Equivariant Nahm Transforms and Minimal Yang–Mills Connections

by

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Dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mathematics in the Graduate School of Duke University

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ABSTRACT

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Abstract

This dissertation examines two different subjects within the study of instantons: the construction of Nahm transforms for instantons invariant under certain group actions; and a generalization of the proof that Yang–Mills minimizers are instantons.

The first Nahm transform examined is the ADHM construction for S^1-invariant instantons on S^4, which correspond to singular monopoles on $\mathbb{R}^3$. In this case, there is a decomposition of the ADHM data in terms of S^1-subrepresentations of $\ker D$. The moduli spaces of S^1-invariant $SU(2)$-instantons are given up to charge 3, and examples of ADHM data for instantons of charge 4 are also provided.

The second Nahm transform considered is for instantons on a certain flat quotient of $\mathbb{R}^4$ with nonabelian fundamental group. Equivalently, one can consider these to be $\mathbb{Z}_2$-invariant instantons on T^4, and the Nahm transform yields instantons invariant under a crystallographic action.

In our study of minimal Yang–Mills connections, we extend results of Bourguignon–Lawson–Simons and Stern, who showed that connections that minimize $\|F_\nabla\|^2$ on homogeneous manifolds must be instantons or have instanton subbundles. We extend the previous arguments by considering variations constructed using conformal vector fields, and also allow these vector fields to be incomplete. We prove a minimality result over a half-cylinder.

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Contents

Chapter 1

Introduction

An *instanton* is a connection ∇ on a hermitian vector bundle E over a four-manifold M such that the curvature is self-dual or anti-self-dual, $F_\nabla = \pm * F_\nabla$. Instantons form a special case of Yang–Mills connections, which are critical points of the Yang–Mills Lagrangian $\frac{1}{2}\|F_\nabla\|_{L^2}^2$. The study of Yang–Mills connections, as well as critical points of other similar Lagrangians, constitutes gauge theory. The Yang–Mills equations originated in physics, where they were proposed as a non-abelian generalization of Maxwell's Equations. Indeed, one can obtain Maxwell's Equations as a special case of Yang–Mills by letting E be a line bundle over Minkowski space.

From a mathematical viewpoint, instantons are a natural object to study when considering vector bundles. The curvature of a connection measures an inherent non-flatness in the bundle E, generalizing Riemannian curvature on the tangent bundle. As such, minimizing $\|F_\nabla\|_{L^2}^2$ can be interpreted as finding a connection on E that is as close to flat as possible. That the minimal curvature may be nonzero is due to topological obstructions. The study of the space of instantons has led to many interesting advances in geometry, including notably being used to prove Donaldson's Theorem regarding the topology of differential four-manifolds. Gauge theory continues to be an active area of research at the intersection of geometry, analysis, topology, and physics.

An important foundational piece in gauge theory is the ADHM construction, which parametrizes $SU(n)$-instantons on S^4 by linear algebraic data satisfying certain quadratic equations. A generalization of this construction yielded the Nahm transform, often described as a Fourier transform for instantons. The transform gives a correspondence between instanton-like objects on two spaces that are dual to each other in some sense. In its various forms, the transform has proved very successful in studying various gauge-theoretic objects, such as monopoles, solutions to Hitchin's equations, and instantons with various periodic conditions. For this work, we concern ourselves in particular with Nahm transforms for instantons invariant under certain group actions.

In Chapter 2, we introduce notation and provide some background information on instantons. We describe the process of dimensional reduction, with an emphasis on monopoles as a reduction of instantons to $\mathbb{R}^3$. We end the chapter by collecting facts about spinor bundles and Dirac operators that will be necessary for our descriptions of the Nahm transform.

In Chapter 3, we describe the Nahm transform in varying levels of generality. We begin with a broad and intuitive look at the Nahm transform over general manifolds, following the viewpoint of [Jar02]. We then specialize to instantons on $\mathbb{R}^4$ satisfying various invariance and periodicity conditions, giving a general heuristic for constructing a Nahm transform in these conditions. The bulk of the chapter is then devoted to the details of constructing the Nahm transform in two specific cases: instantons on a torus T^4, and the ADHM construction on $\mathbb{R}^4$. The first provides a useful foundation for considering the Nahm transform on crystallographic quotients of $\mathbb{R}^4$ in Chapter 5, while the second is directly relevant to the construction in Chapter 4.

In Chapter 4, we describe a correspondence between singular monopoles on $\mathbb{R}^3$ and S^1-invariant instantons on $\mathbb{R}^4$, viewed away from 0 as a non-trivial S^1-fibration. We decompose the ADHM data in terms of S^1-subrepresentations of $\ker D$, and write the corresponding decomposition of the ADHM equations. We evaluate the character of the S^1-representation, first for structure group $SU(2)$ and then more generally for $SU(n)$. We provide a description of moduli spaces of S^1-invariant $SU(2)$-instantons of charge up to 3, and also give examples of S^1-invariant ADHM data in charge 4. Some of the work for this chapter was completed in collaboration with Benoit Charbonneau.

In Chapter 5, we provide a brief introduction crystallographic groups and Bieberbach groups, which are cocompact discrete subgroups of isometries on $\mathbb{R}^n$. We then give an example of such a group B, yielding a flat compact manifold $M = \mathbb{R}^4/B$ with nonabelian fundamental group. A Nahm transform on M is constructed, with the transformed connection being an instanton on $\mathbb{R}^4$ that is invariant under a crystallographic action.

Chapter 6 is independent of the preceeding chapters. We consider the converse to the statement that instantons are minimizers of the Yang–Mills energy. In [BLS79, BL81] it was shown that stable Yang–Mills connections over compact homogeneous spaces and with structure group $G = SU(2)$, $U(2)$, or $SU(3)$ are instantons. In [Ste10] it was shown that upon passing to certain subbundles of the adjoint bundle, the statement is true for arbitrary compact structure group and for complete (but not necessarily compact) homogeneous spaces. We generalize the argument from [Ste10], allowing variations defined with conformal and incomplete vector fields. We then prove a minimality result for Yang–Mills connections on a half-cylinder.

Chapter 2

Preliminaries

2.1 Yang–Mills connections

Let G be a Lie group and let $P \to M$ be a principal G-bundle. For a representation $\rho : G \to GL(V)$, we define the bundle associated to P by ρ to be

$$P \times_\rho V = (P \times V)/\sim \tag{2.1.1}$$

where $\sim$ denotes the equivalence relation $(p, v) \sim (pg, \rho(g^{-1})v)$ for any $g \in G$. We then say a vector bundle E has a *G-structure* (or equivalently, call E a *G-vector bundle*) if it is an associated bundle to some principal G-bundle. We shall always assume that G is compact.

When $G = U(n)$ or $SU(n)$, and absent any additional qualification, we will take ρ to be the standard representation on $\mathbb{C}^n$. With this assumption, a $U(n)$-vector bundle is a hermitian bundle of rank n. An $SU(n)$-vector bundle is a hermitian bundle of rank n equipped with a complex volume form $\nu \in \det(E)$.

A local section σ of P, defined over some open set U, defines a local trivialization of E by

$$\begin{aligned} U \times V &\to E|_U \\ (x, v) &\mapsto [\sigma(x), v]. \end{aligned} \tag{2.1.2}$$

If we fix a basis $\{v_i\}$ for V, we then obtain a local frame for E given by the sections $s_i(x) = [\sigma(x), v_i]$. A frame of E corresponding to a section of P is called a *gauge*. A different choice of gauge over U is given by a different section $\tilde{\sigma}$, which is related to the original frame by $\tilde{\sigma} = \sigma g$ for some $g = g(x)$ a G-valued function. We call g a *gauge transformation.* Since

$$[\sigma', v] = [\sigma g, v] = [\sigma, \rho(g)v], \tag{2.1.3}$$

change of gauge from σ' to σ corresponds to acting by $\rho(g)$ on V.

Remark. In the literature, there is inconsistency in the use of the term 'gauge group.' Depending on the author, it can mean either the group G or the group $\mathcal{G}$ of G-valued functions defining gauge transformations. In this work the issue will be avoided by referring to G as the structure group, and to $\mathcal{G}$ as the group of gauge transformations.

For a G-vector bundle E, we define its adjoint bundle $\mathrm{ad}(E)$ to be the vector bundle associated to the frame bundle P by the adjoint representation of G,

$$\mathrm{ad}(E) = P \times_{\mathrm{ad}} \mathfrak{g}. \tag{2.1.4}$$

For x in the base manifold M, an element $\Phi(x) \in \mathrm{ad}(E)_x$ defines an endomorphism of the fibre E_x via the induced representation $d\rho$ of $\mathfrak{g}$ on E_x. That is, for σ a section of P and $\Phi(x) \in \mathfrak{g}$, we have an $\mathrm{End}(E)$-valued function given by

$$[\sigma(x), \Phi]([\sigma(x), v]) = [\sigma(x), d\rho(\Phi(x))v]. \tag{2.1.5}$$

We view $\Phi(x)$ as a locally $\mathfrak{g}$-valued function such that a gauge transformation acts as $\Phi \mapsto \mathrm{ad}(g)\Phi$. Since we will commonly take G to be a matrix Lie group, we write $\mathrm{ad}(g)\Phi = g\Phi g^{-1}$.

A connection ∇ on E is called a G-connection if, with respect to a local gauge, parallel transport by ∇ is a g-valued function. Equivalently, G-connections are obtained from a connection on P by the associated bundle construction. In the case $G = U(n)$, such connections are precisely those that are compatible with the hermitian inner product on fibres, in the sense that

$$\nabla_X \langle s_1, s_2 \rangle = \langle \nabla_X s_1, s_2 \rangle + \langle s_1, \nabla_X s_2 \rangle \tag{2.1.6}$$

for all $X \in \Gamma(TM)$ and $s_1, s_2 \in \Gamma(E)$. An $SU(n)$-connection is compatible with the hermitian inner product, and additionally has $\nabla_X \nu = 0$, where $\nu \in \det(E)$ is the complex volume form, and where ∇ is taken as the induced connection on $\det(E)$.

Recall that for a connection ∇, we define the connection 1-form with respect to a local frame $s = \{s_i\}$ to be the matrix-valued 1-form A such that

$$\nabla_X s_i = [A(X)]_{ji} s_j. \tag{2.1.7}$$

If ∇ is a G-connection, then parallel transport by ∇ is G-valued, and hence with respect to a local gauge A is $\mathfrak{g}$-valued. Conversely, if A is locally $\mathfrak{g}$-valued, then ∇ is a G-connection. Recall that a connection on P is given as a $\mathfrak{g}$-valued 1-form ω on TP. Given a section $\sigma \in \Gamma(P)$, the pullback $\sigma^*\omega$ defines the connection form A of the corresponding connection on E from the associated bundle construction.

Note that we refrain from saying the connection form is $\mathrm{ad}(E)$-valued since it does not transform appropriately under change of gauge. Connections transform as $\nabla \mapsto g\nabla g^{-1}$, which in terms of connection form A is the transformation

$$A \mapsto gAg^{-1} + gdg^{-1}. \tag{2.1.8}$$

We say two connections ∇^1 and ∇^2 are *gauge equivalent* if there is a gauge transformation such that $\nabla^1 = g\nabla^2 g^{-1}$. In this case, we view the two connections as differing only in choice of local trivialization. We study the connections on E modulo gauge equivalence.

Yang–Mills theory is the study of a special class of connections, to be defined below. The gdg^{-1} term in the transformation of the connection form makes it difficult to study a connection directly in terms its behaviour with respect to a local frame. In particular, even if the connection form vanishes with respect to one gauge, applying a non-constant gauge transformation g yields non-zero connection form gdg^{-1}.

As with connections on the tangent bundle, we study a connection on E by its curvature, generalizing Riemannian curvature. We define the exterior covariant derivative d_∇ on $\Omega^p(E) = \Gamma(\bigwedge^p TM \otimes E)$ such that $d_\nabla(\omega \otimes s) = (d\omega) \otimes s + (-1)^p \omega \otimes \nabla s$. Unlike the exterior derivative, d_∇^2 is in general non-vanishing. We define the curvature of ∇ to be $F_\nabla \in \Omega^2(\mathrm{ad}(E))$, given by

$$F_\nabla s = d_\nabla^2 s, \tag{2.1.9}$$

It can be checked that under change of gauge the curvature transforms as $F_\nabla \to gF_\nabla g^{-1}$, so F_∇ is indeed $\mathrm{ad}(E)$-valued. With respect to a local gauge in which ∇ has connection form A, we have the expression for F_∇ as

$$F_\nabla = dA + \frac{1}{2} A \wedge A. \tag{2.1.10}$$

Here, exterior product of $\mathfrak{g}$-valued forms is the tensor product of the Lie bracket on $\mathfrak{g}$ with the exterior product on forms. Note that the anticommutativity of the Lie bracket combined with graded commutativity for forms means that for $A, B \in \Omega^1(M, \mathfrak{g})$ we have $A \wedge B = B \wedge A$. In particular, we do not have the usual vanishing of $A \wedge A$ like we do for usual differential 1-forms. In coordinates x_i with the connection form $A = A_i dx^i$, the curvature $F_\nabla = F_{ij} dx^i \wedge dx^j$ is given by

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]. \tag{2.1.11}$$

Since we are assuming G to be compact, $\mathrm{ad}(E)$ inherits an ad-invariant inner product on fibres from the ad-invariant inner product on $\mathfrak{g}$. In the $SU(n)$-case, this is given by $\langle A, B \rangle = -\operatorname{trace}(AB)$.

We define the *Yang–Mills Lagrangian* on the space of connections to be

$$\mathcal{Y}(\nabla) = \frac{1}{2} \|F_\nabla\|_{L^2}^2 = \frac{1}{2} \int_M |F_\nabla|^2 \, d\,\mathrm{vol}_M \,. \tag{2.1.12}$$

Note that ad-invariance of the inner product on fibres implies that $|gF_\nabla g^{-1}| = |F_\nabla|$, and so $\mathcal{Y}$ is invariant under gauge transformation. We are interested in critical points of this Lagrangian. Solving the Euler-Lagrange equations gives the *Yang–Mills equation*,

$$d_\nabla^* F_\nabla = 0. \tag{2.1.13}$$

Here $d_\nabla^* = (-1)^{n(p+1)+1} * d_\nabla * : \Omega^p(\mathrm{ad}(E)) \to \Omega^{p-1}(\mathrm{ad}(E))$ is the L^2-adjoint of d_∇. A solution to the Yang–Mills equation is called a *Yang–Mills connection.*

The Yang–Mills equation is a second order non-linear PDE in the components of ∇. Coupled with the Bianchi identity

$$d_\nabla F_\nabla = 0, \tag{2.1.14}$$

which is true for all connections, the Yang–Mills equation is elliptic.

We can also consider a Lagrangian for pairs (∇, Φ) consisting of a connection ∇ and section of $\mathrm{ad}(E)$. We define the *Yang–Mills–Higgs Lagrangian* to be

$$\tilde{\mathcal{Y}}(\nabla, \Phi) = \frac{1}{2} \int_M |F_\nabla|^2 + |d_\nabla \Phi|^2 \, d\,\mathrm{vol}\,. \tag{2.1.15}$$

The Yang–Mills–Higgs Lagrangian is sometimes taken to include a potential function $U(\Phi)$ in the integrand, but for our current purposes we take $U = 0$. We call Φ the *Higgs field.* The Euler–Lagrange equations give the *Yang–Mills–Higgs equations*

$$d_\nabla^* F_\nabla = [d_\nabla \Phi, \Phi], \tag{2.1.16}$$

$$d_\nabla^* d_\nabla \Phi = 0. \tag{2.1.17}$$

2.2 Instantons

In dimension four, there is a special class of solution to the Yang–Mills equation.

Definition 2.2.1. A connection ∇ is said to be an *instanton* if its curvature satisfies the self-duality or anti-self-duality equation,

$$F_\nabla = \pm * F_\nabla. \tag{2.2.2}$$

Observe that because $*\colon \Omega^p \to \Omega^{n-p}$, and because F_∇ is a 2-form, this equation is specific to dimension four. In most of what follows, we will take instantons to have anti-self-dual curvature, the self-dual case being given by changing orientation. The exception is in Chapter 6, where we consider both self-dual and anti-self-dual curvature.

The Yang–Mills equation (2.1.12) and the Bianchi identity (2.1.14) together show that an instanton is always Yang–Mills. Furthermore, on compact manifolds, instantons are minimizers of the Yang–Mills Lagrangian. To see this, note that the (anti)-self-dual parts of the curvature $F_\nabla^\pm = \frac{1}{2}(1 \pm *)F_\nabla$ are orthogonal to each other, and so

$$\|F_\nabla\|^2 = \|F_\nabla^+\|^2 + \|F_\nabla^-\|^2. \tag{2.2.3}$$

On the other hand,

$$\begin{aligned}\int_M \operatorname{trace}(F_\nabla \wedge F_\nabla) &= \int_M \operatorname{trace}(F_\nabla^+ \wedge F_\nabla^+) + \int_M \operatorname{trace}(F_\nabla^- \wedge F_\nabla^-) \\ &= \|F_\nabla^+\|^2 - \|F_\nabla^-\|^2. \end{aligned} \tag{2.2.4}$$

Recall by Chern–Weil theory that

$$\frac{1}{8\pi^2}\int_M \operatorname{trace}(F_\nabla \wedge F_\nabla) = c_2(E) - c_1(E)^2 \tag{2.2.5}$$

is a topological invariant of the bundle, where $c_1(E)$ and $c_2(E)$ denote the first and second Chern classes of E, and we omit evaluation on M from the notation. Therefore, by (2.2.3), (2.2.4), and (2.2.5) we have that

$$\|F_\nabla\|^2 = 8\pi^2(c_2(E) - c_1(E)^2) + 2\|F_\nabla^-\|^2 = -8\pi^2(c_2(E) - c_1(E)^2) + 2\|F_\nabla^+\|^2, \tag{2.2.6}$$

and thus $\|F_\nabla\|^2$ is minimized if either F_∇^+ or F_∇^- vanishes. Note that the converse to the statement does not a priori hold, and indeed is the subject of Chapter 6.

Returning to the convention that instantons have anti-self-dual curvature, a closer look at the above argument shows that instantons satisfy $\|F_\nabla\|^2 = \frac{k}{8\pi^2}$, where $k =$

$-(c_2(E) - c_1(E)^2)$ is an integer. We call k the *charge* of the instanton. The first Chern class is obtained by taking the trace of the curvature, suitably normalized. If the structure group is $SU(n)$, then F_∇ is $\mathfrak{su}(n)$-valued and hence traceless. Therefore, $SU(n)$-vector bundles E have $c_1(E) = 0$, and for an $SU(n)$-instanton the charge is simply $-c_2(E)$.

2.3 Dimensional reduction

From the anti-self-duality equation, we can obtain other gauge theoretic equations of interest on spaces of lower dimension by the process of dimensional reduction. We describe this process over $\mathbb{R}^4$, on which we fix Euclidean coordinates x_i. Moreover, by contractibility we may fix a global frame for E. Let ∇ be an instanton on $\mathbb{R}^4$, and suppose that ∇ is constant in the last $4 - m$ directions $x_{m+1}, \ldots, x_4$ for some $0 \leq m < 4$, by which we mean that the connection form $A = A_i dx^i$ with respect to the global frame is constant in these directions. We can then define a connection ∇_B with connection form $B = \sum_{i=1}^m A_i dx^i$ on $\mathbb{R}^m \subset \mathbb{R}^4$. The leftover components of A define endomorphisms of the bundle, $\Phi_i = A_i$ for $m + 1 \leq i \leq 4$. Writing the instanton equation for A then gives equations in terms of B, $d_B\Phi_i$, and commutators of Φ_i on $\mathbb{R}^{j-1}$. We call these equations the dimensional reduction of the instanton equation to $\mathbb{R}^m$.

As an example, consider an instanton that is constant in one variable x_4. Then we let $B = A_1 dx^1 + A_2 dx^2 + A_3 dx^3$ give a connection on $\mathbb{R}^3$, and let $\Phi = A_4$. Since $\partial_4 A_i = 0$, the curvature of F_A in coordinates is then

$$(F_A)_{ij} = (F_B)_{ij} \qquad \text{if } i, j \neq 4, \tag{2.3.1}$$

$$(F_A)_{i4} = \partial_i \Phi + [A_i, \Phi] = (\nabla_B)_i \Phi \qquad \text{if } i \neq 4. \tag{2.3.2}$$

The anti-self-duality equations in coordinates pair terms that do not involve the index 4 with terms that do. Using equations (2.3.1) and (2.3.2), the anti-self-duality equations become

$$(F_B)_{ij} = -(\nabla_B)_k \Phi \tag{2.3.3}$$

for cyclic permutations (i, j, k) of $(1, 2, 3)$. These equations are expressed succinctly by the *Bogomolny Equation*

$$F_\nabla = - * d_\nabla \Phi. \tag{2.3.4}$$

Definition 2.3.5. A pair ∇, Φ is called a *monopole* if it satisfies the Bogomolny Equation (2.3.4).

The above discussion shows that translation-invariant instantons on $\mathbb{R}^4$ have a corresponding monopole on $\mathbb{R}^3$. From a monopole, constructing a translation-invariant instanton by $\nabla = \nabla_B + \Phi \otimes dx^4$, shows the correspondence is one-to-one.

Remark. The framework given for dimensional reduction worked because we could fix a global frame for E, with respect to which it made sense to say that the connection was constant in certain directions. When the manifold M is not contractible, it may not be possible to fix a global frame (or, as in Chapter 4, we may have reasons to choose other frames), and so we instead use the language of invariant connections. Let H be a group of isometries of M with a lift to an action on the bundle E. Since this action identifies the fibres $E_x \cong E_{hx}$, we can then identify the bundles $h^*E \cong E$. Therefore, we may consider the pullback connection $h^*\nabla$ as a connection on E, and we say ∇ is H*-invariant* if $h^*\nabla = \nabla$ for all $h \in H$.

A monopole is a solution to the Yang–Mills–Higgs equations. Indeed, to see it satisfies (2.1.16), note that if (∇, Φ) is a monopole

$$\begin{aligned} d_\nabla^* F_\nabla &= *d_\nabla * F_\nabla \\ &= - * d_\nabla^2 \Phi \\ &= -[*F_\nabla, \Phi] \\ &= [d_\nabla \Phi, \Phi]. \end{aligned}$$

To see it solves (2.1.17),

$$\begin{aligned} d_\nabla^* d_\nabla \Phi &= - * d_\nabla * d_\nabla \Phi \\ &= *d_\nabla F_\nabla \\ &= 0. \end{aligned}$$

Remark. We could also define monopoles to be solutions to $F_\nabla = *d_\nabla \Phi$. This corresponds to self-dual instantons rather than anti-self-dual.

As a final observation, note that a translation-invariant instanton has constant curvature in the x_4 direction. Therefore, either $F_\nabla = 0$ identically, or $\|F_\nabla\|_{L^2} = \infty$. To obtain non-flat finite-energy instantons, by which we mean $\|F_\nabla\|_{L^2} < \infty$, one could replace the direction x_4 with a copy of S^1 so that the integral of a constant function is no longer infinite. This process can be generalized: monopoles on a manifold M correspond to S^1-invariant instantons on $M \times S^1$. In Chapter 4, we consider a situation where monopoles correspond to finite-energy instantons on $\mathbb{R}^4$ as a non-trivial S^1 fibration over $\mathbb{R}^3$, at the cost of introducing a singularity at 0.

2.4 Dirac operators

Constructing Nahm transforms in Chapters 3 and 4 will necessitate the use of Dirac operators, and so we recall some important facts. For a more in-depth treatment, readers are directed to [LM89] (especially for a thorough treatment of Clifford algebras and spin representations), [BGV04], and [Roe98] (which the author of this dissertation found to be a very accessible introduction to the subject). We assume some familiarity with spin groups, spin structures on manifolds, and the spin representation. As all of our applications will be in dimension 4, we restrict attention to this case.

Let Δ be the spin representation of Spin(4), which is a 4-dimensional representation. Note that Δ is not irreducible as a Spin(4)-representation, but instead decomposes as discussed below. With the standard embedding of $\mathrm{Spin}(4) \subset \mathrm{Cl}(\mathbb{R}^4)$, Δ extends to a left Clifford module. For e_i an orthonormal basis of $\mathbb{R}^4$, let c_i denote Clifford multiplication by $c(e_i)$, and let multiple indices denote Clifford multiplication by the product $c_{i_1 i_2 \ldots i_k} = c_{i_1} c_{i_2} \ldots c_{i_k}$. We will continue using this notation throughout, where more generally e_i will be an orthonormal frame on TM. There is a hermitian inner product on Δ for which Clifford multiplication by c_i is skew-hermitian.

We define a linear map $\bigwedge^k \mathbb{R}^4 \to \mathrm{Cl}(\mathbb{R}^4)$ on basis elements by $c(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}) = c_{i_1 i_2 \ldots i_k}$. Because c_i and c_j anticommute for $i \neq j$, this map is well-defined regardless of the ordering of the indices $i_1, \ldots, i_k$ in the exterior product. This map is not an algebra homomorphism, however, as can be seen by noting $c(e_i)^2 = -1$, while $c(e_i \wedge e_i) = 0$.

Define a volume element on $\mathbb{R}^4$ by $\mathrm{vol} = e_1 \wedge e_2 \wedge e_3 \wedge e_4$. Then note that $c(\mathrm{vol})^2 = c_{1234}^2 = 1$, and so the eigenvalues of $c(\mathrm{vol})$ acting on Δ must be 1 or -1. Let $\Delta^{\pm}$ denote the ∓ 1-eigenspaces of $c(\mathrm{vol})$ (note the sign convention).

Since c_i anticommutes with $c(\mathrm{vol})$, if $v \in \Delta^{\pm}$ then $c_i v \in \Delta^{\mp}$. Since $c_i^2 = -1$, and hence c_i defines an isomorphism $\Delta^+ \xrightarrow{\cong} \Delta^-$. In particular, since $\dim(\Delta) = 4$ and as an eigendecomposition $\Delta = \Delta^+ \oplus \Delta^-$, both Δ^+ and Δ^- must be nonempty and have dimension 2. In general, multiplication by odd elements of $\mathrm{Cl}(\mathbb{R}^4)$ maps $\Delta^{\pm} \to \Delta^{\mp}$, while multiplication by even elements preserves the decomposition. Since

Spin(n) lies inside the even part of the Clifford algebra, each $\Delta^{\pm}$ define Spin(n)-subrepresentations of Δ. In fact, $\Delta^{\pm}$ are irreducible subrepresentations.

We now consider the multiplication by self-dual and anti-self-dual forms on $\Delta^{\pm}$. It can be checked by direct computation that $c(\mathrm{vol})c(e^i \wedge e^j) = -c(*e^i \wedge e^j)$ for $i \neq j$. Therefore, if $v \in \Delta^+$,

$$c(e^i \wedge e^j)v = -c(\mathrm{vol})c(e^i \wedge e^j)v = c(*e^i \wedge e^j)v. \tag{2.4.1}$$

In particular, this shows that $c(e^i \wedge e^j - *e^i \wedge e^j) = 0$ on Δ^+. A similar argument shows that $c(e^i \wedge e^j + *e^i \wedge e^j) = 0$ on Δ^-.

$$\begin{aligned} c(\textstyle\bigwedge^-)|_{\Delta^+} &= 0, \\ c(\textstyle\bigwedge^+)|_{\Delta^-} &= 0, \end{aligned} \tag{2.4.2}$$

where $\bigwedge^2 = \bigwedge^+ \oplus \bigwedge^-$ denotes the decomposition into self-dual and anti-self-dual forms.

Recall the definition of a spin structure on M as a principal Spin(n)-bundle P, with a map from P to the orthonormal frame bundle of M given by conjugation in $\mathrm{Cl}(TM)$. We then define the bundle of spinors to be the associated vector bundle

$$S = P \times_{\mathrm{Spin}(n)} \Delta. \tag{2.4.3}$$

The vector bundles associated to $\Delta^{\pm}$ then give subbundles $S^{\pm}$, called the positive and negative spinor bundles. Clifford multiplication by $X \in \Gamma(TM)$ gives a bundle map $c(X) : S^{\pm} \to S^{\mp}$. Moreover, the bundle S inherits a hermitian inner product from Δ, with respect to which Clifford multiplication is skew-hermitian.

The Levi-Civita connection on M lifts to a connection on the spin structure P, which then induces a connection ∇ on S. This connection is compatible with the hermitian structure on S, as well as with Clifford multiplication in the sense that

$$\nabla_X c(Y)s = c(\nabla_X Y)s + c(Y)\nabla_X s \tag{2.4.4}$$

for $X, Y \in \Gamma(TM)$ and $s \in \Gamma(S)$.

Given a local orthonormal frame e_i, we define the Dirac operator $D : \Gamma(S) \to \Gamma(S)$ locally to be

$$Ds = c_i \nabla_{e_i} s. \tag{2.4.5}$$

This definition does not depend on choice of frame e_i, and so extends to a well-defined global operator.

More generally, let E be a hermitian vector bundle with metric-compatible connection ∇_E. Then tensoring with the spin connection ∇_S on S, we obtain the connection $\nabla_{E\otimes S} = \nabla_E \otimes 1 + 1 \otimes \nabla_S$. Clifford multiplication acts on $E \otimes S$ by $c_i = 1 \otimes c_i$, and we define the Dirac operator on $E \otimes S$ by

$$D_{E\otimes S} = c_i (\nabla_{E\otimes S})_{e_i}. \tag{2.4.6}$$

Henceforth, we will only be considering the Dirac operator on bundles of the form $E \otimes S$, and as such we simplify notation by writing $D = D_{E\otimes S}$.

Let $D^{\pm} = D|_{E\otimes S^{\pm}}$. The spin connection ∇_S preserves the decomposition $S = S^+ \oplus S^-$, while Clifford multiplication maps $S^{\pm} \to S^{\mp}$. As such, taken together in the definition of the Dirac operator, we have that $D^+ : \Gamma(E \otimes S^+) \to \Gamma(E \otimes S^-)$ and $D^- : \Gamma(E \otimes S^-) \to \Gamma(E \otimes S^+)$.

We end our background on Dirac operators by proving the *Lichnerowicz formula*

$$D^2 = \nabla^*\nabla + c(F_\nabla) + \frac{1}{4}R, \tag{2.4.7}$$

where $c(F_\nabla) = \sum_{i<j} F_{ij} c_{ij} \in \mathrm{End}(E \otimes S)$, and where R denotes the scalar curvature on M. By looking at its restriction to S^+, we may begin to see the relation of this formula to the study of instantons. Since $c(\bigwedge^-) = 0$ on S^+, we have that

$$D^-D^+ = \nabla^*\nabla + c(F_\nabla^+) + \frac{1}{4}R. \tag{2.4.8}$$

If $F_\nabla^+ = 0$ and if M is flat, then we have $D^-D^+ = \nabla^*\nabla$. This fact will be very important to our discussion of the Nahm transforms in Chapters 3, 4, and 5.

