Essays on Dynamic Game Theory

by

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Dissertation submitted in partial fulfillment of the
requirements for the degree of Doctor of Philosophy
in the Department of Economics
in the Graduate School of
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ABSTRACT

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Abstract

Chapter 2 considers income-share agreements (ISAs), which recently have been gaining traction as a way for students to finance college education, marketed as a way for students to reduce the down-side risk of winding up in a low-paying job with high student debt. Because ISA payments are a fraction of the on-the-job wage, incentives for both applicant and provider are different from a traditional debt-financed job applicant on the job market. I develop a labor-search model to show how financing affects job-market outcomes such as wages, search duration, and overall utility, set within an equilibrium framework in which the terms and methods of financing are endogenous. I show that ISAs can constitute an important part of the college-financing decision for financially-disadvantaged potential college students, and can act well as a substitute for traditional debt-markets when the cost of college is neither very low nor very high.

Chapter 3 considers a continuous-time organization design problem. Each member's output is an imperfect signal of his underlying effort, and each member's utility from remaining in the organization is endogenous to other members' efforts. Monetary transfers are assumed infeasible. Incentives can be provided only through two channels: expulsion following poor performance and respite following good performance. We derive the steady state distribution of members' continuation utilities for arbitrary values of the initial and maximum continuation utilities and then optimize these values according to organizational objectives. An optimally designed organization can be implemented by associating continuation utilities with a performance-tracking reputation system.

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I am deeply indebted to both Aaron Kolb and Curt Taylor, whose collaboration was vital to this dissertation. Chapter 2 is based on joint work with both Aaron and Curt; I have little doubt that without all of us working together, nothing resembling Chapter 2 could have materialized. Beyond collaboration, both also helped me find my strengths as a researcher and and understanding of economic theory that goes far beyond what can be learned in any classroom.

Most importantly, I would not have been able to make it through this process without the friendship and love of close friends and family. Dan Garrett, Ben Heebsh, and Zach Nolan, among so many others, have been an invaluable source of support in good times and in bad. To mom and dad, no words could ever come close to thanking you for all you have done for me. Everything I have done here—everything I will ever do—is a testament to the two of you.

Chapter 1

Introduction

The study of dynamic games is integral to economists' ability to model complex interactions over time. Decisions that impact an agent's future choice set are plentiful, and without a theory of dynamic games, few real-world decisions could be adequately analyzed. This dissertation presents two models of dynamic games as applied to important questions in labor economics and industrial organization. In the next chapter, I show the effects of a novel form of college financing on a job applicant's labor search decisions, and consider how college students should choose their method of financing their education in order to maximize lifetime income. This process is inherently a dynamic decision problem; a student's choice of financing has a substantial impact on the incentives they face following graduation, and choices that an applicant makes on the job market must be made considering their effects on options later in the process. Subsequently, in the third chapter I show a model of large collective organizations, where the payoff of each agent in a continuum is determined by the actions of all agents in the organization. In order to ensure that agents do their fair share, they gain or lose reputation according to a complex system that allows a "vacation" following continued good results and termination after enough bad results accumulate. Tracking the ever-changing reputations of a continuum of agents requires a new approach to dynamic mechanism design that derives a steady state distribution of agents within the organization from the parameters that define individual agents' contracts in order to make the problem tractable. Finally, the fourth chapter concludes.

The next chapter, titled "Income-Share Agreements on the Job Market: Debt Versus

Equity," addresses income-share agreements (ISAs), which allow students to accept money towards college tuition in return for a fixed portion of their after-graduation earnings. This changes a student's incentives on the job market by lowering the cost of searching for a job relative to a student who has traditional debt financing. In response, I find that students with ISAs will slow their search and have higher nominal wages, but that only students with intermediate levels of financial need benefit, as the costs of ISAs reduce take-home pay for the neediest students below the cost of attending college, while for the least needy student, the costs of debt are sufficiently low that the traditional financing method dominates the new option. However, I find that these results are dependent on the exact nature of ISAs. Given the alignment of incentives between provider and student that does not exist with debt, I show that if providers have the ability to invest in students' job market outcomes, the addition of ISAs to the market can improve college access for the neediest students and in fact speed up the labor search process. Additionally, if there is competition in the market for ISAs sufficient to give bargaining power to students, even the least financially needy students prefer to fully fund their education using income-sharing.

In chapter 3, titled "Communities, Co-ops, and Clubs: Social Capital and Incentives in Large Collective Organizations," my coauthors and I consider a continuous-time organization design problem where each members' output is an imperfect signal of his underlying effort, and each member's utility is a determined by the efforts of other members. Without monetary transfers, we show that incentives can only be provided through allowing respites from hard work following good performance, and expulsion after poor performance. In order to analyze the design of such an organization, we derive the steady state distribution of agents' continuation utilities, which is then optimized according to various objective functions. In order to implement the optimal organization, the planner tracks agents' performance with a reputation system that adjusts based on each member's output.

Chapter 2

Income-Share Agreements on the Job Market: Debt Versus Equity

2.1 Introduction

Income-share agreements—agreements whereby students receive money for tuition and fees in return for a share of their future income after graduation—have grown rapidly in popularity over the last few years. Since first conceived by Milton Friedman in 1955, income-share agreements (ISAs) occasionally have been used as a part of the financial aid landscape in higher education, but never more so than since 2016 when the first large-scale ISA of the modern era, the Back a Boiler-ISA Fund, began at Purdue University. Since then, similar programs have begun across the United States, providing students with an alternative means of financing their collegiate educations.

In this paper, I use a simple microeconomic model to analyze the effect of income-sharing on numerous labor market outcomes. In particular, I find that ISAs put upward pressure on wages but may increase or decrease take-home pay, with increases in take-home pay being associated only with a form of ISA that directly helps students find better jobs. I additionally show that ISAs tend to increase the time it takes for applicants to find jobs, with another exception for the ISAs that invest in students' labor market outcomes. I find sufficient conditions for both positive results to hold for these types of agreements. These types of investment-providing ISAs may also expand access to higher education for students facing particularly high costs of attending. Students are allowed to choose their source of funding endogenously, and I find that in the optimal financial-aid environment, all students utilize income-share agreements to some degree, though income-sharing generally cannot replace debt as a sole source of financial aid. Even non-optimized income-sharing is shown to be an effective financial aid option for students who face intermediate levels of need.

Income-sharing in higher education began in practice with a program run by Yale University in the 1970s, which is widely regarded as having failed due to high inflation and mistakes in the program design Curran (2018). ISAs were revived at Purdue University in 2016, and have since been established at dozens of institutions. Some universities, such as Purdue and the University of Utah offer income-sharing with the help of companies, including Vemo Education, that specialize in providing income-share agreements. Other universities, such as Clarkson University and Colorado Mountain College, provide ISAs directly through the university with funds raised from donors. Others still, such as the University of California, San Diego, have partnered with local businesses that specialize in helping students in particular fields find good jobs, like the San Diego Workforce Partnership. ISAs of the latter two types are often provided only to students who major in specific fields, and the funding provider works more closely with the students; the last type in particular invests heavily in the labor market success of partnered students.

Analyzing the labor-market outcomes of modern income-share agreements empirically remains infeasible, due to the only very recent roll-out of these programs; few students with ISAs have joined the labor force. Until such empirical work is possible, I seek to find the impacts of income-sharing through a straightforward labor search model. Job applicants draw from a wage distribution at a particular rate, and choose either to accept a wage offer or return to searching. By stripping the model down to only the bare essentials, I am able to isolate the effects of income-sharing on the outcomes of interest. When an applicant has an ISA, she does not get to keep the entirety of the offered wage; if an applicant has an

ISA provider who invests in them, she receives job offers faster. This comes from thinking of investment as helping an applicant by placing calls on her behalf to potential employers, helping write résumés and cover letters, or teaching interviewing techniques. With only these factors distinguishing income-sharing from traditional lending in the model, the results are broadly applicable. The impact of ISAs I find are distillations of only the fundamental features of income-share agreements.

Related literature

This paper contributes to a number of active research areas. Most crucially, I add to the small number of vital studies of income-share agreements, beginning with Madonia and Smith (2019), who study the impact of ISAs on tournament poker players' effort, showing empirically that players' returns fall substantially when they sell stakes in their winnings, which can be attributed both to selection into tournaments with more talented opponents and to a reduction in the incentives to play at one's best. The study of ISAs with in an educational context begins with the recent working paper Mumford (2018). Mumford uses data from Purdue University to study what drives selection into ISAs, finding that parents' socioeconomic conditions is the largest factor in students' funding choices, and that, within a given major choice, students with lower grades or aptitudes do not adversely select into ISAs. Furthermore, Mumford shows that ISA enrollment is not driven by students with greater risk-aversion. Like Madonia and Smith, I look at the outcomes of ISA-funded individuals compared with the outcomes of traditionally funded individuals. Like Mumford, I consider educational ISAs and students' choices of funding options. Inspired by his results, the financial background of the students is the primary driver of students' decision calculus in my model. Nonetheless, I am the first to develop a framework that includes selection into an ISA with long-run outcomes. In particular, I am the first to show wage and employment effects of educational ISAs, as insufficient data currently exists for empirical work, given the nascence of income-sharing in the world of higher education.

I also contribute to a literature on optimal student lending design. Recent work in this area, such as that by Britton et al. (2019), Britton and Gruber (2019), and Lochner and Monge-Naranjo (2016), have shown the value of income-contingent loan repayment options for students. While ISAs are not loans, this paper shows that well-designed income-sharing agreements can be preferable to debt under broad circumstances, and can be a part of the optimal college financing landscape.

I adapt techniques from the literature on labor search. Classic papers such as McCall (1970), Mortensen (1970), Gronau (1971), and Lippman and McCall (1976) along with expansions on their models in Mortensen (1986) and Mortensen and Pissarides (1999) serve as the theoretical underpinnings of my model in this paper. I use the McCallian framework, rather than the benchmark Diamond (1982) and Mortensen and Pissarides (1994) model, in order to isolate the impact of ISAs on search behavior itself. This is particularly important as ISAs are newly introduced to a market environment that has long since reached maturity in their absence, while job applicants with ISAs are rare relative to those who are traditionally financed.

In the following section, I lay out a model of labor search that incorporates income-sharing and further enhance it with an investment technology for ISA providers. Section 2.3 analyzes the effects of ISAs on various measures of labor market success and access to college. Students are allowed to select their methods of financing in section 2.4, in which student and social welfare is discussed, followed by the construction of the optimal income-share agreement. Section 2.5 concludes.

2.2 Model

Job search without income-sharing

A risk neutral job applicant discounts time at rate r in continuous time across an infinite horizon.[1]She begins time unemployed, and receives a flow payoff b as long as she remains unemployed. The value $b > 0$ may be thought of as the flow value of the leisure she enjoys by not having a job. At the beginning of time, she also carries debt Δ, which will be endogenized later.

As long as the applicant is unemployed, she draws a wage offer i.i.d. at rate $\alpha > 0$ from an exogenous, stationary distribution $F(w)$ with bounded support. Whenever she draws an offer, she has the choice to accept the offer, earning the drawn wage as a flow payoff forever,[2]or to reject the offer and return to searching for a job. The value to the applicant of accepting an offer w is

$$W(w) = \frac{w}{r} - \Delta. \tag{2.2.1}$$

As the value of accepting a wage offer is increasing in the offered wage, the applicant follows a threshold strategy with reservation wage w_R and the leisure and search value of unemployment must be the value of accepting a job with at the reservation wage, i.e. $U = W(w_R)$. While searching for a job, the applicant enjoys not only the flow value b but additionally the option value of being able to search for a permanent job. Thus, the applicant's flow value of unemployment can be written

$$rU = b + \alpha \int_0^\infty \max\{0, W(w) - U\}\, dF(w). \tag{2.2.2}$$

[1]The analysis in this subsection is directly adapted from McCall (1970) and the related literature described above.

[2]This assumption does not impact the qualitative results of the analysis, in this or in later sections, provided that the wage distribution remains fixed.

At any time before accepting a job, the applicant is earning b, and with probability α gets to choose between searching more, or accepting the value of the drawn wage offer and forgoing further search.

Combining the equations defining the value functions of employment and unemployment, we get the classical reservation wage equation

$$w_R = b + \frac{\alpha}{r} \int_{w_R}^{\infty} (w - w_R)\ dF(w) = b + \frac{\alpha}{r} \int_{w_R}^{\infty} (1 - F(w))\ dw. \tag{2.2.3}$$

The right-hand side of this fixed-point equation is easily shown to be a contraction, and thus there is a unique fixed-point that can serve as the applicant's reservation wage.

It is important for us to note that the optimal search strategy of the applicant is independent of Δ, the debt that the applicant carries, as the debt she carries must be repaid regardless of her employment status.

Job search with income-sharing

We now turn to our main consideration of income-share agreements. An applicant who has accepted an ISA, receiving an amount Δ^E in return for a share ϕ of her future income, is broadly similar to the traditional debt-financed student analyzed above. However, the value to a student with an ISA of accepting a wage offer w is

$$W(w) = (1 - \phi)\frac{w}{r} - \Delta + \Delta^E. \tag{2.2.4}$$

as she must give a fraction of her income to the ISA provider, and has received some upfront payment from the ISA provider.

As before, the value of accepting a wage increases with the wage itself, so a threshold strategy must be optimal. Following the same derivation process, we arrive with a similar reservation wage equation for an ISA-financed student:

$$(1 - \phi)w_R^E = b + (1 - \phi)\frac{\alpha}{r} \int_{w_R^E}^{\infty} (1 - F(w))\ dw. \tag{2.2.5}$$

From this we immediately arrive at the following lemma by implicitly differentiating the reservation wage w_R^E with respect to the income-sharing rate ϕ.

Lemma 2.2.1. *The reservation wage is increasing in the income-sharing rate.*

As the percentage of her income that an applicant must relinquish to the ISA provider increases, the value of working relative to unemployment decreases. The wage that made her indifferent between accepting and declining a job offer is now insufficient, and she prefers to wait for a better offer, even though waiting itself is costly.

Investment in the contact rate

Unlike traditional lenders, income-share providers have an incentive to increase a funded applicant's wages if it is possible to do so. ISA providers are paid more when the students they fund get higher-paying jobs, and they get paid more when the students they fund get jobs sooner, since they discount the future. Providers should look for any possible avenue with which they can ease the search process, and evidence from the San Diego Workforce Partnership suggests that they do so by assisting applicants with résumé writing and helping applicants access a network of potential employers in relevant fields. Using this stylized fact as inspiration, we think of the provider as being able to make costly investments in an applicant's rate of receiving job offers.

Rather than receiving wage offers at rate α, an investment-type ISA funded applicant receives offers at rate $\alpha^N \geq \alpha$, chosen by the ISA provider. The provider faces a cost $c(\alpha^N)$ described by the assumption below:

Assumption 1. $c(\alpha^N)$ *satisfies*

- $c(\alpha) = 0$

- $c'(\alpha^N) \geq 0$
- $c''(\alpha^N) > 0$
- $c'(0) = 0$
- $\lim_{\alpha^N \to \infty} c'\left(\alpha^N\right) = \infty$

These conditions serve to ensure that the ISA always finds some unique amount of investment profitable, but is only willing to invest a finite amount.

The applicant's search problem remains similar to that derived above, with

$$(1-\phi)w_R^N = b + (1-\phi)\frac{\alpha^N}{r}\int_{w_R^N}^{\infty}(1-F(w))\ dw \tag{2.2.6}$$

but there is now a new condition required to define equilibrium, since the provider must choose α^N optimally. The applicant's reservation wage equation describes a unique optimal reservation wage for any given contact rate, so we can treat the provider as optimally choosing the applicant's reservation wage, subject to said reservation wage satisfying equation 2.2.6, and maximizing his own payoff.

Just like the applicant, we can define value functions for the provider. The value $V(w)$ to the provider of the applicant accepting wage w needs to balance the monetary value of the wage itself with the delay caused by searching. The following lemma gives the provider's value of a particular reservation wage.

Lemma 2.2.2. *The value to the provider of the applicant having reservation wage w_R^N is*

$$V(w) = \frac{\phi w_R^N H^N}{r + H^N},$$

in which

$$H^N := \alpha^N \left(1 - F\left(w_R^N\right)\right),$$

Since the applicant starts out unemployed, and the provider is choosing the applicant's reservation wage implicitly through his own choice of α^N, the provider's value of the applicant's unemployment is equal to the value to the provider of the applicant accepting a job at the reservation wage, adjusted to account for the fact that higher reservation wages take longer to implement, net the initial cost of implementing a given contact rate. Thus, the ISA provider's problem is to maximize

$$\max_{w_R^N,\alpha^N} \phi \frac{w_R^N H^N}{r + H^N} - c\left(\alpha^N\right), \text{ s.t. } (1-\phi)w_r^N = b + (1-\phi)\frac{\alpha^N}{r}\int_{w_r^N}^{\infty}(1 - F(w))\, dw. \quad (2.2.7)$$

The ISA's solution to this problem defines a system of two equations which characterize labor-search under an investment-type ISA. Equation 2.2.6 and

$$c'\left(\alpha^N\right) = \phi w_R^N \frac{1 - H^N}{r + H^N}\frac{\partial H^N}{\partial \alpha^N} + \phi \frac{H^N}{r + H^N}\frac{\partial w_R^N}{\partial \alpha^N} \quad (2.2.8)$$

together characterize the ISA provider's optimal choice of contact rate and the applicant's choice of reservation wage. The analysis of the optimal choices of w_R^N and α^N are simpler when the partials of H^N and w_R^N with respect to α^N are not written out in full, so they are not written out here.

The following lemma, comparing investment-type ISAs to non-investment ISAs and traditional student loans, must hold, following from assumption 1 and the applicant's search condition 2.2.6.

Lemma 2.2.3. *If an ISA is able to invest in job applicants, it will do so, and the reservation wage of an applicant who receives investment will be higher than the reservation wage of an applicant with an ISA who does not receive an investment, which is higher again than an applicant funded through traditional loans.*

- $\alpha^N > \alpha$
- $w_R^N > w_R^E > w_R$

The very goal of investment is to increase the reservation wage of the job applicant, so as long as assumption 1 holds, investment will be forthcoming and gross wages will be under for investment-ISAs.

An increase in the income-sharing rate increases the reservation wage for a non-investment ISA, since the flow value of remaining unemployed is relatively more valuable when the value of accepting a given wage decreases. It is less immediate, however, that increasing income-sharing would have a similar effect for investment-type ISAs. The income-sharing rate not only impacts the optimal choice of reservation wage, but also the optimal investment in the contact rate. Increases in the contact rate serve to increase reservation wages, but it may not always be so that increases in income-sharing increase the contact rate. Nonetheless, it is still the case that increases in income-sharing serve to increase the reservation wage even in investment-type ISAs.

Proposition 2.2.1. *The reservation wage w_R^N of an investment-type ISA funded applicant is increasing in the income-share ϕ.*

The intuition for this proposition is straightforward. The reservation wage is increasing in the contact rate, as increasing the contact rate directly increases the option value of continuing to search for more jobs. Since an increase in the income-sharing rate, holding the contact rate fixed, also increases the relative value of searching, it is only possible for an increase in the income-sharing rate to decrease the overall reservation wage if the increase in the income-share *decreases* the contact rate chosen by the provider. The provider would like to maximize the applicant's reservation wage given his cost function and the optimality condition for the applicant. Since changing the income-sharing rate does not change the investment cost, the provider would only decrease the contact rate if the reservation wage chosen by the applicant goes up to such an extent as to outweigh the benefits from reducing the contact rate and saving money. That is, the provider would reduce investment when income-sharing rises only in circumstances in which the reservation wage increases anyway.

2.3 Labor-market outcomes and access to college

The effect of an income-share agreement on labor-market outcomes extends far beyond the reservation wage chosen by job applicants that have been analyzed thus far. Before considering the welfare implications of ISAs, it is instructive to consider the impact of income-share agreements on specific goals that may be of interest to policy-makers.

Take-home pay

Most immediately, the take-home pay of an employee who has signed an ISA, unlike that of an employee without one, is not the wage paid by the employer, but rather a fraction thereof. While ISA-funded job applicants have been shown to have a higher reservation wage than those without ISAs, they may end up with a lower net wage after a portion of their pay is diverted to the ISA provider.

Depending on the draw from the wage distribution that induces an applicant to stop searching and begin employment, an employee with or without an ISA may have a higher take-home pay simply through luck. Therefore, attention is focused on the take-home portion of the applicant's *reservation* wage, rather than her observed wage, as, *ex ante*, this is the income-level that is most closely associated with the applicant's actual welfare. This is the flow value of take-home pay that makes the applicant indifferent between taking the job or continuing to search. It is reasonable to expect that, as income-sharing becomes more extreme, take-home pay would decrease. In the limit as the provider gets to keep nearly all of the earned wages, the boundedness of the wage distribution prevents the applicant from simply waiting for a wage offer sufficiently high so as to make up for the lost income, and all value created by a successful search flow to the provider. In fact, when investment is not an option for the provider, an increase in income-sharing will *always* result in a decrease in take-home pay.

