Essays in Macroeconomics and Entrepreneurship

by

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Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Economics
in the Graduate School of Duke University
2019

ABSTRACT

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Abstract

This dissertation is comprised of three chapters in macroeconomics, entrepreneurship, and heterogeneous agent models. In the first chapter, I answer two main questions — What are the empirical facts regarding entrepreneurial investment choices, and to what extent do the investment choices of entrepreneurs help to explain top wealth inequality. To that end, I document some novel facts about entrepreneurial investment dynamics. I show that these facts are suggestive that capital illiquidity are an important friction with regards to entrepreneurial investment choices. To quantify these frictions, I construct a new general-equilibrium heterogeneous agents model of entrepreneurship that features illiquid investments. I calibrate this model to identifying features of the data, and find a large role for illiquidity. I also find that the investment and savings choices of entrepreneurs help explain a substantial fraction of wealth inequality. Counterfactual analysis shows that the illiquidity friction generates substantial welfare and productivity losses by allocating wealth away from high productivity entrepreneurs to low productivity entrepreneurs, which simultaneously leads to lower wealth inequality. As such, I find that a policy of partial insurance against illiquidity risks can help ameliorate these losses, and simultaneously increases wealth inequality. In my second chapter, I present new evidence regarding the effect of uncertainty shocks on firm startup and exit rates. I document that uncertainty shocks are strongly and negatively correlated with firm startup rates, but essentially uncorrelated with exit rates. I show how my model of illiquid entrepreneurial investments can help explain these facts, and argue that capturing the extensive margin of adjustment to uncertainty shocks is important in amplifying and propagating the ef-

fects of uncertainty shocks. Finally, in my last chapter, I present a new computational algorithm to compute distributions in heterogeneous agent models. I show how this algorithm improves on current methods by reducing the amount of computational memory and time required, and provide a simple and intuitive explanation as to how this algorithm improves on the textbook method.

Acknowledgements

I am indebted to my advisors, Andrea Lanteri and Craig Burnside, for their tireless support and advice throughout this process. This dissertation would have been impossible without their help and guidance, and I could not have asked for better advisors in my career. I would also like to thank my extended "Duke Economics family" — Matthias Kehrig, Cosmin Ilut, Kyle Jurado, Pietro Peretto, Francesco Bianchi, Daniel Xu, and Allan Collard-Wexler for their helpful comments and advice at various points of this journey. I will miss running into you guys at Saladelia. Also, a shout out to Chris Timmins — My career as a macroeconomist would literally not have started without your introduction of Cooper and Haltiwanger (2006) to me in my first summer as a PhD student. Also another shout out to the folks at "213" and Ecoteach — Thank you for helping make this journey much less arduous than it could have been! Outside of Duke, I would like to thank the Federal Reserve Bank of St Louis for hosting me during the writing of my first chapter, and the Kauffman Foundation for providing sponsorship to access the Kauffman Firm Survey at the NORC enclave. To my friends that have at various points walked this journey with me — Thank you for the laughter and joy that has made this possible. Finally, and most importantly, I would like to thank my amazing wife, Amanda Schwantes, for tirelessly supporting me when things seemed impossible (and the numerous puppy and panda photos!).

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1

Introduction

Entrepreneurs are often considered the "engines" of economic growth, making the study of entrepreneurial investment dynamics an important dimension for economic research. This dissertation studies a crucial aspect of entrepreneurial investment dynamics that has not been well studied: That entrepreneurial investment is often illiquid, and that entrepreneurs generally bear the full risk of their businesses during the initial stages of business formation. In the first chapter of this dissertation, I first provide strong empirical evidence suggesting the importance of investment illiquidity for entrepreneurial startup and investment choices. I then construct a calibrated general-equilibrium macroeconomic model to understand the impact of illiquidity on the overall macroeconomy, in particular focusing on its interaction with the wealth distribution. In the second chapter, I then build on this framework to study how innovations to idiosyncratic subjective uncertainty can interact with investment illiquidity to explain a stark fact in the data: that entrepreneurial entry rates are highly negatively correlated with policy uncertainty, but exit rates are essentially uncorrelated with variations in policy uncertainty. I then show how my model of entrepreneurship can amplify and propagate the recessionary effects of uncertainty shocks.

In these two papers, as well as the bulk of newer heterogeneous agent models, the

state space of the model often includes a non-degenerate cross-sectional distribution of idiosyncratic states. As models increase in their complexity, the size of the state space often grows disproportionately, making it challenging to compute the cross-sectional distribution with standard computing resources. In my final chapter, I show how a single simple modification to the standard power iteration method used in macroeconomics can drastically reduce the computational resources and time required required to solve these models. Examples are provided to show how this method is generically applicable to a wide range of models.

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Entrepreneurial Investment Dynamics and the Wealth Distribution

2.1   Introduction

Several studies have documented that entrepreneurs account for a large share of wealth in the economy, and also constitute a large fraction of the population at the top of the wealth distribution[1]. Are business owners rich because they are rewarded for taking risk, or do the wealthy simply pursue entrepreneurship disproportionately regardless of innate talent? Moreover, what investment or disinvestment frictions do these entrepreneurs face, and how do they relate to the efficiency of the distribution of wealth? These questions have potentially important implications for policy reforms intended to promote entrepreneurship or reduce wealth inequality[2].

In this paper, I answer these questions by documenting new facts regarding entrepreneurial investment behavior, and rationalize them using a quantitative macroeconomic model of entrepreneurship. I develop a rich general-equilibrium model of

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[1] For instance, Cagetti and De Nardi (2006) report that entrepreneurs account for 7.6% of the population, own 33% of its wealth, and represent 54% of the top 1% in the wealth distribution. This phenomenon is also reported in Section 2.2 of this paper.

[2] Yellen (2014), for instance, raises this point.

entrepreneurial choice, in which heterogeneous individuals decide, depending on their ability and portfolio of assets, whether to pursue entrepreneurship or become a worker, and how they should allocate their wealth. A key feature of the model is that households face incomplete insurance markets with regards to their income risk, and entrepreneurs face both investment and disinvestment frictions arising from collateral constraints and resale frictions. This rich framework allows me to distinguish between these two frictions, which I use to match the new facts documented in this paper.

This paper makes three main contributions to the literature. First, I provide new direct empirical evidence on entrepreneurial investment behavior. Using data drawn from the Kauffman Firm Survey (KFS), a nationally representative panel survey on startups, I find that the distribution of the returns to capital[3] of young entrepreneurial firms is highly dispersed, and skewed toward firms with low returns to capital (i.e., left-skewed). Moreover, while the returns to capital exhibit significant persistence across the distribution, the persistence is much higher for entrepreneurs with low returns to capital (relative to their high performing counterparts).

The second contribution of this paper is to develop a quantitative macroeconomic model of entrepreneurship that can rationalize and interpret these facts, and thus provide a deeper and nuanced understanding of the relationship between entrepreneurship and wealth inequality. While my calibration exercise focuses only on matching entrepreneurial investment choices, I find that this model broadly replicates the high wealth dispersion in the economy, as well as the high wealth concentration among entrepreneurs. For instance, the top 1% own about 23% of the wealth in the model economy, and about 30% in the US economy; similarly, entrepreneurs own about 32% of the wealth in the model, and about 33% of the wealth in the US economy. This outcome provides support for the argument that entrepreneurial investment and savings choices explain a substantial amount of the wealth dispersion in the data.

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[3] As proxied by the log average revenue product of capital.

Through the lens of my calibrated model, I find that entrepreneurs face substantially more acute downsizing frictions than investment frictions. Specifically, I find that for every unit of capital sold, the entrepreneur loses 69% of the underlying real value of the capital asset. Upon exit, the entrepreneur faces an additional 31% write down on her assets[4]. In contrast, the entrepreneur can collateralize up to 93% of the value of her capital; however, due to the low resale price of capital, only 29% of new investment can be debt financed.

High resale frictions imply that entrepreneurial capital is very illiquid, thus generating illiquidity risk that magnifies the underlying exogenous risks of entrepreneurship that arise from incomplete financial markets. This in turn leads to a poor allocation of talent and capital relative to a counterfactual economy without illiquidity risk. Along the intensive margin, wealthier low productivity entrepreneurs operate larger firms, while poorer high productivity entrepreneurs operate smaller firms; along the extensive margin, poorer high productivity potential entrepreneurs delay entry, while richer low productivity incumbents delay exit. As a result of this, I find that a substantial fraction of wealth is allocated toward low productivity entrepreneurs and away from high productivity entrepreneurs. Consequently, a removal of resale frictions can lead to substantial efficiency gains, in which steady-state aggregate TFP in the entrepreneurial sector increases by 11%, while average welfare, factoring in transitional dynamics, is 4.4% higher in consumption equivalent terms. Moreover, although wealth becomes even more concentrated when capital is fully liquid, this greater wealth concentration reflects an improved allocation of wealth to high productivity entrepreneurs.

The last contribution relates to fiscal policy. In addition to the poor allocation of capital and talent, illiquidity risks also lead entrepreneurs to accumulate more liquid

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[4] While these frictions might seem large, they are comparable to prior research in both the household dynamics (c.f. Diaz and Luengo-Prado (2010), Kaplan and Violante (2014) and Berger and Vavra (2015)) and firm dynamics (c.f. Cooper and Haltiwanger (2006), Bloom (2009), and Gilchrist et al. (2014)) literature.

assets as a buffer stock (relative to the counterfactual economy)[5]. Consequently, fiscal policy that provides partial consumption insurance against this source of risk can improve economic efficiency. For instance, I find that a policy that helps entrepreneurs defray the cost of downsizing[6], when funded by a proportional tax on bond returns, can improve economic outcomes in terms of welfare and total factor productivity. These outcomes accrue from both the direct effect of insurance provision and a general equilibrium effect that raises wages.

2.1.1 Related Literature

This paper contributes to several strands of research on entrepreneurship.

First, this paper is most directly related to the literature that studies how entrepreneurial investment and savings can explain the high wealth inequality in the data. As in Quadrini (2000) and Cagetti and De Nardi (2006), I document that entrepreneurs are overrepresented among the richest households in the United States, and construct a model of entrepreneurial choice and savings to rationalize these facts. These papers find that entrepreneurship can explain the high wealth inequality, by directly calibrating their models to match the empirical wealth distribution. However, they do not assess the performance of their models in matching entrepreneurial investment behavior in the data. In this paper, I provide direct empirical evidence regarding the investment behavior of entrepreneurs, and calibrate my model directly to investment data on entrepreneurs. I contribute to this literature by showing that an entrepreneurship model disciplined only by micro-level evidence on entrepreneurial investment can still broadly replicate the empirical wealth distribution, and thus provide empirical support for this strand of literature.

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[5] This effect is reminiscent of the question studied by Aiyagari (1994), who examines whether precautionary savings lead to savings in excess of the complete markets representative agent model.

[6] This proposed policy is similar in spirit to a capital loss tax subsidy, like in Section 1231 of the US tax code.

Second, this paper contributes to the large literature that studies how financial frictions affect entrepreneurial outcomes. For instance, Cagetti and De Nardi (2006), Buera and Shin (2011), Buera and Shin (2013), Buera et al. (2011), Midrigan and Xu (2014) and Bassetto et al. (2015) have documented that financial frictions can generate substantial ex post misallocation of resources by distorting the allocation of entrepreneurial talent and capital. A connecting thread across this literature is their focus on collateral constraints as the source of financial frictions, and a common assumption that an entrepreneur's business assets are perfect substitutes for liquid assets such as bonds. In this paper, I depart from that framework by distinguishing between liquid savings wealth and illiquid business wealth, and study how asset illiquidity, when interacted with incomplete financial markets, also generates substantial ex post misallocation of resources. Similar to prior literature, asset illiquidity leads to large numbers of undercapitalized high productivity entrepreneurs, as well as larger numbers of lower productivity but rich entrepreneurs. However, unlike the prior literature, the economic mechanism arises from heightened risk as a result of illiquidity. This mechanism thus generates a second form of misallocation, where undercapitalized high productivity entrepreneurs save too much in liquid assets as self-insurance against illiquidity risk. This latter mechanism is absent in the framework employed by prior papers.

Recent papers in the firm dynamics literature, such as that by Hsieh and Klenow (2009), Asker et al. (2014) and Gopinath et al. (2017), have noted that the dispersion in returns to capital can reflect factor misallocation across firms. Moreover, these and other papers have tried to identify specific mechanisms to explain this observation. For instance, Asker et al. (2014) propose a mechanism of dynamically chosen inputs and capital adjustment cost, while Midrigan and Xu (2014) and Gopinath et al. (2017) propose a mechanism of financial frictions (collateral constraints). In general, these frictions are studied in isolation, in the context of a representative household,

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or in partial equilibrium. My paper contributes to this literature by studying both financial and real adjustment frictions in combination, and in the context of a general-equilibrium heterogeneous household model. This framework allows me to connect the dispersion in returns to capital to the dispersion in wealth across households, and in particular, show how real adjustment frictions can affect wealth inequality. Moreover, it allows me to map the effect of capital misallocation into welfare outcomes through both a partial and general equilibrium effect — a dimension of capital misallocation that has not been discussed in prior literature.

This paper also contributes to the broader firm dynamics literature that has sought to identify or estimate the extent of financial and real frictions via a model-driven indirect inference approach (see, for instance, Cooper and Haltiwanger (2006), Asker et al. (2014), Lanteri et al. (2018), and David and Venkateswaran (2018)). This paper adds to that literature by developing and implementing a simple diagnostic to disentangle financial frictions (driven by collateral constraints) and illiquidity effects driven by partial irreversibility by matching the dynamic properties of the distribution of returns to capital. I also show that this strategy is an improvement over standard methods that identify the extent of collateral constraints by targeting the leverage ratios of firms (as, for instance, in Khan and Thomas (2013) and Midrigan and Xu (2014)). I show that explicit targeting of the leverage ratio has a potential to overestimate the tightness of collateral constraints when illiquidity risks are ignored.

Finally, Kaplan and Violante (2014) in a recent strand of literature on household dynamics, have noted that standard "one asset" models, in which households only trade in a fully liquid asset, cannot fully rationalize the consumption dynamics of households. In that literature, the authors propose a new paradigm by which households hold a portfolio of liquid and illiquid assets. Within that framework, illiquidity restricts the ability of households to smooth consumption. This paper relates to that literature, in which, in the context of my framework, illiquid entrepreneurial busi-

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ness assets allow the model to rationalize the investment dynamics of entrepreneurs. However, unlike in their paper, in which the illiquid asset simply pays a deterministic linear return, the illiquid asset in my framework is in fact a firm with stochastic productivity that partly reflects the talent of the entrepreneur. As a result, the inability of households to smooth consumption compounds, in equilibrium, into a misallocation of talent and wealth.

The rest of the paper proceeds as follows. In Section 2.2, I summarize and describe the key facts regarding the investment choices of entrepreneurs. The findings presented in Section 2.2 will provide the motivation for the model, which I describe in Section 3.2. In Section 3.3, I discuss in detail the calibration strategy and outcomes; in particular, I explain how my model can discriminate between illiquidity frictions and collateral constraints, and why this method is preferred over standard methods. Section 2.5 then discusses the implications of the model, and Section 2.6 extends the model to study how a policy of partial insurance can improve allocative efficiency in an economy subject to illiquidity risks. Section 2.7 concludes.

2.2 Stylized Facts

In this section, I document key facts regarding the wealth of entrepreneurs and their investment characteristics. These facts are drawn primarily from the Kauffman Firm Survey, and are used to form the core of the evidence I will present regarding the investment characteristics of entrepreneurs. However, as the Kauffman Firm Survey only samples a single cohort of entrepreneurs, and has relatively sparse evidence on the wealth of their owners, I will supplement these facts with data drawn from the Panel Survey of Income Dynamics.

In the following subsection, I begin by briefly discussing the two data sources and key conceptual definitions of the terms used. Following that, I report on the broad wealth characteristics of entrepreneurs. In particular, I show how entrepreneurs

account for a large share of wealth in the economy, and why it is important to study the frictions that impede the ability of entrepreneurs to reallocate their business investments. Finally, I report the facts regarding entrepreneurial investment behavior. In particular, the evidence will provide support for the proposition that entrepreneurs are likely to be strongly affected by capital illiquidity, and in particular, more so than by financing constraints.

2.2.1 Data and Definitions

The Kauffman Firm Survey

The Kauffman Firm Survey (KFS) is a single cohort panel survey, consisting solely of firms that were formed in the year 2004 and tracked through 2011. The universe of firms considered for survey inclusion was all newly registered firms in 2004 from the Dun and Bradstreet database, followed by a series of conditions that are detailed further in Appendix A.1. "Entrepreneurs", in the context of the KFS, therefore refers to this population of entrant firms.

The KFS is a large survey, and provides extensive details regarding their survey respondents. For the purposes of studying entrepreneurial investment, I focus on the assets and liabilities of the firm. Descriptive summary statistics of this sample[7] are reported in Table 2.1. Since this paper is concerned with entrepreneurial investment behavior, I focus only on the subset of entrepreneurs who simultaneously report both positive revenue and capital stock. For a full description of the data construction, I refer the reader to Appendix A.1. For readers who are more interested in the broader characteristics of this data set (for instance, in firms that are pre-revenue), Robb and Robinson (2014) provide a detailed breakdown of the characteristics of the data.

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[7] Note that all variables are reported in real terms, and certain variables, such as the capital stock, are constructed from a subset of the firm's portfolio of assets. In addition to these adjustments for prices, however, these statistics are computed using "raw" variables without further adjustments. Therefore, the statistics reported here might not correspond to the statistics provided later in this section.

Table 2.1: Summary statistics of the Kauffman Firm Survey.

	10th percentile	Median	90th perentile	Mean	Coefficient of Variation
Revenue	$5,363	$100,000	$1,231,601	$947,550	12.43
Capital stock	$2,534	$10,099	$511,367	$541,430	12.6
Employment	0	1	8	3.67	2.8
Growth rate of capital	-0.89	0.99	3.52	21.8	43.2
Total net liquid asset	$0	$2,500	$49,394	$43,486	79.4

A big advantage of using the KFS in this paper is the ability to directly observe the investment choices of new and privately owned firms (via the firm's balance sheet) over a long period of time. As a result, the findings in the KFS can be directly mapped onto a model that jointly describes heterogeneous household and firm dynamics. In contrast, prior literature that has studied entrepreneurial investment has generally exclusively used household surveys such as the Panel Survey of Income Dynamics[8]. Unlike the KFS, important information on returns to firm assets, such as the marginal product of capital, is not computable using household surveys as households do not report the book value of the firm's asset[9]. Moreover, the PSID sample has substantial attrition. As a result, it is difficult to draw substantive inference on the dynamics of capital accumulation (or decumulation), which requires a long panel survey.[10]

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[8] See for instance, Quadrini (2000) or Cagetti and De Nardi (2006).

[9] The Panel Survey of Income Dynamics typically asks households for their perceived value of their firm. A typical question, such as in the 2013 wave, asks respondents, "How much is your part of the business worth, that is, how much would it sell for?".

[10] There is also a small but growing literature that uses Census microdata, such as the Longitudinal Business Database (LBD) or the Longitudinal Employer-Household Dynamics (LEHD), to study new firm behavior (see, for instance, Choi (2018)) Unfortunately, census microdata in general do not provide much information on the balance sheet of private firms. As a result, while it is possible to match a primary owner to her/his firm, we are not able to study the returns to firm assets using these data sets.

The Panel Survey of Income Dynamics

The Panel Survey of Income Dynamics (PSID) is a nationally representative survey that was conducted annually in the United States from 1968 to 1997, and every two years thereafter. The statistics I report are drawn from the samples from 2003 to 2013, but the main statistics of interest I use in the calibration will be drawn from the 2003 sample, which will keep the sample characteristics as close to the KFS sample as possible. "Entrepreneurs", in the context of the PSID, are defined as all self-employed business owners, following the definition used in Cagetti and De Nardi (2006). In this paper, the primary role of the PSID is to allow me to look at aggregate statistics, such as the entry and exit rate of entrepreneurs, and the number of entrepreneurs in the population. I will also use the wealth supplement of the PSID to examine the wealth distribution, and to make the case that studying the wealth accumulation dynamics of entrepreneurs is indeed important.

How comparable are the KFS and PSID? A key question this paper addresses is how the investment choices of entrepreneurs feed back into the wealth dynamics of these entrepreneurs. Unfortunately, for most of the sample years, the KFS does not provide substantial information regarding the personal wealth of the entrepreneurs. However, as we will see later in this section (specifically, in Figure 2.1), for the years for which wealth information is available, the wealth distribution in the KFS and PSID broadly align. As such, this provides confidence that the KFS is a reliable source for building a model of entrepreneurship and household wealth.

2.2.2 Evidence on the Wealth of Entrepreneurs

In this subsection, I briefly discuss why it is important to develop a framework that matches both an entrepreneur's investment behavior and her wealth accumulation dynamics.
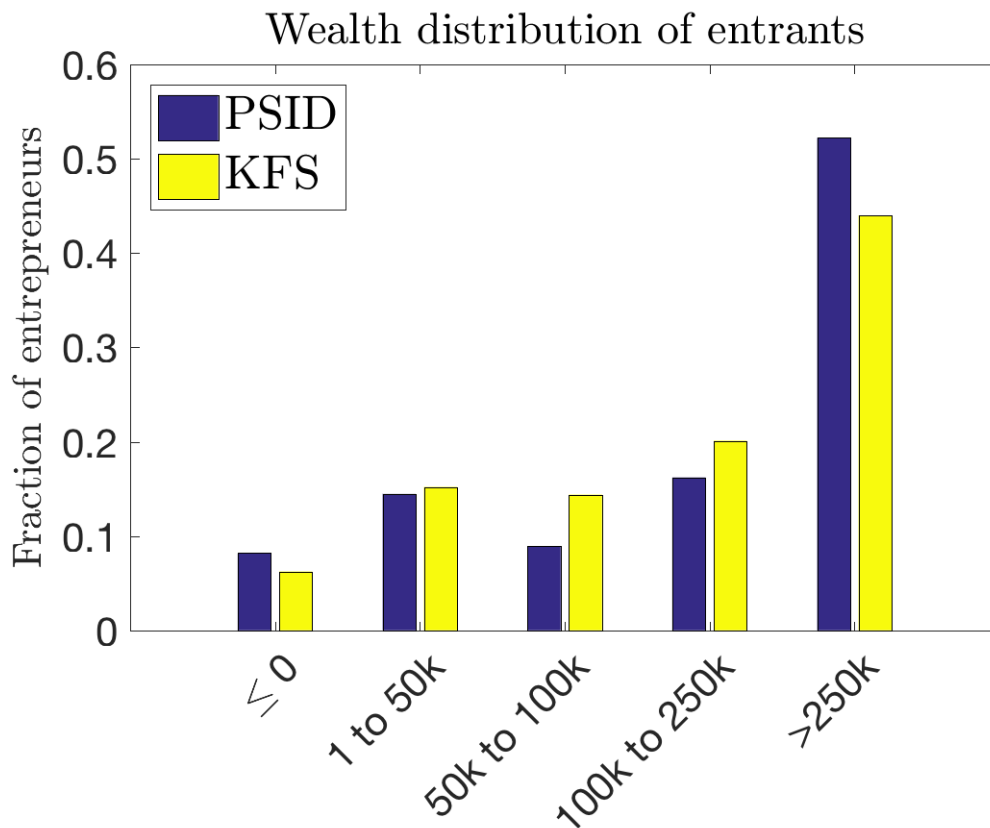
FIGURE 2.1: The number of entrepreneurs within a wealth bin, reported as a fraction of total number of entrepreneurs, in the KFS (yellow bars) and PSID (blue bars).

As has been documented in prior literature (for instance, in Cagetti and De Nardi (2006)), the wealth distribution isn't simply highly concentrated; it is also especially concentrated among entrepreneurs[11]. In particular, entrepreneurs are increasingly overrepresented as we move up the wealth distribution. In Figure 2.1, I report the fraction of all entrepreneurs who belong to a given wealth bin for entrepreneurs in both the KFS and the PSID. As we can see, the majority of entrepreneurs fall into the highest wealth bin[12].

Unfortunately, the KFS is limited in its information regarding the wealth distri-

[11] See in particular, tables 1 and 3 of Cagetti and De Nardi (2006). In their paper, entrepreneurs in 1989 make up 7.6% of the population, but accounted for 33% of its wealth.

[12] In both cases, wealth is constructed as total net wealth. Wealth bins correspond to the five choices given to survey respondents in the KFS in the years 2008 to 2011. For this figure, I constructed the same corresponding wealth bin in the PSID for entrant entrepreneurial households for the years 2003 to 2013, and computed the average over all six waves.

bution of entrepreneurs, as it only provides information for these five categories[13]. To conduct a more extensive analysis, I now turn to the PSID for a finer breakdown of the wealth distribution. Following the same method used in constructing Figure 2.1, I now compute the number of entrepreneurs who make up a given wealth percentile, as a fraction of the total number of individuals within that given percentile. Results are reported in table 2.2. There, we see a similar pattern, by which the fraction of individuals who are entrepreneurs increases as we move up the wealth distribution. For example, in 2003, entrepreneurs make up 7.9% of the population, but represent 53% of the households in the top 1%. The sharp wealth concentration among entrepreneurs suggests that understanding the factors that affect the wealth accumulation dynamics of entrepreneurs could indeed be important in understanding how efficiently wealth is being allocated across the population.

Table 2.2: Fraction of individuals in each wealth percentile who are entrepreneurs, for the years 2003 to 2013

	Number of entrepreneurs in each wealth percentile				
Year	Total population	Top 20%	Top 10%	Top 5%	Top 1%
2003	0.079	0.210	0.278	0.371	0.530
2005	0.071	0.182	0.248	0.319	0.390
2007	0.075	0.184	0.255	0.331	0.421
2009	0.073	0.192	0.254	0.295	0.460
2011	0.073	0.193	0.246	0.326	0.478
2013	0.067	0.170	0.215	0.270	0.384

However, to the extent that an entrepreneur's business wealth is only a small component of her overall wealth, the fact that entrepreneurial capital is illiquid might seem irrelevant. To the contrary, the portfolios of entrepreneurs are in fact highly undiversified. In Figure 2.2, I report the fraction of an entrepreneur's portfolio that comes from business wealth for each percentile range. As we can see, entrepreneurs

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[13] That said, $250,000 in net wealth corresponds to the top 35% of households. This means that about half of entrant entrepreneurs belong to the top 35%.
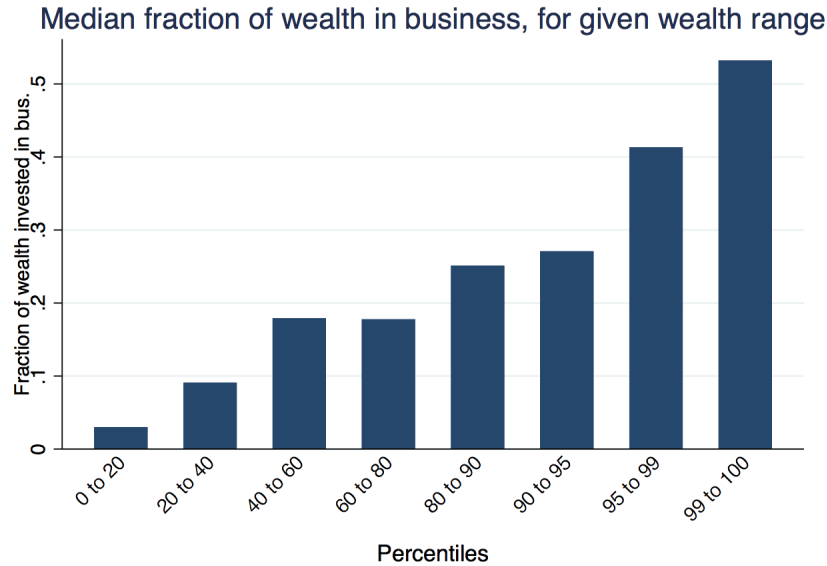
FIGURE 2.2: Fraction of the entrepreneur's portfolio that comes from business wealth, within each wealth percentile range. Values are reported for the median within each percentile range.

are generally underdiversified[14]. Moreover, the level of underdiversification broadly increases as we move up the wealth distribution: The median entrepreneur in the bottom 20% has about 3% of her wealth invested in her business; in contrast, the median entrepreneur in the top 1% has about 53% invested in her wealth. This suggests that the illiquidity of business assets is indeed an important dimension for study.

2.2.3 Evidence on the Investment Dynamics of Entrepreneurs

In this section, I present key facts regarding entrepreneurial investment behavior, both in the cross-section and across time. In particular, I focus on the distribution of investment rates and the distribution of the (log) average revenue product of capital (ARPK), where ARPK is used here as a proxy for the returns to capital. Investment rates are defined as the ratio of current period investment to beginning-of-period capital stock, while ARPK is defined as the ratio of current period value added

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[14] This fact has also been reported in prior research, such as that of Moskowitz and Vissing-Jorgensen (2002).

to beginning-of-period capital stock. In both cases, the variables are residualized measures to account for cross-industry heterogeneity. For a more detailed description of variable creation, I refer the reader to Appendix A.1.

Fact I: Firm-level investment rates are lumpy and volatile

Figure 2.3 shows the distribution of investment rates ($\frac{I}{K}$), and Table 2.3 reports some of the key moments associated with this. The investment distribution presented in Figure 2.3 is winsorized at the 1st and 95th percentiles for clarity of presentation, while the moments in Table 2.3 are constructed for the distribution winsorized at the 1st and 99th percentiles.
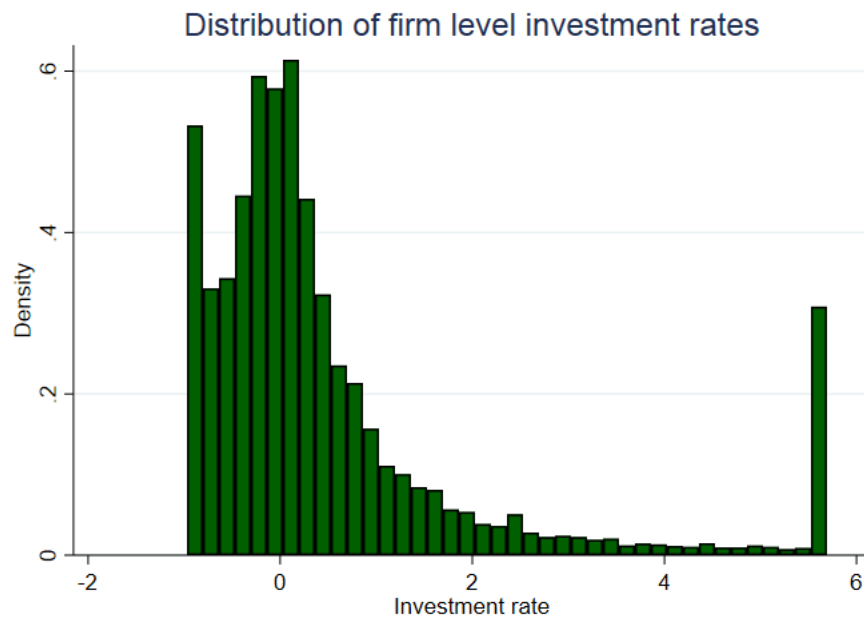
FIGURE 2.3: Distribution of investment rates (winsorized at the 5th and 95th percentiles)

Table 2.3: Selected moments from the distribution of investment rates

Moments of $\frac{I}{K}$	Value
Mean	1.15
Standard deviation	4.74
5th percentile	-0.85
Median	0.096
95th percentile	5.65
Autocorrelation	-0.069
Skewness	6.2
Kurtosis	46.2
% of $\frac{I}{K} > 0$	55.6%
% of $\lvert\frac{I}{K}\rvert < \frac{1}{4}$	30.0%

As Figure 2.3 shows, the distribution of investment rates is highly right-skewed, along with a very long right tail. The 95th percentile of investment rates is around 5.7; that is, at the 95th percentile, firms invest up to 5.7 times their capital stock in the same year. In contrast, the 5th percentile of investment rates is around -0.87, which corresponds to the firm owner's essentially selling all her capital stock. Across the entire sample of firm-year observations, 44.9% of firms are downsizing and 55.1% are increasing their capital stock. Broadly speaking, this observation is reflective of the results on investment moments found in the firm dynamics literature, such as the seminal paper by Cooper and Haltiwanger (2006); that is, investment at the firm level is typically very lumpy. As noted by Cooper and Haltiwanger (2006), lumpy investment behavior at the firm level often indicates that the adjustment cost function of the firm features a significant fixed cost element. The data therefore provide suggestive evidence that entrepreneurial capital is indeed highly illiquid.

One might notice that the investment rate distribution features a standard deviation that is much larger than that in the firm dynamics literature (for example, the standard deviation of investment rates in Cooper and Haltiwanger (2006) is 0.33). This is unsurprising, given the scale that entrepreneurial firms operate at. The median capital stock is $10,099 (the mean is $541,430, reflecting the sizable right-skewness of

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the firm size distribution). At the 95th percentile, where $\frac{I}{K} \approx 5.65$, this means that the median firm is investing around \$57,059. While this is sizable for many individuals, in terms of a business investment, it is small. For example, this could correspond to a small moving firm's simply buying a new truck.

Fact II: The distribution of the average revenue product of capital is highly dispersed and left-skewed

The distribution of log ARPK among entrepreneurial firms is highly dispersed, as we can see in Table 2.4. This observation is similar to the findings in other contexts[15], which has pointed out that the dispersion of ARPK can be indicative of ex post misallocation of capital across firms. In addition to this observation, I find that the distribution of log ARPK is highly left-skewed. In Figure 2.4, I plot the distribution of log ARPK and overlay a normal distribution with the same standard deviation as reference; Table 2.4 below documents some key cross-sectional moments of this distribution. In particular, I report three measures of skewness for the distribution of $\log ARPK$, including two quantile-based skewness measures that are considered to be more robust than the standardized third moment[16]. As we can see, the pooled distribution is left-skewed, regardless of the measure of skewness. This left-skewness is also reflected visually in Figure 2.4. In Appendix A.2.1, I show further that the left-skewness is a robust feature of the data.

Table 2.4: Selected moments from the distribution of residualized $\log ARPK$. Note that the mean is 0 by construction. The Kelly skewness measure uses the 90th and 10th percentiles, while the Bowley measure uses the 75th and 25th percentiles.

Mean	Standard Deviation	Skewness		
		Pearson	Kelly	Bowley
0.00	1.75	-0.39	-0.12	-0.09

[15] See, for instance, Hsieh and Klenow (2009), Asker et al. (2014) and Gopinath et al. (2017).

[16] See, for instance, the discussion in Kim and White (2004).
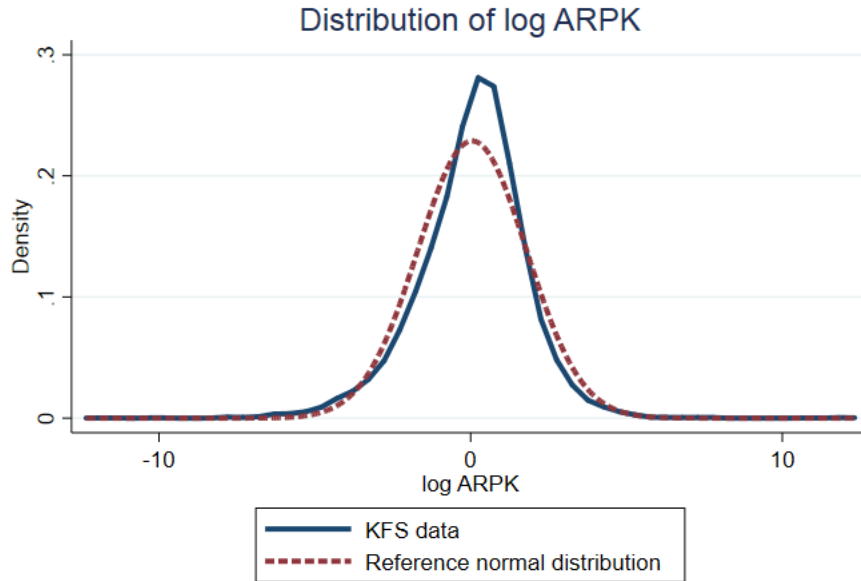
FIGURE 2.4: The distribution of log ARPK. A reference normal distribution is overlaid on the data.

Note that by the definition of ARPK, this measure can serve as a proxy for the firm's return to capital[17]. Therefore, firms in the left tail of the distribution have relatively low returns to capital, and are thus looking to downsize; in contrast, firms in the right tail of the distribution have relatively high returns to capital, and hence are looking to expand. The left-skewness of this distribution therefore suggests that there are relatively more firms with low returns to capital.

As will be further discussed in Section 3.3, this is a relatively surprising finding, given the commonly held notion of "financially-constrained entrepreneurs"[18]. Within that framework, financial constraints exacerbate the difficulty high productivity entrepreneurs encounter in fully capitalizing their firm. As a consequence, this should generate a right-skewed distribution of returns to capital; that is, there are many

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[17] This follows prior literature that uses the marginal revenue product of capital to proxy for the rate of returns to capital (see, for instance, Hsieh and Klenow (2009)). As in that literature, this requires an assumption that firms use a Cobb-Douglas production function. With that assumption, ARPK is then simply proportional to MRPK.

[18] As an example of this large literature on financing constraints and entrepreneurship, I refer the reader to Evans and Jovanovic (1989), Cagetti and De Nardi (2006), and Schmalz et al. (2017).

more firms with high returns to capital. Instead, the data suggest that the opposite might be happening: Many low productivity firms are finding it difficult to downsize. However, to the extent that the distribution of the ex post returns to capital reflect innovations to the productivity process of the firm, the left-skewness of ARPK might simply reflect the fact that many entrepreneurs have bad ideas (i.e., so the distribution of productivity is also left skewed). To provide further evidence that this is not the case, I now present a third fact of entrepreneurial investment.

Fact III: Firm ARPK is (a) highly persistent and (b) more persistent in the left tail of the distribution than the right tail.

To document this fact, I employ a nonparametric approach by estimating a transition matrix[19] for $\log ARPK$. Here, I estimate the persistence in relative rankings by first binning the firms into quintiles (estimated at the industry level) on a year-by-year basis[20], and then estimating the transition matrix (across quintiles) for the entire sample. I report the estimated transition matrix below, with standard errors in parentheses below the estimated value.

To clarify the import of these results, I direct the reader to values along the diagonal, which reflect the probability of staying in quantile q, given that you were in quantile q yesterday. As such, a *null* hypothesis of no persistence should reflect that this probability is 0.2. In contrast, looking across all the values along the diagonal, we see that all values are statistically different from (greater than) 0.20, thus demonstrating that ARPK exhibits persistence. Moreover, there is also a significant asymmetry in the persistence of $\log ARPK$ in the left and right tail: The persistence of $\log ARPK$, conditional on being in the first quintile, is statistically and econom-

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[19] An alternative method is to estimate this asymmetry by estimating the conditional autocorrelation of $\log ARPK$. This exercise is done as a robustness check, and reported in appendix A.2. However, the transition matrix approach is preferred due to its nonparametric assumptions.

[20] I have also estimated the transition matrix by fixing the quintiles to the bins in 2004. The results are very similar. Refer to appendix A.2 for additional robustness checks.

Table 2.5: Transition probabilities of log ARPK. Rows might not sum to 1, as entries have been rounded to two significant figures.

		Quintile tomorrow				
		1	2	3	4	5
	1	0.58 (0.017)	0.24 (0.015)	0.096 (0.010)	0.046 (0.007)	0.042 (0.007)
Quintile today	2	0.23 (0.014)	0.39 (0.016)	0.22 (0.014)	0.11 (0.011)	0.057 (0.007)
	3	0.093 (0.010)	0.21 (0.014)	0.37 (0.016)	0.22 (0.014)	0.010 (0.008)
	4	0.058 (0.008)	0.11 (0.010)	0.21 (0.013)	0.37 (0.016)	0.26 (0.014)
	5	0.057 (0.007)	0.073 (0.009)	0.12 (0.011)	0.28 (0.015)	0.47 (0.017)

ically significantly larger than the persistence of $\log ARPK$, conditional on being in the last quintile.

What is the economic significance of this outcome? As will be further discussed in Section 3.3, a standard frictionless model predicts no persistence in ARPK; therefore, through the lens of that model, any persistence in ARPK must in turn reflect capital reallocation frictions. In the context of an economy in which collateral constraints are the sole source of distortions, the data should then reflect higher persistence in the right tail of the distribution. Since collateral constraints only serve to impede capitalization of the firm, this implies that firms with high ex post returns will persist in this state longer. For firms in the left tail — that is, firms that have relatively low returns — these firms generally are not in need of capitalization. As a result, a model of only collateral constraints will predict that persistence in the left tail should be lower than that in the right tail. The fact that firms with low returns are persisting longer in that state therefore suggests that downsizing frictions might be the predominant source of distortion in this economy.

It is important to note that the proposition to be discussed later in Section 3.3 (regarding the lack of persistence of ARPK in a frictionless model) rests only on the idea that, conditioned on all current measurable state variables, the ex post

distribution of returns to capital is simply a random forecast error. This proposition is very general, and can encompass any Markov productivity process. However, as a simple rationality check, I also estimate the transition matrix for value added, as reported in Table 2.6. Recall that ARPK is simply the ratio of the firm's value added to its capital stock; therefore, to the extent that a productivity process with higher left-tail persistence is generating this result, we should also see that value added would feature higher left-tail persistence. As we can see in Table 2.6, this is not the case in the data: The distribution of value added is in fact higher in the right tail than the left tail.

Table 2.6: Transition probabilities of log value added. Rows may not sum to 1 as the entries have been rounded to two significant figures.

		Quintile tomorrow				
		1	2	3	4	5
Quintile today	1	0.72 (0.013)	0.20 (0.012)	0.046 (0.006)	0.027 (0.005)	0.008 (0.003)
	2	0.22 (0.012)	0.52 (0.014)	0.19 (0.011)	0.05 (0.006)	0.014 (0.003)
	3	0.05 (0.006)	0.22 (0.012)	0.53 (0.014)	0.18 (0.011)	0.02 (0.004)
	4	0.023 (0.005)	0.045 (0.006)	0.22 (0.012)	0.58 (0.014)	0.13 (0.01)
	5	0.002 (0.001)	0.01 (0.003)	0.02 (0.004)	0.16 (0.01)	0.80 (0.01)

2.3   Model

2.3.1   Households and Production

The economy is populated by a continuum of households, each indexed by $i \in [0, 1]$, and a continuum of representative corporate firms. Households and firms are infinitely lived, and time is discrete. As in Bewley (1977), there is no aggregate uncertainty in the economy, but individuals are subject to idiosyncratic shocks.

Preferences and Discounting

All households are endowed with identical time-separable utility function with constant relative risk aversion (CRRA), and discount future utility at rate β. Households value nondurable consumption c, a choice that depends on the household's endowment of other factors and savings choices. The household's lifetime expected utility can therefore be written as

$$V_0 \equiv \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U\left(c_{i,t}\right)$$

$$= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{c_{i,t}^{1-\gamma}}{1-\gamma}$$

where γ is the coefficient of relative risk aversion. The household's objective is to maximize expected lifetime utility by choosing a sequence of consumption $\{c_{i,t}\}_{t=0}^{\infty}$.

Production technology

A household begins each period as either a worker (W) or an entrepreneur (E), as well as with a portfolio of liquid risk-free bonds $(\tilde{b}_{i,t})$, liquid shares issued by corporate firms[21] $(x_{i,t})$, and illiquid physical capital $(k_{i,t})$. Occupational type and asset allocations are chosen in the last period. In this section, I focus only on the production technologies available to the households and firms. I will delay discussion of the asset structure to the next section.

Workers If the individual is a worker, she supplies inelastic labor efficiency units $\theta_{i,t}$ to a spot market, and receives a wage income of $w\theta_{i,t}$. $\theta_{i,t}$ is an idiosyncratic shock to

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[21] As I will explain later, this economy features no aggregate risk. As a result, liquid shares and bonds are perfect substitutes, which I will therefore collectively refer to as liquid risk-free assets.

labor productivity, and follows a Markov process whose evolution depends on the last period occupational type of the individual. If the individual was also a worker in the last period, then $\theta_{i,t}$ is drawn from the conditional distribution $P_{\theta|\theta_{-1}} \equiv Pr\left(\theta|\theta_{i,t-1}\right)$ over labor productivity, where $\theta_{i,t-1}$ is the labor productivity of the individual in the last period. If the individual worker was an entrepreneur in the last period, $\theta_{i,t}$ is drawn from the conditional distribution $P_{\theta|\psi_{-1}^{\theta}} \equiv Pr\left(\theta|\psi_{i,t-1}^{\theta}\right)$, where $\psi_{i,t-1}^{\theta}$ is an initial condition drawn by the entrepreneur prior to making her occupational choice (to be described in detail later). In both cases, θ is drawn from a common support Θ.

