16})$$

$$c^N = z^N (k^N)^{\alpha_{k^N}} (h^N)^{\alpha_{h^N}}. \quad (\text{B.17})$$

From (B.2), we get $p^N = \frac{1-\kappa}{\kappa} \left(\frac{c^T}{c^N} \right)^{1+\eta}$ and write c^N as a function of c^T , p^N , and structure parameters

$$c^N = \left(p^N \frac{\kappa}{1-\kappa} \right)^{\frac{-1}{1+\eta}} c^T = \zeta c^T. \quad (\text{B.18})$$

If we can further assume

$$h^N = \nu h^T. \quad (\text{B.19})$$

Then from (B.11), (B.18), and (B.19); from (B.4) and (B.10); and from (B.8), we get the following equations:

$$\frac{1}{z^T \alpha_{h^T}} \frac{(1+\nu)^{\omega-1}}{[\kappa + (1-\kappa)(\zeta)^{-\eta}]^{-\frac{1}{\eta}-1} \kappa} = (k^T)^{\alpha_{k^T}} (h^T)^{\alpha_{h^T}-\omega} m^{\alpha_m}, \quad (\text{B.20})$$

$$\frac{R+\delta-1}{z^T \alpha_{k^T}} = (k^T)^{\alpha_{k^T}-1} (h^T)^{\alpha_{h^T}} m^{\alpha_m}, \quad (\text{B.21})$$

$$\frac{1+\tau}{z^T \alpha_m} = (k^T)^{\alpha_{k^T}} (h^T)^{\alpha_{h^T}} m^{\alpha_m-1}. \quad (\text{B.22})$$

Simplify the notation by assuming

$$A = \frac{(1+\nu)^{\omega-1}}{[\kappa + (1-\kappa)(\zeta)^{-\eta}]^{-\frac{1}{\eta}-1} \kappa}, B = \frac{\alpha_{k^T} \alpha_{h^T}}{\alpha_{h^T} \alpha_{k^T}}. \quad (\text{B.23})$$

Thus from (B.20), (B.21), and (B.23); and from (B.20), (B.22), (B.23), we get

$$k^T = \frac{A}{\alpha_{h^T}} \frac{\alpha_{k^T}}{R+\delta-1} (h^T)^\omega, \quad (\text{B.24})$$

$$m = \frac{A}{\alpha_{h^T}} \frac{\alpha_m}{1+\tau} (h^T)^\omega. \quad (\text{B.25})$$

Plug (B.23), (B.24), and (B.25) into one of (B.20), (B.21), (B.22), we can derive the solution of h^T , k^T , m , and w for any given A

$$h^T = \left[\frac{1}{z^T} \left(\frac{R + \delta - 1}{\alpha_{k^T}} \right)^{\alpha_{k^T}} \left(\frac{A}{\alpha_{h^T}} \right)^{\alpha_{h^T}} \left(\frac{1 + \tau}{\alpha_m} \right)^{\alpha_m} \right]^{\frac{1}{\alpha_{h^T}(1-\omega)}}, \quad (\text{B.26})$$

$$k^T = \frac{A}{\alpha_{h^T}} \frac{\alpha_{k^T}}{R + \delta - 1} (h^T)^\omega, \quad (\text{B.27})$$

$$m = \frac{A}{\alpha_{h^T}} \frac{\alpha_m}{1 + \tau} (h^T)^\omega, \quad (\text{B.28})$$

$$w = z^T \alpha_{h^T} (k^T)^{\alpha_{k^T}} (h^T)^{\alpha_{h^T}-1} m^{\alpha_m}. \quad (\text{B.29})$$

Plug (B.26), (B.27), and (B.28) back into (B.4), (B.8), and (B.11), it can be shown that (B.4), (B.8), and (B.11) are satisfied. From (B.4) and (B.6), and from (B.5) and (B.7)

$$\frac{r}{w} = \frac{\alpha_{k^T} h^T}{\alpha_{h^T} k^T} = \frac{\alpha_{k^N} h^N}{\alpha_{h^N} k^N}, \quad (\text{B.30})$$

$$\Rightarrow \frac{k^N}{k^T} = \frac{\alpha_{k^N} \alpha_{h^T} h^N}{\alpha_{h^N} \alpha_{k^T} h^T} = B\nu. \quad (\text{B.31})$$

Now we are ready to solve the non-stochastic steady state of the model. First, we solve the relative price of non-tradable good, p^N . Use equations (B.10),

(B.18), (B.23), (B.27), and (B.31), we can rewrite equation (B.5) as

$$\begin{aligned}
\frac{R + \delta - 1}{p^N \alpha_{k^N}} &= z^N (k^N)^{\alpha_{k^N} - 1} (h^N)^{\alpha_{h^N}} \\
&= z^N (B \nu k^T)^{\alpha_{k^N} - 1} (\nu h^T)^{\alpha_{h^N}} \\
&= z^N B^{\alpha_{k^N} - 1} (k^T)^{\alpha_{k^N} - 1} (h^T)^{\alpha_{h^N}} \\
&= z^N B^{\alpha_{k^N} - 1} \left(\frac{A}{\alpha_{h^T}} \frac{\alpha_{k^T}}{R + \delta - 1} (h^T)^\omega \right)^{\alpha_{k^N} - 1} (h^T)^{\alpha_{h^N}} \\
&= z^N B^{\alpha_{k^N} - 1} \left(\frac{A}{\alpha_{h^T}} \frac{\alpha_{k^T}}{R + \delta - 1} \right)^{-\alpha_{h^N}} (h^T)^{\alpha_{h^N} (1 - \omega)}.
\end{aligned} \tag{B.32}$$

Plug (B.26) into (B.32), we get

$$\begin{aligned}
\frac{R + \delta - 1}{p^N \alpha_{k^N}} &= z^N \left(\frac{1}{z^T} \right)^{\frac{\alpha_{h^N}}{\alpha_{h^T}}} B^{\alpha_{k^N} - 1} \left(\frac{R + \delta - 1}{\alpha_{k^T}} \right)^{\alpha_{h^N} \left(\frac{\alpha_{k^T}}{\alpha_{h^T}} + 1 \right)} \\
&\quad \left(\frac{1 + \tau}{\alpha_m} \right)^{\frac{\alpha_m \alpha_{h^N}}{\alpha_{h^T}}}.
\end{aligned} \tag{B.33}$$

Thus we derive the analytical solution of p^N

$$p^N = \frac{\frac{R + \delta - 1}{\alpha_{k^N}}}{z^N \left(\frac{1}{z^T} \right)^{\frac{\alpha_{h^N}}{\alpha_{h^T}}} B^{\alpha_{k^N} - 1} \left(\frac{R + \delta - 1}{\alpha_{k^T}} \right)^{\alpha_{h^N} \left(\frac{\alpha_{k^T}}{\alpha_{h^T}} + 1 \right)} \left(\frac{1 + \tau}{\alpha_m} \right)^{\frac{\alpha_{h^N}}{\alpha_{h^T}} \alpha_m}. \tag{B.34}$$

From the expression, the non-stochastic steady state of p^N is uniquely determined. Once we solve for p^N , we are able to solve for ν , which is our second

step. Consider the investment to GDP ratio as a function of ν

$$\begin{aligned}
\frac{i}{y^T - m} &= \left(\frac{\delta}{R + \delta - 1} \right) \left(\frac{\alpha_{k^T}}{1 - \frac{1}{1+\tau}\alpha_m} \right) \left(1 + \frac{k^N}{k^T} \right) \\
&= \left(\frac{\delta}{R + \delta - 1} \right) \left(\frac{\alpha_{k^T}}{1 - \frac{1}{1+\tau}\alpha_m} \right) \left(1 + \frac{\alpha_{k^N} \alpha_{h^T}}{\alpha_{h^N} \alpha_{k^T}} \nu \right) \\
&= J(1 + B\nu) = J + JB\nu, \\
c^T &= (y^T - m) \left(1 - \frac{i}{y^T - m} - \frac{tb}{y^T - m} \right) \\
&= (y^T - m) \left\{ 1 - J(1 + B\nu) - S_{tb} \left[1 + \frac{p^N y^N}{y^T - m} \right] \right\}.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{c^T}{c^N} &= \frac{y^T - m}{y^N} \left\{ 1 - J(1 + B\nu) - S_{tb} \left[1 + \frac{p^N y^N}{y^T - m} \right] \right\} \\
&= \left(1 - \frac{\alpha_m}{1 + \tau} \right) \frac{y^T}{y^N} \left\{ 1 - J(1 + B\nu) - S_{tb} \left[1 + \frac{p^N}{1 - \frac{\alpha_m}{1+\tau}} \frac{y^N}{y^T} \right] \right\} \\
&= \left(1 - \frac{\alpha_m}{1 + \tau} \right) \frac{\alpha_{h^N} p^N}{\alpha_{h^T} \nu} \left\{ 1 - J(1 + B\nu) - S_{tb} \left[1 + \frac{p^N}{1 - \frac{\alpha_m}{1+\tau}} \frac{\alpha_{h^T}}{\alpha_{h^N}} \frac{\nu}{p^N} \right] \right\}.
\end{aligned}$$

Define $Q = \left(1 - \frac{\alpha_m}{1+\tau} \right) \frac{\alpha_{h^N}}{\alpha_{h^T}}$, then,

$$\begin{aligned}
\frac{c^T}{c^N} &= Q \frac{p^N}{\nu} \left\{ 1 - J(1 + B\nu) - S_{tb} \left[1 + \frac{\nu}{Q} \right] \right\}, \\
\frac{c^T}{c^N} \nu &= Q p^N (1 - J - S_{tb}) - Q p^N J B \nu - p^N S_{tb} \nu, \\
\nu &= \frac{Q(1 - J - S_{tb})}{\frac{c^T}{p^N c^N} + Q J B + S_{tb}}.
\end{aligned}$$

After we solve for p^N and ν , we can solve for other variables accordingly.

$$\zeta = \left(p^N \frac{\kappa}{1 - \kappa} \right)^{\frac{-1}{1+\eta}},$$

$$A = \frac{(1 + \nu)^{\omega-1}}{[\kappa + (1 - \kappa)(\zeta)^{-\eta}]^{-\frac{1}{\eta}-1} \kappa}.$$

B.3 Two Sectors with Homogenous Capital - Final Good

Now the investment good is a final good.

B.3.1 Lagrange and Optimality Conditions

The Lagrangian for the household's problem is:

$$\begin{aligned} \mathcal{L} = & E_0 \sum_{t=0}^{\infty} \theta_t \{ U(c_t, h_t) + \lambda_t [z_t^T F(k_t^T, h_t^T, m_t) - (1 + \tau)m_t + p_t^N z_t^N F(k_t^N, h_t^N) \\ & + \Gamma_t + d_t - R_{t-1}d_{t-1} - a_t^T - p_t^N a_t^N + p_t V(a_t^T, a_t^N) \\ & - p_t(c_t + i_t + \Phi(k_{t+1} - k_t))] + \lambda_t q_t [(1 - \delta)k_t + i_t - k_{t+1}] \\ & + \lambda_t r_t (k_t - k_t^T - k_t^N) + \lambda_t w_t (h_t - h_t^T - h_t^N) \}. \end{aligned}$$

The first order conditions are:

$$\lambda_t = U_c V_T,$$

$$p_t^N = \frac{V_N}{V_T},$$

$$\lambda_t = \beta(\tilde{c}_t, \tilde{h}_t) R_t E_t \lambda_{t+1},$$

$$r_t = z_t^T F_k(k_t^T, h_t^T, m_t),$$

$$r_t = p_t^N z_t^N F_k(k_t^N, h_t^N),$$

$$w_t = z_t^T F_h(k_t^T, h_t^T, m_t),$$

$$w_t = p_t^N z_t^N F_h(k_t^N, h_t^N),$$

$$1 + \tau = z_t^T F_M(k_t^T, h_t^T, m_t),$$

$$q_t = p_t,$$

$$1 = p_t V_T,$$

$$\begin{aligned} \lambda_t p_t [1 + \Phi_{k_{t+1}}(k_{t+1} - k_t)] &= \beta(\tilde{c}_t, \tilde{h}_t) E_t \lambda_{t+1} \\ &\quad \{p_{t+1} [1 - \delta - \Phi_{k_{t+1}}(k_{t+2} - k_{t+1})] + r_{t+1}\}, \end{aligned}$$

$$\frac{U_h}{U_c V_T} = -w_t.$$

In equilibrium, we have the following equilibrium conditions

$$\begin{aligned}
d_t &= R_{t-1}d_{t-1} + a_t^T + m_t - z_t^T F(k_t^T, h_t^T, m_t) \\
a_t^N &= z_t^N F_h(k_t^N, h_t^N), \\
V(a_t^T, a_t^N) &= c_t + i_t + \Phi(k_{t+1} - k_t), \\
\tilde{c}_t &= c_t, \\
\tilde{h}_t &= h_t, \\
k_{t+1} &= i_t + (1 - \delta)k_t, \\
k_t &= k_t^T + k_t^N, \\
h_t &= h_t^T + h_t^N, \\
\Gamma_t &= \tau m_t.
\end{aligned}$$