Lemma 2.3.1. *The applicant's take-home pay in a non-investment ISA, w_R^E, is decreasing in the income-sharing rate ϕ.*

This is easily shown by taking the derivative of the right-hand side of equation 2.2.5 with respect to ϕ:

$$\frac{\partial(1-\phi)w_R^E}{\partial\phi} = \frac{-\alpha}{r}\left[(1-\phi)\left(1-F\left(w_R^E\right)\right) + \int_{w_R^E}^{\infty} 1 - F(w)\,dw\right] < 0. \tag{2.3.1}$$

Since the reservation wage itself is increasing in ϕ, take-home pay is decreasing. By increasing income-sharing, the provider is increasing wages, but is additionally capturing rents from the applicant herself.

However, this bleak result does not carry over into the scenario in which ISA providers invest in applicants.

Proposition 2.3.1. *Introducing income-sharing increases take-home pay.*

By investing in the contact rate of the applicant, the provider increases the reservation wage of the applicant. Increasing the income-sharing rate, then, may cause the reservation wage to increase through two channels—directly, as jobs are less valuable when income-sharing is in effect, and, if the income-sharing has also increased the provider's investment, indirectly by making searching more valuable. The direct effect, we have seen, is insufficient for raising take-home pay, but the contact rate rises enough for the provider to bring enough extra value to the search process, allowing for take-home pay to increase.

A note about investment

In an investment-type ISA, it becomes possible for take-home pay to increase precisely because an option is available to the provider and applicant that is not available through other types of financing. It is natural, then, to consider whether gains from investment are truly a feature of income-share agreements, or rather if investment itself is the proper focus of our attention. A number of points may be raised in defense of the author's viewpoint that investment as a byproduct of the ISA itself, rather than investment in its own right, is the proper focus of study.

When considering the traditional debt-financed job applicant, one would be hard pressed to conclude that the government, through its student lending process, provides students with an explicit advantage in the job search process in practice, nor indeed does any private lender. ISAs, however, have in some circumstances been provided by organizations who provide applicants with résumé assistance and direct access to employer networks. In the world as we observe it, ISAs are in fact investing in applicants in a way that traditional funding sources are not. Indeed, it hardly should be surprising that ISAs invest while traditional lenders do not. Were a traditional lender to be endowed with the same investment technology as an ISA, there would be no incentive for them to use it. Student loan debt is nondischargable through bankruptcy, and thus most borrowers generally are obliged to pay back the same amount of money regardless of their income. Even under income-contingent loan repayment plans, the total amount the borrower is required to repay is capped by the principal of the loan, plus interest, leaving the potential upside to the lender of investing substantially lower than for an ISA provider.

It is possible, too, that job applicants are able to invest on their own in the contact rate, regardless of their method of funding. Services that charge job applicants for the same kinds of improvements to their searching abilities offered by ISAs abound. However, it is likely, that ISA providers may be able to invest in the search process at a cheaper cost than the applicants themselves can, given that they may specialize in such activities, but it is admittedly the case that a student who keeps her entire earnings is more incentivized to invest in her labor-market outcomes than one who keeps only a portion of her wage for herself. However, as long as ISAs can provide investment services more efficiently than students have external access to, or if ISA investment can complement student-led investment rather than substitute away from it, the incentive for such investment is so uniquely aligned between ISA and student that it is impossible to separate the investment from the ISA for the sake of analyzing the overall effects of income-sharing on labor-market outcomes.

Unemployment

A policy-maker may also be interested in the effects of ISAs on unemployment. In our model of the job market, unemployment is frictional. Individuals are unemployed because the search process is not instant; it takes time for applicants to find jobs that they are willing to accept. We have seen that take-home pay is lower among ISA-funded applicants who do not receive investment, and that it may be lower among even those ISA-funded applicants who do. If finding a job takes longer for such an applicant, only to result in a lower take-home pay, then the additional frictional unemployment can be interpreted negatively for the applicant, even if it may potentially be positive overall in conjunction with the provider's share of income. Furthermore, when investment costs are such that the take-home wage of the applicant increases under an ISA, an increase in frictional unemployment does not necessarily end in a worse labor-market outcome for applicants. In view of this, the author does not seek to claim that increases or decreases in unemployment are necessarily good or bad, but instead to simply describe the effects of ISAs on unemployment.

The probability that a debt-financed applicant has not found a job by time t is e^{-Ht}, in which $H := \alpha\left(1 - F\left(w_R\right)\right)$ is the hazard rate of the search process, equal to the probability in a given instant of receiving a wage offer greater than the reservation wage. Similarly for ISA-funded applicants, we can define $H^E := \alpha\left(1 - F\left(w_R^E\right)\right)$ and $H^N := \alpha^N\left(1 - F\left(w_R^N\right)\right)$. The expected length of time until the applicant finds a job, or the expected time for them to remain unemployed, is simply the inverse of the hazard rate, $\frac{1}{H}$, $\frac{1}{H^E}$, or $\frac{1}{H^N}$. An increase in the hazard rate can be thought of as an increase in the speed of finding a job, or equivalently a decrease in unemployment.

Non-investment type ISAs lead to longer unemployment spells than do traditional student loans, and increases in income-sharing lengthen the average time until an applicant finds a job, as shown in the following lemma.

Lemma 2.3.2. *Increasing the income-sharing rate decreases the hazard rate H^E, i.e., $\frac{\partial H^E}{\partial \phi} < 0$.*

The intuition behind this result is straightforward; increasing income-sharing increases the reservation wage of an applicant, making any given drawn wage offer less likely to be accepted. Since the contact rate is being held constant, it takes longer on average for an applicant to find a job that pays sufficiently well, and she remains unemployed longer.

Here we see that for ISA-funded applicants who do not have investments from their provider, it takes longer to find a job on average than a debt-financed applicant, and the job that eventually gets accepted has a lower expected take-home pay on average than that of a debt-financed applicant.

Things are more complicated investment-type ISA funded applicants, however, as the speed of search must decrease when income-sharing is high, though it is also possible for the speed of search to increase, as shown in the next proposition.

Proposition 2.3.2. *Both of the following properties hold for investment-type ISAs:*

- $\lim_{\phi \to \overline{\phi}} H^N = 0$*, in which* $\overline{\phi} = 1 - \frac{b}{\overline{w}}$ *and* $\overline{w}$ *is the supremum of the support of the wage distribution*

- $\lim_{\phi \to 0} \frac{\partial H^N}{\partial \phi} > 0$.

As the income-sharing rate approaches the point for which the applicant would be as well off collecting just the flow value of unemployment as she would be earning the highest possible wage, the applicant's reservation wage approaches this maximal wage $\overline{w}$; there is no incentive for her to take a job that gives her a lower take-home pay than the flow value of unemployment. The probability that any drawn wage offer is higher than her reservation wage approaches zero. However, by assumption 1, the provider is unwilling to increase investment fast enough to keep the hazard rate from collapsing to zero, due to the convexity of the cost function. Thus, in the limit, the applicant never finds a job and remains unemployed forever.

When the income-sharing rate is low, however, it may be possible to have increases in the income-sharing rate actually *increase* the speed of search. When income-sharing is just introduced, the increase in the contact rate speeds up search more than the increase in the reservation wage can slow it down, and unemployment drops as ϕ rises.

College access

College access may not be a labor-market outcome, but is also potentially affected by the presence of ISAs. Given that ISAs supplement the traditional student loan offerings, rather than replacing them, adding ISAs as an option for students cannot reduce college access. However, under some circumstances, the number of students who are able to afford university educations may increase.

Suppose the net cost of college for a given student is K. This cost represents the amount that the student needs to finance through debt or ISAs to meet tuition, fees, and room and board costs, and does not include any money that the student or her family may have saved previously, nor does it include any scholarship or grant money that she may have been awarded. This value varies, then, among even students who attend the same college and who have the same course of study and job-market prospects.

A student is willing to attend college if and only if the cost of college is less than the value of attending. In our model, the value of attending college is exactly the unemployment value of a job applicant, i.e., the value of the stream of take-home pay that the student is minimally willing to accept. Thus, the student will attend college whenever $K < \frac{w_R}{r}$ if she has access to traditional student debt options.

When students have access to ISAs, the value of attending college changes along with the wage prospects. As discussed above, take-home pay goes down with ISAs that do not have investment, so $\frac{w_R^E}{r} < \frac{w_R}{r}$, and students with high net costs of college $K \in \left((1-\phi)\frac{w_R^E}{r}, \frac{w_R}{r}\right]$ are willing to attend college *only* through traditional debt, and would not attend if ISAs were the only available method of financing their education.

However, when ISAs have opportunities for investment, and investment costs are low, it is possible for $(1-\phi)w_R^N > w_R$. In this scenario, the existence of ISAs allows students who would ordinarily not attend college due to high costs to find college newly valuable. For $K \in \left(\frac{w_R}{r}, (1-\phi)\frac{w_R^N}{r}\right]$, students are only willing to attend college with ISAs, and the existence of ISAs with investment increases the number of students who have access to higher education. Because K, the net cost of college of a given student, is likely to be highest among students with the lowest socioeconomic status, investment-type ISAs may provide the greatest benefits to those who have the worst opportunities without them.

2.4 Optimal policies

The analysis to this point takes the choice of financing method as given. In this section, the choice of a student to accept an income-share agreement is endogenized, allowing an understanding of who is likely to benefit from ISAs and to design both the optimal income-share agreement for students and the optimal income-share agreements for providers.

At any given income-sharing rate, a student who chooses to accept an ISA of any amount should opt to accept as much money as the provider is willing to give. Conditional on the percentage of income offered to the provider, the repayment amount that a student is obligated to provide is independent of the amount received upfront, and both the initial sum provided by the ISA and the balance of any debt that the student may have are irrelevant to labor-market outcomes, so there is no reason for the student to turn down money offered to them. It is of vital importance, then, to ask how much money an ISA provider is willing to contribute to any given student. *Ex-ante*, the provider values an income-share agreement at $\phi \frac{w_R^E H^E}{r+H^E}$ if they cannot invest, and at $\phi \frac{w_R^N H^N}{r+H^N} - c\left(\alpha^N\right)$ if he can invest. Thus, a non-investment type ISA provider is willing to provide up to

$$K^E := \phi \frac{w_R^E H^E}{r + H^E} \tag{2.4.1}$$

and an investment-type ISA provider is willing to provide up to

$$K^N := \phi \frac{w_R^N H^N}{r + H^N} - c\left(\alpha^N\right). \tag{2.4.2}$$

When should a student prefer an ISA to debt? The preference over funding options depends on several factors, primarily, the income-sharing rate ϕ and the student's individual net cost of college K. The overall choices students make are characterized in the following two propositions.

Proposition 2.4.1. *When ISAs do not include investment, a student's choice of financing is characterized by the following. For* $\phi \in [0, .5]$,

$$\begin{cases} K \leq \frac{w_R - (1-\phi)w_R^E}{r} & \textit{Debt only} \\ K \in \left(\frac{w_R - (1-\phi)w_R^E}{r}, K^E\right] & \textit{ISA only} \\ K \in \left(K^E, \frac{(1-\phi)w_R^E}{r}\right] & \textit{Maximal ISA, remainder debt} \\ K \in \left(\frac{(1-\phi)w_R^E}{r}, \frac{w_R}{r}\right] & \textit{Debt only} \\ K > \frac{w_R}{r} & \textit{No college.} \end{cases}$$

For $\phi \in (.5, \overline{\phi}]$, *the choice is instead characterized by*

$$\begin{cases} K \leq \frac{w_R - (1-\phi)w_R^E}{r} & \textit{Debt only} \\ K \in \left(\frac{w_R - (1-\phi)w_R^E}{r}, \frac{(1-\phi)w_R^E}{r}\right] & \textit{ISA only} \\ K \in \left(\frac{(1-\phi)w_R^E}{r}, \frac{w_R}{r}\right] & \textit{Debt only} \\ K > \frac{w_R}{r} & \textit{No college.} \end{cases}$$

Students prefer ISAs if the take-home pay that they forfeit through a worse labor-market outcome is less than the amount of debt that they forgo by accepting the ISA. However, there are two problems that prevent ISAs being able to simply replace debt for high-cost students: providers are only willing to provide up to K^E, and the overall value of the labor-market is lower for ISA-funded students than debt-financed students, so the highest-cost individuals *cannot* benefit from ISAs without investment.

When income-sharing is particularly extreme, providers are willing to provide a larger sum of money than the student herself takes home, so the scenario in which the student fully exhausts ISA funds but still requires debt to meet the cost of college ceases to exist.

When an ISA is able to invest, then there are some ϕ for which take-home pay is larger with the ISA than without, so students always prefer funding with an ISA as much as possible.

Regardless of whether or not investment is possible, income-share agreements are chosen by some students. In particular, even when investment does not increase take-home pay, all students facing a cost of college between $\frac{w_R-(1-\phi)w_R^E}{r}$ and $\frac{(1-\phi)w_R^E}{r}$ use ISAs to some degree in their optimal financing package. Furthermore, we can look at how changing the income-sharing rate affects how many students choose ISAs.

Proposition 2.4.2. *For non-investment type ISAs, the measure of the interval of costs such that students use an ISA decreases for all $\phi \in (0, \overline{\phi}]$.*

As ϕ increases, debt becomes a relatively better alternative to ISAs, increasing the minimum cost of college necessary before a student chooses an ISA. The overall value of college decreases with increases in the income-share, so some ISA funded students are required to substitute away towards debt, causing the set of students who fund using any ISA to shrink overall as ϕ goes up.

The optimal ISA

Now that the effects of income-share agreements have been fully characterized, along with the scenarios under which students elect to use them, it is possible to design the optimal ISA. It is impossible to consider the optimal income-sharing rule ϕ on its own, as changing ϕ changes whether or not a given student accepts an ISA, so it is important to balance the intensive-margin welfare effects of changes in income-sharing with extensive-margin effects of students entering or leaving behind ISAs.

A policy designer maximizing student welfare or one maximizing provider welfare wants to choose the ideal income-sharing rate *conditional on the student choosing an ISA.* A student elects to use an ISA only when ISAs provide her with higher welfare than does traditional debt. Since changing ϕ does not affect the welfare implications of the debt-financed labor market, if any ϕ can cause the applicant to use an ISA, the student's welfare is necessarily higher than it would have been under debt-only financing. Similarly, ISA providers only earn revenue if students use their services.

Indeed, if a ϕ can be chosen that allows a student to fully-fund her education with an ISA, fully funding through an ISA must be the student-welfare optimizing choice.

Proposition 2.4.3. *Let*

$$w^* = 2b + \frac{\alpha}{r} \int_{w^*}^{\infty} 1 - F(w)\, dw.$$

Then, when investment is not possible, there exists $\bar{K} \in (0, \frac{w^*}{2r}$ *such that*

- *Any* K *such that* $K \leq \bar{K}$ *can be fully funded by an ISA, and is in the optimal ISA*
- *For any such* K*, the optimal* ϕ *for students satisfies* $K = K^E$
- *For any such* K*, the optimal* ϕ *for providers satisfies* $rK = w_R - (1-\phi)w_R^E$
- *The student-optimal* ϕ *is lower than the provider-optimal* ϕ
- *For* $K \in \left(\bar{K}, \frac{w_R}{r}\right)$*, the optimal* ϕ *for students satisfies* $rK = (1-\phi)w_R^E$
- *For* $K \in \left(\bar{K}, \frac{w_R}{r}\right)$*, the provider is indifferent between all possible choices of* ϕ*.*

When fully funding education through an ISA, the student-optimal income-sharing rate is the lowest rate required to allow full funding, while the provider-optimal income-sharing rate is the highest such rate. On the other hand, if it is impossible to fully fund through an ISA, the student actually prefers a high income-sharing rate. The student has to borrow any remaining cost after the payment she receives from the ISA provider, and the provider is willing to give the student up to the amount she expects to receive from the student in the future. Thus, a student who fully exhausts her ISA options can be thought of as receiving the entire reservation wage, rather than just the take-home portion, as the portion provided to the provider directly corresponds to debt the student does not need to take out. Increasing the income-sharing rate increases the reservation wage, so the student-optimal income-sharing rate would be the one that maximizes the reservation wage while ensuring that the student is able to earn enough to fully pay for her education.

2.5 Conclusion

Income-share agreements have grown in importance in higher education over the last several years, but they remain understudied. In this paper, I have modeled how income-share agreements can affect labor-market outcomes through several different measures, and have shown that these agreements have far reaching impacts that differ from traditional lending in significant ways. As ISA providers continue to expand nationwide, it will be important for regulators, financial aid officers, parents, and above all students to understand ISAs' differences from student debt, so that students can be assured of their individually best options.

My results show that ISAs may vary by type, but that the most common form of ISAs is likely to result in lower take-home pay and longer job searches for the average job applicant. Importantly, ISAs that invest in job applicants can help applicants both find jobs faster and find jobs that pay more than their debt-financed peers. This alignment between the incentives of provider and student is a valuable aspect of the ISA environment, and it would behoove both regulator and student to approach investment-type ISAs differently from those ISAs that do not invest in applicants.

Even for income-share agreements that do lower take-home pay or decrease the speed of finding jobs, students often can benefit through a reduction in the size of their traditional loan obligations. In fact, optimally-chosen personalized ISAs are part of the optimal financial-aid package for every student who wishes to attend college, with students who need relatively little aid being able to fully finance their education with ISAs and students with the greatest need being able to partially substitute away from debt. For ISAs *with* investment, students may prefer standardized, rather than personalized, ISAs to debt even when their need is very low, and students with the greatest need may be able to access higher education despite being unable to do so before. ISAs of all types can be beneficial to have in the financial-aid toolkit, but the most helpful ISAs are the ones that come with the most focus on the individual needs of the student, providing optimally chosen, personalized income-sharing rates or direct investment in the job-market process.

There are likely effects of ISAs that are left out of the analysis in this paper. All decision-makers in this model are risk-neutral, which means that the impact of different risk attitudes among students on selection into ISAs is not discussed. It is reasonable to expect that greater levels of risk aversion would be associated with higher take-up of ISAs relative to debt, since income-share agreements offer a type of insurance against bad labor-market outcomes that is missing from most debt repayment plans. Such a mechanism would suggest that my results likely undervalue the usefulness of ISAs as an option students could benefit from. On the other hand, ISAs may attract adverse selection and moral hazard, in that students who expect to be less productive employees may see ISAs as particularly cheap pathways to affording higher education or that students who have ISAs may put less effort into their schoolwork, their job search, or their job in light of the fact that they do not get to take home the full value of their productivity. This may make firms less willing to hire an ISA-funded applicant at any given wage. While my model cannot capture this dynamic, I conjecture that a reduction in the wage a firm is willing to pay for an ISA funded student would reduce the usefulness of ISAs in serving as effective financial aid. A more thorough treatment of income-sharing including the effects of moral hazard and adverse selection is left for future research.

Chapter 3

Communities, Co-ops, and Clubs: Social Capital and Incentives in Large Collective Organizations[1]

> If the members of a large group rationally seek to maximize their personal welfare, they will *not* act to advance their common or group objectives unless there is coercion to force them to do so, or unless some separate incentive, distinct from the achievement of the common or group interest, is offered to the members of the group individually on the condition that they help bear the costs or burdens involved in the achievement of the group objectives.
> —Mancur Olson, The Logic of Collective Action: Public Goods and the Theory of Groups (1971)

Organizations where members share access to a collectively produced common good are ubiquitous: e.g., communities, co-ops, clubs, and teams. Such collectives are often egalitarian in the sense that output is shared more or less equally among members. This can occur either for technological reasons (when a team wins, all its members enjoy the victory), or for ideological reasons (the output of communal farms such as the Israeli Kibbutzim or North-American Hutterite settlements customarily accrues equally to all adult members Van den Berghe and Peter (1988); Abramitzky (2008)). It is, therefore, often either infeasible or undesirable to provide incentives to members of a collective through explicit monetary channels. Rather, the organization may track the performance of individual members and may use a variety of non-pecuniary incentives such as rewarding good performance with

[1]This chapter was produced in collaboration with Aaron M. Kolb and Curtis R. Taylor

respite or prestige and punishing poor performance with peer pressure or — in extreme cases — expulsion Freeman et al. (2008).

In this paper we analyze a continuous-time organization design problem where a social planner must manage incentives for a continuum of agents — the members — who may make costly contributions to each other's payoffs in the form of contributing to the commonly consumed good. Such effort is not observed directly, but rather with noise, and thus each agent's performance is stochastic. To provide incentives, the designer maintains a (possibly informal) reputation system tied to agents' performance, whereby agents who hit the bottom of the reputation scale are expelled from the organization and agents who hit the top *must* be allowed a respite, that is, to shirk for a nonnegligible amount of time. At the top of the scale, an agent's situation can improve no further, and thus it is impossible to incentivize effort. Between these extreme ends of the scale, agents are expected to exert full effort. Because expulsions and respites are both socially inefficient, each agent's reputation is linked to his stochastic output with the minimal sensitivity consistent with incentive compatibility.