Entrepreneurs If the individual is an entrepreneur, she operates a production technology $f(z,k,l,\bar{l})$ that combines external labor hired from a spot market (l), endowed labor ($\bar{l}$), capital (k), and idiosyncratic entrepreneurial productivity $z_{i,t}$ to produce output $y_{i,t}$. $z_{i,t}$ is an idiosyncratic shock to entrepreneurial productivity, and follows a Markov process whose evolution depends on the last period occupational type of the individual. If the individual was also an entrepreneur in the last period, $z_{i,t}$ is drawn from the conditional distribution $P_{z|z_{-1}} \equiv Pr\left(z|z_{i,t-1}\right)$ over entrepreneurial productivity, where $z_{i,t-1}$ is the entrepreneurial productivity of the individual in the last period. If the entrepreneur was a worker in the last period, she draws $z_{i,t}$ from the conditional distribution $P_{z|\psi_{-1}^{z}} \equiv Pr\left(z|\psi_{i,t-1}^{z}\right)$, where $\psi_{i,t-1}^{z}$ is an initial condition drawn by the worker prior to making her occupational choice (to be described in detail later). In both cases, z is drawn from a common support $\mathbb{Z}$.

The production function is assumed to be of the following form:

$$f\left(z,k,l\right) = z\left(k^{\alpha_e}(\bar{l}+l)^{1-\alpha_e}\right)^{\nu}$$

where $\alpha_e \leq 1$ is the capital intensity and $\nu < 1$ is the span-of-control parameter that determines the degree of decreasing returns to scale. As in Lucas (1978), this cap-

tures the idea that managerial skills become stretched over larger and larger projects. Endowed labor $\bar{l}$ is a parameter and assumed to be constant across households, and is a perfect substitute for externally hired labor l, which costs w per unit of labor efficiency units hired. Finally, I assume that entrepreneurs and firms produce homogeneous outputs. Given these assumptions, we can write the entrepreneur's profit function as

$$\pi\left(z,k\right) = z\left(k^{\alpha_e}(\bar{l}+l)^{1-\alpha_e}\right)^{\nu} - w\left(l-\bar{l}\right)$$

We can further define the optimal profit function for the entrepreneur. Here, labor choice is a static decision and independent of the structure of the rest of the problem. Therefore, we can see that optimal labor demand satisfies

$$l^* = \begin{cases} \bar{l} & \text{if } l^* \leq \bar{l} \\ \left[\frac{(1-\alpha)\nu}{w}\right]^{\frac{1}{1-(1-\alpha)\nu}} z^{\frac{1}{1-(1-\alpha)\nu}} k^{\frac{\alpha\nu}{1-(1-\alpha)\nu}} & \text{if } l^* > \bar{l} \end{cases}$$

and optimal profits is given by

$$\pi^* = \begin{cases} z\left(k^{\alpha}\left(\bar{l}\right)^{1-\alpha}\right)^{\nu} & \text{if } l^* \leq \bar{l} \\ \left[\mathcal{A}\left(w\right) - w\mathcal{A}\left(w\right)^{\frac{1}{(1-\alpha)\nu}}\right] z^{\Theta_z} k^{\Theta_k} + w\bar{l} & \text{if } l^* > \bar{l} \end{cases} \tag{2.1}$$

where $\mathcal{A}(w) \equiv \left[\frac{(1-\alpha)\nu}{w}\right]^{\frac{(1-\alpha)\nu}{1-(1-\alpha)\nu}}$, $\Theta_z \equiv \frac{1}{1-(1-\alpha)\nu}$ and $\Theta_k \equiv \frac{\alpha\nu}{1-(1-\alpha)\nu}$.

Representative corporate firms   In the real economy, a substantial fraction of investment and hiring are done by large corporate firms. Therefore, following Cagetti and De Nardi (2006), I model this sector as a second sector of production populated by a large number of homogeneous firms operating a constant returns to scale production

technology given by

$$Y^C = A \left(K^C \right)^\alpha \left(L^C \right)^{1-\alpha}$$

where A is aggregate TFP and α is the capital share of this sector. Here, I assume that A is constant over time, and as a result there is no aggregate risk in this economy. As in the entrepreneur's choices, the corporate firm's capital stock K^C is determined from last period's choices, while labor L^C is decided in the current period.

2.3.2 Asset Structure

Households have access to 3 types of assets to smooth inter-temporal consumption: liquid bonds $\tilde{b}$, liquid shares issued by the corporate sector x, and illiquid physical capital k.

Illiquid physical capital

Only households that elect to become (or stay) entrepreneurs in the next period can invest in illiquid physical capital k, which depreciates at rate δ_k. The primary purpose of physical capital is as input for entrepreneurial production; however, it also serves a secondary purpose as consumption insurance. In particular, entrepreneurs can use the illiquid asset to smooth consumption by selling off parts of their asset stock in bad times, or as collateral to borrow for consumption.

Capital is illiquid due to buying and selling frictions associated with adjusting the capital stock. Focusing first on resale frictions, I assume that the entrepreneur faces two types of downsizing frictions associated with the entrepreneur's future occupational choice. When the entrepreneur decides to downsize, but still continue in entrepreneurship, she faces a per-unit transaction cost λ such that she recoups $(1-\lambda)$ of the transacted asset. For convenience, I term this choice "partial liquidation", and

define $p_{used}^{PL} \equiv (1 - \lambda)$ as the resale price of capital under partial liquidation.

If the entrepreneur decides to fully exit the business, she has to pay an additional proportional selling cost ζ on top of the earlier transaction cost. As a result, the net return from selling a unit of capital is $p_{used}^{FS} \equiv (1 - \zeta)(1 - \lambda)$, where I denote the preceding term as the resale price of capital under fire sale liquidation. This preceding formulation imposes that small-scale adjustments along the intensive margin are less costly than large changes, and also captures the idea that large exit sales are akin to a "fire sale" of assets[22,23].

In addition to downsizing frictions, I also assume that entrepreneurs face a proportional fixed cost when they choose to expand their capital stock. This modeling assumption parametrizes the disruptions associated with expansion (c.f. Cooper and Haltiwanger (2006) for discussion of this assumption)[24].

Liquid bonds

Bonds are traded among households to smooth intertemporal consumption. They cost one unit of consumption today, and return a risk free rate of r_{t+1} the next period.

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[22] It is important to note that this does not necessarily mean that every entrepreneur who exits a business has to incur a loss. Rather, it says that when an entrepreneur exists her business, the physical capital stock she owns is typically worth less than even the actual depreciated value. On the other hand, this paper has nothing to say about the potential profits to be made on intangible capital, such as brand name or R&D capital.

[23] A concern that readers might have is that this setup allows entrepreneurs to downsize to the smallest possible limit (i.e., $k \approx 0$) before exiting, in order to avoid paying the ζ transaction cost. To ameliorate these concerns, in separate robustness checks I consider a redefinition of "partial liquidation" and "fire sales liquidation" in the following manner: Partial liquidation events occur when the entrepreneur sells a volume of capital less than or equal to an η fraction of the depreciated capital stock, and the entrepreneur has to pay transaction cost λ on that transacted capital. Fire-sale liquidation events are triggered when the entrepreneur sells more than an η fraction of her depreciated capital stock (including the case of exit). As in the main text, the entrepreneur has to pay a (net) per-unit transaction cost of $1 - (1 - \lambda)(1 - \zeta)$. I find that in this setup, there are no material changes to the conclusions drawn in this paper.

[24] In robustness checks, I also assume a fourth friction in the form of an entry friction. Specifically, I assume that workers who want to enter entrepreneurship must start their business with a minimum level of capital k_{min}^w. This assumption reflects the fact that entry typically requires some form of minimum capital investment. In general, this friction does not materially affect the conclusions of this paper.

Households that are borrowing have two options to borrow. First, all households are allowed to borrow via unsecured debt, where they face an adhoc borrowing constraint $\underline{b}$. Second, households that are entrepreneurs also own capital, and they are allowed to use the liquid value of capital as collateral to borrow a larger volume. This debt can be used to finance more investment in their firm, or simply to smooth consumption. Here, the liquid value of capital is simply the resale value of capital, i.e., $(1 - \lambda)(1 - \delta_k)k$. Therefore, all households face the following borrowing constraint:

$$\tilde{b}_{i,t+1} \geq -\varphi(1 - \lambda)(1 - \delta_k)k_{i,t+1} - \underline{b}$$

where $\varphi \in [0, \infty)$, with $\varphi = 0$ representing no collateralized borrowing, and $\varphi \to \infty$ representing no collateral required for borrowing. As is common in this literature[25], φ captures the idea of limited enforceability of debt contracts. Unlike prior literature on entrepreneurship, the fact that capital is illiquid also reduces the amount of collateral available to the entrepreneur, and therefore also potentially worsens the financial conditions of the entrepreneur. Finally, I assume that an intermediation cost ϕ_d must be be paid in addition to interest on the stock of debt; therefore, net payments[26] by the debtor who has a stock of debt $\tilde{b}$ are $(r_t + \phi_d)\tilde{b}$.

Liquid shares

Households can also buy shares (equity) $x_{i,t}$ issued by the corporate firm and earn dividends on those shares. Shares costs p_t units of consumption, and returns a per-unit dividend d_t.

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[25] See, for instance, Buera and Shin (2013).

[26] Note that in general, default does not occur in equilibrium in this model under the benchmark calibration. However, in certain circumstances, such as when studying unexpected shocks, fully leveraged households might find themselves unexpectedly in default, and cannot make their debt payments. For this subset, entrepreneurs are allowed to default on the remaining stock of their debt, but have to exit the entrepreneurial sector. The cost of default D (i.e., all the debt that is unpaid) is transferred in a lump sum in equal proportion to the rest of the households. In practice, $D \approx 0$, even when default is theoretically an option.

Before proceeding further, note that since this economy does not feature any aggregate uncertainty, and that bonds and shares are perfect substitutes (both being fully liquid), return on bonds must be equal to the return on shares in each period. This implies that the following no-arbitrage condition must hold:

$$1 + r_t = 1 + \frac{p_t}{d_t}$$

Moreover, any household's portfolio can be defined as follows:

1. For debtor households (i.e., $\tilde{b}_t < 0$), the household will not be holding any shares (i.e., $x_t = 0$). Moreover, the household's flow of debt payments is simply $(r_t + \phi_d)\tilde{b}_t$.

2. For saver households, the portfolio is indeterminate, i.e., the relevant state variable is simply net liquid assets, as defined by $b_t \equiv \tilde{b}_t + p_t x_t$. Moreover, the flow of return on liquid assets is simply $r_t b_t$.

Given the earlier pair of definitions, it is clear that b, net liquid asset, is the only relevant state variable for the household, as far as the liquid portion of the household's assets are concerned. Moreover, the borrowing constraint can be trivially redefined in terms of b as

$$b_{i,t+1} \geq -\varphi(1 - \lambda)(1 - \delta_k)k_{i,t+1} - \underline{b}$$

Therefore, going forward, I will only refer to liquid asset b when referring to the household's portfolio.

2.3.3 The Problem of the Individual

The household's problem

First denote by "primed variables" as next-period variables, and "unprimed variables" as current variables. At the beginning of the period, an individual is characterized by her occupational type $h \in \{W, E\}$ and her asset holdings (k, b). If the individual is an entrepreneur, she also starts the period with entrepreneurial productivity shock z and an outside option shock ψ^θ. The outside option shock is drawn from an invariant distribution F_{ψ^θ}, and as discussed earlier, will determine the entrepreneur's productivity as a worker if she chooses to transition into labor work in the next period. If the individual is a worker, she starts the period with a labor productivity shock θ and an outside option shock ψ^z. The outside option shock is drawn from an invariant distribution F_{ψ^z}, and as discussed earlier, will determine the worker's productivity as an entrepreneur if she chooses to transition into entrepreneurial work in the next period. Note that if the individual (either an entrepreneur or worker) does not utilize her outside option, she will draw a fresh outside option shock in the next period.

Let the value functions of entrepreneurs and workers be represented by V_e and V_w, respectively, and the adjustment cost function for capital (as discussed earlier) by $\mathcal{C}(k', k, h')$. Note in particular that the adjustment cost function also depends on the individual occupational choice. Entrepreneurial households solve the following recursive problem:

$$V_e\left(\psi^\theta, z, k, b\right) = \max\{V_{ee}, V_{ew}\} \tag{2.2}$$

where V_{ee} and V_{ew} are the value functions of the entrepreneur conditioned on choosing to stay in entrepreneurship or exit into labor work.

The entrepreneur who decides to stay in entrepreneurship solves

$$V_{ee}\left(\psi^\theta, z, k, b\right) = \max_{k',b'} U\left(c\right) + \beta \int_{\psi'_\theta} \int_{z'} V_e\left(\psi^{\theta'}, z', k', b'\right) dP_{z'|z} dF_{\psi^\theta} \qquad (2.3)$$

s.t.

$$\hat{\pi} \equiv \pi^* + \left(1 + r \times 1_{\{b \geq 0\}} + r_d \times 1_{\{b < 0\}}\right) b - \mathcal{C}\left(k', k, h'\right)$$

$$k' > 0$$

$$c = \hat{\pi} - k' - b' \geq 0$$

$$b' \geq -\varphi\left(1 - \lambda\right)\left(1 - \delta_k\right) k' - \underline{b}$$

,while the entrepreneur who exits entrepreneurship solves

$$V_{ew}\left(\psi^\theta, z, k, b\right) = \max_{b'} U\left(c\right) + \beta \int_{\psi'_z} \int_{\theta'} V_w\left(\psi^{z'}, \theta', b'\right) dP_{\theta'|\psi^\theta} dF_{\psi^z} \qquad (2.4)$$

s.t.

$$\hat{\pi} \equiv \pi^* + \left(1 + r \times 1_{\{b \geq 0\}} + r_d \times 1_{\{b < 0\}}\right) b - \mathcal{C}\left(k', k, h'\right)$$

$$k' = 0$$

$$c = \hat{\pi} - k' - b' \geq 0$$

$$b' \geq -\varphi\left(1 - \lambda\right)\left(1 - \delta_k\right) k' - \underline{b}$$

where π^* is the optimal profits of the entrepreneur as given in equation 2.1, and $1_{(\cdot)}$

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is an indicator function that evaluates to 1 when the argument is true.

Workers solve

$$V_w \left(\psi^z, \theta, b \right) = \max \left\{ V_{we}, V_{ww} \right\} \tag{2.5}$$

where V_{we} and V_{ww} are the value functions of the worker conditioned on entering into entrepreneurship or choosing to stay in labor work.

The worker who decides to enter into entrepreneurship solves

$$V_{we} \left(\psi^z, \theta, b \right) = \max_{k', b'} U\left(c\right) + \beta \int_{\psi'_\theta} \int_{z'} V_e \left(\psi^{\theta'}, z', k', b' \right) dP_{z' | \psi^z} dF_{\psi^\theta} \tag{2.6}$$

s.t.

$$c = \theta w + \left(1 + r \times 1_{\{b \geq 0\}} + r_d \times 1_{\{b < 0\}} \right) b - k' - b'$$

$$k' > 0$$

$$b' \geq -\varphi \left(1 - \lambda \right) \left(1 - \delta_k \right) k' - \underline{b}$$

while the worker who decides to stay in labor work solves

$$V_{ww} \left(\psi^z, \theta, b \right) = \max_{b'} U\left(c\right) + \beta \int_{\psi'_z} \int_{\theta' \in \Theta} V_w \left(\psi^{z'}, \theta', b' \right) dP_{\theta' | \theta} dF_{\psi^z} \tag{2.7}$$

s.t.

$$c = \theta w + \left(1 + r \times 1_{\{b \geq 0\}} + r_d \times 1_{\{b < 0\}} \right) b - b'$$

$$b' \geq -\underline{b}$$

The corporate firm's problem

The corporate firm's objective is to maximize lifetime dividend flows for its share-holders. It decides independently how much physical capital to invest, and how much labor to hire at the prevailing wage w. To raise capital for investment, the representative firm can issue equity. Let Π denote the value function of the corporate firm. The representative firm solves the following recursive problem:

$$\Pi\left(K^c\right) = \max_{K^{c\prime}} \pi + \frac{1}{1+r}\Pi\left(K^{c\prime}\right) \tag{2.8}$$

s.t.

$$\pi = Y^c - \left(K^{c\prime} - (1-\delta)K^c\right) - wL^c$$

where π represents current-period dividends paid out to the firms' investors, and the firm discounts future profits at rate $\frac{1}{1+r}$. Here, I assume that the corporate sector does not face any adjustment costs or financial frictions[27]. This market arrangement leads to the standard first-order condition for capital and labor demand:

$$r + \delta = \alpha A \left(\frac{K^c}{L^c}\right)^{\alpha-1} \tag{2.9}$$

$$w = (1-\alpha) A \left(\frac{K^c}{L^c}\right)^{\alpha} \tag{2.10}$$

2.3.4   Definition of Equilibrium

The state space of the model comprises liquid asset holdings $b \in \mathbb{B}$, capital holdings $k \in \mathbb{K}$, occupational choice $h \in \mathbb{H}$, entrepreneurial productivity $z \in \mathbb{Z}$, labor produc-

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[27] I assume this because the primary question of interest in this paper is in the importance of capital reallocation frictions for entrepreneurs. However, this is not to say that adjustment costs or financing frictions are not important for large firms. For recent discourse on this dimension of investment, I refer readers to recent papers, such as those by Khan and Thomas (2013) or Bassetto et al. (2015).

tivity $\theta \in \Theta$, entrepreneurial productivity option $\psi^z \in \Psi^z$, and labor productivity option $\psi^\theta \in \Psi^\theta$. Denote by $\mathbb{S} = \mathbb{B} \times \mathbb{K} \times \mathbb{H} \times \mathbb{Z} \times \Theta \times \Psi^z \times \Psi^\theta$ the complete state space, and $\mathbf{s} \in \mathbb{S}$ the state vector representing each individual agent. I now proceed to define the equilibrium of this model.

Definition 1 *A stationary equilibrium of the model is defined by*

1. *The interest rate r and wage rate w*

2. *Value functions: $V_e, V_{ee}, V_{ew}, V_w, V_{ww}, V_{we}, \Pi$*

3. *Policy functions: $k'(\mathbf{s})$, $b'(\mathbf{s})$, $h'(\mathbf{s})$*

4. *Optimal static profit function of the entrepreneur $\pi^*(\mathbf{s})$*

5. *Adjustment cost function $\mathcal{C}(\mathbf{s})$*

6. *Labor demand $l(\mathbf{s})$ from entrepreneurs and labor supply θ from workers*

7. *Factor demand K^c and L^c from the corporate sector*

8. *Invariant distribution of households $\Lambda(\mathbf{s})$*

 such that

1. *Taking r and w as given, the households' decision rules and value functions, as in equations 3.7, 3.8, 3.9, 3.10, 3.11, and 3.12, solve the individual problems.*

2. *Taking r and w as given, the representative corporate firm's decision rules and value function, as given in equation 3.13, 3.14, and 3.15, solve the firm's problem.*

3. *Factor markets clear, where*

   (a) *Bonds: $\int b' d\Lambda = K^c$*

(b) Labor: $\int \theta 1_{\{h=W\}} d\Lambda = \int l 1_{\{h=E\}} d\Lambda + L^c$

4. *The aggregate resource constraint is satisfied, where*

$$\int c + k' 1_{\{h'=E\}} + b' + \mathcal{C}(k', k, h') 1_{\{h'=E\}} d\Lambda = \int \pi^* 1_{\{h=E\}} + \theta w 1_{\{h=W\}} +$$

$$(1+r)b + (1-\delta_k)k 1_{\{h=E\}} d\Lambda$$

5. *The decision rules of the households, along with the exogenous Markov and i.i.d. processes, generate the time-invariant Markov transition kernel Γ, which, given any initial distribution of households Λ_0, generates the time-invariant distribution Λ; that is,*

$$\Lambda = \Gamma(\Lambda)$$

The method by which I compute the solution to the individual's problem and the stationary equilibrium is documented in Appendix A.3.

2.4 Calibration

In this section, I discuss the calibration strategy and outcomes. I begin with a summary of parameters taken from the literature, parameters estimated directly from the data, and parameters calibrated by matching model-implied moments to their data counterpart. I then discuss the strategy I use to identify the resale frictions and collateral constraints. Finally, I present some evidence as external validation for the model, and in particular, argue that ignoring resale frictions might lead to the wrong inference regarding the importance of financing constraints for entrepreneurs.

2.4.1 Summary of Parameters and Calibration Outcomes

Parametric assumptions for productivity shocks

There are four stochastic processes in this model: entrepreneurial productivity, worker productivity, outside option signal for business productivity, and outside option signal for labor productivity. To take the model to the data, I assume that the four processes follow the following parametric forms;

Entrepreneurial productivity Entrepreneurial productivity is assumed to follow an $\mathrm{AR}(1)$ process of the form $\log z' = (1 - \rho_z)\,\mu_z + \rho_z \log z + \sigma_z \epsilon'_z$, with $\epsilon'_z \sim i.i.d., N(0, 1)$. The process is discretized with a 15-point Markov transition matrix using the method detailed by Tauchen (1986).

Entrepreneurial productivity signal The signal is assumed to be drawn from a distorted invariant distribution of the actual invariant distribution of business productivity. Note that business productivity has the invariant distribution $N(\mu_z, \sqrt{\frac{1}{1-\rho^2}}\sigma_z)$. In the case of the signal, I assume worker households draw a signal from the "twisted" signal distribution $N(\tilde{\mu}_z, \sqrt{\frac{1}{1-\rho_z^2}}\sigma_z)$. $\tilde{\mu}_z$ serves to shift the probability that workers get "good" outside options relative to their current labor productivity.

Labor productivity Labor productivity is assumed to follow an $\mathrm{AR}(1)$ process of the form $\log \theta' = (1 - \rho_\theta)\,\mu_\theta + \rho_\theta \log \theta + \sigma_\theta \epsilon'_\theta$, with $\epsilon'_\theta \sim i.i.d., N(0, 1)$. The process is discretized with a 15-point Markov transition matrix using the method detailed by Tauchen (1986).

Labor productivity signal The signal is assumed to be drawn from a distorted invariant distribution of the actual invariant distribution of labor productivity. Similar to the case of business productivity signals, labor productivity has the invariant distribution

$N(\mu_\theta, \sqrt{\frac{1}{1-\rho_\theta^2}}\sigma_\theta)$; I assume then that entrepreneurial households draw a signal from the "twisted" distribution $N(\tilde{\mu}_\theta, \sqrt{\frac{1}{1-\rho_\theta^2}}\sigma_\theta)$. $\tilde{\mu}_\theta$ serves to shift the probability that entrepreneurs get "good" outside options relative to their current entrepreneurial productivity.

For the rest of the calibration exercise, the following assumptions are made. The model frequency is annual, which corresponds to the frequency in the KFS. As in the literature, many standard parameters (such as the corporate sector's capital share, depreciation rate, and labor income process) are taken from the prior literature. The exact numbers used and their rationale are reported under "Group A" in Table 2.7.

Three parameters are inferred directly from the data: the depreciation rate of the entrepreneur's capital stock δ_k, capital intensity α_k, and returns to scale ν. The depreciation rate for the entrepreneur's capital stock differs from the corporate sector, as the composition of the aggregated "capital" stock is different for an entrepreneur as compared to that of a corporate firm. Consequently, I construct the depreciation rate for the entrepreneur's capital as a weighted average of the individual depreciation rates of the components of the types of capital that make up an entrepreneur's stock of capital. Capital intensity and returns to scale are estimated using a hybrid cost shares approach and production function regression. The exact method by which these three parameters are constructed is relegated to Appendix A.4. The estimated values used in the model are reported below under "Group B" in Table 2.7.

The rest of the 10 parameters are inferred indirectly from the data by jointly calibrating them to identifying moments from the data. A brief description of these parameters, as well as the mapping of data moments to parameters, is summarized under "Group C" in Table 2.8. Note that since the data moments are computed using the investment data of only a single cohort of entrants over 7 years (recall that the KFS is a single cohort panel survey over 7 years), corresponding model moments are

Table 2.7: Fixed and estimated parameters

Parameter	Value	Description	Source
**Group A parameters**			
γ	Risk aversion	2	Standard. See, for instance, Storesletten et al. (2004)
μ_θ	Unconditional mean of labor productivity	0	Normalization
ρ_θ	Persistence of labor productivity	0.90	Floden and Linde (2001), Storesletten et al. (2004)
σ_θ	Conditional variance of labor productivity	0.20	Floden and Linde (2001), Storesletten et al. (2004)
μ_z	Mean of business productivity process	0	Normalization
α	Capital share of corporate sector	0.33	Fixed to value in Cagetti and De Nardi (2006)
A	Corporate sector TFP	1	Normalization
δ	Depreciation rate of corporate sector capital	0.10	Standard
**Group B parameters**			
δ_k	0.15	Depreciation rate of entrepreneur's capital	From data (See Appendix A.4)
α_e	0.423	Capital intensity of entrepreneurial production function	From data (See Appendix A.4)
ν	0.786	Returns to scale of entrepreneurial production function	From data (See Appendix A.4)

also computed using only population moments[28] of a single cohort of entrants tracked for 7 years.

As we can see from the first two rows of Table 2.7, the resale transaction costs are very high. When interpreted as resale prices, the calibration implies that downsizing firms receive about 31 cents on the dollar for every unit of capital sold, while exiting

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[28] In the calibration exercise, all model moments are computed exactly rather than by simulation. This is done by simulating a cohort using the "non-stochastic" simulation method of Young (2010), which thus avoids simulation errors from confounding the analysis.

Table 2.8: Calibrated parameters

Parameter	Value	Description	Identifying moment
Group C parameters			
λ	0.688	Resale transaction cost	Probability that firm stays in quintile 1 of ARPK distribution
ζ	0.314	Exit friction	Skewness of ARPK
φ	0.930	Collateral constraint	Skewness / Probability that firm stays in quintile 5 of ARPK distribution
f_s	0.032	Investment fixed cost	Rate of positive investment reported
ρ_z	0.810	Autocorrelation of productivity shock	Autocorrelation of investment rates
σ_z	0.350	Volatility of productivity shock	Coefficent of variation of investment, firm size distribution
$\bar{l}$	0.211	Entrepreneur's endowed labor	% of firms that are employers
$\tilde{\mu}_z$	0.510	Mean of entrepreneurial prospects signal shock	Fraction of households that are entrepreneurs in steady-state
$\tilde{\mu}_\theta$	0.707	Mean of labor prospects signal shock	Exit rate
β	0.9185	Discount factor	Interest rate of 3.5%

firms only recoup 21 cents on the dollar. As a consequence of these high resale frictions, the economy features heightened risk through an illiquidity risk channel, the consequence of which will be briefly discussed in the latter half of this section and more extensively discussed in Section 2.5. In contrast, we see from the third row of Table 2.8 that the friction coming from credit frictions alone is relatively weak; the entrepreneur is able to collateralize almost 93% of her asset. However, this does not imply that financial frictions in general are weak in this economy. Instead, recall that the net collateral constraint is the product of the collateral constraint parameter (φ) and the resale price of used capital ($1 - \lambda$), i.e., $\varphi \times (1 - \lambda)$. This product comes up to about 0.29, meaning that for every \$1 of new investment made by the

entrepreneur, only 29% of that can come from external loans. This finding is in line with prior literature on financial frictions and entrepreneurship, such as Cagetti and De Nardi (2006); the main difference here being that most of the borrowing frictions arise simply from the low resale value of capital.

In Table 2.10, I report model fit by comparing the data moments I target against model-implied moments. Broadly speaking, the model does a good job of both matching micro-level moments (i.e., from the KFS) and macro-level moments (taken from the PSID). In Table 2.9, we see that the baseline model does not simply replicate the two tail conditional probabilities; it also broadly replicates the characteristics of the full transition matrix of ARPK.

Table 2.9: Transition matrices for ARPK under the baseline calibration. Entries have been rounded to two decimal places for clarity of presentation.

		Quintile tomorrow				
		1	2	3	4	5
	1	0.58	0.25	0.11	0.04	0.01
	2	0.24	0.30	0.24	0.16	0.07
Quintile today	3	0.11	0.22	0.25	0.25	0.17
	4	0.05	0.14	0.22	0.28	0.30
	5	0.03	0.09	0.16	0.27	0.45

As an external validation of the model, I report in Table 2.11 a set of moments that are not targeted (either implicitly or explicitly) by the calibration exercise: the persistence and dispersion of value added, and the time series of the average net leverage ratio of firms[29]. As we can see, the model also matches these moments very well. For instance, the baseline calibration predicts the same asymmetry of persistence in the left and right tail of the distribution of value added[30] (that is, the right tail is more persistent than the left). As such, we see that the greater left-tail persistence of

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[29] I refer the reader to Appendix A.1 for how the net leverage ratio was constructed.

[30] The full transition matrix of value added is further reported in Table A.4 of Appendix A.9. As in the transition matrix for ARPK, the transition probabilities of value added implied by the model also broadly match that of the data.

ARPK in the model is not driven by a counterfactual assumption on the underlying TFP process.

In the next subsection, I will explain how the skewness of the ARPK distribution, as well as its persistence in the two tails of the distribution, can be an informative device in discriminating between downsizing frictions (i.e., partial irreversibility as parametrized by λ and ζ) and financial frictions (i.e., collateral constraints as parametrized by φ). Following that, I will demonstrate that ignoring resale frictions might lead one to overestimate the tightness of the financial frictions faced by entrepreneurs.

Table 2.10: Model fit: Data and corresponding model moments

Moments	Data	Model
Targeted KFS moments		
$Pr(1 \rightarrow 1)$	0.58	0.58
$Pr(5 \rightarrow 5)$	0.47	0.45
Kelly skew $log(Y/K)$	-0.12	-0.11
% +ve investment	56%	50%
CV of I/K	4.1	4.0
Autocorrelation of I/K	-0.069	0.09
% Employer firms	57%	58%
KFS exit rate	10%	30%
Targeted: Aggregate moments		
% of households that are entrepreneurs (PSID)	7.9%	9.0%
% Exit rate (PSID)	34%	32%
% Startup rate (PSID)	3%	3.2 %
Interest rate	3.5%	3.5 %

2.4.2 Identification Strategy

A challenge in the literature of firm dynamics lies in a simple identification strategy to discriminate between *investment*-type frictions, such as collateral constraints, and *disinvestment*-type frictions, such as the resale frictions in this model. In this

Table 2.11: Untargeted moments: KFS vs. Model

Moments	Data	Model
Standard dev of value added	0.21	0.26
Autocorrelation of value added	0.85	0.85
$Pr(1 \rightarrow 1)$, value added	0.72	0.72
$Pr(5 \rightarrow 5)$, value added	0.80	0.75
Average leverage ratio (at year 1)	0.097	0.037
Average leverage ratio (at year 2)	0.047	0.035
Average leverage ratio (at year 3)	0.038	0.026
Average leverage ratio (at year 4)	0.037	0.018
Average leverage ratio (at year 5)	0.038	0.012
Average leverage ratio (at year 6)	0.011	0.008
Average leverage ratio (at year 7)	0.030	0.006
Average leverage ratio (at year 8)	0.014	0.004

subsection, I elaborate on the identification strategy that allows me to discriminate between resale and financial frictions. In particular, I first present two propositions using a standard neoclassical model of firm dynamics that will guide the calibration exercise, following which I discuss comparative statics exercises that document the identification method. In the next subsection, I show why this strategy is important. In particular, I document how simply looking at raw firm-level leverage (such as the debt-to-asset ratio) can be misleading, and lead one to conclude that firms face excessively tight collateral constraints.

Predictions of the standard model without resale frictions

Consider here a canonical model of firm production, in which a firm operates a production function $Y = zK^\alpha$, where z is firm-level TFP, K is capital, Y is output, α is the returns to scale, r is the interest rate, and δ is the user cost of capital (deprecia-

tion)[31]. In addition, I assume that TFP evolves as an AR(1) process as follows:

$$\log z_{t+1} = \rho \log z_t + \epsilon_{t+1}$$

where ϵ_t is any i.i.d. innovation to z. Moreover, assume that the choice of capital at time t is not measurable with respect to time $t+1$ innovations[32]. Note that ϵ can assume any nondegenerate distribution. Finally, also assume that there is no endogenous entry or exit[33], and that the firm is risk-neutral.

Proposition 1 Consider an investment model as discussed above, and in which there are no frictions to capital adjustment. Then (log) ARPK can be expressed as (denoted by a superscript TTB for "time-to-build"),

$$\log ARPK^{TTB} = \vartheta + \epsilon$$

where ϑ is a collection of parameters. As such, the distribution of log ARPK is simply a mean-shifted distribution of the underlying innovations ϵ. Moreover, this implies that log ARPK has no persistence[34].

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[31] All derivations hold if we include labor as an input, but I ignore it in the interest of algebraic clarity. I choose this simple model to illustrate my point, as these models allow me to analytically characterize the skewness and persistence of ARPK. In contrast, a full-scale model does not easily admit an analytical expression.

[32] Trivially, if time t choice of capital is measurable with $t+1$ innovations, a frictionless model would generate a degenerate distribution for ARPK. While additional frictions, such as financial frictions, would lead to a nondegenerate distribution for ARPK, results will be the same as under the assumptions here. I refer the reader to Appendix A.5 for more details.

[33] The fact that there is no exit or entry in this framework will not affect the results, but are assumed as such for clarity in presenting the preceding results.

[34] The proposition here relates to a key point raised by Asker et al. (2014). There, using numerical examples, the authors show that a substantial portion of the dispersion in log MRPK (log ARPK) observed by Hsieh and Klenow (2009) can be explained using a standard canonical model of firm dynamics with time-to-build. My results here are the analytical counterpart to their numerical results.

Proof The derivation of the preceding equation is relegated to Appendix A.5. The result that log ARPK has no persistence comes directly from the equation. Since ϵ is an i.i.d innovation, log ARPK will not feature any persistence.

An additional result here is that the *skewness* of the distribution log APRK is equal to the skewness of the distribution of the underlying innovations. In most standard firm (or entrepreneurial) dynamics models, the innovations are typically distributed Gaussian or Pareto; the implication then is that log ARPK has zero or positive skewness, which is counterfactual to my empirical findings. As I will show in the next section, the illiquid aspect of capital can naturally generate a left-skewed distribution, without having to engineer a left-skewed distribution of innovations.

Proposition 2 Consider an investment model as discussed above, in which firms face collateral constraints. Specifically, consider a linear reduced-form collateral constraint of the following form:

$$b' \geq -\varphi(1-\delta)k'$$

where b' is next-period debt (or savings, if $b' > 0$), k' is next-period capital, and φ captures the strength of financial frictions. Then log ARPK in the collateral constraints model is related to the frictionless model through the following relationship (denoted by a superscript TTB, FF for "time-to-build with financial frictions"):

$$\log ARPK^{TTB,FF} = \log ARPK^{TTB} + \xi_{-1}$$

where ξ_{-1} is a random variable that has a right-skewed distribution (and is predetermined in the last period). As such, if the skewness of $\log ARPK^{TTB}$ is lower than some threshold that is strictly bigger than 0, the distribution of $\log ARPK^{TTB,FF}$ will be more right-skewed than the distribution of the underlying innovations. If

the skewness of $\log ARPK^{TTB}$ is larger than the threshold, than the distribution of $\log ARPK^{TTB,FF}$ is less right skewed than the distribution of the underlying innovations, but it will always be right-skewed. Moreover, the right tail of the distribution will be more persistent than the left tail.

Proof   The proof is relegated to Appendix A.5.

The key takeaway here is that this class of models will *always* generate greater persistence in the right tail of the distribution than the left tail. Moreover, under assumptions of normality in innovations, the resulting distribution of ARPK will be right-skewed; a similar prediction arises when innovations are assumed to be right-skewed, such as a Pareto distribution.

The results in this subsection suggest two things. Firstly, short of imposing a left-skewed TFP process in the model, we see that a model featuring only collateral constraints, such as that of Cagetti and De Nardi (2006), will not be able to replicate the left-skewness seen in the data. Second, and more importantly, regardless of the parametric form assumed by the innovations, the model featuring only collateral constraints will always generate higher persistence in the right tail of the distribution ARPK, relative to the left tail. This implies that a model that ignores resale frictions will not be able to simultaneously capture these two features of the data.

This result occurs because collateral constraints impede capital accumulation, but do not affect capital decumulation. Faced with borrowing constraints, higher productivity firms with low internal funds are unable to fully finance their capital investment for extended periods of time. As a result, these firms persist in a state in which their internal rate of return is higher than the external rate of return. In contrast, since low productivity firms that are looking to downsize are not affected by borrowing constraints, this means that low productivity firms will rapidly downsize to a point at which their internal rate of return matches the external rate of return

(in expectation). As a result, the distribution of ARPK becomes right-skewed, and there is higher persistence in the right tail than the left tail of the distribution.

(a) Skewness (b) Relative persistence of tails
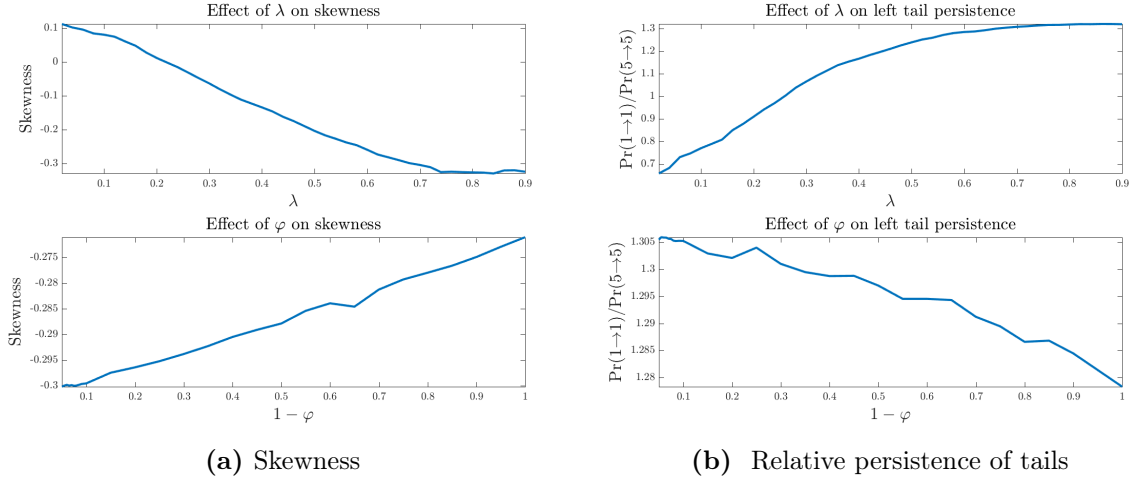
FIGURE 2.5: Effect of λ and φ on skewness and relative persistence of ARPK. Figure (b) reports the ratio of the probability of staying in quintile 1 to the probability of staying in quintile 5 (i.e., $Pr(1 \rightarrow 1)/Pr(5 \rightarrow 5)$). Note that increasing φ represents loosening collateral constraints; therefore, for clarity, I plot the relevant moments against $1 - \varphi$.

Therefore, in order for a model to be able to simultaneously target both outcomes, downsizing frictions, which impede capital decumulation, is crucial. This is exactly the role played by λ in this context. When large firms are hit by a bad shock, λ induces a wait-and-see attitude, since these firms do not wish to take a large haircut on investment. Moreover, mean reversion in TFP implies that these firms are better off holding on to their capital rather than selling it. As a result, increases in λ will result in stronger persistence in the left tail. Since there are also substantially more large firms with relatively lower productivity, λ also increases the persistence of the left tail. This discussion is reflected in Figures 2.5a and 2.5b below, which report the effect of changing λ and φ on skewness and relative persistence, respectively. As we can see, tightening financial conditions increases skewness and decreases left-tail persistence (relative to the right tail), while increasing resale frictions leads to the opposite prediction.

On the whole, how important are downsizing frictions in helping the model to match the skewness and asymmetric persistence noted in the data? Table 2.12 reports the model-implied moments for skewness and tail persistence when the resale frictions are removed, with the net collateral constraint held fixed (i.e., $\lambda = \zeta = 0$ with φ reset to 0.29, labeled "Liquid k"). As we can see, once the two resale frictions are removed, overall skewness rises to 0.04; that is, the distribution is now *right-skewed*. Similarly, left-tail persistence falls to 0.36, whereas right-tail persistence rises to 0.47. In other words, the model now exhibits greater persistence in the right tail than the left tail.

Table 2.12: Effect of illiquidity on skewness and asymmetric persistence: Comparison with counterfactual model with no resale frictions.

Target	Data	Model	
		Baseline	Liquid k
Kelly skew $\left(log\frac{Y}{K}\right)$	-0.12	-0.11	0.04
$Pr(1 \rightarrow 1)$	0.58	0.58	0.36
$Pr(5 \rightarrow 5)$	0.47	0.45	0.47

2.4.3   External Validation: Firm Leverage

As documented earlier, most firms in the KFS carry either zero or very low levels of net debt relative to their physical or pledgeable assets, a fact that was replicated by the baseline calibration of my model. To demonstrate that this prediction is primarily a result of illiquidity frictions, I recomputed the implied leverage ratio under the counterfactual "Liquid k" model, in which resale frictions are set to 0 and the collateral constraint is recalibrated to hold fixed the maximum leverage. Results are reported in figure 2.6, in which I overlay the data (solid black line) onto the predictions of the baseline model (solid blue line) and the counterfactual liquid k model (dashed red line). The horizontal dashed line at the top of the graph reflects the maximum leverage attainable by firms in both versions of the model.

As we can see, even under the counterfactual model, the average leverage held by
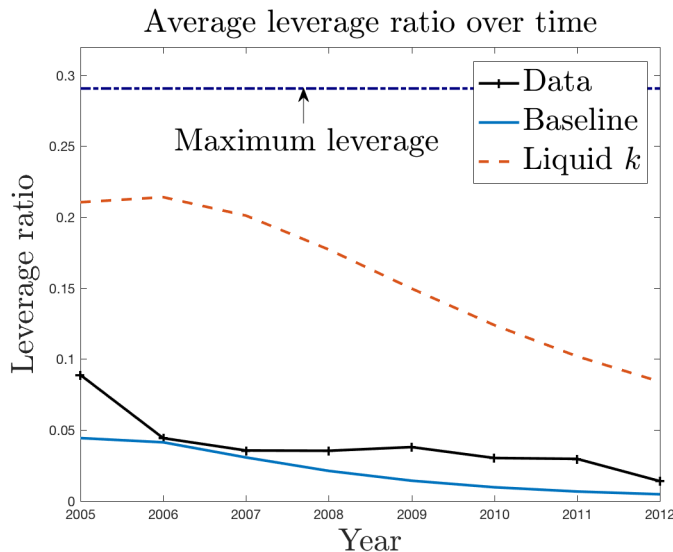
FIGURE 2.6: Time series of average firm-level leverage ratio in the data, and the predictions under the baseline calibration and counterfactual "Liquid k" model. The horizontal dashed line denotes the maximum leverage attainable by firms in the model.

firms is strictly below the maximum leverage allowable to firms. This result reflects the intuition of Moll (2014), who points out that in the face of borrowing constraints, households can circumvent financial frictions through self-insurance. In this context, worker households accumulate bonds prior to entry. As a result, upon entry, the average entrant does not need to expend her full borrowing capacity to finance her investment. Therefore, average leverage is always below the maximum leverage, even under the counterfactual model.

However, we can see clearly from Figure 2.6 that the self-insurance mechanism cannot fully explain the low leverage held by entrepreneurs in the KFS. In fact, the counterfactual model predicts that the average entrant starts with almost twice the leverage of that in the data. Instead, the illiquidity risk channel that arises from resale frictions is crucial in allowing the model to match the low leverage of entrepreneurial firms. In this case, illiquidity risk endogenously leads entrepreneurs to avoid hitting the borrowing constraint. This effect will be explained further in Section 2.5.