We prove (2.4.7) at a fixed point p. Let e_i be an orthonormal frame for TM near p such that $(\nabla_i)_p e_j = 0$ for all i, j. Evaluating the lefthand side,

$$\begin{aligned} D^2 s &= c_i \nabla_i c_j \nabla_j s \\ &= c_i^2 \nabla_i \nabla_i + \sum_{i<j} c_{ij} [\nabla_i, \nabla_j] \end{aligned} \tag{2.4.9}$$

The term $c_i^2 \nabla_i \nabla_i = -\nabla_i \nabla_i$ is the local coordinate expression for $\nabla^* \nabla$. The term $[\nabla_i, \nabla_j]$ gives curvature on $E \otimes S$, which decomposes on the tensor product. We write the curvature on $E \otimes S$ as $F_{ij} \otimes 1 + 1 \otimes K_{ij}$, where F_{ij} is the curvature on E and K_{ij} is the curvature on S. Then $\sum_{i<j} c_{ij} F_{ij} = c(F_\nabla)$, and we are left only to evaluate $c_{ij} K_{ij}$.

Since the spin connection on S is obtained from an equivariant lift of the Levi–Civita connection to the spin bundle, its curvature also lifts. Written in terms of the Riemann curvature tensor ${R_{ijk}}^\ell$, the curvature on TM is given by

$$R(\partial_i, \partial_j) = {R_{ijk}}^\ell e_\ell \otimes e^k = \sum_{k<\ell} {R_{ijk}}^\ell (e_\ell \otimes e^k - e_k \otimes e^\ell). \tag{2.4.10}$$

The element $e_\ell \otimes e^k - e_k \otimes e^\ell \in \mathfrak{so}(T_p M)$ lifts to an element of the Lie algebra of $\mathrm{Spin}(n)$, viewed as the even part of $\mathrm{Cl}(TM)$. We obtain this lift explicitly by first exponentiating

$$\begin{aligned} &\exp(t(e_\ell \otimes e^k - e_k \otimes e^\ell)) \\ &\qquad = \cos t(e_k \otimes e^k + e_\ell \otimes e^\ell) + \sin t(e_\ell \otimes e^k - e_k \otimes e^\ell) \in SO(T_p M). \end{aligned} \tag{2.4.11}$$

This then lifts to $(\cos \frac{t}{2} + \sin \frac{t}{2} c_{k\ell}) \in \mathrm{Spin}(n)$, which upon differentiating shows that $e_\ell \otimes e^k - e_k \otimes e^\ell$ corresponds to the action $\frac{1}{2} c_{k\ell}$ on the spinors.

Therefore, the curvature K_{ij} acting on spinors is given by

$$K_{ij} = \frac{1}{2} \sum_{k<\ell} {R_{ijk}}^\ell c_{k\ell} = \frac{1}{4} {R_{ijk}}^\ell c_{k\ell} \tag{2.4.12}$$

where the second equality above is obtained using the skew-symmetry in k and ℓ of both ${R_{ijk}}^{\ell}$ and $c_{k\ell}$. Taking advantage of skew-symmetry in i and j, we can write

$$\sum_{i<j} c_{ij} K_{ij} = \frac{1}{8} {R_{ijk}}^{\ell} c_{ijk\ell}. \tag{2.4.13}$$

Note that if the indices i, j, k are all distinct, then $c_{ijk} = c_{kij} = c_{jki}$. Such terms involve $({R_{ijk}}^{\ell} + {R_{kij}}^{\ell} + {R_{jki}}^{\ell}) c_{ijk}$, which vanishes by the first Bianchi identity. Therefore, the only nonvanishing terms in (2.4.13) are those where $k = i$ or $k = j$, giving

$$\begin{aligned} \frac{1}{8} {R_{ijk}}^{\ell} c_{ijk\ell} &= \frac{1}{8} \left({R_{iji}}^{\ell} c_{iji\ell} + {R_{ijj}}^{\ell} c_{ijj\ell} \right) \\ &= -\frac{1}{4} \operatorname{Ric}_{j\ell} c_{j\ell}. \end{aligned} \tag{2.4.14}$$

Note that $\operatorname{Ric}_{j\ell}$ is symmetric in j, ℓ, while $c_{j\ell}$ is skew-symmetric when $j \neq \ell$. Therefore, the only nonvanishing terms are when $j = \ell$, and so

$$-\frac{1}{4} \operatorname{Ric}_{j\ell} c_{j\ell} = -\frac{1}{4} \operatorname{Ric}_{jj} c_{jj} = \frac{1}{4} R. \tag{2.4.15}$$

Substituting this back into (2.4.9) finishes the proof of the Lichnerowicz formula (2.4.7).

Chapter 3

The Nahm Transform and the ADHM Construction

The Nahm transform is a powerful tool for studying the moduli space of instantons. This is particularly true for quotients of $\mathbb{R}^4$, where it can be viewed as a generalization of the ADHM construction and where Nahm proved his original transform between monopoles and solutions to the Nahm equations [Nah82].

We begin this chapter with a general overview of the Nahm transform, following the viewpoint in [Jar04]. We then specialize to two cases which will be of relevance in following chapters, taking our base manifold to be $\mathbb{R}^4/\mathbb{Z}^4$ or $\mathbb{R}^4$. The first is often called a Fourier transform for instantons, while the second is the ADHM construction. Both of these cases are covered in [DK90, Chapter 3], although our approach more closely resembles those of [Cha04, Nye01].

3.1 General Nahm transform

Let M be a 4-manifold with spin structure. Let S denote the bundle of spinors, with its usual decomposition $S = S^+ \oplus S^-$ into ± 1-eigenbundles of $c(\mathrm{vol})$, where vol denotes the volume form of M. Let ∇ be an instanton on some hermitian vector bundle $E \to M$. We consider specifically instantons with anti-self-dual curvature, the self-dual case being obtained by a change of orientation. We then form the Dirac operator $D\colon \Gamma(E \otimes S) \to \Gamma(E \otimes S)$ associated to ∇ by

$$D = c_i \nabla_i, \tag{3.1.1}$$

where c_i denotes Clifford multiplication by an element e_i of an orthonormal frame of TM and $\nabla_i = \nabla_{e_i}$. Observe that D anticommutes with $c(\mathrm{vol})$, so setting $D^pm = D|_{\Gamma(E\otimes S^\pm)}$ gives

$$D^\pm \colon \Gamma(E \otimes S^\pm) \to \Gamma(E \otimes S^\mp). \tag{3.1.2}$$

Let B be a vector bundle on M with a family of instantons $\{\nabla_{(B_\xi)}\}_{\xi\in T}$, parametrized by some manifold T. In practice, we often take $\nabla_{(B_\xi)}$ to be flat connections. We then define a corresponding family of twisted connections ∇_ξ on $E \otimes B$ by the tensor product $\nabla_\xi = \nabla \otimes 1 + 1 \otimes \nabla_{B_\xi}$. The twisted connection ∇_ξ is again an instanton, as seen by noting that the curvature is $F_\xi = F_\nabla \otimes 1 + 1 \otimes F_{B_\xi}$.

Let D_ξ denote the Dirac operator associated to ∇_ξ. Recall the Lichnerowicz formula (2.4.8)

$$D_\xi^- D_\xi^+ = \nabla^*\nabla + c(F^+_{E\otimes B}) + \frac{1}{4}R,$$

where R denotes the scalar curvature of M. Since ∇_ξ is an instanton, the anti-self-duality of $F_{(E\otimes B)}$ simplifies the above equation. We see, for example, that if R is greater than some positive constant, then $\ker D_\xi^+ = 0$. Even though M will be flat in our cases of interest, we will use this fact in the $\mathbb{R}^4$ case by taking the conformal identification with S^4. In the T^4 case, we will need to make additional assumptions to ensure $\ker D_\xi^+ = 0$.

When $\ker D_\xi^+ = 0$, we define a bundle $\hat{E}$ over T by defining the fibre

$$\hat{E}_\xi = \ker D_\xi^-. \tag{3.1.3}$$

We can consider the fibres $\hat{E}_\xi$ inside of product bundle $T \times L^2(M, E \otimes B \otimes S^-)$. If $\ker D_\xi^+ = 0$ for all $\xi \in T$ and if D_ξ is Fredholm, the index formula tells us that $\dim(\hat{E}_\xi) = -\operatorname{ind}(D_\xi)$ is constant, and therefore $\hat{E}$ is a bundle. Moreover, as a subbundle of the product bundle, $\hat{E}$ inherits an induced connection from the product connection via orthogonal projection. This bundle $\hat{E}$ and the induced connection are called a *Nahm transform* of ∇.

Remark. There are many constructions of Nahm transforms where D_ξ is not Fredholm for every ξ, resulting in singularities in the transformed bundle $\hat{E}$. In the cases we consider here, we will have that D_ξ is Fredholm or, as in the case of $\mathbb{R}^4$, related conformally to a Fredholm operator.

Although the above construction is quite general, in certain applications the transformed bundle has additional interesting properties. In particular, if T is 4-dimensional, then under certain assumptions the transformed connection will also be an instanton, discussed further in [Jar04, §2.2]. For our purposes, we focus on the Nahm transform on quotients of $\mathbb{R}^4$.

3.2 Invariant and periodic instantons on $\mathbb{R}^4$

There is a particularly nice class of Nahm transforms defined for invariant and periodic instantons on $\mathbb{R}^4$. The flatness and parallelizability of $\mathbb{R}^4$ allows the definition of a family of flat line bundles L_ξ, for which the projections onto $\ker D_\xi$ interact well with differentiation in ξ. In many cases, the result is a correspondence between instanton (or its dimensional reduction) on one quotient of $\mathbb{R}^4$ to an instanton (or its dimensional reduction) on a dual quotient of $(\mathbb{R}^4)^*$. We now give a broad picture of this class of Nahm transforms.

Consider an instanton ∇ on $\mathbb{R}^4$ that is invariant under a subgroup Λ of translations of $\mathbb{R}^4$. Such subgroups are of the form $\mathbb{R}^d \times \mathbb{Z}^m$ for $m + d \leq 4$. Such an instanton is constant in the directions corresponding to $\mathbb{R}^d$, and we may equivalently consider a dimensional reduction of the instanton to $\mathbb{R}^{4-d}$. For example, if $\Lambda = \mathbb{R}$, the connection ∇ is equivalent to a monopole on $\mathbb{R}^3$, while if $\Lambda = \mathbb{R}^3$, the connection ∇ is equivalent to a solution of Nahm's Equations. The $\mathbb{Z}^m$ part of Λ adds periodicity assumptions to these various connections.

First consider the situation where the subgroup of translations is discrete, $\Lambda =$

$\mathbb{Z}^m$. We define our twisting connections on a line bundle $L \to \mathbb{R}^4$, which by the contractibility of $\mathbb{R}^4$ are all topologically trivial. As such, we can choose a global frame for L. For $\xi \in (\mathbb{R}^4)^*$ define the $U(1)$-connection ∇_{L_ξ} to have constant connection form $i2\pi\xi$ with respect to the global frame. We let L_ξ denote L equipped with the connection ∇_{L_ξ}, which are thus parametrized by $(\mathbb{R}^4)^*$. Since such a connection is translation-invariant, it descends to the quotient by Λ, defining a flat connection on the topologically trivial line bundle over $M = \mathbb{R}^4/\Lambda$.

Flat bundles are determined by their holonomy on $\pi_1(M) = \Lambda$. Since parallel translation from the origin to a point $x \in \mathbb{R}^4$ is given by $e^{-i2\pi\xi(x)}$, the line bundle L_ξ is determined by $\xi \in (\mathbb{R}^4)^*$ up to translation by the dual group

$$\Lambda^* = \left\{\zeta \in (\mathbb{R}^4)^* \mid \zeta(x) \in \mathbb{Z} \text{ for all } x \in \Lambda\right\}.$$

An equivalent perspective is to view $L_\xi \to \mathbb{R}^4$ as the result of applying the gauge transformation $g(x) = e^{-i2\pi\xi(x)}$ to the product connection d. Such a gauge transformation descends to the quotient $\mathbb{R}^4/\Lambda$ if and only if $\xi \in \Lambda^*$. Therefore, we may consider L_ξ as being parametrized by the dual quotient $(\mathbb{R}^4)^*/\Lambda^*$. Taking the twisted Dirac operator D_ξ on $E \otimes L_\xi \otimes S$ then allows us to construct the Nahm transform as in the previous section, defining a bundle $\hat{E} \to (\mathbb{R}^4)^*/\Lambda^*$ with $\hat{E}_\xi = \ker D_\xi^-$.

In the case where Λ^* contains a continuous subgroup, translation along these directions in $(\mathbb{R}^4)^*$ yields equivalent fibres $\hat{E}_\xi$. As such, identifying these fibres – as we do when taking the quotient $(\mathbb{R}^4)^*/\Lambda^*$ – involves the dimensional reduction procedure for instantons. In particular, we obtain additional maps in $\mathrm{End}(\hat{E})$ corresponding to the components of the connection in directions that collapse in the quotient. The extreme example in this case is when $\Lambda = 0$, so that $\Lambda^* = (\mathbb{R}^4)^*$. More details about interpreting the Nahm transform in this case, which yields the ADHM construction, can be found in Section 3.4.3.

The case in which $\Lambda \supset \mathbb{R}^d \cong \mathrm{span}\{v_1, \ldots, v_d\}$ fits into the same framework with only some minor adaptations. On descending to the quotient, components of the connection form in the directions v_i are lost, but just as in the dimensional reduction of instantons, we keep the information from the lost components in the form of maps $i2\pi\xi(v_i) \in \mathrm{End}(L_\xi)$. Note that the maps $i2\pi\xi_1(v_i)$ and $i2\pi\xi_2(v_i)$ are inequivalent for $\xi_1(v_i) \neq \xi_2(v_i)$. Then multiplying by some constant gives $(\xi_1 - \xi_2)(\varepsilon v) \notin \mathbb{Z}$. However, $\varepsilon v \in \Lambda$, showing that $\xi_1 - \xi_2 \notin \Lambda^*$. So we can still parametrize these bundles L_ξ by $\xi \in (\mathbb{R}^4)^*/\Lambda^*$. The Dirac operator D_ξ can then be defined by pulling back the spinor bundle S on $\mathbb{R}^4$ by an inclusion $\mathbb{R}^4/\mathbb{R}^d \hookrightarrow \mathbb{R}^4$, and then defining $D_\xi = D_0 - i2\pi c(\xi)$.

In ideal settings, the curvature of the transformed bundle $\hat{E}$ satisfies the ASD equation or an appropriate dimensional reduction. In Section 3.3, where we take $\Lambda = \mathbb{Z}^4$, there are no complications, and we see that $\hat{E}$ is in fact an instanton bundle on the dual torus. The argument for self-duality in the case of the torus should be taken as a guiding heuristic for proving self-duality in other cases. By comparison, in Section 3.4 we consider the ADHM construction, which can be thought of as the case $\Lambda = 0$. Here, the argument for self-duality must be modified to account for asymptotic complications, and the resulting ADHM equations can be thought of as 'ASD + contributions from ∞.'

With this framework, one can quickly conjecture correspondences between different dimensional reductions of instantons. If $\Lambda = \mathbb{R}^d \otimes \mathbb{Z}^m$, then $\Lambda^* = \mathbb{R}^{4-d-m} \otimes \mathbb{Z}^m$. As such, one expects an m-fold periodic solution to a $(4-d)$-dimensional reduction of the instanton equation to correspond to an m-fold periodic solution to a $(d+m)$-dimensional reduction. A thorough survey of these various cases is provided in [Jar04, §3.1 - §3.3]. Since the publication of that survey, more cases have been studied, such as in [CH19].

3.3 Nahm transform on T^4

The simplest case of the Nahm transform occurs when $\Lambda = \mathbb{Z}^4$ is a full-rank lattice, and so $M = T^4$. This case is described in [DK90, 3.2] where it is called a Fourier transform for instantons, and the viewpoint is rooted in the equivalent formulation in terms of holomorphic structures. We repeat the description here, with a viewpoint more centred on Dirac spinors, as it provides a template for Chapter 5. Also, as the analysis is greatly simplified by the compactness of T^4, we can take the Nahm transform in this case as a guiding heuristic for proving anti-self-duality of the transformed connection in other cases.

First, we must make an additional assumption on E and ∇ to ensure that that $\ker D_\xi^+ = 0$.

Definition 3.3.1. An $SU(n)$-vector bundle E with connection ∇ is said to be *without flat factors* if there is no splitting $E = E' \oplus L$ that is compatible with ∇ and with L a flat line bundle.

Since T^4 is flat and ∇ is an instanton, the Licherowicz formula (2.4.8) becomes

$$D_\xi^- D_\xi^+ = \nabla_\xi^* \nabla_\xi. \tag{3.3.2}$$

If there exists a non-zero $\varphi \in \ker D_\xi^+$, then $\nabla_\xi \varphi = 0$. In particular, the pointwise norm $|\varphi|$ must be constant. Therefore, φ is a nowhere vanishing parallel section of $E \otimes S^+$ and thus defines a flat line subbundle of $E \otimes L_\xi$. Taking the tensor product with the dual $L_{-\xi}$ then gives a flat factor of E. As such, if E is without flat factors then $\ker D_\xi^+ = 0$.

Moreover, by compactness D_ξ is Fredholm for all ξ. We can then take the Nahm transform $\hat{E}_\xi = \ker D_\xi^-$, giving a bundle $\hat{E}$ over the dual torus $\hat{T}^4 = (\mathbb{R}^4)^*/\Lambda^*$.

Proposition 3.3.3. *Let ∇ be an $SU(n)$-instanton on a bundle $E \to T^4$ that is without flat factors. Then the transformed connection $\hat{\nabla}$ on $\hat{E} \to \hat{T}$ is an instanton.*

Proof. Since E is without flat factors, we have $\ker D_\xi^+ = 0$. Therefore, $\hat{E}_s = \ker D_\xi^-$ has constant rank determined by $\operatorname{ind} D_\xi$, and is thus a subbundle of the trivial bundle $L^2(E \otimes S^-)$. Moreover, $\nabla_\xi^* \nabla_\xi$ is invertible, and we let $G_\xi = (\nabla_\xi^* \nabla_\xi)^{-1}$ be its Green's operator. Note that by (3.3.2), $G_\xi|_{S^+} = (D_\xi^- D_\xi^+)^{-1}$.

Let $P_\xi \colon L^2(E \otimes S^-) \to \ker D_\xi^-$ be orthogonal projection. Since T^4 is compact, D_ξ^- is the adjoint of D_ξ^+, and we can express

$$P_\xi = 1 - D_\xi^+ G_\xi D_\xi^-. \tag{3.3.4}$$

The connection $\hat{\nabla}$ on $\hat{E}$ is then given by $P_\xi d P_\xi$, where the exterior derivative is taken with respect to the coordinates of ξ. Let $\partial_j := \frac{\partial}{\partial \xi_j}$. We have the following commutator,

$$\left[\partial_j, D_\xi^\pm\right] = i2\pi c_j. \tag{3.3.5}$$

Using the fact that $P_\xi D_\xi^+$ and $D_\xi^- P_\xi$ both vanish,

$$\begin{aligned} \hat{\nabla}_j \hat{\nabla}_k &= P_\xi \partial_j P_\xi \partial_k P_\xi \\ &= P_\xi \partial_j \partial_k P_\xi - P_\xi [\partial_j, D_\xi^-] G_\xi [D_\xi^+, \partial_k] P_\xi \\ &= P_\xi \partial_j \partial_k P_\xi - 4\pi^2 P_\xi c_j G_\xi c_k P_\xi. \end{aligned} \tag{3.3.6}$$

Note that Clifford multiplication by c_j commutes with $\nabla_\xi^* \nabla_\xi$, and so it also commutes with G_ξ. Skew-symmetrizing (3.3.6) then gives

$$\hat{F}_{jk} = [\hat{\nabla}_j, \hat{\nabla}_k] = -8\pi^2 P_\xi c_j c_k G_\xi P_\xi. \tag{3.3.7}$$

By (2.4.2), Clifford multiplication by self-dual forms vanishes on S^-, we have from this expression that the self-dual part of $\hat{F}$ vanishes. □

We can compute the rank of $\hat{E}$ using via index theory. Indeed, since $\ker D_\xi^+ = 0$,

$$\dim \hat{E}_\xi = \dim \ker D_\xi^- = -\operatorname{ind} D = -c_2(E), \tag{3.3.8}$$

where by $c_2(E)$ we mean the second Chern class of E integrated over T^4. Recall that because ∇ is an $SU(n)$-instanton, $c_2(E) = \frac{1}{8\pi^2}\|F_\nabla\|^2$ gives the charge of the instanton.

Note that applying the Nahm transform to $\hat{E} \to \hat{T}^4$ gives a new bundle with connection $\hat{\hat{E}} \to T^4$. This second transformed bundle is in fact isomorphic to the bundle E, showing that the Nahm transform is invertible. This is proved, for example, in [BvB89], and is proved from the perspective of holomorphic structures in [DK90, §3.2]. A proof of the analogous inversion in another construction of a Nahm transform can be found in [Nak93].

Given that the correspondence is invertible, and from computing the indices of the Dirac operators, we have that $SU(n)$-instantons of charge k on T^4 are in one-to-one correspondence with $SU(k)$-instantons of charge n on $\hat{T}^4$.

3.4 ADHM construction

We now turn to the ADHM Construction, first proved in [AHDM78]. This procedure constructs $SU(n)$-instantons on $\mathbb{R}^4$ from a given set of linear algebraic data satisfying certain conditions. Moreover, the construction is invertible, showing that any $SU(n)$-instanton can be produced in this way. The original proof relied on twistor theory, and a very approachable resource describing the construction from this perspective can be found in [Ati79].

Our approach, however, will be from the Nahm transform perspective, and we follow closely the descriptions in [DK90, §3.3] and [Cha04, Chapter 1].

3.4.1 Real ADHM data and equations

We begin by defining the data that will parametrize the instantons.

Definition 3.4.1. *ADHM data* consist of a k-dimensional hermitian vector space V and an n-dimensional hermitian vector space W, equipped with hermitian endomorphisms $a_i \in \operatorname{End}(V)$ for $i = 1, \ldots, 4$, and a linear map $\psi : W \to \Delta^+ \otimes V$.

Recall that Δ^+ is the positive spin representation of Spin(4), as discussed in Section 2.4. Since the positive spinor bundle S^+ is trivial $\mathbb{R}^4$, fibres of S^+ can be globally identified with Δ^+, and as a result it is common in discussions of the ADHM construction to conflate S^+ with Δ^+. In our approach, however, we will make use of a conformal identification with the spinor bundle on S^4, in which case S^+ is no longer trivial, and can no longer make a global identification of fibres $S^+|_x$ with Δ^+. We will therefore maintain the distinction between the two.

Consider Clifford multiplication c_{1j} for $j = 2, 3, 4$ acting on Δ^+. Note that $(c_{1j})^2 = -1$, and by using the fact that $c(\bigwedge^-) = 0$ on Δ^+, we have, for example,

$$c_{12}c_{13}|_{S^+} = c_{23}|_{\Delta^+} = c_{14}|_{\Delta^+}. \tag{3.4.2}$$

The same relation holds when we cyclically permute the indices 2, 3, and 4, and as such this provides a quaternionic structure on S^+. Since $\dim \Delta^+ = 2$, we see that $\operatorname{End}(\Delta^+)$ is spanned by the identity and the elements c_{1j}.

Observe that $\psi\psi^* \in \operatorname{End}(\Delta^+ \otimes V) = \operatorname{End}(\Delta^+) \otimes \operatorname{End}(V)$, and so we can write

$$\psi\psi^* = (\psi\psi^*)_0 \otimes 1 + \sum_{j=2}^{4} (\psi\psi^*)_j \otimes c_{1j}. \tag{3.4.3}$$

With this notation, we write the *ADHM equations*

$$[a_1, a_i] + [a_j, a_k] = (\psi\psi^*)_i, \tag{3.4.4}$$

where (i, j, k) are cyclic permutations of $(2, 3, 4)$. These equations also have a convenient expression in terms of self-dual forms acting on Δ^+. Considering the 1-form $a = a_i e^i$, we can take $(a \wedge a)^+$ to be the self-dual part of $a \wedge a$. Then, letting $(\psi\psi^*)_{\bigwedge^+}$

be the $c(\bigwedge^+)$ part of (3.4.3), we can rewrite (3.4.4) as

$$c((a \wedge a)^+) = (\psi\psi^*)_{\bigwedge^+}. \tag{3.4.5}$$

We define an additional non-degeneracy condition on the data. For $x \in \mathbb{R}^4$ let $Q_x \colon V \otimes \Delta^+ \to V \otimes \Delta^- \oplus W$ be the map

$$Q_x = \begin{bmatrix} c_i \otimes (a_i - x_i) \\ \psi^* \end{bmatrix}. \tag{3.4.6}$$

Definition 3.4.7. The ADHM data (V, W, a_i, ψ) are said to be *valid* if they satisfy the ADHM equations (3.4.4) and if Q_x is injective for all x.

We can also define an equivalence relation on ADHM data, analogous to gauge equivance.

Definition 3.4.8. The ADHM data (V, W, a_i, ψ) and $(V', W', a_i', \psi'$ are said to be *equivalent* if there are there are isometries $v : V \to V'$ and $w : W \to W'$ such that $a_i' = va_iv^{-1}$ and $\psi' = v\psi w^{-1}$.

3.4.2 Complex ADHM data and equations

We can also consider a complexified version of the ADHM data. Although for the remainder of this chapter we will continue working with the real data described in Section 3.4.1, the complex version is convenient for the description in Chapter 4, and so we introduce it here.

Definition 3.4.9. *Complex ADHM data* consist of hermitian vector spaces V and W, with linear maps $\alpha, \beta \in \mathrm{End}(V)$, $\pi : V \to W$, and $\sigma : W \to V$.

Given real ADHM data (V, W, a_i, ψ), we define the equivalent complex ADHM data by taking the same hermitian vector spaces V and W. Let $\alpha = a_1 + ia_2$ and

$\beta = a_3 + ia_4$. Note that from these we can recover the endomorphisms a_i by taking hermitian and anti-hermitian parts of α and β.

Since $(c_{12})^2 = -1$, its eigenvalues are $\pm i$. Since c_{13} anticommutes with c_{12}, it maps the i-eigenspace of c_{12} to the $-i$-eigenspace, and vice versa. In particular, the $\pm i$-eigenspaces of c_{12} in Δ^+ are non-empty, and we can choose a unit i-eigenvector q and a unit $-i$-eigenvector p such that $c_{13}q = p$. We can then choose a basis for Δ^- for which Clifford multiplication $\Delta^+ \to \Delta^-$ is given with respect to the basis (p, q) by

$$\begin{aligned} c_1 &= \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \qquad c_2 = \begin{bmatrix} i & \\ & -i \end{bmatrix}, \\ c_3 &= \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}, \qquad c_4 = \begin{bmatrix} & i \\ i & \end{bmatrix}. \end{aligned} \tag{3.4.10}$$

We obtain the maps π and σ by expressing the map $\psi\colon W \to V \otimes \Delta^+$ with respect to the basis p, q as

$$\psi = \pi^* \otimes p + \sigma \otimes q \tag{3.4.11}$$

With respect to this basis,

$$\psi\psi^* = \begin{bmatrix} \pi^*\pi & \pi^*\sigma^* \\ \sigma\pi & \sigma\sigma^* \end{bmatrix}. \tag{3.4.12}$$

Since, for example, c_{12} maps $p \to ip$ and $q \to -iq$, we see that $(\psi\psi^*)_2 = i\pi^*\pi - i\sigma\sigma^*$. Similarly, $(\psi\psi^*)_3 = -\sigma\pi + \pi^*\sigma^*$ and $(\psi\psi^*)_4 = i\pi^*\sigma^* + i\sigma\pi$. Expanding the commutators $[\alpha, \beta]$, $[\alpha, \alpha^*]$, and $[\beta, \beta^*]$, we then have that the ADHM equations (3.4.4) are equivalent to

$$[\alpha, \beta] + \sigma\pi = 0, \tag{3.4.13}$$

$$[\alpha, \alpha^*] + [\beta, \beta^*] + \sigma\sigma^* - \pi^*\pi = 0. \tag{3.4.14}$$

For coordinates z_1, z_2 on $\mathbb{C}^2$, let $\alpha_z = \alpha - z_1$ and $\beta_z = \beta - z_2$, and define $R_z\colon V \oplus$

$V \oplus W \to V \oplus V$ by

$$R_z = \begin{bmatrix} \alpha_z^* & \beta_z^* & \pi^* \\ -\beta_z & \alpha_z & \sigma \end{bmatrix}. \tag{3.4.15}$$

With respect to the basis p, q for Δ^+, we can write $V \otimes \Delta^+ = (V \otimes p) \oplus (V \otimes q)$, and with respect to this decomposition, and the basis chosen for Δ^-, the map R_z is simply Q_x^* with complex coordinates $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$. The non-degeneracy condition is then equivalent to requiring R_z to be full rank for all $z \in \mathbb{C}^2$.

3.4.3 The ADHM construction as a Nahm transform

Before continuing on to the construction of the correspondence between ADHM data and instantons, it may be helpful to see how the ADHM construction fits within the Nahm transform framework. We can consider $\mathbb{R}^4 = \mathbb{R}^4/\{0\}$, and since the dual subgroup of $\{0\}$ is all of $(\mathbb{R}^4)^*$, we expect in the Nahm transform to construct a bundle over $(\mathbb{R}^4)^*/(\mathbb{R}^4)^* = \{0\}$. A bundle over a single point is simply a vector space V, and the remnants of the connection provide the maps $a_i \in \mathrm{End}(V)$.

To be more precise, we consider twisting by $i2\pi\xi$ for $\xi \in (\mathbb{R}^4)^*$ as usual. Defining the Dirac operators $D_\xi = D + i2\pi c(\xi)$, we let the transformed bundle be $\hat{E}_\xi = \ker D_\xi^-$. Since $(\mathbb{R}^4)^*$ is contractible, we can identify all fibres of $\hat{E}$. Indeed, $e^{-i2\pi\xi}$ defines a global $U(1)$ gauge transformation over $\mathbb{R}^4$ that identifies $\ker D_\xi^- \cong \ker D_0^-$.