Our main insight is that collective organizations feature a particular dynamic feedback effect, and thus their optimal design often involves a novel trade-off not present in standard principal-agent settings. In collective organizations, the value of being a member is endogenously determined as a function of the efforts of all other members. Absent monetary transfers, such an organization can mimic the effect of bonus payments for good performance by allowing agents to shirk occasionally; however, shirking by some agents reduces the value to *all* agents of being in the organization. In turn, this reduces members' fear of being expelled for bad performance, further undermining incentives.

We model organizations as having an exogenous inflow of new members, which is meant to capture the fact that individuals hear about the organization or develop an interest in it over time. The instantaneous effort of each member of the organization is confounded with normally distributed noise, and thus a given agent's performance and continuation payoff are Brownian diffusions. Given the law of motion of agents' continuation values under the reward and punishment scheme described above, we employ methods in stochastic processes to derive the steady state in which the density of organization members at each level of reputation (or continuation utility) remains constant over time. The characteristics of the steady state distribution are governed by two parameters: the highest attainable level of continuation utility, w^*, and the level at which new organization members are inserted, w^0. We then use this steady state to frame the designer's problem directly, which — as discussed further below — may involve a variety of different objective functions.

The steady state distribution exhibits several critical features. First, by definition, the continuation value $w = 0$ is a *resetting boundary* in the sense that the flow of agents being expelled from the organization at that point must equal the flow of new agents joining at the higher level w^0. Second, w^* is a *sticky boundary* or *slow-reflecting barrier* in the sense that an agent reaching this level accumulates a positive (finite) expected measure of time there. Moreover, an agent exerts low effort — *shirks* — if and only if his continuation utility is exactly w^*, and otherwise he provides high effort — *works.* Together these observations imply that the steady state distribution possesses a positive mass of agents at the very top of the scale who are rewarded with a respite from working. Importantly, because we are studying an organization where efforts have externalities, these reward periods affect all members and their incentives by reducing their collective flow benefits. For the organization to be sustainable, agents must earn a positive continuation payoff from membership.[2]This places a natural limit on the fraction of members who feasibly can be permitted to free-ride

[2]Our participation constraint assumes that agents are free to exit the organization at will. This was not the case, for example, in the collective farms of China in the 1960s and 70s, where peasants, who would have preferred to migrate to cities, were compelled to live and work on their assigned rural plantations Zhao (1999).

at any given instant.

Our primary goal is to investigate a social planner's organization design problem; namely, we allow her to choose w^* and w^0 subject to feasibility constraints. To build intuition, and as an intermediate step in solving the problem, it is useful to consider a relaxed problem whereby the planner promises agents a fixed, positive flow payoff from membership in the organization. This results in a bounded set of design parameters, w^0 and w^*, for which the organization generates enough output to cover the promised flow payments. We show that when the principal increases w^0, the starting value to new agents, both the mass of working agents and the mass of shirking agents increase, and in fact, the latter effect dominates with respect to the fraction of agents shirking; the planner may, therefore, be forced to limit the starting utility assigned to new members in order for generated output to cover the promised flow to all members. Increasing the maximum continuation value w^* means promising later, but more frequent, periods of respite; this change in the organization also increases both the mass of working agents and the mass of shirking agents. The latter effect eventually dominates, and the average output of the organization falls. Hence, the principal must also limit the maximum continuation value to agents in the organization, which in turn limits their expected tenure and ultimately the overall size of the collective.

We then consider two possible objective functions for the planner. For instance, a planner who generates revenue primarily through advertising (e.g., a social network) would want to maximize the size of the organization. In this case, the objective is increasing in both w^* and w^0: increasing these values delays the time at which agents are expelled, which leads to a larger organization in the steady state. The feasibility constraint then forces the planner to trade-off higher w^0 with higher w^*. On the other hand, a social planner may — as in a partnership or co-op setting — want to maximize the per-capita output of members, which would involve maximizing the fraction of agents working.[3]We

[3]The literature on the neoclassical theory of labor managed firms uses the term *Illyrian firm*

show that this objective leads to a vanishingly small organization; members are promised an arbitrarily small continuation value with virtually no hope of earning an opportunity to free-ride before being dismissed. Whatever her objective, we note that by identifying continuation utilities with reputation scores, the planner's problem can be thought of as the design of an optimal reputation system.

We discuss applications of our model and related literature in the next section. In Section 3.2 we introduce the formal model along with a broad overview of the steps involved in the analysis. In Section 3.3 we solve for the steady state distribution of continuation utilities (i.e., reputations) of the organization members for arbitrary values of the policy instruments. Section 3.4 characterizes the feasible set of choice variables. In Section 3.5 we discuss the principal's organization design problem for the two possible objectives mentioned above. We summarize our findings and outline some directions for future work in Section 3.6.

3.1 Applications and Related Literature

3.1.1 Applications

Our setting has three key features that distinguish it from standard dynamic contracting models: (i) agents decide whether to exert effort, which has positive externalities, (ii) the organization designer has limited incentive instruments, namely respite and expulsion, to incentivize agents to exert effort, and (iii) the designer chooses how to use these incentive instruments, taking into account how agents will behave in response. Numerous organizations exhibit such traits to some degree.

to refer to those which maximize *dividends* or net revenue per worker Ward (1958); Law (1977). However, see our discussion of the objectives actually pursued by worker co-ops at the end of the following subsection.

Feature (i) can arise in organizations in two ways. First, in some organizations, agents' efforts contribute to a common good which is divided evenly among all members. As noted above, in farming collectives such as the kibbutzim and Hutterite settlements, output — or the proceeds from selling output — is generally shared equally among all adults Van den Berghe and Peter (1988); Abramitzky (2008). Likewise, when members of a club exert effort to promote an event or engage in fundraising activities, all members of the club benefit. Second, in some organizations, such as ride-sharing or room-rental platforms, agents are paired with one another in short-term matches, and each agent's effort benefits solely her current partner. For example, in the case of a ride-sharing platform, positive externalities within a match are associated with the efforts of both drivers and passengers to be prompt and courteous.

Feature (ii) is motivated by the observation that transfers are notably absent as incentive instruments in many organizations. The kibbutzim and Hutterite settlements, having been founded on egalitarian ideals, are opposed to transfers by their very nature. Sharing economy platforms typically do not use bonus payments based on ratings to incentivize their users.[4]

Feature (iii) forms the essence of organizational design. For instance, Uber maintains a five-star rating system for both drivers and passengers, with 5.0 being the maximum possible rating. In September 2018, Uber introduced a policy in Australia and New Zealand, where the average user's rating is 4.5, of deactivating users whose ratings fall below a minimum level Cherney (2018). In an example where the community is interpreted at a national scale, China has introduced a social credit rating system for all its citizens which takes into account both financial behavior (such as credit-card payments) and social behavior (such

[4]To be sure, ride-sharing platforms such as Lyft, Gett, Juno, Uber and Via, do allow passengers to tip drivers through their apps, but a recent survey of over 2600 active Uber drivers revealed that "...tip income was negligible in the majority of cases." Wong (2018)

as volunteer activity and adherence to family-planning limits); this system integrates with already existing blacklisting systems used to restrict activities such as loans and travel Chin and Wong (2016). In both of these examples the function of the rating system is to track performance and incentivize good behavior as defined by the designer.

Perhaps the best overall application of our model is to a worker cooperative or labor-managed firm (LMF).[5]These entities are generally founded on egalitarian principles and therefore exhibit a high degree of equality in pay among members regardless of seniority or skill differentials (see Chapter 6 in Bonin and Putterman (2013) for numerous examples and insightful discussion). Thus, a typical LMF produces output that is shared more or less equally by its members and does not provide incentives through explicit monetary channels. Monitoring in modern LMFs is often performed by peers where: "most workers say that they can detect fellow employees who shirk ... and many report that they would speak to the shirker or report the behavior to a supervisor" (Freeman et al., 2008, p. 1). While termination appears to be more rare in LMFs than in conventional firms Alves et al. (2016), the ultimate threat of dismissal still plays an important incentive role in a variety of co-op settings Albanese et al. (2017). This is especially true for junior or candidate members who often face a probationary period before becoming vested members of the enterprise. However, "shareholders can also be fired for repeated malfeasance" (Craig and Pencavel, 1992, p. 1084). For instance, Bylaw 21 of the shareholder's manual issued by the Fort Vancouver Plywood Company, Inc. provides:

> The board of directors at any regular or special meeting shall have the power by a majority vote to remove from working status any shareholder-worker whom

[5]The most authoritative empirical research on worker cooperatives in the US is Craig and Pencavel (1992) who investigated the behavior of the largest and most durable LMFs in US manufacturing, the plywood firms in the Pacific Northwest. Three other examples of historically prosperous co-ops, worker-owned scavenger companies, taxi cooperatives, and professional partnerships such as legal or accounting firms, are discussed by Russell (1985). Non-profit hospitals were modeled as physician cooperatives by Pauly and Redisch (1973), and university academic departments were modeled as faculty co-ops by James and Neuberger (1981).

> they shall find to be physically or mentally unfit for such work, or who refuses to do his work as outlined by the management.[6]

While it is difficult to document explicit respite policies for good performance as such, (Craig and Pencavel, 1992, p. 1085) report, "Job assignments are varied and sometimes rotated, although, if a particularly attractive position opens up, its allocation is determined by seniority or previous work performance." Thus, the structure and internal governance of LMFs appear to square remarkably well with the three distinguishing features of our model: team production, non-pecuniary incentive instruments, and implementation of incentives via imperfect dynamic monitoring.

The question of *optimal* organization design presupposes an objective function for the designer, and indeed, there has been considerable theoretical debate about the objectives of LMFs ever since the seminal publication by Ward (1958).[7]To address the question empirically, Burdín and Dean (2012) use panel data from 31 Uruguayan industries to estimate the relative weight worker-managed firms place on total employment (firm size) versus dividends (net revenue per worker). Interestingly, the LMFs in their data appear to place relatively high weight on the size of the enterprise, employing systematically more workers than would comparable profit-maximizing firms. In a sense, this observation also

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[6]See `http://courts.mrsc.org/appellate/024wnapp/024WnApp0120.htm`. There is evidence that this provision was indeed sometimes used to dismiss vested members of the cooperative who were detected shirking. In one such case a dismissed worker's foreman testified in court:

> Well, he was placed on the 8-foot green chain during the three days. I observed his work habits and they were very poor. And I received numerous complaints from the other green chain workers. And so I told Mr. McIntyre two or three times he was going to have to improve, work harder, just show more initiative. The following two days I didn't see any initiative at all. The other four guys had to do all the work. And I had gone to the superintendent on a couple of times and asked him to come down and watch and see what kind of problems I had. So he did come down and observed his work habits a couple times before we decided on the pink slip.

[7]The objective functions we consider, namely organizational size and per capita output, are analogous to the two sides of this debate.

squares with our results. Specifically, we show in Proposition 3.5.2 that an organization that endeavors to maximize per capita output without placing any weight on employment must be arbitrarily small.

3.1.2 Literature

Besides the literature discussed in the previous subsection, this paper also contributes to several additional lines of research. The first, on the economics of clubs and other collective organizations more generally, begins with the classic works of Buchanan (1965) and Olson (1971), along with Helsley and Strange (1991), Scotchmer (1985), and Oakland (1972) among others, dealing with optimal group membership, size, and fee structure.

A second literature to which we contribute, on the use of ratings and reputation for incentive compatibility, begins with Holmström (1999) and includes — among others — a recent working paper by Hörner and Lambert (2016). This literature investigates the use of reputation as a means for eliciting the rated agent's cooperation in a setting where the principal has imperfect control over compensation. In order to maintain a high reputation — and thereby a high continuation payoff — agents are required to produce a stream of signals reflective of high effort.[8]Using different terminology, Olszewski and Safronov (2018) study the use of *chips* for incentives in a favor-exchange game between two players.

Another branch of related research studies continuous-time optimal contracting in the context of corporate finance. The pioneering article in this literature is DeMarzo and Sannikov (2006) (hereafter DS), which was followed by a number of related works including Sannikov (2008), Zhu (2013), and Grochulski and Zhang (2016).[9]Each of these papers investigates variations of the DS baseline model which we, too, adapt to our setting.

[8]An alternative use of ratings is studied by Bonatti and Cisternas (2018).

[9]See also the literature on folk theorems in continuous time with imperfect monitoring; e.g., Sannikov (2007), Peski and Wiseman (2015), and Bernard and Frei (2016).

Specifically each of them considers a single agent who may take an action either to produce output or to benefit himself, and solves for the optimal path of continuation values in order to maximize output. DS implement their optimal contract by way of basic financial instruments, while the others abstract from implementation considerations.

We also contribute to a recent burgeoning literature on dynamic incentives in the absence of monetary transfers. For instance, Li et al. (2017) study the dynamic allocation of power between a principal, who has formal authority in the organization, and an agent, who has private information regarding the current set of available projects. Under the optimal relational contract, the agent recommends a project to be completed each period. The more frequently he recommends the principal's favorite project rather than his own, the more continuation utility the agent accumulates. High values of continuation utility are associated with more power in the organization in the sense that the principal is obliged to accept the agent's recommendations with greater frequency. The authors show that this process ultimately converges to one of two (inefficient) absorbing states: a maximum level of continuation utility for the agent that is implemented by always accepting his recommendations or a minimum level of continuation utility that is implemented by never listening to him. Another paper in this vein is Lipnowski and Ramos (2018) who explore delegated authority in an infinite-horizon game with imperfect monitoring. In this setting a principal wishes to fund good projects and not fund bad ones. An agent is privately informed about the state of each project, but prefers the principal to fund them all. In the initial phase of the relationship the agent is granted considerable authority to initiate projects and he accordingly directs the principal to fund all good ones as well as a significant fraction of those that are bad. As time progresses, the agent's goodwill eventually runs dry — the equilibrium of the game enters a phase where the principal rarely delegates authority to the agent, and when she does, the agent recommends only good projects. A third recent paper in this line is Guo and Hörner (2018) which investigates the limits to efficient dynamic allocation in the presence of private persistent information. In this model, a social planner with full power of commitment wishes to supply a perishable good to an agent in those periods when his valuation exceeds the constant provision cost. The agent's valuation is always positive but is not observed or learned, even imperfectly, by the planner. As in Li et al. (2017), the optimal incentive compatible mechanism eventually converges to one of two antipodal (inefficient) situations: the agent either receives the good in every period *ad infinitum* or he never receives it again.

While we too consider dynamic incentives in the absence of monetary transfers, our focus differs from the three papers just outlined in several key respects. First, we consider a continuum of agents who interact with each other rather than focusing on providing incentives for a single agent in isolation. Second, the designer in our setting is interested in the optimal structure of the collective organization as represented by the steady-state equilibrium distribution of continuation utilities of its members. Finally, we study a hidden action model in continuous time rather than a hidden information model in discrete time.

Of course, our paper also contributes to the organizational economics literature. Specifically, because each agent in our model receives a gross expected payoff related to the effort provided by other agents, our setting resembles a dynamic *partnership* in which output is divided equally among the members of the organization as in Farrell and Scotchmer (1988) and Levin and Tadelis (2005), models of favor-trading as in Hauser and Hopenhayn (2010), and of dynamic partnership rematching as in McAdams (2011). This partnership aspect and the methods we use to derive and analyze the steady state are the key features that distinguish our model from other recent work on dynamic relational contracts such as Andrews and Barron (2016).

Finally, Acemoglu and Wolitzky (2018) model a society as a population of agents who can exert effort with positive externalities and who may be punished for exerting low effort; some fraction of the population are elite types who are less vulnerable to this punishment. The authors show that elites may nonetheless prefer to receive equal punishments — so-called equality under the law — as this increases their joint effort which then increases the effort of non-elites. In contrast to their model, ours features imperfect monitoring, which gives rise to an equilibrium distribution of continuation payoffs and effort dynamics that underlie the organizational design problem. Acemoglu and Wolitzky (2018) also assume the existence of an explicit punishment technology, while incentives in our setting are generated either by letting agents enjoy the public good without exerting effort (respite) or by cutting off their access to the public good entirely (expulsion).

3.2 Setup

Time is continuous over an infinite horizon. At each moment there is a positive measure of massless agents present in an organization. Agents, the members, are indexed by a continuous variable i. Each agent is risk neutral and discounts the future at rate r. A member who *exits* the organization receives a payoff of 0 from that point forward, and thus the rule that specifies when to remove an agent from the organization will be a key component of organization design. A flow of new members $\psi > 0$ join the organization at each instant; this will generate turnover of members while allowing us to use steady state methods.

While remaining in the organization, each agent receives a flow utility u which will, in equilibrium, be generated by the collective actions of all members. For now, we can assume u is a constant, exogenously determined flow payoff. At each instant, each agent i chooses an effort level $e^i \in \{H, L\}$, where we refer to choosing H as *working* and L, *shirking*; thus each agent i chooses a stream of effort levels, which is a stochastic process $(e^i_t)_{t\geq 0}$. The flow cost of effort is $c(e^i_t)$, defined by $c(H) = c > 0$ and $c(L) = 0$. Thus, high effort has a flow cost $c > 0$ and low effort has no cost. An agent's effort generates an output stream of contributions to the common good given by a Brownian diffusion

$$dX^i_t = (\mu_{e^i_t})dt + dB^i_t, \tag{3.2.1}$$

where we assume that $\mu_H - c > 0 > \mu_L$, so that high effort is efficient and low effort is not.

The diffusion process X^i admits multiple interpretations. First, in some instances, each agent's output may be clearly delineated from others'; for example, if each agent in a farming collective is responsible for an individual plot of land, each agent's output is clearly defined and can be monitored in isolation. Second, even if output is not clearly delineated — agents may be working in groups and their output jointly determined — the principal might have means of monitoring individual agents' efforts. Indeed, although we follow the convention of calling X^i the *output* stream of agent i, it can also be interpreted as a stream of signals about agent i's unobserved effort. Third, agents might observe (signals of) one another's efforts and report these to the principal. For example, in ride-sharing services, agents are randomly matched and observe noisy signals of their match's effort. With a large number of agents, repeated interactions are rare, and agents may be willing to truthfully report their signals to the principal.

Since agents with the same continuation utilities are essentially identical, we suppress the index i whenever doing so does not create confusion. Below we also speak of *the agent* with the understanding that we are focusing on a single arbitrary member of the organization. A *contract* in this context specifies: (i) a fixed flow utility u, (ii) a removal time τ and (iii) a recommended effort process e. We consider permanent expulsion at τ. Also, an agent is free to leave the organization at any point, but may not rejoin.

In order to implement a contract, the principal assigns each agent a score or *reputation* process, S. Each incoming agent begins with some initial reputation level, and his reputation evolves thereafter according to his output stream. As each agent is motivated solely by the evolution of his continuation payoff, the designer may simply set the reputation process S for each agent equal to his continuation payoff process W. In particular, (i) an agent is removed from the organization when his reputation reaches $w = 0$, (ii) new agents are granted a continuation payoff of $w^0 > 0$, and (iii) since flow payoffs are bounded above and agents discount the future, there is some maximum attainable reputation level w^*. The distribution of agents at each instant thus is characterized by a population distribution over continuation payoffs. We ultimately will allow the principal to directly choose w^0 and w^* as a part of the organization design problem.

From standard results in continuous-time contracting,[10] the following are known:

- While the agent remains in the organization (i.e., $t \leq \tau$), there exists a process β_t representing the sensitivity of the agent's continuation value to output:

$$dW_t = rW_t dt - (u - c(e_t))dt + \beta_t(dX_t - \mu_{e_t} dt).$$

- The contract is incentive compatible if and only if for all $t \leq \tau$ and $W_t \geq 0$, $e_t = H$ implies $\beta_t \geq \lambda$ and $e_t = L$ implies $\beta_t \leq \lambda$, where $\lambda := \frac{c}{\mu_H - \mu_L}$.

So long as the designer's objective is increasing in effort,[11] removing an agent is inefficient (i.e., on path, agents are removed due to bad luck, not because they were shirking at the moment). Therefore, the principal wishes to minimize volatility and thus minimize the sensitivity β_t subject to incentive compatibility for the recommended effort level; it is optimal to set either $\beta_t = \lambda$ to induce working or $\beta_t = 0$ to induce shirking. Hence, when the agent works, his continuation value evolves as

$$\begin{aligned} dW_t &= rW_t dt - (u - c)dt + \lambda(dX_t - \mu_H dt) \\ &= (rW_t - (u - c))dt + \lambda dB_t. \end{aligned} \tag{3.2.2}$$

When the agent shirks, his continuation value evolves deterministically as

$$dW_t = (rW_t - u)dt. \tag{3.2.3}$$

In particular, the agent shirks whenever his reputation reaches its maximum level $w^* < u/r$, where the drift in (3.2.3) is necessarily downward.[12] For later convenience, we

[10] See DeMarzo and Sannikov (2006) Lemmas 2 and 3 or Zhu (2013) Lemmas 3.1 and 3.2.

[11] The designer's objective may be increasing in effort either directly, if effort enters into the objective function, or indirectly, if effort (of other agents) enters into agents' value of being in the organization.