B.3.2 Non-Stochastic Steady State

Use the functional forms, and rewrite the optimal conditions

$$\lambda = U' \left[\kappa (a^T)^{-\eta} + (1 - \kappa) (a^N)^{-\eta} \right]^{-\frac{1}{\eta}-1} \kappa (a^T)^{-\eta-1}, \quad (\text{B.35})$$

$$p^N = \frac{1 - \kappa}{\kappa} \left(\frac{a^T}{a^N} \right)^{1+\eta}, \quad (\text{B.36})$$

$$R = \left(1 + c - \frac{h^\omega}{\omega} \right)^{\beta_1}, \quad (\text{B.37})$$

$$r = z^T \alpha_{k^T} (k^T)^{\alpha_{k^T}-1} (h^T)^{\alpha_{h^T}} m^{\alpha_m}, \quad (\text{B.38})$$

$$r = p^N z^N \alpha_{k^N} (k^N)^{\alpha_{k^N}-1} (h^N)^{\alpha_{h^N}}, \quad (\text{B.39})$$

$$w = z^T \alpha_{h^T} (k^T)^{\alpha_{k^T}} (h^T)^{\alpha_{h^T}-1} m^{\alpha_m}, \quad (\text{B.40})$$

$$w = p^N z^N \alpha_{h^N} (k^N)^{\alpha_{k^N}} (h^N)^{\alpha_{h^N}-1}, \quad (\text{B.41})$$

$$1 + \tau = z^T \alpha_m (k^T)^{\alpha_{k^T}} (h^T)^{\alpha_{h^T}} m^{\alpha_m-1}, \quad (\text{B.42})$$

$$q = p, \quad (\text{B.43})$$

$$1 = p \left[\kappa (a^T)^{-\eta} + (1 - \kappa) (a^N)^{-\eta} \right]^{-\frac{1}{\eta}-1} \kappa (a^T)^{-\eta-1}, \quad (\text{B.44})$$

$$r = p(R + \delta - 1), \quad (\text{B.45})$$

$$h^{\omega-1} = \frac{z^T \alpha_{h^T} (k^T)^{\alpha_{k^T}} (h^T)^{\alpha_{h^T}-1} m^{\alpha_m}}{\left[\kappa (a^T)^{-\eta} + (1 - \kappa) (a^N)^{-\eta} \right]^{\frac{1}{\eta}+1} \kappa^{-1} (a^T)^{\eta+1}}, \quad (\text{B.46})$$

$$h = h^T + h^N, \quad (\text{B.47})$$

$$k = k^T + k^N, \quad (\text{B.48})$$

$$i = \delta k, \quad (\text{B.49})$$

$$d = Rd + a^T + m - z^T F(k^T, h^T, m), \quad (\text{B.50})$$

$$\Gamma = \tau m, \quad (\text{B.51})$$

$$a^N = z^N (k^N)^{\alpha_{k^N}} (h^N)^{\alpha_{h^N}}, \quad (\text{B.52})$$

$$c + i = \left[\kappa (a^T)^{-\eta} + (1 - \kappa) (a^N)^{-\eta} \right]^{-\frac{1}{\eta}}. \quad (\text{B.53})$$

From $p^N = \frac{1-\kappa}{\kappa} \left(\frac{a^T}{a^N}\right)^{1+\eta}$, the domestic non-tradable absorption can be represented as a function of domestic tradable good absorption and relative price of non-tradable good

$$a^N = \left(p^N \frac{\kappa}{1-\kappa}\right)^{\frac{-1}{1+\eta}} a^T = \zeta a^T. \quad (\text{B.54})$$

We can further assume

$$h^N = \nu h^T. \quad (\text{B.55})$$

Then from (B.46), (B.54), and (B.55); from (B.38) and (B.45); and from (B.42), we get

$$\frac{1}{z^T \alpha_{h^T}} \frac{(1+\nu)^{\omega-1}}{[\kappa + (1-\kappa)(\zeta)^{-\eta}]^{-\frac{1}{\eta}-1} \kappa} = (k^T)^{\alpha_{k^T}} (h^T)^{\alpha_{h^T}-\omega} m^{\alpha_m}, \quad (\text{B.56})$$

$$\frac{p(R+\delta-1)}{z^T \alpha_{k^T}} = (k^T)^{\alpha_{k^T}-1} (h^T)^{\alpha_{h^T}} m^{\alpha_m}, \quad (\text{B.57})$$

$$\frac{1+\tau}{z^T \alpha_m} = (k^T)^{\alpha_{k^T}} (h^T)^{\alpha_{h^T}} m^{\alpha_m-1}. \quad (\text{B.58})$$

Define

$$A = \frac{(1+\nu)^{\omega-1}}{[\kappa + (1-\kappa)(\zeta)^{-\eta}]^{-\frac{1}{\eta}-1} \kappa}; B = \frac{\alpha_{k^N} \alpha_{h^T}}{\alpha_{h^N} \alpha_{k^T}}, \quad (\text{B.59})$$

then from (B.56), (B.57), (B.58), and (B.59), we can express k^N and m as functions of h^N

$$k^T = \frac{A}{\alpha_{h^T} p(R+\delta-1)} (h^T)^\omega, \quad (\text{B.60})$$

$$m = \frac{A}{\alpha_{h^T} 1+\tau} (h^T)^\omega. \quad (\text{B.61})$$

Given A , we can derive the following solution:

$$h^T = \left[\frac{1}{z^T} \left(\frac{p(R + \delta - 1)}{\alpha_{k^T}} \right)^{\alpha_{k^T}} \left(\frac{A}{\alpha_{h^T}} \right)^{\alpha_{h^T}} \left(\frac{1 + \tau}{\alpha_m} \right)^{\alpha_m} \right]^{\frac{1}{\alpha_{h^T}(1-\omega)}} \quad (\text{B.62})$$

$$k^T = \frac{A}{\alpha_{h^T}} \frac{\alpha_{k^T}}{p(R + \delta - 1)} (h^T)^\omega, \quad (\text{B.63})$$

$$m = \frac{A}{\alpha_{h^T}} \frac{\alpha_m}{1 + \tau} (h^T)^\omega, \quad (\text{B.64})$$

$$w = \alpha_{h^T} (k^T)^{\alpha_{k^T}} (h^T)^{\alpha_{h^T}-1} m^{\alpha_m}. \quad (\text{B.65})$$

From (B.38) and (B.40); and from (B.39) and (B.41) we get the following relationship between k^T and k^N

$$\frac{r}{w} = \frac{\alpha_{k^T} h^T}{\alpha_{h^T} k^T} = \frac{\alpha_{k^N} h^N}{\alpha_{h^N} k^N}, \quad (\text{B.66})$$

$$\frac{k^N}{k^T} = \frac{\alpha_{k^N} \alpha_{h^T} h^N}{\alpha_{h^N} \alpha_{k^T} h^T} = \frac{\alpha_{k^N} \alpha_{h^T}}{\alpha_{h^N} \alpha_{k^T}} \nu. \quad (\text{B.67})$$

Now we are ready to solve for the non-stochastic steady state of the model. We first solve for the ratio of a^N/a^T . Use equations (B.45), (B.55), (B.59), (B.63), and (B.65), we can rewrite equation (B.39) as

$$\begin{aligned} \frac{p(R + \delta - 1)}{p^N \alpha_{k^N}} &= z^N (k^N)^{\alpha_{k^N}-1} (h^N)^{\alpha_{h^N}} \\ &= z^N (B \nu k^T)^{\alpha_{k^N}-1} (\nu h^T)^{\alpha_{h^N}} \\ &= z^N B^{\alpha_{k^N}-1} (k^T)^{\alpha_{k^N}-1} (h^T)^{\alpha_{h^N}} \\ &= z^N B^{\alpha_{k^N}-1} \left(\frac{A}{\alpha_{h^T}} \frac{\alpha_{k^T}}{p(R + \delta - 1)} (h^T)^\omega \right)^{\alpha_{k^N}-1} (h^T)^{\alpha_{h^N}} \\ &= z^N B^{\alpha_{k^N}-1} \left(\frac{A}{\alpha_{h^T}} \frac{\alpha_{k^T}}{p(R + \delta - 1)} \right)^{-\alpha_{h^N}} (h^T)^{\alpha_{h^N}(1-\omega)}. \end{aligned} \quad (\text{B.68})$$

Plug (B.62) into (B.68), we get

$$\frac{p(R + \delta - 1)}{p^N \alpha_{k^N}} = z^N \left(\frac{1}{z^T} \right)^{\frac{\alpha_{h^N}}{\alpha_{h^T}}} B^{\alpha_{k^N} - 1} \left(\frac{p(R + \delta - 1)}{\alpha_{k^T}} \right)^{\alpha_{h^N} \left(\frac{\alpha_{k^T}}{\alpha_{h^T}} + 1 \right)} \left(\frac{1 + \tau}{\alpha_m} \right)^{\frac{\alpha_m \alpha_{h^N}}{\alpha_{h^T}}}. \quad (\text{B.69})$$

Rearrange (B.69)

$$p^N = \frac{\frac{p(R + \delta - 1)}{\alpha_{k^N}}}{z^N \left(\frac{1}{z^T} \right)^{\frac{\alpha_{h^N}}{\alpha_{h^T}}} B^{\alpha_{k^N} - 1} \left(\frac{p(R + \delta - 1)}{\alpha_{k^T}} \right)^{\alpha_{h^N} \left(\frac{\alpha_{k^T}}{\alpha_{h^T}} + 1 \right)} \left(\frac{1 + \tau}{\alpha_m} \right)^{\frac{\alpha_{h^N}}{\alpha_{h^T}} \alpha_m}}. \quad (\text{B.70})$$

Define $\Lambda = \frac{\frac{R + \delta - 1}{\alpha_{k^N}}}{z^N \left(\frac{1}{z^T} \right)^{\frac{\alpha_{h^N}}{\alpha_{h^T}}} B^{\alpha_{k^N} - 1} \left(\frac{R + \delta - 1}{\alpha_{k^T}} \right)^{\alpha_{h^N} \left(\frac{\alpha_{k^T}}{\alpha_{h^T}} + 1 \right)} \left(\frac{1 + \tau}{\alpha_m} \right)^{\frac{\alpha_{h^N}}{\alpha_{h^T}} \alpha_m}}$, rewrite (B.70) as

$$p^N = \frac{\Lambda p}{p^{\alpha_{h^N} \left(\frac{\alpha_{k^T}}{\alpha_{h^T}} + 1 \right)}}. \quad (\text{B.71})$$

Since $p^N = pV_N$, and $p = \frac{1}{V_T}$, we rewrite (B.71)

$$V_N = \frac{p^N}{p} = \frac{\Lambda p}{p^{\alpha_{h^N} \left(\frac{\alpha_{k^T}}{\alpha_{h^T}} + 1 \right)} p} = \Lambda (V_T)^{\alpha_{h^N} \left(\frac{\alpha_{k^T}}{\alpha_{h^T}} + 1 \right)}. \quad (\text{B.72})$$

We know that

$$\begin{aligned} V_N &= [\kappa + (1 - \kappa) (\zeta)^{-\eta}]^{-\frac{1}{\eta} - 1} (1 - \kappa) \zeta^{-\eta - 1}, \\ V_T &= [\kappa + (1 - \kappa) (\zeta)^{-\eta}]^{-\frac{1}{\eta} - 1} \kappa, \end{aligned}$$

thus

$$\begin{aligned}
& [\kappa + (1 - \kappa) (\zeta)^{-\eta}]^{-\frac{1}{\eta}-1} (1 - \kappa) \zeta^{-\eta-1} \\
&= \Lambda \left\{ [\kappa + (1 - \kappa) (\zeta)^{-\eta}]^{-\frac{1}{\eta}-1} \kappa \right\}^{\alpha_{h^N} \left(\frac{\alpha_{k^T}}{\alpha_{h^T}} + 1 \right)}, \\
\frac{\Lambda \kappa^{\alpha_{h^N} \left(\frac{\alpha_{k^T}}{\alpha_{h^T}} + 1 \right)}}{(1 - \kappa)} \zeta^{\eta+1} &= [\kappa + (1 - \kappa) (\zeta)^{-\eta}]^{\left(\frac{1}{\eta} + 1 \right) \left(\alpha_{h^N} \left(\frac{\alpha_{k^T}}{\alpha_{h^T}} + 1 \right) - 1 \right)}.
\end{aligned} \tag{B.73}$$

Define $\Omega = \left[\frac{\Lambda \kappa^{\alpha_{h^N} \left(\frac{\alpha_{k^T}}{\alpha_{h^T}} + 1 \right)}}{(1 - \kappa)} \right]^{\frac{\eta}{\eta+1}}$, rewrite (B.73) as

$$(\Omega \zeta^\eta) = [\kappa + (1 - \kappa) (\zeta)^{-\eta}]^{\left(\alpha_{h^N} \left(\frac{\alpha_{k^T}}{\alpha_{h^T}} + 1 \right) - 1 \right)}. \tag{B.74}$$

The left side of (B.74) is an increasing function of ζ and the right side of (B.74) is an increasing function of ζ if $\alpha_{h^N} \left(\frac{\alpha_{k^T}}{\alpha_{h^T}} + 1 \right) < 1$. If $\alpha_{h^N} \left(\frac{\alpha_{k^T}}{\alpha_{h^T}} + 1 \right) = 1$, the right side of (B.74) is independent of ζ . If $\alpha_{h^N} \left(\frac{\alpha_{k^T}}{\alpha_{h^T}} + 1 \right) > 1$, the right side of (B.74) is a decreasing function of ζ . It then can be shown that there is only one solution to this equation. After we solve for ζ , we can solve for p , r , and p^N from (B.44), (B.45), and (B.71), respectively.