The Nahm transformed connection is defined to be $\hat{\nabla}_i = P_\xi \frac{\partial}{\partial \xi_i} P_\xi$. Using the gauge transformation $e^{-2i\pi\xi}$ to identify fibres of $\hat{E}$, we have $P_\xi = e^{i2\pi\xi} P e^{-i2\pi\xi}$, and then $\hat{\nabla}$ has constant connection form $a_i = i2\pi P x_i P$. By rescaling, we can omit the $i2\pi$. Since we have identified all fibres of $\hat{E}_\xi$, from now on we only need consider D_0.

The curvature of this constant connection is $\hat{F}_{ij} = [a_i, a_j]$, and so the left-hand side of the ADHM equations (3.4.4) can be thought of as the self-dual part of $\hat{F}_a$. Since the right-hand-side is not zero, the transformed connection is not an instanton, but still instanton-like in that it has prescribed self-dual curvature $\hat{F}^+$. The proof that the transformed constant connection satisfies the ADHM equations proceeds in the same spirit as the argument that the Nahm transform on the torus yields an instanton. However, the non-compactness of $\mathbb{R}^4$ means more care is required in the analysis, and boundary terms in the argument yield the right-hand side of (3.4.4). The map ψ thus captures asymptotic behaviour of the instanton, and can also be viewed as evaluation of $\varphi \in \ker D^-$ at the point at ∞ on S^4 via stereographic projection.

3.4.4 From instanton to ADHM data

We now formalize the previous discussion to show how to associate ADHM data to an $SU(n)$-instanton on a bundle $E \to \mathbb{R}^4$. Let $D\colon \Gamma(E \otimes S) \to \Gamma(E \otimes S)$ be the associated Dirac operator. By stereographic projection, which preserves anti-self-duality because it is conformal, we may consider ∇ as a connection on $S^4 \setminus \{\infty\}$. Uhlenbeck's Removable Singularity Theorem [Uhl82], guarantees that the connection extends smoothly at ∞. We then set the ADHM data $W = E_\infty$, the fibre at ∞ with renormalized inner product $\langle \cdot, \cdot \rangle_W = \pi^2 \langle \cdot, \cdot \rangle_{E_\infty}$.

Dirac spinors on $\mathbb{R}^4$ and S^4

We canobtain some asymptotic information about Dirac spinors by comparing to the Dirac operator D_{S^4} on S^4. Since S^4 and $\mathbb{R}^4$ are conformal with $g_{S^4} = \frac{4}{(1+|x|^2)^2} g_{\mathbb{R}^4}$, we can conformally identify $SO(T\mathbb{R}^4)$ with $SO(TS^4)$ away from ∞ by mapping the frame $e_i \mapsto \tilde{e}_i = \frac{1}{2}(1 + |x|^2)e_i$. This identification lifts to an identification of the spin structures on $\mathbb{R}^4$ and $S^4 \setminus \{\infty\}$. The frame $\{e_i\}$ induces a trivialization of the spinor bundle $S \cong \mathbb{R}^4 \times \Delta$, while the frame $\{\tilde{e}_i\}$ induces a trivialization for the spinor bundle $\tilde{S}$ over S^4 away from ∞, giving $\tilde{S}|_{S^4 \setminus \infty} \cong (S^4 \setminus \infty) \times \Delta$. Taking the conformal identification of $\mathbb{R}^4 \subset S^4$ and identity on Δ then identifies S with $\tilde{S}$. Note that under this identification, the action of c_i on S corresponds to the action of $\tilde{c}_i = c(\tilde{e}_i)$ on $\tilde{S}$. Given this identification of S with $\tilde{S}$, we will henceforth denote by S the spinor bundle either on $\mathbb{R}^4$ or on S^4, as appropriate.

Forming the Dirac operator $D_{S^4} = \tilde{c}_i \nabla_i^{S^4}$ as usual, we have the following relation to the Dirac operator on $\mathbb{R}^4$,

$$D_{S^4} = \left(\frac{2}{1+|x|^2}\right)^{\frac{-1}{2}} D \left(\frac{2}{1+|x|^2}\right)^{\frac{3}{2}}. \tag{3.4.16}$$

Motivated by the above equation, for $\varphi \in L^2(\mathbb{R}^4, E \otimes S)$, let

$$\tilde{\varphi} = (1+|x|^2)^{\frac{3}{2}}\varphi. \tag{3.4.17}$$

Lemma 3.4.18. *The map* $\varphi \mapsto \tilde{\varphi} = (1+|x|^2)^{\frac{3}{2}}$ *gives an isomorphism*

$$\ker D \cap L^2(\mathbb{R}^4, E \otimes S) \xrightarrow{\cong} \ker D_{S^4} \cap L^2(S^4, E \otimes S).$$

Proof. For notational simplicity, we omit the notation $E \otimes S$ and write only the domain, taking for example $L^2(\mathbb{R}^4) = L^2(\mathbb{R}^4, E \otimes S)$. Suppose $\varphi \in L^2(\mathbb{R}^4)$. Since $\mathrm{vol}_{S^4} = O(|x|^{-8})\,\mathrm{vol}_{\mathbb{R}^4}$, we have that $|\tilde{\varphi}|^2\,\mathrm{vol}_{S^4} = O(|x|^{-2})|\varphi|^2\,\mathrm{vol}_{\mathbb{R}^4}$. Since $|\varphi|^2$ is integrable, so is $|\tilde{\varphi}|^2$, showing that $\tilde{\varphi} \in L^2(S^4)$.

If moreover $\varphi \in \ker D$ then (3.4.16) shows that $D_{S^4}\tilde{\varphi} = 0$ as a distribution away from ∞ on S^4. That is,

$$\langle \tilde{\varphi}, D_{S^4}\omega \rangle_{L^2(S^4)} = 0 \tag{3.4.19}$$

for any test function $\omega \in C_0^\infty(S^4 \setminus \{\infty\})$, the space of all smooth sections on S^4 vanishing on a neighbourhood of ∞. To see that $D_{S^4}\tilde{\varphi} = 0$ distributionally on all S^4, we must show (3.4.19) holds for all $\omega \in C^\infty(S^4)$, including when $\omega(\infty) \neq 0$.

Let B_r denote the geodesic ball with radius r centred at $\infty \in S^4$, and let γ_k be a smooth cutoff function supported on $S^4 \setminus B_{\frac{1}{k}}$ such that $\gamma_k = 1$ on $S^4 \setminus B_{\frac{2}{k}}$. Since the distance between $B_{\frac{1}{k}}$ and $S^4 \setminus B_{\frac{2}{k}}$ is $\frac{1}{k}$, we can take γ_k to have $|\nabla\gamma_k| \leq 2k$ pointwise.

Then

$$\begin{aligned}\langle \tilde{\varphi}, D_{S^4}\omega\rangle_{L^2(S^4)} &= \lim_{k\to\infty} \langle \tilde{\varphi}, \gamma_k D_{S^4}\omega\rangle_{L^2(S^4)} \\ &= \lim_{k\to\infty} \big(\langle \tilde{\varphi}, D_{S^4}(\gamma_k\omega)\rangle_{L^2(S^4)} - \langle \tilde{\varphi}, \tilde{c}_i(\nabla_i)\omega\rangle_{L^2(S^4)}\big) \qquad (3.4.20)\end{aligned}$$

The first term above vanishes because $\gamma_k\omega \in C_0^\infty(S^4 \setminus \{\infty\})$. In the second term, $|\nabla_i\gamma_k|$ is bounded by $2k$ and supported on $B_{\frac{2}{k}}$, while ω is bounded. Therefore, for some constant C, the limit from (3.4.20) is bounded by

$$\begin{aligned}\lim_{k\to\infty} |\langle \tilde{\varphi}, \tilde{c}_i(\nabla_i)\omega\rangle_{L^2(S^4)}| &\le \lim_{k\to\infty} Ck \int_{B_{\frac{2}{k}}} |\tilde{\varphi}| \\ &\le \lim_{k\to\infty} Ck \left|B_{\frac{2}{k}}\right| \|\tilde{\varphi}\|_{L^2(S^4)}. \qquad (3.4.21)\end{aligned}$$

Since the volume $\left|B_{\frac{2}{k}}\right| = O(k^{-4})$, we have that the limit (3.4.21) vanishes. Therefore $D_{S^4}\tilde{\varphi} = 0$ distributionally on all S^4, and by elliptic regularity $\tilde{\varphi}$ is in fact a smooth solution.

We have thus shown that $\varphi \mapsto \tilde{\varphi}$ gives a well-defined map $\ker D \cap L^2(\mathbb{R}^4) \to \ker D_{S^4} \cap L^2(S^4)$. To show this map is an isomorphism, we show that the inverse, given by multiplying by $(1+|x|^2)^{-\frac{3}{2}}$, is well-defined.

Since L^2-solutions to $D_{S^4}\tilde{\varphi} = 0$ are smooth by elliptic regularity, in particular, $\tilde{\varphi}$ is bounded. Therefore multiplying by $(1+|x|^2)^{-\frac{3}{2}}$ yields $\varphi = O(|x|^{-3})$ as $x \to \infty$, and hence $\varphi \in L^2(\mathbb{R}^4)$. Moreover, since $(1+|x|^2)^{-\frac{3}{2}}$ is smooth, φ is smooth, and so $D\varphi = 0$ in the ordinary sense.

Therefore the inverse $\tilde{\varphi} \to \varphi$ is well-defined, completing the proof. □

We henceforth omit the intersection with L^2, and assume that $\ker D$ denotes the L^2 kernel unless otherwise specified. We therefore have that $\ker D \cong \ker D_{S^4}$. Since the conformal map preserves the decomposition $S = S^+ \oplus S^-$, we also have $\ker D^\pm \cong \ker D^\pm_{S^4}$.

Since the scalar curvature on S^4 is constant and positive, and since $F_\nabla^+ = 0$, the Lichnerowicz formula (2.4.8) implies that

$$\|D_{S^4}^+ \tilde{\varphi}\|_{L^2(S^4)}^2 = \|\nabla \tilde{\varphi}\|_{L^2(S^4)}^2 + \frac{R}{4}\|\tilde{\varphi}\|_{L^2(S^4)}^2. \tag{3.4.22}$$

In particular, $\ker D_{S^4}^+ = 0$, and so by our isomorphisms, $\ker D^+ = 0$. Of course, to show this, we could have kept our perspective on $\mathbb{R}^4$ and noted that covariant constant φ must be 0, otherwise the $\|\varphi\|_{L^2} = \infty$.

We now turn our attention to $\ker D^-$. Since $D_{S^4}^-$ is Fredholm, we see that $\ker D^-$ is finite-dimensional. In the proof of of Lemma 3.4.18, we showed moreover that elements of $\ker D^-$ are $O(|x|^{-3})$. We define $V = \ker D^-$.

The map $\psi\colon W \to V \otimes \Delta^+$ will be defined as an adjoint to evaluation at ∞, with the factor of Δ^+ arising from a trivialization of S^- near ∞. To make a precise definition of ψ, we must first set up trivializations of the spinor bundle S and describe the change of trivialization map between them.

Consider the neighbourhoods $U_0 = S^4 \setminus \{\infty\}$ and $U_\infty = S^4 \setminus \{0\}$. Let the coordinates x_i on $U_0 = \mathbb{R}^4$ and the coordinates x_i' on $U_\infty = \mathbb{R}^4$ both be obtained by stereographic projection to S^4, so that $x_i' = \frac{x_i}{|x|^2}$. The Euclidean orthonormal frame $e_i = \frac{\partial}{\partial x_i}$ (which we have already been using above) then corresponds conformally to the frame $f_i = \frac{1}{2}(1+|x|^2)e_i$, which is orthonormal with respect to the round metric on $U_0 \subset S^4$. Similarly, the Euclidean orthonormal frame $e_i' = \frac{\partial}{\partial x_i'}$ corresponds to the orthonormal frame $f_i' = \frac{1}{2}(1+|x'|^2)e_i'$ with respect to the round metric on $U_\infty \subset S^4$.

Note that the change of coordinates is orientation-reversing. We will take the orientation on S^4 to be that given by U_0, as we have taken implicitly until now. The local frame $\{f_i'\}$ is then negatively-oriented.

These frames define trivializations of the spinor bundle S through the associated bundle construction. Working first over U_0, the frame $\{f_i\}$ lifts to a section σ of the spin bundle $P|_{U_0}$. Recalling the definition of the spinor bundle as the associated bundle $S = P \times_{\mathrm{Spin}(4)} \Delta$, where $\Delta = \Delta^+ \oplus \Delta^-$ is the spin representation as discussed in Section 2.4, we can define a trivialization of $S|_{U_0}$ by

$$\begin{aligned} \tau_0 : U_0 \times \Delta &\xrightarrow{\cong} S|_{U_0} \\ (x, v) &\mapsto [\sigma(x), v]. \end{aligned} \tag{3.4.23}$$

Because P is a double cover of the oriented orthonormal frame bundle, there is a choice of sign in the lift σ, and hence also in the trivialization of $S|_{U_0}$. We leave the sign undetermined for the time being, and fix it later during the proof of Lemma 3.4.25.

The trivialization over U_∞ is defined similarly, albeit with a slight complication. Since the frame $\{f_i'\}$ is negatively oriented, it does not lift to the spin bundle P. We instead consider the principal Pin(4)-bundle P' which is a double cover of the orthonormal frame bundle. P' can be constructed explicitly as a subbundle of the Clifford bundle $\mathrm{Cl}(TS^4)$. The Clifford module structure of the spin representation Δ then restricts to an action by Pin(4), and as a result we may consider the spinor bundle as the associated vector bundle $S = P' \times_{\mathrm{Pin}(4)} \Delta$. Observe that an element $g \in \mathrm{Pin}(4) \setminus \mathrm{Spin}(4)$ can be expressed as a Clifford product of an odd number of unit vectors, and so Clifford multiplication by g defines a map $\Delta^\pm \to \Delta^\mp$.

We can then lift the frame $\{f_i'\}$ to a section σ' on P'. The corresponding trivialization of $S|_{U_\infty}$ is then given, as in (3.4.23), by

$$\begin{aligned} \tau_\infty : U_\infty \times \Delta &\xrightarrow{\cong} S|_{U_\infty} \\ (x', v) &\mapsto [\sigma'(x'), v]. \end{aligned} \tag{3.4.24}$$

Because σ' is negatively oriented, it differs from a positively oriented frame by Clifford multiplication by some element of $\mathrm{Pin}(4) \setminus \mathrm{Spin}(4)$. Such a Clifford element reverses the grading on the spin representation. Therefore, the trivialization induced by a negatively oriented frame such as σ' associates elements of Δ^+ to sections of S^-, and elements of Δ^- to sections of S^+.

The trivializations for S above are both for the spinor bundle on S^4. We can define a similar trivialization for the spinor bundle on $\mathbb{R}^4$ using the orthonormal frame $\{e_i\}$. As discussed previously, with respect to these trivializations, the conformal identification of spinor bundles on S^4 and $\mathbb{R}^4$ is given by the identity on Δ.

Lemma 3.4.25. *With respect to the trivializations τ_0 defined in (3.4.23) and τ_∞ defined in (3.4.24), the change of trivialization map $\tau_0^{-1} \circ \tau_\infty|_{S^\pm}$ acts on Δ by $\mp c(f_r)$, where f_r denotes the radial geodesic vector field away from 0 in S^4.*

Remark. Let $r = |x|$ on U_0, and denote the corresponding radial vector field ∂_r, so that ∂_r is the unit radial vector field with respect to the Euclidean metric. Multiplying by the conformal factor gives the corresponding radial geodesic vector field $f_r = \frac{1}{2}(1 + r^2)\partial_r$ on S^4.

When we conformally identify the spinor bundles $S^\pm|_{U_\infty}$ and $S^\pm|_{U_0}$ with the spinor bundles for the Euclidean metric on the respective copies of $\mathbb{R}^4$, Clifford multiplication is scaled by an appropriate conformal factor. The corresponding change of trivialization of the spinor bundle with respect to the Euclidean metric is given by $\mp c(\partial_r) = \mp \frac{c(x)}{|x|}$.

Remark. The statement of Lemma 3.4.25 is slightly misleading in that the trivializations of the spinor bundle have so far only been determined up to sign. Therefore, the sign of the change of trivialization is as yet undetermined. We will fix the appropriate sign in the course of the proof.

Proof. Let μ be the pointwise orthogonal transformation on TS^4 away from the points 0 and ∞, defined by $\mu(f_i) = f_i'$. At a point p, μ lifts to some $\tilde{\mu} \in \mathrm{Pin}(4) \subset \mathrm{Cl}(T_pS^4)$. Then $\sigma\tilde{\mu} = \sigma'$ in P', and by the equivalence relation $[pg, v] = [p, gv]$ in the associated bundle construction, such a $\tilde{\mu}$ induces a transformation on Δ giving the change of trivialization map for S. Our proof proceeds by obtaining an explicit lift $\tilde{\mu}$ at a point p.

As remarked above, letting $r = |x|$ on U_0 with corresponding radial vector field ∂_r, we have $f_r = \frac{1}{2}(1+r^2)\partial_r$. Similarly, letting $r' = |x'|$ on U_∞, we have $r' = \frac{1}{r}$. For the radial vector field $\partial_{r'}$, and geodesic radial vector field $f'_{r'} = \frac{1}{2}(1+(r')^2)\partial_{r'}$ on S^4,

$$\begin{aligned} f_r &= \frac{1}{2}(1+r^2)\left(-\frac{1}{r^2}\partial_{r'}\right) \\ &= -f'_{r'}. \end{aligned} \tag{3.4.26}$$

On the other hand, $f_r = \frac{x_i}{|x|} f_i$ and $f'_{r'} = \frac{x'_i}{|x'|} f'_i$, and so also using that $\frac{x_i}{|x|} = \frac{x'_i}{|x'|}$, we have

$$\begin{aligned} \mu(f_r) &= \frac{x_i}{|x|}\mu(f_i) \\ &= \frac{x'_i}{|x'|} f'_i \\ &= f'_{r'} \\ &= -f_r. \end{aligned} \tag{3.4.27}$$

We now turn our attention to how μ acts on vectors orthogonal to f_r. Since f_i is a rescaling of the coordinate vector field ∂_{x_i} and f'_i is a rescaling of $\partial_{x'_i}$, we have that μ is a rescaling of the change of coordinate transformation mapping $\partial_{x_i} \mapsto \partial_{x'_i}$. Note that the change of coordinates identifies the sphere $S^3_r \subset U_0$ with the sphere $S^3_{r'} \subset U_\infty$ by a simple scaling factor, and so the change of coordinate transformation similarly acts on vectors in $T_pS^3_{r(p)} \subset T_pS^4$ by rescaling. The transformation μ must then also act on $T_pS^3_{r(p)}$ as some rescaling. Since μ is orthogonal, this 'rescaling' must in fact fix the vectors in $T_pS^3_{r(p)}$, which are precisely all vectors orthogonal to f_r in T_pS^4. That is, μ has the effect of mapping $f_r \mapsto -f_r$, and fixes all vectors orthogonal to f_r, and so geometrically, μ can be viewed as reflection in the hyperplane $T_pS^3_{r(p)} \subset T_pS^4$.

To lift μ to an action on P', we express it as conjugation by a Clifford element. Let f_{θ_i} be an orthonormal basis of $T_pS^3_{r(p)} \subset T_pS^4$, and define $\tilde{\mu} = c(f_{\theta_1})c(f_{\theta_2})c(f_{\theta_3})$. The map μ is then given on $T_pS^4 \subset \mathrm{Cl}(T_pS^4)$ as conjugation by $\tilde{\mu}$, which can be seen directly by $\tilde{\mu} f_r \tilde{\mu}^{-1} = -f_r$ and $\tilde{\mu} f_{\theta_i} \tilde{\mu}^{-1} = f_{\theta_i}$. Note that we have a choice of orientation for the basis f_{θ_i}, corresponding to a choice of sign of $\tilde{\mu}$. We choose the sign such that $c(\mathrm{vol}_{S^4}) = -c(f_r)\tilde{\mu}$.

On the bundle P', $\tilde{\mu}$ acts by right multiplication, and so we have $\sigma' = \sigma\tilde{\mu}$. Note that this equality involved finally fixing signs of σ and σ'. In terms of the trivializations τ_0 and τ_∞ of S, this gives for $v \in \Delta$ and $p \in S^4$

$$\begin{aligned} \tau_0^{-1} \circ \tau_\infty(p, v) &= \tau_0^{-1}([\sigma'(p), v]) \\ &= \tau_0^{-1}([\sigma(p)\tilde{\mu}, v]) \\ &= \tau_0^{-1}([\sigma(p), \tilde{\mu} v]) \\ &= (p, \tilde{\mu} v). \end{aligned} \tag{3.4.28}$$

Therefore, the change of trivialization from $S|_{U_\infty}$ to $S|_{U_0}$ is given by multiplication by $\tilde{\mu}$ on Δ. Since $\tilde{\mu} = -c(f_r)^2\tilde{\mu} = c(f_r)c(\mathrm{vol}_{S^4})$, and since $S^\pm$ are the ∓ 1-eigenbundles of $c(\mathrm{vol}_{S^4})$ we then have that the change of trivialization for $S^\pm$ is given by multiplication by $\mp c(f_r)$. □

Remark. As observed previously, since $\{f_i'\}$ is negatively oriented, the trivialization of S over U_∞ identifies fibres of S^- with Δ^+. The trivialization over U_0 meanwhile identifies fibres of S^- with Δ^-, and we have, as expected, that the change of trivialization $c(f_r)$ maps Δ^+ to Δ^-.

We are now in a position to define the map $\psi : W \to V \otimes \Delta^+$, recalling that we defined $W = E_\infty$. Consider $\varphi \in \ker D^-$, with its corresponding lift $\tilde{\varphi} \in \ker D^-_{S^4}$. Evaluating at ∞ gives $\tilde{\varphi}(\infty) \in E_\infty \otimes S^-_\infty$, which with respect to our trivialization of the spin bundle can be expressed as $\tilde{\varphi}_\infty \in E_\infty \otimes \Delta^+$. Following the conventions of [DK90] and [Cha04], let η be the skew-form on Δ^+, given on the basis p, q from Section 3.4.2 by $\eta(p, q) = 1$. We then define ψ by its adjoint,

$$\psi^*(\varphi \otimes v) = \eta(\tilde{\varphi}_\infty, v), \tag{3.4.29}$$

for $\varphi \in V$ and $v \in \Delta^+$. The adjoint is defined with respect to the renormalized inner product $\langle \cdot, \cdot \rangle_W = \pi^2 \langle \cdot, \cdot \rangle_{E_\infty}$.

We can obtain asymptotic information about $\varphi \in V$ in terms of $\tilde{\varphi}_\infty$. Extending $\tilde{\varphi}_\infty$ to a section of $E \otimes S^-$ on a neighbourhood of ∞ by radial parallel translation, we have in coordinates $x' = \frac{x}{|x|^2}$ near ∞ that $\tilde{\varphi} = \tilde{\varphi}_\infty + O(|x'|)$ as $x' \to 0$.

We now shift our perspective back to $U_0 = S^4 \setminus \{\infty\}$. Since our trivialization near ∞ identifies S^- with Δ^+, we can consider $\tilde{\varphi}$ to be a section of $E \otimes S^+$, and Lemma 3.4.25 gives us $\tilde{\varphi} = c(f_r)\tilde{\varphi}_\infty + O(|x|^{-1})$ as $x \to \infty$.

Finally, we conformally identify $S^-|_{U_0}$ with $S^-|_{\mathbb{R}^4}$, recalling that under this identification $c(f_r)$ becomes $\frac{c(x)}{|x|}$ by the remark following Lemma 3.4.25. Also multiplying by $(1 + |x|^2)^{-\frac{3}{2}}$ to get our expression for φ, we have the asymptotic expansion as $x \to \infty$

$$\varphi(x) = \frac{c(x)\tilde{\varphi}_\infty}{|x|^4} + O(|x|^{-4}). \tag{3.4.30}$$

Projection and Green's operator

The last pieces of the ADHM data left to define are the endomorphisms a_i. Let P be orthogonal projection onto $\ker D^-$, and let

$$a_i = Px_iP. \tag{3.4.31}$$

Multiplying an element $\varphi \in V$ by x_i results in $x_i\varphi = O(|x|^{-2})$, which is in general not in L^2. It is therefore necessary to check that a_i is in fact well-defined. Pairing with another element $\zeta \in V$ gives the pointwise inner product $\langle x_i\varphi, \zeta \rangle = O(|x|^{-5})$, which is integrable. For a basis $\{\zeta_k\}$ of V, we can define $P = \zeta_k \otimes \zeta_k^*$, and we then have that $Px_i\varphi = \langle x_i\varphi, \psi_k \rangle_{L^2}\psi_k$ is well-defined.

As in Section 3.3, we will be able to obtain commutation results for a_i by writing an expression for P in terms of a Green's $\nabla^*\nabla$, which we take with respect to the Euclidean metric. Observe that the L^2-kernel of $\nabla^*\nabla$ is 0, and we define $G =$

$(\nabla^*\nabla)^{-1}$. As in [DK90, Proposition 3.3.21], G is an integral operator with kernel $k(x,y) = O(|x-y|^{-2})$. A special case of [DK90, Lemma 3.3.5] gives the following lemma.

Lemma 3.4.32. *If s is continuous and $s = O(|x|^{-4})$ then $Gs = O(|x|^{-2})$ as $|x| \to \infty$.*

Proof. For large $|x|$, we estimate the integral

$$Gs(x) = \int_{\mathbb{R}^4} k(x,y)s(y)\,dy \tag{3.4.33}$$

by separating the domain of integration into the regions $B_{\frac{|x|}{2}}(x)$, $B_{\frac{|x|}{2}}(0)$, and their complement $\tilde{B}$.

For $y \in B_{\frac{|x|}{2}}(0)$, we have $|k(x,y)| \le C|y|^{-2}$ and $|s(y)| \le C(1+|y|)^{-4}$, so

$$\begin{aligned}\int_{B_{\frac{|x|}{2}}(x)} |k(x,y)s(y)|\,dy &\le C\int_{B_{\frac{|x|}{2}}(x)} (1+|y|)^{-4}|y|^{-2}\,dy \\ &\le \tilde{C}|x|^{-2}.\end{aligned} \tag{3.4.34}$$

For $y \in B_{\frac{|x|}{2}}(x)$, we have $|k(x,y)| \le C|x-y|^{-2}$, and since $|y| > \frac{|x|}{2}$, we also have $|s(y)| \le C|y|^{-4} \le C|x|^{-4}$. So,

$$\begin{aligned}\int_{B_{\frac{|x|}{2}}(x)} |k(x,y)s(y)|\,dy &\le C|x|^{-4}\int_{B_{\frac{|x|}{2}}(x)} |x-y|^{-2}\,dy \\ &\le \tilde{C}|x|^{-2}.\end{aligned} \tag{3.4.35}$$

Lastly on the complement of the last two regions, we have $|k(x,y)| \le C|y|^{-2}$ and so $|y| > \frac{|x|}{2}$, so

$$\begin{aligned}\int_{\tilde{B}} |k(x,y)s(y)|\,dy &\le C\int_{|y|>\frac{|x|}{2}} |y|^{-6}\,dy \\ &\le \tilde{C}|x|^{-2}.\end{aligned} \tag{3.4.36}$$

From (3.4.34), (3.4.35), and (3.4.36) we have that $Gs(x) = O(|x|^{-2})$. □

For what follows, it will be necessary to have more precise information about the asymptotic behaviour of elements in GV.

Proposition 3.4.37. *For $\varphi \in V$, and with $\tilde{\varphi}_\infty$ as in (3.4.30), for $|x|$ large*

$$G\varphi = \frac{c(x)\tilde{\varphi}_\infty}{4|x|^2} + O(|x|^{-2}). \tag{3.4.38}$$

Proof. For this proof it will be convenient to work using the Euclidean metric on both $U_0 = S^4 \setminus \{\infty\}$ and $U_\infty = S^4 \setminus \{0\}$, obtained conformally from the round metric by stereographic projection from ∞ and 0. Let x be coordinates on U_0 for which the metric $g_0 = \sum_i (dx_i)^2$. Taking the coordinates $x' = \frac{x}{|x|}$ on U_∞, we can express the metric $g_\infty = \sum_i (dx_i')^2$. We then have the conformal relation $g_0 = |x'|^{-4} g_\infty$. We denote by $(\nabla^*\nabla)_0$ and $(\nabla^*\nabla)_\infty$ the Laplacian with respect to the corresponding metrics.