[12] Recall that u/r is the value an agent would get by shirking yet remaining in the organization forever, and is therefore an upper bound on continuation payoffs. Since we will pose the principal's problem directly in terms of a steady state distribution of agents, we cannot have $w^* = u/r$; otherwise, the drift in (3.2.3) would vanish for $W_t = w^*$, making w^* an absorbing state, leading to an organization that grows over time without bound.

define $\rho(w) := rw - u$ so that $\rho(w^*)$ is the (downward) drift at w^*.

To summarize, the contract terms can be expressed in terms of (u, w^0, w^*), where an individual agent's continuation value process W starts at w^0, it evolves according to (3.2.2) when $W_t \in (0, w^*)$ and (3.2.3) when $W_t = w^*$, and the agent is removed when $W_t = 0$. Formally, the process W belongs to a class of diffusions known as *Sticky Brownian Motion*.[13]

3.2.1 Overview of Analysis

Now that the model has been presented, it is helpful to outline the four steps we use to solve the organization design problem.

1. *Determine agents' law of motion*: fixing contract terms (u, w^0, w^*), we characterize the evolution of the state variable — i.e., continuation utility — for each agent as a stochastic process. This step has been performed above.

2. *Find steady state distribution*: we next determine a stationary distribution for continuation utilities under the contract terms (u, w^0, w^*), using the law of motion above and taking into account the inflow and outflow of agents.

3. *Determine feasibility*: since agents in a collective organization exert externalities on one another, the flow payoffs u they earn must be consistent with the steady state distribution of agents and their behavior. This requirement determines a feasible set of organization parameters (u, w^0, w^*) over which the designer can optimize.

4. *Optimize organization parameters*: for a given objective function for the organization designer, we determine the optimal values of the design parameters within the feasible set derived in the third step.

[13] See Harrison and Lemoine (1981) and Zhu (2013).

3.3 The Steady State

A steady state corresponds to a stationary distribution of agents over all possible levels of continuation payoff, as poorly performing agents drop out of the organization, new agents arrive, and as the continuation value of all agents within the system evolve in response to their Brownian output streams. Essentially, a stationary distribution is a distribution of agents' continuation values in $[0, w^*]$ in the organization which is constant over time; any movement of agents away from a particular continuation value (up or down) is exactly offset by movement toward that continuation value by other agents. As we will be interested in the total mass of agents in the organization as well as their distribution, we do not require that the stationary distribution integrate to one. Mathematically, the steady state in our setting is equivalent to a rescaling of the stationary distribution of a process W defined on the interval $[0, w^*]$ as follows:

- When $W_t = 0$, it immediately resets to w^0.
- For $W_t \in (0, w^*)$, it evolves as (3.2.2).
- For $W_t = w^*$, it evolves as (3.2.3).

In other words, the process undergoes resetting (as exiting agents are replaced with new agents) and slow reflection (as agents at the top of the distribution are permitted to shirk).[14]The stationary distribution is scaled so that the flow rate of mass at 0 is ψ, the inflow of agents into the organization.

Proposition 3.3.1 fully characterizes the distribution of continuation utilities in a steady state for a given specification of the promised flow payoff u, the highest achievable

[14]A combination of resetting and (partial) reflection arises Kolb (2019), where the underlying process is a seller's reputation.

continuation utility w^*, and the level at which new agents are admitted to the organization w^0. The proof is given in the appendix.

Define $\gamma(w) := \frac{rw-(u-c)}{\lambda\sqrt{r}}$ and $\text{erf}\{x\} := \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$, the Gauss error function.

Proposition 3.3.1. *In a steady state, the distribution of agents consists of two densities f_-, f_+ and an atom $\nu\{w^*\}$ given by*

- $f_-(w) = e^{\gamma(w)^2}\frac{\psi\sqrt{\pi}}{\lambda\sqrt{r}}\left[\text{erf}\left\{\frac{u-c}{\lambda\sqrt{r}}\right\} + \text{erf}\{\gamma(w)\}\right]$ *for* $w \in [0, w^0]$*;*
- $f_+(w) = e^{\gamma(w)^2}\frac{\psi\sqrt{\pi}}{\lambda\sqrt{r}}\left[\text{erf}\left\{\frac{u-c}{\lambda\sqrt{r}}\right\} + \text{erf}\left\{\gamma(w^0)\right\}\right]$ *for* $w \in [w^0, w^*)$*;*
- $\nu\{w^*\} = \frac{\lambda^2 f_+(w^*)}{2(u-rw^*)}$.

In words, the distribution consists of a density function composed of two segments (with a kink where they meet at w^0) as well as a mass point at the top of the support, w^*; Figure 3.1 illustrates.

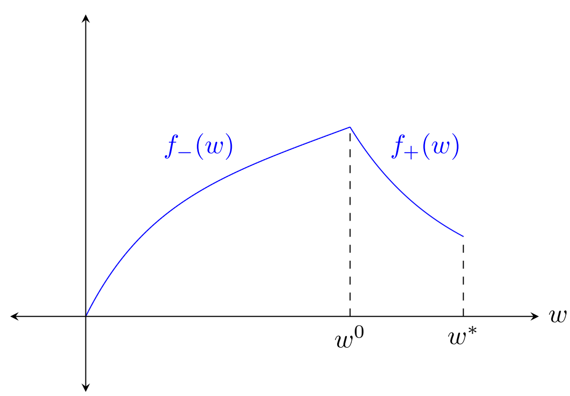

Figure 3.1: Steady state distribution of agents (omitting the atom at w^*).

We make several observations about the stationary distribution. In discrete time, the continuation value of each agent follows a random walk. At generic states $w \notin \{0, w^0, w^*\}$, agents "leaving" the continuation value w are immediately replaced by agents moving up from $w - dw$ or down from $w + dw$. In continuous time, this is stated by the Kolmogorov forward equation (B.1.1), which implies the functional form for the densities $f_-(w)$ and $f_+(w)$ up to two constants. The difference in slopes at the kink at w^0 is due to the exogenous inflow of new agents and must match the outflow of agents at $w = 0$.

The third item of Proposition 3.3.1 is the most important economically. First, note that there is a strictly positive measure of agents at w^* who are shirking; since the process W is a sticky Brownian motion, there is a positive measure of times t at which $W_t = w^*$, and thus the stationary distribution involves an atom of mass at exactly w^*. This means that there is a nontrivial fraction of agents in the organization who are shirking at any time. Second, the size of this atom $\nu\{w^*\}$ is proportional to the "stickiness" $u - rw^*$ of the process at w^*. The greater the maximum continuation value w^*, the more "frequent" the shirking reward must be delivered to sustain it (the continuation value process is stickier at w^*), and thus the larger the measure of agents who are shirking in a steady state.

Given the steady state for exogenous organization design parameter values (u, w^0, w^*), the question now is what values the principal can and should set.

3.4 Feasibility of Organizations

Prior to this point it has sufficed to interpret u as an exogenous flow payoff that the principal can promise the agents. In this section, we endogenize this flow payoff by imposing the *feasibility* constraint that the principal can only promise what flow value the organization itself produces. We obtain, in closed form, a sufficient condition and a necessary condition for the existence of a feasible organization. Moreover, we characterize the way in which feasibility depends on the organization design parameters, and provide further qualitative results about the set of feasible organizations.

To capture in a tractable model the feature that working agents impose positive externalities on the platform and shirking agents impose negative externalities, we specify that the flow payoff each agent receives is the average output of all the other agents. We offer two interpretations for such payoffs. One interpretation is that all agents contribute their efforts high or low to a common good which gets divided evenly among all agents in the organization, and thus agents obtain a flow value equal to the average flow output of its members. A second interpretation is that agents in the organization are randomly and instantaneously matched with one another, and agents obtain a flow value equal to the output of their partner. In expectation, each agent receives the weighted average output of other agents.

Since the average output of all agents is a function of the mass of working agents and mass of shirking agents, and those masses are in turn functions of the promised flow payoff u, the latter is now a fixed point.

We now formally analyze feasibility. Define the measure of agents active (i.e., working) in a steady state by

$$\alpha(u, w^0, w^*) := \int_0^{w^0} f_-(w)\, dw + \int_{w^0}^{w^*} f_+(w)\, dw,$$

and (with some abuse of notation) define the measure of non-active (i.e., shirking) agents at the top of the distribution by

$$\nu(u, w^0, w^*) := \nu\{w^*\}.$$

Denote the fraction of agents working by

$$Q(u, w^0, w^*) := \frac{\alpha(u, w^0, w^*)}{\alpha(u, w^0, w^*) + \nu(u, w^0, w^*)}. \tag{3.4.1}$$

The endogenous flow utility produced when fraction $Q = Q(u, w^0, w^*)$ of agents are working is thus the average output,

$$U(Q) := Q\mu_H + (1 - Q)\mu_L. \tag{3.4.2}$$

where working agents contribute μ_H and shirking agents contribute μ_L.

In order to define feasibility, we state two conditions:

$$0 < w^0 \leq w^* < u/r \tag{3.4.3}$$

$$U(Q(u, w^0, w^*)) = u. \tag{3.4.4}$$

Definition 3.4.1. *An organization* (u, w^0, w^*) *is feasible if* (3.4.3) *and* (3.4.4) *are satisfied.*

The key part of Definition 3.4.1 is the fixed point condition $u = U(Q(u, w^0, w^*))$. This condition says that the flow payoff that each agent receives, which the principal has exogenously promised, must equal the flow payoff generated by the organization. Condition (3.4.3) says that u must be positive, otherwise agents would be better off leaving the organization (individual rationality) than working at all. Moreover, $w^* < \frac{u}{r}$ because the highest level of continuation utility can at most equal the perpetuity value of shirking forever (and as noted earlier, if $w^* = u/r$, w^* becomes an absorbing state). We denote by $\mathcal{S}$ the set of feasible organizations.

3.4.1 Existence and Nonexistence

Since an organization design (u, w^0, w^*) is feasible only if u is generated by the agents in the organization, a feasible organization need not exist. The next proposition ensures that when agents are sufficiently patient, when the cost of effort is sufficiently low, or when effort produces sufficiently high output, a feasible organization exists; that is, there exists a triple (u, w^0, w^*) that satisfies (3.4.3) and (3.4.4).[15]Note that in the extreme case, if agents were perfectly patient, the principal could sustain an arbitrarily large organization with an arbitrarily large fraction of agents working; in that case, the prospect of expulsion alone would incentivize agents to work.

Define

$$r^* := \frac{2(\mu_H - \mu_L)^2}{c} \left(c + \mu_H - 2\mu_L - 2\sqrt{(c - \mu_L)(\mu_H - \mu_L)} \right). \tag{3.4.5}$$

Proposition 3.4.1. *If $r \in (0, r^*)$, then the feasible set is nonempty. The cutoff r^* is increasing in μ_H and decreasing in c, and it satisfies* $\lim_{|\mu_H - c| \to 0} r^* = 0$ *and* $\lim_{\mu_H \to \infty} r^* = \lim_{c \to 0} = \infty$*.*

It is easy to verify that the threshold (3.4.5) is homogeneous of degree two in (μ_H, μ_L, c), and thus the requirement on agents' patience relaxes when these parameters increase by a common multiplicative factor. This fact leads to a model prediction that struggling organizations could benefit from assigning agents tasks of greater importance, for which effort is more difficult but also easier to monitor as the difference between success and failure is larger.

Conversely, we show that an organization *cannot* be sustained if agents are sufficiently impatient. The intuition for this is straightforward. Each agent is motivated to exert high

[15]Given Proposition 3.4.1, whenever we state results which condition on the feasible set being nonempty, it should be understood that this condition applies for a nonempty set of input parameter values.

effort by the threat of eventual removal and by the promise of eventual *vacation*. As he becomes very impatient, the prospect of future sticks and carrots lose their salience, and it becomes impossible to incent high effort. Put another way, when agents become extremely impatient, the principal must reward them with more frequent vacations, but this reduces the overall output of the organization, and eventually there is no positive wage that the principal can promise based on the organization's output.

The following proposition gives a sufficient condition for nonexistence in closed form. For a fixed discount factor, nonexistence obtains when the cost of effort is sufficiently large or the value of high effort is sufficiently small.[16]

Proposition 3.4.2. *If* $r > \frac{2\mu_H(\mu_H - c)}{\lambda^2}$*, there is no feasible organization.*

Note, however, that while the organization must induce a positive flow wage u to prevent all agents from leaving, that wage can be less than c — agents may receive a negative net flow payoff while in the working state, yet prefer to remain in the organization to obtain a positive net flow payoff while enjoying respite. In other words, agents might be "working for the weekend" in the sense that the only instances at which they receive more flow utility being in the organization than out of it are those when they are not working.

Proposition 3.4.3. *For a nonempty set of parameter values, there exists a feasible organization* (u, w^0, w^*) *with* $u < c$*.*

3.4.2 Characterizing the Feasible Set

Having established that the set of feasible organizations can be nonempty, we turn to characterizing this set. As an intermediate step, it is useful to consider a fixed promised

[16]Recall that we have assumed $\mu_L < 0$, so that low effort is inefficient. Without this assumption, there would always exist an (uninteresting) organization in which all agents shirk.

utility u and characterize the set of (w^0, w^*) pairs such that (u, w^0, w^*) satisfies (3.4.3) and $U(Q(u, w^0, w^*)) \geq u$; in other words, it is the set of organizations whose output is at least u when agents receive flow payoff u. We call this the *u-supportive set* and denote it by S^u.[17]

Consider the effect of marginally increasing w^0 for fixed values of w^* and u subject to (3.4.3); the principal implements such a change through later expulsion of agents in a stochastic sense. Since agents enter the organization with higher continuation values, a greater mass of agents in steady state reside at continuation values above w^0, which means there are both more agents in the interval (w^0, w^*) who are working and more agents at w^* who are shirking; that is, both α and ν are increasing in w^0. While it is immediately clear that the organization becomes larger, these forces affect the *average* output of the organization in opposite directions. We show analytically that the increase in shirking dominates, so that the fraction of agents working unambiguously decreases in w^0. This observation implies that the u-supportive set lies below some curve $\bar{w}^0(w^*)$ in (w^*, w^0)-space, defined for w^* in a subset of $(0, u/r)$.

Next, consider fixing u and w^0 while marginally increasing w^*, a change which is implemented by allowing agents to shirk more. This change has no effect on the steady state distribution of agents below w^*, but it expands the distribution to the right, increasing the mass of agents working. The effect of increasing w^* on the fraction of agents shirking is ambiguous; we show that the latter is quasiconcave in the former. For small w^*, say very close to w^0, increasing w^* can improve the average effort in the organization by delaying the time at which new agents get to shirk. On the other hand, as w^* increases, agents must spend a greater amount of time shirking at w^* — the reflecting barrier at w^* becomes stickier, and this eventually outweighs the increase in the mass of working agents. This quasiconcavity implies that the horizontal cross sections of the u-supportive set are intervals.

[17]For visual depictions of the u-supportive set, it is convenient to place w^* on the horizontal axis and w^0 on the vertical axis.

Finally, we note that a "wedge" always exists between the bottom left of the u-supportive set and the 45-degree line. In other words, when new agents start at very low continuation values and are highly likely to be expelled in the very near future, the designer must force them to work for some time before earning respite, or else the output of the organization will not exceed the promised wage.

Figure 3.2 shows the u-supportive set for a fixed value of u.

Proposition 3.4.4. *There exist* $\underline{u}, \overline{u} \in (0, \mu_H)$ *such that if* $u < \underline{u}$ *or* $u > \overline{u}$*, the* u*-supportive set is empty. When it is nonempty, the* u*-supportive set for any* $u \in (0, \mu_H)$ *can be written as* $\{(w^*, w^0) \in \mathbb{R}^2_+ : w^0 \in (0, \bar{w}^0(w^*)]\}$*, where* $\bar{w}^0$ *is a single-peaked function taking values in* $[0, w^*]$*. If* w^* *is such that* $\bar{w}^0(w^*) < w^*$*, then* $\bar{w}^0$ *is continuously differentiable at* w^**. For sufficiently small* $w^0 > 0$*,* $w^* > w^0$ *whenever* (w^*, w^0) *is in the* u*-supportive set.*

From the definitions, if an organization (u, w^0, w^*) is feasible, then (w^0, w^*) is u-supportive. Though not immediate from the definition, a converse to this statement is also true, as reported in the following proposition. The value of these facts are that the u-supportive sets fully determine the feasible set, which aids us in solving the principal's problem in the next section.

Proposition 3.4.5. *An organization* (u, w^0, w^*) *is feasible if and only if* (w^0, w^*) *is* $\tilde{u}$*-supportive for some* $\tilde{u} \in (0, \mu_H)$*. Hence,* $\mathcal{S} = \{(\tilde{u}, w^0, w^*) : (w^0, w^*) \in S^{\tilde{u}}$ *and* $\tilde{u} \in (0, \mu_H)\}$*. There is a nonempty set of parameter values such that* $w^0 < w^*$ *for all* $(u, w^0, w^*) \in \mathcal{S}$ *and* $\mathcal{S}$ *is nonempty.*

The last part of the proposition says that for some parameter settings, the principal must force agents to wait some time before enjoying respite in order to create a feasible organization; in other words, all the u-supportive sets lie below the 45-degree line. We provide a sufficient condition, in closed form, on parameter values for this to be the case.

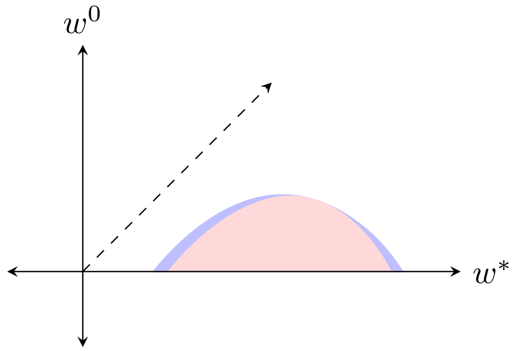

Figure 3.2: A projection of the feasible set (light blue) overlayed with a u-supportive set (light red) for $(\mu_H, \mu_L, c, r, \psi) = (.5, -.5, .14, 1, 1)$ and $u = .46$. The dashed line is the 45-degree line.

Armed with the results of this section, we turn to the principal's problem of choosing among feasible organizations to maximize one of two possible objective functions.

3.5 The Principal's Problem(s)

We now analyze the problem of a principal who seeks to design an organization to maximize an objective $V(u, w^0, w^*)$ over the set of feasible organizations. We focus on two distinct objectives: maximizing the organization's size and maximizing per capita output.

3.5.1 Maximizing Organizational Size

Suppose the organization designer wishes to maximize the total size of the organization. For example, a social networking website might want to attract as many users as possible in order to maximize advertising revenue; whether those users are engaging in high quality interactions with each other is of lesser importance, conditional on their willingness to keep using the platform. The designer's problem is to maximize $V(u, w^0, w^*) = \alpha(u, w^0, w^*) + \nu(u, w^0, w^*)$ subject to (3.4.3) and (3.4.4).

By Proposition 3.4.5, this problem can be solved in a two-step optimization, with the principal first picking an arbitrary $u \in (0, \mu_H)$ and optimizing the size of the organization within the u-supportive set, and then (working backwards) optimizing over u. As argued in the previous section, both α and ν are increasing in both w^0 and w^* within the u-supportive set for any u. Hence, a solution to the first step must lie on the northeastern frontier of the u-supportive set.

In case parameter values are in the set described by the last statement of Proposition 3.4.5, any such solution must lie on the *interior* of this northeastern frontier. To see this, note that the designer can trade off a marginal reduction in w^0 for a (relatively) arbitrarily large increase in w^*. In other words, if the u-supportive set lies below the 45-degree line, the top of this set is flat, and the indifference curve for organizational size which intersects the top of the feasible set must cut into the feasible set, so there exist other points in the

u-supportive set which yield a larger organizational size. This implies that if (u, w^0, w^*) is a size-maximizing organization, (w^0, w^*) does not lie at the top of the union of S^u. Intuitively, the principal finds it worthwhile to limit the lifetime value promised to incoming agents by increasing the performance demands on these agents for earning respite; the resulting increase in the fraction of agents working "buys" the principal the ability to more significantly increase the size of the organization by providing later, but more frequent, respite.

In the second step of the optimization, the principal faces a trade-off, as the u-supportive sets vanish as u becomes sufficiently small or sufficiently large. Hence, any optimal u lies in $[\underline{u}, \overline{u}]$ as defined in Proposition 3.4.4.

Proposition 3.5.1. *If the feasible set is nonempty, there exists a feasible organization which maximizes total size across all feasible organizations, and it lies on the northeastern frontier of its u-supportive set.*

That the u-supportive sets vanish as u becomes sufficiently small highlights a tension facing the principal between organizational size and feasibility. To illustrate this point, consider a fixed (w^0, w^*) pair and vary the flow payoff u exogenously. The lemma below implies that unambiguously, a principal seeking to maximize the organization's size benefits from reducing the payoff u. Effectively, reducing this payoff puts upward pressure on the drift of agents' continuation values, as expressed in (3.2.2). This results in later expulsion of agents and a larger organization overall. However, as the discussion above demonstrates, eventually the average output of the organization deteriorates as the proportion of working agents decreases.