A common approach taken in the literature to discipline the collateral constraint parameter is to target some macroeconomic aggregate variable, such as the debt-to-GDP ratio (Midrigan and Xu (2014)) or aggregate debt-to-asset ratio (Khan and Thomas (2013)). Following that strategy, a similar approach in the context of this paper would be to target the average leverage of firms in the KFS. In Figure 2.7, I report the result of this alternative calibration by overlaying the time series of average leverage predicted by this alternative model onto the predictions reported in figure 2.6 (dot-dash yellow line). As we can see, a calibration in which φ is set to about 0.1 broadly matches the level and trend in the data. Unfortunately, this would lead us to greatly overestimate the actual tightness of financial conditions. Moreover, this match is also only possible at the expense of matching the skewness and relative persistence of ARPK.
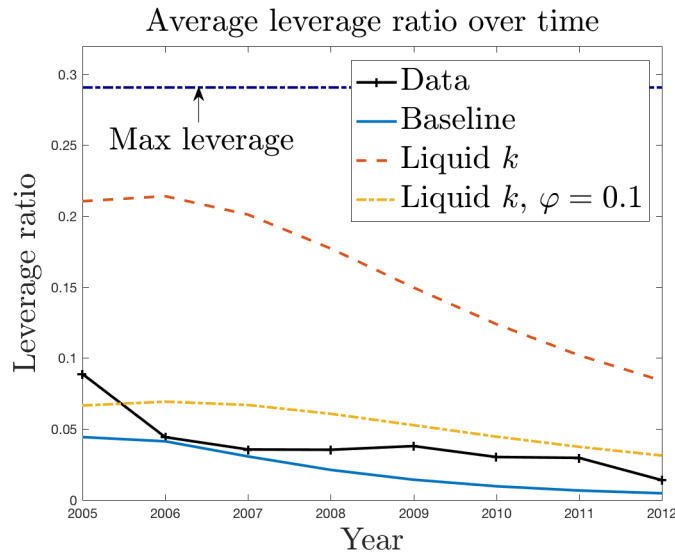
FIGURE 2.7: Time series of average firm-level leverage. This replicates Figure 2.6, and overlays it with the recalibrated counterfactual "Liquid k" model to generate the low leverage ratios observed in the data.

An alternative approach, taken by Cagetti and De Nardi (2006), is to use the wealth distribution to infer the extent of financial frictions in the economy. In their

calibration exercise, the ratio of the median net worth of entrepreneurs to workers serves as an indirect target for the collateral constraint parameter. The authors find that entrepreneurs can collateralize up to 25% of their capital, which is very close to the 29% I find in my calibration. At a first pass, this implies that both approaches generate very similar estimates for the extent of financial constraints faced by entrepreneurs. However, their approach would understate the extent of risks faced by entrepreneurs. Specifically, because their approach does not incorporate illiquidity risks, entrepreneurship would be a less risky prospect, and capital and talent are in fact more efficiently allocated under their framework. For instance, as we already see in Figure 2.6, a lack of illiquidity risk will allow entrepreneurs to take on more debt. As we will see in Section 2.5, this will imply that capital is better allocated toward high productivity entrepreneurs under the counterfactual, even though under both scenarios, the maximum leverage allowable is exactly the same. Moreover, the existence of illiquidity risk implies that a policy that provides partial insurance against this risk can have substantial welfare gains. Such a policy option is further explored in Section 2.6.

2.5   Results

2.5.1   Entrepreneurship and the Wealth Distribution

Table 2.13 below reports the wealth distribution under the baseline calibration (row 1), and contrasts it with the empirical wealth distribution in the United States[35] (row 3). As a reference, I also included the wealth distribution generated by an equivalent

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[35] The first column, which reports the fraction of wealth owned by entrepreneurs, is taken from the 1989 Survey of Consumer Finances (SCF), as computed by Cagetti and De Nardi (2006). The full wealth distribution, reported in the rest of the table, is taken from the 1992 SCF, as reported by Castaneda et al. (2003). In practice, I have recomputed the same statistics using the 2004 SCF, and find very similar results. However, as the goal of this paper is to connect my results to prior literature, I have reported their empirical findings as a reference point. Computations using the 2004 SCF are available upon request.

model in which the option to become an entrepreneur did not exist[36]. The Gini coefficient is also reported as a single summary statistic for the dispersion of wealth.

Table 2.13: Wealth shares for each quantile range of the wealth distribution, for the baseline model, model with no entrepreneurs, and the empirical wealth distribution.

Model/Data	% Owned by entrepreneurs	Concentration of wealth in:								Gini
		Quintile range					Top 10%			
		1st	2nd	3rd	4th	5th	90-95	95-99	99-100	
Baseline	32	1.5	4.2	7.9	15	72	11	23	23	0.68
No entrepreneurs	0	1.2	9.1	25	28	37	15	16	6.1	0.57
Data (USA)	33	-0.39	1.7	5.7	13	79	13	24	30	0.78

As one can see, the model with entrepreneurs is strictly closer to the empirical wealth distribution than the model without entrepreneurs. For instance, the top 1% holds about 23% of the wealth in the baseline economy, closely matching the 30% in the data. In contrast, the model without entrepreneurs predicts that the top 1% only holds around 6% of the wealth. Moreover, the model also replicates the high concentration of wealth among entrepreneurs. In the first column of Table 2.13, we see that entrepreneurs in the baseline calibration own about 32% of total wealth in the economy, closely replicating the data in which entrepreneurs own about 33% of total wealth. In Table 2.14 below, we see that the model also generates the overrepresentation of entrepreneurs among richer households that we saw in Section 2.2.

Table 2.14: Fraction of the population in each wealth percentile range that are entrepreneurs. Data (USA) refers to the 2003 wave of the PSID, and replicates the first row of Table 2.2.

	Number of entrepreneurs in each wealth percentile			
	Top 20%	Top 10%	Top 5%	Top 1%
Baseline	0.18	0.29	0.44	0.74
PSID (2003)	0.21	0.28	0.37	0.53

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[36] This simply reduces to the model in Aiyagari (1994). The discount factor is recalibrated to generate the same interest rate of 3.5%.

From a theoretical perspective, the ability of an entrepreneurship model to replicate the empirical wealth distribution is not new. This has been noted by prior papers, such as Cagetti and De Nardi (2006) in extensive numerical exercises, and also reported in Benhabib et al. (2015) in the form of a theoretical proof. In the latter paper, the authors argue that uninsurable, persistent, and highly volatile returns to capital leads households to accumulate more wealth, thus extending the right tail of the wealth distribution. Entrepreneurial returns, which are a form of capital income risk, bear this feature, and thus lead to a model economy that features a more unequal distribution of wealth.

However, unlike prior literature for which this has been a largely theoretical exercise[37], my results provide empirical support for entrepreneurship as a channel for generating high wealth dispersion. Recall that the key parameters that generate this wealth dispersion were calibrated to match the micro-level data, and the wealth distribution is not a target for calibration. Hence, there is no reason a priori that entrepreneurship should generate high wealth dispersion. This finding therefore provides support for entrepreneurship as a source of top wealth inequality[38].

2.5.2 The Role of Illiquidity Frictions

The Wealth Distribution

Table 2.15 contrasts the economy when capital is fully liquid against the baseline economy, as well as the empirical wealth distribution as a reference. For the model with no resale frictions, following the earlier strategy in Section 3.3, I reset the col-

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[37] For instance, in both Cagetti and De Nardi (2006) and Benhabib et al. (2015), the wealth distribution is an explicit target for calibration, while micro-level investment data are not targeted.

[38] While my baseline calibration generates less wealth dispersion than Cagetti and De Nardi (2006), I reiterate that the results I find are derived entirely from parameter values estimated (or calibrated) off firm-level investment microdata. As a result, key parameters that determine the extent of wealth dispersion (aside from the illiquidity frictions) are also different. For instance, the returns to scale parameter in my model is substantially smaller than that of Cagetti and De Nardi (2006) (0.786 in my baseline calibration vs. 0.88 in Cagetti and De Nardi (2006)). I refer the reader to Appendix A.6 for further discussion of the resulting implications.

lateral constraint parameter to $\tilde{\varphi}$, such that $\tilde{\varphi} = (1 - \lambda)\varphi$, thus keeping the net collateral constraint constant[39]. The no resale frictions model is solved under partial-equilibrium (row 2), where I keep the interest rate r at the baseline calibration, as well as under general-equilibrium (row 3)[40].

Table 2.15: The impact of illiquidity frictions. "Liquid k" refers to the model in which capital is not subject to resale transaction frictions.

	Concentration of wealth in:								
	Quintile range					Top 10%			Gini
Model/Data	1st	2nd	3rd	4th	5th	90-95	95-99	99-100	
Baseline	1.5	4.2	7.9	15	72	11	23	23	0.68
Liquid k (PE)	1.1	3.0	5.7	11	83	14	28	25	0.75
Liquid k (GE)	1.1	3.0	5.7	11	79	9.7	29	31	0.76
Data (USA)	-0.39	1.7	5.7	13	79	13	24	30	0.78

Regardless of general equilibrium effects, one sees right away that illiquidity frictions reduce wealth dispersion. For instance, the Gini coefficient rises to almost 0.76 under the counterfactual calibration, which almost matches the empirical wealth Gini of 0.78. The reduction in wealth dispersion results from the impact of illiquidity frictions on the cross-sectional distribution and dynamics of the idiosyncratic excess returns to wealth[41]. There are two ways by which this happens.

First, illiquidity frictions lower the dispersion of wealth by significantly increasing the relative persistence of the excess returns to wealth for households with low returns, without substantially changing the relative persistence of the returns to wealth for

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[39] This avoids confounding a pure collateral constraint with the illiquidity effect. The collateral constraint effect is explored extensively in Cagetti and De Nardi (2006), where they find that tightening collateral constraints reduces wealth dispersion. This effect also exists in my model.

[40] I report both the GE and PE results, as savings propensity and optimal firm sizes are all endogenously affected by the interest rate, which in turn affects the wealth distribution. The results here suggest that the higher wealth dispersion under a no resale frictions framework is generated from the illiquidity effect, rather than changes in the interest rate.

[41] Idiosyncratic return to capital is defined as $r_{i,t}^k \equiv \alpha_k \nu \frac{y_{i,t}}{k_{i,t}} - \delta$; the idiosyncratic return to bonds as $r_{i,t}^b \equiv r + \phi_d \times \mathbb{I}(b_{i,t} < 0)$; correspondingly, the return to wealth as $r_{i,t}^\omega \equiv \frac{r_{i,t}^k k_{i,t} + r^b b_{i,t}}{\omega_{i,t}}$ and the excess return to wealth as $\tilde{r}_{i,t}^\omega = r_{i,t}^\omega - r$

households with high returns. As returns are broadly decreasing in wealth[42], this result means that it is much more difficult for rich households to sustain a high income through engaging in entrepreneurship. Consequently, the wealth dispersion is lower when capital is illiquid.

This effect is reported in Table 2.16, in which I report the transition probabilities (constructed in quintiles) of the excess returns to wealth. There, we see that under the baseline calibration, households in the first quintile of the distribution have a 60% chance of staying in the first quintile; in contrast, when capital is fully liquid, households only have a 26% chance of staying in the first quintile. Moreover, the rate of mean reversion for low-returns households is substantially lower when capital is illiquid. For the baseline calibration, a household in the first quintile only reverts to the median quintile (i.e., quintile 3) with a probability of 28%, while households revert with a probability of 56% when capital is fully liquid[43]. In contrast, we see that illiquidity frictions do not substantially change the persistence of the rates of returns to wealth of high-returns households (i.e. quintile 5).

Second, illiquidity frictions also lower wealth dispersion by distorting the shape of the distribution of returns to wealth. Referring now to Table 2.17, we see in the last 5 columns that illiquidity frictions reduces the average excess return for households with high returns. Therefore, for households above the median, the average excess return is 37.3%, in contrast to an average of 47.2% for the economy with no illiquidity frictions. While households below the median do earn a higher rate of return on average than the model without illiquidity frictions, this difference is too small to make up for the large difference in the right tail. As such, overall average excess returns to wealth are smaller under the baseline than the economy without resale frictions.

[42] Table 2.16 reports the average wealth relative to the mean for two quintiles. For full results, refer to table A.5 in appendix A.9.2

[43] As will be discussed later, an important reason for this is that low productivity entrepreneurs are unable to easily exit their businesses when capital is illiquid.

Table 2.16: Transition probabilities of the excess returns to wealth. This reports a subset of transition probabilities. Full transition matrices are reported in Table A.5 of Appendix A.9

Model	Average wealth (relative to mean)	Probability of transitioning to:			Probability of exit
		Quintile 1	Quintile 3	Quintile 5	
		When in quintile 1			
Benchmark	2.2	0.60	0.28	0.02	0.24
Liquid k	0.87	0.25	0.56	0.02	0.49
		When in quintile 5			
Benchmark	0.28	0.05	0.20	0.59	0.18
Liquid k	0.26	0.07	0.16	0.56	0.15

Table 2.17: The excess returns to wealth computed at different percentiles, as well as the overall average.

Model	Average	Average ($<$ median)	Average ($\geq$ median)	Excess returns to wealth at percentile:				
				20%	40%	50%	60%	80%
Baseline	18.7%	-0.28%	37.3%	-0.34%	2.51%	5.03%	8.86%	26.2%
Liquid k	22.9%	-1.4%	47.3%	-1.42%	1.66%	6.16%	13.0%	37.9%

The allocation of resources along the wealth distribution

While a lower wealth dispersion might seem desirable, the key mechanism driving lower wealth dispersion is the result of poor allocation of bonds, capital, and talent at the household level. For instance, if we look at Figure 2.8, which plots the average productivity of entrepreneurs across the wealth distribution, we see a stark result, by which average entrepreneurial productivity declines with wealth: The richest entrepreneurs are also the least productive. In contrast, we do not see this result when the economy does not suffer from capital illiquidity. In that economy, entrepreneurs who are wealthy also generally have high productivity[44].

[44] While the richest entrepreneurs still have relatively low productivity, this is simply a result of mean reversion in the exogenous productivity process. That is, rich entrepreneurs became rich because they were once very productive, but in the long run, their productivity falls back to the mean.

If we look ahead to Table 2.19 in Section 2.5.3, we also see that capital is poorly allocated relative to the economy without illiquidity frictions. Relative to the counterfactual model, the average returns of entrepreneurs who make a capital loss and those who make a capital gain is 2.6 percentage points lower and 14.5 percentage points higher, respectively, while the aggregate private equity premium of the baseline model is 2.6 percentage points higher. This implies that low productivity entrepreneurs are holding too much capital, while high productivity entrepreneurs are holding too little capital. Moreover, in the aggregate, there is underinvestment relative to the counterfactual.
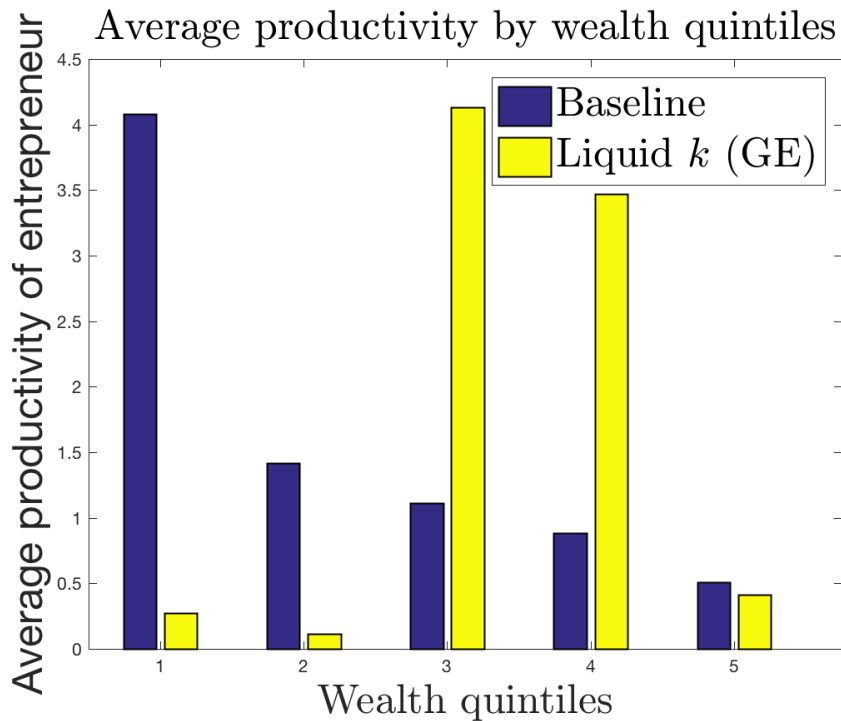
FIGURE 2.8: Average productivity of entrepreneurs for each quintile along the wealth distribution, under the baseline and counterfactual calibrations.

At the aggregate level, the result is costly and can lead to substantial losses in total factor productivity (TFP) and welfare. For instance, in Table 2.18 below, I report the potential welfare and productivity gains to be made if the illiquidity frictions could

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be costlessly removed[45], and the economy is allowed to gradually adjust to the new steady state. As we can see, after the removal of resale frictions, welfare improves and TFP increases. The average productivity of both workers and entrepreneurs increase as well, reflecting a better allocation of talent, and the population of entrepreneurs increases due to a removal of the implicit entry barrier generated by the illiquidity risk. Finally, real wages also increase. This result occurs because of a general equilibrium effect: The improved allocation of resources among entrepreneurs and potential entrants leads to an increase in labor demand from the entrepreneurial sector, and an increase in total savings as richer entrepreneurs are able to accumulate more wealth. As a result, the marginal product of labor in the corporate sector increases, which leads to an increase in wages.

Table 2.18: The change in welfare, TFP, average productivity, size of entrepreneurial and worker population, and wages after the resale frictions are removed. All values are computed relative to the baseline calibration, and reported in percentage terms.

	Welfare change (in % CEV)	TFP	Average productivity	Size of population	Change in wages
Full economy	4.4	11%	-	-	7.3%
Entrepreneurs	7.7	-	5.0%	25.4%	-
Workers	4.1	-	1.0%	-2.5%	-

2.5.3 Illiquidity Frictions: Inspecting the Mechanism

How do illiquidity frictions generate these effects? In general, the observations above result from a combination of a poor allocation of capital, portfolio mix, and talent at the individual level. To better clarify these effects, I now turn to a discussion of the model mechanisms.

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[45] The comparisons of TFP are done across steady states, while welfare is computed taking into account the entire transitional dynamics of the economy. This will allow us to consider these welfare gains as the upper bounds to any welfare gains to be made. I refer the reader to Appendix A.7 for the definitions of welfare and TFP, as well as a full description of how welfare is computed under the transitional path.
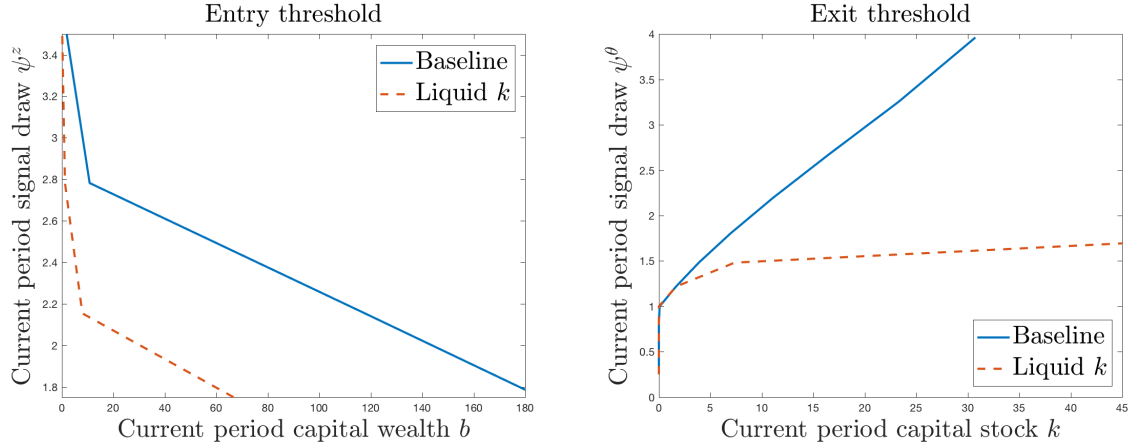
(a) Entry policy in (ψ^θ, k) space. Workers enter into entrepreneurship if they are right of the threshold.

(b) Exit policy in (ψ^z, k) space, holding b and z fixed. Incumbent entrepreneurs exit if they are to the left of the threshold.

FIGURE 2.9: Exit and entry policies under the baseline calibration (solid blue line) and counterfactual economy with no resale frictions (dashed red line). For the counterfactual economy, the interest rate is fixed to the baseline's interest rate to avoid confounding general-equilibrium effects with the effect coming from illiquidity.

Capital illiquidity leads to a fall in the value of entrepreneurship, thus generating an endogenous barrier to entry for potential entrants. This fall in the value of entrepreneurship arises from illiquidity risk. When formulating their entry decision, potential entrants weigh the value of staying a worker — which guarantee a rate of return of r on their assets — against investing in a business in which they run the risk of losing $1 - (1-\lambda)(1-\zeta)$ of their investment if they have to exit the next period. When ζ or λ increases, the losses incurred by entrepreneurs become larger upon a bad shock. The value of entrepreneurship falls from the perspective of workers, thus raising the bar for entry into entrepreneurship. Given the same signal shock (ψ^z), potential entrants have to be richer (in terms of liquid wealth) in order to enter into entrepreneurship, in order to sufficiently self-insure against the illiquidity risk. This effect plays out in Figure 2.9a, in which I plot the entry threshold in (ψ^z, b)-space for the baseline calibration and the counterfactual economy[46]. As we can see, the

[46] Comparisons in Figures 2.9 to 2.11 are all done with the net collateral constraint $\varphi(1-\lambda)$. This

threshold for entry shifts right when capital is more illiquid. At the individual level, this means that it becomes harder for workers to access a potentially high-returns technology, leading to a loss of lifetime income.

Illiquidity also decreases the value of exit, thus generating an endogenous barrier to exit. This happens naturally, since exiting entrepreneurs lose a portion of their capital due to the transaction cost. Consequently, holding fixed the continuation value of entrepreneurship, an entrepreneur must receive a better signal if she is to sell her business. This effect is reflected in Figure 2.9b, in which I plot the exit threshold in (ψ^θ, k)-space, contrasting the baseline economy with the counterfactual economy with no resale frictions. Here, we see clearly that the exit threshold shifts left when capital becomes more illiquid, meaning that for any firm size (in terms of capital stock), the signal required to trigger exit must be higher.

At the individual level, this decrease in exit propensity makes it more difficult for a poor performing entrepreneur to exercise her outside option as a worker. This is especially costly when the entrepreneur receives a good signal, but is unable to exercise it due to the illiquidity. As a result, she is forced to forgo a substantial amount of lifetime income. This mechanism is what primarily drives the high persistence in the left tail of the wealth distribution, as reported in Table 2.16. Looking at the last row, we can see that the exit rate for entrepreneurs in the first quintile is substantially lower under the baseline calibration.

At the aggregate level, this generates a selection effect against high productivity entrants and a selection effect for low productivity incumbents[47]. Moreover, this

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avoids the confounding secondary effect by which changing λ also affects the collateral constraint. The interest rate is also held constant to avoid confounding the effects of prices with the illiquidity effect.

[47] In actuality, this effect on the exit margin at the individual level is not a foregone conclusion. The effect of illiquidity on the choice to exit results from a two-horse race between a fall in the value of exit, discussed in this subsection, and a decrease in the value of entrepreneurship, as discussed in the earlier subsection. At the aggregate level, however, the first effect dominates in steady state. As a result, we see more low productivity entrepreneurs and fewer high productivity entrepreneurs under the baseline calibration. For further discussion of this mechanism, I refer the reader to Appendix

same effect generates a selection against incumbents who could potentially be high productivity workers, since it leads relatively low productivity incumbents to persist as entrepreneurs for too long. As a result, the average productivity of entrepreneurs and workers both fall, as we saw in Table 2.18. Moreover, there are more low productivity entrepreneurs and fewer high productivity entrepreneurs, as we see in Figure 2.10. The illiquidity risk, which renders entrepreneurship more risky, also leads to fewer entrepreneurs under the baseline calibration[48].
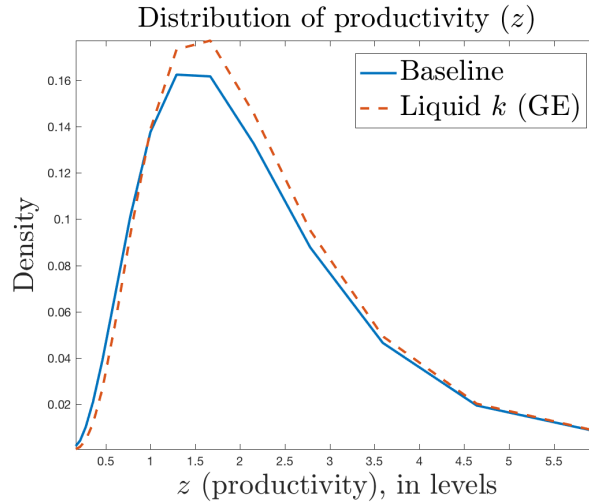
FIGURE 2.10: Distribution of productivities under the baseline calibration (solid blue line) and counterfactual economy (dashed red line). Note that there are also fewer entrepreneurs under the baseline calibration (about 8.9%, against 11.3% in the counterfactual).

The intensive margin

Investment and disinvestment behavior can also be distorted when capital becomes more illiquid, as reflected in Figures 2.11a and 2.11b. Due to the asymmetric purchase and resale price of capital, incumbents become cautious in investing and disinvesting. Entrepreneurs who receive a good productivity shock will invest less than the uncon-

A.8.1.

[48] However, it is important to note that this outcome is also not guaranteed. In fact, the steady state number of entrepreneurs depends on the interaction of the entry and exit rates along a transitional path from one steady state to another. I refer the reader to Appendix A.8.2 for a more detailed discussion.
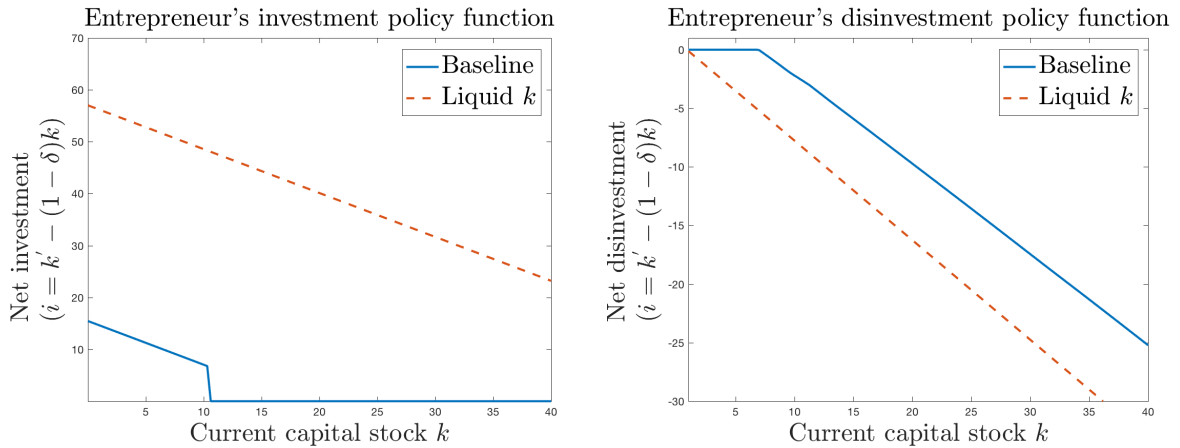
(a) Net investment ($i = k' - (1-\delta)k$) as a function of current capital stock, high z

(b) Net disinvestment ($i = k' - (1-\delta)k$) as a function of current capital stock, low z
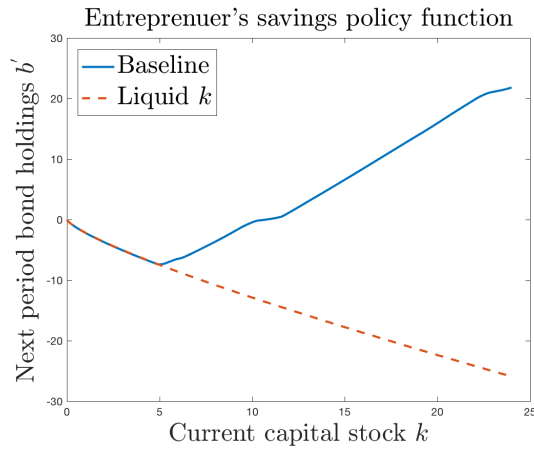
(c) Next-period bond holdings as a function of current capital stock

FIGURE 2.11: Entrepreneur's investment (top left), disinvestment (top right), and savings (bottom) choices under the baseline (solid blue line) and counterfactual (dashed red line) calibrations. Entrepreneurs invest less, disinvest less, and save more (borrow less) when capital is illiquid.

strained optimum, as they want to avoid being trapped with too much capital in the event of a downturn in the following periods; in contrast, entrepreneurs who receive a bad shock will disinvest less than the unconstrained optimum. In the latter case, entrepreneurs are hedging against the event of an upturn in the following periods. Since newly purchased capital is more expensive than used capital, entrepreneurs would like to reduce their expenditure in the case of an upturn. This causes poorly performing

entrepreneurs to hold on to larger capital stocks. At the aggregate level, this leads to a poor allocation of capital across entrepreneurial households: Poor performing entrepreneurs operate firms that are too large, and good performing entrepreneurs operate firms that are too small. This effect can be seen most clearly in Figure 2.12a, in which I plot the average firm size for each level of productivity (z) under the baseline calibration and the counterfactual economy with no resale frictions.
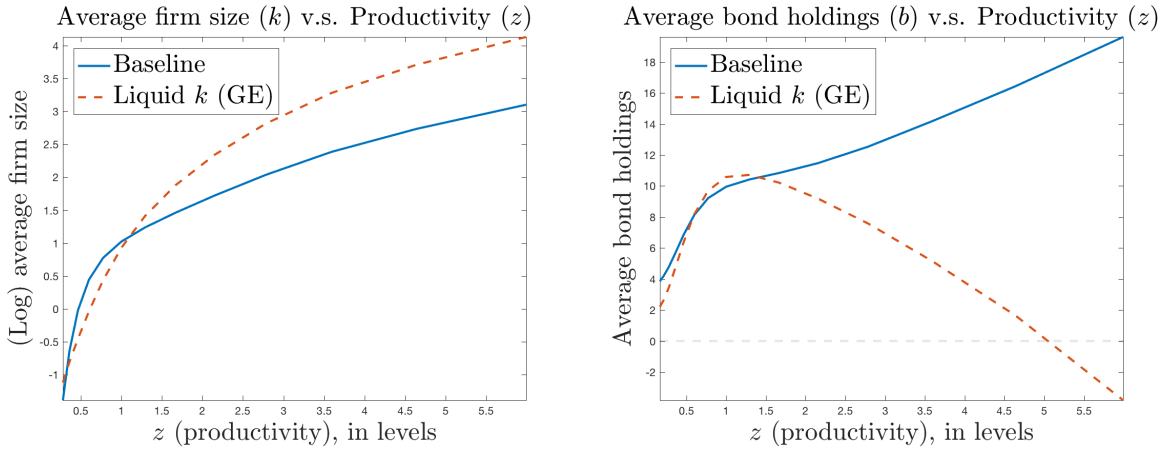
(a) Average firm size (capital, log scale) as a function of productivity (z). We see that low productivity firms are larger, while high productivity firms are smaller under the baseline calibration.

(b) Average bond holdings as a function of productivity (z). High productivity entrepreneurs hold more liquid assets under the baseline calibration as a buffer stock against illiquidity risk.

FIGURE 2.12: Average firm sizes and bond holdings as a function of productivity under the baseline (solid blue line) and counterfactual (dashed red line) calibrations.

In a similar vein, the portfolio of the entrepreneur also becomes poorly allocated. Due to the illiquid nature of capital, households are now less able to use their physical capital stock as a means to smooth consumption. In the case of a model in which bonds and capital are perfect substitutes, the household does not have to worry about losing their investment in the event of a bad shock. In the worst-case scenario in which $z \to 0$, the rate of return to investment is $-\delta$ (i.e., the user cost of capital). In contrast, with $\lambda > 0$ and/or $\zeta > 0$, the household also loses more capital due to the transaction costs. As such, for entrepreneurs in an economy in which capital is more illiquid, entrepreneurial households will demand more bond holdings to insure

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themselves against illiquidity risk. Taken together, entrepreneurial households that are expanding their businesses will, in general, tend to operate smaller firms and hold more liquid assets (or simply borrow less)[49], as we can see in Figure 2.11c.

At the aggregate level, this effect leads to an overaccumulation of liquid assets by entrepreneurs who are trying to insure themselves against the illiquidity risk. In Figure 2.12b, I plot the average bond holdings of entrepreneurs against their productivity levels. Here, we see that across almost all productivity levels, entrepreneurs in the baseline calibration have higher average bond holdings than their frictionless counterparts. Moreover, the average bond holdings are increasing under the baseline calibration, whereas it has a "hump shape" for the counterfactual. This aggregate outcome reflects the results reported in Figure 2.11c. In the counterfactual economy, entrepreneurs are willing to take up more debt to finance their investment; as a result, high productivity entrepreneurs tend to be more indebted, and hence have lower average bond holdings. In contrast, the fear of illiquidity risk drives entrepreneurs under the baseline calibration to hold more liquid assets, thus leading average bond holdings to increase.

For the individual entrepreneur, this poor allocation of assets implies that they are implicitly "leaving a lot of money" on the table. In Table 2.19, I report the average excess returns to capital at different percentiles, the overall average, and the implied equity premium[50]. We see that across the distribution, the rates of return to capital are generally higher under the baseline calibration, suggesting that these

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[49] In fact, taken together, this effect looks very similar to a borrowing constraint: High productivity entrepreneurs operate firms that are "too small", and tend to accumulate more bonds. However, while small firm sizes seen through the lens of financing frictions is an indication that the entrepreneur would like to expand if she could, the smaller firm sizes here is a result of households simply having no desire to expand to the unconstrained optimum. Likewise, where wealth accumulation in the face of financial frictions results from households trying to save out of their constraints, higher (liquid) wealth accumulation here is simply a result of households trying to self-insure against illiquidity risk.

[50] The excess returns to private equity premium can be computed as $\alpha_k \nu \frac{Y^e}{K^e} - \delta_k - r$, where Y^e is the total output of the entrepreneurial sector, and K^e is the total capital allocated to the entrepreneurial sector. I refer the reader to Appendix A.7.1 for further explanation of why this is the appropriate definition.

entrepreneurs "could have" substantially improved on their investment portfolio by investing more in their firms. However, when we focus on the downsize risks, we see that the average entrepreneur who receives a negative return incurs substantially more losses under the baseline calibration. This substantial downsize risk, which is induced by the illiquidity risk, explains why entrepreneurs do not invest more in their firm. In fact, looking at the aggregate risk premium, we see that the cost of illiquidity risk amounts to around 2.6 percentage points in forgone returns.

Table 2.19: Excess returns to capital computed at different cutoffs, as well as the overall average and the aggregate risk premium[51]for entrepreneurial investment.

					Excess returns to capital at percentile:			
Model	Aggregate risk premium	Average	Average $(r_{i,t}^k \leq 0)$	Average $(r_{i,t}^k > 0)$	20%	40%	60%	80%
Baseline	15.6%	31.6%	-7.75%	42.8%	-1.45%	13.0%	28.6%	51.7%
Liquid k	12.7%	18.4%	-5.15%	28.4%	-2.98%	3.42%	12.8%	31.9%

One might recall from Table 2.17 that entrepreneurs below the median in excess returns to wealth earn higher returns to wealth under the baseline calibration. This outcome is in fact reflective of the mechanisms discussed here. Entrepreneurs, faced with illiquidity risk, are more cautious in their investment, reducing their overall exposure to their downsize risk. This allows poor performing entrepreneurs to earn higher returns than their counterparts who do not face any illiquidity risks. However, this also means that they forgo the upside benefits that come with investing in a risky asset, giving up the (unbounded) high returns. As a result, entrepreneurs earn lower average returns to wealth when faced with illiquidity risks.

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[51] For reference, Kartashova (2014) reports a private equity risk premium of 16.5% for the sample period of 2005 to 2007.

2.6 Policy Analysis: Providing Partial Insurance

In Section 2.5, we saw that removing resale frictions can lead to substantial long run gains to welfare and total factor productivity. However, to the extent that these illiquidity frictions (and thus risks) are inherent to the economy, this proposition would not be independently feasible without external intervention. In this section, I propose a simple experiment in which a fiscal authority provides partial insurance against illiquidity risk, and explore its implications for the macroeconomy.

In this experiment, the government commits to a fixed budget G to help entrepreneurs defray the cost of downsizing[52]. Specifically, the government commits to lowering the transaction cost faced by entrepreneurs from λ to $\tilde{\lambda}$ for every unit of used capital transacted; the total cost is then paid for by G. From the perspective of households, this policy has two benefits: First, it increases the resale value of capital, and therefore reduces the amount of illiquidity risk the individual faces; second, it relaxes the collateral constraint. To finance this policy, the government taxes bond returns (i.e., a capital gains tax). The cost is therefore only borne by relatively wealthier households who have positive liquid savings. Using the notation established in Section 3.2, the equilibrium of this model can be defined as follows:

Definition 2 *A stationary competitive equilibrium with partial insurance consists of the interest rate r, wage rate w, used capital resale price $1 - \tilde{\lambda}$, tax rate τ, value functions of households and firms $\{V_e, V_w, \Pi\}$, allocations $\{k', b', l\}$, and distribution of agents Λ over the state space $\mathbb{S}$ such that,*

   1. Taking r, w, $1 - \tilde{\lambda}$, and τ as given, the households' and firms' choices are optimal.

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[52] This proposed policy is similar in spirit to a capital loss tax subsidy, as currently practiced by the Internal Revenue Service (IRS). I refer readers to Section 1231 (as well as Sections 167, 1245, and 1250) of the US tax code for more details.

2. The government's budget, where G is a parameter, balances:

$$G = \tau r \int b \times 1_{\{b \geq 0\}} d\Lambda$$

3. The cost of the policy cannot exceed the government's budget, that is,

$$G \geq \left(\lambda - \tilde{\lambda}\right) \int \left(k' - (1 - \delta) k\right) \times \mathbb{I}_{\{k' - (1-\delta)k < 0\}} d\Lambda$$

$$\mathbb{I}_{\{k' - (1-\delta)k < 0\}} = \begin{cases} 1 & \text{, if } k' - (1 - \delta) k < 0 \\ 0 & \text{, if } k' - (1 - \delta) k \geq 0 \end{cases}$$

4. Factor markets clear,

   (a) Bonds: $\int b' d\Lambda = K^c$

   (b) Labor: $\int \theta h d\Lambda = \int l d\Lambda + L^c$

5. The distribution Λ is time-invariant, given by

$$\Lambda = \Gamma(\Lambda)$$

   Where Γ is the one-period transition operator on the distribution.

The policy proposed is motivated by two reasons. First, recall that due to illiquidity risk, entrepreneurial capital is misallocated relative to the counterfactual economy, in which low productivity entrepreneurs operate firms that are too large, while high productivity entrepreneurs operate firms that are too small. This suggests that a policy that reduces illiquidity risk can potentially bring sizable efficiency gains by appropriately realigning capital allocation with productivity. Second, illiquidity risk drives entrepreneurs to "overaccumulate" liquid wealth in a fashion that is reminiscent of the effect studied by Aiyagari (1995): that is, the effect that precautionary

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savings at the individual level leads to larger aggregate savings relative to a complete markets economy. As such, taxation on bond returns would serve as a simple and direct mechanism to address this effect.

In Table 2.20, I report the welfare gains and TFP of the preceding model relative to the benchmark calibration.

Table 2.20: Effect on welfare, TFP, size of entrepreneurial sector and wealth dispersion. Columns 1, 2, and 3 report the average percentage consumption equivalent variation for workers, entrepreneurs, and the whole economy; column 4 reports the percentage TFP gains relative to the benchmark economy; column 5 ("% Δ E") reports the increase in number of entrepreneurs relative to the baseline; and column 6 reports the wealth dispersion as summarized by the Gini coefficient.

Model	Welfare change			TFP gains	% Δ E	Gini
	Workers	Entrepreneurs	Economy wide			
Illiquidity + Collateral	0.54%	0.78%	0.56%	4.5%	3.8%	0.71
Illiquidity only	0.36%	0.83%	0.40%	4.0%	4.6%	0.70

Table 2.21: Effect on prices. Columns 1 and 2 report the interest rates and wage rates under the baseline model and policy counterfactuals. Column 3 reports the resale value of capital, and column 4 reports the tax rate required to finance this policy.

Model	Interest rate	Wage rate	Resale value	Tax rate
Baseline	3.50%	1.04	0.31	0%
Illiquidity + Collateral	3.17%	1.05	0.57	8.35%
Illiquidity only	3.25%	1.05	0.58	8.15%

Here, I solve two versions of the same model. In the first row ("Illiquidity + Collateral"), I report the results when λ is unconditionally lowered to $\tilde{\lambda}$; hence, the government's policy relaxes both the illiquidity and collateral constraint effect. In the second row ("Illiquidity only"), I report the same experiment, but holding the net collateral constraint fixed; that is, I reset φ such that $\varphi(1 - \lambda)$ is fixed. This isolates the effect of this policy to only the illiquidity effect. This allows me to decompose the effect of fiscal policy into its effect through the illiquidity channel and the collateral constraint channel. I also report average welfare gains for subsets of the population (i.e., workers and entrepreneurs) in order to decompose the effect of this policy across

occupational types. For both exercises, I set $G = 0.01$, which amounts to about 0.5% of steady-state GDP under the benchmark calibration.

The first result we see is that households are, on average, better off under this fiscal policy despite the relatively high tax rate they face. Moreover, TFP increases by about 4%. Given the substantial distortion arising from these resale frictions, this is not surprising. With the new policies, entrepreneurial talent and capital are now better allocated, despite the distortionary tax used to finance the program. As an example, in Figure 2.13, I report (a) the average firm size as a function of productivity, and (b) the distribution of productivities, before and after the reform in which both the illiquidity and collateral effects are alleviated. This figure is the counterpart to Figures 2.10 and 2.12a from the earlier section. Here, we see that under the reform, low productivity firms are now smaller and high productivity firms are larger. Moreover, we also see fewer low productivity firms under the reform, and we see an expansion of the entrepreneurial sector (column 5 of Table 2.20) Finally, we see that the wealth dispersion increases after the reform (column 6 of Table 2.20). This reflects the better allocation of capital toward high productivity entrepreneurs, who are better able to generate and thus accumulate wealth.

Going back to Tables 2.20 and 2.21, we see that these policies benefit both workers and entrepreneurs. Looking first to the workers, we see that even a reform that only alleviates the illiquidity effect also benefits workers, even though they do not directly utilize this policy. This happens because the lifetime utility of workers incorporates both the option to become an entrepreneur and the option to stay a worker. Decreasing illiquidity increases the value of entrepreneurship, and therefore directly increases the welfare of workers. In addition, when we also factor in the collateral effect, we see that worker welfare improves further. This is because workers are better able to enter into entrepreneurship now that they face looser borrowing constraints. Finally, workers are also partly better off because of the general-equilibrium effects of the
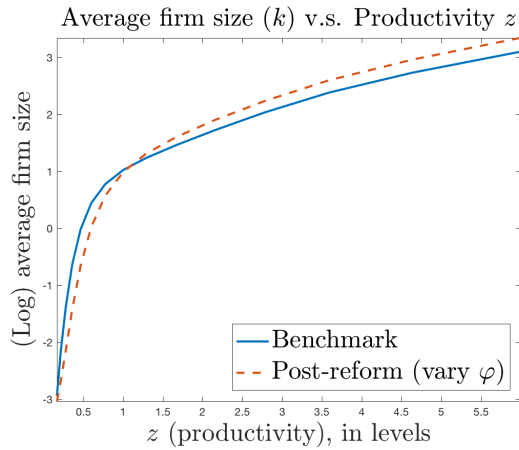
(a) Average firm size (log scale) as a function of productivity (levels).
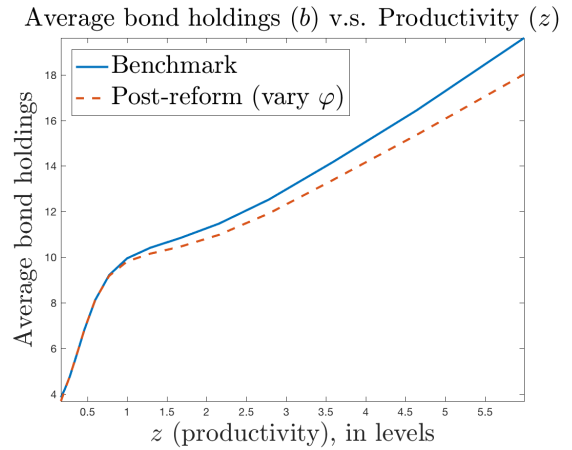
(b) Average bond holdings as a function of productivity (levels).
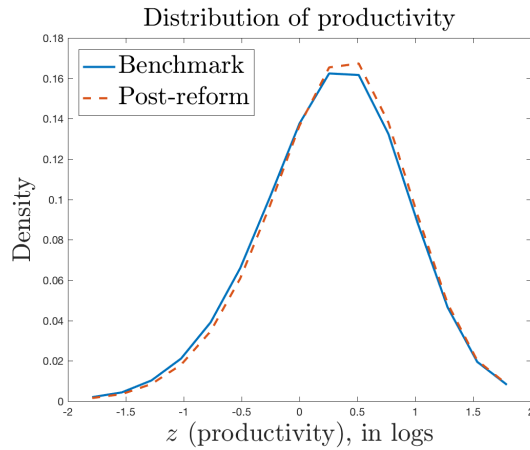
(c) Distribution of productivity (log scale)

FIGURE 2.13: Allocation of capital, bonds and productivity before (solid blue lines) and after the policy reform (dashed red lines). After the reform, low productivity firms are smaller and high productivity firms are larger. High productivity entrepreneurs are also more leveraged, reflecting more efficient borrowing. There are also fewer low productivity firms and more high productivity firms.

policy on prices. With wealth better allocated, the capital-labor ratio increases as households are richer and more households pursue entrepreneurship. Consequently, the wage rate increases, which benefits workers directly.