Recall that for a metric g, the conformal Laplacian in dimension 4 is given by

$$L_g = (\nabla^*\nabla)_g + \frac{1}{6}R \tag{3.4.39}$$

where R is the scalar curvature. If $g' = f^2 g$ for some positive conformal factor f, then the conformal Laplacians are related by $L_{g'} = f^{-3} L_g f$, the exponents of f here being determined by the fact we are considering the dimension 4 case. Since g_0 and g_∞ are both flat, their usual Laplacians are equal to their conformal Laplacians, and since g_0 and g_∞ are related by the conformal factor $f = |x'|^{-2}$, we have

$$(\nabla^*\nabla)_0 = |x'|^6 (\nabla^*\nabla)_\infty |x'|^{-2}. \tag{3.4.40}$$

Since our coordinates are related by $\frac{x}{|x|^2} = x'$, we have $\frac{c(x)}{|x|^2} = c(x')$. We then evaluate

$$
\begin{aligned}
&(\nabla^*\nabla)_0 \frac{c(x)\tilde{\varphi}_\infty}{|x|^2} \\
&= |x'|^6 (\nabla^*\nabla)_\infty |x'|^{-2} \left(c(x')\tilde{\varphi}_\infty\right) \\
&= |x'|^6 \left((\nabla^*\nabla)_\infty \left(\frac{c(x')}{|x'|^2} \right) \tilde{\varphi}_\infty + \frac{c(x')}{|x'|^2} (\nabla^*\nabla)_\infty \tilde{\varphi}_\infty - (\nabla_\infty)_i \frac{c(x')}{|x'|^2} (\nabla_\infty)_i \tilde{\varphi}_\infty \right).
\end{aligned}
\tag{3.4.41}
$$

A straightforward computation gives the first term

$$
(\nabla^*\nabla)_\infty \frac{c(x')}{|x'|^2} = \frac{4c(x')}{|x'|^4}. \tag{3.4.42}
$$

For the second term, since $(\nabla^*\nabla)_\infty \tilde{\varphi}_\infty$ is smooth, we have that as $x' \to 0$,

$$
\frac{c(x')}{|x'|^2} (\nabla^*\nabla)_\infty \tilde{\varphi}_\infty = O(|x'|^{-1}). \tag{3.4.43}
$$

For the last term, note that $(\nabla_\infty)_i \tilde{\varphi}_\infty$ is smooth and $(\nabla_\infty)_i \frac{c(x')}{|x'|^2} = O(|x'|^{-2})$, giving

$$
-\nabla_i \frac{c(x')}{|x'|^2} \nabla_i \tilde{\varphi}_\infty = O(|x'|^{-2}). \tag{3.4.44}
$$

From (3.4.41), we then have (3.4.30),

$$
\begin{aligned}
(\nabla^*\nabla)_0 \frac{c(x)\tilde{\varphi}_\infty}{|x|^2} &= |x'|^6 \left(\frac{4c(x')\tilde{\varphi}_\infty}{|x'|^4} + O(|x'|^{-2}) \right) \\
&= 4|x'|^2 c(x')\tilde{\varphi}_\infty + O(|x'|^4) \\
&= 4\frac{c(x)\tilde{\varphi}_\infty}{|x|^4} + O(|x|^{-4})
\end{aligned}
\tag{3.4.45}
$$

as $x \to \infty$. The asymptotic expansion (3.4.30) then tells us that (3.4.45) is $\varphi(x) + O(|x|^{-4})$. Applying G to both sides gives

$$
G\varphi = \frac{c(x)\tilde{\varphi}_\infty}{4|x|^2} + GO(|x|^{-2}). \tag{3.4.46}
$$

Lemma 3.4.32 then gives the result. $\square$

Since $D^-D^+ = \nabla^*\nabla$, we have that $G|_{S^+} = (D^-D^+)^{-1}$. Therefore, projection onto $V = \ker D^-$ is given by $P = 1 - D^+GD^-$. At first glance, this expression appears to be valid only for sections in L^2, but we would like to apply it to sections of the form $x_i\varphi$ where $\varphi \in V$. To see that it applies, let γ_n be cutoff functions with $\gamma_n = 1$ on B_n and the support of γ_n contained in B_{n+1}. Then

$$\begin{aligned}\langle P\gamma_n x_i\varphi, \psi\rangle_{L^2} &= \int_{\mathbb{R}^4} \langle (1 - D^+GD^-)\gamma_n x_i\varphi, \psi\rangle \\ &= \int_{B_n} \langle (1 - D^+GD^-)x_i\varphi, \psi\rangle + \int_{B_{n+1}\setminus B_n} \langle \gamma_n x_i\varphi, \psi\rangle \\ &\quad - \int_{B_{n+1}\setminus B_n} \langle D^+GD^-\gamma_n x_i\varphi, \psi\rangle. \end{aligned} \tag{3.4.47}$$

Since $\psi, \varphi = O(|x|^{-3})$, and the volume of $B_{n+1} \setminus B_n$ is $O(n^3)$, we evaluate

$$\left| \int_{B_{n+1}\setminus B_n} \langle \gamma_n x_i\varphi, \psi\rangle \right| \leq \int_{B_{n+1}\setminus B_n} Cn^{-5} = O(n^{-2}). \tag{3.4.48}$$

Recall the following pointwise identity

$$\langle D^+s_1, s_2\rangle = \langle s_1, D^-s_2\rangle + \nabla_i\langle c_i s_1, s_2\rangle \tag{3.4.49}$$

for $s_1 \in \Gamma(E \otimes S^+)$ and $s_2 \in \Gamma(E \otimes S^-)$. Then, using that $D^-\psi = 0$,

$$\begin{aligned} - \int_{B_{n+1}\setminus B_n} \langle D^+GD^-\gamma_n x_i\varphi, \psi\rangle &= \int_{S^3_n} \langle c(\partial_r)GD^-x_i\varphi, \psi\rangle \\ &= \int_{S^3_n} \langle c(\partial_r)Gc_i\varphi, \psi\rangle. \end{aligned} \tag{3.4.50}$$

Since c_i commutes with G, and since $G\varphi = O(|x|^{-1})$ by (3.4.38), we have the integrand above is $O(|x|^{-4})$. Since the volume of S^3_n is $O(n^3)$, the integral (3.4.50) is $O(n^{-1})$.

We then see that in the limit $n \to \infty$ in (3.4.47), the boundary terms vanish, leaving

$$\langle Px_i\varphi, \psi\rangle_{L^2} = \langle (1 - D^+GD^-)x_i\varphi, \psi\rangle_{L^2}. \tag{3.4.51}$$

Therefore the expression $P = 1 - D^+GD^-$ is valid also for $x_i\varphi$, where $\varphi \in V$.

With these definitions in place, we can show that we obtain valid ADHM data.

Theorem 3.4.52. *Given an instanton ∇ on $E \to \mathbb{R}^4$, the data $V = \ker D^-$, $W = E_\infty$, a_i as defined in (3.4.31), and ψ as defined in (3.4.29) satisfy the ADHM equations (3.4.4).*

Proof. Let ω_k be an L^2-orthonormal basis for $V = \ker D^-$. For $\varphi \in V$,

$$\begin{aligned} a_i a_j \varphi &= \langle x_i P x_j \varphi, \omega_k \rangle_{L^2} \omega_k \\ &= \lim_{r\to\infty} \int_{B_r} \big(\langle x_i x_j \varphi, \omega_k \rangle + \langle x_i D^+ G D^- x_j \varphi, \omega_k \rangle \big) \, \omega_k. \end{aligned} \tag{3.4.53}$$

The first term inside the integral will vanish upon skew-symmetrizing, so we focus on the second. By (3.4.49) the second term of (3.4.53) is

$$\langle c_i G c_j \varphi, \omega_k \rangle_{L^2} \omega_k + \text{ boundary term.} \tag{3.4.54}$$

Because Clifford multiplication commutes with $\nabla^*\nabla$, it commutes with G, and hence $\langle c_i G c_j \varphi, \omega_k \rangle_{L^2} \omega_k = P c_{ij} G \varphi$.

By (3.4.49), the boundary term of (3.4.54) is the limit $r \to \infty$ of

$$\int_{S_r} \left\langle x_i \frac{c(x)}{|x|} G c_j \varphi, \omega_k \right\rangle \omega_k = \int_{S_r} \left\langle x_i \frac{c(x)}{|x|} c_j G \varphi, \omega_k \right\rangle \omega_k. \tag{3.4.55}$$

The lefthand term of the inner product is $O(1)$ and the righthand term is $O(|x|^{-3})$, and so the only terms that do not vanish in the limit $r \to \infty$ are the leading terms. Evaluating these leading terms, we have $G\varphi = \frac{c(x)}{4|x|^2}\tilde{\varphi}_\infty + O(|x|^{-2})$ and $\omega_k = \frac{c(x)(\tilde{\omega}_k)_\infty}{|x|^4}$. Considering only these terms, (3.4.55) becomes

$$\begin{aligned} \int_{S_r} \left\langle x_i \frac{c(x)}{|x|} c_j \frac{c(x)}{|x|^2} \tilde{\varphi}_\infty, \frac{c(x)}{4|x|^4} (\tilde{\omega}_k)_\infty \right\rangle \omega_k &= \int_{S_r} \frac{1}{4|x|^5} \langle x_i c_j c(x) \tilde{\varphi}_\infty, (\tilde{\omega}_k)_\infty \rangle \omega_k \\ &= \int_{S_r} \frac{1}{4|x|^5} \langle x_i x_k c_j c_k \tilde{\varphi}_\infty, (\tilde{\omega}_k)_\infty \rangle \omega_k. \end{aligned} \tag{3.4.56}$$

By symmetry on S_r, the integral vanishes unless $i = k$, in which case x_i^2 can, again by symmetry, be replaced by $\frac{|x|^2}{4}$. The above integral then becomes

$$\int_{S_r} \frac{1}{16|x|^3} \langle c_j c_i \tilde{\varphi}_\infty, (\tilde{\omega}_k)_\infty \rangle \omega_k. \tag{3.4.57}$$

Upon taking the limit, this becomes $\frac{\pi^2}{8}\langle c_{ji}\tilde{\varphi}_\infty, (\tilde{\omega}_k)_\infty\rangle_{E_\infty}\omega_k$. Taking the renormalized inner product $\langle\cdot,\cdot\rangle_W = \pi^2\langle\cdot,\cdot\rangle_{E_\infty}$, for $i \neq j$

$$[a_i, a_j]\varphi = 2Pc_{ij}G\varphi + \frac{1}{4}\langle c_{ji}\tilde{\varphi}_\infty, (\tilde{\omega}_k)_\infty\rangle_W\omega_k. \tag{3.4.58}$$

Let η be the skew-form used in the definition of Ψ in (3.4.29). Note that

$$\eta(c_{12}p, q) = \eta(-ip, q) = -i = -\eta(p, c_{12}q). \tag{3.4.59}$$

Similar computation shows that $\eta(c_{jk}v, w) = -\eta(v, c_{jk}w)$ for all $j \neq k$ and $v, w \in \Delta^+$. Using the basis p, q for Δ^+, we can then expand the above inner product as

$$\begin{aligned}
\langle c_{ji}\tilde{\varphi}_\infty, (\tilde{\omega}_k)_\infty\rangle\omega_k &= \langle\eta(c_{ji}\tilde{\varphi}_\infty, p), \eta((\tilde{\omega}_k)_\infty, p)\rangle + \langle\eta(c_{ji}\tilde{\varphi}_\infty, q), \eta((\tilde{\omega}_k)_\infty, q)\rangle\omega_k \\
&= \langle\eta(\tilde{\varphi}_\infty, c_{ij}p), \eta((\tilde{\omega}_k)_\infty, p)\rangle + \langle\eta(\tilde{\varphi}_\infty, c_{ij}q), \eta((\tilde{\omega}_k)_\infty, q)\rangle\omega_k \\
&= \langle\psi^*(\varphi\otimes c_{ij}p), \psi^*(\omega_k\otimes p)\rangle + \langle\psi^*(\varphi\otimes c_{ij}q), \psi^*(\omega_k\otimes q)\rangle\omega_k \\
&= \langle\psi\psi^*(\varphi\otimes c_{ij}p), \omega_k\otimes p\rangle + \langle\psi\psi^*(\varphi\otimes c_{ij}q), \omega_k\otimes q\rangle\omega_k \\
&= \mathrm{trace}_{\Delta^+}(\psi\psi^*\circ(1\otimes c_{ij}))\varphi,
\end{aligned} \tag{3.4.60}$$

where we denote $\mathrm{trace}_{\Delta^+} = 1\otimes\mathrm{trace}$ on $\mathrm{End}(E\otimes\Delta^+) = \mathrm{End}(E)\otimes\mathrm{End}(\Delta^+)$. With notation as in (3.4.3), we have that

$$\psi\psi^*\circ(1\otimes c_{ij}) = (\psi\psi^*)_0\otimes c_{ij} + \sum_{k=2}^{4}(\psi\psi^*)_k c_{1kij}. \tag{3.4.61}$$

If, say, $i = 1$, then $c_{1kij} = c_{kj}$, which is trace-free on Δ^+ unless $k = j$. Similarly, if $j = 1$ then $c_{1kij} = -c_{ki}$ is trace-free unless $k = i$. On the other hand, if $k = i$ then $c_{1kij} = -c_{1j}$ is trace-free unless $j = 1$, and if $k = j$ then $c_{1kij} = c_{1i}$ is trace-free unless $i = 1$. If, however, the pairs $\{i, j\}$ and $\{1, k\}$ are disjoint, then $c_{1kij} = \pm c(\mathrm{vol}) = \mp 1$ on Δ^+.

Summarizing, we have that c_{1kij} is trace-free unless the pairs $\{1, k\}$ and $\{i, j\}$ are either equal or disjoint, and we have

$$\operatorname{trace}_{\Delta^+}(\psi\psi^* \circ (1 \otimes c_{ij})) = \begin{cases} 2(\psi\psi^*)_i & \text{if } j = 1, \\ -2(\psi\psi^*)_j & \text{if } i = 1, \\ \mp 2(\psi\psi^*)_k & \text{if } c_{1kij} = \pm c(\text{vol}). \end{cases} \tag{3.4.62}$$

Taking the self-dual part of $a \wedge a$, (3.4.58) gives us

$$[a_1, a_i] + [a_j, a_k] = 2P(c_{1i} + c_{jk})G - (\psi\psi^*)_i \tag{3.4.63}$$

for (i, j, k) cyclic permutations of $(1, 2, 3)$. Since $c(\bigwedge^+)$ is 0 on S^-, we have $2P(c_{1i} + c_{jk})G = 0$. The remaining terms give precisely the ADHM equations (3.4.4). □

Corollary 3.4.64. *For V, W, a_i, ψ the ADHM data corresponding to an instanton ∇ on E, the dimension of V is the charge of the instanton $-c_2(E)$.*

Proof. In the construction, $V = \ker D^-$. Since $\ker D^+ = 0$, by the Atiyah–Singer Index Theorem, $\dim V = -\operatorname{ind} D = -c_2(E)$. □

3.4.5 From ADHM data to instanton

Conversely, given ADHM data V, W, a_i, ψ, we define an $SU(n)$-vector bundle E and an instanton ∇ as follows. Recall the definition, first given in (3.4.6),

$$Q_x = \begin{bmatrix} c_i \otimes (a_i - x_i) \\ \psi^* \end{bmatrix}.$$

By the non-degeneracy condition, Q_x has full rank for all x, and so has kernel of dimension n everywhere. Thus $E_x = \ker Q_x$ defines a rank n subbundle of the trivial bundle $V \otimes \Delta^- \oplus W \to \mathbb{R}^4$. Define ∇ to be the connection induced from the trivial connection by orthogonal projection.

Theorem 3.4.65. *Given ADHM data V, W, a_i, ψ, the connection ∇ defined above is an instanton.*

Proof. Evaluating $Q_x^* Q_x \colon V \otimes \Delta^+ \to V \otimes \Delta^+$,

$$\begin{aligned} Q_x^* Q_x &= -(c_i(a_i - x_i))(c_j(a_j - x_j)) + \psi\psi^* \\ &= \sum_{i=1}^{4} (a_i - x_i)^2 - \sum_{i<j} c_{ij}[a_i, a_j] + \psi\psi^* \\ &= \sum_{i=1}^{4} (a_i - x_i)^2 - c((a \wedge a)^+) + \psi\psi^* \\ &= \sum_{i=1}^{4} (a_i - x_i)^2 + (\psi\psi^*)_0 \otimes 1, \end{aligned} \tag{3.4.66}$$

where we have used the ADHM equations 3.4.4 to obtain the final equality. By the nondegeneracy condition, $Q_x^* Q_x$ is invertible for all x, and so we can define $\Gamma_x = \left(\sum_{i=1}^{4} (a_i - x_i)^2 + (\psi\psi^*)_0\right)^{-1} \in \mathrm{End}(V)$. Writing $\Gamma_x \otimes 1 \in \mathrm{End}(V) \otimes \mathrm{End}(\Delta)$, we have that $(\Gamma_x \otimes 1)|_{\Delta^+} = (Q_x^* Q_x)^{-1}$. Note in particular that Clifford multiplication commutes with Γ_x.

Projection onto $E_x = \ker Q_x$ is given by

$$P_x = 1 - Q_x \Gamma Q_x^*. \tag{3.4.67}$$

The connection on E_x is then $\nabla = P_x d P_x$.

Computing the curvature gives

$$\begin{aligned} \nabla_i \nabla_j &= P_x \partial_i (1 - Q_x \Gamma Q_x^*) \partial_j P_x \\ &= P_x \partial_i \partial_j P_x + P_x c_i \Gamma c_j P_x \\ &= P_x \partial_i \partial_j P_x + P_x c_i c_j \Gamma P_x. \end{aligned} \tag{3.4.68}$$

After skew-symmetrizing, the first term vanishes, leaving

$$[\nabla_i, \nabla_j] = 2 P_x c_{ij} \Gamma P_x. \tag{3.4.69}$$

Note that the c_{ij} are acting on Δ^-, and so the self-dual part vanishes. Therefore, ∇ is an instanton. $\square$

The process described above is the inverse of the process of generating ADHM data from an instanton. Therefore, valid ADHM data are in one-to-one correspondence with $SU(n)$-instantons on $\mathbb{R}^4$.

Chapter 4

The S^1-Invariant ADHM Construction

As discussed in §2.2, monopoles are the 3-dimensional reduction of instantons. As such, monopoles on a space X correspond to instantons on $X \times \mathbb{R}$ or $X \times S^1$. For this chapter, we consider an identification of instantons and monopoles given by an S^1-fibration that is not trivial, namely (up to orientation) the Hopf fibration $S^3 \to S^2$ extended radially. In this case, studied by Pauly [Pau96, Pau98], the monopoles will be on $\mathbb{R}^3$ with singularity at the origin.

Because of this correspondence, in order to study singular monopoles on $\mathbb{R}^3$, we can instead consider the corresponding instanton on $\mathbb{R}^4$, in which case the powerful tool of the ADHM construction (see §3.4) can be applied. Of course, most instantons constructed via ADHM will not be S^1-invariant, so it is necessary to see how the S^1-invariance condition exhibits itself in terms of the ADHM data. In this chapter, we give an S^1-invariant ADHM construction by decomposing $V = \ker D^-$ into S^1-subrepresentations.

The approach taken is similar to that in [BA90]. In that case, they give an S^1-invariant ADHM construction for hyperbolic monopoles by considering the conformal identification $H^3 \times S^1 = \mathbb{R}^4 - \mathbb{R}^2 = S^4 - S^2$.

4.1 Monopoles and S^1-invariant instantons

Identifying $\mathbb{R}^4 = \mathbb{H}$, S^1 acts by multiplication on the right by $e^{i\theta}$. We consider the map

$$\begin{aligned} \pi : \mathbb{H} &\to \operatorname{Im}\mathbb{H} \\ q &\mapsto qi\bar{q}. \end{aligned} \tag{4.1.1}$$

The map π is invariant under the S^1 action and so defines a quotient map.

We will work with this map using complex coordinates. Identifying $\mathbb{H} = \mathbb{C}^2$ by $q = z_1 + z_2 j$, then $e^{i\theta} \in S^1$ acts by $(e^{i\theta} z_1, e^{-i\theta} z_2)$. We can also identify $\mathbb{C} \oplus \mathbb{R} = \Im\mathbb{H}$ by $(z, x) \mapsto xi - zk$, in which case the quotient map (4.1.1) is written

$$\begin{aligned} \pi : \mathbb{C}^2 &\to \mathbb{C} \oplus \mathbb{R} \\ (z_1, z_2) &\mapsto (2z_1 z_2, |z_1|^2 - |z_2|^2). \end{aligned} \tag{4.1.2}$$

Away from the fixed point 0, this defines an S^1 fibration over $\mathbb{R}^3 \setminus \{0\}$. Indeed, up to choice of orientation, fixed only for sign purposes, this is the Hopf fibration $S^3 \to S^2$ extended radially.

We will also use real coordinates, given by $(z_1, z_2) = (x_1 + ix_2, x_3 + ix_4)$ where convenient. In these real coordinates, the vector field ∂_θ in the direction of the action is given by

$$\partial_\theta = -x_2\partial_1 + x_1\partial_2 + x_4\partial_3 - x_3\partial_4. \tag{4.1.3}$$

Let ξ denote twice its metric dual,

$$\xi = 2(-x_2 dx_1 + x_1 dx_2 + x_4 dx_3 - x_3 dx_4). \tag{4.1.4}$$

Letting y be the coordinates on $\mathbb{R}^3$, from (4.1.1) it is easy to see that $|y| = |x|^2$. Moreover, computing pullbacks gives that the forms $\pi^* dy_i$ are all orthogonal to each other and orthogonal to ξ. Since $|\pi^* dy_i| = |\xi| = 2|x|$, we can express the metric on $\mathbb{R}^4$ as

$$\sum_{i=1}^{4} dx_i^2 = \frac{1}{4|x|^2}\left(\sum_{i=1}^{3}(\pi^* dy_i)^2 + \xi^2\right). \tag{4.1.5}$$

We can now describe the correspondence between instantons on $\mathbb{R}^4 \setminus \{0\}$ and monopoles on $\mathbb{R}^3 \setminus \{0\}$. Given an $SU(n)$-vector bundle E' over $\mathbb{R}^3 \setminus \{0\}$ with a monopole (∇^3, Φ), we define a connection ∇^4 on the pullback bundle $E = \pi^* E'$ by

$$\nabla^4 = \pi^* \nabla^3 + \pi^* \Phi \otimes \xi. \tag{4.1.6}$$

Because the fibres of the pullback bundle E are naturally identified along S^1-orbits, for $e^{i\theta} \in S^1$ viewed as a diffeomorphism of $\mathbb{R}^4$, we can identify $(e^{i\theta})^* E = E$. As such, we can view $(e^{i\theta})^* \nabla^4$ as a connection on E. For connections ∇^4 obtained from pairs (∇^3, Φ) as in (4.1.6), ∇^4 is S^1-invariant in the sense that $(e^{i\theta})^* \nabla^4 = \nabla^4$. A more in-depth discussion of the S^1 action on E is given later, when discussing the extension to a bundle with action on S^4.

Lemma 4.1.7. *The pair (∇^3, Φ) is a monopole if and only if ∇^4 is an instanton.*

Proof. Computing the curvature of ∇^4,

$$F_{\nabla^4} = \pi^* F_{\nabla^3} + \pi^*(d_{\nabla^3}\Phi) \wedge \xi + \pi^* \Phi \wedge d\xi. \tag{4.1.8}$$

Observe that $d\xi = 4(dx_1 \wedge dx_2 - dx_3 \wedge dx_4)$ is anti-self-dual, and so

$$F^+_{\nabla^4} = \left(\pi^* F_{\nabla^3} + \pi^*(d_{\nabla^3}\Phi) \wedge \xi\right)^+ . \tag{4.1.9}$$

For a one-form on α on $\mathbb{R}^3$, the expression for the metric (4.1.5) shows clearly that $*_4(\pi^*\alpha \wedge \xi) = \pi^*(*_3\alpha)$. Then evaluating (4.1.9),

$$0 = \frac{1 + *_4}{2} \pi^*(F_{\nabla^3} + *_3 d_{\nabla^3}\Phi). \tag{4.1.10}$$

Clearly if (∇^3, Φ) satisfies the Bogomolny equation (2.3.4), then $F^+_{\nabla^4} = 0$.

Conversely, suppose $F^+_{\nabla^4} = 0$. For any $\beta \in \Omega^2(\mathbb{R}^3)$ note that $*_4 \pi^* \beta = \pi^* *_3 \beta \wedge \xi$, again by (4.1.5). Note, however, that ξ is orthogonal to all pullbacks from $\mathbb{R}^3$, and so we have that if $\pi^*\beta$ is self-dual then $\beta = 0$. Applying this to $\beta = F_{\nabla^3} + *_3 d_{\nabla^3}\Phi$ shows that (∇^3, Φ) satisfies the Bogomolny equation. □

Remark. Because only the self-dual part $d\xi$ vanishes, we do not obtain a correspondence for connections with self-dual curvature. This stands in contrast to the correspondence between monopoles on M and S^1-invariant instantons on $M \times S^1$ (cf. §2.3).

This correspondence is one-to-one: if ∇^4 is an instanton on $E = \pi^* E'$ that is S^1-invariant and $s \in \Gamma(E')$, then $\nabla^4 \pi^* s$ is S^1-invariant, and so is itself the pullback of a section on E'. This allows us to define

$$\pi^*(\Phi s) = \frac{1}{2|x|^2} \nabla^4_{\partial_\theta} \pi^* s, \tag{4.1.11}$$

$$\pi^*(\nabla^3_X s) = \nabla^4_{\pi^* X} \pi^* s. \tag{4.1.12}$$

Here $\pi^* X$ denotes the horizontal lift of X.

Remark. The correspondence between instantons and monopoles implicitly specifies an S^1 action on the bundle E. If a connection is invariant with respect to multiple actions on E, then it will correspond to multiple monopoles. This phenomenon can be seen in §4.3.1.

Pauly [Pau98] studied such monopoles with conditions such that F_{∇^4} has finite energy locally around 0, in which case ∇^4 extends to a smooth connection across the origin by [Uhl82]. In particular, he enforced no conditions on the behaviour of F_{∇^4} as $x \to \infty$. We study such monopoles under the assumption that ∇^4 has finite energy over all $\mathbb{R}^4$.

By stereographic projection, we consider $\mathbb{R}^4 \subset S^4$. The finite energy condition $\|F_{\nabla^4}\|^2 < \infty$ then ensures that E and ∇^4 can also be extended smoothly over the point ∞ [Uhl82], and so we obtain a bundle $E \to S^4$.

Even though the finite energy condition on ∇^4 means that the instanton can be extended smoothly across 0, in general the corresponding monopole will still be singular at 0. The finite energy assumption, however, does enforce a specific behaviour of Φ, both at the singularity at 0 and as a decay condition at ∞. This behaviour of Φ is described in Proposition 4.1.25, and is given in terms of the behaviour of the S^1-action on E, which we describe now.

The S^1-action extends to an action on S^4, with the only fixed points being 0 and ∞. Recall the definition of the pullback bundle

$$E = \pi^* E' = \{(p, v) \in (S^4 \setminus \{0, \infty\}) \times E' \mid v \in E'_{\pi(p)}\}. \tag{4.1.13}$$

With this description, we define a lift of the S^1 action on $S^4 \setminus \{0, \infty\}$ to E by $\rho(e^{i\theta})(x, v) = (e^{i\theta}x, v)$. Equivalently, given a local frame $s' = \{s'_i\}$ on E', we can define the pullback frame on E by $s_i = \pi^* s'_i$. The lift of the S^1-action is given with respect to this frame by $\rho(e^{i\theta})s_i = s_i(e^{i\theta}x)$. Note that, although we have extended the bundle E over all of S^4, the pullback structure, and hence the S^1-action ρ, is defined only away from the points 0 and ∞. Corollary 4.1.23 will show that the S^1-action can be extended to E_0 and E_∞.

Although a pullback frame like the above is convenient for the description of the S^1-action, there are reasons to consider other frames on E. In particular, a singularity of (∇^3, Φ) pulls back to a singularity in the connection form of ∇^4 when written with respect to an S^1-invariant frame. However, since ∇^4 extends smoothly to S^4, there must be a change of frame that removes the apparent singularity in the connection form. This new frame, however, will not in general be S^1-invariant.

We therefore wish to describe the S^1-action on E with respect to more general choice of frames. For $s = \{s_i\}$ a local frame on E such that both x and and $e^{i\theta}x$ are in the domain of s, we denote by $[\rho_x(e^{i\theta})]_s \in \mathbb{C}^{n\times n}$ the frame-dependent expression for $\rho(e^{i\theta})$, so that

$$\rho(e^{i\theta})s_j(x) = [\rho_x(e^{i\theta})]_{kj} s_k(x). \tag{4.1.14}$$

Let us consider in particular a radially covariant constant frame $\tilde{s}$. Fix an orthonormal basis $\{e_i\}$ of E_0, and define $\tilde{s}_i(x)$ by parallel translation of e_i along the radial geodesic from 0 to x. This defines a trivialization of $E|_{S^4\setminus\{\infty\}}$. Given that this frame is not S^1-invariant, we should expect $[\rho_x(e^{i\theta})] \neq 1$ in general. Moreover, the expression $[\rho_x(e^{i\theta})]$ could a priori vary with x. The following lemma shows this is not the case.

Lemma 4.1.15. *Let $\tilde{s}$ be the radially covariant constant frame as defined above. Then the local expression $[\rho_x(e^{i\theta})]_{\tilde{s}}$ is constant in x.*

Proof. For a curve $\gamma : [0,1] \to S^4$, let $\tau_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$ denote parallel transport along γ. Let $f_0 \in E_{\gamma(0)}$ and let f be a covariant constant section along γ with $f(\gamma(0)) = f_0$, and so $\tau_\gamma f_0 = f(\gamma(1))$. Because ∇ is S^1-invariant, $\rho(e^{i\theta})s$ is covariant constant along the curve $e^{i\theta}\gamma$, showing that $\tau_{e^{i\theta}\gamma}\rho(e^{i\theta})f_0 = \rho(e^{i\theta})f(\gamma(1)) = \rho(e^{i\theta})\tau_\gamma f_0$. That is,

$$\tau_{e^{i\theta}\gamma}\rho(e^{i\theta}) = \rho(e^{i\theta})\tau_\gamma. \tag{4.1.16}$$

For the rest of the proof we will work with the frame $\tilde{s}$ and a fixed θ, so for ease of notation we omit these in the local expression, writing $[\rho_x(e^{i\theta})]_{\tilde{s}} = [\rho_x]$. Let $[\tau_\gamma]$ denote the local expression for τ_γ with respect to the frame $\tilde{s}$. Then locally, (4.1.16) becomes

$$[\tau_{e^{i\theta}\gamma}][\rho_{\gamma(0)}] = [\rho_{\gamma(1)}][\tau_\gamma]. \tag{4.1.17}$$

Consider radial geodesic coordinates x for $S^4 \setminus \{\infty\}$ centred at 0, and for $\lambda > 0$ let λx denote scalar multiplication in these coordinates. If γ is a radial geodesic, then since $\tilde{s}$ is radially covariant constant, $[\tau_\gamma] = 1$. Moreover, the curve $e^{i\theta}\gamma$ is also a radial geodesic, and so $[\tau_{e^{i\theta}\gamma}] = 1$ also. Setting γ in (4.1.17) to be the radial geodesic from x to λx then gives

$$[\rho_x] = [\rho_{\lambda x}]. \tag{4.1.18}$$

Therefore, $[\rho_x]$ is radially constant.

Since ∇ is smooth, the connection form A of ∇ with respect to the frame $\tilde{s}$ is bounded over compact subsets $K \subset S^4 \setminus \{\infty\}$. Therefore, there is a constant C_K depending only on the subset K such that

$$|[\tau_\gamma] - 1| < C_K |\gamma|, \tag{4.1.19}$$

where γ is a curve in K and $|\gamma|$ denotes the length of γ.