Lemma 3.5.1. *For $0 < w^0 \leq w^* < u/r$, f_- and f_+ are decreasing in u, pointwise w.r.t. w, and both α and ν are decreasing in u. Hence, total organizational size is decreasing in u.*

An implication of this result is that if the principal's goal is to maximize the organization's size, then whenever the principal is faced with multiple fixed point values of u for a given (w^0, w^*), the principal should choose to implement the *lowest* of these fixed points. In the next section, we consider an objective function of the principal where the opposite is true.

3.5.2 Maximizing Per Capita Output

We now consider a designer whose objective is to maximize steady state output per capita. For example, the designer could be the leader of a research lab who is in continual need of assistants to perform specialized tasks. Resources and credit for publications or discoveries must be shared with each member of the team, and therefore the designer's first order concern is the average effort level, not how many people join the team.

Formally, the designer's problem is

$$\max_{(u,w^0,w^*)\in\mathcal{S}} Q(u,w^0,w^*)\mu_H + (1-Q(u,w^0,w^*))\mu_L$$

subject to (3.4.3) and (3.4.4), which is equivalent to maximizing $Q(u,w^0,w^*)$, or simply u, over the set of feasible organizations.

So that the problem is nontrivial, suppose that the feasible set is nonempty, and consider a particular u such that the u-supportive set is nonempty. By Lemma B.2.1, the per-capita output of the organization is strictly decreasing in w^0, so for any fixed w^*, the principal would like to set w^0 arbitrarily close to 0. Consequently, the organization will have virtually no agents. Since we do not allow $w^0 = 0$, there is an open set problem, which we circumvent by characterizing the supremum of per-capita output over the set of feasible organizations with $w^0 > 0$. We can then identify sequences of organizations for which per-capita output converges to this supremum.

It is useful, then, to consider the limit function $Q^0(u,w^*) := \lim_{w^0\to 0} Q(u,w^0,w^*)$, which is an upper bound on $Q(u,w^0,w^*)$ for $w^0 > 0$. We show that this function, like

$Q(u, w^0, w^*)$ for $w^0 > 0$, is single-peaked in w^*, and we denote its unique maximizer by $w^*_{PC}(u)$. Hence, for fixed u, the sequence of average output associated with any sequence of u-supportive (w^0, w^*) converging to $(0, w^*_{PC}(u))$ converges to the supremum over S^u of $Q(u, w^0, w^*)$. It is worth emphasizing that despite the fact that $w^0 \to 0$, $Q(u, w^0, w^*)$ remains bounded away from 1; although new agents become arbitrarily unlikely to reach the shirking state before being removed from the organization, a positive *fraction* of agents remain shirking.

Next, we identify the optimal u. Since the objective is to maximize the fixed point u itself, it is enough to find the supremum, denoted u_{PC}, of the set of u such that $U(Q^0(u, w^*_{PC}(u))) > u$. Although u_{PC} itself is not attainable in a feasible organization — this would require $w^0 = 0$ which is not permitted — u_{PC} can be approximated arbitrarily closely by a sequence of feasible organizations. These results are summarized in the following proposition.

Proposition 3.5.2. *If the feasible set is nonempty, then there exists a unique triple* $(u_{PC}, w^0_{PC}, w^*_{PC}) = (u_{PC}, 0, w^*_{PC})$ *with the following properties: (i)* u_{PC} *is the supremum of the per capita output across all feasible organizations, and (ii) the designer can approximate* u_{PC} *arbitrarily closely by choosing feasible organizations arbitrarily close to* $(u_{PC}, w^0_{PC}, w^*_{PC})$. *However, approximating* u_{PC} *requires organizations to be made arbitrarily small.*

It is noteworthy that there is a unique (limiting) organization design which maximizes per capita output, but the principal must heavily sacrifice the organization's size to attempt to implement it. The reason is that the organization most effectively limits the fraction of agents shirking by starting agents at extremely low continuation values; this ensures that they are very unlikely to earn the opportunity to shirk before being expelled from the organization. This result is reminiscent of partnership organizations in which senior (i.e., vested) partners recruit junior colleagues on the lowest rung of the ladder and promote virtually none of them. Interestingly we find that such an organization is itself *very* small in steady state, as partners trade off the size of the organization in order to maintain a high percentage of hard-working juniors.

3.6 Discussion

We have proposed a model of organization design in which there is a large number of small agents, whose efforts exert positive externalities but whose interactions are such that high effort can only be incentivized through a central reputation system. Absent transfer payments, agents must be permitted to shirk in some instances after good performance. Under the Brownian monitoring (or output) technology, this implies that agents' continuation values follow a sticky Brownian motion. Using techniques in stochastic calculus, we characterize the steady state of the organization as a stationary distribution over continuation values. Finally, we frame the social planner's optimization problem in terms of the steady state distribution, where she optimizes the design parameters subject to a feasibility constraint. We identify a fundamental trade-off between the size and feasibility of the organization, mediated by the mass of shirking agents.

Our model could also be extended to capture other aspects of real world organizations. For example, we have assumed an exogenous inflow rate of new agents, but one could endogenize the inflow rate, say, to be an increasing function of the starting payoff w^0. We have also assumed that no agents voluntarily leave the organization, but there are several ways that agents could leave an organization in practice. Agents might have idiosyncratic shocks that force them to seperate from the organization, independent of their continuation values; we conjecture that this would simply increase the effective discount factor of agents, and would reduce the size of the feasible set. A more substantively different possibility would be to give agents a positive outside option; this would put a positive lower bound on continuation values and would also restrict the feasible set.

Chapter 4

Conclusion

In this dissertation, I have shown how income-share agreements may be used by some students to reduce their debt from attending college, and how these agreements effect the decision making process of students on the job market. Using a dynamic model of financing selection and labor search, I find that which students may benefit from income-sharing depends strongly on the way that the individual income-sharing agreement works, and that properly designed or regulated, income-share agreements can play a large role in the landscape of higher education.

This dissertation also shows a method for analyzing complex dynamical systems where many individual agents interact by deriving a steady state distribution of the agents' states over time. Applying this method to large collective organizations such as labor managed firms, it is shown that the optimal design involves different trade-offs than in standard settings, and is heavily dependent on the exact objective of the mechanism designer.

Appendix A

Proofs for Chapter 2

A.1 Proofs for section 2.2

Proof of lemma 2.2.2. The probability that the applicant has not gotten a job by time t is $e^{-H^N t}$, and so the probability that the applicant gets a job at exactly time t is equal to $H^N e^{-H^N t}$. Thus, the value to the provider of a wage w can be written

$$V(w) = H^N \int_0^\infty e^{-\left(r+H^N\right)t} \phi w \, dt.$$

Solving this integral yields

$$V(w) = \frac{\phi w H^N}{r + H^N}.$$

□

Proof of proposition 2.2.1. Start by transforming 2.2.6 and 2.2.8 with $\gamma := 1 - \phi$. Then, we take the γ-derivative of these first order equations. Rearranging, we get the system

$$\frac{\partial w_R^N}{\partial \gamma} \left(1 + \frac{\alpha^N \left(1 - F\left(w_R^N\right)\right)}{r}\right) = \frac{-b}{\gamma^2} + \frac{1}{r} \int_{w_R^N}^\infty 1 - F(w) \, dw \, \frac{\partial \alpha^N}{\partial \gamma} \tag{A.1.1}$$

$$\begin{aligned} &\frac{\partial \alpha^N}{\partial \gamma} \left(c''\left(\alpha^N\right)\left(r + H^N\right) + c'\left(\alpha^N\right)\left(1 - F\left(w_R^N\right)\right)\right) = \\ &\quad \frac{-c'\left(\alpha^N\right)}{1-\gamma} + (1-\gamma)\left[\left(1 - H^N\right) \frac{\partial H^N}{\partial \alpha^N} \frac{\partial w_R^N}{\partial \gamma} - w_R^N \frac{\partial H^N}{\partial \alpha^N} \frac{\partial H^N}{\partial \gamma}\right. \\ &\qquad \left. + w_R^N \left(1 - H^N\right) \frac{\partial^2 H^N}{\partial \gamma \partial \alpha^N} + \frac{\partial w_R^N}{\partial \alpha^N} \frac{\partial H^N}{\partial \gamma} + H^N \frac{\partial^2 w_R^N}{\partial \gamma \partial \alpha^N}\right] \end{aligned} \tag{A.1.2}$$

First, consider the point for which $\phi = 0$, or $\gamma = 1$, and so $w_R^N = w_R$ and $\alpha^N = \alpha$. At this point, lemma 2.2.3 implies that both $\frac{\partial w_R^N}{\partial \gamma}$ and $\frac{\partial \alpha^N}{\partial \gamma}$ are negative; when income sharing is introduced, the reservation wage and the contact rate both go up. The reservation wage must be continuous in ϕ, and equivalently in γ, so by the intermediate value theorem, if $\frac{\partial w_R^N}{\partial \phi} < 0$ ever holds, there must be some $\overline{\gamma}$ for which $\left.\frac{\partial w_R^N}{\partial \gamma}\right|_{\gamma=\overline{\gamma}} = 0$. At this point, since by the chain rule, $\frac{\partial w_R^N}{\partial \gamma} = \frac{\partial w_R^N}{\partial \alpha^N}\frac{\partial \alpha^N}{\partial \gamma}$, it must be that $\left.\frac{\partial \alpha^N}{\partial \gamma}\right|_{\gamma=\overline{\gamma}} = 0$, and thus also $\left.\frac{\partial H^N}{\partial \gamma}\right|_{\gamma=\overline{\gamma}} = 0$. Simplifying, then, we get

$$
\begin{aligned}
0 &= \frac{-b}{\overline{\gamma}^2} \\
\frac{c'\left(\alpha^N\right)}{1-\gamma} &= (1-\gamma)\left[w_R^N\left(1-H^N\right)\frac{\partial^2 H^N}{\partial \gamma \partial \alpha^N} + H^N \frac{\partial^2 w_R^N}{\partial \gamma \partial \alpha^N}\right]
\end{aligned}
$$

The first line yields a contradiction, as γ is bounded. Since the reservation wage is increasing in ϕ when $\phi = 0$, and the reservation wage must be continuous in ϕ, it must be that the reservation wage is *always* increasing in ϕ. □

A.2 Proofs for section 2.3

Proof of proposition 2.3.1. In order to ensure that a local increase in take-home pay yields a strictly greater take-home pay than traditional debt financing, we check for a condition local to $\phi = 0$. Again, let $\gamma := 1 - \phi$. Further, let $H := \alpha\left(1 - F\left(w_R\right)\right)$ be the hazard rate of the search process at $\phi = 0$. Then, assuming that the partials of w_R^N and α^N are bounded and using L'Hôpital's Rule when necessary, we find that

$$
\begin{aligned}
\left.\frac{\partial w_R^N}{\partial \gamma}\right|_{\gamma=1}(r+H) &= -rb + \int_{w_R}^{\infty} 1 - F(w)\,dw\ \left.\frac{\partial \alpha^N}{\partial \gamma}\right|_{\gamma=1} \\
\left.\frac{\partial \alpha^N}{\partial \gamma}\right|_{\gamma=1} c''\left(\alpha^N\right)(r+H) &= c''(\alpha)\left.\frac{\partial \alpha^N}{\partial \gamma}\right|_{\gamma=1}
\end{aligned}
$$

From the second equality, we find that either $r + H = 1$, which is generally false, or that $\frac{\partial \alpha^N}{\partial \gamma} = 0$ whenever $\gamma = 1$, which is also false, as $\frac{\partial w_R^N}{\partial \gamma} > 0$ and the chain rule requires the partials of w_R^N and α^N to have the same sign. Our assumption that the partials are bounded at $\gamma = 1$ must then be false. In particular,

$$\lim_{\gamma \to 1} w_R^N = \lim_{\gamma \to 1} \alpha^N = -\infty.$$

Since we can write

$$\frac{\partial \gamma w_R^N}{\partial \gamma} = \gamma \frac{\partial w_R^N}{\partial \gamma} + w_R^N,$$

taking the limit of both sides as $\gamma \to 1$ shows that

$$\lim_{\gamma \to 1} \frac{\partial \gamma w_R^N}{\partial \gamma} = \lim_{\gamma \to 1} \frac{\partial w_R^N}{\partial \gamma} = -\infty$$

and take-home pay always increases when income-sharing is introduced. □

Proof of proposition 2.3.2. Again we let $\gamma := 1 - \phi$. Deriving the hazard rate with respect to γ, we get

$$\frac{\partial H^N}{\partial \gamma} = \frac{\partial \alpha^N}{\partial \gamma} \left(1 - F\left(w_R^N\right)\right) - \alpha^N f\left(w_R^N\right) \frac{\partial w_R^N}{\partial \gamma} \tag{A.2.1}$$

For the first claim in the proposition, we define $\overline{\gamma} := \frac{b}{\overline{w}}$ and look for the limit as $\gamma \to \overline{\gamma}$. Since $w_R^N > b$ must always hold, we see that $\lim_{\gamma \to \overline{\gamma}} w_R^N = \overline{w}$. Additionally, since $\lim_{\alpha^N \to \infty} c'(\alpha^N) = \infty$, it must be that $\lim_{\gamma \to \overline{\gamma}} \alpha^N$ and $\lim_{\gamma \to \overline{\gamma}} \frac{\partial \alpha^N}{\partial \gamma}$ are both finite. Thus,

$$\lim_{\gamma \to \overline{\gamma}} H^N = \lim_{\gamma \to \overline{\gamma}} \alpha^N \left(1 - F\left(\overline{w}\right)\right) = 0.$$

For the second claim, we take first-order conditions from equations 2.2.6 and 2.2.8, yielding equations A.1.1 and A.1.2. Just as in the proof of proposition 2.3.1, we find $\lim_{\gamma \to 1} \frac{\partial \alpha^N}{\partial \gamma} = -\infty$. Simplifying equation A.2.1 with $\frac{\partial w_R^N}{\partial \gamma} = \frac{\partial w_R^N}{\partial \alpha^N} \frac{\partial \alpha^N}{\partial \gamma}$, we get that

$$\frac{\partial H^N}{\partial \gamma} = \frac{\partial \alpha^N}{\partial \gamma} \left(1 - F(w_R^N) - \alpha^N f\left(w_R^N\right) \frac{\partial w_R^N}{\partial \alpha^N}\right).$$

Since the $\lim_{\gamma \to 1} \frac{\partial \alpha^N}{\partial \gamma} = -\infty$, we can see that

$$\operatorname{sgn}\left\{ \left.\frac{\partial H^N}{\partial \gamma}\right|_{\gamma=1} \right\} = -\operatorname{sgn}\left\{ 1 - F(w_R) - \alpha f(w_R) \left.\frac{\partial w_R^N}{\partial \alpha^N}\right|_{\gamma=1} \right\} = -1.$$

□

A.3 Proofs for section 2.4

Proof of proposition 2.4.1. If the student can fully fund through an ISA, an ISA is preferable to debt when

$$\frac{(1-\phi)w_R^E}{r} > \frac{w_R}{r} - K,$$

since debt must be repaid. As long as this repayment amount is sufficiently large, a student would prefer an ISA, but can only accept an ISA for the full amount if

$$K < \frac{\phi w_R^E H^E}{r + H^E}.$$

If the student must take on debt regardless of using an ISA or not, then it must be determined if

$$\frac{(1-\phi)w_R^E}{r} - \left(K - \frac{\phi w_R^E H^E}{r + H^E} \right) > \frac{w_R}{r} - K,$$

since it is preferable to accept as much ISA funding as possible if any ISA funding is accepted, and even with the ISA the student will need to take on some debt. If the student prefers an ISA to debt, she also prefers to partially fund with the ISA than to revert to only using debt. Finally, the applicant is unwilling to use either an ISA if the cost of college is higher than the value of the job-search under an ISA, so for high enough costs of college, the student is unwilling to use an ISA but is still willing to use debt to fund college. □

Proof of proposition 2.4.2. The length of all regions of K in which students elect to use ISAs is

$$\frac{(1-\phi)w_R^E}{r} - \frac{w_R - (1-\phi)w_R^E}{r} = \frac{2(1-\phi)w_R^E - w_R}{r}.$$

Since take-home pay is assumed decreasing in ϕ, this region shrinks with ϕ. □

Proof of proposition 2.4.3. A cost of college K is fully fundable through an ISA as long as

$$K \leq \min \left\{ (\frac{\phi w_R^E H^E}{r + H^E}, \frac{(1-\phi) w_R^E}{r} \right\}.$$

That is, an ISA can fully fund education as long as it provides both enough money to the provider that the provider is willing to pay the cost K, and that it provides enough take-home pay to the student that the student is willing to go to college at all using an ISA. Since both elements of the maximization are continuously differentiable in ϕ, the maximum of the right-hand side, representing the largest amount of money possibly fully fundable by an ISA, is at some point where $\phi < .5$. When $\phi = .5$, $w_R^E = w^*$, so some cost $\bar{K}$ such that $r\bar{K} \leq \frac{1}{2} w^*$ is the maximal fully fundable cost. The student-optimal income-sharing rate is thus the smallest ϕ for which K remains in the fully funded region, or, from proposition 2.4.1, the smallest for which $K \leq \frac{\phi w_R^E H^E}{r + H^E}$. Similarly, the provider-optimal income-sharing rate is the largest ϕ for which K remains in the fully funded region, or, from proposition 2.4.1,

$$\sup \left\{ \phi : rK > w_R - (1-\phi) w_R^E \right\}.$$

For $K > \frac{w^*}{2r}$, the student cannot fully fund her education through an ISA, and so the optimal ISA will involve partially funding through an ISA. When partially funding through an ISA, the student exhausts the amount of money the provider is willing to give, making the provider indifferent between any ISAs or not providing anything at all. Further, the student treats each additional dollar of an ISA provided as decreasing the amount she has to borrow. In this manner, the student's welfare is given by

$$(1-\phi)\frac{w_R^E}{r} - \left(K - K^E\right).$$

Since the reservation wage is increasing in ϕ, the student is best off with the highest possible ϕ such that she remains in the partial-ISA funding region, which, from proposition 2.4.1

and lemma 2.2.1, is the largest ϕ such that $rK \leq (1-\phi)\frac{w_R^E}{r}$ and $\phi \leq .5$. That $\phi \leq .5$, however, is a redundant qualification, since $\phi > .5$ implies that $rK > (1-\phi)\frac{w_R^E}{r}$. $\square$

Appendix B

Proofs for Chapter 3

B.1 Proofs for section 3.3

In this section we derive the stationary distribution. In Lemma B.1.1, we show that the densities must satisfy a standard Kolmogorov forward equation along with a set of boundary conditions. While stationary distributions and their boundary conditions have been treated in numerous texts (for example, see Karlin and Taylor (1981), Gardiner (2009)), we are unaware of an existing result that is sufficiently general to take "off the shelf" by accommodating the sticky reflection, resetting barrier and form of drift for our continuation value process, so we provide a full derivation here which adapts the approach of Harrison and Lemoine (1981).

For future reference, we state the Kolmogorov forward (or Fokker-Planck) equation

$$rf(w) + (rw - (u - c))f'(w) = \frac{\lambda^2}{2} f''(w) \tag{B.1.1}$$

and define the general form of its solution up to two arbitrary constants as follows:

$$f(w) = e^{\gamma(w)^2} \left(C_1 + C_2 \text{erf} \left\{ \gamma(w) \right\} \right), \tag{B.1.2}$$

where the functions γ and erf were defined in Section 3.3.

Lemma B.1.1. *The steady state distribution of agents is characterized by an atom $\nu\{w^*\}$ and piecewise densities f_- and f_+ of the form* (B.1.2) *defined on* $(0, w^0]$ *and* $[w^0, w^*)$*, respectively, subject to the following boundary conditions:*

1. $f_-(0) = 0$.

2. $f_-(w^0) = f_+(w^0)$.

3. $f'_-(0+) = f'_-(w^0-) - f'_+(w^0+)$.

4. $\frac{\lambda^2}{2} f_+(w^*-) + \rho(w^*)\nu\{w^*\} = 0$.

5. $\frac{\lambda^2}{2} f'_-(0+) = \psi$.

Proof of Lemma B.1.1. We follow the approach in (Harrison and Lemoine, 1981, pp. 220-221) who derive the stationary distribution for a sticky Brownian motion with constant (negative) drift on $[0, \infty)$ and sticky reflection at 0; the main modifications are to account for the resetting barrier and state-dependent drift. The infinitesimal generator of the W process is the operator Γ defined by

$$\Gamma h(w) = \lim_{dt \downarrow 0} \frac{\mathbb{E}_w[h(W_{dt})] - h(w)}{dt}.$$

For all $w > 0$ and functions h in a suitable domain, the above limit is well-defined and takes values

$$\Gamma h(w) = \begin{cases} (rw^* - u)h'(w^*) & \text{if } w = w^* \\ (rw - (u - c))h'(w) + \frac{\lambda^2}{2} h''(w) & \text{if } w \in (0, w^*). \end{cases}$$

In particular, the above is valid for all h such that h is twice continuously differentiable and bounded and that $\Gamma h(w)$ is continuous, including at w^*. For $w = 0$, the generator is not defined since the jump from 0 to w^0 is instantaneous. For convenience, define $\mu(w) := rw - (u - c)$ and recall that $\rho(w) := rw - u$.

