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# A VARIATION ON THE DONSKER-VARADHAN INEQUALITY FOR THE PRINCIPAL EIGENVALUE

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Abstract. The purpose of this short note is to give a variation on the classical Donsker-Varadhan inequality, which bounds the first eigenvalue of a second-order elliptic operator on a bounded domain $\Omega$ by the largest mean first exit time of the associated drift-diffusion process via

$$\lambda_1 \geq \frac{1}{\sup_{x\in\Omega} \mathbb{E}_x \tau_{\Omega^c}}.$$

Instead of looking at the mean of the first exist time, we study quantiles: let $d_{p,\partial\Omega} : \Omega \to \mathbb{R}_{\geq 0}$ be the smallest time $t$ such that the likelihood of exiting within that time is $p$, then

$$\lambda_1 \geq \frac{\log{(1/p)}}{\sup_{x\in\Omega} d_{p,\partial\Omega}(x)}.$$

Moreover, as $p \to 0$, this lower bound converges to $\lambda_1$.

## 1. Introduction

We consider, for open and bounded $\Omega \subset \mathbb{R}^n$, solutions of the equation

$$\begin{aligned} -\mathrm{div}(a(x)\nabla u(x)) + \nabla V \cdot \nabla u &= \lambda u \qquad &&\text{in } \Omega \\ u &= 0. &&\text{on } \partial\Omega \end{aligned}$$

Estimating the smallest possible value of $\lambda_1$ for which the equation has a solution is a problem of fundamental importance. Finding upper bounds is, in many instances, rather straightforward by testing with a family of functions – finding lower bounds is substantially more difficult. An important conceptual leap is due to Donsker & Varadhan, who take

$$L = -\mathrm{div}(a(x)\nabla u(x)) + \nabla V \cdot \nabla u$$

and take $-L$ as the infinitesimal generator of a drift-diffusion process (here and in all subsequent steps we always assume sufficient regularity on both the operator and the domain). The maximum mean exit time then serves as a lower bound of the first eigenvalue $\lambda_1$.

**Theorem** (Donsker-Varadhan [5, 6], CPAM 1976)**.**

$$\lambda_1 \geq \frac{1}{\sup_{x\in\Omega} \mathbb{E}_x \tau_{\Omega^c}}.$$

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*Proof.* The proof is simple: note that $w(x) = \mathbb{E}_x \tau_{\Omega^c}$ solves the equation

$$\begin{aligned} -\mathrm{div}(a(x)\nabla w(x)) + \nabla V \cdot \nabla w &= 1 \qquad \text{in } \Omega \\ w &= 0. \end{aligned}$$

By the comparison principle, we have

$$|u(x)| \leq w(x)\, \lambda \max_{x \in \Omega} |u(x)|,$$

and thus, which completes the argument,

$$\lambda w(x) \geq 1.$$

□

The result can be interpreted in two ways: if, perhaps by symmetry considerations, it is possible to roughly predict the location that maximizes the mean first exit time, then the result allows for lower bounds on the eigenvalue $\lambda_1$ and, conversely, knowledge about the eigenvalue $\lambda_1$ guarantees the existence of points in the domain for which the mean first exit time is 'large'. Among other applications, the Donsker-Varadhan estimate is crucially used in the potential theoretic analysis of metastability in [1, 2, 3] (see Lemma 2.1 in [3] where the Lemma is quoted and an improvement in Lemma 2.2 in the same paper) and in Markov state models (see e.g., [8] and references therein).

## 2. The Result

The Donsker-Varadhan inequality is based on the mean value of the first exit time. We will work with quantiles of that distribution instead: for fixed $0 < p < 1$, we define the diffusion distance to the boundary $d_{p,\partial\Omega} : \Omega \to \mathbb{R}_+$ implicitly as

$$\mathbb{P}\left(\text{first exit time} \leq d_{p,\partial\Omega}(x_0)\right) = p,$$

where the probability is taken over drift-diffusion processes generated by $-L$ and started in $x_0$. Our main result is that there is a natural relation between that quantity and the smallest eigenvalue $\lambda_1$ of the differential operator.

