SCHOOLS OF GREEK MATHEMATICAL PRACTICE

by

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Dissertation submitted in partial fulfillment of
the requirements for the degree of Doctor
of Philosophy in the Department of
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This dissertation revolves around a central observation that, although the methodological differences among Greek mathematical writings are striking, these differences do not lie primarily along lines of subject matter or time period. Almost all mathematical works fall clearly into one of two distinct sets of methodological conventions, which are observable from the classical period through late antiquity, in all disciplines. Because these sets of conventions transcend time and subject, and instead seem to be followed consistently by certain authors who interact among themselves in the manner of philosophical traditions, I have interpreted them as schools of mathematical practice.

I have named the schools "systematist" and "heurist" according to the characteristic epistemological orientation of each. The systematist school, of which Euclid is the paradigmatic author, is motivated by the goal of a generalized and systematic treatment of mathematical information. Features of this method include strict conventions of presentation, the idealization of mathematical objects, a preference for universalizing propositions over unique problems, and a general reluctance to work with specific numbers, physical tools, or measurements. The heurist school, in which Archimedes, Heron of Alexandria, and Ptolemy worked, is oriented toward the discovery and development of effective methods of problem-solving. Presentation is less structured and usually more personalized, specific solutions are allowed to stand implicitly for universal principles, and physical phenomena, measurement, tools, and numerical calculations are more commonly included and addressed.

Over the course of this work I will demonstrate the existence of these two schools with a thorough survey of Greek works of theoretical mathematics; I will outline the schools' characteristic features and histories, and show how the influences of philosophical movements and intellectual social networks affected their practices. Each of the four chapters addresses the ways in which the methods of each school were expressed in the four most common mathematical disciplines: geometry, arithmetic, astronomy, and music. The chapters will show not only the evidence for the divisions between the schools in each field, and how they developed, but also that both schools made only minor adaptations to their methodologies according to subject matter. In fact, it can be shown that even when they departed from more traditional mathematical disciplines into areas of research such as mechanics and catoptrics (i.e. fields of science and technology that use mathematical tools but are not essentially governed by mathematical principles), the epistemological and methodological differences between the systematist and heurist schools were retained.

The conclusion will show, first, that the systematist/heurist divide was fundamental and pervasive throughout the history of Greek works on mathematics and related disciplines, but that the divide was largely obscured by the activities of late antique scholars. Second, the conclusion will provide a brief sketch of how the combination of late antique transmission and modern reception of Greek mathematics have affected not only the perception of ancient mathematics (giving undue emphasis to systematist texts, despite evidence that the heurists were the larger school), but also the development of modern mathematical methods.

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ACKNOWLEDGEMENTS

I wish to express my gratitude to those whose support has made this research possible: to my parents, Timothy and Mary Winters, who were my first teachers in classics, and my sisters, Kate, Anne, and Rebecca, who were my first teachers in humility; to the members of the Duke classics department, who warmly accepted a graduate student with a non-traditional background; to my committee, for their kind advice and insight; and to my advisor, Dr. Joshua Sosin, to whose brilliance, scholarly integrity, and dedication to his students I will always aspire.

INTRODUCTION: REVISING THE HISTORY OF GREEK MATHEMATICS

A deeply embedded tradition in the modern study of ancient mathematics groups three Hellenistic authors together as the vanguard of Greek mathematical genius: Euclid, Archimedes, and Apollonius.[1] These three authors, each a near predecessor of the next, wrote extensively (and brilliantly) on geometry between the late fourth and early second centuries BCE, producing works that were admired, studied, and preserved by their successors into posterity. The perception of them as a group of founding fathers who set the standards of their discipline and elevated the western mathematical tradition to its highest expression until the Cartesian revolution is as understandable as it is pervasive.

The argument of this dissertation is that despite this perception, Euclid, Archimedes, and Apollonius are not a group. That is, they are not members of a single group, but rather of two groups, which I am calling schools of practice. I will define and explain the differences between these schools at length in the following chapters. Euclid and Apollonius belong to a school that I have named "systematist," Archimedes to a

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[1] E.g. W. W. Rouse Ball, *A Short Account of the History of Mathematics* (London: MacMillan & co., 1912), 50: "[Alexandria] was particularly fortunate in producing within the first century of its existence three of the greatest mathematicians of antiquity—Euclid, Archimedes, and Apollonius." Or consider T. L. Heath, *A History of Greek Mathematics*, vol. 1 (Oxford: The Clarendon Press, 1921), 1: "Not only are the range and the sum of what the Greek mathematicians actually accomplished wonderful in themselves; it is necessary to bear in mind that this mass of original work was done in an almost incredibly short space of time…" and, after discussing Euclid's work and its origins, and the work of Archimedes, p. 3: "With Apollonius, the main body of Greek geometry is complete…" Much more recently S. Cuomo, *Ancient Mathematics* (London ; New York: Routledge, 2001), 63, divides Hellenistic mathematicians into "little people" and "Big Guys (Euclid, Archimedes, and Apollonius)."

school that I have named "heurist." Archimedes, despite lying between the other two authors in time, does not represent a transition from the work of Euclid to the work of Apollonius. Apollonius, in fact, shares a much closer affinity with Euclid in style and methodology than Archimedes shares with either of them. Archimedes' research interests differed from theirs (ranging into theoretical mechanics and hydrostatics), and when he did work on similar material (such as conic sections, circles, or large numbers), he worked in a very different way. Archimedes was not grouped with the other two in antiquity,[2] nor were his works received or transmitted along the same lines as theirs during the middle ages. The understanding of these three authors as members of a single tradition is, in short, a modern phenomenon.

In the following chapters, I am going to argue that the methodological and stylistic differences between Euclid and Apollonius on the one hand, and Archimedes on the other, are representative of their respective schools, and can be seen throughout the body of extant Greek mathematical texts. I will show that these differences were felt by ancient authors and by later commentators and editors, and that the schools understood themselves as groups to the extent of authors tending to reference members of their own school and ignore those of the other. Because modern scholarship has focused on the

[2] Indeed, there seems to have been a tradition of perceived rivalry between them. The sixth-century CE commentator Eutocius tried to debunk the story that Apollonius had stolen his ideas about conic sections from Archimedes' unpublished work (J. L. Heiberg, ed., *Apollonii Pergaei quae Graece exstant cum Commentariis Antiquis*, vol. 2 (Leipzig: Teubner, 1891), 168.5-17). In addition, as I shall show later, authors who referred most often to Archimedes' work tended to ignore the other two, and vice versa.

history of specific mathematical concepts rather than on the style and methodology of texts, the division between the two schools has been missed.

Instead of recognizing schools of practice, modern understanding has essentially interpreted Euclid's *Elements*, a systematist text, as the perfect paradigm of Greek mathematical writing. Works of the heurist school have thus been seen as either isolated moments of unique genius (Archimedes' *Method*, for example), or as failures to attain the Greek standard of rigor (Heron's *Metrica*). This is partially an inheritance from late antique scholarship that gave more attention on the whole to works of the systematist school—with the result that, though it seems to have begun as a smaller, newer, and more specialized group than the heurists,[3] the systematist school came to dominate the historical record. Thus, much of what is said about "classical Greek mathematics" actually only applies to systematist texts.[4]

In addition to obscuring the contributions of the heurists, the modern failure to recognize the schools has largely elided the influence of philosophy and intellectual culture at large on the practices of theoretical mathematicians. The systematist school grew up around the Platonic academy and was strongly associated with the Pythagorean tradition. Texts of that school sought to remove mathematics from the physical world as

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[3] Many of the heurists' methods bear similarities to the geometry of ancient Babylon and Egypt, indicating a wider mathematical diaspora and an older tradition (see Ch. 1 note 141).
[4] e.g. D. H. Fowler, *The Mathematics of Plato's Academy: A New Reconstruction*, 2nd ed. (Oxford: Oxford University Press, 1999), 10: "My first characteristic of early Greek mathematics is negative: it seems to be completely non-arithmetised." or Morris Kline, *Mathematical Thought from Ancient to Modern Times* (New York: Oxford University Press, 1990), 49: "Classical Greek science, as we shall see, was qualitative."

much as possible, and to generate a universal and systematic exposition of purely intellectual truths. The heurists, on the other hand, understood mathematics as inextricably linked to the observable and physical world, and sought out creative and repeatable problem-solving methods for the handling of specific questions.

By reconstructing the lost history of the Greek mathematical schools, I hope not only to restore awareness of a major ancient reality and an axis of division within Greek intellectual culture as a whole, but also to shed light on a certain modern blind spot: we tend to talk about the study of mathematics in terms of which the systematist school would approve (timelessness, a universal language, absolute truth, etc.). In doing so, we have unwittingly inherited a tradition from late antique Neoplatonism, and it has affected our own modes of discourse and research. The history of the Greek schools shows that the practice of mathematics is contingent on factors ranging from philosophy and religion to social networks and technology. A clearer understanding of this history and these contingencies will, I hope, inform larger conversations about the underlying factors of our own mathematical practice.

Finally, several points of terminological and methodological clarity are in order. First, the terms "systematist" and "heurist" are original, as is the taxonomy they represent. Although authors clearly felt the differences between them, the ancient schools did not name themselves. I have coined the names in order to highlight the attitudes with which each school acquired, investigated, and presented mathematical knowledge. The names do not, however, express the full extent of the differences between the two schools. For example, the systematist focus on a comprehensive demonstration of mathematical principles and the heurist focus on creative and iterative problem-solving methods are signified as essential characteristics by the names I have given to the schools. Just as characteristically, however, the heurists treat mathematical objects as having physical presence, whereas the systematists idealize them—but this difference is not as clearly signified by the names. No single term can cover all such characteristics, because they are a historical accident rather than the necessary result of any mathematical theory or practice.

Likewise, although most original mathematical texts and indeed most authors fall clearly into one school or the other, there is a handful of occasions on which a text will cross the systematist/heurist border. That is, a primarily systematist text might include a small cluster of propositions that use heurist methods, or vice versa.[5] The possible reasons for such crossovers differ from work to work. In the case of the *Division of the Canon* (Ch. 4), for example, the text itself is likely a patchwork of multiple authors, whereas Archimedes purposely shows two styles of proof in his *Quadrature of the Parabola* (Ch. 1) in order to demonstrate the validity of his own methodological innovation. Texts of this kind are rare, and will be examined on a case-by-case basis.

In setting the boundaries of this project, I have had to make a few arbitrary decisions. First of all, I have divided the texts by subject into the four traditional

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[5] Examples of such texts are Archimedes' *Quadrature of the Parabola*, Hypsicles' *Anaphoricus*, and the pseudo-Euclidean *Division of the Canon.*

disciplines of the quadrivium, with a chapter dedicated to each: geometry, arithmetic, astronomy, and music. This organization reflects ancient terminology to an extent, but should not be mistaken for a fundamental mathematical reality. Most texts necessarily cross such boundaries, and some texts (such as Theon of Smyrna's *Useful Mathematics for the Understanding of Plato*) will thus appear in more than one chapter. Second of all, I have restricted the evidence to texts that include original mathematical work by authors who wrote in Greek between the sixth century BCE and the fourth century CE. I have considered handbooks, introductions, and commentaries only insofar as they can inform us of the methods and styles of the works they discuss. The systematist and heurist schools are most meaningful as a way of grouping original works of theoretical mathematics written for other practitioners. Because handbooks, introductions, and commentaries often do not show original methodologies, it is rarely useful to try to place them in one school or another.

On a related note, when I say that Greek mathematicians used different methods, I am not talking about the difference between what we call theoretical and applied mathematics. Many scholars have already observed that some Greeks practiced purely theoretical mathematics and others did not.[6] But it is not the case that the systematists did

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[6] e.g. Kline, *Mathematical Thought from Ancient to Modern Times*, 49-50: "Educated people did not concern themselves with practical problems […] Mathematical thought was thus separated from practical needs […] When the barrier between the cultured and slave classes was breached in the Alexandrian period (300 BCE to about 600 CE) and educated men interested themselves in practical affairs, the emphasis shifted to quantitative knowledge […]." See also Cuomo, *Ancient Mathematics*, 79-105 for a comparison between Euclid and some of the geometers I have called "heurist."

theoretical math and the heurists did applied math. They both did theoretical math. The primary difference in method between the two schools has almost nothing to do with the modern categories of applied and theoretical.[7] Instead, it has to do with how mathematical objects (like lines and figures) are conceptualized, and how knowledge about them is generated. While authors who wrote in the heurist tradition are more likely than their systematist counterparts to have actually written on applied mathematics, I consider all the works treated in the following chapters to be theoretical, because they are all concerned with the acquisition and presentation of mathematical knowledge for its own sake—that is, none of these works turns mathematics to any practical purpose such as land measurement, building, finance, or timekeeping.

With regard to methodology, the argument I am making requires some unique practices. A larger portion of this work must be a descriptive treatment of primary texts, for several reasons. First, the texts themselves are less commonly studied. Second, in writing on mathematics for humanists, the shorthand of mathematical terminology and

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[7] Ancient and late ancient authors recognized a difference between the mathematical knowledge needed for everyday practical purposes and that acquired for its own sake, or for the sake of philosophy, but the ancient authors do not use the categories of theoretical and applied mathematics. e.g. Plato, *Republic* VII 525b: πολεμικῷ μὲν γὰρ διὰ τὰς τάξεις ἀναγκαῖον μαθεῖν ταῦτα, φιλοσόφῳ δὲ διὰ τὸ τῆς οὐσίας ἁπτέον εἶναι γενέσεως ἐξαναδύντι "For it is necessary for a soldier to learn these things [logistic and arithmetic] because of drawing up battle formations, but for a philosopher because he must take care about Being, rising up out of created things," or Eutocius in his commentary on Archimedes' *Measurement of the Circle*: τοῦτο δὲ ἀκριβέστερον μὲν εἶναι δοκεῖ, οὐ χρήσιμον δὲ πρὸς τὸν Ἀρχιμήδους σκοπόν· ἔφαμεν γὰρ αὐτὸν σκοπὸν ἔχειν ἐν τῷδε τῷ βιβλίῳ τὸ σύνεγγυς εὑρεῖν διὰ τὰς ἐν τῷ βίῳ χρείας, "Now, this [number] seems to be more accurate, but it was not useful for Archimedes' purpose; for we said that in this book he has the purpose of finding what is close enough for use in life." Eutocius, "Commentarii in Dimensionem Circuli," in *Archimedis Opera Omnia, cum Commentariis Eutocii.*, ed. E. S. Stamatis and J. L. Heiberg, vol. 3 (Stuttgart: Teubner, 1972), 258.18-22.

notation common to most modern studies of this material is not useful, and ought to give way to verbal explanation. I have tried to relegate the majority of mathematical explanations to the footnotes. Finally, while many of the individual features of the ancient texts have been pointed out in the secondary literature, I am proposing for the first time a taxonomy into which to sort these features. This will require the collection and arrangement of much information that we already know, but which has never been organized. The format of this dissertation will thus necessarily be somewhat repetitive. For similar reasons, the first chapter will be considerably longer than the rest. Since the taxonomy I am suggesting is original, I have to begin by meticulously tracing its contours. These categories, once defined, will hold in the other disciplines (arithmetic, astronomy, etc.) discussed in later chapters. In each discipline, the categories will be expressed slightly differently according to the mathematical subject matter, so it is important to lay down their initial boundaries as carefully as possible.

Finally, I shall avoid modern algebraic notation. Since the time of Descartes, mathematics has used algebraic notation as its principal means of problem-solving. The innovation of Descartes was the division of all space into a three-dimensional numbered grid, so that any spatial object could be described by the set of coordinates that fulfill a given equation. No ancient Greek work of either school used these methods. Any text that uses modern algebraic notation to express Greek geometry is a translation, with all of the risks and benefits that such entails. But in this case it is a translation not only of language but of the methods themselves, and so it raises the question of how far the means of expressing and "doing" mathematics can actually differ before the substance of the text itself is lost. A scholarly debate over whether or not algebraic notation is an appropriate translation choice for the ancient texts has continued for nearly a hundred years.[8] Both sides have their merits, but since my purpose is to examine different methods used by Greeks to solve mathematical problems, I shall keep strictly to their idiom, including the use of lettered diagrams and a good deal of specialized vocabulary, which I shall gloss whenever appropriate.

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[8] For a survey of the debate, see Viktor Blåsjö, "In Defence of Geometrical Algebra," *Archive for History of Exact Sciences* 70, no. 3 (May 1, 2016): 325–59. The major works include: H. G. Zeuthen, *Die Lehre von den Kegelschnitten im Altertum.*, trans. R. J. D. von Fischer-Benzon (Kopenhagen: A.F. Höst & Sohn, 1886); Bernard Vitrac, "Peut-on Parler d'algèbre Dans Les Mathématiques Grecques Anciennes ?," *Mirror of Heritage (Ayene-Ne Miras)* 3 (2005): 1–44; Sabetai Unguru, "On the Need to Rewrite the History of Greek Mathematics," *Archive for History of Exact Sciences* 15, no. 1 (1975): 67–114; Szabó, *Anfänge der griechischen Mathematik.* (München, Wien: Oldenbourg, 1969); and Paul Tannery, *De la Solution géométrique des problèmes du second degré avant Euclide.* (Bordeaux: Impr. de G. Gounouilhou, 1884).

I. GEOMETRY

Greek geometry, it is often claimed, was unlike any contemporary tradition of geometry: it neither followed the conventions of Babylonian mathematics nor used the methods of Egypt, despite the claims of Greeks themselves (including Herodotus) that the Egyptians were their first teachers in the art.[1] Already, in its earliest extant texts Greek geometry appears as an advanced, highly systematic, theoretical study practiced by members of the philosophical schools. This has led many scholars to ascribe an almost mystical genius to the Greeks, and to see in the abstractness and complexity of their mathematics seeds of the modern habit of scientific inquiry.[2]

But by "Greek geometry" here scholars usually mean only geometry as Euclid did it. The particular conventions and standards of rigor practiced by Euclid have so thoroughly informed modern scholarship, that many still ascribe his characteristics to Greek geometry in general. The extant evidence of Greek geometry, however, shows a diversity of methods, of which the style of Euclid is only a single strain. As impressive as the achievements of this strain are, their prominence in the historical record has eclipsed

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[1] Herodotus, *Histories* II.109. Aristotle makes a similar claim in *Metaphysics* 981b21, but he uses the general μαθηματικαί instead of Herodotus' more specific γεωμετρίη. See also Proclus, *In Primum Euclidis Elementorum Librum Commentarii*, ed. Gottfried Friedlein (Leipzig: Teubner, 1873), 64.18-19.

[2] Paul Tannery, *La Géométrie grecque: Comment son histoire nous est parvenue et ce que nous en savons: Essai critique par Paul Tannery. 1. Ptie. Historie génêrale de la Géométrie Élémentaire* (Paris: Gauthier-Villars, 1887), 4.

authors who worked with other methods.[3] Most modern scholarship on Greek mathematics, therefore, focuses on the history of particular concepts, or on the similarities and differences between ancient and modern methods. But I mean to suggest for the first time that there were discernible schools of Greek mathematicians, which were recognized in antiquity but have not been handed down to modernity. I call one school the systematists. This is the group of which Euclid and Apollonius of Perga are members, and it is the style of geometry usually thought of as "Euclidean"[4] and quintessentially Greek. The other school, that of Archimedes and Heron of Alexandria, I call the heurists. They represent perhaps a much larger group of ancient mathematicians, but they have been much more obscured by posterity. In this chapter I shall argue that the Greek authors who wrote and worked in the tradition of Euclid were a small and specialized group, and that the others worked with very different methods and in independent networks; and furthermore that systematist geometry has its origins in the circle of Plato, which may have influenced its distinctive character.

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[3] See again Fowler, *The Mathematics of Plato's Academy*, 10: "My first characteristic of early Greek mathematics is negative: it seems to be completely non-arithmetised." or Kline, *Mathematical Thought from Ancient to Modern Times*, 49: "Classical Greek science, as we shall see, was qualitative."

[4] I use the term "Euclidean" here to describe the style and methods used by Euclid himself, not its meaning for present-day mathematics. In modern mathematics, "Euclidean geometry" refers to geometry which assumes a certain postulate about the nature of space, specifically Euclid's fifth postulate, that parallel lines do not ever meet when produced. Both systematist and heurist geometry are "Euclidean" according to this modern usage. "Non-Euclidean geometry" explores what can be logically claimed about space and spatial objects if the parallel postulate is not necessarily true. The validity of Euclid's parallel postulate has been a topic of debate since antiquity, but non-Euclidean geometry was only developed in the nineteenth century by authors such as Bolyai, Lobachevsky, and Riemann.

I.1 The Systematist School

Geometry is the study of space and spatial objects: straight lines, curves, angles, plane figures, and three-dimensional figures. The etymology of the word suggests that the measurement of spatial objects should be a primary concern, but geometry also asks questions about relative sizes, constructions of figures, and constraints on the dimensions of figures. Included in this study are all extant texts that self-identify as geometrical, and all texts whose primary subject matter is geometrical according to the definition just given. I have not included texts that make use of geometry only in the service of studying the stars, motion, architecture, or any other topic. The texts surveyed in this section will demonstrate the essential characteristics of systematist geometry, and will reveal an initial group of authors consistently worked in the systematist school.

I.1.1 Euclid

Mathematics is done and taught so differently in the modern world that the systematist style is no longer as broadly familiar as it once was. In order to understand what is different about the heurist geometers, we must first have a strong sense of the distinctive systematist characteristics, which are best exemplified in Euclid's *Elements*. Euclid lived and worked in Alexandria in the 4th-3rd centuries BCE, but nearly nothing else is known of his life. He is not the earliest of his school, but the *Elements* is the largest and earliest complete work of systematist geometry, and it had a powerful influence on later mathematicians. The following features of the *Elements* are seen almost universally among the systematists and quite rarely among the heurists, and so I will survey them thoroughly.

First, there are no numbers, variables, or symbolic notation in the text of the *Elements*. Math is done using words, almost entirely in complete sentences. There are no triangles with sides 3, 4, and 5. If one line is equal to another, Euclid says that "it is equal" (ἴση ἐστίν) instead of using an equal sign (=). A proposition in the *Elements* is a block of straightforward prose accompanied by a lettered diagram. At first glance, the lettered diagram is the closest Euclid comes to something like a variable. That is, a proposition about an equilateral triangle may include a diagram of a triangle whose vertices are labeled A, B, and Γ. In the text, the sides of the triangle will be referred to as AB, AΓ, and BΓ. So, one might be inclined, on seeing a sentence such as ἡ AB ἴση ἐστὶν τῇ AΓ, to think of it as equivalent to a modern expression such as $x = y$, in which AB and AΓ, or x and y, are variables that stand for unknown numbers. Nonetheless, in the *Elements* (and in all works of the same style) spatial magnitudes are never represented by numbers. Euclid always treats geometrical objects and numbers separately. In a few of the books of the *Elements* (VII-IX), theoretical numbers are represented by diagrams of straight lines (in order to avoid using specific numbers), and Euclid is clearly aware that there is at least a potential connection between spatial extension and number,[5] but number

[5] E.g. *Elements* X.5, in which he proves that commensurable magnitudes have to one another the same ratio as a number has to another number.

and spatial magnitude are always treated as separate categories.[6] AB and AΓ are not variables, then, because they do not stand for any unknown quantity. They are merely names given to parts of the diagram for ease of expression (so that the author doesn't have to say "the upper left-hand line side of the triangle").

Second, there are no "problems" in the *Elements.* Euclid's geometry does not set specific tasks such as "find the area of a rectangle" or "find the length of the hypotenuse." Instead, each proposition is either a proof or a construction. A proof logically demonstrates a general truth about a type of figure, whereas a problem would obtain specific information about a specific figure. For example, proposition I.6 proves that if any triangle contains two angles equal to each other, the sides opposite those angles will also be equal to each other. This holds true for any given triangle that fulfills the stated requirements. The majority of propositions in the *Elements* are proofs. A construction is a type of proposition that shows that it is possible to "draw" a figure or a part of a figure with specified features.[7] A construction will first give the method for drawing the figure, and then prove that it has the required features. The first proposition of the *Elements*, in fact, is the construction of an equilateral triangle. It first draws a

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[6] The strictness of this separation has led to many claims that Greek geometry in general was not arithmetized, e.g. Fowler, *The Mathematics of Plato's Academy*, 10. However, I will show later that among the characteristics of the heurist group is a tendency to arithmetize geometry, and this tendency is in evidence before Euclid.

[7] This does not refer to physical drawing, because it is not actually possible to physically draw a line as Euclid defines it (a length with no breadth). Drawing here is theoretical. When a line is drawn in this sense it means that it is able to be added to the figure with certain specifications (such as length and position) using no outside tools or measurements, but only the principles of geometry already known and postulated. More on this later.

triangle whose sides are the radii of equal circles, then proves that all of the sides of the triangle are equal.

Finally, Euclid does not comment on his own work. Aside from the propositions, the only other types of content in the *Elements* are definitions of terms, postulates,[8] and common notions.[9] Euclid never explains a given proposition's importance or use, or why a student might want to know it. He does not explain how he arrived at the proofs that he presents, nor how a novice might go about making geometrical discoveries. He does not even explain the order in which the propositions appear or the goal of the book. He only presents the propositions themselves.[10]

The following points about the *Elements* will probably be less foreign to someone with modern mathematical training, but they are nevertheless important and characteristic features of the systematist style. Specifically, it is systematic, regularly structured, rigorous according to clear standards, and idealized. These features will later stand out in contrast to those of the heurists, to varying degrees depending on the author and work.

Euclid is systematic insofar as each part of the *Elements* is a complete whole and integrated into a larger complete whole. Each proposition contains a single complete

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[8] Postulates are a type of preliminary assumption necessary to doing any geometry at all, such as I.P2, that it is possible to continuously extend a straight line and retain its straightness

[9] These are axiomatic statements that are considered so self-evident that they do not qualify as assumptions, because no logical thought could occur without them, such as I.CN1, that things equal to the same thing are also equal to each other.

[10] Other authors in the systematist group are less austere. Apollonius, for example, provides some prolegomena before each book. Still, it is characteristic of the systematists that the body of the mathematical text be presented without interruptions or commentary from the author, and without an explanation of the means of discovery.

argument,[11] parts of which depend on previous propositions, definitions, postulates, and common notions. Almost every proposition informs the arguments of later propositions.[12] The propositions are arranged in thirteen larger books. Each book is broadly unified according to subject matter, and has a cumulative structure beginning with the definitions (and in Book I, the postulates and common notions) necessary for understanding its particular material. The books themselves are also arranged in order of increasing complexity. As more information is accumulated, later books draw on a larger pool of information, and tend to integrate the topics of the previous books into increasingly complex propositions (culminating, in Book XIII, in the construction of the regular solids: the pyramid, cube, octahedron, icosahedron, and dodecahedron). The propositions and books are so conscientiously systematized that at no point is the reader required to look forward to future material in order to understand a proposition.

The *Elements* are also regularly structured: that is, the parts follow clear conventions of ordering even when it is not strictly necessary for their integration into a systematic whole. Each proposition is composed of six discrete parts, whose names

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[11] That is, each proposition only demonstrates one statement without digressions. If an additional argument can be made that is relevant to the proposition, it is attached to the proposition in the form of a porism (also known as a corollary, a minor fact resulting from the proposition to which it is attached) or a lemma (a minor fact that aids only in proving the proposition to which it is attached).

[12] This interdependency has been charted in most editions of the *Elements*, and in Heath's translation (T. L. Heath, *The Thirteen Books of Euclid's Elements* (Cambridge: Cambridge University Press, 1923)) he includes the dependencies (where applicable) of each statement in each proposition.

Proclus provides[13]: πρότασις, ἔκθεσις, διορισμός, κατασκευή, ἀπόδειξις, συμπέρασμα. The *protasis*, or enunciation, clearly states what must be demonstrated. The *ekthesis* states the data, including a preliminary form of the diagram. The *diorismos* restates the general enunciation in terms specific to the data given in the *ekthesis*. The *kataskeuē* makes additional constructions or additions to the diagram in preparation for the *apodeixis*, or argument, which walks the reader through the proof and usually concludes with a restatement of the specific *diorismos*. The *symperasma* is the final conclusion, in which the general *protasis* is restated. Euclid follows this order meticulously, and only occasionally omits parts that are not necessary for clarity or rigor (for example, not every proposition will have a *kataskeuē*, e.g. I.6, and sometimes the *diorismos* is omitted for the sake of brevity if the *protasis* is particularly long, e.g. I.7).

There is also a clear order across propositions within each book, as well as across books within the work as a whole. For example, in Book I, within series of propositions about the same kind of figure (e.g. an isosceles triangle) he always places the propositions about angles before those about lines.[14] Another version of this regular structuring can be seen in the diagrams. The textual transmission of the diagrams is not unproblematic, but they clearly tend to over-specification and predictable orientations in

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[13] Proclus, *In Primum Euclidis Elementorum Librum Commentarii,* (*In prim. Euc.*) ed. Friedlein, 203.1-15. In the following passages, 203.17 to 205.12, he gives full details about when the various parts of a proposition might be left out.

[14] For example, *Elements* I.5/I.6, I.9/I.10, I.18/I.19, etc. The later propositions depend on the earlier ones, but he could have reversed the order without a loss of rigor. A similar point could be made for most books of the *Elements*, e.g. XI.2/XI.3.

the manuscript tradition.[15] All of these forms of structuring, wherein clear conventions are followed even when they are not strictly necessary, contribute to a sense of purposeful orderliness that is characteristic of all works in the systematist tradition, and very few outside of it.

Next, the *Elements* are rigorous according to clear standards. That is, the text is precise and regular in how conclusions are reached and verified. In fact, the systematization and structuring described above are major factors in the way Euclid follows and/or establishes conventions of rigor. We may tend to assume that mathematics is the discipline of formal rigor *par excellence*, but this too had to be invented and developed, and as we shall see, Greek authors show that certain methods of demonstration were sufficient for some practitioners, but not for others. The systematists in particular show different standards of rigor from those of the heurists, and several examples from Euclid himself can show what these standards were.[16]

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[15] Over-specification means that the diagram shows a more regular version of the figure than is necessary for the proof, e.g. an isosceles triangle when any triangle would do. By predictable orientations I mean that rectilinear figures will usually have a base parallel to the line of the text, circles will have a diameter either parallel or perpendicular to the text, etc. There is, in fact, only a single manuscript (Tehran, Sipahsaler 540) that fails to follow these conventions. For more on this, see Gregg De Young, "Mathematical Diagrams from Manuscript to Print: Examples from the Arabic Euclidean Transmission," *Synthese; Dordrecht* 186, no. 1 (May 2012): 26, and Ken Saito, "Traditions of the Diagram, Tradition of the Text: A Case Study," *Synthese; Dordrecht* 186, no. 1 (May 2012): 7–20. The first chapter of Reviel Netz, *The Shaping of Deduction in Greek Mathematics : A Study in Cognitive History* (Cambridge ; New York: Cambridge University Press, 1999): 12-57 also gives an extensive survey of diagrams in the manuscript tradition. Christián C. Carman, "Accounting for Overspecification and Indifference to Visual Accuracy in Manuscript Diagrams: A Tentative Explanation Based on Transmission," *Historia Mathematica* 45, no. 3 (August 1, 2018): 217–36, experimentally models the process by which diagrams may have become overspecified in transmission.

[16] Euclid himself is never explicit about his standards of rigor or the bases of the distinctions he makes between types of first principles or types of propositions. The explanations of these we have mostly from

First, logical rigor: any logical step in a proof must be evident from what has already been established, without requiring an intuitive step or uncertain inference. The cumulative nature of the *Elements* ensures that any information needed to prove a proposition has already been given in the preceding material before the argument is made, and that no outside knowledge is necessary. Any reader who understands Greek and has read the *Elements* from the beginning has everything needed to follow any subsequent proof.

Another feature of Euclid's logical rigor is seen in the way diagrams are used in the text. Geometrical diagrams are a potential source of logical laxity, because they cannot be the things that they represent. Euclid is very clear about this: his second definition of Book I, that a line is a breadthless length (γραμμὴ δὲ μῆκος ἀπλατές), precludes any visible diagram from depicting a line as he defines it. In his use of diagrams, he shows an awareness of their limitations as sources of information.[17] According to the authors of several contemporary attempts to model Euclid's geometry

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Proclus. Furthermore, the textual transmission of the *Elements* is such that there is some doubt surrounding the specific terms used for some distinctions (e.g. "common notions" – see note 22). However, there is sufficient evidence both within the text itself and among the other works of Euclid and of authors in his group to convince most scholars (Tannery is the only frequent skeptic) that these distinctions were recognized and intentional.

[17] The question of how the original diagrams were drawn does not affect this point. Netz, *The Shaping of Deduction in Greek Mathematics,* showed that the text itself implies that diagrams were present for reference by the reader. He also showed that the text-diagram pair were closely interdependent, and neither could be reconstructed without the other. It is safe to assume that the original text did include diagrams, which would have been sufficiently determined by the requirements of each proposition so as not to differ in substance from any later rendition of the diagram. As for what we do know about the ancient diagrams, there are two papyrus fragments that contain parts of the *Elements* (*P.Mich.* III 143 and *P.Oxy*. I 29), and only one (*P.Oxy.* I 29) contains a diagram. This diagram is unlabeled and very roughly drawn (the lines do not meet neatly, some are retraced).

using systems of symbolic logic[18], the only arguments derived from the diagrams are topological, not metrical. That is, the diagram can provide information about the arrangement of points and lines relative to each other only when those arrangements would not be altered by a distortion of the diagram.[19] It cannot provide any information about the relative sizes of lines or figures (metrical information) that cannot be derived from the text itself. So for example, it may be inferred from a diagram that two lines intersect, but not whether the angle that they form is greater or less than a given angle. This way, the reader is always assured that, although the diagram can never be an accurate representation of the object it stands for, it will not affect the soundness of the argument, because no substantive information depends on being read directly from the diagram.

Finally, the systematist practice of eschewing the use of numbers or calculations in geometry is also in the service of logical rigor. By not assigning numbers to the lengths of lines or the areas of figures, Euclid avoids both a necessity to estimate and a loss of generality. Certain geometrical magnitudes, such as the diagonal of a square or the

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[18] See especially K. Manders, "The Euclidean Diagram," in *The Philosophy of Mathematical Practice*, by Paolo Mancosu (Oxford; New York: Oxford University Press, 2008), 80–133. For logical modeling of Euclid's geometry, see John Mumma, "Intuition Formalized: Ancient and Modern Methods of Proof in Elementary Geometry" (Ph.D. Dissertation, Carnegie Mellon University, 2006); Nathaniel Miller, *Euclid and His Twentieth Century Rivals: Diagrams in the Logic of Euclidean Geometry* (Stanford: Center for the Study of Language and Information, 2007); and especially Jeremy Avigad, Edward Dean, and John Mumma, "A Formal System for Euclid's Elements," *The Review of Symbolic Logic* 2, no. 04 (December 2009): 700–768.

[19] A distortion of the diagram would include changes to the orientation or size of any of the elements of the diagram, provided that the conditions stipulated in the *ekthesis* and *kataskeuē* are still met.

circumference of a circle, cannot be expressed as integers, fractions, or finite sums of fractions (assuming the other magnitudes such as the side of the square and the radius of the circle are expressed as integers, fractions, or finite sums of fractions). If expressed numerically, they would have to be estimated and could not therefore be precisely determined. Furthermore, choosing particular numbers as examples (such as assigning lengths of 5 and 3 to the sides of a rectangle) runs the risk of a false generalization. The claim of the enunciation may be true under one set of numbers but not another.[20] By working without specific numbers, Euclid ensures that his definitions of figures are always precise and generalizable, which strengthens the arguments derived from them.

Systematist practice also keeps to a principle of exhaustive rigor insofar as it makes meticulous distinctions between types and cases of arguments, claims, and propositions, and makes sure to address each type and case. For example, within each proposition, a clear distinction is made between specific and general statements. In fact, one of the features which defines the parts of the propositions is whether they are making a general or specific statement, or stepping from one to the other. The *protasis* is always a general statement, applying to any iteration of the diagram that fulfills the given conditions. The *ekthesis* produces a specific case of the general conditions. The *diorismos*

[20] Imagine, for an extreme example, a proposition that claimed that the sides of right triangles are always mutually commensurable, and attempted to prove it numerically. If the sides were assigned the right values, such as 3, 4, 5, and 5, 12, 13, or any other of the so-called "Pythagorean triples" of integers that fulfill the equation $a^2+b^2=c^2$, the completely false enunciation would be confirmed by the numbers. Those numbers demonstrate specific cases of mutually commensurable sides, but they cannot demonstrate a generality.

almost perfectly mimics the *protasis* and adds no new information to the proof, but is nevertheless required as a specific claim about the case given in the *ekthesis*.[21] Furthermore, the *diorismos* is frequently restated before the *symperasma*, which means that the confirmation of the specific case is used as a platform to step to the general conclusion. The specific and general claims are treated as distinct, and both must be made for the argument to be complete.

Another example of exhaustive rigor are the distinctions between the types of first principles set out in Book I: ὅροι (definitions), αἰτήματα (postulates), and κοιναί ἔννοιαι (common notions).[22] These are all essential principles that are unprovable and yet

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[21] This is Proclus' interpretation of the *diorismos* (Proclus, *In prim. Euc.* 205.10-11, and 208.21), and it is consonant with the rest of Euclid's mathematics with respect to rigor. There is another possible definition of the term *diorismos*, given by Proclus at 202.2-5: καὶ τοῦτο μάλιστα ἐν τοῖς διορισμοῖς ἐξετάζουσα, εἰ ἀδύνατον τὸ διὰ τοῦτο ζητούμενον ἢ δυνατόν, καὶ μέχρι τίνος ἐγχωρεῖ καὶ ποσαχῶς. Here the *diorismos* is an expression of the limiting cases of a problem, and whether it is possible or impossible under given conditions. For example, Heath, *A History of Greek Mathematics*, vol. 1, 371 cites *Elements* I.22, wherein a triangle must be constructed out of three lines equal to three given lines. Appended to the *protasis* is the stipulation that any two of the lines taken together must be greater than the third line. This is a *diorismos* in the sense of a limiting condition of the general enunciation. However, Tannery, *La Géométrie Grecque*, 149 note, argues that this is an incorrect use of the term by Proclus, who may have made the mistake because both the limiting addition to the *protasis* and the proper *diorismos* after the *ekthesis* begin with the same words, δεῖ δή. I agree with Tannery for two reasons: first, of our two major sources for these definitions, Proclus uses *diorismos* in both senses, but Pappus (who is earlier by about 100 years) only in the first sense; and second, the addition to the *protasis* is always a general or definitional requirement that has already been previously proven. For example, it was proved in *Elements* I.20 that in *any* triangle, two sides taken together are greater than the third. Once proved, this theorem is available as a general truth. There is therefore no qualitative difference between the first part of the *protasis* and the appended stipulation. They are both generalities, and there is no need to distinguish them (contra see Ken Saito and Nathan Sidoli, "The Function of Diorism in Ancient Greek Analysis," *Historia Mathematica* 37, no. 4 (November 2010): 579–614).

[22] There is some disagreement about which common notions Euclid actually included and whether he called them "common notions" or "axioms." For a summary of the arguments see Heath, *The Thirteen Books of Euclid's Elements,* vol. 1, 221-232. But whatever the specifics, there was a set of common notions at the beginning of the *Elements* that were distinguished from the postulates, on the grounds that they are general truths not specific to geometry.

necessary for other proofs, but they are distinguished from each other by type: definitions naturally cannot be proven; postulates and common notions are both axiomatic statements, but the postulates are specific to geometry and the common notions are general truths.[23] Many of the books of the *Elements* contain definitions, but the postulates and common notions appear only at the beginning of Book I. They are considered sufficient first principles for all of geometry.

Another important distinction that Euclid maintains is the separation of lemmas (λήμματα) and porisms (πορίσματα) from the propositions themselves and from each other. This allows the readers to be certain, first that within a proposition all statements will serve a single proof of a single enunciation, and second that it will be clear whether a small complementary proposition is *serving* the main proof (in which case it is a lemma) or *resulting* from it (a porism).

Finally, Euclid is usually careful to prove all distinct cases of a proposition if they are not self-evident.[24] For example, in propositions I.35-40, he proves the following cases of the same basic principle:

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[23] The common notions are (*Elements* I.CN1-5):
Things equal to the same thing are equal to each other.
If equals are added to equals, the wholes are equal.
If equals are subtracted from equals, the remainders are equal.
Things that coincide with one another are equal.
The whole is greater than the part.

[24] Proclus in his commentary on I.35 particularly praises Euclid for his rigor in this respect, and especially in proving the most difficult case of the proposition (Proclus, *In. prim. Euc.* 395-396 and 399). It is also true that Euclid does not always provide proofs of all possible cases. Avigad, Dean, and Mumma ("A Formal System for Euclid's Elements") have observed along with Heath (*Elements* vol. 1 p. 59) that Euclid will often prove only the most difficult case of a proposition, apparently considering the other cases proved *a fortiori*, but he "*does* explicitly introduce case splits when they are needed" Avigad, Dean, and Mumma,

35. Parallelograms on the same base and in the same parallels are equal to one another.

36. Parallelograms on equal bases and in the same parallels are equal to one another.

37. Triangles on the same base and in the same parallels are equal to one another.

38. Triangles on equal bases and in the same parallels are equal to one another.

39. Equal triangles on the same base and on the same side are also in the same parallels.

40. Equal triangles on equal bases and on the same side are also in the same parallels.

Even though Euclid does not always work through all possible cases, his care in producing separate proofs of six very closely related propositions such as these shows a concern with exhaustive rigor which will almost never be seen in the heurist group (see the section below on Heron's *Metrica*).

System, structure, and rigor all stand out as essential characteristics of the *Elements*, but there are texts of the heurist tradition that have one or more of these qualities. The practice that separates systematist geometry most completely from all the

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"A Formal System for Euclid's Elements," p. 37. The point is that concern with proving multiple cases at all is already an indication that exhaustive rigor is a priority in the *Elements*, unlike in its heurist counterparts.

heurist styles is the idealization of geometry. The systematist school treats geometrical objects as non-physical, purely mental ideas.

The practices of idealization contribute even more strongly to the character of systematist geometry than those of system, structure, and rigor. Idealized geometry is not the same thing as theoretical geometry. It is certainly true that Euclid is “theoretical” rather than practical, meaning that the *Elements* are not a manual by which one can learn how to allot land or construct a building, or even how to solve geometrical problems.[25] But many heurists are equally theoretical. Archimedes’ *Quadrature of the Parabola* is hardly an arsenal of practical problem-solving tools, but as we shall see it is not in the same vein as the *Elements*. Euclid is theoretical in a particular way, for the purpose of idealization. Proof-based geometry, as opposed to problem-based, is not ordered toward acquiring skills, but toward understanding truths. If it is true, as I am arguing, that Euclid and his colleagues conceive of the geometric project as purely intellectual, and its object non-physical, then the only sensible approach is that of idealization. That is, Euclid is “abstract”, not only insofar as he never treats a specific square or circle (but rather a paradigm that could be any square or circle), but also insofar as his objects are explicitly non-physical.

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[25] Since he uses no numbers, Euclid cannot pose problems such as “find the area of a right triangle if its sides are 3, 4, and 5”—he certainly never poses any problem of this kind. Furthermore, Euclid is never transparent about the method by which one might discover the arguments he sets out. He gives only results, not a means of achieving them. In fact, it seems to be another characteristic of the systematist authors in general to be unconcerned with teaching a repeatable or transferable mathematical process. Heurists such as Heron (and Diophantus, as we will see in a later chapter) do this frequently.

Euclid shows in several ways that he understands geometrical figures to be non-physical. He defines them in a way that would be impossible for physical objects. "A point is that which has no part."[26] "A line is a breadthless length."[27] He never mentions real-world methods of construction, such as a compass or a straight-edge. When Euclid constructs figures, he does not describe a necessarily physical process. For example, in the first three postulates at the beginning of Book I, although he speaks of "drawing" (ἀγαγεῖν) and "producing" (ἐκβαλεῖν) a straight line, and even of "describing" (γράφεσθαι) a circle, his own language shows that he is not thinking of real physical actions. Not only has he already defined lines as inherently unable to be physically visualized (because they have no breadth), but the postulates themselves say that a straight line can be drawn from *any* point to any other (ἀπὸ παντὸς σημείου ἐπὶ πᾶν σημεῖον), and that a circle can be described with *any* radius and center (παντὶ κέντρῳ καὶ διαστήματι). These conditions clearly include instances that would be physically impossible to draw. Finally, once Euclid has defined and constructed the figures, he does not treat them as though they had physical properties. He demonstrates general and precise, but never empirical, information about them. The figures are not subject to change or the passing of time, nor are they measured[28], weighed, or counted. In all but

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[26] Euclid, *Elements* I.D1: Σημεῖόν ἐστιν, οὗ μέρος οὐθέν.
[27] Euclid, *Elements* I.D2: Γραμμὴ δὲ μῆκος ἀπλατές.
[28] Books II, IV, X, XIII, and parts of Book VI contain material that could be characterized as "metrical" (see W. R. Knorr, *The Evolution of the Euclidean Elements: A Study of the Theory of Incommensurable Magnitudes and Its Significance for Early Greek Geometry* (Dordrecht, Holland; Boston: D. Reidel Pub. Co., 1975), 6-7). However, the only applicable sense of the word "metrical" is "concerned with size." These books compare areas, lines, and angles to each other only in relative terms: one area may be equal to

four propositions and five definitions,[29] they are not even moved.[30] Heurist authors frequently treat geometric figures in these ways, and this will in fact prove to be the main difference between the two sets of practices.

In short, the systematist method is characterized by non-symbolic, non-subjective, proof-based, structured, systematic, and rigorous treatment of idealized spatial objects. The cumulative effect of these characteristics is to present geometry as a realm of exact and universal abstract knowledge: there is no estimation or guesswork, any claim holds

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another, twice another, one and a half times another, etc. This is not the same thing as measurement, which compares magnitudes not to each other but to an arbitrary standard unit. Euclid does not do this. We will see later that, while Euclid and those of his tradition may say that one area is equal to another, only those outside his tradition may say that an area is e.g. “five units”.

[29] XIII.13-16. In every case a semicircle is being rotated about its axis. The verb used is (περι)φέρεσθαι. It is noteworthy that only semicircles are explicitly moved in the propositions *Elements.* In a later chapter I will show how in a certain strain of astronomy, related to systematist geometry, it is understood that circular and spherical motion does not compromise the idealization of mathematical objects as linear or irregular motion does. The definitions referred to are 14, 18, 20, 21, and 23 of Book XI, the first book on solid geometry. The same word, περιφέρεσθαι, is used to describe the revolutions of a semicircle (14), a right triangle (18, 20), and a rectangle (21, 23), which produce a sphere, cone, and cylinder respectively. These definitions are conceptual only and the imagined motion of the generating plane figures does not bear on any proof argument.

[30] In I.4 Euclid places a triangle on top of another in order to show that it is perfectly congruent to another triangle (he makes later use of this technique in I.8, and in III.24 with segments of circles). This proposition has been a frequent topic of controversy especially in the development of modern mathematics. See e.g. Arthur Schopenhauer, *Die Welt Als Wille Und Vorstellung*, 3. verb. und beträchtlich verm. Aufl. (Zürich: Haffman, 1988), II.130, and Hermann von Helmholtz, “The Origin and Meaning of Geometrical Axioms,” *Mind* 1, no. 3 (July 1876): 319. For more, see Heath*, The Thirteen Books of Euclid’s Elements,* vol. 1, 224-231 and 248-250). The controversy rests on whether the thought experiment of applying one triangle to another is rigorous. There is no postulate on which to base this method, with the possible exception of Common Notion 4, which does not include angles such as Euclid uses here; it must also be assumed that the triangle is not deformed by the motion. However, Euclid does not use a verb of motion in the proposition. He says that the one triangle will be “applied” or “fitted” (ἐφαρμοζόμενον) to the other, and that a point of one will be “placed” (τιθέμενον) at the corresponding point of the other. He doesn’t talk about any mechanical process such as sliding, rolling, pushing, or pulling. He uses verbs that seem general and static compared to those used by Archimedes, for example, in his construction of a spiral (ἰσοταχέως περιφέρεσθαι, περιάγεσθαι) when he has a line and a point move certain distances in the same time: *Spirals* definition 1 (J. L. Heiberg and E. S. Stamatis, eds., *Archimedis Opera Omnia, cum Commentariis Eutocii.*, vol. 1 (Stuttgart: Teubner, 1972), 44.18-20). Heath (*The Thirteen Books of Euclid’s Elements* vol. 1, 225), disagrees with me about this and interprets *Elements* I.4 as using verbs of motion.

for any figure that meets the stipulated conditions, and the imprecision and complexity associated with physical objects and processes have been removed. Euclid's *Elements* was not alone or first in its methods, but part of a school of practice, the members of which shared methods, stylistic conventions, and most of all a conception of fully idealized geometry.

I.1.2 Example of a Systematist Proposition

As a general paradigm of what a systematist proposition looks like, consider Euclid's *Elements* I.17[31]:

> Παντὸς τριγώνου αἱ δύο γωνίαι δύο ὀρθῶν ἐλάσσονές
> εἰσι πάντῃ μεταλαμβανόμεναι. *Protasis*
> Ἔστω τρίγωνον τὸ ΑΒΓ· λέγω, ὅτι τοῦ ΑΒΓ τριγώ- *Ekthesis*
> νου αἱ δύο γωνίαι δύο ὀρθῶν ἐλάττο-
> νές εἰσι πάντῃ μεταλαμβανόμεναι. *Diorismos*
> Ἐκβεβλήσθω γὰρ ἡ ΒΓ ἐπὶ τὸ Δ. *Kataskeuē*
> Καὶ ἐπεὶ τριγώνου τοῦ ΑΒΓ ἐκτός
> ἐστι γωνία ἡ ὑπὸ ΑΓΔ, μείζων ἐστὶ
> τῆς ἐντὸς καὶ ἀπεναντίον τῆς ὑπὸ
> ΑΒΓ. κοινὴ προσκείσθω ἡ ὑπὸ ΑΓΒ· αἱ ἄρα ὑπὸ ΑΓΔ,
> ΑΓΒ τῶν ὑπὸ ΑΒΓ, ΒΓΑ μείζονές εἰσιν. ἀλλ' αἱ ὑπὸ

[31] Text from Euclid, *Elementa*, ed. J. L. Heiberg and E. S. Stamatis (Leipzig: Teubner, 1969).

ΑΓΔ, ΑΓΒ δύο ὀρθαῖς ἴσαι εἰσίν· αἱ ἄρα ὑπὸ ΑΒΓ,

ΒΓΑ δύο ὀρθῶν ἐλάσσονές εἰσιν. ὁμοίως δὴ δείξομεν,

ὅτι καὶ αἱ ὑπὸ ΒΑΓ, ΑΓΒ δύο ὀρθῶν ἐλάσσονές εἰσι

καὶ ἔτι αἱ ὑπὸ ΓΑΒ, ΑΒΓ. (15) *Apodeixis*

Παντὸς ἄρα τριγώνου αἱ δύο γωνίαι δύο ὀρθῶν ἐλάσ-

σονές εἰσι πάντῃ μεταλαμβανόμεναι· ὅπερ ἔδει δεῖξαι. *Symperasma*

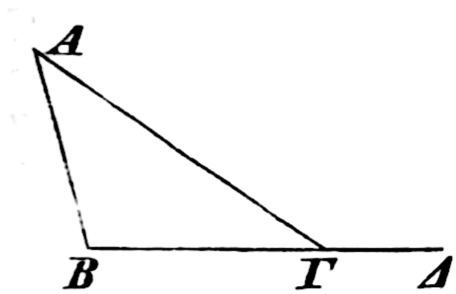

Figure 1: Euclid, *Elements* I.17. Diagram from Euclid, *Elementa*, ed. J. L. Heiberg and H. Menge (Leipzig: Teubner, 1883), 45.

Protasis: "In every triangle, two angles taken together in any way whatsoever are less than two right angles." Note the generalizing terms Παντὸς and πάντῃ

Ekthesis: "Let there be a triangle, ΑΒΓ." This is all the apparatus necessary to meet the requirements of the *protasis*, because it applies to every triangle without further specification. The imperative Ἔστω is a standard formula used to signal the *ekthesis*.[32]

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[32] There are many formulaic words and phrases used to signal the parts of a proposition, the most common being λέγω ὅτι, which appears over a thousand times in works of the systematist school. For more on such formulas, see below, ch. 1 notes 153-156.

Diorismos: “I say that in triangle ΑΒΓ, two angles taken together in any way whatsoever are less than two right angles.” The formula λέγω ὅτι almost always introduces the *diorismos* in Euclid and systematist texts. The only step here is to move from the general statement about “every triangle” in the *protasis* to the specific statement about this triangle, ΑΒΓ. The rest of the phrasing remains the same.

Kataskeuē: “For let ΒΓ be extended to Δ.” This is the only modification of the *ekthesis* required to proceed with the apodeixis. This step is justified by Book I postulate 2, which states that any straight line can be continually produced in a straight line.

Apodeixis: “And since the angle ΑΓΔ is an exterior angle of triangle ΑΒΓ, it is greater than the interior and opposite angle ΑΒΓ [proved in I.16]. Let the angle ΑΓΒ be added in common [i.e. to angles ΑΓΔ and ΑΒΓ]; so the angles ΑΓΔ and ΑΓΒ are greater than the angles ΑΒΓ and ΒΓΑ [note that ΑΓΒ and ΒΓΑ are the same angle]. But the angles ΑΓΔ and ΑΒΓ are equal to two right angles [proved in I.13]. So the angles ΑΒΓ and ΒΓΑ are less than two right angles. And similarly we will show that the angles ΒΑΓ and ΑΓΒ are less than two right angles, and also the angles ΓΑΒ and ΑΒΓ.” Each step is sequential, and no step is taken which has not already been shown to be allowed. Once the proof is made for one set of angles, it is extended to the other two sets, thus fulfilling the requirements of the *diorismos*. The formula ὁμοίως δὴ δείξομεν introduces an extension of the proof to greater generality, of which the *apodeixis* would include the same steps, and so is not given. The same formula frequently introduces a porism.[33]

Symperasma: "So, in every triangle, two angles taken together in any way whatsoever are less than two right angles. Which it was required to prove." Aside from the transitional word ἄρα and the concluding formula ὅπερ ἔδει δεῖξαι, which appears at the end of every proposition in the *Elements*, the phrasing is exactly the same as the *protasis*.

I.1.3 Euclid's Predecessors

We have very little extant text from geometers who came before Euclid, which may be an indication of the popularity that the *Elements* enjoyed even from the beginning. There is some ancient testimony about the mathematicians who contributed to the substance of the *Elements*. Proclus, Eutocius, Simplicius, and Pappus all give brief histories of geometry or various geometrical methods and problems, as well as preserving similar material from earlier writers such as Geminus and Eudemus.[34] Further information can be extracted from the prolegomena to extant mathematical works. Some

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[33] See below, ch. 1 notes 153-156 for ways in which formulas are used by the systematist school as opposed to the heurist school.

[34] Relevant passages are: Proclus *In prim. Euc.* 64.3-68.24; Eutocius, "Commentarii in Libros de Sphaera et Cylindro," in *Archimedis Opera Omnia, cum Commentariis Eutocii*, ed. E. S. Stamatis and J. L. Heiberg, vol. 3 (Stuttgart: Teubner, 1972), 54.26-106.24; Simplicius, *In Aristotelis physicorum libros octo commentaria*, ed. H. Diels, Commentaria in Aristotelem Graeca 9 (Berlin: Reimer, 1882), 61.1-68.32; and Pappus, *Collectionis quae supersunt*, ed. Friedrich Hultsch (Berlin: Weidmann, 1876), Book IV 270-303, Book VII 634-642.

modern scholarship has attempted to trace the sources of Euclid's knowledge, with mixed success.[35]

Proclus begins his summary with Thales of Miletus, saying that he discovered many propositions and the underlying principles of many others, and that Thales' methods were not consistently generalizing/universalizing.[36] Later in the commentary, he attributes to Thales the proof that the diameter bisects the circle,[37] part of *Elements* I.5 (that isosceles triangles have equal angles at the base),[38] *Elements* I.15 (that when two straight lines intersect, opposite angles are equal) and *Elements* I.25 (that in triangles the greater base contains the greater angle), both on the additional authority of Eudemus.[39]

Proclus in his summary says that Pythagoras transformed geometry into a liberal study, "examining its principles from the beginning and tracking down the theorems immaterially and intellectually,"[40] and that he discovered the theory of proportionals (the subject of Book V of the *Elements*) and the construction of cosmic figures (the subject of Book XIII of the *Elements*).[41]

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[35] Tannery, *La Géométrie Grecque*, Knorr, *The Evolution of the Euclidean Elements*, Fowler, *The Mathematics of Plato's Academy*, to name a few. The evidence is based partly on the testimonies of the commentators mentioned above, as well as Plato and Aristotle, and partly on a kind of reverse-engineering of the process of mathematical discovery, which is why many of the questions, such as whether there was a foundations crisis connected with incommensurability, or whether Greeks engaged in "geometric algebra," generally remain unresolved.
[36] Proclus, *In prim. Euc.* 65.7-11.
[37] Proclus, *In prim. Euc.* 157.10-11.
[38] Proclus, *In prim. Euc.* 250.20-251.2.
[39] Proclus, *In prim. Euc.* 299.1-5, 352.13-16.
[40] Proclus, *In prim. Euc.* 65.15-21.
[41] This may be only a reflection of Pythagoras' status as a legendary founder in mathematics. But if there is truth to it, a similar attribution can also be found in Diogenes Laertius, *Vitae Philosophorum* VIII.11-12, ed. H. S. Long (Oxford: Clarendon Press, 1964): τοῦτον καὶ γεωμετρίαν ἐπὶ πέρας ἀγαγεῖν, Μοίριδος

The first of Euclid's predecessors about whom Proclus gives more specific information is Oenopides of Chios.[42] Plato mentions Oenopides and Anaxagoras as mathematicians, though there they seem to be related to astronomy rather than geometry.[43] Proclus attributes Euclid's propositions *Elements* I.12[44] and I.23[45] to Oenopides, and associates him and his successors with the development of the distinction between a theorem and a problem,[46] which is an important part of the systematist tradition.

Next, Proclus states that Hippocrates of Chios (ca. 470-410 BCE) was the first to compile a book of *elements*.[47] Hippocrates is chiefly known for developing methods of squaring lunes, which are crescent-moon-shaped figures enclosed by two arcs of circles.[48]

πρῶτον εὑρόντος τὰς ἀρχὰς τῶν στοιχείων αὐτῆς, ὥς φησιν Ἀντικλείδης ἐν δευτέρῳ Περὶ Ἀλεξάνδρου (*FGrHist* 140 F 1). μάλιστα δὲ σχολάσαι τὸν Πυθαγόραν περὶ τὸ ἀριθμητικὸν εἶδος αὐτῆς. ("And [they also say] that this man [Pythagoras] brought geometry to the limits, once Moiris had first found the beginnings of its elements, as Antikleides says in the second book *On Alexander*. And [they say] that Pythagoras busied himself with the arithmetical form of geometry.")