Now a measure ν is a stationary distribution on $[0, w^*]$ for W if and only if for all $t \geq 0$ and all admissible h, we have

$$\int_{[0,w^*]} h(w)\nu(dw) = \int_{[0,w^*]} \mathbb{E}_w[h(W_t)]\nu(dw). \tag{B.1.3}$$

Essentially, this condition says that any statistic of a stationary distribution is unchanging over time. In order to characterize a steady state distribution, we want to transform the right side of the above expression into terms involving only $h''(w)$ and $h'(w^*)$. Now we must have and $\nu\{0\} = 0$ since otherwise the outflow of agents would be of higher order than the inflow. Hence, expanding the right hand side using Ito's formula, restricting the integral to $w \in (0, w^*]$ so that the generator is defined, we have

$$\int_{[0,w^*]} \mathbb{E}_w[h(W_t)]\nu(dw) = \int_{(0,w^*]} \left(h(w) + \mathbb{E}_w \left[\int_0^t \Gamma h(W_s)ds + \sum_{0<s\leq t} \Delta h(W)_s \right] \right) \nu(dw)$$

where $\Delta h(W)_s := h(W_s) - h(W_{s-})$ for $s > 0$. Subtracting the left hand side of (B.1.3) from this, we have

$$\begin{aligned} 0 &= \int_{(0,w^*]} \mathbb{E}_w \left[\int_0^t \Gamma h(W_s)\, ds + \sum_{0<s\leq t} \Delta h(W)_s \right] \nu(dw) \\ &= \int_{(0,w^*]} \mathbb{E}_w \left[\int_0^t \Gamma h(W_s)\, ds \right] \nu(dw) + \int_{(0,w^*]} \mathbb{E}_w \left[\sum_{0<s\leq t} \Delta h(W)_s \right] \nu(dw). \end{aligned}$$

Dividing through by t and taking limits as $t \to 0$ yields

$$\begin{aligned} 0 &= \int_{(0,w^*]} \Gamma h(w)\nu(dw) + \lim_{t\to 0} \frac{1}{t} \int_{[0,w^*]} \mathbb{E}_w \left[\sum_{0<s\leq t} \Delta h(W)_s \right] \nu(dw) \\ &= \int_{(0,w^*]} \Gamma h(w)\nu(dw) + \frac{\lambda^2}{2} f'_-(0)(h(w^0) - h(0)). \end{aligned} \tag{B.1.4}$$

To obtain the second term of (B.1.4), note that $\Delta h(W)_s > 0$ only when $W_{s-} = \lim_{t\to s} W_t = 0$, and in these cases we have $\Delta h(W)_s = h(W_s) - h(W_{s-}) = h(w^0) - h(0)$. To a first order approximation, the second term of (B.1.4) is thus the expectation, over starting points w, of the size of a single jump, $h(w^0) - h(0)$, times the probability that the process starting from w reaches 0 (the probability of 2 or more jumps may be ignored since once the process resets at w^0 it is very far away from 0). After taking the limit, the second term above is $h(w^0) - h(0)$ times the flow rate of mass hitting 0, which is $\lambda^2 f'_-(0+)/2$.

The first term of (B.1.4) can be expanded as

$$\begin{aligned}&\int_{(0,w^*)} \left[\mu(w)h'(w) + \frac{\lambda^2}{2}h''(w)\right]\nu(dw) + \rho(w^*)h'(w^*)\nu\{w^*\} \\ &= \int_{(0,w^*)} \mu(w)h'(w)\nu(dw) + \int_{(0,w^*)} \frac{\lambda^2}{2}h''(w)\nu(dw) + \rho(w^*)h'(w^*)\nu\{w^*\}.\end{aligned} \tag{B.1.5}$$

We now focus on the first term of (B.1.5). The integral can be split into two regions, $(0, w^0)$ and $[w^0, w^*)$, where the stationary distribution has a density $f_\pm$ of the form (B.1.2). Then, by writing $h'(w) = h'(w^0) - \int_w^{w^0} h''(y)dy = h'(w^0) + \int_{w^0}^w h''(y)dy$, the first term of (B.1.5) is equivalent to

$$\begin{aligned}&\int_{(0,w^0)} \mu(w)\left[h'(w^0) - \int_{(w,w^0)} h''(y)dy\right] f_-(w)dw \\ &\quad + \int_{[w^0,w^*)} \mu(w)\left[h'(w^0) + \int_{(w^0,w)} h''(y)dy\right] f_+(w)dw \\ &= h'(w^0)\int_{(0,w^*)} \mu(w)\nu(dw) - \int_{(0,w^0)} \left[\int_{(0,w)} \mu(y)f_-(y)dy\right] h''(w)dw \\ &\quad + \int_{[w^0,w^*)} \left[\int_{[w,w^*)} \mu(y)f_+(y)dy\right] h''(w)dw.\end{aligned}$$

Substituting the above into (B.1.5), the first term of (B.1.4) becomes

$$\begin{aligned}\int_{(0,w^*]} \Gamma h(w)\nu(dw) &= \int_{(0,w^0)} h''(w)\left[\frac{\lambda^2}{2}f_-(w) - \int_{(0,w)} \mu(y)f_-(y)dy\right] dw \\ &\quad + \int_{[w^0,w^*)} h''(w)\left[\frac{\lambda^2}{2}f_+(w) + \int_{[w,w^*)} \mu(y)f_+(y)dy\right] dw \\ &\quad + h'(w^0)\int_{(0,w^*)} \mu(w)\nu(dw) \\ &\quad + \rho(w^*)h'(w^*)\nu\{w^*\}.\end{aligned} \tag{B.1.6}$$

The first two terms of (B.1.6), having integrals involving $h''(w)$ as coefficients, are all set. Take the last two terms of (B.1.6) and add back in the second term on the RHS of (B.1.4) to write the RHS of (B.1.4) as the sum of the first two terms of (B.1.6) and

$$\frac{\lambda^2}{2} f'_-(0)(h(w^0) - h(0)) + h'(w^0) \int_{(0,w^*)} \mu(w)\nu(dw) + \rho(w^*)h'(w^*)\nu\{w^*\}. \tag{B.1.7}$$

As noted, the goal is to transform the h involvement above into h'' and $h'(w^*)$ terms. For the first term of (B.1.7), integrate the derivatives twice and exchange the order of integration to get

$$\begin{aligned}
\frac{\lambda^2}{2} f'_-(0)(h(w^0) - h(0)) &= \frac{\lambda^2}{2} f'_-(0) \int_0^{w^0} h'(w)dw \\
&= \frac{\lambda^2}{2} f'_-(0) \int_0^{w^0} \left[h'(w^*) - \int_w^{w^*} h''(y)dy \right] dw \\
&= \frac{\lambda^2}{2} f'_-(0) \left(w^0 h'(w^*) - \int_0^{w^0} \left[\int_w^{w^*} h''(y)dy \right] dw \right) \qquad \text{(B.1.8)} \\
&= \frac{\lambda^2}{2} f'_-(0) \left(w^0 h'(w^*) - \int_0^{w^0} w h''(w)dw - \int_{w^0}^{w^*} w_0 h''(w)dw \right).
\end{aligned}$$

For the second term of (B.1.7), we have

$$\begin{aligned}
h'(w^0) \int_{(0,w^*)} \mu(w)\nu(dw) &= \left(h'(w^*) - \int_{w^0}^{w^*} h''(y)dy \right) \int_{(0,w^*)} \mu(w)\nu(dw) \\
&= h'(w^*) \int_{(0,w^*)} \mu(w)\nu(dw) - \int_{w^0}^{w^*} h''(w) \left[\int_{(0,w^*)} \mu(y)\nu(dy) \right] dw
\end{aligned} \tag{B.1.9}$$

where we have swapped w and y as variables of integration for later convenience. Plugging (B.1.8) and (B.1.9) back into (B.1.7) and adding back in the first two terms of (B.1.6), we can write (B.1.4) as

$$0 = h'(w^*)M^* + \int_{(0,w^0)} h''(w)M_-(w)dw + \int_{[w^0,w^*)} h''(w)M_+(w)dw, \tag{B.1.10}$$

where we define

$$M^* := \frac{\lambda^2}{2} f'_-(0) w^0 + \int_{(0,w^*)} \mu(w)\nu(dw) + \rho(w^*)\nu\{w^*\},$$
$$M_-(w) := -\frac{\lambda^2}{2} f'_-(0) w + \frac{\lambda^2}{2} f_-(w) - \int_0^w \mu(y) f_-(y) dy,$$
$$M_+(w) := -\frac{\lambda^2}{2} f'_-(0) w^0 + \frac{\lambda^2}{2} f_+(w) + \int_w^{w^*} \mu(y) f_+(y) dy - \int_{(0,w^*)} \mu(y)\nu(dy).$$

Equation (B.1.10) is exactly what we are after. It allows us to completely characterize the steady state distribution. Specifically, because $h'(w^*)$ and $h''(w)$ are completely free (up to the differentiability conditions), the expressions attached to them must all vanish:

$$M^* = 0$$
$$M_-(w) \equiv 0$$
$$M_+(w) \equiv 0.$$

Equation (B.1.10) has several implications. First, from $M''_-(w) = 0$ and $M''_+(w) = 0$, we recover the Kolmogorov forward equation (B.1.1) on the left and right pieces. In addition,

1. $M_-(0+) = 0$ implies $f_-(0+) = 0$
2. $M_-(w^0) = M_+(w^0)$ implies $f_-(w^0) = f_+(w^0)$
3. $M'_-(w^0-) = M'_+(w^0+)$ implies $f'_-(0+) = f'_-(w^0-) - f'_+(w^0+)$
4. $M^* + M_+(w^*) = 0$ implies $\frac{\lambda^2}{2} f_+(w^*-) + \rho(w^*)\nu\{w^*\} = 0$.

Finally, since the outflow of agents must equal the inflow, we have the condition $\frac{\lambda^2}{2} f'_-(0+) = \psi$ which pins down the scale of the distribution. □

Proof of Proposition 3.3.1. By Lemma B.1.1, the steady state distribution of agents can be described by the densities $f_\pm(w) = e^{\gamma(w)^2}(C_1^\pm + C_2^\pm \mathrm{erf}\{\gamma(w)\})$ and an atom $\nu\{w^*\}$ subject to the stated constraints.

As $f_-(0) = 0$, we have:

$$0 = e^{\frac{(u-c)^2}{\lambda^2 r}} \left(C_1^- + C_2^- \operatorname{erf}\left\{ \frac{-(u-c)}{\lambda\sqrt{r}} \right\} \right)$$

Because the error function has odd symmetry, this means that

$$C_1^- - C_2^- \operatorname{erf}\left\{ \frac{u-c}{\lambda\sqrt{r}} \right\} = 0$$

Knowing too that $f'_-(0) = \frac{2\psi}{\lambda^2}$, and by differentiating $f_-(w)$, we get:

$$\begin{aligned}
\frac{2\psi}{\lambda^2} &= 2\gamma(0)\gamma'(0)e^{\gamma(0)^2} \left(C_2^- \operatorname{erf}\left\{ \frac{u-c}{\lambda\sqrt{r}} \right\} + C_2^- \operatorname{erf}\left\{ \gamma(0) \right\} \right) \\
&\quad + e^{\gamma(0)^2} C_2^- \gamma'(0) \operatorname{erf}'\{\gamma(0)\} \\
&= 2e^{\gamma(0)^2} C_2^- \gamma'(0) \frac{e^{-\gamma(0)^2}}{\sqrt{\pi}} = \frac{2C_2^- \sqrt{r}}{\lambda\sqrt{\pi}} \\
\implies C_2^- &= \frac{\psi\sqrt{\pi}}{\lambda\sqrt{r}}.
\end{aligned}$$

Thus the lower segment of the distribution function is

$$f_-(w) = e^{\gamma(w)^2} \left(\frac{\psi\sqrt{\pi}}{\lambda\sqrt{r}} \left[\operatorname{erf}\left\{ \frac{u-c}{\lambda\sqrt{r}} \right\} + \operatorname{erf}\left\{ \gamma(w) \right\} \right] \right). \tag{B.1.11}$$

Since $f_-(w)$ and $f_+(w)$ must agree at w^0, we set the lower and upper f functions equal at w^0 to get

$$\begin{aligned}
f_+(w^0) = f_-(w^0) &= e^{\gamma(w^0)^2} \left(\frac{\psi\sqrt{\pi}}{\lambda\sqrt{r}} \left[\operatorname{erf}\left\{ \frac{u-c}{\lambda\sqrt{r}} \right\} + \operatorname{erf}\left\{ \gamma(w^0) \right\} \right] \right) \\
&= e^{\gamma(w^0)^2} (C_1^+ + C_2^+ \operatorname{erf}\left\{ \gamma(w^0) \right\})
\end{aligned}$$

and thus, by rearranging terms, we find that

$$C_1^+ = \frac{\psi\sqrt{\pi}}{\lambda\sqrt{r}} \left[\operatorname{erf}\left\{ \frac{u-c}{\lambda\sqrt{r}} \right\} + \operatorname{erf}\left\{ \gamma(w^0) \right\} \right] - C_2^+ \operatorname{erf}\left\{ \gamma(w^0) \right\}.$$

Therefore,

$$f_+(w) =$$
$$e^{\gamma(w)^2}\left(\frac{\psi\sqrt{\pi}}{\lambda\sqrt{r}}\left[\text{erf}\left\{\frac{u-c}{\lambda\sqrt{r}}\right\}+\text{erf}\left\{\gamma(w^0)\right\}\right]-C_2^+\text{erf}\left\{\gamma(w^0)\right\}+C_2^+\text{erf}\left\{\gamma(w)\right\}\right)$$

and, differentiating both $f_+(w)$ and $f_-(w)$, we get

$$f'_+(w) = 2\gamma(w)\gamma'(w)f_+(w) + e^{\gamma(w)^2}C_2^+\text{erf}'\{\gamma(w)\}\gamma'(w)$$

and

$$f'_-(w) = 2\gamma(w)\gamma'(w)f_-(w) + \frac{\psi\sqrt{\pi}}{\lambda\sqrt{r}}e^{\gamma(w)^2}\text{erf}'\{\gamma(w)\}\gamma'(w).$$

Because $f'_-(0+) = f'_-(w^0-) - f'_+(w^0+)$, it must be that

$$C_2^+ = \frac{\frac{\psi\sqrt{\pi}}{\lambda\sqrt{r}}\left[e^{\gamma(w^0)}\text{erf}'\{\gamma(w^0)\}\gamma'(w^0) - e^{\gamma(0)^2}\text{erf}'\{\gamma(0)\}\gamma'(0)\right]}{e^{\gamma(w^0)^2}\text{erf}'\{\gamma(w^0)\}\gamma'(w^0)} = 0$$

Thus, the upper segment of the distribution function is

$$f_+(w) = e^{\gamma(w)^2}\frac{\psi\sqrt{\pi}}{\lambda\sqrt{r}}\left[\text{erf}\left\{\frac{u-c}{\lambda\sqrt{r}}\right\}+\text{erf}\left\{\gamma(w^0)\right\}\right] \tag{B.1.12}$$

Finally, completing the derivation of the distribution of agents in the steady state, the mass of agents at w^*, $\nu\{w^*\}$, satisfies

$$\nu\{w^*\} = \frac{\lambda^2 f_+(w^*)}{2(u - rw^*)}.$$

□

B.2 Proofs for section 3.4

Lemma B.2.1. *For all fixed u, w^* such that $0 < w^* < u/r$ and all $w^0 \in (0, w^*]$, both $\alpha(u, w^0, w^*)$ and $\nu(u, w^0, w^*)$ are increasing in w^0 and $Q(u, w^0, w^*)$ is strictly decreasing in w^0.*

Proof. We have $Q = \frac{\alpha}{\alpha+\nu} = \frac{1}{1+\frac{\nu}{\alpha}}$ which is decreasing iff $\frac{\nu}{\alpha}$ is increasing, which is true iff $\frac{\nu_{w^0}}{\nu} > \frac{\alpha_{w^0}}{\alpha}$. Using $X := \frac{u-c}{\lambda\sqrt{r}}$ and $Y := \frac{\psi\sqrt{\pi}}{\lambda\sqrt{r}}$ and expanding, these quantities are

$$\begin{aligned}
\nu =& \frac{\lambda^2}{2(u-rw^*)} f_+(w^*) = \frac{\lambda^2}{2(u-rw^*)} e^{\gamma(w^*)^2} Y \left(\operatorname{erf}\{X\} + \operatorname{erf}\{\gamma(w_0)\}\right) \\
\nu_{w^0} =& \frac{\psi}{u-rw^*} e^{\gamma(w^*)^2 - \gamma(w^0)^2} > 0 \\
\alpha =& \int_0^{w^0} f_-(w)\,dw + \int_{w^0}^{w^*} f_+(w)\,dw \\
=& \int_0^{w^0} e^{\gamma(w)^2} Y \left[\operatorname{erf}\{X\} + \operatorname{erf}\{\gamma(w)\}\right] dw \\
&+ \int_{w^0}^{w^*} e^{\gamma(w)^2} Y \left[\operatorname{erf}\{X\} + \operatorname{erf}\left\{\gamma(w^0)\right\}\right] dw \\
\alpha_{w^0} =& \int_{w^0}^{w^*} e^{\gamma(w)^2} Y \frac{2e^{-\gamma(w^0)^2}}{\sqrt{\pi}} \frac{\sqrt{r}}{\lambda}\,dw \\
=& \frac{2\sqrt{r}e^{-\gamma(w^0)^2}}{\lambda\sqrt{\pi}\left[\operatorname{erf}\{X\} + \operatorname{erf}\{\gamma(w^0)\}\right]} \int_{w^0}^{w^*} f_+(w)\,dw > 0.
\end{aligned}$$

Define $Z := \frac{2\sqrt{r}e^{-\gamma(w^0)^2}}{\lambda\sqrt{\pi}[\operatorname{erf}\{X\}+\operatorname{erf}\{\gamma(w^0)\}]} > 0$ to be the constant outside the integral of the last expression. By canceling terms, we have $\frac{\nu_{w^0}}{\nu} = Z$, whereas

$$\frac{\alpha_{w^0}}{\alpha} = Z \frac{\int_{w^0}^{w^*} f_+(w)\,dw}{\int_0^{w^0} f_-(w)\,dw + \int_{w^0}^{w^*} f_+(w)\,dw} < Z, \tag{B.2.1}$$

so we are done. □

The next lemma implies that for any fixed $w^0 > 0$, there is a single (possibly empty) interval of w^* values for which the organization is feasible.

Lemma B.2.2. *For all fixed u, w^0 such that $0 < w^0 < u/r$ and for all $w^* \in [w^0, u/r)$, $\alpha(u, w^0, w^*)$ is increasing in w^* and Q is quasiconcave in w^*. In particular, Q is eventually decreasing, and if $Q_{w^*} = 0$ for some value of w^*, then Q_{w^*} is decreasing in w^* at that point.*

Proof. To abbreviate, we use prime notation to denote the derivatives with respect to w^*, and as f_+ and its derivatives are to be evaluated at w^*, we suppress dependence on w^*. Immediately, we have $\alpha' = f_+ > 0$, and from $\nu = \frac{\lambda^2}{2}\frac{f_+}{u-rw^*}$ we have $\nu' = \frac{\lambda^2}{2}\left[\frac{(u-rw^*)f_+' + rf_+}{(u-rw^*)^2}\right]$. Now $Q = \frac{\alpha}{\alpha+\nu} = g\left(\frac{\nu}{\alpha}\right)$ where $g(x) := \frac{1}{1+x}$. Taking derivatives, we have $Q' = g'\left(\frac{\nu}{\alpha}\right)\left(\frac{\nu}{\alpha}\right)'$. Now Q is decreasing if and only if $\left(\frac{\nu}{\alpha}\right)' > 0$ which is equivalent to $\alpha\nu' - \nu\alpha' > 0$. Expanding yields

$$\begin{aligned}\alpha\nu' - \nu\alpha' &= \alpha\frac{\lambda^2}{2}\left[\frac{(u-rw^*)f_+' + rf_+}{(u-rw^*)^2}\right] - \frac{\lambda^2}{2}\frac{f_+^2}{u-rw^*} \\ &= \frac{\lambda^2}{2(u-rw^*)}\left\{\alpha\left[f_+' + \frac{rf_+}{(u-rw^*)}\right] - f_+^2\right\}.\end{aligned}$$

Inside the braces, as $w^* \to u/r$, α and f_+ have positive, finite limits and f_+' has a finite limit, while $\alpha\frac{rf_+}{u-rw^*} \to +\infty$. Hence the expression tends to $+\infty$ as $w^* \to u/r$, and in particular, it is positive for sufficiently large w^*. We conclude that Q is eventually decreasing.

