Essays on Dynamic Incentives

by

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David McAdams

Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Economics
in the Graduate School of Duke University
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Abstract

This thesis consists of three essays on dynamic contracts or games with incomplete information. In Chapter 1, I study concealing losses in dynamic relationships. I investigate a continuous-time reputation game, where an agent privately observes a Poisson process of losses and chooses whether to disclose them to the principal or to conceal them. By disclosing a loss, the agent passes on the consequences to the principal, which is an observable event. By concealing a loss, the agent bears the consequences himself, and the principal is unaware of the occurrence. The agent's type is private and uncertain: he is either strategic or honest. An honest agent faces a lower rate of losses and always discloses them. Both the principal and the agent enjoy a flow of benefits from the relationship while it is active. The principal, however, may unilaterally end the relationship at any time, and she prefers to maintain a relationship only with the low-frequency (honest) type. She learns about the agent's type through the pattern of disclosed losses. I characterize a class of equilibria with an intuitive structure consisting of a milking phase at high reputation and a building phase at low reputation. In the milking phase, every loss is passed through and is tolerated by the principal. In the building phase, the strategic type of agent probabilistically passes through losses and the principal randomly terminates the relationship when she incurs one. Applications include filing insurance claims, employees who make costly mistakes and friends or colleagues who ask for favors.

In Chapter 2[1], I employ novel methods to investigate optimal project management in a setting plagued by unavoidable setbacks. The contractor can cover up delays from shirking either by making false claims of setbacks or by postponing the reports of real ones. The sponsor induces work and honest reporting via a soft deadline and a reward for completion. Late-stage setbacks trigger randomization between cancellation and extension. Thus the project may run far beyond its initial schedule, generating arbitrarily large overruns, and yet be canceled. Absent commitment to randomize, the sponsor grants the contractor more time to complete the project.

In Chapter 3[2], I study the optimal incentive scheme for a long-term project with both moral hazard and adverse selection. The moral hazard issue is due to the fact that the agent's effort, which increases the arrival rate of a Poisson process, is not observable by the principal. In addition, the agent's effort cost, which needs to be reimbursed by the principal, is also the agent's private information. This gives rise to the adverse selection problem. The principal needs to design the optimal menu of contracts, each of which is chosen by the agent with a specific effort cost. I fully characterize the optimal menu in the case of two types of agents. Specifically, the agent with a lower cost is offered a probation contract, which confirms the agent's type if there is an arrival during a probation period; the agent with the higher cost is offered a sign-on-bonus contract with an immediate direct initial payment. I then explore the more general case with continuous types of agents. In particular, I provide an easy-to-compute upper bound on the principal's utility. The upper bound computation also yields a feasible menu of probation and sign-on-bonus contracts, and the corresponding lower bound it generates. I further provide a condition which can be used to verify whether the upper and lower bounds coincide, implying the

[1] This is collaborated work with Curtis Taylor, Mark Westerfield and Felix Zhiyu Feng.

[2] This is collaborated work with Feng Tian, Peng Sun and Izak Duenyas.

optimality of our feasible menu of contracts. Numerical studies confirm that the verification condition almost always holds for commonly used probability distributions of effort costs.

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Acknowledgements

I am deeply indebted to my advisor Curtis Taylor for his continuous guidance, encouragement and support in the past six years. I was fortunate enough to take Curt's game theory course at Duke. His lectures stimulated my interests in dynamic contracts and reputations, and more importantly, equipped me with the technical tools to do frontier research in economic theory. After I determined to pursue my research on dynamic incentives, Curt became the natural choice of my primary advisor. Talking with Curt, I was constantly impressed by the depth of his economic thinking, the sharpness of his mind and the novelty of his research ideas. Whenever I bring up a new research idea, Curt always gives the most helpful comments. He has been not only encouraging and inspiring, but also outspoken about the limitation of my research ideas.

I am grateful for two of my dissertation committee, Peng Sun and Arjada Bardhi. My story with Peng started after attending his course on Dynamic Programming at Fuqua Business School. Interested in applying dynamic methods to economic theory, we naturally started collaborating on a coauthored paper on dynamic contract design. The experience of collaborating with Peng is invaluable to me for a lifetime, where I not only learned a lot from him, but also learned together with him. Peng has inspired me with his in-depth understanding of real-world business problems. The most exciting part of working with Peng is that we share great devotions to research. Whenever an exciting idea popped up, we would talk as early as eight in the morning, or until late at night.

Arjada joined Duke when I was writing my second-year paper. My research interests overlap with her expertise, so I invited her to my dissertation committee and she happily agreed. I then took her course on continuous-time methods, which turned out to be tremendously useful in my research afterwards. I will never forget the time when she wrote down a whole page of comments regarding my job market talk, and patiently explained each item to me. I benefited tremendously from her suggestions.

I would also like to thank my friend and coauthor Feng Tian. We met each other as master cohorts at Duke University who share common pursuits to become scholars in economics and management. We became long-term friends and coauthors ever since, helping each other both in research and in life. Feng has been the role model of dedication to research, who has constantly provided me with inspiration. Looking forward, we will continue as long-term friends and coauthors.

Aside from the people I just mentioned, I have also been fortunate to receive helpful comments on my research from many others, including Philipp Sadowski, Todd Sarver, Huseyin Yildrim, Attila Ambrus, Felix Feng, Shouqiang Wang, Jinge Liu, Lin Zhao, Xinyue Bei, and participants at the Duke economic theory lunch group.

Finally, I would like to thank my family for their unconditional love and support over the past six years, especially my beloved wife Tongrong Wang, who has accompanied me through happiness and hardship, and my parents, who have sacrificed a lot to help me succeed, and no words can express my gratitude.

1

Concealing Losses in Dynamic Relationships

I investigate a continuous-time reputation game, where an agent privately observes a Poisson process of losses and chooses whether to disclose them to the principal or to conceal them. By disclosing a loss, the agent passes on the consequences to the principal, which is an observable event. By concealing a loss, the agent bears the consequences himself, and the principal is unaware of the occurrence. The agent's type is private and uncertain: he is either strategic or honest. An honest agent faces a lower rate of losses and always discloses them. Both the principal and the agent enjoy a flow of benefits from the relationship while it is active. The principal, however, may unilaterally end the relationship at any time, and she prefers to maintain a relationship only with the low-frequency (honest) type. She learns about the agent's type through the pattern of disclosed losses. I characterize a class of equilibria with an intuitive structure consisting of a milking phase at high reputation and a building phase at low reputation. In the milking phase, every loss is passed through and is tolerated by the principal. In the building phase, the strategic type of agent probabilistically passes through losses and the principal randomly terminates the relationship when she incurs one. Applications include filing insurance claims, employees who make costly mistakes and friends or colleagues who ask for favors.

1.1 Introduction

To set the stage, consider the following motivating examples:

- Auto insurance: Upon the occurrence of an accident, the policyholder decides either to file a claim or to settle privately. If a claim is filed, then the loss from the accident will be covered by the insurance company. However, the company may cancel or tailor the terms of the policy if too many claims are filed. Unreported traffic accidents are common. According to Davis et al. (2015), 15.2 percent of car crashes go unreported to insurance companies.

- Social relationships: Reputation can be interpreted as trust or social capital. Needs arrive randomly over time after which an individual either seeks help from others (e.g., a loan of money or a equipment) or makes efforts to resolve them independently. In the short run, asking for help is most convenient, but in the long run, it might compromise one's reputation of being reliable or trustworthy. Asking one's friends or colleagues for help too frequently may ultimately jeopardize the relationship.

- Employment: Employees make occasional mistakes that generate losses. Whenever such mistakes are made, workers learn them before supervisors become aware. so the employees can decide either to hide the mistakes while bearing the losses on their own or to report them to their bosses. If losses occur too frequently, the employee's competence might be called into question, which could lead to termination.

In accordance with these examples, in this paper I explore a continuous-time reputation game, where an agent privately observes a Poisson process of losses and chooses whether to disclose them to the principal or to conceal them. Losses are costly and recurring. By disclosing a loss, the agent passes on the consequences to the principal, which is an observable event. By concealing a loss, the agent bears the consequences on his own, and the principal is unaware of the occurrence. The agent's type is private and uncertain: he is either strategic or honest. An honest agent (i.e., a safe driver or a reliable partner/employee) faces a lower Poisson rate of losses and always discloses them. Both the principal and the agent enjoy a flow of benefits from the relationship while it is active. The principal may end the relationship at any time. She collects information about the agent's type through the frequency of disclosed losses and prefers to maintain a relationship only with an honest agent; i.e., if she knew the agent was strategic, then she would terminate the relationship immediately. Therefore, the strategic type of agent may have incentives to conceal losses in order to maintain a reputation as the honest type who incurs less frequent losses.

I define reputation as the public belief that the agent is the honest type and show that the equilibria of the game exhibit phases of building and milking his reputation. When the agent's reputation is high, the game is in the milking phase, where the principal is tolerant to any loss reported by the agent, and the agent runs down his reputation by passing on all losses to the principal. When the agent's reputation is low, the game enters the building phase, where the principal penalizes the agent by random termination of the relationship upon disclosed losses, and the agent endeavors to raise his reputation by personally absorbing some losses. The agent's strategy is monotone in his reputation: the lower his reputation, the less likely he will disclose losses; i.e., the harder he tries to restore his reputation.

I begin the analysis with a non-strategic full disclosure benchmark, where all losses are disclosed by both types of agent immediately upon occurrence. In this setting, the principal faces a standard two-armed bandit problem. The safe arm is to terminate the relationship, and the risky arm is to continue experimenting with the agent. It is optimal for the principal to implement a threshold strategy, where she terminates the relationship whenever the agent's reputation (i.e., her belief that he is the type with less frequent losses) falls weakly below a critical level. As usual, this threshold is lower than the myopic threshold because of the value of the information from experimentation, as the principal can collect more information about the agent's type only by retaining him.

When the agent can strategically conceal losses, I show that multiple equilibria exist, all exhibiting the reputational building-milking structure. The equilibria also all have the same threshold belief that marks the transition between the building and milking phases. Moreover, this threshold coincides with the one in the full disclosure benchmark. The agent's payoff varies across equilibria, hinging on how harsh the principal implements the penalty within the milking phase: for a more lenient termination policy, the agent enjoys a higher payoff. Remarkably, the principal's equilibrium payoff is the same across all equilibria and corresponds to her payoff under the full-disclosure benchmark, because her value is insured by the reputational concerns of the strategic agent.

There exist closed-form solutions for the agent's value as a function of his reputation. Interestingly, during the building phase, his value function has a constant elasticity in reputation, and during the milking phase, it has a constant elasticity in the likelihood of reputation. With constant elasticity, his value increases by a fixed percentage in response to a one percent increase in reputation (or likelihood function of reputation [1]). Correspondingly, upon disclosure of losses, the reputation or likelihood of reputation decreases by a fixed fraction within the building or milking phases. This implies a disclosed loss always decreases the agent's value by a fixed fraction in each phase.

To generalize, I examine two extensions. The first extension explores the opposite case from my baseline model, where the honest type is now more likely to suffer losses compared with the strategic type, implying that the reputation (of the agent being honest) is *bad*. I show that the equilibrium structure under bad reputation is similar to the baseline model, which also consists of a building phase and a milking phase. The difference is that the strategic agent in the bad reputation setting

[1] Reputation is the public belief p that the agent is honest, and its likelihood function is $p/(1-p)$.

conceals losses to differentiate himself from the honest type, as opposed to imitating the honest type in the baseline model. A particularly stark equilibrium emerges under bad reputation, where the strategic agent conceals every loss, and once a loss is disclosed, the agent is identified as an honest agent and is terminated. The second extension verifies the robustness of my equilibria when the agent can take *preemptive* actions to prevent losses rather than *reactive* actions to conceal them as in the baseline version of the model. Up to a re-normalization of parameters, every equilibrium in my baseline model is equivalent to an equilibrium under preemptive actions.

The paper is organized as follows. In Section 2, I review related literature. In section 3, I introduce the model setup and define the equilibrium concept. I formulate both player's optimization problems in Section 4. In Section 5, I explore the full-disclosure benchmark. Next, Section 6 contains the main analysis. Following that, welfare implications are provided in section 7. Finally, in Section 8, I discuss the extensions and I conclude the paper in section 9. All proofs and some purely technical results can be found in the appendix.

1.2 Literature

This paper belongs to the extensive and growing literature on reputation games. My main contribution is to introduce endogenous signal disclosure that affects reputation, as opposed to the exogenous signals common in the literature. That is, I allow costly manipulation of the Poisson history.

The paper that is closest to mine is Pei (2016). In his model there are three players – the agent's performance is observed by an intermediary, who then decides whether to disclose it to the public. My model differs in two critical ways. First, my disclosure of signals (i.e, losses) is decided by the agent on his own, who faces a trade-off between bearing an instantaneous cost and damaging his future reputation. In Pei (2016) the intermediary makes disclosure decisions, trading off public trust and motivating the agent via the ensuing reputational concerns. Second,the signal in Pei's model is conclusive good news, and hence one revealed signal ends the game. However, in my baseline model, signals are inconclusive bad news. The inconclusiveness of signals makes the dynamics of reputation in my model much richer.

The original research on reputation are the celebrated *gang of four* articles, Kreps and Wilson (1982) and Milgrom and Roberts (1982). The key trade-off in my model resembles theirs, in that the agent chooses between a myopic option and a long-term option. I differentiate by analyzing such a trade-off in a continuous-time framework with Poisson arrivals. Furthermore, because the posterior belief in the pioneering papers changes monotonically, there is no dynamics of "building and milking" reputation, whereas such dynamics are endemic in my model because posterior beliefs either drift up continuously or jump down discretely.

The reputation in my model derives from the principal possessing incomplete information over the agent's fixed type; that is, from adverse selection. Recent papers by Board and Meyer-ter Vehn (2013), Dilmé (2019) focus on reputation games with moral hazard. That is, in their models, agents may exert effort to change their types. For a more exhaustive literature review on reputation models, see Pei (2016).

The principal's problem in my model is a multi-armed bandit problem with inconclusive bad news, which is the single-player version of Keller and Rady (2015). Congruent with their results, the principal's optimal policy in my model is a cutoff strategy which is more tolerant than that of a myopic player, due to the value of experimentation. However, their model focuses on the equilibrium of the multi-player game, while my model focuses on the reputation interaction between a principal and an agent.

My model also relates to the literature on career concerns, in the sense that a low-type agent exerts effort (i.e., conceals losses) to imitate a high-type agent. This behavior leads to a signal-jamming effect. That is, the difference between the two types becomes narrower, making it harder for the principal to learn the agent's type. Bonatti and Hörner (2017) study career concerns with exponential learning. In their paper, agents exert effort to earn higher wages which are based on assessed ability and expected output. In my model, incentives are created by the future benefits from the relationship, rather than payments. Again, in my model, the inconclusive signal setting provides richer reputational dynamics as the agent's reputation may rise and fall.

The extension of my model with preemptive effort can be considered a reputation-game counterpart to Biais et al. (2010a). These authors solve for an optimal dynamic contract when a firm hires an agent to reduce the Poisson rate of losses. The incentives in their model are provided through monetary transfers as rewards and downsizing as punishments. By contrast, in my model, efforts are incentivized solely by reputation effects, where a good performance increases the principal's belief that the agent is honest, which in turn increases the agent's expected future payoff.

The feature that the agent may either disclose or conceal a Poisson arrival is similar to Green and Taylor (2016b). In a recent working paper, Feng et al. (2021) also study optimal contracts to induce agents to self-report bad news, which are setbacks. In these papers, the principal can always induce truthful reports with commitment power through verifiable terms in an enforceable contract. However, truth-telling fails in my paper because the principal lacks such commitment power by assumption.

1.3 Model

Consider a reputation game over a continuous time horizon. There are two players, one principal, and one agent in a partnership that the principal can irrevocably terminate at any time $\tau > 0$. If termination occurs, both players receive outside option values normalized to 0. The agent is of two possible private types: either *honest* ($\theta = 1$) or *strategic* ($\theta = 0$). The prior probability that the agent is honest is $p_0 \in (0, 1)$. Both parties enjoy flow benefits from remaining in the partnership, which is B for the principal and b for the agent. However, there is *risk* during the partnership. Formally, the risk is a Poisson process of losses that is privately observable to the agent [2]. Different types of agents are heterogeneous in their levels of Poisson risk: the honest agent's Poisson rate is λ_0; the strategic agent's Poisson rate is $\lambda > \lambda_0$, that is, losses occur more frequently for the strategic agent.

Upon the occurrence of a loss, the agent decides either to *disclose* it or to *conceal* it. By disclosing a loss, the agent passes it on to the principal (e.g., file claims, report accidents, or seek help), whereby the loss becomes publicly observable and the principal suffers an instantaneous cost of $K > 0$. On the contrary, if the agent conceals a loss, he privately bears an immediate cost of $k > 0$, and the loss remains unobserved (e.g., settle privately or solve independently). For the honest agent, arrivals are always disclosed.[3]

[2] Throughout I use the terms "arrival" and "loss" interchangeably.

[3] Hereinafter "agent" refers specifically to the strategic agent, as the honest agent's strategy is degenerate.

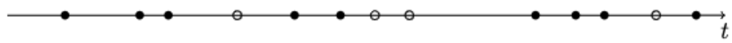


Figure 1.1: **A Sample Path of Disclosed and Concealed Losses**

In Figure 1.1, each circle represents an arrival of a loss. If the agent discloses it, it is a filled circle; otherwise if the agent conceals it, it is a hollow circle. Only filled circles are observable to the principal, while the agent observes all circles. Next, I make the following assumption.

Assumption 1.3.1.

$$\lambda K > B > \lambda_0 K$$

Assumption 1.3.1 says, for the principal, the strategic agent is inefficient, while the honest agent is efficient. If the principal knew the agent were strategic, and if the agent always disclose losses, the principal would prefer to terminate the relationship. However, the strategic agent prefers remaining in the relationship to collect private flow benefits, hence the conflict of interest. Moreover, due to the agent's type being private, the principal has to learn it over time through observing disclosed losses. This creates incentives for the strategic agent to conceal losses to prolong the relationship. Both players are risk-neutral, and the principal's discount rate is $\rho > 0$, while the agent's discount rate is $r > 0$. [4] Last, I assume

Assumption 1.3.2.

$$b \geqslant (\lambda + r)k$$

The private cost for the agent to conceal a loss is sufficiently low. This assumption is to avoid trivial equilibria where the agent always prefers to disclose losses. [5]

[4] First, ρ and r cannot be 0 because if players were perfectly patient, their expected payoffs would diverge. Second, the principal is often assumed to be more patient than the agent ($\rho < r$). However, there is no need to impose this in my setting, as I do not allow monetary transfers.

[5] To see this, the agent can gain $\frac{b-\lambda k}{r}$ by concealing all losses. He would prefer to disclose if this value is smaller than his cost k.

History and Pure Strategy

In this section, I define histories and strategies. A Poisson filtration of a Poisson counting process N_t is

$$\mathcal{F}_t^N = \{t_1, t_2, ..., t_n\}. \tag{1.1}$$

where there are in total $n \in \mathbb{N}$ arrivals with $t_1 \leqslant t_2 \leqslant ... \leqslant t_n$ being each of the arrival times. A public history h_t is the publicly observed filtration of the Poisson history $h^t = \left\{\mathcal{F}_t^N\right\}$. A private history $\tilde{h}_t$ of the agent is the history up to time t of the Poisson process $\tilde{N}$ he privately observes, together with his type and his history of past actions a^t

$$\tilde{h}^t = \{\mathcal{F}_t^{\tilde{N}}, \theta, a^t\}. \tag{1.2}$$

The set of all private histories is denoted $\tilde{\mathcal{H}}$. Next, I define the agent's strategy. A *pure strategy* is a function

$$a : \tilde{\mathcal{H}} \to \{0, 1\}, \tag{1.3}$$

where $a = 1$ represents that the agent discloses a loss, and $a = 0$ that he conceals a loss. Furthermore, I say that a public history h^t is *consistent* with a private history $\tilde{h}^t$ if the Poisson arrivals that are disclosed in the private history match the observed public history. The principal's strategy is a $\mathcal{F}_t^N$ -stopping time τ upon which the game ends and both parties receive utility normalized to 0.

Mixed Strategies and Payoffs

A *mixed strategy* for the agent is a function

$$\sigma : \tilde{\mathcal{H}} \to [0, 1]; \quad \mathbb{P}(a = 1) = \sigma, \tag{1.4}$$

that is, if an agent plays a mixed strategy σ, it means he discloses a loss with probability σ and conceals a loss with probability $1 - \sigma$.

Next, I define payoffs. Let $V(\sigma, \tau)$ and $U(\sigma, \tau)$ be the principal's and the agent's expected payoff respectively. Conditional on public history h^t and prior belief p_0, the principal's expected future payoff is given by

$$V_t(\sigma, \tau | h^t, p_0) = \mathbb{E}^{\theta,\sigma,\tau}\left[\int_{s=t}^{\tau} e^{-\rho(s-t)}\left(Bds - KdN_s\right)\bigg| h^t, p_0\right]\mathbb{1}_{t\leqslant\tau}. \tag{1.5}$$

Condition on private history $\tilde{h}^t$, the agent's expected payoff is given by

$$U_t(\sigma, \tau | \tilde{h}^t) = \mathbb{E}^{\sigma,\tau}\left[\int_{s=t}^{\tau} e^{-r(s-t)}\left(bds - k(d\tilde{N}_s - dN_s)\right)\bigg| \tilde{h}^t\right]\mathbb{1}_{t\leqslant\tau}. \tag{1.6}$$

There are two major differences between the principal's and the agent's payoffs. First, the principal takes an expectation over the agent's uncertain type θ, while the agent does not, due to the agent's type being his private information. Second, the principal bears a loss upon a disclosed public arrival dN_t, while the agent bears a loss upon a concealed arrival, which is the difference $d\tilde{N}_t - dN_t$ between the privately observed arrival and the disclosed public arrival.

Equilibrium and MPE

As is well known, the set of public perfect equilibria (PPE) in dynamic games is typically very large. In PPE, strategies depend on the public history, but the history itself is infinite-dimensional. Therefore, I refine PPE and solve for a Markov perfect equilibrium. Let the public belief that the agent is honest be

$$p(h^t) = \mathbb{P}_t(\theta = 1 | h^t). \tag{1.7}$$

Definition 1.3.3. *A Markov perfect Bayesian equilibrium (MPBE) is a set of strategies including the principal's stopping strategy τ^* and the agent's mixed strategy σ^* that satisfies*

1. ***Markov Property**: For any two consistent public/private histories $(h^{t_1}, \tilde{h}^{t_1})$ and $(h^{t_2}, \tilde{h}^{t_2})$ such that $p(h^{t_1}) = p(h^{t_2})$*

$$\sigma^*|_{\tilde{h}^{t_1}} = \sigma^*|_{\tilde{h}^{t_2}}, \quad \tau^*|_{h^{t_1}} = \tau^*|_{h^{t_2}}.$$

2. ***Best Response***: *τ^* and σ^* are mutual best responses.*

$$\tau^* \in \arg\max_{\tau} \quad V(\sigma^*, \tau) \tag{1.8}$$

$$\sigma^* \in \arg\max_{\sigma} \quad U(\sigma, \tau^*) \tag{1.9}$$

3. ***Belief Consistency***: *The belief p_t updates according to Bayes' Rule with respect to the probability measures induced by σ^*, τ^*.*

The Markov property says we can regard the public belief p as the only state variable, which is the sufficient statistic for both players' strategies and payoffs. Because both players' strategies depend only on the public belief, we avoid complications arising from higher-order beliefs. Next, the best response property requires that no player can benefit from unilaterally deviating taking the opponent's strategy as given. Last, the public belief evolves according to Bayes' rule to satisfy belief consistency.