42 Proclus, *In prim. Euc.* 65.21-66, 80.15-20, 283.7-8, 333.5-9

43 Plato, *Erastae* 132a-b: ἐτυγχανέτην οὖν δύο τῶν μειρακίων ἐρίζοντε, περὶ ὅτου δέ, οὐ σφόδρα κατήκουον. ἐφαινέσθην μέντοι ἢ περὶ Ἀναξαγόρου ἢ περὶ Οἰνοπίδου ἐρίζειν· κύκλους γοῦν γράφειν ἐφαινέσθην καὶ ἐγκλίσεις τινὰς ἐμιμοῦντο τοῖν χεροῖν ἐπικλίνοντε καὶ μάλ' ἐσπουδακότε. "And there happened to be two of the boys arguing about something, but I didn't hear very well. Indeed, they seemed to be arguing either about Anaxagoras or about Oenopides; for they seemed to be drawing circles and representing certain inclinations by leaning upon* their hands, and they were very intent." (*this is the LSJ translation for this attestation, but I think it should read instead "making angles with their hands" or "bending their hands" for the passage to make more sense. If they are doing astronomy, and they have circles drawn in the dirt, it makes sense for them to incline their hands at the angle they are trying to show. Ancient astronomy at its geometrical core is about circles intersecting at different angles.)

44 Proclus, *In prim. Euc.* 283.7-8.

45 Proclus, *In prim. Euc.* 333.5-9.

46 Proclus, *In prim. Euc.* 80.15-20.

47 Proclus, *In prim. Euc.* 66.7-8. Literally "the first of those remembered or mentioned" (πρῶτος γὰρ ὁ Ἱπποκράτης τῶν μνημονευομένων καὶ στοιχεῖα συνέγραψεν.).

48 To "square" a figure means to find a square that is equal in area to that figure. At 422.24-26, Proclus says that he thinks that the ancients were led to investigate the squaring of the circle because of *Elements* I.45, a problem of constructing a parallelogram equal to a given irregular rectilinear figure on a given rectilinear

Squaring the lune was a part of the ongoing research on squaring the circle, a problem well-known, widespread, and important enough to be a matter of concern to nearly every ancient geometer at one point or another. Simplicius, in his commentary on Aristotle's *Physics*, quotes Eudemus' account of Hippocrates' work in full.[49] The style and methodology of the proofs are essentially systematist,[50] but this could be the the result of bias on the part of Eudemus, who was a nearer contemporary of Euclid (he died around the time of Euclid's *floruit*, 300 BC). Finally, Proclus attributes to Hippocrates the discovery of what he calls "reduction" (ἀπαγωγή),[51] which is the process of extracting from a complicated proposition a simpler theorem whose results imply the proof of the larger theorem. Eutocius in his commentary on Archimedes' *Sphere and Cylinder II*[52] says that Hippocrates' discovery was specifically the reduction of the problem of doubling a cube to the problem of finding two mean proportionals.[53] This reduction is given as a definition in Book V of the *Elements*: Definition 10 of Book V uses it to define a triplicate ratio, which is a ratio of two cubes with reference to their sides.[54]

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angle. (Ἐκ τούτου δὲ οἶμαι τοῦ προβλήματος ἐπαχθέντες οἱ παλαιοὶ καὶ τὸν τοῦ κύκλου τετραγωνισμὸν ἐζήτησαν.)

[49] Simplicius, *in Aristotelis physicorum libros octo commentaria*, 61.1-68.32.

[50] Heath comments, in particular, that the language of the passage demonstrates how early the technical terminology of Euclid's geometry was in use. (Heath, *A History of Greek Mathematics*, vol. 1, 187.)

[51] Proclus, *In prim. Euc.* 213.7-11.

[52] Eutocius, "Commentarii in Libros de Sphaera et Cylindro," 88.17-23.

[53] *b* and *c* are mean proportionals between *a* and *d* if a:b :: b:c :: c:d. If two mean proportionals are found between a number and its double, then the cube on the first mean proportional will be double the cube on the first number.

[54] Euclid, *Elements* V.D10: Ὅταν δὲ τέσσαρα μεγέθη ἀνάλογον ᾖ, τὸ πρῶτον πρὸς τὸ τέταρτον τριπλασίονα λόγον ἔχειν λέγεται ἤπερ πρὸς τὸ δεύτερον, καὶ ἀεὶ ἑξῆς ὁμοίως, ὡς ἂν ἡ ἀναλογία ὑπάρχῃ. "When four magnitudes are proportional, the first is said to have to the fourth the triplicate ratio of that which it has to the second, and so on continuously in the same way, whatever the proportion may be."

Plato is next on Proclus' list, and he is connected to Euclid by a network of students and teachers, but there is no direct influence on the *Elements*, at least in the sense of propositions that could be traced back to Plato's mathematical work. Proclus does claim that Euclid was a Platonist, which is why he ordered the *Elements* to the construction of the regular solids in Book XIII.[55] However, this is not the same as direct mathematical influence. It is also true that in the dubiously authored *Epistulae* the Euclidean definition of the circle is used, but it is far from verbatim, and besides this definition seems to be the most common.[56] Plato's connection to systematist geometry will be discussed in more depth later.

Eudoxus of Cnidos, whom Proclus calls "a companion of those around Plato,"[57] is reported to have been a major producer of the subject material that became the *Elements*, even though we have little reliable evidence about his geometrical work.[58] Proclus says

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[55] Proclus, *In prim. Euc.* 68.20-23: καὶ τῇ προαιρέσει δὲ Πλατωνικός ἐστι καὶ τῇ φιλοσοφίᾳ ταύτῃ οἰκεῖος, ὅθεν δὴ καὶ τῆς συμπάσης στοιχειώσεως τέλος προεστήσατο τὴν τῶν καλουμένων Πλατωνικῶν σχημάτων σύστασιν, "[Euclid] was Platonist in his purpose and at home in that philosophy, which is why he even set up the end of the whole *Elements* to be the construction of the so-called Platonic figures."

[56] Plato, *Epistulae* 342b6-342c1: λόγος δ' αὐτοῦ τὸ δεύτερον, ἐξ ὀνομάτων καὶ ῥημάτων συγκείμενος· τὸ γὰρ ἐκ τῶν ἐσχάτων ἐπὶ τὸ μέσον ἴσον ἀπέχον πάντῃ, λόγος ἂν εἴη ἐκείνου ᾧπερ στρογγύλον καὶ περιφερὲς ὄνομα καὶ κύκλος. "The second thing is the account of it (the circle), made out of nouns and verbs; for, the distance from the extremities to the center being everywhere equal, would be the account of that whose name is round and circumference and circle." Compare to Euclid, *Elements* I.D15: Κύκλος ἐστὶ σχῆμα ἐπίπεδον ὑπὸ μιᾶς γραμμῆς περιεχόμενον [ἣ καλεῖται περιφέρεια], πρὸς ἣν ἀφ' ἑνὸς σημείου τῶν ἐντὸς τοῦ σχήματος κειμένων πᾶσαι αἱ προσπίπτουσαι εὐθεῖαι [πρὸς τὴν τοῦ κύκλου περιφέρειαν] ἴσαι ἀλλήλαις εἰσίν. "The circle is a plane figure contained by one line (which is called the circumference) upon which, from a point among those lying inside of the figure, all the straight lines falling (upon the circumference of the circle) are equal to each other.

[57] ἑταῖρος δὲ τῶν περὶ Πλάτωνα, Proclus, *In prim. Euc.* 67.3.

[58] There is a greater body of evidence about Eudoxus' astronomical theory. For more on this, see Heath, *A History of Greek Mathematics (HGM)* vol. 1, 322-325 and 329-335.

that he "first increased the multitude of so-called universal theorems, added another three proportions to the existing three, and furthered to a multitude the work on the section, which got its start from Plato, also using analysis on it."[59] The first scholium to Book V of the *Elements* says that "some say the book is the discovery of Eudoxus, the teacher of Plato."[60] Archimedes attributes two specific Euclidean propositions to Eudoxus, stating that his theorems on solids are outstanding, namely that every pyramid is one third of a prism with the same base and height (*Elements* XII.7 porism), and that that every cone is a third of a cylinder with the same base and height (*Elements* XII.10).[61] Despite this sparse evidence, Proclus ascribes to Eudoxus a great deal of influence on the *Elements* when he says that Euclid wrote them by "collecting the elements, arranging many of the [theorems] of Eudoxus, perfecting many of Theaetetus', and even bringing to irrefutable proof those more weakly demonstrated by his predecessors."[62] The modern tradition has

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[59] Proclus, *In prim. Euc.* 67.3-8: πρῶτος τῶν καθόλου καλουμένων θεωρημάτων τὸ πλῆθος ηὔξησεν καὶ ταῖς τρισὶν ἀναλογίαις ἄλλας τρεῖς προσέθηκεν καὶ τὰ περὶ τὴν τομὴν ἀρχὴν λαβόντα παρὰ Πλάτωνος εἰς πλῆθος προήγαγεν καὶ ταῖς ἀναλύσεσιν ἐπ' αὐτῶν χρησάμενος. Proclus does not give further information on what the universal theorems might be, or what the "section" is (the same word is used for conic sections, cutting solids into parts in specific ratios, and cutting magnitudes in general into the golden section, also called "extreme and mean ratio").
[60] Σ Euc. *Elements* [ad V], 1.7-8 [Euclid, *Elementa*, ed. J. L. Heiberg and E. S. Stamatis (Leipzig: Teubner, 1969), vol. 5]: τὸ δὲ βιβλίον Εὐδόξου τινὲς εὕρεσιν εἶναι λέγουσι τοῦ Πλάτωνος διδασκάλου.
[61] Archimedes, *On the Sphere and Cylinder I*, prolegomena, in *Archimède*, ed. Charles Mugler, vol. 1 (Paris: Les Belles Lettres, 1971) 9.3-7.
[62] Proclus, *In prim. Euc.* 68.7-10: ὁ τὰ στοιχεῖα συναγαγὼν καὶ πολλὰ μὲν τῶν Εὐδόξου συντάξας, πολλὰ δὲ τῶν Θεαιτήτου τελεωσάμενος, ἔτι δὲ τὰ μαλακώτερον δεικνύμενα τοῖς ἔμπροσθεν εἰς ἀνελέγκτους ἀποδείξεις ἀναγαγών.

followed suit, and Eudoxus is still considered to have been one of the greatest mathematicians of antiquity.[63]

Theaetetus seems to have been nearly as important a contributor to the material that became the *Elements* as Eudoxus was. Proclus credits him, along with Leodamas of Thasos and Archytas of Tarentum, with increasing the theorems of geometry and grouping them more scientifically,[64] and he says later that Euclid perfected many of Theaetetus' theorems.[65] Pappus' commentary on *Elements* Book X (preserved only in Arabic) attributes a large part of the theory of proportions to Theaetetus.[66] Scholium 62 to Book X claims that proposition X.9 is Theaetetus' discovery.[67] Plato has Theaetetus himself recount this discovery in the dialogue named after him (147d-148b), which may be the source of the scholiast's information. The *Suda* says that Theaetetus was the first to construct the regular solids, the subject of Book XIII of the *Elements*,[68] although the first scholium to Book XIII says that the Pythagoreans discovered the cube, pyramid, and

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[63] Although it is outside the scope of this summary, historians of mathematics have attempted to trace in as much detail as possible the specific contributions of Eudoxus to the theory underlying the *Elements*. For more on this see Heath, *HGM* vol. 1, 325-329 and Knorr, *The Evolution of the Euclidean Elements*, 252-285.
[64] Proclus, *In prim. Euc.* 66.14-18.
[65] Proclus, *In prim. Euc.* 68.6-10.
[66] Menso Folkerts and Maria Grazia Albiani, "Theaetetus," *Brill's New Pauly*, October 1, 2006.
[67] Σ Euc. *Elements* [ad X.9], 62: Τὸ θεώρημα τοῦτο Θεαιτήτειόν ἐστιν εὕρημα… "This theorem is a Theaetetean discovery."
[68] *Suda* s.v. Theaetetus: Ἀθηναῖος, ἀστρολόγος, φιλόσοφος, μαθητὴς Σωκράτους, ἐδίδαξεν ἐν Ἡρακλείᾳ. πρῶτος δὲ τὰ πέντε καλούμενα στερεὰ ἔγραψε. γέγονε δὲ μετὰ τὰ Πελοποννησιακά. "Athenian, astronomer, philosopher, student of Socrates, he taught in Herakleia. And he first wrote on the five so-called solids. And he was born after the Peloponnesian wars." [Adler, ed., *Suidae Lexicon*, 4 vols., Lexicographi Graeci, 1.1-1.4 (Leipzig: Teubner, 1928), 93.]

dodecahedron, and that only the octahedron and icosahedron are Theaetetus' original work.[69]

The final direct predecessor to Euclid and the *Elements*, according to Proclus, was Hermotimus of Colophon. There is no other certain mention of him in Greek writings. Proclus says only that he advanced the work of Eudoxus and Theaetetus, discovered many of the propositions in the elements, and contributed to the theory of loci.[70]

I.1.4 Other Systematists Before Euclid

Proclus mentions many other names in his summary of pre-Euclidean geometry, but there is not sufficient evidence that any of these writers had a direct influence on the *Elements.* There is evidence, however, that there were other authors working in the systematist style before Euclid himself.

Menaechmus is mentioned by Proclus as a pupil of Eudoxus and an associate of Plato.[71] He is said to have discovered the conic sections, the curved lines known as the parabola, hyperbola, and ellipse.[72] Menaechmus' work is described in more detail by Eutocius in his commentary on Book II of Archimedes' *Sphere and Cylinder,* where he gives a history of the ancient problem of doubling a cube. Eutocius reproduces the

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[69] Σ Euc. *Elements* [ad XIII], 1: τρία δὲ τῶν προειρημένων ε σχημάτων τῶν Πυθαγορείων ἐστίν, ὅ τε κύβος καὶ ἡ πυραμὶς καὶ τὸ δωδεκάεδρον, Θεαιτήτου δὲ το τε ὀκτάεδρον καὶ τὸ εἰκοσάεδρον. "Three of the aforementioned five figures are from the Pythagoreans, the cube, the pyramid, and the dodecahedron, and the octahedron and the icosahedron are from Theaetetus."

[70] The theory of loci is a way of conceiving of lines and curves as a range of possible locations for a point that is subject to certain constraints. For example, a circle in this theory can be described as the locus of all points that are a certain distance (the radius) away from a fixed point (the center).

[71] Proclus, *In prim. Euc.* 67.9-12.

[72] Proclus, *In prim. Euc.* 111.20-25.

solutions of several famous geometers, and two solutions attributed to Menaechmus follow systematist methods.[73] Menaechmus' solutions are construction proofs, demonstrating that it is possible to find two magnitudes that are "mean proportionals" between two given magnitudes. They also observe the full presentation structure of a systematist proof, although any stylistic similarities here may carry less weight than methodological similarities, because the presentation of these solutions could have been adapted by Eutocius more easily than the methods themselves.

Archytas of Tarentum was also associated with Plato and the Academy. He is credited by Proclus along with Leodamas and Theataetus with increasing the theorems and bringing them into a more "scientific" state.[74] Eutocius provides Archytas' solution to the finding of two mean proportionals after Menaechmus'.[75] Archytas' method is difficult to categorize as strictly systematist without qualification, because his construction of the diagram requires the theoretical rotation of two figures. It is not usual for motion to be employed in a systematist proof. However, the rotations required by Archytas are not truly mechanical, because they are not time-dependent. There is no difference between these rotations and, for example, "extending" a line or "describing" a circle. They are ways of describing spatial extension which require no conditions of material existence to be placed on the figures. Euclid himself employs these types of

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[73] Eutocius, "Commentarii in Libros de Sphaera et Cylindro," 78.13-84.11.
[74] Proclus, *In prim. Euc.* 66.14-18
[75] Eutocius, "Commentarii in Libros de Sphaera et Cylindro." 84.12

rotations in several places in the *Elements*.[76] Therefore, since in all other respects Archytas' method is systematist, he ought to be included as part of this group.

I.1.5 Systematists After Euclid: Apollonius of Perga

At the end of the third century BCE Apollonius of Perga published eight books on the geometry of conic sections. Conic sections are the lines generated by slicing a cone with a plane surface. We call these lines the parabola, the hyperbola, and the ellipse,[77] names which we received from Apollonius.[78] The study of conics was reportedly invented by Menaechmus, although we do not have any work of his showing this, except for his use of intersecting parabolas and hyperbolas in his solution for the doubling of the cube. Proclus and Eutocius credit Menaechmus as the inventor of the conics, both citing an epigram of Eratosthenes to this effect.[79] Euclid also wrote a work on conics, which is lost.[80] Apollonius refers to this work in Book I of his own *Conics*. The study of the conic

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[76] See above, ch. 1 note 29 for more on the rotation of figures in the *Elements*.
[77] Other conic sections are the circle (the section parallel to the base of the cone) and the triangle (the section perpendicular to the base from the vertex of the cone), neither of which is the focus of any treatise on conic sections, since they were an established part of elementary geometry long before conics were developed, and since neither requires a cone for its construction.
[78] Apollonius' predecessors in the field used the terms "section of a right angled cone," "section of an obtuse angled cone," and "section of an acute angled cone" for the parabola, hyperbola, and ellipse respectively (Pappus, *Collectionis quae supersunt*, 672.22-24). Proclus, *In prim. Euc.* 419.15-24, says the words parabola, hyperbola, and ellipse came from the Pythagoreans, but only through their research on the application of areas, and not as names for the curves themselves.
[79] Proclus, *In prim. Euc.* 111.20-25. Eutocius, "Commentarii in Libros de Sphaera et Cylindro," 96.17. The relevant line of Eratosthenes' epigram reads: μηδὲ Μεναιχμείους κωνοτομεῖν τριάδας διζήσῃ. "Do not seek to produce by conic sections the three Menaechmean lines.
[80] For a summary of Euclid's extant and lost works see Heath, *HGM* vol. 1, 421-446. For Euclid's *Conics* specifically see *ibid*. 438-439.

sections thus seems to have a strong (though not exclusive) grounding in the systematist tradition, which Apollonius is clearly continuing.

Apollonius is a systematist first and foremost in his commitment to completeness and generality in his subject matter. He articulates this priority in the preface to his first book.[81] His initial definitions are noteworthy in this respect. His construction of the cone from which all of the sections are generated uses non-mechanical rotation of a line infinitely extended in both directions about the vertex: that is, his generating cone is not bounded by circular bases, so that sections of any size can potentially be taken. Furthermore, his construction allows for cones other than "right cones," in which the axis is perpendicular to the circular base, and Apollonius makes sure that his definitions are valid for any type of cone.[82] Producing the double cone also allows him to treat paired hyperbolae and eventually conjugate hyperbolae (two pairs of hyperbolae with the same asymptotes), and thus to treat the topic as completely as possible. Heath notes that Apollonius, in defining the parts of the cone and its sections (such as the diameter, vertex, and axis), explicitly makes these definitions applicable to any curve, not just the parabola,

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[81] Apollonius, *Conics* I Prolegomena 23-26: περιέχει δὲ τὸ μὲν πρῶτον τὰς γενέσεις τῶν τριῶν τομῶν καὶ τῶν ἀντικειμένων καὶ τὰ ἐν αὐταῖς ἀρχικὰ συμπτώματα ἐπὶ πλέον καὶ καθόλου μᾶλλον ἐξειργασμένα παρὰ τὰ ὑπὸ τῶν ἄλλων γεγραμμένα. "The first (book) contains the generation of the three sections and the opposite-facing (hyperbolae), and their principal properties, worked out more fully and universally than in the writings of others."

[82] According to Eutocius' commentary on the *Conics* (Eutocius, "Commentaria in Conica," in *Apollonii Pergaei quae Graece exstant cum Commentariis Antiquis*, ed. J. L. Heiberg, vol. 2 (Leipzig: Teubner, 1891), 168.17-170.3), Geminus reports that the ancients constructed only right cones by rotating a right triangle about one of its sides, and that they generated the sections from different cones with a right, acute, or obtuse angle at the vertex. If this is true, Apollonius' construction, which produces all sections from the same cone, is a major step in generalization.

hyperbola, and ellipse.[83] Finally, he describes the conic sections as generally as possible: they can all be generated from any single cone, cut in different ways; their properties are articulated in the propositions with reference not only to the principle axes of the curves, but to any of the diameters;[84] and when a certain property belongs to two or more of the sections, he demonstrates the property for each case to which it applies.[85]

In addition to the substance, Apollonius also adheres to the methods and forms of systematist mathematics. He never uses numerical or mechanical methods in his propositions, his subject matter is thematically arranged book to book and sequentially arranged by difficulty within each book[86], and each logical step in each proof is derivable either from an earlier proposition in the *Conics* or from a proposition in the *Elements.* Each proposition also follows the Euclidean form as articulated by Proclus, including *protasis*, *ekthesis*, *diorismos*, *kataskeuē*, *apodeixis*, and *symperasma*. Apollonius observes this structure even when it results in some very difficult reading, for example *Conics* I.12, constructing the hyperbola, in which the *protasis* alone is twenty-two lines long.[87]

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[83] Heath, *HGM* vol. 2, 134.
[84] In the study of conics, a diameter is a line that bisects, with whatever orientation, all parallel lines within the curve. There are infinite diameters. An axis bisects *at right angles* all parallel lines within the curve. Apollonius' use of diameters rather than axes in his propositions demonstrates that the properties of the conic sections are not special to the axes, but are generally true for any diameter.
[85] e.g. *Conics* I.33 and I.34, in which Apollonius proves a tangency property for the parabola, then for the cases of the hyperbola and ellipse, specifying all three.
[86] Apollonius gives a summary of the contents and purpose of each book (*Conics* I Prolegomena 21-49). He characterizes the final four books as περιουσιαστικώτερα, which could be translated as "superfluous," but might be better as "fuller (in treatment)." The word seems to reemphasize his commitment to exhaustiveness.
[87] Heath (*HGM* vol. 2, 175) notes that Apollonius' strict adherence to the Euclidean structure of a proposition, as well as his exhaustive treatment of the particular cases of general principles, is a major deterrent to potential readers of the *Conics*, and may be responsible for the relative obscurity of the book.

Finally, Apollonius seems to be working within his school to make the work of other systematists, particularly Euclid, more complete. Pappus in the *Synagoge* says that Apollonius wrote the *Conics* by filling out and adding to the four books of Euclid's *Conics*.[88] In the prolegomena to Book I, Apollonius says that in discovering the new material that he will present in Book III, he realized that Euclid had not fully worked out a theory of three- and four-line locus problems.[89] So not only was he using Euclid's *Conics* as a direct reference, but he was also working within the parameters of systematist geometry to further his school's goal of exhaustiveness.

Moreover, Euclid is the only predecessor on conics whom Apollonius uses. He ignores the work of Aristaeus, an older contemporary of Euclid, who wrote a lost work of five books on solid locus problems,[90] as well as Archimedes on conic sections.[91] Both seem to have been important contributors to the field, and so it is noteworthy that Apollonius should ignore them entirely, especially since he mentions other geometers in

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[88] Pappus, *Collectionis quae supersunt*, 672.18-20.

[89] *Conics* I Prolegomena 31-37. Many properties of the conic sections can be understood by a type of locus problem (for a definition, see note 70). The sections themselves are sometimes called "solid loci" as opposed to "plane loci" because they are generated by a solid figure, the cone. Three- and four-line locus problems deal with the ratios between areas cut off by certain important lines in a conic section.

[90] Pappus (*Collectionis quae supersunt,* 672.20-24) lists Aristaeus' *Solid Loci* with Euclid's and Apollonius' *Conics* as a major work on the conic sections. He also mentions that Aristaeus uses the older terminology for the curves (that is, "sections of a right, obtuse, and acute cone" instead of "parabola, hyperbola, and ellipse"). In another passage, though of disputed genuineness (Pappus, *Collectionis quae supersunt*, 677.25–678.6), Pappus claims that Euclid acknowledged the value of Aristaeus' contribution without trying to surpass him or to cover the same ground. It therefore seems unlikely that Apollonius was unaware of Aristaeus' work.

[91] Archimedes treats conics primarily in his *Quadrature of the Parabola*, but the geometry of conic sections also features heavily in *Conoids and Spheroids*, and in the *Method.*

his prolegomena, such as Conon, Eudemus, Philonides, Naucrates, and Nicoteles.[92] Heath, in his translation of the *Conics*, conjectures that Apollonius ignored Archimedes' *Quadrature of the Parabola* mostly because it was not sufficiently elementary.[93] Neither, however, are the later books of the *Conics*, and even in the earlier books there is material that closely relates to that of Archimedes. What can be said of Archimedes is that he approaches his subject matter with methods and priorities that differ fundamentally from those of the systematist group (more on this later); and although we cannot say with certainty that Aristaeus' methods were different from Euclid's, Heath himself could find no other explanation for Apollonius' simultaneous omission of Aristaeus and inclusion of Euclid.[94]

Apollonius should, for all these reasons, be considered an unambiguous member of the systematist tradition. Features of Apollonius' text, both in his presentation of the material and in the sources he cites, become clearer when viewed as the result of this association. He is strict in his adherence to the method and style exemplified by Euclid, and his neglect of Aristaeus and Archimedes may be a clue about awareness of the division between the systematists and heurists among practitioners themselves. Now it

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[92] Apollonius also neglects to mention Menaechmus, who seems to have worked in the systematist tradition. However, Menaechmus might easily have been left out for the same reason it is rare to see geometers cite works earlier than the *Elements*: Euclid superseded him.
[93] T. L. Heath, *Apollonius. Treatise on Conic Sections.* (Cambridge: W. Heffer & Sons, 1961), lix.
[94] Heath, *HGM* vol. 2, 119.

remains for us to consider other late members of the systematist group before moving on to the heurists.

I.1.6 Other Systematists After Euclid

Apollonius' *Conics* is one of the larger and better-studied geometrical works of the systematist school, but there are other authors who came after Euclid, from whom we have fragments, attestations, and entire books showing that the systematist method and style were practiced consistently over a long span of time. For the sake of space, I will summarize the works only of authors who were writing original geometrical texts, omitting authors of secondary texts such as introductions, collections, and commentaries, because their mathematical methods and styles are naturally informed by the works they write about. These secondary texts will nevertheless return in a significant way in a later chapter.

Hypsicles worked in the latter half of the second century BCE, about one hundred years after Apollonius. He wrote a short geometrical work that is included in many manuscripts of Euclid as the fourteenth book of the *Elements*,[95] because it builds on the construction of the regular solids in the thirteenth book. He claims in the prolegomena that he is correcting and completing work of a similar nature by Apollonius. Hypsicles compares three of the regular solids, the cube, the dodecahedron, and the icosahedron,

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[95] There is also often appended a fifteenth book, which seems, from the prolegomena to its third section, to have been written by a student of Isidore of Miletus, the architect of Hagia Sophia, sometime in the sixth century CE. This is beyond the scope of the present chapter, but the style of the fifteenth book also generally seems to be systematist.

with respect to their sides, surfaces, and volumes. His object, methods, and style are clearly systematist. That is, he mostly observes the proposition structure and its signaling phrases, such as λέγω ὅτι to introduce the *diorismos*, or ὁμοίως δὴ δείξομεν, ὅτι… to introduce a similar proposition or another case of the general principle. He uses no methods of measurement or mechanical processes, and he provides general proofs, never specific cases standing for all. He does not use numbers, with two exceptions,[96] which are used only to count the "pyramids" contained in the volume of a dodecahedron or icosahedron, and this is not what we mean by a "numerical" method.

Zenodorus was the author of a treatise on isoperimetric figures,[97] of which we have fragments in several slightly differing versions.[98] His date is uncertain, but he mentions Archimedes, who died at the end of the third century BCE, and he is quoted by Pappus in the fourth century CE. He is usually placed somewhere in the second century BCE, because his style (in every version) is very close to that of Euclid.[99] Zenodorus'

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[96] Hypsicles, "Hypsiclis Liber sive Elementorum Liber XIV qui fertur," in *Euclidis Elementa*, ed. J. L. Heiberg and E. S. Stamatis, vol. 5 (Leipzig: Teubner, 1977), 1–22, section 10, lines 26 and 28.

[97] Isoperimetric figures are polygons with equal perimeters but different forms. The main point of Zenodorus' work was to show that, of any isoperimetric regular polygons (i.e. equal sides, equal angles, and equal perimeters), such as a square, pentagon, and hexagon, the one with the greatest number of sides encloses the greatest area. He shows also that a circle with the same perimeter as any regular polygon will enclose a greater area than that polygon.

[98] Pappus quotes him extensively in Book V of the *Synagoge,* though without attribution (Pappus, *Collectionis quae supersunt*, 308.9 ff.). Theon of Alexandria, about fifty years later, quotes Zenodorus with attribution in his commentary on Ptolemy's *Almagest* (Theon of Alexandria, "Commentaria in Ptolemaei Syntaxin Mathematicam I-IV," in *Commentaires de Pappus et de Théon d'Alexandrie sur l'Almageste,* Vols. 2-3, ed. A. Rome, vol. 2, Studi e Testi 72, 106 (Vatican City: Biblioteca Apostolica Vaticana, 1936), 354 ff.). The third version is from an introduction to Ptolemy's *Almagest*, of unknown authorship (contained in Pappus, *Collectionis quae supersunt*, 1138-1165).

[99] E.g. Heath, *HGM,* vol. 2, 207, supposes a date between 200 and 90 BCE. By "style," authors such as Heath and Thomas mean that he uses Euclidean proof structure and phrasing (Ivor Thomas, trans., *Greek*

methods are systematist, and he uses no numbers. His references to Archimedes are restricted to the *Sphere and Cylinder* and the first proposition of *Measurement of the Circle*, which are Archimedes' most famous works, and his most systematist in style, as we shall see. Between the two principal versions of Zenodorus' text, those quoted by Pappus and by Theon of Alexandria, there are minor differences in phrasing, but the vocabulary and structure of the propositions is mostly the same.[100] Theon's strongly systematist orientation may have colored his own treatment of Zenodorus' text, but Pappus does not seem to have had similarly strong priorities, and his version also follows the systematist forms.

Theodosius wrote three books on spherical geometry in the first century BCE. Euclid's *Elements* contains remarkably little material about spheres, and Theodosius seems to be filling in the gap.[101] Several factors in addition to his mathematical methods[102] demonstrate Theodosius' place in the systematist school. He includes a thorough set of definitions at the beginning of the work, and scrupulously follows the

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Mathematical Works, Volume II*: Aristarchus to Pappus*, Loeb Classical Library 362 (Cambridge, MA: Harvard University Press, 1941), 386 note b). Toomer places Zenodorus in the early 2nd century BCE (G. J. Toomer, *Diocles. On Burning Mirrors: The Arabic Translation of the Lost Greek Original* (Berlin; New York: Springer-Verlag, 1976), 2).

[100] An exception is that Pappus' version uses περιφέρεια for the circumference of a circle, and Theon's uses περίμετρος

[101] The reason for the *Elements'* omission of spherical geometry is mysterious. It may be, as Heath supposes (*HGM* vol. 2, 247) that spherical geometry was too closely associated with astronomy to be considered part of elementary geometry. My own opinion is that spherical geometry was simply too advanced to be considered elementary, as was the case with conic sections.

[102] Theodosius is systematist in his methods according to the criteria enumerated in the introduction. He uses no numerical solutions or mechanical constructions, he proves general principles over specific cases, he assumes nothing without proof that cannot be found in the *Elements* or in a previous proposition of his own, and his propositions are systematically arranged and sequential.

proposition structures, vocabulary, and phrasing.[103] He defines the sphere in the same way as Euclid defines the circle (a surface on which all the lines falling from one inner point are equal).[104] Second, he seems to be consciously referring to parts of the systematist tradition. Book I of the *Sphaerica* is loosely modeled on Book III of the *Elements* (the book on circles), in that it proves roughly equivalent propositions for spheres as for circles in roughly the same order.[105] Book II continues in a similar vein, in which circles on the sphere are equivalent to chords in a circle. In addition to drawing on the material from Book III of Euclid, Theodosius provides the proofs for many pre-Euclidean propositions that are used by Euclid and others of his school.[106] Finally, Theodosius distances himself from any physical understanding of his mathematics. His avoidance of astronomical terminology looks purposeful when, in Book III, he takes interest in certain specific circles on the surface of a sphere that can only be references to celestial phenomena.[107] His avoidance of any language of motion in the treatise looks

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[103] Examples include λέγω ὅτι at the beginning of every *diorismos*, ὁμοίως δὴ δείξομεν to extend an example to greater generality, and ἐκ δὴ τούτου φανερόν to introduce a porism.
[104] Theodosius, *Sphaerica* I Definition 1: Σφαῖρά ἐστι σχῆμα στερεὸν ὑπὸ μιᾶς ἐπιφανείας περιεχόμενον, πρὸς ἣν ἀφ' ἑνὸς σημείου τῶν ἐντὸς τοῦ σχήματος κειμένων πᾶσαι αἱ προσπίπτουσαι εὐθεῖαι ἴσαι ἀλλήλαις εἰσίν. Theodosius, *Sphaerica*, ed. J. L. Heiberg, Abhandlungen der Gesellschaft der Wissenschaften zu Göttingen, Philol.-Hist. Kl. N.F., 19.3 (Berlin: Weidmann, 1927).
[105] For a summary of the parallels between Book I of the *Sphaerics* and Book III of the *Elements*, see Heath *HGM* vol. 2, 247-248.
[106] Euclid's *Phaenomena* and Autolycus' *On the moving sphere* either assume or directly quote many of the propositions that Theodosius proves. For a comprehensive list, see Heath, *HGM* vol. 2, 252. Autolycus' connection to the systematist school will be discussed in a later chapter, because his works are more explicitly astronomical.
[107] That is, Theodosius never mentions the ecliptic, the horizon, or the sphere of the fixed stars. Instead, he speaks of great circles, circles at an incline, parallel small circles, etc. For a summary, see Heath, *HGM* vol. 2, 249.

similarly purposeful, though it does more to establish him as a geometer in general (as opposed to an astronomer) than as a systematist specifically.[108] Finally, Theodosius uses no numbers and performs no calculations, to the extent that his investigations are, as Heath comments, "useless for any practical purpose."[109]

Menelaus, who wrote another *Sphaerica* sometime in the late first or early second century CE,[110] is a little more difficult to categorize on stylistic grounds than Theodosius, because his work does not survive in Greek.[111] However, in content Menelaus is clearly systematist, and if the various translations are essentially faithful, he also avoids explicit astronomical language, keeping strictly to the static geometry of the sphere. Specifically, Book I of the *Sphaerica* deals with spherical triangles,[112] which Pappus says Menelaus called "τρίπλευρα" as distinct from "τρίγωνα," plane triangles.[113] Book I of the *Sphaerica* generally proves for spherical triangles the same propositions that Book I of the *Elements* proves for plane triangles.[114] Book II revisits many of the propositions of Book III of Theodosius' *Sphaerica*, and often extends or generalizes them. Book III deals with the

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[108] In a later chapter, we will see many systematist authors, including Euclid himself, who incorporate the language and geometry of motion into their astronomical works.
[109] Heath, *HGM* vol. 2, 249.
[110] Menelaus's date is roughly determined by references to him in Ptolemy and Plutarch.
[111] A Latin translation is available: Menelaus, *Sphaericorum libri III*, trans. Edmund Halley (sumptibus Academicis, 1758).
[112] Spherical triangles are those lying on the surface of a sphere, formed by the intersections of arcs of great circles. Great circles are those which bisect the sphere.
[113] Pappus, *Collectionis quae supersunt*, 476.16-17.
[114] For a list of equivalent propositions see Heath, *HGM* vol. 2, 263, and for a more thorough explanation, see Axel Anthon Björnbo, *Studien über Menelaos' Sphärik: Beiträge zur Geschichte der Sphärik und Trigonometrie der Griechen*, Abhandlungen zur Geschichte der mathematischen Wissenschaft mit Einschluß ihrer Anwendungen, 14 (Leipzig: Teubner, 1902), 33-51.

ratios between chords, and thus represents some of the earliest extant trigonometry.[115] All of this material is presented with systematist methods as far as the textual history allows us to tell. That is, Menelaus eschews numerical or mechanical reasoning, he enunciates fundamental principles including initial definitions, and he generalizes and gives multiple cases of propositions where appropriate.[116] In matters of form, Menelaus is slightly less scrupulous than Theodosius. He assumes propositions from sources other than the *Elements*,[117] and in the propositions equivalent to those for plane triangles in *Elements* Book I, he does not always follow the same logical lines as Euclid, which Theodosius did when he extended *Elements* III to spheres. Even in the absence of the original Greek, however, Menelaus' systematist method is clear enough.

I.2 The Heurist School

As we have seen, the geometers who worked within the systematist tradition have a marked style: from their methods of proof and modes of presentation to their more philosophical priorities concerning generality, exhaustiveness, and rigor, they show a specific and recognizable character. The heurists are not so unified. What we can say

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[115] Chords are the straight lines connecting the end points of any circular arc. Their length is determined by the angle at the center of the circle. Chords serve as a way of expressing the ratios between parts of circles in terms of straight lines, so they can be compared more easily to rectilinear figures. In modern trigonometry, the chord is essentially replaced by the sine function.

[116] For example Menelaus' *Sphaerica* II.11 extends Theodosius' *Sphaerica* III.13 to greater generality, and Menelaus' *Sphaerica* III provides multiple cases of the first proposition.

[117] E.g. Menelaus assumes two propositions from Ptolemy in III.1 (Heath, *HGM* vol. 2, 266-268).

generally about their methods and stylistic choices will be unevenly applicable to any given author. They show a much greater variety of practices than the systematist authors.

However, it is not the case that the heurists' identity is merely negative. In addition to their own preferred methods, standards of verification, and vocabulary, they have a way of engaging with the systematist tradition that shows both respect and criticism. Moreover, there is evidence that members of both the systematists and the heurists were at least somewhat socially bonded to each other within each group. That is, they were not just scattered mathematicians who happened to work in the same way: they knew each other and showed some awareness of being part of a tradition.

In terms of their method and style the heurists' most essential quality as mathematicians is a kind of flexibility in places where systematists are rigid. That is not to say at all that the quality of their work is lower. The heurists simply have different priorities from the universalizing theory and formal rigor of their counterparts. These priorities are manifest in several different ways.

First, heurists are less concerned with generality and universal principles. They are much less likely to demonstrate that a certain principle applies to all cases and variations of the diagram. In fact they are less likely to prove general principles at all, often preferring to show a method by which a type of problem may be solved. A common model of the heurist style of proof is to find a specific answer to a problem that is not universally framed, perform a "check," e.g. by reinserting the numerical answer into the original problem, and allowing it to be implied or intuitively understood that the problem is repeatable with any numbers.[118] That is, their methods are usually more problem-based than proof-based, more heuristic than systematic. One might go so far as to say that they treat geometry more as a descriptive than a prescriptive science.

Similarly, they do not idealize their mathematics. They comfortably treat geometrical figures as physical objects, both conceptually and actually. They frequently use conceptual mechanical methods which construct figures using time-dependent motion, or which theoretically "weigh" and "balance" figures as Archimedes does, as well as methods that physically manipulate the diagram as a means of proof, such as constructions in which a figure can only be produced by means of a certain tool or device. Likewise, they are likely to use numerical methods and notation.[119] They may employ conceptual numerical methods which "measure" a figure by assigning numerical units to its dimensions, or physical methods such as the use of a ruler to measure parts of the diagram itself. These methods should not give the impression, however, that the heurists were merely a "practical" school as opposed to the "theoretical" systematist school. Most heurist works are highly theoretical in the sense of being ordered toward conceptual mathematical knowledge that serves no immediate utilitarian purpose. An author in this school might be more inclined to write a work of practical mathematics, but

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[118] We will see especially good examples of this in Heron, as well as among mathematicians such as Diophantus in later chapters.

[119] An example of a numerical method would be describing a square as a four-sided figure whose sides are all three units, as opposed to a four-sided figure whose sides are all equal to each other. In this simple example, the difference may seem trivial, but the implications for problem solving are profound. Later, I will compare two treatments of similar figures, one with numerical methods and one without. The difference is striking.

that is not the primary difference between the groups. Heuristic methods and physical conceptualization of geometric objects do not imply practical geometry as opposed to theoretical; they are simply a different theory.

As to style, the heurists' sense of completeness and structure does not seem to be as stringent as their systematist peers. They more often write works with a specific goal (such as proving a single principle, solving a particular problem, or demonstrating a certain method), rather than "covering" a topic from its first principles to its grandest implications. They are less likely to include propositions that don't serve the main goal of the work simply for the sake of including all related information. Their works thus tend to be more linear or recursive and less systematic in structure. They also seem less likely to write books for the sake of "filling in the gaps" in someone else's treatment of a subject.

Finally, on a smaller scale, the heurists' stylistic choices are often more flexible. They rarely observe the systematist proposition structure,[120] and they often use a different verbal style.[121] For example, the phrase ἐκ δὴ τούτου φανερόν to introduce a porism is commonly found in Euclid, Apollonius, and Theodosius, but it appears only once in Archimedes (in his most systematist-style work, *On the Sphere and Cylinder*) and not in the work of any other heurist.[122] Likewise, the phrase ὁμοίως δὴ δείξομεν to extend or generalize a proposition is found hundreds of times among Euclid, Apollonius,

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[120] Archimedes is really the only exception to this, and only in certain of his works, as we shall see.
[121] For more on these verbal formulas, see below, ch. 1 notes 153-156.
[122] Archimedes, *Sphere and Cylinder* I.28.

Theodosius, and Hypsicles, but only twice in Archimedes' *Sphere and Cylinder*[123] and once in Heron's *Metrica*.[124] More generally, the heurists may have a less detached authorial voice, or they may directly address the reader within a proposition (e.g. Heron's *Metrica*). In some cases, they even seem to have different vocabulary for the same topics.[125] Though they may not have a single recognizable writing style, the heurists are united in their noticeable departure from the distinctive systematist style.

Thus, while they show greater variety in their practices than the systematists, the heurists do not seem to acknowledge any internal dividing lines among themselves. Their identity as a group can be seen in their stylistic and methodological choices, from which it is clear that they are all drawing (albeit each with his particular priorities) from the same pool of practices.

I.2.1 Heron of Alexandria

Among authors who worked in the heurist tradition Heron of Alexandria is both exemplary and well represented.[126] His date has been fixed in the mid-first century, based on an eclipse that he claims to have observed in 62 CE.[127] Heron wrote on a variety of

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[123] Archimedes, *Sphere and Cylinder* I.6, I.11.
[124] Heron, *Metrica* III.19.
[125] For example, there seems to be a divide amongst later geometers, along systematist/heurist lines, about the names of the conic sections.
[126] Archimedes is the other major heurist geometer. He has a good deal more extant work than Heron, is much earlier, and became a more important figure, but both his geometry and his relationship to the systematist tradition are much more complicated than Heron's. I will therefore leave him for the next section, when the differences between the two schools have been clearly and sufficiently exemplified.
[127] Heron's date was the subject of a long debate until Neugebauer's discovery of this reference. O. Neugebauer, *Uber eine Methode zur Distanzbestimmung Alexandria-Rom bei Heron*, Historisk-filologiske meddelelser, XXVI, 2, 7 (Copenhagen: Levin & Munksgaard, Ejnar Munksgaard, 1938).

topics besides geometry, including mechanics and catoptrics, but many of his works suffered a great deal in transmission. His best-preserved work, the *Metrica*, is a geometrical treatise in three books which amply illustrates the priorities of the heurist school.

Of all Heron's works, the *Metrica* has come to us most intact.[128] Most of his other geometrical treatises were heavily edited and epitomized by the Byzantine scholars who used them,[129] but the *Metrica* has retained its original form. It also shows the fullest extent of Heron's original geometrical thought, since most of his other works are based on it.[130] The first book deals with plane figures (most problems are about finding the area), the second with solids, and the third with problems of dividing various figures into parts that have a certain ratio to one another.

The *Metrica* differs from systematist works in almost every particular. The *Metrica* is problem-based rather than proof-based,[131] meaning that instead of demonstrating general principles about figures, it sets out ways of finding information

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[128] Heath, *HGM* vol. 2, 316-317. Part of the evidence for this is corroboration of the text by quoted fragments in other writers, in addition to references to Archimedes and Eudoxus in the prolegomena, the Greek form in which the fractions are written (as opposed to the Egyptian), and the fact that it seems to have been less widely disseminated and therefore less altered (the text was not discovered until 1896 by Richard Schöne, in a single manuscript from Constantinople).

[129] Heath, *HGM* vol. 2, 307-308.

[130] For a thorough list see Heath, *HGM* vol. 2, 318-320.

[131] These are my terms, in accord with modern interpretations of the mathematics in question. Heron himself says that he is giving "geometrical proofs" (γεωμετρικὰς ἀποδείξεις) in several places (e.g. about the first six problems of *Metrica* I, at *Metrica* I.6, 25-28. [H. Schöne et al., eds., *Heronis opera quae supersunt omnia* vol. 3 (Leipzig: Teubner, 1899)]). But ἀποδείξις here never corresponds to how Proclus defines the term with respect to Euclid's geometry. Heron uses the word to refer to the parts of his problems that deal specifically with geometric figures, as opposed to the parts that deal only with calculations.

about a specific figure with certain specifications. To this end, it uses numerical analysis and calculations throughout, assigning arbitrary numbers to various dimensions of the figures and then finding the desired quantity based on those.[132] No text that otherwise follows systematist forms and methods uses specific numbers or numerical notation to do geometry. The problems are also not generalizing, but are specific. That is, when Heron presents and solves a problem about a figure, he uses not an abstract, paradigmatic example of the figure, but one with numerically specified dimensions. The first problem of the *Metrica*, for example, is to find the area of a rectangle with a length of 5 units (μονάδες) and a width of 3 units. Heron solves the problem using these numbers, and never shows that his solution would be equally valid with any other numbers.[133]

Next, the problems in the *Metrica* are almost never structured like a systematist proposition.[134] They include no formulaic introductory or concluding statements.[135]

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[132] By "numbers" I refer to the Greek numbering system of letters with a superscript bar.

[133] He does, however, say that the same reasoning would apply if the figure were a square (κἂν τετράγωνον δὲ ᾖ τὸ χωρίον, ὁ αὐτὸς ἁρμόσει λόγος. *Metrica* I.1, 12-13). This is a nod to broader applicability, but it does not change the specificity of the method itself. Heron solves the problem by dividing up each side into its units, dropping perpendiculars to the opposite side, and counting the squares in the middle. It may be clear enough to us, as it probably was to Heron, that this method would work for a rectangle with any dimensions. But his solution is not explicitly generalized, as was the practice among the systematists.

[134] *Metrica* I.32 and III.18 are the only problems that fully follow the format of a systematist proof. I.15, I.27-29, II.7, II.10, III.10-17, III.19, and III.23 use systematist methods and mostly follow the form, but they contain no *protasis* or *symperasma*, and only I.27, II.7, II.10, and III.18 use λέγω ὅτι to introduce the *diorismos*. III.23 also uses the λέγω ὅτι phrase, but, although the geometry uses no numbers, Heron does not even show the claim he makes. He simply notes that Archimedes already proved it.

[135] Some of the problems do have introductory statements, but they do not follow any standard formula, as systematist propositions do. For example, *Metrica* I.15 begins: Ἔστω τραπέζιον τὸ ΑΒΓΔ δοθεῖσαν ἔχον ἑκάστην τῶν πλευρῶν καὶ ὀρθὴν τὴν ὑπὸ ΒΓΔ γωνίαν. ὅτι δοθεῖσά ἐστιν ἡ ἀπὸ τοῦ Α κάθετος ἀγομένη ἐπὶ τὴν ΓΔ. "Let there be a trapezium ABCD having each of the sides given and the right angle BCD. That the perpendicular dropped from A to CD is given." A short proof follows, which aside from the lack of a *protasis* is very much in the style of Euclid's *Data* (that is, a systematist-style argument about whether a

Instead, most problems begin with ἔστω δὴ and the name of some figure, and then the problem is posed to find (εὑρεῖν), to examine (ἐπισκέψασθαι, only in I.4), to measure (μετρῆσαι), to remove (ἀφελεῖν, of cutting off part of a figure), or to cut (τεμεῖν) some part of the figure, such as the area or a side. And instead of a single unified argument, like a systematist proposition, Heron's problems in the *Metrica* are usually recursive with two parts. In the first, he works with the geometry, examining a figure and solving a problem about it. In the second, he walks the reader through the calculations without reference to the figure, usually directly instructing the reader with imperative verbs (a practice which is never seen among the systematists).

The *Metrica* is not systematic inasmuch as it does not build on itself internally to the degree found in systematist works. Sometimes the text will refer to earlier problems, but this nearly always occurs in transitional comments rather than the problems themselves.[136] The early problems do not lay theoretical foundations for the later ones, because the goal of the *Metrica* is not the development of a theoretical system (as is the goal of the *Elements*, for example). Instead, its goal is to demonstrate the methods and theory used for solving a certain type of problem, so its structure is more recursive than cumulative. It presents mostly the same type of problem again and again for increasingly

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magnitude is "given"). The proof is then followed by what Heron calls the "analysis," which has more or less the form of the other problems in *Metrica* I.

[136] For example, the paragraph immediately following *Metrica* I.16 in which Heron explains that of any four-sided figure with known sides the area can be found by dividing it into two triangles and following the method he demonstrated in *Metrica* I.8 for finding a the area of a triangle with known sides.

complicated figures. However, while a systematist work would give multiple cases of the same principle for the sake of exhaustiveness, the *Metrica* gives only one variation of each type of figure to be analyzed.[137] Another structural feature of the *Metrica* that is not seen in the systematist tradition is the way Heron creates sections and signals their relative importance by addressing the reader in the body of the text. Not only does he instruct the reader on the calculations in the "synthesis" parts of the problems (as well as using signaling sentences such as "the synthesis will proceed as follows"), but he also inserts paragraphs of commentary between problems to inform the reader of what he is doing or where more information can be found. For example, after *Metrica* I.16, there is a short paragraph explaining (among other things) that now that he has covered three- and four-sided figures, he will write about equal-sided and equal-angled figures up to the dodecagon, since this continues up to the circumference of a circle.[138] Paragraphs like this occur after *Metrica* I.6, I.16, I.25, II.12, II.15, III.9, and III.19, as well as in the prolegomena and concluding statements of each book, but there is a great deal more of

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[137] For example, in *Elements* I.35-38, Euclid proves, in order, that parallelograms on the same base and in the same parallels are equal, that parallelograms on equal bases and in the same parallels are equal, that triangles on the same base and in the same parallels are equal, and that triangles on equal bases and in the same parallels are equal. Heron makes no gesture toward this type of thoroughness, and only gives one specific case of each type of area to be found (i.e. one right triangle, one isosceles triangle, one rectangle, etc.).

[138] Heron, *Metrica* I.16, 47-51: καὶ τὰ μὲν περὶ τῶν τριπλεύρων καὶ τετραπλεύρων ἐπὶ τοσοῦτον εἰρήσθω, ἑξῆς δὲ περὶ τῶν ἰσοπλεύρων τε καὶ ἰσογωνίων εὐθυγράμμων γράψομεν ἄχρι τοῦ δωδεκαγώνου, ἐπειδὴ τοῦτο συνεγγίζει μᾶλλον τῇ τοῦ κύκλου περιφερείᾳ. "And let thus much be said about three-sided and four-sided figures, and from here we will write about equilateral and equiangular rectilinear figures up to the dodecagon, since this much more closely approaches the circumference of a circle."

this type of commentary woven into the problems themselves.[139] Thus, Heron supplements the looseness of structure in both the problems and the work as a whole with much guiding and signaling language, whereas the systematists keep the reader oriented through strict adherence to form.

Finally, Heron's sources of information show his heurist associations. With respect to other authors, he refers to works by Archimedes twenty-one times in the course of the *Metrica*, but although he occasionally makes use of systematist styles and materials, he never mentions Euclid by name.[140] The *Metrica*, unlike most systematist works, does not refer only to itself or the *Elements*, but both invokes the authority of outside sources and assumes a certain amount of prior knowledge on the side of the reader. For instance, there are no definitions or first principles given, nor does Heron justify his methodological first step in every problem, which is to assign numerical units to lines, planes, and solids (e.g. a rectangle with sides of three and five units). This last point may seem trivial, since in modern mathematics it is customary to assign numbers

[139] As for example, the introduction to *Metrica* I.9, 1-4: Ἐπεὶ οὖν ἐμάθομεν τριγώνου τῶν πλευρῶν δοθεισῶν εὑρεῖν τὸ ἐμβαδὸν ῥητῆς οὔσης <τῆς> καθέτου, ἔστω μὴ ῥητῆς ὑπαρχούσης τῆς καθέτου τὸ ἐμβαδὸν εὑρεῖν. "Since we have learned, therefore, to find the area of a triangle with given sides when the existence of a perpendicular is established, let there be [a problem] to find the area when the existence of a perpendicular is not established."

[140] The other names mentioned by Heron in the *Metrica* are Eudoxus, a systematist, whom he praises in the prolegomena to *Metrica* I; Plato, from whom we have no work of geometry (although, as seen above, most of the systematists were associated in some way with the Academy), in *Metrica* II.15 where Heron calls the regular solids "the solids of Plato" (referring to the categorization of the solids as elements of the universe in the *Timaeus*); and Dionysodorus in *Metrica* II.13. We do not have any work by Dionysodorus except for two ambiguous fragments in Eutocius, so we do not know his associations, but Heron credits to him a proof about the torus or "spire" (σπεῖρα), which Heath (*HGM* vol. 2, 219) argues must have been proved along the lines of Archimedes' *Method*. If there is anything to that, Dionysodorus would also have been a heurist.

(or variables that stand for unknown numbers) to lines, planes, and solids. But it was not always customary. Euclid and his associates do not do it, and David Fowler is not alone in claiming that there was not a Greek tradition in this kind of geometry until long after Euclid.[141] And even if it were trivial to assign numbers to geometrical magnitudes, it is no less trivial to state explicitly that a line may be drawn from one point to another – Euclid nevertheless states it. Heron omits this type of theoretical groundwork altogether.

Now some have claimed that Heron is not in fact doing geometry but a subdiscipline of it, sometimes called mensuration.[142] The work is, after all, called *Metrica* and not *Geometrica*, and Heron says in several places, including the prolegomena to Book I, that he is giving the measurements (μετρήσεις) of various figures. But Heron did write a *Geometrica*,[143] whose problems are of exactly the same nature as those of the *Metrica*, and indeed are even more oriented toward measurement, since they use specific units of measurement such as feet and stades, whereas the *Metrica* uses only abstract, unspecified μονάδες. More importantly, Heron himself claims to be doing geometry. He says for example that the first six problems of Book I (which have all the heurist features

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[141] There were, however, traditions of this kind of geometry in Babylon and Egypt, which some have argued that Heron may be drawing from (e.g. B. L. van der Waerden, *Geometry and Algebra in Ancient Civilizations* (Berlin; New York: Springer-Verlag, 1983), 183. For summaries, see O. Neugebauer, *The Exact Sciences in Antiquity*, 2nd ed. (New York: Harper & Brothers, 1962), 34-36, 42-48, 78-80; Kline, *Mathematical Thought from Ancient to Modern Times*, 10-11, 18-21. For some further specifics on Babylonian geometry, see van der Waerden, *Geometry and Algebra in Ancient Civilizations*, 56-62.
[142] Heath, *HGM* vol. 2, 198, see also Cuomo, *Ancient Mathematics* 166. Wilbur Richard Knorr, *The Ancient Tradition of Geometric Problems* (Boston: Birkhäuser, 1986), 156 calls it "metrical geometry."
[143] In Schöne et al., *Heronis opera quae supersunt omnia*, vol. 4. The authorship of the *Geometrica* has been contested, but it is clearly based on the first book of the *Metrica*, since the problems are on the same material and in the same order.

I described) were done by “reckoning the geometrical proofs,” as opposed to analysis and synthesis by numbers.[144] Even the first sentence of the book is Ἡ πρώτη γεωμετρία, ὡς ὁ παλαιὸς ἡμᾶς διδάσκει λόγος, περὶ τὰς ἐν τῇ γῇ μετρήσεις καὶ διανομὰς κατησχολεῖτο, ὅθεν καὶ γεωμετρία ἐκλήθη, “The first geometry, as the ancient account teaches us, was concerned with measurements and divisions on the earth, whence it was called geometry.” Furthermore, the authors whose work Heron claims to be continuing, Archimedes and Eudoxus, were geometers. The specific propositions he cites from Archimedes as examples of the work he is continuing are from the *Sphere and Cylinder*, which is done in the style of pure systematist geometry. Heron, therefore, does not seem to acknowledge a difference between geometry and his practice. He understands it as the same discipline, he is just doing it in a different way.

It is said also that Heron’s *Metrica* was practical or applied, as opposed to theoretical geometry.[145] Heron does highlight the usefulness of geometry in his prolegomena, and says that he is writing this work because of the indispensable nature of the subject.[146] Useful, however, is not opposed to theoretical. In fact, the *Metrica* is the

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[144] Heron, *Metrica* I.6, 25-28: Μέχρι μὲν οὖν τούτου ἐπιλογιζόμενοι τὰς γεωμετρικὰς ἀποδείξεις ἐποιησάμεθα, ἑξῆς δὲ κατὰ ἀνάλυσιν διὰ τῆς τῶν ἀριθμῶν συνθέσεως τὰς μετρήσεις ποιησόμεθα. “Up until this point we have solved problems by reckoning the geometrical proofs, from now on we will make the measurements according to analysis through the synthesis of numbers.” Analysis and synthesis here are referring to some additional numerical methods Heron introduces in the following problems. The basic method of solving the problems remains the same.

[145] Heath, *HGM* vol. 2, 198; Cuomo, *Ancient Mathematics,* 164-166.

[146] Heron, *Metrica* I prolegomena 3-5: χρειώδους δὲ τοῦ πράγματος τοῖς ἀνθρώποις ὑπάρχοντος ἐπὶ πλέον προήχθη τὸ γένος, “And since it was a useful thing for people, in time the discipline was advanced,” and 22-25: ἀναγκαίας οὖν ὑπαρχούσης τῆς εἰρημένης πραγματείας καλῶς ἔχειν ἡγησάμεθα συναγαγεῖν, ὅσα τοῖς πρὸ ἡμῶν εὔχρηστα ἀναγέγραπται καὶ ὅσα ἡμεῖς προσεθεωρήσαμεν. “Therefore, since the matter

most abstract of Heron's mathematical works. Not only does he frequently cite the geometrical principles informing his calculations, but in this work alone he uses abstract μονάδες instead of actual units of measurement. In fact, the unspecified μονάδες act somewhat like algebraic variables in that they can be substituted for any unit of measurement. These are some of Heron's ways of implicitly articulating a methodological theory. In other places he is more explicit about his theoretical work: he begins I.8, for example, by telling the reader that there is a universal method for calculating the area of a triangle of known sides.[147] Besides the generalizable (if not explicitly generalized) methodology, the subject matter of the *Metrica* speaks against it being a purely practical work. Some later problems especially deal with extremely complicated figures whose usefulness is not very clear. It is hard to imagine, for example, that the methods for finding the volumes of an icosahedron and a dodecahedron in II.8 and II.9 are broadly applicable to practical situations. So it does not seem to me that Heron is doing practical or applied geometry – most of the *Metrica* is actually quite impractical, and he does not apply it to any concrete situations. The difference between his geometry and Euclid's is not that Euclid's is theoretical and Heron's is applied – they are both theoretical, but with different methods and practices. The division of Greek

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under discussion is indispensable, I was convinced to collect as many beneficial things as were written by my predecessors, and as many as I observed in addition."

[147] Heron, *Metrica* I.8, 1-3: Ἔστι δὲ καθολικὴ μέθοδος ὥστε τριῶν πλευρῶν δοθεισῶν οἱουδηποτοῦν τριγώνου τὸ ἐμβαδὸν εὑρεῖν χωρὶς καθέτου. "And there is a universal method for how to find the area of any triangle when the three sides are given, without a perpendicular."

geometry into "theoretical" and "applied," is of limited value, and fails to properly describe authors such as Heron and Archimedes.