Second step, we calibrate the parameter β_1 for the baseline case when $\tau = 10\%$. In this step, solve for $\nu = \frac{H^N}{H^T}$ by assuming $S_{tb} = 2\%$ or 3.6% . (In the simulation, we get the calibration value of β_1 by assuming $S_{tb} = 2\%$). Use

the definitions , we get

$$\begin{aligned}
\frac{a^T}{a^N} &= \frac{y^T - m}{y^N} \left[1 - \frac{\frac{tb}{y^T - m + p^N y^N}}{\frac{y^T - m}{y^T - m + p^N y^N}} \right] \\
&= \left(1 - \frac{\alpha_m}{1 + \tau} \right) \frac{y^T}{y^N} \left[1 - S_{tb} \frac{\left(1 - \frac{\alpha_m}{1 + \tau} \right) y^T + p^N y^N}{\left(1 - \frac{\alpha_m}{1 + \tau} \right) y^T} \right] \\
&= \left(1 - \frac{\alpha_m}{1 + \tau} \right) \frac{y^T}{y^N} \left[1 - S_{tb} \frac{\left(1 - \frac{\alpha_m}{1 + \tau} \right) \frac{y^T}{y^N} + p^N}{\left(1 - \frac{\alpha_m}{1 + \tau} \right) \frac{y^T}{y^N}} \right] \\
&= \left(1 - \frac{\alpha_m}{1 + \tau} \right) \frac{y^T}{y^N} (1 - S_{tb}) - S_{tb} p^N, \\
\frac{y^N}{y^T} &= \frac{(1 - S_{tb}) \left(1 - \frac{\alpha_m}{1 + \tau} \right)}{S_{tb} p^N + \frac{a^T}{a^N}}, \\
\frac{\alpha_{h^T}}{\alpha_{h^N}} \frac{1}{p^N} \nu &= \frac{(1 - S_{tb}) \left(1 - \frac{\alpha_m}{1 + \tau} \right)}{S_{tb} p^N + \frac{a^T}{a^N}}, \\
\nu &= p^N \frac{\alpha_{h^N}}{\alpha_{h^T}} \frac{(1 - S_{tb}) \left(1 - \frac{\alpha_m}{1 + \tau} \right)}{S_{tb} p^N + \frac{a^T}{a^N}}, \\
\nu &= \frac{\alpha_{h^N}}{\alpha_{h^T}} \frac{(1 - S_{tb}) \left(1 - \frac{\alpha_m}{1 + \tau} \right)}{S_{tb} + \frac{a^T}{p^N a^N}} \\
&= Q \frac{(1 - S_{tb})}{S_{tb} + \frac{a^T}{p^N a^N}}, \\
\zeta &= \left(p^N \frac{\kappa}{1 - \kappa} \right)^{\frac{-1}{1 + \eta}}, \\
A &= \frac{(1 + \nu)^{\omega - 1}}{\left[\kappa + (1 - \kappa) (\zeta)^{-\eta} \right]^{\frac{-1}{\eta} - 1} \kappa}.
\end{aligned}$$

Now we can solve for other variables accordingly and get the calibration of β_1 .

In the last step, we solve the model. Since in the comparison exercise, the values for the deep parameters (structural parameters) should be keep

constant. We need to keep β_1 constant as we change the value of τ . Here, the variable we have to adjust to accommodate the change of τ is S_{tb} . From (B.39), we get

$$\begin{aligned}
\frac{p(R + \delta - 1)}{p^N \alpha_{k^N}} &= B^{\alpha_{k^N} - 1} \left(\frac{A}{\alpha_{h^T}} \frac{\alpha_{k^T}}{p(R + \delta - 1)} \right)^{-\alpha_{h^N}} (h^T)^{\alpha_{h^N}(1-\omega)}, \\
\left(\frac{A}{\alpha_{h^T}} \frac{\alpha_{k^T}}{p(R + \delta - 1)} \right)^{\alpha_{h^N}} &= \frac{B^{\alpha_{k^N} - 1} p^N \alpha_{k^N}}{p(R + \delta - 1)} (h^T)^{\alpha_{h^N}(1-\omega)}, \\
\frac{A}{\alpha_{h^T}} \frac{\alpha_{k^T}}{p(R + \delta - 1)} &= \left(\frac{B^{\alpha_{k^N} - 1} p^N \alpha_{k^N}}{p(R + \delta - 1)} \right)^{\frac{1}{\alpha_{h^N}}} (h^T)^{(1-\omega)}, \\
A (h^T)^{(\omega-1)} &= \frac{p(R + \delta - 1) \alpha_{h^T}}{\alpha_{k^T}} \left(\frac{B^{\alpha_{k^N} - 1} p^N \alpha_{k^N}}{p(R + \delta - 1)} \right)^{\frac{1}{\alpha_{h^N}}}, \\
\frac{(1 + \nu)^{\omega-1} (h^T)^{(\omega-1)}}{[\kappa + (1 - \kappa) (\zeta)^{-\eta}]^{-\frac{1}{\eta} - 1} \kappa} &= \frac{p(R + \delta - 1) \alpha_{h^T}}{\alpha_{k^T}} \left(\frac{B^{\alpha_{k^N} - 1} p^N \alpha_{k^N}}{p(R + \delta - 1)} \right)^{\frac{1}{\alpha_{h^N}}}, \\
\frac{h^{\omega-1}}{[\kappa + (1 - \kappa) (\zeta)^{-\eta}]^{-\frac{1}{\eta} - 1} \kappa} &= \frac{p(R + \delta - 1) \alpha_{h^T}}{\alpha_{k^T}} \left(\frac{B^{\alpha_{k^N} - 1} p^N \alpha_{k^N}}{p(R + \delta - 1)} \right)^{\frac{1}{\alpha_{h^N}}}.
\end{aligned} \tag{B.75}$$

Since we have solved for p , p^N , and ζ , we can solve for h from (B.75). Once we get h , we can solve for c from (B.37) and solve for w from (B.46):

$$\begin{aligned}
c &= R^{\frac{1}{\beta_1}} - 1 + \frac{h^\omega}{\omega}, \\
w &= \frac{h^{\omega-1}}{[\kappa + (1 - \kappa) (\zeta)^{-\eta}]^{-\frac{1}{\eta} - 1} \kappa}.
\end{aligned}$$

From the market clearing condition (B.53), we have

$$\begin{aligned}
c + i &= (\kappa + (1 - \kappa)\zeta^{-\eta})^{-\frac{1}{\eta}} \frac{1}{\zeta} c^N, \\
c &= Gc^N - i, \\
c &= G \frac{wh^N}{\alpha_{h^N}} - \delta \left(\frac{\alpha_{k^N}}{\alpha_{h^N} p^N} \frac{w}{r} h^N + \frac{\alpha_{k^T}}{\alpha_{h^T} r} \frac{w}{r} h^T \right), \\
c &= G_1 h^N - G_2 h^N - G_3 h^T, \\
c &= (G_1 - G_2) h^N - G_3 (h - h^N), \\
c &= (G_1 - G_2 + G_3) h^N - G_3 h, \\
\Rightarrow h^N &= \frac{G_3 h + c}{G_1 - G_2 + G_3}, \\
h^T &= \frac{(G_1 - G_2) h - c}{G_1 - G_2 + G_3}.
\end{aligned}$$

Then the rest variables can be solved accordingly.

B.4 Two Sectors with Heterogenous Capital

B.4.1 Lagrange and Optimality Conditions

The Lagrangian for the household's problem is:

$$\begin{aligned}
\mathcal{L} &= E_0 \sum_{t=0}^{\infty} \theta_t \{ U(c_t, h_t) + \lambda_t \{ z_t^T F(k_t^T, h_t^T, m_t) - (1 + \tau) m_t \\
&\quad + p_t^N z_t^N F(k_t^N, h_t^N) + \Gamma_t + d_t - R_{t-1} d_{t-1} - c_t^T - p_t^N c_t^N \\
&\quad + p_t V(c_t^T, c_t^N) - p_t [c_t + i_t^T + \Phi(k_{t+1}^T - k_t^T) + i_t^N + \Phi(k_{t+1}^N - k_t^N)] \} \\
&\quad + \lambda_t q_t^T [(1 - \delta^T) k_t^T + i_t^T - k_{t+1}^T] + \lambda_t q_t^N [(1 - \delta^N) k_t^N + i_t^N - k_{t+1}^N] \\
&\quad + \lambda_t w_t (h_t - h_t^T - h_t^N) \}.
\end{aligned}$$

The first order conditions are:

$$\begin{aligned}
U_c &= \lambda_t p_t, \\
U_h &= -w_t \lambda_t, \\
\lambda_t &= \beta(\tilde{c}_t, \tilde{h}_t) R_t E_t \lambda_{t+1}, \\
1 &= p_t V_T, \\
p_t^N &= p_t V_N, \\
p_t^N &= \frac{V_N}{V_T} \text{ or combined as this equation,} \\
w_t &= z_t^T F_h(k_t^T, h_t^T, m_t), \\
w_t &= p_t z_t^N F_h(k_t^N, h_t^N), \\
1 + \tau &= z_t^T F_M(k_t^T, h_t^T, m_t), \\
p_t &= q_t^T = q_t^N, \\
0 &= \lambda_t q_t^T [1 + \Phi_{k_{t+1}^T}(k_{t+1}^T - k_t^T)] \\
&\quad - \beta(\tilde{c}_t, \tilde{h}_t) E_t \lambda_{t+1} \{q_{t+1}^T [1 - \delta + \Phi_{k_{t+1}^T}(k_{t+2}^T - k_{t+1}^T)] \\
&\quad - z_{t+1}^T F_k(k_{t+1}^T, h_{t+1}^T, m_{t+1})\}, \\
0 &= \lambda_t q_t^N [1 + \Phi_{k_{t+1}^N}(k_{t+1}^N - k_t^N)] \\
&\quad - \beta(\tilde{c}_t, \tilde{h}_t) E_t \lambda_{t+1} \{q_{t+1}^N [1 - \delta + \Phi_{k_{t+1}^N}(k_{t+2}^N - k_{t+1}^N)] \\
&\quad - p_{t+1}^N z_{t+1}^N F_k(k_{t+1}^N, h_{t+1}^N)\}.
\end{aligned}$$

Notice that, compared with the optimal conditions in the case where the investment good is measured in tradable good, the optimal conditions for this case in which the investment good is measured in final good are different only in the optimal conditions for the investment and capital accumulation. In the previous case, the price of investment is 1, while in this case, the price of investment is p_t , which are always equal to the price of good used as the

measure base.