Belief Dynamics

For ease of notation, denote $p_t = p(h^t)$. To characterize the dynamics of the public belief, consider the following two possible events between $[t, t + dt)$ when the current belief is at p_t:

1. After a disclosed arrival ($dN_t = 1$), let $j(p_t)$ be the posterior belief. Applying Bayes' rule yields

$$\begin{aligned} j(p_t) &= \mathbb{P}(\theta = 1 | dN_t = 1; p_t) \\ &= \frac{p_t \mathbb{P}(dN_t = 1|\theta = 1)}{p_t \mathbb{P}(dN_t = 1|\theta = 1) + (1 - p_t)\mathbb{P}(dN_t = 1|\theta = 0)} \\ &= \frac{p_t \lambda_0 dt}{p_t \lambda_0 dt + (1 - p_t)\lambda\sigma(p_t)dt} = \frac{p_t \lambda_0}{p_t \lambda_0 + (1 - p_t)\lambda\sigma(p_t)}. \end{aligned} \tag{1.10}$$

Note that the strategic agent will disclose an arrival with Poisson rate $\lambda\sigma(p_t)$, which follows the *thinning property* [6] of Poisson processes. Formally,

$$\begin{aligned}\mathbb{P}(dN_t = 1|\theta = 0) &= \mathbb{P}(dN_t = 1|d\tilde{N}_t = 1; \theta = 0)\mathbb{P}(d\tilde{N}_t = 1|\theta = 0) \\ &= \sigma(p_t)\lambda dt. \end{aligned} \tag{1.11}$$

2. Absent a disclosed arrival, let p_{t+dt} be the posterior belief,

$$\begin{aligned} p_{t+dt} =& \mathbb{P}(\theta = 1|dN_t = 0; p_t) \\ =& \frac{p_t \mathbb{P}(dN_t = 0|\theta = 1)}{p_t \mathbb{P}(dN_t = 0|\theta = 1) + (1 - p_t)\mathbb{P}(dN_t = 0|\theta = 0)} \\ =& \frac{p_t(1 - \lambda_0 dt)}{p_t(1 - \lambda_0 dt) + (1 - p_t)(1 - \lambda\sigma(p_t)dt)}. \end{aligned} \tag{1.12}$$

As dt tends to 0, the continuous limit of (1.12) implies

$$p'_t = \lim_{dt \to 0} \frac{p_{t+dt} - p_t}{dt} = p_t(1 - p_t)(\lambda\sigma(p_t) - \lambda_0). \tag{1.13}$$

To summarize the belief dynamics,

$$dp_t = p_t(1 - p_t)(\lambda\sigma(p_t) - \lambda_0)dt + [j(p_t) - p_t]\, dN_t \tag{1.14}$$

That is, the belief drifts continuously over time absent disclosed arrivals, and makes a discrete jump from p_t to $j(p_t)$ upon an arrival.

Lemma 1.3.1. *p'_t is increasing in σ; $j(p_t)$ is decreasing in σ.*

Note that $\sigma(.) \in [0, 1]$ represents the fraction of losses the agent decides to disclose. In other words, σ measures the extent to which the agent is milking from his reputation, while $(1 - \sigma)$ measures the extent to which the agent is building his reputation. Indeed, if the agent fully discloses all his losses, it means that he is maximally depleting his reputation in exchange for private benefits. With full disclosure, the loss rate gap $(\lambda\sigma - \lambda_0)$ between the strategic and the honest agent is also at

[6] After thinning a Poisson λ process with a thinning probability $\sigma \in [0, 1]$, the resulting stochastic process is still a Poisson process but with intensity $\sigma\lambda$.

its maximum, which leads to the principal's fastest learning speed. As is stated in Lemma 1.3.1, the larger σ is, the worse the posterior belief will be after a loss is disclosed, and the more optimistic the principal will be absent losses.

1.4 Formulation of Value Functions

In this section I derive both player's value functions. I start by defining

$$V(p_t) = V(\sigma, \tau; h^t, p_0); \quad U(p_t) = U(\sigma, \tau; \tilde{h}^t), \tag{1.15}$$

where $(h^t, \tilde{h}^t)$ are consistent public and private histories and $p_t = p(h^t)$ is the public belief. By writing $V(.)$ and $U(.)$ as uni-dimensional functions of belief p, I am able to make the value functions tractable.

1.4.1 Principal's Utility

Let $V^c(p)$ be the principal's continuation value if she decides not to terminate, then the Hamilton–Jacobi–Bellman (HJB) equation follows

$$V^c(p_t) = Bdt - \bar{\lambda}(p_t)Kdt + \underbrace{(1 - \bar{\lambda}(p_t)dt)}_{\text{no loss}} V(p_{t+dt}) + \underbrace{\bar{\lambda}(p_t)dt}_{\text{loss}} e^{-\rho dt} V\left(j(p_t)\right), \tag{1.16}$$

where $\bar{\lambda}(p) = \lambda_0 p + \lambda(1-p)$ is the average rate of a Poisson arrival when the belief is p. The first two terms are the expected payoff that the principal collects between $[t, t + dt)$; the third term corresponds to the event where no loss is disclosed; the last term corresponds the event of a disclosed loss. Furthermore, the principal decides when to terminate by choosing stopping time τ.

$$V(p_t) = \max_{\tau} \quad V^c(p_t)\mathbb{1}_{t\leqslant\tau}. \tag{1.17}$$

The solution to (1.17) is

$$\begin{cases} \tau = t & \text{if} \quad V^c(p_t) < 0 \\ \tau > t & \text{if} \quad V^c(p_t) > 0. \end{cases} \tag{1.18}$$

Clearly, the principal will terminate if her continuation value is strictly negative, and will retain the agent if her continuation value is strictly positive. This leaves out the knife-edge case where $V^c(p_t) = 0$, in which the principal is indifferent and hence she could *randomly* terminate the agent. That is, the principal immediately terminates the agent with probability $\mathbb{P}(\tau = t | \tau \geqslant t; p_t) = \alpha(p_t)$. Here we consider Markov strategies $\alpha : [0,1] \to [0,1]$ that depend only on the belief p_t.[7] It is worth noting that randomization will not change the belief p_t, as both types of agent with the same history are terminated with identical probabilities. Randomization is introduced to provide incentives for the agent to care about his reputation.

1.4.2 Agent's Utility

I derive the agent's HJB equation by exhausting the two possible events within the next time duration dt: with probability λdt the loss occurs, and then the agent decides whether to conceal it or to disclose it; with the remaining probability $1 - \lambda dt$ there is no loss and no termination and the belief evolves and becomes p_{t+dt}.

$$\begin{aligned} U(p_t) = bdt + \underbrace{\lambda dt}_{\text{loss}} \max \left\{ \overbrace{U(p_t) - k}^{\text{conceal}}, \overbrace{(1 - \alpha(p_t))U(j(p_t))}^{\text{disclose}} \right\} \\ + \underbrace{(1 - \lambda dt)}_{\text{no loss}} e^{-rdt} U(p_{t+dt}). \end{aligned} \tag{1.19}$$

Additionally, the following best response conditions must hold:

$$\sigma = \begin{cases} 1, & U(p) - (1 - \alpha(p))U\left(j(p)\right) \leqslant k \\ 0 < \sigma < 1, & U(p) - (1 - \alpha(p))U\left(j(p)\right) = k \\ 0, & U(p) - (1 - \alpha(p))U\left(j(p)\right) \geqslant k. \end{cases} \tag{BR}$$

The condition (BR) is the required consistency between the maximization and the definition of mixed strategy σ in (1.19). Note that if the difference of the agent's

[7] Both the principal's strategy $\alpha(.)$ and the agent's strategy $\sigma(.)$ are functions of belief p_t, but I will sometimes suppress them by writing α, σ for ease of notation.

value between before and after disclosing a loss is smaller than his private cost, then it is optimal for the agent to disclose $\sigma = 1$; on the contrary, if the agent plays a mixed strategy $\sigma \in (0, 1)$, it has to be that he is indifferent between concealing and disclosing the arrival; otherwise, the agent conceals arrival for sure and $\sigma = 0$.

1.5 Full Disclosure Benchmark

I start by considering the benchmark case where the strategic agent fully discloses his losses. I explore the principal's optimal stopping strategy. Letting $\sigma(p) \equiv 1$ in (1.14) yields the belief dynamics

$$dp_t = p_t(1 - p_t)(\lambda - \lambda_0)dt + [j(p_t) - p_t]\, dN_t, \tag{1.20}$$

where $j(p_t) = \frac{\lambda_0 p_t}{\lambda_0 p_t + \lambda(1-p_t)}$. Both strategic and honest agents are assumed to fully disclose their losses, however, they differ in arrival rates as $\lambda > \lambda_0$. Therefore, the principal faces a single-player multi-armed bandit problem as in Keller and Rady (2015) (hereinafter K&R). The safe arm in my model is to terminate the relationship and receive an outside option of zero, and the risky arm is to retain the agent and continue experimenting with him through the losses he discloses over time.

Lemma 1.5.1. *If $\sigma \equiv 1$, and the principal is myopic, her optimal stopping strategy is*

$$\tau = \inf\{t : p < p^m\},$$

where $p^m := \frac{\lambda K - B}{(\lambda - \lambda_0)K}$ is the principal's myopic belief threshold.

At the myopic threshold of belief p^m, the principal's expected flow payoff for the next instant is zero, conditional on the agent fully disclosing all losses. For any belief lower than p^m, the expected value for myopic agent to continue is negative, and she would terminate immediately.

Theorem 1.5.1. *If $\sigma \equiv 1$ and $0 < \rho < \infty$, the principal's optimal strategy is a threshold strategy with $\bar{p} \in (0, p^m)$,*

$$\tau = \inf\{t : p_t < \bar{p}\}. \tag{1.21}$$

Intuitively, the principal terminates if and only if the public belief hits a lower bound. Moreover, her optimal threshold is relatively more tolerant than that of a myopic principal, because of the value of information as in K&R. Specifically, apart from its productive value, continuing with the relationship has additional informational value, as the principal can collect more information over time, which helps her to learn the agent's type more accurately. To see this, consider for some p such that $V(p) > 0$ and $V(j(p)) = 0$. The principal's HJB equation (1.16) yields the following differential equation:

$$V'(p)p(1-p)(\lambda - \lambda_0) = (\lambda_0 p + \lambda(1-p) + \rho)\, V(p) + (\lambda_0 p + \lambda(1-p))\, K - B. \tag{1.22}$$

This is solved by

$$V(p) = \left[\frac{B - \lambda K}{\rho + \lambda}(1-p) + \frac{B - \lambda_0 K}{\rho + \lambda_0} p\right] + C_1 p \left(\frac{p}{1-p}\right)^{\frac{\lambda_0 + \rho}{\lambda - \lambda_0}}, \tag{1.23}$$

where $C_1 > 0$ is a constant of integration. The first term represents the expected productive value of the agent, and the second term represents its extra informational value. Let $\bar{p}$ be the switching point, which can be pinned down by the following value matching condition. [8]

$$0 = V(\bar{p}) = \left[\frac{B - \lambda K}{\rho + \lambda}(1-\bar{p}) + \frac{B - \lambda_0 K}{\rho + \lambda_0} \bar{p}\right] + C_1 \bar{p} \left(\frac{\bar{p}}{1-\bar{p}}\right)^{\frac{\lambda_0 + \rho}{\lambda - \lambda_0}} \tag{VM}$$

Although there is no closed-form solution for $\bar{p}$, it is uniquely determined by (VM). In the following analysis, I will regard $\bar{p}$ as a known constant.

[8] As Keller and Rady (2015) argues, smooth pasting fails at $\bar{p}$. This is because $\bar{p}$ is not a regular boundary. That is, beliefs can only jump across it discretely rather than drift across it continuously, which leads to a kink at the boundary.

1.6 Equilibrium Analysis

In this section, I solve for Markov perfect equilibria. I show that multiple MPBE exist which all feature a similar structure. Specifically, there are two phases depending on the reputation, a building phase and a milking phase. In the milking phase, the agent's reputation is high enough such that the principal does not terminate the relationship upon a disclosed loss, and the agent milks from his reputation by disclosing every loss he encounters. In the building phase, the agent builds his reputation by partial disclosure of his losses, and the principal randomly terminates the agent with a belief-dependent probability each time a loss is disclosed.

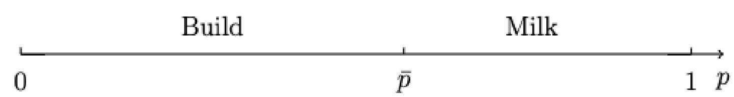


Figure 1.2: **Building and Milking Phases**

First of all, the agent never plays the pure strategy of concealing all losses under any circumstances. Intuitively, if the agent conceals every loss that he encounters, then any disclosed loss is a perfect signal that the agent is honest. Therefore, the principal will never terminate the agent. This will lead to the agent's maximum utility of b/r, which makes the agent strictly prefer to disclose losses. Hence it is never the best response for the agent to fully conceal all losses. Next, I analyze the rest two cases, namely full disclosure and partial disclosure. It turns out that an equilibrium consists of both full disclosure phase and partial disclosure phase. Therefore, I will analyze these two phases separately, and then knit them together.

Full Disclosure and No Termination

Consider the case where the agent fully discloses all his losses $\sigma(p) = 1$ and the principal never terminates $\alpha(p) = 0$.

Proposition 1.6.1. *If $\sigma(p) = 1$ and $\alpha(p) = 0$, then the agent's value function $U(p)$ follows*

$$U(p) = \frac{b}{r} - C\left(\frac{1-p}{p}\right)^{\gamma}, \tag{1.24}$$

where the constant $C \geqslant 0$ is to be determined. This solves the delay differential equation (A.7)*. Here $\gamma > 0$ is the unique positive root of the following characteristic equation*

$$\left(\frac{\lambda}{\lambda_0}\right)^{\gamma} - 1 = \frac{r}{\lambda} + \left(1 - \frac{\lambda_0}{\lambda}\right)\gamma. \tag{CE}$$

The properties of the agent's value function under full disclosure are described in Lemma A.3.3. It is not surprising that the agent's value function is increasing in the belief (i.e., reputation) p, because at higher beliefs he can stay and collect private benefits for a longer duration. In the extreme case, when the belief is 1, the agent is never terminated and enjoys the maximal value of b/r. At any belief p, the agent's value is linear in a power function of the likelihood of reputation. Because the parameter γ measures the percentage change of the agent's value in response to a change in the likelihood $p/(1-p)$, its economic interpretation is an *elasticity.* [9] A larger γ means the agent's value is more sensitive to the changes in the likelihood, hence the changes in belief as well. [10]

Lemma 1.6.1. *When $\sigma(p) = 1$ and $\alpha(p) = 0$, then $U(p) - U(j(p))$ is decreasing in p.*

Lemma 1.6.1 says the penalty for disclosing losses will shrink as the belief increases. In other words, it becomes increasingly tempting for the agent to milk from

[9] The notion of elasticity is fitting here because posterior beliefs are multiplied by a constant fraction after a disclosed loss when $\sigma = 1$, as $\frac{j(p)}{1-j(p)} = \frac{\lambda_0}{\lambda}\frac{p}{1-p}$.

[10] For detailed comparative statics of γ with regard to the model primitives, see Section 1.7.

his reputation at higher beliefs. Indeed, as the agent cares more about the present than the future, prolonging the relationship would yield lower marginal value at higher beliefs. In comparison, the cost of maintaining the reputation stays constant regardless of the belief. Therefore, as long as the agent is willing to disclose a loss at a certain belief, it has to be that he is willing to disclose losses at higher beliefs as well.

Partial Disclosure and Random Termination

Consider the case where the principal randomly terminates the agent once a loss is disclosed and the agent plays a mixed strategy between disclosing and concealing his losses. That is, $\alpha(p) > 0$ and $0 < \sigma(p) < 1$. Random termination requires the principal to be indifferent about whether to terminate $V^c(j(p)) = 0$, by (1.16),

$$bdt - \bar{\lambda}(p_t)Kdt + (1 - \bar{\lambda}(p_t)dt)V(p_{t+dt}) + \bar{\lambda}(p_t)dte^{-rdt}V\left(j(p_t)\right) = 0. \qquad (1.25)$$

From the principal's indifference condition, the agent's mixed strategy $\sigma(p)$ can be derived.

Lemma 1.6.2. *If $\alpha(p) > 0$ and $0 < \sigma(p) < 1$, then the agent mixes with probability*

$$\sigma(p) = \frac{B - \lambda_0 Kp}{\lambda K(1-p)}.$$

The properties of $\sigma(.)$ are characterized in Lemma A.3.4 .First of all, since $\sigma(.)$ is increasing in p, the agent will conceal losses more frequently as the belief decreases. Concealing losses can be thought of as the agent building reputation because in the absence of a disclosed loss the belief that he is the honest type drifts up. The lower the belief is, the more the agent tries to build his reputation. On the contrary, the higher the belief is, the more the agent milks his reputation. At the threshold belief $\bar{p}$, the agent fully discloses his losses, which marks the transition boundary to the milking phase. Although the strategic agent is imitating the honest agent, he will only do so partially in the sense that his expected rate of disclosed losses $\lambda\sigma$ is still higher than that of the honest agent λ_0. Every time the agent discloses a loss, the posterior belief shrinks by a fixed fraction $\frac{\lambda_0 K}{B} \in (0,1)$, which is the ratio between the honest agent's expected cost and the principal's flow benefit. Next, I characterize the agent's value function $U(p)$ by plugging his mixed strategy from Lemma 1.6.2 into his HJB equation (1.19).

Proposition 1.6.2. *If $0 < \sigma(p) < 1$ and $\alpha(p) > 0$, then the agent's value function follows*

$$U(p) = \frac{b - \lambda k}{r} + Dp^{\eta}, \tag{1.26}$$

where $\eta = \frac{rK}{B - \lambda_0 K} \geqslant 0$ and the constant $D \geqslant 0$ is to be determined.

Furthermore, the properties of the agent's value function under the mixed strategy are articulated in Lemma A.3.6. As in the full disclosure case, under the mixed strategy the agent's value function is also increasing in the belief p. As the agent is indifferent between disclosing and concealing losses, this means he is willing to conceal losses for higher future benefits. In the extreme case, when the belief goes to 0, the agent achieves his minimum payoff equivalent to him bearing all losses on his own, which yields $(b - \lambda k)/r$. At any belief p, the difference between $U(p)$ and the agent's minimum utility is a constant times the belief to the power of η. Therefore, the agent's value function has a constant elasticity η over the belief p.

Equilibrium

Under the full and partial disclosure cases, the agent's respective value functions have been explicitly derived. In this section, I construct a class of equilibria by knitting together these cases. I begin with the principal's strategy.

Lemma 1.6.3. *If the agent's strategy follows*

$$\sigma(p) = \begin{cases} 1, & \text{if} \quad p > \bar{p} \\ \dfrac{B - \lambda_0 K p}{\lambda K(1-p)}, & \text{if} \quad p \leqslant \bar{p}, \end{cases} \tag{1.27}$$

then the principal's value function satisfies

$$\begin{cases} V(p) = V^c(p) > 0 & \text{if} \quad p > \bar{p} \\ V(p) = V^c(p) = 0 & \text{if} \quad p \leqslant \bar{p}. \end{cases} \tag{1.28}$$

To understand Lemma 1.6.3, consider it as a variation of Theorem 1.5.1. In the full disclosure benchmark, the principal's continuation value $V^c(p)$ is positive for $p > \bar{p}$ and negative for $p < \bar{p}$. Here, the agent's strategy is identical to the full disclosure benchmark for $p > \bar{p}$, which renders the principal's value to be identical. As a result, it is the best response for the principal to not terminate when $p > \bar{p}$. However, for $p < \bar{p}$, the agent's strategy differs from the full disclosure benchmark, as he conceals losses with non-negative probability. His mixing probability is calibrated such that the principal's continuation value $V^c(p)$ is exactly 0, such that she is indifferent between whether to terminate. It is important to note that the principal's threshold $\bar{p}$ is identical to her stopping threshold in the full disclosure benchmark.

Next, I focus on the incentives from the agent's side. In particular, I construct the agent's value function $U(p)$ on $p \in [0, 1]$. This value function is constructed by matching the values of the piece-wise value functions derived in (1.24) and (1.26) at the threshold $\bar{p}$.

$$\frac{b - \lambda k}{r} + D\bar{p}^{\eta} = \frac{b}{r} - C\left(\frac{1-\bar{p}}{\bar{p}}\right)^{\gamma} \tag{1.29}$$

Definition 1.6.3. *The **feasible** set of constants (C, D) is defined as*

$$\mathcal{S} := \{(C, D) | \text{(A.24)}, \text{(A.23)}, \text{(1.29)} \textit{ are satisfied}\}.$$

The set S contains the feasible combinations of integration constants (C, D). Each combination corresponds to one possible equilibrium. Therefore, there are typically multiple equilibria, all of the same structure. Notice that the value matching condition at $\bar{p}$ implies that (C, D) has a one-to-one correspondence. Intuitively, a large C and small D means that the principal is relatively lenient with the agent, while a small C and a large D means that the principal is relatively harsh.

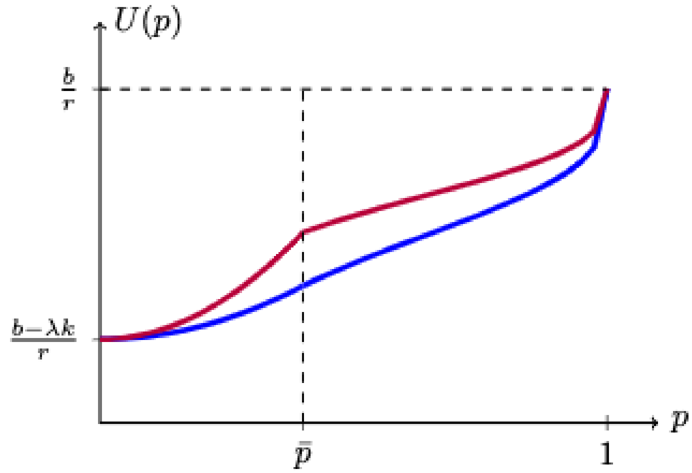


Figure 1.3: **Agent's Value under General MPE**

Theorem 1.6.4. *For each feasible* (C, D)*, the game has a unique MPBE where the agent's strategy is characterized in* (1.27) *and the principal's strategy is characterized by*

$$\alpha(p) = \begin{cases} 0, & p \geqslant \dfrac{\lambda\bar{p}}{\lambda\bar{p} + \lambda_0(1-\bar{p})} \\[2ex] 1 - \dfrac{\frac{b}{r} - C\left(\frac{\lambda}{\lambda_0}\right)^{\gamma}\left(\frac{1-p}{p}\right)^{\gamma}}{\frac{b-\lambda k}{r} + D\left(\frac{\lambda_0 p}{\lambda_0 p + \lambda(1-p)}\right)^{\eta}}, & \bar{p} \leqslant p < \dfrac{\lambda\bar{p}}{\lambda\bar{p} + \lambda_0(1-\bar{p})} \\[2ex] 1 - \dfrac{b - (\lambda + r)k + rDp^{\eta}}{b - \lambda k + rD\left(\frac{\lambda_0 K}{B}\right)^{\eta} p^{\eta}}, & p < \bar{p} \end{cases}$$

Below $\bar{p}$, the game is in the building phase, where the agent conceals some of his losses to make the principal indifferent about termination when experiencing a loss, and the principal randomly terminates the agent with a probability such that he is indifferent between concealing and disclosing. Above $\bar{p}$, the game transitions to the milking phase, where the agent fully discloses his losses and the principal never terminates because her value is strictly positive. As depicted in Figure 1.3, there are multiple possible value functions for the agent, which can be ranked from high to low. It is worth noting that even though the agent's value varies across equilibria, the principal's value is always the same, coinciding with her value in the full disclosure benchmark. This is because the principal's expected payoff is 0 in the building phase due to the agent's mixing. In the full disclosure benchmark,the principal terminates when reputation is below $\bar{p}$ and does not terminate when reputation is above $\bar{p}$. This gives the identical payoff to the principal as in every MPE.

In the following sections, I characterize and discuss two extreme equilibria: the agent's worst and best.

The Agent's Worst Equilibrium

If the principal implements the harshest penalty for disclosing losses when $p < \bar{p}$, the agent faces the worst equilibrium. That is, his expected payoff is the lowest for any belief p among all equilibria.

Theorem 1.6.5. *The agent's worst MPBE occurs when he plays* (1.27) *and the principal plays*

$$\alpha(p) = \begin{cases} 0, & p \geqslant \dfrac{\lambda\bar{p}}{\lambda\bar{p} + \lambda_0(1-\bar{p})} \\[2ex] \dfrac{\lambda k}{b - \lambda k}\left[\left(\dfrac{\lambda}{\lambda_0}\right)^{\gamma}\left(\dfrac{\bar{p}}{1-\bar{p}}\right)^{\gamma}\left(\dfrac{1-p}{p}\right)^{\gamma} - 1\right], & \bar{p} \leqslant p < \dfrac{\lambda\bar{p}}{\lambda\bar{p} + \lambda_0(1-\bar{p})} \\[2ex] \dfrac{rk}{b - \lambda k}, & p < \bar{p} \end{cases}$$

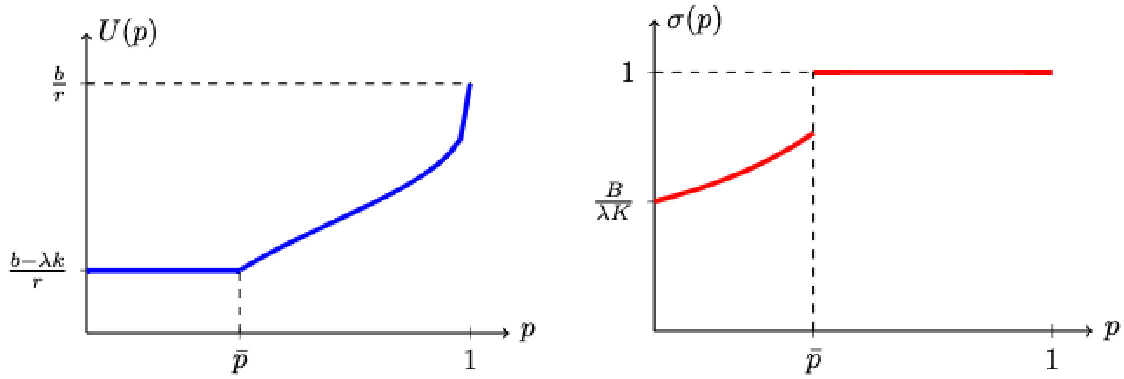


Figure 1.4: Agent's Worst MPE

The agent's value function and mixed strategy are depicted in Figure 1.4. Because the principal uses the harshest punishment, the agent receives his worst equilibrium payoff. Mathematically, this equilibrium corresponds to the minimum value of the parameter $D = 0$. It is the worst for the agent, in the sense that once the game enters the milking phase, the agent's payoff is his worst possible value $\frac{b-\lambda k}{r}$, which is the value the agent can get by concealing all losses in the future in order to stay and collect private benefits.