Unsurprisingly, entrepreneurs also benefit from this policy, since it directly affects the allocation of capital among entrepreneurs. However, we see that a policy that addresses both the illiquidity and collateral effects actually delivers lower welfare gains for entrepreneurs than a policy that strictly pays for partial insurance only.
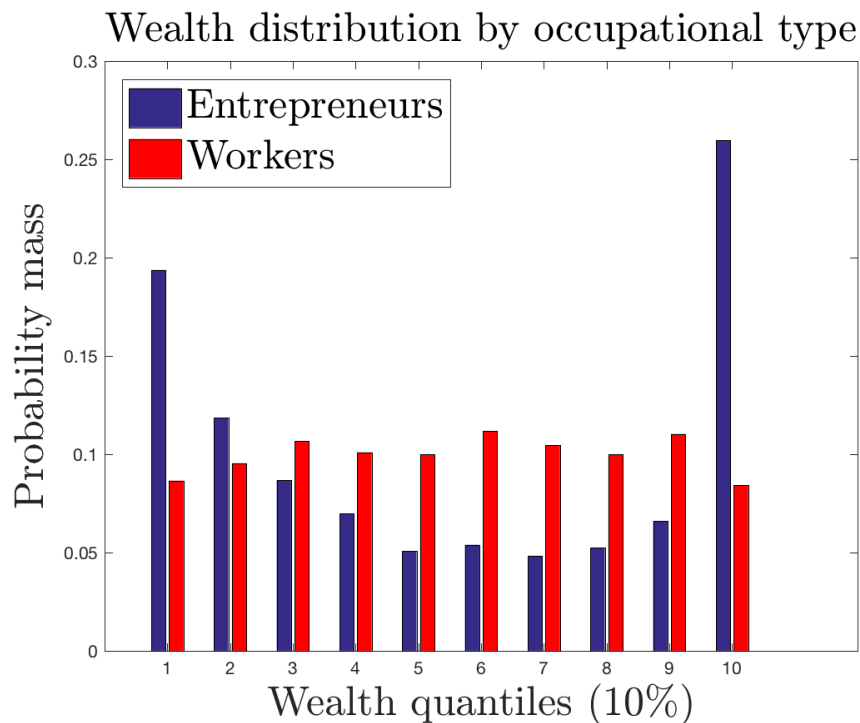
Figure 2.14: Distribution of wealth in deciles in the benchmark economy, for workers and entrepreneurs. The distribution is renormalized by the appropriate measure (i.e., number of entrepreneurs and workers). Wealth is noticeably more concentrated among entrepreneurs.

Why is this so? This result is best understood when we put into context who the taxpayers are and who benefits from the two policies. In Figure 2.14, we see that wealth is substantially more concentrated among the entrepreneur population than the worker population. Consequently, a disproportionate amount of the tax burden falls on entrepreneurs. However, rich entrepreneurs do not directly benefit from an alleviation of collateral constraints, since they are already financially unconstrained. In contrast, entrepreneurs across the wealth distribution will benefit from alleviation of the illiquidity effect, since even very rich entrepreneurs can use this policy when they downsize. Consequently, they prefer a policy in which all tax receipts are put toward decreasing the level of illiquidity.

This effect can be seen when we overlay the distribution of welfare gains on the wealth distribution for entrepreneurs under the two policies, as in Figure 2.15. We see that entrepreneurs in the bottom 50% generally prefer the "Illiquidity + Collateral"
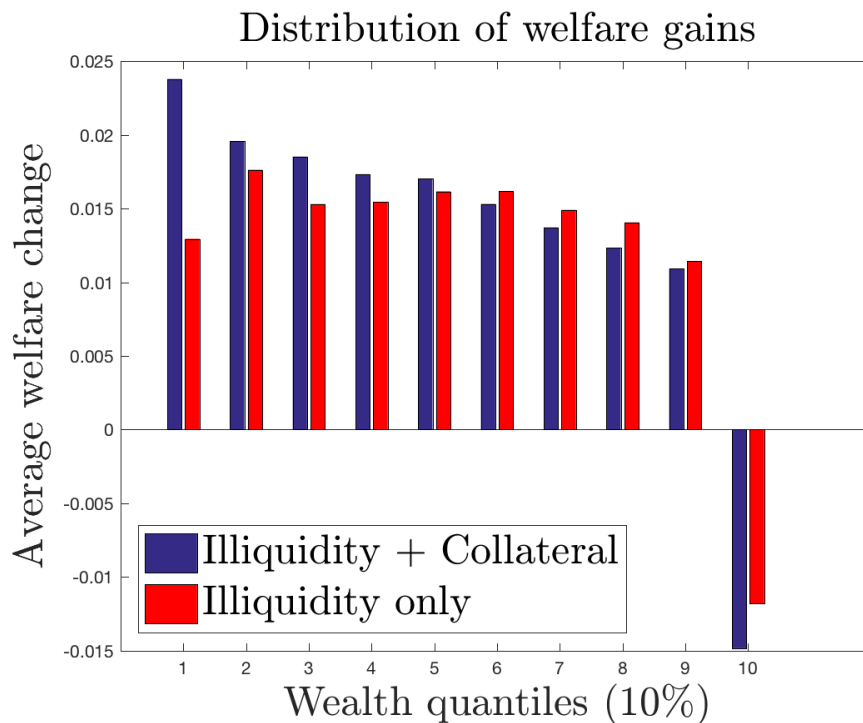
FIGURE 2.15: Distribution of welfare gains (in consumption equivalent variation) across the wealth distribution (in deciles) for entrepreneurs, under the two policies.

policy over the "Illiquidity only" policy, since these are the entrepreneurs who are most likely to be financially constrained. In contrast, entrepreneurs above the median prefer the "Illiquidity only" policy, reflecting the fact that they do not face especially tight borrowing constraints. The richest 10% of entrepreneurs suffer a welfare loss, as the bulk of the policy is financed by them. However, the welfare loss under the "Illiquidity only" policy is smaller, reflecting a preference for this policy. Combining this result with the wealth distribution in Figure 2.14, we can see immediately why the average welfare gains are higher for an entrepreneur under the "Illiquidity only" policy.

Finally, in the third column, we see that average welfare gains are larger by about 0.16 percentage points if the same budget is used to alleviate both the collateral and illiquidity constraint effects, rather than just the illiquidity effects. A naive decomposition therefore shows that about 29% of the full impact of the policy can

be attributed to the alleviation of collateral constraints, which is a sizable amount. However, the bulk of the policy's impact still derives through the illiquidity effect.

2.7 Conclusion

In this paper, I studied extensively the investment dynamics of entrepreneurs and their relationship with the distribution of wealth, capital, and entrepreneurial productivity. The paper makes four key contributions.

The first contribution is in documenting empirically that the distribution of the rates of return to capital among early-stage entrepreneurs, as proxied by the log average revenue product of capital, is highly left-skewed. Moreover, entrepreneurs with low rates of return persist in this state for longer periods of time than their high-return counterparts. This observation runs contrary to prior literature on entrepreneurial investment, which emphasizes collateral constraints as a key driver of entrepreneurial investment behavior. As shown in this paper, financing constraints generate right-skewness and right-tail persistence, which is counterfactual to my empirical findings.

The second contribution is to build a parsimonious model of entrepreneurship to rationalize these findings. Specifically, I show that these facts can be easily reconciled within a standard macroeconomic model of entrepreneurship once we explicitly model entrepreneurial capital as an illiquid asset. When capital is no longer a perfect substitute for bonds, poor performing entrepreneurs hold on to capital for extended periods of time due to a real options value effect. At the aggregate level, this manifests as a left-skewed distribution of rates of return to capital, as well as a left-tail persistence.

The paper's third contribution is to connect entrepreneurial investment — and the dispersion in returns to capital — back to the wealth distribution. My paper provides empirical support for the mechanism theorized in the prior literature regarding entrepreneurship and wealth dispersion; that is, entrepreneurial investment and savings explain the large wealth dispersion and concentration in the economy.

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However, I obtain two important results that deviate from prior literature. When entrepreneurial capital is illiquid, the average returns to wealth are lowered for all households. Consequently, the resulting wealth distribution features lower wealth inequality than what has been reported in the literature. Moreover, wealth dispersion in my model arises in part from a misallocation of capital. Relative to a counterfactual economy where capital is fully liquid, there are substantially more poor performing wealthy entrepreneurs. This leads to welfare and TFP losses relative to the counterfactual economy.

The last contribution relates to fiscal policy. Here, I show that a government program that subsidizes the sale of entrepreneurial capital can substantially improve welfare and TFP. Moreover, this program is fully funded by a tax on liquid bond returns, suggesting that a reallocation of wealth can bring about important welfare benefits.

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Uncertainty Shocks and Entrepreneurship

3.1   Introduction

What is the effect of uncertainty shocks on entrepreneurship, and how does it affect the overall population of households, as well as the broader aggregate economy? In this paper, I bring in new evidence regarding the relationship between uncertainty shocks and entrepreneurial startup and firm exit rates, and then construct a parsimonious general-equilibrium incomplete markets model of entrepreneurship and uncertainty shocks to explain these facts. Using this model, I then argue that a combination of perceived heightened uninsurable risks faced by entrepreneurial households, and an extensive margin of adjustment for entrepreneurial firms, can magnify and prolong recessions driven by uncertainty shocks.

I first begin by providing some new stylized evidence on the effect of uncertainty shocks on the extensive margin of adjustment for entrepreneurs. In figures 3.1 and 3.2, I plot a time series of the entry and exit rates of entrepreneurial firms, i.e., startup formation and firm exit[1], along with the economic policy uncertainty index

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[1] The entry and exit rates are computed using both the BDS and the PSID. In the case of the BDS, "entrepreneurial firms" are defined as firms with employment sizes of less than 250 employees, and the entry and exit rates pertain to these firms. In the case of the PSID, which is a survey of

constructed by Baker et al. (2016). Two stylized facts emerge: First, we see that entry is highly negatively correlated with economic policy uncertainty, regardless of whether we are computing entry using the aggregated data from the BDS (correlation of -0.64), or microdata from the PSID (correlation of -0.46); Second, we see that exit is only very weakly correlated with economic policy uncertainty, a statement which again holds true when we compute exit using the BDS (correlation of 0.04) or the PSID (correlation of 0.04).

In fact, looking to the microdata in the PSID, panel probit regressions reveal that a one standard deviation increase in (log) policy uncertainty (evaluated at the mean) leads to a statistically significant decrease of 0.28 percentage points in the entry propensity[2]. Given that entry rate in the PSID is only 3% on average, this is an economically significant number. In contrast, the same increase in economic policy uncertainty has only statistically insignificant effects on exit propensities.
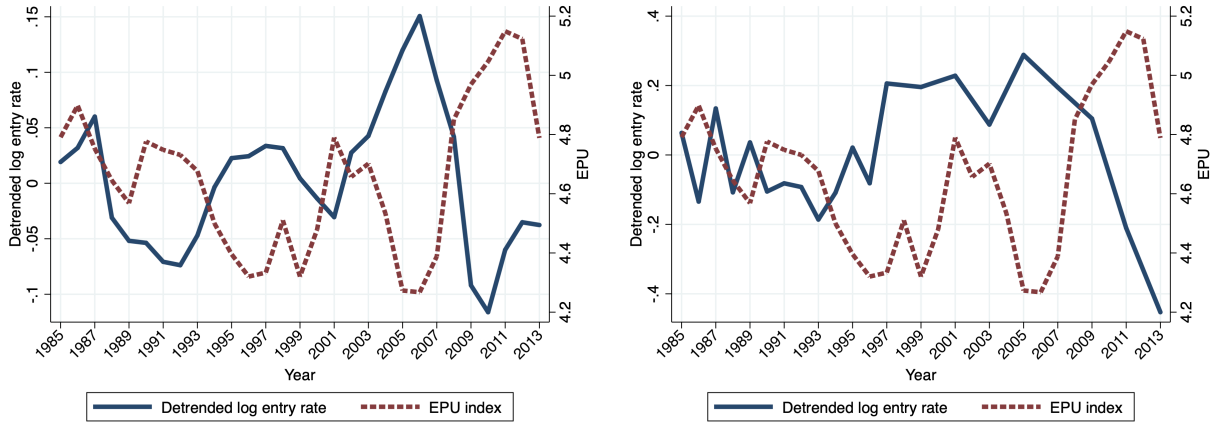
(a) BDS, $corr(entry, EPU) = -0.64$ (b) PSID, $corr(entry, EPU) = -0.46$

FIGURE 3.1: Entry rates in the BDS (left) and PSID(right) and economic policy uncertainty. Entry rates are in logs and detrended using a linear trend.

Why does the entry and exit margin vary so differently with respect to an uncertainty shock? A contribution of this paper is to explain these facts using a model of

households, an "entrepreneurial firm" is synonymous with an individual who runs a business. For further discussion of their exact definitions, I refer the reader to Appendix A

[2] The entry propensity refers to the probability that an individual enters into entrepreneurship.
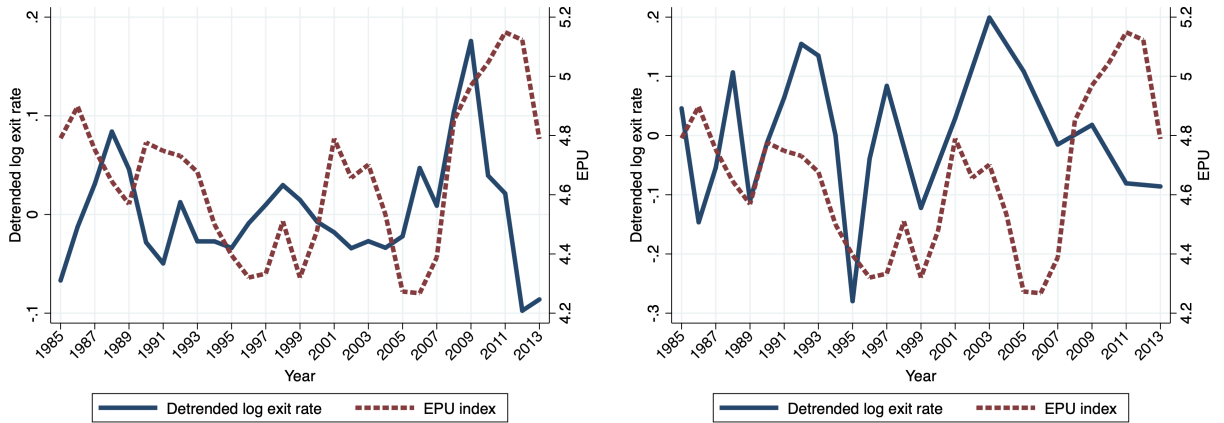
(a) BDS, $corr(exit, EPU) = 0.04$ **(b)** PSID, $corr(exit, EPU) = 0.04$

FIGURE 3.2: Exit rates in the BDS (left) and PSID(right) and economic policy uncertainty. Exit rates are in logs and detrended using a linear trend.

endogenous entrepreneurial choice, where entrepreneurial capital is illiquid as in Tan (2018a). As heavily emphasized by the literature, when capital adjustment faces a significant non-convex (fixed) cost element, the real options effect generated by the interaction of the non-convex adjustment costs and uncertainty can induce a strong "wait-and-see" effect[3]. Transient shocks to uncertainty about future profitability can therefore greatly decrease the value of starting a business, leading to a fall in entry propensities. However, along the exit margin, uncertainty shocks exert two opposing forces. Along one dimension, heightened uncertainty generates a similar wait-and-see effect that decreases exit propensities for some entrepreneurs. This happens because the real options effect increases the value of delaying disinvestment decisions (of which exit is a form of). However, low performing entrepreneurs also perceive a higher probability of receiving negative profitability shocks if they stay in business. In contrast, exiting their business allows entrepreneurs to draw fresh ideas in the next period that are uncorrelated with their current business prospects. As a result, the continuation value of entrepreneurship falls relative to their outside option, driving up exit propensities. At the aggregate level, these two competing forces manifest themselves

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[3] See, for instance, Bloom (2009), Bachman and Bayer (2014) and Bloom et al. (2018)

as an ambiguous effect on exit, thus leading to a low correlation between exit and uncertainty.

The adjustment along the extensive margin, in response to an uncertainty shock, has important implications for business cycle dynamics. The entrepreneurial sector shrinks after an uncertainty shock, decreasing overall labor demand; in contrast, the labor supply increases as more individuals pursue labor work. As a consequence, this generates a much sharper fall in the wage rate, relative to a model where the extensive margin of adjustment was shut down. Moreover, as it takes time for the entrepreneurial sector to rebuild and return to its steady-state size, output from the entrepreneurial sector is persistently depressed for a long period of time. The combined fall in the size of this sector, along with the long recovery time, generates a much deeper and longer recession than a similar model without an extensive margin of adjustment. Qualitatively, this suggests that models that abstract from the extensive margin might under-estimate the impact of uncertainty shocks.

3.1.1 Related literature

This paper relates to several strands of literature that study the effect of uncertainty and uncertainty shocks on entrepreneurial risk taking, household savings, and firm investment.

First, this paper is most directly related to the firm dynamics literature that has studied the impact of uncertainty shocks on firm investment. Similar to Bloom (2009), Bachman and Bayer (2013), Bachman and Bayer (2014), Gilchrist et al. (2014), Basu and Bundick (2017), and Bloom et al. (2018), I find that uncertainty shocks can generate a steep recession and sharp declines in investment. Unlike their paper, my results pertain to an economy where firms are fully owned by individual households, and where these business risks are uninsurable. As a result, I find that the precautionary savings motive that arises during an uncertainty shock also depresses consumption

during impact, and does not feature the "volatility overshoot" documented in Bloom et al. (2018). Moreover, this outcome also arises due to a standard consumption smoothing mechanism, rather than due to nominal rigidities as in Basu and Bundick (2017).

Second, this paper also relates to a recent set of literature that has emphasized the importance of including an extensive margin of adjustment in otherwise standard heterogeneous firm dynamics models. As in recent papers such as that by Clementi and Palazzo (2016) and Lee and Mukoyama (2018), interactions between the extensive and intensive margin of adjustment have an important channel in magnifying and propagating business cycle shocks. Similar to the prior literature, I find that a model with an active extensive margin of adjustment generates a recession that is 20% deeper than one without such a margin of adjustment. Unlike this literature, however, the main focus of this paper is on the impact of uncertainty shocks in a model of endogenous entry and exit, while the prior literature explores only the impact of first moment shocks (such as TFP shocks). Moreover, this paper emphasizes a new mechanism in which uncertainty affects entry and exit — specifically, the interaction of an innovation to the spread of outside options with partial irreversibility of investment. Finally, this paper emphasizes the role of precautionary savings and incomplete markets, which departs from the prior literature such as Clementi and Palazzo (2016), which looks at primarily a model of complete markets and risk-neutral firms.

Finally, this paper also relates more broadly to the macroeconomics literature that has emphasized the role of small or entrepreneurial firms in business cycles, such as that by Gertler and Gilchrist (1994), Rampini (2004), and Bassetto et al. (2015). The main focus of these papers is the reaction of the entrepreneurial sector to first moment shocks or financial shocks, while this paper primarily addresses the impact of shocks to idiosyncratic uncertainty (i.e., a second moment shock).

The rest of this paper is organized as follows. In Section 2, I describe the model

environment which I will use to analyze the impact of an uncertainty, and in Section 3, I briefly discuss the model calibration. In Section 4, I present and discuss the model results. Section 5 concludes this paper, and discusses future work.

3.2 Model

This section describes the baseline model which I will use to analyze the effect of an uncertainty shock. The model is an incomplete markets entrepreneurship model, following the setup in Tan (2018a)[4]. The uncertainty shock will be modeled as an unexpected (temporary) one time shock that increases the level of uncertainty as it pertains to the entrepreneur's profits.

3.2.1 Households and Production

The economy is populated by a continuum of households, each indexed by $i \in [0, 1]$, as well as a continuum of representative corporate firms. Households and firms are infinitely lived, and time is discrete. As in Bewley (1977), individuals are subject to idiosyncratic shocks; moreover, individuals are also subjected to aggregate shocks in the form of uncertainty shocks.

Preferences and Discounting

All households are endowed with identical time-separable utility function with constant relative risk aversion (CRRA), and discount future utility at rate β. Households value non-durable consumption c, a choice which depends on the household's endowment of other factors and savings choices. The household's lifetime expected utility

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[4] I follow this setup given its ability to successfully capture the investment behavior of entrepreneurs.

can therefore be written as

$$V_0 \equiv \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U\left(c_{i,t}\right)$$

$$= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{c_{i,t}^{1-\gamma}}{1-\gamma}$$

where γ is the coefficient of relative risk aversion. The household's objective is to maximize expected lifetime utility by choosing a sequence of consumption $\{c_{i,t}\}_{t=0}^{\infty}$.

Production Technology

An individual begins each period as either a worker (W) or an entrepreneur (E), as well as a portfolio of liquid risk-free assets $(b_{i,t})$ and illiquid physical capital $(k_{i,t})$[5]. Occupational type and asset allocations are chosen in the last period. In this section, I will focus only on the production technologies available to the households and firms, delaying a discussion of the asset structure to the next section.

Workers   If the individual is a worker, she supplies inelastic labor efficiency units $\theta_{i,t}$ to a spot market, and receives a wage income of $w_t \theta_{i,t}$. $\theta_{i,t}$ is an idiosyncratic shock to labor productivity, and follows an AR(1) process that depends on the individual's last period occupation. Specifically, if the individual was also a worker in the last period, then θ evolves as follows:

$$\log \theta_{i,t} = \mu^\theta \left(1 - \rho^\theta\right) + \rho_\theta \log \theta_{i,t-1} + \sigma^\theta \epsilon_{i,t} \tag{3.1}$$

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[5] As discussed in Tan (2018a), the entrepreneurs in fact hold a mix of liquid bonds, liquid stocks, and illiquid capital. However, the lack of aggregate risk in this economy implies that bonds and stocks are perfect substitutes, and for the sake of brevity, I will collectively refer to them as liquid risk-free assets.

where $\theta_{i,t-1}$ is the worker's productivity last period; and if the individual was an entrepreneur, then it evolves as

$$\log \theta_{i,t} = \mu^\theta \left(1 - \rho^\theta\right) + \rho_\theta \log \psi^\theta_{i,t-1} + \sigma^\theta \epsilon_{i,t} \tag{3.2}$$

where $\psi^\theta_{i,t-1}$ is an initial condition drawn by the individual prior to switching into labor work . $\psi^\theta_{i,t-1}$ is assumed to be drawn from the mean-shifted invariant distribution of θ:

$$\log \psi^\theta_{i,t-1} \sim N \left(\mu_{\psi^\theta}, \frac{(\sigma^\theta)^2}{1 - (\rho^\theta)^2}\right) \tag{3.3}$$

In other words, $\psi^\theta_{i,t-1}$ is drawn from a normal distribution with the same variance as that of θ, but a (potentially) different mean.

Entrepreneurs If the individual is an entrepreneur, she operates a production technology that combines external labor hired from a spot market (l), endowed labor $(\bar{l})$, capital (k), common productivity A^z_t and idiosyncratic entrepreneurial productivity $z_{i,t}$ to produce homogeneous output $y_{i,t}$ as given by the production function below:

$$y_{i,t} = A^z_t z_{i,t} \left(k_{i,t}^{\alpha_e}(\bar{l} + l_{i,t})^{1-\alpha_e}\right)^\nu$$

,where $\alpha_e \leq 1$ is the capital intensity, and $\nu < 1$ is the span-of-control parameter that determines the degree of decreasing returns to scale[6]. A^z_t is a common shock to entrepreneurial productivity. In the baseline model, this is assumed to be constant. $z_{i,t}$ is an idiosyncratic shock to entrepreneurial productivity, and follows an AR(1) process that depends on the last period occupational type of the household. If the

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[6] As in Lucas (1978), this captures the idea that managerial skills become stretched over larger and larger projects. The endowed labor $\bar{l}$ is a parameter and assumed to be constant across households, and is a perfect substitute for externally hired labor l, which costs w per unit of labor efficiency units hired.

individual was also an entrepreneur in the last period, then z evolves as follows:

$$\log z_{i,t} = \mu_t^z \left(1 - \rho^z\right) + \rho_z \log z_{i,t-1} + x_t^z \sigma^z \epsilon_{i,t} \tag{3.4}$$

where $\log z_{i,t-1}$ was the entrepreneur's productivity in the last period; and if she was a worker, then it evolves as

$$\log z_{i,t} = \mu_t^z \left(1 - \rho^z\right) + \rho_z \log \psi_{i,t-1}^z + x_t^z \sigma^z \epsilon_{i,t} \tag{3.5}$$

where $\psi_{i,t-1}^z$ is an initial condition drawn by the individual prior to becoming an entrepreneur. $\psi_{i,t-1}^z$ is assumed to be drawn from the following distribution:

$$\log \psi_{i,t-1}^z \sim N\left(\mu^{\psi^z}, \frac{\left(x_{t-1}^{\psi^z}\sigma^z\right)^2}{1 - (\rho^z)^2}\right) \tag{3.6}$$

Uncertainty shocks

Uncertainty shocks appear in this model through two channels: x_t^z, as it appears in equation 3.4 and 3.5, and $x_t^{\psi^z}$, as it appears in equation 3.6. These two shocks have different roles. x_t^z increases the conditional volatility of the entrepreneur's and entrant's productivity, which makes continuation or entry into entrepreneurship riskier when it increases in the current period. On the other hand, $x_t^{\psi^z}$ increases the spread of the distribution of initial conditions that the potential entrants can draw prior to entry. This means that workers who decide to wait one period before entering also face greater risk; likewise, current entrepreneurs who decide to exit also face greater risks.

Note that μ_t^z, the unconditional mean of the z processes, is also allowed to vary with time. In the baseline model, μ_t^z is simply adjusted to ensure that the shocks to z are mean-preserving spreads in levels; that is, $\mu_t^z = \frac{1}{2}\left(x_t^z - x_0^z\right)^2 \sigma_z^2$. This avoids the

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effect where an uncertainty shock, especially a large shock, drastically increases the average productivity of the entrepreneur[7].

Representative corporate firms

In the real economy, a substantial fraction of investment and hiring are done by large corporate firms. Therefore, following Cagetti and De Nardi (2006), I model this sector as a second sector of production populated by a large number of homogeneous firms operating a constant returns to scale production technology given by

$$Y_t^C = A_t \left(K_t^C\right)^\alpha \left(L_t^C\right)^{1-\alpha}$$

where A_t is aggregate TFP and α is the capital share of this sector. A_t is allowed to potentially vary over time, but the process is assumed to be deterministic. As a result, there is no aggregate risk in this economy. As in the entrepreneur's choices, the corporate firm's capital stock K^C is determined from last period's choices, while labor L^C is decided in the current period. The corporate firm funds its investment by issuing claims to its dividend flows (i.e. shares). Finally, I also assume that the corporate firms produce the same homogeneous outputs as entrepreneurial firms.

Asset Structure

As discussed earlier, households have the option to hold a portfolio of illiquid physical capital and liquid risk-free assets, conditioned on their occupational choice.

Illiquid physical capital   Only households who elect to become (or stay) entrepreneurs the next period can save in illiquid physical capital k, which depreciates at rate δ_k. Capital is illiquid because of buying and selling frictions associated with adjusting the

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[7] This issue arises due to the log-normal assumption. The mean of the ψ^z process, however, does *not* need to be adjusted. This is because what matters for the worker is simply the relative starting positions of the initial conditions, rather than the actual levels.

capital stock. On the resale side, I assume that the entrepreneur faces two types of downsizing frictions which are associated with the entrepreneur's future occupational choice. When the entrepreneur decides to downsize, but still continue in entrepreneurship, she faces a per-unit transaction cost λ, such that she recoups $(1 - \lambda)$ of the transacted asset. If the entrepreneur decides to fully exit the business, she has to pay an additional proportional selling cost ζ on top of the earlier transaction cost. As a result, the net return from selling a unit of capital when exiting is $= (1 - \zeta)(1 - \lambda)$. This implies that small scale changes along the intensive margin are less costly than large changes along the extensive margin. This asymmetry therefore also influences how entrepreneurs react to an uncertainty shock in terms of exit.

On the investment side, entrepreneurs face a proportional fixed cost when they choose to expand their capital stock. This modeling assumption is similar to that taken by Bloom et al (2018), and in the face of an uncertainty shock, generates an additional wait-and-see attitude with regards to investment.

Liquid risk-free assets All individuals can trade liquid risk-free assets b to smooth inter-temporal consumption. For savers, they cost one unit of consumption today, and return a risk free rate of r_{t+1} the next period; for borrowers, every unit of debt taken on costs the individual an interest rate of $r_{t+1} + \phi_d$ tomorrow. ϕ_d is an intermediation cost that borrowers have to bear. For individuals who are borrowing, they have two options to borrow. Firstly, all individuals are allowed to borrow via unsecured debt, where they face an ad-hoc borrowing constraint $\underline{b}$. Secondly, individuals that are entrepreneurs also own capital, and they are allowed to use the liquid value of capital as collateral to borrow a larger volume. This debt can be used to finance more investment into their firm, or simply for smoothing consumption. Here, the liquid value of capital is simply the resale value of capital, i.e. $(1 - \lambda)(1 - \delta_k) k$. Therefore,

all households face the following borrowing constraint

$$b_{i,t+1} \geq -\varphi \left(1 - \lambda\right) \left(1 - \delta_k\right) k_{i,t+1} - \underline{b}$$

where $\varphi \in [0, \infty)$, with $\varphi = 0$ representing no collateralized borrowing, and $\varphi \to \infty$ representing no collateral required for borrowing. As is common in this literature[8], φ capture the idea of limited enforceability of debt contracts, while as discussed in Tan (2018a), $(1 - \lambda)$ further tightens the individual's borrowing constraints due to the lower resale value of capital.

3.2.2 Individuals' Problems

The household's problem

At the beginning of the period, an individual is characterized by her occupational type $h_{i,t} \in \{W, E\}$ and her asset holdings $(k_{i,t}, b_{i,t})$. If the individual is an entrepreneur, she also starts the period with entrepreneurial productivity shock $z_{i,t}$, and an outside option shock $\psi_{i,t}^{\theta}$ as discussed earlier. If the individual is a worker, she starts the period with a labor productivity shock $\theta_{i,t}$, and an outside option shock $\psi_{i,t}^{z}$. For brevity, denote the vector of idiosyncratic state variables for the entrepreneurs as $\vec{se}_{i,t} \equiv \left(\psi_{i,t}^{\theta}, z_{i,t}, k_{i,t}, b_{i,t}\right)$, the vector of idiosyncratic state variables for the workers as $\vec{sw}_{i,t} \equiv \left(\psi_{i,t}^{z}, \theta_{i,t}, b_{i,t}\right)$, and the vector of aggregate state variables as $\Omega_t \equiv \left(x_t^z, x_t^{\psi^z}, r_t, w_t\right)$.

Let the value functions of entrepreneurs and workers be represented by V^e and V^w respectively, and the adjustment cost function for capital (as discussed earlier) by $\mathcal{C}\left(k_{t+1}, k_t, h_{t+1}\right)$. Entrepreneurial households solve the following recursive problem

$$V^e\left(\vec{se}_{i,t}; t\right) = \max\left\{V^{ee}\left(\vec{se}_{i,t}; t\right), V^{ew}\left(\vec{se}_{i,t}; t\right)\right\} \tag{3.7}$$

---

[8] See for instance, Buera and Shin (2013).

where V^{ee} and V^{ew} are the value functions of the entrepreneur conditioned on choosing to stay in entrepreneurship, or exit into labor work. Note that the time subscripts are made explicit, as the distribution is not stationary due to the presence of aggregate (deterministic) shocks.

The entrepreneur who decides to stay in entrepreneurship solves

$$V^{ee}\left(\vec{se}_{i,t};t\right) = \max_{k_{i,t+1},b_{i,t+1}} U\left(c_{i,t+1}\right) + \beta \int_{\psi^{\theta}_{i,t+1}} \int_{z_{i,t+1}} V^{e}\left(\vec{se}_{i,t+1};t+1\right) dP_{z_{t+1}|z_t} dF_{\psi^{\theta}_t}$$

(3.8)

s.t.

$$\hat{\pi}_{i,t} \equiv y_{i,t} - w_t l_{i,t} + \left(1 + r_t \times 1_{\{b_{i,t} \geq 0\}} + r_d \times 1_{\{b_{i,t} < 0\}}\right) b - \mathcal{C}\left(k_{i,t+1}, k_{i,t}, h_{i,t+1}\right)$$

$$k_{i,t+1} > 0$$

$$c_{i,t} = \hat{\pi} - k_{i,t+1} - b_{i,t+1} \geq 0$$

$$b_{i,t+1} \geq -\varphi\left(1 - \lambda\right)\left(1 - \delta_k\right) k_{i,t+1} - \underline{b}$$

,while the entrepreneur who exits entrepreneurship solves

$$V^{ew}\left(\vec{se}_{i,t};t\right) = \max_{b_{i,t+1}} U\left(c_{i,t}\right) + \beta \int_{\psi^z_{i,t+1}} \int_{\theta_{i,t+1}} V^w\left(\vec{sw}_{i,t};t+1\right) dP_{\theta_{i,t+1}|\psi^\theta_{i,t}} dF_{\psi^z_{i,t+1}} \quad (3.9)$$

s.t.

$$\hat{\pi}_{i,t} \equiv y_{i,t} - w_t l_{i,t} + \left(1 + r_t \times 1_{\{b_{i,t} \geq 0\}} + r_d \times 1_{\{b_{i,t} < 0\}}\right) b - \mathcal{C}\left(k_{i,t+1}, k_{i,t}, h_{i,t+1}\right)$$

$$k_{i,t+1} = 0$$

$$c_{i,t} = \hat{\pi}_{i,t} - k_{i,t+1} - b_{i,t+1} \geq 0$$

$$b_{i,t+1} \geq -\varphi\left(1 - \lambda\right)\left(1 - \delta_k\right) k_{i,t+1} - \underline{b}$$

where $1_{(.)}$ is an indicator function.

For workers, they solve

$$V^w\left(\vec{sw}_{i,t};t\right) = \max\left\{V^{we}\left(\vec{sw}_{i,t};t\right), V^{ww}\left(\vec{sw}_{i,t};t\right)\right\} \quad (3.10)$$

where V^{we} and V^{ww} are the value functions of the worker conditioned on entering into entrepreneurship, or choosing to stay in labor work.

The worker who decides to enter into entrepreneurship solves

$$V_{we}\left(\vec{s^w}_{i,t};t\right) = \max_{k_{i,t+1},b_{i,t+1}} U\left(c_{i,t}\right) + \beta \int_{\psi^\theta_{i,t+1}} \int_{z_{i,t+1}} V^e\left(\vec{s^e}_{i,t+1};t+1\right) dP_{z_{i,t+1}|\psi^z_{i,t}} dF_{\psi^\theta_{i,t+1}}$$

(3.11)

s.t.

$$c_{i,t} = \theta_{i,t}w_t + \left(1 + r_t \times 1_{\{b_{i,t}\geq0\}} + r_d \times 1_{\{b_{i,t}<0\}}\right) b_{i,t} - k_{i,t+1} - b_{i,t+1}$$

$$k_{i,t+1} > 0$$

$$b_{i,t+1} \geq -\varphi\left(1-\lambda\right)\left(1-\delta_k\right) k_{i,t+1} - \underline{b}$$

while the worker who decides to stay in labor work solves

$$V^{ww}\left(\vec{s^w}_{i,t};t\right) = \max_{b_{i,t+1}} U\left(c_{i,t}\right) + \beta \int_{\psi^z_{i,t+1}} \int_{\theta_{i,t+1}\in\Theta} V^w\left(\vec{s^w}_{i,t};t+1\right) dP_{\theta_{i,t+1}|\theta_{i,t}} dF_{\psi^z_{i,t}}$$

(3.12)

s.t.

$$c_{i,t} = \theta_{i,t}w_t + \left(1 + r_t \times 1_{\{b_{i,t}\geq0\}} + r_d \times 1_{\{b_{i,t}<0\}}\right) b_{i,t} - b_{i,t+1}$$

$$b_{i,t+1} \geq -\underline{b}$$

The corporate firm's problem

The corporate firm objective is to maximize lifetime dividend flows for its shareholders. It decides independently on how much physical capital to invest, and how much

labor to hire at the prevailing wage w. To raise capital for investment, the representative firm can issue equity. Let Π denote the value function of the corporate firm. The representative firm solves the following recursive problem

$$\Pi\left(K_t^c\right) = \max_{K_{t+1}^c} \pi + \frac{1}{1+r_{t+1}}\Pi\left(K_{t+1}^c\right) \tag{3.13}$$

$$s.t.$$

$$\pi = Y_t^c - \left(K_{t+1}^c - (1-\delta)K_t^c\right) - w_t L_t^c$$

where π represents current period dividends paid out to the firms' investors, and the firm discounts future profits at rate $\frac{1}{1+r_{t+1}}$. This market arrangement leads to the standard first order condition for capital and labor demand:

$$r_t + \delta = \alpha A_t \left(\frac{K_t^c}{L_t^c}\right)^{\alpha-1} \tag{3.14}$$

$$w_t = (1-\alpha) A_t \left(\frac{K_t^c}{L_t^c}\right)^{\alpha} \tag{3.15}$$

3.2.3 Equilibrium

Uncertainty shocks: Pure uncertainty and volatility effects

As discussed in Bloom (2009), uncertainty shocks exert their forces on the aggregate economy through two distinct channels: First through a pure uncertainty channel which work through the individual's expectations (i.e. individuals face higher uncertainty); and second through a realized volatility effect where the higher dispersion in shocks pushes individuals towards both tails of the distribution. As there is sparse evidence regarding how much of uncertainty faced by entrepreneurs amount to subjective uncertainty (i.e. expectations that may not align fully with the actual dispersion of the stochastic process) or real uncertainty (i.e. real volatility effects), moving for-

ward, I will discuss in detail both scenarios. To fix ideas and notation, I will define both equilibriums in this section first, and then discuss in greater detail in the next section the setup of these shocks.

Subjective uncertainty equilibrium

Using the notation from earlier, that is letting s_t^e and s_t^w denote the state vector of idiosyncratic states faced by the entrepreneur and worker respectively, and $\Omega_t = \left\{ r_t, w_t, x_t^z, x_t^{\psi^z} \right\}$ the state vector of aggregate variables; and letting $\tilde{x}_t^z$ and $\tilde{x}_t^{\psi^z}$ denote the individual's perception of the shock to the volatility of z such that the aggregate state vector perceived by the household is $\tilde{\Omega}_t = \left\{ r_t, w_t, \tilde{x}_t^z, \tilde{x}_t^{\psi^z} \right\}$, the equilibrium is defined as follows

Definition 3 *A sequential equilibrium with subjective uncertainty is defined by*

1. *A sequence of realized shocks to volatility x_t^z and $x_t^{\psi^z}$*

2. *A sequence of shocks to the individual's perception of the path of the actual shocks: $\tilde{x}_t^z$ and $\tilde{x}_t^{\psi^z}$, where in general, $\tilde{x}_t^z \neq x_t^z$ and $\tilde{x}_t^{\psi^z} \neq x_t^{\psi^z}$*

3. *A sequence of interest rates and wage rate $\{r_t, w_t\}_{t \geq 0}$*

4. *A sequence of value functions: $\left\{ V_{i,t}^e \left(s_t^e, \tilde{\Omega}_t \right), V_{i,t}^w \left(s_t^w, \tilde{\Omega}_t \right), \Pi_t \left(r_t, w_t \right) \right\}_{t \geq 0, i \in [0,1]}$*

5. *A sequence of policy functions: $\left\{ k_{i,t} \left(s_t^e, s_t^w, \tilde{\Omega}_t \right), b_{i,t} \left(s_t^e, s_t^w, \tilde{\Omega}_t \right), h_{i,t} \left(s_t^e, s_t^w, \tilde{\Omega}_t \right) \right\}_{t \geq 0, i \in [0,1]}$*

6. *A sequence of labor demand $\left\{ l_{i,t} \left(s_t^e, \tilde{\Omega}_t \right) \right\}_{t \geq 0, i \in [0,1]}$ from entrepreneurs, and labor supply $\{\theta_{i,t}\}_{t \geq 0, i \in [0,1]}$ from workers*

7. *A sequence of factor demand $\{K_t^c \left(r_t, w_t \right), L_t^c \left(r_t, w_t \right)\}_{t \geq 0}$ from the corporate sector*

8. A sequence of distributions of individuals $\left\{ \Lambda_t \left(s_t^e, s_t^w, \tilde{\Omega}_t, \Omega_t \right) \right\}_{t \geq 0}$

such that

1. Taking $\{r_t, w_t\}_{t \geq 0}$ as given, the households' decision rules and value functions, as in equations 3.7, 3.8, 3.9, 3.10, 3.11, and 3.12, solve the individual problems.

2. Taking $\{r_t, w_t\}_{t \geq 0}$ as given, the representative corporate firm's decision rules and value function, as given in equation 3.13, 3.14, and 3.15, solve the firm's problem.

3. Factor markets clear, where for all t

   (a) Bonds: $\int b_{i,t} d\Lambda_t = K_t^c$

   (b) Labor: $\int \theta_{i,t} 1_{\{h_{i,t}=W\}} d\Lambda_t = \int l 1_{\{h_{i,t}=E\}} d\Lambda_t + L_t^c$

4. The aggregate resource constraint is satisfied, where

$$\int c_{i,t} + k_{i,t+1} 1_{\{h_{i,t+1}=E\}} + b_{i,t+1} + \mathcal{C}(k_{i,t+1}, k_{i,t}, h_{i,t+1}) 1_{\{h_{i,t+1}=E\}} d\Lambda_t$$

$$= \int \pi_{i,t}^* 1_{\{h_{i,t}=E\}} + \theta_{i,t} w_t 1_{\{h_{i,t}=W\}} + (1 + r_t) b_{i,t} +$$

$$(1 - \delta_k) k_{i,t} 1_{\{h_{i,t}=E\}} d\Lambda_t$$

5. The decision rules of the households, along with the exogenous Markov and iid processes, generate the sequence of Markov transition kernels Γ_t which, given any initial distribution of households Λ_0, generates the sequence of distributions $\{\Lambda_t\}_{t \geq 0}$; that is,

$$\Lambda_{t+1} = \Gamma_t (\Lambda_t)$$

where in particular, in steady state, the distribution Λ is time invariant, and prices are constant.

At this point, it is important to point out that the individual's decision rule depend on $\tilde{\Omega}_t$, while the exogenous processes themselves depend on Ω_t; as a result, the transition kernel Γ_t arises as a combination of both the individual's perception of z, as well as the actual evolution of z.

Real uncertainty equilibrium

The equilibrium under real uncertainty is defined similarly, with the only modification being that $\tilde{x}_t^z = x_t^z$; that is, the beliefs of the entrepreneurs fully align with the actual realized volatility process. As a result, the only modification to the equilibrium definition is that $\tilde{\Omega}_t = \Omega_t$. For the sake of brevity, I will defer the formal definition of equilibrium to the appendix.

3.3   Calibration

3.3.1   Steady-state model

Given that the goal of this paper is to study the impact of uncertainty shocks on entrepreneurial choices, it is crucial that the model is able to simultaneously replicate the micro-level investment behavior of entrepreneurs and entrants, as well as the broader distribution of wealth across entrepreneurs and workers.