B.4.2 Non-Stochastic Steady State

Use the functional forms, and rewrite the equilibrium conditions

$$\lambda = U' \left[\kappa (c^T)^{-\eta} + (1 - \kappa) (c^N)^{-\eta} \right]^{-\frac{1}{\eta}-1} \kappa (c^T)^{-\eta-1}, \quad (\text{B.76})$$

$$p^N = \frac{1 - \kappa}{\kappa} \left(\frac{c^T}{c^N} \right)^{1+\eta}, \quad (\text{B.77})$$

$$R = \left\{ 1 + c - \frac{h^\omega}{\omega} \right\}^{\beta_1}, \quad (\text{B.78})$$

$$r = z^T \alpha_{k^T} (k^T)^{\alpha_{k^T}-1} (h^T)^{\alpha_{h^T}} m^{\alpha_m}, \quad (\text{B.79})$$

$$r = p^N z^N \alpha_{k^N} (k^N)^{\alpha_{k^N}-1} (h^N)^{\alpha_{h^N}}, \quad (\text{B.80})$$

$$w = z^T \alpha_{h^T} (k^T)^{\alpha_{k^T}} (h^T)^{\alpha_{h^T}-1} m^{\alpha_m}, \quad (\text{B.81})$$

$$w = p^N z^N \alpha_{h^N} (k^N)^{\alpha_{k^N}} (h^N)^{\alpha_{h^N}-1}, \quad (\text{B.82})$$

$$1 + \tau = z^T \alpha_m (k^T)^{\alpha_{k^T}} (h^T)^{\alpha_{h^T}} m^{\alpha_m-1}, \quad (\text{B.83})$$

$$q = p, \quad (\text{B.84})$$

$$1 = p \left[\kappa (c^T)^{-\eta} + (1 - \kappa) (c^N)^{-\eta} \right]^{-\frac{1}{\eta}-1} \kappa (c^T)^{-\eta-1}, \quad (\text{B.85})$$

$$r = p(R + \delta - 1), \quad (\text{B.86})$$

$$h^{\omega-1} = \frac{z^T \alpha_{h^T} (k^T)^{\alpha_{k^T}} (h^T)^{\alpha_{h^T}-1} m^{\alpha_m}}{\left[\kappa (c^T)^{-\eta} + (1 - \kappa) (c^N)^{-\eta} \right]^{\frac{1}{\eta}+1} \kappa^{-1} (c^T)^{\eta+1}}, \quad (\text{B.87})$$

$$h = h^T + h^N, \quad (\text{B.88})$$

$$k = k^T + k^N, \quad (\text{B.89})$$

$$i = \delta k, \quad (\text{B.90})$$

$$d = Rd + c^T + m - z^T F(k^T, h^T, m), \quad (\text{B.91})$$

$$\Gamma = \tau m, \quad (\text{B.92})$$

$$c^N = z^N (k^N)^{\alpha_{k^N}} (h^N)^{\alpha_{h^N}}, \quad (\text{B.93})$$

$$c + i = \left[\kappa (c^T)^{-\eta} + (1 - \kappa) (c^N)^{-\eta} \right]^{-\frac{1}{\eta}}. \quad (\text{B.94})$$

It can be shown the steady states in this case are the same as those in the case when the capital is sector homogenous.

B.5 Unconditional Welfare Cost

The derivation of the unconditional welfare cost is similar to that in SGU (2004). The conditional lifetime welfare of household, V_0 , is defined as

$$V_0(x_0, \sigma) = U(C_0(x_0, \sigma), H_0(x_0, \sigma)) \\ + \beta(\tilde{C}_0(x_0, \sigma), \tilde{H}_0(x_0, \sigma))E_0V_1(x_1(x_0, \sigma), \sigma).$$

The unconditional lifetime welfare of household, EV , can be written as

$$EV(\sigma) = EU(C(x, \sigma), H(x, \sigma)) + E\beta(\tilde{C}(x, \sigma), \tilde{H}(x, \sigma))V(x'(x, \sigma), \sigma).$$

The parameter σ controls the size of volatility of the economy; the variable x represents the set of state variables. The E denotes the unconditional expectation operator. Other variables have their usual meanings. Define the unconditional welfare associated with the reference shock process r as

$$EV^r(\sigma) = EU(C^r(x, \sigma), H^r(x, \sigma)) + E\beta(\tilde{C}^r(x, \sigma), \tilde{H}^r(x, \sigma))V^r(x'(x, \sigma), \sigma). \quad (\text{B.95})$$

where the variables c^r and h^r denote the unconditional expectations for consumption and hours under the exogenous shock process r . Similarly, the unconditional welfare associated with the alternative exogenous shock process a is defined as

$$EV^a(\sigma) = EU(C^a(x, \sigma), H^a(x, \sigma)) + E\beta(\tilde{C}^a(x, \sigma), \tilde{H}^a(x, \sigma))V^a(x'(x, \sigma), \sigma).$$

The welfare cost ratio, λ , is implicitly defined by

$$EV^a(\sigma) = EU((1 - \lambda)C^r(x, \sigma), H^r(x, \sigma)) \\ + E\beta((1 - \lambda)\tilde{C}^r(x, \sigma), \tilde{H}^r(x, \sigma))V^a(x'(x, \sigma), \sigma). \quad (\text{B.96})$$

The ratio λ represents the constant ratio of consumption across states. Following the same argument in SGU (2004), the welfare cost ratio, λ , is a function of σ only.

$$\lambda = \lambda(\sigma).$$

Approximate λ up to the second order with respect to σ around its non-stochastic steady state

$$\lambda(\sigma) = \lambda(0) + \lambda_\sigma(0)\sigma + \frac{1}{2}\lambda_{\sigma\sigma}(0)\sigma^2 = \frac{1}{2}\lambda_{\sigma\sigma}(0)\sigma^2. \quad (\text{B.97})$$

The result in the equation (105) follows from the same argument in SGU (2004), λ vanishes as $\sigma \rightarrow 0$, i.e. $\lambda(0) = 0$. Our goal is to find $\lambda_{\sigma\sigma}(0)$.

B.5.1 Notation Simplification

To simplify the notation, we use the following rules to simplify my notations: $C = e^{c^r(x,\sigma)}$, $c = c^r(x, \sigma)$, $c_\sigma = \frac{\partial c^r(x,\sigma)}{\partial \sigma}$, $\bar{c}_\sigma = \frac{\partial c^r(\bar{x},0)}{\partial \sigma}$, $V',r = V^r(x'(x, \sigma), \sigma)$ and et al. The expression of $(\bar{x}, 0)$ in the derivatives, such as $\frac{\partial S^r(\bar{x},0)}{\partial x_i}$ denotes that the derivatives have been evaluated at the non-stochastic steady states. We also define the following proxy variables

$$S^r(x, \sigma) = U(C^r(x, \sigma), H^r(x, \sigma)) = U(e^{c^r(x,\sigma)}, e^{h^r(x,\sigma)}),$$

$$S^a(x, \sigma) = U((1 - \lambda(\sigma))C^r(x, \sigma), H^r(x, \sigma)) = U((1 - \lambda(\sigma))e^{c^r(x,\sigma)}, e^{h^r(x,\sigma)}),$$

$$T^r(x, \sigma) = \beta(\tilde{C}^r(x, \sigma), \tilde{H}^r(x, \sigma))V^r(x'(x, \sigma), \sigma),$$

$$T^a(x, \sigma) = \beta((1 - \lambda(\sigma))\tilde{C}^r(x, \sigma), \tilde{H}^r(x, \sigma))V^a(x'(x, \sigma), \sigma).$$

Then the equations (B.95) and (B.96) can be rewritten as

$$EV^r(\sigma) = ES^r(x, \sigma) + ET^r(x, \sigma), \quad (\text{B.98})$$

$$EV^a(\sigma) = ES^a(x, \sigma) + ET^a(x, \sigma). \quad (\text{B.99})$$

B.5.2 Second Order Approximation

The right sides of (B.98) and (B.99) are known. To get $\lambda_{\sigma\sigma}(0)$, we need to know $ES^r(x, \sigma)$, $ET^r(x, \sigma)$, $ES^a(x, \sigma)$, and $ET^a(x, \sigma)$. To achieve this, we approximate each term in the right sides up to the second order.

$$\begin{aligned}
S^r &= \bar{S}^r + \sum_{i=1}^{n_x} \bar{S}_{x_i}^r (x_i^r - \bar{x}_i) + \frac{1}{2} \bar{S}_{\sigma\sigma}^r \sigma^2 \\
&\quad + \frac{1}{2} \sum_{i=1}^{n_x} \sum_{j=1}^{n_x} \bar{S}_{x_i x_j}^r (x_i^r - \bar{x}_i)(x_j^r - \bar{x}_j) + \bar{S}_{\sigma}^r \sigma \\
&= \bar{S}^r + \sum_{i=1}^{n_x} \bar{S}_{x_i}^r (x_i^r - \bar{x}_i) + \frac{1}{2} \bar{S}_{\sigma\sigma}^r \sigma^2 \\
&\quad + \frac{1}{2} \sum_{i=1}^{n_x} \sum_{j=1}^{n_x} \bar{S}_{x_i x_j}^r (x_i^r - \bar{x}_i)(x_j^r - \bar{x}_j), \tag{B.100}
\end{aligned}$$

$$\begin{aligned}
T^r &= \bar{T}^r + \sum_{i=1}^{n_x} \bar{T}_{x_i}^r (x_i^r - \bar{x}_i) + \frac{1}{2} \bar{T}_{\sigma\sigma}^r \sigma^2 \\
&\quad + \frac{1}{2} \sum_{i=1}^{n_x} \sum_{j=1}^{n_x} \bar{T}_{x_i x_j}^r (x_i^r - \bar{x}_i)(x_j^r - \bar{x}_j) + \bar{T}_{\sigma}^r \sigma \\
&= \bar{T}^r + \sum_{i=1}^{n_x} \bar{T}_{x_i}^r (x_i^r - \bar{x}_i) + \frac{1}{2} \bar{T}_{\sigma\sigma}^r \sigma^2 \\
&\quad + \frac{1}{2} \sum_{i=1}^{n_x} \sum_{j=1}^{n_x} \bar{T}_{x_i x_j}^r (x_i^r - \bar{x}_i)(x_j^r - \bar{x}_j). \tag{B.101}
\end{aligned}$$

The variables with bar denote non-stochastic steady states. The first order derivatives with respect to σ are zero. All the derivatives are evaluated at the non-stochastic steady states. We have similar expression for S^a and T^a .

Take unconditional expectation on both sides of (B.100) and (B.101), we get

$$\begin{aligned}
ES^r &= \bar{S}^r + \sum_{i=1}^{n_x} \bar{S}_{x_i}^r E(x_i^r - \bar{x}_i) + \frac{1}{2} \bar{S}_{\sigma\sigma}^r \sigma^2 \\
&\quad + \frac{1}{2} \sum_{i=1}^{n_x} \sum_{j=1}^{n_x} \bar{S}_{x_i x_j}^r E(x_i^r - \bar{x}_i)(x_j^r - \bar{x}_j), \tag{B.102}
\end{aligned}$$

$$\begin{aligned}
ET^r &= \bar{T}^r + \sum_{i=1}^{n_x} \bar{T}_{x_i}^r E(x_i^r - \bar{x}_i) + \frac{1}{2} \bar{T}_{\sigma\sigma}^r \sigma^2 \\
&\quad + \frac{1}{2} \sum_{i=1}^{n_x} \sum_{j=1}^{n_x} \bar{T}_{x_i x_j}^r E(x_i^r - \bar{x}_i)(x_j^r - \bar{x}_j), \tag{B.103}
\end{aligned}$$

$$\begin{aligned}
ES^a &= \bar{S}^a + \sum_{i=1}^{n_x} \bar{S}_{x_i}^a E(x_i^a - \bar{x}_i) + \frac{1}{2} \bar{S}_{\sigma\sigma}^a \sigma^2 \\
&\quad + \frac{1}{2} \sum_{i=1}^{n_x} \sum_{j=1}^{n_x} \bar{S}_{x_i x_j}^a E(x_i^a - \bar{x}_i)(x_j^a - \bar{x}_j), \tag{B.104}
\end{aligned}$$

$$\begin{aligned}
ET^a &= \bar{T}^a + \sum_{i=1}^{n_x} \bar{T}_{x_i}^a E(x_i^a - \bar{x}_i) + \frac{1}{2} \bar{T}_{\sigma\sigma}^a \sigma^2 \\
&\quad + \frac{1}{2} \sum_{i=1}^{n_x} \sum_{j=1}^{n_x} \bar{T}_{x_i x_j}^a E(x_i^a - \bar{x}_i)(x_j^a - \bar{x}_j). \tag{B.105}
\end{aligned}$$

The next step is to find the explicit expressions for all the first order and second derivatives, the relevant components of (B.102), (B.103), (B.104), and (B.105).