Let $x, y \neq 0$, and let $K \subset S^4 \setminus \{\infty\}$ be a ball centred at 0 containing both x and y. Let $\gamma_{y,x}$ denote a length-minimizing geodesic from x to y. Since $S^4 \setminus \{0, \infty\}$ is connected, it is sufficent to show that $[\rho_x]$ is locally constant. We may thus assume without loss of generality that $\gamma_{y,x}$ is contained in K and does not pass through 0. The geodesic distance $|\gamma_{y,x}|$ between x and y is bounded by the Euclidean distance $|x - y| = \sqrt{\sum (x_i - y_i)^2}$ in the geodesic coordinates, and so we obtain

$$|[\tau_{\gamma_{y,x}}] - 1| < C_K |\gamma_{y,x}| < C_K |x - y|. \tag{4.1.20}$$

Scaling by $\lambda > 0$ gives $|\lambda x - \lambda y| = \lambda |x - y|$. Therefore, for fixed x and y,

$$[\tau_{\gamma_{\lambda y, \lambda x}}] = 1 + O(\lambda) \tag{4.1.21}$$

as $\lambda \to 0$. Since the S^1-action preserves the lengths $|x|$, $|y|$, and $|\gamma_{y,x}|$, we have similarly that $[\tau_{e^{i\theta}\gamma_{\lambda y, \lambda x}}] = 1 + O(\lambda)$.

From (4.1.17),

$$\begin{aligned} [\rho_{\lambda x}] &= [\tau_{e^{i\theta}\gamma_{\lambda y, \lambda x}}]^{-1} [\rho_{\lambda y}] [\tau_{\gamma_{\lambda y, \lambda x}}] \\ &= [\rho_{\lambda y}] + O(\lambda). \end{aligned} \tag{4.1.22}$$

From (4.1.18), however, $[\rho_{\lambda x}]$ and $[\rho_{\lambda y}]$ are both constant in λ. Therefore $[\rho_{\lambda x}] - [\rho_{\lambda y}]$ is both constant in λ and $O(\lambda)$, hence it must be 0. Therefore $[\rho_x] = [\rho_y]$, and so $[\rho_x]$ is constant in x. $\square$

Because $[\rho_x]$ is constant in the frame $\tilde{s}$, it can be extended continuously to the fibre E_0. By defining a similar frame that is radially covariant constant from ∞, we can extend the action to E_∞. This proves the following Corollary.

Corollary 4.1.23. *The S^1-action $\rho(e^{i\theta})$ on $E|_{S^4 \setminus \{0,\infty\}}$ extends continuously to an S^1-action on E over all of S^4.*

Since 0 and ∞ are fixed points of the S^1 action, we have that $\rho(e^{i\theta})$ maps E_0 to E_0 and E_∞ to E_∞. Therefore, the fibres E_0 and E_∞ are representations of S^1. Since S^1 is abelian, these representations diagonalize, so that after some local changes of trivialivization we have

$$\rho(e^{i\theta})_{E_0} = \begin{bmatrix} e^{ik_1\theta} & & & \\ & e^{ik_2\theta} & & \\ & & \ddots & \\ & & & e^{ik_n\theta} \end{bmatrix}, \qquad \rho(e^{i\theta})_{E_\infty} = \begin{bmatrix} e^{i\ell_1\theta} & & & \\ & e^{i\ell_2\theta} & & \\ & & \ddots & \\ & & & e^{i\ell_n\theta} \end{bmatrix}. \tag{4.1.24}$$

We will refer to the coefficients k_i and ℓ_i of the exponents as the *weights* of the representation. We henceforth assume the weights are ordered by $k_1 \geq k_2 \geq \cdots \geq k_n$ and $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_n$. Since ρ preserves the $SU(n)$-structure of the vector bundle, as can be seen by looking in an S^1-invariant frame, we have

$$\sum_j k_j = \sum_j \ell_j = 0.$$

Remark. Even though Lemma 4.1.15 shows that $[\rho_x]_s$ is constant in a frame s defined everywhere away from ∞, note that this frame need not in general extend across ∞, and so the constant action need not extend across ∞. When we then choose a frame s' around ∞ with respect to which $[\rho_x]_{s'}$ is constant, it is not in general true that $[\rho_x]_s = [\rho_x]_{s'}$, and so the extensions of the S^1 action at 0 and ∞ will in general not be isomorphic as S^1-representations. In particular, we should not expect that $k_j = \ell_j$.

The weights k_j and ℓ_j can be obtained directly from the monopole as the eigenvalues of the leading term of Φ, at 0 for the weights k_j and at ∞ for the weight ℓ_j.

Proposition 4.1.25. *Let (∇^3, Φ) and ∇^4 be a monopole and instanton as in the preceeding discussion, with $\|F_{\nabla^4}\|^2 < \infty$. Then in some trivialization near 0,*

$$\Phi = \frac{1}{2|y|} \begin{bmatrix} ik_1 & & \\ & \ddots & \\ & & ik_n \end{bmatrix} + O(1),$$

and in some trivialization near ∞,

$$\Phi = \frac{1}{2|y|} \begin{bmatrix} i\ell_1 & & \\ & \ddots & \\ & & i\ell_n \end{bmatrix} + O(|y|^{-2}).$$

Proof. Let $e_1, \ldots, e_n$ be a basis for E_0 with respect to which ρ_{E_0} is diagonal as in (4.1.24). Let s be the frame on $E|_{S^4 \setminus \{\infty\}}$ defined by parallel translation of e_i along radial geodesics from 0. By Lemma 4.1.15, we then have the frame-dependent expression $[\rho_x(e^{i\theta})]_s$ is constant in x.

Let (z_1, z_2) be complex coordinates for the stereographic projection $\mathbb{C}^2 = \mathbb{R}^4 = S^4 \setminus \{\infty\}$, and let $U = \{z_1 \neq 0\}$. Any element of U can be written as $(z_1, z_2) = e^{i\theta} \cdot (\lambda, w)$, where $e^{i\theta} = \frac{z_1}{|z_1|} \in S^1$, $\lambda = |z_1| \in \mathbb{R}_{>0}$, and $w = \frac{z_1 z_2}{|z_1|} \in \mathbb{C}$. The slice $\mathbb{R}_{>0} \times \mathbb{C}$ then parametrizes all of the S^1-orbits in U. Starting with the frame s along the slice $\mathbb{R}_{>0} \times \mathbb{C}$, we can use the S^1-action to extend to an S^1-invariant frame $(t_1, \ldots, t_n)$. More precisely, we define

$$t_j(e^{i\theta} \cdot (\lambda, w)) = \rho(e^{i\theta}) s_j(\lambda, w) = e^{ik_j\theta} s_j(e^{i\theta} \cdot (\lambda, w)). \tag{4.1.26}$$

We can define a similar S^1-invariant frame away from $z_2 = 0$.

Let A be the connection form of ∇ with respect to the frame s and $\tilde{A}$ be the connection form with respect to the frame t. Since the change of gauge from s to t is

$$g(e^{i\theta} \cdot (\lambda, w)) = \begin{bmatrix} e^{-ik_1\theta} & & \\ & \ddots & \\ & & e^{-ik_n\theta} \end{bmatrix},$$

we obtain

$$\begin{aligned} \tilde{A}_{\partial_\theta} &= gA_{\partial_\theta}g^{-1} + g\partial_\theta g^{-1} \\ &= gA_{\partial_\theta}g^{-1} + \begin{bmatrix} ik_1 & & \\ & \ddots & \\ & & ik_n \end{bmatrix}. \end{aligned} \tag{4.1.27}$$

Let $r = |z|$, and ∂_r give the unit radial vector field on $\mathbb{C}^2 = \mathbb{R}^4$. Since the frame s was obtained by radial parallel translation, we have $A_{\partial_r} = 0$ and A vanishes in all directions at the origin. As such, since $|\partial_\theta| = r$,

$$|\partial_r A_{\partial_\theta}| = |F_\nabla(\partial_r, \partial_\theta)| \leq r\|F_\nabla\|_{L^\infty}. \tag{4.1.28}$$

Therefore $|A_{\partial_\theta}| \leq r^2\|F_\nabla\|_{L^\infty}$. From (4.1.27) and the correspondence with Φ given in (4.1.11),

$$\begin{aligned} \pi^*\Phi &= \frac{1}{2r^2}\tilde{A}_{\partial_\theta} \\ &= \frac{1}{2r^2}\begin{bmatrix} ik_1 & & \\ & \ddots & \\ & & ik_n \end{bmatrix} + O(1) \end{aligned} \tag{4.1.29}$$

In coordinates on $\mathbb{R}^3$,

$$\Phi = \frac{1}{2|y|}\begin{bmatrix} ik_1 & & \\ & \ddots & \\ & & ik_n \end{bmatrix} + O(1) \tag{4.1.30}$$

as $y \to 0$.

Performing a similar argument in the stereographic coordinates $\tilde{x}$ near ∞, in an S^1-invariant trivialization we have

$$A_{\partial_\theta} = \begin{bmatrix} i\ell_1 & & \\ & \ddots & \\ & & i\ell_n \end{bmatrix} + O(|\tilde{x}|^2) \tag{4.1.31}$$

as $\tilde{x} \to 0$. Since $|\tilde{x}| = |x|^{-1}$, we can compute in the coordinates x

$$\pi^*\Phi = \frac{1}{2|x|^2} \begin{bmatrix} i\ell_1 & & \\ & \ddots & \\ & & i\ell_n \end{bmatrix} + O(|x|^{-4}). \tag{4.1.32}$$

In coordinates on $\mathbb{R}^3$,

$$\Phi = \frac{1}{2|y|} \begin{bmatrix} i\ell_1 & & \\ & \ddots & \\ & & i\ell_n \end{bmatrix} + O(|y|^{-2}) \tag{4.1.33}$$

as $y \to \infty$. □

4.2 S^1-invariant ADHM construction

In this section we specialize the ADHM construction from §3.4 to the case where ∇ is S^1-invariant. We do this by decomposing the space $V = \ker D^-$ into subrepresentations of different weights. We can compute the dimensions of these subrepresentations using equivariant index theory.

Proposition 4.2.1. *The S^1-action lifts to an action on the spinor bundle S over S^4. Moreover, this action preserves the decomposition $S = S^+ \oplus S^-$.*

Proof. Let $\mu_\theta\colon S^4 \to S^4$ denote the action by $e^{i\theta} \in S^1$. For an element of the frame bundle $s_x \in F_{SO}(S^4)_x$, pushforward $(\mu_\theta)_* s_x$ gives a lift of $\mu_\theta(x)$. Choose an element of the spin bundle $p_x \in P_x$ above s_x. For small θ there is a unique lift γ of $(\mu_\theta)_* s_x$ to P such that $\gamma(0) = p_x$. Continuing in θ then defines a section, and hence a trivialization of P, along the curve $\mu_\theta(x)$. If $\gamma(2\pi) = \gamma(0)$, then there is a trivialization over the entire S^1 orbit in S^4, and hence a lift of the S^1 action. Note that since $(\mu_{2\pi})_* s_x = s_x$, the lift $\gamma(2\pi)$ must be one of two elements in P_x above s_x. Contracting S^1 orbits in S^4 to the fixed point 0, by homotopy it is then enough to exhibit a lift on the fibre P_0. Note that we could instead contract to the fibre at ∞, and the construction of a lift there would be similar to that described below.

We construct the lift explicitly on $\mathrm{Spin}(T_0S^4) \subset \mathrm{Cl}(T_0S^4)$. For an orthonormal basis of T_0S^4, let $c_1, \ldots, c_4$ be the corresponding elements of $\mathrm{Cl}(T_0S^4)$. We then define

$$\gamma(\theta) = \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}c_1c_2\right)\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}c_3c_4\right). \tag{4.2.2}$$

Note that, despite the presence of $\frac{\theta}{2}$, the above expression is unchanged on replacing θ by $\theta + 2\pi$, and therefore γ is a well-defined map $S^1 \to \mathrm{Spin}(T_0S^4)$.

We now verify that $\gamma(\theta)$ is a lift of $(\mu_\theta)_*$. For $i = 1, 2$ note that c_i commutes with c_3c_4, but anticommutes with c_1c_2. Therefore, $\gamma(\theta)$ acts on $c_i \in T_0S^4$ by

$$\begin{aligned}
\gamma(\theta)c_i\gamma(\theta)^{-1} &= \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}c_1c_2\right) c_i \left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}c_1c_2\right) \\
&= \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}c_1c_2\right)^2 c_i \\
&= (\cos\theta + \sin\theta c_1c_2)c_i.
\end{aligned}$$

It is a simple computation to show that μ_θ does indeed act on $\mathrm{span}\{c_1, c_2\}$ by multiplication by $(\cos\theta + \sin\theta c_1c_2)$. A similar computation shows that $\gamma(\theta)$ acts correctly on c_3 and c_4. Therefore γ defines a lift at 0, and thus a lift everywhere on S^4.

This action then gives an action on the spinor bundle S which respects the Clifford action on S in the sense that, for $s \in S_p$ and $X \in T_p S^4$

$$\gamma(\theta)c(X)s = c((\mu_\theta)_* X)\gamma(\theta)s. \tag{4.2.3}$$

Recall that the decomposition $S = S^+ \oplus S^-$ is the ∓ 1-eigenvalue decomposition with respect to $c(\mathrm{vol})$, where vol is the volume form on S^4. Since vol is invariant under the S^1-action, this decomposition is also preserved under the action. $\square$

We can compute the action on the fibres $S_0^\pm$ and $S_\infty^\pm$.

Lemma 4.2.4. *The S^1 action is trivial on S_0^+ and S_∞^-, and has weights ± 1 on S_∞^+ and S_0^-.*

Proof. At 0, the action is given explicitly by (4.2.2), and when expanded becomes

$$\gamma(\theta) = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}c_{1234} + \cos\frac{\theta}{2}\sin\frac{\theta}{2}(c_{12} - c_{34}), \tag{4.2.5}$$

On S^+, the volume form c_{1234} acts as -1 while anti-self-dual forms such as $c_{12} - c_{34}$ act trivially. Thus, the S^1-action on S^+ is simply 1. On S^-, the volume form acts as 1 and c_{34} acts as $-c_{12}$, giving that

$$\gamma(\theta)|_{S^-} = \cos\theta + \sin\theta c_{12}. \tag{4.2.6}$$

Since $c_{12}^2 = -1$, it diagonalizes on (the two-dimensional) S^- as $\pm i$, from which we see that $\gamma(\theta)|_{S^-}$ diagonalizes as $e^{\pm i\theta}$.

Around ∞, we note that the action appears the same in stereographic coordinates, and so we obtain the same explicit expression for $\gamma(\theta)|_{S_\infty}$. The same argument as at 0 then applies, needing only to switch S^+ and S^- because the map from coordinates x near 0 to stereographic coordinates $\tilde{x}$ near ∞ is orientation-reversing. $\square$

If ∇ is an S^1-invariant connection, then the associated Dirac operator $D = c_i\nabla_i$ acting on $E \otimes S$ is S^1-equivariant. Likewise, the operators $D^\pm \colon S^\pm \otimes E \to S^\mp \otimes E$ are S^1-equivariant.

As a result, $V = \ker D^-$ carries an S^1-action. We can then decompose $V = \bigoplus_j V_{m_j}$, where V_{m_j} is a subrepresentation on which S^1 acts as $e^{im_j\theta}$. Note that in this notation, V_{m_j} is not an irreducible representation (which are all one-dimensional) but instead a collection of all the irreducible subrepresentations of the same weight.

Recall from Sections 3.4.2 and 3.4.4 that the ADHM data are given by maps $\alpha = Pz_1$ and $\beta = Pz_2$, where by z_i we mean multiplication by z_i, and P denotes orthogonal projection to V. Since P is S^1-equivariant, for $\phi \in V_m$ we have

$$\rho(e^{i\theta})(\alpha\phi) = P(e^{i\theta}z_1)(e^{im\theta}\phi) = e^{i(m+1)\theta}\alpha\phi. \tag{4.2.7}$$

As such, $\alpha\colon V_m \to V_{m+1}$, and similarly $\beta\colon V_m \to V_{m-1}$. We then write $\alpha_m = \alpha|_{V_m}$ and $\beta_m = \beta|_{V_m}$.

As part of the ADHM data, there was an additional vector space W and maps $\pi : V \to W$ and $\sigma : W \to V$. We decompose these similarly, letting $\pi_m = \pi|_{V_m}$, and $\sigma_m = \mathrm{proj}_{V_m}\sigma$. Restricting to the $V_m \to V_m$ part of the ADHM equations (3.4.13) and (3.4.14) gives

$$\alpha_{m-1}\beta_m - \beta_{m+1}\alpha_m + \sigma_m\pi_m = 0, \tag{4.2.8}$$

$$\alpha_{m-1}\alpha_{m-1}^* - \alpha_m^*\alpha_m + \beta_{m+1}\beta_{m+1}^* - \beta_m^*\beta_m + \sigma_m\sigma_m^* - \pi_m^*\pi_m = 0. \tag{4.2.9}$$

On the other hand, for $m \neq p$, terms involving α and β vanish in the $V_m \to V_p$ part of the ADHM equations, giving additional equations only for π and σ,

$$\sigma_p\pi_m = 0, \tag{4.2.10}$$

$$\sigma_p\sigma_m^* - \pi_p^*\pi_m = 0. \tag{4.2.11}$$

These S^1-invariant ADHM equations, together with the same non-degeneracy condition that R_z from (3.4.15) is full rank, can then be used to construct S^1-invariant instantons.

In order to construct examples, it remains only to see what decompositions of V are indeed possible. In the same way that the dimension of V can be computed using index theory, the dimension of each V_m can be found using equivariant index theory. For this purpose, it is necessary to introduce some concepts from equivariant K-theory. It should be noted, however, that in the case of interest, because the fixed points of the action are isolated points, many of the topological considerations do not arise, and the index computation involves only the characters of certain representations. The definitions below leading up to the Atiyah–Segal–Singer Fixed Point Theorem follow mainly from Lawson–Michelson [LM89, §§ III.9, III.14].

Let M be a manifold equipped with an action by a compact Lie group G. In a similar way to how $K(M)$ is defined to be formal sums of equivalence classes of vector bundles, we define $K_G(M)$ to be formal sums of G-vector bundles, now where equivalence is given by G-equivariant isomorphisms. We also define the representation group $R(G)$ to be the abelian group generated by irreducible representations of G, or equivalently $R(G) = K_G(\{\mathrm{pt}\})$.

Our application of equivariant index theory will be on a fixed point set, and so it is sufficient to consider the case where the G-action on M is trivial. Therefore each fibre of a G-vector bundle is a representation of G. Consider for a moment a representation V of G, and denote by V_i the irreducible representations of G. Any finite-dimensional representation of G can be decomposed into a sum of irreducible subrepresentations, and by Schur's Lemma $\mathrm{Hom}_G(V_i, V) = \mathbb{C}^{d_i}$ where d_i is the number of copies of V_i showing up in the decomposition. Therefore, we have that

$$V = \bigoplus_i \mathrm{Hom}_G(V_i, V) \otimes V_i.$$

Applying this to each fibre of a G-vector bundle E and varying the base point,

$$E = \bigoplus_i \mathrm{Hom}_G(E_i, E) \otimes E_i, \tag{4.2.12}$$

where E_i is the (topologically) trivial irreducible G-bundle $E_i = M \times V_i$. Extending this equivalence to formal sums of bundles, we then have for trivial G-actions on M

that

$$K_G(M) \cong K(M) \otimes R(G). \tag{4.2.13}$$

We define the equivariant Chern character ch_G to be the $R(G)$-valued characteristic class obtained by composing this isomorphism with $\mathrm{ch} \otimes 1$. For a representation V of G, there is a character (of the representation) $\chi_V \colon G \to \mathbb{C}$ defined by taking the trace of $g \colon V \to V$. Composing this with ch_G gives a map

$$\mathrm{ch}_g \colon K_G(M) \to H^\bullet(M). \tag{4.2.14}$$

Also, given a G-bundle E, define $\lambda_{-1}(E) = [\Lambda^{\mathrm{even}}(E)] - [\Lambda^{\mathrm{odd}}(E)] \in K_G(M)$.

Given a G-invariant elliptic operator $P : C^\infty(E) \to C^\infty(F)$, we define the $R(G)$-valued G-index to be

$$\mathrm{ind}_G(P) = [\ker P] - [\mathrm{coker}\, P]. \tag{4.2.15}$$

As above, we can take the character of this representation to define

$$\mathrm{ind}_g(P) = \mathrm{trace}(g|_{\ker P}) - \mathrm{trace}(g|_{\mathrm{coker}\, P}). \tag{4.2.16}$$

Additionally, define the symbol class $\sigma(P)$ of P to be

$$\sigma(P) = [\pi^* E] - [\pi^* F] \in K_G(TM), \tag{4.2.17}$$

where $\pi \colon TM \to M$ denotes projection.

Remark. The symbol class $\sigma(P)$ should more correctly be viewed as an element of $K_G(DM, \partial DM)$, classes of vector bundles that are equivalent on the boundary of the disk bundle of M. By ellipticity, the symbol of P gives such an equivalence between $\pi^* E$ and $\pi^* F$ over ∂DM. For the following, however, we will be pulling back $\sigma(P)$ to a point, and so the distinction will be unimportant.

Let $F_g \subset M$ denote the fixed point set of an element $g \in G$, and let $i \colon TF_g \hookrightarrow TM$ be inclusion, and let $\pi \colon TF_g \to F_g$ be projection. Let N_g be the normal bundle of F_g.

With the notation above, we have the Atiyah–Singer–Segal Fixed Point Theorem,

$$\operatorname{ind}_g(P) = (-1)^d \left(\frac{\operatorname{ch}_g i^*\sigma(P)}{\operatorname{ch}_g(\lambda_{-1}\pi^*(N_g \otimes \mathbb{C}))} \hat{A}(F_g)^2 \right) [F_g]. \tag{4.2.18}$$

where d is the dimension of F_g.

Proposition 4.2.19. *Let ∇ be an S^1-invariant instanton on a bundle E where $k_1, \ldots, k_n$ and $\ell_1, \ldots, \ell_n$ are the weights of the S^1-action on E_0 and E_∞. Then the character of the representation* $\ker D^-$ *is*

$$\chi_{\ker D^-} = \frac{(e^{ik_1\theta} + \cdots + e^{ik_n\theta}) - (e^{i\ell_1\theta} + \cdots + e^{i\ell_n\theta})}{(e^{i\theta/2} - e^{-i\theta/2})^2}. \tag{4.2.20}$$

Proof. Since, for $\theta \neq 0$, the fixed point set consists of two isolated points, $F_\theta = \{0, \infty\}$, the various vector bundles in (4.2.18) are all merely pairs of vector spaces. As such, ch_g is merely the character of the representation. Therefore, evaluating first at 0 and making use of Lemma 4.2.4 to compute the character of the representations of S_0^+ and S_0^-,

$$\begin{aligned}\operatorname{ch}_\theta(i^*\sigma)_0 &= \chi_{E_0 \otimes S_0^+}(\theta) - \chi_{E_0 \otimes S_0^-}(\theta) \\ &= (\chi_{E_0} \cdot (\chi_{S_0^+} - \chi_{S_0^-}))(\theta) \\ &= (e^{ik_1\theta} + \cdots + e^{ik_n\theta})(-e^{i\theta} + 2 - e^{-i\theta}).\end{aligned} \tag{4.2.21}$$

Similarly, at ∞

$$\operatorname{ch}_\theta(i^*\sigma)_\infty = (e^{i\ell_1\theta} + \cdots + e^{i\ell_n\theta})(e^{i\theta} - 2 + e^{-i\theta}). \tag{4.2.22}$$

The denominator may be explicitly computed from the action on $(N_\theta)_p = T_pS^4 \otimes \mathbb{C}$

$$\begin{aligned}\operatorname{ch}_\theta(\lambda_{-1}\pi^*(N \otimes \mathbb{C})) &= \chi_{\Lambda^{\text{even}} T_pS^4 \otimes \mathbb{C}}(\theta) - \chi_{\Lambda^{\text{odd}} T_pS^4 \otimes \mathbb{C}}(\theta) \\ &= 6 + e^{2i\theta} + e^{-2i\theta} - 4e^{i\theta} - 4e^{-i\theta} \\ &= (e^{i\theta/2} - e^{-i\theta/2})^4.\end{aligned} \tag{4.2.23}$$

Note also that $\hat{A}(\mathrm{pt}) = 1$.

Since $\ker D^+ = 0$, assembling (4.2.21), (4.2.22), and (4.2.23) in (4.2.18) gives

$$\begin{aligned}\chi_{\ker D^-}(\theta) &= -\operatorname{ind}_\theta(D^+)\\ &= \frac{((e^{ik_1\theta} + \cdots + e^{-ik_n\theta}) - (e^{i\ell_1\theta} + \cdots + e^{i\ell_n\theta}))(e^{i\theta} - 2 + e^{-i\theta})}{(e^{i\theta/2} - e^{-i\theta/2})^4}.\end{aligned}$$

Canceling by the common factor $(e^{i\theta/2} - e^{-i\theta/2})^2$ gives the result. □

4.3 Structure group $SU(2)$

If the structure group of E is $SU(2)$, then E_0 and E_∞ are each 2-dimensional, and so the S^1 actions each have two weights. Since these weights must sum to 0, they are $k, -k$ at 0 and $\ell, -\ell$ at ∞. In this case, the character of the representation $V = \ker D^-$ from (4.2.20) simplifies, giving the following proposition.

Proposition 4.3.1. *Let ∇ be an S^1-invariant instanton with weights k on E_0 and ℓ on E_∞. Then the character of the S^1-representation on V is*

$$\chi_V = \sum_{m=-(k-1)}^{(k-1)} (k - |m|)e^{im\theta} - \sum_{m=-(\ell-1)}^{(\ell-1)} (\ell - |m|)e^{im\theta}. \tag{4.3.2}$$

Proof. From (4.2.20), the character of the representation is

$$\chi_V = \frac{(e^{ik\theta} + e^{-ik\theta}) - (e^{i\ell\theta} + e^{-i\ell\theta})}{(e^{i\theta/2} - e^{-i\theta/2})^2}. \tag{4.3.3}$$

Subtracting 2 from $(e^{ik\theta} + e^{-ik\theta})$ and adding 2 to $-(e^{i\ell\theta} + e^{-i\ell\theta})$, these terms are then squares. The above then evaluates to

$$\begin{aligned}\chi_V &= \left(\frac{e^{i\frac{k}{2}\theta} - e^{-\frac{k}{2}\theta}}{e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}}\right)^2 - \left(\frac{e^{i\frac{\ell}{2}\theta} - e^{-\frac{\ell}{2}\theta}}{e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}}\right)^2\\ &= (e^{i\frac{k-1}{2}\theta} + e^{i\frac{k-2}{2}\theta} + \cdots + e^{-i\frac{k-1}{2}\theta})^2\\ &\qquad - (e^{i\frac{\ell-1}{2}\theta} + e^{i\frac{\ell-2}{2}\theta} + \cdots + e^{-i\frac{\ell-1}{2}\theta})^2\end{aligned} \tag{4.3.4}$$

The proposition is then given by seeing that

$$
\begin{aligned}
&(e^{i\frac{k-1}{2}\theta} + e^{i\frac{k-2}{2}\theta} + \cdots + e^{-i\frac{k-1}{2}\theta})^2 \\
&\quad = e^{i(k-1)\theta} + 2e^{i(k-2)\theta} + \cdots + (k-1)e^{i\theta} + k \\
&\qquad + (k-1)e^{-i\theta} + \cdots + 2e^{-i(k-2)\theta} + e^{-i(k-1)\theta},
\end{aligned}
$$

and recognizing that (4.3.2) is simply a more succinct expression for these coefficients.

□

The decomposition $V = \bigoplus V_m$ into subrepresentations V_m of weight m can be obtained from the character, with $\dim(V_m)$ given by the coefficient of $e^{im\theta}$ in χ_V. We then have the following results about such instantons and their weights.

Corollary 4.3.5. *If E admits an S^1-invariant instanton, then $k \geq \ell$.*

Proof. From (4.3.2), the dimension of V_0, which must be nonnegative, is $k - \ell$. □

We can write the dimensions a little more simply as

$$
\dim(V_m) = \begin{cases} k - \ell & \text{if } |m| \leq \ell, \\ k - |m| & \text{if } \ell < |m| \leq k-1, \\ 0 & \text{if } |m| \geq k. \end{cases} \tag{4.3.6}
$$

Corollary 4.3.7. *An S^1-invariant instanton on E has charge $k^2 - \ell^2$.*

Proof. By Corollary 3.4.64, the charge of an instanton is equal to $\dim(V)$, which can be found by summing $\dim(V_m)$. □

Corollary 4.3.8. *There are no S^1-invariant $SU(2)$-instantons with charge $2 \pmod 4$.*

Proof. By Corollary 4.3, the charge is a difference of squares, and so cannot be 2 (mod 4). □

Before continuing on to computing examples with various different weights, which will comprise the rest of this section, we prove a lemma that is helpful in verifying the non-degeneracy of the ADHM data.

Lemma 4.3.9. *The non-degeneracy condition for S^1-invariant ADHM data to be valid (see Definition 3.4.7) need only be verified at $z = 0$. Moreover, the non-degeneracy condition holds if* $\ker \alpha \cap \ker \beta \cap \ker \pi = 0$.

Proof. Working with complex data, the non-degeneracy condition is that

$$R_z = \begin{bmatrix} \alpha_z^* & \beta_z^* & \pi^* \\ -\beta_z & \alpha_z & \sigma \end{bmatrix}$$

is full rank for all z. Equivalently, $R_z R_z^*$ is invertible for all z. Evaluating and making use of the ADHM equations,

$$\begin{aligned} R_z R_z^* &= \begin{bmatrix} \alpha_z^* \alpha_z + \beta_z^* \beta_z + \pi^* \pi & -\alpha_z^* \beta_z^* + \beta_z^* \alpha_z^* + \pi^* \sigma^* \\ -\beta_z \alpha_z + \alpha_z \beta_z + \sigma \pi & \beta_z \beta_z^* + \alpha_z \alpha_z^* + \sigma^* \sigma \end{bmatrix} \\ &= \begin{bmatrix} \alpha_z^* \alpha_z + \beta_z^* \beta_z + \pi^* \pi & 0 \\ 0 & \beta_z \beta_z^* + \alpha_z \alpha_z^* + \sigma^* \sigma \end{bmatrix} \end{aligned} \tag{4.3.10}$$

The ADHM equations then also tell us that the two remaining submatrices are equal to each other, and so it suffices to check the non-degeneracy of $\alpha_z^* \alpha_z + \beta_z^* \beta_z + \pi^* \pi$, which as a sum of positive semi-definite matrices is itself positive semi-definite. Moreover, a vector is in the kernel of the sum if and only if it is in the kernel of each of $\alpha_z^* \alpha_z$, $\beta_z^* \beta_z$, and $\pi^* \pi$ individually. This is equivalent to the vector being in the kernel of α_z, β_z, and $\ker \pi$ simultaneously.