An interesting feature of this equilibrium is that the principal puts the agent exactly one loss away from termination during the building phase. That is, the agent's expected payoff is equal to his worst possible value $(b-\lambda k)/r$ as he is always indifferent between disclosing and concealing losses, receiving a payoff equal to concealing all losses.

Agent's Best Equilibrium

Next I explore the other extreme case for the agent, where the principal implements the least harsh termination scheme, and the agent enjoys his highest value at any belief p among all possible equilibria. Naively, one might suppose that the situation where the agent is never terminated is the least harsh equilibrium. However, such an equilibrium does not exist because lack of termination gives the agent incentives to disclose all losses, which ultimately makes the principal unwilling to retain him at low beliefs. This means that a building phase must exist in an equilibrium. Following the previous structure, the agent's best equilibrium corresponds to the case where the termination is exactly enough to induce the agent to conceal losses in the milking phase.

Theorem 1.6.6. *The agent's best MPBE occurs when he plays* (1.27) *and the principal plays*

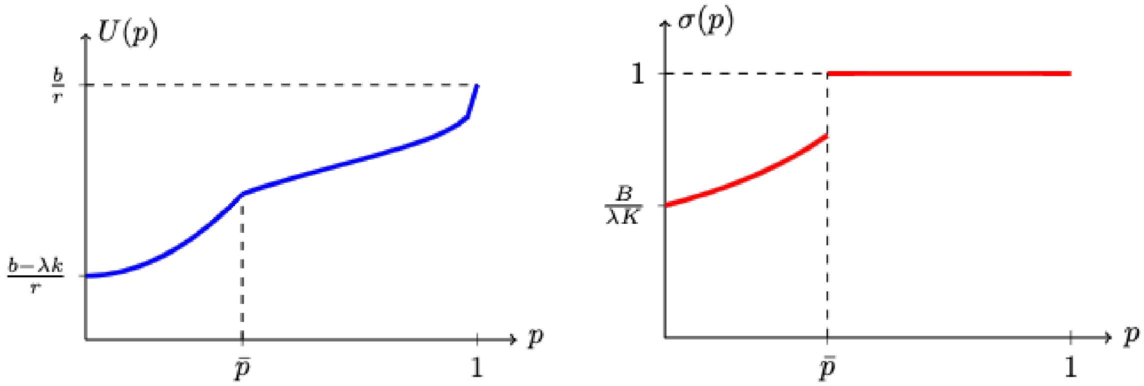
$$\alpha(p) = \begin{cases} 0, & p \geqslant \dfrac{\lambda\bar{p}}{\lambda\bar{p} + \lambda_0(1-\bar{p})} \\[2ex] 1 - \dfrac{\frac{b}{r} - \underline{C}\left(\frac{\lambda}{\lambda_0}\right)^{\gamma}\left(\frac{1-p}{p}\right)^{\gamma}}{\frac{b-\lambda k}{r} + \bar{D}\left(\frac{\lambda_0 p}{\lambda_0 p + \lambda(1-p)}\right)^{\eta}}, & \bar{p} \leqslant p < \dfrac{\lambda\bar{p}}{\lambda\bar{p} + \lambda_0(1-\bar{p})} \\[2ex] 1 - \dfrac{b - (\lambda + r)k + r\bar{D}p^{\eta}}{b - \lambda k + r\bar{D}\left(\frac{\lambda_0 K}{B}\right)^{\eta} p^{\eta}}, & p < \bar{p}, \end{cases}$$

where $\bar{D} = \frac{k}{\bar{p}^{\eta}\left(1-\left(\frac{\lambda K}{B}\right)^{\eta}\right)}$ *and* $\underline{C} = \left(\frac{\bar{p}}{1-\bar{p}}\right)^{\gamma}\left[\frac{\lambda k}{r} - \frac{k}{\left(1-\left(\frac{\lambda K}{B}\right)^{\eta}\right)}\right]$.

The agent's value function and mixed strategy are depicted in Figure 1.5. Because the principal is employing the most lenient punishment, the agent enjoys his best equilibrium. Mathematically, this corresponds to the maximum value of the parameter D. No other equilibrium can yield a better payoff for the agent because his value function would otherwise be so steep in the building phase that he would have incentives to fully conceal losses.

1.7 Welfare Analysis

In this section I analyze the welfare implications of the baseline model. I start by comparing the equilibrium outcomes with the full disclosure benchmark. I then



(a) Agent's Value Function $U(p)$ (b) Agent's Mixed Strategy $\sigma(p)$

Figure 1.5: **Agent's Best MPE**

provide comparative statics to see how the equilibrium structure responds to changes in parameter values. Finally I explore the long-term dynamics to determine how the agent behaves in the long run.

Who Benefits from Concealing Losses

Comparing the equilibrium of the baseline model to the full disclosure benchmark, the only difference is that losses may be concealed in the building phase. The question of interest is how will the agent's behavior of concealing losses affect the welfare of each party including the principal, the honest agent, and the strategic agent. It turns out that the possibility of concealing losses is a Pareto improvement over the full-disclosure benchmark.

Let $V_0(p), U_0^\theta(p)$ be the full disclosure benchmark value of the principal and the agent with type $\theta \in \{0,1\}$, and $V(p), U^\theta(p)$ be their corresponding values in the baseline model.

Proposition 1.7.1. *The principal is unaffected but both types of agent strictly benefit* *from the ability of the strategic agent to conceal losses.*

$$V_0(p) = V(p); \quad U_0^\theta(p) < U^\theta(p)$$

There are three welfare implications that are worth mentioning in Proposition 1.7.1. First of all, the principal enjoys the same value between the full disclosure benchmark and the baseline model. The agent's behavior of concealing losses puts the principal in the exact situation as she faces in the full disclosure benchmark: she enjoys an expected payoff of 0 during the building phase, which is identical to what she gets by terminating the relationship; she faces an agent who fully discloses all his losses during the milking phase; the threshold is exactly the same as the full disclosure benchmark. In other words, the agent's behavior of concealing losses happens to *subsidize* the principal such that she is indifferent to the full disclosure benchmark.

Second, the honest agent also benefits from concealment of losses by the strategic agent. This is because the principal becomes relatively more tolerant regarding termination in the baseline model. Without concealing losses, the principal terminates for sure once the belief drops below the threshold, but with concealing losses, the principal implements a more lenient termination policy of random termination. This allows the honest agent to persist in the system for a longer expected time, as the probability that he is incorrectly terminated becomes lower.

Third, the strategic agent also benefits from concealing losses. By Assumption (1.3.2), the strategic agent considers the relationship to be relatively more valuable than the cost of concealing one single loss. Therefore, he prefers to conceal losses to maintain the relationship, which turns out to improve his welfare, as the principal responds by using a more lenient termination policy in the baseline model compared to the full disclosure benchmark.

To sum up, although concealing losses is often viewed as dishonest or unethical, I show that, in the context of my model, the ability to do so actually weakly benefits all parties.

Comparative Statics

In this section I present comparative statics for the equilibrium regarding changes in model parameters. Notice that the global shape of the agent's value function $U(p)$ hinges on the endogenous parameter γ. I call γ the root of the characteristic equation (CE) regarding the dynamic system (A.7). Rewrite (CE) in the form of an implicit function $F(\gamma; \lambda_0, \lambda, r) = 0$ where

$$F(\gamma; \lambda_0, \lambda, r) := \left(\frac{\lambda}{\lambda_0}\right)^{\gamma} - 1 - \frac{r}{\lambda} - \left(1 - \frac{\lambda_0}{\lambda}\right)\gamma. \tag{1.30}$$

Lemma 1.7.1. *$F(\gamma; \lambda_0, \lambda, r)$ is HD0 in (λ_0, λ, r).*

Lemma 1.7.1 says, the characteristic root γ stays constant if the three parameters (λ_0, λ, r) are scaled together proportionally. This implies that the agent's value function is robust regarding the choice of time units, as only makes sense. Next, I explore how the change of a single model parameter would alter the equilibrium, specifically on γ and η, the elasticities of the agent's value function $U(p)$.

Proposition 1.7.2. $\frac{\partial \gamma}{\partial r} > 0; \quad \frac{\partial \gamma}{\partial \lambda_0} < 0$

First, the more patient the agent is, the less elastic his milking phase value function is, i.e., the more drastically his value function responds to changes in the belief. To see this, observe that every time the agent discloses a loss, his continuation value decreases as his posterior belief jumps closer to the building phase, where he starts to conceal losses. Concealing a loss is relatively more costly for an impatient agent than for a patient one, as an impatient agent assigns more weight to the instantaneous cost of concealing losses. Second, if the discrepancy of arrival rates between the strategic agent and the honest agent becomes larger, then the strategic agent's value function will be more elastic. With a larger discrepancy in arrival rates, the strategic agent and the honest agent will therefore perform more differently. Hence it is easier for the principal to learn the agent's type. This means that once a loss is disclosed, there will be a higher penalty.

Proposition 1.7.3. $\frac{\partial \eta}{\partial r} > 0; \quad \frac{\partial \eta}{\partial \lambda_0} > 0$

As in the milking phase, the agent's value function during the building phase also has an elasticity that is increasing in the discount rate r. The intuition is similar: a loss is more costly to an impatient agent than to a patient one. Because the agent plays a mixed strategy, he is indifferent between disclosing and concealing a loss. Clearly, it is more costly to conceal a loss if the agent is less patient, which explains why his value function is more sensitive. What is different from the milking phase is that the agent's value function now has an elasticity that is increasing in the honest agent's arrival rate λ_0. To see this, consider the extreme case where losses never occur for the honest agent ($\lambda_0 = 0$). There is no opportunity for the strategic agent to disclose even a single loss because once he does, he will reveal his type, regardless of the current belief. Therefore, the strategic agent's value function is least elastic during the building phase when $\lambda_0 = 0$. As λ_0 increases, losses occur more frequently for the honest agent, which also implies that it is easier for the strategic agent to imitate the honest type. Therefore, the strategic agent's value function becomes more sensitive to the belief during the building phase.

Long Term Dynamics

In this section I explore the dynamics of the log likelihood of beliefs. As will be shown later, the likelihood representation yields clean and intuitive reputation dynamics.

Formally, define

$$x_t := \log\left(\frac{p_t}{1-p_t}\right). \tag{1.31}$$

First, consider the milking phase. Conditional on the agent's type, I can rewrite the dynamics of the belief p_t in (1.14) as a compensated martingale of the likelihood x_t.

- If $\theta = 1$,

$$dx_t = \left(\lambda - \lambda_0 + \lambda_0 \log(\frac{\lambda_0}{\lambda})\right) dt + \log(\frac{\lambda_0}{\lambda})\left(dN_t - \lambda_0 dt\right). \tag{1.32}$$

- If $\theta = 0$,

$$dx_t = \left(\lambda - \lambda_0 + \lambda \log(\frac{\lambda_0}{\lambda})\right) dt + \log(\frac{\lambda_0}{\lambda})\left(dN_t - \lambda dt\right). \tag{1.33}$$

Unlike the dynamics of the belief p_t in (1.14) where the changes in belief over time and upon arrivals depend on the current belief p_t itself, the dynamics of the likelihood x_t exhibit a simple linear form. That is, whenever there is no Poisson arrival, the reputation x_t drifts up with slope $(\lambda - \lambda_0)$; upon each Poisson arrival, the reputation takes a downward jump equal to a constant $\log\left(\frac{\lambda}{\lambda_0}\right)$. Neither the upward drift nor downward jump is dependent on the current likelihood x_t. Figure 1.6 shows two typical sample paths of x_t when the agent is honest ($\theta = 1$) or strategic ($\theta = 0$) respectively.

As suggested above, the dynamics of the likelihood x_t yield a broken linear graph. The green broken line represents a typical trajectory of the likelihood when the agent is honest. With less frequent losses, the trajectory increases on average. Formally, x_t is a *submartingale* for an honest agent. The red broken line represents a typical trajectory of the likelihood when the agent is strategic. Since losses occur more frequently to a strategic agent, the trajectory decreases on average. That is, x_t is a *supmartingale* for a strategic agent. Since the game moves to the building phase

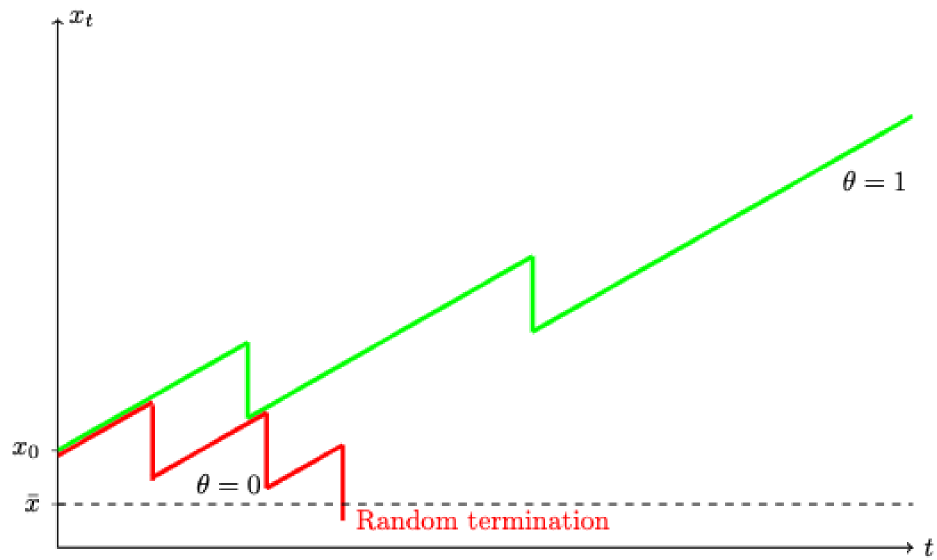


Figure 1.6: Sample Trajectories of Reputation

once the belief drops below the threshold $\bar{p}$, the dynamics of the likelihood x_t will also change after crossing the threshold $\bar{x} = \log\left(\frac{\bar{p}}{1-\bar{p}}\right)$.

Next, I explore the expected duration of the relationship. Let τ denote the termination time, which is an $\mathcal{F}_t^N$-stopping time. Define the expected duration $E[\tau]$ as the expectation of length that the relationship lasts. Though it is complicated to quantify $E[\tau]$, the next proposition provides a qualitative characterization of it.

Proposition 1.7.4.

$$\mathbb{E}[\tau|\theta = 1] = \infty,$$

$$\mathbb{E}[\tau|\theta = 0] < \infty.$$

This proposition describes the long run dynamics of the game. Given the principal's optimal strategy, the expected stopping time of the honest agent goes to infinity, while the expected stopping time of the strategic agent is finite. This means that an honest type may remain in the system eternally, while a strategic type is terminated almost surely.

1.8 Extensions

1.8.1 Bad Reputations

In Assumption 1.3.1, I assumed that the strategic type of agent is less efficient than the honest type, creating incentives for the strategic agent to conceal losses to imitate his honest counterpart. In this extension I explore the opposite case where the strategic type is more efficient than the honest type.

Assumption 1.8.1.

$$\lambda K < B < \lambda_0 K$$

Contrasting with the situation under Assumption 1.3.1, under Assumption 1.8.1 the principal would like to keep the strategic agent and terminate the honest one. In other words, the reputation of being an honest type is bad for the strategic type. Interestingly, the strategic type still has an incentive to conceal arrivals, but for the opposite reason. Rather than imitating the honest type, now the strategic type conceals arrivals to differentiate himself, which echoes Ely and Välimäki (2003).

Theorem 1.8.2. *There exists an MPBE $\mathcal{E}^*$ such that for some threshold $\bar{p} \in (0,1)$,*

$$\sigma^*(p) \equiv 0$$

$$\tau^* = \inf\{t : p_t > \bar{p}\}.$$

The equilibrium $\mathcal{E}^*$ has an intuitive interpretation. If the principal believes that the strategic agent conceals all losses, a single disclosed loss is a perfect signal of the agent being honest, which triggers termination. Anticipating this, the strategic agent will indeed prefer to conceal all his losses, resulting in continuation payoff $\frac{b-\lambda k}{r} \geqslant k$. Comparing this equilibrium with that of the baseline model, it is easy to see that the underlying incentives are opposite. In the baseline model, the strategic agent pools (imperfectly) with the honest type (i.e., to maintain a reputation as being the more efficient type). In the current variant of the model, the strategic agent is dynamically separating from the honest agent. Hence the principal learns his type asymptotically.

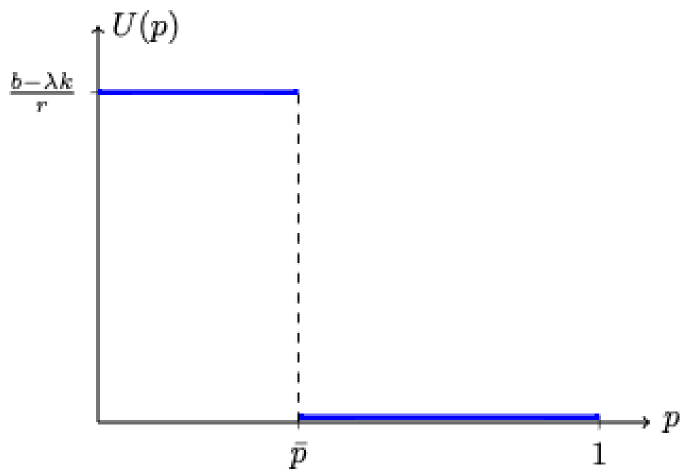


Figure 1.7: **Agent's Value Function when Fully Concealing Losses**

Figure 1.7 depicts the agent's value function, which is a step function featuring a discontinuity at $\bar{p}$, the principal's termination threshold. Formally,

$$U(p) = \begin{cases} \dfrac{b - \lambda k}{r}, & p \leqslant \bar{p} \\ 0, & p > \bar{p} \end{cases} \tag{1.34}$$

Since the strategic agent always conceals losses $\sigma^*(p) \equiv 0$, the posterior belief once a loss is disclosed (1.10) becomes

$$j(p) = \frac{\lambda_0 p}{\lambda_0 p + \lambda(1-p)\sigma} = \frac{\lambda_0 p}{\lambda_0 p + \lambda(1-p)0} = 1 \tag{1.35}$$

To verify that concealing all losses is the strategic agent's best response, (BR) is satisfied because

$$U(p) - U(j(p)) = \frac{b - \lambda k}{r} - 0 \geqslant \frac{(\lambda + r)k - \lambda k}{r} = k \tag{1.36}$$

In other words, The principal faces an exponential bandit problem with conclusive bad news. Suppose for some p such that $V(p) > 0$ and $V(j(p)) = 0$, the principal's value function follows

$$V'(p)p(1-p)(0-\lambda_0) = (\lambda_0 p + \rho)\, V(p) + \lambda_0 pK - B. \tag{1.37}$$

This is solved by

$$V(p) = \left[\frac{B-\lambda K}{\rho}(1-p) + \frac{B-\lambda_0 K}{\rho+\lambda_0}p\right] + C_2 p\left(\frac{p}{1-p}\right)^{\frac{\lambda_0+\rho}{-\lambda_0}}. \tag{1.38}$$

Again, the first term represents the productive value of the agent, while the second term corresponds to the informational value from learning the agent's type. Here $\bar{p}$ can be pinned down by the following value matching condition.

$$0 = V(\bar{p}) = \left[\frac{B-\lambda K}{\rho}(1-p) + \frac{B-\lambda_0 K}{\rho+\lambda_0}p\right] + C_2 p\left(\frac{p}{1-p}\right)^{\frac{\lambda_0+\rho}{-\lambda_0}} \tag{VM'}$$

In general, coordination issues arising from the interaction between the principal and agent may lead to the existence of multiple equilibria other than $\mathcal{E}^*$. Suppose the principal expects the agent not to fully conceal arrivals. Her posterior belief after observing a loss will not be as pessimistic and will not necessarily trigger immediate termination. Hence the agent may have incentives not to fully conceal losses. Therefore, other equilibria can be characterized as follows.

Theorem 1.8.3. *There exist multiple MPBE where* $\tau = \inf\{t : p_t > \bar{p}\}$ *and*

$$\sigma(p) = \begin{cases} 1, & 0 < p < p^* \\ 0, & p^* \leqslant p < 1 \end{cases} \tag{1.39}$$

for $\forall p^*$ *that satisfy* $0 < p^* < \bar{p} < 1$.

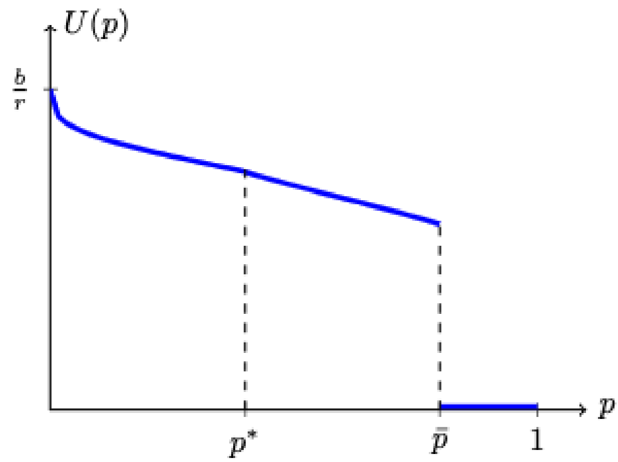


Figure 1.8: **Agent's Value Function under Bad Reputation**

1.8.2 Preemptive Actions

In the baseline model, the agent's actions are *reactive*, in the sense that only *after* the occurrence of a loss does the agent react by deciding whether to disclose it. In this section, I explore an extension of my baseline model where actions are taken *preemptively*. That is, rather than costly manipulating the public Poisson history, the agent can actually prevent losses from occurring by taking costly preemptive actions. I show that every MPBE of the baseline model is also an equilibrium of this extension, up to a parametric re-normalization. The underlying logic is that the reputation effect works only through the frequency of *public* losses, whether manipulated preemptively or reactively. The key is to establish equivalence between preemptive and reactive actions.

The model setup is identical to the baseline model, except for the agent's action. Now all losses that occur are publicly observable. At any point of time t, the strategic agent determines his level of preemptive action $(1 - \sigma_t) \in [0, 1]$ with a linear flow cost $c_t = c(1 - \sigma_t)$. The corresponding arrival rate of losses is $\lambda_t = \lambda\sigma_t$. If the agent chooses the maximal level of preemptive action $\sigma_t = 0$, he can prevent all losses by reducing λ_t to 0. To the contrary, if he chooses the minimal level of preemptive action $\sigma_t = 1$, his arrival rate of losses is $\lambda_t = \lambda > \frac{B}{K}$, which is inefficient for the principal by Assumption 1.3.1. First of all, the game from the principal's side is identical to the baseline model, so the dynamics of her value function continues to follow (1.16). Next, for the agent, let $\sigma(p_t)$ be the Markov strategy, then his Hamiltonian-Jacobi-Bellman (HJB) equation becomes

$$U(p_t) = \max_{\sigma(p_t)\in[0,1]} \left\{ (b - c_t)dt + \underbrace{\lambda\sigma(p_t)dt}_{\text{loss}}(1 - \alpha(p_t))U(j(p_t)) \right.$$
$$\left. + \underbrace{(1 - \lambda\sigma(p_t))dt}_{\text{no loss/termination}} e^{-rdt}U(p_{t+dt}) \right\}, \tag{1.40}$$

where $j(p_t) = \frac{\lambda_0 p_t}{\lambda_0 p_t + \lambda\sigma(p_t)(1-p_t)}$ is the posterior belief upon a disclosed loss. The agent chooses his level of preemptive action within a continuum $(1-\sigma_t) \in [0, 1]$. During the next infinitesimal dt, the first term represents the agent's net flow payoff; the second term corresponds to the event that a loss occurs; and The last term corresponds to no loss.

Theorem 1.8.4. *Every MPBE in the baseline model is also an equilibrium under preemptive actions iff $k = \frac{c}{\lambda}$.*

The equivalence condition $k = \frac{c}{\lambda}$ has an intuitive interpretation. The instantaneous cost c for the maximal level of preemptive action is equal to the expected flow cost λk from concealing losses. Moreover, interior levels of the preemptive action are equivalent to mixed strategies between disclosing and concealing losses under linear effort costs.