The methodological differences between heurists and systematists emerge clearly from a comparison of Euclid's *Elements* I.41 and Heron's *Metrica* I.3. Both handle the same geometrical material (a triangle and a parallelogram that have the same base and lie within the same parallel lines) in very different ways.

Euclid, *Elements* I.41[148]

Ἐὰν παραλληλόγραμμον τριγώνῳ βάσιν τε ἔχῃ τὴν αὐτὴν καὶ ἐν ταῖς αὐταῖς παραλλήλοις ᾖ, διπλάσιόν ἐστι τὸ παραλληλόγραμμον τοῦ τριγώνου.

Παραλληλόγραμμον γὰρ τὸ ΑΒΓΔ τριγώνῳ τῷ ΕΒΓ βάσιν τε ἐχέτω τὴν αὐτὴν τὴν ΒΓ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἔστω ταῖς ΒΓ, ΑΕ· λέγω, ὅτι διπλάσιόν ἐστι τὸ ΑΒΓΔ παραλληλόγραμμον τοῦ ΒΕΓ τριγώνου.

Ἐπεζεύχθω γὰρ ἡ ΑΓ. ἴσον δή ἐστι τὸ ΑΒΓ τρίγωνον τῷ ΕΒΓ τριγώνῳ· ἐπί τε γὰρ τῆς αὐτῆς βάσεώς ἐστιν αὐτῷ

[148] Text from Euclid, *Elementa*, ed. J. L. Heiberg and E. S. Stamatis (Leipzig: Teubner, 1969).

τῆς ΒΓ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΒΓ, ΑΕ. ἀλλὰ τὸ ΑΒΓΔ παραλληλόγραμμον διπλάσιόν ἐστι τοῦ ΑΒΓ τριγώνου· ἡ γὰρ ΑΓ διάμετρος αὐτὸ δίχα τέμνει· ὥστε τὸ ΑΒΓΔ παραλληλόγραμμον καὶ τοῦ ΕΒΓ τριγώνου ἐστὶ διπλάσιον.

Ἐὰν ἄρα παραλληλόγραμμον τριγώνῳ βάσιν τε ἔχῃ τὴν αὐτὴν καὶ ἐν ταῖς αὐταῖς παραλλήλοις ᾖ, διπλάσιόν ἐστι τὸ παραλληλόγραμμον τοῦ τριγώνου· ὅπερ ἔδει δεῖξαι.

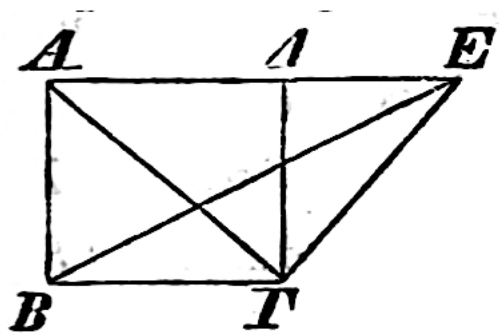

Figure 2: Euclid, *Elements* I.41. Diagram from Euclid, *Elementa*, ed. J. L. Heiberg and H. Menge (Leipzig: Teubner, 1883), 97.

If a parallelogram both has the same base as a triangle and is within the same parallels, the parallelogram is the double of the triangle. For let the parallelogram ΑΒΓΔ have the same base, ΒΓ, as the triangle ΕΒΓ, and let it be within the same parallels, ΒΓ and ΑΕ. I say that the parallelogram ΑΒΓΔ is the double of the triangle ΒΕΓ. For let ΑΓ be joined. The triangle ΑΒΓ is then equal to the triangle ΕΒΓ. For it [ΑΒΓ] is both of the same base, ΒΓ, as it [ΕΒΓ], and is within the same parallels, ΒΓ and ΑΕ. But the parallelogram ΑΒΓΔ is the double of the triangle ΑΒΓ, for the diameter ΑΓ cuts it in two. And so, the parallelogram ΑΒΓΔ is also the double of the triangle ΕΒΓ. If, then, a parallelogram both has the same base as a triangle and is within the same parallels, the parallelogram is the double of the triangle. Which it was required to show.

Heron, *Metrica* I.3[149]

Ἔστω τρίγωνον ἰσοσκελὲς τὸ ΑΒΓ ἴσην ἔχον τὴν ΑΒ τῇ ΑΓ καὶ ἑκατέραν <τῶν> ἴσων μονάδων ι. τὴν δὲ ΒΓ [τῇ ΑΓ <καὶ> ἑκατέραν τῶν ἴσων μονάδων ι <τὴν δὲ ΒΓ>] μονάδων ιβ. εὑρεῖν αὐτοῦ[ς] <τὸ ἐμβαδὸν.> ἤχθω κάθετος ἐπὶ τὴν ΒΓ ἡ ΑΔ. καὶ διὰ μὲν τοῦ Α τῇ ΒΓ παράλληλος ἤχθω ἡ ΕΖ, διὰ δὲ τῶν Β, Γ τῇ ΑΔ παράλληλοι ἤχθωσαν αἱ ΒΕ, Γ<Ζ>· διπλάσιον ἄρα ἐστὶν τὸ ΒΓΕΖ παραλληλόγραμμον τοῦ ΑΒΓ τριγώνου· βάσιν τε γὰρ αὐτῷ ἔχει τὴν αὐτὴν καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστίν. καὶ ἐπεὶ ἰσοσκελές ἐστι καὶ κάθετος ἦκται ἡ ΑΔ, ἴση ἐστὶν ἡ ΒΔ τῇ ΔΓ. καὶ ἔστιν ἡ ΒΓ μονάδων ιβ· ἡ ἄρα ΒΔ ἐστὶ μονάδων ς. ἡ δὲ ΑΒ μονάδων ι· ἡ ἄρα ΑΔ ἔσται

[149] Text from Schöne et al., *Heronis opera quae supersunt omnia,* vol. 3, 8-10.

μονάδων η, ἐπειδήπερ τὸ ἀπὸ τῆς ΑΒ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΒΔ ΔΑ· <ὥστε καὶ> ἡ ΒΕ ἔσται μονάδων η. ἡ δὲ ΒΓ ἐστὶ μονάδων ιβ. τοῦ ἄρα ΒΓΕΖ παραλληλογράμμου τὸ ἐμβαδόν ἐστι μονάδων ϟς· ὥστε τοῦ ΑΒΓ τριγώνου τὸ ἐμβαδόν ἐστι μονάδων μη. ἡ δὲ μέθοδός ἐστιν αὕτη· λαβὲ τῶν ιβ τὸ ἥμισυ· γίνονται ς· καὶ τὰ ι ἐφ’ ἑαυτά· γίνονται ρ. ἄφελε τὰ ς ἐφ’ ἑαυτά, ἅ ἐστι λς· γίγνονται λοιπὰ ξδ. <τούτων πλευρὰ γίνεται η·> τοσούτου ἔσται ἡ ΑΔ κάθετος. <καὶ τὰ ιβ ἐπὶ τὰ η· γίνονται> ϟς. τούτων τὸ ἥμισυ. <γίνονται μη· τοσούτων ἔσται τὸ ἐμβαδὸν τοῦ τριγώνου>.

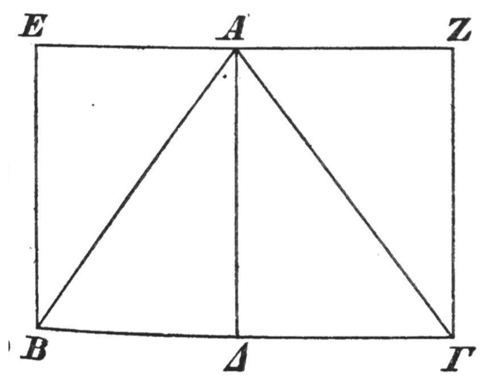

Figure 3: Heron, *Metrica* I.3. Diagram from H. Schöne et al., eds., *Heronis opera quae supersunt omnia,* vol. 3 (Leipzig: Teubner, 1899), 9.

Let there be an isosceles triangle, ΑΒΓ, having the side AB equal to the side ΑΓ, and each of 10 equal units, and side ΒΓ of 12 units. To find its area. Let ΑΔ be drawn perpendicular to BC. And through A let EZ be drawn parallel to BC, and through B and Γ let BE and ΓZ be drawn parallel to AΔ. Then the parallelogram BΓEZ is the double of the triangle ABΓ. For it [BΓEZ] both has the same base as it [ABΓ] and is within the same parallels. And since it is isosceles and AΔ was drawn perpendicular, BΔ is equal to ΔΓ. And BΓ is of 12 units. BΔ is then of 6 units. And AB is of 10 units. AΔ will then be of 8 units, since the square formed from AB is equal to those formed from BΔ and ΔA. And so, BE will also be of 8 units. And BΓ is of 12 units. The area of the parallelogram BΓEZ, then, is of 96 units. And so, the area of the triangle ABΓ is of 48 units. And the method is this: take half of 12, this gives 6. And multiply 10 by itself, this gives 100. Subtract 6 multiplied by itself, which is 36, this gives 64. The side [i.e. square root] of this is 8. The perpendicular AΔ will be of this quantity. And multiply 12 by 8, this gives 96. Take half of it, this gives 48. The area of the triangle will be of this quantity.

The stylistic differences between these two passages are prominent. The proposition from the *Elements* contains all of the canonical parts. It begins with a completely generalized enunciation of what is to be proved (the *protasis*, lines 1-3). Then the specific demonstrative case is set out (the *ekthesis*, i.e. the parallelogram ABCD and the triangle EBC, lines 4-6), and the general enunciation is claimed for the specific case (the *diorismos*, lines 6-8, beginning with the formulaic λέγω, ὅτι). The argument of the proof (lines 9-16) includes both a *kataskeuē* (the additional construction of AC, line 9) and an ἀπόδειξις which draws directly from the information provided by the *kataskeuē* (e.g. lines 14-15). The generalized statement is then restated (the *symperasma*, lines 17-

19) and followed by the formulaic ending ὅπερ ἔδει δεῖξαι. In *Metrica* I.3, on the other hand, there is no generalized formulation of a proposition or a task. Instead a specific instance of a problem is given right at the outset, and this specific case is never linked to a general statement. Furthermore, Heron is linear where Euclid is recursive. Heron never looks back to remind the reader what has been accomplished. Most especially, this problem has two parts with significantly different styles: the geometrical first part from lines 1-18, and then the numerical calculation (the "method") from lines 18-24.[150] The language of the first part is slightly more formal, especially in using third person imperatives. The second part uses second person singular imperatives and more contracted syntax (fewer particles, verbs elided more often). The systematist proposition, on the other hand, maintains the same voice throughout.

Moreover, the propositions derive and handle the information that they use differently. *Elements* I.41 is part of a self-contained system of information, and all its logical steps are dependent on axioms or previous propositions within the *Elements* itself. For example, in lines 9-13, ABC is proved to be equal to EBC because the triangles fulfill the conditions of *Elements* I.37. Again, in lines 13-16, the ABCD is proved to be twice the triangle ABC because AC is a diameter of ABCD, and therefore bisects it, as was proved in I.34. And all of the relevant terms and methods were defined and articulated at

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[150] Note, however, that the geometrical part of the problem also uses numbers and makes calculations. It just does this with explicit reference to the geometrical figure, whereas the second part gives only the details of the arithmetical operations.

the beginning of the book. The *Metrica*, on the other hand, gives no definitions or methodological principles, and *Metrica* I.3 draws on a good deal of outside knowledge. Most prominently, it draws on *Elements* I.41 for the argument that the parallelogram BCEZ is twice the triangle ABC. Heron also assumes in this problem the ability to draw (a) a perpendicular (*Elements* I.11), (b) a parallel (*Elements* I.31), and (c) a line equal to a given line (*Elements* I.2). It is not unusual that a mathematician should draw on Euclid, but normally the mathematicians who do so signal their dependence by adhering to the systematist style (i.e. the proposition structure and formulaic phrasing), while Heron makes no nod to the systematist tradition either stylistically or methodologically, despite his obvious familiarity with it. Euclid is only one of the sources of information that Heron uses without explicit proof or justification. The entire method of the *Metrica* is based on an application of numbers to geometrical objects, which Heron assumes to be understood.

With regard to the place of these passages in the larger structures of the works, *Elements* is part of a series of propositions (I.33-I.45) demonstrating the constraints placed by parallel lines on triangles or parallelograms contained within them, which serves the overall goal of Book I, namely an exhaustive and logically sequential catalogue of the properties of triangles and parallelograms. On the other hand, Book I of the *Metrica* is about finding the areas of different shapes, and though it ranges impressively from basic rectilinear shapes to conic sections and sections of spheres, it is not exhaustive or cumulative in the same way as the *Elements*.

In comparison to the systematist tradition, what we can see in *Metrica* is a different way of doing geometry: using numbers, solving specific problems without general proof, taking exemplary cases instead of exhausting the variations of a problem, and proceeding in a series of more or less isolated problems, from simpler to more complicated, instead of following a unified structure.

I.2.2 Archimedes

Archimedes worked in the early third century BCE, after Euclid but before Apollonius, and long before Heron. He was from Syracuse, but he had contacts in Alexandria, among whom were Eratosthenes and Conon the astronomer. He was a polymath, and his works on arithmetic, astronomy, mechanics, and hydrostatics will be considered in other chapters, but even within the field of pure geometry, he was the most versatile mathematician whose work has come down to us. Although he wrote several works that draw on the systematist style, Archimedes belongs to the heurist tradition. Like Heron, he was clearly familiar with systematist geometry, but he certainly worked outside it, as we shall see in an overview of the different methods employed by Archimedes in his purely geometrical works,[151] including numerical/arithmetical

[151] "Purely geometrical" works are those whose goal is the analysis of geometrical figures as spatial objects. I have excluded *On Floating Bodies* and *On Plane Equilibriums*, because their goals are the elucidation of the principles of hydrostatics and mechanics, respectively. The geometry used in these works is clearly a means to an end. I have also excluded *The Sand-Reckoner* and *The Cattle Problem*, as these are works of arithmetic (and a little astronomy). Finally, I have not considered the fragmentary *Stomachion*, because there is not enough text remaining to speak conclusively on either method or style. However, the *Stomachion* appears to be a single problem (more like a puzzle) about rearranging the parts of a square that has been dissected in a certain way. There is certainly no parallel to this kind of geometrical puzzle in systematist works, and no particular systematist phrasing stands out in the surviving text.

methods, mechanical curves, and Archimedes' own mechanical method. Archimedes was one of the first geometers (possibly the first) to obviously distinguish themselves from the systematist school of practice, which was rising to dominance at the time.

I.2.2.1 Archimedes' Systematist (and almost-systematist) Works: *Sphere and Cylinder, Conoids and Spheroids*

On the Sphere and Cylinder is Archimedes' longest and most systematist-style treatise. He uses no numbers or mechanical methods, makes generalizing proofs, includes definitions and postulates at the outset, and follows the proposition format.[152] He uses the systematist phrase λέγω ὅτι (and variations) in the *diorismos* thirty-one times in this work.[153] This is his only geometrical work in which we find the phrase ὅπερ ἔδει δεῖξαι

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[152] Heiberg and Stamatis, *Archimedis Opera Omnia, cum Commentariis Eutocii,* vol. 1, 2-229. Definitions and postulates, 6-10.

[153] Archimedes *Sphere and Cylinder (SC)* I.1, 2, 3, 7, 7 (alternative proof), 8-11, 14 (Heiberg 62.14 and 64.28), 16-22, 24, 25, 33 (Heiberg 120.18 and 122.25), *SC* II.1 (synthesis), 2, 4 (synthesis), 5 (synthesis), 7 (synthesis), 8 (Heiberg 210.18 and 212.9), 8 (alternative proof), and 9. Archimedes also uses φημὶ δὴ ὅτι three times in *SC*, all in the alternative proof of II.8. φημὶ δὴ ὅτι appears in no other work of Archimedes, and it is not a common systematist phrase (e.g. it appears in none of the geometrical works of Euclid or Apollonius). λέγω ὅτι (including its variations) does not appear more than six times in any other work of Archimedes, and this difference cannot be accounted for by a difference in the number of propositions in each work. Furthermore, aside from three instances in the *Method* (1, 9, and 15), all these examples appear in works of the systematist style: six times in *Plane Equilibriums* (I.4, 5, 7, 11, 12, and 13 alternative proof), once in the first proposition of *Measurement of the Circle*, twice in the last six propositions of *Quadrature of the Parabola* (16 and 22), and three times in *Conoids and Spheroids* (4, 21, and 28). Much more regularly, Archimedes uses δεικτέον ὅτι (and its variations) for this purpose, but the distribution is different. δεικτέον ὅτι appears only 13 times in *Sphere and Cylinder* (I.6, 12, 13, 15, 26, 32, 35, 37, 38, 41, 44, II.8, and II.8 alternative), but 19 times in *Conoids and Spheroids* (1-6, 12, 14, 21-27, and 29-32), 15 times in *Spirals* (1, 2, 10-12, 14-16, 18, 19, and 24-28), and 10 times in *Plane Equilibriums* (I.1, 6, 8, 13, II.1, 2, and 7-10), as well as three times in *Quadrature of the Parabola* (5, 21, and 24), once in *Sand Reckoner* (Heiberg vol. 2, 242.4-5), and 10 times in *Floating Bodies* (I.6, 7, 8, 9, II.1, 2, 7, 8, and 10 [Heiberg vol. 2 402.8-9 and 408.10]).

as part of the *symperasma*,[154] or in which ἐκ δὴ τούτου φανερόν introduces a porism,[155] and his only work of any kind in which ὁμοίως δὴ δείξομεν is used to extend or generalize a proposition.[156] Finally, *Sphere and Cylinder* is the only work in which Archimedes explicitly cites Euclid.[157] Its only heurist feature is that it is somewhat problem-based in outlook. Its goal as stated in the prolegomena is to prove a certain set of ratios between the sizes of solid figures. Thus, it is not exhaustively covering an area of inquiry, as do the works of Euclid, Apollonius or Theodosius, but rather it is answering specific questions. Nevertheless, it is difficult to consider *Sphere and Cylinder* as anything other than systematist.

No other work of Archimedes is as unambiguous as *Sphere and Cylinder* in both method and style, but *On Conoids and Spheroids* shares many of the qualities of a systematist treatise. It uses no numerical methods or mechanical curves, it makes universal claims about generalized figures, and it includes a brief exposition of definitions and first principles at the outset. The form of the propositions usually includes a *protasis* and *ekthesis*, as well as some form of *diorismos* before the body of the argument and a *symperasma* afterward.

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[154] Archimedes, *SC* II.2. This phrase also appears three times in *Plane Equilibriums* (II.3, 8, and 9).
[155] Archimedes, *SC* I.28. This phrase also appears at at *Plane Equilibriums* I.5 por. 2. There are several variations on ἐκ δὴ τούτου φανερόν in *SC* and in other works. Τούτων δὴ δεδειγμένων φανερὸν: *SC* I.12 por. 1; Φανερὸν δὲ ἐκ τῶν ἀποδεδειγμένων: *SC* I.12 por. 2; Ἐκ τούτου δὲ φανερὸν: *SC* I.31, 40 por. 2, *Conoids and Spheroids* 6, *Spirals* 21; Προδεδειγμένων δὲ τούτων φανερὸν: *SC* I.34; Καὶ φανερὸν: *SC* I.29, II.2; Ἐκ τούτου οὖν φανερὸν: *Spirals* 10; Ἐκ τούτου φανερόν: *Spirals* 23.
[156] Archimedes, *SC* I.6, 11, and at I.44 with οὖν instead of δὴ. The similar phrase ὁμοίως δὲ δείξομεν is used once at *Method* 14, but there are textual problems. See Heiberg, vol. 2, 496.30.
[157] Archimedes, *SC* I.2, Heiberg 13.6.

However, there are several ways in which *Conoids and Spheroids* departs from systematist methods and forms. First, the organization of the text seems less systematic. The definitions and postulates are not given as separate enumerated lists, as they are in *Sphere and Cylinder* and in most systematist works, but are mixed into the preliminary remarks in continuous prose. Also in the prolegomena, Archimedes gives a program of the work, which includes a set of "things put forward for consideration,"[158] the majority of which he expresses as indirect questions, introduced by διὰ τί.[159] That is, he presents *Conoids and Spheroids* as primarily problem-based.[160] The results of Archimedes' problem-based approach are seen also in the organization of the propositions, which he arranges apparently in the shortest path for dealing with the questions he wants to answer. Thus, for example, the first propositions are not construction problems or definition theorems, as they are in Apollonius and Euclid, but proofs of the summation of series of magnitudes, necessary for later propositions.[161] Likewise, Archimedes does not provide proofs of the propositions on conics he invokes, if they were proved by other authors.[162]

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[158] The phrase he uses is προεβάλλετο δὲ τάδε θεωρῆσαι (Archimedes, *Conoids and Spheroids (CS)* prolegomena, Heiberg 248.11, 250.24, 254.21)

[159] Archimedes, *Conoids and Spheroids (CS)* prolegomena, Heiberg 248.11 and 15, 250.24, 252.3, 254.22, and 256.10.

[160] Compare, for example, the emphasis on exhaustive coverage in the prolegomena to Apollonius' *Conics*.

[161] Archimedes, *CS* 1 and 2 sum series of lines and rectangular areas, respectively.

[162] e.g. Archimedes, *CS* 3, in which he states a preliminary *protasis* in the first paragraph, claims that it was proved in the "elements of conics" (ἐν τοῖς κωνικοῖς στοιχείοις Heiberg 270.24), and then immediately moves on to a new *protasis*, the proof of which depends on the first unproved statement.

Apollonius, by contrast, is exhaustive in his subject matter, and provides full proofs of every proposition even where he could direct the reader to other authors.[163]

Stylistically, *Conoids and Spheroids* differs from *Sphere and Cylinder* in ways that cannot be accounted for merely by the Doric dialect, which *Sphere and Cylinder* lost in transmission.[164] *Conoids and Spheroids* prefers the decidedly heurist phrase δεικτέον ὅτι as opposed to λέγω ὅτι in the *diorismos*,[165] and it uses no other formulaic systematist phrases except for one instance of ἐκ τούτου δὲ φανερόν, introducing the work's only porism.[166]

Finally, *Conoids and Spheroids* uses a certain method of proof which, while not necessarily heurist, is associated with mechanical methods developed by Archimedes himself. This method is a type of analysis by convergence, seen in propositions 19-22 and 25-30 of *Conoids and Spheroids.* A curvilinear solid figure, such as a hyperboloid, is divided into segments by inscribing or circumscribing sections of cylinders. These sections, by being continuously bisected, can become smaller and smaller until they

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[163] For example, Apollonius, *Conics* III.17, proves what Archimedes states in the *protasis* of *CS* 3. As a later author, Apollonius had not only Archimedes' work available to him, but also the works of Euclid and Aristaeus to which Archimedes frequently refers. Apollonius provides similar proofs in *Conics* I.20, I.35, and I.46, which Archimedes skips the proofs of in *Quadrature of the Parabola* 1-3, referring the reader again to the "elements of conics."
[164] The works of Archimedes that have presumably lost their original Doric dialect in the manuscript transmission are: *Sphere and Cylinder, Plane Equilibriums, Measurement of the Circle,* and *Method*.
[165] A comparison of the frequency of δεικτέον ὅτι as opposed to λέγω ὅτι amongst geometers shows λέγω ὅτι overwhelmingly preferred among systematists, appearing over 1,000 times in their works, while δεικτέον ὅτι appears slightly more than fifty times.
[166] Archimedes, *CS* 6. It is notable that *Conoids and Spheroids* only provides one porism, because it supports the sense that the work is directed toward the solution of specific problems, rather than the exhaustive treatment of a subject area.

approximate the volume of the solid by an infinitesimal difference. Two features of this method of proof make it look heurist. First, Archimedes in the *Method* (proposition 10) shows how he used his own mechanical method to discover these very propositions. The division of a figure in this way involves the conception of the figure as "made up" of an infinite number of infinitely thin planes, a physical construction that the systematist school avoids. This method is associated by late testimony with Democritus,[167] whom Archimedes credits in the *Method* with the first articulation of two propositions from *Sphere and Cylinder*,[168] and who is noticeably ignored by systematist geometers.

In addition to the physical/mechanical associations of dividing up the solid figure in this way, Archimedes' form of the convergence argument does not follow systematist conventions. Archimedes uses the cases of inscribed and circumscribed segments in separate propositions, whereas Euclid, when making a convergence argument, takes inscribed and circumscribed figures together.[169] Archimedes employs this same method in the *Measurement of the Circle, Spirals,* and in the later propositions of *Quadrature of the Parabola*. Wilbur Knorr associated the method with Eudoxus,[170] who was a major contributor to the *Elements* and almost certainly worked in the systematist style, from

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[167] Plutarch, *De communibus notitiis adversus Stoicos* 1079e1-9 (*Plutarchi Moralia*, ed. R. Westman and M. Pohlenz, 2nd ed., vol. 6.2 (Leipzig: Teubner, 1959)). For more on the association between Democritus' and Archimedes' ideas of the division of figures, see Heath, *HGM* vol. 1, 180.
[168] Archimedes, *Method* prolegomena, Heiberg vol.2, 430.1-9.
[169] E.g. Euclid, *Elements* XII.2.
[170] For a detailed description of the differences between the Euclidean convergence method and the one Archimedes uses, see Knorr, *The Ancient Tradition of Geometric Problems* 153-154 and W. R. Knorr, "Archimedes and the Elements: Proposal for a Revised Chronological Ordering of the Archimedean Corpus," *Archive for History of Exact Sciences* 19, no. 3 (1978), 219-223.

what we can know of him. It is therefore not that Archimedes' convergence argument is heurist in and of itself, but rather that he had available to him a much more streamlined version of the same argument, from Euclid himself, and instead used the longer and older version.

Conoids and Spheroids, then, does not fall squarely under the criteria of the systematist school of practice, but the features that keep it from fitting easily into the one school do not conclusively place it in the other. However, in comparison with *Sphere and Cylinder*, and considering his other works, *Conoids and Spheroids* does provide evidence that Archimedes did not adhere to systematist conventions even where they were available and familiar to him, and that his neglect of certain stylistic conventions was accompanied by similarly unconventional methods.

I.2.2.2 Archimedes' Numerical Methods: *Measurement of the Circle*

Measurement of the Circle is a short work of only three propositions, which state the following:

1. Every circle is equal to a right triangle of which one of the legs is equal to the radius and the other to the circumference.
2. The area of the circle has a ratio of 11:14 to the square on the diameter.
3. The circumference of the circle has to its diameter a ratio less than 22:7 but greater than 223:71 (i.e. the approximation of π).

There are some problems with the text,[171] but as it has come to us *Measurement of the Circle* is an unusual combination of systematist and heurist methods, which probably cannot be accounted for by accidents of transmission. The first proposition is entirely systematist in form and method. No numbers are used, nor any construction or form of proof not seen in the *Elements.* Not only is the first proposition the only one to include a *diorismos*, but it is even introduced with λέγω ὅτι. The other two propositions, on the other hand, are clearly heurist in form. While they give a *protasis*, *apodeixis*, and *symperasma*, they do not include any kind of *diorismos*, and they combine into a single sentence what would otherwise be an *ekthesis* and a *kataskeuē* separated by a *diorismos*.[172]

Even more important, *MC* 2 and 3 use numbers throughout, including in the *protasis* (which makes the form look even less systematist). In fact, the stated goal of both of these propositions is to express a geometrical relationship as a numerical relationship. This is recognized as theoretically possible in the *Elements*,[173] but it is never used with specific numbers in any otherwise systematist text. Furthermore, the proposition in the *Elements* that allows geometrical magnitudes to have numerical ratios

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[171] Archimedes, *Measurement of the Circle (MC)* Heiberg vol. 1 232-243, see especially notes on 237, 239, and 243. See also T. L. Heath, *The Works of Archimedes* (Cambridge: Cambridge University Press, 1912), 93. The biggest problem with the text is that proposition 2 depends on proposition 3, so they appear to be out of order.

[172] It is true that Proclus allows for the *diorismos* to be occasionally omitted for brevity's sake if the *protasis* is particularly long, but in this case, neither proposition has a particularly long *protasis* – certainly no longer than that of the first proposition, in which the *diorismos* is preserved.

[173] Euclid, *Elements* X.5: commensurable magnitudes have to each other the ratio of a number to a number.

to one another specifies that the magnitudes must be commensurable: that is, they must have a common subdivision (what we would call a common factor). But the circumference and diameter of a circle are not commensurable, which is why we see in *MC* 3 another methodological choice that is never made by systematist geometers: approximation. The goal of *MC* 3 is to show not what the ratio is, but that it lies somewhere within a specified range. This is, in the end, a necessary outcome of doing geometry with numbers. Heron also has to approximate frequently,[174] although he seems to have carefully chosen his numbers to avoid this where possible.

The change in method from the first to the later propositions in *Measurement of the Circle* is accompanied by an equally stark change in style. The correlation between non-systematist methods and a loosening of the systematist style is seen in other texts as well, both from Archimedes and from other authors. It therefore looks as though the difference between the two schools of practice was sufficiently important that it was maintained by Archimedes even within a single work.

I.2.2.2 Archimedes' Mechanical Curves: *On Spirals*

Spirals could be described as heurist material with a somewhat systematist form. The work is about as systematist in style as *Conoids and Spheroids,* but is significantly less systematist in substance. Both of these (along with *Sphere and Cylinder*) are later works by Archimedes, according to Knorr, who attributes their tendency toward

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[174] e.g. Heron, *Metrica* I.8, where he estimates the square root of 720 to be $26^5/_6$.

formalism as an attempt by Archimedes to appease his correspondent, Dositheus, and make him more accepting of Archimedes' unconventional methods.[175]

A mechanical curve is a line whose extension in space cannot be described without recourse to motion. These curves are not found in any otherwise systematist text, and later we will see more evidence that the use of mechanical curves or not was a significant dividing line between the two schools of practice.[176] Archimedes' spirals do not require an instrument for their construction, as some mechanical curves do, but they do depend on the assumption of two simultaneous motions with certain constraints. Given a circle and its radius, a point on the radius must move from the center of the circle to the circumference at a fixed speed. In the same amount of time, the radius must revolve about the center of the circle and return to its initial position just as the moving point reaches the circumference. Under these constraints, the motion of the point will describe a spiral line within the circle. Archimedes seems to take steps to justify the use of a mechanical curve, by beginning the work with two short propositions showing how

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[175] Knorr, "Archimedes and the Elements," 248. The formalism to which Knorr refers is one of the practices I have classified as systematist: it includes the stylistic elements I have discussed, such as the strict proposition format and the formulaic phrases like λέγω ὅτι. Almost nothing is known of the life of Dositheus, so his mathematical affiliations cannot be perfectly determined. On the one hand, he was a student of Conon, who seems to have been a heurist (based on Archimedes' praise [*QP* prolegomena Heiberg vol. 2, 262.1-13] and Apollonius' criticism of him [*Conics* IV prolegomena 14-38], as well as his work in mathematical-observational astronomy). On the other hand, Knorr's argument is convincing, and if as Folkerts suggests ("Dositheus [3]," *Brill's New Pauly*, October 1, 2006) Dositheus is the same as the Pelusian biographer of Aratus (a theoretical astronomer in the tradition of Eudoxus), then through Aratus he also has some Platonist and systematist associations.

[176] According to Simplicius, Iamblichus attributes to Apollonius the use of a mechanical curve to square the circle, but we have no text of this and no additional evidence for it. (Simplicius, *In Aristotelis Categorias Commentarium*, ed. K. Kalbfleisch, Commentaria in Aristotelem Graeca 8 (Berlin: Reimer, 1907), 192.18-25.

uniform motion of points produces lines that are proportional to the time of motion.[177] This supports Knorr's theory that Archimedes tried to work in a more formal style, similar to that of Euclid, in his correspondence with Dositheus.[178] The mechanical construction of the spiral is the clearest heurist feature of *Spirals*.

Aside from the use of a mechanical curve, the methods of *Spirals* are mostly systematist. There are no numbers or other mechanical methods, and the propositions are generalizing. Where Archimedes uses a convergence argument, in propositions 21-23, he simultaneously inscribes and circumscribes the converging figures.[179] On the other hand, Knorr shows (based on the testimony of Pappus) that *Spirals* is a formalized version of earlier research by Archimedes, done similarly to the propositions of the *Method*.[180] This parallels what we saw in *Conoids and Spheroids*, where a previously heurist treatment of a problem was revamped to suit the priorities of Archimedes' correspondent, Dositheus.

In terms of the organization of the text, *Spirals* is similar to *Conoids and Spheroids* in being problem-based – the spirals are useful for analyzing the area of a circle (the subject of the last five propositions, 24-28), and all of the other propositions are ordered to this end. *Spirals* does include a list of definitions, but it comes after a set of

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177 Archimedes, *Spirals* 1 and 2, Heiberg vol. 2, 13-16.
178 Knorr, "Archimedes and the Elements," 248, 255. The formalism Knorr recognizes in Archimedes' later texts is what I have classified as the stylistic preferences and generalizing language of the systematist school. It does not reach to the level of systematic and exhaustive treatment of the material, nor to the level of idealized conceptualization of mathematical objects, and so it does not speak to fundamental methodology (e.g. whether to use mechanical curves).
179 See above, ch. 1 note 170 for sources on Archimedes' convergence method.
180 Knorr, "Archimedes and the Elements," 244.

eleven preliminary propositions, in which Archimedes proves various propositions about motion, proportion, and circles.[181] These preliminary propositions have little or nothing to do with the spirals, but they are crucial for the analysis of the circle, which is the actual goal of the work. Here, the spiral is not treated as an object of inquiry, but as a tool.[182]

The similarities hold with respect to style. Most propositions contain the systematist parts, and *Spirals* includes six porisms, which are a somewhat systematist feature, and three of which are introduced by the formulaic ἐκ τούτου φανερόν.[183] However, *Spirals* uses only δεικτέον ὅτι to introduce a *diorismos*, never λέγω ὅτι.

The formal style of *Sphere and Cylinder*, *Conoids and Spheroids*, *Spirals*, and the final propositions of *Quadrature of the Parabola*, which we will see in the next section, are best explained by the influence of Dositheus, to whom they are all addressed.[184] This formalism is a feature of systematist geometry, which Archimedes is imitating primarily in style, but also to an extent in method (e.g. avoiding numbers). Ultimately, however, the underlying goals of these works, and the analytical methods (mechanics, convergence as

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[181] Although this does not seem to be a common practice in any of the works we see before Archimedes, he says in the prolegomena that he is including these necessary propositions “as in the other works of geometry” (ὡς καὶ τῶν ἄλλων τῶν γεωμετρουμένων) (Archimedes, *Spirals* prolegomena, Heiberg vol. 2 12.4-5). He is either referring to his own works (although none of these begins with a set of preliminary propositions before a list of definitions), or we have here evidence for an older stylistic practice that was not preserved in the systematist tradition.
[182] Compare this, for example, to Theodosius’ treatment of spherical geometry or Apollonius’ treatment of conic sections.
[183] Archimedes, *Spirals* 10, 21, 23. Heron does not use them, nor does any other heurist geometer, and neither do the texts in which Archimedes does not appear to be actively trying to write in the systematist style. Euclid, Apollonius, and Theodosius, on the other hand, all include porisms.
[184] See above, ch. 1 notes 175 and 178 for more on Knorr’s argument.

in the *Method*, etc.) employed in the last three, indicate that Archimedes did not belong to the systematist school.

I.2.2.4 Archimedes' Innovation in Mechanical Methods: *Method, Quadrature of the Parabola*

If a distinguishing mark of systematist geometry is a reluctance to treat geometrical figures as physical objects, then the geometry of Archimedes in these next two works is clearly heurist. Both the *Method* and *Quadrature of the Parabola* involve a way of analyzing the size and dimensions of a figure by theoretically weighing it against another figure or line. This method is unique to Archimedes and is his invention, as he makes clear in the prolegomena to the *Method*. Knorr argues that he developed the method through his work attempting to formalize the principles of mechanics in *Equilibrium of Planes*.[185] Archimedes, moreover, is aware that he departs here from normal geometry. He distinguishes between the mechanical and the geometrical proofs of the same theorems in the prolegomena to both works,[186] and reserves the word ἀπόδειξις for the geometrical proofs. Also in both works, he presents the proofs by the mechanical method first, with the more systematist-style geometrical proofs at the end.[187]

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[185] Knorr, "Archimedes and the Elements," 245.

[186] Archimedes, *Method*, Heiberg vol. 2 428.18–430.26, and *Quadrature of the Parabola* (*QP*) Heiberg vol. 2 264.25–266.4.

[187] The geometrical proofs are the convergence arguments in the style seen in *Conoids and Spheroids*. The last six propositions of *QP* are "geometrical" proofs, as is *Method* 14. *Method* 15 is too fragmentary to be perfectly certain, but the extant fragments do not show any vocabulary of centers of gravity or equilibrium. The remaining propositions of the *Method* are lost.

The *Method* is a fragmentary work considered by Knorr to be one of Archimedes' latest.[188] It is addressed to Eratosthenes, another heurist geometer, with whom Archimedes seems to have corresponded periodically, and with whom he may have had a sense of rivalry.[189] The *Method* occasionally uses systematist phrases (three times λέγω ὅτι and once ὁμοίως δὲ δείξομεν), and it does read somewhat like a systematist treatise.[190] It begins with a set of postulates, and the propositions usually contain the main parts.[191] On the other hand, Archimedes is not strict about the divisions between the *ekthesis*, *diorismos*, and *kataskeuē* (e.g. *Method* 5 combines the first and last and includes no *diorismos*). Furthermore, the propositions begin with transitional statements that make the whole text read like a narrative or an extended letter (similarly to Heron's *Metrica*), which detracts from the sense of the work as a formal treatise.

On the Quadrature of the Parabola is an earlier work than any of the others considered so far, except probably for the *Measurement of the Circle*.[192] It is Archimedes' first work addressed to Dositheus, and it is the least systematist of these as well. Knorr postulates that, in their first correspondence, Archimedes took the risk of presenting unconventional material, unsure of Dositheus' reaction (which, judging by the

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[188] Knorr, "Archimedes and the Elements," 265-269.
[189] The first sentence of the *Method* says that Archimedes sent to Eratosthenes the προτάσεις of certain propositions without the demonstrations, challenging Eratosthenes to prove them (these were all propositions that Archimedes had investigated by means of his mechanical method). Archimedes, *Method* prolegomena, Heiberg vol. 2 426.4-7. Archimedes also sent the arithmetical *Cattle Problem* to Eratosthenes, again in the form of a challenge (Heiberg vol. 2 528.2-4, 532.6-9).
[190] λέγω ὅτι: Archimedes, *Method* 1, 9, 15. ὁμοίως δὲ δείξομεν: Archimedes, *Method* 14.
[191] Archimedes, *Method* postulates 1-11 (Heiberg vol. 2 430.28-434.12).
[192] Knorr, "Archimedes and the Elements," 269.

formality of his later works, was not positive).[193] This uncertainty may also account for the sharp change in the style of the work from the first seventeen propositions, which are very informal, to the final six, which are almost systematist.

The first three propositions, which have the flavor of necessary preliminaries (in lieu of definitions and postulates), are given only as προτάσεις, and Archimedes simply says that they are proved in the "elements of conics."[194] *QP* 4-17 are probably Archimedes' most heurist geometrical work, along with the *Method* and the last two propositions of *Measurement of the Circle*. They do not include generalizing προτάσεις or συμπεράσματα, but proceed immediately to the *ekthesis* of the specific case under consideration.[195] *QP* 18-24, however, observe the formal and generalizing parts. These are the only propositions in the work that include a proper *diorismos* (which does not also serve as the first enunciation of the goal of the proposition), and only in this section of the work does Archimedes use the phrases λέγω ὅτι or δεικτέον ὅτι.[196] Perhaps most illustrative of all, after *QP* 17 Archimedes gives a brief list of definitions of the parts of a parabolic segment (words which he has been using all along), before proceeding to the

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[193] Knorr, "Archimedes and the Elements," 248.

[194] ἐν τοῖς κωνικοῖς στοιχείοις, Archimedes *QP* prolegomena, Heiberg vol. 2 268.3. It is uncertain whether Archimedes was referring to the lost *Conics* of Euclid, or that of Aristaeus, or some other lost work.

[195] An exception is *QP* 17, which does begin with a generalizing statement, but after a transitional phrase (τούτου δεδειγμένου φανερόν, ὅτι… "with this demonstrated, it is clear that…") similar to those used in the *Method*, and again detracting from the formality. Really, it makes *QP* 17 look more like a porism than a proposition in its own right. The last six propositions do not use such transitional phrases, except in the instance of a porism, such as in *QP* 20 (δεδειγμένου δὲ τούτου δῆλον, ὅτι…).

[196] In the heurist-style propositions, when Archimedes expresses what he is trying to prove, he says φαμί with an indirect statement instead of a ὅτι clause (*QP* 6, 8, 9, 10, 12, 14, 15, 16). Similar clauses occur in *Conoids and Spheroids* 25, 27, and 29, *Spirals* 13, and *Equilibrium of Planes* I.9 and II.6.

rest of the propositions. The effect is jarring, as if he concluded the heurist section and then started the systematist section as though from the beginning.

The point of *Quadrature of the Parabola* is the theorem proved in *QP* 17 (in heurist style) and *QP* 24 (in systematist style), that the area of any segment of a parabola is four thirds of the triangle with the same base and equal height. *QP* 18-24 work up to the proof of this through clearly systematist methods, always generalizing, employing no numbers or mechanics. *QP* 4-17 use the Archimedean mechanical method of balancing the parabolic segment against a theoretical weight. The fact that the same theorem is proved twice shows that the differences in method and style between these sections is not a result of certain methods being better suited to certain types of geometrical inquiry, but is a choice to be made by the author according to the priorities and standards he prefers. Archimedes is the only author who treats the same material with the different methods of both schools, and is thus an exception that actually does prove the rule.

In order to illustrate the difference between Archimedes' mechanical method and the methods of systematist geometry, consider the following two propositions about parabolas, Archimedes' *Quadrature of the Parabola* 14, and Apollonius' *Conics* I.42:[197]

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[197] I have chosen to compare *QP* 14 with a proposition from Apollonius, rather than from a later proposition from *QP*, because Apollonius is an unambiguously systematist author who, in all the work we have from him, observes the methods and formalities of that school. Even at his most systematist in style, Archimedes never fully conforms. For example, Archimedes' summation of a geometric series in *QP* 23 is done differently from Euclid's summation of the same kind of series, even though the requirements of the proof are equivalent (See Heath, *Works of Archimedes,* xlvii).

Archimedes, *Quadrature of the Parabola* 14[198]

Ἔστω τμᾶμα τὸ ΒΘΓ περιεχόμενον ὑπὸ εὐθείας καὶ ὀρθογωνίου κώνου τομᾶς. Ἔστω δὴ πρῶτον ἁ ΒΓ ποτ' ὀρθὰς τᾷ διαμέτρῳ, καὶ ἄχθω ἀπὸ μὲν τοῦ Β σαμείου ἁ ΒΔ παρὰ τὰν διάμετρον, ἀπὸ δὲ τοῦ Γ ἁ ΓΔ ἐπιψαύουσα τᾶς τοῦ κώνου τομᾶς κατὰ τὸ Γ· ἐσσεῖται δὴ τὸ ΒΓΔ τρίγωνον ὀρθογώνιον. Διῃρήσθω δὴ ἁ ΒΓ ἐς ἴσα τμάματα ὁποσαοῦν τὰ ΒΕ, ΕΖ, ΖΗ, ΗΙ, ΙΓ, καὶ ἀπὸ τᾶν τομᾶν ἄχθωσαν παρὰ τὰν διάμετρον αἱ ΕΣ, ΖΤ, ΗΥ, ΙΞ, ἀπὸ δὲ τῶν σαμείων, καθ' ἃ τέμνοντι αὗται τὰν τοῦ κώνου τομάν, ἐπεζεύχθωσαν ἐπὶ τὸ Γ καὶ ἐκβεβλήσθωσαν. Φαμὶ δὴ τὸ τρίγωνον τὸ ΒΔΓ τῶν μὲν τραπεζίων τῶν ΚΕ, ΛΖ, ΜΗ, ΝΙ καὶ τοῦ ΞΙΓ τριγώνου ἔλασσον εἶμεν ἢ τριπλάσιον, τῶν δὲ τραπεζίων τῶν ΖΦ, ΗΘ, ΙΠ καὶ τοῦ ΙΟΓ τριγώνου μεῖζον [ἐστιν] ἢ τριπλάσιον.

(179) Διάχθω γὰρ εὐθεῖα ἁ ΑΒΓ, καὶ ἀπολελάφθω ἁ ΑΒ ἴσα τᾷ ΒΓ, καὶ νοείσθω ζύγιον τὸ ΑΓ· μέσον δὲ αὐτοῦ ἐσσεῖται τὸ Β· καὶ κρεμάσθω ἐκ τοῦ Β, κρεμάσθω δὲ καὶ τὸ ΒΔΓ ἐκ τοῦ ζυγοῦ κατὰ τὰ Β, Γ, ἐκ δὲ τοῦ θατέρου

[198] Text from *Archimède*, ed. Charles Mugler, vol. 2 (Paris: Les Belles Lettres, 1971), 178-181.

μέρεος τοῦ ζυγοῦ κρεμάσθω τὰ Ρ, Χ, Ψ, Ω, Δς χωρία κατὰ τὸ Α, καὶ ἰσορροπείτω τὸ μὲν Ρ χωρίον τῷ ΔΕ τραπεζίῳ οὕτως ἔχοντι, τὸ δὲ Χ τῷ ΖΣ τραπεζίῳ, τὸ δὲ Ψ τῷ ΤΗ, τὸ δὲ Ω τῷ ΥΙ, τὸ δὲ Δς τῷ ΞΙΓ τριγώνῳ· ἰσορροπήσει δὴ καὶ τὸ ὅλον τῷ ὅλῳ· ὥστε τριπλάσιον ἂν εἴη τὸ ΒΔΓ τρίγωνον τοῦ ΡΧΨΩΔς χωρίου. Καὶ ἐπεί ἐστιν τμᾶμα τὸ ΒΓΘ, ὃ περιέχεται ὑπό τε εὐθείας καὶ ὀρθογωνίου κώνου τομᾶς, καὶ ἀπὸ μὲν τοῦ Β παρὰ τὰν διάμετρον ἆκται ἁ ΒΔ, ἀπὸ δὲ τοῦ Γ ἁ ΓΔ ἐπιψαύουσα τᾶς τοῦ κώνου τομᾶς κατὰ τὸ Γ, ἆκται δέ τις καὶ ἄλλα παρὰ τὰν διάμετρον ἁ ΣΕ, τὸν αὐτὸν ἔχει λόγον ἁ ΒΓ ποτὶ τὰν ΒΕ, ὃν ἁ ΣΕ ποτὶ τὰν ΕΦ· ὥστε καὶ ἁ ΒΑ ποτὶ τὰν ΒΕ τὸν αὐτὸν ἔχει λόγον, ὃν τὸ ΔΕ τραπέζιον ποτὶ τὸ ΚΕ. Ὁμοίως δὲ δειχθήσεται ἁ ΑΒ ποτὶ τὰν ΒΖ τὸν αὐτὸν ἔχουσα λόγον, ὃν τὸ ΣΖ τραπέζιον ποτὶ τὸ ΛΖ, ποτὶ δὲ τὰν ΒΗ, ὃν τὸ ΤΗ ποτὶ τὸ ΜΗ, ποτὶ δὲ τὰν ΒΙ, ὃν τὸ ΥΙ ποτὶ τὸ ΝΙ. Ἐπεὶ οὖν ἐστι τραπέζιον τὸ ΔΕ τὰς μὲν ποτὶ τοῖς Β, Ε σαμείοις γωνίας ὀρθὰς ἔχον, τὰς δὲ πλευρὰς ἐπὶ τὸ Γ νευούσας, ἰσορροπεῖ δέ τι χωρίον αὐτῷ τὸ Ρ κρεμάμενον ἐκ τοῦ ζυγοῦ κατὰ τὸ Α οὕτως ἔχοντος τοῦ τραπεζίου ὡς νῦν κεῖται, καὶ ἔστιν ὡς ἁ ΒΑ ποτὶ τὰν

ΒΕ, οὕτως τὸ ΔΕ τραπέζιον ποτὶ τὸ ΚΕ, μεῖζον ἄρα (180) ἐστὶν τὸ ΚΕ χωρίον τοῦ Ρ χωρίου : δέδεικται γὰρ τοῦτο. Πάλιν δὲ καὶ τὸ ΖΣ τραπέζιον τὰς μὲν ποτὶ τοῖς Ζ, Ε γωνίας ὀρθὰς ἔχον, τὰν δὲ ΣΤ νεύουσαν ἐπὶ τὸ Γ, ἰσορροπεῖ δὲ αὐτῷ χωρίον τὸ Χ ἐκ τοῦ ζυγοῦ κρεμάμενον κατὰ τὸ Α οὕτως ἔχοντος τοῦ τραπεζίου ὡς νῦν κεῖται, καὶ ἔστιν ὡς μὲν ά ΑΒ ποτὶ τὰν ΒΕ, οὕτως τὸ ΖΣ τραπέζιον ποτὶ τὸ ΖΦ, ὡς δὲ ά ΑΒ ποτὶ τὰν ΒΖ, οὕτως τὸ ΖΣ τραπέζιον ποτὶ τὸ ΛΖ· εἴη οὖν κα τὸ Χ χωρίον τοῦ μὲν ΛΖ τραπεζίου ἔλασσον, τοῦ δὲ ΖΦ μεῖζον· δέδεικται γὰρ καὶ τοῦτο. Διὰ τὰ αὐτὰ δὴ καὶ τὸ Ψ χωρίον τοῦ μὲν ΜΗ τραπεζίου ἔλασσον, τοῦ δὲ ΘΗ μεῖζον, καὶ τὸ Ω χωρίον τοῦ μὲν ΝΟΙΗ τραπεζίου ἔλασσον, τοῦ δὲ ΠΙ μεῖζον, ὁμοίως δὲ καὶ τὸ Δς χωρίον τοῦ μὲν ΞΙΓ τριγώνου ἔλασσον, τοῦ δὲ ΓΙΟ μεῖζον. Ἐπεὶ οὖν τὸ μὲν ΚΕ τραπέζιον μεῖζόν ἐστι τοῦ Ρ χωρίου, τὸ δὲ ΛΖ τοῦ Χ, τὸ δὲ ΜΗ τοῦ Ψ, τὸ δὲ ΝΙ τοῦ Ω, τὸ δὲ ΞΙΓ τρίγωνον τοῦ Δς, φανερὸν ὅτι καὶ πάντα τὰ εἰρημένα χωρία μείζονά ἐστι τοῦ ΡΧΨΩΔς χωρίου. Ἔστιν δὲ τὸ ΡΧΨΩΔς τρίτον μέρος τοῦ ΒΓΔ τριγώνου· δῆλον ἄρα ὅτι τὸ ΒΓΔ τρίγωνον ἔλασσόν ἐστιν ἢ τριπλάσιον τῶν ΚΕ, ΛΖ, ΜΗ, ΝΙ τραπεζίων καὶ τοῦ ΞΙΓ τριγώνου. Πάλιν, ἐπεὶ τὸ μὲν ΖΦ τραπέζιον ἔλασσόν ἐστι τοῦ Χ χωρίου, τὸ δὲ ΘΗ τοῦ Ψ, τὸ δὲ ΙΠ τοῦ Ω, τὸ δὲ ΙΟΓ τρίγωνον τοῦ Δς, φανερὸν ὅτι καὶ πάντα τὰ εἰρημένα ἐλάσσονά ἐστι τοῦ ΔςΩΨΧ χωρίου· φανερὸν οὖν ὅτι καὶ τὸ ΒΔΓ τρίγωνον μεῖζόν ἐστιν ἢ τριπλάσιον

(181) τῶν ΦΖ, ΘΗ, ΙΠ τραπεζίων καὶ τοῦ ΙΓΟ τριγώνου, ἔλασσον δὲ ἢ τριπλάσιον τῶν προγεγραμμένων.

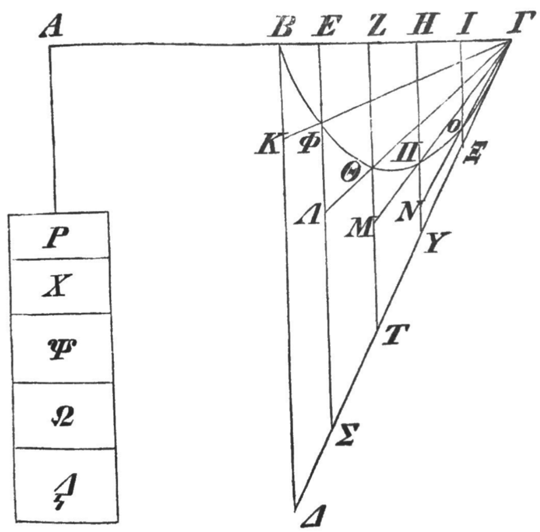

Figure 4: Archimedes, *Quadrature of the Parabola* 14. Diagram from Heiberg, J. L., ed., *Archimedis Opera Omnia, Cum Commentariis Eutocii*, Vol. 2 (Stuttgart: Teubner, 1913), 289.

Let there be a segment ΒΘΓ contained by the section of an upright and right-angled cone. First let ΒΓ be perpendicular to the diameter[199], and let ΒΔ be drawn from point Β parallel to the diameter, and from Γ let ΓΔ be drawn touching the section of the cone at Γ; therefore, ΒΓΔ will be a right triangle. Then let ΒΓ be divided into any number of equal segments, ΒΕ, ΕΖ, ΖΗ, ΗΙ, ΙΓ, and upon the section let ΕΣ, ΖΤ, ΗΥ, ΙΞ be drawn parallel to the diameter, and from the points at which they themselves cut the section of the cone, let them be joined to Γ and produced. I say then that the triangle ΒΔΓ is less than the triple of the trapezia ΚΕ, ΛΖ, ΜΗ, ΝΙ[200], and the triangle ΞΙΓ, and is greater than the triple of the trapezia ΖΦ, ΗΘ, ΙΠ, and the triangle ΙΟΓ.

For let the straight line ΑΒΓ be extended, and let ΑΒ be taken from it equal to ΒΓ, and let ΑΓ be considered a balance; and Β will be its midpoint; and let it be suspended from Β, and also let ΒΔΓ be suspended from the beam at Β and Γ, and from the other side of the beam let the areas Ρ, Χ, Ψ, Ω, and Δς be suspended at Α, and let the area Ρ be in equilibrium with the trapezium ΔΕ thus arranged, and area Χ with the trapezium ΖΣ, and Ψ with ΤΗ, and Ω with ΥΙ, and Δς with the triangle ΞΙΓ; then the whole will also be in equilibrium with the whole; so the triangle ΒΔΓ would be the triple of the area ΡΧΨΒΔς. And since ΒΓΘ is the segment which is contained by a straight line

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[199] Conic sections have multiple diameters. A diameter is any line that, within the section, bisects a chord drawn at a certain angle, and all chords parallel to it. Archimedes generally uses the word “diameter” to refer only to the axis, the line through the vertex of the section that bisects chords at a right angle.
[200] When a four-sided figure is referred to with two letters, the letters signify diagonally opposite corners of the figure.

and a section of a right cone, and BΔ is drawn from B parallel to the diameter, and ΓΔ is drawn from Γ touching the section of the cone at Γ, and a certain other line ΣE is drawn parallel to the diameter, BΓ has the same ratio to BE that ΣE has to EΦ; so also, BA has the same ratio to BE that the trapezium ΔE has to KE. And likewise, AB will be shown to have the same ratio to BZ that the trapezium ΣZ has to ΛZ, and [the same ratio] to BH that [the trapezium] TH has to MH, and to BI that YI has to NI. Therefore, since there is the trapezium ΔE having its right angles at the points B and E, and its sides inclining toward Γ, a certain area P hanging from the beam at A will be in equilibrium with it, the trapezium being arranged as it now lies, and as BA is to BE, so is the trapezium ΔE to KE, then the area KE is greater than the area P: for this has been proven. And again, [there is] the trapezium ZΣ having its right angles at Z and E, and ΣT inclining toward Γ, the area X hanging from the beam at A is in equilibrium with it, the trapezium being arranged as it now lies, and as AB is to BE, so is the trapezium ZΣ to ZΦ, and as AB is to BZ, so is the trapezium ZΣ to ΛZ; therefore, the area X would be less than the trapezium ΛZ, and greater than ZΦ; for this has been proven. Because of the same things, the area Ψ is also less than the trapezium MH, and greater than ΘH, and the area Ω is less than the trapezium NOIH, and greater than ΠI, and likewise also the area Δς is less than the triangle ΞIΓ, and greater than ΓIO. Since, therefore, the trapezium KE is greater than the area P, and ΛZ than X, and MH than Ψ, and NI than Ω, and the triangle ΞIΓ than Δς, it is clear that all the specified areas are greater than the area PXΨΩΔς. And PXΨΩΔς is the third part of the triangle BΓΔ; then it is clear that the triangle BΓΔ is less than the triple of the trapezia KE, ΛZ, MH, NI, and the triangle ΞΙΓ. Again, since the trapezium ZΦ is less than the area X, and ΘH than Ψ, and ΙΠ than Ω, and ΙΟΓ than Δς, it is clear also that all the specified [areas] are less than the area ΔςΩΨX. It is therefore clear also that the triangle ΒΔΓ is greater than the triple of the trapezia ΦΖ, ΘΗ, ΙΠ, and the triangle ΙΓΟ, and less than the triple of those written before.

Apollonius, *Conics* I.42[201]

Ἐὰν παραβολῆς εὐθεῖα ἐπιψαύουσα συμπίπτῃ τῇ διαμέτρῳ, καὶ ἀπὸ τῆς ἁφῆς εὐθεῖα καταχθῇ ἐπὶ τὴν διάμετρον τεταγμένως, ληφθέντος δέ τινος ἐπὶ τῆς τομῆς σημείου καταχθῶσιν ἐπὶ τὴν διάμετρον δύο εὐθεῖαι, καὶ ἡ μὲν αὐτῶν παρὰ τὴν ἐφαπτομένην, ἡ δὲ παρὰ τὴν ἀπὸ τῆς ἁφῆς κατηγμένην, τὸ γινόμενον ὑπ’ αὐτῶν τρίγωνον ἴσον ἐστὶ τῷ περιεχομένῳ παραλληλογράμμῳ ὑπό τε τῆς ἀπὸ τῆς ἁφῆς κατηγμένης καὶ τῆς ἀπολαμβανομένης ὑπὸ τῆς παραλλήλου πρὸς τῇ κορυφῇ τῆς τομῆς. ἔστω παραβολή, ἧς διάμετρος ἡ ΑΒ, καὶ ἤχθω ἐφαπτομένη τῆς τομῆς ἡ ΑΓ, καὶ τεταγμένως κατήχθω ἡ ΓΘ, καὶ ἀπό τινος σημείου τυχόντος κατήχθω ἡ ΔΖ,

[201] Text from J. L. Heiberg, ed., *Apollonii Pergaei quae Graece exstant cum Commentariis Antiquis*, vol. 1 (Leipzig: Teubner, 1891).

καὶ διὰ μὲν τοῦ Δ τῇ ΑΓ παράλληλος ἤχθω ἡ ΔΕ, διὰ δὲ τοῦ Γ τῇ ΒΖ ἡ ΓΗ, διὰ δὲ τοῦ Β τῇ ΘΓ ἡ ΒΗ. λέγω, ὅτι τὸ ΔΕΖ τρίγωνον ἴσον ἐστὶ τῷ ΗΖ παραλληλογράμμῳ.