Since α_z and β_z are triangular, it is clear that α_z is injective if $z_1 \neq 0$ and β_z is injective if $z_2 \neq 0$. Therefore non-degeneracy holds everywhere away from 0, and to check the condition at 0 it suffices to check that $\ker \alpha$, $\ker \beta$, and $\ker \pi$ share no nonzero vectors. □

4.3.1 $k = \ell$

We begin with simplest example, where the S^1 action has equal weights at 0 and ∞. By Corollary 4.3, the charge is then 0, and so the instanton is trivial, as is $E = M \times \mathbb{C}^2$. Since we have a global identification of fibres, we can take the S^1-action to be globally

$$r_\theta = \begin{bmatrix} e^{ik\theta} & \\ & e^{-ik\theta} \end{bmatrix}.$$

Curiously, this does not mean that the monopole is trivial. Although the connection form with respect to the obvious trivialization on $E = M \times \mathbb{C}^2$ vanishes, we note that this trivialization is not S^1-invariant. To obtain an S^1-invariant trivialization, consider the gauge transformation

$$g(z) = \frac{1}{|z_1|^k} \begin{bmatrix} \bar{z}_1^k & \\ & z_1^k \end{bmatrix}$$

away from $z_1 = 0$. Away from $z_2 = 0$ we can use a similar gauge transformation involving z_2. We note that $g(e^{i\theta} \cdot z) r_\theta g(z) = 1$ and so this trivialization is indeed S^1-invariant. We can then compute the Higgs field,

$$\begin{aligned} \pi^*\Phi &= \frac{1}{2|z|^2} A_{\partial_\theta} \\ &= \frac{1}{2|z|^2} g \partial_\theta g^{-1} \\ &= \frac{1}{2|z|^2} \begin{bmatrix} ik & \\ & -ik \end{bmatrix}. \end{aligned}$$

In the coordinates y on $\mathbb{R}^3$ we then have

$$\Phi = \frac{1}{2|y|} \begin{bmatrix} ik & \\ & -ik \end{bmatrix}$$

This example demonstrates that the correspondence between monopoles and S^1-invariant instantons implicitly takes into account the S^1 action under consideration.

Since the trivial instanton on the trivial $SU(2)$-bundle is S^1-invariant with respect to many different S^1-actions, it corresponds to many different monopoles.

4.3.2 $k = 1$, $\ell = 0$

The simplest case of a nontrivial instanton is with weights $k = 1$ and $\ell = 0$, which must therefore have charge 1. In this case $V = V_0$, and so $\alpha = \beta = 0$. Up to gauge, the ADHM data can be parametrized by $\lambda \in \mathbb{R}_{>0}$, giving

$$\pi = \begin{bmatrix} \lambda \\ 0 \end{bmatrix},$$
$$\sigma = \begin{bmatrix} 0 & \lambda \end{bmatrix}.$$

These data correspond to the charge 1 instantons centred at the origin in $\mathbb{R}^4$. That these are all the charge S^1-invariant instantons of charge 1 can be directly verified: they are all S^1-invariant as shown below, while no other instanton of charge 1 can be S^1 invariant since the charge density $|F_\nabla|^2$ must be centred at a fixed point of the S^1-action.

These instantons can be seen explicitly, following the description in [Ati79, Chapter II], using quaternionic coordinates and identifying the imaginary quaternions with $\mathfrak{su}(2)$. The connection form is then

$$A = \operatorname{Im}\left(\frac{\bar{x}dx}{\lambda^2 + |x|^2}\right).$$

Note that pulling back by the S^1-action does not result in the same form. This is a result of the frame itself not being S^1-invariant, and so we apply the gauge transformation $g = \frac{x}{|x|}$ to make it so. The new connection form is then

$$\begin{aligned}\tilde{A} &= gAg^{-1} + gdg^{-1} \\ &= \operatorname{Im}\left(\frac{\lambda^2 dx\,\bar{x}}{2|x|^2(\lambda^2 + |x|^2)}\right).\end{aligned}$$

Pulling back by the S^1 action shows that this connection form is S^1-invariant, at the cost of being singular at the origin.

We can then compute the Higgs field

$$\begin{aligned}\pi^*\Phi &= \frac{1}{2|x|^2}\tilde{A}_{\partial_\theta} \\ &= \frac{\lambda^2(xi\bar{x})}{2|x|^4(\lambda^2+|x|^2)}.\end{aligned}$$

Noting that $(xi\bar{x}) = \pi(x)$ under the identification of $\mathbb{R}^3$ with the imaginary quaternions, we thus have that

$$\Phi = \frac{\lambda^2 y}{2|y|^2(\lambda^2+|y|)},$$

where here $y = y_1 i + y_2 j + y_3 k$. Diagonalizing with the gauge transformation $h = \frac{|y|i+y}{\sqrt{2|y|(|y|+y_1)}}$ gives

$$\tilde{\Phi} = h\Phi h^{-1} = \frac{i\lambda^2}{2|y|(\lambda^2+|y|)}.$$

4.3.3 $k = 2$, $\ell = 1$

Because of Corollary 4.3.8, the next simplest case arises for charge 3 instantons, which occurs when $k = 2$ and $\ell = 1$. In this case $V = V_1 \oplus V_0 \oplus V_{-1}$. Each of these subrepresentations is one-dimensional, and so we can think of α_m, β_m as simply complex numbers, and π_m and σ_m^* as elements of $E_\infty = \mathbb{C}^2$.

Proposition 4.3.11. *S^1-invariant instantons with weights $k = 2, \ell = 1$ are parametrized by $\eta_1, \eta_2 \in E_\infty$ satisfying $\eta_1 \neq \pm\eta_2$ and $\langle \eta_1, \eta_2\rangle \in \mathbb{R}$. Acting by $SU(2)$ on η_1, η_2 gives gauge equivalent solutions.*

The ADHM for these parameters are given by

$$\pi = \begin{bmatrix} \eta_1 & 0 & \eta_2 \end{bmatrix}, \qquad \sigma = \begin{bmatrix} \eta_2^\perp \\ 0 \\ \eta_1^\perp \end{bmatrix},$$

where we take $\bullet^\perp$ *to denote* $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \mapsto \begin{bmatrix} \bar{w}_2 & -\bar{w}_1 \end{bmatrix}$. *The* α *and* β *are given by*

$$\alpha_0 = \alpha_{-1} = \frac{1}{2}\sqrt{|\eta_1|^2 - |\eta_2|^2 + \sqrt{(|\eta_1|^2 - |\eta_2|^2)^2 + 4|\eta_2^\perp \eta_1|^2}}.$$

If $\alpha_0 = 0$, *which happens only if* $\eta_2^\perp \eta_1 = 0$ *and* $|\eta_2|^2 > |\eta_1|^2$, *we take* $\beta_1 = \beta_0 = \sqrt{|\eta_2|^2 - |\eta_1|^2}$, *otherwise* $\beta_1 = \beta_0 = \frac{\eta_2^\perp \eta_1}{\alpha_0}$.

Proof. For such an S^1-invariant instanton, the ADHM matrix R can be written

$$R(z) = \begin{bmatrix} -\bar{z}_1 & & & -\bar{z}_2 & \beta_1^* & & \pi_1^* \\ \alpha_0^* & -\bar{z}_1 & & & -\bar{z}_2 & \beta_0^* & \pi_0^* \\ & \alpha_{-1}^* & -\bar{z}_1 & & & -\bar{z}_2 & \pi_{-1}^* \\ z_2 & & & -z_1 & \alpha_0 & & \sigma_1 \\ \beta_1 & z_2 & & & -z_1 & \alpha_{-1} & \sigma_0 \\ & \beta_0 & z_2 & & & -z_1 & \sigma_{-1} \end{bmatrix} \tag{4.3.12}$$

By Lemma 4.3.9, non-degeneracy need only be verified for $R(0)$.

The proof then proceeds by successive claims.

Claim 1. At least one of π_1, σ_1^* is non-zero, and at least one of π_{-1}, σ_{-1}^* is non-zero.

To verify this claim, note that if both $\pi_1 = \sigma_1^* = 0$, then by (4.2.9),

$$|\alpha_0|^2 = |\beta_1|^2.$$

On the other hand, by (4.2.8),

$$\alpha_0 \beta_1 = \sigma_1 \pi_1 = 0,$$

and since $\alpha_0, \beta_1 \in \mathbb{C}^{1\times 1}$, these imply that $\alpha_0 = \beta_1 = 0$. But then looking at the first row of $R(0)$, we see that the nondegeneracy condition is not satisfied.

A similar argument shows that at least one of π_{-1}, σ_{-1} is non-zero.

Claim 2. At least one of π_1, π_{-1} is non-zero. Similarly, at least one of σ_1, σ_{-1} is non-zero.

Suppose to the contrary that $\pi_1 = \pi_{-1} = 0$. Then by Claim 1, both σ_1 and σ_{-1} are non-zero. By (4.2.11), we have

$$\sigma_1 \sigma_{-1}^* = \pi_1^* \pi_{-1} = 0,$$

and therefore σ_1 and σ_{-1} are orthogonal in $\mathbb{C}^2$. By (4.2.10),

$$\sigma_1 \pi_0 = 0, \qquad \sigma_{-1} \pi_0 = 0.$$

Therefore, π_0 is a third orthogonal element of $\mathbb{C}^2$, and so must be 0.

Furthermore, from (4.2.8), we have

$$\alpha_0 \beta_1 = 0, \qquad \alpha_{-1} \beta_0 = 0.$$

Therefore the first three rows of $R(0)$ contain at most two nonzero entries, and so the nondegeneracy condition is not satisfied. Therefore the claim is verified.

Claim 3.

$$\pi_0 = \sigma_0^* = 0.$$

We will show $\pi_0 = 0$. The argument for σ_0 is similar.

Suppose first that $\sigma_1 = 0$. Then by Claims 1 and 2, both π_1 and σ_{-1} are nonzero. But then by 4.2.10 and 4.2.11,

$$\pi_1^* \pi_0 = \sigma_1 \sigma_0^* = 0, \qquad \sigma_{-1} \pi_0 = 0.$$

Thus π_0 is orthogonal to π_1 and σ_{-1}, which are themselves a pair of nonzero orthogonal elements of $\mathbb{C}^2$, and so $\pi_0 = 0$.

A similar argument holds in the case $\sigma_{-1} = 0$. Suppose then that σ_1 and σ_{-1} are both nonzero, and assume to the contrary that π_0 is also nonzero. Since they are both orthogonal to π_0^*, they must be colinear. Since π_1^* is orthogonal to σ_{-1} and π_{-1}^* is orthogonal to σ_1, then π_1 and π_{-1} are als colinear. But then by 4.2.11,

$$\pi_1^* \pi_{-1} = \sigma_1 \sigma_{-1}^*,$$

and since the righthand side is nonzero, neither π_1 nor π_{-1} is zero. The equations (4.2.10) for all m together then show that $|\pi_i| = |\sigma_i|$.

Note however that (4.2.8) and (4.2.9), and the orthogonality of σ_1 and π_1, give that

$$\alpha_0 \beta_1 = 0, \qquad |\alpha_0|^2 - |\beta_0|^2 = 0,$$

which imply that $\alpha_0 = \beta_1 = 0$, and so the first column of $R(0)$ is 0, contradicting nondegeneracy.

Thus, in any case, $\pi_0 = 0$, verifying the claim..

Let us fix bases in V and E_∞ and write

$$\pi_1 = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}, \qquad \pi_{-1} = \begin{bmatrix} \omega_3 \\ \omega_4 \end{bmatrix}.$$

By orthogonality, we then have

$$\sigma_1 = \lambda_1 \begin{bmatrix} \bar{\omega}_4 & -\bar{\omega}_3 \end{bmatrix}, \qquad \sigma_{-1} = \lambda_{-1} \begin{bmatrix} \bar{\omega}_2 & -\bar{\omega}_1 \end{bmatrix}$$

for some $\lambda_1, \lambda_{-1} \neq 0$.

Claim 4. $|\lambda_1| = |\lambda_{-1}| = 1$.

To verify the claim, first consider the case where $\sigma_1 \pi_1 \neq 0$. Then since π_0 and σ_0 are both 0, summing up (4.2.8) over all m gives

$$\sigma_1 \pi_1 = -\sigma_{-1} \pi_{-1}. \tag{4.3.13}$$

Expanding this in coordinates shows that $|\lambda_1| = |\lambda_{-1}|$. Then summing up (4.2.9) gives

$$|\pi_1|^2 + |\pi_{-1}|^2 = |\sigma_1|^2 + |\sigma_{-1}|^2, \tag{4.3.14}$$

which shows that $|\lambda_1| = 1$.

If, on the other hand, σ_1 or π_1 is 0, then the first statement can be trivially satisfied, and the second statement is again obtained from (4.3.14).

Consider now the case where $\sigma_1\pi_1 = 0$ but neither σ_1 nor π_1 vanish. Then by orthogonality, we must have that π_1 and π_{-1} are colinear, and taking the absolute value of (4.2.10) gives

$$|\pi_1||\pi_{-1}| = |\sigma_1||\sigma_{-1}|. \tag{4.3.15}$$

When added to or subtracted from (4.3.14), which was valid independent of the assumptions made prior to it, gives

$$(|\pi_1| + |\pi_{-1}|)^2 = (|\sigma_1| + |\sigma_{-1}|)^2, \tag{4.3.16}$$

$$(|\pi_1| - |\pi_{-1}|)^2 = (|\sigma_1| - |\sigma_{-1}|)^2. \tag{4.3.17}$$

Since $|\pi_1| \neq |\sigma_1|$, otherwise $\alpha_0\beta_1 = 0$ would imply $\alpha_0 = \beta_1 = 0$ violating non-degeneracy, it must be the case that $|\pi_1| = |\sigma_{-1}|$ and $|\pi_{-1}| = |\sigma_1|$, hence $|\lambda_1| = |\lambda_{-1}| = 1$.

Note that multiplying the basis vector for V_{-1} by $e^{i\theta}$ changes λ_{-1} by $e^{-i\theta}$ while changes λ_1 by $e^{i\theta}$. So by fixing a basis vector for V_{-1} we can assume that $\lambda_1 = \lambda_{-1} = 1$. Expanding shows that $\sigma_1\pi_1 = -\overline{\sigma_{-1}\pi_{-1}}$, and so comparing with (4.3.13), which is valid regardless of the assumptions prior to it, we see that $\sigma_1\pi_1$ and $\sigma_{-1}\pi_{-1}$ are real. Expanding (4.2.11) shows that $\pi_1^*\pi_{-1} = \sigma_1^*\sigma_{-1}$ are real.

Claim 5. $|\alpha_i|$ and $|\beta_i|$ are determined by π_1, π_{-1}.

From above, σ_1 is determined by π_{-1}, and if $\pi_1\sigma_1 = 0$, then one of α_0, β_1 is zero, and the absolute value of the other is determined by $|\pi_1|^2 - |\sigma_1|^2$ using (4.2.9). If, on the other hand, $\pi_1\sigma_1 \neq 0$, then neither α_0 nor β_1 vanish. We then have by (4.2.8) and (4.2.9) that $|\alpha_0|^2$ satisfies the quadratic

$$|\alpha_0|^4 + (|\sigma_1|^2 - |\pi_1|^2)|\alpha_0|^2 - |\sigma_1\pi_1|^2 = 0. \tag{4.3.18}$$

Since the discriminant is positive, roots exist, and moreover the inequality $|\alpha_0|^2 \geq |\pi_1|^2 - |\sigma_1|^2$ obtained from (4.2.9) determines which root is valid. Therefore $|\alpha_0|$ is determined, and hence $|\beta_1|$ is determined. By a similar argument $|\alpha_{-1}|$ and $|\beta_0|$ are also determined, finishing the claim.

From the equality $\alpha_0\beta_1 = \sigma_1\pi_1$, we in fact have that α_0 and β_1 are determined up to multiplication by $e^{i\theta}$ for α_0 and multiplication by $e^{-i\theta}$ for β_1, and similarly for α_{-1} and β_0. Rotating the basis elements for V_1 and V_0 shows that these give equivalent data, and so α_i and β_i are in fact determined up to equivalence.

Setting $\pi_1 = \eta_1$ and $\pi_{-1} = \eta_2$ gives the desired parametrization. All other ADHM data are then determined by the arguments above, and it can be checked that these data satisfy the ADHM equations. It thus remains only to show that these data satisfy the nondegeneracy condition.

Note that if $\eta_1 = \pm\eta_2$, then $\pi_1\sigma_1 = 0$ and $|\pi_1|^2 - |\sigma_1|^2 = 0$, so $\alpha_0 = \beta_1 = 0$, which does not satisfy non-degeneracy. If $\eta_1 \neq \pm\eta_2$, then the requirement that $\langle\eta_1, \eta_2\rangle \in \mathbb{R}$ then implies that either $\pi_j\sigma_j \neq 0$ or $|\pi_j|^2 - |\sigma_j|^2 \neq 0$, for each of the indices $j = 1, -1$.

If $\pi_1\sigma_1 \neq 0$ (and hence also $\pi_{-1}\sigma_{-1} \neq 0$), then all of $\alpha_0, \alpha_{-1}, \beta_1, \beta_0$ are nonzero. The kernel of α and β then clearly intersect only at 0, and so the nondegeneracy condition is satisfied.

If, on the other hand, $\pi_1\sigma_1 = 0$, in which case $|\pi_1|^2 - |\sigma_1|^2 \neq 0$. Assume that $|\pi_1|^2 - |\sigma_1|^2 > 0$. Then (4.2.9) implies that $|\alpha_0|^2 > 0$, and so in particular $\alpha_0 \neq 0$. We also have that $|\pi_{-1}|^2 - |\sigma_{-1}|^2 = |\sigma_1|^2 - |\pi_1|^2 < 0$, and (4.2.9) then implies that $\alpha_{-1} \neq 0$. Thus $\ker\alpha = V_1$. On the other hand, $|\pi_1|^2 - |\sigma_1|^2 > 0$ implies $\pi_1 \neq 0$, and so $V_1 \cap \ker\pi_1 = 0$. A similar argument applies in the case $|\pi_1|^2 - |\sigma_1|^2 < 0$, showing instead that $\beta_i \neq 0$. In either case, the nondegeneracy condition is satisfied. □

4.3.4 $k = 2,\ \ell = 0$

When the weights are $k = 2$ and $\ell = 0$, we have a charge 4 instanton, and the decomposition $V = V_1 \oplus V_0 \oplus V_{-1}$, with $\dim V_1 = \dim V_{-1} = 1$ and $\dim V_0 = 2$. This gives our first example of a space with a subrepresentation V_m of dimension greater than 1. Below we give the ADHM data for a one-parameter family of instantons with such weights. In particular, the moduli space of such instantons is nonempty.

The ADHM data are given, for $\lambda \in \mathbb{R}_{>0}$ by

$$\begin{gathered} \alpha_0 = \begin{bmatrix} \lambda & 0 \end{bmatrix}, \qquad \alpha_{-1} = \begin{bmatrix} \lambda \\ 0 \end{bmatrix} \\ \beta_0 = \begin{bmatrix} 0 & \lambda \end{bmatrix}, \qquad \beta_1 = \begin{bmatrix} 0 \\ \lambda \end{bmatrix} \\ \sigma_0 = \begin{bmatrix} 0 & \lambda \\ \lambda & 0 \end{bmatrix}, \qquad \pi_0 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \\ \pi_1 = \sigma_1 = \pi_{-1} = \sigma_{-1} = 0. \end{gathered} \tag{4.3.19}$$

One can check then directly that the ADHM equations (3.4.13) and (3.4.14) are satisfied. Moreover, since π_0 is injective on V_0, α_{-1} is injective on V_{-1}, and β_1 is injective on V_1, by Lemma 4.3.9 these give nondegenerate data.

4.4 Structure group $SU(n)$

Consider now the general case of structure group $SU(n)$.

Proposition 4.4.1. *The character of the S^1 representation on $V = \ker D^-$ is*

$$\chi_{\ker D^-}(\theta) = \sum_m \left(\sum_{k_i < m} (m - k_i) - \sum_{\ell_i < m} (m - \ell_i) \right) e^{im\theta}. \tag{4.4.2}$$

Before proving this proposition, we first note that there are finitely many nonzero terms in this sum. Indeed, if m is smaller than all weights k_i and ℓ_i, then the inner sums are empty. On the other hand, if m is larger than all weights, then the inner sums are over all possible weights. The inner sums then evaluate to

$$\sum_{k_i}(m-k_i) - \sum_{\ell_i}(m-\ell_i) = nm - \sum k_i - nm + \sum \ell_i = 0.$$

Therefore, the only nonzero coefficients occur between the smallest and largest of the weights.

Proof. The result follows from (4.2.20) once we have the following equation for formal Laurent series:

$$(x-2+x^{-1})\sum_m \sum_{k_i<m}(m-k_i)x^m = \sum_{k_i} x^{k_i}. \tag{4.4.3}$$

The coefficient of x^m on the lefthand side of (4.4.3) is

$$\sum_{k_i<m-1}(m-1-k_i) - 2\sum_{k_i<m}(m-k_i) + \sum_{k_i<m+1}(m+1-k_i). \tag{4.4.4}$$

When $k_i < m-1$ above, the three terms cancel. For $k_i = m-1$, the the first term vanishes, leaving

$$\sum_{k_i=m-1}(-2(m-k_i)+(m+1-k_i)) = 0,$$

while for $k_i = m$ only the last term remains leaving

$$\sum_{k_i=m}(m+1-k_i) = \sum_{k_i=m} 1.$$

This establishes (4.4.3), and finishes the proof. □

Recall that we order the weights $k_1 \geq \cdots \geq k_n$ and $\ell_1 \geq \cdots \geq \ell_n$.

Corollary 4.4.5. *If E admits an S^1-invariant instanton, then $k_1 \geq \ell_1$ and $k_n \leq \ell_n$.*

Proof. If $k_n > \ell_n$, then $k_i \geq \ell_n + 1$ for all i. Then the dimension of V_{ℓ_n+1} is

$$\sum_{k_i<\ell_n+1}((\ell_n+1)-k_i) - \sum_{\ell_i<\ell_n+1}((\ell_n+1)-\ell_i) = -\sum_{\ell_i=\ell_n} 1 < 0.$$

Since dimensions must be nonnegative, this shows $k_n \leq \ell_n$.

If $k_1 < \ell_1$, then $k_i \leq \ell_1 - 1$ for all i. Then the dimension of $V_{\ell_1 - 1}$ is

$$\sum_{k_i < \ell_1 - 1} ((\ell_1 - 1) - k_i) - \sum_{\ell_i < \ell_1 - 1} ((\ell_1 - 1) - \ell_i)$$

Note that if we add $-\sum_{\ell_i = \ell_1} ((\ell_1 - 1) - \ell_i)$ to the above, we are now summing over all of weights k_i and ℓ_i, which evaluates to 0. Therefore, the dimension of $V_{\ell_1 - 1}$ is

$$\sum_{\ell_i = \ell_1} (-1) < 0,$$

contradicting nonnegativity of the dimension. □

Corollary 4.4.6. *An S^1-invariant instanton on E has charge*

$$\sum_{k_i} k_i^2 - \sum_{\ell_i} \ell_i^2. \tag{4.4.7}$$

Proof. The charge of the instanton is the dimension of V, which can be computed by summing the dimensions of V_m. Alternatively, one can evaluate $\chi_V(0)$ using (4.2.20) and L'Hopital's Rule. □

Corollary 4.4.8. *If E admits an S^1-invariant instanton then*

$$\sum_{k_i} k_i^2 \geq \sum_{\ell_i} \ell_i^2.$$

Chapter 5

The Nahm Transform on a Bieberbach Manifold

In Section 3.3 we considered the Nahm transform on quotients of $\mathbb{R}^4$ by a subgroup of translations. In this chapter, we consider quotients more generally by subgroups of the isometries on $\mathbb{R}^4$ that include a rotational part. In Section 5.1 we define crystallographic groups and Bieberbach groups. In Sections 5.2 and 5.3 we give an example of a Bieberbach group B and construct a Nahm transform for instantons on the quotient $\mathbb{R}^4/B$.

5.1 Crystallographic and Bieberbach groups

We begin with a brief description of Bieberbach groups in general, following [Cha86, Chapter I]. Let $\mathcal{M}$ be the group of rigid motions on $\mathbb{R}^n$. These consist of rotation by an element of $r \in O(n)$ followed by translation by some $v \in \mathbb{R}^n$. Taking the rotational part of a given rigid motion gives a homomorphism $r\colon \mathcal{M} \to O(n)$.

We consider subgroups of $G \subset \mathcal{M}$. We say G is *uniform* if the quotient $\mathbb{R}^n/G$ is compact. If G is discrete, then the orbits of G consist of discrete points. If G is both discrete and uniform, we say G is *crystallographic*. If, in addition, G is torsionfree, we say it is a *Bieberbach* subgroup of $\mathcal{M}$. For discrete subgroups G, being torsion free is equivalent to G acting freely on $\mathbb{R}^n$. Therefore, if G is Bieberbach, then $\mathbb{R}^n/G$ is a compact flat manifold with fundamental group G.

For a subgroup G, the rotational parts $r(G)$ form a subgroup of $O(n)$, while the kernel of r is the normal subgroup $\Lambda \subset G$ consisting of pure translations. Note that if $r(G)$ and Λ are both discrete – in which case $r(G)$ is in fact finite, since $O(n)$ is compact – then G is discrete. If, moreover, Λ is a (full-rank) lattice, then $\mathbb{R}^n/G$ is a quotient of T^n and hence compact. These two properties together show that G is crystallographic. Bieberbach's First Theorem (see, e.g. [Cha86, p. 17]) gives the converse.

Theorem 5.1.1 (Bieberbach's First Theorem)**.** *Let G be a crystallographic subgroup of $\mathcal{M}$. Then $r(G)$ is finite and Λ spans $\mathbb{R}^n$.*

As a result of Bieberbach's First Theorem and the fact that $\Lambda \subset G$ is normal, if G is crystallographic then $\mathbb{R}^n/G \cong T^n/r(G)$.

5.2 A Bieberbach quotient of T^4

For the rest of the chapter, we focus on a specific example of a Bieberbach group. Consider the coordinates x_i on $\mathbb{R}^4$ with orthonormal basis $e_i = \frac{\partial}{\partial_i}$. Let $\mu \in O(n)$ be given by $\mu(x_1, x_2, x_3, x_4) = (x_2, x_1, x_4, x_3)$. For $i = 1, \ldots, 4$, let $g_i = (\mu, e_i)$ denote the isometry given $x \to \mu x + e_i$. Let B denote the group generated by these g_i.

Elements of B are then of the form (μ^k, v) for $k = 0, 1$ and $v \in \mathbb{Z}^4$. The rotational part is $r(B) = \{1, \mu\} = \mathbb{Z}_2 \subset O(n)$. Let $\sigma(x) = \sum_i x_i$, and note that applying σ to the translational part of a group element gives a homomorphism $\sigma\colon B \to \mathbb{Z}$. Since for each of the generators, $\sigma(g_i) = 1$, we see that the parity of $\sigma(\mu^k, v)$ matches the parity of k. Thus, the purely translational subgroup is $\Lambda = \{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid \sigma(x) \in 2\mathbb{Z}\}$. Since these are discrete and Λ spans $\mathbb{R}^4$, B is a crystallographic subgroup of $\mathcal{M}$.

Moreover, we can see that B is torsionfree (and hence Bieberbach) as follows. Suppose (μ^k, v) has finite order. If $k = 0$, then $v \in \Lambda$, and $(1, v)^m = (1, mv)$, which is zero only if $v = 0$. On the other hand, if $k = 1$ and $(\mu, v)^m = (1, 0)$, then m is even and $(\mu, v)^m = (1, (m/2)(v + \mu v))$. Therefore $v = -\mu v$, and hence $\sigma(v) = 0$. But the parity of σ must match the parity of k providing a contradiction.

Therefore $M = \mathbb{R}^4/B$ is a compact flat manifold. Moreover, recalling our homomorphism $r : B \to \mathbb{Z}_2$, since $\Lambda = r^{-1}(1)$ it is a normal subgroup of B. Therefore, M is also a quotient of $T^4 = \mathbb{R}^4/\Lambda$ by $B/\Lambda = \mathbb{Z}_2$. Let $q\colon T^4 \to M$ denote the quotient. Furthermore M is oriented since $\mu \in SO(4)$. In fact, M is parallelizable, as can be seen by considering the following vector fields

$$
\begin{gathered}
(\partial_1 + \partial_2),\\
(\partial_3 + \partial_4),\\
\cos(\pi\sigma(x))(\partial_1 - \partial_2) + \sin(\pi\sigma(x))(\partial_3 - \partial_4),\\
-\sin(\pi\sigma(x))(\partial_1 - \partial_2) + \cos(\pi\sigma(x))(\partial_3 - \partial_4).
\end{gathered}
$$

Note that these vector fields are invariant under the action of B, and thus descend to M. Moreover they are orthogonal.

5.3 Nahm transform on a Bieberbach manifold

To construct a Nahm Transform, we consider a space of flat bundles on M. As in Section 3.2, we parametrize such bundles by elements of $(\mathbb{R}^4)^*$. However, only elements of $(\mathbb{R}^4)^*$ that are invariant under the action of B give well-defined 1-forms on M. Ultimately, we will consider rank 2 twisting bundles parametrized by all of $(\mathbb{R}^4)^*$, but for now we consider line bundles parametrized by the elements of $(\mathbb{R}^4)^*$ that do descend to M. Letting $e^i \in (\mathbb{R}^4)^*$ be the dual basis to e_i, it is convenient to define a new basis for $(\mathbb{R}^4)^*$,

$$
\begin{aligned}
v^1 &= \frac{1}{2}(e^1 + e^2), \qquad & w^1 &= \frac{1}{2}(e^1 - e^2),\\
v^2 &= \frac{1}{2}(e^3 + e^4), \qquad & w^2 &= \frac{1}{2}(e^3 - e^4).
\end{aligned}
\tag{5.3.1}
$$

Note that v^1 and v^2 are invariant under the action of B, and so give well-defined constant 1-forms on the quotient M. On the other hand, w^1 and w^2 are not invariant as acting by μ introduces a sign change.

Let $\xi = \xi_i v^i \in (\mathbb{R}^4)^*$, which as a combination of v^i is well-defined as a form on M. Let ∇_ξ be the connection on the trivial line bundle on $\mathbb{R}^4$ with connection form $i2\pi\xi$. Since ∇_ξ descends to the quotient, it defines a flat connection on a topologically trivial line bundle L on M. When equipped with the connection ∇_ξ, we shall denote the line bundle L_ξ.