It is edifying to compare this extension of the model with Biais et al. (2010a). Both settings feature the principal incentivizing the agent to exert effort to reduce the arrival rate of losses. The difference lies in that incentives in Biais et al. (2010a)are provided by monetary payments while here effort is induced through reputation effects. In short, this extension can be viewed as a reputational counterpart of Biais et al. (2010a). The fact that there is no commitment power in my model renders it a continuous-time dynamic game instead of a dynamic contract setting.

1.8.3 Two Strategic Types

It is natural to extend the baseline model by making both types of agents strategic. That is, even the honest agent can incur a private cost of k to conceal a loss. One answer to what can happen in this case is as follows.

Proposition 1.8.5. *If both types of agent are strategic, there exists an equilibrium where all losses are concealed and the principal terminates surely upon any disclosed loss.*

This equilibrium structure resembles the worst equilibrium of the bad reputation model, where one disclosed loss signals an agent to be the principal's disfavored type, which triggers immediate termination. This creates sufficient incentives for both types of agent to conceal losses, as the net present value by remaining in the relationship outweighs the instantaneous cost of concealing a loss. However, the distinction is that disclosure of losses occurs with probability zero when both types of agents are strategic, while in the bad reputation case, losses will be disclosed by the honest type. If the honest type of agent were able to conceal losses, he would have done so. Unlike the bad reputation case, with two strategic types there is no disclosure of losses in the equilibrium, which means that any disclosure is off the equilibrium path. The principal assigns an off-equilibrium-path belief of 0, which creates incentives for an agent to conceal losses.

1.8.4 Concealing without Solving

In the baseline model I assume that once a loss is concealed, it is neither observable, nor has any direct economic impact on the principal. Therefore, to conceal a loss is equivalent to incurring it. This setting is fit for the car insurance, friendship and employment examples, however, there are cases where concealing a loss does not resolve it, but simply makes it unobservable. For example, government agencies may take measures to conceal adversary news when a public relation emergency occurs. In this case, the principal (the general public) still suffers from the loss, but is just unaware of it. In other words, rather than solving the problem, the action of concealing losses is now considered a maleficence with negative social impact. I show that under this setting, the building-milking equilibrium structure still holds. However, there is an interesting difference that the principal will start random termination at the myopic belief p^m.

Formally, once a loss occurs and the agent decides to conceal it, he incurs a fixed cost k to make it unobservable to the principal. However, the principal still bears a loss K without knowing its occurrence. The rest of the setting is identical to the baseline model. First of all, the principal and the agent's value function now become

$$V_t(\sigma, \tau | h^t, p_0) = \mathbb{E}^{\theta,\sigma,\tau}\left[\int_{s=t}^{\tau} e^{-\rho(s-t)}\left(Bds - Kd\tilde{N}_s\right)\middle| h^t, p_0\right]\mathbb{1}_{t\leqslant\tau}. \tag{1.41}$$

A similar equilibrium structure still holds where the agent builds his reputation when it is low, and milks it when it is high. However, the threshold between the building and milking phase is now different. Previously, the principal started random termination at belief $\bar{p}$, which is relatively more tolerant than the myopic threshold because of the informational value due to experimentation. However, when the agent is able to conceal losses, the strategic agent's behavior of imitating the honest agent offsets such effect. That is, it becomes too hard, even impossible for the principal to learn the agent's type.

1.9 Conclusion

It is common to conceal losses in relationships. Policyholders choose not to report traffic accidents to their insurance companies; employees endeavor to hide their mistakes from supervisors; and individuals refrain from asking their friends and colleagues for too many favors. In this paper I model such decisions in a reputational framework. Agents conceal losses to build their reputations in order to forestall termination, allowing them to enjoy relational benefits in the future.

In the baseline model, I explore the reputation effect when a strategic type of agent has incentives to imitate an honest type. The reputation effect arises from the fact that the principal is learning about the agent through observing his disclosed losses, and an agent with a bad enough history will be terminated. I show that the equilibrium of this game features a building phase and a milking phase. Specifically, the strategic agent builds his reputation by imitating the honest agent through partial disclosure of his losses when the principal's belief is low, and the strategic agent milks his reputation through full disclosure of his losses when the principal's belief is high. This result is reminiscent of the bad news case in Dilmé (2019), where people milk their reputation by switching to low action when it is high, and switching to high action to build reputation when it is low. Returning to the auto insurance example, because concealing losses only occurs in the building phase, the 15.2% unreported car accidents arguably come from the policyholders with bad driving records, a potentially testable implication of my model.

In the opposite case where the reputation of being honest is bad for the strategic agent, the principal terminates the agent whenever the belief exceeds a certain threshold. Here a stark equilibrium exists in which the strategic agent conceals all losses to avoid the penalty of being terminated upon the disclosure of a single identifying loss. I also explored an extension where the agent's action is preemptive rather than reactive, where the agent can reduce the arrival rate of losses by exerting effort over time. I showed that the equilibrium in the baseline model still holds for preemptive actions modulo a change of variables.

By introducing Poisson losses and allowing costly manipulation of the observable history, I generate novel reputation dynamics that feature building and milking phases. The dynamics are particularly rich because of the inconclusiveness of the Poisson signals, as both types of agent can experience losses. It is natural to consider extensions of my model by introducing more frictions through imperfect observability of outcomes, or less frictions through monitoring technologies. Yet another way to extend my model is to analyze costly manipulation regarding other underlying stochastic processes, such as Brownian motion. It would be particularly interesting in this regard to consider a Brownian model where a manager has incentives to divert funds when reputation concerns exist. I leave such extensions for future work.

2

Setbacks, Shutdowns and Overruns

We employ novel methods to investigate optimal project management in a setting plagued by unavoidable setbacks. The contractor can cover up delays from shirking either by making false claims of setbacks or by postponing the reports of real ones. The sponsor induces work and honest reporting via a soft deadline and a reward for completion. Late-stage setbacks trigger randomization between cancellation and extension. Thus the project may run far beyond its initial schedule, generating arbitrarily large overruns, and yet be canceled. Absent commitment to randomize, the sponsor grants the contractor more time to complete the project.

2.1 Introduction

Broad and expanding swaths of the modern economy are dedicated to the planning and execution of projects, "temporary endeavor[s] undertaken to create a unique product, service or result ... The development of software for an improved business process, the construction of a building or bridge, the relief effort after a natural disaster, the expansion of sales into a new geographic market all are projects."[1] Given the current and growing significance of this mode of production, it is important to understand its intrinsic characteristics and – in particular – how best to improve its efficacy. Indeed, the annals of project management are rife with jobs that ran notoriously over time and over budget, some of which were ultimately canceled by their sponsors resulting in little if any residual value.

For example, what might be "the most highly publicized software failure in history" (Goldstein, 2005) is the FBI's contracting debacle with SAIC to develop a virtual-case-file-(VCF) system. Irigoyen (2017) summarizes the VCF project failure as going through "significant management and implementation problems and cost overruns, which culminated in the cancellation of the project in 2005, with little to show for the USD170 million investment." The FBI Director at the time, Robert S.

[1] Excerpted from What is Project Management? (Project Management Institute, 2020)

Mueller, III, testified before a Congressional subcommittee that he was disheartened by "the setbacks which have plagued this program..."[2]

The FBI is hardly alone in its project management woes. For instance, "According to a 2017 report from the Project Management Institute, 14 percent of IT projects fail. However, that number only represents the total failures. Of the projects that didn't fail outright, 31 percent didn't meet their goals, 43 percent exceeded their initial budgets, and 49 percent were late" (Greene, 2019). The same pattern exists in large scale construction and industrial manufacturing, as illustrated by the high profile cases listed in Table 1

Table 1

Project	Time Overrun	Cost Overrun	Source
Boston's Big Dig	9 years	190%	Haynes (2007)
Sydney Opera House	10 years	1400%	Wild (2015)
Boeing 787 DreamLiner	3 years	200%	Shenhar et al. (2016)
Berlin's Brandenberg Airport	10 years	300%	Brandt (2020)

Most concerning of all, setbacks can cause a project to be canceled leaving the sponsor with huge bills and often nothing more. A prime recent example is South Carolina's V.C. Summer nuclear power plant construction project, canceled in 2017 after a series of major setbacks and cost overruns, saddling taxpayers with a bill of $9 billion and "nothing to show for it" (Lacy, 2019).

In this paper we argue that schedule overruns and project cancellations are not always the product of incompetence or inattention, but – at least to some degree – are unavoidable consequences of optimal project governance in the face of agency frictions. In particular, we introduce a model of project development in which setbacks arise naturally as part of the production process. Examples include discovering: adverse site conditions (construction), a design feature doesn't work as intended

[2] https://archives.fbi.gov/archives/news/testimony/fbis-virtual-case-file-system.

(manufacturing), or incompatibility of certain off-the-shelf subroutines (software engineering). Due to unforeseeable contingencies such as these, the amount of time and resources required to complete the project are necessarily uncertain.

In our model, as in practice, the sponsor (the principal) must hire a contractor (the agent) to run the project on her behalf. Both parties are risk-neutral, but the agent is protected by limited liability. Setbacks arrive randomly according to a Poisson process with known intensity. There is a flow cost of running the project, and the project is completed whenever a span of time $\bar{X}$ passes without the arrival of a setback. The first-best policy in this environment is straightforward. The project should be started and run until completed if and only if the value of the finished project to the sponsor exceeds the flow cost of operation times the expected duration.

The contractual setting we investigate is marked by both hidden actions and hidden states. The principal is unable to observe the progress of the project or the occurrence of setbacks herself, and must rely on unsubstantiated reports from the agent. However, delivery of the completed and working project is verifiable – the principal can use the software, fly the plane, or occupy the building once it is complete. Because the principal cannot observe the status of the unfinished project, the agent may surreptitiously divert the flow of operating capital to garner private benefits instead of advancing the project. The combination of hidden actions and hidden states gives the agent broad scope for committing moral hazard. Specifically, he may cover up the interruption of progress associated with resource diversion either by submitting false reports of setbacks or delaying the reports of real ones. Thus, the principal's problem is to write a contract, contingent only on the passage of time and project delivery, that induces the agent to work efficiently and report honestly.

The crucial incentive constraint is what we label the *No-Postponed-Setbacks* (NPS) condition. This constraint requires that whenever a setback occurs, the agent prefers to report it immediately rather than divert resources for any length of time and report it later. We show that (NPS) always binds under an optimal incentive scheme. This has several important implications. First, it implies that the agent also prefers not to cover up resource diversion with claims of false setbacks; that is, binding (NPS) is necessary and sufficient for incentive compatibility. Second, it allows us to fully characterize the optimal contract which turns out to correspond to what the U.S. General Service Administration labels as a *cost-plus-award-fee contract*.[3]

Under the optimal contract, the principal gives the agent a type of soft deadline, S_0, that we term a *time budget* for delivering the completed project and commits to pay the running cost until the project is completed or canceled. Operationally, the time budget counts down deterministically with calendar time unless and until a setback renders project completion in the remaining time infeasible; i.e., $S_{t-} < \bar{X}$. At this point a binary randomization procedure is invoked under which the project is either cancelled, $S_t = 0$, without payment to the agent or he is granted a minimally feasible schedule extension, $S_t = \bar{X}$. The probability of project extension is set to $\frac{S_{t-}}{\bar{X}}$ so that the expected duration of the contract at the point of randomization is the same as the value remaining on the time budget when the setback was reported. Subsequent reports of setbacks are treated analogusly. Hence, the endogenously determined initial value of the time budget, S_0, is the expected amount of time the agent is initialy awarded to complete the project. If one or more schedule extensions results in the project running longer than S_0, then an *overrun* (in time and operating cost) is said to occur. Given that the first-best policy is to run the project until it is finished, overruns are not in themselves problematic. However, a substantial

[3] "A cost-plus-award-fee contract is a cost-reimbursement contract that provides for a fee consisting of (a)a base amount (which may be zero) fixed at inception of the contract and (b) an award amount, based upon a judgmental evaluation by the Government, sufficient to provide motivation for excellence in contract performance." (The U.S. General Service Administration FAR 16.401). We comment on this implementation more formally in the discussion following Proposition 2.5.2 below.

overrun that ultimately ends in project cancelation obviously involves a large waiste of resources.

Under the optimal implementation via a time budget, the contract ultimately ends for one of two reasons, either because the agent delivers the completed project or because it is canceled by the principal. If the agent delivers the completed project, then he is paid a *linear* reward consisting of a fixed fee plus an incentive award proportional to any value remaining on the time budget.

while the optimal incentive contract – time budget and terminal payment scheme – induces the agent to work diligently and report honestly, it may, nevertheless, result in the type of unfortunate outcomes observed in our leading examples. In other words, schedule and cost overruns and cancellations that yield no useful output, are *features* of an optimal contract. These features obtain because randomization is necessary: if the contract had a deterministic deadline, then a late-stage setback would render project completion impossible and the agent would *shirk out the clock*. On the other hand, if there was some sequence of reports that enabled the project to run indefinitely, then the agent would make those reports and shirk forever. The only incentive compatible solution is random termination, which yields the possibility of both overruns and (more importantly) inefficient cancellations.

Our analysis utilizes novel methods to characterize the principal's value function. In particular, we identify two fundamental martingales and invoke the optional stopping theorem. The principal's expected payoff equals the probability of project completion times the first-best value of the project net of expected agency rents. The probability of project completion is increasing, concave, and approaches 1 as the length of the soft deadline, S, tends to infinity. On the other hand, agency rents increase linearly in S. Hence, there exists a unique optimal initial time budget $S_0 = S^*$ to assign to the agent at project inception.

Interestingly, the principal's value function is a concave polynomial with kinks at

$S = n\bar{X}$. These kinks imply that the optimal initial time budget S^* is an integer multiple of $\bar{X}$ for a non-negligible set of parameters – there are *focal contract lengths* that are multiples of the best-case duration. We use these observations to identify the conditions under which $S^* = \bar{X}$ is optimal; this is a *short-leash* contract in which the soft deadline equals the expected duration of the project and every reported setback results in cancellation with positive probability. Although a short-leash contract has an expected duration of $\bar{X}$, the support of the stopping time is unbounded due to the possibility of multiple project extensions. Hence, even when the principal commits to keep the agent on a short leash, arbitrarily large cost and schedule overruns occur with positive probability, and may still result in wasteful inefficient cancelation. Importantly, every optimal contract possesses a short-leash phase that is triggered whenever a setback occurs sufficiently late in the schedule; i.e., $S_{t-} < \bar{X}$.

After fully characterizing the optimal contract of our baseline model, we go on to consider two generalizations. First, as noted above, the optimal contract of the baseline model requires the principal to commit to randomized extension or cancellation with explicit probabilities. However, absent such commitment, the principal would strictly prefer to keep the agent working on the project by extending it rather than canceling it. So, we vary the baseline model by relaxing the assumption of full commitment and investigate a setting in which randomization is feasible but not verifiable. In this case, randomization by the principal is incentive compatible if and only if the agent, upon receiving an extension, himself randomizes between continuing to work and shirking away the granted time. While relaxing commitment is clearly harmful to the principal, we use the martingale methods for characterizing the value function to show that she optimally grants the agent a *longer* initial schedule in this setting. Intuitively, she does so in order to raise the likelihood that the project will be completed before the problematic short-leash phase of the schedule is ever reached. Put differently, granting larger S^* not only increases the chance the agent will successfully complete the project, it decreases the chance that lack of commitment will ever come into play. Under this version of the contract, the principal commits to let the agent run the project for an initial interval of time after which his continued employment may be regarded as *at-will.*

In the second generalization, we consider a setback environment that encompasses numerous possible variations of the baseline model. These include: multi-stage projects where setbacks occur within stages, fractional setbacks where the magnitude of each mishap is a random variable between 0 and 1, and setbacks of bounded scope where the amount of progress lost is capped. In this general environment, we show that the agency frictions associated with limited liability and private information regarding the state of progress are necessarily controlled by optimal contracts with a similar structure. Instead of a time budget that counts down deterministically absent a setback, the principal offers an expected time budget that counts down on average. However, the principal's incentive devices remain control of time (i.e. termination) and a payment upon completion.

Although we view our findings as primarily applied in scope, it is worth highlighting our most novel methodological contributions and insights. First, as noted above, identification of two economicly fundamental martingales (time budget plus calendar time and the agents utility) and use of the optional stopping theorem allows us to interpret the principal's value function in a remarkably clean and intuitive way. In particular, our analysis emphasizes the role of endowing the agent with a finite expected schedule (the time budget). Second, unlike the stationary *breakthroughs* setting underpinning most models with endogenous deadlines, we use an innovative itterative procedure to solve a model with a continuum of privately observed states (progress) and an unbounded number of potential *breakdowns* (setbacks). This necessitates optimization over a non-degenerate two dimensional space, namely the agent's continuation utility and the current state of progress. Application of the itterative procedure also reveals that the principal's payoff (as a function of the expected time remaining) is an nth-order concave polynomial on the interval $((N-1)\bar{X}, n\bar{X})$ for $n = 1, 2, \ldots$ with concave kinks at each transition point. More, the linearity of the value function on $(0, \bar{X})$ and the kink at $\bar{X}$ imply that discrete randomization between project cancelation and extension occurs with positive probability on the path of play. Finally, we explore relaxing the principal's ability to commit to such randomization, and analyze a version of the model in which it is incentive compatible for her to mix. We believe that these techniques are broadly applicable across a variety of dynamic incentive environments.

2.2 Related Literature

The literature on the optimal provision of incentives in dynamic environments is extensive and active. Pioneering articles responsible for moving it forward at various stages include Spear and Srivastava (1987b), Phelan and Townsend (1991), Quadrini (2004), Clementi and Hopenhayn (2006), DeMarzo and Sannikov (2006), Biais et al. (2007), Sannikov (2008), and Williams (2011).

Bergemann and Hege (1998) investigate venture capital financing in a discreet-time model where the arrival of revenues depends on whether the project is good or bad and whether the entrepreneur (agent) works or shirks. The dynamic agency costs may be high and lead to an inefficient early termination of the project. Toxvaerd (2006) considers a setting in which a finite number of observable arrivals are needed in order to complete a project. In his setting, the agent is risk averse and the optimal contract trades off optimal risk-sharing for incentive provision, but does not involve deadlines or inefficient termination. Biais et al. (2010b) analyze a model in which large observable losses may arrive via a Poisson process, and an agent must exert hidden effort in order to minimize the likelihood of their arrival. In contrast with these papers, the Poisson shocks in our model are privately observed by the agent and arise as an unavoidable consequence of the production process – that is they are *discoveries*. The potential for their occurrence essentially gives the agent cover to commit moral hazard; i.e., to make plausible excuses for why project completion has been delayed.[4]

Most closely related are four recent papers (one publication and three working papers) that explore the optimal deadline for a project in the context of dynamic agency. The published article is Green and Taylor (2016c) which in some ways explores the mirror immage of the setting investigated here. In Green and Taylor (2016c) the project is completed once two Poisson breakthroughs occur. The agent hired to run the project privately observes the occurrence of the first breakthrough, or what the authors call *progress*. As in our setting, the agent can surreptitiously divert the principal's flow of investment in the project for private benefit, which delays progress. However, once the agent reports a breakthrough, there is no turning back, which limits his ability for further manipulation. The optimal contract in Green and Taylor (2016c) features an intermediate deadline such that the agent is subjected to termination at a constant rate if he has not reported progress by that point. If the agent reports progress during this probationary phase, then he is given a relatively short deterministic amount of time to finish.

We consider a richer complementary setting where progress advances smoothly and in which a potentially infinite number of discrete setbacks may occur en route to completion. Thus our agent may repeatedly report the occurrence of false setbacks or repeatedly postpone reporting the occurrence of real ones, or any combination of these. The agent in Green and Taylor (2016c) is eventually subjected to smooth stochastic termination for *not reporting* a discrete breakthrough; whereas the agent in our model eventually faces a discrete probability of termination for *reporting* the

[4] Other papers that share some features with the environment we study include: Lewis (2012), Mason and Välimäki (2015), Rahmani et al. (2017), Vasama (2017), and Hoffmann et al. (2020).

occurance of setbacks. There is no consideration of limited commitment or short-leash contracts in Green and Taylor (2016c),and they do not employ our martingale methods, (although these techniques can be applied to their setting).

The critical difference between the two papers is in the type of enterprise they investigate. The projects analyzed in Green and Taylor (2016c) involve innovative breakthroughs (e.g., multi-stage scientific discoveries). By contrast, The kind of projects we consider feature deterministic progress subject to unforeseen setbacks (e.g., construction). The uncertain element in Green and Taylor (2016c) is precisely how to advance the project, whereas the uncertain elements in our setting are what exactly can go wrong. The optimal contracts across the two settings share some broad similarities, but they differ in important ways due to the differences in the underlying environments.

In an insightful working paper, Madsen (2021) studies how an organization should optimally manage a project of uncertain scope when advised by an expert with private information about the project's state who prefers to prolong his employment. In this model, a project turns from "good" to "bad" stochastically over time. The agent is a "advisor", who possesses private information regarding whether the project quality has changed and must be incentivized to report this. Mayer (2021) presents a dynamic contracting model in which a project succeeds if it survives until the completion date. While the project is in operation, the agent exerts unobservable precautionary effort in order to reduce the arrival rate of a privately observed failure shock that will kill the project before it reaches completion. As in Madsen (2021), the principal must provide incentives for the agent to report that the project has gone bad and should be terminated. Yet a third recent paper featuring a single privately observed transition is Curello and Sinander (2021). Similar in spirit but opposite in application to Madsen (2021), in this model, a technological breakthrough occurs exogenously at some random time witnessed only by the agent. The principal would like to adopt the innovation as soon as possible, but the agent prefers the *status quo* technology. Hence, the agent must be incentivized, through non-monetary means, to disclose the arrival of the innovation.

Our setting clearly differs from those studied in these three papers along a number of salient dimensions. Rather than the arrival of a single irreversable transition, our agent may observe numerous setbacks, none of which render project completion infeasible. Indeed, it is common knowledge from the outset that finishing the project is efficient. Our agent is not an advisor hired to monitor whether project quality has changed – His expertise resides in the ability to operate the project itself. This provides him with an informational advantage that the principal manages through implementation of a dynamic delivery-contingent contract.

2.3 The Model and The First-Best

2.3.1 The Project

A risk-neutral principal (she) hires a risk-neutral agent (he) over an infinite horizon to work on a project. The principal has deep pockets, and the agent has no wealth and is thus protected by limited liability. The project requires accumulated progress $\bar{X}$ before it is completed; $\bar{X}$ is the project's *scope*. As the agent works on the project, *progress* X_t accumulates deterministically. However, *setbacks* occur, following a Poisson process with arrival rate λ, which is the setback *frequency*. A setback at t resets progress from X_t to 0. When progress reaches $\bar{X}$, the project is complete and results in a monetary payoff of R to the principal. While the project is in operation, the principal must pay a flow cost of c to keep it running.

Two points are worth highlighting. First, for simplicity we assume that an incomplete project has no value to the principal. Second, setbacks result naturally as a result of unforeseeable contingencies, and, in particular, setbacks are not due to the negligence or indolence of the agent.[5]

The potential for moral hazard in this setting stems from the ability of the agent to surreptitiously divert the resource flow c to his own private benefit and cover the resulting cessation in progress by misinforming the principal about the occurrence or timing of setbacks. Formally, the project's true progress follows

$$dX_t = a_t(dt - X_t dN_t), \tag{2.1}$$

where $a_t \in \{0, 1\}$ denotes the agent's private action. $a_t = 0$ represents shirking, which corresponds to diversion of the resource stream c, while $a_t = 1$ represents working, which corresponds to using the funds to develop the project. Shirking yields the agent a private flow benefit of b:

Assumption 2.3.1. *$b < c$, so shirking (or diversion) is socially inefficient.*

Whenever the agent shirks, progress on the project remains constant; i.e., setbacks are discovered only if the agent is working. Both the principal and agent are perfectly patient and possess outside options of zero.[6]

2.3.2 The First-Best

If the agent's actions are publicly observable, then the principal can induce his compliance without incurring additional cost. Clearly, if it is worth starting the project in the first place, then it is worth running it until it is eventually completed.

[5] In other words, we model setbacks as discoveries resulting from working on the project, not from shirking. Unforeseeable contingencies are discovered that make the required time and resources uncertain. One interpretation is that there is a path to complete the project, and its length must be discovered through trial and error. Another is that there are many ex-ante equivalent paths by which the project may be completed, and each fails with probability $1 - e^{-\lambda \bar{X}}$. If, instead, setbacks occurred as a consequence of shirking, then the first-best could be implemented by setting a hard deadline of $\bar{X}$ and paying the agent a fixed award upon project delivery. In any case, we investigate the possibility of partial setbacks in Section 2.7.

[6] Our results hold if the principal and agent share a subjective discount rate, $r > 0$. The optimal contract is a time budget (soft deadline) with random extensions and termination, just as with $r = 0$. In fact, our economy with $r = 0$, including the principal's payoffs and policies, is attained as the limit of economies as $r \to 0$. We outline the solution to the contracting problem with discounting in Appendix B.5.