ἐπεὶ γὰρ τῆς τομῆς ἐφάπτεται ἡ ΑΓ, καὶ τεταγμένως κατῆκται ἡ ΓΘ, ἴση ἐστὶν ἡ ΑΒ τῇ ΒΘ· διπλασία ἄρα ἐστὶν ἡ ΑΘ τῆς ΘΒ. τὸ ΑΘΓ ἄρα τρίγωνον τῷ ΒΓ παραλληλογράμμῳ ἐστὶν ἴσον. καὶ ἐπεί ἐστιν, ὡς τὸ ἀπὸ ΓΘ πρὸς τὸ ἀπὸ ΔΖ, ἡ ΘΒ πρὸς ΒΖ διὰ τὴν τομήν, ἀλλ’ ὡς μὲν τὸ ἀπὸ ΓΘ πρὸς τὸ ἀπὸ ΔΖ, τὸ ΑΓΘ τρίγωνον πρὸς τὸ ΕΔΖ τρίγωνον, ὡς δὲ ἡ ΘΒ πρὸς ΒΖ, τὸ ΗΘ παραλληλόγραμμον πρὸς τὸ ΗΖ παραλληλόγραμμον, ἔστιν ἄρα ὡς τὸ ΑΓΘ τρίγωνον πρὸς τὸ ΕΔΖ τρίγωνον, τὸ ΘΗ παραλληλόγραμμον πρὸς τὸ ΖΗ παραλληλόγραμμον. ἐναλλὰξ ἄρα ἐστίν, ὡς τὸ ΑΘΓ τρίγωνον πρὸς τὸ ΒΓ παραλληλόγραμμον, τὸ ΕΔΖ τρίγωνον πρὸς τὸ ΗΖ παραλληλόγραμμον. ἴσον δὲ τὸ ΑΓΘ τρίγωνον τῷ ΗΘ παραλληλογράμμῳ· ἴσον ἄρα ἐστὶ τὸ ΕΔΖ τρίγωνον τῷ ΗΖ παραλληλογράμμῳ.

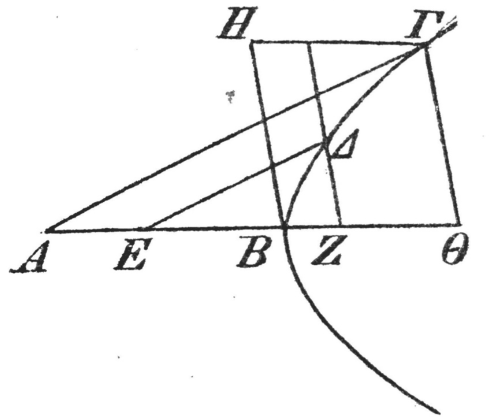

Figure 5: Apollonius, *Conics* I.42. Diagram from Heiberg, J. L., ed., *Apollonii Pergaei quae Graece exstant cum Commentariis Antiquis*, Vol. 1 (Leipzig: Teubner, 1891), 129.

If a straight line tangent to a parabola falls upon the diameter[202], and from the tangent[203] point a straight line is drawn ordinate-wise, and, some point being taken on the section[204], two straight lines are drawn to the diameter, and one of them is parallel to the tangent, the other parallel to the line drawn from the tangent point[205], the triangle

[202] A diameter, again, is any line that divides a parabola in such a way that it bisects all the chords within the parabola that cross the diameter at a certain angle. The lines within the parabola that are bisected by the diameter are said to be drawn "ordinate-wise."

[203] Unlike ἐπιψαύουσα, the word ἁφῆς does not carry the special meaning of "tangent," i.e. touching a curve at only one point. It is also used of the point where a line meets a curve but would cross it if extended, e.g. line 5. I am translating it as "tangent point" where it refers to a tangent point and as "contact point" in any other cases.

[204] "Section" always means conic section (i.e. slice cut out of a cone or conic surface), in this case the parabola.

[205] I.e. the line drawn ordinate-wise to the diameter.

resulting from them[206] is equal to the parallelogram contained by the line drawn from the contact point[207] and the part taken away by the parallel line to the vertex[208] of the section.

Let there be a parabola, whose diameter is AB, and let a tangent to the section, AΓ, be taken, and let ΓΘ be drawn ordinate-wise, and from some random point let ΔZ be drawn [also ordinate-wise], and let ΔE be taken through Δ parallel to AΓ, and let ΓH be taken through Γ parallel to BZ, and let BH be taken through B parallel to ΘΓ. I say that the triangle ΔEZ is equal to the parallelogram HZ.

For since AΓ is tangent to the section, and ΓΘ is drawn ordinate-wise, AB is equal to BΘ, and so AΘ is the double of ΘB. The triangle is then equal to the parallelogram BΓ. And since, because of the section, ΘB is to BZ as the square [formed] from ΓΘ is to the square from ΔZ, but the triangle AΓΘ is to the triangle EΔZ as the square from ΓΘ is to the square from ΔZ, and the parallelogram HΘ is to the parallelogram HZ as ΘB is to BZ, then the parallelogram ΘH is to the parallelogram ZH as the triangle AΓΘ is to the triangle EΔZ. Alternately, then, the triangle EΔZ is to the parallelogram HZ as the triangle AΘΓ is to the parallelogram BΓ.

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[206] I.e. the triangle formed from the diameter, the ordinate-wise line through the random point to the diameter, and the line parallel to the tangent drawn from the random point to the diameter.
[207] I.e. the ordinate-wise line through the random point.
[208] I.e. the part of the diameter whose termini are the point where the ordinate-wise line from the random point meets the diameter and the vertex, where the diameter crosses the parabola.

Since Archimedes' method is an innovation of his own, his work differs significantly from Apollonius'. Archimedes does not invoke the work of any previous geometers in *QP* 14, whereas in *Conics* I.42 above, Apollonius draws on three separate propositions from Euclid.[209] However, this is part of a larger pattern of Archimedes making surprisingly less use of systematist methods that were available to him in his work.[210]

Note also the difference in structure and language. Archimedes provides no *protasis*. The *diorismos*, if it may be called that, serves the purpose of the *protasis* by expressing the goal of the proposition for the first time. It is also phrased completely differently from the systematist λέγω ὅτι (*Conics* I.42, 15), as Archimedes uses φαμί with an accusative-infinitive construction (*QP* 14, 178.11). The lack of a *protasis* gives *QP* 14 a sense of being less generalizing in its methods than *Conics* I.42. That is, Archimedes proves what he states in lines 178.11-15, which is about the specific triangle and section that he constructs and names ΒΔΓ and ΒΘΓ. He never explicitly asserts the truth of the information about this triangle and section for any triangle or section.

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[209] He refers to *Elements* I.41 in lines 12-13, *Elements* VI.20 in lines 14-15, and *Elements* VI.1 in lines 15-16

[210] For example, Heath, *Works of Archimedes,* xl-xli, explains that the application of areas is used much less frequently in Archimedes than in Apollonius, and when it is used, it is not used in the general form developed in the *Elements*. Archimedes also shows an independence from Euclid in his use of the theory of proportion. Heath attributes much of this to the innovation of Archimedes' particular genius, but sometimes it is clear that he is simply using the same body of material in a different mode of practice. For example, Apollonius and Archimedes use different convergence methods, of which Archimedes' is actually less efficient. This difference surely cannot be attributed to a special brilliance of Archimedes over Apollonius, but it can be attributed to a different set of customary practices.

The most significant difference between these two propositions, however, lies in the math itself. Both propositions seek the dimensions of the parabola (or segment of a parabola), specifically as can be known from the relationships among the diameters, ordinates, and tangents. The triangle formed from a diameter, an ordinate, and a tangent is the central reference point of each argument. Both propositions also show how the particular curvature of a parabola affects the ratios between the characteristic lines and the figures formed on them. However, while Apollonius uses purely systematist geometrical methods, Archimedes draws on his theory of weight distribution and centers of gravity, which he developed in *Equilibrium of Planes.* In lines 179.1-10, Archimedes constructs a straight line to be imagined as a balance, and theoretically suspends from it, in postulated equilibrium, both the triangle ΒΔΓ and a series of unspecified weights. In order for this method to work, geometrical lines and figures must be treated as objects with mass, oriented in a particular way with reference to the surface of the earth, and subject to the same rules of gravity as the objects of everyday experience. This is contrary to the systematist tradition of geometry (unbroken, and nearly unquestioned, since Plato) that conceives of geometric objects as eternal, unchanging, non-physical, and only truly accessible in thought.

Archimedes, though clearly of the heurist school, was familiar with Euclid and his associates and their methods, and he shows in multiples places that he is capable of working within their paradigm. However, he never does so with the faithfulness to form and style observed by Apollonius, Theodosius, or others of the systematist school. Where he does use systematist methods this seems to be an artefact of his correspondence with Dositheus. He drew on older methods that are not seen in the systematist school, as well as innovating his own. He strongly influenced later heurist geometers, as we have already seen that he influenced Heron. Because of his early date and his works like *Quadrature of the Parabola*, which show an awareness of different conventions for handling the same material, Archimedes strikingly illustrates the distinction between the two schools in the earlier stages of their development.

I.2.3 Other Heurist Geometers

As with the early systematists, we have only fragments and testimonia of the heurist geometers before the third century BCE, which are nevertheless enough to establish that there was a tradition before the most famous practitioners. Archimedes is the first of his school from whom we have complete extant works, but we have already seen evidence that he was drawing on certain practices from before the time of Euclid.

Democritus, who wrote on philosophy, astronomy, arithmetic, and geometry, lived in the fifth century BCE and was a contemporary of Socrates. While Aristotle mentions him favorably many times, his name never appears in the works of Plato, and according to Aristoxenus—by way of Diogenes Laertius—Plato wished to burn all of Democritus' books.[211] None of his mathematical writing survives, but the testimonia of Archimedes, Heron, and Plutarch give a sense of his geometrical thinking. Plutarch

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[211] Diogenes Laertius, *Vitae Philosophorum* IX.40.4-8.

ascribes to Democritus a physical understanding of geometrical figures as being composed of an infinite number of infinitesimal parts, also generally known as "indivisibles".[212] Heath sees a strong connection between this conception and Archimedes' method of analysis by division of figures, seen in the *Method* and *Conoids and Spheroids*.[213] Archimedes, in the prolegomena to the *Method*, credits Democritus with having asserted the truth (though without proof) of two propositions in *Sphere and Cylinder*, which further substantiates the sense of connection between them.[214] Heron (or Diophantus, or whoever the author is) says in the *Definitions*, that Democritus' first principles of mathematical hypotheses were "indivisibles and voids."[215] These small hints do not add up to a very clear picture of Democritus' methods, but they are consistent among them, and it is noteworthy that the only geometers to mention Democritus at all are heurists.

Another contemporary of Socrates, Hippias of Elis, is credited by Proclus with using a certain curve called the "quadratrix" (τετραγωνίζουσα) to trisect any angle.[216] Proclus says elsewhere that Hippias wrote a treatise on the quadratrix explaining its properties,[217] and Heath considers him to be the inventor of the curve, though Nicomedes

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[212] Plutarch, *De communibus notitiis adversus Stoicos* 1079e1-9.
[213] Heath, *HGM* vol. 1, 180. See also the section on Archimedes' *Conoids and Spheroids* above.
[214] Archimedes, *Method* prolegomena, Heiberg vol. 2, 430.
[215] *Definitiones* 8.9-11 (Heiberg vol. 4, 138). Concerning the authorship of the *Definitiones*, see note 271 below, as well as W. R. Knorr, "Arithmêtikê Stoicheiôsis: On Diophantus and Hero of Alexandria," *Historia Mathematica* 20, no. 2 (May 1, 1993): 180–192.
[216] Proclus, *In prim. Euc.* 272.1-12
[217] Proclus, *In prim. Euc.* 356.6-12

and Dinostratus are mentioned by various authors as using it.[218] Pappus, though he does not credit Hippias,[219] gives the construction of the quadratrix, which is mechanical. Like the spiral of Archimedes, the quadratrix is generated by the motion of a point at the intersection of two lines moving with uniform motion in the same time.[220] Hippias is thus the earliest geometer for whom we have evidence of the use of mechanical curves.[221]

A rough contemporary of Archimedes, Philo of Byzantium wrote almost exclusively on mechanics, although Eutocius attributes to him a solution to the problem of doubling the cube.[222] The solution involves placing a ruler upon a diagram of a rectangle and circle, and turning it about a point until it meets the lines extended from the sides of the rectangle, a clearly heurist technique. Philo's solution is very similar to the one given by Heron in his own *Belopoeica* and the fragmentary *Mechanics*, which Eutocius reproduces just before Philo's.[223]

Another third-century contemporary, Eratosthenes, worked in Alexandria and corresponded with Archimedes. Eratosthenes' interests were wide-ranging, but he seems

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[218] Heath, *HGM* vol. 1, 226.
[219] Pappus, *Collectionis quae supersunt* Book IV (Hultsch 252 ff).
[220] Given the top right quadrant of a circle inscribed in a square, the radius starting from the vertical position must revolve through the right angle to the horizontal position, and in the same time the top side of the square must slide down to coincide with the bottom side. The intersection of the radius and the side as they both move describes the quadratrix.
[221] In the passage cited above, Pappus says that Dinostratus used the quadratrix for the squaring of the circle. Dinostratus was the brother of Menaechmus and a contemporary of Plato, and Proclus says in his summary that the two of them, along with Amyclas, made geometry "more perfect." Nothing else is known of Dinostratus, however, and Pappus' source (probably Sporus – Heath, *HGM* vol. 1, 226) is later and less reliable than Proclus'.
[222] Eutocius, "Commentarii in Libros de Sphaera et Cylindro," 60-64.
[223] Eutocius, "Commentarii in Libros de Sphaera et Cylindro," ("Comm. SC") 58-60. For a full explanation of these solutions, see Heath, *HGM* vol. 1, 262-264.

to have valued his work in geometry. He wrote an epigram praising his solution to the doubling of the cube.[224] His solution, given by both Pappus and Eutocius, uses a device wherein models of plane figures (parallelograms in Eutocius' reproduction, triangles in Pappus') could slide over one another along sets of grooves in parallel rulers.[225] When the figures were arranged in a certain way, the line joining their intersections cut off perpendicular lines that were in continuous proportion, so that the two middle lines were the mean proportionals between the first and last (recall that the problem of doubling the cube was understood to be equivalent to the problem of finding two mean proportionals). In Eutocius' reproduction, the solution is given in the context of a letter supposedly written by Eratosthenes to Ptolemy III Euergetes. The letter might be a forgery,[226] but it includes a genuine epigram by Eratosthenes at the end, in which he invites the reader to ignore the difficult solutions of Archytas, Menaechmus, and Eudoxus. All of these authors are associated by Proclus with the development of the *Elements*, and can be considered systematist geometers. The letter reproduced by Eutocius places all three of them at the Academy with Plato, and criticizes them for finding solutions that were

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[224] Reproduced in Eutocius, "Comm. SC," 96.10-27.

[225] This solution is given in Eutocius, "Comm SC," 88-96, and Pappus *Collectionis quae supersunt* Book III, Hultsch 56-58. For a full explanation of the construction and a diagram of the device, see Heath, *HGM* vol. 1, 258-259.

[226] Wilamowitz argues that the letter is a forgery and the epigram is genuine in "Ein Weihgeschenk des Eratosthenes," *Kleine Schriften*, ed. Paul Maas, vol. 2 (Berlin: Weidmann, 1935), 48-70. Knorr, *Ancient Tradition of Geometric Problems*, 17-20, argues that the letter is genuine.

written as proofs (ἀποδεικτικῶς) and could not be put to any practical use (χειρουργῆσαι δὲ καὶ εἰς χρείαν πεσεῖν μὴ δύνασθαι).[227]

Later in the third century, Nicomedes wrote a treatise on curves called conchoids (according to Proclus and Eutocius) or cochloids (according to Pappus).[228] Nicomedes used the cochloids to solve the doubling of the cube, and he criticized Eratosthenes' solution as "unmanageable and lacking in geometrical skill."[229] The treatise is now lost, but Eutocius and Pappus both give essentially the same method for constructing a cochloid.[230] A ruler is set up so as to be able to slide one end along another ruler, while the other end is constrained in a kind of sliding rotation by a bar perpendicular to the second ruler.[231] Nicomedes may have considered this a less complicated and more geometrical solution than Eratosthenes' because, even though they are both mechanical solutions using devices, Nicomedes constructs a mechanical curve within a geometric diagram in order to cut lines in the desired ratios, whereas in Eratosthenes' solution the mean proportional lines can be measured straight from the device.

Diocles was a younger contemporary of Apollonius, and wrote in the second century BCE a treatise on burning mirrors, of which some fragments are preserved in

[227] Eutocius, "Comm SC," 90. The letter makes a half-exception for Menaechmus, saying that his solution could be put to use, but only with difficulty.
[228] Heath, *HGM* vol. 1, 238 considers cochloids to be the original name.
[229] ὡς ἀμηχάνοις τε ἅμα καὶ γεωμετρικῆς ἕξεως ἐστερημένοις. Eutocius "Comm SC," 98.6-7.
[230] Pappus *Collectionis quae supersunt* Book IV (Hultsch 242-244), Eutocius "Comm SC," 98-100.
[231] For a full explanation of the construction and a diagram of the device, see Heath, *HGM* vol. 1, 238-239.

Eutocius,[232] and others in the Arabic recension of al-Haytham.[233] Diocles used a curve called a cissoid (named for its resemblance to the leaves of a plant of the same name) to solve the doubling of the cube. The cissoid is produced within a quadrant of a circle, by tracing the path of the intersection of two lines as their end points slide along the circumference of the circle cutting off equal arcs.[234] In the solution quoted by Eutocius, a ruler is used to join various points along the path and estimate the shape of the cissoid. The mean proportionals for the solution are cut off by the cissoid on perpendiculars dropped from the diameter of the circle to the circumference.

We cannot say much about the style, proposition format, or textual structure of the works of these authors, because they are all fragmentary and contained in the later commentaries.[235] Their methods, however, unite and distinguish them from the systematist geometers: they use mechanical curves, sliding rulers, approximations, and measurements. These authors also tend to have a connection with the physical sciences and practical mathematics. Diocles also wrote on parabolic mirrors, Philo on mechanics, and Eratosthenes on geography and astronomy. This does not mean, however, that their geometrical works were considered to be "applied" (as opposed to "pure") geometry, or to be less rigorous than their more abstract counterparts. It merely suggests that their

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[232] Eutocius "Comm. SC," 66.8-70.5
[233] Toomer, *Diocles on Burning Mirrors*.
[234] For an explanation and diagram, see Heath, *HGM* vol. 1, 264-265.
[235] For a good assessment of the textual integrity of the fragments preserved in Eutocius, see W. R. Knorr, *Textual Studies in Ancient and Medieval Geometry* (Boston: Birkhäuser, 1989), 77-130.

other scientific studies might have directed them to the discovery of new methods of approaching geometry, while the authors of the systematist school observed more rigid conventions.

I.3 Additional Remarks

Evidence of specific mathematical practices before the time of Euclid is sparse, but there is enough to get a sense of when and why the two schools of practice may have formed. The methodological differences reflect broader attitudes and approaches to geometry, and to the mathematical arts in general. The heurists, with their use of numbers, approximation, mechanical curves, measurement, and instrumental methods, treat geometry as a physical thing that can be worked with and shaped. The systematists, on the other hand, through formalism, generalization, systematization, and abstraction, present their geometry as something purely intellectual, timeless, precise, and perfect, a conception of mathematics that remains dominant to this day.[236] Traces of these different

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[236] Roger Penrose, *The Road to Reality: A Complete Guide to the Laws of the Universe* (New York: Alfred A. Knopf, 2005), 10: "For the first time, with mathematical proof, it was possible to make significant assertions of an unassailable nature, so that they would hold just as true even today as at the time that they were made, no matter how our knowledge of the world has progressed since then. The **truly timeless nature** of mathematics was beginning to be revealed." (emphasis mine). Peter Hilton, "Mathematics in Our Culture," in *Mathematics from the Birth of Numbers*, by Jan Gullberg (New York; London: W. W. Norton & Company, 1997), xxi: "Genuine mathematics, then, its methods and its concepts, by contrast with soulless calculation, constitutes one of the finest expressions of the human spirit. The great areas of mathematics […] have undoubtedly arisen from our experience of the world around us, in order to systematize that experience, to give it order and coherence […]. However, within each of these areas, and between these areas, progress is often made with **no reference to the real world**, but in response to what might be called the mathematician's apprehension of the natural dynamic of mathematics itself." (emphasis mine). Albert Einstein, *Relativity: The Special and General Theory*, trans. R. Lawson, 15th ed. (New York: Random House, 1952), 4: "Geometrical ideas correspond to more or less exact objects in nature, and these last are undoubtedly the exclusive cause of the genesis of those ideas. Geometry ought to refrain from such a course, in order to give to its structure the **largest possible logical unity**." (emphasis mine).

attitudes in authors from whom we have little or no mathematical writing can therefore help to place them in one school or the other, and to sketch something of the history of the schools. Moreover, to the extent that we can establish the relationships between mathematical writers, how and whether they used each other's work, and how their works were transmitted, we can see that the schools were not merely an unconnected set of authors who happened to do math in similar ways. These were social groups, and the members of each school not only adhered to the school's conventions, but also interacted professionally amongst themselves while usually ignoring members of the other school.

I.3.1 Early Days: It's all Plato's fault

Before the third century BCE, geometers seem to group along lines of their relationships to Plato's Academy. As we saw from the summary of Proclus, the majority of contributors to the body of work that became the *Elements* were associated with the Academy, and the earliest authors with clear heurist associations, Democritus and Hippias, were viewed very unfavorably by Plato. Proclus also claims that Euclid himself was a Platonist. Proclus' own Platonism could be a source of bias with regard to Euclid, but his claim finds support elsewhere.[237] Plato's own writings about mathematics, and other testimonia about him, show that he and the members of the systematist school were on the same page.

[237] For a good summary, see Cuomo, *Ancient Mathematics,* 51-56.

Aristotle says in the *Metaphysics* that Plato considered mathematical objects to be intermediate between perceptible objects and Forms, because although they are many where the Forms are one,[238] mathematical objects are ἀΐδια καὶ ἀκίνητα, "eternal and unmoved."[239] In Book 7 of the *Republic*, Plato enumerates the essential mathematical disciplines, and says that for practical purposes very little mathematics is needed, but that the deeper study of mathematics will lift the mind to the form of the good.[240] Plato is also scrupulous about abstraction, and about not treating geometry physically. He says that when mathematicians use diagrams in their arguments, they are not really thinking about the diagrams but about what they represent, "trying to see those things which one could not see except by intellect."[241] These are only a very few examples of many which illustrate Plato's general attitude. More specifically, Heath argues that Plato is the origin of the Eucidean definitions of a line (a breadthless length) and a straight line (a line which lies evenly with the points on itself), as well as contributing to the definition of a point (that which has no part).[242] And he used the systematist definition of a circle in the *Parmenides* and that of the sphere the *Timaeus*.[243]

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[238] That is, there are many mathematical triangles: right, acute, and obtuse triangles of various proportions, none of which is a physical or perceptible object. But there is only one Form of triangle.
[239] Aristotle, *Metaphysics* 987b14-18.
[240] Plato, *Republic* VII 526d-e.
[241] Plato, *Republic* VI 510e-511a: ζητοῦντες δὲ αὐτα ἐκεῖνα ἰδεῖν ἅ οὐκ ἂν ἄλλως ἴδοι τις ἢ τῇ διανοίᾳ.
[242] Heath, *HGM* vol. 1, 293-294, based on language in the *Meno* and *Parmenides*, as well as on testimony from Aristotle.
[243] Plato, *Parmenides* 137e, *Timaeus* 33b.

The strong associations of the systematist tradition with the Academy help explain why mechanical curves and instrumental methods were a major point of division between the two schools. Hippias, the inventor of the quadratrix and the first we know of to use mechanical curves, is very unfavorably represented in the two dialogues in which he appears.[244] Plutarch says in the *Quaestiones Conviviales* that Plato reprimanded those around Eudoxus, Archytas, and Menaechmus for trying to solve the doubling of the cube using mechanical and instrumental methods.[245] Tannery rejected this passage, because Plutarch is a late source and because the preserved solutions of Menaechmus and Archytas do not use any such methods, nor is there other evidence that Eudoxus did so.[246] Heath, on the other hand, considers the passage well-founded, based on Plato's other writings about mathematics.[247]

Plutarch says also that it was because of Plato that mechanics was divorced from geometry and became part of the art of war, overlooked by philosophy; that Eudoxus and Archytas made beginnings in solving geometrical problems through mechanical methods, but that Plato became incensed at this and called them destroyers and corruptors of the good of geometry, because they turned it "away from the incorporeal and mental things,

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[244] Plato's *Hippias Major* and the *Hippias Minor* both portray Hippias as absurdly arrogant.
[245] Plutarch, *Quaestiones Conviviales* 718e7-10: διὸ καὶ Πλάτων αὐτὸς ἐμέμψατο τοὺς περὶ Εὔδοξον καὶ Ἀρχύταν καὶ Μέναιχμον εἰς ὀργανικὰς καὶ μηχανικὰς κατασκευὰς τὸν τοῦ στερεοῦ διπλασιασμὸν ἀπάγειν ἐπιχειροῦντας.
[246] Tannery, *Geometrie Grecque* 79-80.
[247] Heath, *HGM* vol. 1, 287.

toward the perceptible."[248] Whatever Plutarch's reliability with regard to specific events, his picture of Plato's attitude is consonant with Plato's writings, and his timing of the rift between mechanics and geometry is substantiated. Aristotle says in the *Posterior Analytics* that it is not permissible to use mechanical principles in proofs about solid geometry (such as doubling the cube), because mechanics is subordinate to solid geometry and so cannot prove its principles.[249] The use of mechanical methods was thus already a point of contention by the time of Aristotle.

In the case of numerical methods, such as we saw in Archimedes and Heron, we can see a similar story. The older mathematical traditions of Babylon and Egypt dealt with geometry using numbers, measurements, and numerical ratios.[250] The banishment of number from geometry seems to be a Greek invention, and specifically one of the systematist school. But even within the Greek tradition, number and geometry did not become separated until around the time of Plato, or a little bit later. The Pythagoreans, perhaps as far back as Pythagoras himself, worked with figured numbers, which marry geometry and arithmetic in precisely the way we see used many centuries later by

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[248] Plutarch, *Marcellus* 14.11-12: ἐπεὶ δὲ Πλάτων ἠγανάκτησε καὶ διετείνατο πρὸς αὐτούς, ὡς ἀπολλύντας καὶ διαφθείροντας τὸ γεωμετρίας ἀγαθόν, ἀπὸ τῶν ἀσωμάτων καὶ νοητῶν ἀποδιδρασκούσης ἐπὶ τὰ αἰσθητά, καὶ προσχρωμένης αὖθις αὖ σώμασι πολλῆς καὶ φορτικῆς βαναυσουργίας δεομένοις, οὕτω διεκρίθη γεωμετρίας ἐκπεσοῦσα μηχανική, καὶ περιορωμένη πολὺν χρόνον ὑπὸ φιλοσοφίας, μία τῶν στρατιωτίδων τεχνῶν ἐγεγόνει.
[249] Aristotle, *Posterior Analytics* I, 76a23-25 (ch. 9) and 78b36-38 (ch. 13). The argument about proofs, genres, and their principles is made over the course of chapters 7-13 (75a38-79a16).
[250] See Heath, *HGM* vol. 1, 121-128.

Heron.[251] The Pythagoreans also dealt a great deal with numerical ratios through their work on music,[252] and we have from Diogenes Laertius (quoting Anticleides) that Pythagoras particularly concerned himself with the arithmetical aspect of geometry.[253]

So whence the rift? According to a scholium to *Elements* X, "The Pythagoreans approached the investigation of commensurability in the beginning, having first discovered it from their consideration of numbers; for although the unit is the common measure of all (numbers), they were not able to find a common measure when it came to magnitudes. The cause of this is that in every case a number of any kind whatsoever, being divided by any way of cutting it whatsoever, leaves some smallest part which does not admit cutting; but no magnitude divided indefinitely leaves a part which, because it is smallest, does not admit cutting."[254] It is impossible to use numbers to precisely represent ratios between incommensurable magnitudes, such as the diameter of the circle and its circumference, or the diagonal of a square and its side. The number would go on forever. This would not necessarily bother a geometer for whom complete precision and

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[251] Figured numbers are sets of numbers arrived at by arranging units into equilateral shapes such as a triangle or a square. By uniformly increasing the sides of the figure, you get a set of "triangular" or "square" numbers, all of which span the area of the figure. To us, the square numbers are most familiar: 1, 4, 9, 16, 25, etc. For more on the Pythagorean figured numbers, see Heath, *HGM* vol. 1, 76-85.
[252] For more see Heath, *HGM* vol. 1, 85-90.
[253] Diogenes Laertius, *Vitae Philosophorum* VIII.12: μάλιστα δὲ σχολάσαι τὸν Πυθαγόραν περὶ τὸ ἀριθμητικὸν εἶδος αὐτῆς.
[254] Σ Euc. *Elements* [ad X], 1.21-29: ἦλθον δὲ τὴν ἀρχὴν ἐπὶ τὴν τῆς συμμετρίας ζήτησιν οἱ Πυθαγόρειοι πρῶτοι αὐτὴν ἐξευρόντες ἐκ τῆς τῶν ἀριθμῶν κατανοήσεως. κοινοῦ γὰρ ἁπάντων ὄντος μέτρου τῆς μονάδος καὶ ἐπὶ τῶν μεγεθῶν κοινὸν μέτρον εὑρεῖν οὐκ ἠδυνήθησαν. αἴτιον δὲ τὸ πάντα μὲν καὶ ὁποιονοῦν ἀριθμὸν καθ' ὁποιασοῦν τομὰς διαιρούμενον μόριόν τι καταλιμπάνειν ἐλάχιστον καὶ τομῆς ἀνεπίδεκτον, πᾶν δὲ μέγεθος ἐπ' ἄπειρον διαιρούμενον μὴ καταλιμπάνειν μόριον, ὃ διὰ τὸ εἶναι ἐλάχιστον τομὴν οὐκ ἐπιδέξεται.

perfection were not a top priority, but it would bother a Platonist. Most scholars believe that at some point around the time of Plato there was a kind of "crisis" of incommensurability in mathematics that led to the development of a geometry without numbers.[255] The title of Democritus' *On irrational lines and solids*—which is all that survives of it—shows that they were known in the fifth century BCE.[256] We also know that Theaetetus and Eudoxus contributed heavily to the (non-numeric) geometrical theory of incommensurability as seen in Euclid's *Elements* Book V.[257] Morris Kline attributes the crisis and separation of number and geometry to the work of Eudoxus.[258]

If the problem of incommensurability was at the root of the arithmetic/geometry rift, it is unclear why a single person need have been the cause, but the motive force of the process seems once again to be centered at the Academy. We can see the result in Aristotle's *Posterior Analytics*, where he says that arithmetic and geometry are different genres, so that it is not possible to adapt an arithmetical proof to the attributes of magnitudes, unless those magnitudes are numbers.[259] Plato himself always differentiates between calculation (λογισμός[260]) and geometry when he discusses mathematics (e.g.

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[255] Heath, *HGM* vol. 1, 155: there was a fully developed theory of irrationals before Eudoxus; Kline, *Mathematical Thought from Ancient to Modern Times*, 49-50, Knorr, *Evolution of the Euclidean Elements*, 286-288; contra see Fowler, *The Mathematics of Plato's Academy*, 356-367.
[256] Heath, *HGM* vol. 1, 156.
[257] Knorr, "The Geometry of Incommensurability: Theaetetus and Eudoxus" in *Evolution of the Euclidean Elements*, 252-297.
[258] Morris Kline, *Mathematical Thought from Ancient to Modern times*, 49.
[259] Aristotle, *Posterior Analytics* 75b3-6: ὧν δὲ τὸ γένος ἕτερον, ὥσπερ ἀριθμητικῆς καὶ γεωμετρίας, οὐκ ἔστι τὴν ἀριθμητικὴν ἀπόδειξιν ἐφαρμόσαι ἐπὶ τὰ τοῖς μεγέθεσι συμβεβηκότα, εἰ μὴ τὰ μεγέθη ἀριθμοί εἰσι.
[260] Plato's use of the word λογισμός as opposed to ἀριθμητική is notable, because it means calculation and more specifically seems to refer to practical calculation, possibly such as we see in Heron.

throughout Book 7 of the *Republic*), which is consonant with his general attitude about practical crafts – another possible factor contributing to the removal of number from geometry. On the other hand, in one passage of Plato we see geometry that looks like Heron's (*Meno*, 82b-85b). Socrates shows Meno's slave that you cannot double a square by doubling its side, you must double it by constructing a square on its diagonal. In his demonstration, he draws the square with a side of two feet, and invites the slave to calculate (λογισάμενος εἰπέ – 82d4) that the whole square is four feet. This is precisely what Heron does in the first proposition of the *Metrica*, only slightly less difficult, and using specific units of feet instead of the abstract units that Heron employs. Now, Plato may be using this type of practical geometry to show that Socrates is talking down to the slave boy, or he may be using it naturally if the number/geometry split had not fully occurred at this time. Either way, the passage demonstrates that what we see in the *Metrica* is not a late arrival. The systematist school did not receive from their predecessors a geometry devoid of number; they created it themselves.

I.3.2 The Middle Years: Hellenistic and Agonistic

The systematist and heurist schools were essentially formed by the beginning of the third century BCE. The methodological differences between the groups are more pronounced from Euclid and Archimedes onward, where we also have more evidence for them. In addition to the differences of practice already discussed, we can observe the two schools as social groups through their relationships to each other and to each other's work.

Archimedes and Heron both give credit to Democritus, which no systematists do, and Heron is the only geometer to mention Hippias (though only in the dubiously authored *Definitiones*).[261] Archimedes clearly considered Eratosthenes a colleague (perhaps a rival), who was a heurist himself and severely criticized the geometers at the Academy. Another friend and colleague of Archimedes was Conon the astronomer, whom he praises warmly in many of his works. In the prolegomena to *Quadrature of the Parabola* Archimedes tells Dositheus that he was going to send the book to Conon before the latter's untimely death. *Quadrature of the Parabola* is one of Archimedes' most heurist works in both substance and style, and we have already seen that he presented his later missives to Dositheus in a format much closer to the apparently preferred systematist style. We can therefore conjecture that Conon, in his geometrical writings, was not part of the systematist school. If this is so, then it is notable that Apollonius criticizes Conon[262] as not showing proper mastery of proofs regarding the question of at how many points it is possible for conic sections to touch each other. His criticism is lukewarm, and he shows a respect for the usefulness of Conon's work in determining the limits of possibility of such problems, but he recounts a bitter controversy between Conon and another geometer named Nicoteles of Cyrene (of whom no other evidence remains), in which Nicoteles declared that Conon's findings were useless for determining

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[261] *Definitiones* 136.1. Hippias is listed with Thales and Pythagoras as one of the founding fathers of Greek geometry.
[262] Apollonius, *Conics* IV prolegomena 14-31.

limits of possibility. Apollonius, while he finds Nicoteles unfair on this point, does allow that his conflict with Conon was reasonable, on account of the insufficiency of the latter's proofs.

Apollonius' relationship to the work of Archimedes also shows the tendency for one school to ignore the other. Apollonius does not mention any of Archimedes' theorems on conic sections, even though they were relevant to his work. Heath considers this strange, and proposes that perhaps Archimedes' theorems were not considered part of the elementary material of conics.[263] But Apollonius does not restrict himself to elementary theory beyond Book IV, and given all of the other geometers whom he mentions in his prolegomena, it remains strange that Archimedes should not be among them. Eutocius quotes Heracleides, the biographer of Archimedes, as saying that Archimedes invented the theory of conics and Apollonius plagiarized when he found that Archimedes hadn't published it.[264] Eutocius himself does not believe this story (the theory of conic sections is considerably older than Archimedes), and Heiberg thinks that it may have arisen because many of Apollonius' propositions are quoted by Archimedes as already known.[265] Although Heracleides' story is certainly factually inaccurate, it does show that there was a sense even among ancient authors that Apollonius did not treat Archimedes' work with respect.

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[263] Heath, *Apollonius. Treatise on Conic Sections,* lix.
[264] Eutocius, "Commentaria in Conica," 168.
[265] Apollonius, *Conica,* Heiberg vol. 2, 168.

The other heurists also group around Archimedes. Knorr sees so little influence of Apollonius on the later 3rd-2nd century geometers such as Nicomedes and Diocles, and so much influence of Archimedes, that he argues for pushing their dates back closer to Archimedes' time.[266] Heron refers to Archimedes many more times than he mentions anyone else, and he draws on a large body of Archimedes' work, including some treatises that are now lost. But he only mentions Apollonius twice, once in the *Definitiones*, a commentary on the state and history of geometry that has probably been misattributed to him,[267] and once in the *Stereometrica*.[268] Likewise, all of Heron's references to Euclid are from the *Definitiones*, with the exception of two in the *Geometrica* and three in the *Stereometrica*, both texts heavily compromised by transmission.[269]

These associations are not hard lines, but they do indicate that the methodological differences between the two schools seem to have correlated with a social grouping.

I.3.3 The Later Years: It's all Neoplatonism's fault

The role of the late antique commentators will be discussed in greater depth later, but suffice it to say here that, despite the strong presence of the heurist tradition during the Hellenistic period, considerably less of it is preserved than the systematist. Even in the case of Archimedes, Eutocius tells us that he is the first to write such a commentary.

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[266] Knorr, *Ancient Tradition of Geometric Problems,* 210.
[267] Knorr has argued that the *Definitiones* are more likely the work of Diophantus. Knorr, "Arithmêtikê Stoicheiôsis."
[268] *Definitiones* 137.4. *Stereometrica* II.34.
[269] *Definitiones* prolegomena 5, 104.1, 116.1, 125.2, 128.1, 136.1, 136.9, 136.56, 137.4; *Geometrica* 17, 22; *Stereometrica* I.13, II.57 and 60.

He restricts himself to Archimedes' most systematist works, *Sphere and Cylinder*, *Equilibrium of Planes* (which is a work of mechanics done very much in the systematist style), and *Measurement of the Circle*. And even *Measurement of the Circle* is a stretch for him. Cuomo points out that Eutocius feels the need to demonstrate that it is possible to arithmetize geometrical objects, as if attempting to justify Archimedes in this regard. On the other hand, in his commentary on Apollonius, Eutocius focuses his efforts on how Apollonius generalizes and shows multiple cases, and he draws the readers' attention to the sequential arrangement of the first ten propositions of the *Conics*.[270]

If, therefore, modern scholars of Greek mathematics have tended to disproportionately focus on works of the systematist school, they are not the first to do so. Systematist geometry, rooted as it is in the philosophical priorities of the Platonic Academy, naturally found favor with the Neoplatonists of late antiquity. These later authors of commentaries, collections and textbooks laid the groundwork for a perspective which treats systematist geometry as the norm from which others diverged (or even as the standard from which others fell short). Later, I will show that the Neoplatonic leanings of Theon of Alexandria, Geminus, Simplicius, Serenus, Proclus, Eutocius, and other late commentators led not only to a greater preservation of systematist—that is, Platonic—geometry, but a preference for systematist practices amongst mathematicians, which has lasted down to the present day.

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[270] Cuomo, *Greek Mathematics,* 231.

II. ARITHMETIC

In the field of arithmetic, significantly fewer texts and testimonies have survived than in geometry or astronomy. It seems to have been a topic of less interest to both schools, although there are more works of heurist arithmetic than systematist, especially works written for mathematicians rather than for beginning students. There are several theories as to the scarcity of arithmetical works. Arithmetic may have been considered a more basic discipline than the others, better for beginning students than as a field of research in its own right.[1] It may also have had more limited scope than, for example, geometry, whose objects are more varied than numbers are. Finally, some scholars have proposed that notation may have presented difficulties, because Greek numerical notation was not ideally suited for representing patterns, the recognition of which is critical for generating theorems.[2] Greek notation was also not standardized, especially beyond the basic integers. The representation of fractions, for example, varied widely. Whatever the reason may have been, arithmetic seems to have carried less importance than geometry among theoretical mathematicians in general. However, the differences between

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[1] For example, Plato's remarks in the *Laws* about teaching arithmetic to small children as they do in Egypt by means of games, having them divide apples among their friends, etc. (819a8-819d3)

[2] Martin Luther D'Ooge, Frank Egleston Robbins, and Louis Charles Karpinski, *Nicomachus of Gerasa, Introduction to Arithmetic*, University of Michigan Studies. Humanistic Series ; v. 16 (Ann Arbor: University of Michigan Press, 1926), 66-70. Because Greeks represented numbers as the letters of the alphabet, there is no similarity between, for example, 1 and 10 or between 10 and 100. Arithmetical tricks that are common in modern mathematics (such as knowing that every multiple of five ends in 0 or 5) were not available to the Greeks.

systematist and heurist methods are no less clear, nor do they seem to have been less strictly observed by practitioners.

In addition, certain differences and features that we do not see in geometry appear in arithmetical works. Among the systematists, we see a relaxation of formalism in certain contexts, along with a greater interest in philosophical concerns. Among the heurists, we see an exploration of notation and repetition as problem-solving methods. These features appear in arithmetic because they are rooted both in the historical context of the discipline (e.g. the interests of the Pythagorean school in number) and in the epistemological character of the material itself. For example, since numbers are groups of discrete units (as opposed to lines and figures, which are continuously divisible), pattern recognition is more central to the study of arithmetic than to geometry. This principle is reflected in the heurist school through iterative problem solving, and in the systematist school through illustrative examples in addition to (or instead of) proofs.

I have included in this chapter a consideration of some works that fall outside the scope of a normal mathematical treatise: the arithmetical epigrams of the Greek Anthology, and the *Theologoumena Arithmeticae*, a Neopythagorean work on the philosophy of number. These works are associated with the heurist and systematist schools respectively, and they show both the breadth of arithmetic as a topic as well as the ways in which each school's mode of thinking is expressed at or beyond the edges of the genre. They also reflect the interests and perceptions of educated non-mathematicians, and show that the division between the two modes of thought extends outside the modern definition of the strictly mathematical treatise.

II.1 The Systematist School

The field of arithmetic would reasonably be expected to loosen the systematist stricture against working with specific numbers, and in most cases it did. However, the works written for practicing mathematicians follow essentially the same conventions we saw in geometry. Instead of specific numbers, they use abstract numbers represented by single letters in the text. If diagrams are included, the abstract numbers are represented by straight lines labeled with letters. Apollonius' theory of large numbers, preserved by Pappus, uses specific numbers at the end of the work as an illustrative example of what he has just proved generally.[3] In texts for non-mathematicians, generalized theorems are stated, but demonstrative examples with specific numbers replace traditional proofs.

There are two discernable strains within systematist arithmetic. The first, represented by Euclid and Apollonius, concentrates on number theory in and of itself, without any overt philosophical agenda. This is the tradition that Diophantus engages with in *On Polygonal Numbers*. The second strain is the Neopythagorean arithmetic represented by Nicomachus of Gerasa and Theon of Smyrna. Such works have a strong philosophical frame, within which the math is a means to an end, such as the better

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[3] Pappus, *Collectionis quae supersunt (Synagoge)*, II, 18

understanding of Plato. Later works such as the anonymous *Theologoumena Arithmeticae* take this philosophical tendency so far as to nearly abandon mathematics altogether.

Two major methodological features unite the number theorists and Neopythagoreans as systematists. The first is the generalizing and comprehensive nature, and systematic arrangement, of their texts. On the one hand, the Neopythagoreans do not follow the traditional structure of the proposition, partly because the texts we have are for beginning students, and their aim is to quickly provide the reader with necessary terminology and basic theorems. But they do express each theorem with a general *protasis*, and their works attempt to cover the whole of arithmetic and music (Theon includes astronomy, and intended to cover geometry as well), and to systematically classify types of numbers, ratios, and proportions. They demonstrate almost no interest in teaching students ways of solving problems or making calculations (heurist interests).[4] Rather than showing the mathematical reasoning behind the theorems, they provide examples that demonstrate each theorem using the simplest numbers possible. In some cases this leads to inaccuracy,[5] but the Neopythagoreans were nonetheless respected

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[4] One exception is that Eratosthenes' method for finding prime numbers is reproduced by Nicomachus (*Introduction to Arithmetic* I.13).
[5] E.g. Nicomachus' assertion that only odd numbers are relatively prime (*Introduction to Arithmetic* I.13).

mathematicians.[6] Nicomachus was translated into Latin by Apuleius,[7] and Iamblichus wrote a commentary on his *Introduction to Arithmetic*.[8]

The other unifying factor between the two strains is the same idealizing approach to numbers that the systematist geometers take to lines and figures. All of these authors use methods and language that remove conceptual number as far as possible from any numbered thing. The number theorists achieve this by avoiding specific numbers, and abstracting numbers so far as to represent them by lettered straight lines. The Neopythagoreans take a more explicit approach. Both Nicomachus and Theon of Smyrna devote sections of their prolegomena to the difference between number itself and number in any given instantiation,[9] and both assert the ontological priority of the former. This idealizing conceptualization of number is a key feature of all systematist arithmetic, and

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[6] For example, Marinus reports that Proclus considered himself to stand in the line of great mathematicians by virtue of a dream he had, in which it was revealed to him that he possessed the soul of Nicomachus: ὅτι τῆς Ἑρμαϊκῆς εἴη σειρᾶς σαφῶς ἐθεάσατο καὶ ὅτι τὴν Νικομάχου τοῦ Πυθαγορείου ψυχὴν ἔχοι, ὄναρ ποτὲ ἐπίστευσεν. Marinus, *Life of Proclus*, 28.699-701 (Rita Masullo, *Marino Di Neapoli. Vita Di Proclo* (Naples: M. D'Auria, 1985), 57-93).
[7] We have testimonies of Apuleius' translation from Cassiodorus (*Institutiones* II.4.7) and Isidore (*Etymologiae* III.2.1). For more on this, see S. J. Harrison, *Apuleius: A Latin Sophist* (Oxford: Oxford University Press, 2000).
[8] Iamblichus, *In Nicomachi Arithmeticam Introductionem Liber*, ed. H Pistelli and U. Klein (Leipzig: Teubner, 1894).
[9] Nicomachus *Introduction to Arithmetic* I.1, I.2, I.VI. Theon of Smyrna, *Expositio Rerum Mathematicarum ad legendum Platonem utilium (Useful Mathematics)*, ed. E. Hiller (Leipzig: Teubner, 1878), 19-20.

is consistent with the association of both the number theorists and the Neopythagoreans with Platonism.[10] The same idealization of number is often articulated by Plato himself.[11]

In this section, I will show that the two primary authors of each strain, the number theorists (Euclid and Apollonius) and the Neopythagoreans (Nicomachus and Theon), are all working with the methods of the systematist school.

II.1.1 Euclid

Books VII-IX of the *Elements* are devoted to number theory.[12] Euclid defines all relevant terms at the beginning of Book VII, from unit, number, even, and odd, to prime, square, and perfect numbers. These twenty-two definitions serve all three of the books on number. No postulates or common notions are provided. Book VII deals primarily with the parts (we would say "factors") of numbers, and in it Euclid shows methods of finding greatest common divisors and least common multiples. Book VIII covers plane and solid numbers, and proportions, and Book IX covers series of numbers.

As in the rest of the *Elements*, no specific numbers appear anywhere in the three books. All of the propositions are expressed and proved in the abstract. The numbers are

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[10] The associations of Euclid and Apollonius are detailed in chapter 1, those of Nicomachus and Theon will be explained below.

[11] E.g. *Philebus* 56d4-56e8, on the difference between the arithmetic that counts different units for each thing, such as two armies or two oxen, and the arithmetic of the philosophers that has only one unit for all.

[12] The material of Books II and V is mostly transferable to arithmetic. Book V covers proportion theory and Book II exposits a type of geometry in which figures are scaled up and down. The geometry of Book II is the primary basis for the idea that the Greeks practiced "geometric algebra," that is a rendering of algebraic concepts in geometric terms. Although the material of Books II and V overlaps with certain algebraic problems, it is not explicitly about numbers or number theory, as Books VII-IX are.

never represented as symbols, but as straight lines of various lengths. No metrical inferences are made from the diagrams,[13] and the lengths are never given specific values.

The effect of Euclid's treatment is to place number theory within the *Elements* as part of geometry. Not only do they follow the same style, but Books VII-IX also form part of the whole system of the *Elements*. They draw on material from earlier books, especially Book V, and they are used in turn in Book X, which is critical for Books XI-XIII. In other words, arithmetic in the *Elements* is subsumed under geometry, forming less than a quarter of the work as a whole. It is treated as a sub-discipline, whereas earlier Pythagoreans treated arithmetic as the primary mathematical science. This may reflect Plato's greater emphasis on geometry.[14] The later Neopythagoreans, especially Nicomachus, reassert the epistemological priority of arithmetic over geometry.[15]

II.1.2 Apollonius of Perga

No complete arithmetical texts remain from Apollonius, but we do have fragments and attestations about his work in this area from the extant half of Book II of Pappus' *Synagoge*, which preserves paraphrases of ten theorems (15-25) from an unnamed book. Like Archimedes in the *Sand Reckoner*, Apollonius apparently developed a way of expressing very large numbers in terms of powers of myriads called double

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[13] See Manders, "The Euclidean Diagram." 80–133, and above, ch. 1, 20-21.

[14] Although in *Republic* VII (524d-526c) Plato lists arithmetic first among the mathematics that the citizens must study, the majority of the mathematical examples he provides in the dialogues to illustrate philosophical points are taken from geometry.

[15] Nicomachus *Introduction* I.3, I.4. Theon of Smyrna *Useful Mathematics* 3-5.

myriads (10000^2), triple myriads (10000^3), etc., as well as a method of multiplying them. Unlike Archimedes, however, Apollonius appears to have presented his results using systematist methods.

Pappus explains Apollonius's text as well as directly quoting it,[16] so the evidence for Apollonius' method is somewhat obscured, but it seems that Apollonius was working similarly to Euclid: representing unspecified numbers by straight lines labeled with letters. For example, these propositions generally begin with "Let there be two numbers, A and B," or something similar, rather than specifying units.[17] There are some specific numbers in the passages, which mostly seem to be Pappus' own examples. Three times during the course of the exposition, Pappus concludes the unspecified part of the proposition by saying, "But it is clear through the numbers,"[18] implying that the numbers were not given by Apollonius. Seven times,[19] Pappus concludes the numerical part of a proposition by saying that the "γραμμικόν" was shown either by Apollonius or ἐκ τοῦ στοιχείου, "from the element" (meaning either a previous proof or the first principles of the work[20]). The neuter γραμμικόν ("linear" or "geometrical") refers to the theorem. This

[16] So we infer from how frequently Pappus refers to Apollonius in the third person during these passages (nine times), saying "…as Apollonius showed" or something similar. *Synagoge* II.16, 17 (twice), 18, 19, 23, 25 (three times). It is also clear that Pappus is not directly quoting because he sometimes gives commentary such as Ἐπὶ δὲ τοῦ ιθ´ θεωρήματος ("on the 19th theorem"), with which he introduces *Synagoge* II.18. Pappus opens II.17, 18, 23, and 24 with analogous sentences.
[17] e.g. *Synagoge* II.19: Ἀλλὰ δὴ ἔστωσαν δύο ἀριθμοὶ οἱ A B…
[18] Ἔστι δὲ φανερὸν διὰ τῶν ἀριθμῶν. Book II.16, 21, 23; p.6.1, 12.9, 16.10.
[19] *Synagoge* II.16, 18, 19, 21, 22, 23, and 25, in which he uses γραμμικῶς instead of γραμμικόν.
[20] στοιχείον in this case cannot refer to Euclid's *Elements*, because to do so it would have to appear in the plural.

language means that there was a geometrical proof of what Pappus has just shown with numbers. Consider, for example, *Synagoge* II.18:[21]

ιη´ Ἐπὶ δὲ τοῦ ιθ´ θεωρήματος. Ἔστω τις ἀριθμὸς ὁ Α ἐλάσσων μὲν ἑκατοντάδος μετρούμενος δὲ ὑπὸ δεκάδος, καὶ ἄλλοι ὁσοιδηποτοῦν ἀριθμοὶ ἐλάσσονες δεκάδος οἷον οἱ Β Γ Δ Ε, καὶ δέον ἔστω τὸν ἐκ τῶν Α Β Γ Δ Ε στερεὸν εἰπεῖν.

Ἔστω γὰρ καθ’ ὃν μετρεῖται ὁ Α ὑπὸ τῆς δεκάδος ὁ Ζ, τουτέστιν ὁ πυθμὴν τοῦ Α, καὶ εἰλήφθω ὁ ἐκ τῶν Ζ Β Γ Δ Ε στερεὸς καὶ ἔστω ὁ Η· λέγω ὅτι ὁ διὰ τῶν Α Β Γ Δ Ε στερεὸς δεκάκις εἰσὶν οἱ Η.

Καὶ ἔστι φανερὸν διὰ τῶν ἀριθμῶν· τοῦ γὰρ Α ὑποκειμένου, φέρ’ εἰπεῖν, μονάδων κ´ καὶ τοῦ Β μονάδων γ´ καὶ τοῦ Γ μονάδων δ´ καὶ τοῦ Δ μονάδων ε´ καὶ τοῦ Ε μονάδων ϛ´, ὁ ἐξ αὐτῶν στερεὸς γίνεται μονάδες ͵ζϛ´. ἀλλὰ καὶ τοῦ Ζ ὄντος μονάδων β´, ὅς ἐστι πυθμὴν τοῦ Α, ὁ ἐκ τούτου καὶ τῶν Β Γ Δ Ε στερεὸς δεκάκις γενόμενος ἔσται μονάδες ͵ζϛ´, ἴσος τῷ ἐκ τῶν Α Β Γ Δ Ε στερεῷ. τὸ δὲ γραμμικὸν ὑπὸ τοῦ Ἀπολλωνίου δέδεικται.

[21] Text from Pappus, *Collectionis quae supersunt*, ed. Friedrich Hultsch, vol. 1 Berlin: Weidmann, 1876, 8.12-28. (translation mine).

18. On the 19th theorem.[22] Let there be a number A less than a hundred, and measured by ten, and as many other numbers as one pleases less than ten, such as B, Γ, Δ, E, and let it be necessary to determine the product of A, B, Γ, Δ, and E.

For let there be Z, less than ten, by which A is measured, that is the base of A, and let the product of Z, B, Γ, Δ, and E be taken, and let it be H. I say that the product of A, B, Γ, Δ, and E is ten times H.

And it is clear through the numbers. For with the proposed A being, let us say, of 20 units, and B of 3 units, and Γ of 4 units, and Δ of 5 units, and E of 6 units, the product of these is 7200. But also with Z being of 2 units, which is the base of A, the product of this and B, Γ, Δ, and E when taken ten times will be 7200, equal to the product of A, B, Γ, Δ, and E. And the linear theorem is proved by Apollonius.

The first two paragraphs of this passage may be a direct quotation or paraphrase of Apollonius. The language is consistent with systematist work. For example, he uses the formula λέγω ὅτι, and he refers to ten and one hundred as the abstract ἡ ἑκατοντάς and ἡ δεκάς, as later systematists such as Nicomachus and Theon of Smyrna do (Diophantus, a later heurist, does not use these terms), rather than the more concrete ἑκατόν and δέκα.

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[22] Each theorem in Pappus' text is numbered in this way, with the label of the proposition being one less than the proposition it is said to be "on" (e.g. "16. On the 17th proposition." and so on). The reason for this is unclear, especially since the beginning of the book is lost.

The final paragraph is almost certainly Pappus' own work, and not only because he says that Apollonius proved the linear theorem. The language changes abruptly, becoming less formal and less structured (e.g. the genitives absolute, the disappearance of the third person imperatives, ὅς ἐστι instead of τουτέστιν). He also uses the phrase φέρ' εἰπεῖν, which is not attested in authors as early as Apollonius.

The implication of this passage is that Apollonius provided a proof of this theorem using a linear diagram, in the style of Euclid, which Pappus has omitted and replaced with a demonstration "through the numbers."

The section in *Synagoge* II where Apollonius seems to have departed from normal systematist methods is appended to the last proposition.[23] Pappus says that these theorems provide a method by which Apollonius can numerically account for a hexameter verse by multiplying the number represented by each letter in the verse in order. He says that Apollonius gave the hexameter at the beginning (of the book, presumably) as Ἀρτέμιδος κλεῖτε κράτος ἔξοχον ἐννέα κοῦραι. Pappus then walks through the calculation and expresses the final result using Apollonius' means of expressing large numbers (double, triple, quadruple myriads, etc.).

It is clear from Pappus' text that the body of Apollonius' work in this book is done using systematist methods, but this puzzle of calculating the value of a line of poetry seems like the kind of task a heurist might set himself as a platform for showing

[23] Pappus, *Synagoge* II, 18.

off an ingenious method of solution. However, no heurist texts present this kind of problem, though they do devise many and various puzzles.[24] There is, however, a Pythagorean tradition of calculating the value of words and names as a form of prophecy.[25]

Besides the connection to Pythagorean mysticism, Apollonius' word puzzle is methodologically sound in the systematist tradition. As we will see in the following two authors, the use of examples to illustrate general theorems is a part of systematist arithmetic, and not only because pattern recognition, central to number theory, is difficult to achieve without considering specific numbers. One of the priorities of the school is to work with idealized objects, and specific numbers do not violate this priority to the same degree that measured lines would. Assigning a specific number to a length gives it a physical quality, constraint by measurement, which changes both its nature and how math about it is done. An idealized line as the systematist geometers conceived of it is continuously divisible, but a numerically measured line is composed of units and must be

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[24] In the Greek Anthology XIV, poems 20 and 105 employ the fact that letters stand for numbers as part of their riddles, but there is no calculation involved whatsoever.