Next, suppose $Q' = 0$, which is equivalent to $\alpha\nu' - \nu\alpha' = 0$. As noted above, $\alpha' > 0$ and hence if $Q' = 0$ we have $\nu' > 0$. Further differentiation yields $Q'' = g''\left(\frac{\nu}{\alpha}\right)\left[\left(\frac{\nu}{\alpha}\right)'\right]^2 + g'\left(\frac{\nu}{\alpha}\right)\left(\frac{\nu}{\alpha}\right)''$. Using $Q' = 0$ and $g' < 0$, we have $Q'' < 0$ if and only if

$$\left(\frac{\nu}{\alpha}\right)'' > 0 \iff \alpha^2\nu'' - \alpha\nu\alpha'' > 2\alpha\alpha'\nu' - 2\nu(\alpha')^2 \iff \nu''\alpha' > \alpha''\nu'.$$

where we have used that $\alpha = \alpha'\frac{\nu}{\nu'}$ to obtain the last inequality. We now establish this last inequality. Differentiate v' to obtain

$$\begin{aligned}\nu'' &= \frac{\lambda^2}{2}\left[\frac{(u-rw^*)^3 f_+'' + 2r(u-rw^*)^2 f_+' + 2r^2(u-rw^*)f_+}{(u-rw^*)^4}\right] \\ &> \frac{\lambda^2}{2}\left[\frac{(u-rw^*)^3 f_+'' + r(u-rw^*)^2 f_+' + r^2(u-rw^*)f_+}{(u-rw^*)^4}\right] \\ &> \frac{\lambda^2}{2}\left[\frac{(u-rw^*)f_+'' + rf_+'}{(u-rw^*)^2}\right],\end{aligned}$$

where in the first inequality we have used from $v' > 0$ that $(u - rw^*)f'_+ + rf_+ > 0$ and in the second inequality we have used $(u - rw^*)f_+ > 0$. Recall that $\alpha' = f_+$ and $\alpha'' = f'_+$, and thus a sufficient condition for our desired inequality $\nu''\alpha' > \alpha''\nu'$ is

$$\frac{\lambda^2}{2}\left[\frac{(u - rw^*)f''_+ + rf'_+}{(u - rw^*)^2}\right] f_+ > f'_+ \frac{\lambda^2}{2}\left[\frac{(u - rw^*)f'_+ + rf_+}{(u - rw^*)^2}\right] \iff f''_+ f_+ > (f'_+)^2.$$

Using $f_+(w^*) = e^{\gamma(w^*)^2} Y[\text{erf}\{X\} + \text{erf}\{\gamma(w^0)\}]$ and (by definition) $\gamma(w) = \frac{rw-(u-c)}{\lambda\sqrt{r}}$, this inequality can be written as

$$[(2\gamma\gamma')^2 f_+ + 2(\gamma')^2 f_+]f_+ > (2\gamma\gamma')^2(f_+)^2$$

which clearly holds. We conclude that $Q' = 0$ implies $Q'' < 0$, so Q is quasiconcave in w^*. □

Proof of Proposition 3.4.1. For any fixed w^0 and w^* with $0 < w^0 \leq w^*$, $\alpha(u, w^0, w^*)$ and $f_+(w^*)$ (where dependence of the latter on u and w^0 has been suppressed) have finite limits as $u \downarrow rw^*$, while $\nu = \frac{\lambda^2}{2(u-rw^*)} f_+(w^*) \to +\infty$, and thus

$$\begin{aligned}\lim_{u\downarrow rw^*} Q(u, w^0, w^*) &= \lim_{u\downarrow rw^*} \frac{\alpha(u, w^0, w^*)}{\alpha(u, w^0, w^*) + \nu(u, w^0, w^*)} \\ &= 0.\end{aligned}$$

In what follows, we show there exists $w^* \in (0, (\mu_H - c)/r)$ such that for sufficiently small w^0, when $u = rw^* + c$, output strictly exceeds u, and hence by the intermediate value theorem, there exists a fixed point u. Take $w^* \in (0, \mu_H - c)/r)$ and $u = rw^* + c \in (0, \mu_H)$.

Then

$$\alpha(u, w^0, w^*)|_{u=rw^*+c}$$
$$= \int_0^{w^0} Y \exp(r(w-w^*)^2/\lambda^2) \left[\operatorname{erf}\left\{\frac{\sqrt{r}w^*}{\lambda}\right\} + \operatorname{erf}\left\{\frac{\sqrt{r}(w-w^*)}{\lambda}\right\}\right] dw$$
$$+ \int_{w^0}^{w^*} Y \exp(r(w-w^*)^2/\lambda^2) \left[\operatorname{erf}\left\{\frac{\sqrt{r}w^*}{\lambda}\right\} + \operatorname{erf}\left\{\frac{\sqrt{r}(w^0-w^*)}{\lambda}\right\}\right] dw,$$
$$\nu(u, w^0, w^*)|_{u=rw^*+c} = \frac{\lambda^2}{2c} Y \left[\operatorname{erf}\left\{\frac{\sqrt{r}w^*}{\lambda}\right\} + \operatorname{erf}\left\{\frac{\sqrt{r}(w^0-w^*)}{\lambda}\right\}\right].$$

Now $\lim_{w^0 \downarrow 0} Q(rw^*+c, w^0, w^*)$ is of the form $\frac{0}{0}$, and by L'Hôpital's rule,

$$\lim_{w^0 \downarrow 0} Q(rw^*+c, w^0, w^*) = \lim_{w^0 \downarrow 0} \frac{\alpha_{w^0}(rw^*+c, w^0, w^*)}{\alpha_{w^0}(rw^*+c, w^0, w^*) + \nu_{w^0}(rw^*+c, w^0, w^*)}, \qquad \text{where}$$
$$\alpha_{w^0}(rw^*+c, w^0, w^*) = \int_{w^0}^{w^*} Y \exp(r(w-w^*)^2/\lambda^2) \frac{2\sqrt{r}}{\lambda\sqrt{\pi}} \exp(-r(w^0-w^*)^2/\lambda^2)$$
$$\nu_{w^0}(rw^*+c, w^0, w^*) = \frac{\lambda^2}{2c} Y \frac{2\sqrt{r}}{\lambda\sqrt{\pi}} \exp(-r(w^0-w^*)^2/\lambda^2).$$

Taking limits and simplifying, we obtain

$$\lim_{w^0 \downarrow 0} Q(rw^*+c, w^0, w^*) = \frac{\int_0^{w^*} \exp(r(w-w^*)^2/\lambda^2)\, dw}{\int_0^{w^*} \exp(r(w-w^*)^2/\lambda^2)\, dw + \frac{\lambda^2}{2c}}$$
$$=: \hat{Q}(w^*).$$

Next, we show that there exists $w^* \in (0, (\mu_H - c)/r)$ such that

$$\hat{Q}(w^*)\mu_H + (1 - \hat{Q}(w^*))\mu_L > u = rw^* + c. \tag{B.2.2}$$

This inequality is equivalent to

$$\int_0^{w^*} \exp(r(w-w^*)^2/\lambda^2)\, dw(\mu_H - c - rw^*) + \frac{\lambda^2}{2c}(\mu_L - c - rw^*) > 0.$$

The integrand above is bounded below by 1, so the left hand side as a whole is bounded below by

$$w^*(\mu_H - c - rw^*) + \frac{\lambda^2}{2c}(\mu_L - c - rw^*), \tag{B.2.3}$$

which can be written as $g(w^*)$ where $g(x) := Ax^2 + Bx + c$, $A := -r$, $B := \mu_H - c - \frac{\lambda^2 r}{2c}$ and $C := \frac{\lambda^2}{2c}(\mu_L - c)$. Note that g is a concave quadratic function with $g(0) = \frac{\lambda^2}{2c}(\mu_L - c) < 0$, and hence if g has any real roots, either both are positive or both are negative. Moreover, both roots are positive if and only if their sum is positive. Now g has real roots with a positive sum if and only if both of the following conditions hold:

$$0 < B^2 - 4AC = \left(\mu_H - c - \frac{\lambda^2 r}{2c}\right)^2 + 4r\frac{\lambda^2}{2c}(\mu_L - c) \qquad \text{and} \tag{B.2.4}$$

$$0 < -\frac{B}{A} \iff 0 < B = \mu_H - c - \frac{\lambda^2 r}{2c}. \tag{B.2.5}$$

Using $\lambda = \frac{c}{\mu_H - \mu_L}$, these inequalities expand to, respectively,

$$0 < \left(\mu_H - c - \frac{cr}{2(\mu_H - \mu_L)^2}\right)^2 + 2\frac{cr}{(\mu_H - \mu_L)^2}(\mu_L - c) \qquad \text{and} \tag{B.2.6}$$

$$0 < \mu_H - c - \frac{cr}{2(\mu_H - \mu_L)^2}. \tag{B.2.7}$$

Collecting r terms, the right side of (B.2.6) is a convex quadratic in r,

$$h(r) := \frac{c^2}{4(\mu_H - \mu_L)^4}r^2 + \left(\frac{-c^2 + 2c\mu_L - c\mu_H}{(\mu_H - \mu_L)^2}\right)r + (\mu_H - c)^2$$

with sign pattern $+, -, +$. It follows that $h(0) > 0$ and $h'(0) < 0$. The inequality (B.2.7) is equivalent to

$$r < \bar{r} := \frac{2(\mu_H - c)(\mu_H - \mu_L)^2}{c}.$$

For $r = \bar{r}$, the first term on the right side of (B.2.6) vanishes while the second term is negative, so $h(\bar{r}) < 0$. It follows that h has two real roots, both positive. Now $h(r)$ is decreasing for all $r \in [0, \bar{r}]$ and in this interval, $h(r) \geq 0$ if and only if $r < r^*$, where r^* is the lower of the two roots of h, given explicitly by (3.4.5). Hence, for $r < r^*$, (B.2.6) and (B.2.7) hold, and therefore g has two positive roots.

It is easy to verify that $g'(x)|_{x=(\mu_H - c)/r} < 0$, and since g is concave, the two roots of g must lie in $(0, (\mu_H - c)/r)$, which implies there exists w^* in this interval such that $g(w^*) > 0$.

Retracing the earlier steps, (B.2.2) holds for such w^* and hence by continuity there exists $w^0 \in (0, w^*)$ such that $Q(u, w^0, w^*)\mu_H + (1 - Q(u, w^0, w^*))\mu_L > u$ for $u = rw^* + c$. By the intermediate value theorem, there exists $u \in (rw^*, rw^* + c)$ such that average output under (u, w^0, w^*) is exactly u, and a feasible organization exists.

We claim that r^* is increasing in μ_H and decreasing in c. For μ_H it suffices to show that the term $c + \mu_H - 2\mu_L - 2\sqrt{(c - \mu_L)(\mu_H - \mu_L)}$ is increasing in μ_H. By direct computation, its derivative w.r.t. μ_H is $1 - \frac{\sqrt{c-\mu_L}}{\sqrt{\mu_H - \mu_L}} > 0$ as $\mu_H > c$. For c, we have

$$\begin{aligned}
\frac{\partial}{\partial c} r^* &= \frac{2(\mu_H - \mu_L)^2}{c^2}\left(-\mu_H + 2\sqrt{(c-\mu_L)(\mu_H-\mu_L)} + 2\mu_L - c\frac{\mu_H - \mu_L}{\sqrt{(c-\mu_L)(\mu_H-\mu_L)}}\right) \\
&< \frac{2(\mu_H - \mu_L)^2}{c^2}\left(-\mu_H + 2\sqrt{(c-\mu_L)(\mu_H-\mu_L)} + 2\mu_L - c\right) \\
&= \frac{4(\mu_H - \mu_L)^2}{c^2}\left(\sqrt{(c-\mu_L)(\mu_H-\mu_L)} - \frac{(\mu_H - \mu_L) + (c - \mu_L)}{2}\right)
\end{aligned}$$

which is negative by applying the Arithmetic Mean–Geometric Mean inequality to the pair of positive numbers $(c - \mu_L, \mu_H - \mu_L)$.

Now r^* is continuous in (μ_H, μ_L, c) and vanishes when $\mu_H = c$, giving the limit result for $|\mu_H - c| \to 0$. As $\mu_H \to \infty$, note that both the first factor and the second factor are positive and tend to infinity as $\mu_H \to \infty$. As $c \to 0$, the first factor tends to $+\infty$ and the second factor tends to $\left(0 + \mu_H - 2\mu_L - 2\sqrt{(0 - \mu_L)(\mu_H - \mu_L)}\right) > 0$, giving the last two limits in the proposition. □

The following lemma shows the existence of a "wedge" in the graph of the feasible set.

Lemma B.2.3. *For all $u \in (0, \mu_H)$, for sufficiently small $w^0 > 0$, $w^* > w^0$ whenever (w^*, w^0) is in the u-supportive set.*

Proof. We show that $\lim_{w^0 \to 0} Q(u, w^0, w^0) = 0$. We have $Q(u, w^0, w^0) = \frac{\alpha(u, w^0, w^0)}{\alpha(u, w^0, w^0) + \nu(u, w^0, w^0)}$

and we show that $\lim_{w^0 \to 0} \frac{\alpha(u,w^0,w^0)}{\nu(u,w^0,w^0)} \to 0$. Expanding,

$$\begin{aligned}
\lim_{w^0 \to 0} \frac{\alpha(u, w^0, w^0)}{\nu(u, w^0, w^0)} &= \lim_{w^0 \to 0} \frac{\int_0^{w^0} f_-(w)\, dw}{\frac{\lambda^2}{2(u-rw^0)} f_-(w^0)} \\
&= \frac{\lim_{w^0 \to 0} f_-(w^0)}{\lim_{w^0 \to 0} \frac{d}{dw^0}\left[\frac{\lambda^2}{2(u-rw^0)} f_-(w^0)\right]}.
\end{aligned}$$

The numerator has limit $f_-(0) = 0$, while the denominator has limit

$$\begin{aligned}
&\lim_{w^0 \to 0} \frac{Y\lambda^2}{2\sqrt{\pi}(u - rw^0)^2} \Big[e^{\gamma(w^0)^2} \sqrt{\pi} r \left(\text{erf}\{X\} + \text{erf}\left\{\gamma(w^0)\right\}\right) \\
&\quad + 2(u - rw^0)\gamma'(w^0) \left(1 + e^{\gamma(w^0)^2} \sqrt{\pi} \left(\text{erf}\{X\} + \text{erf}\left\{\gamma(w^0)\right\}\right) \gamma(w^0) \right) \Big] = \frac{\psi}{u} > 0,
\end{aligned}$$

where we have used that $\lim_{w^0 \to 0} \left(\text{erf}\{X\} + \text{erf}\left\{\gamma(w^0)\right\}\right) = \text{erf}\{X\} + \text{erf}\{-X\} = 0$. It follows that $\lim_{w^0 \to 0} \frac{\alpha(u,w^0,w^0)}{\nu(u,w^0,w^0)} = 0$, as desired. □

Lemma B.2.4. *The limit $Q^0(u, w^*) := \lim_{w^0 \to 0} Q(u, w^0, w^*)$ is well-defined, and $Q^0(u, w^*)$ is a differentiable, single-peaked function of w^* which is maximized at some $w^*_{PC}(u) \in (0, u/r)$. Moreover, $Q^0(u, w^*_{PC}(u)) < 1$.*

Proof. As $\lim_{w^0 \to 0} Q(u, w^0, w^*)$ is of the form 0/0, we use L'Hôpital's rule. Using the expressions from the proof of Lemma B.2.1,

$$\begin{aligned}
\lim_{w^0 \to 0} Q(u, w^0, w^*) &= \lim_{w^0 \to 0} \frac{\alpha_{w^0}(u, w^0, w^*)}{\alpha_{w^0}(u, w^0, w^*) + \nu_{w^0}(u, w^0, w^*)} \\
&= \frac{\int_0^{w^*} e^{\gamma(w)^2} dw}{\int_0^{w^*} e^{\gamma(w)^2} dw + \frac{\lambda^2}{2(u-rw^*)} e^{\gamma(w^*)^2}} \\
&=: Q^0(u, w^*).
\end{aligned}$$

It is clear that $Q^0(u, w^*)$ is twice continuously differentiable. We argue that $Q^0(u, w^*)$ is single-peaked in w^*, i.e., that $\frac{\partial^2 Q^0(u,w^*)}{(\partial w^*)^2} < 0$ whenever $\frac{\partial Q^0(u,w^*)}{\partial w^*} = 0$. Define $\alpha_0(w^*) := \int_0^{w^*} e^{\gamma(w)^2} dw$ and $\nu_0(w^*) := \frac{\lambda^2}{2(u-rw^*)} e^{\gamma(w^*)^2}$, so that $Q^0(u, w^*) = \frac{\alpha_0(w^*)}{\alpha_0(w^*)+\nu_0(w^*)}$. By arguments in the proof of Lemma B.2.2, it is enough to show that $\frac{\nu_0''}{\nu_0'} > \frac{\alpha_0''}{\alpha_0'}$ whenever $\left(\frac{\nu_0}{\alpha_0}\right)' = 0$, i.e., whenever $\alpha_0 \nu_0' = \nu_0 \alpha_0'$. Define $f_0(w^*) := e^{\gamma(w^*)^2}$. The rest of the proof of single-peakedness is then isomorphic to the proof of Lemma B.2.2, since $f_0(w^*)$ is a positive constant multiple of $f_+(w^*)$ (since $w^0 > 0$ in Lemma B.2.2).

Next, it is straightforward to verify that $Q^0(u, 0) = 0$ and $\lim_{w^* \to u/r} Q^0(u, w^*) = 0$ and that $Q^0(u, w^*) > 0$ for all $w^* \in (0, u/r)$, and since $Q^0(u, w^*)$ is single-peaked in w^*, it attains its maximum on $[0, u/r]$ at some unique $w^*_{PC}(u) \in (0, u/r)$. Finally, it is clear from inspection that $Q^0(u, w^*) < 1$ for all $w^* \in (0, u/r)$, so in particular, $Q^0(u, w^*_{PC}(u)) < 1$. $\square$

Proof of Proposition 3.4.2. Suppose $r > \frac{2\mu_H(\mu_H - c)}{\lambda^2}$. By Lemma B.2.1, if a feasible platform exists, then a feasible platform exists for arbitrarily low $w^0 > 0$. Fixing u and w^0, by Lemma B.2.2, the set of w^* such that (w^0, w^*) is u-supportive is an interval $[\underline{w}, \overline{w}]$. By Lemma B.2.3, for sufficiently low $w^0 > 0$, $\underline{w} > w^0$ which implies that $Q(u, w^0, \underline{w}) = Q(u, w^0, \overline{w})$, and by Lemma B.2.2 there exists a unique $w^* \in [\underline{w}, \overline{w}]$ such that $\frac{\partial}{\partial w^*} Q(u, w^0, w^*) = 0$, which implies $\alpha_{w^*} \nu = \alpha \nu_{w^*}$. At such a w^*, we have

$$\begin{aligned}
\frac{Q(u, w^0, w^*)}{1 - Q(u, w^0, w^*)} = \frac{\alpha}{\nu} = \frac{\alpha_{w^*}}{\nu_{w^*}} &= \frac{f_+(w^*)}{\frac{\lambda^2}{2(u - rw^*)^2}[(u - rw^*)f_+'(w^*) + rf_+(w^*)]} \\
&= \frac{f_+(w^*)2(u - rw^*)^2}{\lambda^2[(u - rw^*)2\gamma(w^*)\gamma'(w^*)f_+(w^*) + rf_+(w^*)]} \\
&= \frac{2(u - rw^*)^2}{2(u - rw^*)(rw^* - (u - c)) + r\lambda^2}.
\end{aligned}$$

The platform (u, w^0, w^*) is not feasible if $\frac{Q(u, w^0, w^*)}{1 - Q(u, w^0, w^*)} < \frac{u - \mu_L}{\mu_H - u}$; we now show this inequality holds. Using the expression above this is equivalent to

$$\frac{2(u - rw^*)^2}{2(u - rw^*)(rw^* - (u - c)) + r\lambda^2} < \frac{u - \mu_L}{\mu_H - u}. \tag{B.2.8}$$

The denominator on the left hand side of (B.2.8) must be positive, otherwise we would have $\nu = 0$ which is impossible with $w^* < u/r \leq \mu_H/r < \infty$. It follows that (B.2.8) is equivalent to

$$0 > 2(u - rw^*)^2(\mu_H - u) - [2(u - rw^*)(rw^* - (u - c)) + r\lambda^2](u - \mu_L). \tag{B.2.9}$$

The right hand side above is a quadratic function of w^* which is convex since it has coefficient $2r^2(\mu_H - \mu_L) > 0$ on $(w^*)^2$. Evaluated at $w^* = u/r$, it simplifies to $-r\lambda^2(u - \mu_L) < 0$. Evaluated at $w^* = 0$, it simplifies to $k(u) := 2u^2[\mu_H - c - \mu_L] + 2uc\mu_L - r\lambda^2(u - \mu_L)$. We now show that $k(u) < 0$. Note that k is a convex, quadratic function of u with $k(0) = r\lambda^2\mu_L < 0$. Moreover, $k(\mu_H) = -(\mu_H - \mu_L)[r\lambda^2 - 2\mu_H(\mu_H - c)] < 0$ by assumption. Hence by convexity in w^*, (B.2.9) holds for all u, w^* such that $0 < rw^* < u < \mu_H$, completing the proof. □