Suppose that the project is operated until completed and let F^{FB} be the value to the principal at inception. Because the time between setbacks is exponentially distributed with intensity λ, and the principal pays a flow cost c, we have the recursive relationship:

$$F^{\mathrm{FB}} = \int_0^{\bar{X}} \lambda e^{-\lambda X}(F^{\mathrm{FB}} - cX)dX + e^{-\lambda \bar{X}}(R - c\bar{X}), \tag{2.2}$$

where the integral in this expression corresponds to the possibility that a setback occurs before the project is finished, resetting progress X to 0, at which point the project must re-start. Integrating and solving yields

$$F^{\mathrm{FB}} = R - \frac{c}{\lambda}\left(e^{\lambda \bar{X}} - 1\right). \tag{2.3}$$

The first-best value is easily interpreted. Because the project is operated until it is complete, the principal eventually obtains R. Her expected cost when initiating the project is the flow cost c times the project's total expected duration $\frac{1}{\lambda}\left(e^{\lambda \bar{X}} - 1\right)$. It is straightforward to verify that expected duration increases in $\bar{X}$ and λ and that $\lim_{\lambda \to 0} \frac{1}{\lambda}\left(e^{\lambda \bar{X}} - 1\right) = \bar{X}$. Thus, a project with larger scope or higher frequency of setbacks has a longer expected duration, while a project for which setbacks never occur has a deterministic duration of $\bar{X}$.

It follows immediately that the first-best policy is to start the project and run it until completed if and only if the right side of (2.3) is positive. However, in the second-best, incentivizing the agent involves paying him rents, so a somewhat stronger assumption on the gross value of the project to the principal is required:

Assumption 2.3.2.

$$R - \frac{c+b}{\lambda}\left(e^{\lambda \bar{X}} - 1\right) > 0. \tag{2.4}$$

As we show in Corollary 2.5.4 below, this condition is both necessary and sufficient for the principal to be willing to hire the agent to run the project. Interestingly, although Assumption 2.3.2 implies that the principal is willing to incur the flow cost $c + b$ until the project is eventually completed, this is not the outcome implemented by an optimally designed contract.

2.4 Unobservable Progress and Incentive Compatibility

2.4.1 Contracts and Reports

The agent's expertise and decision to work on the project allow him to privately observe its state at each instant. The principal, however, must rely on status reports from the agent. We assume:

Assumption 2.4.1. *The principal cannot observe the agent's choice of action $a_t \in \{0, 1\}$, the state of the project X_t, or the occurrence of setbacks. The principal can observe project completion only upon delivery, which is contractually verifiable.*

This assumption allows the agent extensive latitude to commit malfeasance without detection. For instance, he could shirk for some time and then falsely claim a setback to cover up the lack of progress; or, following a real setback, the agent could shirk for a spell before reporting it.

However, the fact that the principal knows how long it takes to complete the project in the absence of a setback and that a completed project is verifiable does place some discipline on the agent's actions and reports. We assume

Assumption 2.4.2. *If the agent is verifiably detected misallocating resources or lying about the project's progress, then he is terminated at that point without severance.*

In other words, if the agent deviates from the principal's recommended actions, (shirks or lies), then the outcome generated must be consistent with *some* feasible path under the recommended actions. For example, the agent cannot shirk for an $\bar{X}$-interval of time without reporting a false setback or he will be fired for not delivering the completed project.

The agent makes a report of the project's current state, $\hat{X}_t$. Given the project's true evolution (2.1), reporting the path of $\hat{X}$ implicitly reports actions ($\hat{a}$) and setbacks ($\hat{N}$), with

$$d\hat{X}_t = \hat{a}_t(dt - \hat{X}_t d\hat{N}_t). \tag{2.5}$$

In fact, as long as the agent implicitly reports working, he needs only report the occurrence of setbacks with the understanding that "no news is good news" regarding progress.

The principal possesses two instruments for providing incentives so that the agent faithfully runs the project and honestly reports progress and setbacks. She can cancel the project prior to completion (i.e., *fire* the agent), or she can provide the agent with a reward when the completed project is delivered. We also allow the principal to provide the agent with rewards based on reported project status; however, we will show that because both parties are risk-neutral and are equally patient, it is without loss of generality to backload all monetary payments into a single reward granted upon successful completion.

Definition 2.4.3 (Contract)**.** *Denote the probability space as* $(\Omega, \mathcal{F}, P)$*, and the filtration as* $\{\mathcal{F}_t\}_{t\geqslant 0}$ *generated by the history of reports* $\{\hat{X}_t\}_{t\geqslant 0}$*. Contingent on the filtration, a contract specifies a stopping time* τ *when the contract is terminated (by completion or cancellation), a terminal reward* K_τ *to the agent, and cumulative intermediate rewards* $\{C\}_{t\geqslant 0}$*. All quantities are assumed to be integrable and measurable under the usual conditions.*

Contracts are characterized using the agent's continuation utility as the state variable. Given a contract, the agent chooses actions $\{a_t\}_{t\geqslant 0}$ and reports $\{\hat{X}\}_{t\geqslant 0}$. His continuation utility is the expected value of the reward from project completion plus private benefits from any shirking:

$$W_t^{a,X} = E_{a,X}\left[\int_t^{\tau} b(1-a_s)ds + \int_t^{\tau} dC_s + K_\tau \middle| \mathcal{F}_t\right]. \tag{2.6}$$

The principal's objective function F_t is the expected value of the benefit from a completed project net of the expected operating cost and the expected reward to the agent:

$$F_t^{a,X} = E_{a,X}\left[-\int_t^{\tau} c ds + R_\tau - \int_t^{\tau} dC_s - K_\tau \middle| \mathcal{F}_t\right], \tag{2.7}$$

where $R_\tau = R$ if the project is completed and 0 if it is not.

Before we characterize general incentive compatibility, we can simplify the contracting space:

Lemma 2.4.2 (High Action and Prizes)**.** *The principal will always choose to implement the high action ($a_t = 1$). The principal will pay the agent only upon successful completion of the project ($K_\tau > 0$ iff success; $dC_t = 0$).*

The first result holds because it is always more efficient to award the agent intermediate consumption than to implement inefficient shirking. The second result holds because both the principal and agent are equally patient and so payments can always be delayed.

A contract is incentive compatible if the agent chooses the high action and accurately reports the status of the project:

Definition 2.4.4 (Incentive Compatibility)**.** *A contract is incentive compatible if the agent maximizes his objective (2.6) by choosing $a_t = 1$ and $\hat{X}_t = X_t$ for all $t \geqslant 0$.*

Then, in an incentive compatible contract, the agent's continuation utility is the expected value of the terminal prize. The principal's utility is the expected payoff of the project minus the running cost and expected prize.

A contract is optimal if it maximizes the principal's objective function within the class of feasible, incentive compatible contracts:

Definition 2.4.5 (Optimal Contract)**.** *A contract is optimal if it maximizes the principal's objective function (2.7) over the set of contracts that 1) are incentive compatible, 2) grant the agent his initial level of utility W_0, and 3) honor limited liability, $W_t \geqslant 0$.*

2.4.3 Incentive Compatibility

In this subsection, we introduce a necessary incentive constraint, the *No-Postponed-Setbacks* (NPS) condition. This constraint provides the necessary incentives for the agent to report any setbacks immediately, rather than delaying the report and shirking in the meantime. Later, we will show that this constraint is also sufficient to prevent any other deviation.

We now summarize the evolution of the agent's continuation utility, W:

Lemma 2.4.4 (Incentive Compatibility)**.** *Given any contract and any sequence of the agent's choices, there exists a predictable, finite, non-negative process J_t $(0 \leqslant t \leqslant \tau)$ such that W_t evolves according to*

$$dW_t = J_t\left(\lambda dt - dN_t\right). \tag{2.8}$$

Between setbacks, J is deterministic. A necessary condition for incentive compatibility is that between setbacks, we have:

$$J_{t+\delta} \geqslant J_t + b\delta + \int_0^{\delta} \lambda J_{t+s} ds, \quad \forall \delta \in (0, \bar{X} - X). \tag{NPS}$$

The contract is terminated if $W_t = 0$.

The agent's continuation utility under an incentive compatible contract is a martingale. Hence, it drifts up deterministically at rate $\lambda J_t dt$ as the agent accumulates progress toward project completion and earning the prize, but it jumps down by J_t whenever there is a setback that wipes out the accumulated progress. To understand the (NPS) incentive constraint, suppose the state of progress is $X_t \in (0, \bar{X})$ when a setback occurs. Consider two possible paths the agent might take at this point:

- [Work] The agent reports the setback immediately, and then works as desired.
- [Shirk] The agent delays reporting the setback and shirks for time $\delta \leqslant \bar{X} - X_t$. Then, he reports a (bigger) setback and works as desired.

A critical feature of the shirk path is that after the postponed setback is finally reported, the agent has dissipated his persistent private information about the status of the project. The agent and the principal both believe that X_t is 0 and have the same information about the project and contract going forward. Thus, the agent's continuation utility and the principal's beliefs about it coincide.

Now, we compare the two paths, with working first. Since working is optimal, the agent's continuation utility is a martingale, and we have[7]

$$E\left[W_\tau\right] - W_{t-} = -J_t \ . \tag{2.9}$$

The only difference between the agent's expected utility when the project ends and his utility at $t-$ is the jump down from reporting the setback and realizing the loss in progress.

Next, we consider shirking. In this case, the change in continuation utility is

$$E\left[W_\tau\right] - W_{t-} = \int_0^\delta \lambda J_{t+s} ds - J_{t+\delta} \ , \tag{2.10}$$

with an additional private benefit due to shirking of $b\delta$. The first term accounts for the upward drift in the principal's beliefs about the agent's continuation utility

[7] We adopt the standard convention of indexing the value of a process immediately prior to a jump with $t-$.

as he (falsely) reports progress while shirking; and the second term captures the jump down in the principal's beliefs about the agent's continuation utility when he finally stops shirking and reports a larger setback than actually occurred. Adding the private benefit $b\delta$ to (2.10), comparing to (2.9), and re-arranging, we obtain (NPS). This constraint simply says that the value from the work path is at least as high as the value from the shirk path; i.e., the agent prefers to face the music immediately rather than to postpone reporting a setback.

The (NPS) constraint requires that the agent's loss of utility between setbacks is at least equal to the time he could have spent shirking between them. Thus, there is a round trip penalty imposed on the agent between any two truthfully reported setbacks. We call this a "round trip" because the agent goes from $X = 0$ through some path and back to $X = 0$. To see the penalty, imagine that the agent starts at time t with $X_t = 0$ and works until $t + \delta$ and $X_{t+\delta} = \delta$ when he receives a setback. With truthful reporting, (NPS) implies that the agent's continuation utility is

$$W_{t+\delta} = W_t + \int_0^\delta \lambda J_{t+s} ds - J_{t+\delta} \leqslant W_t - J_t - b\delta = W_t - b\delta, \tag{2.11}$$

where we have used in the final step that a setback at $X = 0$ has no effect and no penalty ($J_t = 0$ if $X_t = 0$). If the (NPS) constraint binds, then the agent's utility-drop between setbacks grows linearly with time: after each setback, the agent is re-started with a continuation utility that is lower by the amount of time he could have spent shirking.

2.4.5 Termination and Randomization

We now consider how the agent is terminated when the project is still incomplete.

First, termination is required. Imagine not; then, there is some path of X that would result in the project being funded without end. However, the agent could simply mimic that path with his reports while shirking, and thus obtain infinite utility. The agent would prefer this to any incentive compatible path.

Put differently, the NPS round trip penalty implies that between any two setbacks, the agent loses continuation value at least proportional to the elapsed time. However, termination must occur if $W_t = 0$ because, given limited liability, termination is the only way for the principal to deliver $W_t = 0$. Because the agent's initial utility W_0 is finite, the agent must also eventually run out of time.

Second, termination is random and not deterministic. We reason based on the NPS round trip utility penalty (2.11) and the fact that the agent has limited liability. Imagine that a setback occurs at t resulting in $W_t \in (0, b\bar{X})$. In this case, if the agent continues to work and makes progress $\delta > \frac{W_t}{b}$ before suffering another setback, then the drop in his continuation utility required by the (NPS) constraint would result in $W_{t+\delta} < 0$, which is not feasible. The agent would prefer to shirk rather than to report the second setback. What can the principal do about this? One option is simply to terminate the contract at t and give the agent a severance payment of W_t. However, allowing the agent to shirk or giving the agent a severance payment is never optimal (Lemma 2.4.2).

Instead, there is a better alternative: the principal can use randomization to either fire the agent without severance (generating $W_t = 0$) or increase W_t enough to restore incentive compatibility. Randomization preserves the agent's expected continuation utility – and thus the principal's expected payout to the agent – but (unlike paying severance) it allows for a positive probability that the project will be completed.

To preserve consistency and incentive compatibility, the agent must be randomly assigned a utility equal to 0 or greater than $b\bar{X}$. We assume now (and verify in Proposition 2.5.4) that the principal's value function is concave so that she wishes to use the least disperse randomization procedure possible.

Lemma 2.4.6. *If $W_t < b\bar{X}$ following a reported setback, then incentive compatibility and concavity of the principal's value function require that the agent is assigned utility of $b\bar{X}$ with probability $p = \frac{W_t}{b\bar{X}}$ or utility 0 with probability $1 - p$.*

2.5 Optimal Contract: A Time Budget

2.5.1 The Principal's Problem

The concept of a *time budget* plays a crucial role in the implementation of an optimal contract. A time budget is a stochastic deadline that counts down deterministically absent a setback and may jump (due to extension or termination) with the report of a setback. Formally we have the following:

Definition 2.5.1 (Time Budget)**.** *A time budget is a non-negative process S_t satisfying $dS_t = -dt$ absent a reported setback. The principal initially grants the agent S_0 and then cancels the project iff $S_t = 0$ and the project has not succeeded. Thus, a time budget creates a random stopping time τ when the contract is terminated (on the events of project completion or cancellation).*

Our first result is that the (NPS) constraint is binding and sufficient, and thus the optimal contract can be implemented with a time budget and prize-upon-completion. In other words, the agent's loss of utility between setbacks evolves linearly with time, and this is enough to generate full effort and prevent any mis-reporting by the agent. Intuitively, the best the agent can do by lying to the principal is to gain time to divert resources from the project (shirk), and reducing the agent's expected prize by the amount he could have diverted is enough to deter such malfeasance. In turn, this means that the optimal contract can be implemented as a time budget:

Proposition 2.5.2. *The optimal contract has the following properties:*

i. The (NPS) constraint binds, and the agent's utility penalty for reporting a *setback J_t is a function of X_t only:*

$$J_t = J(X_t) = \frac{b}{\lambda}\left(e^{\lambda X_t} - 1\right). \tag{2.12}$$

ii. The contract can be implemented with a time budget which is set such that $bS_0 = W_0$ and the agent is terminated if $S_t = 0$ and the project is not delivered. If $S_{t-} < \bar{X}$ and a setback is reported, then S_t is set to either 0 with probability $1-p$ or $\bar{X}$ with probability p where

$$p = \frac{S_{t-}}{\bar{X}}. \tag{2.13}$$

These probabilities imply that $S_t + t$ is a martingale.

iii. The agent's continuation utility under the optimal contract is

$$W_t = bS_t + J(X_t). \tag{2.14}$$

If the agent completes the project at time τ, he receives a reward of

$$K_\tau = bS_\tau + \frac{b}{\lambda}\left(e^{\lambda \bar{X}} - 1\right). \tag{2.15}$$

This result says that it is optimal for the principal to assign the agent an amount of time S_0 to complete the project. The agent works on the project and makes continuous progress reports including the occurrence of any setbacks. If $S_t < \bar{X}$ remains on the clock and a setback is reported, then there is not enough time remaining to complete the project. At this point, the randomization procedure is invoked in which the project is either canceled with probability $1 - \frac{S_{t-}}{\bar{X}}$, or the schedule is extended to $S_t = \bar{X}$ with the complementary probability. Any subsequent setbacks are treated analogously, until the project is ultimately either canceled or completed.

The payment to the agent for delivering the completed project at time τ consists of a fixed reward $\frac{b}{\lambda}\left(e^{\lambda \bar{X}} - 1\right)$ plus a *bonus* that is proportional to the remaining time on the schedule bS_τ. The bonus term is the inverse of the NPS constraint: because the agent's utility declines between setbacks, he must receive an incentive payment if the project succeeds before he reports another setback. Note that if the agent ever receives an extension (setting the time budget back to $S_t = \bar{X}$) then the project cannot be completed early, and the agent will just receive the fixed reward.

The fixed payment, which is equal to $J(\bar{X})$, is calibrated so that the agent is indifferent between working on the project when $X = 0$ and shirking; i.e., it is the minimal reward for project completion that will induce the agent to start working with no progress in hand. As the agent works and X increases, the likelihood of project completion without a setback rises, and the agent's corresponding utility from the partially completed project, $J(X)$, accordingly grows exponentially. Hence, if a setback occurs at progress level X_t, the agent's expected utility drops by $J(X_t)$, but if he manages to push the project from 0 to $\bar{X}$ without incurring a setback, then he receives the fixed payment of $J(\bar{X})$ plus any bonus he is due for early completion.

As noted in the introduction, the implementation of the optimal incentive mechanism characterized in Proposition 2.5.2 is a cost-plus-award-fee contract. In particular, the principal commits: (i) to cover the operating cost of the project $c\tau$, (ii) to pay a fixed fee $J(\bar{X})$ upon project completion, and (iii) to pay an incentive award bS_τ for early delivery.

It is worth noting that optimal incentives can be implemented with less stringent reporting requirements. Rather than requiring continuous progress reports, at project inception the principal can announce a *soft deadline* $T = S_0 - \bar{X}$ and then commit to fund the project until this date *no-questions-asked*. If the agent delivers the completed project at $\tau \leqslant T$, he receives K_τ as given in the proposition. Once the soft deadline has passed, the principal requires setbacks to be reported, and she follows the random termination procedure specified in Proposition 2.5.2 from that point on.[8]

[8] The general structure outlined in Proposition 2.5.2 remains qualitatively intact under various

2.5.2 The Initial Value of the Project

Given the form of the optimal contract as a time budget and terminal reward, we can derive the principal's value function $F(S, X)$. We are most interested in her valuation of a given time budget when starting from scratch: $F(S, 0)$.[9] As we will see, the value function is not everywhere differentiable with respect to S; instead the value function has kinks at multiples of $\bar{X}$, and the optimal time-budget has *focal lengths* at those points. Instead of the usual techniques involving PDEs and dynamic programming, we will use two martingales and the optional stopping theorem. This allows for a more fundamental understanding of the contract and its characteristics.

In order to simplify the principal's problem, we will define two useful auxiliary functions, with slight abuses of notation.[10] The first function, $\pi(S)$, is the probability that the agent is eventually successful in completing the project. The second function, $\sigma(S)$, is the expected remaining time until contract termination (with either success or failure). Both functions assume that the principal starts with $S_t = S$ and $X_t = 0$. Since the project can be completed early, the expected time to completion is weakly less than the time budget ($\sigma(S) \leqslant S$). We have

$$\pi(S) = \mathrm{E}_t \left[1_{X_\tau = \bar{X}} | S_t = S, X_t = 0\right] \tag{2.16}$$

$$\sigma(S) = \mathrm{E}_t \left[\tau - t | S_t = S, X_t = 0\right], \tag{2.17}$$

where τ is the contract stopping time (Definitions 2.4.3 and 2.5.1). These two functions capture the loss to the principal from the second-best contract. In the first-best, the agent runs the project as long as necessary to complete it. In the second-best,

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alternative model specifications. For example, suppose for any setback that occurs, there is a probability that it is an un-fixable dead end. Then, the optimal contract is unchanged except that the agent receives a severance payment equaling bS_{t-} for the report of a fatal setback. In Sections 2.6 and 2.7 we discuss additional modifications that carry useful economic insights: lack of commitment and partial setbacks.

[9] For $X > 0$ we have $F(S, X) = \int_0^{\bar{X}-X} \lambda e^{-\lambda t} \left(F(S-t, 0) - ct\right) dt + e^{-\lambda(\bar{X}-X)}(R - c(\bar{X} - X))$.

[10] These methods are described without discounting in the text; however the same procedure can be followed with discounting, as we outline in Appendix B.5.

the principal imposes a stochastic time limit (the time budget) which reduces both the probability of success and the time allowed.

We can now significantly simplify the principal's problem. From the principal's and agent's payoffs (2.6 and 2.7), and using the auxiliary functions we have just defined, we have

$$F(S, X = 0) = \pi(S)R - W(S, X = 0) - c\,\sigma(S), \tag{2.18}$$

where $W(S, X = 0)$ is the agent's continuation value. From the incentive compatibility condition (2.14), we have $W(S, X = 0) = bS$, leaving us with

$$F(S, X = 0) = \pi(S)R - bS - c\,\sigma(S). \tag{2.19}$$

This gives us a very intuitive representation of the principal's value of the project. It is the expected reward, minus the expected agency rents granted the agent by the time budget, and minus the expected direct running cost.

We continue by applying the optional stopping theorem[11] to two salient martingales in order to relate the probability of success to the expected time remaining. First, the agent's continuation utility $W_t = bS_t + J(X_t)$ is a martingale that can only stop at two boundaries, project success and failure. Second, the randomization probabilities (2.13) imply that $S_t + t$ is a martingale, so the optional stopping theorem shows that $S_0 + 0 = \mathrm{E}\left[S_\tau + \tau\right]$. Thus we have

$$bS_0 = W_0 = \mathrm{E}\left[W_\tau\right] = \mathrm{E}\left[bS_\tau + J(X_\tau)\right] = bS_0 + \mathrm{E}\left[-b\tau + J(X_\tau)\right] \tag{2.20}$$

$$0 = -b\sigma(S_0) + \mathrm{E}\left[J(X_\tau)\right] \tag{2.21}$$

$$= -b\sigma(S_0) + \pi(S_0)\mathrm{E}\left[J(\bar{X})\right] + (1 - \pi(S_0))\mathrm{E}\left[J(0)\right] \tag{2.22}$$

$$= -b\sigma(S_0) + \pi(S_0)J(\bar{X}), \tag{2.23}$$

where the first line follows from the optional stopping theorem, the second line follows from the definition of $\sigma(S)$, the third line from the optional stopping theorem, and

[11] As a reminder, Doob's optional stopping theorem shows that the expectation of a martingale at a stopping time is equal to the current value of the martingale. Our setting fits the version of this result given in Theorem 5.3.1 of Cohen and Elliott (2015).

the fourth line from $J(0) = 0$ (2.12). Re-arranging and generalizing to any time with $X_t = 0$, we have

$$\pi(S) = \frac{b\sigma(S)}{J(\bar{X})} = \frac{\lambda\,\sigma(S)}{e^{\lambda\bar{X}} - 1}. \tag{2.24}$$

The intuition for this result comes from the martingale property of the agent's continuation utility. The agent's utility $W_t = bS_t + J(X_t)$ counts up as progress is obtained and down as time passes. Those two changes must cancel out on average to maintain incentives. Thus, the passage of time is exactly matched by an increase in the probability that the agent receives the constant part of his reward, $J(\bar{X})$ – and the average time to completion must be proportional to the probability of success.

When S is short, the project is very likely to be canceled and both $\pi(S)$ and $\sigma(S)$ are relatively small. On the other hand, a long time budget implies a small probability of cancelation, and in the limit as $S \to \infty$, the project is never canceled so that the expected time to termination corresponds to the expected time to completion; i.e., $\lim_{S\to\infty} \pi(S) = 1$. Plugging (2.24) into (2.19), we can simplify the principal's value function further:

Proposition 2.5.3. *The principal's initial valuation of a given time budget S is*

$$F(S,0) = \left(\frac{\lambda R}{e^{\lambda\bar{X}} - 1} - c\right)\sigma(S) - bS \tag{2.25}$$

$$= \pi(S)\left(R - \frac{c}{\lambda}\left(e^{\lambda\bar{X}} - 1\right)\right) - bS. \tag{2.26}$$

This value function is concave and hump-shaped in S.

This result has an intuitive interpretation. Plugging in the first-best value (2.3 and 2.26), we can write

$$F(S,0) = \pi(S)F^{\text{FB}} - bS, \tag{2.27}$$

Written this way, we see that asymmetric information harms the principal for two related reasons. First, $\pi(S)$ is the probability that the agent eventually delivers a completed project starting with an initial time budget of S – the probability that the project is not inefficiently canceled. The second way in which the principal is harmed is that she has to pay the expected agency rent of $W_t = bS_t$.

Intuitively, the larger the time budget S, the more likely it is that the agent will complete the project; i.e., $\pi(S)$ is increasing and $\lim_{S\to\infty} \pi(S) = 1$. But, of course the principal will not commit to pay the agent an unboundedly large rent to obtain a payoff bounded by F^{FB}. In particular, she faces a tradeoff when setting the initial time budget between higher probability of project completion, $\pi(S)$, and paying higher agency rents, bS. This tradeoff manifests in the hump shape of the value function $F(S,0)$. At low levels of S both the principal and agent prefer a larger time budget. However, as S grows, diminishing marginal returns to the probability of project completion $\pi(S)$ are eventually dominated by the linear agency cost bS, and $F(S,0)$ peaks at some critical value S^* beyond which it decreases.