**Theorem.** *Let $0 < \delta < 1$. If $|u(x)| = \delta \|u\|_{L^\infty}$, then*

$$d_{p,\partial\Omega}(x) \geq \frac{\log\left(\delta/p\right)}{\lambda}.$$

We are not aware of this result being in the literature. Related statements seem to have first appeared in [7, 9], a discrete analogue was given by Cheng, Rachh and the second author in [4]. A particularly interesting case is given by taking the limit $\delta \to 1$ and reorganizing the estimate. Note that a similar estimate could also be obtained in the discrete case, we leave the details to the reader.

**Corollary** (Donsker-Varadhan for Quantiles)**.**

$$\lambda_1 \geq \frac{\log(1/p)}{\sup_{x \in \Omega} d_{p,\partial\Omega}(x)}.$$

*Moreover, the right-hand side converges to $\lambda_1$ as $p \to 0$.*

We observe two major differences that become relevant when estimating $d_{p,\partial\Omega}(x)$ with a Monte Carlo method:

(1) instead of having to compute a mean (which, especially for heavy-tail distributions, can be difficult), it suffices to estimate the likelihood of exiting within a fixed time $t$. The desired outcome is a Bernoulli variable with likelihood $p$ – the problem thus reduces to estimating the parameter in a $\{0,1\}$ Bernoulli distribution and adjusting time $t$, which is more stable.
(2) By decreasing the value of $p$, the result can be arbitrarily refined – the difficulty being that estimating the parameter becomes more computationally costly as $p \to 0$, as one needs more simulations to ensure that there are enough samples in the $p-$th quantile to give a stable estimation of the Bernoulli parameter. In practice, the available amount of computation will impose a restriction on the value of $p$ that can be reasonably estimated with a certain degree of confidence.

## 3. Proofs

### 3.1. **Proof of the Theorem.**

*Proof.* We assume w.l.o.g. that $u(x) > 0$ and recall that $0 < \delta < 1$ is defined via $\delta\|u\|_{L^\infty} = u(x)$. We use $\omega(t)$ to denote drift-diffusion process started in $x$ and running up to time $t$. Since

$$-\mathrm{div}(a(x)\nabla u(x)) + \nabla V \cdot \nabla u = \lambda_1 u,$$

we have that

$$u(x) = e^{\lambda t}\mathbb{E}_\omega\left(u(\omega(t))\right)$$

with the convention that $u(\omega(t))$ is 0 if the drift-diffusion processes leaves $\Omega$ at some point in the interval $[0,t]$. Let now $t = d_{p,\partial\Omega}(x)$, in which case we see that

$$\mathbb{E}_\omega\left(u(\omega(t))\right) \le p\|u\|_{L^\infty} + (1-p)0.$$

Altogether, we obtain

$$\delta\|u\|_{L^\infty} = u(x) = e^{\lambda d_{p,\partial\Omega}(x)}\mathbb{E}_\omega\left(u(\omega(t))\right) \le e^{\lambda d_{\partial\Omega}(x)}p\|u\|_{L^\infty}$$

from which the statement follows. □

### 3.2. **Proof of the Corollary.**

*Proof.* It remains to show that the lower bound is asymptotically sharp as $p \to 0^+$. Let $x \in \Omega$ be arbitrary and let $\delta_x$ be the Dirac distribution centered at $x$. We are interested in the long-time behavior of applying the drift-diffusion process to these initial conditions; denoting the eigenpairs of the differential operator by $(\lambda_k, \phi_k)$, we can use the spectral theorem to estimate

$$\int_\Omega e^{(-\mathrm{div}(a(x)\nabla \cdot) + \nabla V \cdot \nabla \cdot)t}\delta_x dz = \int_\Omega \sum_{k=1}^{\infty} e^{-\lambda_k t}\left\langle \delta_x, \phi_k\right\rangle \phi(z)dz.$$

The specral gap implies that, as $t \to \infty$,

$$\int_\Omega \sum_{k=1}^{\infty} e^{-\lambda_k t}\left\langle \delta_x, \phi_k\right\rangle \phi(z)dz = \phi_1(x)e^{-\lambda_1 t} + o\left(e^{-\lambda_1 t}\right).$$