In particular, as discussed in Tan (2018a), entrepreneurs face very stark disinvestment frictions that make downsizing difficult; that is, entrepreneurial investment is very illiquid. The same effect also means that in steady-state, the distribution of productivity is decreasing in wealth. This implies that the effect of a subjective uncertainty shock can potentially have extremely large effects on the aggregate economy.

For instance, as discussed in Bloom et al. (2018), the real options value effect generated by these non-convexities in capital adjustment cost can lead to sudden drops in investment during periods of heightened uncertainty. In the context of entrepreneurs who face incomplete markets, this effect could be even more severe since an increase in uncertainty corresponds to an increase in the extent of uninsurable risks. This in turns drives up the illiquidity risks that potential entrants and entrepreneurs face. Moreover, the joint distribution of wealth and productivity means that high productivity entrepreneurs are also hit hardest by the uncertainty shock, since these are also individuals with low buffer stock of savings.

The calibration therefore follows from Tan (2018a), and are reported in tables 3.1 and 3.1 respectively. I refer the reader to Tan (2018a) for the exact strategy which was used to calibrate these parameters.

3.3.2 Uncertainty shocks

Evidence on the extent of an uncertainty shock faced by entrepreneurs is relatively sparse. In light of this fact, I consider three calibration strategies.

First, I consider an innovation to the individual's perceived uncertainty that matches the percentage growth in economic policy uncertainty between 2007 and 2008; this amounts to an innovation of 46%. In other words, I assume that individuals face a one time shock to $\tilde{x}^z$ and $\tilde{x}^\psi$, where $\tilde{x}^z$ and $\tilde{x}^\psi$ rise from their (normalized) steady-state value of 1 to 1.46 within a single period, before reverting back to unity for the rest of the experiment. This process is reflected in figure 3.3 below. However, as the uncertainty shock is only subjective, x^z and x^ψ remain at their steady-state. Given that the evidence provided in the introduction pertains to the effect of individual perception of uncertainty (i.e., the EPU index captures an individual's uncertainty about the future rather than realized volatility), I believe that this is an appropriate assumption.

Table 3.1: Fixed and estimated parameters

Parameter	Value	Description	Source
Group A parameters			
γ	Risk aversion	2	Standard. See for instance, Storesletten et al (2004)
μ_θ	Unconditional mean of labor productivity	0	Normalization
ρ_θ	Persistence of labor productivity	0.90	Floden and Linde (2001), Storesletten et al (2004)
σ_θ	Conditional variance of labor productivity	0.20	Floden and Linde (2001), Storesletten et al (2004)
μ_z	Mean of business productivity process process	0	Normalization
α	Capital share of corporate sector	0.33	Fixed to value in Cagetti and De Nardi (2006)
A	Corporate sector TFP	1	Normalization
δ	Depreciation rate of corporate sector capital	0.10	Standard
$\bar{c}$	Consumption value of home production	0	See text
Group B parameters			
δ_k	0.15	Depreciation rate of entrepreneur's capital	From data (See appendix)
α_e	0.423	Capital intensity of entrepreneurial production function	From data (See appendix)
ν	0.786	Returns to scale of entrepreneurial production function	From data (See appendix)

In an alternate calibration, I also consider a subjective uncertainty shock, but utilize the calibration in Bloom et al. (2018). In this case, the authors assume that individuals face a 410% elevation of their idiosyncratic uncertainty process; that is $\tilde{x}_t^z$ spikes from a steady-state level of 1 to 4.1 for a year. I find that the results are qualitatively similar, but leads to unrealistic predictions with respect to its impact on the real economy. For instance, gross positive investment falls by 100% upon impact.

Table 3.2: Calibrated parameters

Parameter	Value	Description	Identifying moment
Group C parameters			
λ	0.688	Resale transaction cost	Probability that firm stays in quintile 1 of ARPK distribution
ζ	0.314	Exit friction	Skewness of ARPK
φ	0.930	Collateral constraint	Skewness / Probability that firm stays in quintile 5 of ARPK distribution
f_s	0.032	Investment fixed cost	Rate of positive investment reported
ρ_z	0.810	Autocorrelation of productivity shock	Autocorrelation of investment rates
σ_z	0.350	Volatility of productivity shock	Coefficent of variation of investment, firm size distribution
$\bar{l}$	0.211	Entrepreneur's endowed labor	% of firms that are employers
$\tilde{\mu}_z$	0.510	Mean of entrepreneurial prospects signal shock	Fraction of households that are entrepreneurs in steady-state
$\tilde{\mu}_\theta$	0.707	Mean of labor prospects signal shock	Exit rate
β	0.9185	Discount factor	Interest rate of 3.5%

The results from this experiment are therefore relegated to the appendix.

In a third calibration, I consider the case of a shock where uncertainty is "real", as defined in the earlier section. In this calibration, the large volatility effect produces a large increase in the spread of realized productivities, which in turn generates a large increase in the population of very high productivity entrepreneurs. As aggregate output (and hence labor demand) is convex in the spread of productivities (i.e., the Abel-Hartman-Ooi effect), the large increase in realized volatility means that the entrepreneurial sector dominates all labor demand in the economy, therefore shutting down the corporate sector. Given this unrealistic outcome, the results are therefore also relegated to the appendix.
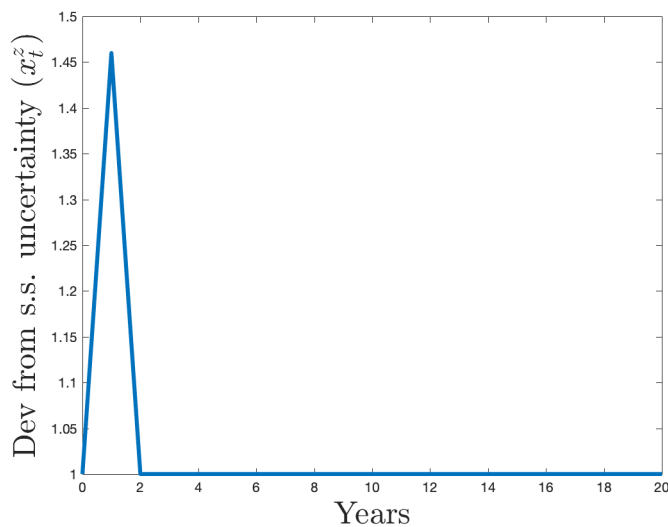
FIGURE 3.3: Path of uncertainty shock

In future work, a goal of this paper is to document a better calibration with regards to the uncertainty shock.

3.4 Results

Figure 3.4 reports the impact of a subjective uncertainty shock on the aggregate economy for various outcomes. First directing the reader to figures 3.4a and 3.4b, we see that a positive innovation to subjective uncertainty leads to a large fall in the entry rate (of around 17.6%), while a relatively low increase in the exit rate (around 2.3%). This is consistent with the evidence provided earlier, where we saw that entry rates were strongly and negatively correlated with economic policy uncertainty, whereas exit rates were only weakly (and positively correlated) so.

The model is able to replicate this feature of the data due to a combination of the investment real options effect driven by the partial irreversibility of entrepreneurial investment, and an outside option effect relating to how entrants draw their fresh ideas (the initial condition shock ψ^z). In the case of entry, partial irreversibility of investment implies that uncertainty shocks lower the continuation value of entrepreneurship

from the view point of potential entrants. This effect means that some potential entrants might choose to delay entry when uncertainty is higher — a real options effect that has been well documented in the literature. However, this is not the only channel in the model. Compounding the real options effect is that fact that uncertainty shocks imply that potential entrants who wait might potentially receive an even better outside option draw in the next period. This effect increases the value of being a worker in the next period[9]. The two effects therefore complement each other, leading to a large decline in the entry rate during a transient uncertainty shock.

Uncertainty shocks have a much weaker effect on the exit margin, because the real options effect now compete with the outside option effect. When uncertainty is higher, entrepreneurs delay disinvestment decisions, as the value of holding on to capital is higher than selling it. This implies that some marginal entrepreneurs who might have exited in steady-state now choose to delay exit. In contrast, the outside option effect means that exiting now becomes a more attractive prospect for very low productivity entrepreneurs. While these entrepreneurs might have chosen to stay in entrepreneurship in steady-state, the uncertainty shock implies that if they shut down their business, they might be able to obtain a much better entrepreneurial idea (as a worker) in the next period[10]. As a result, the uncertainty shock has a purging effect that pushes very low productivity entrepreneurs towards exit. Since an uncertainty shock simultaneously decreases exit propensity for some entrepreneurs while increasing it for others, the net impact is much weaker. In the baseline calibration, we see that the latter effect is slightly stronger, therefore generating a weak increase in exit rates.

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[9] This effect is actually very similar to the "locally risk loving effect" that Vereshchagina and Hopenhayn (2009) discusses in their paper. In the context of this paper, the marginal entrant delays entry because the increase in uncertainty over future signals increases the value of waiting.

[10] This effect, while also present even if entrepreneurs stay in business, is much weaker, since productivity conditioned on staying is persistent, whereas an exiting entrepreneur can draw a completely new idea in the next period from the unconditional distribution.

Figure 3.4c reports the impact of a subjective uncertainty shock on GDP. As in Bloom et al. (2018), an uncertainty shock has a large impact on GDP; at the trough of the recession, GDP falls by about 0.6%[11]. As we can see from figure 3.4d, the large fall in aggregate GDP is driven by a steep decline of output in the entrepreneurial sector; at the trough of the recession, entrepreneurial output falls by almost 8.6%. As a result, although entrepreneurial output in this economy accounts for only about 21% of GDP, the large fall in output of this sector generates a large recession.

(a) Entry rate   **(b)** Exit rate   **(c)** Aggregate GDP

(d) Entrepreneurial sector GDP   **(e)** Consumption   **(f)** Net investment

(g) Gross investment per en-**(h)** Gross disinvestment per en-**(i)** Liquid assets per en-trepreneur   trepreneur   trepreneur
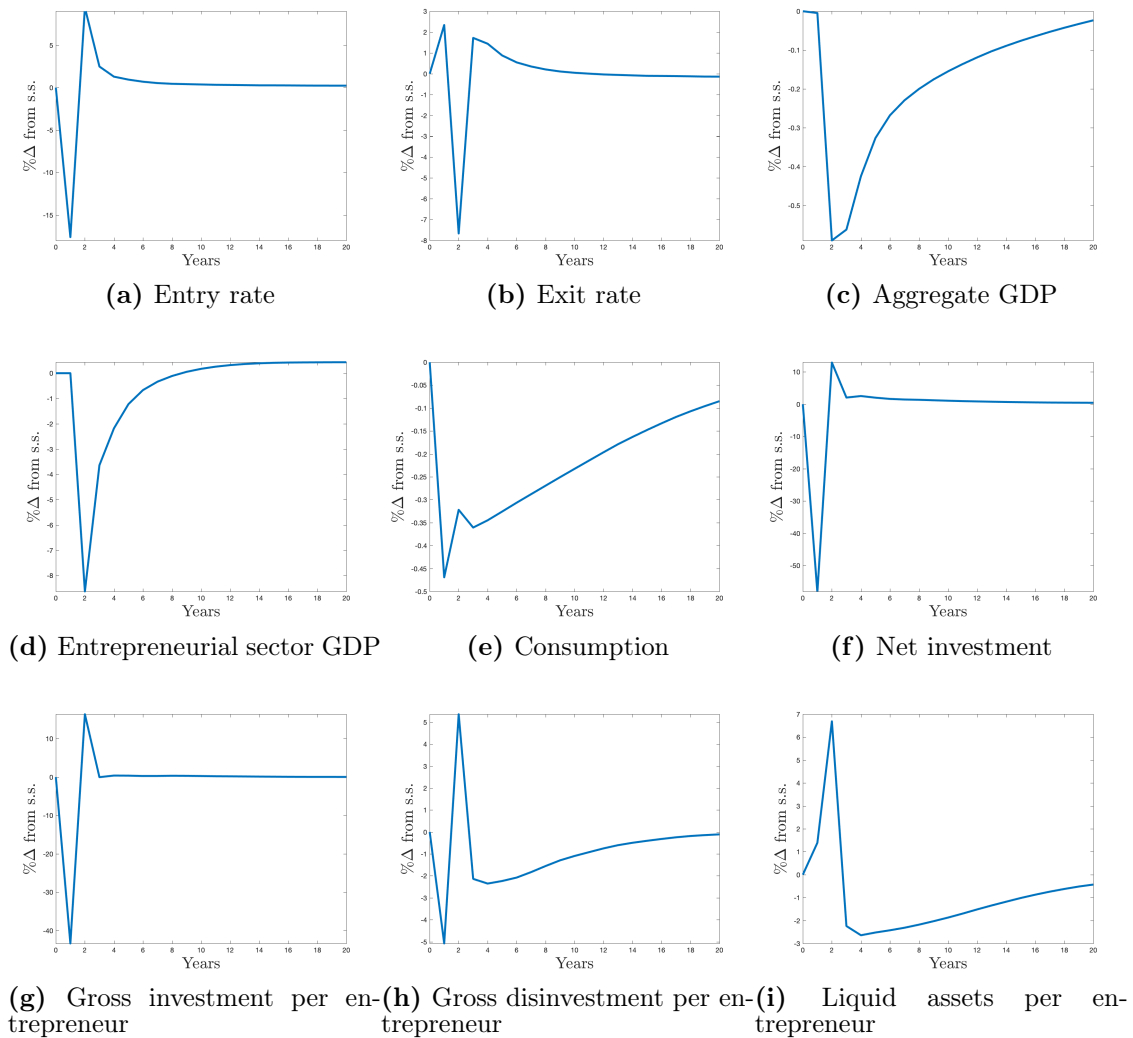
FIGURE 3.4: Effect of an uncertainty shock on the aggregate economy.

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[11] Remarkably, if the calibration of Bloom et al. (2018) was used, then GDP falls by about 2.5%, which almost matches the finding in their paper (see figure 6 of their paper).

Moreover, notice in figure 3.4e that consumption falls upon impact by about 0.5%, and stays depressed for a long period before converging to the steady-state. This is very different from the result in Bloom et al. (2018), who document that an uncertainty shock generates a small "consumption overshoot" at the initial stages of the shock. The result here documents why it is important to model uninsurable risks, as well as a two asset model. Unlike in models of complete markets and single assets, the sharp fall in investment (which also happens in this economy, as we can see in figure 3.4f) does not have to translate directly into an increase in consumption. Instead, the uncertainty shock increases the precautionary savings motives of the household. This leads households to cut back on investment into the risky asset — as we see in figure 3.4f, where net investment plunges by about 58% — while simultaneously shifting their income into the risk-free asset — as we see in figure 3.4i, where the liquid asset holdings per entrepreneur spike by 6.7%[12]. As a result, consumption falls upon impact, and stays depressed for a long period of time.

Why does the entrepreneurial sector respond so strongly to an uncertainty shock? A part of the reason stems from the standard investment real options effect, which leads to sharp declines in investment and overall capital reallocation. For instance, if we look to figure 3.4g and 3.4h, we see that the net investment per surviving firm[13] plunges by about 43% upon impact, while gross disinvestment also falls. While investment quickly increases after the shock, the desire to smooth consumption means that entrepreneurs take time to accumulate capital. In contrast to a risk-neutral firm — for instance, as in Bloom et al. (2018) — risk-averse households do not immediately pivot their portfolio of assets towards the risky asset even after the shock has

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[12] In fact, this effect is similar to that documented in Bayer et al. (ming).

[13] Going forward, all variables will be aggregate variables, unless I explicitly note that they are normalized variables. Due to the changes in the measure of active entrepreneurs, some aggregate variables such as investment change due to both the extensive and intensive margins. Therefore, it becomes very important to be clear which of the changes are due to intensive margins of changes (as in this example), and extensive margins of adjustment.

dissipated. As a result, as we see in figures 3.4g and 3.4i, while entrepreneurs do draw down on their liquid savings to finance their investments after the shock, they do so slowly.

However, the investment real options effect, while powerful in magnifying and propagating the effects of an uncertainty shock, is not the only channel through which uncertainty shocks play out. Instead, the interaction of the uncertainty shock with the extensive margin also has a large impact in amplifying the recessionary effects of uncertainty shocks. This channel is the focus of the next subsection.

3.4.1   The Role of the Extensive Margin

In figure 3.5a, I plot the change in GDP for the baseline economy (solid blue line) against a counterfactual economy where individuals do not have an occupational choice (dashed red line) — that is to say, an economy where the measure of workers and entrepreneurs stay constant. In the figure, we see that when there is no extensive margin of adjustment, the uncertainty shock generates a shallower recession that also recovers faster after the shock has passed.
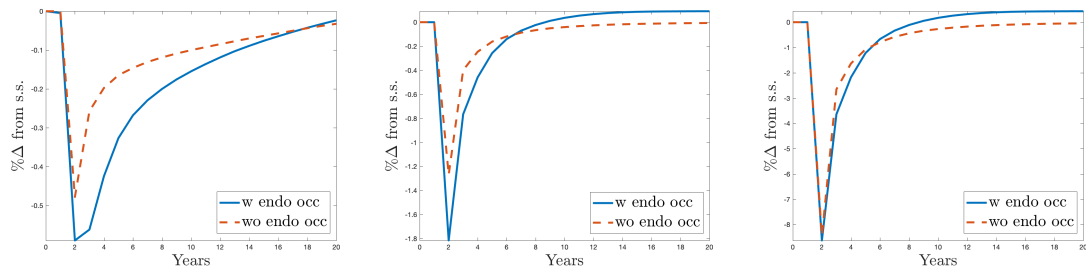
(a) Total change in GDP   (b) Change in GDP if corporate sector output was fixed   (c) Change in GDP of entrepreneurial sector

FIGURE 3.5: Effect of an uncertainty shock: The extensive margin

To understand the role of the extensive margin, I plot in figure 3.5b the change in aggregate GDP if the output in the corporate sector was held constant — in other words, this is the contribution of the entrepreneurial sector to the fall in aggregate

GDP. We can see clearly that the entrepreneurial sector has a much larger impact under the baseline model. If we look to figure 3.5c, we see that the impact of an uncertainty shock on the entrepreneurial sector itself is almost the same for both models. However, as the entrepreneurial sector constitutes a larger share of output under the baseline calibration (21% vs 15%), this translates into a sharper and more prolonged recession.

(a) Total labor supply   **(b)** Entrepreneurial sector labor demand   **(c)** Wages
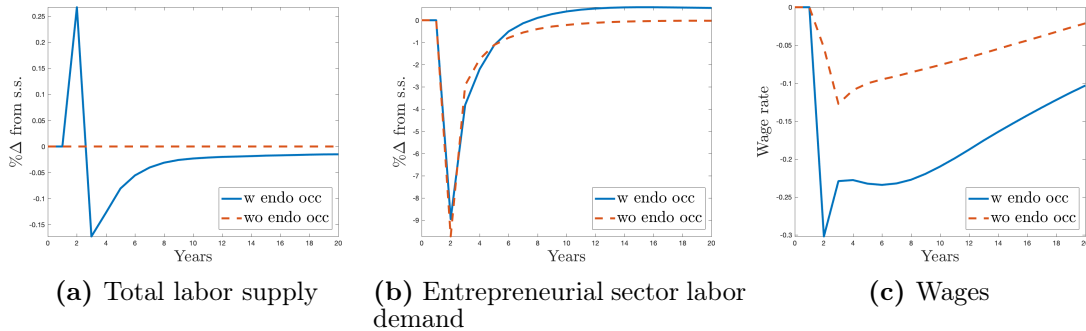
FIGURE 3.6: Effect of the uncertainty shock on labor and wages.

An additional dimension through which the extensive margin of adjustment exacerbates the recessionary effect of an uncertainty shock is through its effect on the labor markets. Along the intensive margin, the uncertainty shock leads to a "misallocation" of capital due to the wait-and-see effect on investment. As a result, labor demand from the entrepreneurial sector declines, as labor is complementary with capital. This in turn leads to a decline in wages in equilibrium, as we see in figure 3.6c. This effect is present in both the model with and without endogenous occupational choice, as we can see in 3.6b. However, because more individuals now pursue labor work in the baseline model, labor supply shifts out (figure 3.6a), which amplifies the decline in wages. As a result, the initial decline in wages in the baseline model is more than double that of the counterfactual model[14].

[14] As the corporate sector serves to absorb the residual labor supply, and households do not have an intensive margin of adjustment for labor, there is in fact no structural unemployment is this model. Moreover, because savings also increase, this then implies that the corporate sector undergoes a

While uncertainty shocks result in sharper recessions when the extensive margin of adjustment is active, it is important to note that this impact is an economically efficient outcome. To see this, I plot in figures 3.7a and 3.7b the response of output per entrepreneur and entrepreneur sector TFP respectively. We see that for the counterfactual model, the average output per entrepreneur plunges by about 8.4%, while TFP falls by about 1.5%, upon impact of the uncertainty shock. In contrast, the impact for the baseline model is much more tempered.
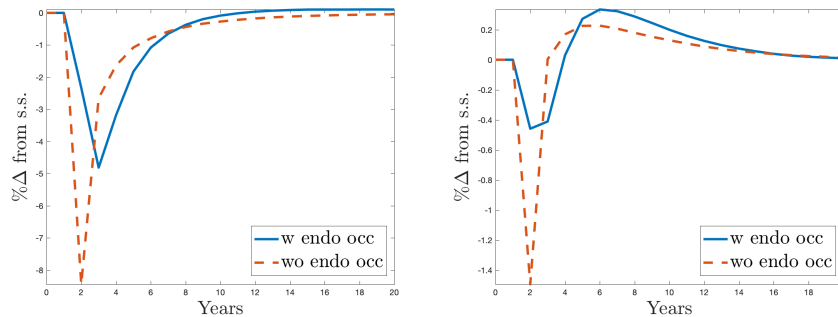
(a) Output per surviving entrepreneur   (b) TFP of entrepreneurial sector

FIGURE 3.7: Uncertainty shocks lead to sharp falls in average output per entrepreneur, but has a cleansing effect when the extensive margin is taken into account.

The reason why this happens is because the extensive margin of adjustment serves as an outside option for entrepreneurs to fall back on. This means that on impact, the uncertainty shock has a cleansing effect that drives out low productivity entrepreneurs to utilize their outside option. While this leads to a sharp decline in entrepreneurial output due to the exit of entrepreneurs, this also means that surviving entrepreneurs are in fact more productive than if they did not have the option to exit. For instance, if we look to figure 3.7a, we see that the fall in GDP per entrepreneur is only about half

small expansion upon impact of the shock, and interest rate rises. In ongoing work, I consider two extensions to this basic model. First and foremost, I incorporate uncertainty shocks that also affect the corporate sector, where I extend the model to a heterogeneous corporate sector composed of entrepreneurs who are able to "take their firm public". In this extension, I therefore break the direct link between wages and interest rates imposed by a frictionless representative corporate firm (i.e. the aggregate capital-labor ratio is no longer a sufficient statistic for market clearing.). Second, I extend the model to include frictional employment.

of that of the counterfactual economy; likewise, if we look to the change in aggregate TFP in the entrepreneurial sector, the measure-adjusted TFP of the baseline economy falls by only about 0.5%, in contrast to the counterfactual which falls by around 1.5%.

3.4.2 Aggregate welfare

Uncertainty shocks have a counterintuitive effect on welfare. In table 3.3, I report the change in welfare for entrepreneurs and workers due to the uncertainty shock. The welfare changes are computed with reference to a counterfactual scenario where the economy remains in steady-state, reported in percent consumption equivalent variation terms.

In column 1, I report the welfare change for the baseline economy where movements along the extensive margins are active. Here, we see a surprising outcome — uncertainty shocks generate a recession, but *increases* welfare for entrepreneurs and workers. In other words, households actually prefer being hit by an uncertainty shock. In column 2, I report the welfare change for the economy where the extensive margin of adjustment is inactive. Here, we see that entrepreneurs still benefit (in welfare terms) from the uncertainty shock, but workers now would pay up to 0.05% of lifetime consumption to avoid the shock.

Column 3 reveals the reason why we see such a counterintuitive result, and also explains why the two economies have such different responses. In column 3, I re-compute the welfare changes assuming that the real economy evolves as in under the baseline calibration, but that the *uncertainty shock never hits the households' expectations formations process*. In other words, this reports the effect on welfare due to real changes in the economy. Here, we see that both workers and entrepreneurs are hurt (in a welfare sense) by the recessionary effects of the uncertainty shock. This tells us that the actual impact of the shock is large and negative, but due to a corresponding change in the perception held by entrepreneurs and workers, individuals in

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the economy actually perceive higher welfare.

Table 3.3: Effects of an uncertainty shock on welfare. Columns 1 and 2 report the average change in welfare for each sub-group of households, in percent consumption equivalent variation. Column 3 reports the change in welfare under the baseline calibration, with the household's perceived uncertainty adjusted to their pre-shocked levels, but keeping the responses of the economy constant.

	w/ endo occ	w/o endo occ	w/ endo occ, Prob adjusted
Entrepreneurs	0.35	0.25	-0.05
Workers	0.39	-0.05	-0.1

This outcome relates to the non-convexity arising from the extensive margin of adjustment. For individuals who are on the margin of adjustment, the value function is locally convex, which generates a locally risk-loving region[15]. For workers who are on the verge of entry, the larger uncertainty regarding future signals implies that workers who delay entry have a higher probability of drawing an even better signal tomorrow, whereas the worst case scenario is that the worker simply remains a worker. In other words, since the downsize risk is bounded below whereas the upside risk is fully unbounded, an increase in the spread of the shock strictly improves the worker's perceived welfare. A similar effect plays out for low productivity entrepreneurs who are on the exit margin. For them, their downsize risk is simply taking home their labor income, while the upside risk is that they get to start afresh with a potentially good new business idea. This outcome in fact explains why workers are worse off under the counterfactual model without endogenous occupational choice, and entrepreneurs also see a lower welfare gain. Since workers do not have the opportunity to enter into entrepreneurship, they simply bear the full brunt of the recession without the benefit of the change in the upside risk. Entrepreneurs also benefit less, since they are not able to exit and draw a fresh business idea.

Finally, we see that entrepreneurs perceive higher welfare with or without an extensive margin of adjustment. This arises because entrepreneurial production is

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[15] This is similar to the mechanism discussed in Vereshchagina and Hopenhayn (2009)

convex with respect to the realization of entrepreneurial productivity, and therefore expected output in the next period is increasing in the volatility of productivity. Here, the downsize risks for surviving entrepreneurs is bounded below by 0 (the lowest possible state of z, given the assumption of a log-normal process), while the upside risk is unbounded. As a result, entrepreneurs perceive that expected future business income is in fact increasing in the volatility of entrepreneurial productivity, and hence have higher welfare.

3.5   Conclusion and Future Work

This paper provides an analysis of the role of uncertainty shocks for entrepreneurs, and show how the recessionary effects of uncertainty shocks are transmitted from the entrepreneurial sector to the broader aggregate economy. In particular, it argues that the empirical relationship of entry rates, exit rates, and economic and policy uncertainty can be rationalized through a model where entrepreneurial investment is partially irreversible, and entrepreneurs and workers face time-varying outside options. Moreover, it documents the importance of the extensive margin in amplifying and propagating the recessionary effects of a positive innovation to uncertainty. In future drafts, I will consider three key extensions.

First, the evidence I present is relatively stylized. As such, in ongoing work, the goal is to better identify what constitutes empirically an uncertainty shock from the perspective of entrepreneurs.

Second, the subjective uncertainty shock in this paper is modeled as an adhoc shock to the perceived risks faced by individuals. In ongoing work, I explore variations of this framework by including a process of learning a la David et al. (2016) and David and Venkateswaran (2018), and modeling the uncertainty shock as a shock to the posterior variance of beliefs as in Senga (2018). The goal of this extension is to endogenize the adhoc assumption made in this draft. Moreover, in this draft, the

uncertainty shock arises as a symmetric increase of the spread of productivity risks. As documented recently by Bloom et al. (2016), recessions are typically characterized by a negative skewness shock — in other words, entrepreneurs should face a higher probability of negative idiosyncratic draws than positive draws. Future extensions of this paper will incorporate this insight.

Finally, the assumption of a representative corporate firm with frictionless capital and labor adjustment leads to a overly tight link between interest rates and wages, as the (corporate) capital-labor ratio is the only market clearing object. In ongoing work, I extend my model to allow entrepreneurs to endogenously incorporate their firms, and as a result, the corporate sector is composed of heterogeneous corporate firms. In this framework, the direct link between wages and interest rates are broken, as corporate capital and labor are now two distinct market clearing objects.

4

A Fast and Low Computational Memory Algorithm for Computing Distributions in Heterogeneous Agent Models

4.1   Introduction

Heterogeneous agent models in macroeconomics generally do not admit closed form representations of the cross-section distribution of agents, or it's associated law of motion. As a result, an approximation is required. One common approach is to approximate the continuous distribution as a discretized probability mass function (PMF), and consequently, approximate the Markov kernel associated with the evolution of this distribution as a Markov transition matrix.

To put things concretely, let $\lambda(x, t)$ represent the true cross-sectional distribution at time t (and x the full set of state variables), and $T(x'|x, t)$ represent the Markov kernel that operates on this distribution at time t [1], such that the true law of motion

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[1] This definition is deliberately broad, and allows for a time varying distribution / transition, to capture the fact that this algorithm is useful for solving both stationary (for instance, models that extend the Bewley-Huggett-Aiyagari framework) and non-stationary heterogeneous agent (for instance, models that extend the Krusell and Smith (1998) framework) models

evolves as

$$\lambda(x', t+1) = \int_x T(x'|x,t)\lambda(x,t)dx$$

Then an appropriate approximation can be represented as

$$\hat{\lambda}(x', t+1) = \sum_x \hat{T}(x'|x,t)\hat{\lambda}(x,t)dx$$

where the integral is replaced by a sum to denote that $\hat{T}$ is a Markov matrix[2].

Unfortunately, $\hat{T}$ is often large, and this method quickly becomes unfeasible when the state space increases. This happens because the construction of $\hat{T}$ itself can be very slow, and require huge computational memory; moreover, operating $\hat{T}$ on $\hat{\lambda}$ can also be costly in terms of computational time and memory. This can make it unfeasible for individuals without access to large computing resources to solve such models.

The purpose of this paper is to propose a simple new method to approximate T. This algorithm improves on the textbook method by greatly reducing the memory requirements and computational time. For instance, using this algorithm, it is possible to solve a model with more than 9 million states on a laptop computer with 4gb of RAM. Moreover, the speed up over the textbook method increases with the state space, making this procedure increasingly attractive as the model becomes more complicated[3]. This method thus makes it feasible to solve large scale heterogeneous agent models even with modest computational power.

The rest of the paper is presented as follows. In section 2, I denote the generic class of problems for which this method is suitable for, discuss the issues associated with

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[2] See for instance, pg 728 to 729 of Ljungqvist and Sargent (2012) and the associated Matlab codes

[3] I utilize this procedure in a companion paper, Tan (2018a). In that paper, the cross-sectional distribution has around 9 million states; the actual computation requirement is only around 3 gb of RAM.

the standard textbook method, and explain the algorithm I propose. In section 3, I provide three examples of how this method works, and document the improvement over the standard textbook method; I conclude in section 4.

4.1.1   Relevant literature

Heterogeneous agent models   Heterogeneous agent models have become increasingly complex, requiring increasing computational power to solve them. While a lot of papers have focused on algorithms that reduce computational time in solving the individual's problem (i.e. the dynamic programming step), few have suggested algorithms to reduce the computational time required in solving for the stationary or transitional distribution. This paper adds to the latter literature by proposing an extremely simple method to reduce computational burden in the latter step[4].

Parallel computing   Many researchers in economics are adopting the method of indirect inference to estimate their models of heterogeneous agents. This method typically requires the evaluation of the objective function multiple times. A way to speed up this procedure is to parallelize the evaluation of the objective function. However, the textbook method involved in solving for the distribution makes it difficult to parallelize the algorithm due to the large computational memory that each fork in the parallel thread requires. This paper, which proposes a simple method to reduce memory demands, make it feasible to estimate large scale heterogeneous agent models in parallel.

4.2   Algorithm

In this section, I first define the problem of interest and the notations. Following that, I summarize the issues associated with the textbook method. Finally, I describe the

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[4] To the best of my knowledge, Young (2010) is the only paper that directly addresses this problem.

algorithm for my proposed method. As the model setup here is deliberately general, readers might find this section difficult to follow. If readers simply wish to apply this method, I suggest that readers skip straight to section 3 where I discuss three examples.

4.2.1 Model setup

Consider the following (potentially non-stationary) dynamic program:

$$V_t\left(x, y, z; \theta, \eta, \zeta\right) = \max_{x', \tilde{y}} \pi\left(x', \tilde{y}; x, y, z, \theta, \eta, \zeta\right) + \beta \mathbb{E}\left[\max_{z'} V_{t+1}\left(x', y', z'; \theta', \eta', \zeta'\right) | \theta, \eta, \zeta'\right]$$

$$s.t.$$

$$\theta' \sim h^\theta\left(\theta'|\theta, t\right)$$

$$\eta' \sim h^\eta\left(\eta'|\eta, t\right)$$

$$\zeta' \sim h^\zeta\left(\zeta'|\zeta, t\right)$$

$$x' = g^x\left(x, y, z; \theta, \eta, \zeta, t\right)$$

$$\tilde{y} = g^y\left(x, y, z; \theta, \eta, \zeta, t\right)$$

$$y' \sim h\left(y'|\tilde{y}, \eta, t\right)$$

$$z' = g^z\left(x', \tilde{y}, z; \theta, \eta, \zeta', t\right)$$

Given the notation, the assumptions are

1. x, y, z are the endogenous state variables of the dynamic program. Without loss of generality, they each individually represent a unique scalar state variable[5].

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[5] They can also represent vectors of state variables, but for the sake of clarity, we should simply

They can either be continuous or discrete.

2. θ, η, ζ are the exogenous state variables of the dynamic program. They each also individually represent a vector of state variables.

3. π is a return function.

4. g^i is an optimal policy function for variables $i \in \{x, y, z\}$.

5. θ, η and ζ are Markov processes, and h^j is the associated Markov kernel for each process $j \in \{\theta, \eta, \zeta\}$. The Markov kernel is assumed to be potentially time-varying[6]. For simplicity, they are assumed to be uncorrelated.

6. x' and $\tilde{y}$ are chosen using only information on θ, η and ζ in time t.

7. y' is the stochastic result of choice $\tilde{y}$ and the markov shock η'.

8. z' is chosen after information about "tomorrow's" ζ is revealed (i.e. ζ'). It is also chosen after x and $\tilde{y}$ decisions are made, but prior to the shock to y is realized.

Let λ_t denote the distribution of agents at time t. Then the set of policy and transition functions $\{g^i, h^j\}_{i,j}$ together induces a Markov kernel T_t that describes the law of motion of the distribution of agents:

$$\lambda_{t+1}\left(x', y', z'; \theta', \eta', \zeta'\right) = \int_{x,y,z,\theta,\eta,\zeta} T_t\left(x', y', z'; \theta', \eta', \zeta' | x, y, z; \theta, \eta, \zeta\right) \ldots$$

$$\lambda_t\left(x, y, z; \theta, \eta, \zeta\right) dx dy dz d\theta d\eta d\zeta$$

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think of x, y, and z as scalars.

[6] For instance, in life cycle models with income risk is age dependent, this kernel would be time varying.

In a stationary equilibrium, this reduces to

$$\lambda\left(x', y', z'; \theta', \eta', \zeta'\right) = \int_{x,y,z,\theta,\eta,\zeta} T\left(x', y', z'; \theta', \eta', \zeta' | x, y, z; \theta, \eta, \zeta\right) \dots$$

$$\lambda\left(x, y, z; \theta, \eta, \zeta\right) dx dy dz d\theta d\eta d\zeta$$

where the time subscripts are dropped to indicate that λ is an invariant distribution.

4.2.2 Textbook method

In the textbook method, the Markov kernel T is replaced with a discrete approximation $\hat{T}$ (i.e. a Markov transition matrix); similarly, the probability density function λ is replaced by a discrete approximation $\hat{\lambda}$ (i.e. a probability mass function). Let N_i be the number of nodes used for variable $i \in \{x, y, z, \theta, \eta, \zeta\}$. In that case, the dimensions of $\hat{\lambda}$ is $N_\lambda = N_x \times N_y \times N_z \times N_\theta \times N_\eta \times N_\zeta$, and the corresponding dimension of $\hat{T}$ is $N_\lambda \times N_\lambda$. Except for the case of small scale models, this method is computationally expensive for two reasons:

High memory usage

Consider a case where we use 50 grid points for the endogenous states, and 5 for exogenous states. The total number of states is $\sim$ 15.6 million, and correspondingly, $\hat{T}$ is a 15.6 million by 15.6 million matrix. This would require $\sim 1.9 \times 10^6$ gb of RAM - a memory requirement that is obviously not feasible.

Using sparse matrices as an alternative   In practice, $\hat{T}$ is very sparse, and a practical alternative is to store $\hat{T}$ as a sparse matrix. While this is feasible for smaller scaled systems, the smallest possible memory required for this example is still about 15.6gb

of RAM (with a density of $nnz = 15.6 \times 5^3$ million)[7]. Actual operations using this matrix, such as multiplication, would incur an additional cost. This is still unfeasible for most standard computers.

Speed

Using the textbook method can be very slow for two reasons:

1. The construction of $\hat{T}$ can be very slow even when we use a sparse matrix optimized for memory allocation.

2. The standard algorithm requires that we operate $\hat{T}$ on $\hat{\lambda}$. This operation involves multiplying a $15.6 \times 10^6 \times 15.6 \times 10^6$ matrix with a 15.6×10^6 vector. This operation is very slow.

   (a) In the case of computing the stationary distribution, this typically involves operating on this multiplication process a few hundred times.

   (b) In the case of computing the transitional distribution, this operation is often repeated for a few thousand times (in both perfect foresight models and models with aggregate risk).

4.2.3   Proposed method

The proposed method uses the observation that in the case of economic models, the exogenous processes evolve ***independently*** of the endogenous processes.

To be precise, let $\Gamma_x\left(x, y, z, \theta, \eta, \zeta, t\right)$, $\Gamma_{\tilde{y}}\left(x, y, z, \theta, \eta, \zeta, t\right)$, $\Gamma_z\left(x, y, z, \theta, \eta, \zeta', t\right)$ denote the time t Markov kernels induced by choices x', $\tilde{y}$ and z' respectively. Then, due to the independence of Γ and h, we can decompose T_t as follows:

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[7] This corresponds to when one solves exactly on a grid, and also ignores the computational overhead of storing a sparse matrix. See for example, 729 of Ljungqvist and Sargent (2012). More sophisticated methods such as that proposed by Young (2010) would incur an even higher memory cost. nnz refers to the number of non-zeros in a sparse matrix.

$$\lambda_{t+1}\left(x', y', z'; \theta', \eta', \zeta'\right) = \int_{x,y,z,\theta,\eta,\zeta} T_t\left(x', y', z'; \theta', \eta', \zeta' | x, y, z; \theta, \eta, \zeta\right) \ldots$$

$$\lambda_t\left(x, y, z; \theta, \eta, \zeta\right) dx dy dz d\theta d\eta d\zeta$$

$$= \int_{\theta,\eta,\tilde{y}} h_t\left(\theta' | \theta\right) h_t\left(\eta' | \eta\right) h_t\left(y' | \tilde{y}, \eta\right) \ldots$$

$$\ldots \left(\int_z \Gamma_{z,t} \left(\int_\zeta h_t\left(\zeta' | \zeta\right) \left(\int_{x,y} \Gamma_{x,t} \Gamma_{\tilde{y},t} \lambda_t dx dy \right) d\zeta \right) dz \right) d\tilde{y} d\eta d\theta$$

Next, let $\hat{\Gamma}_x\left(x, y, z, \theta, \eta, \zeta, t\right)$, $\hat{\Gamma}_{\tilde{y}}\left(x, y, z, \theta, \eta, \zeta, t\right)$, $\hat{\Gamma}_z\left(x, y, z, \theta, \eta, \zeta', t\right)$ denote the discrete approximation of the Markov kernels described above; similarly, let $\hat{h}$ represent the discrete approximation (or exact values) to the kernel generated by the exogenous Markov processes[8]. $\hat{T}_t$ can now be expressed as a tensor product of the preceding functions:

$$\hat{T} = \hat{h}_\theta \otimes \hat{h}_\eta \otimes \hat{h}_\zeta \otimes \hat{h}_{y' | \tilde{y}, \eta} \otimes \hat{\Gamma}_x \otimes \hat{\Gamma}_{\tilde{y}} \otimes \hat{\Gamma}_z$$

As discussed earlier, $\hat{T}_t$ is costly to construct and to operate with. However, due to the nesting structure above, we can in fact obtain the following sparser matrices:

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[8] For instance, a common approach in macroeconomics is to approximate a Gaussian AR(1) process using the Tauchen (1986) method

$$\hat{T}_1 \equiv \hat{\Gamma}_x \otimes \hat{\Gamma}_{\tilde{y}}$$

$$\hat{T}_2 \equiv \hat{h}_\zeta$$

$$\hat{T}_3 \equiv \hat{\Gamma}_z$$

$$\hat{T}_4 \equiv \hat{h}_\theta \otimes \hat{h}_\eta \otimes \hat{h}_{y'|\tilde{y},\eta}$$

Therefore, when computing $\hat{\lambda}_{t+1}$, we simply loop through the recursion:

$$\lambda^0 \equiv \hat{\lambda}_t \tag{4.1}$$

$$\lambda^1\left(x', \tilde{y}, z; \theta, \eta, \zeta\right) = \hat{T}_1 \lambda^0\left(x, y, z; \theta, \eta, \zeta\right) \tag{4.2}$$

$$\lambda^2\left(x', \tilde{y}, z; \theta, \eta, \zeta'\right) = \hat{T}_2 \lambda^1\left(x', \tilde{y}, z; \theta, \eta, \zeta\right) \tag{4.3}$$

$$\lambda^3\left(x', \tilde{y}, z'; \theta, \eta, \zeta'\right) = \hat{T}_3 \lambda^2\left(x', \tilde{y}, z; \theta, \eta, \zeta'\right) \tag{4.4}$$

$$\lambda^4\left(x', y', z'; \theta', \eta', \zeta'\right) = \hat{T}_4 \lambda^3\left(x', \tilde{y}, z'; \theta, \eta, \zeta'\right) \tag{4.5}$$

$$\hat{\lambda}_{t+1} = \lambda^4 \tag{4.6}$$

To understand how the nesting structure works, the reader needs to focus carefully on reading the arguments to the λ function, and keep track of how the state variables evolve. In line 4.2, the kernel $\hat{T}_1$ only operates on x and y (moving the masses to x' and $\tilde{y}$); as such, the density of $\hat{T}_1$ is simply $nnz \approx 2 \times N_\lambda$. The "2" appears in the calculation because we have two variables that are being operated on, i.e. x and y. In line 4.3, $\hat{T}_2$ operates only on the exogenous process ζ (i.e. from ζ to ζ'). As a result, this is simply a $N_\zeta \times N_\zeta$ matrix. In line 4.4, $\hat{T}_3$ operates on z (i.e. from z to z'), so $\hat{T}_3$ has density $nnz \approx N_\lambda$. Finally in line 4.5, $\hat{T}_4$ operates on θ, η and $\tilde{y}$, and so has

density of $nnz \approx N_\theta \times N_\theta \times N_\eta \times N_\eta \times N_{\tilde{y}}$.

Low memory usage

Using the same preceding example in section 4.2.2, T_1 is $nnz \approx 2 \times 15.6 \times 10^6$, T_2 is $nnz = 5^2$, T_3 is $nnz \approx 15.6 \times 10^6$ and T_4 is $nnz = 5^2 \times 5^2 \times 50^2 \approx 15.6 \times 10^6$, which means total $nnz \approx 4 \times 1.56 \times 10^6$. From 15.6gb of RAM, we have reduced the problem to $4 \times 1.56 = 6.24gb$ of RAM - a large memory requirement, but feasible even using a simple laptop.

Fast

This method provides large speed gains over the textbook method as it avoids constructing a large and dense matrix. However, it incurs a penalty by the repeated recursion, whereas the textbook method only necessitates a single matrix multiplication. For very small scale models, this penalty wins out, making my algorithm slower than the textbook method. However, for most cases when memory requirement is not trivial, this method beats out the textbook method significantly. I defer the reader to section 3 for a treatment on when this method is faster than the textbook method.