2.5.3 The Value Function Characterization

We now characterize the expected duration of the contract $\sigma(S)$ and use that to analyze the principal's value function.

In the randomization region ($S_t \leqslant \bar{X}$), $X_t = 0$ triggers immediate randomization and implies that the project is never completed with extra time remaining. The randomization probabilities (2.13) imply that the time budget is also the expected time remaining, $\sigma(S) = S$. This feature yields the following result:

Lemma 2.5.4. *For $S \leqslant \bar{X}$, we have $\sigma(S) = S$ and*

$$F(S,0) = S\left(\tfrac{R}{J(\bar{X})} - c - b\right) = S\left(\frac{\lambda R}{e^{\lambda \bar{X}} - 1} - c - b\right). \tag{2.28}$$

It is also possible to solve explicitly for $\sigma(S)$ if $S > \bar{X}$ by applying an iterative procedure that is detailed in the appendix. Using (2.25), we can then obtain the principal's value function in closed form as a piecewise polynomial in S. The important properties of $\sigma(S)$ are summarized as follows:

Proposition 2.5.4 (Value Function Properties). *$\sigma(S)$ is continuous in S. For all $n \geqslant 1$, we have*

(i) For $S \in ((n-1)\bar{X}, n\bar{X}]$, $\sigma(S)$ is a concave, increasing polynomial of order n.

(ii) $\lim_{S\to\infty} \frac{\partial}{\partial S}\sigma(S) = 0$ and $\lim_{S\to\infty} \sigma(S) = \frac{1}{\lambda}\left(e^{\lambda\bar{X}} - 1\right)$.

(iii) $\lim_{S\uparrow n\bar{X}} \frac{\partial}{\partial S}\sigma(S) > \lim_{S\downarrow n\bar{X}} \frac{\partial}{\partial S}\sigma(S) > 0$.

The first observation is a straightforward implication of the iterative procedure: each round adds a higher order term but the function is always concave and increasing. Combining this observation with Lemma 2.5.4 we see that

$$\sigma(S) \begin{cases} = S, & \text{if } S \leqslant \bar{X} \\ < S, & \text{if } S > \bar{X}. \end{cases}$$

In other words, the expected duration of the project at inception, $\sigma(S_0)$, is weakly less than the initial time budget, S_0, allotted to the agent. $\sigma(S_0) < S_0$ means that the principal builds some slack or *slippage* time into the contract: she initially gives the agent more expected time to complete the project than its actual expected duration. This implies that whenever the project is extended such that $\tau > S_0$, a schedule and cost overrun must have happened. That is, the project runs longer than its initial expected duration of $\sigma(S_0)$ and costs more than the initial estimate of $c\sigma(S_0) + bS_0$.

The second observation in Proposition 2.5.4 is a restatement of the fact that the marginal benefit from increasing S becomes arbitrarily small as $\pi(S)$ goes to 1. This along with the fact that the marginal cost remains constant at b implies that there exists a unique S^* at which $F(S, 0)$ is maximized.

The third observation identifies kinks in the value function at positive integer multiples of $\bar{X}$. One implication is that it is optimal for the principal to assign an initial time budget equal $n\bar{X}$ for a non-negligible set of parameters. In other words, the time budget has *focal lengths* that are integer multiples of the minimal time to completion. In practice, project managers often report three estimates for completion time: Best-case, worst-case, and most-likely. A particularly salient situation is when the principal holds the agent to the best-case scenario in expectation, $S^* = \bar{X}$, and responds to any reported setback by either canceling the project or resetting the clock. We analyze such a contract in the following subsection.

2.5.5 A Short-Leash Contract

Here, we use the first kink in the principal's value function to derive the conditions under which it is optimal for her to set an initial time budget of $S_0 = S^* = \bar{X}$. We call these *short-leash* contracts. Applying the iterative procedure in the appendix gives

$$\lim_{S \uparrow \bar{X}} \frac{\partial}{\partial S}\sigma(S) = 1 > 1 - e^{-\lambda\bar{X}} = \lim_{S \downarrow \bar{X}} \frac{\partial}{\partial S}\sigma(S). \tag{2.29}$$

Using this observation to evaluate the derivative of the value function in (2.25) directly yields the following result:

Corollary 2.5.5. *The optimal contract involves the minimal initial time budget of $S_0 = S^* = \bar{X}$ and a fixed prize of $\frac{b}{\lambda}\left(e^{\lambda\bar{X}} - 1\right)$ iff*

$$\frac{c+b}{\lambda}\left(e^{\lambda\bar{X}} - 1\right) < R < \frac{c + \frac{b}{1-e^{-\lambda\bar{X}}}}{\lambda}\left(e^{\lambda\bar{X}} - 1\right). \tag{2.30}$$

The first inequality is a restatement of Assumption 2.3.2 (that the project is contractually feasible), and it ensures that $F(S,0)$ is increasing for $S < \bar{X}$. The second inequality then ensures that $F(S,0)$ is decreasing for $S > \bar{X}$. Hence, when (2.30) holds, the kink in the value function at $S = \bar{X}$ corresponds to the peak and it is optimal for the principal to set an initial time budget of $S^* = \bar{X}$. In other words, she should keep the agent on a short leash, granting him in expectation only the minimal amount of time necessary to complete the project, requiring him to report every setback, and canceling the project with positive probability each time one is reported.

The key parameters in (2.30) are b, the agent's per-period benefit from shirking, and λ, the expected frequency of setbacks. If b is too large, then the left inequality in (2.30) fails and moral hazard precludes the project from ever getting off the ground (i.e., Assumption 2.3.2 is violated). On the other hand, if b is too small, then the right inequality fails. In this case, moral hazard is less concerning, and the principal prefers to give the agent more than the minimal initial time to complete the project.

Intuitively, a short-leash contract is optimal if λ is sufficiently small. To see this, we calculate the limit as $\lambda \to 0$ in (2.30) to obtain

$$(c+b)\bar{X} < R < \infty,$$

which holds by Assumption 2.3.2. Hence, when the expected frequency of setbacks is small enough, the principal allows no slack in the schedule, committing to only the minimal rent of $b\bar{X}$ necessary to induce the agent to work on the project. This makes sense – when setbacks are relatively unlikely, the project is completed in the minimal time, $\bar{X}$, with high probability. In other words, cancellation is of little concern, and the principal prefers to promise the agent only the minimal rent necessary.

At inception of a short-leash project, the expected duration is $\sigma(S = \bar{X}) = \bar{X}$, and the expected cost to the principal is $(c+b)\bar{X}$. However, if the agent reports a setback with S_t left on the schedule, then he is granted an extension of $\bar{X} - S_{t-}$ (i.e., the clock is reset) with probability $\frac{S_{t-}}{\bar{X}}$. Hence, the support of the stopping time τ is unbounded, implying that the project may run arbitrarily long, incur arbitrarily large costs, and yet may still be canceled.

In fact, it is possible to determine explicitly the probabilities of extensions, cancellations, and overruns, and we do so for a short-leash contract. Figure 2.1 plots the probabilities of these events. Define $\mu \equiv \lambda \bar{X}$ to be the expected number of setbacks experienced while the project is in operation. Then,

1. $P_{\text{OT}}(\mu) = e^{-\mu}$ is the probability that the project is completed on time (left panel, red dotted curve). The value of this function decreases from 1 to 0 because as the expected number of setbacks rises, the probability that none occur falls. When setbacks are a virtual certainty, the project cannot be completed on time.

2. $P_{\text{EC}}(\mu)$ is the probability that the project is canceled early, before its initial expected duration of $\bar{X}$ (left panel, blue solid curve).[12] This function increases from 0 because as the expected number of setbacks rises, it becomes ever more likely that the project will not survive the requisite randomizations before time $\bar{X}$ has elapsed. Indeed, $\lim_{\mu \to \infty} P_{\text{EC}}(\mu) = 1$ because a steady stream of setbacks must result in early project cancellation for any $\bar{X} > 0$.

3. $P_{\text{OR}}(\mu) = 1 - P_{\text{EC}}(\mu) - P_{\text{OT}}(\mu)$ is the probability of an overrun, $\Pr\{\tau > \bar{X}\}$: the probability that the project ends, either from completion or cancellation, after the initial expected duration $\bar{X}$ (right panel, black solid line). It is low for small values of μ because the project will most likely be completed on time. It rises until achieving a maximum of approximately 0.39 when $\mu = 3.34$ and then decreases as the probability of early cancellation becomes ever more likely. We break this into two sub-possibilities: overruns for which the project is eventually completed P_{ORC}, and overruns followed by eventual cancellation P_{ORF}.[13]

[12] This can be obtained analytically from $P_{\text{EC}}(\mu) = p(x = 1; \mu)$, where $p(x; \mu)$ is the solution to the second-order ODE $p''(x; \mu) + \mu p'(x; \mu) + \mu p(x; \mu) = \mu$ with boundary conditions $p(x = 0; \mu) = p'(x = 0; \mu) = 0$.

[13] A project that is completed is either completed on time or completed after an overrun, so we can solve from the probability of success of any kind, $\pi(\bar{X}) = \frac{\lambda \bar{X}}{e^{\lambda \bar{X}} - 1} = \frac{\mu}{e^{\mu} - 1} = P_{OT}(\mu) + P_{ORC}(\mu)$. Then, $P_{ORF} = P_{OR} - P_{ORC}$ because an overrun will result in either cancellation or completion.

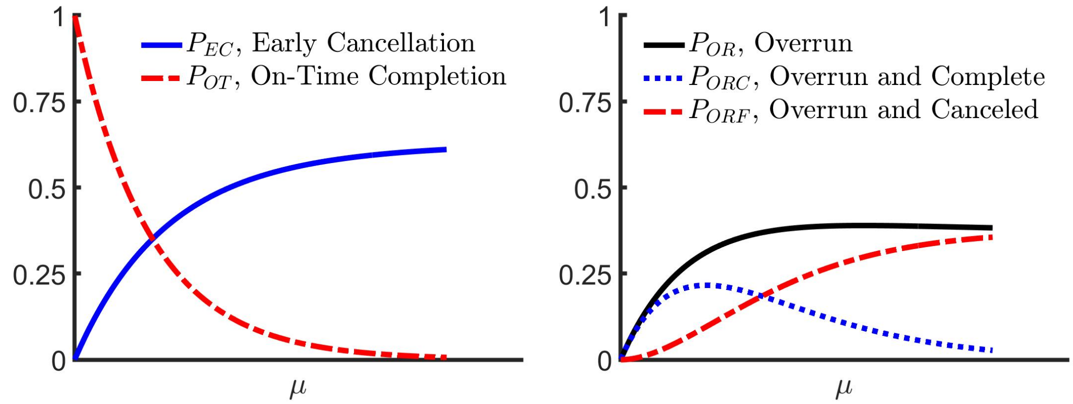

Figure 2.1: Probability of an Overrun Under a Short-leash Contract

The left panel of this figure plots the probability of early cancellation (P_{EC}, blue solid line) and the probability of on-time completion (P_{OT}, red dash-dot line). The right panel of this figure plots the probability of an overrun (P_{OR}) $\mu \equiv \lambda\bar{X}$ is the expected number of setbacks experienced while the project is in operation.

2.5.6 The Value of Randomization

We can now address the value to the principal of the reporting process: why not simply assign a deadline of S_0 and stick with it? The answer is that the value of reporting derives directly from application of the randomization procedure once the contract enters the short-leash phase, $S_t \leqslant \bar{X}$. Assume instead that the principal does offer a fixed deadline of S_0 and the prize (2.15) upon completion, and then does not solicit or review reports. The agent will use high effort until experiencing a setback at $S_t < \bar{X}$ and will then shirk for the remainder of the contract.

Under the optimal contract, when $S_t \geqslant \bar{X}$, the time budget behaves just like a deadline, counting down naturally (i.e. $dS_t = -dt$), and the principal effectively ignores all reports. In fact, the (NPS) constraint binds, implying that the agent is always indifferent between working or shirking, and hence, willing to work for $S_t \geqslant \bar{X}$. However, if a setback occurs at $S_t < \bar{X}$ and the deadline is fixed, then it is impossible for the agent to complete the project and receive payment in the remaining time. Because he can only realize any positive utility through the final reward at the time of project completion if he chooses to work, he prefers to shirk out the clock and report a last-second setback to obtain bS_t. In other words, a fixed deadline and prize can be used to induce full effort until the short-leash phase of the contract. At that point, there isn't enough time left to complete the project if a setback occurs, and the agent postpones reporting the setback until the end of the contract and shirks. The reports are useful precisely because they enable the principal to create a soft deadline instead of a hard one: the principal stochastically extends the agent's time or terminates him, in place of inefficient shirking.

2.6 The Role of Commitment

A potential criticism of the optimal contract characterized in Proposition 2.5.2 is that it involves the principal committing to randomly cancel the project if a setback is reported when $S_t < \bar{X}$. In particular, if the agent will continue to work after receiving an extension, then the principal strictly prefers extending the project to canceling it, and – lacking commitment to randomize – she will grant an extension with probability 1. On the other hand, if the agent believes that the project will always be extended, then he will shirk and report false setbacks *ad infinitum*, leading the principal to prefer project cancelation. This is the familiar logic underpinning a mixed-strategy equilibrium. To relax the required level of commitment, we modify the baseline model in this section by assuming that randomization by either party is possible but not contractually verifiable. All other aspects of the environment remain intact.

Consider again the situation in which a setback results in $W_t \in (0, b\bar{X})$. As we noted in subsection 2.4.5, one way to give the agent his promised continuation utility is simply to cancel the project and make a severance payment. This option has value $-W_t$ to the principal. We showed that the principal could do better by randomizing between project cancellation without severance ($W_t = 0$) and the minimally feasible project extension ($W_t = b\bar{X}$). Indeed, because the agent continues to work if granted an extension, this randomization has strictly positive net value to the principal. However, as per the argument in the previous paragraph, this scenario can only be implemented when randomization is verifiable.

Nevertheless, it is possible to implement an outcome that delivers expected value of zero to the principal, which still dominates payment of severance. To see how, suppose that rather than working with probability 1 when granted an extension, the agent randomizes at that moment between continuing to work and shirking out the clock, two options over which he is indifferent. If the agent randomizes such that the principal's expected value from extending the project is 0, then she will be indifferent between canceling the project and extending it, and will be willing to randomize herself. In terms of payoffs, the only difference between this setting and the baseline model is that the occurrence of a setback inside the randomization region drops the principal's value function down to zero. The agent's continuation utility W_t is still a martingale and (NPS) still binds.

Proposition 2.6.1 (Non-verifiable Randomization)**.** *If randomization is not verifiable, then the optimal contract for the principal can still be implemented with a time budget and the same prize structure K_τ. If $S_{t-} < \bar{X}$ and a setback is reported, then the principal extends the schedule to $\bar{X}$ with probability $\frac{S_{t-}}{\bar{X}}$ and cancels the project with probability $1 - \frac{S_{t-}}{\bar{X}}$. Upon receiving an extension, the agent randomizes between shirking out the clock with probability $q = \frac{\hat{F}(\bar{X},0)}{\hat{F}(\bar{X},0)+c\bar{X}}$, and continuing to work with probability $1 - q$, where*

$$\hat{F}(S = \bar{X}, 0) \equiv Re^{-\lambda \bar{X}} - \frac{c+b}{\lambda}\left(1 - e^{-\lambda \bar{X}}\right). \tag{2.31}$$

Note that this version of the contract does not need to specify specific probabilities of cancellation or extension – it needs only to specify that the principal has the right to cancel or extend the project *at will*. This is somewhat more general than the contract characterized in Proposition 2.5.2 where the exact probabilities of cancellation and extension were necessarily an explicit part of the agreement.

The inability to commit to explicit probabilities harms the principal. In particular, $\hat{F}(S = \bar{X}, 0)$ is the value she derives from a short-leash contract when she cannot commit and

$$\hat{F}(S = \bar{X}, 0) = F(S = \bar{X}, 0) \left(\frac{1 - e^{-\lambda \bar{X}}}{\lambda \bar{X}} \right). \tag{2.32}$$

The fraction on the right side of this equation is less than 1 since $\lambda \bar{X} > 0$. A setback during the course of a short-leash contract under full commitment still leaves the principal with a positive expected payoff as shown in Corollary 2.5.4, whereas a setback in the randomization region absent commitment results in an expected payoff of zero. Although the inability to commit lowers the principal's value for the project, the above expression implies that Assumption 2.3.2 remains necessary and sufficient for feasibility of contracting when the principal cannot commit to randomize.

Note that while we specified that the agent randomizes between continuing to work and shirking out the clock after receiving an extension, there are other strategies the agent could pursue that are expected payoff equivalent for both parties. For example, while shirking the agent could mimic nature by randomly reporting setbacks at rate λ, continuing to shirk after each and every extension. Relative to shirking out the clock, this would increase the variance of τ, though not its expected value.

Perhaps surprisingly, the lack of commitment to randomize leads the principal to grant the agent *more* initial time:

Proposition 2.6.2 (Optimal Initial Time Budget)**.** *Define S^* to be the principal's optimal initial time budget when the randomization is verifiable, and $\hat{S}^*$ to be the principal's optimal initial time budget when the randomization is not verifiable. As-*

sume that the agent's initial outside option value is sufficiently low such that his participation constraint is met in both economies. Then, $S^* \leqslant \hat{S}^*$.

Because the inability to commit to explicit randomization harms the principal (i.e. $\hat{F}(S,0) < F(S,0)$), a reasonable conjecture is that she would prefer to grant the agent less time. After all, for any given value of S, lack of commitment implies a lower probability of project completion and reduces the initial value of the project to the principal. It is, therefore, somewhat surprising that she responds by devoting more time and money to the less valuable project. In addition, the agent benefits from the principal's lack of commitment because his expected payoff is proportional to schedule length.

The intuition is actually straightforward. Lack of commitment power only harms the principal if a setback occurs in the short-leash randomization region, $S_t < \bar{X}$. By granting the agent a longer initial time budget, the principal raises the probability that the project will be completed before the lack of commitment becomes a problem. That is, she reduces the likelihood that her inability to commit will even come into play. In a sense, the principal doubles down on the part of the contract to which she can commit (the length of the schedule) in order to reduce the impact of the part to which she cannot (explicit probabilities of project cancellation and extension).

2.7 Partial Setbacks

Our baseline model assumes that setbacks wipe out progress completely. An important question in this regard is how general is our implementation using a time-budget and a prize-upon-completion? There are many ways to model *partial setbacks*. In this section we show that our contract form – a type of time-budget and reward – is part of any optimal contract across a wide range of setback-style models.

To accomplish this, we assume a general setback structure:

$$dX_t = a_t(dt - Y_t dN_t), \tag{2.33}$$

where $Y_t = Y(X_t, \theta_t) \in [0, X_t]$ is a function of the state of the project augmented by an additional random variable θ_t. With this framework, we can match several examples of partial setbacks:

- Let the project have two stages, 0 to $\tilde{X}$ and $\tilde{X}$ to $\bar{X}$. We set $Y_t = X_t$ if $X_t \leqslant \tilde{X}$ and $Y_t = X_t - \tilde{X}$ if $X_t \geqslant \tilde{X}$, meaning that setbacks wipe out progress within a stage only, but the principal cannot observe the stage or progress within stage.

- Let the project have setbacks of random size, so that $Y_t = \theta_t X_t$, where θ_t are i.i.d with $\theta \sim G(\theta)$.

- Let setbacks have a maximum size, so that $Y_t = \max(X_t, \bar{Y})$. Thus, setbacks can wipe out the early development of a project (the phase with $X_t \leqslant \bar{Y}$) but not the entire project if near completion.

We assume that the project is always worth continuing (analogously to Assumption 2.3.2), and that the principal can voluntarily re-set progress to zero[14] Then, the arguments in Lemma 2.4.2 remain valid in this setting:

Lemma 2.7.1 (High Action and Prizes)**.** *The principal will always choose to implement the high action ($a_t = 1$). The principal will pay the agent only upon successful completion of the project ($K_\tau > 0$ iff success; $dC_t = 0$).*

This lemma is crucial because it means that the principal continues to use only two incentive devices in the more general setting: termination and a prize upon completion. Then, to show that something like a time budget is necessary, we need

[14] Setting $X_t = 0$ is value destroying, but it is a convenient tool to allow the principal to re-start the contract in the proof of Lemma 2.7.1.

only to specify the termination time under an incentive compatible contract. To deliver utility of W_0 in an incentive compatible way *requires* terminating the agent after enough time has gone by. The intuition for this result is simple: if there were a path of the project that would enable the agent to continue operating for long enough, then the agent would simply report that path while shirking and gain more utility than the optimal contract promises.

To see the result, fix a contract and define τ^R to be the stopping time associated with a particular sequence of reports: the time the agent either successfully reports completion of the project or is terminated. Since the agent can only report paths that are possible, the space of possible paths and the space of possible reports are the same. For any such contract that delivers utility W_0 to the agent, we must have $\mathrm{E}[\tau^R] \leqslant W_0/b$, otherwise the agent could make this sequence of reports while shirking to gain utility greater than W_0. Thus, while τ^R is not bounded, its expectation is bounded for incentive compatible contracts, and the agent either completes the project or is fired in finite time with probability 1.

Then, define

$$\sigma_t = \mathrm{E}_t[\tau] - t, \tag{2.34}$$

where τ is the stopping time associated with truthful reporting, which makes σ_t analogous to $\sigma(S)$ (2.17). Because $\mathrm{E}_t[\tau]$ is a conditional expectation and hence a martingale, $\mathrm{E}[d\sigma_t] = -dt$. More, σ_t is equal to zero whenever the project is completed or terminated. To proceed, we define the notion of an *expected* time budget:

Definition 2.7.1 (Expected Time Budget)**.** *An expected time budget is a non-negative process σ_t satisfying $E[d\sigma_t] = -dt$. The principal initially grants the agent σ_0 and then, if the project does not succeed, cancels the project iff $\sigma_t = 0$. An expected time budget recovers the random stopping time τ from termination of the contract (on the events of project completion or cancellation).*

Now, the principal's problem can be simplified under partial setbacks in a manor analogous to that in the baseline section. Denote the probability of success by π_t:

$$\pi_t = \mathrm{E}_t \left[1_{X_\tau = \bar{X}} \right]. \tag{2.35}$$

Then, from the principal's and agent's payoffs (2.6 and 2.7), we have that the principal's value can be written

$$F_t = \pi_t R - W_t - c\sigma_t, \tag{2.36}$$

which is analogous to (2.18). This is sufficient to show:

Proposition 2.7.2. *Assume that setbacks follow the structure in (2.33), and that the contract offered is incentive compatible and delivers the agent an initial utility of W_0. Then, the contract terminates the agent when an expected time budget hits zero, unless the project is complete, in which case the agent is awarded a prize which equals his terminal utility. The principal's payoff is given by (2.36).*

This result generalizes the notion of time-budgets to models with partial setbacks: any optimal contract can be implemented with an expected time budget and a (path dependent) prize-upon-completion. The expected time budget captures termination, and the prize captures the agent's terminal utility, and those are the only two incentive devices that the principal uses (Lemma 2.7.1). We have not assumed that an optimal contract exists, and a full characterization of an optimal contract would require fixing a specific model of partial setbacks from the class described by (2.33). Instead, Proposition 2.7.2 tells us that regardless of the particular selection we fix, an optimal contract is always implemented with an expected time budget (soft deadline) and a reward for successful completion, just as in our baseline model.

2.8 Conclusion

At a very general level, projects are usually viewed as possessing three defining features, scope, schedule, and budget – the so-called "iron triangle." (Wyngaard et al., 2012) The scope of a project is the quality of the deliverable, be it a software application, a power plant, or a doctoral thesis; The schedule is the time allotted to production of the deliverable; and the budget is the monetary or other physical resources committed to it. However, because projects are, by definition, at least somewhat unique, their implementation typically involves considerable uncertainty. In this paper we held scope fixed, and presented a model of project implementation focusing on what appears to be the most common sources of project uncertainty, schedule setbacks and the concomitant cost overruns.

Whether a project is under taken in-house (e.g., the Boeing Dreamliner) or outsourced (e.g., South Carolina's V.C. Summer nuclear plant or the FBI's virtual-case-file system), its progress will almost surely be hampered to some degree by agency frictions. To study this, we embedded a natural model of production with random setbacks into a dynamic agency environment and solved for the optimal contract from the principal's perspective. This analysis yielded a number of novel insights and conclusions. Among the most robust are: 1) an optimal contract can always be implemented with a time budget and a terminal payment corresponding to a cost-plus-award-fee contract; 2) penalties for reports of setbacks or delays are generally more severe the later they occur in project development; and 3) mishaps that are reported near the end of the allotted schedule either result in project cancelation or minimally feasible project extension.