B.5.3 Expressions for Derivatives

The first order derivatives of $S^r(x, \sigma)$ are

$$\begin{aligned}
S_{x_i}^r &= U_C^r C^r c_{x_i}^r + U_H^r H^r h_{x_i}^r, \\
S_{\sigma}^r &= U_C^r C^r c_{\sigma}^r + U_H^r H^r h_{\sigma}^r.
\end{aligned}$$

Evaluated at non-stochastic steady state,

$$\bar{S}_{x_i}^r = \bar{U}_C \bar{C}^r \bar{c}_{x_i}^r + \bar{U}_H \bar{H}^r \bar{h}_{x_i}^r, \quad (\text{B.106})$$

$$\bar{S}_\sigma^r = \bar{U}_C \bar{C}^r \bar{c}_\sigma^r + \bar{U}_H \bar{H}^r \bar{h}_\sigma^r = 0. \quad (\text{B.107})$$

The second order derivatives of $S^r(x, \sigma)$ are

$$\begin{aligned} S_{x_i x_j}^r &= U_{CC}^r C^r c_{x_j}^r C^r c_{x_i}^r + U_{CH}^r H^r h_{x_j}^r C^r c_{x_i}^r + U_C^r C^r \left(c_{x_j}^r c_{x_i}^r + c_{x_i x_j}^r \right) + \\ &\quad U_{HC}^r C^r c_{x_j}^r H^r h_{x_i}^r + U_{HH}^r H^r h_{x_j}^r H^r h_{x_i}^r + U_H^r H^r \left(h_{x_j}^r h_{x_i}^r + h_{x_i x_j}^r \right), \\ S_{\sigma\sigma}^r &= U_{CC}^r (C^r)^2 (c_\sigma^r)^2 + U_{CH}^r H^r h_\sigma^r C^r c_\sigma^r + U_C^r C^r (c_\sigma^r c_\sigma^r + c_{\sigma\sigma}^r) + \\ &\quad U_{HC}^r C^r c_\sigma^r H^r h_\sigma^r + U_{HH}^r (H^r)^2 (h_\sigma^r)^2 + U_H^r H^r (h_\sigma^r h_\sigma^r + h_{\sigma\sigma}^r). \end{aligned}$$

Evaluated at the non-stochastic steady state,

$$\begin{aligned} \bar{S}_{x_i x_j}^r &= \bar{U}_{CC} \bar{C}^r \bar{c}_{x_j}^r \bar{C}^r \bar{c}_{x_i}^r + \bar{U}_{CH} \bar{H}^r \bar{h}_{x_j}^r \bar{C}^r \bar{c}_{x_i}^r \\ &\quad + \bar{U}_C \bar{C}^r \left(\bar{c}_{x_j}^r \bar{c}_{x_i}^r + \bar{c}_{x_i x_j}^r \right) + \bar{U}_{HC} \bar{C}^r \bar{c}_{x_j}^r \bar{H}^r \bar{h}_{x_i}^r \\ &\quad + \bar{U}_{HH} \bar{H}^r \bar{h}_{x_j}^r \bar{H}^r \bar{h}_{x_i}^r + \bar{U}_H \bar{H}^r \left(\bar{h}_{x_j}^r \bar{h}_{x_i}^r + \bar{h}_{x_i x_j}^r \right), \end{aligned} \quad (\text{B.108})$$

$$\bar{S}_{\sigma\sigma}^r = \bar{U}_C \bar{C}^r \bar{c}_{\sigma\sigma}^r + \bar{U}_H \bar{H}^r \bar{h}_{\sigma\sigma}^r. \quad (\text{B.109})$$

The first order derivatives of $S^a(x, \sigma)$ are

$$S_{x_i}^a = U_C^a C^r (1 - \lambda) c_{x_i}^r + U_H^a H^r h_{x_i}^r,$$

$$S_\sigma^a = U_C^a C^r ((1 - \lambda) c_\sigma^r - \lambda_\sigma) + U_H^a H^r h_\sigma^r = 0.$$

Evaluated at the non-stochastic steady state,

$$\bar{S}_{x_i}^a = \bar{U}_C \bar{C}^r \bar{c}_{x_i}^r + \bar{U}_H \bar{H}^r \bar{h}_{x_i}^r, \quad (\text{B.110})$$

$$\bar{S}_\sigma^a = \bar{U}_C \bar{C}^r (\bar{c}_\sigma^r - \bar{\lambda}_\sigma) + \bar{U}_H \bar{H}^r \bar{h}_\sigma^r = 0. \quad (\text{B.111})$$

The second order derivatives of $S^a(x, \sigma)$ are

$$\begin{aligned}
S_{x_i x_j}^a &= U_{CC}^a [(1 - \lambda) C^r]^2 c_{x_j}^r c_{x_i}^r + U_{CH}^a H^r h_{x_j}^r (1 - \lambda) C^r c_{x_i}^r \\
&\quad + U_C^a (1 - \lambda) C^r \left(c_{x_j}^r c_{x_i}^r + c_{x_i x_j}^r \right) + U_{HC}^a (1 - \lambda) C^r c_{x_j}^r H^r h_{x_i}^r \\
&\quad + U_{HH}^a H^r h_{x_j}^r H^r h_{x_i}^r + U_H^a H^r \left(h_{x_j}^r h_{x_i}^r + h_{x_i x_j}^r \right) \\
S_{\sigma\sigma}^a &= U_{CC}^a [(1 - \lambda) C^r c_\sigma^r - C^r \lambda_\sigma]^2 + U_{CH}^a H^r h_\sigma^r [(1 - \lambda) C^r c_\sigma^r - C^r \lambda_\sigma] \\
&\quad + U_C^a (1 - \lambda) [(C^r c_\sigma^r)^2 + C^r c_{\sigma\sigma}^a] - U_C^a C^r (2c_\sigma^r \lambda_\sigma + \lambda_{\sigma\sigma}), \\
&\quad + U_{HC}^a [(1 - \lambda) C^r c_\sigma^r - C^r \lambda_\sigma] H^r h_\sigma^r + U_{HH}^a (H^r h_\sigma^r)^2 \\
&\quad + U_H^a H^r (h_\sigma^r h_\sigma^r + h_{\sigma\sigma}^a).
\end{aligned}$$

Evaluated at non-stochastic steady state, the second order derivatives of $S^a(x, \sigma)$ are

$$\begin{aligned}
\bar{S}_{x_i x_j}^a &= \bar{U}_{CC} (\bar{C}^r)^2 \bar{c}_{x_j}^r \bar{c}_{x_i}^r + \bar{U}_{CH} \bar{H}^r \bar{h}_{x_j}^r \bar{C}^r \bar{c}_{x_i}^r + \bar{U}_C \bar{C}^r \left(\bar{c}_{x_j}^r \bar{c}_{x_i}^r + \bar{c}_{x_i x_j}^r \right) \\
&\quad + \bar{U}_{HC} \bar{C}^r \bar{c}_{x_j}^r \bar{H}^r \bar{h}_{x_i}^r + \bar{U}_{HH} \bar{H}^r \bar{h}_{x_j}^r \bar{H}^r \bar{h}_{x_i}^r + \bar{U}_H \bar{H}^r \left(\bar{h}_{x_j}^r \bar{h}_{x_i}^r + \bar{h}_{x_i x_j}^r \right),
\end{aligned} \tag{B.112}$$

and

$$\begin{aligned}
\bar{S}_{\sigma\sigma}^a &= \bar{U}_{CC} (\bar{C}^r \bar{c}_\sigma^r - \bar{C}^r \bar{\lambda}_\sigma)^2 + \bar{U}_{CH} \bar{H}^r \bar{h}_\sigma^r [\bar{C}^r \bar{c}_\sigma^r - \bar{C}^r \bar{\lambda}_\sigma] \\
&\quad + \bar{U}_C \left[(\bar{C}^r \bar{c}_\sigma^r)^2 + \bar{C}^r \bar{c}_{\sigma\sigma}^a \right] \\
&\quad - \bar{U}_C \bar{C}^r (2\bar{c}_\sigma^r \bar{\lambda}_\sigma + \bar{\lambda}_{\sigma\sigma}) + \bar{U}_{HC} [\bar{C}^r \bar{c}_\sigma^r - \bar{C}^r \bar{\lambda}_\sigma] \bar{H}^r \bar{h}_\sigma^r \\
&\quad + \bar{U}_{HH} (\bar{H}^r \bar{h}_\sigma^r)^2 + \bar{U}_H \bar{H}^r (\bar{h}_\sigma^r \bar{h}_\sigma^r + \bar{h}_{\sigma\sigma}^a) \\
&= \bar{U}_C [\bar{C}^r \bar{c}_{\sigma\sigma}^a - \bar{C}^r \bar{\lambda}_{\sigma\sigma}] + \bar{U}_H \bar{H}^r \bar{h}_{\sigma\sigma}^a.
\end{aligned} \tag{B.113}$$

Notice that if we follow SGU (2004) notation $x'(x, \sigma) = z(x, \sigma) + \sigma \varepsilon'$, we have

$$\begin{aligned} x'_{l,x_i} &= z'_{l,x_i}, \\ x'_{l,\sigma} &= z'_{l,\sigma} + \varepsilon', r_l, \end{aligned}$$

The first order derivatives of $T^r(x, \sigma)$ are

$$\begin{aligned} T_{x_i}^r &= \beta_C^r C^r c_{x_i}^r V'^{r,r} + \beta_H^r H^r h_{x_i}^r V'^{r,r} + \beta^r \sum_{l=1}^{n_x} V'_{x_l}{}^{r,r} z'_{l,x_i}, \\ T_{\sigma}^r &= \beta_C^r C^r c_{\sigma}^r V'^{r,r} + \beta_H^r H^r h_{\sigma}^r V'^{r,r} + \beta^r \sum_{l=1}^{n_x} V'_{x_l}{}^{r,r} (z'_{l,\sigma} + \varepsilon_l'^{r,r}) + \beta^r V_{\sigma}^{\prime,r}. \end{aligned}$$

Evaluated at the non-stochastic steady state

$$E\bar{T}_{x_i}^r = \bar{\beta}_C \bar{C}^r \bar{c}_{x_i}^r \bar{V}'^{r,r} + \bar{\beta}_H \bar{H}^r \bar{h}_{x_i}^r \bar{V}'^{r,r} + \bar{\beta} \sum_{l=1}^{n_x} \bar{V}'_{x_l}{}^{r,r} \bar{z}'_{l,x_i}, \quad (\text{B.114})$$

$$E\bar{T}_{\sigma}^r = \bar{\beta}_C \bar{C}^r \bar{c}_{\sigma}^r \bar{V}'^{r,r} + \bar{\beta}_H \bar{H}^r \bar{h}_{\sigma}^r \bar{V}'^{r,r} + \bar{\beta} \sum_{l=1}^{n_x} \bar{V}'_{x_l}{}^{r,r} \bar{z}'_{l,\sigma} + \bar{\beta} \bar{V}_{\sigma}^{\prime,r} = 0. \quad (\text{B.115})$$

The second order derivatives of $T^r(x, \sigma)$ are

$$\begin{aligned}
T_{x_i x_j}^r &= \left(\beta_{CC}^r C^r c_{x_j}^r + \beta_{CH}^r H^r h_{x_j}^r \right) C^r c_{x_i}^r V'^{,r} \\
&+ \beta_C^r \left(C^r c_{x_j}^r c_{x_i}^r V'^{,r} + C^r c_{x_j x_i}^r V'^{,r} + C^r c_{x_i}^r \sum_{l=1}^{n_x} V_{x_l, r}^{\prime, r} z_{l, x_j}^r \right) \\
&+ \left(\beta_{HC}^r C^r c_{x_j}^r + \beta_{HH}^r H^r h_{x_j}^r \right) H^r h_{x_i}^r V'^{,r} \\
&+ \beta_H^r \left(H^r h_{x_j}^r h_{x_i}^r V'^{,r} + H^r h_{x_j x_i}^r V'^{,r} + H^r h_{x_i}^r \sum_{l=1}^{n_x} V_{x_l, r}^{\prime, r} z_{l, x_j}^r \right) \\
&+ \left(\beta_C^r C^r c_{x_j}^r + \beta_H^r H^r h_{x_j}^r \right) \sum_{l=1}^{n_x} V_{x_l, r}^{\prime, r} z_{l, x_i}^r \\
&+ \beta^r \sum_{l=1}^{n_x} \left(\sum_{n=1}^{n_x} V_{x_l, r, x_n}^{\prime, r} z_{n, x_j}^r \right) z_{l, x_i}^r + \beta^r \sum_{l=1}^{n_x} V_{x_l, r}^{\prime, r} z_{l, x_j, x_i}^r,
\end{aligned}$$

$$\begin{aligned}
T_{\sigma\sigma}^r &= \left(\beta_{CC}^r C^r c_{\sigma}^r + \beta_{CH}^r H^r h_{\sigma}^r \right) C^r c_{\sigma}^r V'^{,r} \\
&+ \beta_C^r \left\{ C^r c_{\sigma}^r c_{\sigma}^r V'^{,r} + C^r c_{\sigma\sigma}^r V'^{,r} + C^r c_{\sigma}^r \left[\sum_{l=1}^{n_x} V_{x_l, r}^{\prime, r} \left(z_{l, \sigma}^r + \varepsilon_{l, r}^{\prime, r} \right) + V_{\sigma}^{\prime, r} \right] \right\} \\
&+ \left(\beta_{HC}^r C^r c_{\sigma}^r + \beta_{HH}^r H^r h_{\sigma}^r \right) H^r h_{\sigma}^r V'^{,r} \\
&+ \beta_H^r \left\{ H^r h_{\sigma}^r h_{\sigma}^r V'^{,r} + H^r h_{\sigma\sigma}^r V'^{,r} \right\} \\
&+ \beta_H^r H^r h_{\sigma}^r \left[\sum_{l=1}^{n_x} V_{x_l, r}^{\prime, r} \left(z_{l, \sigma}^r + \varepsilon_{l, r}^{\prime, r} \right) + V_{\sigma}^{\prime, r} \right] \\
&+ \left(\beta_C^r C^r c_{\sigma}^r + \beta_H^r H^r h_{\sigma}^r \right) \left[\sum_{l=1}^{n_x} V_{x_l, r}^{\prime, r} \left(z_{l, \sigma}^r + \varepsilon_{l, r}^{\prime, r} \right) + V_{\sigma}^{\prime, r} \right] \\
&+ \beta^r \sum_{l=1}^{n_x} \left[\sum_{n=1}^{n_x} V_{x_l, r, x_n}^{\prime, r} \left(z_{n, \sigma}^r + \varepsilon_{n, r}^{\prime, r} \right) + V_{x_l, r, \sigma}^{\prime, r} \right] \left(z_{l, \sigma}^r + \varepsilon_{l, r}^{\prime, r} \right) \\
&+ \beta^r \sum_{l=1}^{n_x} V_{x_l, r}^{\prime, r} z_{l, \sigma\sigma}^r + \beta^r \left[\sum_{l=1}^{n_x} V_{\sigma, x_l, r}^{\prime, r} \left(z_{l, \sigma}^r + \varepsilon_{l, r}^{\prime, r} \right) + V_{\sigma\sigma}^{\prime, r} \right].
\end{aligned}$$