[25] This practice is called "gematria," a particular Greek version of which is "isopsephy," the calculation of numerically equivalent words, phrases, or lines of poetry. Similar practices are found also in Egypt, and in many Hebrew writings. For more on this, see the section below on the *Theologoumena Arithmeticae,* as well as Franz von Dornseiff, *Das Alphabet in Mystik und Magie*, Stoicheia : Studien zur Geschichte des Antiken Weltbildes und der griechischen Wissenschaft; Hft. VII (Leipzig, Berlin: Teubner, 1925), 91-118, and Rodney Ast and Julia Lougovaya, "The Art of the Isopsephism in the Greco-Roman World," in *Ägyptische Magie*, ed. A. Jördens (Wiesbaden: Harrassowitz, 2015), 82–98. There is further testimony about this practice among the Pythagoreans from Hippolytus, *Refutation of all heresies* IV.3-14. He gives details that follow Apollonius' practice closely, including the use of πυθμήν to denote the factor less than ten of a number that has ten as a factor. M. Marcovich, *Hippolytus. Refutatio Omnium Haeresium*, Patristische Texte und Studien 25 (Berlin: De Gruyter, 1986).

expressed in terms of them. There is not such a change in the conception of number when moving from abstract to specific. An abstract number and the number five are both groups of units, and both are handled in the same way in calculation. The substantive divide for most systematist authors seems to have been between numbers themselves and numbered things, rather than between unspecified and specified numbers. The following systematist authors of the Neopythagorean strain are explicit about this.[26]

II.1.3 Nicomachus of Gerasa

Nicomachus's *Introduction to Arithmetic*, written in the late 1st or early 2nd c. AD,[27] is the first example of the Pythagorean branch of systematist arithmetic. Unlike Euclid's number books, Nicomachus' *Introduction* is dedicated to arithmetic entirely distinct from geometry. There are no lettered diagrams, and specific numbers are used in illustrative examples rather than formal proofs. This would tend to distance the *Introduction* from systematist geometry, but there is good reason to consider the *Introduction* as systematist arithmetic. For one thing, its purpose is to give an exhaustive survey of arithmetic from its first principles to its most complex propositions, and also to situate arithmetic within the philosophical and mathematical disciplines. Nicomachus argues both that arithmetic is crucial to the philosopher, because it is knit into the

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[26] Nicomachus *Introduction to Arithmetic* I.1, I.2, I.VI. Theon of Smyrna *Useful Mathematics* 19-20 (Hiller).
[27] D'Ooge 71-72. The date is fixed by the fact that Apuleius translated his work into Latin (attested by Cassiodorus), and that Nicomachus mentions Thrasyllus, a writer on music under Tiberius.

arrangement of the cosmos,[28] and that arithmetic is prior to all other mathematics, because it can exist without geometry, astronomy, and music, but they cannot exist without arithmetic.[29] Furthermore, Nicomachus' frequent philosophical remarks are strongly Platonist in character. In his conception of the creation of the cosmos through a demiurge, in his division of the world into eternal beings and the changeable objects that partake of them, and even in his adoption of Pythagorean mysticism toward number, Nicomachus draws on Plato's example. He also frequently quotes Platonic dialogues.[30]

In addition to the philosophical slant and the systematic and exhaustive aims of the work as a whole, the *Introduction* follows most of the systematist conventions of method that are already familiar from geometry. After the philosophical prolegomena, the work begins with definitions of both number and numbered things, as well as odd and even (chapters 6 and 7). New terms are defined as they are introduced. All arithmetical claims are first expressed as general enunciations, and are then illustrated using specific numbers.[31] The material is built up sequentially. Chapters 8-10 of Book I deal with even

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[28] Nicomachus, *Introduction to Arithmetic* I.3.6, I.6.1.
[29] Nicomachus, *Introduction to Arithmetic* I.4-5.
[30] E.g. *Introduction to Arithmetic* I.3.5-7, in which Nicomachus produces examples and long quotations from the *Epinomis* and the *Republic* to show that mathematics is necessary for philosophy, or I.2.2 in which he quotes from the *Timaeus* to show the difference between truly real eternal things and the contingent things that partake of them.
[31] E.g. *Introduction to Arithmetic* I.8.4-5, which introduces even-times even numbers. In section four it is defined as a number which can be successively divided in half until unity is reached (i.e. powers of two). In section five, Nicomachus uses the example of 64, divided to 32, 16, 8, 4, 2, and finally 1. For more complicated examples Nicomachus sometimes provides tables, e.g. I.9.9, which shows a method of generating odd-times even numbers by multiplying odd numbers by even-times even numbers.

numbers classed by their factors,[32] chapters 11-13 with odd numbers classed by their factors (including primes and an explanation of Eratosthenes' Sieve for finding them),[33] and chapters 14-16 with even numbers classed by the sum of their factors.[34] The remaining chapters of Book I (17-23) deal with numbers relative to one another, defining and illustrating the various ratios.[35] Book II builds from ratios to combinations of ratios, then to figured numbers including plane and solid numbers, and finally to proportions and series of numbers. This is the same type of cumulative and sequential arrangement of the text that is seen in the major systematist geometers. Nicomachus frequently signals his intention to lay out the material systematically in his various introductory remarks.[36]

The textual features that seem discordant with systematist methods are easily explained by differences of genre between the *Introduction* and, for example, the *Elements*. While the *Elements* will give its reader a thorough education in the basics of geometry and number theory, it is equally useful to practicing mathematicians as a compendium of rigorously demonstrated theorems. The *Introduction*, on the other hand, is designed only as a pedagogical tool for non-practitioners. First, Nicomachus says in the

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[32] These classes are the even-times even (explained above), even-times odd (doubles of odd numbers), and odd-times even (odd numbers multiplied by powers of two).

[33] These classes are prime (only factor is 1), composite (factors other than 1), and relative primes (pairs of numbers with no common factor but 1). It is unclear why Nicomachus classes relative primes as odd, since pairs of one even and one odd number frequently have no common factor but 1 (e.g. 4 and 9).

[34] These classes are superabundant (the sum of a number's factors is greater than the number), deficient (the sum of the factors is less than the number), and perfect (the sum of the factors equals the number).

[35] The broad categories are: multiple (n=ax), submultiple (n=x/a), superparticular (n=x+x/a), superpartient (n=x+bx/a), multiple superparticular (n=ax+x/b), and multiple superpartient (n=ax+bx/c). Subgroups of these ratios are given in each chapter.

[36] e.g. Nicomachus, *Introduction to Arithmetic* II.1.1, II.6.1.

introductory remarks to II.6 that he has “measured out by selection what is appropriate and easy to comprehend for the condition of students just beginning.”[37] Second, its subject matter consists almost entirely of definitions and illustrations. Relatively few theorems are articulated, and very few compared with those in the *Elements*. Nicomachus leaves a great deal out, to the point where the correctness of what he says is questionable. For example, he classifies composite numbers as a species of odd numbers, when according to his own definition all even numbers above two are also composite.[38] This choice is understandable if Nicomachus’ plan is to provide three species each of even and odd (and then even again) so as to help the student mentally organize, but it does not hold up as a general principle. Third, the *Introduction* is expressly concerned with the philosophical significance of arithmetic, more than the practice of it. The first six chapters are devoted to philosophical matters, and the rest of the work includes frequent philosophical interludes. This parallels the work of Theon of Smyrna, who wrote a work of “mathematics useful for the understanding of Plato.”[39] The works of these two authors correspond much more closely in subject matter than either does with Euclid, making it likely that they are meant to make students of philosophy conversant with the basic terms

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[37] Μέχρι μὲν οὖν τοῦδε ἱκανῶς περὶ τοῦ πρὸς ἕτερόν πως ἔχοντος ποσοῦ διειλέγμεθα συμμετρησάμενοι κατ’ ἐκλογὴν τὰ προσήκοντα καὶ εὐπερίληπτα τῇ τῶν ἄρτι εἰσαγομένων ἕξει. II.6.1. Mansfeld takes issue with the translation of D’Ooge, which renders the phrase τῇ τῶν ἄρτι εἰσαγομένων ἕξει as “to the nature of the matters thus introduced” rather than, “for the condition of students just beginning.” (Jaap Mansfeld, *Prolegomena Mathematica: From Apollonius of Perga to Late Neoplatonism* (Leiden; Boston: Brill, 1998), 85 note 300).
[38] Nicomachus, *Introduction to Arithmetic* I.12.1.
[39] Theon of Smyrna, *Expositio Rerum Mathematicarum ad legendum Platonem utilium (Useful Mathematics)*, ed. E. Hiller (Leipzig: Teubner, 1878).

and ideas of number theory, rather than to be resources for mathematicians. Finally, Nicomachus wrote a much more extensive *Art of Arithmetic*, now lost, which must have dealt more deeply with theorems and problems than the *Introduction* does. So, if the *Introduction* is a drastically abridged version of a more complete work, it is probable that Nicomachus has simply omitted the proofs of theorems in this case, rather than that he is part of a different tradition of mathematical practice.

II.1.4 Theon of Smyrna

Roughly contemporary with Nicomachus, Theon wrote a handbook called *Useful Mathematics for the Understanding of Plato*. Like Nicomachus' *Introduction to Arithmetic*, *Useful Mathematics* is a compilation of basics, meant not to generate new knowledge but to familiarize students and non-mathematicians with terms and principles. The primary topics covered in Book I are arithmetic in part one and music in part two. Book II covers astronomy and will be dealt with in the following chapter.

Book I of Theon's *Useful Mathematics* covers roughly the same ground as Nicomachus' *Introduction*, with some notable exceptions. He includes a treatment of side and diagonal numbers (which Nicomachus neglects when dealing with figured numbers),[40] and a section on instrumental music which precedes that on mathematical music.[41] In general, Theon is less detailed than Nicomachus. For example, in the

[40] Theon, *Useful Mathematics* 42.10-45.8.
[41] Theon, *Useful Mathematics* 46.20-58.12.

discussion of even-times even numbers, he lists only 32, 64, and 128 as illustrations, and does not show the method of producing them, as Nicomachus does.[42]

Theon's style of presentation is very similar to Nicomachus's. He opens with a long introduction justifying the study of mathematics for philosophical ends. His arithmetical material begins with a discussion of conceptual number vs. numbered things and unity as the source of all numbers. All theorems are expressed as general principles, and illustrations from specific numbers are presented instead of proofs.

Theon is even clearer than Nicomachus about his intention to produce a text only for non-mathematicians. In the introduction, he explicitly says that he will concisely present what is necessary for a student to read and understand Plato if that student has entirely missed out on mathematical study.[43] Theon is as concerned as Nicomachus, if not more so, with the philosophical significance of arithmetic, and takes up a good deal of space with long quotations from Plato. The *Useful Mathematics*, then, should not be considered under the same methodological standard as a work for practitioners. Like the *Introduction*, it is systematist arithmetic for beginners.

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[42] Theon, *Useful Mathematics* 25.5-18.

[43] Theon, *Useful Mathematics* 1.10-16: ὥστε δὲ τοὺς διημαρτηκότας τοῦ ἐν τοῖς μαθήμασιν ἀσκηθῆναι, ὀρεγομένους δὲ τῆς γνώσεως τῶν συγγραμμάτων αὐτοῦ μὴ παντάπασιν ὧν ποθοῦσι διαμαρτεῖν, κεφαλαιώδη καὶ σύντομον ποιησόμεθα τῶν ἀναγκαίων καὶ ὧν δεῖ μάλιστα τοῖς ἐντευξομένοις Πλάτωνι μαθηματικῶν θεωρημάτων παράδοσιν […] "And so that those who missed being trained in mathematics, but are yearning for an understanding of his [Plato's] writings, may not fail in everything they desire, we shall lay out a summary and shortcut of the necessities and an exposition of the mathematical theorems which are especially necessary for those about to encounter Plato…"

II.1.5 Theologoumena Arithmetica

This anonymous work is a compilation of Neopythagorean number philosophy excerpted largely from Nicomachus, who wrote a larger work of the same name, and a work by Anatolius, *On the Decade*. It must therefore be placed in the late third century CE or later. It was once attributed to Iamblichus, who wrote a commentary on Nicomachus' *Introduction to Arithmetic*, but the attribution is now generally considered false.[44]

The structure of the *Theologoumena Arithmeticae* is straightforward, one book describing the mathematical, philosophical, and mystical properties of each of the numbers one through ten. Within each book, however, there is almost no discernable structure. One scholar has commented that it reads almost like a student's notes.[45] There is very little discussion of the mathematical properties of the numbers,[46] and the majority of the space is dedicated instead to their mystical and philosophical properties. For example, five is said "to be called 'marriage' as it is from masculine and feminine,"[47] meaning that it is the sum of the first even and the first odd, two and three. It is later noted that five is the number of the regular solids, which Plato's *Timaeus* says are the

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[44] Victor De Falco, ed., *Theologoumena Arithmeticae* (Leipzig: Teubner, 1922), viii.
[45] R. A. H. Waterfield, "Emendations of [Iamblichus], *Theologoumena Arithmeticae* (De Falco)," *The Classical Quarterly* 38, no. 1 (1988), 215.
[46] It is always noted whether numbers are even or odd, and there is the occasional mathematical comment, such as in Book V *On the Pentad*, "When squared it always contains itself, for five times five is 25." τετραγωνιζομένη ἀεὶ περιέχει ἑαυτήν, πεντάκις γὰρ πέντε κε΄. De Falco, *Theologoumena* 31.1-2.
[47] διὸ καὶ γάμος καλεῖται ὡς ἐξ ἄρρενος καὶ θήλεος. De Falco, *Theologoumena* 30.19.

four elements and the universe.[48] Names of the gods are assigned to some numbers (six is said to be called Amphitrite because it is twice three[49]), as well as natural forces (two is called movement, genesis, change, divisibility, and many others[50]).

The *Theologoumena Arithmeticae* represents the Neopythagorean number mysticism taken to the point of near independence from actual arithmetic. Nevertheless, it is a part of the systematist tradition. Nicomachus is quoted extensively in the text, and many parallel passages can be found both in Theon of Smyrna and in the commentaries on Plato's *Timaeus* by Proclus and Chalcidius.[51] The philosophical interest in arithmetic is stronger in the Neopythagorean strain of the systematist school, but it is not isolated there. The school as a whole is largely motivated by a Platonist organization of reality and knowledge, the mysticism of which is not entirely absent even from Euclid and Apollonius. Euclid's names for the classifications of numbers (e.g. "perfect" numbers) reflect earlier Pythagorean philosophy as well as Platonism, [52] and Apollonius' calculation of the numerical value of a hexameter is a form of isopsephy, which was strongly associated with Pythagorean number mysticism (see note 25). Iamblichus in his *Life of Pythagoras* reports that Pythagoras himself gave prophecies through numbers in this way.[53]

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[48] De Falco, *Theologoumena* 31.4-7.
[49] De Falco, *Theologoumena* 49.21-22.
[50] De Falco, *Theologoumena* 8.2-3.
[51] For parallel passages see De Falco, *Theologoumena* xvi-xvii.
[52] For more on this, see Heath, *HGM* vol. 1 70-76.
[53] Iamblichus *Life of Pythagoras* 147. For more on isopsephy see Franz von Dornseiff, *Das Alphabet in Mystik und Magie*,

II.2 The Heurist School

While systematist arithmetic attempted a comprehensive explanation of the properties of number in the abstract, heurist arithmetic was concerned with creative and efficient problem-solving methods. All authors in this school use specific numbers in their texts, and none of them shows an interest in the philosophical distinction between numbered things and number considered in the abstract. This does not mean, however, that their arithmetic is less theoretical than that of the systematists, or that it should be taken as “practical” or “applied” mathematics. In this section I will show that heurist arithmetic can be seen as a theory of problem solving, rather than a catalogue of descriptive theorems.

One of the distinguishing features of heurist arithmetic as problem-solving is an interest in notation. Archimedes’ *Sand Reckoner*, Eratosthenes’ *Sieve*, and Diophantus’ *Arithmetica* all devote space to the problem of representing numerical information efficiently. They do not use symbolic notation as we do in modern algebra. Rather, they find various ways of compressing information in a way that makes it easier to organize and perceive more at a glance. Archimedes’ method is purely conceptual, defining classes of numbers in which the unit of each subsequent class is the highest number expressible in the previous class. He names these classes so that a number of any given size can be represented by a few words. Eratosthenes, in the *Sieve*, marks each multiple with superscript letters indicating its factors, so that relative primes as well as pure primes can be found at a glance. Diophantus devises an extensive catalogue of abbreviations that are used almost symbolically to represent various arithmetical operations and variables. The *Arithmetica* approaches the efficiency of symbolic algebra in this way, and Diophantus employs some of the same notation in *On Polygonal Numbers*. These various forms of information compression contrast starkly with the highly formulaic methods of the systematists. Each mode has its advantages. On the one hand, the systematists' formulaic language and standard proposition structure orient the reader very easily in the text. The familiarity of the form allows the substance to take the foreground. On the other hand, the heurists' flexibility and creativity opens the door for methodology itself to lead to new ways of thinking about arithmetic. Archimedes' *Sand Reckoner*, for example, by allowing the unit to be redefined for each class of numbers, makes "unit" a versatile concept rather than the absolute, atomic unity described by Nicomachus. This is a powerful conceptual tool, of which modern physics makes extensive use.[54] We will see similar thinking at work in astronomy, with the sexagesimal system.

The other characteristic feature of heurist arithmetic is an iterative methodology, in which the principle behind the solution is shown to be sound through repetition and extension rather than through the explicit generalization of the systematists. The framing of the problem in Archimedes' *Sand Reckoner* extends his method for expressing large numbers to the extremes of astronomical distances and powers of millions, and he shows

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[54] For example, the speed of light is frequently set to 1 in particle physics to simplify calculations of energy.

that this method can be "carried on this way forever"[55] to numbers greater than the multitude of grains of sand it would take to fill the entire universe. Eratosthenes' *Sieve* functions by repeatedly eliminating subsequent multiples from the range of numbers in which primes are sought.[56] Diophantus' *Arithmetica* solves many similar problems with minor variations, showing the versatility of a basic methodological framework.[57]

As systematist arithmetic contains the sub-genre of Neopythagorean philosophy of number, heurist arithmetic also includes a group of texts at the edges of the mathematical genre, the arithmetical epigrams. Forty-five of these are collected in Book XIV of the Greek Anthology, and we have others from Archimedes and Diophantus.[58] These epigrams are short word problems, calling for the numerical determination of one or more unknowns (problems that we would solve today with basic algebra or systems of equations). They often include literary or historical references, sometimes more quotidian situations such as ordering bricks or diluting wine. I will explain in this section why I consider these to be texts of the heurist school.

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[55] Archimedes, *Sand Reckoner* (Mugler) ii 147.8 ἀεὶ οὕτως προαγόντων ἐς τὰς μυριακισμυριοστᾶς περιόδου μυριακισμυριοστῶν ἀριθμῶν μυρίας μυριάδας.

[56] E.g. write out all the odd numbers from 100 to 200 and cross out every multiple of 3, every multiple of 5, then 7, etc. The primary source for the method is Nicomachus, *Introduction to Arithmetic*, I.13.2-9. See the section below on Eratosthenes, for more on the *Sieve.*

[57] E.g. Book IV of the *Arithmetica* is all about numbers that obey the Pythagorean theorem, and all the problems are minor variations on solving the Pythagorean theorem numerically. Diophantus, *Opera Omnia*, ed. Paul Tannery (Leipzig: Teubner, 1893). See the section below on Diophantus for more details.

[58] While Archimedes is the author of the *Cattle Problem*, the epigram in *Arithmetica* V.30 is of unknown authorship, quoted by Diophantus.

II.2.1 Archimedes

As we saw in the previous chapter, Archimedes used numbers in some of his geometrical work.[59] Two of his other works, the *Sand Reckoner* and the *Cattle Problem*, explicitly address problems of number itself using heurist methods, though in very different literary styles and with different goals. Both of these treatises are problem-based, neither pointing to any exhaustive system. Both also use specific numbers determined by the constraints of the problems, rather than abstracted numbers or numbers discussed in terms of their qualities and characteristics, as we find in systematist texts.

II.2.1.1 The *Sand Reckoner*

In the *Sand Reckoner*, Archimedes sets out to calculate the number of grains of sand it would take to fill the universe to the sphere of the fixed stars. In order to do this, he needs numbers larger than could be expressed in the system of notation available to him,[60] so he develops a special system for expressing very large numbers. He proposes that the numbers expressible by common notation (1 to 100,000,000) be called the "first numbers" (πρῶτοι ἀριθμοί). Then, for higher numbers, he postulates a second order of numbers (δεύτεροι ἀριθμοί), of which one hundred million, the myriad-myriad, will be the new unit. That is, he set one hundred million equal to the "one" of the second numbers, and could thus express up to a myriad-myriad of those new units, i.e. one hundred million squared. Then that number (a myriad-myriad [myriad-myriads]) will be

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[59] *Measurement of the Circle* 2, 3; *Method* 14.

[60] The largest number easily expressible in normal Greek notation was a myriad myriads, or 100,000,000.

the unit for the third order of numbers, and so on, up to one hundred million orders of numbers. He continues: the first hundred million orders can be called the first "period" (περίοδος) of orders, and the highest number of that period (the myriad-myriad of the myriad-myriadth order) can be set as the unit for the second period of orders, and so on up to a hundred million periods of a hundred million orders of a hundred million numbers each. This process can be extended indefinitely, to express a number of any conceivable size. Once this new notation system is explained, Archimedes uses it to express the solution to the original problem, the number of grains of sand necessary to fill the universe.

The *Sand Reckoner* is a quintessentially heurist treatise. It provides a specific solution to a single problem. The solution is implicitly transferable to other problems, but is neither framed in universalizing language nor placed within a larger system of information. Specific numbers, apparently arbitrarily chosen, are used instead of abstract conditions in the framing of the problem.[61] The notation system Archimedes invents is not a number theory, but rather a problem-solving tool, like the mechanical curves, numerical examples, and estimation techniques used by the heurist geometers.

The *Sand Reckoner* also demonstrates why the division of Greek mathematics into "theoretical" and "practical" disciplines is insufficiently descriptive. Archimedes is not a systematist, because his goal is not a generalized theory or comprehensive system,

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[61] Archimedes postulates the size of a grain of sand, and takes the astronomical dimensions from a lost treatise by Aristarchus.

and his conceptualization of the mathematics is in no way idealizing. But the *Sand Reckoner* is certainly a theoretical rather than a practical work. The mathematics it contains may be more anchored to the physical world than Euclid's number theory, but even the astronomical problem is designed to be as unrealistic an idea as possible. The problem acts primarily as a framework within which to showcase the innovative notation system for large numbers, and it calls for a solution at the extreme limits of countable things in order to show that the notation can be extended indefinitely. If the point had been to provide a practical method for counting small objects in large containers, the problem could have been framed for any number of plausible practical situations.[62] Instead, Archimedes has devised an iterative, transferable problem-solving method that shows a theoretical principle of arithmetic (i.e. the flexibility of the definition of "unit") without a formal proof.

II.2.1.2 The *Cattle Problem*

The *Cattle Problem* has an entirely different style from the *Sand Reckoner.* It is an arithmetical epigram written in 22 elegiac couplets, in which the reader is asked to calculate the number of the cattle of the sun. The cattle are divided into four herds, white, black, tawny, and dappled, with cows and bulls of each color. Certain ratios of cows to bulls in various herds are then given, along with the condition that the white and black

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[62] Consider, for example, the *Stereometrica* or *Geometrica*, true works of practical arithmetic based on Heron. They provide methods for calculating the sizes of everyday objects such as ships and amphitheaters, and include specific units of measurement.

bulls together were arrayed in a square, while the tawny and dappled bulls together were arrayed in a triangle. The solution must be a set of eight numbers (the bulls and cows of each color) that meet all of the given constraints. Modern algebra, which was unavailable to Archimedes and Eratosthenes (to whom the letter containing the poem was addressed[63]) would solve the *Cattle Problem* using a system of equations. But the exact solution is so complicated that it was not achieved until the last century, with the help of computers.[64]

There has been some debate about whether Archimedes intended the *Cattle Problem* to be solved.[65] On the one hand, there is a solution, and it is not impossible that Archimedes was able to find and express it, especially given the system he devised in the *Sand Reckoner* for handling large numbers. On the other hand, the solution is so large and obscure that it is equally possible that Archimedes designed it to be unsolvable. His addressee in the letter is Eratosthenes, with whom he seems to have some kind of rivalry,

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[63] This is according to the brief prose heading preceding the epigram. The heading is a later composition, but there is no reason to doubt its accuracy. For a brief discussion of the issue, see Geoffrey C. Benson, "Archimedes the Poet: Generic Innovation and Mathematical Fantasy in the Cattle Problem," *Arethusa* 47, no. 2 (June 3, 2014): 170-171.
[64] The *Cattle Problem* was discovered in 1773 by G.E. Lessing, whose proposed solution was incorrect: G.E. Lessing, *Sämtliche Schriften.*, ed. Karl Lachmann and Franz Muncker, vol. 12 (Stuttgart: G.J. Göschen, 1924). In 1880, the problem was solved imprecisely by A. Amthor, "Das Problema Bovinum des Archimedes," *Zeitschrift f. Math. u. Physik (Hist.-Litt.Abtheilung)* 25 (1880): 153–171. Amthor's solution was precise to four significant digits, but he showed that each of the eight numbers must be approximately 206,545 digits long. The answer was calculated precisely in 1965 by H. C. Williams, R. A. German, and C. R. Zarnke, "Solution of the Cattle Problem of Archimedes," *Mathematics of Computation* 19, no. 92 (1965): 671–674.
[65] Ilan Vardi, "Archimedes' Cattle Problem," *The American Mathematical Monthly*; Washington 105, no. 4 (April 1998), 317-318. See also Reviel Netz, *Ludic Proof: Greek Mathematics and the Alexandrian Aesthetic* (Cambridge; New York: Cambridge University Press, 2009), 167-168, and Benson, "Archimedes the Poet." 185-186.

and the poem itself is couched as a challenge, the outcome of which will gauge the reader's intelligence and education.

The *Definitiones*, a late commentary on Euclid, groups the *Cattle Problem* with the field of logistic.[66] It describes logistic as having as its object countable things instead of number itself. A scholium on Plato's *Charmides*, which is very similar to the passage from the *Definitiones*, claims that the aim of logistic is "that which is common in life and useful for business."[67] These comments are critical of logistic, especially coming from a Platonist standpoint. Since the distinction between number itself and numbered things is very important in systematist arithmetic, it is unsurprising that these texts should criticize the *Cattle Problem* as "practical." However, in each passage it is noted that logistic pronounces on sensible things as though they were absolute objects.[68] It is clear that this is intended as a further criticism, viewed as a failure on the part of logisticians to distinguish between sensible and absolute objects. But the effect of these comments is to show that mathematicians such as Archimedes considered themselves to be doing not practical calculation, but theoretical arithmetic.

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[66] *Definitiones* 135.5 (Schöne et al., *Heronis opera quae supersunt omnia* vol. 4). The *Definitiones* were originally ascribed to Heron, but Knorr has argued that Diophantus is the author (Knorr, "Arithmêtikê Stoicheiôsis.").

[67] Σ Plato *Charm.* [ad 165e] 14-15: τέλος δὲ αὐτῆς τὸ κοινωνικὸν ἐν βίῳ καὶ χρήσιμον ἐν συμβολαίοις. W. C. Greene, ed., *Scholia Platonica* (Haverford, Pennsylvania: American Philological Association, 1938).

[68] *Definitiones* 135.5: ὑποτιθεμένη δὲ τὸ μὲν ἓν ὡς μονάδα, τὸ δὲ ἀριθμητὸν ὡς ἀριθμόν, οἷον τὰ τρία τριάδα εἶναι καὶ τὰ δέκα δεκάδα ("and presenting one as the unit and the numbered thing as number, such as that three is the triad and ten is the decade"), and Σ Plato *Charm.* [ad 165e] 15¨ εἰ καὶ δοκεῖ περὶ τῶν αἰσθητῶν ὡς τελείων ἀποφαίνεσθαι ("and it seems to pronounce on perceptible objects as though they were absolute").

II.2.2 Eratosthenes

Among the fragments of Eratosthenes' mathematical works is a piece of arithmetic which he famously developed, called the *Sieve* (κόσκινον). The *Sieve* is a method for finding prime numbers, and it is given in detail in Nicomachus' *Introduction to Arithmetic*, I.13.2-9, in the course of his discussion of relative primes. All of the odd numbers from three to the highest number desired are written out, and then all composite (non-prime) numbers are "marked by certain signs."[69] First, every multiple of three is marked. Then, returning to the beginning, every multiple of every successive number is marked. Eventually only prime numbers are left unmarked. The marks indicate what number measures the marked number. Marked numbers can therefore be checked for common factors at a glance, and relative primes can be found just as easily.

The *Sieve*, although time-consuming to set up, is an efficient method of solution. It requires no calculation whatsoever, merely repeated counting. The iterative method most likely reduces the margin for error compared to calculation, and is presumably transferable to any expressible range of numbers. That is, if only the prime numbers between 1000 and 1500 are desired, it is not necessary to write out all of the numbers from three. Within the target range one could find the first multiple of three and mark every subsequent multiple, then do the same for five, and so on. Finally, the *Sieve* uses

[69] Nicomachus, *Introduction to Arithmetic* I.13.7: ἐὰν οὖν σημείοις τισὶν ἐπιστίξῃς τοὺς ἀριθμούς, "If you mark the numbers with certain signs…" The signs are not specified, but it is implied that they are small subscript or superscript numbers indicating the factor, such as marking every third number with a small γ.

presentation and notation to compress information and facilitate pattern recognition. The marks will fall on the numbers in predictable patterns, and all of the prime factors of a number will be listed above or below the number. This will allow the user to determine information such as common factors, common multiples, primes, and relative primes simply by reading off the page, without calculation. Despite the loss of the original text, the method of the *Sieve* is a clear example of the heurist approach to problem-solving.

II.2.3 Diophantus of Alexandria

Active in the third century CE,[70] Diophantus is one of the latest mathematicians under consideration. The *Arithmetica*, his major work, originally contained thirteen books, of which I-III and VIII-X have survived in Greek, and IV-VII in Arabic.[71] Like Nicomachus, Diophantus shows the tendency of later authors to draw more freely from different methodological traditions. However, in his essential approach to the arrangement and presentation of mathematics, Diophantus uses heurist methods.

II.2.3.1 The *Arithmetica*

The prolegomena to the *Arithmetica* in Book I are brief, only two paragraphs. The work is addressed to a Dionysius, who is encouraged to proceed through the material despite its apparent difficulty, because his eagerness and Diophantus' demonstration will

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[70] There is some doubt about this, but no serious challenges. See Norbert Schappacher, "Diophantus of Alexandria: A Text and Its History," *UFR de Mathématiques et d'Informatique*, 2001, 2-4.
[71] For a discussion of the placement of the Arabic books in the text as a whole, see Jacques Sesiano, *Books IV to VII of Diophantus' Arithmetica in the Arabic Translation Attributed to Qusṭā Ibn Lūqā* (New York; Heidelberg: Springer-Verlag, 1982), 3-8.

make it easy to learn. No philosophical justification is given for the study, as in Nicomachus or Theon of Smyrna. There follows a short list of definitions, of square and cube numbers as well as higher powers, and then a list of the abbreviations that Diophantus will use for various terms. For example, instead of writing "square," he will write Δ^{Y}, for δύναμις.[72] He then names various fractional parts in general terms. For example, any fraction that is one over a number, he will call τὸ ἀριθμοστόν, a fraction that is one over a square, he will call τὸ δυναμοστόν, etc.[73]

There are no proofs in the *Arithmetica.* It is a collection of problems with specific numerical solutions, which do not explicitly demonstrate any general mathematical principles.[74] Diophantus' method is very similar to Heron's in the *Metrica*.[75] A task is set (using an infinitive such as εὑρεῖν or διελεῖν), specific numbers are assigned that fit the initial conditions, and then the problem is solved through a series of calculations. The phrasing of these calculations is extremely flexible. Diophantus sometimes uses third-person imperatives, sometimes first person singular or plural indicatives, sometimes second person singular. The tense shifts between present and future. Often, all of these options are seen within a single calculation.[76] His presentation is even looser than that of

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[72] Diophantus, *Arithmetica* I, 4.14-16 (Tannery).
[73] Diophantus, *Arithmetica* I, 6.14-15 (Tannery).
[74] In the three number books of Euclid's *Elements*, by contrast, only eight of the one hundred and two propositions are expressed as problems, all of them demonstrate general principles, and of course there are no specific numbers used at all.
[75] Their vocabulary is also very similar. For example, Heron is the only other author who uses the word δυναμοδύναμις to refer to the fourth power of a number (*Metrica* I.17.17, I.17.25).
[76] e.g. *Arithmetica* IV.29

Heron, who keeps to third-person imperatives in the geometrical part of his problems, and switches to second person singular imperatives in the numerical calculation. Once a solution is reached, Diophantus will conclude with a short phrase such as καὶ ποιοῦσι τὰ τῆς προτάσεως ("and they [the numbers] meet the conditions"). There is a great deal of variation in this concluding statement.[77] There is much more variation of language in the *Arithmetica* than in any systematist text and many heurist ones.[78]

In one respect, besides the opening definitions, Diophantus writes almost as a systematist. When he first poses each problem, he does so without specifics. That is, he begins with a general statement of the problem, such as "to divide a proposed number into two numbers with a given difference".[79] A number is then immediately specified, and all of the calculations are carried out on the specific numbers. Although these generalized enunciations somewhat resemble the *protaseis* of systematist proofs, they do not serve the same function. In systematist proofs (including the arithmetical ones we have from Euclid), the *protasis* makes a universal claim that is then demonstrated through a diagram that meets all of the conditions without being specified in any way that would compromise the generality of the proof. The *protasis* is supported by the generality of the

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[77] Other frequent examples include καὶ ποιοῦσι τὸ πρόβλημα ("and they solve the problem"), καὶ μένει τὰ τῆς προτάσεως ("and the conditions hold"), καὶ ποιοῦσι τὸ ἐπίταγμα ("and they satisfy the requirements"), and even occasionally καὶ ἡ ἀπόδειξις φανερά ("and the proof is clear" e.g. *Arithmetica* III.18), although ἀπόδειξις here does not have the systematist meaning, but refers to the reinsertion of the solution into the original conditions to verify that they are satisfied.

[78] In the calculations, Diophantus is almost conversational at times, using phrases such as ἔρχομαι ἐπὶ τὸ ἐξ ἀρχῆς ζητούμενον ("I come to the thing sought from the beginning") in *Arithmetica* III.11, or ὅταν δέ τι τοιοῦτο ζητῇς ("When you are looking for something like this") in *Arithmetica* IV.33.

[79] *Arithmetica* I.1: τὸν ἐπιταχθέντα ἀριθμὸν διελεῖν εἰς δύο ἀριθμοὺς ἐν ὑπεροχῇ τῇ δοθείσῃ.

diagram, which can be drawn as any possible instantiation of the initial conditions without changing the text.[80] In the *Arithmetica*, on the other hand, the enunciation of the problem gets no generalizing support from the calculation. Each problem lists specific operations done on specific numbers. The order or nature of these operations could easily change if the numbers themselves were different, and the text of the problem would certainly change.[81] In other words, the problems are not written so as to demonstrate that the numbers chosen for the calculation could be any numbers whatsoever.

In addition to the method of solution and the method of presentation, the arrangement of the material in the *Arithmetica* is also heurist. The problems are not arranged sequentially or cumulatively, though they generally increase in difficulty through the books.[82] There is also no sense of an exhaustive body of information to be covered on various topics. The problems within each book are loosely related by theme (e.g. Book VI addresses numbers that obey the Pythagorean theorem and Book V square numbers). But the problems neither prove propositions nor demonstrate unique methods, so that they do not "cover ground" in the way that systematist proofs do. Scholars have

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[80] See especially Netz, *The Shaping of Deduction in Greek Mathematics*, 19-57 and especially 51-56.

[81] For example, in the very first proposition (*Arithmetica* I.1), to divide a given number into two numbers with a given difference, Diophantus chooses 100 for the given number and 40 for the given difference. If he had chosen an odd number instead of 100, he would have gotten a fractional solution. This could involve an extra line of text for converting fractions, or it could be left as is, since the problem is extremely simple. But unlike the text of a proposition in the *Elements*, the text of this problem would change. An even better example is *Arithmetica* III.10, where Diophantus explicitly says that if 13 were a square number the problem would be easy from there.

[82] Alain Bernard and Jean Christianidis, "A New Analytical Framework for the Understanding of Diophantus's *Arithmetica* I–III," *Archive for History of Exact Sciences* 66, no. 1 (January 1, 2012): 1–69, e.g. p. 58 about *Arithmetica* II.17 and II.18 fitting badly with the series of problems in which they are included.

noticed that it is nearly impossible to group the problems in each book by method of solution, or to articulate general or easily transferable methods of solution.[83] Some scholars have proposed that the progressive difficulty of the problems, and the repetitive and heuristic presentation of their solutions, amount to a didactic style in which the reader is trained to find solutions with flexibility and efficiency.[84] This is a different model from the systematist presentation, which broadly speaking is aimed at acquiring information rather than skills.

Diophantus makes no attempt to hide this, and in fact says it explicitly in the preface to Book I:[85]

Νῦν δ’ ἐπὶ τὰς προτάσεις χωρήσωμεν ὁδόν, πλείστην ἔχοντες τὴν ἐπ’ αὐτοῖς τοῖς εἴδεσι συνηθροισμένην ὕλην. πλείστων δ’ ὄντων τῷ ἀριθμῷ καὶ μεγίστων τῷ ὄγκῳ, καὶ διὰ τοῦτο βραδέως βεβαιουμένων ὑπὸ τῶν παραλαμβανόντων αὐτὰ καὶ ὄντων ἐν αὐτοῖς δυσμνημονευτῶν, ἐδοκίμασα τὰ ἐν αὐτοῖς ἐπιδεχόμενα διαιρεῖν, καὶ μάλιστα τὰ ἐν ἀρχῇ ἔχοντα στοιχειώδως ἀπὸ ἁπλουστέρων ἐπὶ σκολιώτερα διελεῖν ὡς προσῆκεν. οὕτως γὰρ

[83] e.g. Hermann Hankel, *Zur Geschichte der Mathematik in Alterthum und Mittelalter* (Leipzig: Teubner, 1874), 164-165. Heath gives an extensive survey of attempts to classify the problems of the *Arithmetica*, as well as his own classification (T. L. Heath, *Diophantus of Alexandria, a Study in the History of Greek Algebra* (Cambridge: University Press, 1910), 54-98).

[84] Bernard and Christianidis, "A New Analytical Framework for the Understanding of Diophantus's Arithmetica I–III," 63-64. Cyrus Hettle, "The Symbolic and Mathematical Influence of Diophantus's Arithmetica," *Journal of Humanistic Mathematics* 5, no. 1 (January 30, 2015), 146-147. Jean Christianidis, "The Way of Diophantus: Some Clarifications on Diophantus' Method of Solution," *Historia Mathematica* 34, no. 3 (August 2007): 289–305.

[85] Diophantus *Arithmetica* I, 15.25-16.7 (Tannery).

εὐόδευτα γενήσεται τοῖς ἀρχομένοις, καὶ ἡ ἀγωγὴ αὐτῶν μνημονευθήσεται, τῆς πραγματείας αὐτῶν ἐν τρισκαίδεκα βιβλίοις γεγενημένης.

"Now let us proceed down the path of the propositions, containing a very great amount of material gathered together into these types.[86] Since they are very great in number and very voluminous, and since, because of this, they are confirmed slowly by those receiving them, and are difficult for them to remember, I tried to divide up what is undertaken by them [i.e. the students], and especially to divide what is elementary in the beginning from simpler to more complicated, as is appropriate. For thus it will become easier for beginners, and their method will be remembered, the treatment being divided into thirteen books."

Diophantus states clearly that his purpose is the exposition of a methodology, and that many problems will be repetitions and variations on a few core types of problems, moving from easier to harder not under a logical sequence, but in order to more effectively teach the method.

II.2.3.2 *On Polygonal Numbers*

On Polygonal Numbers is much shorter than the *Arithmetica*. It is fragmentary, and systematist in style. It contains four numbered propositions and a body of additional material under the heading τὸ ὑπερτεθὲν δεῖξαι ("to show what is left over").[87]

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[86] In the preceding three paragraphs Diophantus has given several very broad categories of problems into which most of the problems will fall.
[87] Diophantus, *On Polygonal Numbers (OPN)*, 466.20 (Tannery)

Polygonal numbers are series of numbers, each of which, when represented as a regular array of points, can form a regular polygon. The most familiar polygonal numbers are the square numbers: 1, 4, 9, 16, 25, 36, etc. Each of these numbers can be represented as a square grid of dots. There are also series of triangular, pentagonal, hexagonal numbers, and so on, as well as solid numbers such as cubes.

There is a long tradition of the study of polygonal numbers in the Pythagorean strain of the systematist school. Speusippus, Plato's successor as the head of the Academy, is said to have written a book *On Pythagorean Numbers* that dealt with both plane and solid polygonal numbers, as well as with the Platonic solids.[88] Nicomachus and Theon of Smyrna both devote sections of their treatises to polygonal numbers.[89]

On Polygonal Numbers is strikingly similar to the texts in which Archimedes emulated the systematist style. For example, although the language is more formal (the systematist concluding statement, ὅπερ ἔδει δεῖξαι, occurs four times[90]), Diophantus, like Archimedes, prefers δεικτέον ὅτι to λέγω ὅτι.[91] Likewise, Diophantus observes the main structure of a systematist proof, but is not scrupulous about including every part, or about clearly demarcating the proofs as individual units. Each proposition begins with a generalized *protasis*, but props 3, 4, and the "remainder" are long, not clearly structured,

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[88] This testimony comes from the *Theologoumena Arithmeticae*, 82, 10–85, 23 (De Falco)
[89] Nicomachus, *Introduction to Arithmetic* II.7-11. Theon *Useful Mathematics* 27-41.
[90] *OPN* 1, 3 (twice), 5 (474.9).
[91] λέγω ὅτι appears twice: *OPN* 4, remainder (466.23), while δεικτέον ὅτι appears three times: *OPN* 1, 2, 3. See above, ch. 1 note 165.

and include proofs of multiple theorems.[92] This is similar to Archimedes' run-on proofs in *Method* and *Conoids and Spheroids*.[93]

On Polygonal Numbers is problem-based in the way that Archimedes' *Spirals* or *Quadrature of the Parabola* are. Diophantus is setting out to prove a certain method of conversion between higher polygonal numbers and squares.[94] He is not providing an exhaustive set of theorems about polygonal numbers by any means. The fragment that remains is the beginning of the work, and Diophantus immediately advances toward the particular theorem he wants to prove.

In addition, after the remainder is appended a section that lacks proofs and shows how to perform the calculations for converting a polygonal number into a square.[95] Its heading reads Διδασκαλικώτερον δὲ ὑποδείξομεν καὶ τοῖς βουλομένοις εὐχερῶς ἀκούειν τὰ ζητούμενα διὰ μεθόδων, "And we shall also show a more instructive way for those who wish to easily learn what is sought through methods."[96] This final section is highly reminiscent of Heron's *Metrica*, in which he includes the "method" of calculation after each geometrical theorem. Diophantus' language also changes in this section, becoming

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[92] *OPN* 3 proves two cases of the same theorem. The "remainder" proves four theorems before concluding, for example.
[93] e.g. *Conoids and Spheroids* 15, which proves two different theorems, *Conoids and Spheroids* 16 which proves three, *Method* 2 and 3, each of which prove two.
[94] He explains in the second paragraph that his treatise will show that any polygonal number, multiplied by a certain number according to the number of angles of the polygon, and added to a similarly determined square, will produce a square number. Diophantus, *On Polygonal Numbers*, 450.11-16 (Tannery).
[95] Diophantus, *On Polygonal Numbers*, 474.10-480.22 (Tannery)
[96] Diophantus, *On Polygonal Numbers*, 474.10-11 (Tannery)

more similar to the *Arithmetica* in its relaxation of formalism and use of personal language (frequent first-person plural verbs, etc.).

Finally, Diophantus occasionally includes specific numbers and abbreviations in the main text (such as the symbol for a square number).[97] This does not necessarily divide him from all systematist authors (Nicomachus uses specific numbers), but since he is imitating the formal systematist style and is clearly not writing for non-experts, the numbers and symbols do appear out of place.

There is no hint in the two opening paragraphs about whether Diophantus is writing to a patron, colleague, or student. He begins immediately with a brief definition of polygonal numbers and how they are generated, and no correspondent is addressed. So it is not clear whether Diophantus is adopting a systematist style for the sake of his reader, as seems to be the case with Archimedes, or because of the Pythagorean history of the subject matter.

There is little to account for the similarity between the styles of Archimedes' treatises for Dositheus and Diophantus' *On Polygonal Numbers* besides either direct emulation or a 600-year continuity in style among heurist authors writing systematist texts. The two authors were not from the same location (Diophantus was associated with Alexandria), nor are their more overtly heurist texts particularly similar. The *Arithmetica* bears a stronger resemblance to Heron's *Metrica* than to any of Archimedes' works, and

[97] E.g. *OPN* 1, 452.15 (Tannery), where we see $\delta^{\kappa\iota\varsigma}$ for "four times" and $\square^{o\nu}$ for "square".

is entirely different from *On Polygonal Numbers* (except for the final Heronian "method" section of *OPN*). Also, Diophantus' variation on systematist style is certainly not the result of an ignorance of original systematist source material, especially if, as Knorr proposed, he is the true author of the *Definitiones*, an introduction to Euclid's geometry formerly attributed to Heron.[98] The tradition of systematist geometry maintained its style into late antiquity, so it cannot be that time obscured the systematist conventions. Serenus, in the fourth century CE, follows the systematist proof structure and formulaic language much more strictly than Diophantus in the third century. In short, it may be that Archimedes set or followed a set of stylistic choices (preference of δεικτέον ὅτι over λέγω ὅτι, conversational interludes, flexible proposition structures, and run-on propositions) specifically for heurist authors imitating the systematist style, and that Diophantus imitated these choices six hundred years later.

II.2.4 **The Greek Anthology*, Book XIV***

Math problems written as epigrams are known from as early as the Hellenistic period.[99] Book XIV of the Greek Anthology contains 45 arithmetical epigrams. The majority of the epigrams, 116-146, are attributed to an author named Metrodorus, a grammarian in the late fifth and early sixth centuries. Heath is confident that Metrodorus was only the collector of these poems, and that many of them are ancient.[100] Most of

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[98] Knorr, "Arithmêtikê Stoicheiôsis."
[99] Archimedes' *Cattle Problem* is the major example, along with Eratosthenes' epigrams on the doubling of the cube, reproduced in Eutocius, "Commentarii in Libros de Sphaera et Cylindro," 88.8-10 and 96.10-27.
[100] Heath, *HGM* vol. 2, 442.

these epigrams include mythological references, some to famous ancients such as Homer and Hesiod (147), or Pythagoras (1). One refers to Diophantus himself (126). The rest present their problems as everyday situations, such as ordering bricks from a brick-maker (136).

The epigrams in the Greek Anthology are not theoretical mathematics, as they only pose specific problems and make no attempt to demonstrate a method of solution. But neither are they properly termed practical or applied mathematics. We have many examples of practical arithmetic at work, especially among works formerly attributed to Heron, such as the *Geometrica*, *Stereometrica*, and *De Mensuris*.[101] These are truly practical math: they detail solutions to problems that would arise in life and business, and they express their solutions not only in numbers but in concrete units, such as "feet" or "fingers". The epigrams of the Greek Anthology are not practically useful in this way at all. They are clearly recreational. In addition to being clever poems displaying literary skill and education, they are contained in Book XIV among riddles, reported oracles, and other puzzles. The mythological references also diminish the likelihood of practical application, and even where business situations are postulated, the problems are drastically over-complicated compared to the situation. That is, they are nearly all problems of analysis (what would become algebra): sets of complex constraints on one or

[101] For a discussion of the authorship of these treatises, see Heiberg's introduction (Schöne et al., *Heronis opera quae supersunt omnia* vol. 5, xxi-xxxvii).

more unknown quantities. For a very simple example, in epigram 129 a sailor asks the captain how much longer they have to travel, and the sailor's answer is,

"Ναῦτα, μέσον Κριοῖο μετώπου
Κρηταίου Σικελῆς τε Πελωρίδος ἑξάκι μέτρα
χίλια· δοιῶν δ' αὖτε παροιχομένοιο δρόμοιο
πέμπτων διπλάσιον Σικελὴν ἐπὶ πορθμίδα λείπει."[102]

"Sailor, between the face of Cretan Creus and Sicilian Peloris lie six thousand measures; and there remains the double of two fifths of the distance already passed to the Sicilian strait."

Clearly, this is not meant to represent a real-life interaction, or to provide a useful or efficient way to express travel distances.

Now, the methods one would use to solve these problems could be used in commercial, legislative, or agricultural matters, but if the poems were meant to demonstrate these methods, then they would provide solutions,[103] as do the extant prose works explicitly intended for practical purposes. The division of mathematics into theoretical and practical categories fails to meet the evidence in the case of these

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[102] Hermann Beckby, ed., *Anthologia Graeca*, Tusculum-Bücherei. Zweisprachige Antike Taschenausgaben (München: Heimeran, 1958), XVI.129 (translation mine).

[103] Although we do not have the original presentation of these poems, and so cannot be perfectly certain that no solutions were provided, the other evidence we have suggests that the poems were not meant to be accompanied by detailed calculations. Archimedes provides no clues about the solution of the *Cattle Problem*, and Diophantus' epigram is an anomaly within the *Arithmetica*. None of the other propositions is presented this way, and Diophantus quotes the poem as though it were an old riddle, because he explains the meaning of it before setting up a solution using his own methods. That is, he does not present the epigram as though the solution were previously known.

epigrams. They are mental exercises in problem-solving, part recreation and part education.

The arithmetical epigrams as a subcategory within the heurist school of practice show the extent of the cultural penetration of the heurist way of thinking. Although they lie at the edges of the mathematical genre, they are numerically specified, problem-based mathematics aimed at effective solutions rather than universal axioms. Of known mathematicians, only those associated with heurist methods wrote such epigrams.[104] We have the *Cattle Problem* from Archimedes, and a problem of wine measures from Diophantus. *Arithmetica* V.30 begins with an eight-line epigram about a sailor who mixes together eight-drachma and five-drachma measures of wine and arranges them for his fellow sailors, with the total value as a square number. Diophantus transfers the conditions of the problem into his own language and then solves it. Besides Archimedes and Diophantus, I know of no other mathematical author who wrote or reproduced arithmetical epigrams.

II.3 Additional Remarks

This survey of arithmetic has begun to show what will become clearer in the following chapters, that each mathematical discipline influenced the expression of

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[104] There is an epigram attributed to Euclid in some manuscripts, included in Cougny's appendix to the Greek Anthology (E. Cougny, *Epigrammatum Anthologia Palatina cum Planudeis et Appendice Nova*, vol. 3 (Paris: Didot, 1890), 563) and reproduced in J. L. Heiberg and H. Menge, eds., *Euclidis Opera Omnia*, vol. 8 (Leipzig: Teubner, 1916), 286. However, it is clear both from the style (which is consistent with the other epigrams of the Greek Anthology) and from the MS transmission (see note at Heiberg and Menge, *Euclidis Opera Omnia* vol. 8, 286) that Euclid is not the author.

systematist and heurist methods within it. For instance, systematist texts do not entirely avoid using specific numbers in this field, at least for introductory texts, since number is the primary subject matter. Heurist arithmetic shows an iterative principle of problem-solving that is not present in heurist geometry, and is connected to the way number is indefinitely extended. That is, while geometrical objects are scaled up and down continuously (e.g. *Elements* I.P2, to produce a finite straight line continuously in a straight line),[105] number is scaled up and down by the repeated addition or subtraction of discrete parts. So for example, Eratosthenes' *Sieve* is an effective method of analyzing ranges of numbers because it involves not complex calculations, but simple repeated counting and marking.

Similarly, in arithmetic the systematists and heurists diverge more widely over notation than in geometry. While most systematist arithmetic keeps to the same abstract formal conventions of systematist geometry, there is no heurist arithmetic outside of the epigrams that does not make heavy use of information compression as a problem-solving method. It is even possible to see the epigrams themselves as an exercise in information compression, since the format requires all of the necessary conditions for a solution to be given in as few metrical lines as possible.

Finally, in both schools we see sub-genres of mathematical writing that is aimed at a generally educated audience rather than at practitioners: the Neopythagorean

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[105] The exception to the continuous scaling of geometrical objects is when they are numerically determined, as in Heron's *Metrica.*

philosophical textbooks on the systematist side, and the arithmetical epigrams on the heurist. These sub-genres reflect the status of arithmetic as the most fundamental mathematical discipline, and therefore the most easily accessed by non-experts. Numbers are epistemologically prior to geometrical objects, because arithmetic can be done without geometry but geometry cannot be done without numbers. As Nicomachus explains, the number three can be explained without a triangle, but a triangle cannot be explained without the number three.[106]

These types of discipline-specific methodological differences will become more pronounced in the second half of the quadrivium, music and astronomy, where the already advanced mathematics is further complicated by its connection to observable phenomena.

[106] Nicomachus, *Introduction to Arithmetic* I.4.4-5.

III. ASTRONOMY

The study of astronomy presents a complication that geometry and arithmetic do not have to deal with: observed physical phenomena. So far we have seen a dividing line between the systematists and heurists over whether mathematical objects are idealized or understood physically. In geometry, for example, time-dependent curves were avoided by systematists and embraced by heurists. With astronomy, however, systematist texts can no longer avoid engaging with motion and time.

Similarly, since the objects of astronomy are observed celestial bodies such as the sun and moon, the division between the systematist tendency toward generalization and the heurist concern with solving specific problems is naturally blurred a bit. Both systematist and heurist texts may address questions such as the length of the year or the rising times of constellations. As we shall see, however, they are likely to ask and answer these questions in different ways.

Neither division, idealized vs. physical mathematics or generalized vs. specific outlooks, disappears entirely from the astronomical work of the two schools. Instead, these divisions fade to the background somewhat, while the issue of the epistemological priority of theory vs. observation takes center stage. All Greek astronomical texts operate under an assumption that the celestial bodies move in uniform circular motion. The positioning and ordering of the planets was debatable, including the position of the earth (Aristarchus developed a heliocentric model of the universe, and there is a great deal of variation in Presocratic cosmological models), but regular motion in circles seems to have been a universal precept. Naturally, close and accurate observation reveals that all the planets seem to deviate from such motion to a greater or lesser extent, and a major division between the systematist and heurist schools can be seen in whether and how this deviation is addressed. While systematist texts tend to either ignore anomalies or emphasize the imperfection of our observations compared to the necessary perfection of the universe, heurist texts give priority to observation and, especially in the work of Hipparchus and Ptolemy, are ready to make drastic changes to the theory in order to bring it in line with observed data.

In this chapter, I will show the development of the schools' astronomical methods through works of mathematical astronomy. I have omitted certain material for brevity's sake. For example, aside from the following brief overview of Presocratic cosmology, I have not addressed the non-mathematized cosmological models and physical theories of didactic poetry or natural philosophy. Nor have I included handbooks or introductions, such as the works of Geminus and Cleomedes, except for Theon of Smyrna's *Useful mathematics for the understanding of Plato*, which contains some original propositions in the systematist style. Finally, I have avoided the ancillary disciplines of optics and catoptrics (the study of light, perspective, mirrors, etc.). These disciplines seem to have been understood as a natural part of astronomy, and usually traveled with astronomical texts in the manuscript tradition.[1] However, for the purposes of this chapter, I have confined the survey to works that deal directly with astronomical phenomena using mathematical methods.

III.1 Presocratics and Pythagoreans

We have no extant works of Greek mathematical astronomy from before the fourth century BCE[2], but fragments and testimonia give a sense of the cosmological theories that were in circulation earlier. As we shall see in the following chapter on music, these early, non-mathematical theories laid a foundation that systematists and heurists received and worked with in different ways.

The development of Presocratic cosmology was largely motivated by tensions between theories that foregrounded observation and physical causality, and those that foregrounded metaphysics, ontology, and epistemology. The Milesian Presocratics (Thales, Anaximander, Anaximenes) seem to have been much concerned with the former.

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[1] See Colin Webster, "Euclid's Optics and Geometrical Astronomy," *Apeiron* 47, no. 4 (2014): 526–551, and Bernardo Machado Mota, "The Astronomical Interpretation of Catoptrica," *Science in Context; Cambridge* 25, no. 4 (December 2012): 469–502, for more on these topics.

[2] This does not mean that there were no mathematical works before this time. In the Aetian doxographic texts we have multiple testimonies about Presocratic philosophers who "agree with mathematicians." For example, one passage states that Alcmaeon "and the mathematicians" claim that the planets move east to west, opposite to the fixed stars (Ps. Plutarch, "Placita Philosophorum," in J. Mau, ed., *Plutarchi Moralia*, (Leipzig: Teubner, 1971), 50–153, 889c4-6; Ps. Galen, "De Historia Philosophica," in H. Diels, ed., *Doxographi Graeci* (Berlin: Reimer, 1879), 58.3-5; Eusebius, "Preparatio Evangelica," in K. Mras, ed., *Eusebius Werke*, vol. 8, Die Griechischen Christlichen Schriftsteller, 43.2 (Berlin: Akademie Verlag, 1956), 15.47.2. It is not certain that the mathematicians referred to in this passage are understood to be contemporary with Alcmaeon, or whether the doxographer is simply noting that later mathematicians agree with the philosopher who came before them. Still, Herodotus claims that the Greeks had already learned the use of astronomical measuring tools from the Babylonians (*Histories* II.109), and so it is hard to believe that there was no mathematical astronomy in practice before the fourth century, even if no texts survive.

Aristotle credits Thales as the originator of philosophical inquiry into material causes,[3] and Theon of Smyrna, paraphrasing Eudemus, says that Thales investigated eclipses, the periods of solstices, and the length of the year.[4] In the same passage, Theon says that Anaximander declared that the earth was not fixed, but moved about the center of the cosmos; and that Anaximenes claimed that the moon took its light from the sun. Multiple other testimonies contribute to the impression of the early Milesians as primarily focused on physical phenomena and their causes (placing water, air, or boundless motion as the first cause of the universe, for example).[5] Heraclitus followed in their footsteps, claiming fire as the fundamental physical substance, declaring the experiences of the senses to be preferable as sources of information, and criticizing Xenophanes and Pythagoras as having learned much but understood little.[6] Later still, Leucippus, Democritus, and the atomists reacted against the metaphysics of the Eleatic philosophers with a staunchly materialistic and non-teleological view of the universe.[7]

Xenophanes, Parmenides, Empedocles, and the Eleatics on the one hand, and Pythagoras, Anaxagoras, and the Pythagoreans on the other hand all seem to have

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[3] Aristotle, *Metaphysics* 983b20-22.
[4] Theon of Smyrna, *Useful Mathematics,* 198.
[5] A complete catalogue of these testimonies is beyond the scope of this chapter, but for more general information, see Patricia Curd, "Presocratic Philosophy," in *The Stanford Encyclopedia of Philosophy*, ed. Edward N. Zalta (Metaphysics Research Lab, Stanford University, 2019); G. S. Kirk, J. E. Raven, and M. Schofield, *The Presocratic Philosophers: A Critical History with a Selection of Texts*, 2nd ed (Cambridge; New York: Cambridge University Press, 1983). For testimonies and fragments specifically related to Presocratic cosmology, see T. L. Heath, *Greek Astronomy* (London: Dent, 1932), 1-39.
[6] Daniel W. Graham, "Heraclitus," In *The Stanford Encyclopedia of Philosophy*, ed. Edward N. Zalta (Metaphysics Research Lab, Stanford University, 2019).
[7] Sylvia Berryman, "Democritus," in *The Stanford Encyclopedia of Philosophy*, ed. Edward N. Zalta (Metaphysics Research Lab, Stanford University, 2016).

subscribed to some version of the view that mere observed appearances are fundamentally deceptive, and that the real truth behind the cosmos is perfect, eternal, and unchanging. Among the Eleatics this view was manifested as a mistrust of phenomena and an insistence on the impossibility of real change or imperfection in the heavens.[8] Among the Pythagoreans, the idea arose that mathematics, particularly "number," is the perfect reality underneath the physical veil. Notably, the Pythagoreans do not seem to have meant that measurement (or in other words, the numerical *description* of observed phenomena) is the key to knowledge—rather, they meant that an understanding of mathematical principles, such as perfect numbers and ratios, illuminates the nature that mere physical observation would obscure. Aristotle particularly criticizes Pythagorean cosmology for forcing phenomena to suit their theories instead of seeking explanations to account for phenomena.[9] He also notes that they were determined to make their system of numerology, harmonics, and cosmology complete and internally consistent, and therefore insisted that there are ten planets rather than the observed nine (they invented an invisible "counter-earth" to do it), because ten is the perfect number.[10]

Plato drew heavily on the Pythagorean tradition, and the systematist mathematics that grew up in and around the Academy bear many markings of its influence. This is not to say, however, that the tension in Presocratic philosophy between reliance on and

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[8] Curd, "Presocratic Philosophy," see especially sections on Xenophanes and Parmenides.
[9] Aristotle, *De Caelo*, 293a18.
[10] Aristotle, *Metaphysics*, 986a1.

mistrust of observed phenomena coincides in an unproblematic way with the attitudes and methods of the heurists and systematists respectively. Both schools received the previous traditions in their own ways.