Recall that a flat line bundle on a manifold M is classified by the $U(1)$-representation of $\pi_1(M) = B$ determined by its holonomy. A representative of the loop in $\pi_1(M)$ corresponding to the generator $g_i \in B$ is given by the curve te_i, for $0 \leq t \leq 1$. From the description $\nabla_\xi = d + i2\pi$ with respect to the given frame, parallel transport along, say, te_1 is given by $e^{-it2\pi\xi(e_1)} = e^{-it\pi\xi_1}$. Similar computations show that the representation ρ of B corresponding to holonomy is given on generators by

$$\begin{aligned} \rho(g_1) = \rho(g_2) = e^{-i\pi\xi_1}, \\ \rho(g_3) = \rho(g_4) = e^{-i\pi\xi_2}. \end{aligned} \tag{5.3.2}$$

We see that the line bundle L_ξ is defined up to translation by $2\mathbb{Z}$ in either ξ_1 or ξ_2.

Consider now the pullback q^*L_ξ on T^4. Since the lattice Λ is generated by products $g_i g_j$, holonomy on q^*L_ξ is given by composition of the appropriate holonomies from (5.3.2). All such holonomies are either of the form $e^{i2\pi\xi_i}$ or $e^{i\pi(\xi_1+\xi_2)}$. As such, in addition to translation by $2\mathbb{Z}\times 2\mathbb{Z}$, we have that the pullbacks are also invariant up to translation by (k, k) for $k \in \mathbb{Z}$. That is, on T^4 the pullbacks satisfy $q^*L_\xi \cong q^*L_{\xi+v^1+v^2}$. As pullback bundles, however, we have a natural identification of the 2 fibres over a $\mathbb{Z}_2$-orbit in T^4, and as such a natural lift of the $\mathbb{Z}_2$ action to the pull-back bundle. It is this choice of $\mathbb{Z}_2$ action that distinguishes q^*L_ξ from $q^*L_{\xi+v^1+v^2}$ on T^4. Let $b \in B$, and let $\mathrm{sgn} : B \to \{1, -1\} = \mathbb{Z}_2 \cong B/\Lambda$ be the quotient homomorphism. That is

$\mathrm{sgn}(b) = 1$ if b acts trivially on T^4, and is -1 otherwise. Let $\alpha(b) : q^*L_\xi \to q^*L_\xi$ be the natural action on L_ξ induced by the pullback structure. Then under the identification of $q^*L_\xi = q^*L_{\xi+v^1+v^2}$ as line bundles on T^4, the action of B/Λ obtained from the pullback structure of $q^*L_{\xi+v^1+v^2}$ is $\mathrm{sgn}(b)\alpha(b)$.

Given these line bundles, we define a rank 2 vector bundle $V_\xi = L_\xi \oplus L_{\xi+v^1+v^2}$ over M. Since L_ξ and $L_{\xi+v^1+v^2}$ are topologically trivial for all ξ, all V_ξ are topologically trivial rank 2-bundles. Even though we could define a global smooth frame on V_ξ, it is instead convenient to use the following. Consider $q^*V_\xi = q^*L_\xi \oplus q^*L_{\xi+v^1+v^2}$ on T^4. Since $q^*L_\xi \cong q^*L_{\xi+v^1+v^2}$ on T^4, we can then choose a frame for q^*V_ξ such that the connection form of the pullback connection $q^*(\nabla_\xi \oplus \nabla_\xi)$ is $i2\pi\xi$ on each component. The natural $\mathbb{Z}_2$-action on the fibres of q^*V_ξ obtained from the pullback structure is then given by $1 \oplus \mathrm{sgn}(b)$. This frame then pushes down to M, albeit not smoothly. One advantage of this frame is the expression for the connection,

$$\nabla_\xi \oplus \nabla_{\xi+v^1+v^2} = d + i2\pi \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{5.3.3}$$

We define a further twisting of this bundle as follows. Let $\tau = i2\pi\tau_i w^i$. As stated above, τ is not invariant under B but instead changes by a sign when multiplied by generators. Viewed as a map with respect to the frame for q^*V_ξ described above, multiplication by the form τ then maps a section that is trivial under the $\mathbb{Z}_2$ action to one that changes sign under the action, and vice versa. Therefore, on M we can view τ as a $\mathrm{Hom}(L_\xi, L_{\xi+v^1+v^2}) \oplus \mathrm{Hom}(L_{\xi+v^1+v^2}, L_\xi)$-valued form. With respect to the direct sum definition of V_ξ, $\tau(\partial_i)$ defines an off-diagonal element of $\mathrm{End}(V_\xi)$. Adding $i2\pi\tau$ to the connection on V_ξ then gives the connection

$$\nabla_{\xi,\tau} = d + i2\pi \begin{bmatrix} \xi & \tau \\ \tau & \xi \end{bmatrix}. \tag{5.3.4}$$

Observe that $\nabla_{\xi,\tau}$ is a flat connection. Let $V_{\xi,\tau}$ denote the bundle V_ξ equipped with $\nabla_{\xi,\tau}$.

Computing the holonomy along the curve te_1, parallel transport is given by

$$\exp\left(-i\pi t\begin{bmatrix}\xi_1 & \tau_1\\ \tau_1 & \xi_1\end{bmatrix}\right) = e^{-i\pi t\xi_1}\begin{bmatrix}\cos(\pi t\tau_1) & -i\sin(\pi t\tau_1)\\ -i\sin(\pi t\tau_1) & \cos(\pi t\tau_1)\end{bmatrix}. \tag{5.3.5}$$

Note that upon reaching $t = 1$, the fibre $(V_{\xi,\tau})_{e_1}$ is identified with the fibre at 0 in our given frame by multiplication by -1 in the second direct summand. Therefore, holonomy along the curve te_1 is given by

$$\rho_1(\xi,\tau) = e^{-i\pi\xi_1}\begin{bmatrix}\cos(\pi\tau_1) & -i\sin(\pi\tau_1)\\ i\sin(\pi\tau_1) & -\cos(\pi\tau_1)\end{bmatrix}. \tag{5.3.6}$$

Similarly, we can compute holonomy along the curve te_2, giving

$$\rho_2(\xi,\tau) = e^{-i\pi\xi_1}\begin{bmatrix}\cos(\pi\tau_1) & i\sin(\pi\tau_1)\\ -i\sin(\pi\tau_1) & -\cos(\pi\tau_1)\end{bmatrix}, \tag{5.3.7}$$

and similar expressions can be computed for holonomy in the directions e_3 and e_4.

From the holonomy computation, we can see that the bundles are invariant under the shift of parameters $\xi_i, \tau_i \mapsto \xi_i + 1, \tau_i + 1$. Together with the translations in each variable by 2, this gives that the bundle $V_{\xi,\tau}$ is defined up to translations $\hat{\Lambda}$ generated by

$$\hat{\Lambda} = \mathrm{span}_{\mathbb{Z}}\{v^1 + w^1, v^1 - w^1, v^2 + w^2, v^2 - w^2\}. \tag{5.3.8}$$

We have an additional action on the parameters ξ and τ by $\mathbb{Z}_2$. Note that

$$\begin{aligned}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\rho_1(\xi,\tau)\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} &= e^{-i\pi\xi_1}\begin{bmatrix}-\cos\pi\tau_1 & -i\sin\pi\tau_1\\ i\sin\pi\tau_1 & \cos\pi\tau_1\end{bmatrix}\\ &= \rho_1(\xi + (v_1 + v_2), -\tau).\end{aligned} \tag{5.3.9}$$

Conjugation by the same matrix has the same effect on the other holonomies, showing that the bundles $V_{\xi,\tau}$ and $V_{(\xi+v^1+v^2),-\tau}$ are equivalent up to a gauge transformation. Indeed, let

$$g_0(x) = e^{i2\pi(v_1+v_2)(x)}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} \tag{5.3.10}$$

acting on q^*V_ξ with respect to our given frame. Note that for $b \in B$ we have

$$\begin{aligned} g_0(bx) &= e^{i2\pi(v_1+v_2)(bx)} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ &= \operatorname{sgn}(b) \left(1_E \otimes e^{i2\pi(v_1+v_2)(x)} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & \operatorname{sgn}(b) \end{bmatrix} g_0(x) \begin{bmatrix} 1 & 0 \\ 0 & \operatorname{sgn}(b) \end{bmatrix}. \end{aligned} \tag{5.3.11}$$

Therefore, $g_0(bx)$ is obtained from $g_0(x)$ by conjugation by the $\mathbb{Z}_2$-action $1 \oplus \operatorname{sgn}(b)$ on q^*V_ξ, and so $g_0(x)$ is well-defined gauge transformation on the quotient. Note moreover that taking the gauge transformation of the connection form of $\nabla_{\xi,\tau}$,

$$g_0 \left(i2\pi \begin{bmatrix} \xi & \tau \\ \tau & \xi \end{bmatrix} \right) g_0^{-1} + g_0 dg_0^{-1} = i2\pi \begin{bmatrix} \xi & -\tau \\ -\tau & \xi \end{bmatrix} + i2\pi \begin{bmatrix} v^1+v^2 & 0 \\ 0 & v^1+v^2 \end{bmatrix}, \tag{5.3.12}$$

which is the connection form for $\nabla_{(\xi+v^1+v^2),-\tau}$. We then have that the $\mathbb{Z}_2$-action $(\xi,\tau) \mapsto (\xi + v^1 + v^2, -\tau)$ corresponds to the identification of the corresponding bundles by the gauge transformation g_0.

Although we will continue working with the parameters ξ, τ, let us return for a moment to the coordinates of $(\mathbb{R}^4)^*$ with respect to the dual basis $e^i = dx^i$, and let $\{\hat{x}_i\}$ be coordinates with respect to this basis. In these coordinates, $\hat{\Lambda}$ is the standard integer lattice $\bigoplus \mathbb{Z}e^i$. The map $\xi + \tau \mapsto \xi - \tau$ is expressed as the map μ from Section 5.2, given by $\mu(\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4) = (\hat{x}_2, \hat{x}_1, \hat{x}_4, \hat{x}_3)$. Writing $a = v^1 + v^2 = \frac{1}{2}(e^1+e^2+e^3+e^4)$, we have that the $\mathbb{Z}_2$ action is given on $\hat{T}^4 = (\mathbb{R}^4)^*/\hat{\Lambda}$ by $\hat{x} \mapsto \mu\hat{x}+a$.

Considering instead the universal cover $(\mathbb{R}^4)^*$, where the isometry $g : \hat{x} \mapsto \mu\hat{x} + a$ and $\hat{\Lambda}$ generate a group $\hat{B}$. Note that we only need g and three of the generators Λ to generate $\hat{B}$, as $\sum e_i = g^2$. While $\hat{B}$ is a crystallographic group, it is not Bieberbach.

Indeed, there are four 2-tori of fixed points in $\hat{T}^4$ given by translating the 2-torus $\{t(e^1+e^2)+s(e^3+e^4) \mid s,t \in \mathbb{R}\}$ by each of the four elements

$$\begin{aligned} &\frac{1}{2}(e^1+e^3), \qquad \frac{1}{2}(e^1+e^4), \\ &\frac{1}{2}(e^2+e^3), \qquad \frac{1}{2}(e^2+e^4). \end{aligned} \tag{5.3.13}$$

In order to avoid dealing with orbifolds, we maintain the viewpoint of having a $\mathbb{Z}_2$-action on $\hat{T}^4$ rather than descending to the quotient.

We now use these $V_{\xi,\tau}$ to construct a Nahm transform. Consider an $SU(2)$-vector bundle E on M with an instanton ∇. Let $\nabla_{E\otimes V_{\xi,\tau}}$ be the twisted connection on $E\otimes V_{\xi,\tau}$ obtained by tensoring with the flat connection $\nabla_{\xi,\tau}$. Suppose moreover that $E\otimes V_{\xi,\tau}$ is without flat factors for all ξ,τ. Note that $\nabla_{E\otimes V_{\xi,\tau}}$ is still an instanton. Since M is parallelizable, $SO(M)$ is trivial. Therefore M has a spin structure, and hence a spinor bundle S. As usual, let $D^{\pm}_{\xi,\tau}$ denote the $S^{\pm}\to S^{\mp}$ parts of the Dirac operator associated to $\nabla_{E\otimes V_{\xi,\tau}}$.

By slight abuse of notation, we will take Clifford multiplication to be given on $V_{\xi,\tau}\otimes S$ by

$$\begin{aligned} c(v_i) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes c(v_i), \\ c(w_i) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes c(w_i) \end{aligned} \tag{5.3.14}$$

Since the $\mathrm{End}(V_{\xi,\tau})$ parts all commute with each other, we have the usual rules for Clifford multiplication. Moreover, we can recover the $\mathrm{End}(V_{\xi,\tau})$ parts by taking the $\mathbb{Z}_2$-invariant and $\mathbb{Z}_2$-skew-invariant parts of the Clifford multiplication. That is, for any $\eta \in \bigwedge(\mathbb{R}^4)^*$ we have

$$c(\eta) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes c\left(\frac{\eta+\mu^*\eta}{2}\right) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes c\left(\frac{\eta-\mu^*\eta}{2}\right). \tag{5.3.15}$$

Because μ is orientation-preserving, if η is self-dual then $\mu^*\eta$ is also. Therefore, the usual fact that $c(\bigwedge^+) = 0$ on S^- still holds in our new notation. Also, with this notation we have $D_{\xi,\tau} = D + i2\pi c(\xi + \tau)$.

Just as in the case of the Nahm Transform on the torus, we have $\ker D^+_{\xi,\tau} = 0$. Indeed, as before, the Lichnerowicz formula $D^- D^+ = \nabla^*\nabla + c(F^+)$ tells us that an element in the kernel of $D^+_{\xi,\tau}$ would have to be parallel with respect to $\nabla_E \otimes \nabla_{\xi,\tau}$, but such a section would provide a flat factor of the bundle $E \otimes V_{\xi,\tau}$.

Let $\hat{E}$ be the bundle over $\hat{T}^4$ given by $\hat{E} = \ker D^-_{\xi,\tau}$, with connection $\hat{\nabla}$ induced from orthogonal projection from $L^2(E \otimes V_{\xi,\tau} \otimes S^-)$.

Proposition 5.3.16. *The transformed connection $\hat{\nabla}$ on $\hat{E}$ is an instanton. Moreover, $\hat{E}$ is equipped with a $\mathbb{Z}_2$ bundle action that covers the action $x \mapsto \mu x + a$ on $\hat{T}^4$ described above.*

Proof. Seeing that $\hat{\nabla}$ is an instanton is completely analogous to the same result for the Nahm transform on a torus in Proposition 3.3.3. As before, we let $G_{\xi,\tau} = (\nabla^*_{\xi,\tau}\nabla_{\xi,\tau})^{-1}$. Since both $\mathrm{End}(V_{\xi,\tau})$ parts of the Clifford multiplication in (5.3.14) are constant and commute with the connection form of $V_{\xi,\tau}$, Clifford multiplication $c(v_i)$ and $c(w_i)$ commute with $\nabla^*\nabla_{\xi,\tau}$. Therefore $c(v_i)$ and $c(w_i)$ commute with $G_{\xi,\tau}$. Let $P_{\xi,\tau} = 1 - D^+_{\xi,\tau} G_{\xi,\tau} D^-_{\xi,\tau}$ be projection onto $\ker D^-_{\xi,\tau}$. The same computation as for the Nahm transform in Proprosition 3.3.3 then applies. Thus, as in (3.3.7)

$$\hat{F}_{\eta,\omega} = -8\pi^2 P_{\xi,\tau} c(\eta) c(\omega) G_{\xi,\tau} P_{\xi,\tau}, \tag{5.3.17}$$

where $\eta \neq \omega$ are any two of v^1, v^2, w^1, w^2. Because Clifford multiplication by self-dual forms is trivial on S^-, we have that $\hat{F}$ is anti-self-dual, hence $\hat{\nabla}$ is an instanton.

We now show that the connection is invariant under the $\mathbb{Z}_2$ action. Let $g_0(x)$ be the gauge transformation from (5.3.10), which we recall identifies $\nabla_{\xi,\tau} = g_0 \nabla_{\xi+v^1+v^2,-\tau} g_0^{-1}$. By tensoring with 1_E and 1_S, we consider g_0 to be a gauge transformation of $E \otimes V \otimes S$.

We define lift of the $\mathbb{Z}_2$-action on $\hat{T}^4$ to the trivial bundle $\hat{T}^4 \times L^2(E \otimes V \otimes S)$ by

$$(\hat{x}, s) \mapsto (\mu\hat{x} + a, g_0 s). \tag{5.3.18}$$

Note that $D_{\xi,\tau} = g_0(D_{\xi+v^1+v^2,\tau})g_0^{-1}$, and so g_0 defines a map taking $\ker D_{\xi,\tau}$ to $\ker(D_{\xi+v^1+v^2})$. Therefore $P_{\xi,\tau} = g_0(P_{\xi+v^1+v^2,-\tau})g_0^{-1}$. Since g_0 does not depend on ξ or τ, it commutes with differentiation in ξ and τ, and we have

$$\begin{aligned} P_{\xi,\tau}dP_{\xi,\tau} &= g_0(P_{\xi+v^1+v^2,-\tau})g_0^{-1}dg_0(P_{\xi+v^1+v^2,-\tau})g_0^{-1} \\ &= g_0(P_{\xi+v^1+v^2,-\tau})d(P_{\xi+v^1+v^2,-\tau})g_0^{-1} \end{aligned} \tag{5.3.19}$$

That is, $\hat{\nabla}$ is invariant by the $\mathbb{Z}_2$ action, up to conjugation by the $\mathbb{Z}_2$ action on $\hat{E}$. $\square$

Chapter 6

Minimal Energy Yang–Mills

As discussed in 2.2, instantons are global minimizers of the Yang–Mills energy $\|F_\nabla\|^2$. A natural question is then to ask whether the converse statement is true. In certain circumstances, the answer is yes. Bourguignon, Lawson, and Simons [BLS79, BL81] used the second variational inequalities of $\nabla + t\iota_X F^{\pm}_\nabla$ to show that, for $G = SU(2), U(2)$, or $SU(3)$, if ∇ is a G-connection that is a stable critical point of the energy, then it is an instanton. Using a related variation involving the flow along X (of which the Bourguignon–Lawson–Simons variation is the linear term), Stern [Ste10] extended the result for higher rank structure group over complete (but not necessarily compact) homogeneous spaces, in which case the adjoint bundle has an instanton subbundle and an anti-instanton subbundle, and these subbundles commute.

In this chapter, we generalize this argument over certain manifolds of cohomogeneity one. This necessitates taking the Taylor expansion of the variation studied by Stern, but the argument otherwise follows closely. In this chapter, we also allow the use of conformal vector fields X, as opposed to simply Killing fields.

For this chapter we will consider connections that minimize $\|F^-\|^2$. In the case of a complete manifold, we have $\|F^-\|^2 = \|F\|^2 + p_1(E)[M]$ where $p_1(E)[M]$ is the constant obtained by integrating the Pontrjagin form of E. Therefore, minimizing $\|F^-\|^2$ is equivalent to minimizing $\|F\|^2$. Due to the presence of boundary terms, this is no longer true in general for incomplete manifolds.

For the variation $\nabla + t\iota_X F^+_\nabla$ to be valid, we also make the assumption, for vector fields X under consideration, that $\iota_X \mathcal{L}^k_X F^+_X$ is in $L^2 \cap L^4$ for all F^+_X for all k.

6.1 Conformal vector fields

A vector field X is called *conformal* if $\mathcal{L}_X g = \alpha^2 g$, where g is the metric on M. Equivalently, the flow $\varphi_{X,t}$ along the vector field is a conformal map for all t.

Let $\mathcal{L}$ denote the Lie derivative extended to $\mathrm{ad}(E)$-valued forms using the parallel transport of ∇. This Lie derivative satisfies a generalization of Cartan's magic formula,

$$\mathcal{L}_X = \iota_X d_\nabla + d_\nabla \iota_X. \tag{6.1.1}$$

In dimension 4, the Hodge star $*\colon \Omega^2 \to \Omega^2$ is conformally invariant, and therefore $[\mathcal{L}_X, *] = 0$. In particular, the decomposition $\Omega^2 = \Omega^+ \oplus \Omega^-$ is preserved by $\mathcal{L}_X$. This leads to the following two propositions, which will be helpful in simplifying the variations of the curvature.

Proposition 6.1.2. *If X is conformal and ∇ is Yang–Mills, then $d_\nabla \iota_X F_\nabla^+$ is self-dual.*

Proof. By formula (6.1.1),

$$d_\nabla \iota_X F_\nabla^+ = \mathcal{L}_X F_\nabla^+ + \iota_X d_\nabla F_\nabla^+.$$

By the Yang–Mills equation and the Bianchi identity, $d_\nabla F_\nabla^+ = 0$. Since X is conformal, $\mathcal{L}_X$ preserves self-duality. □

We take the exterior product on $\mathrm{ad}(E)$-valued forms to be the Lie bracket on $\mathrm{ad}(E)$ tensored with exterior product on forms. For $\omega \in \Omega^\bullet(\mathrm{ad}(E))$ let $e(\omega)$ denote left exterior multiplication by ω.

Proposition 6.1.3. *For any $r \geq 1$,*

$$d_\nabla \mathcal{L}_X^r F_\nabla^+ = -\sum_{k=0}^{r-1} \mathcal{L}_X^k e(\iota_X F_\nabla) \mathcal{L}_X^{r-1-k} F_\nabla^+. \tag{6.1.4}$$

Proof. We begin by noting that

$$\begin{aligned}[d_\nabla, \mathcal{L}_X] &= d_\nabla \iota_X d_\nabla + d_\nabla^2 \iota_X - \iota_X d_\nabla^2 - d_\nabla \iota_X d_\nabla \\ &= e(F_\nabla)\iota_X - \iota_X e(F_\nabla) \\ &= -e(\iota_X F_\nabla) \end{aligned} \tag{6.1.5}$$

In the case $r = 1$,

$$d_\nabla \mathcal{L}_X F_\nabla^+ = -e(\iota_X F_\nabla) F_\nabla^+ + \mathcal{L}_X d_\nabla F_\nabla^+.$$

Since $d_\nabla F_\nabla^+ = 0$, we have obtained the desired equality.

If the statement holds for r, then

$$\begin{aligned} d_\nabla \mathcal{L}_X^{r+1} F_\nabla^+ &= -e(\iota_X F_\nabla)\mathcal{L}^r F_\nabla^+ + \mathcal{L}_X d_\nabla \mathcal{L}_X^r F_\nabla \\ &= -\sum_{k=0}^{r} \mathcal{L}_X^k e(\iota_X F_\nabla)\mathcal{L}^{r-k} F_\nabla^+. \end{aligned}$$

Therefore the statement holds by induction. □

The following definition will be necessary for obtaining local information from L^2 variational inequalities.

Definition 6.1.6. Let V be a vector space of vector fields on M, and let V be equipped with an inner product. Evaluation at a point p gives a map $P_p\colon V \to T_pM$. Let $\tilde{P}_p$ be the restriction of P_p to the orthogonal complement of its kernel. We say V is *conformally spanning (with conformal factor α)* if V consists entirely of conformal vector fields, and for all p the map $\tilde{P}_p$ is a conformal map with conformal factor $\alpha(p)$.

Note that if V is a conformally spanning set, then in particular the vector fields in V span T_pM for every p.

Remark. If M admits a conformally spanning set consisting entirely of vector fields that are complete, meaning their flow is defined for all time, then M is in fact conformal to a homogeneous space. This can be seen using the following theorem of Alekseevskiĭ and Ferrand.

Theorem (Alekseevskiĭ [Ale72], Ferrand [Fer96]). *Let $C(M)$ be the group of conformal transformations of a Riemannian manifold M. If M is not conformal to S^n with the round metric or $\mathbb{R}^n$ with the Euclidean metric, then $C(M)$ can be reduced to the group of isometries by some conformal transformations.*

In the case M is conformal to S^n or $\mathbb{R}^n$, it is conformal to a homogeneous space, and otherwise, the conformally spanning set V generates the isometries needed to show that M is homogeneous.

Note, however, that this argument fails in the case that the vector fields in M are not complete, since in this case they do not generate diffeomorphisms of M. We study here a particular example of where incomplete vector fields are natural to consider, namely when the manifold M is itself incomplete.

6.2 Vanishing commutators

The goal is to prove a generalization of the minimality result of [Ste10], the precise formulation of which is given in Section 6.3. To do this we consider variations of a Yang–Mills connection ∇ of the form

$$\nabla^{n,t} := \nabla + B_n(t), \tag{6.2.1}$$

where

$$B_n(t) := \sum_{k=1}^{n} \frac{t^k}{k!} \iota_X \mathcal{L}_X^{k-1} F_\nabla^+. \tag{6.2.2}$$

Note that letting $n = 1$ gives the variation considered in [BLS79, BL81], while in general this is simply the n-th Taylor expansion of the variation considered in [Ste10]. Using the Taylor expansion allows the consideration of vector fields X for which the flow is not defined for all time. These variations will be used to show the vanishing of certain commutators of derivatives of F_∇^- and F_∇^+.

The first step is to compute variational inequalities, beginning by computing the curvature of $\nabla^{n,t}$, which we denote $F^{n,t}$.

$$F^{n,t} = F_\nabla + d_\nabla B_n(t) + \frac{1}{2} B_n(t) \wedge B_n(t). \tag{6.2.3}$$

Expanding the middle term using (6.1.1) and (6.1.4) gives

$$\begin{aligned}
d_\nabla B_n(t) &= \sum_{k=1}^{n} \frac{t^k}{k!} d_\nabla \iota_X \mathcal{L}_X^{k-1} F_\nabla^+ \\
&= \sum_{k=1}^{n} \frac{t^k}{k!} \mathcal{L}_X^k F_\nabla^+ - \sum_{k=1}^{n} \frac{t^k}{k!} \iota_X d_\nabla \mathcal{L}_X^{k-1} F_\nabla^+ \\
&= \sum_{k=1}^{n} \frac{t^k}{k!} \mathcal{L}_X^k F_\nabla^+ + \sum_{k=1}^{n} \frac{t^k}{k!} \sum_{\ell=0}^{k-2} \iota_X \mathcal{L}_X^\ell (\iota_X F_\nabla \wedge \mathcal{L}_X^{k-\ell-2} F_\nabla^+) \\
&= \sum_{k=1}^{n} \frac{t^k}{k!} \mathcal{L}_X^k F_\nabla^+ - \sum_{k=1}^{n} \frac{t^k}{k!} \sum_{\ell=0}^{k-2} \sum_{m=0}^{\ell} \binom{\ell}{m} \iota_X \mathcal{L}_X^m F_\nabla \wedge \iota_X \mathcal{L}_X^{k-m-2} F_\nabla^+ \\
&= \sum_{k=1}^{n} \frac{t^k}{k!} \mathcal{L}_X^k F_\nabla^+ - \sum_{k=1}^{n} \frac{t^k}{k!} \sum_{m=0}^{k-2} \sum_{\ell=m}^{k-2} \binom{\ell}{m} \iota_X \mathcal{L}_X^m F_\nabla \wedge \iota_X \mathcal{L}_X^{k-m-2} F_\nabla^+ \\
&= \sum_{k=1}^{n} \frac{t^k}{k!} \mathcal{L}_X^k F_\nabla^+ - \sum_{k=1}^{n} \frac{t^k}{k!} \sum_{m=0}^{k-2} \binom{k-1}{m+1} \iota_X \mathcal{L}_X^m F_\nabla \wedge \iota_X \mathcal{L}_X^{k-m-2} F_\nabla^+
\end{aligned}$$

The last step uses the identity $\sum_{p=q}^{r} \binom{p}{q} = \binom{r+1}{q+1}$. Shifting the index m by 1,

$$d_\nabla B_n(t) = \sum_{k=1}^{n} \frac{t^k}{k!} \mathcal{L}_X^k F_\nabla^+ - \sum_{k=1}^{n} \sum_{m=1}^{k-1} \frac{t^k}{m!(k-m-1)!k} \iota_X \mathcal{L}_X^{m-1} F_\nabla \wedge \iota_X \mathcal{L}_X^{k-m-1} F_\nabla^+. \tag{6.2.4}$$

Turning our attention to the last term of (6.2.3), we now have

$$\begin{aligned}\frac{1}{2}B_n(t)\wedge B_n(t) &= \frac{1}{2}\left(\sum_{m=1}^{n}\frac{t^m}{m!}\iota_X\mathcal{L}_X^{m-1}F_\nabla^+\right)\wedge\left(\sum_{\ell=1}^{n}\frac{t^\ell}{\ell!}\iota_X\mathcal{L}_X^{\ell-1}F_\nabla^+\right)\\ &= \frac{1}{2}\sum_{k=1}^{n}\sum_{m=1}^{k-1}\frac{t^k}{m!(k-m)!}\iota_X\mathcal{L}_X^{m-1}F_\nabla^+\wedge\iota_X\mathcal{L}_X^{k-m-1}F_\nabla^+\\ &\qquad +\frac{1}{2}\sum_{k=n+1}^{2n}\sum_{m=k-n}^{n}\frac{t^k}{m!(k-m)!}\iota_X\mathcal{L}_X^{m-1}F_\nabla^+\wedge\iota_X\mathcal{L}_X^{k-m-1}F_\nabla^+\end{aligned}\tag{6.2.5}$$

Using commutativity of $\mathrm{ad}(E)$-valued 1-forms,

$$\begin{aligned}\iota_X\mathcal{L}_X^{m-1}F_\nabla^+\wedge\iota_X\mathcal{L}_X^{k-m-1}F_\nabla^+ &= \left(\frac{m}{k}+\frac{k-m}{k}\right)\iota_X\mathcal{L}_X^{m-1}F_\nabla^+\wedge\iota_X\mathcal{L}_X^{k-m-1}F_\nabla^+\\ &= \frac{m}{k}\iota_X\mathcal{L}_X^{k-m-1}F_\nabla^+\wedge\iota_X\mathcal{L}_X^{m-1}F_\nabla^+\\ &\qquad +\frac{k-m}{k}\iota_X\mathcal{L}_X^{m-1}F_\nabla^+\wedge\iota_X\mathcal{L}_X^{k-m-1}F_\nabla^+.\end{aligned}$$

In the sum $k=1,\ldots,n$ in (6.2.5), the m and $k-m$ terms then combine, and we obtain the sum

$$\sum_{k=1}^{n}\sum_{m=1}^{k-1}\frac{t^k}{m!(k-m-1)!k}\iota_X\mathcal{L}_X^{m-1}F_\nabla^+\wedge\iota_X\mathcal{L}_X^{k-m-1}F_\nabla^+,$$

which cancels with part of the sum in (6.2.4). Assembling the terms from (6.2.4) and (6.2.5),

$$\begin{aligned}F^{n,t} &= F_\nabla+\sum_{k=1}^{n}\frac{t^k}{k!}\mathcal{L}_X^kF_\nabla^+\\ &\quad -\sum_{k=1}^{n}\sum_{m=1}^{k-1}\frac{t^k}{m!(k-m-1)!k}\iota_X\mathcal{L}_X^{m-1}F_\nabla^-\wedge\iota_X\mathcal{L}_X^{k-m-1}F_\nabla^+\\ &\quad +\frac{1}{2}\sum_{k=n+1}^{2n}\sum_{m=k-n}^{n}\frac{t^k}{m!(k-m)!}\iota_X\mathcal{L}_X^{m-1}F_\nabla^+\wedge\iota_X\mathcal{L}_X^{k-m-1}F_\nabla^+.\end{aligned}\tag{6.2.6}$$

6.2.1 Zeroth order

We first consider the variation (6.2.1) with $n=1$, which will eventually show us that commutators of components of F^+ and F^- must vanish.