There are numerous avenues available for future research. For example, expanding our current treatment to incorporate common strategies for dealing with the time pressure created by unanticipated setbacks seems promising. The completion of projects is frequently time-sensitive as noted by Lewis and Bajari (2011) who investigate the procurement of highway construction projects where completion delays can have large social costs. In this vein, exploring the possibility of speeding up production through *fast-tracking* (running several phases in parallel) or *crash-*

ing (deploying more resources) to make up for unanticipated delays is a potentially important consideration. Finally, there is the question of scope itself. Throughout we supposed that the project was either incomplete (worth zero to the sponsor) or complete (worth a fixed amount). In reality, the ultimate quality of many projects varies along a continuum. Indeed, *scope creep* on the part of sponsors (demanding a higher quality deliverable than originally specified) is often cited as a contributing factor to project failure.[15] We leave these considerations and others for future work, having judged this particular project to be deliverable as complete.

[15] Ely and Szydlowski (2020) considers a dynamic moral hazard problem in which the principal uses scope creep to entice the agent to exert effort on a project that he would not have agreed to work on if he had known the full scope of the project at the outset.

3

Dynamic Moral Hazard with Adverse Selection

We study the optimal incentive scheme for a long-term project with both moral hazard and adverse selection. The moral hazard issue is due to the fact that the agent's effort, which increases the arrival rate of a Poisson process, is not observable by the principal. In addition, the agent's effort cost, which needs to be reimbursed by the principal, is also the agent's private information. This gives rise to the adverse selection problem. The principal needs to design the optimal menu of contracts, each of which is chosen by the agent with a specific effort cost. We fully characterize the optimal menu in the case of two types of agents. Specifically, the agent with a lower cost is offered a probation contract, which confirms the agent's type if there is an arrival during a probation period; the agent with the higher cost is offered a sign-on-bonus contract with an immediate direct initial payment. We then explore the more general case with continuous types of agents. In particular, we provide an easy-to-compute upper bound on the principal's utility. The upper bound computation also yields a feasible menu of probation and sign-on-bonus contracts, and the corresponding lower bound it generates. We further provide a condition which can be used to verify whether the upper and lower bounds coincide, implying the optimality of our feasible menu of contracts. Numerical studies confirm that the verficiation condition almost always holds for commonly used probability distributions of the effort cost.

3.1 Introduction

Many business environments involve situations where an agent is supposed to exert effort and obtain results over time for a principle. For example, a firm's R&D department (principal) funds researchers (agent) for extended durations of time, hoping to generate "breakthrough results." Similarly, many companies expect their employees in charge of business development activities to acquire new target customers. In politics, many firms hire lobbying agencies in the hope of influencing politicians to pass legislation benefiting the firm. In all these situations, the agent's activities are hard to observe, leading to a dynamic moral hazard problem. Furthermore, the agent's capabilities to achieve results, reflected in its operating cost, may be only known by the agent, and not observable to the principal. For example, firms may have a hard time estimating how much expenditure it will take the researcher team to achieve breakthroughs, or how much lobbying expenditure it will take a lobbyist to achieve legislative results. An agent may either claim a higher expenditure than necessary, or, if under-funded, may choose not to exert effort. This therefore gives rise to a principal-agent problem with both dynamic moral hazard and adverse selection.

In particular, we consider a risk-neutral setting in which a principal hires an agent to create positive results (new businesses, research breakthroughs, favorable legislation, etc.). When the agent exerts effort, positive results arrive according to a Poisson process with a given rate. This instantaneous arrival rate is a positive constant if and only if the agent exerts costly effort. If the agent shirks, the arrival rate drops to zero. A distinct feature of our model, compared to previous literature, is that the agent's cost rate (of exerting effort) is private information, which represents heterogeneity in the cost/outcome ratio of agents. Thus, an agent with a lower cost structure is able to obtain more positive results for the same effort than an agent with a higher cost structure. However, the principal does not have a priori knowledge of an agent's cost structure and could be stuck with either a high cost/incapable agent or an agent that is capable but does not exert the right effort if the principal does not provide the right incentives.

We formulate the problem as a continuous-time dynamic moral hazard problem with adverse selection. The continuous-time dynamic moral hazard literature originates from Sannikov (2008), which considers a principal hiring an agent to affect the cash flow of a project by controlling the drift of a Brownian motion. Sannikov (2008) develops a continuation utility formulation (similar to the "promised utility" framework in Spear and Srivastava (1987a)) to transform the contract design problem into a dynamic programming problem in which the agent's continuation utility is the state variable. This formulation has been further applied to the Poisson model by Biais et al. (2010c). However, their Poisson arrival is "bad news" that causes losses to the principal, rather than "good news" that is beneficial to the principal in our model. Some recent papers also consider "good" Poisson arrivals (Sun and Tian, 2018; Green and Taylor, 2016a; Shan, 2017). However, the operating cost of the agent in these papers is known by the principal.

Following standard results in mechanism design in Laffont and Martimort (2009), the principal should provide a menu of contracts, such that an agent with a specific cost chooses a particular contract from this menu. Following the revelation principle (Myerson, 1981), it is without loss of generality for us to consider direct mechanisms. In traditional adverse selection models, such mechanisms only involve allocation and payment decisions that depend on the agent's type. In our setting, however, after the agent reports the operating cost, the two players still face a dynamic moral hazard game. That is, payments and allocation (and contract termination) should also depend on the agent's dynamic performance, which is stochastic in nature. Therefore, the principal needs to optimize over menus of dynamic incentive compatible contracts that motivate agents with different operating costs to continuously exert effort before contract termination. Furthermore, one type of agent should not have an incentive to choose a dynamic contract for another type. Consequently, the optimal design problem can no longer be formulated as a classic dynamic program. In this paper, we contribute by providing a solution approach based on deterministic continuous time optimal control.

There have been previous attempts on the problem with both dynamic moral hazard and adverse selection. Ma (1991) focuses on renegotiation and actions with long-term effects, whereas we give the principal full commitment power on contracting, and hence the issue of renegotiation does not exist. Mayer (2020) and Rong et al. (2021) both consider dynamic moral hazard problems with adverse selection where an agent is hired to exert effort to reach a single breakthrough. Hence, their model only considers a single arrival while we consider an infinite horizon Poisson process. Furthermore, the adverse selection in their model comes from the information about the arrival (timing of the arrival or the status of the arrival) but not a characteristic of the agent (capability of the agent in our model). Similarly, Chen et al. (2018) considers an infinite horizon Poisson model where the adverse selection also comes from the feature of the arrivals. Cvitanić et al. (2013) and Santibáñez et al. (2020) study continuous-time moral hazard problems in infinite horizon with adverse selection under Brownian and Poisson stochasticity, respectively. To solve the adverse selection problem, they adopt the methodology of a credible set regarding the agents' continuation and temptation values. Rather than resorting to their method, which involves stochastic differential equations with variational inequalities, we formulate our optimization problem with a deterministic optimal control approach. Our formulation enables us to provide closed-form solutions of optimal menu of contracts with intuitive implementations, such as *probation contract* and *sign-on-bonus* contract. Another distinct feature of our paper from previous papers is that we can tackle the continuous type problem by taking advantage of our deterministic optimal control approach, while they only consider the two type problem.

Dynamic contracting problems have been the subject of recent Operations Research studies (see, for example, Zhang, 2012a,b; Li et al., 2013). A particularly relavent stream of papers applies dynamic incentive design to project management settings (Kwon et al., 2010; Chen et al., 2015; Dawande et al., 2019). In these papers, a project manager designs short-term contracts for multiple independent agents (contractors). In comparison, we focus on designing long-term contracts for a single agent. Further, including both moral hazard and adverse selection on the agent's effort cost distinguishes our paper from the aforementioned literature. It is worth mentioning two recent papers, which also consider both moral hazard and adverse selection issues. Chen et al. (2020) studies a principal who periodically offers agents "limited-term" non-monetary rewards in order to induce agents' effort over the long-run. The reward's value to each agent is the agent's private information. The paper is focused on designing near-optimal "limited-term" stationary policies. Zorc et al. (2019) considers a delegated search model where the agent's search effort and the findings from the search process are private information. They adopt the framework of Plambeck and Zenios (2000), which considers a risk averse agent who can borrow money. In comparison, our risk-neutral agent is cash-constrained.

Another related strand of literature combines dynamic moral hazard with learning. Unlike our private information setting where the principal can elicit truthful information, under their setting with learning, the true state is unobservable to either party, and hence the contract has to update both parties' belief (see, for example, Bhaskar, 2012; Kwon, 2011). Halac et al. (2016) further considers long-term contracting problems that involve adverse selection, moral hazard, and learning. In another paper, Bimpikis et al. (2019) studies when and what information a contest designer should disclose in a contest design setting, which affects how much participants learn about the feasibility of the innovation over time.

In our model, the principal offers the agent a menu of contracts, with each item in the menu designed specifically for an agent with a particular operating cost. In each contract, the principal keeps track of the agent's performance score, which is the continuation utility as in Sannikov (2008). The performance score takes an upward jump once a Poisson arrival occurs and keeps decreasing between arrivals. There are two absorbing states, a lower threshold of zero at which the contract terminates, and an upper threshold at which the agent will never be terminated, is rewarded with a monetary payment for each future arrival. If the agent is of two possible types, with the cost being high or low, when the high-cost agent is too costly, he is asked to leave the system to avoid inefficiency. In order to induce truthful revelation of high cost, we should adopt a *pay-to-leave* contract: an immediate payment that is equal to what the agent can get from mimicking the low-cost agent. This immediate payment is the information rent rewarded to the agent. If the high-cost agent is less costly, he can possibly be hired. Yet it could be that even though he is hired, the agent would still prefer to mimic the low-cost agent. Hence, the principal should use a *sign-on-bonus* contract which provides the agent an immediate payment to induce truth telling. The contract to the low-cost agent only reimburses the low operating cost. This implies that the high-cost agent cannot afford to exert effort, and therefore is unable to generate any arrival by mimicking the low-cost one. Consequently, the first arrival is the particular evidence that differentiates the low-cost agent from the high-cost one who attempts to mimic without working. Therefore, the principal gives the low-cost agent a *probation contract*, which starts with a probation period for the low-cost agent to prove his type. We believe our model is especially appropriate in settings like new business generation, R&D or legislative lobbying where 'arrivals' are likely to be rare but significant.

If an arrival occurs during the probation period, the identity of the low-cost agent has been confirmed. Hence, to design the menu of contracts, we can focus on designing the length of the probation period and the magnitude of the sign-on bonus. This enables us to transform the original contract design problem into a deterministic optimal control problem. If there is a continuum of possible cost levels, we formulate an easy-to-compute upper bound optimization problem to the original problem thanks to the deterministic optimal control formulation. This optimization problem further provides a way for us to design a menu of contracts. Furthermore, we show that if the solution in the upper bound calculation satisfies a simple condition, then the upper and lower bounds match, which implies that our contract design is in fact optimal. Our numerical study illustrates that the condition is often satisfied with commonly used distributions. In this case, the principal designs a menu with a continuum of different items, each of which has the form of *sign-on-bonus contract* or *probation contract*.

Our paper is one of the first papers that combine the continuous-time moral hazard problem and the adverse selection problem. Luckily, we can solve the two-type problem and show that the optimal contracts take a simple and intuitive structure. Furthermore, ours is the first paper that can tackle the continuous-type problem. Although the space of dynamic contracts is enormous, the optimal contracts only take two intuitive and easy-to-implement forms both in two-type and continuous-type problems.

The rest of the paper is organized as follows. We introduce the model at the beginning of Section 3.2. In Section 3.3, we present three contracts that are candidates for optimal contracts when the agent type is unknown by the principal. In Section 3.4, we solve the optimal contracts for the two-type case. In Section 3.5, we consider the contract design problem for the continuous-type case.

3.2 Model

A principal contracts an agent to increase the arrival rate of a Poisson process over an infinite time horizon. At any point of time t, the agent can privately choose to either work or shirk. Whenever working, the agent incurs a constant flow of cost, and generates Poisson arrivals with an instantaneous rate μ. Shirking costs the agent nothing, and also generates no arrivals. Each arrival yields a revenue R to the principal, and is observable to both the principal and the agent. Therefore, we denote $\nu_t \in \{0, \mu\}$ to represent the agent's effort level at time t, such that $\nu_t = 0$ represents shirking and $\nu_t = \mu$ working. Further denote a right-continuous counting process $N = \{N_t\}_{t \geqslant 0}$ to represent the total number of arrivals up to time t, which generates a filtration $F^N = \{F^N_t\}_{t \geqslant 0}$. Therefore, the instantaneous arrival rate of the counting process at time t is ν_t, and the left-continuous effort process $\nu = \{\nu_t\}_{t \geqslant 0}$ is F^N-predictable.

The agent's capability, reflected by his operating cost per unit of time, is uncertain to the principal a priori. That is, a more capable agent can generate arrivals with a lower operating cost. In this paper, we use "capability" and "cost" interchangably. We assume that the operating cost is the agent's private information, and stays the same throughout the time horizon. The common prior distribution of the operating cost has a support $\mathcal{C}$. In this paper we consider C to be either a binary set $\{g, b\}$, or a continuous interval. We also refer to the operating cost $c \in C$ as the agent's *type*. We assume that the principal needs to cover the operating cost because the agent has limited liability and is cash constrained, a standard assumption in the dynamic contracting literature. In particular, at any point in time, in order for the agent to exert effort, the principal needs to reimburse for the agent's reported operating cost c. (This situation is pretty common in the contexts such as R&D and lobbying where the principals have to provide a continuous flow of payments to the agents to let them operate. It may take the form of retainers in the case of lobbyists or a fixed amount of repetitive payments in the case of contract R&D.)[1]

Therefore, if the agent's type is c but pretends to be of a better (lower-cost) type $c' < c$, and the principal only pays operating cost c', then this agent is not able to

[1] The charging of retainers by lobbyists is common, see for example, https://lobbyit.com/pricing/, https://arnoldpublicaffairs.com/faq/ and https://lobbying101.wordpress.com/about-lobbyists/how-much-do-they-charge/. Furthermore, it is common that R&D projects are funded for long durations of time and may not bring any results at the end. For example, 50% of registered clinical trials are never published in full and at least 50% of published reports are not sufficiently clear, complete, or accurate for others to interpret, use, or replicate the research correctly, see https://blogs.bmj.com/bmj/2016/01/14/paul-glasziou-and-iain-chalmers-is-85-of-health-research-really-wasted/ and https://www.fiercebiotech.com/special-report/2019-s-top-15-clinical-trial-flops-and-a-dishonorable-mention. Other famous failure examples include VCF developed by FBI (https://en.wikipedia.org/wiki/VirtualCaseFile) and South Carolina's nuclear power plant construction project. (https://en.wikipedia.org/wiki/Nukegatescandal).

generate any arrivals. If the agent shirks, a fraction $\rho \in (0, 1]$ of the operating cost payment by the principal can be diverted as a shirking benefit to the agent. For ease of exposition, we assume that $\rho = 1$, such that all the operating cost can be converted to the agent's shirking benefit. Following Green and Taylor (2016a), we assume whenever the agent's operating cost is c, the principal has to pay the agent at least a flow of c at any point in time in order for the enterprise to continue operating. That is, whenever the payment flow from the principal drops below c, the operations stop completely and there will be no more future arrivals. This further implies that a high-cost agent who mimicks a low cost agent can enjoy the low operating cost as a shirking benefit until contract termination.[2]

Because the agent knows the operating cost in the beginning of the time horizon, following the *Revelation Principle*, it is without loss of generality to consider direct mechanisms (see, for example, Myerson, 1986; Pavan et al., 2014). In our context, the principal designs a menu of contracts $\Gamma_C = \{\gamma^c\}_{c \in C}$, such that type c agent chooses contract γ^c. Any contract $\gamma^c = (L^c, \tau^c)$ includes an F-predictable payment process L^c, and a F-random time τ^c representing contract termination. When stressing the operating cost c is not necessary, we also use notation $\gamma = (L, \tau)$ without superscripts to represent a generic contract. As for the contract termination time τ, if $\tau = \infty$, the contract continues throughout the infinite time horizon.

Specifically, for the payment process $L = \{L_t\}_{t \geqslant 0}$, at each time epoch $t \geqslant 0$, L_t represents the cumulative payment from the principal to the agent up to time t. For simplicity of expressions, in the rest of the paper we consider $dL_t = \ell_t dt + I_t$, in which ℓ_t represents the flow, and I_t the instantaneous payment at time t. Limited

[2] Shirking and misuse of research funds are surprisingly common in R&D settings, see for example, https://www.chron.com/news/houston-texas/article/Prof-accused-of-spending-NASA-grants-on-cars-1722521.php, https://www.nbcnews.com/news/us-news/philadelphia-professor-accused-spending-185-000-grant-funds-strip-clubs-n1118571, https://www.newsweek.com/fund-meant-vaccine-research-misused-least-145m-unrelated-expenses-almost-decade-1564954, and https://www.theguardian.com/higher-education-network/2015/mar/27/research-grant-money-spent.

liability of the agent and the assumption that the agent is cash constrained imply that payment is from the principal to the agent but not the other way around, or,

$$L_{t_1} \geqslant L_{t_2},\ \forall t_1 \leqslant t_2.$$

Furthermore, before contract termination, the payment needs to cover the operating cost, or,

$$L^c_{t_2} - L^c_{t_1+} \geqslant c(t_2 - t_1),\ \forall t \in (t_1, t_2],\ t_2 \leqslant \tau,$$

where we use notation $X_{t+} := \lim_{s \downarrow t} X_t$ to represent the right limit of any left-continuous process $\{X_t\}_{t \geqslant 0}$ at time t. Similarly, we define notation $X_{t-} := \lim_{s \uparrow t} X_t$. For simplicity of expressions, in the rest of the paper we consider $dL_t = \ell_t dt + I_t$, in which ℓ_t represents the flow, and I_t the instantaneous payment at time t. Therefore, the aforementioned constraints on payments can be summarized in the following *limited liability* (LL) constraint for all contract $\gamma^c = (L^c, \tau^c) \in \Gamma_C$,

$$I_t \geqslant 0,\quad \ell^c_t \geqslant c,\quad \forall t \in [0, \tau] \text{ and } c \in C. \tag{LL}$$

Both the principal and the agent discount future costs, and payments with a discount rate r. Without loss of generality, and for simplicity of expressions, we normalize time unit such that

$$\mu + r = 1. \tag{3.1}$$

In order to formally define direct mechanisms, we need to start with expressing the agent's utility.

Agent utility

Given a dynamic contract $\gamma = (L, \tau)$ and an effort process ν, the expected discounted utility of the agent with an operating cost c is

$$u(\gamma, \nu; c) = \mathrm{E}^{\nu}\left[\int_0^{\tau} e^{-rt}(dL_t - c1_{\nu_t = \mu} dt)\right], \tag{3.2}$$

in which the expectation E^{ν} is taken with respect to probabilities generated from the effort process ν.

Next, we describe the agent's cash constraint. The agent's resource to conduct the project is provided solely by the principal, and insufficient resources would render the agent unable to exert effort. Formally, any effort process of a type c agent facing a contract $\gamma = (L, \tau)$ needs to satisfy,

$$\nu_t = \mu, \text{ only if } \ell_s \geqslant c, \forall s \leqslant t. \tag{3.3}$$

Use $N(\gamma, c)$ to denote the set of all F^N-predictable effort processes ν that satisfy condition (3.3) for a type c agent facing a contract γ. Further use $\mathfrak{N}(\gamma, c) \subseteq N$ to denote the set of *best-response effort processes*, that is,

$$u\big(\gamma, \nu; c\big) \geqslant u(\gamma, \nu'; c), \ \forall \nu \in \mathfrak{N}(\gamma, c) \text{ and } \nu' \in N. \tag{3.4}$$

We denote F_t^N-predictable effort process $\nu^0 = \{\nu_t^0\}_{t \geqslant 0}$ to be the *always shirking process* such that $\nu_t^0 = 0$ almost surely for all t before contract termination. Similarly, we denote F_t^N-predictable effort process $\bar{\nu} = \{\bar{\nu}_t\}_{t \geqslant 0}$ to be the *always exerting effort process* such that $\bar{\nu}_t = \mu$ almost surely for all t before contract termination. Define quantities

$$\beta_c := \frac{c}{\mu}, \quad \text{and} \quad \bar{w}_c = \frac{\mu \beta_c}{r}. \tag{3.5}$$

A simple contract that induces the agent to always exert effort is to pay the agent a constant β_c for each arrival besides reimbursing the operating cost rate c. That is, we can express such a simple contract as $\bar{\gamma}^c = (L^c, \tau^c)$ with $dL_t^c = \beta^c dN_t + c dt$ and $\tau^c = \infty$. One can verify that the corresponding agent's utility is

$$u(\bar{\gamma}^c, \bar{\nu}, c) = \bar{w}_c.$$

Although this simple contract is not optimal, the quantities β_c and $\bar{w}_c$ are useful for describing the optimal contracts.

Furthermore, the revelation principal implies that we can focus on direct mechanisms. Therefore, we need the following *Truth-Telling* (TT) contraint on the menu Γ_C, which ensures that an agent with operating cost c indeed chooses contract γ^c from the menu.

$$u(\gamma^c, \nu^c; c) \geqslant u\left(\gamma^{c'}, \nu; c\right), \ \forall c, c' \in C, \ \nu^c \in N(\gamma^c, c), \nu \in N(\gamma^{c'}, c). \tag{TT}$$

It is standard to consider the agent's continuation utility (also called *promised utility*) at time t, defined as (see, for example, Biais et al., 2010c),

$$W_t(\gamma, \nu; c) = \mathrm{E}^\nu \left[\int_t^\tau e^{-r(s-t)} (dL_s - c1_{\nu_s=\mu} ds) \middle| \mathcal{F}_t^N \right] 1_{t<\tau}. \tag{3.6}$$

In this literature it is standard to assume that the principal has the commitment power to a long term contract, while the agent does not need to commit to staying in the contract. That is, we need the following *Individual Rationality* (IR) constraint to guarantee participation before contract termination,

$$W_t(\gamma, \nu; c) \geqslant 0, \ \forall t \in [0, \tau], \ c \in C. \tag{IR}$$

The following result depicts the dynamics of the process W_t, and provides an equivalent condition to the best response effort process.

Lemma 3.2.1. *For any contract γ, effort process ν, and operating cost c, there exists an F^N-adaptive process H_t such that*

$$dW_t(\gamma, \nu; c) = \{[rW_{t-}(\gamma, \nu; c) - \nu_t H_t + c1_{\nu_t=\mu}]dt + H_t dN_t - dL_t\} 1_{0\leqslant t<\tau}. \tag{PK}$$

Furthermore, the following defined effort process is a best response to contract γ, or, $\{\nu_t\}_{t\in[0,\tau]} \in \mathfrak{N}(\gamma, c)$, in which

$$\nu_t = \begin{cases} \mu, & \text{if } H_t \geqslant \beta_c, \\ 0, & o.w. \end{cases} \tag{IC}$$

Lemma 3.2.1 implies that the principal can motivate a type c agent to exert effort if and only if each arrival yields an upward jump of at least β_c in the agent's promised utility. Later in the paper we show that in the optimal contract, the *incentive compatibility* (IC) constraint may not always be binding. That is, for certain operating cost c and time t, we need $H_t > \beta_c$, greater than the minimum amount necessary to induce effort.

Principal utility.

Denote $U(\gamma, \nu)$ to represent the principal's total expected discounted utility from a contract γ while the agent's type is c and uses an effort process $\nu \in N(\gamma, c)$. That is,

$$U(\gamma, \nu) := \mathrm{E}^{\nu}\left[\int_0^{\tau} e^{-rt}\left(RdN_t - dL_t\right)\right]. \tag{3.7}$$

Now we define $\mathcal{U}(\Gamma_{\mathcal{C}}) := \mathrm{E}\left[U(\gamma^c, \nu^c)\right]$ to represent the principal's total expected discounted utility from the *menu* of contracts $\Gamma_{\mathcal{C}}$ when the agent's effort process $\nu^c \in \mathfrak{N}(\gamma, c)$ satisfies (IC). The principal's contract design problem is

$$\begin{aligned} Z(C) := \sup_{\Gamma_{\mathcal{C}}} \quad & \mathcal{U}(\Gamma_{\mathcal{C}}) \\ \text{s.t. } & (\mathrm{LL}), (\mathrm{PK}), (\mathrm{IC}), (\mathrm{IR}), \text{ and } (\mathrm{TT}). \end{aligned} \tag{3.8}$$

Note that the expectation in the objective function is taken with respect to the operating cost c, while constraints (LL), (PK), (IC) and (IR) are for all $c \in C$. In contrast, the constraint (TT) is for all pairs of operating costs c and c', which implies that the maximization problem (3.8) cannot decouple in c. Finally, the objective function value $Z(C)$ is the principal's optimal expected utility.

3.3 Implementable Contracts

In this section, we present all possible contract forms that will appear in an optimal menu of contracts before rigorously deriving them in the next section. Note that the space of the dynamic contracts could be enormous. Here we greatly narrow down the possibilities to two structures: *sign-on-bonus* contract (including *pay-to-leave* contract as a special case) and *probation* contract, all of which are mathematically tractable and possess managerial interpretations. In later sections, we will formally show how to derive the optimal contracts and verify that these three contract structures suffice.