III.2 The Systematist School

Most of the characteristics of systematist astronomy are similar to those of geometry and arithmetic, but the subject of astronomy introduces certain exceptions and complications, whether by necessity or simply through the school's development. For example, there is still the preference for a formulaic, impersonal style and propositions that follow a regular format, but systematist astronomical texts are more likely to include compound propositions (that is, propositions whose *protaseis* make multiple statements that are all proved in the same block of text). Numerical methods and notation are still mostly eschewed, but when the subject matter involves the reckoning of months or years, sometimes numbers are used to express them. Again, there is the strong tendency to de-emphasize the physical aspects of the mathematics, such as tools and measurements, but since astronomy is the study of physical things, that tendency is not always observed strictly. In order to say anything about specific planets and constellations, authors had to at least acknowledge that they were dealing with an object of perception. They used concepts such as visibility, motion, and time, which are avoided in geometrical and arithmetical works.

Despite such exceptions, systematist astronomy is not fundamentally observational or predictive. That is, systematist texts do not produce tables of astronomical measurements or provide a means of calculating where and when a star or planet will appear on a given day. Rather, given a theory of uniform circular motion in a geocentric universe and certain well-known patterns (like the movement of the sun around the zodiac circle, or “ecliptic”), these texts compare lengths of days, periods between solstices, rising and setting times of stars or planets, and other such phenomena, in terms usually no more specific than “greater,” “less,” or “equal.” In other words, systematist astronomy does not treat the phenomena as epistemologically prior; it treats the axiom of the perfection of the cosmos and the theory of uniform circular motion as prior. Mostly the phenomena are ignored. But if the phenomena and the theory seem to disagree, the phenomena are mistrusted. Thus, for example, we see from the systematists the first developments in optics,[11] the study of how we see objects and how our observations may be deceived,[12] as a part of the body of astronomical knowledge.[13] In a similar vein is the later tradition (mostly among Neoplatonist writers) about astronomers trying to “save the phenomena.”[14]

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[11] The earliest complete text is Euclid, “Optica,” in *Euclidis Opera Omnia*, ed. J. L. Heiberg, vol. 7 (Leipzig: Teubner, 1895), 2–120. M. F. Burnyeat, “Archytas and Optics,” *Science in Context* 18, no. 1 (March 2005): 35–53, argues that Archytas of Tarentum was actually the founder of the discipline, and points out that optics is known to have been studied at the Academy.

[12] For more on the philosophy of early optics, see Sylvia Berryman, “Euclid and the Sceptic: A Paper on Vision, Doubt, Geometry, Light and Drunkenness,” *Phronesis* 43, no. 2 (July 1, 1998): 176–196.

[13] For more on the relationship between optics and astronomy, see Webster, “Euclid’s Optics and Geometrical Astronomy.” A similar argument is made for catoptrics, the study of mirrors, in Mota, “The Astronomical Interpretation of Catoptrica.”

[14] The earliest uses of this phrase come from Hipparchus’ commentary on Aratus and Eudoxus. The majority of attestations come from Theon of Smyrna, Simplicius (quoting Eudemus or Geminus, usually), Plutarch, and Proclus.

The idea of "saving the phenomena" is essentially an acknowledgement that if the movements of celestial bodies are accurately observed, they do not appear to move with uniform circular motion. Eccentricities, changes of velocity, and retrogradations can be observed, for example. To "save the phenomena" is to account for such apparent aberrations with a theory of uniform circular motion. Simplicius (drawing from Eudemus) claimed that the challenge of saving the phenomena was initially issued by Plato, and inspired Eudoxus' theory of concentric spheres.[15] The challenge is consistent with the apparent founding principles of systematist astronomy, in which observation and phenomena are considered to be untrustworthy compared to a simple and perfect theory. Although no early systematist authors use the phrase, saving the phenomena seems to have been a motivating principle of systematist work. Hipparchus speaks with apparent sarcasm in his commentary of Eudoxus and Aratus trying to save the phenomena.[16]

Perhaps the most important source for the systematist tradition is the *Little Astronomy*, an ancient collection of elementary texts on astronomy and spherical geometry that included works of Theodosius, Autolycus, Euclid, Menelaus, Aristarchus, and Hypsicles.[17] There is some debate over whether the collection and its name (which we have from a scholion to Pappus *Synagoge* VI and another to Ptolemy's *Almagest*) are

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[15] Simplicius, *In Aristotelis de Caelo Commentaria*, ed. J. L. Heiberg (Berlin: Reimer, 1894), 788.

[16] Hipparchus, *In Arati et Eudoxi Phaenomena Commentariorum Libri III*, ed. C. Manitius (Leipzig: Teubner, 1894), II.3.23 and II.3.28. Aratus wrote a didactic poem, the *Phaenomena*, in which, according to Hipparchus, he "followed" Eudoxus (Hipparchus, *In Arati et Eudoxi*, II.3.29). Aratus' poem is not a mathematical text, so it has not been included in this survey, but it is clear that he stands in the Pythagorean/Eleatic tradition of cosmology.

[17] The oldest version of the collection is found in *Vat. Graec.* 204, a 9th-10th c. Byzantine manuscript.

truly of ancient origin,[18] but even if it was compiled in late antiquity, the *Little Astronomy* represents a recognition of systematist texts as a cohesive group. All the works except one employ systematist style and methodology to the strictest standards, including the avoidance of specific numbers. The exception is Aristarchus' *Sizes and distances of the sun and moon*, which is an early heurist text (roughly contemporary with Archimedes) that will be discussed further below.

III.2.1 Eudoxus

From the assumptions made in the works of Autolycus and Euclid about spherical geometry and its applications to astronomy, it is clear that there was a tradition of systematist geometry/astronomy that predates our extant texts.[19] The two names that come up in every attempt to reconstruct this tradition are Eudoxus (primarily), and Callippus (as a brief footnote at best).

We have extensive fragments of Eudoxus, paraphrased mostly by Hipparchus and other commentators on his *Phaenomena*.[20] Their accuracy, however, is questionable, and

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[18] For more on the *Little Astronomy*, including a brief discussion of whether or not it was truly an ancient collection, see James Evans, *The History and Practice of Ancient Astronomy* (Oxford University Press, 1998), 89-91, and O. Neugebauer, *A History of Ancient Mathematical Astronomy*, Studies in the History of Mathematics and Physical Sciences 1 (Berlin: Springer-Verlag, 1975), 768-769.

[19] The stylistic similarities between Euclid and Autolycus, especially where their material overlaps, suggests that they are drawing on the same tradition, not independently paraphrasing or interpreting math done in another way. For more, see R. S. D. Thomas and J. L Berggren, *Euclid's Phaenomena: A Translation and Study of a Hellenistic Treatise in Spherical Astronomy*, Sources and Studies in the History and Philosophy of Classical Science 4 (New York: Garland, 1996), 6-7.

[20] F. Lasserre, *Die Fragmente des Eudoxos von Knidos* (Berlin: De Gruyter, 1966). There is also a treatise (including a short acrostic poem) on papyrus called the *Ars Astronomica*, usually thought to be spurious, though F. Blass, "Eudoxi Ars Astronomica qualis in Charta Aegyptiaca superest: Praefatio," *Zeitschrift für Papyrologie und Epigraphik* 115 (1997): 79–101, argues for at least the possibility that Eudoxus was the

not enough remains to definitively describe his methods.[21] He does seem to have been one of the first authors to attempt a mathematical rendering of Presocratic cosmology,[22] but the nature of the mathematics is mostly unclear. Aristotle describes Eudoxus' cosmos as a series of concentric spheres on which the stars and planets were carried, to which Callippus added additional spheres in order to account for certain phenomena.[23] Simplicius, referencing Eudemus, claims that Eudoxus developed his theory in order to meet Plato's challenge to find uniform and ordered movements which "save" (διασῴζω—that is, which account for) the celestial phenomena.[24] Eudoxus' theory (including Callippus' emendations) seems to have been purely geometrical, for Aristotle feels the need to improve on it by introducing a mechanical explanation of the spheres and their movements.[25] However, the fragments we do have do not suggest that Eudoxus was working primarily with the methods of spherical geometry, as Autolycus and Euclid were.[26]

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author. For more on the Eudoxus Papyrus, including an argument against Eudoxan authorship based on subject matter, see Neugebauer, *A History of Ancient Mathematical Astronomy*, 686-689.

[21] See especially Alan C. Bowen and Bernard R. Goldstein, "Hipparchus' Treatment of Early Greek Astronomy: The Case of Eudoxus and the Length of Daytime," *Proceedings of the American Philosophical Society* 135, no. 2 (1991): 233–54.

[22] J. L Berggren, "The Relation of Greek Spherics to Early Greek Astronomy," in *Science and Philosophy in Classical Greece*, ed. Alan C. Bowen, Sources and Studies in the History and Philosophy of Classical Science 2 (New York: Garland, 1991), 231.

[23] Aristotle, *Metaphysics*, 1073b17-1074a15. Aristotle is not specific about the phenomena that Callippus was trying to account for, but they may have had to do with planetary retrogradation, which was a problem for Eudoxus' theory (Berggren, "The Relation of Greek Spherics to Early Greek Astronomy," 231).

[24] Simplicius, *In Aristotelis de Caelo Commentaria*, 788. For more on "saving the phenomena," see section on Theon of Smyrna below.

[25] Aristotle, *Metaphysics*, 1073b17-1074a15. See also Heath, *Greek Astronomy*, xlviii-xlix.

[26] Neugebauer, *A History of Ancient Mathematical Astronomy*, 761. See also Thomas and Berggren, *Euclid's Phaenomena*, 6-8.

Fragments remain of two works by Eudoxus, the *Enoptron* and the *Phaenomena*, preserved mostly (and earliest) by Hipparchus, for the purpose of disputing them. They both seem to have covered the same or similar material, specifically the relative lengths of day and night at different times of the year. Hipparchus reports that in the *Enoptron* Eudoxus gives the ratio of 5:3 for the length of day to night on the summer solstice, while in the *Phaenomena* he gives the ratio of 12:7. Although nothing remains of the method by which Eudoxus supposedly arrived at these ratios, Bowen and Goldstein have demonstrated that they can be derived purely arithmetically, using methods known in Eudoxus' time.[27] That is, the ratios do not depend on observation for their derivation. Indeed, the ratio 12:7 in particular is problematic from any perspective of measurement or observation, because it would require that the day be divided into nineteen parts, which would make it incompatible with, for example, the Babylonian observational tradition, which divided the ecliptic into twelve equal parts and 360 degrees.[28] Bowen and Goldstein point out, however, that there is no evidence in our earliest Greek astronomical texts (the majority of which are systematist, incidentally) that such a system was in use or was a matter of concern.[29]

In short, while we lack the evidence to place Eudoxus firmly in either of the schools, we know that he was part of the circle of mathematicians connected to Plato's

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[27] Bowen and Goldstein, "Hipparchus' Treatment of Early Greek Astronomy," 237-238.
[28] Bowen and Goldstein, "Hipparchus' Treatment of Early Greek Astronomy," 248-249.
[29] Bowen and Goldstein, "Hipparchus' Treatment of Early Greek Astronomy," 241-248.

Academy, and that in his other mathematical work he was instrumental in the formation of the systematist school. Though his influence on the development of systematist astronomy cannot be demonstrated, the weight of the evidence suggests that he at least was working with mathematical methods that did not take into account much of physical observation.

III.2.2 Autolycus of Pitane

Autolycus lived and worked during the second half of the fourth century BCE, and his two treatises, the *Moving Sphere* and *Risings and Settings*, are therefore probably the oldest complete Greek mathematical texts that have come down to us.[30] Both works were preserved in the *Little Astronomy*. The *Moving Sphere*, because it treats more elementary material, is generally considered to be the older of the two.[31]

The *Moving Sphere* could almost be considered a work of spherical geometry rather than astronomy, were it not for the focus on the motion of the sphere from the perspective of an observer at its center. The use of words for rising, setting, and the horizon make it clear that the sphere is the observable heavens and the center is the earth.[32] But otherwise, no specific celestial phenomena are mentioned—not even the

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[30] On the other hand, there is very little evidence for the biographies of Autolycus and Euclid, and the chronology of their work is not totally certain. One or more of Euclid's texts may be slightly older. Either way, it is clear that neither author was a direct influence on the other (see section on Euclid's *Phaenomena* below), and that both were working with an already established set of systematist conventions.

[31] For a history of the texts, see G. Aujac, *Autolycus de Pitane: La Sphère en Mouvement; Levers et Couchers Héliaques*, ed. Jean-Pierre Brunet and Robert Nadal (Paris: Les Belles lettres, 1979), 28-40.

[32] *Moving Sphere* 4-12 assume an observer in the center and speak of the visible and invisible parts of the sphere.

words “sun” or “star.” Instead, Autolycus speaks of “points” on the sphere, when they will rise and set at the horizon, and what circles or arcs they will trace when the sphere is moved. The book consists of twelve propositions, each of which meticulously follows the format of a systematist proof, including *protasis, ekthesis, diorismos, kataskeuē,* and *apodeixis.* Only propositions 1 and 10 include a generalized *symperasma*, but the others, though they are specific to the labeled points of each diagram, are phrased to echo the *protasis*. The text begins in every manuscript with a definition of uniform motion, but the authenticity of the definition is disputed, and it may have been only a marginal scholion that became integrated with the text.[33] The propositions are done in general terms using lettered diagrams and no numbers, and although it is a short book, it is sequentially arranged and thorough in treating case variations.[34]

Risings and Settings is comprised of two books, of thirteen and eighteen propositions respectively. There is a great deal of overlap between Book I and the first eight propositions of Book II, which has caused some scholars to question whether or not both books are by Autolycus or if Book II is his revision of his own work.[35] *Risings and Settings* is more overtly astronomical than the *Moving Sphere*, as it discusses the appearances and movements of the sun and stars. The stars are not named, however, but

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[33] Aujac, *Autolycus de Pitane: La Sphère en Mouvement; Levers et Couchers Héliaques*, 42 note 1.
[34] e.g. *Moving Sphere* 4-6, concerning the motion of great circles (i.e. circles cut from the sphere through its center) that lie perpendicular, parallel, and oblique to the horizon.
[35] For a discussion of this issue, see Aujac, *Autolycus de Pitane: La Sphère en Mouvement; Levers et Couchers Héliaques*, 21, and Neugebauer, *A History of Ancient Mathematical Astronomy*, 751.

are rather labeled with letters in the diagrams as though they were geometrical points. No measurements or observed positions are given to stars in the text; instead, they are categorized generally as falling on, north, or south of the ecliptic (the circle of the zodiac and the path of the sun). Here as in the other systematist texts, even observed physical phenomena are treated as abstractly as possible, and are faded into the background in favor of a generalizing theory.

Another feature of *Risings and Settings* that recalls the earlier traditions of the Eleatics and Pythagoreans is the distinction drawn between "true" (ἀληθιναί) and "apparent" (φαινόμεναι) risings and settings of the stars in the definitions and axioms given at the beginning of Book I. The first proposition of Book I proves that apparent risings and settings of morning occur later than the true risings and settings (and that apparent risings and settings of evening are earlier than the true ones). A systemic mistrust of observation is thus built into the work from the beginning.

Stylistically, *Risings and Settings* is nearly as rigorously systematist as the *Moving Sphere*. No numbers or measurements are invoked, the parts of a proposition are followed, and multiple cases are exhaustively observed.[36] The only noticeable differences are the slightly more physical terminology (referring specifically to the sun, the zodiac circle, the stars, etc.), and the rarer occurrence of variations on λέγω ὅτι introducing the

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[36] Neugebauer complains in particular about *Risings and Settings* II.10-18, in which the same principle in different configurations is "repeated without mercy for all nine theorems." (Neugebauer, *A History of Ancient Mathematical Astronomy*, 762.)

diorismos, which in fact is only a feature of Book II.[37] Neither difference indicates any real departure from systematist usage. The vocabulary is demanded by the subject matter, but does not correspond to an increased concern with observation or problem-based approaches. Likewise, the slight stylistic shift in Book II may be accounted for by its being a minor revision and expansion of Book I (particularly if it is the work of a later author). Since they appear together in the manuscript tradition, Book II may have been understood as a supplement to the more formal Book I.

The work of Autolycus clearly shows not only that systematist conventions of style, generalizing method, and idealizing conceptualization were fairly uncompromising even when the subject matter was physical, but also that these conventions were solidly in place before the composition of Euclid's *Elements*, and were not necessarily originated in the field of geometry.

III.2.3 Euclid

Euclid's *Phaenomena* consists of eighteen propositions and a brief introduction.[38] The content of the work is similar to Autolycus' *Risings and Settings*, its primary goal being to describe the locations and relative rising and setting times of fixed stars and arcs

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[37] In the *Moving Sphere*, variations on λέγω ὅτι appear at least once per proposition, twenty times in all; in *Risings and Settings*, the phrase occurs only 29 times in over twice the number of propositions, and almost all of those in Book I. In fact, each proposition of *Risings and Settings* I contains at least one occurrence of λέγω ὅτι, but it only occurs seven times in all of *Risings and Settings* II (propositions 1, 2, 4, 5, 9, 14, and 15).

[38] The whole text has been assembled from two versions, one of which is cut off in the middle of proposition 16, and the other of which only differs from the first in propositions 9 and onward. For more, see Heiberg and Menge, *Euclidis Opera Omnia*, vol. 8.

of the ecliptic. Like Autolycus, Euclid is notably non-specific in his descriptions. He names no specific stars in the propositions,[39] and mentions only the tropics, the equator, and the halves of the ecliptic "following Capricorn" or "following Cancer."

Stylistically and methodologically, the *Phaenomena* is unambiguously systematist, but not very consistent with Euclid's other work. No numbers are used, and the text is structured sequentially from simple demonstrations to complex proofs that rely on previous material, but the proposition format is a bit unusual (for Euclid, not for systematist work in general): not only is the *symperasma* often omitted,[40] but most "propositions" are actually compounds containing several parts. The *protasis* makes a few distinct statements, which are then proved in turn by multiple simple sub-propositions, one for each statement. Each of these sub-propositions independently follows the normal proposition format, albeit somewhat abridged. For example, *Phaenomena* Proposition 2 contains three sub-propositions[41] (and two more by interpolation[42]). The *protasis* makes each of these three (five) smaller statements. A general *ekthesis* produces a diagram sufficient for all the parts. Then each of the sub-

[39] Except in *Phaenomena* 1, in which Euclid mentions the constellations of Cancer and Capricorn, Leo and Aquarius. Both pairs lie opposite one another on the ecliptic, so that one sets while the other rises. But *Phaenomena* 1 is unusual in a couple of ways, and will be treated separately from the others, below.

[40] While it is not at all unusual for systematist works to neglect the *symperasma* (as it simply restates the *protasis* almost verbatim), Euclid normally retains it.

[41] Pappus, incidentally, speaks of *Phaenomena* 2 in two places (*Synagoge* VI, 474 and 594), and in both he mentions only the first statement of the *protasis*. On the other hand, his paraphrases of the *protaseis* of *Phaenomena* 11 and 12 are accurate to the text we have, so we cannot suppose that his text had different divisions of propositions.

[42] Euclid, *Phaenomena* 2, lines 6-10 and 86-135 are interpolated. Pappus did not have them in his text, apparently (see Thomas and Berggren, *Euclid's Phaenomena*, 15).

propositions contains its own *diorismos*, *apodeixis*, and *symperasma*, as well as an additional short *ekthesis* specific to that sub-proposition if needed.[43] Propositions 2, 4, 5, 7, 9, 11, 12, 14, and 16 are compound in this way.

The compound propositions give the impression that the texts we have may be recensions or abridgements, rather than Euclid's original work. Not only is it highly unusual for Euclid to write compound propositions at all, but the lack of *symperasmata* to echo the compound *protaseis* (while each sub-proposition does include an appropriate *symperasma* for its part) suggests an attempt at efficiency of presentation rather than the meticulous partitioning of logical units that is characteristic of, for example, the *Elements.* Furthermore, the propositions that are not compound do include *symperasmata*. It seems, then, that in the *Phaenomena* propositions that use the same diagram are compounded, while those with a unique diagram remain distinct and follow the normal structure. While this is a choice we almost never see in Euclid's other work, it is consistent with the demands of an abridgement.[44]

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[43] While none of the sub-propositions in *Phaenomena* 2 requires a *kataskeuē*, one is provided when needed in other propositions (e.g. *Phaenomena* 7, lines 28-30).

[44] Euclid, *Optics* 46 and *Data* 81 and 93 are the only other instances of compound propositions I have discovered. There are a few examples of multi-part propositions in the *Elements*, *Data,* and *Optics*, but they are not compound propositions in the same sense, because they do not make multiple statements in their *protaseis* which they then prove in sub-propositions. Rather, these other multi-part propositions prove one *protasis* in two phases, each of which may require an additional bit of *ekthesis, diorismos,* and *apodeixis*, but neither of which proves a separate statement. The phases act only as steps, broken down to make a complicated proof of a single statement more intelligible.

The other evidence of abridgement is the tacit presupposition in the *Phaenomena* of many theorems of spherical geometry and astronomy.[45] The gaps were felt by Pappus, who wrote a commentary and an extensive set of lemmas to the text in Book VI of the *Almagest*.[46]

There are two other passages which are even more strikingly inconsistent with the Euclidean norm: the introduction and the first proposition. The introduction is a description of the observed risings and settings of certain stars and the fixed patterns of their movements, all of which form an evidentiary basis for supposing the cosmos to be spherical.[47] After this supposition comes a brief set of definitions of the horizon, the meridian, the tropics, great circles, the time of revolution of the cosmos, and the passage of an arc.[48] The *Phaenomena* is the only one of Euclid's (securely attributed) works to include an introduction beyond a simple list of definitions and postulates,[49] and the style

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[45] The full list of presuppositions would go far beyond the scope of this chapter, but an excellent summary is available in Thomas and Berggren, *Euclid's Phaenomena*, 19-32.

[46] Thomas and Berggren, *Euclid's Phaenomena*, 13-18, address the possible changes to the text after Pappus' commentary, including the likelihood of a recension by Theon of Alexandria, and the writing of the introduction.

[47] Euclid, *Phaenomena*, Proem 1-65.

[48] Euclid, *Phaenomena,* Proem 66-118. The last two definitions are omitted from *Vat. Graec*. 204, the earliest and most authoritative manuscript.

[49] The *Division of the Canon* includes such an introduction, but it is clear that Euclid is not the author of the introduction or the work itself. See ch. 4, section on the *Division of the Canon.*

of the introduction is more consistent with that of later editors and scholiasts.[50] Scholars have argued that the introduction is not the work of Euclid, but a later addition.[51]

The first proposition is unusual for several different reasons. First of all, it invokes an observational tool, the diopter, and mentions two pairs of specific constellations in the zodiac: Cancer and Capricorn, Leo and Aquarius. The purpose of the first proposition is to demonstrate that the earth is in the center of the cosmos. A diopter is placed at a point representing earth, and through it Cancer will be observed rising on the eastern horizon as Capricorn sets in the west. Because both are observed through the same diopter, the proposition argues, the points representing Cancer, Capricorn, and the Earth will be connected by a straight line. Because the line cuts off exactly half of the ecliptic, it must be a diameter of the sphere of the fixed stars (i.e. the cosmos as a whole). The same process is then followed for the constellations Leo and Aquarius, and the proposition concludes that, since two or more diameters of the cosmos cross at the earth, the earth must lie at the center.

The other unusual feature of the first proposition is its format and style. Its simple *protasis* is mirrored by a *symperasma*, and a brief *ekthesis* establishes the position of the earth and the horizon, but the body of the proof is irregular. First of all, it contains no

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[50] E.g. the introduction to Euclid's *Optics* that appears only in the recension once thought to be the work of Theon of Alexandria (J. L. Heiberg, ed., "Opticorum Recensio Theonis," in *Euclidis Opera Omnia*, vol. 7 (Leipzig: Teubner, 1895), 144–246) and the introduction to Theodosius' *Days and Nights*, which is the work of a scholiast (Theodosius, "De Diebus et Noctibus," in *Theodosii de Habitationibus Liber, de Diebus et Noctibus Libri Duo*, ed. R. Fecht, Abhandlungen Der Gesellschaft Der Wissenschaften Zu Göttingen, Philol.-Hist. Kl. N.F., 19.4 (Berlin: Weidmann, 1927), 54–154).
[51] E.g. Thomas and Berggren, *Euclid's Phaenomena*, 756.

diorismos at all. Every other proposition in the book contains a *diorismos*, including each sub-proposition of compound propositions. Next, the first proposition consists of two almost identical parts, the Cancer/Capricorn proof and the Leo/Aquarius proof. This is a bit unusual in itself, but more so is the language of the parts. Each part contains a *kataskeuē* and *apodeixis*, but the *kataskeuē* is expressed as "positioning" the diopter and "observing" the constellations, rather than the normal instructions to draw a line or divide a segment. While the language is in part a necessary result of the more physical method of proof (itself highly unusual), it nevertheless stands apart not only from Euclid's other works, but even from the rest of the *Phaenomena*. The first proposition is the only one in the book to use the word θεωρέω, "to observe," and it is the only one besides proposition 7 to use a first-person reference at all (ἡμετέρα ὄψις, *Phaenomena* 1.3); and *Phaenomena* 7 uses an entirely different phrase.[52] The non-Euclidean introduction, however, does use the phrase τῆς ὄψεως ἡμῶν (line 66). The word ὄψις is only used in the introduction and the first proposition.

It should be noted that, despite the uniqueness of the first proposition, it generally follows a systematist format. And though the language and method of proof are more physical than we see in most other systematist works (the rest of the *Phaenomena* included), there is still a generalization that bespeaks at least an attempt at a systematist

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[52] *Phaenomena* 7 uses the phrase ὑπὲρ ἡμᾶς several times to indicate the varying positions of the ecliptic above the earth (us). This is, of course, aside from the normal systematist phrase ὁμοίως δὴ δείξομεν, which appears several times throughout the work and is very common in Euclid.

outlook: no numbers or specific measurements are invoked, the earth is taken as a point, the diopter is placed on it at an unspecified location, and the simultaneous rising and setting of opposite signs of the zodiac is such a well-known phenomenon that it hardly counts as an appeal to any kind of metrical or other heurist problem-solving method. Nevertheless, it seems possible that the first proposition, like the introduction, is not the work of Euclid.

All in all, it is clear that the *Phaenomena* as we have it is different from the original text, though the extent and timeline of the changes are not certain. By the time of Pappus the propositions were numbered as we have them, so if an editor condensed the compound propositions or added the first proposition and introduction, it must have been done before the fourth century. On the other hand, the strangeness of the first proposition and the compound propositions may indicate that Euclid was not, after all, the author of the original (or he was, but departed substantially from his normal style, for some reason). The rigorous conventions of systematist mathematics meant that nearly all systematist texts shared a broad similarity in style. Euclid's fame would make his name likely to be appended to anonymous but obviously systematist works. I will not draw certain conclusions here, but attention to stylistic and methodological matters within each school could clearly contribute to discussions of authorship and textual history.

III.2.4 Theodosius

In the second or first century BCE, in addition to his *Sphaerica* on spherical geometry, Theodosius wrote two astronomical treatises, *Habitations* and *Days and*

Nights. Both are systematist in style and method, except for a couple of passages that make use of specific numbers.

Habitations[53] concerns the length of days at the north pole, the equator and the "habitable" regions between them. It includes no introduction, definitions, or postulates, but consists of twelve propositions, of which only 10 and 12 invoke specific numbers. In both cases, they are the number of days during which the sun is above or below the horizon. No explicit calculations are performed with these numbers, except in *Habitations* 10 when 30 is added to 187 to get 217 days during which the sun is visible above the horizon at the north pole. Implicitly, the days of the year are understood to add up to 365¼, so the remaining 148¼ days during which the sun is below the horizon are tacitly subtracted from that total. This use of specific numbers, however, does not constitute any kind of numerical problem-solving method. The numbers of days are merely counted (the way the sides of a polygon or might be counted in Euclid's *Elements*—days are naturally countable, so no act of measurement actually takes place), and are declared to be greater than a seven-month and less than a five-month respectively. The numerical notation, while unusual in most systematist texts we have seen so far, is appropriate to the large numbers being used, and is otherwise avoided

[53] Theodosius, "De Habitationibus," in *Theodosii de Habitationibus Liber, de Diebus et Noctibus Libri Duo*, ed. R. Fecht, Abhandlungen Der Gesellschaft Der Wissenschaften Zu Göttingen, Philol.-Hist. Kl. N.F., 19.4 (Berlin: Weidmann, 1927), 14–42.

except for the last two sentences of *Habitations* 12, in which the number 30 is expressed as λ̅ rather than τριάκοντα, as it is in the rest of that proposition.[54]

Habitations 7, 8, 10, 11, and 12 are compound like those mentioned in Euclid's *Phaenomena* above. That is, their *protaseis* have multiple parts, and each of those parts is proved separately with its own *diorismos*, *ekthesis/kataskeuē* (if necessary), and *apodeixis*. Other than this and the numerical notation discussed above, *Habitations* is an unambiguously systematist text, laying out an exhaustive series of theorems that cover the known world, without invoking tools or measurement.

Days and Nights is a slightly longer treatise in two books, one of twelve and one of nineteen propositions,[55] concerning the patterns of the sun's movement. Book I deals with the relative lengths of days and nights depending on the position of the sun with respect to the tropics and the equator. Book II deals with the lengths of days and nights taken together compared at different times of the year, with the position of the sun with respect to the tropics and the meridian at different times of the day and year, and with whether or not a year consists of a complete revolution of the sun around the ecliptic.

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[54] The symbol λ̅ is also used in *Habitations* 10 line 39, when 30 is added to 187 to produce the total number of days of visible sunlight.

[55] This is according to the Greek text of Fecht, *Theodosii de Habitationibus Liber, de Diebus et Noctibus Libri Duo*, which followed *Vat. Graec.* 204 almost exclusively. In the Arabic tradition, the propositions are numbered differently, and Book II contains 21 propositions, three of which (II.10, 12, and 13) do not correspond to the Greek text at all. (Paul Kunitzsch and Richard Lorch, "Theodosius, De Diebus et Noctibus," *Suhayl. International Journal for the History of the Exact and Natural Sciences in Islamic Civilisation*, 10 (2011), 11-12). There were clearly multiple recensions of the text, and we cannot be certain about its original structure, so it is possible that the frequency of compound propositions is the result of editing. But no evidence that I have found so far indicates such a drastic revision, so the safest assumption would seem to be that the compound propositions were original to the text.

Like *Habitations*, *Days and Nights* is systematist in style and method, with many compound propositions. However, three propositions of Book II break with systematist convention by introducing numerical methods. *Days and Nights* II.17, 18, and 19 use specific numbers, almost all for the 365 (and some fraction[56]) days of the year, but II.18 uses other numbers for some basic calculations, to show that under the assumption of a certain length of the year, the sun would require nineteen years to complete a revolution. These three propositions are using numerical methods, dividing the ecliptic into discrete parts and making calculations on them.

Days and Nights is notable for its referencing of other astronomical works. It shares with Euclid's *Phaenomena* a term for the movement or "interchange" (ἐξαλλαγή) of the ecliptic between the visible and invisible hemispheres that is otherwise unattested in astronomy.[57] Two propositions of *Days and Nights* (II.17 and II.18) explicitly cite the work of other mathematicians: Callippus, Meton, and Euctemon. While it is not unheard of for systematist texts to cite other mathematicians, the strict format of the proposition makes it unusual to see such attributions in the body of the text rather than in the proem.[58] On the other hand, Callippus was associated with the work of Eudoxus, and he also expanded on the work of Meton and Euctemon. While we cannot be totally certain that

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[56] *Days and nights* II.17 uses 365¼ as the number of days in a year, a value Theodosius attributes to Callippus, while II.18 uses $365^{5/19}$, a value attributed to Meton and Euctemon, and unnamed others.
[57] See Neugebauer, *A History of Ancient Mathematical Astronomy*, 758-759 for a discussion of this term and concept.
[58] E.g. Apollonius in the prolegomena the *Conics* mentions several contemporary mathematicians, but their names do not appear in the propositions at all.

these authors wrote in the systematist style, they are all associated by Simplicius with attempts to save the phenomena.[59]

Days and Nights II.17 and II.19 have an additional strange feature. Their *protaseis* are phrased as extensions of propositions rather than as the simple statements that one would normally expect. Many propositions, when they are extended by porisms or within the body of the proof to show that the main *protasis* holds for additional or more generalized circumstances, a signaling phrase will be added at the outset (ὁμοίως δὴ δείξομεν is most common, but ὅτι…δείξομεν οὕτως is fairly frequent, or a similar phrase with a different verb such as γνωσόμεθα or ῥησήθεται). In *Days and Nights* II.17 and 19, these signaling phrases are added to the *protaseis* themselves.[60] These phrases do not appear in any other *protasis* in Theodosius' work, and they give the impression that the final three propositions are addenda or extensions, rather than independent propositions.

The final three propositions of *Days and Nights*, though they include unusual features, nevertheless essentially follow the systematist proposition format and fit sequentially into the structure of the work as a whole. They support the conclusion of the final proposition (II.19), that if the fraction of a day by which a year exceeds 365 days is not commensurable with a revolution of the sun, the revolution will never return to its

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[59] E.g. Eudemus in Simplicius, *Commentary on Aristotle's* De Caelo II.12 (F. Wehrli, "Eudemos von Rhodos," in *Die Schule Des Aristoteles*, vol. 8 (Basel: Schwabe, 1969), 11–72, fr. 149), mentions all three of these authors together, regarding their attempts to save the phenomena.

[60] διὰ πόσων… ῥησήθεται οὕτως in II.17 and ὅτι…δείξομεν οὕτως in II.19.

original place at its original point during the year.[61] This is a natural porism of the propositions that have come before, showing that if the year contains a complete rotation of the sun, the days and nights will be equal one year to the next (II.15), and if it does not contain a complete rotation, they will not be equal (II.16). Given the unusual phrasing and methods of these final propositions, they may have been addenda or scholia that were integrated into the final text, or they may have been illustrative examples of the principles demonstrated by the earlier propositions, such as we saw in Apollonius' work on large numbers in chapter 2. Either way, they fit with the rest of the text, despite not being phrased as independent propositions, and they provide additional evidence that systematist astronomy was in some places pushing the boundaries of what could be done without appeal to observational or numerical methods.

III.2.5 Theon of Smyrna

Nearly half of Theon's *Useful mathematics for the understanding of Plato* concerns astronomy.[62] Like the rest of the text, the section on astronomy is a summary of knowledge collected by the middle of the second century CE. Theon draws heavily on the work of Adrastus of Aphrodisias, a Peripatetic philosopher, and briefly cites

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[61] That is, say we begin tracing the path of the sun around the ecliptic at, for example, one degree Aries on March 21st at exactly nine in the morning. If the year exceeds 365 days by a fraction that is not commensurable with the circumference of the ecliptic, then the sun will never again be found in Aries on March 21st at exactly nine in the morning (it may vary from that time or place by only an infinitesimal amount, but will never be exact).

[62] Theon of Smyrna, *Useful mathematics,* 120-205.

Eratosthenes, Hipparchus, Aratus, Callippus, and Aristotle, as well as “the Pythagoreans” generally.

While most of the material presented is attributed to other authors, the few passages Theon does claim as his own work are set out as systematist propositions.[63] After a long exposition of Adrastus’ work on the path of the sun,[64] Theon notes that Hipparchus wondered why the eccentric theory and the epicyclic theory produce the same path, and that Adrastus shows how the eccentric model follows in agreement with the epicyclic model.[65] He sets out to show how the epicyclic model follows the eccentric, and signals that this is his own contribution by saying ἐγώ φημι (“I say”).[66] This is the only use of ἐγώ in the book, and the only place I have found where Theon so unambiguously claims a theorem as his own. This sentence serves as the *protasis* of the following three propositions (he proves the same thing in two different ways, then proves the converse), all of which otherwise strictly observe the compound proposition format and style,

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[63] Theon of Smyrna, *Useful mathematics,* 166-172. J. Dupuis, *Théon de Smyrne; Exposition des Connaissances Mathématiques utiles pour la Lecture de Platon.* (Bruxelles: Culture et Civilisation, 1892), 275 note 2, interprets these passages as a quotation of Adrastus, but it is not clear to me why, given the abrupt shift at 166.12, ὡς δὲ ἐγώ φημι.

[64] Theon of Smyrna, *Useful mathematics*, 146-165.

[65] Theon of Smyrna, *Useful mathematics*, 166. The eccentric and epicyclic theories are two different models of the sun’s path. According to the eccentric model, the sun revolves on a circular path whose center is offset from the center of the cosmos (i.e. the earth). In the epicyclic model, the sun is carried on an epicycle, a small circle whose center is carried around the circumference of a larger circle centered on the earth. Both models account for the same phenomena. Ptolemy presents both models in the *Almagest*, III.3-5 (J. L. Heiberg, ed., *Claudii Ptolemaei Opera quae exstant Omnia* (Leipzig: Teubner, 1898), vol. 1, 216-252).

[66] Theon of Smyrna, *Useful mathematics*, 166.12-13. The full sentence is ὡς δὲ ἐγώ φημι καὶ τῇ κατὰ ἔκκεντρον ἡ κατ’ ἐπίκυκλον, “And thus I say also the epicyclic path [follows] the eccentric.”

including third person imperatives in the *ekthesis* and *kataskeuē*,[67] λέγω ὅτι to signal the *diorismos*,[68] and ὅπερ ἔδει δείξαι to signal the *symperasma*.[69] Theon uses geometrical methods and no numbers, except to place the sun's apogee at five and a half degrees Gemini,[70] a piece of information that is not otherwise used in the proofs.

Theon's own work is presented as systematist, and his outlook is consistently in accord with the systematists' idealized understanding of mathematics throughout the rest of the text. In one more example of this outlook, he shows a recurring interest throughout the astronomy section in how various theories "save the phenomena," a phrase that he uses fifteen times.[71] Though the function of the *Useful Mathematics* as a layperson's introductory handbook may have colored the manner of presentation in most places, it is clear that in astronomy as in the other areas of mathematics, Theon is working from a systematist perspective.

III.3 The Heurist School

Just as in geometry and arithmetic, heurist astronomy shows a more flexible prose style, an inclination for numerical methods, a foregrounding of instruments and measurements, and a greater tendency toward a physical conceptualization of the objects. These practices (with the exception of style) are all somewhat more natural to astronomy,

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[67] e.g. Theon of Smyrna, *Useful mathematics*, 166.14-167.3.
[68] e.g. Theon of Smyrna, *Useful mathematics*, 167.3 and 15, 168.12.
[69] e.g. Theon of Smyrna, *Useful mathematics*, 169.8.
[70] Theon of Smyrna, *Useful mathematics*, 169.14 and 171.6.
[71] Theon of Smyrna, *Useful mathematics*, 150.20, 154.12, 157,12, 158.11, 160.12, 161.3, 163.3, 164.13, 166.5, 175.1 and 15, 176.4, 177.8, 180.9, and 182.8.

and are therefore somewhat less strongly differentiated from the systematist school, though the differences persist. Where a systematist text might use numbers when reckoning days, months, or other countable quantities, for example, a heurist text will apply numerical methods even to geometrical magnitudes such as distance or size. However, the primary distinguishing feature that arose from heurist astronomy is the emphasis given to observation.

The practice of dated observation of astronomical phenomena seems to have come to Greece (probably from Babylon) in the early third century BCE.[72] We know from Ptolemy[73] that Aristarchus of Samos made observations of the solstices, as did the astronomers Callippus, Meton, and Euctemon (about whom we know very little else apart from their work on calendars[74]). Eratosthenes used observational methods to calculate the circumference of the earth, as did Archimedes and Hipparchus in calculating the length of the year.[75] Hipparchus, as we shall see below, placed a great deal of emphasis on observation, and gave it priority over theory as a source of truth. Ptolemy included not only multiple long tables of observed data, but also the designs and instructions for use of several measuring instruments. All heurist texts, in short, share a focused on observation as a fundamental piece of methodology.

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[72] Bernard R. Goldstein and Alan C. Bowen, "The Introduction of Dated Observations and Precise Measurement in Greek Astronomy," *Archive for History of Exact Sciences* 43, no. 2 (1991): 96.
[73] Ptolemy, *Almagest*, III.1 (Heiberg, vol. 1, 203-209).
[74] A description of the so-called Metonic cycle of 19 years is found in Geminus' *Introduction to the Phenomena* (G. Aujac, *Géminos. Introduction aux Phénomènes* (Paris: Les Belles Lettres, 1975), 8.50), and in Ptolemy's *Almagest* III.1 (Heiberg, vol. 1, 203-209).
[75] Ptolemy, *Almagest*, III.1 (Heiberg, vol. 1, 195).

On the other hand, the inherently physical and observational nature of astronomy does result in the occasional blurring of the boundaries between systematist and heurist methods. We have seen before that it was not unheard of for texts belonging mostly to one school to adopt certain features of the other: Nicomachus, for example, used numerical notation for his illustrative examples in the *Introduction to Arithmetic*, a text that is otherwise systematist in outlook. Archimedes, in *Spirals*, used an adaptation of the systematist proposition style to present a heurist methodology. It is rare, however, to see texts in which systematist and heurist elements are better balanced, as they are in two of the works I have included in this section, Aristarchus' *Sizes and distances of the sun and moon*, and Hypsicles' *Anaphoricus*.

Neither of these two works can be easily placed in one school or the other. Aristarchus' *Sizes and distances* is markedly systematist in style, but uses numbers, measurements, and approximation in most of the propositions. Hypsicles' *Anaphoricus* begins with three propositions of systematist arithmetic, but the rest of the text is heurist-style numerical and observational astronomy. These hybrid texts may indicate some of the lines along which the two schools came to differentiate themselves. Astronomy seems to have presented methodological challenges that may have exposed certain limitations of the systematist method, and may have prompted such authors as Aristarchus (who was clearly fluent with systematist work) and Hypsicles (whose geometrical work is unequivocally systematist) to jump fences.

Whatever the reason, there are more works of heurist astronomy than systematist, and heurist methods and style seem to have triumphed over the field in the work of Hipparchus and Ptolemy.

III.3.1 Aristarchus

Aristarchus of Samos was an older contemporary of Archimedes, working mostly in the first half of the third century BCE. He was a student of Strato of Lampsacus, Theophrastus' successor as head of the Lyceum. We know from Archimedes' *Sand Reckoner* that Aristarchus proposed a heliocentric model of the universe, but the work in which that model appeared has been lost. Aristarchus' extant work, *Sizes and distances of the sun and moon*, is included in most manuscripts of the Little Astronomy.[76] It consists of six axioms and eighteen propositions concerning the relative sizes of the sun, moon, and earth, and the distances of the sun and moon from the earth and from each other.

Aristarchus' text is reminiscent of Archimedes' early works, especially *Measurement of the Circle*. That is, the prose style and proposition formats are consistent with systematist conventions, but aside from the first two propositions, the methodology is not. For one thing, Aristarchus makes extensive use of specific numbers (including numerical notation) and numerical methods, especially estimation. For example, proposition 10 of *Sizes and distances* states that the ratio of the sun's volume to the moon's is greater than 5832 to 1 but less than 8000 to 1. Proposition 11 states that the

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[76] T. L. Heath, *Aristarchus of Samos, the Ancient Copernicus* (Oxford: Clarendon Press, 1913).

moon's diameter is greater than 1/30th but less than 2/45ths of the distance from the moon to the earth.[77]

Aristarchus' break from systematist methods consists not only in using numerical notation much more frequently than normal systematist texts, but particularly in assigning numerical values to continuous magnitudes. Theodosius, for example, uses numbers to express the days in a year—a discrete quantity, naturally countable. Aristarchus, however, uses numbers (and imprecise numbers at that) to express ratios between continuous magnitudes that are not proved to be commensurable.[78]

Aristarchus also places more emphasis on observation than the normal systematist astronomers. He uses the "eye" (ὄψις) of the observer as a point of reference in half of his propositions—far more than any systematist.[79] Similarly, in proposition 4, the entire *protasis*, that the circle dividing the dark and light sides of the moon is a great circle, is qualified by the phrase πρὸς αἴσθησιν, "according to perception." The proof rests on the condition that small enough angles are not perceptible to our eye, and the point of the proposition is that the difference between the true circle and a great circle is not

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[77] Cf. Archimedes, *Measurement of the Circle* 3, which states that the circumference of a circle to its diameter has a ratio greater than $3\frac{10}{71}$ but less than $3\frac{1}{7}$.
[78] Euclid, *Elements* X.5 shows that magnitudes have the ratio of a number to a number if the magnitudes are commensurable.
[79] Aristarchus, *Sizes and distances* 3, 4, 5, 6, 8, 9, 11, 12, 13, as well as Hypothesis 3. Euclid's *Phaenomena*, on the other hand, only invokes the eye in the prolegomena and the first proposition. Autolycus and Theophrastus do not invoke it at all. Theon uses the eye once in a mathematical proof (*Useful Mathematics* 190), but not in the propositions he claimed as his own.

perceptible. This explicit appeal to perception as the object of the proof is not seen in any systematist astronomical text at all.

As it is one of the oldest extant texts using heurist methods, it is tempting to search *Sizes and distances* for origins of the heurist school. In particular, the Peripatetic emphasis on observation and the physical world may have informed Aristarchus' choices. The retention of systematist style indicates that Aristarchus was familiar with the systematist school, and may suggest that the heurists branched off from the systematists in order to explore new mathematical methods and modes of inquiry. On the other hand, Archimedes, whose works are contemporary with *Sizes and distances*, has no certain ties to the Peripatetic school or to the Platonic academy. Some scholars have noted that Archimedes' presentation of first principles in *Sphere and Cylinder* are closer to Aristotle's conception of first principles than that in Book I of Euclid's *Elements*.[80] This connection is tenuous, however, and an origin of the heurist school in the Lyceum cannot be firmly established.

III.3.2 Archimedes

Archimedes' only astronomical work, the *Sand Reckoner*, contains his system for expressing large numbers,[81] which he developed in order to make the proposed

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[80] Henry Mendell, "Aristotle and First Principles in Greek Mathematics: Supplement to Aristotle and Mathematics," in *The Stanford Encyclopedia of Philosophy*, ed. Edward N. Zalta (Metaphysics Research Lab, Stanford University, 2019). See also T. L. Heath, *Mathematics in Aristotle* (Oxford: Clarendon Press, 1949), 56-57.
[81] See above, ch. 2.

calculations of the size of the universe and the number of grains of sand required to fill it. Primarily, the *Sand Reckoner* is an arithmetical work. Archimedes states in the proem[82] that his purpose is to show that numbers larger than any conceivable quantity can be expressed. His example of the greatest conceivable quantity is the number of grains of sand if the entire universe were filled with them, because, as he says, he wants to refute those who say that the grains of sand on earth are infinite, or at least that there is no number that could express them. This arithmetical project requires the size of the universe to be known, and so Archimedes begins with some astronomical calculations.[83] Although the astronomy is ancillary to the main project, the methods and style of presentation are heurist, as is the rest of the *Sand Reckoner*.

The work is written in continuous prose with a great deal of personal language. Archimedes contrasts himself with previous mathematicians, including Eudoxus and Aristarchus,[84] and recounts his own thought process.[85] Demonstrations are signaled simply by lines such as τούτων δὲ ὑποκειμένων δείκνυται καὶ τάδε ("these things supposed, the following are also proved"),[86] rather than by the systematist proposition format. Numbers and numerical notation are used throughout, and Archimedes appeals to observation and experiment for critical data (the angle subtended by the sun's diameter

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[82] Archimedes, "Arenarius," in *Archimède*, ed. Charles Mugler, vol. 2, 3 vols. (Paris: Les Belles Lettres, 1971), 134.15-135.2.
[83] Archimedes, "Arenarius," 135-145.
[84] Archimedes, "Arenarius," 137.1-12.
[85] E.g. Archimedes, "Arenarius," 137.14-15, αὐτὸς δὲ ἐπισκεψάμενος τόνδε τὸν τρόπον ἐπειράθην ὀργανικῶς λαβεῖν τὰν γωνίαν ("myself considering this method, I tried to get the angle experimentally")
[86] E.g. Archimedes, "Arenarius," 143.24, or 140.13-14 (Πεπιστευμένων δὲ τούτων δείκνυται…)

according to an observer on earth).[87] He seems to have designed this particular experiment himself. It uses an instrument consisting of a ruler which measures the angles at which the sun is blocked to the observer's eye or not by a cylinder attached to the end of the ruler.

Archimedes draws heavily on the work of Aristarchus in the *Sand Reckoner*, using Aristarchus' calculations on the sizes of the sun and moon from *Sizes and distances*,[88] as well as his lost heliocentric model of the universe, which gives a larger size for the diameter of the universe than any other model. Archimedes' use of Aristarchus' numbers and innovative astronomical model further reinforces the picture of Aristarchus as an early heurist.

III.3.3 Eratosthenes

No astronomical works of Eratosthenes remain, but his famous calculation of the circumference of the earth is reported by Cleomedes in his *Caelestia*,[89] an astronomical handbook of uncertain date, written some time after 50 BCE.[90] Eratosthenes drew on principles of optics and geometry, as well as experimental methods, to show that the circumference of the earth must be 250,000 stades. He used a simultaneous measurement

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[87] Archimedes, "Arenarius," 137.14-140.13.

[88] Archimedes, "Arenarius," 137.3-6, which corresponds to proposition 9 of Aristarchus, *Sizes and distances.*

[89] Cleomedes, *Caelestia (Μετέωρα)*, ed. R. Todd (Leipzig: Teubner, 1990), I.7.64-110.

[90] For more on Cleomedes' date, see R. Todd and Alan C. Bowen, *Cleomedes' Lectures on Astronomy : A Translation of The Heavens*, Hellenistic Culture and Society 42 (Berkeley: University of California Press, 2004), 2-4.

of sundial shadows in the cities of Syene and Alexandria, whose distance apart (5,000 stades) was shown by the shadows to be 1/50th part of the earth's curvature.

As with Eratosthenes' works in other fields, the style of his presentation cannot be determined. But his calculation of the earth's circumference shares with his other reported projects a focus on creative problem-solving, and the use of tools, measurements, and physical considerations. Adding to this his long professional association with Archimedes, the evidence suggests that Eratosthenes' work belonged to the heurist school.

III.3.4 Hypsicles

Hypsicles worked in Alexandria during the second century BCE. His continuation of Euclid's *Elements*, discussed in Chapter 1, was clearly written in the systematist tradition. His astronomical work, the *Anaphoricus*, is a hybrid text—not in the manner of Aristarchus' *Sizes and distances*, which combines systematist style with heurist methodology, but rather by parts. The first three propositions are systematist in both style and method, while the rest of the work is heurist in both.

The *Anaphoricus* develops a method for calculating the rising times of the signs of the zodiac according to the ratio of the longest to the shortest day in any given location.[91] It is the first text that divides the ecliptic into 360 degrees (in fact, it defines

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[91] E.g. in Alexandria, according to the text, the ratio of the longest day to the shortest is 7:5. This ratio determines the time it takes for a sector of the ecliptic to rise above the horizon.

the term “degree,” μοῖρα), and that uses sexagesimal notation. Both these practices seem to have originated in the near east.[92] Although Hypsicles’ method of calculation was superseded by the trigonometric calculation of tables of chords in subsequent decades, the *Anaphoricus* was an innovative work when it was published. It was included and preserved in the Little Astronomy.

There is a marked difference in both style and method between the first three propositions and the rest of the book. The first three propositions prove arithmetical principles about summing series of numbers, which are used in the second part of the work. Each contains the conventional parts and language of a systematist proposition, and is accompanied by a lettered diagram. The propositions use no specific numbers or numerical notation, but are entirely generalized.

After the third proposition, there are no more clear divisions in the text, which runs in continuous prose to the end of the work. Numerical notation is immediately adopted, and specific numbers are assigned to the objects under consideration. The first sentence of this section[93] divides the ecliptic into 360 spatial degrees, and the next divides the time of revolution of the ecliptic into 360 temporal degrees. In the following paragraph[94] the location of observation is posited as Alexandria, and the specific ratio of

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[92] Clemency Montelle, “The Anaphoricus of Hypsicles of Alexandria,” in *The Circulation of Astronomical Knowledge in the Ancient World*, ed. John M. Steele, Time, Astronomy, and Calendars, vol. 6, (Brill, 2016), 287–315.
[93] M. Krause and Victor De Falco, eds., “Hypsikles. Die Aufgangszeiten der Gestirne.,” in *Abhandlungen Der Akademie der Wissenschaften in Göttingen, Philol.-Hist. Kl.*, 3 62 (Berlin: Weidmannsche Buchhandlung, 1966), 34–40, line 55.
[94] “Hypsikles. Die Aufgangszeiten der Gestirne,” line 63.

the longest to the shortest day is given as 7:5, as determined by measurement with a gnomon. The rising times of parts of the ecliptic are calculated in temporal degrees using the principles of series summation proved in the first three propositions, and then the method is further applied to specific signs of the zodiac. Finally, the difference between successive rising times is calculated, and Hypsicles states that other rising times may be determined from that information.[95]

Hypsicles was obviously familiar with systematist conventions and capable of working within them, as he shows in the first part of the *Anaphoricus* and in his Book XIV of the *Elements*. He does not comment on why he chooses to abandon these conventions in the *Anaphoricus*, but the change in style does coincide with the adoption of numerical and observational astronomical practices. The latter part of the *Anaphoricus* is concerned with demonstrating a method of calculation that is implicitly transferrable to any location where the ratio between the longest and shortest days is known, following the heurist focus on developing problem-solving techniques instead of demonstrating universal principles. The coincidence of heurist style, mathematical methods, and epistemological goals in the second part of the *Anaphoricus*, especially following the obviously systematist first part, indicates that a difference between the two modes of doing mathematics was felt by ancient authors, even if the modes were not explicitly named.

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[95] "Hypsikles. Die Aufgangszeiten der Gestirne," line 160.

III.3.5 Hipparchus

During the second century BCE, Hipparchus made observations of solstices and equinoxes, solar eclipses, the motion of the moon, and the positions of stars, records of which are preserved in Ptolemy's *Almagest*, for which Hipparchus's work was a primary source. All of Hipparchus' works are lost or fragmentary except for his *Commentary on the Phaenomena of Aratus and Eudoxus*.[96] Enough remains, however, to show that Hipparchus was closely focused on measurement, observation, data analysis, and the development of effective and transferrable problem-solving methods. He was polemically critical of his systematist predecessors, but he continued and attempted to improve on the work of Aristarchus. In short, Hipparchus worked in the heurist tradition.

The *Commentary on the Phaenomena*, written in three books of continuous prose, criticizes Aratus' *Phaenomena*, a 3rd century BCE didactic poem based on the astronomical work of Eudoxus. Hipparchus claims, in an epistolary proem addressed to a layperson,[97] Aischrion, that he will lay out everything said well or badly by Aratus.[98] But in fact he is mostly critical of Aratus and Eudoxus, as well as Attalus, a previous (enthusiastic) commentator on the *Phaenomena*, especially in the first book of the *Commentary*. In the second book, Hipparchus sets out to correct Aratus' account of the

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[96] Hipparchus, *In Arati et Eudoxi Phaenomena Commentariorum Libri III*, ed. C. Manitius (Leipzig: Teubner, 1894).
[97] Aischrion's identity is not certain, but Hipparchus addresses him as though he were a layperson. See Jessica Lightfoot, "Hipparchus' Didactic Journey: Poetry, Prose, and Catalogue Form in the *Commentary on Aratus and Eudoxus*," *Greek, Roman, and Byzantine Studies* 57, no. 4 (2017): 943.
[98] Hipparchus, *In Arati et Eudoxi* I.1.2: πᾶν καθόλου τὸ καλῶς ἢ κακῶς λεγόμενον

risings and settings of the northern constellations of the zodiac. In the third book, he abandons Aratus altogether[99] and simply provides a catalogue of the risings and settings of southern zodiac constellations, according to observation.

Hipparchus' criticism of Eudoxus' and Aratus' work is based primarily on inconsistencies between their claims and the observed phenomena. In fact, in the proem to Book I, he rebukes Attalus, the commentator on the *Phaenomena* whom he deemed "most accurate,"[100] for attempting to reconcile the observed phenomena with Aratus' theory when addressing inaccuracies in the text, rather than admitting that the theory was incorrect.[101] Hipparchus positions himself in contrast to Attalus as a commentator who will point out the inaccuracies in the *Phaenomena*, and will rely on observation alone as the measure of correctness. In Book II, he speaks sarcastically of his predecessors' attempts to "save the phenomena."[102]

The rest of Hipparchus' work, though fragmentary, all appears to have been equally focused on observation, numerical methods, and problem-solving. He is credited by Theon of Alexandria, for example, with producing a table of chords (varyingly precise numerical values for the lengths of lines subtended by different angles within a circle—an early form of trigonometry and a useful tool for finding numerical solutions to

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[99] Aratus' name is not even mentioned in Book III of the *Commentary*.
[100] Hipparchus, *In Arati et Eudoxi* I.1.3: ἐπιμελέστατα.
[101] Hipparchus, *In Arati et Eudoxi* I.3.2-4. For a discussion of these passages in the context of Hellenistic scholarship more broadly, see Lightfoot, "Hipparchus' Didactic Journey," 948-950.
[102] Hipparchus, *In Arati et Eudoxi* II.3.23 and II.3.28.

geometrical problems).[103] Even if the attribution is false, the analysis of lunar motion attributed to Hipparchus by Ptolemy[104] would have required the use of the same numerical methods.[105] Again, both Ptolemy and Theon of Smyrna recount that Hipparchus refused to attempt a model of planetary motion, because he did not possess sufficient observational data.[106]

Finally, Hipparchus' relationship to his predecessors reinforces the impression that authors of heurist texts behaved as a group positioned against the systematists. Such a claim could be made merely on the grounds of his treatment of Eudoxus, Aratus, and Attalus in the *Commentary*, but in addition, Hipparchus seems to have made extensive use of the work of heurist astronomers, and little to none of systematist works. Hipparchus in the *Commentary* uses the Babylonian system that we first saw in Hypsicles (including the division of the ecliptic into 360 degrees, and the use of the word μοῖρα, "degree"), even though it is clear from the text that Aratus and Eudoxus did not use any Babylonian material.[107] Some scholars have argued that Hipparchus in his work on the precession of solstices and equinoxes made use of the observational data of Timocharis and Aristyllus (two astronomers mentioned frequently by Ptolemy), whose data collections may have been part of a concerted effort to correct and apply measurement

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[103] There is a debate as to whether Theon's attribution is believable. See G. J. Toomer, "The Chord Table of Hipparchus and the Early History of Greek Trigonometry," *Centaurus* 18, no. 1 (1974): 6–28.
[104] Ptolemy, *Almagest*, IV.11 (Heiberg, vol. 1, 338-348).
[105] For specifics on these numerical methods, see Dennis W. Duke, "Hipparchus' Eclipse Trios and Early Trigonometry," *Centaurus* 47, no. 2 (2005): 163–177.
[106] Ptolemy, *Almagest,* IX.2 (Heiberg, vol. 2, 210). Theon of Smyrna, *Useful Mathematics*, 188.
[107] Bowen and Goldstein, "Hipparchus' Treatment of Early Greek Astronomy," 245.

and number to the merely verbal accounts of the phenomena made by Eudoxus, Aratus, and others.[108] In a fragment of Hipparchus' work on the precession of solstices and equinoxes, quoted by Ptolemy, he says that he and Archimedes both made calculations of the length of the year, which were both off by up to a quarter of a day.[109] Finally, Hipparchus seems to have made use of the work of Aristarchus and Eratosthenes. He wrote a work *Sizes and distances of the sun and moon* in which he significantly increased Aristarchus' value for the size of the sun, but stayed within his range for the size of the moon.[110] According to Ptolemy, he also used Aristarchus' observation of a summer solstice in his work *On the length of the year*, in which he noted the precession of solstices and equinoxes.[111] It is noteworthy that Ptolemy refers in this passage (and one other) to a group of scholars associated with Aristarchus, οἱ περὶ Ἀρίσταρχον.[112] However, the observations of Callippus, Meton, and Euctemon (and the scholars around them) are also listed as sources for Hipparchus, and Ptolemy notes that Hipparchus considered the observational methods of all these predecessors to be in need of improvement.[113] Also according to Ptolemy, Hipparchus agreed with Eratosthenes' ratio

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[108] Goldstein and Bowen, "The Introduction of Dated Observations and Precise Measurement in Greek Astronomy," 118.
[109] Ptolemy, *Almagest*, III.1 (Heiberg, vol. 1, 194-195).
[110] Theon of Smyrna, *Useful Mathematics*, 197, reports Hipparchus' numbers as: the sun is 1880 times the size of the earth (by volume), and the earth is 27 times greater than the moon. Aristarchus' values, by contrast, are: the sun is between 254 and 368 times the size of the earth, and the earth is between 15.8 and 31.5 times the size of the moon.
[111] Ptolemy, *Almagest*, III.1 (Heiberg, vol. 1, 206-207).
[112] Ptolemy, *Almagest*, III.1 (Heiberg, vol. 1, 203 and 206).
[113] Ptolemy, *Almagest*, III.1 (Heiberg, vol. 1, 203 and 205). Since no work of Meton and Euctemon has survived, it is impossible to assign them a secure place in either school. They are used as a source by both Theodosius and Hipparchus (according to Ptolemy), so their associations are not clear.

for the arc of the ecliptic between solstices,[114] and some scholars argue that Hipparchus adopted Eratosthenes' position for the meridian that passes through Alexandria.[115] Although most of these associations are fairly slight, they are suggestive of a larger picture of Hipparchus' place in heurist astronomy, especially in the absence of any clear associations with the systematist school (other than to criticize them).