Proof of Proposition 3.4.3. Define $\chi := \frac{2(\mu_H - c)(\mu_H - \mu_L)^2}{e^{(\mu_H - \mu_L)^2}(c - \mu_L) + 2(\mu_H - c)(\mu_H - \mu_L)^2} \in (0, 1)$. We show that in particular, if $r < \chi$, there exists a feasible platform with $u < c$. As in the proof of Proposition 3.4.1, $\lim_{u \downarrow rw^*} Q(u, w^0, w^*) = 0$ for all (w^0, w^*) with $0 < w^0 \le w^*$, and hence for sufficiently small $u > rw^*$, we have $\mu_H Q(u, w^0, w^*) + \mu_L(1 - Q(u, w^0, w^*) < u$. Using $u = c$, we first establish the existence of (w^0, w^*) with $0 < w^0 \le w^* < u/r$ such that output strictly exceeds u. We have

$$\begin{aligned}\alpha(u, w^0, w^*)|_{u=c} &= \int_0^{w^0} Y \exp(rw^2/\lambda^2) \left[\text{erf}\,\{0\} + \text{erf}\left\{\frac{\sqrt{r}w}{\lambda}\right\}\right] dw \\ &\quad + \int_{w^0}^{w^*} Y \exp(rw^2/\lambda^2) \left[\text{erf}\,\{0\} + \text{erf}\left\{\frac{\sqrt{r}w^0}{\lambda}\right\}\right] dw \\ \nu(u, w^0, w^*)|_{u=c} &= \frac{\lambda^2}{2(c - rw^*)} Y \exp(r(w^*)^2/\lambda^2) \left[\text{erf}\,\{0\} + \text{erf}\left\{\frac{\sqrt{r}w^0}{\lambda}\right\}\right].\end{aligned}$$

Recall that a platform's output exceeds u if and only if $\frac{Q(u,w^0,w^*)}{1-Q(u,w^0,w^*)} > \frac{u-\mu_L}{\mu_H-u}$. By the usual use of L'Hôpital's rule, for $u = c$, we have

$$\begin{aligned}\lim_{w^0 \downarrow 0} \frac{Q(u, w^0, w^*)}{1 - Q(u, w^0, w^*)} &= \frac{\int_0^{w^*} \exp(rw^2/\lambda^2) dw}{\frac{\lambda^2}{2(c - rw^*)} \exp(r(w^*)^2/\lambda^2)} \\ &= \frac{2(c - rw^*)}{\lambda^2} \int_0^{w^*} \exp[r(w^2 - (w^*)^2)/\lambda^2] dw.\end{aligned}$$

Evaluating at $w^* = c/\sqrt{r}$ (which is less than u/r since $r < 1$), the last expression becomes

$$\frac{2(c - c\sqrt{r})}{\lambda^2} \int_0^{c/\sqrt{r}} \exp[(rw^2 - c^2)/\lambda^2] dw,$$

which, using that $r < 1$, is bounded below by $\frac{2(c - c\sqrt{r})}{\lambda^2}\frac{c}{\sqrt{r}}e^{-c^2/\lambda^2} = 2(1/\sqrt{r} - 1)(\mu_H - \mu_L)^2 e^{-(\mu_H - \mu_L)^2}$. Now this expression is strictly decreasing in r, and comparing it to $\frac{c-\mu_L}{\mu_H - c}$ and solving for r yields the sufficient condition stated; when this condition holds, by continuity in w^0, we have for sufficiently small w^0 that $(w^0, w^*) = (w^0, c/\sqrt{r})$ supports u with $\mu_H Q(u, w^0, w^*) + \mu_L(1 - Q(u, w^0, w^*) > u$ for $u = c$. From this and the observation at the beginning of the proof about $u \downarrow rw^*$, the intermediate value theorem gives existence of $u < c$ such that $\mu_H Q(u, w^0, w^*) + \mu_L(1 - Q(u, w^0, w^*) = u$, so we conclude there exists a feasible organization exists with $u < c$. □

Proof of Proposition 3.4.4. We must only prove the first claim, since the rest of the proposition is a summary of Lemmas B.2.1, B.2.2 and B.2.3. First, observe that in the proof of Proposition 3.4.2, no organization with flow payoff u is feasible if (B.2.9) holds for all $w^* \in (0, u/r)$. By the arguments there, the maximum value (over $w^* \in (0, u/r)$) of the RHS tends to a negative limit as $u \to 0$, and hence by continuity, there exists $\underline{u} \in (0, \mu_H)$ such that for all $u \in (0, \underline{u})$, no u-supportive organization exists. To establish $\overline{u}$, recall that if an organization is feasible, then $\frac{Q(u,w^0,w^*)}{1-Q(u,w^0,w^*)} \geq \frac{u-\mu_L}{\mu_H - u}$, and since Q is strictly decreasing in w^0, it must be that $\lim_{w^0 \to 0} \frac{Q(u,w^0,w^*)}{1-Q(u,w^0,w^*)} \geq \frac{u-\mu_L}{\mu_H-u}$. Using now familiar calculations, for any $u \in (c, \mu_H)$, the LHS of this inequality simplifies to

$$\begin{aligned}
&\frac{2(u - rw^*)}{\lambda^2} \int_0^{w^*} e^{\gamma(w)^2 - \gamma(w^*)^2} dw \\
&= \frac{2(u - rw^*)}{\lambda^2} \int_0^{w^*} \exp\left[\frac{r(w + w^*) - 2(u - c)}{\lambda\sqrt{r}} \frac{\sqrt{r}(w - w^*)}{\lambda}\right] dw \\
&\leq \frac{2(u - rw^*)}{\lambda^2} \int_0^{w^*} \exp\left[\frac{2(u - c)(w^* - w)}{\lambda^2}\right] dw \\
&\leq \frac{2u}{\lambda^2} w^* \exp\left[\frac{2(u - c)w^*}{\lambda^2}\right] \\
&\leq \frac{2\mu_H}{\lambda^2}\frac{\mu_H}{r} \exp\left[\frac{2(\mu_H - c)\mu_H}{r\lambda^2}\right],
\end{aligned}$$

which is a finite bound independent of u. But as $u \to \mu_H$, $\frac{u-\mu_L}{\mu_H-u} \to \infty$, so the necessary condition fails. Hence there exists $\overline{u} \in (c, \mu_H)$ such that the u-supportive set is empty for all $u > \overline{u}$. ☐

Define $r^{**} : [0,1] \to [0,\infty)$ by

$$r^{**}(\beta) := \frac{(\mu_H - c)^2\beta\left\{3(1-\beta)c + \sqrt{3(1-\beta)c[c(1-\beta)(3+2\beta) + 2\beta(\beta\mu_H - \mu_L)]}\right\}}{3\lambda^2[\beta(\mu_H - c) + c - \mu_L]}.$$

Note that $r^{**}(\beta) > 0$ for all $\beta \in (0,1)$. The following lemma provides an alternative sufficient condition for existence; it can be optimized by choosing $\beta = \arg\max_{\tilde{\beta}\in[0,1]} r^{**}(\tilde{\beta})$.

Lemma B.2.5. *Fix any $\beta \in (0,1)$. If $r < r^{**}(\beta)$, then a feasible platform exists.*

Proof. The proof follows the proof of Proposition 3.4.1 through the step where we have reduced the existence problem to establishing the inequality

$$\int_0^{w^*} \exp(r(w - w^*)^2/\lambda^2)\, dw(\mu_H - c - rw^*) + \frac{\lambda^2}{2c}(\mu_L - c - rw^*) > 0$$

for some $w^* \in (\mu_H - c)/r$. In deriving $r^{**}(\beta)$, we use a sharper lower bound on the integrand, but use a stronger sufficient condition in a later step. Specifically, we have $\exp(r(w - w^*)^2/\lambda^2) \geq 1 + r(w - w^*)^2/\lambda^2$, and hence it suffices to show

$$\begin{aligned} 0 &< (\mu_H - c - rw^*) \int_0^{w^*} [1 + r(w - w^*)^2/\lambda^2] dw + \frac{\lambda^2}{2c}(\mu_L - c - rw^*) \\ &= w^*[1 + r(w^*)^2/(3\lambda^2)](\mu_H - c - rw^*) + \frac{\lambda^2}{2c}(\mu_L - c - rw^*). \end{aligned}$$

Evaluating the expression above at $w^* = \beta(\mu_H - c)/r$ and multiplying through by r^2 yields an inequality involving a quadratic in r:

$$-\frac{\lambda^2[\beta(\mu_H - c) + c - \mu_L]}{2c} r^2 + \beta(1-\beta)(\mu_H - c)^2 r + \frac{\beta^3(1-\beta)(\mu_H - c)^4}{3\lambda^2} > 0.$$

The left hand size is strictly positive at $r = 0$ and is concave. Its unique positive root is $r^{**}(\beta)$, giving the result. ☐

Define $\hat{r} := \frac{(\mu_H-\mu_L)^2\left[3(\mu_H-\mu_L)-\sqrt{(\mu_H-\mu_L)(\mu_H-9\mu_L)}\right]\left(\mu_H+3\mu_L+\sqrt{(\mu_H-\mu_L)(\mu_H-9\mu_L)}\right)^2}{8c^2\left[(\mu_H-\mu_L)+\sqrt{(\mu_H-\mu_L)(\mu_H-9\mu_L)}\right]}$, which is a positive real number as $\mu_H > \mu_L$.

Lemma B.2.6. *If $r > \hat{r}$, then any feasible platform involves $w^0 < w^*$. Moreover, there is a nonempty set of values for (μ_H, μ_L, c, r) such that $r > \hat{r}$ holds while the set of feasible platforms is nonempty.*

Proof. We first show that for $r > \hat{r}$, there is no feasible platform with $w^0 = w^*$. Fix any $u \in (0, \mu_H)$, and consider the function $w^0 \mapsto Q(u, w^0, w^0) = \frac{\int_0^{w^0} f_-(w)dw}{\int_0^{w^0} f_-(w)dw+\nu(w^0)}$. From the proof of Lemma B.2.3, $\lim_{w^0\to 0} Q(u, w^0, w^0) = 0$. On the other hand, taking $w^0 \uparrow u/r$, $\int_0^{w^0} f_-(w)dw$ has a finite limit while $\nu(w^0) = \frac{\lambda^2 f_-(w^0)}{2(u-rw^0)} \uparrow +\infty$, so $\lim_{w^0\uparrow u/r} Q(u, w^0, w^0) = 0$ as well. Now $Q(u, w^0, w^0)$ attains its maximum at some $w^0 \in (0, u/r)$ where the first order condition $0 = \frac{d}{dw^0}Q(u, w^0, w^0) \iff \alpha = \frac{\nu\frac{d\alpha}{dw^0}}{\frac{d\nu}{dw^0}}$ reduces to

$$\alpha = \int_0^{w^0} f_-(w)dw = \frac{f_-(w^0)^2(u - rw^0)}{(u - rw^0)f'_-(w^0) + rf_-(w^0)}. \tag{B.2.10}$$

We now show that at any point satisfying this first order condition, $U(Q(u, w^0, w^0)) = \mu_H Q(u, w^0, w^0) + \mu_L(1 - Q(u, w^0, w^0)) < u$; to do this, we prove the equivalent inequality $\frac{\alpha}{\nu} < \frac{u-\mu_L}{\mu_H-u}$. By (B.2.10),

$$\frac{\alpha}{\nu} = \frac{\frac{f_-(w^0)^2(u-rw^0)}{(u-rw^0)f'_-(w^0)+rf(w^0)}}{\frac{\lambda^2 f_-(w^0)}{2(u-rw^0)}} = \frac{2}{\lambda^2}\frac{f_-(w^0)(u - rw^0)^2}{(u - rw^0)f'_-(w^0) + rf_-(w^0)} < \frac{2(u - rw^0)^2}{\lambda^2 r},$$

where we have used that $f'_-(w^0) > 0$. It suffices then to show that $\frac{2(u-rw^0)^2}{\lambda^2 r} < \frac{u-\mu_L}{\mu_H-u}$, or equivalently $r > \frac{2(u-rw^0)^2(\mu_H-u)}{\lambda^2(u-\mu_L)}$. Since $w^0 \in (0, u/r)$, we have $(u - rw^0)^2 < u^2$, and thus it suffices to show that $r > j(u) := \frac{2u^2(\mu_H-u)}{\lambda^2(u-\mu_L)}$. Our claim is that $\hat{r} = \max_{u\in(0,\mu_H)} j(u)$, so that the condition $r > \hat{r}$ as originally stated is sufficient. Now $j'(u) = -\frac{2}{\lambda^2}\frac{u[2u^2-u(\mu_H+3\mu_L)+2\mu_H\mu_L]}{(u-\mu_L)^2}$. It is easy to verify that there is a unique $u \in (0, \mu_H)$ at which $j'(u) = 0$ and where the maximum is attained, namely $u^* := \frac{1}{4}\left[\mu_H + 3\mu_L + \sqrt{(\mu_H + 3\mu_L)^2 - 16\mu_H\mu_L}\right]$, and $j(u^*) = \hat{r}$.

To prove the second claim of the lemma, it suffices to show that for some $\beta \in (0,1)$ there exists an instance of (μ_H, μ_L, c) such that $\hat{r} < r^{**}(\beta)$ and thus the set of feasible organizations with $r > \hat{r}$ is nonempty by Lemma B.2.5. For $\beta = 1/2$, $(\mu_H, \mu_L, c) = (1/32, -1, 7/1024)$ is one such instance. □

Proof of Proposition 3.4.5. The "only if" direction is trivial, as we can take $\tilde{u} = u$. For the "if" direction, suppose that $(w^0, w^*) \in S^u$ for some $u \in (0, \mu_H)$, with $U(Q(u, w^0, w^*)) \geq u$; if equality holds, we are finished, so suppose there is strict inequality. Now U is continuous and Q is continuous (in particular in u), and hence $\tilde{u} \mapsto U(Q(\tilde{u}, w^0, w^*)) - \tilde{u}$ is continuous. By assumption, $U(Q(\tilde{u}, w^0, w^*)) - \tilde{u} > 0$ for $\tilde{u} = u$, and $\lim_{\tilde{u} \downarrow rw^*} = U(0) - rw^* < 0$, so by the intermediate value theorem there exists $\tilde{u} \in (rw^*, u)$ such that $U(Q(\tilde{u}, w^0, w^*)) = \tilde{u}$. The second statement of the proposition is merely a translation of the first statement into notation, and the third statement is a summary of Lemma B.2.6. □

B.3 Proofs for section 3.5

Proof of Proposition 3.5.1. First consider any fixed u such that S^u is nonempty and consider a sequence of organizations in S^u for which organizational size converges to the supremum of organizational size across organizations in S^u. Note that the set of such u is a compact subset of $[\underline{u}, \overline{u}]$. This sequence cannot involve $w^0 \to 0$, since then organizational size vanishes. Hence, it lies in a compact subset of S^u and must have a subsequence which converges to a point (u, w^0, w^*) in this compact subset; (u, w^0, w^*) then maximizes the organizational size across feasible organizations. Since organizational size is increasing in w^0, this point must satisfy $w^0 = \overline{w}^0(w^*)$, and moreover, it must lie on the northeastern frontier of the u-supportive set, since organizational size is increasing in w^*. We argue that (u, w^0, w^*) satisfies (3.4.4) and is thus feasible. Clearly, this must be true if $w^0 < w^*$, otherwise (u, w^0, w^*) would not be on the northeastern frontier (there would exist $\tilde{w}^* > w^*$ such that $(u, w^0, \tilde{w}^*)$ is also u-supportive). Now if the top of the northeastern frontier includes a point on the 45-degree line, that point is the limit of some sequence of points on the northeastern frontier lying below the 45-degree line, and by continuity, it must be that (3.4.4) is satisfied at the top. So we conclude that any size-maximizing organization in the u-supportive set is feasible.

By the maximum theorem, the function mapping u to the maximum organizational size over organizations in S^u is continuous. Since the set of u for which S^u is nonempty is compact, this function attains its maximum and hence there exists a size-maximizing feasible organization, $(u_{OS}, w^0_{OS}, w^*_{OS})$. This organization also maximizes organizational size within $S^{u_{OS}}$ and hence it lies on the northeastern frontier of $S^{u_{OS}}$. □

Proof of Lemma 3.5.1. Fix $0 < w^0 \leq w^*$ and consider any u_1, u_2 with $u_2 > u_1 > w^* r$. Let prime notation denote derivatives with respect to w, and consider f_-. Recall that the boundary conditions $f_-(0; u) = 0$ and $f'_-(0+; u) = \frac{2\psi}{\lambda^2}$ are independent of u. Subtracting the ODE (B.1.1) for u_2 from the one for u_1, we have $f''_-(0+; u_1) - f''_-(0+; u_2) = \frac{2}{\lambda^2}(u_2 - u_1)f'(0; u_1) > 0$. Hence, for sufficiently small $w > 0$, $f_-(w; u_1) > f_-(w; u_2)$ and $f'_-(w; u_1) > f'_-(w; u_2)$. We claim that the relationship $f'_-(w; u_1) > f'_-(w; u_2)$, and hence $f_-(w; u_1) > f_-(w; u_2)$, extends to all $w > 0$. If not, let w' be the smallest $w > 0$ for which $f_-(w; u_1) = f_-(w; u_2)$, and let w'' be the smallest $w \in (0, w')$ such that $f'_-(w; u_1) = f'_-(w; u_2) > 0$. Again using subtracting the ODEs, these facts imply that $f''_-(w''; u_1) > f''_-(w''; u_2)$. But this implies that for all $w \in (0, w'')$, $f'_-(w; u_1) < f'_-(w; u_2)$, a contradiction. Hence $f_-(w; u_1) > f_-(w; u_2)$ and $f'_-(w; u_1) > f'_-(w; u_2)$ for all $w > 0$. Turning to f_+, note that both of those inequalities hold in particular at w^0, and hence the third boundary condition from Lemma B.1.1 implies $f'_+(w^0; u_1) > f'_+(w^0; u_2)$. Now $f_+(w^0; u_1) = f_-(w^0; u_1) > f_-(w^0; u_2) = f_+(w^0; u_2)$, and a similar argument to that above shows that $f_+(w; u_1) >$

$f_+(w;u_2)$ and $f'_+(w;u_1) > f'_+(w;u_2)$ for all $w \in (w^0, w^*]$, establishing the claim for f_+. Since $\alpha = \int_0^{w^0} f_-(w)\,dw + \int_{w^0}^{w^*} f_+(w)\,dw$, where we have established that the integrands are pointwise decreasing in u, α is decreasing in u. Finally, $\nu = \frac{\lambda^2}{2(u-rw^*)} f_+(w^*)$; both factors are decreasing in u, and so is ν. □

Proof of Proposition 3.5.2. As argued in the main text, there is no solution to the maximization problem since w^0 is restricted to be positive. Since Q is decreasing in w^0 and $w^*_{PC}(u)$ maximizes $Q(u, w^*)$ (see Lemma B.2.4), we have $V(u, w^0, w^*) \leq Q^0(u, w^*_{PC}(u))$ for all feasible (u, w^0, w^*). Now define u_{PC} to be the supremum of the set of u such that $U(Q^0(u, w^*_{PC}(u))) > u$, which is well-defined and less than $\overline{u}$ as defined in Proposition 3.4.4. Define $w^*_{PC} = w^*_{PC}(u_{PC})$ as the unique maximizer of $Q^0(u_{PC}, w^*)$ with respect to w^*.

Let $\tilde{u}$ denote the supremum of the set of u such that (u, w^0, w^*) is feasible for some w^0, w^*. We show that $u_{PC} = \tilde{u}$. First, note that $u_{PC} \geq \tilde{u}$. Indeed, $\tilde{u} \geq u$ for all feasible (u, w^0, w^*), and $Q^0(u, w^*_{PC}(u)) \geq Q^0(u, w^*) > Q(u, w^0, w^*) = u$, so $u < u_{PC}$. Second, note that $u_{PC} \leq \tilde{u}$, since if $u \leq u_{PC}$ is such that $U(Q^0(u, w^*_{PC}(u))) > u$, there exists by continuity an organization (u, w^0, w^*) such that $U(Q(u, w^0, w^*)) > u$ and hence a feasible organization (u', w^0, w^*) with $u' > u$; since $\tilde{u} \geq u'$, and u can be taken arbitrarily close to u_{PC}, we have $u_{PC} \leq \tilde{u}$. We conclude that u_{PC} is the supremum of per capita output over the set of feasible organizations, and hence u_{PC} can be approximated arbitrarily closely using a sequence of feasible organizations.

Finally, since per capita output is decreasing in w^0 and since w^*_{PC} is the unique maximizer of $Q(u_{PC}, w^*)$ wrt w^*, it follows that if a sequence of feasible organizations converges to a vector other than $(u_{PC}, 0, w^*_{PC}(u_{PC}))$, the principal's value converges to a value less than u_{PC}, so any sequence of feasible organizations for which per capita output converges to u_{PC} must converge to $(u_{PC}, 0, w^*_{PC})$. □

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