4.2.4 Alternative - A simple for-loop

Recall that a key limiting factor for approximating T directly is the size of the matrix required to represent $\hat{T}$ - the computer memory requirements are simply too large. However, a simple fix is clearly at hand: Instead of computing $\hat{T}$, one can simply loop through all the state variables. Based on a sample of publicly available code[9], this is exactly the approach taken by some researchers.

In fact, my proposed algorithm could be considered an improvement upon the "for-loop" method. In terms of memory requirements, the recursive structure means

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[9] See for instance, the publicly available codes for Young (2010) and Bloom et al. (2018)

my method requires about the same computational memory as that using for-loops. However, by utilizing sparse matrices, it provides a potential speed up as it utilizes specialized linear algebraic programs (such as LAPACK in Fortran) to compute $\hat{T}\lambda$. In contrast, user-written for-loops seem unlikely to be faster than specialized programs; in fact, taken to its logical conclusion, they could only be as fast as these programs.[10]

4.2.5 Examples

For the rest of the paper, I provide three simple examples to show the speed up gains with respect to the standard algorithm. The examples are chosen to show the three broadest cases in heterogeneous agent models with which this algorithm is applicable.

1. A model where all state variables are chosen prior to the realization of all shocks; and once they are chosen, they are brought into the next period with no more changes: i.e. a model that features only x and θ. I use the Aiyagari (1994) model for exposition.

2. A model, where in addition to the features in (1), also feature a second state variable that is chosen prior to the realization of all shocks, but feature some stochastic transformation before the realization of the next period: i.e. a model that also features y. I use a modified version of the model in Gavazza and Lanteri (2018), which features stochastic depreciation, for exposition.

3. A model where the state variable is chosen *after* the realization of the exogenous state; i.e. when z is active. Here, I solve a modified version of Chang and Kim (2007), where the labor participation choice is made *prior* to the realization of

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[10] To test this hypothesis, I also compare my algorithm against the "for-loop" algorithm in Python using Python's new "just-in-time" (JIT) compiler (via the Numba package) In practice, the JIT compiler should deliver for-loop speeds comparable to that of compiled languages such as Fortran. I find that my algorithm is still faster than a for-loop method that has been "enhanced" by Numba's JIT compiler.

labor income shocks, but asset choices are made *after* labor income shocks are realized.

In all three cases, I only solve for the stationary equilibrium versions of the model, and abstract from aggregate shocks. In general, the computational cost associated with dynamic heterogeneous agent models, at least which respect to the construction of the T matrix, is the same. For stationary models, we operate on a fixed $\hat{T}$ until a stationary distribution $\hat{\lambda}$ is reached; in dynamic models, we operate $\hat{\lambda}_t$ on $\hat{T}_t$ along the simulation / transition path. As such, any computational gains that manifests in the stationary equilibrium will transfer to a model with aggregate dynamics[11].

4.3   Examples

4.3.1   Aiyagari (1994)

The Aiyagari model considers a household that receives uninsurable labor income shock θ that follows an AR(1) Markov process. The household inherits bond holdings a at period t, and then chooses how much bonds a' to bring into period $t+1$.

The individual's problem can be succinctly expressed as

$$V\left(a, \theta\right) = \max_{a'} U\left(c\right) + \beta \mathbb{E}\left[V\left(a', \theta'\right) | \theta\right]$$

$$s.t.$$

$$c = \exp\left(\theta\right) w + \left(1 + r\right) a - a'$$

$$\theta' = \rho\theta + \sigma\epsilon'$$

$$\epsilon' \sim N\left(0, 1\right)$$

---

[11] In fact, the computational gains are often greater for dynamic models, as we have to repeatedly create $\hat{T}_t$ along the transition path. This means that the memory / time savings from my method will greatly speed up the simulation.

Given optimal bond policy $a' = g(a, \theta)$ (and the kernel generated by this process $\Gamma(x'|x, \theta)$), and the evolution of labor income (with the associated kernel $h(\theta'|\theta)$), we can define the evolution of the distribution of households λ as

$$\lambda(\theta', a') = \int T(a', \theta'|a, \theta) \lambda(\theta, a) \, d\theta db$$

where $T \equiv \Gamma h$ is the full transition kernel for the distribution.

Textbook algorithm   The textbook algorithm can be described as follows:

Step 1. Grid construction.

1. Construct a grid for a and θ; call this $\mathcal{G}_a = \{a_1, ...a_n, ..., a_N\}$ and $\mathcal{G}_\theta = \{\theta_1, ..., \theta_m, ..., \theta_M\}$. In other words, the grid for a has N nodes, and the grid for θ has M nodes.

2. Construct the tensor grid $\mathcal{G}_{\theta,a} = \mathcal{G}_a \otimes \mathcal{G}_\theta$. Let $i \in \{0, 1, ..., NM\}$ denote the linear index in this tensor grid; moreover, let $n(i)$ and $n(i)$ denote the index value of a and θ that corresponds to the linear index i. Finally, the tensor grid is ordered such that θ varies along the first dimension, and a varies in the second dimension.

Step 2. Constructing law of motions.

1. Discretize the law of motion for the exogenous process θ on the grid $\mathcal{G}_\theta$. Let the transition matrix be denoted as $\hat{h}$, where the (m', m) entry to $\hat{h}$ is the transition probability from state m to state m'.

2. Solve the individual's problem, and obtain a policy function for next period bond holdings $a' = g(\theta, a)$.

Step 3. Build up the transition matrix $\hat{T}$. $\hat{T}$ is an $MN \times MN$ matrix, where for each entry (j, i),

$$T(j, i) = \hat{h}\left(m(j), m(i)\right) \times \mathcal{B}\left(a' = g(\theta_{m(i)}, a_{n(i)})\right)$$

$$\mathcal{B}\left(a'\right) = \begin{cases} \frac{a' - a_{n(j)-1}}{a_{n(j)} - a_{n(j)-1}} & \text{if } a' \in \left(a_{n(j)-1}.a_{n(j)}\right) \\ \frac{a' - a_{n(j)}}{a_{n(j)+1} - a_{n(j)}} & \text{if } a' \in \left(a_{n(j)}.a_{n(j)+1}\right) \\ 0 & \text{otherwise} \end{cases}$$

The construction of $\mathcal{B}$ corresponds to the method in Young (2010).

Step 4. Solve for the stationary distribution using a power iteration method.

1. Initialize some guess for λ. Call this $\lambda^{(0)}$. λ is a column vector, where entry i corresponds to the probability mass corresponding to households holding assets $a_{n(i)}$ and having productivity $\theta_{m(j)}$. It therefore also conforms to the tensor grid $\mathcal{G}_{\theta,a}$.

2. Using T, compute $\lambda^{(1)} = T \times \lambda^{(0)}$. Compute $d = \sup ||\lambda^{(0)} - \lambda^{(1)}||$. If d is smaller than some tolerance, stop. Otherwise, update $\lambda^{(0)}$ with $\lambda^{(1)}$ and compute $\lambda^{(2)}$. Repeat this recursion for T steps until d is below the tolerance. The final distribution $\lambda^{(T)}$ is the steady-state distribution.

New algorithm This paper improves upon steps 3 and 4 of the textbook algorithm:

Step 3 (NEW). Construct a matrix Γ that has the same indexing as T, where for each

entry (j, i),

$$\Gamma(j, i) = \mathcal{B}\left(a' = g(\theta_{m(i)}, a_{n(i)})\right)$$

$$\mathcal{B}\left(a'\right) = \begin{cases} \frac{a' - a_{n(j)-1}}{a_{n(j)} - a_{n(j)-1}} & \text{if } a' \in \left(a_{n(j)-1}.a_{n(j)}\right) \\[2ex] \frac{a' - a_{n(j)}}{a_{n(j)+1} - a_{n(j)}} & \text{if } a' \in \left(a_{n(j)}.a_{n(j)+1}\right) \\[2ex] 0 & \text{otherwise} \end{cases}$$

Step 4 (NEW). Solve for the stationary distribution using a power iteration method.

1. Initialize some guess for $\lambda^{(0)}$.

2. Iterate on λ as follows: For some iteration t,

   (a) $\tilde{\lambda} = \Gamma \times \lambda^{(t)}$. Note that $\tilde{\lambda}$ is still an $MN \times 1$ column vector.

   (b) Reshape $\tilde{\lambda}$ into a $M \times N$ matrix to conform to h.

   (c) Update: $\lambda^{(t+1)} = h \times \tilde{\lambda}$

3. Compute $d = \sup ||\lambda^{(t)} - \lambda^{(t+1)}||$. If d is smaller than some tolerance, stop; otherwise, repeat step 4.

Results

Here, I compare the speed (figure 4.1) and memory requirements (figure 4.2) of my algorithm against the standard textbook method. To conduct this analysis, I discretized the θ state space with increasingly fine grids: $N_\theta \in \{3, 5, 7, 9, 11, 13, 15, 17\}$; likewise, I discretized the a state space with increasingly fine grids: $N_a \in \{30, 50, 100, 200, 300, ..., 1000\}$, and repeatedly solve the model using all combinations of these grids. For speed comparisons, I solve the same model 100 times, and take the time average of that.

From the following figures, we see that we almost never want to use the textbook

method when solving the Aiyagari model. My proposed method is superior in time and memory requirements.
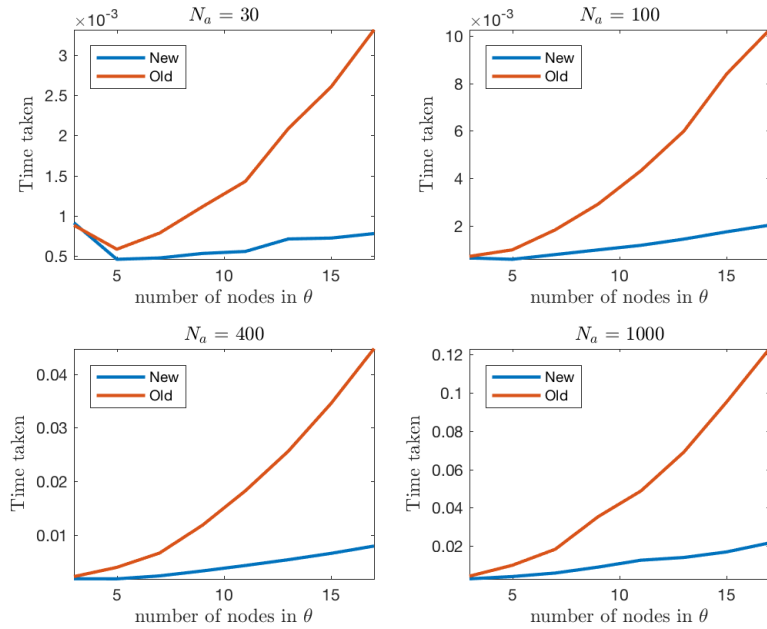
(a) Time taken (in seconds) for different sizes of the state space: N_a refers to the number of nodes in a
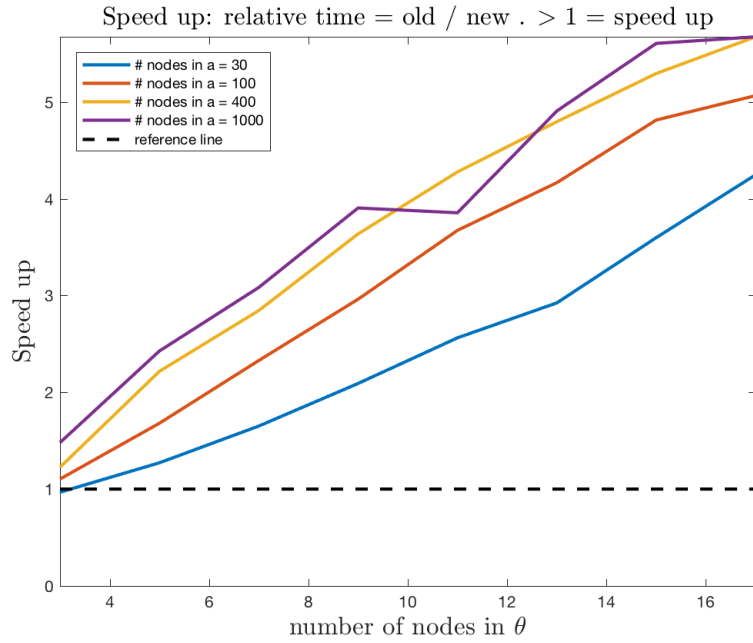
(b) Speed up, computed in relative times. >1 implies speed up ; <1 implies slow down.

FIGURE 4.1: Solution times for textbook method ("Old") v.s. proposed method ("New")
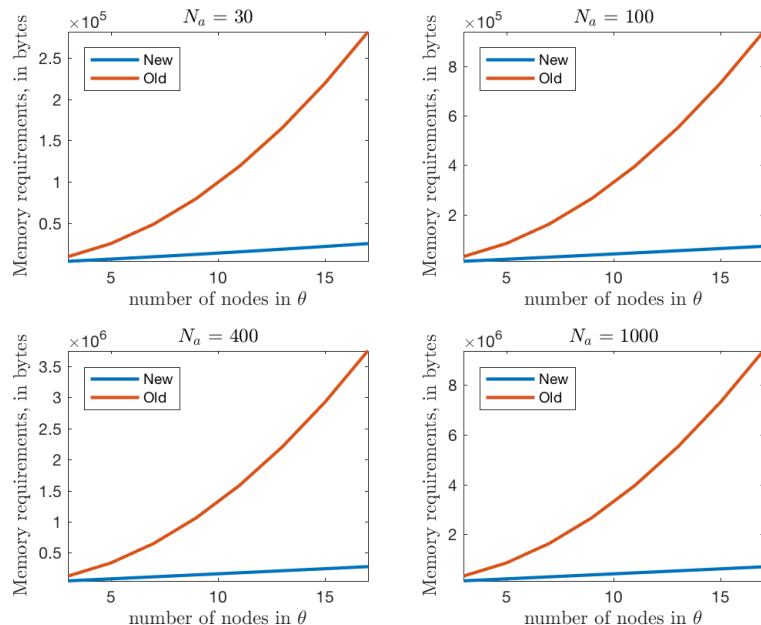
FIGURE 4.2: Memory requirements for textbook method ("Old") v.s. proposed method ("New"). y-axis units are in bytes.

4.3.2 Gavazza and Lanteri (2018)

The Gavazza and Lanteri (2018) framework extends the Aiyagari model to include durable assets. Here, households inherit a car type and bonds in period t. There are K car types: $d_k \in \{d_1, d_2, ..., d_K, d_{K+1}\}$, with $K+1$ denoting "no car". The household then chooses bond holdings for period $t+1$, as well as the type of car to drive in period t ($\tilde{d}$). At the end of the period, the car receives a stochastic depreciation shock: with δ_k probability, a car type k depreciates into a car type $k + 1$. For type K, the car is destroyed with probability 1. Type $K + 1$ refers to the utility gained from having no car.

The individual's problem can be written succinctly as

$$V\left(a, d_k, \theta\right) = \max_{\tilde{d}_k, a'} U\left(c, \tilde{d}_k\right) + \beta \mathbb{E}\left[V\left(a', d'_k, \theta'\right) | \theta.\tilde{d}_k\right]$$

s.t.

$$c = \exp\left(\theta\right) + \left(1 + r\right) a + p_{d_k} \mathbb{I}\left(d_k\right) - a' - p_{\tilde{d}_k} \mathbb{I}\left(\tilde{d}_k\right)$$

$$d'_k = \begin{cases} \tilde{d}_k & w.p.\ 1 - \delta_k \\ \tilde{d}_{k+1} & w.p.\ \delta_k \end{cases}$$

$$\theta' = \rho\theta + \sigma\epsilon'$$

$$\epsilon' \sim N\left(0, 1\right)$$

where the exogenous transition matrix for cars can be written as $K + 1$ by $K + 1$ matrix:

$$D = \begin{bmatrix} 1 - \delta_1 & \delta_1 & 0 & 0 & 0 & 0 \\ 0 & 1 - \delta_2 & \delta_2 & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 - \delta_n & \delta_n \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

Given the following:

1. Optimal bond policy $a' = g^a\left(a, d, \theta\right)$ and car choice policy $\tilde{d} = g^{\tilde{d}}\left(a, d, \theta\right)$; and the joint kernel $\Gamma\left(a', \tilde{d} | a, d, \theta\right)$ generated by these two processes

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2. The evolution of labor income as $\theta' \sim h(\theta'|\theta)$, and the stochastic evolution of car types D

We can define the evolution of the distribution of households as

$$\lambda(\theta', a', d') = \int_{\theta,a} \sum_{\tilde{d},d} T(a', d', \theta'|a, d, \theta) \lambda(\theta, a, d) \, d\theta da$$

where here, $T \equiv GhD$ is the full Markov kernel generated. The summation term above refers to the fact that car types are discrete and follow a discrete state Markov transition.

Textbook algorithm   The textbook algorithm can be described as follows:

Step 1. Grid construction.

1. Construct a grid for a and θ; call this $\mathcal{G}_a = \{a_1, ...a_n, ..., a_N\}$ and $\mathcal{G}_\theta = \{\theta_1, ..., \theta_m, ..., \theta_M\}$. In other words, the grid for a has N nodes, and the grid for θ has M nodes. Finally, let $\mathcal{G}_d$ denote the grid for cars. As discussed earlier, we also assume that there are $K + 1$ car types, so the grid of car types has $K + 1$ notes.

2. Construct the tensor grid $\mathcal{G}_{\theta,d,a} = \mathcal{G}_a \otimes \mathcal{G}_d \otimes \mathcal{G}_\theta$. Let $i \in \{0, 1, ..., M(K+1)N\}$ denote the linear index in this tensor grid; moreover, let $n(i)$, $m(i)$, and $k(i)$ denote the index value of a, θ, and d that corresponds to the same linear index i. Finally, the tensor grid is ordered such that θ varies along the first dimension, d varies in the second dimension, and a varies in the third dimension.

Step 2. Constructing law of motions.

1. Discretize the law of motion for the exogenous process θ on the grid $\mathcal{G}_\theta$. Let the transition matrix be denoted as $\hat{h}$, where the (m', m) entry to $\hat{h}$ is the transition probability from state m to state m'.

2. Solve the individual's problem, and obtain a policy function for next period bond holdings $a' = g^a(\theta, d, a)$ and car choice $\tilde{d} = g^{\tilde{d}}(\theta, d, a)$

Step 3. Build up the transition matrix $\hat{T}$. $\hat{T}$ is an $M(K+1)N \times M(K+1)N$ matrix, where for each entry (j, i),

$$T(j, i) = \hat{h}\left(m(j), m(i)\right) \times D\left(k(j), k(i)\right) \times \ldots$$

$$\mathcal{I}\left(\tilde{d} = g^{\tilde{d}}\left(a_{n(i)}, d_{k(i)}, \theta_{m(i)}\right)\right) \times \mathcal{B}\left(a' = g(\theta_{m(i)}, d_{k(i)}, a_{n(i)})\right)$$

$$\mathcal{B}\left(a'\right) = \begin{cases} \frac{a' - a_{n(j)-1}}{a_{n(j)} - a_{n(j)-1}} & \text{if } a' \in \left(a_{n(j)-1}.a_{n(j)}\right) \\[2ex] \frac{a' - a_{n(j)}}{a_{n(j)+1} - a_{n(j)}} & \text{if } a' \in \left(a_{n(j)}.a_{n(j)+1}\right) \\[2ex] 0 & \text{otherwise} \end{cases}$$

$$\mathcal{I}\left(\tilde{d}\right) = \begin{cases} 1 & \text{if } \tilde{d} = d_{k(j)} \\[2ex] b & \text{otherwise} \end{cases}$$

Step 4. Solve for the stationary distribution using a power iteration method.

1. Initialize some guess for λ. Call this $\lambda^{(0)}$. λ is a column vector, where entry i corresponds to the probability mass corresponding to households holding assets $a_{n(i)}$, inheriting a start-of-period car type $d_{k(i)}$, and having productivity $\theta_{m(j)}$. It therefore also conforms to the tensor grid $\mathcal{G}_{\theta, d, a}$.

2. Using T, compute $\lambda^{(1)} = T \times \lambda^{(0)}$. Compute $d = \sup ||\lambda^{(0)} - \lambda^{(1)}||$. If d is smaller than some tolerance, stop. Otherwise, update $\lambda^{(0)}$ with $\lambda^{(1)}$ and compute $\lambda^{(2)}$. Repeat this recursion for T steps until d is below the tolerance. The final distribution $\lambda^{(T)}$ is the steady-state distribution.

New algorithm This paper improves upon steps 3 and 4 of the textbook algorithm:

Step 3a (NEW). Construct a matrix Γ that has the same indexing as T, where for

each entry (j, i),

$$\Gamma(j, i) = \mathcal{I}\left(\tilde{d} = g^{\tilde{d}}\left(a_{n(i)}, d_{k(i)}, \theta_{m(i)}\right)\right) \times \mathcal{B}\left(a' = g(\theta_{m(i)}, d_{k(i)}, a_{n(i)})\right)$$

where $\mathcal{B}$ and $\mathcal{I}$ have the same definitions as before.

Step 3b (NEW). Construct a matrix H, where

$$H = D \otimes \hat{h}$$

H is a $M(K+1) \times M(K+1)$ matrix.

Step 4 (NEW). Solve for the stationary distribution using a power iteration method.

1. Initialize some guess for $\lambda^{(0)}$.

2. Iterate on λ as follows: For some iteration t,

   (a) $\tilde{\lambda} = \Gamma \times \lambda^{(t)}$. Note that $\tilde{\lambda}$ is still an $M(K+1)N \times 1$ column vector.

   (b) Reshape $\tilde{\lambda}$ into a $M(K+1) \times N$ matrix to conform to H.

   (c) Update: $\lambda^{(t+1)} = H \times \tilde{\lambda}$

3. Compute $d = \sup \|\lambda^{(t)} - \lambda^{(t+1)}\|$. If d is smaller than some tolerance, stop; otherwise, repeat step 4.

Results

Like before, I discretized the θ and a state space with increasingly fine grids: $N_\theta \in \{3, 5, 7, ..., 17\}$ and $N_a \in \{30, 50, 100, 200, 300, ..., 1000\}$, and repeatedly solve the model using all combinations of these grids. Following Gavazza and Lanteri (2018), I use 5 states in car types. I then compare the speed (figure 4.3) and memory requirements (figure 4.4) of my algorithm against the standard textbook method, where the speed is computed by solving the model 100 times and taking the average.
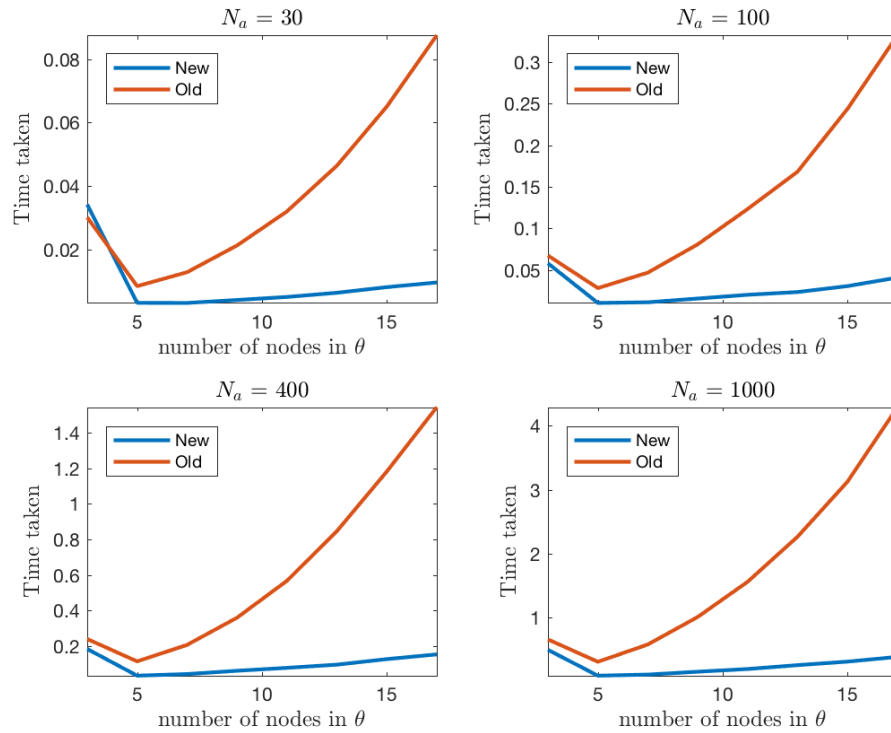
(a) Time taken (in seconds) for different sizes of the state space: N_a refers to the number of nodes in a
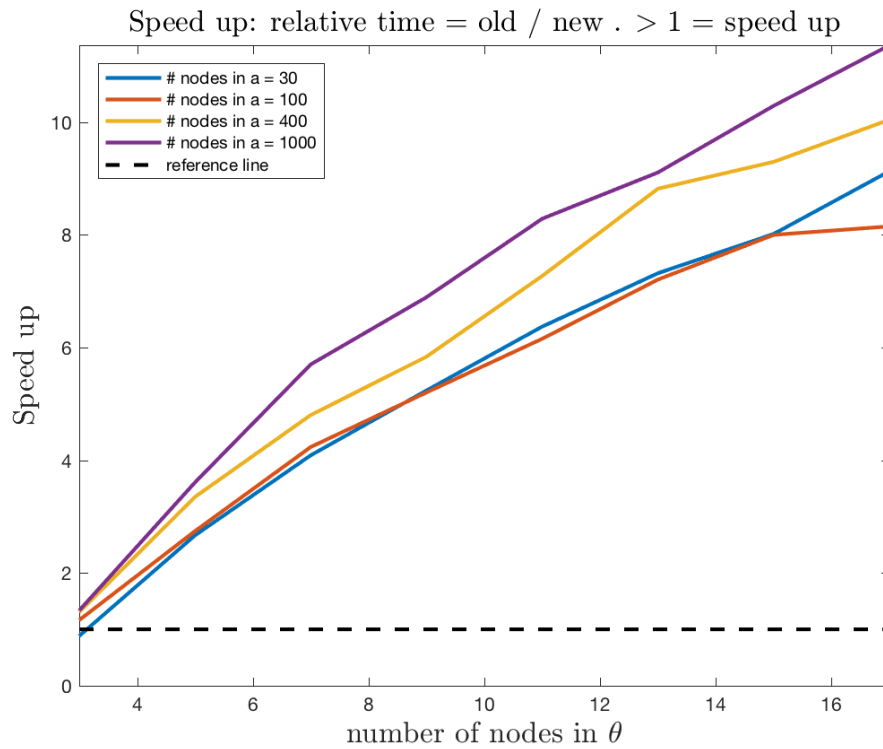
(b) Speed up, computed in relative times. >1 implies speed up ; <1 implies slow down.

FIGURE 4.3: Solution times for textbook method ("Old") v.s. proposed method ("New")

Again, from the following figures, we see that the new method strictly dominates the standard textbook method.

FIGURE 4.4: Memory requirements for textbook method ("Old") v.s. proposed method ("New"). y-axis units are in bytes.

4.3.3   (Modified) Chang and Kim (2007)

The Chang and Kim (2007) extends the basic Aiyagari (1994) model to include an extensive margin in labor supply. In their model, the household observes labor productivity, then chooses their labor supply decision (as well as asset holdings). I modify their model as follows:

1. In period t, the household inherits an asset position (a) and productivity type θ.

2. The household chooses where to work or not (o)

3. The "next period" productivity (θ') is revealed. The actual labor income depends on this productivity.

4. The household makes its "next period" asset choice (a')

The household's problem can be succinctly written as

$$V\left(a,\theta\right) = \max_{o} \left\{ \mathbb{E} \max_{a'} \left[U\left(c,o\right) + \beta V\left(a',\theta'\right) | \theta \right] \right\}$$

$$s.t.$$

$$c = \bar{w} + o \times \exp\left(\theta'\right) + \left(1+r\right)a - a'$$

$$\theta' = \rho\theta + \sigma\epsilon'$$

$$\epsilon' \sim N\left(0,1\right)$$

Given the following:

1. optimal bond policy $a' = g^a\left(a,o,\theta'\right)$, and an associated kernel $G\left(a'|a,\theta';o\right)$

2. optimal working policy $o = g^o\left(a,\theta\right)$, and an associated kernel $H\left(o|a,\theta\right)$

3. the evolution of labor income as $\theta' \sim h\left(\theta'|\theta\right)$,

We can define the evolution of the distribution of households as

$$\lambda\left(\theta',a'\right) = \int T\left(a',\theta'|a,\theta\right) \lambda\left(\theta,a\right) d\theta da$$

where T is the full generated Markov kernel.

Using a standard' textbook algorithm, we would approximate the RHS as

$$\hat{\lambda}\left(\theta',b'\right) = \sum_{\theta,b} \hat{T}\left(b',\theta'|b,\theta\right) \hat{\lambda}\left(\theta,b\right)$$

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Using the algorithm proposed in this paper, we would approximate the RHS as

$$\hat{\lambda}\left(\theta', b'\right) = \sum_{b,h} \hat{G}\left(b, \theta'; h\right) \sum_{\theta} \hat{f}\left(\theta'|\theta\right) H\left(b, \theta\right) \hat{\lambda}\left(\theta, b\right)$$

Step-by-step, the above yields a recursion

$$\lambda_1\left(\theta, b, h\right) = \hat{H}\left(h|b, \theta\right) \hat{\lambda}\left(\theta, b\right)$$

$$\lambda_2\left(\theta', b, h\right) = \sum_{\theta} \hat{f}\left(\theta'|\theta\right) \lambda_1\left(\theta, b, h\right)$$

$$\hat{\lambda}\left(\theta', b'\right) = \sum_{b,h} \hat{G}\left(b'|b, \theta'; h\right) \lambda_2\left(\theta', b, h\right)$$

Results

Here, we see in figure 4.5 that the textbook method is in fact superior to my proposed method when the state space is smaller. This happens as this specification of a dynamic programming problem requires multiple recursive steps per iteration on the distribution when using the proposed method. Consequently, there is a larger penalty associated with this recursion. This penalty overwhelms the memory gains for the models with smaller state spaces, leading to the textbook method winning out. However, as we expand the state space, the latter effect becomes more important. Consequently, the proposed method eventually becomes superior to the textbook method when the state space increases.
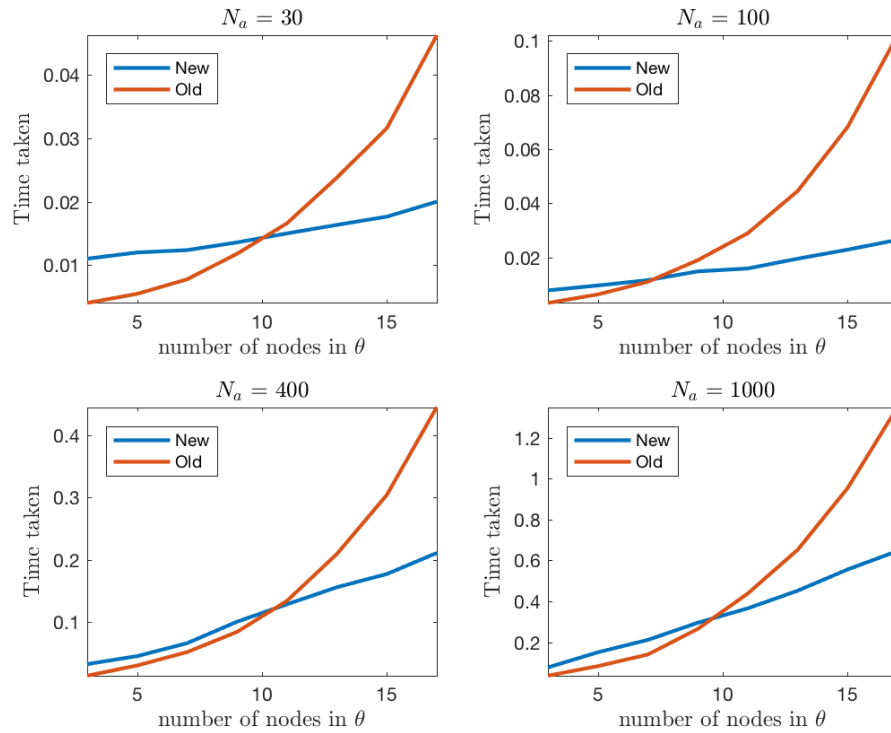
(a) Time taken (in seconds) for different sizes of the state space: N_a refers to the number of nodes in a
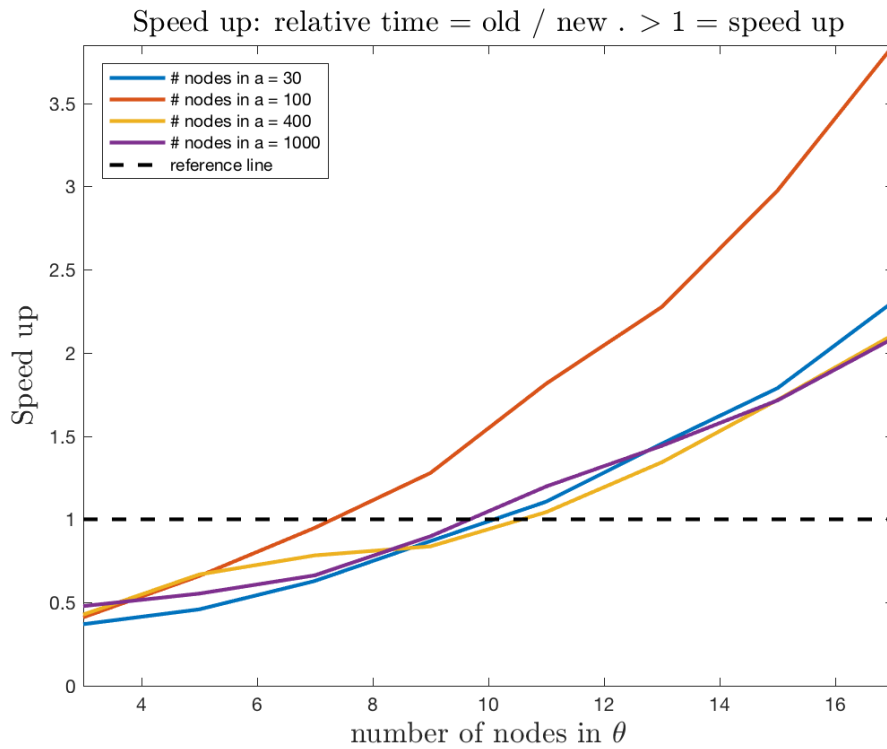
(b) Speed up, computed in relative times. >1 implies speed up ; <1 implies slow down.
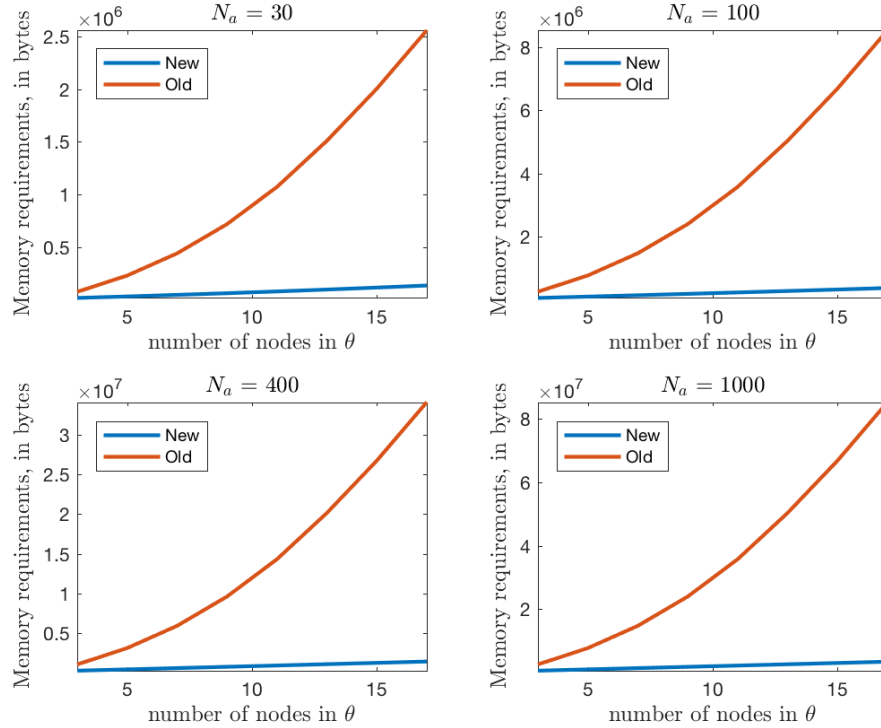FIGURE 4.5: Solution times for textbook method ("Old") v.s. proposed method ("New")

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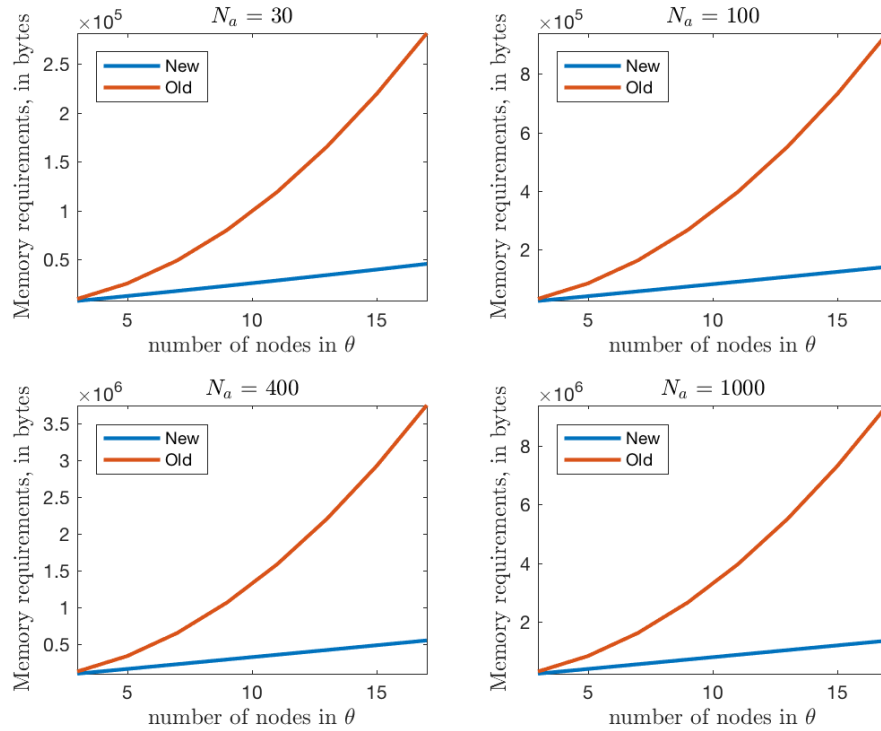
FIGURE 4.6: Memory requirements for textbook method ("Old") v.s. proposed method ("New"). y-axis units are in bytes.

4.4   Conclusion

In this paper, I present a simple numerical algorithm that reduces the memory requirements for solving heterogeneous agent models. The method is also generally faster than textbook treatments; in particular, large gains will be achieved for models with larger state spaces. Given the increasing numerical complexity of heterogeneous agent models, this proposed method, with its efficient use of computational resources, will be a huge advantage for economists.

5

Conclusion

In this dissertation, I study a crucial aspect of entrepreneurial investment dynamics: That entrepreneurial investment is often illiquid, and that entrepreneurs generally bear the full risk of their businesses during the initial stages of business formation. In the first chapter of this dissertation, I provided strong empirical evidence suggesting the importance of investment illiquidity for entrepreneurial startup and investment choices, and constructed a calibrated general-equilibrium macroeconomic model to understand the impact of illiquidity on the overall macroeconomy. In the second chapter, I built on this framework to study how innovations to idiosyncratic subjective uncertainty can interact with investment illiquidity to explain a stark fact in the data: that entrepreneurial entry rates are highly negatively correlated with policy uncertainty, but exit rates are essentially uncorrelated with variations in policy uncertainty. I then showed how my model of entrepreneurship can amplify and propagate the recessionary effects of uncertainty shocks. Finally, in my last chapter, I discuss a new algorithm to solve heterogeneous agents models. I showed how a single simple modification to the standard power iteration method used in macroeconomics can drastically reduce the computational resources and time required to solve these models.

Appendix A

Appendix to Chapter 1

A.1 Data Appendix I: Data Construction

In this section, I briefly discuss the types of firms that are included in the KFS survey. I then discuss how the measures for capital stock, revenue, value added, average revenue product of capital, investment rates, and net leverage ratios are constructed. In addition, I describe the residualization method used to avoid confounding cross-industry heterogeneity in my results.

A.1.1 Survey Inclusion

As discussed in the main text, the universe of firms considered for survey inclusion in the KFS was all newly registered firms in 2004 from the Dun and Bradstreet database. However, given that the focus of the KFS is on new entrepreneurs, this universe is too broad, capturing a wide range of firms from newly registered subsidiaries to established firms spun off from family inheritances. Therefore, for actual inclusion into the survey, a firm must then satisfy at least one of the following conditions:

1. The business was started as independent business, or by the purchase of an existing business, or by the purchase of a franchise in the 2004 calendar year.

2. The business was *not* started as a branch or a subsidiary owned by an existing business, that was inherited, or that was created as a not-for-profit organization in the 2004 calendar year.

3. The business had a valid business legal status (sole proprietorship, limited liability company, subchapter S corporation, C-corporation, general partnership, or limited partnership) in 2004.

4. The business reported at least one of the following activities:

   (a) Acquired employer identification number during the 2004 calendar year

   (b) Organized as sole proprietorships, reporting that 2004 was the first year they used Schedule C or Schedule C-EZ to report business income on a personal income tax return

   (c) Reported that 2004 was the first year they made state unemployment insurance payments

   (d) Reported that 2004 was the first year they made federal insurance contribution act payments

All firms that satisfy at least one of these conditions then make up the sample population of the KFS.

A.1.2 Variable Construction

Capital stock

In order to construct the average revenue product of capital, I first need to construct the capital stock of the firm. The KFS provides the researcher the balance sheet of the firm, and it provides a breakdown of the type of capital asset that the entrepreneurial firm owns[1].

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[1] The full range of asset types are product inventories, land and buildings and structures, vehicles, equipment or machinery, other properties, cash, and "others".

However, as in most standard models, I consider only a single generic capital asset of interest. As such, in order render the results comparable, I construct a representative single asset, real capital stock, $K_{i,t}$, using the nominal value of capital assets as follows:

$$K_{i,t} = \sum_s \frac{K_{i,s,t}}{P_{s,t}}$$

where $P_{s,t}$ is the relative price of each capital type s and vintage t. Subscript i indexes the firm. The relative prices are taken from the BEA. For the aggregated capital stock, I only consider the firm's holdings of product inventories, land and buildings and structures, vehicles, equipment or machinery, and other properties. The value of product inventories are deflated using the GDP deflator.

Revenue and Value added

Construction of the average revenue product of capital also necessitates the construction of a measure of the firm's real value added. A two step adjustment is used to transform nominal revenue into real value added. The first step is straightforward; nominal revenue is deflated by the GDP deflator to obtain real revenue.

Unfortunately, constructing real value added from real revenue is less straightforward. The KFS does not provide information on the firm's material expenses; consequently, one cannot simply subtract out the material cost to retrieve value added.

Instead, following the literature, I assume that the firm revenue production function is given by a Cobb-Douglas function of the form

$$Y_{i,j,t}^R = z_{i,j,t} K_{i,j,t}^{\beta_k^j} L_{i,j,t}^{\beta_l^j} M_{i,j,t}^{\beta_m^j}$$

where i indexes a firm, j indexes the industry, and t indexes time. z here refers to aggregated TPFR (i.e. it summarizes the firm, industry and aggregate level shocks). The capital (K), labor (L) and material (M) intensity parameters $(\beta_k^j, \beta_l^j, \beta_m^j$ respectively) are allowed to vary across industries, but are restricted to sum to unity.

Let P_m^j denote the cost of materials for industry j. Then value added is

$$Y_{i,j,t} = Y_{i,j,t}^R - P_m^j M_{i,j,t}$$

As the KFS does not provide information on $P_m^j M_{i,j,t}$, the goal is to infer $P_m^j M_{i,j,t}$.