Evaluated at the non-stochastic steady state,

$$\begin{aligned}
\bar{T}_{x_i x_j}^r &= \left(\bar{\beta}_{CC} \bar{C}^r \bar{c}_{x_j}^r + \bar{\beta}_{CH} \bar{H}^r \bar{h}_{x_j}^r \right) \bar{C}^r \bar{c}_{x_i}^r \bar{V}'^r \\
&+ \bar{\beta}_C \left(\bar{C}^r \bar{c}_{x_j}^r \bar{c}_{x_i}^r \bar{V}'^r + \bar{C}^r \bar{c}_{x_j x_i}^r \bar{V}'^r + \bar{C}^r \bar{c}_{x_i}^r \sum_{l=1}^{n_x} \bar{V}'^r_{x_l, r} \bar{z}_{l, x_j}^r \right) \\
&+ \left(\bar{\beta}_{HC} \bar{C}^r \bar{c}_{x_j}^r + \bar{\beta}_{HH} \bar{H}^r \bar{h}_{x_j}^r \right) \bar{H}^r \bar{h}_{x_i}^r \bar{V}'^r \\
&+ \bar{\beta}_H \left(\bar{H}^r \bar{h}_{x_j}^r \bar{h}_{x_i}^r \bar{V}'^r + \bar{H}^r \bar{h}_{x_j x_i}^r \bar{V}'^r + \bar{H}^r \bar{h}_{x_i}^r \sum_{l=1}^{n_x} \bar{V}'^r_{x_l, r} \bar{z}_{l, x_j}^r \right) \\
&+ \left(\bar{\beta}_C \bar{C}^r \bar{c}_{x_j}^r + \bar{\beta}_H \bar{H}^r \bar{h}_{x_j}^r \right) \sum_{l=1}^{n_x} \bar{V}'^r_{x_l, r} \bar{z}_{l, x_i}^r \\
&+ \bar{\beta} \sum_{l=1}^{n_x} \left(\sum_{n=1}^{n_x} \bar{V}'^r_{x_l, r, x_n'} \bar{z}_{n, x_j}^r \right) \bar{z}_{l, x_i}^r + \bar{\beta} \sum_{l=1}^{n_x} \bar{V}'^r_{x_l, r} \bar{z}_{l, x_j, x_i}^r, \quad (\text{B.116})
\end{aligned}$$

$$\begin{aligned}
E\bar{T}_{\sigma\sigma}^r &= (\bar{\beta}_{CC}\bar{C}^r\bar{c}_\sigma^r + \bar{\beta}_{CH}\bar{H}^r\bar{h}_\sigma^r)\bar{C}^r\bar{c}_\sigma^r\bar{V}'^r_{',r} \\
&+ \bar{\beta}_C \left\{ \bar{C}^r\bar{c}_\sigma^r\bar{c}_\sigma^r\bar{V}'^r_{',r} + \bar{C}^r\bar{c}_{\sigma\sigma}^r\bar{V}'^r_{',r} \right\} \\
&+ \bar{\beta}_C\bar{C}^r\bar{c}_\sigma^r \left[\sum_{l=1}^{n_x} \bar{V}'^r_{x_l',r} \left(\bar{z}_{l,\sigma}^r + E\varepsilon_l'^r \right) + \bar{V}'^r_{\sigma} \right] \\
&+ (\bar{\beta}_{HC}\bar{C}^r\bar{c}_\sigma^r + \bar{\beta}_{HH}\bar{H}^r\bar{h}_\sigma^r)\bar{H}^r\bar{h}_\sigma^r\bar{V}'^r_{',r} \\
&+ \bar{\beta}_H \left\{ \bar{H}^r\bar{h}_\sigma^r\bar{h}_\sigma^r\bar{V}'^r_{',r} + \bar{H}^r\bar{h}_{\sigma\sigma}^r\bar{V}'^r_{',r} \right\} \\
&+ \bar{\beta}_H\bar{H}^r\bar{h}_\sigma^r \left[\sum_{l=1}^{n_x} \bar{V}'^r_{x_l',r} \left(\bar{z}_{l,\sigma}^r + E\varepsilon_l'^r \right) + \bar{V}'^r_{\sigma} \right] \\
&+ (\bar{\beta}_C\bar{C}^r\bar{c}_\sigma^r + \bar{\beta}_H\bar{H}^r\bar{h}_\sigma^r) \left[\sum_{l=1}^{n_x} \bar{V}'^r_{x_l',r} \left(\bar{z}_{l,\sigma}^r + E\varepsilon_l'^r \right) + \bar{V}'^r_{\sigma} \right] \\
&+ \bar{\beta} \sum_{l=1}^{n_x} E \left[\sum_{n=1}^{n_x} V'_{x_l',x_n',r} \left(\bar{z}_{n,\sigma}^r + \varepsilon_n'^r \right) + \bar{V}'^r_{x_l',r,\sigma} \right] \left(\bar{z}_{l,\sigma}^r + \varepsilon_l'^r \right) \\
&+ \bar{\beta} \sum_{l=1}^{n_x} \bar{V}'^r_{x_l',r} \bar{z}_{l,\sigma\sigma}^r + \bar{\beta} \left[\sum_{l=1}^{n_x} \bar{V}'^r_{\sigma,x_l',r} \left(\bar{z}_{l,\sigma}^r + E\varepsilon_l'^r \right) + \bar{V}'^r_{\sigma\sigma} \right] \\
&= \bar{\beta}_C\bar{C}^r\bar{c}_{\sigma\sigma}^r\bar{V}'^r_{',r} + \bar{\beta}_H\bar{H}^r\bar{h}_{\sigma\sigma}^r\bar{V}'^r_{',r} + \bar{\beta} \sum_{l=1}^{n_x} \bar{V}'^r_{x_l',r} \bar{z}_{l,\sigma\sigma}^r + \bar{\beta}\bar{V}'^r_{\sigma\sigma} \\
&+ \bar{\beta} \sum_{l=1}^{n_x} \sum_{n=1}^{n_x} V'_{x_l',x_n',r} E\varepsilon_n'^r \varepsilon_l'^r. \tag{B.117}
\end{aligned}$$

The first order derivatives of $T^a(x, \sigma)$ are

$$\begin{aligned}
T_{x_i}^a &= \beta^a(1-\lambda)C^r c_{x_i}^r V'^{',a} + \beta_H^a H^r h_{x_i}^r V'^{',a} + \beta^a \sum_{l=1}^{n_x} V'_{x_l',a}{}^a z_{l,x_i}^a, \\
T_\sigma^a &= \beta_C^a C^r [(1-\lambda)c_\sigma^r - \lambda_\sigma] V'^{',a} \\
&+ \beta_H^a H^r h_\sigma^r V'^{',a} + \beta^a \sum_{l=1}^{n_x} V'_{x_l',a}{}^a \left(z_{l,\sigma}^a + \varepsilon_l'^a \right) + \beta^a V_\sigma'^{',a}.
\end{aligned}$$

Evaluated at the non-stochastic steady state

$$E\bar{T}_{x_i}^a = \bar{\beta}_C \bar{C}^r \bar{c}_{x_i}^r \bar{V}'^a + \bar{\beta}_H \bar{H}^r \bar{h}_{x_i}^r \bar{V}'^a + \bar{\beta} \sum_{l=1}^{n_x} \bar{V}'_{x_i, a}{}^a \bar{z}_{l, x_i}^a, \quad (\text{B.118})$$

$$\begin{aligned} E\bar{T}_\sigma^a &= \bar{\beta}_C \bar{C}^r (\bar{c}_\sigma^r - \lambda_\sigma) \bar{V}'^a + \bar{\beta}_H \bar{H}^r \bar{h}_\sigma^r \bar{V}'^a \\ &+ \bar{\beta} \sum_{l=1}^{n_x} \bar{V}'_{x_i, a}{}^a \bar{z}_{l, \sigma}^a + \bar{\beta} \bar{V}'_\sigma^a = 0. \end{aligned} \quad (\text{B.119})$$

The second order derivatives of $T^a(x, \sigma)$ are

$$\begin{aligned} T_{x_i x_j}^a &= \left(\beta_{CC}^a (1 - \lambda) C^r c_{x_j}^r + \beta_{CH}^a H^r h_{x_j}^r \right) (1 - \lambda) C^r c_{x_i}^r V'^a \\ &+ \beta_C^a (1 - \lambda) \left(C^r c_{x_j}^r c_{x_i}^r V'^a + C^r c_{x_j x_i}^r V'^a + C^r c_{x_i}^r \sum_{l=1}^{n_x} V'_{x_i, a}{}^a z_{l, x_j}^a \right) \\ &+ \left(\beta_{HC}^a (1 - \lambda) C^r c_{x_j}^r + \beta_{HH}^a H^r h_{x_j}^r \right) H^r h_{x_i}^r V'^a \\ &+ \beta_H^a \left(H^r h_{x_j}^r h_{x_i}^r V'^a + H^r h_{x_j x_i}^r V'^a + H^r h_{x_i}^r \sum_{l=1}^{n_x} V'_{x_i, a}{}^a z_{l, x_j}^a \right) \\ &+ \left(\beta_C^a (1 - \lambda) C^r c_{x_j}^r + \beta_H^a H^r h_{x_j}^r \right) \sum_{l=1}^{n_x} V'_{x_i, a}{}^a z_{l, x_i}^a \\ &+ \beta^a \sum_{l=1}^{n_x} \left(\sum_{n=1}^{n_x} V'_{x_l, a}{}^a z_{n, x_j}^a \right) z_{l, x_i}^a + \beta^a \sum_{l=1}^{n_x} V'_{x_l, a}{}^a z_{l, x_j x_i}^a, \end{aligned}$$

$$\begin{aligned}
T_{\sigma\sigma}^a &= [\beta_{CC}^a ((1-\lambda)C^r c_\sigma^r - C^r \lambda_\sigma) + \beta_{CH}^a H^r h_\sigma^r] ((1-\lambda)C^r c_\sigma^r - C^r \lambda_\sigma) V'^{,a} \\
&+ \beta_C^a [(1-\lambda)C^r c_\sigma^r c_\sigma^r + (1-\lambda)C^r c_{\sigma\sigma}^a - 2C^r c_\sigma^r \lambda_\sigma - C^r \lambda_{\sigma\sigma}] V'^{,a} \\
&+ \beta_C^a [(1-\lambda)C^r c_\sigma^r - C^r \lambda_\sigma] \left[\sum_{l=1}^{n_x} V'_{x_l^{\prime,a}} \left(z_{l,\sigma}^a + \varepsilon_l^{\prime,a} \right) + V'_\sigma \right] \\
&+ [\beta_{HC}^a ((1-\lambda)C^r c_\sigma^r - C^r \lambda_\sigma) + \beta_{HH}^a H^r h_\sigma^r] H^r h_\sigma^r V'^{,a} \\
&+ \beta_H^a \left\{ H^r h_\sigma^r h_\sigma^r V'^{,a} + H^r h_{\sigma\sigma}^a V'^{,a} \right\} \\
&+ \beta_H^a H^r h_\sigma^r \left[\sum_{l=1}^{n_x} V'_{x_l^{\prime,a}} \left(z_{l,\sigma}^a + \varepsilon_l^{\prime,a} \right) + V'_\sigma \right] \\
&+ [\beta_C^a ((1-\lambda)C^r c_\sigma^r - C^r \lambda_\sigma) + \beta_H^a H^r h_\sigma^r] \left[\sum_{l=1}^{n_x} V'_{x_l^{\prime,a}} \left(z_{l,\sigma}^a + \varepsilon_l^{\prime,a} \right) + V'_\sigma \right] \\
&+ \beta^a \sum_{l=1}^{n_x} \left[\sum_{n=1}^{n_x} V'_{x_l^{\prime,a}, x_n^{\prime,a}} \left(z_{n,\sigma}^a + \varepsilon_n^{\prime,a} \right) + V'_{x_l^{\prime,a}\sigma} \right] \left(z_{l,\sigma}^a + \varepsilon_l^{\prime,a} \right) \\
&+ \beta^a \sum_{l=1}^{n_x} V'_{x_l^{\prime,a}} z_{l,\sigma\sigma}^a + \beta^a \sum_{l=1}^{n_x} V'_{\sigma x_l^{\prime,a}} \left(z_{l,\sigma}^a + \varepsilon_l^{\prime,a} \right) + \beta^a V'_{\sigma\sigma}.
\end{aligned}$$