This means that, asymptotically as $t \to \infty$, the survival probability is maximized by starting in the point in which the first eigenfunction assumes a global maximum. Conversely, as $p \to 0$, we have that

$$(1 + o(1))\|\phi_1\|_{L^\infty} \exp\left(-\lambda_1 \sup_{x \in \Omega} d_{p,\partial\Omega}(x)\right) = p$$

and taking the logarithm then implies

$$(1 + o(1))\lambda_1 \sup_{x \in \Omega} d_{p,\partial\Omega}(x) = \log(1/p).$$

□

The main idea of the argument is that $p \to 0$ naturally corresponds to $t \to \infty$. The spectral theorem implies that long-time asymptotics is essentially given by the first eigenvalue and the first eigenfunction via

$$e^{-tL} f \sim e^{-\lambda_1 t} \langle f, \phi_1 \rangle \phi_1$$

and this is how we indirectly obtain estimates on $\lambda_1$.

## 4. Numerical Examples

4.1. **Unit interval.** A toy example is given by

$$\begin{aligned} -\Delta u &= \lambda u \qquad \text{in } [0,1] \\ u(0) &= 0 = u(1). \end{aligned}$$

The ground state is $u(x) = \sin \pi x$ and $\lambda_1 = \pi^2 \sim 9.86..$ – the Donsker-Varadhan estimate requires us to solve $-\Delta w = 1$, which easily gives $w(x) = x/2 - x^2/2$ and from which we get the lower bound $\lambda \geq 8$. In comparison, our bound for various values of $p$ (computed exactly by solving the equation in closed form) are

| $p$ | 1/2 | 1/4 | $10^{-1}$ | $10^{-2}$ | $10^{-8}$ | Donsker-Varadhan |
|---|---|---|---|---|---|---|
| lower bound | 7.28 | 8.40 | 8.92 | 9.39 | 9.74 | 8 |

4.2. **Unit interval with a quadratic potential.** Let us consider a $1D$ example with a quadratic potential $V = \frac{1}{2}x^2$ on $[-1, 1]$:

$$\begin{aligned} -\Delta u + x\nabla u &= \lambda u \qquad \text{in } [-1,1] \\ u(-1) &= 0 = u(1). \end{aligned}$$

The ground state is $u(x) = 1 - x^2$ with $\lambda = 2$. The mean first exit time $w$ solves

$$-\Delta w + x\nabla w = 1 \qquad \text{in } [-1,1]$$

with Dirichlet boundary condition. Solving the equation by central difference scheme with mesh size $h = 10^{-4}$ yields the Donsker-Varadhan estimate $\lambda \geq 1.678$. To use our bound for various value of $p$, we simulate $10^4$ paths using an Euler-Maruyama scheme with time step size $t = 10^{-4}$ starting at the origin (thanks to the symmetry), the following lower bounds are obtained.

| $p$ | 0.5 | 0.3 | 0.2 | 0.1 | 0.05 | Donsker-Varadhan |
|---|---|---|---|---|---|---|
| lower bound | 1.522 | 1.675 | 1.740 | 1.799 | 1.834 | 1.678 |

4.3. **Unit disk.** Finally, we estimate the ground state of the Laplacian on the unit disk in $\mathbb{R}^2$, which is given by the first nontrivial zero of the Bessel function $\lambda_1 \sim 2.40\dots$ while the Donsker-Varadhan estimate gives

$$w(x) = 1/2 - (x^2+y^2)/2 \quad \text{and thus} \quad \lambda_1 \geq 2.$$

Suppose we could not solve any of these equations in closed form (as is usually the case): using the symmetry of the domain, it suffices to take Brownian motion started in the origin. Discrete Brownian motion with step size (in time) $t = 10^{-4}$ and $10^4$ paths give the following estimates for a lower bound on $\lambda_1$

| $p$ | 0.5 | 0.4 | 0.3 | 0.2 | 0.1 | Donsker-Varadhan |
|---|---|---|---|---|---|---|
| lower bound | 1.68 | 1.85 | 2.04 | 2.19 | 2.37 | 1.96 |

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