To do so, I follow the preceding literature by assuming that capital is fixed one period in advanced, but the firm is able to adjust its labor and material inputs contemporaneously[2]. Given this, the firm's optimal material choice will yield the first order condition:

$$P_m^j = \beta_m^j \frac{Y_{i,j,t}^R}{M_{i,j,t}}$$

$$\implies Y_{i,j,t} = (1 - \beta_m^j) \times Y_{i,j,t}^R$$

Therefore, given the material share parameter, we can immediately back out value added. For the purposes of this paper, I assume that the entrepreneurial firms have the same material share as the industry that they operate in. The material share at the industry level can be estimated directly from national accounting data (the NIPA KLEMS tables). Denoting the estimated material share as $\hat{\beta}_m^j$, I back out an estimate of value added as[3]

$$Y_{i,j,t} = (1 - \hat{\beta}_m^j) \times Y_{i,j,t}^R$$

For the rest of this paper, revenue refers to this measure of value added, unless explicitly stated otherwise.

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[2] See for instance, Olley and Pakes (1996)

[3] One might justifiably be concerned that this measurement of value added is very noisy, and introduces extra measurement error. As a robustness check, I repeat the empirical exercises using the "raw" revenue measure. The qualitative results obtained using the value added measure is replicated when I use the raw revenue measure.

Average revenue product of capital

Having constructed both capital and revenue, the (log) average revenue product of capital is simply

$$\log ARPK \equiv \log\left(\frac{Y_{it}}{K_{i,t-1}}\right)$$

t indicates the survey year. This timing convention is adopted as the KFS only surveys the firm at the start of the following year for information regarding the current year. That is to say, if the survey report is for the year 2004, it was in fact surveyed in 2005. Therefore, K_{2004} is in fact the end of period capital stock, while Y_{2004} is the revenue for 2004. The convention chosen here hence matches the standard timing convention to construct average products of capital as the ratio of the total revenue to the start of period capital stock.

For the main body of the paper, the moments I report are constructed using a pooled measure of $\log ARPK$. In practice, moments constructed using pooled $\log ARPK$ might not be a good measure since there is likely to be large heterogeneity in capital share and returns to scale across industries [4]. However, due to the small sample size of the KFS relative to the number of industries[5], there is simply insufficient statistical power to draw any useful inference if one restricted analysis only to the 6 digit NAICS industry level. Instead, I address this issue through two methods.

Residualized ARPK   All the benchmark empirical results utilize a residualized measure of $\log ARPK$ rather than the raw measure. Here, the pooled $\log ARPK$ variables are residualized by regressing the raw $\log ARPK$ on two digit NAICS industry level fixed effects and time dummies. The residuals of this regression then form my measure of $\log ARPK$ for analysis. This avoids the issue where permanent differences across

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[4] See for instance, Burnside (1993), who estimates the returns to scale for different industries in the United States

[5] There were 3140 firms in 2004 when the survey started; by 2011, there were only 1630 firms left. In contrast, there are 659 six digit level NAICS code industries.

industries (such as heterogeneity in capital share and returns to scale) would distort the distribution, and thus introduce spurious correlations and moments. Moreover, it also removes some of the common aggregate shocks that might distort the distribution of $\log ARPK$ over time. The results presented in the following sub-sections are constructed using this residualized measure[6].

Industry level moments   The biggest concern about my findings relate to the fact that there is substantial permanent heterogeneity across industries that the residualization process is unable to fully purged. To address these concerns, I also directly investigate the relevant moments at the two digit industry level. Due to the smaller sample size, only one industry showed statistically significant results; however, most industries showed economically significant results. Moreover, the qualitative findings at the aggregate level also holds broadly across industries. The results at the industry level are reported in this appendix.

Investment rates

To construct investment and investment rates, I use the perpetual inventory method as follows:

$$I_{i,s,t} = \frac{K_{i,s,t}}{P_{s,t}} - (1 - \delta_s)\frac{K_{i,s,t-1}}{P_{s,t-1}}$$

$$I_{i,s,t} = \sum_s \frac{K_{i,s,t-1}}{\sum_s K_{i,s,t-1}} I_{i,s,t}$$

$$i_{i,s,t} = \frac{I_{i,t}}{K_{i,t-1}}$$

where $I_{i,s,t}$ refers to investment levels of firm i for capital type s and vintage t. I allow for depreciation of each type of capital (δ_s) to differ according to the BEA

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[6] As a robustness check, I also construct extended residualized measures of $\log ARPK$ by using more regressors that could potentially systematically distort the distribution of $\log ARPK$ (for instance, the legal form of the firm, or the gender of the primary owner of the firm). I find that my benchmark results are robust to alternative measures. Results computed using these extended measures are available upon request.

depreciation schedule. Gross investment at the firm level is then constructed as a weighted average of the firm's investment for each capital type. Investment rate is then constructed by scaling the gross investment by the total lagged capital stock.

Just as in the case of $\log ARPK$, using i_{it} "as is" poses a potential problem that the investment series is biased by unobserved aggregate shocks that are orthogonal to the idiosyncratic shocks or frictions that I am interested in studying. To purge the effect of aggregate shocks on investment, I construct a residualized investment series by regressing investment rates on year and two digit NAICS code fixed effects, and construct the relevant investment moments using the residuals.

Net liquid asset

The real net liquid asset of the firm is computed as the difference between the total cash holdings and the business debt of the firm, deflated by the GDP deflator.

Net leverage ratios

Net leverage ratio at the firm level is defined as the ratio of the net business debt of the firm to the capital stock of the firm. Here, net debt is simply defined as $D_{i,t} = \max\{0, Debt_{i,t} - Cash_{i,t}\}$; that is, for firms who have net positive liquid asset holdings, the debt level is simply 0. Therefore, net leverage is simply defined as $\mathcal{L}_{i,t} = \frac{D_{i,t}}{K_{i,t}} = \frac{\max\{0, Debt_{i,t} - Cash_{i,t}\}}{K_{i,t}}$.

The results in table 2.11 were then computed using a winsorized measure of the leverage ratios, where distribution is winsorized at the 1st and 99th percentiles. Moments computed using the un-winsorized measures, as well as alternative measures that only include firms that are in debt (i.e. $D_{i,t} > 0$) are available upon request.

A.2   Data Appendix II: Robustness Checks

In this section, in an effort to ensure the robustness of my empirical findings, I re-examine the data moments I reported through multiple cuts of the data.
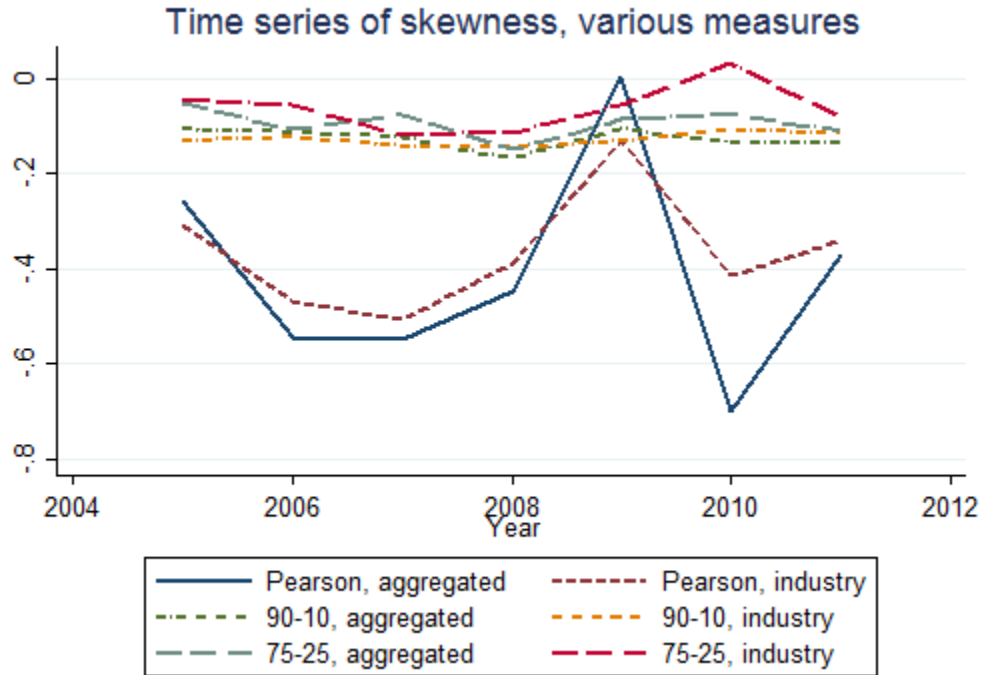
FIGURE A.1: Time series of various measures of skewness.

A.2.1   Skewness: Across Time and Definitions

The skewness moments that I report in the main text takes the entire distribution as a pooled sample. To show that this left-skewness is in fact a robust feature of the data, I also estimated the skewness period-by-period over the entire sample, again for all three measures of skewness. These results are reported in figure A.1. From the graph, we see that regardless of the choice of skewness measure and time period, the distribution of log ARPK is left-skewed.

A.2.2   Asymmetric Persistence: Using Alternative Definitions

The estimation of the persistence of relative rankings was estimated using the full sample, and the quantiles shift over time as I estimated the quantiles period-by-period[7]. This might lead the reader to raise a few concerns, specifically:

1. Quantile construction. The results could be an artifact of the shifting quantiles.

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[7] The quantiles themselves are still estimated at the industry level, not across the pooled sample.

2. Sample selection. The results could be biased by very small firms. Very small firms could be operated by individuals with no desire to maximize profits[8], and these firms are also most likely to be in the left tail. As a result, the higher left tail persistence that I report in the main text could be driven by these firms.

To address these problems, I report in table A.1 below a series of robustness checks. I address the first point by simply fixing the quantiles to the ones computed in period 1 (year 2005), and estimate the transition probabilities. To address the second point, I conduct the original estimation using an increasingly stricter cut off for firms. In the first specification, I consider stricter cut-offs defined in terms of assets. I first conduct the estimation only for firms reporting more than $5000 in assets, and then for firms with more than $10,000 in assets, repeating the process up to $50,000 in assets. In a second specification, I conduct the cut-off using revenue. I start with a threshold of $10,000, and repeat up to $100,000. In table A.1, I only report the persistence in the left tail (i.e. probability of staying in rank 1) and the right tail (i.e. probability of staying in rank 5) since these are the key moments of interest. As one can see, the asymmetry (i.e., higher left-tail persistence) is a robust feature of the data.

Table A.1: Persistence of relative rankings for firms in the first and fifth quintiles under different cut off assumptions. For fixed quintiles, the quintiles are fixed to the cutoffs from year 2005. For the other versions, the quintile is constructed using the full sample as in the main text, but the estimation is done using only firms that pass the asset cut-off. The standard errors are in parenthesis.

Model	Fixed quantiles	Assets $\geq$				Revenue $\geq$				
		$5k	$10k	$20k	$50k	$10k	$20k	$50k	$75k	$100k
$Pr(1 \rightarrow 1)$	0.61	0.56	0.57	0.58	0.60	0.53	0.54	0.55	0.53	0.52
	(0.017)	(.019)	(0.020)	(0.021)	(0.023)	(0.02)	(0.02)	(0.03)	(0.03)	(0.03)
$Pr(5 \rightarrow 5)$	0.49	0.49	0.51	0.49	0.44	0.48	0.48	0.49	0.50	0.51
	(0.017)	(.026)	(.031)	(0.038)	(0.059)	(0.02)	(0.02)	(0.2)	(0.02)	(0.02)

---

[8] See for instance, Hurst and Pugsley (2015) for a discussion on this class of entrepreneurs.

Here, I report the results for the persistence in relative rankings measure at the industry level. I only report the probability of staying in the same quintile for the first and last quintile; the results of the full estimation of the transition matrices are available on request. The blue diamonds correspond to the probability of staying in the first quintile, given that the firm was in the first quintile. The red diamonds correspond to the probability of staying in the last quintile, given that the firm was in the last quintile. Each pair corresponds to a single two digit NAICS industry code. As one can see, the asymmetric persistence does not just apply to the pooled sample. The vast majority of industries also feature economically significant asymmetry.
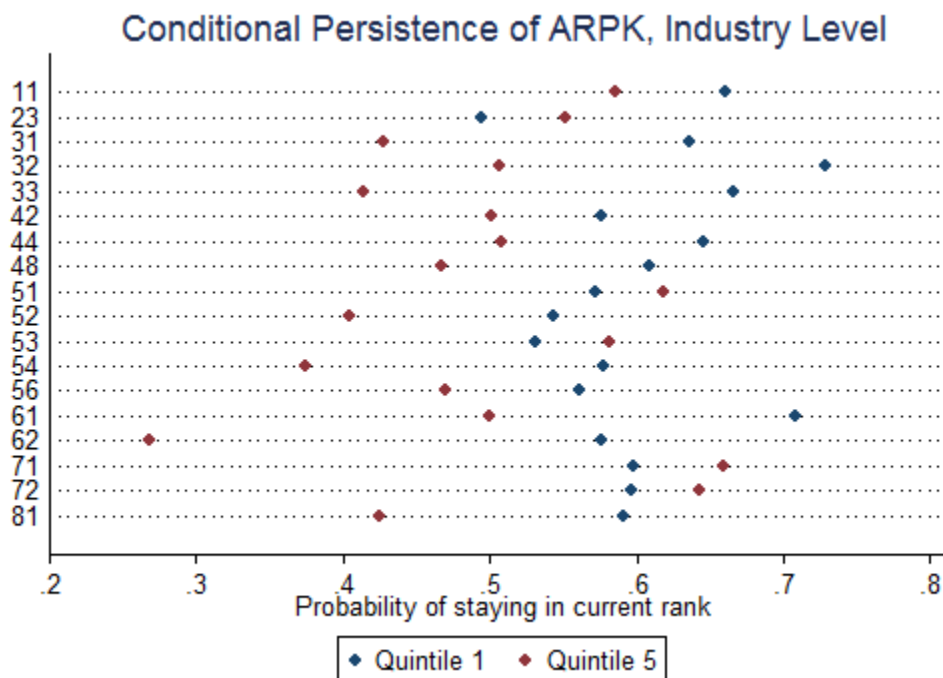
FIGURE A.2: Probability of staying in same quintile. Blue corresponds to the first quintile, red to the last quintile.

A.2.4 Conditional Autocorrelation as Measure of Asymmetric Persistence

A potential issue with estimating transition matrices, and using them to understand mobility at the tails of the distribution, is that transition matrices tend to overstate

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immobility at the tails (relative to other parts of the distribution). This happens because individuals can only move in one direction when they are at either ends of the distribution (i.e., the floor-ceiling effect).

To ensure that my results are not an artifact of this effect, I conduct a second analysis where I estimate the autocorrelation of $\log ARPK$ conditional on the firm position in the distribution. Specifically, I estimate a regression of the form

$$\log ARPK_{i,t} = \alpha + \sum_{q=1}^{5} \rho_q \log ARPK_{i,t-1} + \varepsilon_{it}$$

where α is the intercept term, and ρ_q is a coefficient that depends on the $\log ARPK$ quantile q that the firm is in currently. As table A.2 shows, ρ_1, which is the autocorrelation of $\log ARPK$ when the firm is in quantile 1, is much larger than ρ_5. A Wald test also rejects the null that the two estimates are equivalent, as such supporting the evidence that there is greater persistence in $\log ARPK$ at the bottom quintile relative to the top quintile.

Table A.2: Regression result of $\log ARPK_{i,t} = a + \rho_q \log ARPK_{i,t-1} + \varepsilon_{it}$. $q = 1$ refers to the first quintile, $q = 2$ the second quintile, and so on. Standard errors in parentheses.

	Dependent variable: $\log ARPK_{i,t}$
a	0.075 (0.037)
ρ_1	0.643 (0.034)
ρ_2	0.897 (0.058)
ρ_3	0.672 (0.109)
ρ_4	0.697 (0.052)
ρ_5	0.443 (0.035)

Table A.3: Conditional autocorrelation for firms in the first and fifth quintiles under different cut off assumptions. The standard errors are in parenthesis.

Model	Assets $\geq$						Revenue $\geq$				
	$5k	$10k	$20k	$50k	$75	$100	$10k	$20k	$50k	$75k	$100k
ρ_1	0.68	0.70	0.72	0.73	0.75	0.77	0.70	0.74	0.79	0.78	0.77
	(0.04)	(0.03)	(0.03)	(0.04)	(0.04)	(0.04)	(0.04)	(0.04)	(0.05)	(0.06)	(0.07)
ρ_5	0.40	0.36	0.33	0.26	0.25	0.28	0.41	0.41	0.42	0.42	0.42
	(0.04)	(0.04)	(0.05)	(0.06)	(0.07)	(0.06)	(0.04)	(0.04)	(0.04)	(0.04)	(0.04)

A.2.5   Conditional Autocorrelation: Industry Level

Here, I report the results for the conditional autocorrelation at the industry level. I only report the autocorrelation for firms in the bottom and top quintiles; the results of the full estimation are available on request. The blue diamonds correspond to the autocorrelation when the firm is in the bottom quintile; the red diamonds to the autocorrelation when the firm is in the top quintile. Each pair corresponds to a single two digit NAICS industry code. As in the result earlier, there exist also large differences in in conditional autocorrelation at the industry level.
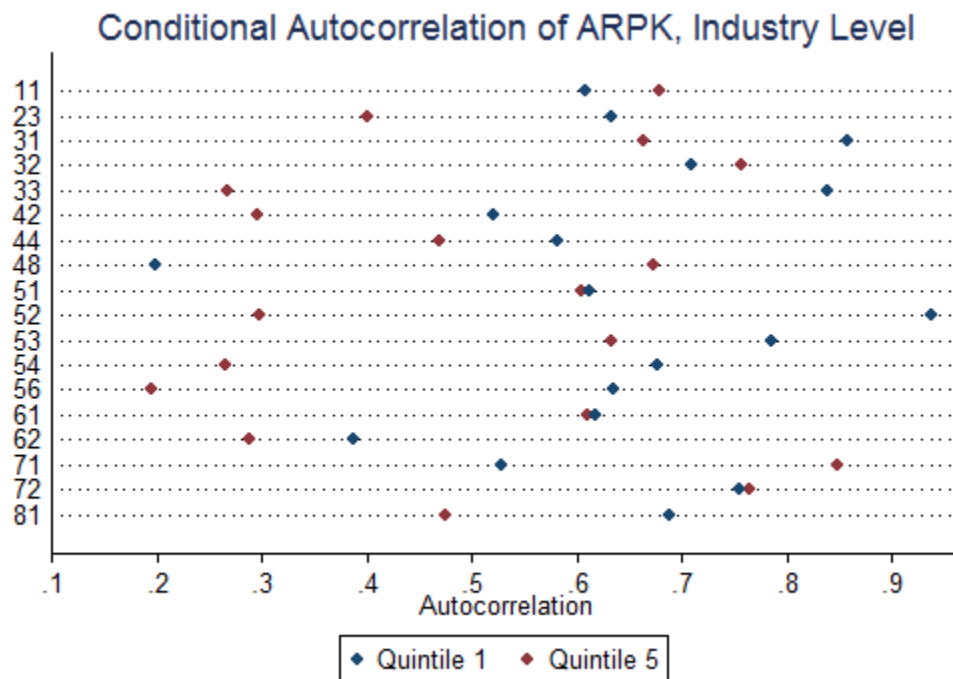
FIGURE A.3: Autocorrelation conditional on being in the first quintile (blue), and the last quintile (red).

A.3 Computational Appendix

In this section, I detail in subsection A.3.1 the solution method used in solving the individual's problem, and in subsection A.3.2 the method used in solving for the stationary equilibrium. The algorithm used in solving for the stationary equilibrium might be of interest to researchers interested in doing non-stochastic simulation with a large number of state variables, as in this paper.

A.3.1 Solving the individual's problem

The household's problem is broken down into a multi-stage problem in order to render it numerically feasible to solve. The steps are detailed below.

First stage: Occupational choice

Define the intermediate value functions V_{ee}, V_{ew}, V_{we}, V_{ww} as the value functions of (1) an entrepreneur who has chosen to stay as an entrepreneur, (2) an entrepreneur who

decides to exit, (3) a worker who decides to become an entrepreneur, and (4) a worker who decides to stay a worker; moreover, let this be the value function associated with the household who has already made its optimal portfolio-savings choice.

Given this definition, we see that the optimal occupational choice of an entrepreneur is given by

$$V_e = \max_{h'} \left(1 - h'\right) V_{ee} + h' \times V_{ew}$$

and the optimal occupational choice of a worker is given by

$$V_w = \max_{h'} \left(1 - h'\right) V_{we} + h' \times V_{ww}$$

where $h' = 1$ if the household chooses to become a worker tomorrow, and $h' = 0$ if she chooses to become a entrepreneur.

Second stage: Savings-investment

The investment-savings problem is described for each for the four groups of households below. Here, for convenience, I shall define the interest rate function $\tilde{r} \equiv r \times 1_{\{b \geq 0\}} + r_d \times 1_{\{b < 0\}}$.

Entrepreneur: Exit

For an entrepreneur who decides to exit, her problem is simply

$$V_{ew}\left(\psi^\theta, z, k, b\right) = \max_{b'} U\left(c\right) + \beta \mathbb{E}\left[V_w\left(\psi^{z'}, \theta', b'\right) | \psi^\theta\right]$$

$$s.t.$$

$$c + b' = \pi^* + (1 + \tilde{r}) b + (1 - \zeta)(1 - \lambda)(1 - \delta) k$$

$$b' \geq \underline{b}$$

To reduce the dimensionality of this problem, I approximate

$$\hat{V}_{ew}\left(\psi^\theta, b'\right) \equiv \mathbb{E}\left[V_w\left(\psi^{z'}, \theta', b'\right) | \psi^\theta\right]$$

using a cubic spline in the b' direction. I then solve the individual's problem using

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the monotone condition.

Entrepreneur: Stay

For the entrepreneur who stays an entrepreneur, it is more convenient to define another set of intermediate value functions. Let V_{ee}^{I}, V_{ee}^{D} and V_{ee}^{0} denote the value functions of an entrepreneur who has decided to (1) invest in his business, (2) disinvest, and (3) keep the same capital stock into the next period.

For an investing entrepreneur, his problem is

$$V_{ee}^{I}\left(\psi^{\theta},z,k,b\right)=\max_{k',b'}U\left(c\right)+\beta\mathbb{E}\left[V_{e}\left(\psi^{\theta'},z',k',b'\right)|z\right]$$

$$s.t.$$

$$c+\left(k'-\left(1-\delta\right)k\right)+b'+f_{s}k=\pi^{*}+\left(1+\tilde{r}\right)b$$

$$k'>\left(1-\delta\right)k$$

$$b'\geq-\varphi\left(1-\lambda\right)\left(1-\delta\right)k'-\underline{b}$$

For a disinvesting entrepreneur, his problem is

$$V_{ee}^{D}\left(\psi^{\theta},z,k,b\right)=\max_{k',b'}U\left(c\right)+\beta\mathbb{E}\left[V_{e}\left(\psi^{\theta'},z',k',b'\right)|z\right]$$

$$s.t.$$

$$c+\left(1-\lambda\right)\left(k'-\left(1-\delta\right)k\right)+b'=\pi^{*}+\left(1+r\right)b$$

$$0<k'<\left(1-\delta\right)k$$

$$b'\geq-\varphi\left(1-\lambda\right)\left(1-\delta\right)k'-\underline{b}$$

Finally, for an entrepreneur who is holding the same (depreciated) capital stock,

his problem is

$$V_{ee}^0\left(\psi^\theta, z, k, b\right) = \max_{k',b'} U\left(c\right) + \beta\mathbb{E}\left[V_e\left(\psi^{\theta'}, z', k', b'\right)|z\right]$$

$$s.t.$$

$$c + b' = \pi^* + \left(1 + r\right)b$$

$$k' = \left(1 - \delta\right)k$$

$$b' \geq -\varphi\left(1 - \lambda\right)\left(1 - \delta\right)k' - \underline{b}$$

This formulation posses three problems. Firstly, the state space spanned by $\mathbb{K} \times \mathbb{B}$ is not orthogonal due to the financing constraints, which makes standard numerical methods not feasible. Secondly, as the portfolio choice here is non-trivial, computational time is substantially increased. Moreover, due to the asymmetric adjustment cost, the type of adjustment itself is conditional on the current state of capital holdings. Finally, the state space of the entrepreneur is spanned by four variables, which also increases computational time substantially. To address these issues, I consider the following modifications to the problem.

Orthogonality problem

Firstly, I address the orthogonality problem by defining a new variable, net liquid asset, a as follows:

$$a \equiv b + \varphi\left(1 - \delta\right)\left(1 - \lambda\right)k + \underline{b} \geq 0$$

The state space of the entrepreneur $\mathbb{S}^E$ is now defined by $\left(\psi^\theta, z, k, a\right) \in \Psi^\theta \times \mathbb{Z} \times \mathbb{K} \times \mathbb{A} = \mathbb{S}^E$, which is orthogonal, since both a and k are only restricted by $a \geq 0$ and $k \geq 0$.

Using this new definition, I address the numerical challenges arising from the non-trivial portfolio choice problem and dimensionality problem using the following strategy.

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Portfolio choice and dimensionality problem

Firstly, note that conditioned on deciding to *stay* as an entrepreneur (i.e. V_{ee}), the current signal shock ψ^θ is not informative of the next period's signal as it is IID. As such, the relevant state space for the entrepreneur who is staying as an entrepreneur is simply the subspace $\mathbb{S}^{EE} = \mathbb{Z} \times \mathbb{K} \times \mathbb{A} \subset \mathbb{S}^E$. This cuts down on one dimension.

Secondly, note that because of the IID nature of ψ^θ, the conditional expectation $\mathbb{E}\left[V_e\left(\psi^{\theta'}, z', k', a'\right) | z\right]$ is defined only over the three dimensional subspace $\mathbb{S}^{EE}$. Specifically, we can write the following definition:

$$\hat{V}_{ee}\left(z, k', a'\right) \equiv \mathbb{E}\left[V_e\left(\psi^{\theta'}, z', k', a'\right) | z\right]$$

To evaluate the continuation value of k' and a', I then approximate $\hat{V}_e$ using a two dimensional cubic spline along the dimensions of k' and a'.

Thirdly, we can take advantage of the linear separability of the cost function to reduce the dimensionality of the state space. Specifically, note that conditional on either form of adjustment, we can rewrite the household's budget as

$$c + b' + k' + \mathcal{C}_1\left(k'\right) = \pi^* + (1 + \tilde{r})b + (1 - \delta)k - \mathcal{C}_2\left(k\right)$$

$$\Leftrightarrow c + a' - \varphi\left(1 - \delta\right)\left(1 - \lambda\right)k' + k' + \mathcal{C}_1\left(k'\right) = \pi^* + (1 + \tilde{r})\left(a - \varphi\left(1 - \delta\right)\left(1 - \lambda\right)k\right)$$

$$+ (1 - \delta)k - \mathcal{C}_2\left(k\right)$$

where $\mathcal{C}\left(k', k\right) = \mathcal{C}_1\left(k'\right) + \mathcal{C}_2\left(k\right)$ is the total cost associated with this adjustment.

When the firm is adjusting upwards (i.e. investing, denoted henceforth by I), we get

$$\mathcal{C}_1\left(k'\right) = 0$$

$$\mathcal{C}_2\left(k\right) = f_s k$$

When the firm is adjusting downwards (i.e. disinvesting, denoted henceforth by

D), we get

$$\mathcal{C}_1\left(k'\right) = -\lambda k'$$

$$\mathcal{C}_2\left(k\right) = \lambda\left(1 - \delta\right)k$$

Therefore, we have two budget constraints given by

$$c + a' - \varphi\left(1 - \delta\right)\left(1 - \lambda\right)k' + k' = \pi^* + \left(1 + \tilde{r}\right)a + \left(1 - \delta - f_s\right.$$

$$\left. - \left(1 + \tilde{r}\right)\varphi(1 - \delta)(1 - \lambda)\right)k \quad (\text{ if } I)$$

$$c + a' + \left(1 - \varphi(1 - \delta)\right)\left(1 - \lambda\right)k' = \pi^* + \left(1 + \tilde{r}\right)a$$

$$+ \left(1 - \left(1 + \tilde{r}\right)\varphi\right)\left(1 - \lambda\right)(1 - \delta)k \quad (\text{ if } D)$$

Also, note that the budget for no adjustment (i.e. $k' = \left(1 - \delta\right)k$) is given by

$$c + b' = \pi + \left(1 + \tilde{r}\right)b$$

$$c + b' = \pi + \left(1 + \tilde{r}\right)a$$

$$- \left(1 + \tilde{r}\right)\varphi(1 - \delta)\left(1 - \lambda\right)k$$

$$\Leftrightarrow c + b' + \varphi(1 - \delta)\left(1 - \lambda\right)k' - \varphi(1 - \delta)\left(1 - \lambda\right)k' = \pi + \left(1 + \tilde{r}\right)a$$

$$- \left(1 + \tilde{r}\right)\varphi(1 - \delta)\left(1 - \lambda\right)k$$

$$\Leftrightarrow c + a' - \varphi(1 - \delta)\left(1 - \lambda\right)\left(1 - \delta\right)k = \pi + \left(1 + \tilde{r}\right)a$$

$$- \left(1 + \tilde{r}\right)\varphi(1 - \delta)\left(1 - \lambda\right)k$$

$$\Leftrightarrow c + a' = \pi^* + \left(1 + \tilde{r}\right)a$$

$$- \left(\tilde{r} + \delta\right)\varphi(1 - \delta)\left(1 - \lambda\right)k$$

Next, we can define by x the total liquid value of the portfolio. That is, let

$$x^I \equiv \pi^* + \left(1 + \tilde{r}\right)a + \left(1 - \delta - f_s - \left(1 + \tilde{r}\right)\varphi(1 - \delta)(1 - \lambda)\right)k \quad (\text{ if } I)$$

$$x^D \equiv \pi^* + \left(1 + \tilde{r}\right)a + \left(1 - \left(1 + \tilde{r}\right)\varphi\right)\left(1 - \lambda\right)(1 - \delta)k \quad (\text{ if } D)$$

$$x^O \equiv \pi^* + \left(1 + \tilde{r}\right)a - \left(\tilde{r} + \delta\right)\varphi(1 - \delta)\left(1 - \lambda\right)k$$

At this point, it is important to note that because $x = x\left(z, k, a\right)$, for any generic

(z, k, a)-triple, it is unlikely that $x^I = x^D$ numerically, due to their differing defini-tions. However, because this total liquid value is a sufficient state variable for the (z, k, a)-triple, it does not matter that $x^I \neq x^D$ for the same generic (z, k, a)-triple.

Now, define by $\left\{ g_I^a(x), g_I^k(x) \right\}$ and $\left\{ g_D^a(x), g_D^k(x) \right\}$ the optimal <u>unconstrained</u> savings-investment policy (i.e. choice of a' and k') when the household is faced with the budget constraints defined by x^I and x^D respectively (and $\tilde{V}_{ee}^I$ and $\tilde{V}_{ee}^D$ the value functions associated with this policy). That is, let them solve the two problems

$$\tilde{V}_{ee}^I = \max_{g_I^a, g_I^k} U(c) + \beta \mathbb{E}\left[V_e\left(\psi^{\theta'}, z', k', a' \right) | z \right]$$

$$s.t.$$

$$c + + g_I^a + (1 - \varphi(1 - \delta)(1 - \lambda)) g_I^k = x^I$$

$$g_I^k > 0$$

$$g_I^a \geq 0$$

and

$$\tilde{V}_{ee}^D = \max_{g_D^a, g_D^k} U(c) + \beta \mathbb{E}\left[V_e\left(\psi^{\theta'}, z', k', a' \right) | z \right]$$

$$s.t.$$

$$c + g_D^a + (1 - \lambda - \varphi(1 - \delta)(1 - \lambda)) g_D^k = x^D$$

$$g_D^k > 0$$

$$g_D^a \geq 0$$

Notice that for this sub-problem, the household does not care about the indi-vidual components of the current portfolio, and only cares about its total value x. Consequently, $\left\{ g_I^a(x), g_I^k(x) \right\}$ and $\left\{ g_D^a(x), g_D^k(x) \right\}$ are only functions of x.

Finally, let $\left\{ g_0^a, g_0^k \right\}$ denote the optimal savings policy when the household is not adjusting. In particular $g_0^k = (1 - \delta) k$.

First, for convenience, denote the following indicator functions as follows:

$$
\mathcal{I}^I = \begin{cases} 1 & \text{(if } g_I^k > (1-\delta)\,k) \\ 0 & \text{(otherwise)} \end{cases}
$$

$$
\mathcal{I}^D = \begin{cases} 1 & \text{(if } g_I^k < (1-\delta)\,k) \\ 0 & \text{(otherwise)} \end{cases}
$$

$$
\mathcal{I}^v = \begin{cases} 1 & \text{(if } \tilde{V}_{ee}^I > \tilde{V}_{ee}^D) \\ 0 & \text{(otherwise)} \end{cases}
$$

$$
\mathcal{I}^0 = \begin{cases} 1 & \text{(otherwise)} \\ 0 & \text{(if } V_{ee}^0 \geq \max\{V_{ee}^I, V_{ee}^D\} \end{cases}
$$

Now, denote by $g^a\left(\psi^{\theta'}, z, k, a\right)$ and $g^k\left(\psi^{\theta'}, z, k, a\right)$ the optimal *constrained* savings-investment policy functions. They must satisfy the following condition: For any portfolio state (z, k, a) [9], we have:

$$
\begin{bmatrix} g^a \\ g^k \end{bmatrix} = \left[\mathcal{I}^I \times \mathcal{I}^v \times \begin{bmatrix} g_I^a \\ g_I^k \end{bmatrix} \circ \left(x^I\right) + \mathcal{I}^D \times (1-\mathcal{I}^v) \times \begin{bmatrix} g_D^a \\ g_D^k \end{bmatrix} \circ \left(x^D\right) \right] +
$$

$$
\left(1 - \mathcal{I}^0 \times \left[\mathcal{I}^I \times \mathcal{I}^v + \mathcal{I}^D \times (1-\mathcal{I}^v)\right]\right) \times \begin{bmatrix} g_0^a \\ g_0^k \end{bmatrix} \circ \left(\psi^{\theta'}, z, k, a\right)
$$

Similarly then, the value function for the entrepreneur who chooses to stay an entrepreneur is

$$
V_{ee} = \left[\mathcal{I}^I \times \mathcal{I}^v \times \tilde{V}_{ee}^I + \mathcal{I}^D \times (1-\mathcal{I}^v) \times \tilde{V}_{ee}^D\right] +
$$

$$
\left(1 - \mathcal{I}^0 \times \left[\mathcal{I}^I \times \mathcal{I}^v + \mathcal{I}^D \times (1-\mathcal{I}^v)\right]\right) \times V_{ee}^0
$$

Worker: Entry into entrepreneurship

Conditional on choosing to become an entrepreneur, the worker's problem is

---

[9] Note the dependence on the portfolio state, not just x. In general, for some (z, k, a), $x_I \neq x_D \neq x_0$.

$$V_{we}\left(\psi^z,\theta,a\right)=\max_{k',a'}U\left(c\right)+\beta\mathbb{E}\left[V_e\left(\psi^{\theta'},z',k',a'\right)|\psi^z\right]$$

$$s.t.$$

$$c+k'+a'-\varphi(1-\delta)\left(1-\lambda\right)k'=\theta w+\left(1+r\right)a$$

$$k'>0$$

$$a'\geq 0$$

Note that in this case, $a=b$ since $k=0$, but $a'=b'+\varphi\left(1-\lambda\right)k'$.

As in the entrepreneur who stays an entrepreneur, I approximate

$$\hat{V}_{we}\left(\psi^z,k',a'\right)\equiv\mathbb{E}\left[V_e\left(\psi^{\theta'},z',k',a'\right)|\psi^z\right]$$

using a two dimensional cubic spline in the k' and a' directions.

Worker: Stay

Conditional on staying a worker, the household's problem is

$$V_{ww}\left(\psi^z,\theta,b\right)=\max_{b'}U\left(c\right)+\beta\mathbb{E}\left[V_w\left(\psi^{z'},\theta',b'\right)|\theta\right]$$

$$s.t.$$

$$c+b'=\theta w+\left(1+r\right)b$$

$$b'\geq 0$$

Note that in this case, $a=b$ and $a=b'$ since $k=k'=0$.

As in the entrepreneur who exits, I approximate

$$\hat{V}_{ww}\left(\theta,b'\right)\equiv\mathbb{E}\left[V_w\left(\psi^{z'},\theta',b'\right)|\theta\right]$$

using a cubic spline along the b' direction.

A.3.2 Solving for the stationary distribution

Given the policy functions and law of motion of exogenous processes from the preceding section, the time-t distribution evolves as

$$\lambda_{t+1}(\mathbf{s}') = \int T_t(\mathbf{s}'|\mathbf{s})\lambda_t(\mathbf{s})dx$$

where, as in the main text, I use $\mathbf{s}$ to denote the full vector of state variables. T here is the Markov kernel induced by the policy functions and exogenous Markov processes. In the stationary distribution, this reduces to

$$\lambda(\mathbf{s}') = \int T(\mathbf{s}'|\mathbf{s})\lambda(\mathbf{s})dx$$

where the only difference is that the time subscripts are dropped: i.e. λ is an invariant measure. A direct method to solve for the invariant distribution is to approximate both λ and T on a grid; in particular, Young (2010) proposes a straightforward method to approximate T as a Markov transition matrix $\hat{T}$.

Unfortunately, a direct construction of $\hat{T}$ is not feasible when the state space is too large. In particular, this model features many points of non-convexity (discrete choices, non-convex adjustment costs). As a result, a large number of nodes are required in order to solve this model to a sufficient degree of accuracy. To that end, I approximate the distribution using 200 grid points in a and b, and 15 nodes in z, θ, ψ^z and ψ^θ. Coupled with the two discrete states in occupational choice, this comes up to $\approx 9 \times 10^6$ grid points. The matrix $\hat{T}$ is therefore $\approx 9 \times 10^6 \times 9 \times 10^6$ with a density of about 0.25% (which amounts to around $15^2 \times 9 \times 10^6$ non-zero grid points; or about 16.2 gb in RAM). Given that the power method used to compute λ involves iterating on $T \times \lambda$, this method is both extremely memory intensive and slow.

Instead of approximating T, I follow the method documented in a companion note (c.f. Tan (2018b)). There, I discuss how I utilize the fact that the kernel T is simply the product of two independent Markov kernels G and H, where H iterates on the

exogenous state variables ($\mathbf{z}$), and G iterates on the endogenous state variables ($\mathbf{y}$):

$$T = HG$$

$$\implies \lambda(\mathbf{y}', \mathbf{z}') = \int_{\mathbf{z}} H(\mathbf{z}'|\mathbf{z}) \int_{\mathbf{y}} G(\mathbf{y}'|\mathbf{y}, \mathbf{z}) \lambda(\mathbf{y}, \mathbf{z}) d\mathbf{y} d\mathbf{z}$$

G is then approximated using the method in Young (2010), whereas H is the transition matrix associated with all the exogenous forcing processes. G has the same dimensions as $\hat{T}$, but now only has a density of $\approx 0.001\%$. This method greatly reduces the computational burden to around 2 gb in RAM. The stationary distribution can be computed in under 100 seconds (holding fixed the same convergence tolerance).

A.4   Parameters Estimated from the Data

In this section, I explain how the depreciation rate, capital intensity, and returns to scale are estimated from the data.

A.4.1   Depreciation rate

The depreciation rate of capital is different for an entrepreneur and for the corporate sector. The rationale here is that the depreciation rate of the corporate sector is a weighted average of the entire range of capital types that exist in the economy. This leads most papers to typically set δ to numbers between 6% to 10%. In contrast, the illiquid capital asset data drawn from the KFS only encompasses a few classes of capital: Structures/buildings/land, vehicles, equipment, inventory and accounts receivables. As such, I construct δ_k as a weighted average of the stock of these capital, and arrive at 15% depreciation (with the individual depreciation rates taken from the BEA).

To construct α_e and ν, I utilize a two-step approach. Firstly, I followed the cost shares approach utilized in Asker et al (2014) to compute a consistent estimator for the labor share. When labor is a variable input and chosen contemporaneously, the first order condition for labor yields (using notation from section 3)

$$\beta^l \equiv (1 - \alpha_e)\nu = \frac{wL}{Y}$$

where β_l is the labor share. Following Asker et al (2014), I first construct a data set of labor shares at the individual level i, and estimate, for each two digit NAICS industry j, the *median* labor share:

$$\hat{\beta}^l_j = median\{\beta^l_{i,j}\}$$

Given a value for ν, I can then back out the capital share using the identity:

$$\hat{\beta}^k_j = \nu - \hat{\beta}^l_j$$

which in turn gives the capital intensity:

$$\hat{\alpha}_e = \frac{\hat{\beta}^k_j}{\nu}$$

To compute the returns to scale ν, note that for an employer firm, the capital stock and revenue is connected through the following equation:

$$\log Y = \Theta_0 + \Theta_k \log K + \Theta_z z \log z$$

$$\Theta_k \equiv \frac{\alpha_e \nu}{1 - (1 - \alpha_e)\nu}$$

Θ_k can be consistently estimated using a quantile regression (at the median). This estimate, along with the labor share, then returns us estimates for α_e and ν.

As discussed in the main body of the paper, Y corresponds to an imputed measure of value added.

A.5 Proofs of Results in Section 2.4.2

In section 2.4.2, I argued that a model with no resale frictions will, in general, not be able to replicate both the left skewness and asymmetric persistence observed in the data. In this section, I derive the analytical results for this.

A.5.1 Proof of Proposition 1 in Section 2.4.2

Here, I show that the standard model of only time-to-build as a friction will not generate the left-skewness and conditional persistence of log ARPK as observed in the KFS. Consider the standard firm problem below (in recursive notation):

$$\Pi(k, z) = \max_{k'} D + \frac{1}{1+r} \mathbb{E}\left[\Pi(k', z')|z\right]$$

$$s.t.$$

$$D = zk^\alpha + (1-\delta)k - k'$$

$$\log z' = \rho \log z + \epsilon'$$

where $'$ denotes variables for the next period. ϵ' is any iid random variable and ρ captures persistence in TFP z. Here, investment has a "time-to-build' element', since the payoff to investment today (k') is only realized tomorrow. D here is per-period dividend flow, and the firm's problem is to maximize lifetime dividend flow.

The first order condition for capital yields

$$\mathbb{E}\left[z'\left(K'\right)^{\alpha-1}|z\right] = \frac{r+\delta}{\alpha}$$

$$\implies \log K' = \frac{1}{\alpha-1}\left(\log\left(\frac{r+\delta}{\alpha}\right) - \log \mathbb{E}\left[z'|z\right]\right)$$

$$\implies \log K' = \frac{1}{\alpha-1}\left(\log\left(\frac{r+\delta}{\alpha}\right) - \rho \log z - \log \mathbb{E}\left[\exp\left(\epsilon'\right)|z\right]\right)$$

$$\Leftrightarrow \log K = \frac{1}{\alpha-1}\left(\log\left(\frac{r+\delta}{\alpha}\right) - \rho \log z_{-1} - \log \mathbb{E}\left[\exp\left(\epsilon\right)|z_{-1}\right]\right)$$

Where for the second last line, since $z' = z^\rho \exp(\epsilon')$, I used the relationship $\log \mathbb{E}[z'|z] = \log(z^\rho \mathbb{E}[\exp(\epsilon')|z]) = \rho \log z + \log \mathbb{E}[\exp(\epsilon')|z]$. The last line is simply a change of time notation, with $_{-1}$ denoting "last period" variables.

Recall that log ARPK is also simply defined as

$$\log ARPK = \log\left(\frac{Y}{K}\right)$$

$$= \log z + (\alpha - 1)\log K$$

Combining the two equations, we get

$$\log ARPK = \log z - \rho \log z_{-1} + \log\left(\frac{r+\delta}{\alpha}\right) - \log \mathbb{E}[\exp(\epsilon)|z_{-1}]$$

Defining $\vartheta \equiv \log\left(\frac{r+\delta}{\alpha}\right) - \log \mathbb{E}[\exp(\epsilon)|z_{-1}]$, and recalling that $\log z = \rho \log z_{-1} + \epsilon$, we can simplify the above relation to

$$\log ARPK = \rho \log z_{-1} + \epsilon + \log\left(\frac{r+\delta}{\alpha}\right) - \rho \log z_{-1} - \log \mathbb{E}[\exp(\epsilon)|z_{-1}]$$

$$= \log\left(\frac{r+\delta}{\alpha}\right) - \log \mathbb{E}[\exp(\epsilon)|z_{-1}] + \epsilon$$

$$= \vartheta + \epsilon \quad \square$$

As discussed in the main text, the distribution of $\log ARPK$ simply inherits the distribution of ϵ, scaled by a constant.

Skewness

Since the distribution of $\log ARPK$ is simply the distribution of ϵ, the skewness of $\log ARPK$ will simply be equal to that of ϵ.