Take unconditional expectation and Evaluate at the non-stochastic steady state,

$$\begin{aligned}
ET_{x_i x_j}^a &= \left(\bar{\beta}_{CC} \bar{C}^r \bar{c}_{x_j}^r + \bar{\beta}_{CH} \bar{H}^r \bar{h}_{x_j}^r \right) \bar{C}^r \bar{c}_{x_i}^r \bar{V}'^a \\
&+ \bar{\beta}_C \left(\bar{C}^r \bar{c}_{x_j}^r \bar{c}_{x_i}^r \bar{V}'^a + \bar{C}^r \bar{c}_{x_j x_i}^r \bar{V}'^a + \bar{C}^r \bar{c}_{x_i}^r \sum_{l=1}^{n_x} \bar{V}'^a_{x_i^l, a} \bar{z}_{l, x_j}^a \right) \\
&+ \left(\bar{\beta}_{HC} \bar{C}^r \bar{c}_{x_j}^r + \bar{\beta}_{HH} \bar{H}^r \bar{h}_{x_j}^r \right) \bar{H}^r \bar{h}_{x_i}^r \bar{V}'^a \\
&+ \bar{\beta}_H \left(\bar{H}^r \bar{h}_{x_j}^r \bar{h}_{x_i}^r \bar{V}'^a + \bar{H}^r \bar{h}_{x_j x_i}^r \bar{V}'^a + \bar{H}^r \bar{h}_{x_i}^r \sum_{l=1}^{n_x} \bar{V}'^a_{x_i^l, a} \bar{z}_{l, x_j}^a \right) \\
&+ \left(\bar{\beta}_C \bar{C}^r \bar{c}_{x_j}^r + \bar{\beta}_H \bar{H}^r \bar{h}_{x_j}^r \right) \sum_{l=1}^{n_x} \bar{V}'^a_{x_i^l, a} \bar{z}_{l, x_i}^a \\
&+ \bar{\beta} \sum_{l=1}^{n_x} \left(\sum_{n=1}^{n_x} \bar{V}'^a_{x_i^l, a, x_n^l, a} \bar{z}_{n, x_j}^a \right) \bar{z}_{l, x_i}^a + \bar{\beta} \sum_{l=1}^{n_x} \bar{V}'^a_{x_i^l, a} \bar{z}_{l, x_j x_i}^a, \quad (\text{B.120})
\end{aligned}$$

$$\begin{aligned}
E\bar{T}_{\sigma\sigma}^a &= [\bar{\beta}_{CC} (\bar{C}^r \bar{c}_\sigma^r - \bar{C}^r \bar{\lambda}_\sigma) + \bar{\beta}_{CH} \bar{H}^r \bar{h}_\sigma^r] (\bar{C}^r \bar{c}_\sigma^r - \bar{C}^r \bar{\lambda}_\sigma) \bar{V}'^a \\
&+ \bar{\beta}_C [\bar{C}^r \bar{c}_\sigma^r \bar{c}_\sigma^r + \bar{C}^r \bar{c}_{\sigma\sigma}^a - 2\bar{C}^r \bar{c}_\sigma^r \bar{\lambda}_\sigma - \bar{C}^r \bar{\lambda}_{\sigma\sigma}] \bar{V}'^a \\
&+ \bar{\beta}_C (\bar{C}^r \bar{c}_\sigma^r - \bar{C}^r \bar{\lambda}_\sigma) \left[\sum_{l=1}^{n_x} \bar{V}'_{x_l^a} \left(\bar{z}_{l,\sigma}^a + E\varepsilon_l'^a \right) + \bar{V}'_{\sigma^r} \right] \\
&+ [\bar{\beta}_{HC} (\bar{C}^r \bar{c}_\sigma^r - \bar{C}^r \bar{\lambda}_\sigma) + \bar{\beta}_{HH} \bar{H}^r \bar{h}_\sigma^r] \bar{H}^r \bar{h}_\sigma^r \bar{V}'^a \\
&+ \bar{\beta}_H \left\{ \bar{H}^r \bar{h}_\sigma^r \bar{c}_\sigma^r \bar{V}'^a + \bar{H}^r \bar{h}_{\sigma\sigma}^a \bar{V}'^a \right\} \\
&+ \bar{\beta}_H + \bar{H}^r \bar{h}_{\sigma\sigma}^a \bar{V}'^a + \bar{H}^r \bar{h}_\sigma^r \left[\sum_{l=1}^{n_x} \bar{V}'_{x_l^a} \left(\bar{z}_{l,\sigma}^a + E\varepsilon_l'^a \right) + \bar{V}'_{\sigma^r} \right] \\
&+ [\bar{\beta}_C (\bar{C}^r \bar{c}_\sigma^r - \bar{C}^r \bar{\lambda}_\sigma) + \bar{\beta}_H \bar{H}^r \bar{h}_\sigma^r] \left[\sum_{l=1}^{n_x} \bar{V}'_{x_l^a} \left(\bar{z}_{l,\sigma}^a + E\varepsilon_l'^a \right) + \bar{V}'_{\sigma^r} \right] \\
&+ \bar{\beta} \sum_{l=1}^{n_x} E \left[\sum_{n=1}^{n_x} \bar{V}'_{x_l^a, x_n^a} \left(\bar{z}_{n,\sigma}^a + \varepsilon_n'^a \right) + \bar{V}'_{x_l^a \sigma} \right] \left(\bar{z}_{l,\sigma}^a + \varepsilon_l'^a \right) \\
&+ \bar{\beta} \sum_{l=1}^{n_x} \bar{V}'_{x_l^a} \bar{z}_{l,\sigma\sigma}^a + \bar{\beta} \sum_{l=1}^{n_x} \bar{V}'_{\sigma x_l^a} \left(\bar{z}_{l,\sigma}^a + E\varepsilon_l'^a \right) + \bar{\beta} \bar{V}'_{\sigma\sigma} \\
&= \bar{\beta}_C \bar{C}^r \bar{c}_{\sigma\sigma}^a \bar{V}'^a - \bar{\beta}_C \bar{C}^r \bar{\lambda}_{\sigma\sigma} \bar{V}'^a + \bar{\beta}_H \bar{H}^r \bar{h}_{\sigma\sigma}^a \bar{V}'^a + \bar{\beta} \sum_{l=1}^{n_x} \bar{V}'_{x_l^a} \bar{z}_{l,\sigma\sigma}^a \\
&+ \bar{\beta} \bar{V}'_{\sigma\sigma} + \bar{\beta} \sum_{l=1}^{n_x} \sum_{n=1}^{n_x} \bar{V}'_{x_l^a, x_n^a} E\varepsilon_n'^a \varepsilon_l'^a. \tag{B.121}
\end{aligned}$$

In deriving the expression for $\bar{\lambda}_{\sigma\sigma}$, I have taken use of the following: (A) in the non-stochastic steady state, we have the following: $\bar{\beta}_C^r = \bar{\beta}_C^a$, $\bar{\beta}_H^r = \bar{\beta}_H^a$, and et al due to the facts that $\bar{\lambda} = 0$. Thus in (B.102) and (B.104), the corresponding coefficients in front of the unconditional expectations are the same. So do the coefficients in (B.103) and (B.105). (B) We assume that $\bar{V}'_{x_l^a}$

is equal to $\bar{V}_{x_i^a}$. (C) The value for $E(x_i - \bar{x}_i)(x_j - \bar{x}_j)$ comes from:

$$\begin{aligned}
E(x_i - \bar{x}_i)(x_j - \bar{x}_j) &= E(x_i - Ex_i + Ex_i - \bar{x}_i)(x_j - Ex_j + Ex_j - \bar{x}_j) \\
&= E(x_i - Ex_i)(x_j - Ex_j) + (Ex_i - \bar{x}_i)(Ex_j - \bar{x}_j).
\end{aligned} \tag{B.122}$$

Plug (B.102)-(B.122) into (B.98) and (B.99), we get the following:

$$\begin{aligned}
&EV^r(\sigma) - EV^a(\sigma) \\
&= ES^r - ES^a + ET^r - ET^a \\
&= \sum_{i=1}^{n_x} \bar{S}_{x_i}^r E(x_i^r - \bar{x}_i) + \frac{1}{2} \bar{S}_{\sigma\sigma}^r \sigma^2 + \frac{1}{2} \sum_{i=1}^{n_x} \sum_{j=1}^{n_x} \bar{S}_{x_i x_j}^r E(x_i^r - \bar{x}_i)(x_j^r - \bar{x}_j) - \\
&\quad \left[\sum_{i=1}^{n_x} \bar{S}_{x_i}^a E(x_i^a - \bar{x}_i) + \frac{1}{2} \bar{S}_{\sigma\sigma}^a \sigma^2 + \frac{1}{2} \sum_{i=1}^{n_x} \sum_{j=1}^{n_x} \bar{S}_{x_i x_j}^a E(x_i^a - \bar{x}_i)(x_j^a - \bar{x}_j) \right] + \\
&\quad \sum_{i=1}^{n_x} \bar{T}_{x_i}^r E(x_i^r - \bar{x}_i) + \frac{1}{2} \bar{T}_{\sigma\sigma}^r \sigma^2 + \frac{1}{2} \sum_{i=1}^{n_x} \sum_{j=1}^{n_x} \bar{T}_{x_i x_j}^r E(x_i^r - \bar{x}_i)(x_j^r - \bar{x}_j) - \\
&\quad \left[\sum_{i=1}^{n_x} \bar{T}_{x_i}^a E(x_i^a - \bar{x}_i) + \frac{1}{2} \bar{T}_{\sigma\sigma}^a \sigma^2 + \frac{1}{2} \sum_{i=1}^{n_x} \sum_{j=1}^{n_x} \bar{T}_{x_i x_j}^a E(x_i^a - \bar{x}_i)(x_j^a - \bar{x}_j) \right].
\end{aligned} \tag{B.123}$$

Define

$$A_2 = \sum_{i=1}^{n_x} (\bar{S}_{x_i}^r + \bar{T}_{x_i}^r) E(x_i^r - \bar{x}_i) - \sum_{i=1}^{n_x} (\bar{S}_{x_i}^a + \bar{T}_{x_i}^a) E(x_i^a - \bar{x}_i),$$

$$B_2 = \frac{1}{2} \sum_{i=1}^{n_x} \sum_{j=1}^{n_x} (\bar{S}_{x_i x_j}^r + \bar{T}_{x_i x_j}^r) E(x_i^r - \bar{x}_i)(x_j^r - \bar{x}_j) -$$

$$\frac{1}{2} \sum_{i=1}^{n_x} \sum_{j=1}^{n_x} (\bar{S}_{x_i x_j}^a + \bar{T}_{x_i x_j}^a) E(x_i^a - \bar{x}_i)(x_j^a - \bar{x}_j),$$

$$C_2 = EV^r(\sigma) - EV^a(\sigma),$$

$$D_2 = \bar{S}_{\sigma\sigma}^r + \bar{T}_{\sigma\sigma}^r - \bar{S}_{\sigma\sigma}^a - \bar{T}_{\sigma\sigma}^a,$$

$$E_2 = \bar{\beta}_C \bar{C}^r \bar{c}_{\sigma\sigma}^r \bar{V}'^r + \bar{\beta}_H \bar{H}^r \bar{h}_{\sigma\sigma}^r \bar{V}'^r + \bar{\beta} \sum_{l=1}^{n_x} \bar{V}'^r_{x_l^r} \bar{x}'^r_{l,\sigma\sigma} + \bar{\beta} \bar{V}'^r_{\sigma\sigma}$$

$$+ \bar{\beta} \sum_{l=1}^{n_x} \sum_{n=1}^{n_x} V'_{x_l^r, x_n^r} E \varepsilon_n'^r \varepsilon_l'^r - \bar{\beta} \sum_{l=1}^{n_x} \sum_{n=1}^{n_x} V'_{x_l^r, x_n^r} E \varepsilon_n'^a \varepsilon_l'^a$$

$$- \bar{\beta}_C \bar{C}^r \bar{c}_{\sigma\sigma}^a \bar{V}'^a - \bar{\beta}_H \bar{H}^r \bar{h}_{\sigma\sigma}^a \bar{V}'^a - \bar{\beta} \sum_{l=1}^{n_x} \bar{V}'^a_{x_l^a} \bar{x}'^a_{l,\sigma\sigma} - \bar{\beta} \bar{V}'^a_{\sigma\sigma},$$

$$F_2 = \bar{U}_C \bar{C}^r \bar{c}_{\sigma\sigma}^r + \bar{U}_H \bar{H}^r \bar{h}_{\sigma\sigma}^r - \bar{U}_C \bar{C}^r \bar{c}_{\sigma\sigma}^a - \bar{U}_H \bar{H}^r \bar{h}_{\sigma\sigma}^a,$$