Proposition 6.2.7. *Suppose M admits a conformally spanning space V. If ∇ is a minimizer of $\|F_\nabla^-\|^2$, and if there is an orthonormal basis $\{X_k\}_{k=1}^d$ of V for which $\iota_{X_k}F_\nabla^+ \in L^2 \cap L^4$, then for each $X \in V$,*

$$\langle F_\nabla^-, \iota_X F_\nabla^+ \wedge \iota_X F_\nabla^+ \rangle_{L^2} = 0.$$

Proof. Consider the variation $\nabla^{1,t}$ for some fixed conformal vector field X. Expanding the curvature gives

$$\begin{aligned} F^{1,t} &= F_\nabla + d_\nabla B_1(t) + \frac{1}{2} B_1(t) \wedge B_1(t) \\ &= F_\nabla + t d_\nabla \iota_X F_\nabla^+ + \frac{t^2}{2} \iota_X F_\nabla^+ \wedge \iota_X F_\nabla^+ \end{aligned} \tag{6.2.8}$$

By Proposition 6.1.2, we have that $d_\nabla \iota_X F_\nabla^+$ is self-dual. Therefore, taking the ASD part,

$$\|(F^{1,t})^-\|^2 = \|F_\nabla^-\|^2 + t^2 \langle F_\nabla^-, \iota_X F_\nabla^+ \wedge \iota_X F_\nabla^+ \rangle_{L^2} + O(t^4). \tag{6.2.9}$$

Since $\|F_\nabla^-\|^2$ is minimized, we must then have

$$\langle F_\nabla^-, \iota_X F_\nabla^+ \wedge \iota_X F_\nabla^+ \rangle_{L^2} \geq 0 \tag{6.2.10}$$

Let $\{X_j\}_{j=1}^d$ be an orthonormal basis for V. Inequality (6.2.10) holds for each X_j since they are all conformal. To show equality in (6.2.10), fix a point $p \in M$ and consider the sum of the pointwise inner products

$$s(p) = \sum_{j=1}^d \langle F_\nabla^-, \iota_{X_j} F_\nabla^+ \wedge \iota_{X_j} F_\nabla^+ \rangle_p. \tag{6.2.11}$$

The above sum is the trace of a quadratic form on V, and is thus invariant under orthogonal transformations on V. In particular, for some conformal factor α and orthonormal basis $\{e_i\}$ for T_pM,

$$s(p) = \sum_{i=1}^4 \alpha^2 \langle F_\nabla^-, \iota_{e_i} F_\nabla^+ \wedge \iota_{e_i} F_\nabla^+ \rangle_p.$$

One can check that if $\varphi \in \Omega^2(M)$ is self-dual, then $\sum_{i=1}^4 \iota_{e_i}\varphi \wedge \iota_{e_i}\varphi$ is also self-dual. Therefore, $s(p)$ must vanish. Integrating $s(p)$ then gives the vanishing of the sum of the L^2 inner products,

$$\sum_{j=1}^{d} \langle F_\nabla^-, \iota_{X_j} F_\nabla^+ \wedge \iota_{X_j} F_\nabla^+ \rangle_{L^2} = 0. \tag{6.2.12}$$

Since each term of the sum individually is nonnegative by (6.2.10), each term must individually vanish. Since $\{X_j\}$ spans V, this gives the desired result. □

Proposition 6.2.13. *Suppose M and ∇ satisfy the conditions of Proposition 6.2.7. Then for all X in the conformally spanning space V,*

$$e^*(\iota_X F_\nabla^-) F_\nabla^+ = 0.$$

Proof. Consider a new variation $\tilde{\nabla}^t = \nabla + B_1(t) + t^{\frac{3}{2}}\psi$ for $\psi \in \Omega^1(\mathrm{ad}(E))$ with compact support. Then

$$F_{\tilde{\nabla}^t} = F_\nabla + d_\nabla B_1(t) + t^{\frac{3}{2}} d_\nabla \psi + \frac{1}{2} B_1(t) \wedge B_1(t) + t^{\frac{3}{2}} B_1(t) \wedge \psi + \frac{t^3}{2} \psi \wedge \psi.$$

Using Proposition 6.2.7 along with the fact that ∇ is Yang–Mills, we obtain

$$\|F_{\tilde{\nabla}^t}^-\|^2 = \|F_\nabla^-\|^2 + 2t^{\frac{5}{2}} \langle F_\nabla^-, \iota_X F_\nabla^+ \wedge \psi \rangle_{L^2} + O(t^3).$$

Then, since $\|F_\nabla^-\|^2$ is minimized,

$$\langle F_\nabla^-, \iota_X F_\nabla^+ \wedge \psi \rangle_{L^2} \geq 0.$$

Replacing ψ by $-\psi$ gives the opposite inequality, and thus

$$\langle F_\nabla^-, \iota_X F_\nabla^+ \wedge \psi \rangle_{L^2} = 0$$

for arbitrary ψ. □

Proposition 6.2.14. *Suppose M and ∇ satisfy the conditions of Proposition 6.2.7. Then*

$$[F^+_{jk}, F^-_{m\ell}] = 0$$

for all indices j, k, m, ℓ.

Proof. From Proposition 6.2.13 we have that the commutator vanishes whenever there is a repeated index, say $k = m$. Then applying self-duality and anti-self-duality relations gives the vanishing of the commutator for other indices. □

As shown in [BLS79] and [BL81], the vanishing of the commutator in Proposition 6.2.14 is enough to show that the connection is an instanton in the case when $G = SU(2), U(2)$, or $SU(3)$ and under certain additional boundary conditions. For higher rank structure groups, we must also consider, as in [Ste10], commutators of derivatives of F^+ and F^-. We do this in the next section.

6.2.2 Higher order

We now establish the vanishing of commutators of the form $[\nabla^k F^+_\nabla, \nabla^\ell F^-_\nabla]$. Propositions 6.2.15 and 6.2.18 below are analogous to Propositions 6.2.7 and 6.2.13 from the previous section. Together, these will form an inductive argument, with Proposition 6.2.14 being the base case.

Proposition 6.2.15. *Suppose M admits a conformally spanning space V. Let ∇ is a minimizer of $\|F^-_\nabla\|^2$, and suppose there is an orthonormal basis $\{X_k\}_{k=1}^d$ of V for which $\iota_{X_k}\mathcal{L}^m_{X_k}F^+_\nabla \in L^2 \cap L^4$ for $0 \leq m \leq n$. Suppose moreover that*

$$[\nabla^j F^+, \nabla^\ell F^-] = 0$$

for $j + \ell < n$. Then for $X \in V$,

$$\langle F^-_\nabla, \iota_X \mathcal{L}^n_X F^+_\nabla \wedge \iota_X \mathcal{L}^n_X F^+_\nabla\rangle = 0.$$

Proof. As in the proof of Proposition 6.2.7, we first prove that the L^2-inner product is non-negative. We do so by considering the variation $\nabla^{n+1,t}$ defined by (6.2.1) and (6.2.2).

In the inductive hypothesis $[\nabla^k F^+, \nabla^\ell F^-] = 0$, we may replace the covariant derivatives with Lie derivatives since their difference results in lower order terms that also vanish. Therefore the terms $\iota_X \mathcal{L}_X^{m-1} F_\nabla^- \wedge \iota_X \mathcal{L}_X^{k-m-1} F_\nabla^+$ in the expansion of $F^{n+1,t}$ vanish, and so by (6.2.6),

$$F^{n+1,t} = F_\nabla + \sum_{k=1}^{n+1} \frac{t^k}{k!} \mathcal{L}_X^k F_\nabla^+ \\ + \frac{1}{2} \sum_{k=n+2}^{2n+2} \sum_{m=k-n-1}^{n+1} \frac{t^k}{m!(k-m)!} \iota_X \mathcal{L}_X^{m-1} F_\nabla^+ \wedge \iota_X \mathcal{L}_X^{k-m-1} F_\nabla^+.$$

Furthermore, since the inner product is ad-invariant, we see that

$$\langle F_\nabla^-, \iota_X \mathcal{L}_X^k F_\nabla^+ \wedge \iota_X \mathcal{L}_X^\ell F_\nabla^+ \rangle = 0$$

whenever either $k < n$ or $\ell < n$. Moreover, $\mathcal{L}_X^k F_\nabla^+$ is self-dual, and so expanding $\|(F^{n+1,t})^-\|^2$ gives

$$\|(F^{n+1,t})^-\|^2 \\ = \|F_\nabla^-\|^2 + \frac{t^{2n+2}}{((n+1)!)^2} \langle F_\nabla^-, \iota_X \mathcal{L}_X^n F_\nabla^+ \wedge \iota_X \mathcal{L}_X^n F_\nabla^+ \rangle + O(t^{2n+4}). \quad (6.2.16)$$

Then since $\|F_\nabla^-\|^2$ is minimized,

$$\langle F_\nabla^-, \iota_X \mathcal{L}_X^n F_\nabla^+ \wedge \iota_X \mathcal{L}_X^n F_\nabla^+ \rangle \geq 0. \quad (6.2.17)$$

For $X \in V$, let $S(X) = \langle F_\nabla^-, \iota_X \mathcal{L}_X^n F_\nabla^+ \wedge \iota_X \mathcal{L}_X^n F_\nabla^+ \rangle$. Equality in (6.2.17) will be obtained by taking the average of S over the unit sphere in V. Let X_k be an orthonormal basis of V. Then $S(y^j X_j)$ is a homogeneous polynomial of degree $2n+2$

in the variables $y^1, \dots, y^d$. Integrating over the unit sphere in $\mathbb{R}^d$,

$$\int_{|y|=1} S(y^j X_j)\, dS^{d-1} = \int_{|y|=1} y^m y^p y^I y^J \langle F^-_\nabla, \iota_{X_m} \mathcal{L}^I_X F^+_\nabla \wedge \iota_{X_p} \mathcal{L}^J_X F^+_\nabla \rangle\, dS^{d-1}$$

By symmetry of $S^{d-1} \subset \mathbb{R}^d$, the above integral is a pairwise contraction in the indices of y. The following demonstrates that all such contractions must vanish.

As noted earlier, we may replace Lie derivatives by covariant derivatives, and we may also commute derivatives, since the remainders all involve lower order commmutators of F^+ and F^-, which vanish by assumption. Thus, if m contracts with an index in I, we may rearrange to obtain the term $\iota_{X_m} \nabla_{X_m} F^+ = d^*_\nabla F^+ = 0$. If, on the other hand, m contracts with p, then $\iota_m \mathcal{L}_I F^+_\nabla \wedge \iota_m \mathcal{L}_J F^+_\nabla$ is self-dual, and the inner product vanishes.

We are then left only with the terms where m contracts with some index in J. Let J' be the indices in J other than m. Then, modulo lower order terms, all such contractions of m with J involve $\mathcal{L}_{J'} \iota_p \nabla_m F^+_\nabla$. Since $d_\nabla F^+_\nabla = 0$,

$$\nabla_m F^+_{pk} = -\nabla_p F^+_{km} - \nabla_k F^+_{mp},$$

and so $\iota_p \nabla_m F^+_\nabla = \iota_m \nabla_p F^+_\nabla - \nabla(F^+_{mp})$, modulo terms that vanish in the inner product. The first term on the right side becomes the case $m = p$ from before, and the inner product vanishes. We are then left only with terms of the form

$$\langle F^-_{k_1 k_2}, [\mathcal{L}_I F^+_{m\ell_1}, \mathcal{L}_{J'} \nabla_{\ell_2} F^+_{mp}] \rangle = \langle \mathcal{L}_I F^+_{m\ell_1}, [\mathcal{L}_{J'} \nabla_{\ell_2} F^+_{mp}, F^-_{k_1 k_2}] \rangle,$$

where each of k_1, k_2 contract with one of ℓ_1, ℓ_2. Note then that

$$[\nabla_{\ell_2} \mathcal{L}_{J'} F^+_{mp}, F^-_{k_1 k_2}] = \nabla_{\ell_2} [\mathcal{L}'_J F^+_{mp}, F^-_{k_1 k_2}] - [\mathcal{L}_{J'} F^+_{mp}, \nabla_{\ell_2} F^-_{k_1 k_2}].$$

The first term on the righthand side vanishes since the commutator is of lower order, while the second term vanishes since ℓ_2 contracts with one of k_1, k_2, and $d_\nabla^* F^- = 0$.

We have thus shown that the integral of $S(X)$ over the sphere in V vanishes. By (6.2.17), however, we have $S(X) \geq 0$ for all $X \in V$, and therefore it must be that $S(X) \equiv 0$ identically. □

Proposition 6.2.18. *Suppose the conditions of Proposition 6.2.15 are satisfied. Then*

$$e^*(\iota_X \mathcal{L}_X^n F_\nabla^+) F_\nabla^- = 0.$$

Proof. Let ψ be an arbitrary compactly supported ad(E)-valued 1-form. Similarly to the proof of Proposition 6.2.13, consider a new variation

$$\tilde{\nabla}^t = \nabla^{n+1,t} + t^{n+2}\psi. \tag{6.2.19}$$

Then the curvature is

$$\tilde{F}^t = F^{n+1,t} + t^{n+2} d_{\nabla^{n+1,t}}\psi + \frac{t^{2n+4}}{2}\psi \wedge \psi. \tag{6.2.20}$$

From Proposition 6.2.15 and its proof,

$$(F^{n+1,t})^- = F_\nabla^- + O(t^{n+2}) \tag{6.2.21}$$

$$\|(F^{n+1,t})^-\|^2 = \|F_\nabla^-\|^2 + O(t^{2n+4}) \tag{6.2.22}$$

Therefore,

$$\|\tilde{F}^t\|^2 = \|F_\nabla^-\|^2 + 2t^{n+2}\langle F_\nabla^-, d_{\nabla^{n+1,t}}\psi\rangle_{L^2} + O(t^{2n+4}). \tag{6.2.23}$$

Evaluating,

$$\langle F_\nabla^-, d_{\nabla^{n+1,t}}\psi\rangle_{L^2} = \langle F_\nabla^-, d_\nabla \psi\rangle_{L^2} + \sum_{k=1}^{n+1} \frac{t^k}{k!}\langle F_\nabla^-, (\iota_X \mathcal{L}_X^{k-1} F_\nabla^+) \wedge \psi\rangle_{L^2}. \tag{6.2.24}$$

The first term of (6.2.24) vanishes because ∇ is Yang–Mills. Additionally, all terms in (6.2.24) involving $\mathcal{L}_X^{k-1} F_\nabla^+$ for $k \leq n$ vanish by the inductive assumption. Therefore, (6.2.23) becomes

$$\|\tilde{F}^t\|^2 = \|F_\nabla^-\|^2 + \frac{2t^{2n+3}}{(n+1)!}\langle F_\nabla^-, (\iota_X \mathcal{L}_X^n F_\nabla^+) \wedge \psi\rangle + O(t^{2n+4}). \tag{6.2.25}$$

Since $\|F_\nabla^-\|^2$ is minimized,

$$\langle F_\nabla^-, (\iota_X \mathcal{L}_X^n F_\nabla^+)\psi\rangle = 0. \tag{6.2.26}$$

Since ψ was arbitrary, $e^*(\iota_X \mathcal{L}_X^n F_\nabla^+)F_\nabla^- = 0$. □

Proposition 6.2.27. *Suppose the conditions of Proposition 6.2.15 are satisfied. Then*

$$[\nabla^k F_\nabla^+, \nabla^\ell F_\nabla^-] = 0$$

for all $k + \ell < n + 1$.

Proof. Fom Proposition 6.2.18, we see

$$\sum_j [\nabla_k^n F_{kj}^+, F_{jm}] = 0,$$

where k is not summed over. Combining this with the Yang–Mills equation, the Bianchi identity, and the self-duality and anti-self-duality relations allows one to algebraically deduce the result. The proof proceeds exactly as in Section 4.3 of [Ste10], and so will not be repeated here. □

Proposition 6.2.27 forms the inductive step, and so along with the base case in Proposition 6.2.14, we have proved the following theorem.

Theorem 6.2.28. *Suppose M admits a conformally spanning space V. If ∇ is a minimizer of $\|F_\nabla^-\|^2$, and if there is an orthonormal basis $\{X_k\}_{k=1}^d$ of V for which $\iota_{X^k}\mathcal{L}_{X_k}^m F_\nabla^+ \in L^2 \cap L^4$ for all m, then*

$$[\nabla^j F_\nabla^+, \nabla^\ell F_\nabla^-] = 0$$

for all j, ℓ.

6.3 Instanton subbundles

As shown in [BLS79, BL81], in the case $G = SU(2), U(2)$, or $SU(3)$, if even the zeroth order commutator from Proposition 6.2.14 vanishes, and assuming boundary conditions to allow an integration by parts argument, then either F_∇^+ or F_∇^- must identically vanish, and so the connection must be an instanton (or anti-instanton). Examples of this case are discussed in Sections 6.4 and 6.5.

For higher rank structure groups G, however, this no longer need be the case as $\mathfrak{g}$ may be large enough to accommodate two nontrivial commuting algebras. Following [Ste10], with the argument reproduced here for completion, we can show instead that $\mathrm{ad}(E)$ has an instanton and an anti-instanton subbundle.

Theorem 6.3.1. *Suppose M is an analytic manifold that admits a conformally spanning space V, and let ∇ be a connection for which $\|F_\nabla^-\|_{L^2}^2$ is minimized, and suppose moreover that for an orthonormal basis $\{X_k\}_{k=1}^d$ of V, the interior products $\iota_{X_k}\mathcal{L}_{X_k}^n F_\nabla^+$ are in L^2 and L^4 for all n. Then there exist subbundles K^+, K^- of $\mathrm{ad}(E)$ that are preserved by ∇ and such that the restriction of ∇ to K^+ is an instanton and to K^- is an anti-instanton. Moreover,*

$$[K^+, K^-] = 0.$$

Proof. Let $K^\pm$ be the subsheaf of $\mathrm{ad}(E)$ generated by the coordinate components of $F_\nabla^\pm$ and all its derivatives. We first want to show that $K^\pm$ is in fact a subbundle.

Fix a point $p \in M$ and let $f_1, \ldots, f_k$ generate $K_p^\pm$. Since $f_1, \ldots, f_k$ are each generated by $F_\nabla^\pm$ and its derivatives, each can be extended to a section of $K_p^\pm$. Thus it remains only to verify that these sections generate $K^\pm$. For some other section f_{k+1}, consider $\eta = f_1 \wedge \cdots \wedge f_k \wedge f_{k+1}$. By assumption η must vanish at p. Moreover, derivatives of η are all also exterior products of $k+1$ elements of $K_p^\pm$, and so must also vanish at p. The Yang–Mills and Bianchi identity together show that $F_\nabla^\pm$ are solutions to an elliptic equation, and thus are analytic. Since η vanishes to all orders, by analyticity it must identically vanish. Therefore, f_{k+1} is generated by $f_1, \ldots, f_k$, and since f_{k+1} was arbitrary, we have that $K^{\pm}$ is a subbundle.

Next observe that $K^{\pm}$ is preserved by ∇ by definition, and $[K^+, K^-] = 0$ by Theorem 6.2.28. Moreover, as curvature acts on $\mathrm{ad}(E)$ by commutation, and $[F_\nabla^-, K^+] = 0$, we have that the curvature on K^+ is simply F_∇^+. Therefore ∇ on K^+ is self-dual, and similarly ∇ on K^- is anti-self-dual. □

Remark. In the case where M is incomplete, it would be natural to impose conditions on the behaviour of ∇ near singular points or boundaries of M. One can then consider connections that minimize $\|F_\nabla^-\|^2$ among a certain class of connections. As long as the variations $\nabla^{n,t}$ defined in (6.2.1) remains within this class of connections, Theorem 6.3.1 may still be applied.

6.4 Minimal Yang–Mills on a compact manifold

As mentioned previously, when the structure group $G = SU(2)$, $U(2)$, or $SU(3)$ we can obtain vanishing results even with just the vanishing zeroth order commutator given by Proposition 6.2.14. The essential idea is that for these structure groups the Lie algebra is not large enough to accomodate the commuting subalgebras $K_p\pm$ from the proof of Theorem 6.3.1.

First assume M to be compact, a noncompact case being given in Section 6.5. As discussed in Section 6.1, a compact M admitting a conformally spanning space V must be homogeneous, and so we do not obtain any new examples beyond what was proved in [BLS79] and [BL81].

We do observe, however, that the vector fields used in [BLS79] over S^4 were conformal and not Killing, while those vector fields used in [Ste10] were Killing. The author hopes that a unified perspective considering all conformal vector fields may yield improved stability results in the future.

Theorem 6.4.1. *Let ∇ be a G-connection on a compact homogeneous four-manifold M with positive Ricci curvature, and suppose $G = SU(2)$, $U(2)$, or $U(n)$. If ∇ is a minimizer for $\|F_\nabla\|^2$, it is either an instanton or an anti-instanton.*

Proof. As in the proof of Theorem 6.3.1 let $K_p^\pm$ be the subalgebra of $\mathrm{ad}(E)_p$ generated by the components of $K_p^\pm$.

Because M is compact, from the Chern–Weil argument in Section 2.2 we have that $\|F_\nabla\|^2$ is minimized if and only if $\|F_\nabla^-\|^2$ is minimized. Since the Killing vector fields on M are a conformally spanning space, by Proposition 6.2.14 we then obtain that

$$[K_p^+, K_p^-] = 0. \tag{6.4.2}$$

We now show that at least one of $K_p^\pm$ must be abelian.

If $\mathfrak{g} = \mathfrak{su}(2)$, and if, say K_p^+ has two non-commuting elements, then K_p^+ must be all of $\mathfrak{su}(2)$. In this case K_p^- must be 0, which is abelian. Performing the same argument with K_p^- shows that at least one of $K_p^\pm$ is abelian.

If $\mathfrak{g} = \mathfrak{su}(3)$, and if, say, K_p^+ contains two non-commuting elements, then K_p^+ must contain $\mathfrak{su}(2)$ as a subalgebra. Then the centralizer of K_p^+, which contains K_p^- is at most one-dimensional, and hence abelian. Replacing K_p^+ with K_p^- above shows that at least one of $K_p^\pm$ is abelian.

Lastly, if $\mathfrak{g} = \mathfrak{u}(2)$, we can embed it in $\mathfrak{su}(3)$ and obtain the result from above.

Therefore, at every point $p \in M$ we have at least one of $[K_p^+, K_p^+]$ or $[K_p^-, K_p^-]$ must vanish. In particular, a zero set of at least one of them must be dense in some open set. Because the generators of $K_p^\pm$ are solutions to an elliptic equation, they must be analytic, and if the zero set $[K_p^\pm, K_p^\pm]$ is dense in an open set, it must vanish everywhere.

Suppose that it is $[K_p^+, K_p^+]$ that vanishes everywhere. Because $d_\nabla F_\nabla^+$ and $d_\nabla^* F_\nabla^+$ all vanish, a Bochner argument gives that (as in [BL81, Theorem 3.10])

$$0 = \nabla^*\nabla F_\nabla^+ + (\mathrm{Ric} \wedge I + 2R)F_\nabla^+ + 2\hat{F}F_\nabla^+ = 0, \tag{6.4.3}$$

where $\hat{F}F_\nabla^+ = -[F_{ij}, F_{k\ell}^+]e^j \wedge e^\ell$. Note, however, that because $[K_p^+, K_p^+]$ and $[K_p^-, K_p^+]$ both vanish, then $\hat{F}F_\nabla^+ = 0$.

Note then that (6.4.3) tells us that

$$0 = \|\nabla F_\nabla^+\|^2 + \langle(\mathrm{Ric} \wedge I + 2R)F_\nabla^+, F_\nabla^+\rangle. \tag{6.4.4}$$

Since M has positive Ricci curvature, we then have that $F_\nabla^+ = 0$, and hence ∇ is an anti-instanton.

The case where $[K_p^-, K_p^-]$ vanishes is similar, and shows that ∇ is an instanton. $\square$

6.5 Minimal Yang–Mills on a cylindrical manifold with bounded end

Consider $M = \mathbb{R}_{>0} \times N$, where N is a homogeneous manifold of dimension 3 with nonnegative Ricci curvature, where we take r to be the coordinate on $\mathbb{R}_{>0}$. Equip M with the product metric $g = dr^2 + g_N$.

Since N is homogeneous, the set of Killing vector fields on N restricted to a point p span T_pN. We can fix an inner product on the space V_N of Killing fields on N for which restriction to a point is an isometry on the orthogonal complement to its kernel. Let $\{X_k\}_{k=1}^d$ be an orthonormal basis for V_N. Let $V = \mathrm{span}\{\partial_r\} \cup \{X_k\}_{k=1}^d$, and note that this is a conformally spanning set for M.

We will again consider the case $G = SU(2)$, $U(2)$, or $SU(3)$. The case proceeds similarly to Theorem 6.4.1, and note that the use of the Bochner formula necessitates integration by parts. As such we consider connections satisfying $|F_\nabla| = O(1)$ and $|\nabla_{\partial_r} F_\nabla| = O(r)$ as $r \to 0$.

Proposition 6.5.1. *Suppose M is as above and $G = SU(2)$, $U(2)$, or $SU(3)$. Suppose ∇ is a connection such that F_∇ satisfies the condition $|F_\nabla| = O(1)$ and $|\nabla_{\partial_r} F_\nabla| = O(r)$ as $r \to 0$. If ∇ minimizes $\|F^-_\nabla\|^2$, then F_∇ is either self-dual or anti-self-dual.*

Proof. At a point $p \in M$ let $K^\pm_p$ be the subalgebra of $\mathfrak{g}$ generated by the components of $F^\pm_\nabla$. Using the conformally spanning set V, by Proposition 6.2.14 we have that $[K^+_p, K^-_p] = 0$. The argument from Theorem 6.4.1 shows that one of $K^\pm_p$ is abelian, and again by analyticity we have that one of $[K^+_p, K^+_p]$ or $[K^-_p, K^-_p]$ must vanish identically on M.

Suppose that it is $[K^+_p, K^+_p]$ that vanishes everywhere. Because $d_\nabla F^+_\nabla$ and $d^*_\nabla F^+_\nabla$ all vanish, we again have the Bochner Formula (6.4.3)

$$0 = \nabla^*\nabla F^+_\nabla + (\operatorname{Ric} \wedge I + 2R)F^+ + 2\hat{F} F^+_\nabla = 0,$$

and once again $\hat{F} F^+_\nabla = 0$. Then,

$$\begin{aligned} 0 &= \int_M \langle \nabla^*\nabla F^+_\nabla, F^+_\nabla \rangle + \langle (\operatorname{Ric} + 2R) F^+_\nabla, F^+_\nabla \rangle \\ &= \|\nabla F^+_\nabla\|^2 + \langle (\operatorname{Ric} + 2R) F^+_\nabla, F^+_\nabla \rangle_{L^2} + \lim_{r \to 0} \int_{\{r\} \times N} \langle \nabla_r F^+_\nabla, F^+_\nabla \rangle \, d\operatorname{vol}_N . \end{aligned} \tag{6.5.2}$$

By our decay assumptions, the boundary integral vanishes, while the remaining terms are non-positive. In particular, we must have $\nabla F^+_\nabla = 0$, and so $|F^+_\nabla|$ is constant. Therefore, $F^+_\nabla = 0$, otherwise this would violate the finite L^2 condition. Therefore ∇ is an anti-instanton.

The case where $[K^-_p, K^-_p]$ vanishes is similar, and shows that ∇ is an instanton. □

Chapter 7

Conclusions

Here we summarize our results, and present potential future directions of research.

In Chapter 4, we adapted the ADHM construction for S^1-invariant instantons on S^4. We proved that the ADHM data decomposes into S^1-subrepresentations of $V = \ker D$, and this decomposition is given in terms of the weights of the S^1-action on the fibres at 0 and ∞. Using this framework, we then found the moduli spaces of S^1-invariant $SU(2)$-instantons of charge up to 3, and exhibited an example of charge 4 demonstrating that this moduli space is non-empty. One could extend this research in different ways:

- One could compute the Higgs fields and connections of monopoles given their corresponding ADHM data, and as such gain more explicit examples of singular monopoles.

- The instanton-monopole correspondence examined here was a special case of a correspondence between multi-monopoles and S^1-invariant instantons on multi-Taub-NUT spaces. The analogue of the ADHM construction in this case is given in terms of bow diagrams, described in [Che11] and [CLHS16]. One might consider whether one can decompose the bow diagrams as an S^1-representation to obtain a description of the S^1-invariant instantons on multi-Taub-NUT spaces.

In Chapter 5, we constructed a Nahm transform on the quotient of $\mathbb{R}^4$ by a specific Bieberbach group B. This construction resulted in a transformed connection that was invariant by a crystallographic action. This project was exploratory in nature, and so leads to many possible questions for future research.

- A first step in extending this work would be to determine if the construction given is invertible, and if so construct the inverse.
- One might also attempt to construct similar Nahm transforms for quotients by other Bieberbach groups, or more generally by crystallographic groups. One could also consider similar constructions for dimensional reductions, and construct a Nahm transform for, say, crystallographic-invariant monopoles.
- In addressing the previous question, one might wish to look for general properties of such Nahm transforms. A particular question in this regard would be to determine the appropriate dual space for $\mathbb{R}^4/B$ for a general Bieberbach group B.

The final chapter consisted of extendinng the proof that minimal Yang–Mills connections are instantons to a broader class of manifold.

- A direct question raised by this work is whether there is a natural boundary condition for instantons which is preserved by the variations considered. In that case, one would obtain a minimality result for instantons that are minimal among those satisfying the boundary condition.
- One might try to weaken the assumptions on the conformal vector fields used in order to generalize the result yet again.

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