Persistence

Since ϵ is i.i.d, $\log ARPK$ is also i.i.d; that is, $\log ARPK$ exhibits no persistence. Moreover, although the result here is derived for homogeneous parameters, long run differences in α, δ or r across industries will not affect the persistence of $\log ARPK$.

Using the same framework for section A.5.1, we can extend it with collateralized borrowing, and write the firm's problem as

$$\Pi\left(b, k, z\right) = \max_{k', b'} Y + \left(1 - \delta\right) k + \left(1 + r\right) b - k' - b' + \frac{1}{1 + r} \mathbb{E}\left[\Pi\left(b', k', z'\right) | z\right]$$

$$s.t.$$

$$Y = z k^{\alpha}$$

$$\log z' = \rho \log z + \epsilon'$$

$$b' \geq -\varphi k'$$

$$Y + \left(1 - \delta\right) k + \left(1 + r\right) b \geq k' + b'$$

where $b < 0$ indicates that the firm is *borrowing*, and $\varphi < 1$ in order for the collateral constraint to be meaningful (if $\varphi \geq 1$, then the firm can always borrow enough to hit the optimal firm size). The last constraint indicates that the firm cannot issue equity. Re-stated in terms of Lagrange multipliers and defining net wealth $\omega \equiv Y + \left(1 - \delta\right) k + \left(1 + r\right) b$, we obtain the firm's problem as

$$\Pi\left(\omega, z\right) = \max_{k', b'} \omega - k' - b' + \frac{1}{1 + r} \mathbb{E}\left[\Pi\left(\omega', z'\right) | z\right] + \mu\left(b' + \varphi k'\right) + \kappa\left(\omega - k' - b'\right)$$

$$s.t.$$

$$\omega' = z' k'^{\alpha} + \left(1 - \delta\right) k' + \left(1 + r\right) b'$$

$$\log z' = \rho \log z + \epsilon'$$

The envelope condition is

$$\frac{\partial \Pi}{\partial \omega} = 1 + \kappa$$

and the first order conditions for capital and bond are (respectively):

$$0 = -1 + \frac{1}{1+r}\mathbb{E}\left[\frac{\partial \Pi'}{\partial \omega'}\left(\alpha z' k'^{\alpha-1} + (1-\delta)\right)|z\right] + \mu\varphi - \kappa$$

$$0 = -1 + \frac{1}{1+r}\mathbb{E}\left[\frac{\partial \Pi'}{\partial \omega'}\left(1+r\right)|z\right] + \mu - \kappa$$

First, combining the first order conditions for bonds with the envelope conditions, we get

$$0 = -1 + \frac{1}{1+r}\mathbb{E}\left[(1+\kappa')\left(1+r\right)|z\right] + \mu - \kappa$$

$$\implies \mathbb{E}\left[\kappa'|z\right] = \kappa - \mu$$

Next, combining the first order conditions for capital with the envelope conditions, we get

$$\frac{1}{1+r}\mathbb{E}\left[(1+\kappa')\left(\alpha z' k'^{\alpha-1} + (1-\delta)\right)|z\right] = 1 + \kappa - \mu\varphi$$

Next, we can express k' in terms of the Lagrange multipliers and z. For convenience, I will move all equations one "time step" back (so $'$ variables become "un-primed", and "un-primed" variables are now $_{-1}$).

I now re-write the first order conditions for capital as

$$k^{\alpha-1}\left(\mathbb{E}\left[z|z_{-1}\right] + \mathbb{E}\left[\kappa z|z_{-1}\right]\right) = \frac{r+\delta}{\alpha} + (\kappa_{-1} - \mu_{-1}\varphi)\left(\frac{1+r}{\alpha}\right) - \mathbb{E}\left[\kappa|z_{-1}\right]\left(\frac{1-\delta}{\alpha}\right)$$

Next, using the first order conditions for bonds, we note that

$$\left(\kappa_{-1} - \mu_{-1}\varphi\right)\frac{1+r}{\alpha} - \mathbb{E}\left[\kappa|z_{-1}\right]\left(\frac{1-\delta}{\alpha}\right) = \left(\kappa_{-1} - \mu_{-1}\varphi\right)\frac{1+r}{\alpha} - \left(\kappa_{-1} - \mu_{-1}\right)\left(\frac{1-\delta}{\alpha}\right)$$

$$= \kappa_{-1}\frac{r+\delta}{\alpha} - \mu_{-1}\frac{\varphi\left(1+r\right)-1+\delta}{\alpha}$$

$$= \frac{r+\delta}{\alpha}\left(\kappa_{-1} - \mu_{-1}\frac{\varphi\left(1+r\right)-1+\delta}{r+\delta}\right)$$

$$= \frac{r+\delta}{\alpha}\left(\kappa_{-1} - \mu_{-1}\tilde{\varphi}\right)$$

where $\frac{\varphi(1+r)-1+\delta}{r+\delta} \equiv \tilde{\varphi} \in \left[\frac{-1+\delta}{r+\delta}, 1\right]$, as $0 \leq \varphi \leq 1$. Putting this relationship back into the first order conditions for capital, we get

$$k^{\alpha-1} = \left[\frac{r+\delta}{\alpha} + \frac{r+\delta}{\alpha}\left(\kappa_{-1} - \mu_{-1}\tilde{\varphi}\right)\right]\left[\mathbb{E}\left[z|z_{-1}\right] + \mathbb{E}\left[\kappa z|z_{-1}\right]\right]^{-1}$$

Where we have obtained an implicit solution for k in terms of z and the Lagrange multipliers. Substituting this back into the definition of $\log ARPK$,

$$\implies \log ARPK = \log z + (\alpha - 1) \log k$$

$$= \log z + \log \left(\frac{r+\delta}{\alpha} + \frac{r+\delta}{\alpha} \left(\kappa_{-1} - \mu_{-1} \tilde{\varphi} \right) \right)$$

$$- \log \left(\mathbb{E} \left[z | z_{-1} \right] + \mathbb{E} \left[\kappa z | z_{-1} \right] \right)$$

$$= \log z + \log \left[\left(\frac{r+\delta}{\alpha} \right) \left(1 + \kappa_{-1} - \mu_{-1} \tilde{\varphi} \right) \right]$$

$$- \log \left[\left(\mathbb{E} \left[z | z_{-1} \right] \right) \left(1 + \frac{\mathbb{E} \left[\kappa z | z_{-1} \right]}{\mathbb{E} \left[z | z_{-1} \right]} \right) \right]$$

$$= \log z + \log \left(\frac{r+\delta}{\alpha} \right) + \log \left(1 + \kappa_{-1} - \mu_{-1} \tilde{\varphi} \right) - \log \mathbb{E} \left[z | z_{-1} \right]$$

$$- \log \left(1 + \frac{\mathbb{E} \left[\kappa z | z_{-1} \right]}{\mathbb{E} \left[z | z_{-1} \right]} \right)$$

$$= \log z - \log \mathbb{E} \left[z | z_{-1} \right] + \log \left(\frac{r+\delta}{\alpha} \right) + \log \left(1 + \kappa_{-1} - \mu_{-1} \tilde{\varphi} \right) - ...$$

$$\log \left(1 + \frac{\mathbb{E} \left[\kappa | z_{-1} \right] \mathbb{E} \left[z | z_{-1} \right] + cov_{z_{-1}} \left(\kappa, z \right)}{\mathbb{E} \left[z | z_{-1} \right]} \right)$$

$$= \log z - \log \mathbb{E} \left[z | z_{-1} \right] + \log \left(\frac{r+\delta}{\alpha} \right) + \log \left(1 + \kappa_{-1} - \mu_{-1} \tilde{\varphi} \right) - ...$$

$$\log \left(1 + \mathbb{E} \left[\kappa | z_{-1} \right] + \frac{cov_{z_{-1}} \left(\kappa, z \right)}{\mathbb{E} \left[z | z_{-1} \right]} \right)$$

$$= \log z - \log \mathbb{E} \left[z | z_{-1} \right] + \log \frac{r+\delta}{\alpha} + \log \left(1 + \kappa_{-1} - \mu_{-1} \tilde{\varphi} \right) -$$

$$\log \left(1 + \kappa_{-1} - \mu_{-1} + \frac{cov_{z_{-1}} \left(\kappa, z \right)}{\mathbb{E} \left[z | z_{-1} \right]} \right)$$

Where I went from the second last line to the last line using the first order conditions for bonds ($\mathbb{E} \left[\kappa | z_{-1} \right] = \kappa_{-1} - \mu_{-1}$)

Finally, noting that

$$\log z - \log \mathbb{E}\left[z|z_{-1}\right] = \rho \log z_{-1} + \epsilon - \log \mathbb{E}\left[z_{-1}^{\rho} \exp\left(\epsilon\right)|z_{-1}\right]$$

$$= \rho \log z_{-1} + \epsilon - \rho \log z_{-1} - \log \mathbb{E}\left[\exp\left(\epsilon\right)|z_{-1}\right]$$

$$= \epsilon - \log \mathbb{E}\left[\exp\left(\epsilon\right)|z_{-1}\right]$$

and that under the model of only time-to-build and no collateral constraints,

$$\log ARPK^{TTB} = \log \frac{r+\delta}{\alpha} - \log \mathbb{E}\left[\exp\left(\epsilon\right)|z_{-1}\right] + \epsilon$$

We obtain for $\log ARPK$:

$$\log ARPK = \log ARPK^{TTB} + \log\left(1 + \kappa_{-1} - \mu_{-1}\tilde{\varphi}\right) - \log\left(1 + \kappa_{-1} - \mu_{-1} + \frac{cov_{z_{-1}}\left(\kappa, z\right)}{\mathbb{E}\left[z|z_{-1}\right]}\right)$$

Now, note that since $\frac{-1+\delta}{r+\delta} \leq \tilde{\varphi} \leq 1$, we see that $\kappa_{-1} - \mu_{-1}\tilde{\varphi} \geq \kappa_{-1} - \mu_{-1}$. Moreover,

since $cov_{z_{-1}}\left(\kappa, z\right) < 0$ and $\mathbb{E}\left[z|z_{-1}\right] > 0$, $\frac{cov_{z_{-1}}(\kappa,z)}{\mathbb{E}[z|z_{-1}]} < 0$. As such, we see that

$$\log\left(1 + \kappa_{-1} - \mu_{-1}\tilde{\varphi}\right) - \log\left(1 + \kappa_{-1} - \mu_{-1} + \frac{cov_{z_{-1}}\left(\kappa, z\right)}{\mathbb{E}\left[z|z_{-1}\right]}\right) \equiv \xi_{-1} > 0$$

where I write ξ_{-1} to note that this is a pre-determined variable.

Rewriting the equation for $\log ARPK$ again, and defining $\log ARPK^{TTB} \equiv A$

for notational ease, we get the following simple linear relation between the "time to

build" ARPK and the actual ARPK:

$$\log ARPK = A + \xi_{-1}$$

□

Skewness

To derive the unconditional skewness of $\log ARPK$, note that (and using the notation

$\mathcal{S}\left(X,Y,Y\right)$ for the co-skewness between X and Y):

$$skewness\left(\log ARPK\right) = skewness\left(A + \xi_{-1}\right)$$

$$= \frac{1}{\sigma_A^3 + \sigma_{\xi_{-1}}^3}[\sigma_A^3 \mathcal{S}\left(A\right) + 3\sigma_A^2 \sigma_\xi \mathcal{S}\left(A, A, \xi_{-1}\right)$$

$$+ 3\sigma_A \sigma_\xi^2 \mathcal{S}\left(A, \xi_{-1}, \xi_{-1}\right) + \sigma_\xi^3 \mathcal{S}\left(\xi_{-1}\right)]$$

Note that while A depends on ϵ ("today's innovation"), ξ_{-1} depends on ϵ_{-1} ("yesterday's innovation"). As such, $A \perp \xi_{-1}$, which implies that $3\sigma_A^2 \sigma_\xi \mathcal{S}\left(A, A, \xi\right) + 3\sigma_A \sigma_\xi^2 \mathcal{S}\left(A, \xi, \xi\right) = 0$: i.e. they have 0 co-skewness, since they are independent. As such, the skewness of $\log ARPK$ reduces to

$$skewness\left(\log ARPK\right) = \frac{1}{\sigma_A^3 + \sigma_{\xi_{-1}}^3}\left[\sigma_A^3 \mathcal{S}\left(A\right) + \sigma_\xi^3 \mathcal{S}\left(\xi_{-1}\right)\right]$$

Finally, recall that $\mathcal{S}\left(A\right) = \mathcal{S}\left(\epsilon\right)$ (and likewise for the standard deviation); therefore, we obtain the final relation:

$$skewness\left(\log ARPK\right) = \frac{1}{\sigma_\epsilon^3 + \sigma_{\xi_{-1}}^3}\left[\sigma_\epsilon^3 \mathcal{S}\left(\epsilon\right) + \sigma_\xi^3 \mathcal{S}\left(\xi_{-1}\right)\right]$$

Note that if $\mathcal{S}\left(\xi_{-1}\right) > -\frac{\sigma_\epsilon^3}{\sigma_{\xi_{-1}}^3}\mathcal{S}\left(\epsilon\right)$, then the distribution will always be right-skewed. This is trivially true for all $\mathcal{S}\left(\epsilon\right) > 0$, since $\mathcal{S}\left(\xi_{-1}\right) > 0$ (ξ_{-1} has a left-truncated distribution since it is strictly bigger than 0). Moreover, if $\mathcal{S}\left(\xi_{-1}\right) < -\frac{\sigma_\epsilon^3}{\sigma_{\xi_{-1}}^3}\mathcal{S}\left(\epsilon\right)$, then the skewness of log ARPK is trivially higher than the skewness of the innovations, although the distribution of log ARPK will be left-skewed. $\square$

Persistence

Here, I show how the model with collateral constraints produces asymmetric persistence in ARPK. To proceed, I first show how there is unconditional autocorrelation, and then show how this autocorrelation is asymmetric.

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Persistence We can compute the autocovariance of $\log ARPK$ to understand its persistence. Specifically,

$$cov(\log ARPK, \log ARPK_{-1}) = cov\left(A + \xi_{-1}, A_{-1} + \xi_{-2}\right)$$
$$= cov\left(A, A_{-1}\right) + cov\left(A, \xi_{-2}\right) + cov\left(\xi_{-1}, A_{-1}\right) + cov\left(\xi_{-1}, \xi_{-2}\right)$$
$$= cov\left(\xi_{-1}, A_{-1}\right) + cov\left(\xi_{-1}, \xi_{-2}\right)$$
$$> 0$$

So we see that $\log ARPK$ exhibits persistence (positive autocorrelation).

Asymmetric Persistence We can now show that the model with collateral constraints delivers asymmetric persistence, where the right tail exhibits greater persistence than the left tail.

To do so, recall that $\xi \in [0, \infty)$, while $A \in \mathbb{R}$. As such, we can always define some arbitrary quantile of the distribution q^- such that the following holds:

1. $\forall \log ARPK < q^-$, $Pr(\xi_{-1} = 0) = 1$ such that $\log ARPK = A$

2. $\forall \log ARPK > 1 - q^-$, $Pr(\xi_{-1} = 0) < 1$

Moreover, by the definition of a quantile, $Pr(\log ARPK < q^-) = Pr(\log ARPK > 1 - q^-)$.

Next, consider 2 types of firms: i and j, where $\log ARPK_i < q^-$ and $\log ARPK_j > 1 - q^-$.

Note that for firm i,

$$cor(\log ARPK_t, \log ARPK_{t+1}) = cor(A_t, A_{t+1}) + cor(A_t, A_{t+1} + \xi_t)$$
$$= cor(A_t, \xi_t)$$

where this correlation term reflects the probability that a firm becomes financially constrained given it's productivity shock. This reflects that fact that even for large

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firms, there is a non-zero probability that they will become financially constrained.

For firm j,

$$cor(\log ARPK_t, \log ARPK_{t+1}) = cor(A_t + \xi_{t-1}, A_{t+1} + \xi_t)$$

$$= cor(A_t, A_{t+1}) + cor(A_t, \xi_t) + cor(\xi_{t-1}, A_{t+1}) + cor(\xi_{t-1}, \xi_t)$$

$$= cor(A_t, \xi_t) + cor(\xi_{t-1}, \xi_t)$$

Here, $cor(\xi_{t-1}, A_{t+1}) = 0$ since the probability of being financially constrained "yesterday" does not depend on future productivity shocks. However, $cor(\xi_{t-1}, \xi_t) > 0$ as noted earlier. Therefore, we see that firms of type i exhibit lower autocorrelation than firms of type j. In other words, the autocorrelation in the right tail of the distribution is greater than that in the left tail. $\square$

A.5.3 Alternative model where choice of capital at time t is measurable with respect to time $t+1$ innovations

In the main text as well as the proofs discussed here, the framework assumes that the choice of capital at time t is not measurable with respect to time $t+1$ innovations. These proofs were derived keeping in mind the same timing restrictions assumed by my model.

In practice, the majority of entrepreneurship model in macroeconomics has assumed that the choice of capital at time t is measurable with respect to time $t+1$ innovations, as for instance, in Cagetti and De Nardi (2006), Buera and Shin (2013), and Midrigan and Xu (2014). The goal of this subsection is to demonstrate that, in that framework, the financial frictions model will still derive only a right-skewness and higher right-tail persistence.

Proposition 3 Consider an investment model with collateral constraints, where the choice of capital at time t is measurable with respect to time $t+1$ innovations. Specifically, consider a linear reduced form collateral constraint of the following form

$$k \leq (1 + \varphi) a$$

where k is working capital and a are net assets of the firm, and φ capture the extent of financial frictions. Then the distribution of (log) ARPK has a left-truncated support, given by (denoted with a superscript FF for financial frictions)

$$\log ARPK^{FF} \in [\log(r+\delta) - \log\alpha, \infty)$$

As such, the distribution of log ARPK is right-skewed regardless of the distribution of the underlying innovations. Moreover, log ARPK is also persistent; the persistence is also higher in the right tail of the distribution than the left tail.

Proof Consider the following model:

$$\Pi(a, z) = \max_k D + \frac{1}{1+r}\mathbb{E}\left[\Pi(a', z')\,|z\right]$$

$$s.t.$$

$$D = Y - (r+\delta)k + (1+r)a$$

$$Y = zk^\alpha$$

$$k \le (1+\varphi)a$$

$$\log z' = \rho\log z + \epsilon'$$

In this framework, there is a financial constraint as firms cannot rent as much capital as they desire. Instead, they either fully finance using internal assets (a), or they utilize the debt markets to finance their investments up to φ of their assets. $\varphi \in [0, \infty)$ captures how much the individual can leverage on his own wealth. If $\varphi = 0$, no borrowing is possible; if $\varphi \to \infty$, there is no borrowing constraint.

Denote by μ the Lagrange multiplier on the borrowing constraint. The first order condition on capital yields

$$\alpha\frac{Y}{K} = r + \delta + \mu$$

$$\implies \log ARPK = \log(r+\delta+\mu) - \log(\alpha)$$

Note that for firms that are unconstrained, $\mu = 0$. Therefore, we see that the support for $\log ARPK$ is given by $[\log \frac{r+\delta}{\alpha}, \infty)$ $\square$.

Skewness

Since the distribution is left-truncated, the distribution is naturally right-skewed. This result is independent of the distribution of the underlying innovations.

Persistence

The distribution will exhibit greater persistence in the right tail than the left tail. To see this, consider, at time t, two firms i and j. For firm i, assume that the firm is unconstrained, so $\mu = 0$; consequently, the ARPK of firm i evaluates to

$$\log ARPK_{i,t} = \log(r + \delta) - \log(\alpha)$$

For firm j, assume that the firm is constrained, so $\mu > 0$; consequently, the ARPK of firm j evaluates to

$$\log ARPK_{j,t} = \log(r + \delta + \mu_{j,t}) - \log(\alpha)$$

Note that $\log ARPK_j > \log ARPK_i$; that is, firm j is in the right tail of the distribution, whereas firm i is in the left tail.

The autocorrelation of $logARPK$ for firm i is trivially 0:

$$corr(\log ARPK_{i,t}, \log ARPK_{i,t+1}) = corr\left(\log(r + \delta) - \log(\alpha), \log(r + \delta + \mu_{i,t+1}) - \log(\alpha)\right)$$

$$= 0$$

In contrast, for firm j, the autocorrelation is strictly positive:

$$corr(\log ARPK_{j,t}, \log ARPK_{j,t+1}) = corr(\log(r + \delta + \mu_{j,t}) - \log(\alpha),$$

$$\log(r + \delta + \mu_{j,t+1}) - \log(\alpha))$$

$$> 0$$

This holds true trivially because μ is always strictly bigger than 0.

Consequently, we see that there is greater persistence in the right tail than the left tail of the distribution[10].

A.6   Returns to Scale and Wealth Dispersion

Here, I explain why the returns to scale is also an important factor in understanding the contribution of entrepreneurial investment and risk taking to the wealth distribution.

To understand this, we have to first understand what constitutes capital income risk in a model of entrepreneurship. Recall that gross capital income in a model of entrepreneurship is simply production income. For a simplified model without labor, gross capital income is then simply $y = zk^\nu$. In a static one asset model without any frictions, we know that optimal choice of k is simply $(\frac{z}{r+\delta})^{\frac{1}{1-\nu}}$, so gross capital income is just $z^{1+\frac{\nu}{1-\nu}}(\frac{1}{r+\delta})^{\frac{\nu}{1-\nu}}$. The dispersion and persistence in (log) capital income is then:

$$var(\log(y_t)) = var(\log(z^{1+\frac{\nu}{1-\nu}}))$$

$$= \left(1 + \frac{\nu}{1-\nu}\right)^2 var(\log(z))$$

$$cov(\log(y_t), \log(y_{t-1})) = \left(1 + \frac{\nu}{1-\nu}\right)^2 cov(\log(z_t), \log(z_{t-1}))$$

where we see right away that (i) the **volatility** of capital income depends on *both* the volatility of TFP as well as the returns to scale, and (ii) the **persistence** of capital income depends only on the persistence of TFP, but it's covariance depends on *both* the persistence of TFP and returns to scale. Crucially, ν inflates the volatility and covariance of capital income at an exponential rate. For instance, consider a TFP process with a small unconditional variance of 0.0335 and a returns to scale of

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[10] One should note that this proposition is true to the extent that we measure the persistence in the tails using the conditional autocorrelation. If one instead uses the transition matrix approach, due to the mass point at $r + \delta$ in this framework, a similar argument cannot be easily made; in fact, the transition matrix might even reflect higher left tail persistence due to the clustering of unconstrained firms at $r + \delta$. However, this is unlikely to be a concern, given the fact that we do not see such a clustering in the data.

0.88^{11}: The resulting volatility of capital income is inflated to almost 1.34, a nearly 40 times increase over the underlying productivity process. On one level, this is in fact an important reason why models of entrepreneurship have been so successful in matching the wealth distribution: For even small values of underlying productivity risk, a positive returns to scale can greatly inflate the dispersion of capital income, thus also increasing the dispersion in wealth.

As an example of the importance ν plays in determining the dispersion of wealth, I report in figure A.4 below a comparative statics exercise that reports the Gini coefficient for every $\nu \in (0.68, 0.88)$. The vertical dashed line is a reference line corresponding to the basline ν in my model. Notice that the returns to scale for the baseline calibration is too small, relative to the standard calibration such as that in Cagetti and De Nardi (2006), to fully match the high empirical wealth inequality.
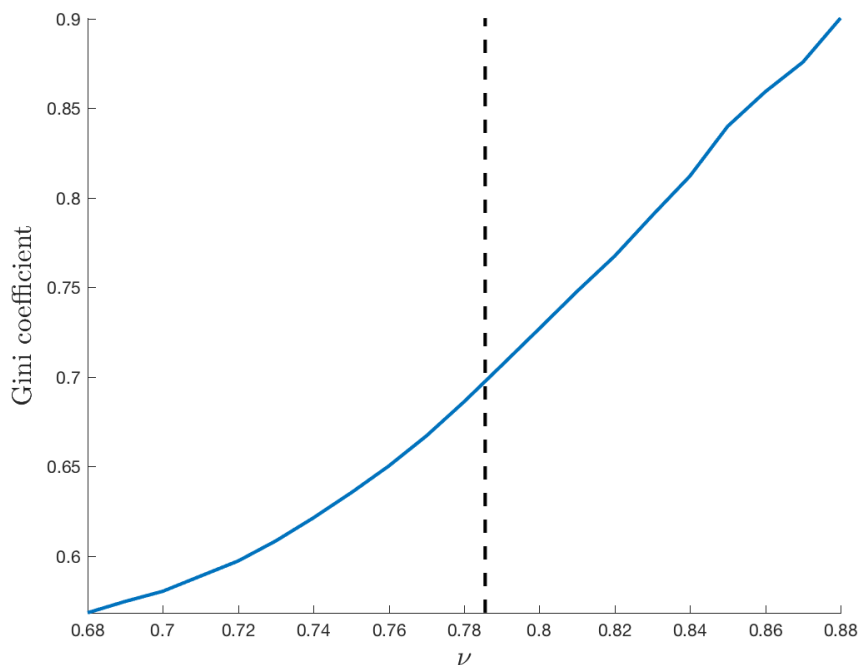

FIGURE A.4: Effect of ν on wealth inequality.

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[11] These numbers in fact correspond to the calibration chosen by Cagetti and De Nardi (2006)

A.7 Model Definitions

A.7.1 Definition of excess private equity risk premium

Recall that the entrepreneur earns income by operating the following production technology

$$y_{i,t} = z_{i,t} \left(k_{i,t}^{\alpha_k} \left(\bar{l} + l \right)^{1-\alpha_k} \right)^{\nu} \times 1^{1-\nu}$$

where we can interpret the last term in the production function as returns to the entrepreneur's fixed factor, normalized to 1. Given that assumption, the production function is constant returns to scale in labor, capital and the fixed factor, and revenue can thus be decomposed as

$$y_{i,t} = \underbrace{F_{i,t}^k k_{i,t}}_{\text{Payments to capital}} + \underbrace{F_{i,t}^l \bar{l}}_{\text{to E's labor}} + \underbrace{F_{i,t}^{\nu}}_{\text{to E's fixed factor}} + \underbrace{F_{i,t}^l \times \max\left\{0, l_{i,t} - \bar{l}\right\}}_{\text{to outside labor}} \quad \text{(A.1)}$$

which in turn implies that the factor payments to capital at the individual level is simply defined as

$$F_{i,t}^k = \alpha_k \nu \frac{y_{i,t}}{k_{i,t}}$$

Since capital depreciates at rate δ_k, the user cost of capital is simply $r + \delta_k$. The **individual** excess returns to capital is then simply

$$\tilde{r}_{i,t}^k = F_{i,t}^k - r - \delta_k$$

To define the **aggregate** excess returns to capital, we can use equation A.1 to decompose the aggregate output in the entrepreneurial sector (Y^e) in terms of their aggregate factor payments:

$$Y^e = \int y_{i,t} di$$

$$= \underbrace{\int F_{i,t}^k k_{i,t} di}_{\text{Aggregate payments to capital}} + \int F_{i,t}^l \bar{l} di + \int F_{i,t}^{\nu} di + \int F_{i,t}^l \times \max\left\{0, l - \bar{l}\right\} di$$

where the aggregate payments to capital reduces to

$$\int F_{i,t}^k k_{i,t} di = \alpha_k \nu \int y_{i,t} di$$

$$= \alpha_k \nu Y^e$$

Since aggregate investment to private equity is simply $K^e = \int k_{i,t} di$, the aggregate excess returns to capital is trivially

$$\tilde{R}^k = \alpha_k \nu \frac{Y^e}{K^e} - \delta_k - r$$

A.7.2 Definitions of Productivity and Welfare

Welfare

Welfare here is measured as the consumption equivalent variation between a reference economy (given by a superscript "R") and the current economy of interest (given by a superscript "N"). In particular, for any household i, the consumption equivalent variation μ_i solves the following problem:

$$V((1 + \mu_i)c_i^R) = V(c^N) \tag{A.2}$$

To compute μ_i for each household, I then simply solve the preceding equation for each point in the state space $\mathbb{S}$. Given this definition, $\mu_i > 0$ implies that the household prefers the new economy over the reference economy.

Having derive the entire distribution of μ_i, I then compute aggregate (average) welfare as

$$\widehat{\mu} = \int \mu_i d\Lambda \tag{A.3}$$

Moreover, I am also able to compute welfare changes for subsets of the economy $\Lambda_s \subseteq \Lambda$. For instance, if I wish to compute the average welfare change for only workers, I can compute

$$\widehat{\mu^w} = \frac{\int \mu_i \times \mathbb{I}_w d\Lambda}{\int \mathbb{I}_w d\Lambda} \tag{A.4}$$

where $\mathbb{I}_w$ is an indicator function that equates to 1 if that household is a worker, and 0 otherwise. The denominator is necessary in order to re-normalize the distribution (i.e. compute the conditional average), in order to avoid conflating the changes in the measure of workers (which is endogenous) and actual changes in the agent's welfare (or distribution of CEV).

Aggregate TFP

"Aggregate TFP" here refers specifically to the aggregate TFP in the entrepreneur sector. Since the corporate sector is composed of representative firms, there cannot be any TFP losses with respect to that sector, and hence I choose not to discuss it here[12].

Here, my concept of TFP stems from the perspective of a statistician who only observes the aggregate capital stock, labor input, and output of the entrepreneur sector. In this case, we can derive TFP $\overline{Z}$ as

$$\overline{Z} \equiv \frac{Y^e}{K_e^{\alpha_e \nu} L_e^{(1-\alpha_e)\nu}} \tag{A.5}$$

Average productivity of entrepreneur

As the model exhibits endogenous entry and exit, the steady-state distribution of *observed* idiosyncratic TFP is different from the underlying distribution of shocks. The selection effects can either increase (positive selection) or decrease (negative selection) the average productivity of entrepreneurs, and in turn, can have powerful effects on aggregate TFP. For this paper, I define the average productivity of an entrepreneur as

$$\overline{Z^e} = \frac{\int z_i \times \mathbb{I}_e d\Lambda}{\int \mathbb{I}_e d\Lambda} \tag{A.6}$$

---

[12] This does not mean that the corporate sector is not important in this model. In fact, the amount of (effective) capital allocated to the corporate sector reflects the change in the precautionary savings behavior of the households, which is an important dimension of this model relative to models of complete markets

where $\mathbb{I}_e$ is an indicator function that equates to 1 if that household is an entrepreneur, and 0 otherwise.

A.7.3 Computing Welfare Changes Along Transition Path

This subsection describes how welfare is computed along the transition path in section 2.5.

To do so, I first solve for the steady-state distribution of households and value functions under the benchmark calibration. For notational convenience, denote the initial steady-state distribution as Λ_0, and the value functions as $V_e\left(a, k, z, \psi_\theta; 0\right)$ and $V_w\left(a, \theta, \psi_z; 0\right)$, where the "0" denotes that the distribution and value functions are at "time-0". Finally, denote the initial adjustment cost parameters as $\bar{\lambda}$, $\bar{\zeta}$, and $\bar{f}_s$, and the collateral constraint as $\bar{\varphi}$.

Next, at time $t = 1$, I assume that households receive information that from $t \geq 2$ onwards, all the frictions that render capital illiquid is suddenly and costlessly removed. In other words, I set $\lambda = 0$, $\zeta = 0$ and $f_s = 0$. In addition, I assume that this change is entirely unforeseen by the households, prior to the release of this information. The economy is then allowed to transition to the new steady-state where capital is fully liquid; moreover, households now have perfect foresight over the entire transition path of the economy.

Stated in recursive terms, this amounts to the following dynamic program. For entrepreneurs,

$$V_e\left(\psi^\theta, z, k, b; t\right) = \max_{h',k',b',l} U\left(c\right)$$

$$+ \left(1 - h'\right) \times \beta \int_{\psi'_\theta} \int_{z'} V_e\left(\psi^{\theta'}, z', k', b'; t+1\right) dP_{z'|z} dF_{\psi^\theta}$$

$$+ h' \times \beta \int_{\psi'_z} \int_{\theta'} V_w\left(\psi^{z'}, \theta', b'; t+1\right) dP_{\theta'|\psi^\theta} dF_{\psi^z}$$

s.t.

$$\hat{\pi} \equiv zf\left(k, \iota\bar{l} + l\right) - wl + \left(1 + r \times 1_{\{b \geq 0\}} + r_d \times 1_{\{b < 0\}}\right) b$$

$$- \mathcal{C}\left(k', k, h', h; f_{s_t}, \lambda_t, \zeta_t, \varphi_{t-1}\right)$$

$$c = \max\left\{\hat{\pi}, 0\right\} - k' - b' \geq 0$$

$$k' \begin{cases} > 0 & \text{if } h' = 0 \\ = 0 & \text{if } h' = 1 \end{cases}$$

$$h' = 1 \quad (\text{ if } \hat{\pi} < 0)$$

$$b' \geq -\varphi_t\left(1 - \lambda_{t+1}\right)\left(1 - \delta\right) k' - \underline{b}$$

where

$$\lambda_t = \begin{cases} \bar{\lambda} & \text{if } t \leq 1 \\ 0 & \text{if } t \geq 2 \end{cases}$$

$$\zeta_t = \begin{cases} \bar{\zeta} & \text{if } t \leq 1 \\ 0 & \text{if } t \geq 2 \end{cases}$$

$$f_{s_t} = \begin{cases} \bar{f}_s & \text{if } t \leq 1 \\ 0 & \text{if } t \geq 2 \end{cases}$$

$$\varphi_t = \bar{\varphi} \frac{\left(1 - \bar{\lambda}\right)}{\left(1 - \lambda_{t+1}\right)}$$

Note the explicit definition of the adjustment function $\mathcal{C}$ on the time-varying parameters, as well as the adjustment of φ to fix the level of collateral constraint. For

workers,

$$V_w\left(\psi^z,\theta,b;t\right) = \max_{h',k',b'} U\left(c\right)$$

$$+\left(1-h'\right)\times\beta\int_{\psi'_\theta}\int_{z'}V_e\left(\psi^{\theta'},z',k',b';t+1\right)dP_{z'|\psi^z}dF_{\psi^\theta}$$

$$+h'\times\beta\int_{\psi'_z}\int_{\theta'\in\Theta}V_w\left(\psi^{z'},\theta',b';t+1\right)dP_{\theta'|\theta}dF_{\psi^z}$$

$$s.t.$$

$$c+k'+b' = \theta w\bar{l} + \left(1+r\times 1_{\{b\geq 0\}} + r_d\times 1_{\{b<0\}}\right)b$$

$$k'\begin{cases} >0 & \text{if } h'=0 \\ =0 & \text{if } h'=1 \end{cases}$$

$$b' \geq -\varphi_t\left(1-\lambda_{t+1}\right)\left(1-\delta\right)k' - \underline{b}$$

The associated equilibrium conditions follows: A sequential competitive equilibrium of the model consist of the path of interest and wage rates $\{r_t, w_t\}_{t\geq 0}$, value functions of households and firms $\{V_e(\cdot;t), V_w(\cdot;t), \Pi(\cdot;t)\}_{t\geq 0}$, allocations $\{k_t, b_t, l_t\}_{t\geq 0}$ and distribution of agents Λ_t over the state space $\mathbb{S}$ such that,

1. Taking $\{r_t, w_t\}_{t\geq 0}$ as given, the households' and firms' choices are optimal.

2. Markets clear, such that for every period t

   (a) Bonds: $\int b_t d\Lambda_t = K_t^c$

   (b) Labor: $\int \theta h_t d\Lambda_t = \int l d\Lambda + L_t^c$

3. The distribution Λ_t evolves as

$$\Lambda_t = \Gamma_t\left(\Lambda_{t+1}\right)$$

Where Γ_t is the one-period transition operator on the distribution at time t induced by the policy functions of the households and firms.

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For the partial equilibrium example, I fix interest rates and wages to the time 0 prices.

To compute welfare, I then compare the time 0 value functions with the time 1 value functions; that is to say, I compare the welfare of a household that lives in the benchmark economy forever with that of an equivalent household who lives in the counter-factual economy, taking into account the entire transitional dynamics associated with moving from the initial economy to the new economy. The individual consumption equivalent variation, welfare changes for sub-populations, as well as the aggregate welfare changes are computed as discussed in the main text, using the time 0 distribution of households for aggregation.

A.8 Further Discussion of Model Mechanisms

A.8.1 Exit Margins: A two horse race

In the main text, I noted that the effect of illiquidity on the exit margin at the individual level can be ambiguous, but not so at the aggregate level. I discuss the reasons in this subsection.

At the individual level, the overall effect can be ambiguous, since illiquidity decreases both the value of exit and value of staying in entrepreneurship. When the former effect dominates, we see a result like that in figure 2.9b, where the exit threshold shifts left. In contrast, when the latter effect dominates, we will see a result like in figure A.5a, where the threshold shifts *right*. This means that exit propensity actually increases when assets become more illiquid.

However, the effect on the aggregate rate of churning is not ambiguous: The rate of churning will always decrease when capital is more illiquid. This is because in steady-state, the exit rate must always equate the entry rate. Since greater illiquidity always leads to lower entry propensity (by lowering the value of entrepreneurship), in equilibrium, the exit rate must also be lower, leading to a decrease in the overall rate of reallocation at the extensive margin (i.e. the equilibrium rate at which households

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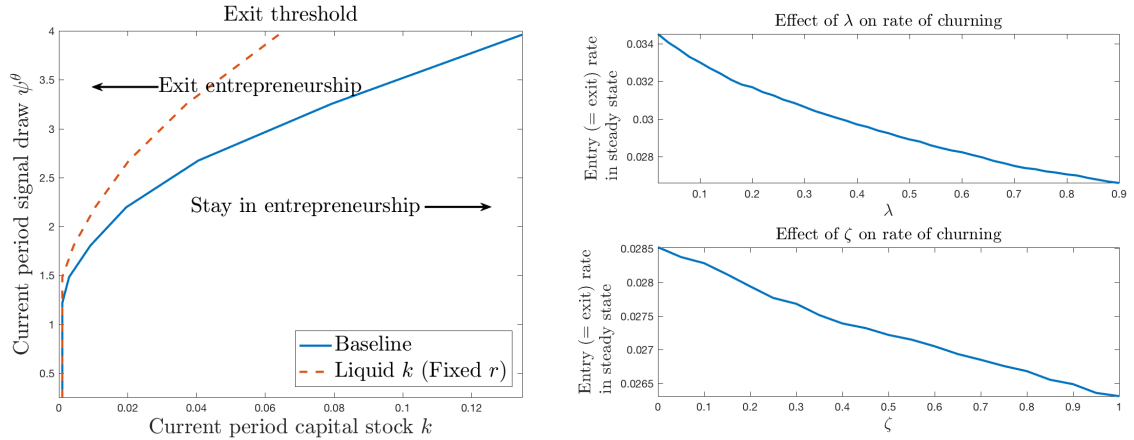
(a) Exit policy in (ψ^θ, k) space, holding z fixed. **(b)** Effect of λ and ζ on steady state entry and exit rates

FIGURE A.5: Effect of illiquidity on the exit threshold (left) and churning along the extensive margin (right).

switch occupations). Indeed, this is reflected in figure A.5b, which plots the number of households who are entering entrepreneurship in steady-state, as a function of ζ (bottom figure) and λ (top figure). This means that in steady-state, illiquidity will always lead to fewer exits, thus generating the aggregate effect we see in the main text.

A.8.2 Interaction of Exit and Entry Rates Along Transition Path

In the main text, I noted that while we observe fewer entrepreneurs under the baseline calibration, this result is not a forgone conclusion. Here, I explain further why this is so, and why we do find that the population of entrepreneurs is lower under the baseline calibration.
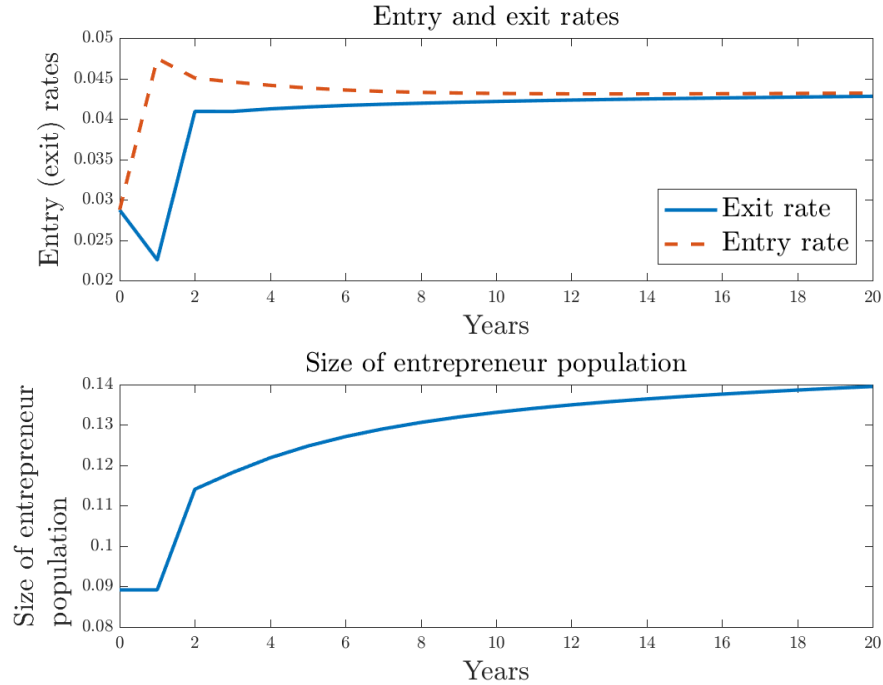
FIGURE A.6: Response of the entrepreneur population size, and entry and exit rates, to a sudden removal of all illiquidity frictions.

Unlike their impact on the entry and exit rates, the impact of capital illiquidity on the fraction of households that are entrepreneurs in steady state can be ambiguous for two reasons. Firstly, the rate of job switching simply gives us an idea on how active reallocation is being pursued, but does not give us any information on the number of entrepreneurs in steady state. Secondly, since the exit and entry rate both simultaneously decrease in steady-state, the steady state number of entrepreneurs depends on which of the two forces are stronger along the transition path towards the steady state. If the exit rate falls at a slower rate than the entry rate, the number of entrepreneurs will decline along the transition path towards the new steady state; on the other hand, if the exit rate falls at a faster rate than the entry rate, the number of entrepreneurs will rise as the economy transitions to the new steady state.

This point can be best illustrated by studying the transition path of the baseline economy to the counter-factual economy where capital is fully liquid. Here, I consider a perfect foresight equilibrium where the transaction costs are suddenly removed, and

in figure A.6, trace out the equilibrium responses of the aggregate entry and exit rates, as well as the size of the entrepreneur population. We see that along the transition path, the entry rate rises faster than the exit rate. This is a case where, at the individual level, the value of entrepreneurship rises faster than the value of exit[13] As a consequence, the steady state population of entrepreneurs is larger under the counter-factual calibration than the baseline calibration.

A.9   Other Figures and Tables

A.9.1   Transition matrices of model: Value added

Table A.4: Transition probabilities of value added under the benchmark calibration.

		Quintile tomorrow				
		1	2	3	4	5
	1	0.72	0.23	0.04	0.01	0.000
Quintile today	2	0.23	0.44	0.26	0.07	0.003
	3	0.04	0.27	0.40	0.25	0.04
	4	0.01	0.06	0.26	0.46	0.21
	5	0.00	0.00	0.03	0.22	0.75

---

[13] Since illiquidity frictions are *removed*, the value of entrepreneurship and exit both rises.

A.9.2 Transition matrices of wealth returns

Table A.5: Transition probabilities of the excess returns to wealth. This replicates table 2.16 in the main text, reporting the full transition probabilities.

Model	Average wealth (relative to mean)	Probability of transitioning to:				
		Quintile 1	Quintile 2	Quintile 3	Quintile 4	Quintile 5
		When in quintile 1				
Baseline	2.2	0.60	0.02	0.28	0.08	0.02
Liquid k	0.87	0.25	0.02	0.56	0.16	0.02
		When in quintile 2				
Baseline	1.4	0.15	0.03	0.58	0.21	0.02
Liquid k	2.6	0.27	0.05	0.5	0.16	0.02
		When in quintile 3				
Baseline	0.69	0.12	0.01	0.47	0.30	0.10
Liquid k	0.86	0.26	0	0.37	0.24	0.12
		When in quintile 4				
Baseline	0.49	0.08	0.01	0.38	0.25	0.30
Liquid k	0.43	0.15	0.00	0.34	0.24	0.28
		When in quintile 5				
Baseline	0.28	0.05	0.00	0.20	0.16	0.59
Liquid k	0.26	0.07	0.00	0.16	0.21	0.56

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Biography

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