Thus we can rewrite (B.123) as

$$2(C_2 - A_2 - B_2) = D_2 \sigma^2. \quad (\text{B.124})$$

As we have shown in (B.109), (B.113), (B.117), and (B.121)

$$\begin{aligned}
\bar{S}_{\sigma\sigma}^r &= \bar{U}_C \bar{C}^r \bar{c}_{\sigma\sigma}^r + \bar{U}_H \bar{H}^r \bar{h}_{\sigma\sigma}^r, \\
\bar{S}_{\sigma\sigma}^a &= \bar{U}_C \bar{C}^r \bar{c}_{\sigma\sigma}^a + \bar{U}_H \bar{H}^r \bar{h}_{\sigma\sigma}^a - \bar{U}_C \bar{C}^r \bar{\lambda}_{\sigma\sigma}, \\
\bar{T}_{\sigma\sigma}^r &= \bar{\beta}_C \bar{C}^r \bar{c}_{\sigma\sigma}^r \bar{V}'^r + \bar{\beta}_H \bar{H}^r \bar{h}_{\sigma\sigma}^r \bar{V}'^r + \bar{\beta} \sum_{l=1}^{n_x} \bar{V}'_{x_l}{}^r \bar{x}'_{l,\sigma\sigma}{}^r + \bar{\beta} \bar{V}'_{\sigma\sigma}{}^r \\
&\quad + \bar{\beta} \sum_{l=1}^{n_x} \sum_{n=1}^{n_x} V'_{x_l, x_n}{}^r E \varepsilon_n'{}^r \varepsilon_l'{}^r, \\
\bar{T}_{\sigma\sigma}^a &= \bar{\beta}_C \bar{C}^r \bar{c}_{\sigma\sigma}^a \bar{V}'^a + \bar{\beta}_H \bar{H}^r \bar{h}_{\sigma\sigma}^a \bar{V}'^a + \bar{\beta} \sum_{l=1}^{n_x} \bar{V}'_{x_l}{}^a \bar{x}'_{l,\sigma\sigma}{}^a + \bar{\beta} \bar{V}'_{\sigma\sigma}{}^a \\
&\quad - \bar{\beta}_C \bar{C}^r \bar{\lambda}_{\sigma\sigma} \bar{V}'^a + \bar{\beta} \sum_{l=1}^{n_x} \sum_{n=1}^{n_x} V'_{x_l, x_n}{}^r E \varepsilon_n'{}^a \varepsilon_l'{}^a,
\end{aligned}$$

and in the non-stochastic steady state, $\bar{V}'^r = \bar{V}'^a$, so,

$$\begin{aligned}
\bar{S}_{\sigma\sigma}^r - \bar{S}_{\sigma\sigma}^a &= \bar{U}_C \bar{C}^r \bar{c}_{\sigma\sigma}^r + \bar{U}_H \bar{H}^r \bar{h}_{\sigma\sigma}^r - \bar{U}_C \bar{C}^r \bar{c}_{\sigma\sigma}^a - \bar{U}_H \bar{H}^r \bar{h}_{\sigma\sigma}^a + \bar{U}_C \bar{C}^r \bar{\lambda}_{\sigma\sigma} \\
&= F_2 + \bar{U}_C \bar{C}^r \bar{\lambda}_{\sigma\sigma}, \\
\bar{T}_{\sigma\sigma}^r - \bar{T}_{\sigma\sigma}^a &= \bar{\beta}_C \bar{C}^r \bar{c}_{\sigma\sigma}^r \bar{V}'^r + \bar{\beta}_H \bar{H}^r \bar{h}_{\sigma\sigma}^r \bar{V}'^r + \bar{\beta} \sum_{l=1}^{n_x} \bar{V}'_{x_l}{}^r \bar{x}'_{l,\sigma\sigma}{}^r + \bar{\beta} \bar{V}'_{\sigma\sigma}{}^r \\
&\quad + \bar{\beta} \sum_{l=1}^{n_x} \sum_{n=1}^{n_x} V'_{x_l, x_n}{}^r E \varepsilon_n'{}^r \varepsilon_l'{}^r - \bar{\beta} \sum_{l=1}^{n_x} \sum_{n=1}^{n_x} V'_{x_l, x_n}{}^r E \varepsilon_n'{}^a \varepsilon_l'{}^a \\
&\quad - \bar{\beta}_C \bar{C}^r \bar{c}_{\sigma\sigma}^a \bar{V}'^a - \bar{\beta}_H \bar{H}^r \bar{h}_{\sigma\sigma}^a \bar{V}'^a - \bar{\beta} \sum_{l=1}^{n_x} \bar{V}'_{x_l}{}^a \bar{x}'_{l,\sigma\sigma}{}^a \\
&\quad - \bar{\beta} \bar{V}'_{\sigma\sigma}{}^a + \bar{\beta}_C \bar{C}^r \bar{\lambda}_{\sigma\sigma} \bar{V}'^a \\
&= E_2 + \bar{\beta}_C \bar{C}^r \bar{\lambda}_{\sigma\sigma} \bar{V}'^a,
\end{aligned}$$

and

$$D_2 = E_2 + F_2 + \left(\bar{U}_C \bar{C}^r + \bar{\beta}_C \bar{C}^r \bar{V}'^a \right) \bar{\lambda}_{\sigma\sigma}. \quad (\text{B.125})$$

The welfare cost, which is measure by $\bar{\lambda}_{\sigma\sigma}$, is thus given by:

$$\bar{\lambda}_{\sigma\sigma} = \frac{\frac{2(C_2 - A_2 - B_2)}{\sigma^2} - E_2 - F_2}{\bar{U}_C \bar{C}^r + \beta_C \bar{C}^r \bar{V}'^r, a}. \quad (\text{B.126})$$

B.6 Solving Dynamic Stochastic General Equilibrium

This appendix shows the general steps to solve dynamic stochastic general equilibrium model by using perturbation method. There are many other methods that can be used to solve the DSGE models, such as policy function iteration, value function iteration, and etc. The advantage is perturbation method is that it can easily handle a model with many state variables. It also explains the MATLAB programs that are used to solve the DSGE models.

B.6.1 Theoretical Steps

Since it is mostly likely that we cannot solve the DSGE system analytically, we need to use linear approximation skills. The first skill we want to use is the first order approximation. To some problems, we need to solve the problem by using the second order approximation, for example, when we are going to the evaluate the welfare effects of the model. This is because given the same non-stochastic steady states, different policies with only different second terms will give the same welfare results if we only use the first order approximation. Woodford (2002) discusses the cases where the first order approximation to the policy functions and the second order approximation to the welfare level are appropriate.

1. Set up the dynamic stochastic general equilibrium model. The usual dynamic stochastic general equilibrium model can be expressed by the

following equation:

$$Ef(s', s, c', c) = 0$$

After the first order approximation, list all the equilibrium conditions in the following form:

$$AEx_{t+1} = Bx_t$$

where

$$x_t \equiv \begin{bmatrix} s_t \\ c_t \end{bmatrix}$$

The solution takes the following forms:

$$s = g(s) \tag{B.127}$$

$$c' = h(s) + \sigma\tilde{\eta}\epsilon \tag{B.128}$$

The variable s denotes state variables. In my paper, state variables include: endogenous but predetermined variables: capital and debt; and exogenous state variables: productivity, world interest rate, and country spread. The variable c denotes choice variables. In my paper, choice variables include: consumption, hours, and etc. The number of choices is assumed to be n_c .

2. We can apply the Schur decomposition method to the above linear system equation to get the following

$$qAz z' E x_{t+1} = qBz z' x_t$$

or

$$aE y_{t+1} = b y_t$$

where $a = qAz$, $b = qBz$, and $y_t = z' x_t$. And a and b are triangle matrix.

3. Partition the system, we get

$$\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} E y_{t+1} = \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix} y_t$$

where y_{2t} is of the order $n_c \times 1$. Rearrange the order of $\frac{b(i,i)}{a(i,i)}$ in an order such that the largest $\frac{b(i,i)}{a(i,i)}$ is at the right bottom corner of the matrix. Focusing the new a, b , we know that if $\frac{b(n,n)}{a(n,n)}$ is less than one, the solution to the difference equation is not uniquely determined, since any initial conditions will converge the steady state. It is possible that self-fulfilling equilibria exist. Here we pay attention to the case where the solutions is uniquely determined by assuming that some $\frac{b(n,n)}{a(n,n)}$ is larger than one. In that case, we the system will explode unless initially, $w(n)$ is zero. In the same logic, we can knock down that it is true that all the values of $w(i = n - ny - 1)$ to $w(n)$ are zero.

4. We thus have the following

$$a_{22} E y_{t+1}^2 = b_{22} y_t^2$$

Here we put terminal condition on y_{t+1} , such that $\lim_{j \rightarrow \infty} E_t y_{t+j}^2 < \infty$. Given that $|\frac{a_{22}}{b_{22}}| < 1$, the only solution is $y_t^2 = 0$.

5. Since

$$y_t^2 = 0 = z'_{12} s_t + z'_{22} c_t$$

we get the solution for the choice variables:

$$c_t = -z'_{22}{}^{-1} z'_{12} s_t$$

6. Go one step ahead, since $y_t^2 = 0$, we have

$$\begin{aligned} a_{11}E y_{t+1}^1 &= b_{11}y_t^1 \\ \Rightarrow a_{11}E(z'_{11}s_{t+1} + z'_{12}c_{t+1}) &= b_{11}z'_{11}s_t + z'_{12}c_t \end{aligned}$$

Given the fact that c_{t+1} is a function of s_{t+1} and s_{t+1} is known at period t , we can solve for s_{t+1} as

$$s_{t+1} = \left[z'_{11} - z'_{21}z'_{22}{}^{-1}z'_{12} \right] a_{11}^{-1}b_{11} \left[z'_{11} - z'_{21}z'_{22}{}^{-1}z'_{12} \right] s_t$$

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B.6.2 MATLAB Programs

Here are the practical steps.

1. Solve the non-stochastic steady state of the model. This is done by the program called `stepxxss.m`, where xx are numbers representing different models.

2. Solve the model symbolically. This is done by the program called `stepxx.m`. This requires Symbolic function of MATLAB. This program calls for the program called `anal_deriv.m`.

3. Solve the model numerically. This is done by the program called `stepxxrun.m`. This program calls for the following programs: `stepxxss.m`, `num_eval.m`, `gx_hx.m`, `mom.m`, `gxx_hxx.m`, `gss_hss.m`, and `unconditional_mean.m`.

4. Solve for the unconditional welfare costs. This is done by the program called `wcxx.m`. It calls for `stepxxrun`, `auxt.m`, and `ir.m`.

5. There are other programs called.

6. Put all the programs in the same folder. Run `wcxx.m`. It takes about 10 minutes to calculate the unconditional welfare costs of sudden stops with the two levels of trade openness. The first number is the welfare cost of sudden stop with 0 tariff rate. The second number is the welfare costs of sudden stops with 10% tariff rate. The negative value means that sudden stops are good.

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Biography

I was born in a small county of Huanggang Region, Hubei Province, China. I got my first state award when I was nine years old mainly because I sat still in all the classes. Six years later, I got admitted as the best student to the best high school in the province where I spent the first half of my high school life. I spent the second half in Beijing as a student of the National Experimental Class of Physics. There I became a fan of football and bridge. In that class, I met many extremely talented people from all over the country and some of them become my life-long friends. I then entered Peking University without taking the national entrance exam. At Peking University, I studied economics and gave myself the English name: Arthur - in memory of King Arthur for his Round Table. I thought I should have some real experience before I pursue my Ph.D. degree in Economics: I then worked for the Bank of China, Head Office, for the next five years. This decision turned out to be a bad one. It almost permanently deviated me from the academic field until I met my dearest wife. I resigned from the Bank and followed my wife to study in USA.