III.3.6 Ptolemy

The *Almagest*, or *Syntaxis*,[116] was written between 138 and 161 CE, and is undoubtedly the most important astronomical text of the ancient world. Ptolemy's model of the cosmos did not even begin to be superseded until the writings of Copernicus, and even then it was many more years before the heliocentric model could rival Ptolemy's in accuracy.[117] Pappus and Theon of Alexandria both wrote commentaries on the *Almagest,* which are partially extant.[118]

Ptolemy addresses the *Almagest* to a Syrus of whom nothing else is known. The work contains thirteen books, covering astronomical theory from the mathematical

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[114] Ptolemy, *Almagest,* I.12 (Heiberg, vol. 1, 67-68). For further discussion of this point, including arguments to the contrary, see G. J. Toomer, *Ptolemy's Almagest* (New York: Springer-Verlag, 1984), 63 note 75.
[115] This argument, however, is highly theoretical. See Toomer, *Ptolemy's Almagest*, 225 note 16.
[116] The Greek name of the text is Μαθηματικὴ Σύνταξις. The title *Almagest* is derived from the Arabic translations by which the text was preserved through the middle ages. For a brief discussion of the history of the title, see Toomer, *Ptolemy's Almagest*, 2. For a more thorough discussion, and a history of the text in the Syriac and Arabic traditions, see Paul Kunitzsch, *Der Almagest: die Syntaxis mathematica des Claudius Ptolemäus in arab.-latein. Uberlieferung* (Wiesbaden: Harrassowitz, 1974).
[117] It was Kepler's introduction of elliptical orbits that finally brought the heliocentric model better in line with the phenomena.
[118] A. Rome, *Commentaires de Pappus et de Théon d'Alexandrie sur l'Almageste*, vol. 1, 2, Studi e Testi 54, 72 (Città del Vaticano: Biblioteca Apostolica Vaticana, 1931).

preliminaries, through the motion of the sun and moon, to the positions and motions of the fixed stars and the planets. It ends, after a table of the first and last visibilities of the planets, with a very brief epilogue (ten lines), also addressed to Syrus.[119]

Although Ptolemy includes several propositions in the systematist style, the *Almagest* is primarily a heurist text in style, method, and outlook. Some scholars have argued that Ptolemy was trying to synthesize or at least reconcile the cosmological views and epistemological categories of Aristotle and Plato,[120] which, if true, may lie behind Ptolemy's limited inclusiveness of systematist content. In the following chapter on music, we shall see more marked attempts by heurist authors to reconcile Neopythagorean and Peripatetic ideas.

The style of the *Almagest* is consistently heurist: continuous prose that directly addresses the concerns of useful, efficient problem-solving, the role of instruments and observation, and the assessment of results through some sort of "check."[121] Where Ptolemy incorporates systematist propositions, they are embedded within heurist methodology, and are not differentiated from the rest of the text except by their style. For example, in *Almagest* I.10,[122] Ptolemy presents six short propositions that set up his calculation of the table of chords in I.11. These propositions are done mostly in the

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[119] Ptolemy, *Almagest,* XIII.11 (Heiberg, vol. 2, 608).
[120] E.g. Jacqueline Feke, "Ptolemy's Defense of Theoretical Philosophy," *Apeiron* 45, no. 1 (January 1, 2012): 61–90. See also Toomer, *Ptolemy's Almagest*, 35-38 for some discussion in the footnotes of Ptolemy's philosophical perspective, which Toomer takes as essentially Aristotelian.
[121] Checking results as a means of verification (as opposed to proof) is a common heurist practice as seen, for example, in Heron's *Metrica* and Diophantus' *Arithmetica*.
[122] Ptolemy, *Almagest*, I.10 (Heiberg, vol. 1, 32-45).

systematist style, except that none of them includes a *protasis* or *symperasma*. Within each proposition, between the *ekthesis* and the conclusion of the *apodeixis*, numbers and other heurist methods are avoided, and systematist language is used (such as λέγω ὅτι in the *diorismos* and, in one instance,[123] ὅπερ ἔδει δεῖξαι after the *apodeixis*). However, these propositions are interspersed with numerical calculations in the looser heurist style, and on the whole are designed to prove the validity of Ptolemy's method of calculating chords. They serve a heurist project: a numerical calculation of a geometrical quantity (the chord) to within a small margin of error. The passage introducing these propositions[124] makes Ptolemy's priorities very clear:

"So, for immediate use we will next make an ordered table of their [the chords'] length, first dividing the circumference into 360 cuts, then laying out the straight lines extending under the circumference by increments of half a part: that is, they are of a certain number of cuts just as when the diameter is divided into 120 cuts, according to what will appear from these calculations to be useful for the arithmetic. First we will

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[123] Ptolemy, *Almagest*, I.10 (Heiberg, vol. 1, 37).

[124] Ptolemy, *Almagest*, I.10 (Heiberg, vol. 1, 31-32): Πρὸς μὲν οὖν τὴν ἐξ ἑτοίμου χρῆσιν κανονικήν τινα μετὰ ταῦτα ἔκθεσιν ποιησόμεθα τῆς πηλικότητος αὐτῶν τὴν μὲν περίμετρον εἰς τξ τμήματα διελόντες, παρατιθέντες δὲ τὰς ὑπὸ τὰς καθ' ἡμιμοίριον παραυξήσεις τῶν περιφερειῶν ὑποτεινομένας εὐθείας, τουτέστι πόσων εἰσὶν τμημάτων ὡς τῆς διαμέτρου διὰ τὸ ἐξ αὐτῶν τῶν ἐπιλογισμῶν φανησόμενον ἐν τοῖς ἀριθμοῖς εὔχρηστον εἰς ρκ τμήματα διῃρημένης. πρότερον δὲ δείξομεν, πῶς ἂν ὡς ἔνι μάλιστα δι' ὀλίγων καὶ τῶν αὐτῶν θεωρημάτων εὐμεθόδευτον καὶ ταχεῖαν τὴν ἐπιβολὴν τὴν πρὸς τὰς πηλικότητας αὐτῶν ποιοίμεθα, ὅπως μὴ μόνον ἐκτεθειμένα τὰ μεγέθη τῶν εὐθειῶν ἔχωμεν ἀνεπιστάτως, ἀλλὰ καὶ διὰ τῆς ἐκ τῶν γραμμῶν μεθοδικῆς αὐτῶν συστάσεως τὸν ἔλεγχον ἐξ εὐχεροῦς μεταχειριζώμεθα. καθόλου μέντοι χρησόμεθα ταῖς τῶν ἀριθμῶν ἐφόδοις κατὰ τὸν τῆς ἑξηκοντάδος τρόπον διὰ τὸ δύσχρηστον τῶν μοριασμῶν ἔτι τε τοῖς πολυπλασιασμοῖς καὶ μερισμοῖς ἀκολουθήσομεν τοῦ συνεγγίζοντος ἀεὶ καταστοχαζόμενοι, καὶ καθ' ὅσον ἂν τὸ παραλειπόμενον μηδενὶ ἀξιολόγῳ διαφέρῃ τοῦ πρὸς αἴσθησιν ἀκριβοῦς.

show how we might make an efficient and swift apprehension of their sizes through the same theorems, as few as possible, so that not only may we have the sizes of the straight lines laid out unchecked, but also we may manage the checking of them readily through the methodical confirmation of them by geometry. And generally, we shall use the numerical methods according to the sexagesimal way, because of the difficulty of use of the fractions,[125] and we shall follow the multiplications and divisions always aiming at approximation, so long as the remainder does not differ in any noteworthy amount from the accuracy of perception."

The emphasis, here as in the rest of the *Almagest*, is on an efficient and iterative problem-solving method for a numerical calculation in which estimation is acceptable and results are checked against observation. This passage places all the systematist-style propositions in the service of a heurist methodology and outlook. The result of this chapter is the production of the table of chords,[126] a reference tool for streamlining the calculation of other values later on, such as the distances between celestial bodies.

In addition to the emphasis on efficient numerical problem-solving, the *Almagest* shows a characteristically heurist interest in instruments and observations. Ptolemy includes instructions for the designs of four different tools, two for observing the distance from the equator to a solstitial point,[127] a spherical astrolabe or armillary sphere,[128] and a

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[125] That is, the traditional system of fractions as opposed to the sexagesimal system of degrees.
[126] Ptolemy, *Almagest*, I.11 (Heiberg, vol. 1, 48-63).
[127] Ptolemy, *Almagest*, I.12 (Heiberg, vol. 1, 64-67).
[128] Ptolemy, *Almagest*, V.1 (Heiberg, vol. 1, 351-354).

tool for observing the moon's parallax;[129] and he discusses many other instruments, including their strengths and weaknesses as means of verification.[130] The *Almagest* is full of tables of data, including both observations of phenomena[131] and predictions which may be confirmed by observation.[132] Observation is taken as an authoritative confirmation of theory throughout the *Almagest*, and is at least as important as a sound theory for determining truth value. For example, in *Almagest* III.3, Ptolemy discusses the apparent deviation of the sun from uniform circular motion.[133] The observed deviation does not call the theory of uniform circular motion into question, because it can be explained within the theory by either an epicyclic or an eccentric model of the sun's orbit around the center of the universe. However, since both models explain the phenomenon equally well, Ptolemy does not give preference to either of them. That is, so long as the phenomena are accounted for within the terms of uniform circular motion, the specific theoretical model of the sun's orbit does not matter.

In one respect, the *Almagest* may be thought to resemble a systematist text: that is, in the cumulative structure and exhaustive treatment of the material. However, although the first book does lay the theoretical and methodological groundwork for what comes after, the ordering of the material is geared toward usefulness for the student at least as

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[129] Ptolemy, *Almagest*, V.12 (Heiberg, vol. 1, 403-406).
[130] E.g. Ptolemy, *Almagest*, V.14 (Heiberg, vol. 1, 417), in which he describes a diopter used by Hipparchus, and discusses the inaccuracies of the device and how he himself circumvented them.
[131] E.g. Ptolemy, *Almagest*, VII.4 (Heiberg, vol. 2, 38-169), the table of constellations in the northern hemisphere.
[132] E.g. Ptolemy, *Almagest*, XIII.4 (Heiberg, vol. 2, 582-586), the planetary latitude tables.
[133] Ptolemy, *Almagest*, III.3 (Heiberg, vol. 1, 216-217).

much as toward logical sequence. In fact, in the programmatic chapter of the introduction,[134] Ptolemy emphasizes both the ease for the student and the priority of observation (rather than theoretical first principles) as a foundation for the work. Additionally, although most parts of later books do depend on earlier materials, there are occasional instances where Ptolemy uses information that he has not yet proved or established.[135] Finally, Ptolemy admits that his treatment of astronomy is not totally exhaustive from first principles to final results. He says in the introduction that he is writing for students who have already progressed in the field, and will therefore not go into detail about material that his predecessors have treated sufficiently.[136] And he claims in the epilogue to have dealt with "almost" (σχεδόν) all of what ought to be considered.[137] On the other hand, Ptolemy also says that he is making a conscious effort to be as complete as possible, but merely wishes to avoid unmanageable length.[138] In these respects, then, the *Almagest* shares certain features of most systematist work.

The final aspect of the *Almagest* to consider is its relation to predecessors in both schools. First, the systematists: Ptolemy mentions Apollonius of Perga twice, when he presents two lemmas concerning the anomaly of the sun that will be useful in dealing with planetary retrogradation.[139] He attributes these lemmas to Apollonius, and presents

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[134] Ptolemy, *Almagest,* I.2 (Heiberg, vol. 1, 8-9).
[135] See Toomer, *Ptolemy's Almagest*, 5-6 for an overview and a couple of examples.
[136] Ptolemy, *Almagest,* I.1 (Heiberg, vol. 1, 8).
[137] Ptolemy, *Almagest*, XIII.11 (Heiberg, vol. 2, 608).
[138] Ptolemy, *Almagest,* I.1 (Heiberg, vol. 1, 8).
[139] Ptolemy, *Almagest*, XII.1 (Heiberg, vol. 2, 450-464).

them “in summary” (ἐξ ἐπιδρομῆς), but in systematist style (though again without *protasis* or *symperasma*).[140] Aristarchus, whom I have considered systematist in style but heurist in substance, is mentioned three times, twice in reference to his followers (οἱ περὶ Ἀρίσταρχον), all three times in reference to the observation of the solstices.[141] Among the heurists, Eratosthenes is mentioned once, when Ptolemy says that he and Hipparchus (and Ptolemy himself) derived the same ratio for the arc between solstices to the meridian.[142] Hipparchus is by far Ptolemy’s most commonly used source, mentioned by name ninety-two times in the *Almagest*. In general, Ptolemy relies on Hipparchus’ observations and calculations, and rarely has anything critical to say of him. For example, Ptolemy defends Hipparchus for not writing a theory of planetary motion in addition to his catalogue of observations, because he did not have sufficient data to do so.[143] Ptolemy calls him a “great lover of truth” (φιλαληθέστατον) and praises his reticence in this matter as a form of intellectual integrity, though he admits that it is a deficiency in Hipparchus’ work. Connected to Hipparchus are the astronomers frequently used by him, Callippus, Timocharis and Aristyllus, all of whom Ptolemy mentions frequently as sources of observations and calculations. Finally, Archimedes is mentioned twice, once in connection with the length of the solstices (where Ptolemy quotes Hipparchus claiming

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[140] In this case, “in summary” apparently means that Ptolemy is combining two lemmas into one with a common diagram. Neugebauer believed that this combination was the work of Apollonius himself (Neugebauer, *A History of Ancient Mathematical Astronomy*, 264), but Toomer argues that it was Ptolemy’s own invention (Toomer, *Ptolemy’s Almagest*, 556 note 3).
[141] Ptolemy, *Almagest*, III.1 (Heiberg, vol. 1, 203 and 206).
[142] Ptolemy *Almagest*, I.12 (Heiberg, vol. 1, 68). The ratio is 11:83.
[143] Ptolemy, *Almagest*, IX.2 (Heiberg, vol. 2, 210-211).

that he and Archimedes similarly erred in calculation by about a fourth of a day),[144] and once in the calculation of the size of the moon, when he invokes Archimedes as an authority in the matter, justifying his own assumption of ratio of the moon's circumference to diameter, because it falls within Archimedes' lower and upper bounds in *Measurement of the Circle*.[145] Aside from the ambiguous figures of Meton and Euctemon,[146] and a "Theon" whose identity is uncertain,[147] I have found no other astronomers mentioned. The credited sources of the *Almagest* come overwhelmingly from the heurist tradition.

Despite the incorporation of a certain amount of systematist content, then, the *Almagest* is a heurist text in all fundamentals. However, the inclusion of systematist-style propositions and the concern for cumulative structure and an exhaustive treatment of the material could indicate that Ptolemy was consciously attempting to cross boundaries or synthesize the two traditions.

III.4 Additional Remarks

The development of Greek astronomy shows that a clear division between the schools remained even in an inherently physical and observational discipline. The systematist texts still found ways to de-emphasize the physical aspects of the study (such as measurement and instruments), and to privilege a theory of regular motion over the

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[144] Ptolemy, *Almagest*, III.1 (Heiberg, vol. 1, 195).
[145] Ptolemy, *Almagest*, VI.7 (Heiberg, vol. 1, 512-513).
[146] Ptolemy, *Almagest*, III.1 (Heiberg, vol. 1, 203-207).
[147] See Toomer, *Ptolemy's Almagest*, 456 note 83 for more on this Theon.

irregularities revealed by measurement-based observation, while the heurists fully embraced the possibilities of tools, numerical methods, etc. As we saw from the early hybrid texts of Aristarchus and Hypsicles, it is possible that the methods of heurist astronomy drew authors away from working in systematist tradition. The heurist school seems to have flourished in this field of study, and its methods had become the norm by the time of Ptolemy. Astronomy, then, is the only mathematical discipline in which posterity has unequivocally favored the heurist school.

IV. MUSIC

The mathematics of music differ significantly in two ways from other mathematical disciplines. First, music is tied more closely to empirical phenomena than geometry or arithmetic. Numbers and lines can be defined without invoking physical perceptions, but pitches and intervals are inherently material. Even astronomy is less physical than music, because the observable phenomena are exclusively visible and occur in patterns that can be closely modeled by geometrical figures. Music, however, is auditory, not necessarily patterned, and does not lend itself to visual modeling. Math about music is an attempt at a description; it cannot be deduced from first principles by means of pure logic. For example, in the absence of the wave mechanics on which modern acoustic theory is based, there is no definition of pitch from which the 2:1 ratio of the octave can be derived.[1] Instead of being derived, the ratio must be shown through sense perception, such as a string divided and plucked. The empirical nature of music clashes somewhat with the preference of the systematist school for idealized, non-

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[1] Wave mechanics understands pitch as a function of frequency, which makes the mathematics consistent across different physical means of producing sound. That is, the musical ratios of the Pythagoreans can only be visually shown on strings of uniform thickness and tension, because the frequency of the transverse wave on the string has an uncomplicated relationship to the frequency of the compression wave heard by the ear. Nicomachus' story of striking anvils with hammers whose weights were in the musical ratios would not produce the associated intervals. Wave mechanics can mathematically account for this difference and show that the ratios of the frequencies of a note and its octave are 2:1 in both cases. Nevertheless, even wave mechanics is essentially descriptive with regard to pitch as it is heard. The auditory perception of frequency as "high" or "low", "consonant" or "dissonant", cannot be mathematically accounted for.

physical mathematics, and their practices seem to have been affected by that tension, as we shall see.

The second difference is related to the first: a mathematical theory is not necessarily implied in either the practice or the theory of music. One cannot do geometry without doing math, but one can "do" music without doing math. One can even do music theory without doing math, as we shall see from the school of Aristoxenus. The evidence for Greek musical practice shows that terminology, notation, and rules of composition were already well-established before the development of a mathematical description.[2] The history of Greek mathematical music theory is therefore more complicated than the history of geometry or arithmetic, because mathematics is only a part of the study of music, and a late part at that. The practices of the systematist and heurist schools had to map not only onto pre-existing musical theories, but onto pre-existing schools of musical theory: those of Pythagoras and Aristoxenus. In this chapter I will show that, while the early Pythagoreans (whose description in the testimonia strongly resembles the systematist school, to which they partly gave rise) were the first to mathematize music theory, their initial opposition was not from a rival mathematical school but from the mostly non-mathematical peripatetic school of Aristoxenus. When the heurist school took up mathematical music, they seem to have undertaken to reconcile the Pythagorean and Aristoxenian theories.

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[2] Jacques Chailley, *La notation archaïque grecque d'après Aristide Quintilien* (Paris: Les belles lettres, 1973), 32-34.

IV.1 Some Basics of Greek Music

From the evidence of ancient theorists, and of papyri and inscriptions containing musical texts, scholars have reconstructed the musical structures used by ancient composers and musicians.[3] Greek musical scales were constructed from tetrachords, sets of four notes spanning the interval of a fourth. While the outer notes were fixed a fourth apart, there were three genera of tetrachords according to the arrangement of the interior intervals: diatonic, chromatic, and enharmonic. The diatonic tetrachord, comprised of two tones and a semitone, is still in use as a normal part of the scale in Western music. The chromatic tetrachord was two semitones and an interval of a tone and a half. The enharmonic tetrachord included a *pyknon* (a series of three closely packed notes at the bottom) of approximately two quarter tones, and a full ditone. The intervals within the tetrachord were not precisely determined. They could vary according to the tuning of the instrument or the preference of the theorist. Essentially, the genera were classified according to how closely the first three notes were packed together, from diatonic (most spread out) to enharmonic (most closely packed).

Two successive tetrachords, either conjunct (when the final note of one tetrachord is the initial note of the other) or disjunct (when the tetrachords are separated by some

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[3] For thorough overviews, see Stefan Hagel, *Ancient Greek Music: A New Technical History* (Cambridge: Cambridge University Press, 2010), 1-96, and M. L. West, *Ancient Greek Music* (Oxford; New York: Clarendon Press, 1992), 160-217.

interval), made up a scale.[4] A scale spanning an octave contained two tetrachords and a single tone, but scales did not always span a full octave. Some did not even span a full seventh – that is, they consisted not of two tetrachords but of one tetrachord plus one or two additional notes.[5] The span of a scale is usually called its ambitus.

The combination of the genera of the tetrachords, the ambitus of a scale, and its starting pitch constitute a mode. The modes had different names, such as Lydian, Phrygian, Dorian, etc.[6] While the exact pitches of each mode are not totally certain (and were not strictly fixed as keys are in modern Western music[7]), it is clear that some were consistently lower (Dorian, Phrygian) or higher (Lydian) than others. Modes were often associated with certain feelings or moral qualities, such as solemnity (Dorian),[8] piety (Phrygian),[9] or effeminacy (Lydian).[10] Because of these associations, there is a tradition in the Pythagorean school of using music as moral therapy.[11]

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[4] Nicomachus claims (*Manual of Harmonics* 5, in Carl Jan, *Musici Scriptores Graeci* (Berlin; New York: De Gruyter, 1995), 236-265.) that tetrachords were normally conjunct until Pythagoras disjoined them. Tetrachords disjoined by a tone span an octave, which is an important interval in Pythagorean music theory because its ratio is 2:1, the primary consonance (more on this in the section on Pythagorean theory below).
[5] For more on these irregular scales, see the section on Damonian scales in West, *Ancient Greek Music,* 174-177.
[6] For a catalogue of modes, see West, *Ancient Greek Music*, 177-184.
[7] West, *Ancient Greek Music*, 172-174.
[8] E.g. Pindar fr. 67, Plato *Republic* 399a-c.
[9] Plato, *Republic* 399a-c.
[10] Plato *Republic* 398e.
[11] E.g. Iamblichus *Life of Pythagoras* 110-112, Aristides Quintilianus *De Musica* 2.19, Plutarch *Isis and Osiris* 384a, etc.

The primary problems for music theorists were the division of the tetrachord, the mathematical determination of various intervals, and the classification of intervals as consonant or dissonant.

IV.2 Pythagoreans and Aristoxenians

The earliest testimony attributing the invention of mathematical music theory to Pythagoras comes from Nicomachus' *Manual of Harmonics*.[12] In the story, Pythagoras was considering whether there could be some rational aid to the sense of hearing, as the ruler and compass are an aid to sight or the balance to touch. He heard anvils being struck at a smithy, recognized musical intervals in the sounds, and found that the intervals were caused by the ratios of the hammers' weights. Although this story cannot be factually true (weight would not account for the intervals[13]), evidence does suggest that the mathematical treatment of music can be traced to the early Pythagoreans.[14] Fragments of early Pythagorean harmonics remain from Philolaus and Archytas, as well as testimonies from Plato, Aristotle, Aristoxenus, and Ptolemy.[15]

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[12] Nicomachus, *Manual of Harmonics,* 6.1-27. We have other testimonies from Porphyry, *Commentary on Ptolemy's Harmonics*, 30.1-5, in Ingemar Düring, *Porphyrios Kommentar zur Harmonielehre des Ptolemaios*, vol. 2, Göteborgs Högskolas Årsskrift, XXXVIII (Göteborg: Elanders boktryckeri aktiebolag, 1932). Plato tells a similar story of the ratios of musical intervals being discovered by Glaucus through striking discs of different thicknesses (*Phaedo* 108d4). He says the discs were designed by Hippasus, whom Theon of Smyrna identifies as a Pythagorean in his account (*Useful Mathematics,* 59.4-21).

[13] Increasing the weight of the hammer would only increase the volume of the sound, not the pitch. The size of the anvil would affect the pitch, but even assuming that the anvils were perfectly cubical and of uniform density, the ratio of the sizes would not directly correspond to the ratio of the interval.

[14] For a survey of evidence, including arguments contra, see A. Barker, *Greek Musical Writings*, vol. 2, 2 vols., Cambridge Readings in the Literature of Music (Cambridge; New York : Cambridge University Press, 1984-1989., 1989), 28-29, with a survey of early Pythagorean texts, 30-44.

[15] Barker, *Greek Musical Writings*, vol. 2, 30-44, contains translations of all these fragments.

The premise of Pythagorean mathematical music theory was that musical intervals could be expressed as ratios of positive integers. The numbers of a ratio corresponded to the lengths of the strings on which the respective notes were played (this only works accurately on strings of uniform thickness[16]). Ratios were classified by type, as follows[17]:

1. Multiple: nx:x, e.g. 6:2
2. Submultiple: x/n:x, e.g. 2:6
3. Superparticular (epimoric): (x+1):x, e.g. 6:5
4. Superpartient (epimeric): (x+x/n):x, e.g. 8:6

Additional variations are combinations of these terms, such as multiple superpartient (mx+x/n), etc.[18] The Pythagoreans asserted that only multiple and superparticular ratios could represent consonant intervals.[19] The consonant musical intervals, expressed in ratios, were:

1. Octave: 2:1
2. Fifth: 3:2

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[16] Some ancient theorists tried to account for the inability of auloi or other types of instruments to accurately reproduce the ratios shown on strings, e.g. Ptolemy, *Harmonics*, I.8, in Ingemar Düring, *Die Harmonielehre des Klaudios Ptolemaios* (Göteborg: Elanders, 1930); others simply denied the inability, e.g. Theon of Smyrna, *Useful Mathematics* 61.1-11.

[17] I have given the names of the ratios as they have been translated into English through the Latin tradition, as well as the Greek variants on superparticular and superpartient. I have also included a shorthand expression of each ratio in modern algebraic notation, for the reader's convenience. The modern notation does not reflect Greek practice. While the ratios were named by Greek theorists, their generalized forms were usually given either in words (e.g. a multiple ratio is the ratio of any number to a multiple of that number) or by examples (e.g. 6:2 is a multiple ratio because six is a multiple of two).

[18] For more on the Pythagorean classification of ratios, see Nicomachus, *Introduction to Arithmetic* I.17-23.

[19] Porphyry gives a thorough overview of the Pythagorean theory of ratios in his *Commentary on Ptolemy's Harmonics*, 107.15-108.21.

3. Fourth: 4:3

The consonance and identifying ratios of these intervals were undisputed in mathematical music theory. The whole tone, with a ratio of 9:8, was not considered a consonance despite having a superparticular ratio.[20] Likewise, the semitone and quarter tone, though they were the subject of much debate with regard to their ratios, were never considered consonant.[21]

The three consonant intervals, the octave, fourth, and fifth, became central not only to Pythagorean musical theory, but to Pythagorean mathematics. A fifth and a fourth played successively span an octave, which can be shown mathematically through the combination of their ratios ($4:3 \times 3:2 = 4:2 = 2:1$).[22] The three ratios make up the *tetractys*, a Pythagorean name for the numbers one through four expressed as the triangular number 10.[23] It is arranged in four rows of one, two, three, and four dots respectively, producing an equilateral triangle with sides of four dots each. Each row is in a musical ratio to any row adjacent to it. A great deal of Pythagorean mysticism concerned the *tetractys*.[24]

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[20] Theon of Smyrna classifies the tone as an origin of a concord, but not a concord itself. Theon of Smyrna, *Useful mathematics,* 49.2-5.
[21] For a review of the debate on the ratios of semitones and smaller intervals, see Aristides Quintilianus, *De Musica Libri Tres* III.1, ed. Winnington-Ingram (Leipzig: Teubner, 1963), 94-96.
[22] The combination of ratios is the same as the multiplication of fractions.
[23] See the section on Diophantus' *On Polygonal Numbers* in ch. 2.
[24] The most famous text on the subject is the *Theologoumena Arithmeticae*; see also A. Chaignet, *Pythagore et la Philosophie Pythagoricienne.*, vol. 2 (Paris: Academie des sciences morale et politiques, 1873), 96-117.

While few fragments remain of early Pythagorean writings, those that do show a clear affinity with the systematist school.[25] However, the work of harmonics that is most clearly systematist in style as well as substance, the pseudo-Euclidean *Division of the Canon*, cannot be certainly identified as Pythagorean.[26] In general, the relationship between the Pythagorean school and systematist mathematics is complicated by the imperfect integration of Pythagorean ideas into the Platonist tradition with which most authors of systematist texts are associated.

The Pythagorean music theorists were often accused by other authors of preferring the mathematical perfection of their theory to the evidence of the senses and of musical practice. For example, Ptolemy criticizes Archytas for dividing tetrachords into intervals using superparticular ratios that do not accord with sense perception.[27] Among early writers, the main critics of the Pythagoreans were associated with the early Peripatetic school: Aristoxenus, whose work is best preserved (discussed further below), and Theophrastus, who succeeded Aristotle as head of the Lyceum. Theophrastus wrote a work *On Music*, a long fragment of which is preserved in Porphyry.[28] In the fragment, he

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[25] Philolaus, Fr. 6 (Stobaeus, *Anthologium*, ed. O. Hense and C. Wachsmuth (Leipzig: Teubner, 1884), I.21 7d) relates the principle of harmony to the construction of the universe out of limiting and unlimited elements, before defining a list of musical intervals in terms of ratios. Porphyry, in his *Commentary on Ptolemy's Harmonics* (107.15-108-21), describes the process by which some early Pythagoreans compared ratios in order to determine which was more consonant. The process involves reducing ratios to the lowest possible terms, called *pythmenes*. This is the same process and terminology used by Apollonius in his system for expressing and multiplying large numbers (see chapter 2). A fragment of Archytas (Porphyry, *Commentary on Ptolemy's Harmonics* 93.6-17) begins with definitions of the arithmetic, geometric, and harmonic means.

[26] See the section below on systematist harmonics for more details.

[27] Ptolemy, *Harmonics* I.13.

[28] Porphyry, *Commentary on Ptolemy's Harmonics*, 61.23-65.15

argues stringently against any numerical conception of music.[29] Aristotle himself, however, recognizes not only that mathematical harmonics and harmonics based only on hearing are different, but that they each have shortcomings.[30] In fact, in several passages he seems to have accepted the idea of intervals as ratios,[31] and to accept the Pythagorean idea that number is instantiated in all physical things.[32] The primary opposition to Pythagorean harmonics, then, seems to have been Aristoxenus and the later Peripatetic school rather than Aristotle himself.

Aristoxenus' two extant works, the *Elements of Rhythm* and the *Elements of Harmonics*, both treat aspects of music theory, but only the *Elements of Harmonics* is relevant to mathematical music.[33] It is a theoretical treatise in three books, the last of which contains some example problems showing the usefulness of Aristoxenus' theory.

The *Elements of Harmonics* incorporates little to no mathematics. Some scholars have characterized Aristoxenus' approach as geometrical, because musical intervals are described in vaguely spatial terms, but his theory does not use any geometrical methods or principles. The term διάστημα, "interval", is universal in music theory, as is the

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[29] For a concise summary of Theophrastus' argument, see A. Barker, "Music and Mathematics: Theophrastus against the Number-Theorists," *Proceedings of the Cambridge Philological Society,* 23, no. 203 (January 1, 1977), 13-14.
[30] Aristotle, *Posterior Analytics* 78b34-79a6.
[31] E.g. Aristotle, *Posterior Analytics* 90a14-23, *De Sensu* 448a9-13, *De Anima* 426a27-b7.
[32] E.g. Aristotle, *Metaphysics* 1090a1-37. In this passage, Aristotle exonerates the Pythagoreans for their statement that all things are number on the grounds that they do not postulate numerical Forms, but rather observe number at work in physical things like musical intervals. That is, Aristotle claims that the Pythagoreans do not believe that number exists in the abstract. He seems to be alone in this view.
[33] The *Elements of Harmonics* may be a conglomerate of two or more treatises by Aristoxenus. See Barker, *Greek Musical Writings,* 120-123 for a discussion.

concept of διαίρεσις, the division of intervals into smaller intervals. But although both terms are drawn from the language of spatial measurement, they never describe spatial relations more complex than relative linear distances.[34] Aristoxenus describes intervals as made up of a certain number of tones (τόνοι), which in turn consist of dieses (διέσεις), the smallest musical intervals in each scale.[35] The exact size of a diesis changes depending on the scale. For example, the diesis of a diatonic scale is a semitone, while the diesis of an enharmonic scale is approximately a quarter-tone. The parts of music theory that Pythagoreans would express using ratios, such as comparison, division, and composition of intervals, Aristoxenus expresses by the addition or subtraction of tones and dieses. There is no numerical notation involved, nor any mathematical theory more complex than addition. That is, the spatial terminology of Aristoxenus does not import any of the geometrical theory with which it is usually associated, as the Pythagorean application of ratio does with numerical theory. Rather, Aristoxenus' terminology is used as a kind of adopted technical language to describe something that otherwise does not have its own technical vocabulary.

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[34] Aristoxenus criticizes Lasus and the school of Epigonus for attributing breadth (πλάτος) to notes (Aristoxenus, *Elementa Harmonica* I.3, ed. R. da Rios (Rome: Polygraphica, 1954), 18-22). That is, he claims that notes must not be considered as having any extension. This argument would make individual notes equivalent to points under a geometrical conception of music, but spatial thinking is not followed in the rest of the work. Aristoxenus describes the pitches of notes, for example, as ὀξύ and βαρύ, "shrill" and "deep" (literally "sharp" and "heavy"), rather than in any geometrical way. It seems more likely that he was using the common vocabulary at hand, than that he had devised a specifically geometrical framework for thinking about music.

[35] Aristoxenus, *Elementa Harmonica* II.46 (da Rios, 57.1-12).

While Aristoxenus mostly ignores the Pythagoreans and mathematical music, he is actively critical of them throughout much of Book II. He carefully qualifies the Pythagorean idea of certain scales improving moral character.[36] He claims that in his study of harmonics, he will provide proofs of his musical theory that accord with the phenomena – unlike his predecessors.[37] He says that the Pythagorean theory of sound as "certain ratios of numbers and relative swiftness,"[38] is an example of Pythagoreans "speaking irrelevantly and turning away from perception as being inaccurate,"[39] as well as "very much at variance with the phenomena."[40] Further on, he writes that mathematical methods (especially those used by geometers) are not appropriate to music, because geometers make no use of their sense perception,[41] whereas the student of music must train the accuracy of his senses. Finally, he criticizes his forerunners for failing to distinguish between the melodious and unmelodious, and he explicitly criticizes "the school of Pythagoras of Zakynthos" for failing to completely enumerate the differences between the various scales.[42]

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[36] Aristoxenus, *Elementa Harmonica* II.31 (da Rios, 41.1-2): τὸ δ' ὅτι καθ' ὅσον μουσικὴ δύναται ὠφελεῖν, "to whatever extent music is able to help [moral character]."
[37] Aristoxenus, *Elementa Harmonica* II.32 (da Rios, 41.13-42.7).
[38] Aristoxenus, *Elementa Harmonica* II.32 (da Rios, 41.21): λόγους δέ τινας ἀριθμῶν εἶναι καὶ τάχη πρὸς ἄλληλα.
[39] Aristoxenus, *Elementa Harmonica* II.32 (da Rios, 41.19-20): οἱ μὲν ἀλλοτριολογοῦντες καὶ τὴν μὲν αἴσθησιν ἐκκλίνοντες ὡς οὖσαν οὐκ ἀκριβῆ.
[40] Aristoxenus, *Elementa Harmonica* II.32 (da Rios, 42.2-3): ἐναντιωτάτους τοῖς φαινομένοις.
[41] Aristoxenus, *Elementa Harmonica* II.32 (da Rios, 42.13-21). The claim that geometers make no use of the senses indicates that Aristoxenus here is specifically criticizing systematist mathematicians, who were associated with Platonism and Pythagoreanism, and who actively downplayed the role of perception in their mathematics, unlike the heurists.
[42] Aristoxenus, *Elementa Harmonica* II.36 (da Rios, 46.6-16).

Many later authors followed the non-mathematical course of Aristoxenus and Theophrastus. Archestratus (3rd c. BCE) worked within the Peripatetic framework, though he differed from Aristoxenus on certain points of content.[43] Cleonides (1st c. CE) wrote an *Introduction to Harmonics* that follows Aristoxenus very closely both in method and content.[44] Bacchius Geron, the author of a 4th c. CE dialogue on music theory, also followed Aristoxenus.[45]

The conflict between the schools of Pythagoras and Aristoxenus was described and analyzed by ancient authors for many subsequent centuries. Ptolemais (1st c. BCE) calls the Pythagoreans *kanonikoi* and the Aristoxenians *mousikoi*, and describes efforts by a third group, called the *mathematikoi*, to adopt what was most accurate from both schools.[46] Didymus (1st c. CE) wrote a work of music theory entitled *On the Pythagoreans and Aristoxenians*, in which he tried to reconcile their theories.[47] Later, Ptolemy's *Harmonics* made a similar attempt.[48] Aristides Quintilianus, one of the most

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[43] Testimony from Porphyry, *Commentary on Ptolemy's Harmonics,* 26.26-30, and Philodemus, *On music*, 137.13-27 (D. Delattre, ed., *Philodème de Gadara, Sur la Musique, Livre IV* (Paris: Les Belles Lettres, 2007) vol. 2), indicates that Archestratus was seen as departing from Aristoxenus' ideas while still operating within his school. For more, see A. Barker, "Musical Theory and Philosophy: The Case of Archestratus," *Phronesis* 54, no. 4/5 (2009): 390–422.
[44] A. Barker, "Aristides Quintilianus and Constructions in Early Music Theory," *The Classical Quarterly* 32, no. 1 (1982): 187 calls Cleonides "slavish" in his adherence to Aristoxenus.
[45] Otto Steinmayer, "Bacchius Geron's 'Introduction to the Art of Music,'" *Journal of Music Theory* 29, no. 2 (1985): 271–298.
[46] Quoted by Porphyry, *Commentary on Ptolemy's Harmonics* 22.22-23.22.
[47] Porphyry *Commentary on Ptolemy's Harmonics* 25.5-6, 5.11-13. Our evidence for Didymus' work comes primarily from Ptolemy's *Harmonics* and Porphyry's commentary on it.
[48] Ptolemy, *Harmonics* I.5-12.

complete sources on the history of music theory, also recognizes characteristics of the early schools and their disagreements.[49]

In Book II of the *Elements of Harmony*, Aristoxenus gives detailed instructions for producing certain musical intervals by adding and subtracting to a given pitch known intervals such as fourths, fifths, and octaves.[50] He also specifies how many tones are contained within given intervals.[51] Although his methods are not presented as mathematical analysis, his results can be numerically compared with the ratios of intervals specified by other theorists.[52] In fact, certain later authors (most notably Ptolemy) laid these comparisons out in tables.[53]

In the course of such comparisons, Aristoxenus's values are taken as representative of his school, and those of Archytas (whom Ptolemy describes as the Pythagorean most skilled in mathematics[54]) are taken as representative of the Pythagoreans. Modern scholarship has generally classified later authors as either Pythagorean or Aristoxenian based on whether their numerical values for intervals accord more closely with those of Archytas or Aristoxenus. Such a classification, however,

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[49] The dates of Aristides Quintilianus are uncertain, but he probably wrote in the 3rd or 4th centuries CE. He describes the mathematics of the Pythagoreans at *De Musica* II.1-2, and the systems of Aristoxenus at *De Musica* I.10, as well as comparing mathematical methods using the monochord and the helicon at *De Musica* III.3. For more on the evidence for older music theory in Aristides Quintilianus, see Barker, "Aristides Quintilianus and Constructions in Early Music Theory."
[50] Aristoxenus, *Elementa Harmonica* II.55-58 (da Rios, 68.10-72.6).
[51] Aristoxenus, *Elementa Harmonica* II.55-58 (da Rios, 68.10-72.6).
[52] If a tone can be expressed as a certain numerical value by virtue of being a part of a whole, then all intervals can be expressed as numbers of "parts." Pythagorean ratios can also be translated to parts of wholes and compared numerically.
[53] Ptolemy, *Harmonics* II.14-15.
[54] Ptolemy, *Harmonics* I.13.

obscures the methodologies of the authors in favor of their results. The work of the systematist and heurist schools in the development of music theory, while partly visible through numerical comparisons, is much more apparent through the methodological features that divide the schools in other disciplines.

IV.3 The Systematist School

Although the systematist school has some roots in the school of Pythagoras, the two are not coextensive. Ptolemais, author of *Pythagorean Elements of Music*, describes two subgroups of the Pythagoreans.[55] One group wanted to do away with perception altogether, and study music using reason alone. These were the Pythagoreans who were most keen on arguing with the *mousikoi*, who give precedence to perception, and of whom the school of Aristoxenus was a subgroup.[56] The other Pythagorean branch she identifies (there are some textual problems here[57]) accepts perception as a starting point for reason, but considers reason to be working independently once it gets started.[58] This description is consistent with her earlier description of the *kanonikoi*, the "Pythagorean harmonicists," who use perception only as a starting point, and from it derive a theory of ratios and intervals which the *mathematikoi* later accept.[59]

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[55] Porphyry *Comm. Ptol. Harm.* 25.10-30.
[56] Porphyry, *Comm. Ptol. Harm.* 24.1-6, 26.2-5.
[57] Barker, *Greek Musical Writings* 242 n148
[58] Porphyry, *Comm. Ptol. Harm.* 25.25-30.
[59] Porphyry, *Comm Ptol. Harm.* 23.17-21.

The *kanonikoi*, who begin from a perceptual starting point (the monochord, or canon), which they later ignore in their arguments about ratios, sound very similar to the systematist geometers who produce diagrams using the ruler and compass, which are then backgrounded out of physical significance. Panaetius the Younger, the author of a treatise *Concerning the Ratios and Intervals in Geometry and Music*, also says that the early investigations of the Pythagoreans into music were followed by "specialists in mathematics," who studied musical ratios on the monochord.[60] In this section, I will argue that the *kanonikoi* are doing systematist harmonics. The methodology of this mathematical-specialist subset of Pythagorean musical texts maps onto the same systematist practices seen in geometry and arithmetic.

IV.3.1 The **Division of the Canon**

While the early Pythagoreans show many of the qualities that distinguish systematist work (such as a concern for exhaustive and systematic presentation of theory, abstraction of concepts, etc.), the first text[61] that is unambiguously systematist in method and style appears in the *Division of the Canon*. This work has a complicated manuscript history, and has variously been attributed to Euclid and Cleonides, as well as left

[60] Quoted by Porphyry, *Comm. Ptol. Harm.* 66.16-30.

[61] Assuming its estimated date of composition corresponds roughly to Euclid's *floruit*. The date and authorship of the *Division of the Canon* are contested in both ancient and modern sources. See André Barbera, *The Euclidean Division of the Canon: Greek and Latin Sources*, Greek and Latin Music Theory (Lincoln: University of Nebraska Press, 1991), 3-13.

anonymous.[62] Scholars have disagreed on even whether it is the work of a single author.[63] I will argue on both stylistic and methodological grounds that it is not.

The *Division of the Canon* as it most often appears[64] is composed of four sections: an introduction, the propositions, the enharmonic passage, and the problem on division of the monochord.[65] The introduction briefly echoes the usual Pythagorean theories of sound and musical ratios,[66] including a definition of pitch in terms of the density or rarity of motion, that closely approaches more modern theories.[67] The very existence of an introduction is an initial clue that the author of the work was not Euclid. The only other Euclidean text to include an introduction is the *Phaenomena*, whose introduction is considered to be the later addition of a different author.[68]

The eleven propositions are written very much in the systematist style.[69] They include the canonical parts of a geometrical proposition and use normal systematist

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[62] Barbera, *The Euclidean Division of the Canon*, 3-13.
[63] Barbera, *The Euclidean Division of the Canon*, 13-22.
[64] Porphyry and Boethius reproduce it only partially. Both their texts are available for comparison in Barbera, *The Euclidean Division of the Canon*, 188-257.
[65] Aside from the numbering of the propositions, these sections are not labeled. They are marked as distinct by stylistic changes in the writing and by changes in subject matter.
[66] For overviews of Pythagorean acoustic theory, see especially B. L. van der Waerden, "Die Harmonielehre der Pythagoreer," *Hermes* 78 (1943): 192–197, and Walter Burkert, *Lore and Science in Ancient Pythagoreanism*, trans. Edwin L. Minar, Jr. (Cambridge: Harvard University Press, 1972), 379-382.
[67] Pitch has been experimentally shown to correspond to the frequency of a sound wave, which can be accurately described as a density or rarity of motion.
[68] e.g. Thomas and Berggren, *Euclid's Phaenomena*, 12: "on the balance of the evidence, the introduction did not belong to the treatise originally as written by Euclid."
[69] The eleventh proposition is consistent in language style, but deviates slightly from normal systematist writing in containing proofs of multiple principles without separating them. Barbera, *The Euclidean Division of the Canon*, 155 note 43, explains that this is the practice of most of the manuscripts. However, the text of Heiberg and Menge, *Euclidis Opera Omnia*, vol. 8, divides *Division of the Canon* 11 into six

language, such as third-person imperatives in the *ekthesis* and λέγω ὅτι (and occasionally φημί) introducing the *diorismos*.[70] But while some scholars have taken this as evidence of Euclid's authorship, the text's adherence to systematist forms is much looser than any of Euclid's firmly attributed works.[71] For one thing, numerical notation is used in at least one proposition.[72] For another, personal language such as ἐμάθομεν, which occurs four times in the *Division of the Canon*, but is only used one other time in the entire Euclidean corpus, is uncharacteristic of Euclid's style. [73]

Despite these few stylistic deviations from Euclid himself, the propositions are systematist in all fundamentals. They are all proofs of principles rather than solutions to problems, and the language is generalized.[74] In many manuscripts, they are accompanied

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sub-sections, 11-16, labels the enharmonic passage as sections 17 and 18, and the monochord problem as sections 19 and 20.

[70] Every proposition begins the *ekthesis* with ἔστω, for example (though the enharmonic passage and the monochord problem do as well). The *diorismoi* of *Division of the Canon (DC)* 1, 2, and 6 begin with φημί, and those of 4, 5, 6 (alternate proof), 7, 8, and 9 with λέγω ὅτι.

[71] Jan, *Musici Scriptores Graeci*, 115, says that the *Division of the Canon* must have the same author as that of the *Elements*, because it uses the phrases ὅπερ ἔδει δεῖξαι and φημὶ δή. However, the former phrase does not appear in the *Division of the Canon*, and the latter does not appear in the *Elements*. The phrase φημὶ δή appears only seven times in the entire Euclidean corpus (excluding the *Division of the Canon*), and only in the *Optics* (once in the original text and four times in the recension of Theon of Alexandria) and the *Catoptrica* (also the recension of Theon of Alexandria).

[72] In *DC* 9, a list of consecutive numbers in sesquioctave ratio is given. In some MSS the numbers are written out in words (Barbera 147), but it is nevertheless out of character for Euclid to use numerical examples at all. He does so nowhere else. There are two other places where numerical notation might be used, but they are less certain. In *DC* 2 an example of multiple ratios (4:8:16) is given in only six of the MSS, and in *DC* 3 an example of a superparticular ratio (2:3 or 12:13) is given in most MSS. These examples, since they are not important to the proposition, are not connected syntactically, and are never used in the argument, could very easily have been notes that became integrated into the text.

[73] The word ἐμάθομεν occurs in *DC* 2, 4, 9, and 11. The only other attestation of the word in the Euclidean corpus is *Elements* X.10.

[74] That is, the *protaseis* are expressed in terms of any interval fitting certain theoretical criteria, and do not require specific numbers. For example, the *protasis* of *DC* 3 states that no mean numbers will fall proportionately within a superparticular interval: Ἐπιμορίου διαστήματος οὐδεὶς μέσος, οὔτε εἷς οὔτε πλείους, ἀνάλογον ἐμπεσεῖται ἀριθμός. Not only are no specific numbers used in the proposition, but the

by diagrams in which pitches are represented by lettered straight lines, as numbers are in Euclid's *Elements* VII-IX.[75] Most notably, the propositions do not talk explicitly about the monochord as a device. It is implied in the construction of ratios, but as a physical object it is left in the background, as the ruler and compass are in systematist geometry.

The other two sections, the enharmonic passage and the division of the monochord, do not read clearly as systematist mathematics. In fact, both seem heurist; they are problems rather than proofs, and they foreground the physical device of the monochord as a means of solving problems. The enharmonic passage deals with two features of the enharmonic scale. In the first half, a method is given for finding two of the notes of the scale, the *paranetai* and the *lichanos*.[76] The second half demonstrates that two other notes, the *parhypatai* and the *tritai*, do not divide the *pyknon* into equal intervals. Both parts use physical terms, such as the "loosening" and "tightening" of strings, as part of the problem-solving, whereas the previous propositions talked only of static ratios. Likewise, the section on the division of the monochord goes into physical detail about the device. It sets a problem "to divide the canon according to the so-called

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language of the *protasis* is explicitly comprehensive (e.g. οὐδεὶς μέσος, οὔτε εἷς οὔτε πλείους, "no mean, either one or many").

[75] For more on the diagrams associated with the text, see Barbera, *The Euclidean Division of the Canon*, 40. Porphyry's text does not include a tradition of diagrams, but Boethius' does.

[76] The pitches of Greek musical scales were not standardized, so the names of the notes indicate relative positions within a given scale rather than any specific note. This is why they must be "found" for any given scale. For more on the notes and arrangements of scales and modes, see West, *Ancient Greek Music*, 219-223.

immutable system."[77] The passage mentions the string of the monochord explicitly,[78] talks about "cutting" intervals, and notes "sounding,"[79] and uses the first person aorist throughout.[80]

The enharmonic passage and the monochord problem do not match the rest of the work, and are entirely inconsistent with anything written by Euclid. It seems more likely that they are excerpts from heurist musical theorists that were added to the *Division of the Canon* on the basis of similarity of theme (much as the *Little Astronomy* is a collection of works of varying style, register, and methodology, unified under the single heading of subject matter). Moreover, these passages are missing from an important branch of the textual tradition. Besides the individual manuscripts, the *Division of the Canon* is reproduced within two other works: Porphyry's *Commentary on Ptolemy's Harmonics*,[81] and Boethius's *De Institutione Musica*.[82] Porphyry includes only the propositions, Boethius the introduction and the first nine propositions.[83] Neither includes the enharmonic passage or the monochord problem.

The propositions in the *Division of the Canon*, then, can be taken as a systematist text anchored to the Pythagorean roots of that school by the acoustic theory of the introduction. The enharmonic passage and the monochord problem fit into the pattern of

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[77] The immutable system is a standardized scale with fixed notes.
[78] χορδῆς, *DC* 19.3 (line numbers from Heiberg and Menge, *Euclidis Opera Omnia*, vol. 8).
[79] E.g. ἔτεμον, *DC* 19.11, 20.2, συμφωνεῖν, *DC* 19.13, 20.9, 20.12.
[80] Recall from ch. 1 that consistent use of personal language is usually a heurist characteristic.
[81] Porphyry, *Commentary on Ptolemy's Harmonics*, I.5.
[82] Boethius, *De Institutione Musica*, IV.1-2, in Latin translation.
[83] Both these versions of the text are reproduced in Barbera, *The Euclidean Division of the Canon*.

heurist musical theory. There are no other early works on music written in the style of early systematist geometers, but there are several authors whose methodological and philosophical practices are consistent with the systematist school.

IV.3.2 Archytas

Ptolemy calls Archytas the member of the Pythagorean school "most concerned with music."[84] He was an associate of Plato, and as a geometer probably worked within the systematist school.[85] Theon of Smyrna groups him as a harmonicist with Eudoxus, saying that those who followed them believed that the ratios of consonances were in numbers.[86] The two fragments of musical work remaining from Archytas betray almost nothing of his mathematical methods, but he does speak of music as one of the μαθήματα, as Plato does.[87]

Archytas' first musical fragment[88] runs along the same lines as the introduction of the *Division of the Canon*. He states that sound is produced by some kind of percussion, and associates high pitch with rapid, energetic motion and low pitch with slow, weak motion.[89]

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[84] Ptolemy, *Harmonics* I.13.9.
[85] See above, ch. 1 section on Archytas.
[86] Theon of Smyrna, *Useful Mathematics*, 61.12-16.
[87] Archytas, 35B1 D.-K. (Porphyry *Comm. Ptol. Harm.* 56.5-57.27). Plato, *Republic* VII 522c-531c.
[88] H. Diels and W. Kranz, *Die Fragmente Der Vorsokratiker*, vol. 1 (Berlin: Weidmann, 1951), 431.
[89] This acoustic theory was considered to be characteristic of the Pythagorean school. E.g. the account of Adrastus preserved in Theon of Smyrna, *Useful Mathematics*, 50, which attributes the theory to the Pythagoreans in general, rather than any specific author.

In the second fragment,[90] Archytas explicitly connects music with mathematical ratios. He gives brief definitions of the three means (arithmetic, geometric, and harmonic) that are relevant to musical theory. An interest in proportion theory independent of specific intervallic ratios is consistent with systematist practice, and is not found either in heurist mathematics or in works that are Pythagorean but not apparently systematist.[91]

Ptolemy preserves in his *Harmonics* Archytas' ratios for the intervals within each genus of tetrachord.[92] Several sets of interval ratios are similarly preserved by Ptolemy, and, although the methods that produced them are lost, the results may map broadly onto the divisions between the schools. The evidence is circumstantial, but generally Archytas' ratios are associated with Pythagoreanism.[93] Those authors who differ significantly from Archytas' results (such as Eratosthenes and Ptolemy himself) definitely worked in the heurist tradition in other areas.[94]

IV.3.3 Nicomachus of Gerasa

In addition to the chapters on harmonics in Nicomachus's *Introduction to Arithmetic*, his very short *Manual of Harmonics* has survived as the only complete treatise on music between the *Division of the Canon* and Ptolemy's *Harmonics*. He also

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[90] Diels and Kranz, *Die Fragmente Der Vorsokratiker*, 435.
[91] For example, no mentions of the three means are found in the fragments of Philolaus, or in the account of Thrasyllus given by Theon of Smyrna (*Useful Mathematics* 47-49, 87-93), or in Aelianus the Platonist as quoted by Porphyry (*Comm. Ptol. Harm.* 33-37). They are, however, thoroughly investigated by Euclid (though not under the specific names), and both Nicomachus and Theon of Smyrna devote space to explaining them.
[92] Ptolemy, *Harmonics* II.14-15 (35A16 D.-K.).
[93] For more on this association, see Barker, *Greek Musical Writings,* vol. 2, 48-50.
[94] See the section below on Ptolemy for more on the tables of intervals.

wrote a longer *Introduction to Harmonics*, which is now lost.[95] Neither of his extant texts shows much in the way of mathematical methods, because they are broad-strokes overviews for the beginning student. But Nicomachus, as a Neopythagorean and Neoplatonist, shared the same conceptual framework about the nature and purpose of mathematics that seems to underlie systematist work.

The *Manual of Harmonics* contains almost nothing mathematical. Most of the chapters are dedicated to sketching the history and philosophy of Pythagorean music, with special reference to Plato and Philolaus.[96] Nicomachus also draws on an Aristoxenian theory which had become ubiquitous by that time,[97] about the difference between the speaking voice and singing voice.[98] The basic intervallic ratios are introduced in *Manual of Harmonics* 5, including the statement (undemonstrated) that the ratio of the fifth is composed of the ratios of a fourth and a whole tone. The famous but impossible account of Pythagoras' discovery of these ratios is given in *Manual of Harmonics* 6, in which Nicomachus assigns numbers in the required ratios to the fixed notes of the eight-note diatonic scale (which he claims is Pythagoras's innovation).[99]

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[95] Nicomachus declares his intention to write the longer *Introduction* in *Manual of Harmonics* 1.
[96] *Manual of Harmonics* 8 and 9 are dedicated to Plato's *Timaeus* and Philolaus's *On nature*.
[97] See Barker, *Greek Musical Writings*, 246, and Flora R. Levin, *The Manual of Harmonics of Nicomachus the Pythagorean* (Grand Rapids, MI: Phanes Press, 1994), 40-41.
[98] Nicomachus, *Manual of Harmonics* 2.
[99] The numbers are: 6—*hypate*, 8—*mese*, 9—*paramese*, and 12—*nete*. Thus the octave = *nete* : *hypate* = 12:6 = 2:1; the fourth = *mese* : *hypate* = 8:6 = 4:3 and *nete* : *paramese* = 12:9 = 4:3; the fifth = *paramese* : *hypate* = 9:6 = 3:2 and *hypate* : *mese* = 12:8 = 3:2; and the tone = *paramese* : *mese* = 9:8.

Manual of Harmonics 8, which covers the harmonic theory in Plato's *Timaeus*, gives an overview of the arithmetical properties of the intervallic ratios relative to each other, as well as naming the harmonic and arithmetic means.[100] This is the most detailed mathematical information in the *Manual*, and it is given in the context of the cosmology (introduced by Nicomachus without mathematics in *Manual of Harmonics* 3), in which the demiurge constructs the universe according to certain ratios, of which musical concords are a kind of ontological echo. In other words, the most mathematical chapter is the one least connected to the actual physical phenomena of music.

The remainder of the *Manual* is devoted to the construction of the musical scales. In *Manual of Harmonics* 10, Nicomachus briefly explains the principle that in a physical instrument, the longer string or column of air will produce a lower note. *Manual of Harmonics* 11 and 12 deal with the full double octave that is considered to be the effective limit of the human vocal range, and with the differences among the diatonic, chromatic, and enharmonic genera. Mathematical principles are rarely invoked in these chapters. When they are, it is usually to explain a tension between Pythagorean theory and conventional musical practice.[101] All in all, the *Manual of Harmonics* has more in common with early Pythagorean musical writings in general than with systematist mathematics in particular.

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[100] The *mese*, 8, is the harmonic mean between the *hypate* and the *nete* (6 and 12), and the *paramese*, 9, is the arithmetic mean between them.
[101] For example in *Manual of Harmonics* 11, Nicomachus explains that certain notes must be added to the scale in order that the double octave may truly be a quadruple proportion, as the theory dictates.

Nicomachus's *Introduction to Arithmetic* II.21-29 are dedicated to a thorough overview of proportion theory, which he relates directly to the study of harmonics.[102] *Introduction to Arithmetic* II.26, 27, and 29 in particular discuss musical specifics such as the intervallic ratios, the division of the canon, and the proportional series 6-8-9-12, which includes all the major consonances and "can truly be called harmony."[103] There is nothing in these chapters about actually constructing musical scales. Even the canon and the aulos are only mentioned in one sentence as visual examples rather than matters for the reader's attention.[104] The focus is entirely on the underlying mathematics.

In the presentation of material, the end of Book II follows the rest of the *Introduction to Arithmetic*. General principles are stated and then illustrated using examples. Any connection between the math and physical phenomena is usually avoided or backgrounded. There is a focus on systematizing the information given: categorizing type of proportions, commenting on the order of presentation, discussing the similarities and differences between successive concepts, etc. In short, these chapters are to harmonic theory what the work as a whole is to arithmetic: not a systematist treatise, but an introduction for beginners written with a systematist outlook.

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[102] Nicomachus, *Introduction to Arithmetic,* II.21, says that this theory will be relevant to music, astronomy, and geometry, but he only goes into details about music in the subsequent chapters.
[103] Nicomachus, *Introduction to Arithmetic,* II.29.
[104] Nicomachus, *Introduction to Arithmetic,* II 27.

IV.3.4 Theon of Smyrna

A substantial portion[105] of Theon's *Useful Mathematics for the Understanding of Plato* deals with musical theory in a similar vein to Nicomachus's *Introduction*, but with more attention to the construction of musical scales. In a programmatic passage,[106] Theon explains that the study of music must follow arithmetic immediately, because we cannot comprehend music and harmony in the cosmos (which is the ultimate goal) without first understanding music in numbers. But although he insists in the same passage that the study of music in instruments is not necessary, Theon begins the musical sections with long quotations from Thrasyllus, an earlier Pythagorean, and Adrastus, a Peripatetic, on acoustic theory and the structure of scales.[107]

Theon begins his discussion of mathematical harmonics with the *tetractys*.[108] He recounts the discoveries of Lasus of Hermione and Hippasus of Metapontum (both Pythagoreans) in connecting the consonant ratios of the *tetractys* with music through experimentation with strings, weights, and bowls filled with liquid. But he says that it was the followers of Eudoxus and Archytas who said that "the ratio of consonances is in numbers, though they also agreed that the ratios are in movements."[109]

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[105] Theon, *Useful Mathematics*, 47-96
[106] Theon, *Useful Mathematics*, 16-17.
[107] Theon, *Useful Mathematics*, 46-49 (Thrasyllus), 49-58 (Adrastus). It is not entirely clear where the excerpt from Adrastus ends. There is a break in the text on page 59, and on 61 Theon explicitly picks up Adrastus again, implying that at some point he had stopped quoting him.
[108] Theon, *Useful Mathematics*, 58-59.
[109] Theon, *Useful Mathematics*, 61: οἱ δὲ περὶ Εὔδοξον καὶ Ἀρχύταν τὸν λόγον τῶν συμφωνιῶν ἐν ἀριθμοῖς ᾤοντο εἶναι, ὁμολογοῦντες καὶ αὐτοὶ ἐν κινήσεσιν εἶναι τοὺς λόγους.

I have argued that Eudoxus and Archytas were both geometers of the early systematist school. The way Theon sets them apart from other Pythagoreans in this passage suggests that they also represented a strain of harmonic theory that agreed with Pythagorean acoustics but was more mathematical. It is a hint that, despite a good deal of overlap, early systematist harmonics was recognizably distinct from pure Pythagoreanism.

The remainder of Theon's harmonic work is very similar to Nicomachus's. He devotes much space to the classification of different ratios and proportions, drawing heavily on the work of Adrastus.[110] He gives a long excerpt from Thrasyllus on the division of the canon according to the ratios of the *tetractys*,[111] and transitions through the *tetractys* to a discussion of the regular solids, and thence to cosmology, leaving harmonics behind. Like Nicomachus, Theon shows almost no interest in the production of playable scales; instead, his focus is on systematizing the numerical theory, and relating it to the other mathematical fields.

IV.4 The Heurist School

Scholars both ancient and modern have indicated that not long after the rivalry between the Pythagorean and Aristoxenian schools developed, certain mathematicians began trying to reconcile them.[112] These mathematicians used the arithmetical tools

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[110] Theon, *Useful Mathematics*, 61-87.
[111] Theon, *Useful Mathematics*, 87-93.
[112] E.g. the passages from Ptolemais, Didymus, and Ptolemy above. See also: Barker, *Greek Musical Writings*, vol. 2, 2-10.

(including the monochord) developed by the Pythagoreans, as well as the Aristoxenian empiricism and attention to musical practice, to design a harmonic theory that was both physically grounded and fully mathematized. Not only are the priorities of these authors consistent with those of the heurist school, but a couple of the authors wrote heurist-style texts in other disciplines.[113] Common features of the heurist harmonicists are a special concern with numerically modelling various scales, the use of more complicated instruments in problem-solving, and departure from the interval ratios attributed to Archytas. Like heurist texts of geometry or arithmetic, those on harmonics show a preference for developing problem-solving methods rather than articulating principles, and tend to foreground the use of physical devices. Stylistically, the texts are less structured and less formulaic than those of systematist school.

IV.4.1 The **Division of the Canon***: Enharmonic and Monochord Problems*

The final two passages in the *Division of the Canon* differ markedly in style and scope from the introduction and the eleven propositions that precede them. While the introduction and propositions are dedicated to the articulation of general principles of the field, and employ formal and abstract language, the enharmonic and monochord passages are designed to walk the reader narratively through the solutions to specific problems. The monochord as a problem-solving tool is in focus, and the language is accordingly more physical (the passages talk of tightening and loosening strings, etc., vocabulary that

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[113] Ptolemy, most notably in astronomy, and Eratosthenes in geometry and arithmetic.

is absent from the propositions). Neither problem appears in the reproductions of the *Division of the Canon* in the works of Porphyry or Boethius.

The enharmonic passage[114] (so-called because it presupposes the enharmonic tetrachord) contains two parts. The first is a brief solution to the problem of constructing the fixed notes of a scale using consonant ratios. The second part demonstrates that the notes within the tetrachord cannot divide it evenly. The parts run together, and the passage is not structured as the earlier propositions are (e.g. there no *protasis*, nor a λέγω ὅτι after a *diorismos*, etc.). The vocabulary of tightening and loosening (ἐπιτείνω and ἀνίημι, respectively) is used throughout.

The monochord passage[115] solves the problem of marking the canon with the fixed notes of a scale. While there is a nod to formalism at the beginning of the passage (a *protasis* is given in the infinitive form used for problems in systematist geometry), the language soon shifts into a streaming narrative. The author solves the problem as a series of steps told in the first person. The language of this passage is also physically grounded: notes are said to "sound" (συμφωνέω) as certain intervals rather than to be in a certain ratio to one another, as they are in the propositions, and the author speaks of "cutting" (τέμνω) the monochord into different lengths. Neither word appears anywhere else in the *Division of the Canon.*

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[114] Barbera, *The Euclidean Division of the Canon*, 172-176. (Heiberg and Menge, *Euclidis Opera Omnia*, vol. 8: *DC* 17-18.)
[115] Barbera, *Euclidean Division of the Canon*, 178-184. (Heiberg and Menge, *Euclidis Opera Omnia*, vol. 8: *DC* 19-20.)

Because of the stylistic differences and their absence from Porphyry and Boethius, scholars have argued that the two final problems were a late addition to the text.[116] They appear to be the work of different authors, based on the vocabulary and narrative style, and they are almost certainly not written by the author of the propositions. The stylistic and methodological choices are consistent with heurist works.

IV.4.2 Eratosthenes

Although no extant text remains, we have testimonia about Eratosthenes' musical work from Ptolemy, Nicomachus, and Theon of Smyrna. In the fields of geometry and arithmetic, we have already seen that Eratosthenes worked with heurist methods. His methodology in harmonics is more difficult to show, but the few pieces of information we do have indicate that he disagreed with the Pythagorean school on certain key points, especially in the places where Pythagorean music seems to have become systematist math.

First of all, Theon recounts that Eratosthenes rejected the conception of intervals as mere numeric ratios,[117] on the grounds that ratios only exist between similar quantities, and intervals are not quantities. Rather, they are the space between two termini. This can be shown (according to Eratosthenes, according to Theon) by comparing the half to the double, and seeing that the intervals are the same (the octave in both cases) but the ratios

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[116] Barbera, *Euclidean Division of the Canon*, 38-39, 46, 60-62, and 13-22 for a survey of scholarship on the authorship of the *Division of the Canon* in general.
[117] Theon of Smyrna, *Useful Mathematics*, 81-82.

are not. This report indicates first of all that Eratosthenes disagreed with a fundamental principle of systematist harmonics—that is, that intervals are numerical ratios—and second of all that he disagreed on empirical grounds. This is consistent with his preference in other fields for demonstrations based on visual or physical problem-solving methods.

Nicomachus criticizes Eratosthenes' division of the canon, saying that he, along with Thrasyllus, "misrepresented" the Pythagorean division.[118] Nicomachus himself proposes to follow the division of Timaeus of Locris, "whom even Plato followed closely."[119] The implication is that Eratosthenes was working with the monochord as a problem-solving tool, and was engaging with the mathematical aspects of music, but in a way that departed from Pythagorean and systematist results.

The other testimonia about Eratosthenes come from Ptolemy's *Harmonics.* In a series of tables, Ptolemy lists the ratios of tetrachord intervals in the scales of all three genera according to Archytas, Aristoxenus, Eratosthenes, Didymus, and himself.[120] In every case, Eratosthenes' numbers are closer to Aristoxenus' than to those of Archytas. While this does not prove that Eratosthenes worked with a heurist methodology in music as he did in his other works, it does suggest that he favored the Aristoxenian approach

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[118] Nicomachus, *Manual of Harmonics* 11.
[119] Nicomachus, *Manual of Harmonics* 11 (ὁ Λοκρὸς Τίμαιος, ᾧ καὶ Πλάτων παρηκολούθησεν). Note also that Nicomachus admits a disjunction between Timaeus, a true Pythagorean, and Plato, whose work is not expected to follow Pythagorean results very closely. In these kinds of statements we can see places where systematist mathematics distinguishes itself from Pythagoreanism.
[120] Ptolemy, *Harmonics* II.14-15

over the Pythagorean one. However, the testimonia of Nicomachus and especially Theon indicate that Eratosthenes engaged with music more as a mathematician than Aristoxenus himself did. That is, Eratosthenes' major criticism of Pythagorean music was not *that* it was mathematical (as Aristoxenus' was), but that the mathematics did not properly map onto the phenomena.

IV.4.3 Didymus

From Porphyry we have the title of Didymus's work, *On the difference between the Aristoxenians and the Pythagoreans*.[121] The work is mentioned as a parallel to the writing of Ptolemais, implying that Didymus also positioned himself outside the Aristoxenian and Pythagorean schools as a critic of them both. Unlike the fragments from Ptolemais, however, Didymus's work seems to have been more concerned with mathematical specifics. At least, Ptolemy includes his numerical tetrachord divisions in the tables along with those of Eratosthenes and Archytas. Didymus's numbers are closer to those of Eratosthenes for the chromatic and diatonic genera, but closer to those of Archytas for the enharmonic.

In addition to including him in the tables of tetrachord divisions, Ptolemy recounts an attempt by Didymus to improve the monochord as a problem-solving tool.[122] His improvement apparently consisted in finding a way to make the placement of the

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[121] Porphyry, *Commentary on Ptolemy's Harmonics* 25.3
[122] Ptolemy, *Harmonics* II.13.

bridge more convenient, and measuring the lengths of the divisions from both endpoints instead of only one. Ptolemy considers the improvement a failure, because Didymus was not able to do any math more complicated than what could done before with the monochord, but he does concede that it solves the problems caused by "continuous" or "movable" bridges.[123]

Ptolemy is also critical of Didymus's specific numerical divisions of tetrachords. He says that Didymus did not pay sufficient attention to perception. He particularly takes issue with Didymus's chromatic and diatonic scales. The division of the chromatic tetrachord he uses, according to Ptolemy's description of it, was entirely unharmonic.[124]

Didymus, then, was not following the Aristoxenian school unquestioningly, but neither was he following the Pythagorean school. He departs from the Pythagorean divisions (as represented by Archytas) of the chromatic and diatonic tetrachords, the two divisions which Ptolemy says Didymus made himself.[125] Furthermore, Ptolemy's criticism of Didymus's ratios is not that Didymus allowed theory to lead him blind (as the charge against Pythagoreans and systematists usually runs), but that his use of the monochord as a tool was insufficiently subtle. Ptolemy is arguing in the passage for the merits of his own eight-stringed canon, so he says that Didymus and his other

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[123] Movable bridges divide the string of the monochord at a point (or the close physical equivalent of a point). The problem solved by Didymus seems to be the complication of measuring all cut-off lengths of string against the length of the whole, instead of the lengths of each other. Measuring from both ends of the canon means that the position of the bridge does not have to be changed in order to cut off a new note.

[124] Ptolemy, *Harmonics* II.13. μηδαμῶς ἐμμελοῦς

[125] Ptolemy, *Harmonics* II.13. The implication is that Didymus borrowed his enharmonic division.

predecessors, by making all of the divisions on a single string, were using a "very difficult method," that makes it impossible to consider all intervals simultaneously.[126]

The musical work of Didymus appears to have been deliberately positioned outside the debate between the Aristoxenian and Pythagorean schools. It was clearly mathematical in nature, and it foregrounded the monochord as a problem-solving tool. It was criticized by Ptolemy as weak in perceptual accuracy, but only because of an insufficient tool, not because of theoretical restrictions (for which the systematists are criticized). Altogether, the description of Didymus looks very much like Ptolemais' description of the *mathematikoi*, and he seems to have been working with heurist methods.

IV.4.4 Ptolemy

Ptolemy's *Harmonics* is the only complete musical treatise that can be characterized as heurist. Written in the second century, it invites comparison with the works of Theon of Smyrna and Nicomachus. Ptolemy is more technical than either of the others, since the *Harmonics* is framed as a comprehensive treatment of the subject, rather than as an introductory work, but the difference is not so complete that no useful comparison can be made between their mathematical and philosophical presentations.

First of all, while Theon and Nicomachus both take a conspicuously Pythagorean/Platonic stance, Ptolemy spends Book I of the *Harmonics* criticizing both

[126] Ptolemy, *Harmonics* II.13: κατασκελεστέραν μέντοι ποιεῖ τὴν μέθοδον.

Pythagorean and Aristoxenian theory in turn. Like Ptolemais and Didymus, Ptolemy positions himself outside the two major old schools of musical theory, as a mathematical expert who can improve them both. Also like Ptolemais, he charges the Aristoxenians with relying too much on perception at the expense of reason and perception alike, and the Pythagoreans with ignoring perception almost entirely.[127]

In general, Ptolemy is more focused on perception and problem-solving than any systematist work: he defines harmonics as a "perceptive function,"[128] and he frequently invokes perception as part of the means by which a principle can be demonstrated.[129] However, he does not hold up perception as the ultimate standard. Reason is to be given a higher priority when hearing cannot judge accurately. Ptolemy calls perception a "messenger" for the rational faculties,[130] and says that the purpose of a theory is to demonstrate that nature has been made according to reason and purpose, particularly those parts of nature which are most beautiful (the parts perceivable by sight and hearing) because they are most closely related to reason.[131]

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[127] Ptolemy, *Harmonics* I.2.
[128] Ptolemy, *Harmonics* I.1: δύναμις καταληπτική. The translation of δύναμις as "faculty" is the translation choice of Jon Solomon, for a justification of which see Jon Solomon, *Ptolemy* Harmonics*: Translation and Commentary*, Mnemosyne, Bibliotheca Classica Batava. Supplementum 203 (Leiden; Boston; Köln: Brill, 2000), 2.
[129] E.g. in his criticisms of Aristoxenian theory in *Harmonics* I.9-11. Ptolemy also says at *Harmonics* I.1 that perception discovers what is approximate and accepts what is exact, while reason accepts what is approximate and discovers what is exact. There is a mutual but hierarchical relationship between the two faculties.
[130] Ptolemy, *Harmonics* I.1.
[131] Ptolemy, *Harmonics* I.2.

Ptolemy's balanced approach to reason and perception can be seen especially in his foregrounding of instruments as problem-solving tools. This is one of the characteristics of the *Harmonics* that most marks it as a heurist text. Unlike Theon and Nicomachus, who give an account of instruments as a way of illustrating established Pythagorean theory,[132] Ptolemy treats instruments as tools of investigation. In addition to describing the use of the monochord in solving problems, he critiques its effectiveness. He explains why the aulos and other musical instruments are not effective tools for the measurement of intervallic ratios.[133] He gives details for the design and use of the helicon,[134] the eight-stringed canon,[135] and his own invention, a fifteen-stringed canon,[136] and explains the benefits and shortcomings of each in solving problems.

In Book II of the *Harmonics*, Ptolemy compares the mathematical results in the measurements of intervals of Archytas, Aristoxenus, Didymus, Eratosthenes, and himself. He assesses the methods of his predecessors, taking Archytas as a the best of the Pythagoreans and Aristoxenus as the best of his school. Eratosthenes and Didymus are characterized as having departed in significant ways from both earlier schools, and they

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[132] Theon and Nicomachus talk about instruments in two ways. The first is in terms of acoustic theory, in which they echo Archytas' passages (and those of the *Division of the Canon*) on sound and the percussion of air. The second is as illustrations of the Pythagorean intervallic ratios. In this context the instruments are taken as examples, not demonstrations, and the examples are frequently incorrect (e.g. Nicomachus' story of the weighted strings). In neither case do they use the instruments as a tool of discovery or demonstration, as Ptolemy does.

[133] Ptolemy, *Harmonics* I.8.

[134] Ptolemy, *Harmonics* II.2. The helicon is a four-stringed canon with an extended bridge that lies obliquely over the strings.

[135] Ptolemy, *Harmonics* I.11.

[136] Ptolemy, *Harmonics* III.1.

are generally praised for that. Ptolemy proposes with his on results to improve on the valuable work they did.

Book III is where the *Harmonics* begins to look slightly less like a heurist text, but only slightly. The book starts with a detailed description of Ptolemy's fifteen-stringed canon and its virtues and uses in generating and measuring the double-octave. But then he transitions into a lengthy philosophical exposition on the relationship between harmonic theory and cosmology, spirit, and moral virtue. This last part of Book III seems Platonist in its outlook, and echoes several of the ideas in the *Timaeus.*[137] There has been doubt since the middle ages as to whether Ptolemy wrote the final eight chapters of the book.[138] If he did, he did not devote as much care to them as to the rest of the work, for many details are openly left out.[139] Either way, while it is unusual for a heurist text to express an interest in philosophy (especially Platonist philosophy), Neoplatonism was ubiquitous by the time of Ptolemy, and so should not necessarily be considered as a strong factor placing him in one school of practice or the other. This is especially true

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[137] For example, Ptolemy, *Harmonics* III.4 compares the parts of the soul to harmonic divisions and astronomical phenomena. For more, see Cristian Tolsa, "Philosophical Presentation in Ptolemy's Harmonics: The Timaeus as a Model for Organization," *Greek, Roman, and Byzantine Studies* 55, no. 3 (May 12, 2015): 688–705.

[138] See Solomon, *Ptolemy* Harmonics, xxx-xxxi; Düring, *Die Harmonielehre des Klaudios Ptolemaios*, lxx-lxxxviii; N. G. Wilson, *Scholars of Byzantium* (London: Duckworth, 1996), 266; Franz Boll, *Studien über Claudius Ptolemäus: Ein Beitrag zur Geschichte der griechischen Philosophie und Astrologie.* (Leipzig: Teubner, 1894), 65.

[139] E.g. Ptolemy, *Harmonics* III.15, "we have left more information for more leisurely hours," ἀλλὰ πρὸς τὸ κατεπεῖγον τῆς χρείας τὴν προτεθεῖσαν ἀρκεῖν ἡγησάμενοι ἔφοδον τὰ πλείω σχολαζούσαις παρήκαμεν ὥραις.

given Ptolemy's positioning of himself as the mathematician whose work can command all other existing practices.

In general, Ptolemy's stylistic choices also place him in the heurist school. He solves most problems with a methodological narrative, rather than as individual propositions. He uses personal language and makes digressions and heavy commentary on the mathematical problems and on other authors who have tried to solve them. And of course, as already mentioned, he prominently centralizes physical tools and phenomena in his methods.

In two passages, Ptolemy follows a systematist proposition style. The first is at I.5, where he quotes certain Pythagorean demonstrations of consonant ratios, complete with straight-line diagrams.[140] The other instances seem almost perverse in their composition, because they use language that usually signals systematist methods, but methods that are far from the abstract and idealized concepts of systematist mathematics. Book II begins with a series of five propositions concerning the divisions of tetrachords and the ratios within them. These propositions are arranged sequentially with minimal commentary, and, though pared down at the ends, mostly follow the structure of Euclidean propositions (the phrase λέγω ὅτι is even used four times, and δείκτεον ὅτι once).[141] They do not include diagrams, but they nevertheless read more like systematist

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[140] Ptolemy, *Harmonics* I.5.

[141] λέγω ὅτι appears one other time, in the second proposition of *Harmonics* III.1. This proposition belongs to a pair of propositions about tuning strings of different thicknesses, in order to use the fifteen-stringed canon as a problem-solving tool.

math than anything else. However, in introducing these propositions Ptolemy makes a direct appeal to the evidence of the senses as the starting point for the demonstrations. At the beginning of the chapter, he proposes to set out the tetrachords "as cithara players tune them". He says that, assuming only the commonly accepted ratios (such as 4:3 for the fourth and 9:8 for the tone), he will demonstrate the ratios within the tetrachords "from the tunings done by sense alone."[142]

In short, the only passages in which Ptolemy does favor systematist style mostly highlight his heurist orientation. The concreteness of his theory (as opposed to systematist abstraction), the focus on instruments and tools, the narrative style and personal language, and the detailed criticism of his predecessors all resemble the texts of Archimedes, Heron, Eratosthenes etc. much more than those of Euclid.

IV.5 Additional Remarks

The mathematical theory of music is complicated by its relationship to practice and to practical theory. Mathematicians could not make the field for themselves, as they could with geometry and arithmetic. They had to find ways of engaging mathematically with existing non-mathematical subject matter. Given the sparsity of extant texts, any picture we can form of how they did this must be made cautiously. However, among the texts we do have we can see the division between the systematist heirs of Pythagorean theory, with their idealized ratios that do not meet the demands of perception, and the

[142] Ptolemy, *Harmonics* II.1: διὰ μόνης τῆς αἰσθήσεως.

heurist problem-solvers attempting to reconcile the Pythagorean and Aristoxenian harmonic traditions through more accurate tools and a more even balance between reason and the senses. So far, wherever mathematics has gone, the systematist/heurist divide in methodology, philosophy, and style have gone with it.

Conclusion: A Revised History of Greek Mathematics

The textual evidence for Greek mathematical practice clearly shows a division between the two schools in style, methodology, and conceptual premises, that is consistent across time periods and disciplines. Moreover, the intertextual references that remain indicate that this division was felt and even cultivated by practitioners, though there is no evidence that the schools were named in antiquity. The preceding chapters have outlined the broadest contours of the systematist and heurist schools as they appear in the four most studied mathematical subjects. Detailing further subtleties and subdivisions is the work of a much larger project, but a rough history of the schools can be reconstructed as follows.

History of the Schools

Among the Presocratic philosophers about whose mathematical work we have testimonia, all four subjects of the quadrivium were a matter of concern, in varying degrees and distributions. For example, Thales and Pythagoras are both said to have worked on geometry. The early Pythagoreans developed arithmetical and musical theories, though it is not clear that other Presocratics seriously studied such things. Nearly everyone at the time was working on astronomy, or at least cosmology. Certain differences in attitude, or in how the work of these authors is received by later commentators, seem to map broadly onto some of the essential characteristics of the more fully realized systematist and heurist schools. The focus on astronomical observation and physical causality of the Milesian philosophers and atomists, as opposed to the Eleatic and Pythagorean mistrust of observation and the tendency to posit non-observable theories (such as the counter-earth), was noted in Chapter 3. Proclus criticizes Thales as insufficiently scientific or generalizing; but Pythagoras he praises as the first to study geometry as a liberal art, "examining its principles from the beginning and tracking down the theorems immaterially and intellectually."[1]

Eudemus' account of the work of Hippocrates of Chios (ca. 470-410 BCE) implies that systematist practices in geometry were already in use during the fifth century.[2] Likewise, Proclus' report of the treatise of Hippias of Elis on the quadratrix, a time-dependent curve used to trisect an angle, is evidence of heurist methods in use at the same time. In the fourth century, Eudoxus developed the systematist geometry that would eventually become Euclid's *Elements*. While we have no accounts of fourth-century heurist authors (the century being rather dominated by Plato and Eudoxus), Plato's own dialogues do indicate that he was aware of mathematical methods that were not similar to those of Hippocrates. For instance, in the *Meno*, the reasoning with which Socrates walks the slave boy through the doubling of the square is far from a systematist proof. In fact, it uses the same style of numerical geometry found in Heron's *Metrica*.[3] And while Plato is

[1] Proclus, *In prim. Euc.* 65.15-21.
[2] This is assuming that Eudemus, who wrote his histories of arithmetic, geometry, and astronomy during the fourth century BCE, did not fundamentally alter Hippocrates' methods and presentation.
[3] Specifically, the square to be doubled is constructed with sides of two feet (πόδες) each, and all the subsequent reasoning is done in reference to these numerical values.

almost certainly not advocating this heurist style as an ideal type of mathematical reasoning (since it is appropriate for a young slave with no formal education at all), he is at least demonstrating that it is familiar.

The influence of Plato on the formation of systematist mathematics is reported by multiple sources. Through his own philosophical interests, he seems to have encouraged mathematicians to think about idealized intellectual objects, generalized statements, universal truths, and, in the Pythagorean tradition, subjects of cosmic significance, such as the regular solids or perfect musical ratios. At least, it is evident that the mathematicians of the fourth and third centuries who wrote in the systematist style were all connected to the Academy and the Pythagorean tradition. It is possible, however, to overstate the importance of Plato himself to mathematics. He did not write any works of mathematics that we know of, and his concern in the dialogues is always with mathematics as a type or paradigm for philosophical inquiry—he does not approach it as a practitioner.

At any rate, by the end of the fourth century, the traits of systematist mathematics are linked to Platonist ideas, and appear mature and standardized in the work of Euclid and Autolycus of Pitane. These methods and stylistic conventions (such as the avoidance of physical or numerical tools, and the formal proposition with the lettered diagram) continue to be used in original works even until late antiquity. In the fourth century CE, Serenus of Antinoöpolis is still practicing systematist geometry.[4] The changes to the Platonist tradition over the centuries, including the rise of Neopythagoreanism, seem to have had relatively little effect other than the greater interest in number theory seen in the handbooks of Nicomachus of Gerasa and Theon of Smyrna during the first and second centuries CE.

While it is clear that heurist methods existed before the third century BCE, they are never codified and formalized in the same way the systematist methods are. Some early heurist texts, such as Archimedes' *Conoids and Spheroids* and Aristarchus' *Sizes and distances*, lie very close to the systematist style, while others, such as Archimedes' *Sand Reckoner*, do not resemble any systematist work. By the second century BCE, there seems to be an even clearer division between the schools' writing styles; we see Hypsicles in the *Anaphoricus* entirely abandoning the proposition format for his numerical astronomy, while Theodosius in the *Sphaerica* remains systematist in all particulars. In general, the introduction of heurist methods seems to correspond with an informality of style. Later heurist mathematicians, such as Heron, Ptolemy, and Diophantus, write much more conversationally than their systematist counterparts, and make free use of personal language in their texts.

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[4] Serenus, *Opuscula*, ed. J. L. Heiberg (Leipzig: Teubner, 1896). We have two remaining treatises from Serenus, one on the sections of a cylinder and one on the sections of a cone. He also wrote a commentary on Apollonius' *Conics*, but it has been lost.

The heurist school seems to have been heavily influenced, beginning in the second century BCE, by Babylonian mathematics. This is especially clear in the work of Hipparchus and Hypsicles, who incorporate the Babylonian sexagesimal system and 360-degree division of the ecliptic into their astronomical work. Heron's numerical geometry in the *Metrica* bears a strong similarity to the Babylonian geometrical tradition.[5] Diophantus' *Arithmetica* likewise draws on Babylonian forms, including in the organization of the first book.[6]

In contrast to the systematist school, it is not clear whether the heurists were strongly influenced by a particular philosophical movement. It might be tempting to look to the Lyceum, but Aristotle's ideas are not always so clearly divergent from Plato's as to have inspired a separate mathematical tradition. Indeed, many mathematical examples given in works of the Organon[7] resemble systematist mathematics more closely than Plato's geometry in the *Meno* does. Besides, many authors of the Peripatetic school were specifically mistrustful of mathematics or had little use for it. Eudemus' interest in mathematics was the exception among Peripatetics, not the rule. Theophrastus and Aristoxenus both argued forcefully against a mathematical conception of music theory;

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[5] See above, ch. 1 note 145.

[6] Piedad Yuste, "Ecuaciones Cuadráticas y Procedimientos Algorítmicos. Diofanto y las Matemáticas en Mesopotamia: (Quadratic Equations and Algorithmic Procedures. Diophantus and Mesopotamian Mathematics)," *Theoria. Revista de Teoría, Historia y Fundamentos de La Ciencia* 23, no. 2 (May 2008): 219–244.

[7] E.g. *Posterior Analytics* II 94a-24-35, in which Aristotle almost provides a proposition proving that the angle in a semicircle is right, or in *Prior Analytics* I 76b39-77a3, in which he argues that geometers do not refer to any particular line drawn in a diagram, but to what they represent.

while Diogenes Laertius attributes books on numbers, geometrical researches, and astronomy to Theophrastus,[8] none has survived; of the later Peripatetics, none had a reputation for mathematical work.

Instead, we see many instances of heurist texts trying to reconcile Pythagorean or Platonist mathematical ideas with the non-mathematical theories and observations of natural philosophers. This in music theory we saw heurist authors openly criticizing the Pythagoreans and Aristoxenians, and positioning themselves outside of both. Ptolemy does this especially explicitly in his *Harmonics*, and similarly presents himself as a master of both traditions in the *Almagest*, which includes some systematist propositions, but whose methodology is almost entirely based on numerical methods and instruments. The corpus of Archimedes can also be seen as spanning the systematist tradition and the methods of more physical arts. His *Quadrature of the Parabola* even goes back and re-proves the major propositions after they have already been demonstrated using mechanical methods. Again, this does not mean that the heurists were in any way doing practical mathematics. Practical mathematics can be seen in Heron's *De mensuris*[9] or the *Corpus Agrimensorum*.[10] What the heurist school developed was a form of theoretical mathematics that more carefully took into account the knowledge and methods of observational and physical sciences.

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[8] Diogenes Laertius, *Vitae Philosophorum* V.48-50.
[9] Heron, "De mensuris," in Schöne et al., *Heronis opera quae supersunt omnia*, vol. 5, 164-218.
[10] Carl Thulin, ed., *Corpus Agrimensorum Romanorum* (Leipzig: Teubner, 1913).

Additional Mathematical Subjects

Although it is both convenient and informative to divide mathematics into sub-disciplines, as I have done in following the quadrivium, it is important to recognize that the boundaries of these sub-disciplines (like the boundaries of mathematics itself) are neither rigidly defined nor strictly observed in practice. For instance, certain texts mix or cross topics. Hypsicles' *Anaphoricus*, though presented as an astronomical text, begins with three propositions of pure arithmetic, dealing only with the summation of series of numbers. Other sub-disciplines defy classification. Works on "spheric" such as those by Menelaus and Theodosius seem to belong equally to geometry and astronomy. Further complications arise as mathematics draws closer to the practical arts. In the fields of astronomy and music, it becomes necessary to consider whether a text on the subject is mathematical or not. Aristoxenus, as I showed in Ch. 4, uses some language of spatial distance and measurement in his *Elements of Harmony*; but I do not consider his work mathematical, because he does not use any mathematical theory or method beyond the addition or subtraction of "tones", nor does he use any mathematical notation, nor does he write in any style that we see in other mathematical texts. Aristoxenus himself even argues that mathematical methods are inappropriate to the study of music.[11]

In the present work I have kept to texts of theoretical mathematics that fall within the obvious contours of the quadrivium, in order to show that the differences between the

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[11] Aristoxenus, *Elementa Harmonica* II.32 (da Rios, 42.13-21).

schools are clear and pervasive throughout the most populated areas of mathematics. There are other ancient fields of study, however, in which mathematics was brought to bear on physical arts and sciences, still presented as theory rather than practical instruction. And the differences that mark the systematist and heurist schools are still evident in these fields of optics, catoptrics, and mechanics.

Optics and Catoptrics

Optics and catoptrics are closely related subjects, the former concerning geometric modeling of light and visual perception, the latter concerning the effects of mirrors. Both were ancillary to astronomy in antiquity. They were not always studied mathematically,[12] but when they were, the systematist/heurist divide is evident and maps onto what we see in authors' other works.

Euclid's *Optics*[13] is a predictably systematist text, using no numbers, beginning with a set of definitions, following the proposition format, and so on. He discusses the physical theory of sight only very sparely in the definitions. His *Catoptrics*,[14] though generally understood to be a recension, possibly by Theon of Alexandria, retains the core systematist style and methods. Ptolemy's *Optics*,[15] on the other hand, though it is only

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[12] For example, we have texts on physical optics that do not rely on mathematical proofs or principles: Damianus' *Optical Hypotheses*, and a fragmentary *Optics* that has been dubiously assigned to Geminus. Both can be found in R. Schöne, *Damianos Schrift Über Optik* (Berlin: Reichsdrukerei, 1897).
[13] Euclid, "Optica," in Heiberg and Menge, *Euclidis Opera Omnia*, vol. 7, 2–120.
[14] Euclid, "Optica," in Heiberg and Menge, *Euclidis Opera Omnia*, vol. 7, 286–342.
[15] A. Lejeune, ed., *L'optique de Claude Ptolémée dans la version latine d'après l'arabe de l'émir Eugène de Sicile*, vol. 8, Université de Louvain. Recueil de Travaux d'histoire et de Philologie 4 (Louvain: Bibliothèque de l'Université, Bureaux du recueil, 1956).

available in Latin from an Arabic translation, is clearly heurist, even where Ptolemy's familiarity with both traditions is as evident as in his other works. In the five books of his *Optics* (of which the first and part of the fifth are missing), Ptolemy includes an extensive discussion of the physical theory of sight (Book II), a problem-oriented section on catoptrics, in which he solves different versions of finding the point of reflection on a mirror (Books III and IV), and a series of tables on angles of refraction, along with details on different refracting media (Book V). Likewise, Heron's *De Speculis* (preserved only in Latin except for a small Greek fragment)[16] devotes the majority of space to physical theory and problems, though the first seven propositions (chapters 4-10) prove some basic principles of reflection in plane, convex, and concave mirrors, in a compressed but essentially systematist style. The first three chapters deal with causes and phenomena of sight, reflection, and refraction. The final eight chapters are all problems of constructing mirrors that will reflect in various, increasingly elaborate ways, even up to creating complex optical illusions. These are not practical instructions from which one could learn how to make mirrors—they are thought experiments in which a theory is illustrated through problem-solving.

[16] Heron, "De Speculis," in Schöne et al., *Heronis opera quae supersunt omnia*, vol. 2, 303–373.

Mechanics

Like optics, theoretical mechanics (the study of force and motion) was not always done mathematically.[17] The majority of the Peripatetic *Mechanical Problems* are only concerned with physical causes, and make no use of magnitudes, ratios, angles, or other mathematical apparatus, such as diagrams.[18] Notably, when mechanics was done mathematically, it was never by authors who primarily worked in the systematist tradition. We have texts remaining from Archimedes,[19] Heron of Alexandria,[20] and Pappus, who preserves (among other things) theorems that he claims were proved by Ptolemy.[21]

Since mechanics is a physical science, unconnected to cosmology, prophecy, or other areas of interest to Pythagorean mysticism or Platonist philosophy, it is consistent with the characters of the schools that only authors of the heurist tradition took an interest

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[17] We also have texts of practical mechanics, such as those by Biton of Pergamum and Apollodorus of Damascus on the construction of siege engines, which can be found, with others, in R. Schneider, *Griechische Poliorketiker*, Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Philol.-Hist. Kl., 10.1 (Berlin: Weidmann, 1908). The distinction I draw is that practical mechanics is instructional, intended to guide the construction of machinery, whereas theoretical mechanics argues, proves, and illustrates the principles governing force and motion in physical objects.

[18] I. Bekker, ed., *Aristotelis Opera*, vol. 2 (Berlin: Reimer, 1831), 847a11-858b31. Several sections of this text do use mathematical methods: *Mechanical Problems* 1, 2, 3, 5, 23, 24, and 25 (848b1-850b9, 850b30-851a37, 854b17-857a4). In all cases the style is heurist.

[19] The *Equilibrium of Planes (De planorum aequilibriis)* and *Floating Bodies (De corporibus fluitantibus)*, in Mugler, *Archimède*, vols. 2 and 3.

[20] Most of Heron's mechanical works are practical: the *Dioptra*, the *Pneumatica*, the *Automata*, and the *Cheirobalistra* all deal with the construction and use of machines. Of a more theoretical orientation, and done mathematically, are the *Baroulkos*, preserved in Pappus, *Synagoge* VIII.11 (Hultsch, 1060.1-1068.17), and the first book of his *Mechanics*, preserved in Arabic with a German translation in Schöne et al., *Heronis opera quae supersunt omnia*, vol. 2, 1-93.

[21] Pappus, *Synagoge* VIII.1-10 (Hultsch, 1030.5-1058.26).

in it. And since all objects are conceptualized physically and subject to force, motion, and time, and physical causality is essential to the proof of any proposition, the methodology of mechanics could be considered fundamentally heurist. There is, however, some variation in style among the texts mentioned. About Heron's style in the *Mechanics* we can say very little, because it exists almost exclusively in an Arabic translation,[22] but the *Baroulkos* is perfectly consistent with Heron's other heurist work: continuous prose, numerical methods, and an approach to theory through problem-solving.[23] The mechanical texts of Archimedes, though, as well as the rest of Pappus' *Synagoge* VIII (much of which is attributed to Ptolemy) keep surprisingly close to systematist style and method: geometrical lettered diagrams, eschewing the use of numbers, propositions that contain all or most of the Euclidean parts, and a general focus on proving principles rather than solving problems. Archimedes even gives a brief list of axioms at the beginning of his *Equilibrium of Planes*.[24]

This leaning toward systematist style in theoretical mechanics may have to do with the systematist school's association with Platonist philosophy. There is some evidence of authors of mechanical texts attempting to legitimize the study of a subject so near the practical arts by showing that it is compatible with philosophy and with

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[22] Schöne et al., *Heronis opera quae supersunt omnia*, vol. 2, 1-93. Some excerpts in Greek also appear in Pappus, *Synagoge* VIII.11 (Hultsch, 1114.23-1134.11, and Schöne et al., *Heronis opera quae supersunt omnia*, vol. 2, 256-299).
[23] The *Baroulkos* is in Pappus, *Synagoge* VIII.11 (Hultsch, 1060.1-1068.17).
[24] Mugler, *Archimède*, vol. 2, 80.1-81.2.

geometry according to the systematist conception. Archimedes, in the *Quadrature of the Parabola* and the *Method*, distinguishes between "mechanical" and "geometrical" proofs of his theorems.[25] He includes both types, and when he presents the geometrical proofs, he writes closer to the systematist style, as if to emphasize that he is versed in both styles, and that his less formal mechanical method will demonstrate the same results as the more formal geometrical propositions. Heron, in the prolegomena to his *Belopoeica*,[26] appeals to philosophy as a justification for the writing of a work on war machines. He claims that the greatest part of philosophy is dedicated to teaching peace of mind (ἀταραξία), and that mechanics has surpassed philosophy in this subject because nothing produces greater peace of mind than the safety of the city, which is secured by the construction of artillery.[27] Pappus, in the prolegomena to Book VIII of the *Synagoge*, also argues that mechanics is a valuable subject of study for philosophers,[28] and insists that geometry is not harmed by being applied to physical things.[29] Given the formalism, rigor, and philosophical associations of systematist math (especially geometry), it is likely that the systematist style was perceived as a means of elevating mathematical writing, and of granting legitimacy to a subject of dubious intellectual standing, like mechanics.

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[25] See above, ch. I section I.2.2.4, "Archimedes' Innovation in Mechanical Methods."
[26] H. Diels and E. Schramm, eds., *Herons Belopoiika*, Abhandlungen der Preussischen Akademie der Wissenschaften, Philosoph.-Hist. Kl. 2 (Berlin: Reimer, 1918).
[27] Diels and Schramm, *Herons Belopoiika*, 1.1-2.14. For more on Heron's rhetoric in this passage, see S. Cuomo, "The Machine and the City: Hero of Alexandria's *Belopoeica*," in *Science and Mathematics in Ancient Greek Culture*, ed. C. J. Tuplin and T. E. Rihil (Oxford: Oxford University Press, 2002) 165-177.
[28] Pappus, *Synagoge* VIII (Hultsch, 1022.3-8).
[29] Pappus, *Synagoge* VIII (Hultsch, 1023.21-23): γεωμετρία γὰρ οὐδὲν βλάπτεται, σωματοποιεῖν πεφυκυῖα πολλὰς τέχνας, διὰ τοῦ συνεῖναι αὐταῖς…

In optics, catoptrics, and mechanics, then, we can see the persistence of the systematist/heurist divide. In optics and catoptrics, which lie closer to the well-established mathematical field of astronomy, the dividing line predictably separates authors according to the traditions in which they work in other fields (Euclid to the systematist school, Heron to the heurist, etc.). In mechanics, however, we can see authors who normally work in the heurist school adopting or approaching a systematist style in order to present material that is alien to usual systematist practices. This gives us a hint that even as early as the third century BCE the systematist school may have been perceived by the intellectual elite as the more prestigious or respectable of the two. If a preference for systematist mathematics was already in place as early as the time of Archimedes, it was only strengthened by the activity of late antique scholars.

Commentaries and Compilations

Mathematical commentaries, histories, and compilations were being written as early as the fourth century BCE, and from the outset they seem to have given the majority of their attention to works of the systematist school. Eudemus' histories of arithmetic, geometry, and astronomy are lost except for testimonia and fragments, but it is clear from what remains that, although Eudemus did not exclude texts that lay outside of strict systematist parameters,[30] either Eudemus or the later authors who excerpted him (such as

[30] For example, we know from Proclus that Eudemus included works using time-dependent curves in his summary of attempts to square the circle or double the cube.

Proclus, Simplicius, and Geminus) gave the majority of space to the Pythagoreans and early systematists such as Eudoxus and Hippocrates of Chios.

Euclid's *Elements* seems to have been the subject of extensive commentaries, editions, and expansions since at least the second century BCE, with Hypsicles' *Book XIV*.[31] Proclus alludes to a multitude of unnamed earlier commentators on the *Elements*, dealing with issues such as the difference between postulates and axioms.[32] Another addition to the *Elements*, Book XV, is of dubious authorship but may have been the work of Isidore of Miletus or one of his students.[33] The pseudo-Heronian *Definitions*, possibly the work of Diophantus, serves as an introduction to and summary of the major concepts in the *Elements* (the author calls it a *syntaxis* in the proem).[34] We know that Heron wrote a commentary on the *Elements*,[35] as did Pappus,[36] possibly also Porphyry.[37] Theon of Alexandria produced a widespread edition of the text, and of course we have Proclus' commentary on Book I.

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[31] E. S. Stamatis, "Hypsiclis Liber sive Elementorum Liber XIV qui fertur," in *Euclidis Elementa*, vol. 5 (Leipzig: Teubner, 1977), 1–22.

[32] E.g. Proclus, *In prim. Euc.*, 181-182. See also *ibid.* 84, 200, 289, 328, and 432.

[33] E. S. Stamatis, "Euclidis Elementorum qui fertur Liber XV," in *Euclidis Elementa*, vol. 5 (Leipzig: Teubner, 1977), 23–38. For a discussion of authorship, see Heath, *HGM* vol. 1, 421.

[34] Schöne et al., *Heronis opera quae supersunt omnia*, vol. 4. For the matter of authorship, see Knorr, "Arithmêtikê Stoicheiôsis."

[35] The commentary of Heron on the *Elements* is attested in the commentary of al-Nayrīzī (10th c.): R. O. Besthorn et al., *Codex leidensis 399, I. Euclidis Elelmenta ex interpretatione al-Hadschdschadschii cum commentariis al-Narizii.*, 2 vols. (Copenhagen: Library Gyldendal, 1893).

[36] We have an Arabic translation of Pappus' commentary on Book X: William Thomson, *The Commentary of Pappus on Book X of Euclid's Elements* (Cambridge: Harvard University Press, 1930).

[37] For a discussion of the possible commentary of Porphyry, see Heath, *The Thirteen Books of Euclid's Elements,* 24.

The *Phaenomena* of Aratus is another early work (third century BCE) that received significant scholarly attention. Though the *Phaenomena* is a didactic poem and not mathematical, it was based on the prose work of the same name by Eudoxus, an important figure in the development of systematist mathematics. Attalus' original commentary is lost, but Hipparchus' scathing criticism of both the *Phaenomena* and Attalus' praise of it implies that the majority opinion favored systematist astronomy in the second century BCE. The *Commentary on the Phaenomena of Eudoxus and Aratus* is a jumping-off point for Hipparchus to show the superiority of his own heurist mathematical astronomy.

Handbooks and introductions became very popular forms of mathematical writing between the first century BCE and the second century CE. Because such texts are not intended for practitioners so much as for beginning students or laypersons, they are not written as technical mathematical works, and so it is difficult to classify them as either systematist or heurist based on style (which is genre-specific) or methodology (which they usually do not show, instead favoring a concise presentation of results only). Even the epistemological orientation and conceptualization of mathematical objects can be difficult to discern in texts that are not so overtly Neopythagorean or Neoplatonist as those of Nicomachus or Theon of Smyrna. Geminus' *Introduction to the Phaenomena* (first century BCE) is an early example of an astronomical handbook.[38] He mentions very

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[38] G. Aujac, *Géminos. Introduction aux Phénomènes* (Paris: Les Belles Lettres, 1975).

few previous astronomers by name besides Aratus,[39] and though more attention is given to systematist than to heurist authors, the *Introduction to the Phaenomena* itself cannot really be classified as either. Cleomedes' *Caelestia* (dated sometime between the first century BCE and the second century CE)[40] is even less mathematically oriented than Geminus' *Introduction*. Cleomedes draws primarily from the work of natural philosophers such as Aristotle and Posidonius, but he does mention Aratus' *Phaenomena* and Eratosthenes' calculation of the size of the Earth, and in this balance he slightly favors the heurist Eratosthenes.[41] Such public-facing works can tell us little more other than which authors were widely enough known to be cited in an introduction to the subject. The other major introductory texts of the second century CE, Nicomachus' *Introduction to Arithmetic* and Theon of Smyrna's *Useful Mathematics for the Understanding of Plato*, have already been discussed at length in other chapters. Although they include some discussions of heurist works such as Eratosthenes' *Sieve*, they are primarily systematist in the texts they survey and in their general orientation.

The fourth century CE saw the important work of Pappus and Theon of Alexandria. Pappus, in the early fourth century, wrote both the *Synagoge* and commentaries on Euclid's *Elements* (an Arabic translation remains of the commentary on

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[39] Eudoxus and Eratosthenes are mentioned in *Introduction to the Phaenomena* VIII.20-24.

[40] Cleomedes, *Caelestia (Μετέωρα)*, ed. R. Todd (Leipzig: Teubner, 1990). For a discussion of the date of the work, see Todd and Bowen, *Cleomedes' Lectures on Astronomy*, 2-4.

[41] Aratus is mentioned three times (*Caelestia* I.3, I.7, and II.1), Eratosthenes five (*Caelestia* I.7, four times, and II.1 once).

Book X[42]) and Ptolemy's *Almagest* (we have the Greek commentary on Books IV and V).[43] The *Synagoge* is eclectic, and includes many fragments of heurist texts that are otherwise preserved only later by Eutocius, such as Nicomedes' doubling of the cube by means of the cochloid curve, or Eratosthenes' instrument for finding mean proportionals. It also contains a commentary on the systematist texts of the *Little Astronomy*, that proposes to solve some difficulties in those works, though Pappus is not, on the whole, critical of them. He mostly follows the systematist style of those works in Book VI of the *Synagoge*. Finally, Pappus is our only source for the arithmetical work of Apollonius, which he both presents in Apollonius' systematist style and "translates" or illustrates with numerical examples.

Theon of Alexandria, working in the late fourth century, also produced commentaries and recensions of the works of both Euclid and Ptolemy. His commentaries on the *Almagest*[44] and Ptolemy's *Handy Tables*[45] are extant. His edition of the *Elements* is probably that which is preserved in the majority of manuscripts of Euclid, but the transmission of the *Elements* is too complex for us to certainly distinguish an original.[46] Recensions of Euclid's *Optics* and *Catoptrics*, in which the systematist style

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[42] Thomson, *The Commentary of Pappus on Book X of Euclid's Elements*.
[43] Rome, *Commentaires de Pappus et de Théon d'Alexandrie sur l'Almageste*, vol. 1.
[44] Rome, *Commentaires de Pappus et de Théon d'Alexandrie sur l'Almageste*, vol. 2.
[45] J. Mogenet and A. Tihon, *Le "grand Commentaire" de Théon d'Alexandrie aux Tables Faciles de Ptolémée*, 2 vols., Studi e Testi 315, 340 (Città del Vaticano: Biblioteca Apostolica Vaticana, 1985, 1991), and A. Tihon, *Le "petit Commentaire" de Théon d'Alexandrie aux Tables Facile de Ptolémée*, Studi e Testi 282 (Città del Vaticano: Biblioteca Apostolica Vaticana, 1978).
[46] See W. R. Knorr, "The Wrong Text of Euclid: On Heiberg's Text and Its Alternatives," *Centaurus* 38 (1996): 208–276, and W. R. Knorr, "On Heiberg's Euclid," *Science in Context* 14, no. 1-2 (June 2001):

and methods are preserved, are also ascribed to Theon. By the fourth century, then, the systematist *Elements* and the heurist *Almagest* seem to have loomed over the landscape of mathematical scholarship, but in general the majority of scholarly attention was going to systematist texts. Euclid's extended corpus, Apollonius, Eudoxus and Aratus, and the *Little Astronomy* received a great deal of space, while apart from Ptolemy's astronomical and musical works, no heurist text seems to have gained a dedicated commentary until Eutocius wrote on Archimedes in the early sixth century.

The strain of Neoplatonism that developed with the work of Plotinus in the third century came to uncontested dominance by incorporating and synthesizing nearly all earlier Greek philosophy, with the exceptions of Epicureanism and Stoic materialism. Even Aristotle came to be understood primarily as a student of Plato and a bearer of the Platonist tradition. This philosophical syncretism, exclusive only of ideas with a strong physical or materialist orientation, gave a noticeable systematist slant to the mathematical scholarship of the subsequent centuries. Iamblichus (3^{rd}-4^{th} c. CE), a student of Porphyry (who was himself a student of Plotinus), strongly promoted the Neopythagorean arithmetical mysticism seen in Nicomachus over a century earlier. At the Neoplatonist school of fourth-century Alexandria, Theon produced his editions of Euclidean works and his commentaries on Ptolemy, who, though methodologically heurist, framed his work in Platonic and Aristotelian philosophical terms, and showed familiarity and comfort with

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133–143. For a briefer summary of textual issues with the *Elements*, see Cuomo, *Ancient Mathematics*, 125-135.

systematist style. In the fifth century, Proclus produced his commentary on Book I of Euclid's *Elements*. Even the work of Eutocius, apparently the first commentator on a heurist author other than Ptolemy, is oriented more toward systematist than heurist values.

In the early sixth century CE, Eutocius produced commentaries on Apollonius' *Conics*[47] and on Archimedes' *Sphere and Cylinder*, *Measurement of the Circle*, and *Equilibrium of Planes*.[48] Little is known of his life, and his connections to Neoplatonism are tenuous. He dedicated his commentary on the first book of the *Sphere and Cylinder* to Ammonius, a student of Proclus who wrote a lost commentary on Nicomachus. In the proem, Eutocius presents himself as the first to write a commentary on the *Sphere and Cylinder*, saying that he found no one before him who had done it; and although he makes no such claim about the *Measurement of the Circle* or *Equilibrium of Planes*, he does appear to be the first and only ancient commentator on any of Archimedes' works. At least, we know of no others. Eutocius' commentaries on Archimedes demonstrate an unusual interest in heurist work for the time. He preserves fragments of many other heurist authors, such as Heron, Nicomedes, Eratosthenes, Philo, and Diocles. Nevertheless, he leans toward systematist priorities in several ways. First of all are the works of Archimedes that Eutocius comments on. The *Sphere and Cylinder* is

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[47] J. L. Heiberg, ed., *Apollonii Pergaei quae Graece exstant cum Commentariis Antiquis*, vol. 2 (Leipzig: Teubner, 1891).
[48] J. L. Heiberg and E. S. Stamatis, eds., *Archimedis Opera Omnia, cum Commentariis Eutocii.*, vol. 3 (Stuttgart: Teubner, 1972).

Archimedes' most systematist text in both style and method. Of the three propositions of the *Measurement of the Circle*, the first is essentially systematist in style, and uses no overtly heurist methods as the other two propositions do. The *Equilibrium of Planes*, though its subject is mechanical, is nevertheless written in a stricter systematist style than any other of Archimedes' works except *Sphere and Cylinder*. Finally, Eutocius' only other commentary is on Apollonius' rigorously systematist *Conics*.[49] In the proem to this commentary, Eutocius defends Apollonius from the charge of Archimedes' biographer Heraclius that he stole the material for the *Conics* from Archimedes' unpublished work. He presents Apollonius' derivation of the conic sections from a single cone (instead of from right, acute, or obtuse cones) as a methodological break from the tradition of which Archimedes was a part. All in all, while Eutocius is an outlier in late antiquity for his interest in a heurist author at all, he still leans toward the systematist tradition even within his treatment of heurist texts.

Transmission, Reception, and Modern Mathematics

The story of late antique mathematics, then, is a story of the gradual predominance of systematist texts and perspectives. This predominance was eventually passed down to early modern scholarship, and has become embedded in modern mathematics and mathematical history. The Arabic and Latin translators of the middle

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[49] Heiberg, *Conica*, vol. 2, 168-360.

ages preserved works of both schools,[50] and the history of transmission provides some hints that the difference between the schools was still felt, if not clearly articulated. Works of the same school tended to be transmitted together. For example, the majority of manuscripts of Archimedes traveled not only with the commentaries of Eutocius, but also with Heron's *De mensuris*.[51] William of Moerbeke, the thirteenth-century translator of the works of Aristotle and Archimedes, also translated Heron's *Catoptrica* and Ptolemy's *Analemma*,[52] but translated no systematist works at all.[53] On the systematist side, the works of the *Little Astronomy* consistently traveled together,[54] and Euclid and Apollonius were particularly linked in the Arabic tradition, possibly through the influence of Hypsicles' *Book XIV*.[55]

The Renaissance saw a revival of the study of Greek and Platonist philosophy. The first editions of the texts of Euclid's *Elements*, Ptolemy's *Almagest*, the major works of Archimedes with Eutocius' commentaries, and the first four books of Apollonius' *Conics* were produced in the 16th century. The complex development of Renaissance Neoplatonism and its relation to mathematical scholarship are far beyond the scope of

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[50] For a conspectus, see Richard Lorch, "Greek-Arabic-Latin: The Transmission of Mathematical Texts in the Middle Ages," *Science in Context* 14, no. 1–2 (June 2001): 313–331.
[51] Heiberg and Stamatis, *Archimedis Opera Omnia, cum Commentariis Eutocii*, vol. 3, ix-xv. See also Heath, *The Works of Archimedes*, xxiii-xxviii.
[52] J. L. Heiberg, ed., *Claudii Ptolemaei Opera quae exstant Omnia* (Leipzig: Teubner, 1898), vol. 2, 194-216. The *Analemma* is a text on a spherical diagram that predicts the position of the sun.
[53] Lorch, "Greek-Arabic-Latin," 319. See also Marshall Clagett, "William of Moerbeke: Translator of Archimedes," *Proceedings of the American Philosophical Society* 126, no. 5 (1982): 356–366.
[54] J. L. Heiberg, *Litterargeschichtliche Studien über Euklid* (Leipzig: Teubner, 1882), 152. See also Heath, *Aristarchus of Samos, the Ancient Copernicus*, 317-321.
[55] Heath, *The Thirteen Books of Euclid's Elements,* vol. 1, 5-6.

this project, but the sheer numbers of editions, translations, and commentaries on the *Elements* alone produced during the 16th century speak to a widespread enthusiasm for Greek mathematics and for Euclid's style in particular.[56] This enthusiasm has lasted down to the present day—it was only recently that Euclid's *Elements* ceased to be the primary school text for teaching geometry. Albert Einstein's public facing book on the theory of relativity begins: "In your schooldays most of you who read this book made acquaintance with the noble building of Euclid's geometry, and you remember—perhaps with more respect than love—the magnificent structure, on the lofty staircase of which you were chased about for uncounted hours by conscientious teachers."[57] Euclid, of course, is not the only Greek mathematician to have been preserved in the modern popular imagination, but his work is easily the most familiar.[58] His enduring popularity is certainly due to the accessibility and comprehensiveness of the *Elements*, but it is probably also a factor behind the view of modern scholarship that the style in which he wrote was quintessentially Greek[59]—that it was the style of Greek mathematics done to

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[56] For a list of these editions, translations, and commentaries, see Heath, *The Thirteen Books of Euclid's Elements,* vol. 1, 97-113.
[57] Einstein, *Relativity: The Special and General Theory*, 3.
[58] Cf. Heath, *HGM*, vol. 1, 358: "Scarcely any other book except the Bible can have circulated more widely the world over, or been more edited and studied."
[59] This attitude is pervasive. For one pointed example, consider this passage from Flaumenhaft's introduction to Taliaferro's translation of Apollonius' *Conics*: "Readers of Apollonius will gain a knowledge of the character of classical mathematics through an introductory treatment of the conic sections. It is in the classical study of the conic sections that the modern reader can most easily see both the achievement of classical mathematics and the difficulty that led Descartes and his followers to turn away from classical mathematics. It has to do with ratio, and with notions of number and of magnitude. For Apollonius, as for Euclid before him, the handling of ratios is founded upon a certain view of the relation between numbers and magnitudes. When Descartes made his new beginning, almost two millennia later, he said that the ancients were handicapped by their having a scruple against using the terms of arithmetic in

the highest standard, rather than one option among stylistic traditions. A recognition of the systematist and heurist schools as distinct traditions will in no way diminish the importance of Euclid or the other systematist authors in the history of mathematics in the west; but hopefully it will lift heurist texts out of the shadow of comparison to an imagined Euclidean standard, and open avenues of inquiry into the intersections of style, method, and philosophy that clearly influenced the development of mathematics in the ancient world, and continue to do so today.

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geometry" (Apollonius, *Conics Books I-III*, trans. R. Catesby Taliaferro (Santa Fe, N.M.: Green Lion Press, 2000), xxii). Note that no mention is made either of Archimedes' very different work on geometry (and conics in particular), nor of the fact that actually many ancient mathematicians applied the terms of arithmetic to geometry. It is only that these mathematicians belonged to the unrecognized heurist school and were considered inferior to the supposed Greek standard of Euclid and the (also unrecognized) systematists.

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