3.3.1 Sign-on-bonus contract

First, we introduce the so-called *sign-on-bonus* contract. In particular, we allow the principal to pay a sign-on-bonus in the beginning.

Definition 3.3.1. *For any initial promised utility $w \geqslant 0$ and sign-on-bonus $B \geqslant 0$, define a* sign-on-bonus contract $\gamma^c_{\mathsf{B}}(w, B) = (L^c, \tau^c_B)$*, which pays the agent* $dL_0 = B + \max\{w - \bar{w}_c, 0\}$ *at time* 0*, and then generates a promised utility process* W^c_t *according to*

$$dW^c_t = \left[r(W^c_{t-} - \bar{w}_c)dt + \min\left\{\bar{w}_c - W^c_{t-}, \beta_c\right\} dN_t\right] 1_{W_{t-} \geqslant 0} \tag{3.9}$$

following $W^c_0 = \min\{w, \bar{w}_c\}$*. Furthermore, the payment process* L^c_t *follows*

$$dL^c_t = \left[cdt + (W^c_{t-} + \beta_c - \bar{w}_c)^+ dN_t\right] 1_{W_{t-} \geqslant 0} \tag{3.10}$$

and the termination time τ^c_B *is according to*

$$\tau^c_B = \min\{t : W^c_{t-} = 0\}. \tag{3.11}$$

According to contract $\gamma^c_{\mathsf{B}}(w, B)$, the principal pays the agent a sign-on-bonus B at the beginning if the initial promised utility w is below the upper bound $\bar{w}_c$. A special case $\gamma^c_{\mathsf{B}}(0, B)$ gives the agent a bonus B without asking the agent to work. Such a *pay-to-leave* contract may be useful if the agent's operating cost c is so high that it is not worth hiring the agent. The initial payment induces the agent to truthfully reveal his type, as will be evident later in the paper.

As long as $w \leqslant \bar{w}_c$, the promised utility W^c_t starts from $W^c_0 = w$, and its dynamics (3.9) is consistent with (PK) with $H_t = \beta_c$. That is, the promised utility takes an upward jump of β_c upon each arrival, and gradually decreases at rate $r(\bar{w}_c - W^c_{t-})$ as long as $W^c_{t-} < \bar{w}_c$. An instantaneous payment occurs when the promised utility W^c_t jumps above $\bar{w}_c$. After that, the promised utility stays at $\bar{w}_c$ and the principal pays the agent β_c for each future arrival, in addition to the flow payment cdt, which reimburses the operating cost. In this case the termination time τ^c is infinity. If W^c_t does not reach $\bar{w}_c$ but decreases to 0 instead, the contract is terminated. Therefore, contract $\gamma^c_{\mathsf{B}}(w, B)$ motivates the agent to always exert effort before contract termination by setting $H^c_t = \beta_c$ at all times in the (IC) constraint.

The sign-on-bonus generalizes the optimal contract structure for the case without adverse selection (Sun and Tian, 2018), by adding a sign-on-bonus B and allowing the initial promised utility w to be higher than $\bar{w}_c$. That is, the optimal contract for the known cost case is another special case of the sign-on-bonus contract. We call this special case, $\gamma^c_{\mathsf{B}}(w, 0)$ for some initial value w, a *standard* contract. In the next section we show exactly when a sign-on bonus contract structure is optimal, and the corresponding optimal w and B values.

3.3.2 Probation contract

The next contract structure is more intricate, and it is important to understand why it may arise. First, recall that it is necessary for the principal to pay a flow rate of at least c in order to induce effort from a type c agent. Consider, for simplicity of illustration, an agent whose true operating cost is c' can only cheat by mis-reporting type $c < c'$. Assume that the principal then pays the agent a flow cost of c with the intention of reimbursing the operating cost. In reality, the agent cannot afford to exert effort, but rather pretends to work while collecting the payment c over time. When there is no effort, there is no arrival. In order to mitigate cheating, the principal should not allow such a no-arrival state to last forever, but rather allow a finite *probation* period. If there is no arrival during this probation period, the agent will be terminated at the end of probation. On the other hand, an arrival during this period would reveal that the agent's true type is indeed c. In this case, uncertainty of the agent's type is resolved, and the principal can follow the contract structure $\gamma^c_{\mathsf{B}}(w,0)$ of Definition 3.3.1 after the first arrival. In order to ensure effort during probation for an agent who has truthfully reported his type, the promised utility needs to take an upward jump of at least β_c, and possibly higher, at the first arrival.

We now specify and derive the dynamics of the probation contract. The agent is only paid $dL_t = cdt$ to cover the operating cost during the probation period. A key element of this probation contract is a threshold $z \geqslant 0$. If the first arrival occurs at time t with $W^c_t \geqslant z$, the promised utility jumps up by exactly β_c. If $W^c_t < z$, on the other hand, the promised utility jumps to $z + \beta_c$ upon an arrival at time t, which means the magnitude of the jump is higher than β_c. Therefore, when the promised utility W^c_t is below the threshold z, it evolves according to (PK) with $H_t = z + \beta_c - W^c_t$, that is,

$$\frac{dW^c_t}{dt} = rW^c_t - \mu(z + \beta_c - W^c_t) = W^c_t - \mu(z + \beta_c). \tag{3.12}$$

Consider a probation period with length τ, that is, $W^c_\tau = 0$. With this boundary condition, we have a closed-form solution to (3.12), which is

$$W^c_t = \mu(z + \beta_c)\left(1 - e^{t-\tau}\right). \tag{3.13}$$

Consequently, if $z < \bar{w}_c$ the time it takes for W^c_t to decrease from the threshold z to 0 is

$$\tau_z := \ln \frac{\mu(z + \beta_c)}{r(\bar{w}_c - z)}. \tag{3.14}$$

If $\tau \leqslant \tau_z$, then (3.13) fully specifies the dynamics of the promised utility before the first arrival or end of probation, and

$$W^c_0(\tau, z) = \mu(z + \beta_c)\left(1 - e^{-\tau}\right), \quad \text{for } \tau < \tau_z. \tag{3.15}$$

If $\tau > \tau_z$, on the other hand, then (3.13) only captures the dynamics from $t = \tau_z$ to $t = \tau$, when $W_t^c \leqslant z$. For the initial period of time from $t = 0$ to $t = \tau - \tau_z$, the promised utility $W_t^c > z$, and evolves according to (PK) with $H_t = \beta_c$, that is,

$$\frac{dW_t^c}{dt} = rW_t^c - \mu\beta_c = r(W_t^c - \bar{w}_c).$$

With boundary condition $W_{\tau-\tau_z}^c = z$, we once again have a closed form solution for $t \in [0, \tau - \tau_z]$,

$$W_t^c = \bar{w}_c - (\bar{w}_c - z)e^{r(t+\tau_z-\tau)}. \tag{3.16}$$

Overall, if $\tau > \tau_z$, at time 0, we have

$$W_0^c(\tau, z) = \bar{w}_c - (\bar{w}_c - z)e^{-r(\tau-\tau_z)}, \quad \text{for } \tau < \tau_z. \tag{3.17}$$

and, if $\tau \leqslant \tau_z$,

If $z \geqslant \bar{w}_c$, on the other hand, W_t^c evolves according to (3.13) from the very beginning. In this case the initial promised utility is still (3.15).

The probation period ends if either the first arrival occurs or W_t^c becomes zero, whichever happens first. Therefore, we further define the first arrival time as

$$\tau_1^N := \min\{t \mid dN_t = 1\}.$$

With the above set up we present the following definition.

Definition 3.3.2. *For any probation time period $\tau \geqslant 0$ and threshold $z \geqslant 0$, we define a* probation contract *$\gamma_{\mathrm{P}}^c(\tau, z) = (L^c, \tau^c)$, which pays $dL_t^c = c dt$ and generates a promised utility process W_t^c that evolves according to the following rules for $t \in [0, \tau)$ if $\tau_1^N > \tau$.*

- *If $z < \bar{w}_c$, then W_t^c follows* (3.16) *for $t \in [0, \max\{\tau - \tau_z, 0\})$, and* (3.13) *for $t \in [\max\{\tau - \tau_z, 0\}, \tau]$, starting from* (3.17) *if $\tau > \tau_z$ and* (3.15) *if $\tau \leqslant \tau_z$.*

- *If $z \geqslant \bar{w}_c$, then W_t^c follows* (3.13) *for $t \in [0, \tau]$, starting from* (3.15).

If $\tau_1^N \leqslant \tau$, then the aforementioned dynamics lasts until $\tau^c = \tau_1^N$. After that point, the contract continues with $\gamma_{\mathsf{B}}^c\left(\max\left\{W_{\tau_1^N}^c, z\right\} + \beta_c, 0\right)$ by resetting time τ_1^N to 0.

In the next section we demonstrate when this contract structure may be optimal, as well as how to specify the corresponding τ and z values. Here, we use a figure to better illustrate this contract structure and the associated dynamic.

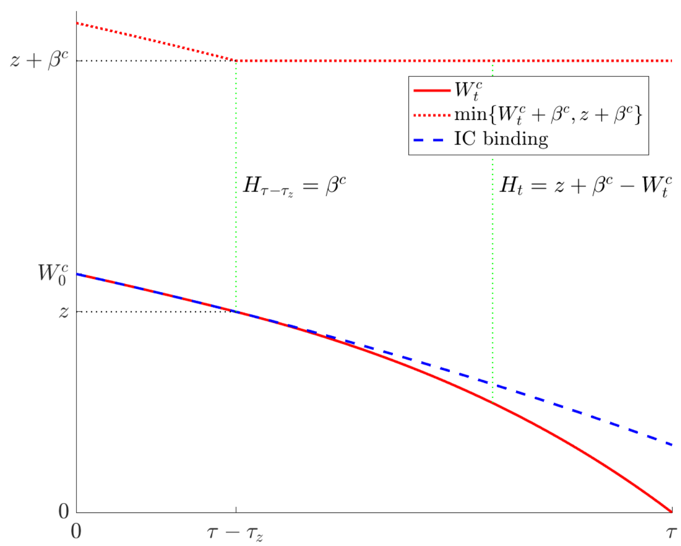

Figure 3.1: **Sample Trajectories of Promised Utility Before First Arrival**

Figure 3.1 gives an illustrative example of contract $\gamma_{\mathsf{P}}^c(\tau, z)$ for the case that $\tau > \tau_z$. The agent's promised utility trajectory W_t^c starts from W_0^c. Over time, if no arrival has occurred, the agent's promised utility drifts down, following the solid curve. If the first arrival occurs before τ, the promised utility jumps up to the dotted curve $\min\{W_t^c + \beta_c, z + \beta_c\}$. Conceptually, the difference $H_t^c = \min\{\beta_c, z + \beta_c - W_t^c\}$ represents the scale of upward jump in the agent's promised utility upon the first arrival. It is fixed and equal to β_c before time $\tau - \tau_z$. After $\tau - \tau_z$, however, the jump $H_t^c = z + \beta_c - W_t^c > \beta_c$, and the (IC) constraint is not binding. After this first arrival, the contract $\gamma_{\mathtt{P}}^{c}(\tau, z)$ sets H_t^c to β_c. Finally, the dashed curve, which overlaps with the solid curve when $t < \tau - \tau_z$, characterizes the movement of the promised utility following the dynamic (3.9) of the regular contract when H_t^c is kept at β_c. The figure implies that allowing the upward jump H_t^c to be higher than β_c effectively shrinks the probation period.

3.3.3 Principal and agent utilities

To conclude this section, we present results on the agent's and principal's utilities under the aforementioned contract structures. First, we formally establish the agent's utility under the sign-on-bonus contract and the probation contract.

Proposition 3.3.3. *(i) For any $w \geqslant 0$ and $B \geqslant 0$, we have $\bar{\nu} \in \mathfrak{N}(\gamma_{\mathtt{B}}^{c}(w, B), c)$, and*

$$u\Big(\gamma_{\mathtt{B}}^{c}(w, B), \bar{\nu}; c\Big) = w + B.$$

(ii) For any $\tau \geqslant 0$ and $z \geqslant 0$, we have $\bar{\nu} \in \mathfrak{N}(\gamma_{\mathtt{P}}^{c}(\tau, z), c)$, and

$$u\Big(\gamma_{\mathtt{P}}^{c}(\tau, z), \bar{\nu}; c\Big) = W_0^c(\tau, z),$$

in which $W_0^c(\tau, z)$ follows (3.17) and (3.15).

We will formally show in the next section that after the type c becomes known, it is optimal for the principal to follow the standard contract $\gamma_{\mathtt{B}}^{c}(w, 0)$. In this case, Proposition 3.3.3 verifies that the agent's utility under the standard contract $\gamma_{\mathtt{B}}^{c}(w, 0)$ is indeed w. The principal's value function, $F_c(w)$, as a function of the promised utility w, is defined by the following differential equation,[3]

$$r(w-\bar{w}_c)F_c'(w) = (c-\mu R) + F_c(w) - \mu F_c\left(w+\beta_c\right), \ \forall w \in [0,\bar{w}_c], \quad (3.18)$$

$$\text{with boundary conditions} \quad F_c(0) = 0 \quad \text{and} \quad F_c(w) = \frac{\mu R - c}{r} - w \text{ for } w \geqslant \bar{w}_c. \quad (3.19)$$

Proposition 3.3.4. *If $R \geqslant \beta_c$, the differential equation* (3.18) *with boundary condition* (3.19) *has a unique solution, $F_c(w)$, which is strictly concave on $[0,\bar{w}_c)$ and $F_c'(w) \geqslant -1$; furthermore, we have $F_c(w) = U(\gamma_{\mathsf{B}}^c(w,0),\bar{\nu})$. If $R < \beta_c$, on the other hand, define $F_c(w) = -w$. Overall, we have $Z(\{c\}) = \max_{w\geqslant 0} F_c(w)$.*

Proposition 3.3.4 implies that the function $F_c(w)$ is indeed the principal's value under contract $\gamma_{\mathsf{B}}^c(w,0)$, which is the optimal contract if the cost is known to be c. Furthermore, the maximizer of function $F_c(w)$ is the agent's utility under the optimal contract when the cost c is known to the principal.

In the next section, we consider the simple case when there are only two types of agents. We will prove that it is sufficient to only consider the sign-on-bonus and probation contract structures to construct an optimal menu of contracts, and we will show how to determine the parameters in these optimal contracts.

3.4 Two-Type Case

It is natural to first consider the simple case when there are only two types $\mathcal{C} = \{g,b\}$ with $g < b$. We can completely solve this two-type case, and the insights and results we derived in this section will also be useful in Section 3.5 when we consider the continuous-type case. The prior probabilities of types g and b are p and $1-p$, respectively. We refer to the agent with the lower cost g as the *good* agent, and b as the *bad* agent. In this section we require the following assumption.

[3] This differential equation is similar to Equation (12) in Sun and Tian (2018). The main difference is due to differences in model set-ups. In Sun and Tian (2018) the principal can pay reimburse the effort cost at a later time, while in our setting the principal needs to reimburse the effort cost immediately.

Assumption 3.4.1.

$\beta_g \leqslant R$, or, equivalently, $\mu R \geqslant g$.

This assumption guarantees that the good agent is efficient, which means that this agent generates a positive societal value whenever exerting effort. With this assumption, we exclude the trivial case where both agents are inefficient, because it is obviously dominant for the principal to offer a null contract with no payment and immediate termination in that trivial case. Note that we have not made any assumption on the bad agent's cost b yet. Indeed the bad agent can either be efficient ($\beta_b \leqslant R$) or inefficient ($\beta_b > R$), which leads to somewhat different optimal contract structures, as will be discussed later in this section.

Following the problem set up in the last section, the menu Γ offered by the principal contains two items, (γ^g, γ^b). The objective function of the principal's contract design problem (3.8) becomes

$$U\left(\Gamma_{\{g,b\}}\right) = pU(\gamma^g, \nu^g) + (1-p)U(\gamma^b, \nu^b). \tag{3.20}$$

In the remainder of this section, we demonstrate the construction of the optimal menu of contracts in steps. First, we construct an optimization problem, which provides an upper bound for the optimization (3.8), in Subsection 3.4.1. Then, in Subsection 3.4.4, we construct a menu of contracts based on the optimal solution to the upper bound optimization problem, and show that this menu of contracts achieves the upper bound, and therefore is indeed the optimal menu. The first two subsections are focused on technical results constructing the optimal menu of contracts. Finally, in Subsection 3.4.7 we discuss additional economic insights of our optimal solution.

3.4.1 Upper bound optimization

In this subsection we present a new optimization problem, which provides an upper bound to the original contract design problem (3.8). In the following result, we use functions F_g and F_b as defined in (3.18)-(3.19) for $c \in \{g, b\}$.

Proposition 3.4.2. *The following optimization problem yields an upper bound to the optimal value of the contract design problem* (3.8)*. That is,* $Y \geqslant Z(\{g,b\})$*, where*

$$Y := \max_{w_g, w_b, \tau, \xi} p \cdot G(w_g, \tau) + (1-p)\xi, \tag{3.21}$$

$$s.t.\ w_g \geqslant w_b \geqslant g(1-e^{-r\tau})/r, \tag{3.22}$$

$$\tau \geqslant 0, \tag{3.23}$$

$$\xi \leqslant F_b(w_b), \tag{3.24}$$

$$\xi \leqslant \frac{w_g - w_b}{b-g}(\mu R - b)^+ - w_b, \tag{3.25}$$

in which we define operator $(x)^+ := \max\{x, 0\}$*, and function* $G(w,\tau)$ *through the following optimal control problem, if* $\tau < \infty$*,*

$$G(w,\tau) := \max_{W_t, H_t} \int_0^\tau \mu e^{-t}[R + F_g(W_t + H_t)]dt - g(1-e^{-\tau}), \tag{3.26}$$

$$s.t.\ \frac{dW_t}{dt} = rW_{t-} - \mu H_t,\ for\ t \in [0,\tau];\ W_0 = w,\ W_\tau = 0,$$

$$H_t \geqslant \beta_g,\ \forall t \in [0,\tau];$$

if $\tau = \infty$*, with a slight abuse of notation, we define*

$$G(w,\tau) := \max_{W_t, H_t} \int_0^\infty \mu e^{-t}[R + F_g(W_t + H_t)]dt - g, \tag{3.27}$$

$$s.t.\ \frac{dW_t}{dt} = rW_{t-} - \mu H_t,\ for\ t \geqslant 0;\ W_0 = w,$$

$$W_t \geqslant 0, H_t \geqslant \beta_g,\ \forall t \geqslant 0.$$

It is instructive to explain the terms in the optimization problem (3.21)-(3.25). First, the decision variables w_g and w_b represent the utilities of type g and b agent under their respective contracts. The decision variable τ is the duration of the probation period for the type g agent. And the variable ξ represents the principal's expected utility facing a type b agent. The constraint (3.22) states that the good agent's utility, w_g, needs to be as good as or better than the bad agent's w_b. Furthermore, the last inequality in (3.22) states that w_b needs to be no less than the total discounted expected operating cost that the agent would receive by pretending to be a good agent. This is because receiving the operating cost g without working yields a utility $\displaystyle\int_0^\tau g e^{-rt} dt = g(1 - e^{-r\tau})/r$.

Next, the term $G(w_g, \tau)$ represents the principal's expected utility if the agent is the good type (with a low operating cost g). We explain the meaning of $G(w_g, \tau)$ in the following remark.

First, $G(w_g, \tau)$ calculates the principal's expected utility where the agent always exerts effort. The reason we can focus on full-effort contract is that any contract in which the agent shirks can be improved by a direct payment and no shirking. According to the optimal control problems (3.26) and (3.27), the principal designs a contract with an initial promised utility w and a (probation) time period τ. The decision variables include the promised utility process W_t, and the upward jump H_t associated with a potential arrival if $t \leqslant \tau$. During this period of time, if an arrival occurs, with rate μ, then the principal receives a revenue R, and the promised utility jumps to $W_t + H_t$, at which point the principal follows the contract that keeps the IC constraint binding, i.e., $\gamma_{\mathtt{B}}^g(W_t + H_t, 0)$ and earns a future utility $F_g(W_t + H_t)$, following Lemma 3.3.4. Recall $\mu + r = 1$, which explains the term $e^{-t} = e^{-(\mu+r)t}$. The constraints further captures the (PK), (IC) and (IR) constraints. Finally, the second term of the objective function in (3.26), $g(1 - e^{-\tau})$, captures the total discounted operating cost that the principal needs to pay before the first arrival or the end of the period, whichever comes first. Here, again, we use $\mu + r = 1$ as the effective discount rate.

Finally, we focus on constraints (3.24) and (3.25). First, constraint (3.24) states that the principal's utility ξ is upper bounded by $F_b(w_b)$ when offering the type b agent a promised utility w_b, consistent with Proposition 3.3.4. Finally, constraint (3.25) ensures a type g agent does not pretend to be of type b, which is elaborated in the following remark. Should the type g agent receive the type b contract, the agent is able to exert effort, and receive the same trajectory of payments as a type b agent. In addition to receiving the w_b reward, the type g agent also collects the extra operating cost $b-g$ for the duration of the contract. This (discounted) duration can be calculated as the (discounted) societal utility, $\xi + w_b$, divided by the societal utility rate, $\mu R - b$, if $\mu R > b$. This implies the following inequality,

$$w_g \geqslant w_b + (b-g)\frac{\xi + w_b}{\mu R - b}, \text{ or, equivalently, } \xi + w_b \leqslant \frac{w_g - w_b}{b-g}(\mu R - b). \quad (3.28)$$

If $\mu R \leqslant b$, on the other hand, the societal value of hiring the agent is negative, and, therefore, $\xi + w_b \leqslant 0$. Constraint (3.25) captures both cases of $\mu R > b$ and $\mu R \leqslant b$.

So far we have provided intuitive interpretations of various components of the optimization problem (3.21)-(3.25). This optimization plays a central role in our contract design problem. In the next subsection we convert the non-convex optimization problem (3.21)-(3.25) into an equivalent convex optimization problem, and obtain a menu of contracts based on its optimal solution. We further show that the performance of such a menu of contracts indeed achieves the upper bound Y of $Z(\{g, b\})$. Therefore, this menu of contracts is optimal. In our construction, each contract in the menu is either a probation contract or a sign-on-bonus contract defined in the previous section.

In order to solve the optimization (3.21)-(3.25), we first solve the deterministic optimal control problem (3.26). The solution approach is based on the Pontryagin minimum principle, as illustrated in the proof of the following Lemmas, presented in the Appendix.

Lemma 3.4.2. *For any $\tau \in [0, \infty)$, define thresholds*

$$\breve{\omega}(\tau) := \frac{1 - e^{-r\tau}}{r} g, \quad \text{and} \quad \hat{\omega}(\tau) := \frac{1 - e^{-\tau}}{r + \mu e^{-\tau}} g. \tag{3.29}$$

(i) If $w \in [\breve{\omega}(\tau), \hat{\omega}(\tau))$, then there exists a unique value $\mathrm{z}(w, \tau) \in [0, \hat{\omega}(\tau))$ (also referred to as z for simplicity) such that

$$w = \bar{w}_g - (\bar{w}_g - \mathrm{z}) e^{r(\tau_{\mathrm{z}} - \tau)}, \tag{3.30}$$

where $\bar{w}_g$ and τ_{z} are defined in (3.5) and (3.14), respectively, with $c = g$ and $z = \mathrm{z}$. Furthermore, the following W_t and H_t solves the optimization $G(w, \tau)$ in (3.26),

$$W_t = \begin{cases} \bar{w}_g - (\bar{w}_g - \mathrm{z}) e^{r(t + \tau_{\mathrm{z}} - \tau)}, & \text{for } t \in [0, \tau - \tau_{\mathrm{z}}], \\ \mu(\mathrm{z} + \beta_g)(1 - e^{t - \tau}), & \text{for } t \in [\tau - \tau_{\mathrm{z}}, \tau], \end{cases} \tag{3.31}$$

and

$$H_t = \begin{cases} \beta_g, & \text{for } t \in [0, \tau - \tau_{\mathrm{z}}], \\ \mathrm{z} + \beta_g - W_t, & \text{for } t \in [\tau - \tau_{\mathrm{z}}, \tau]. \end{cases} \tag{3.32}$$

(ii) If $w \geqslant \hat{\omega}(\tau)$, then define

$$\mathrm{z}(w, \tau) := \frac{w}{\mu(1 - e^{-\tau})} - \beta_g. \tag{3.33}$$

For any $t \in [0, \tau]$, the following H_t and W_t solves the optimization $G(w, \tau)$ in (3.26),

$$W_t = \mu(\mathrm{z} + \beta_g)(1 - e^{t - \tau}), \quad \text{and} \quad H_t = \mathrm{z} + \beta_g - W_t. \tag{3.34}$$

(iii) If $w < \breve{\omega}(\tau)$, the optimization problem is infeasible, or, by convention, $G(w, \tau) = -\infty$.

Similarly, we solve optimization problem (3.27) in the following Lemma.

Lemma 3.4.3. *If $\tau = \infty$, then define*

$$\mathrm{z}(w, \tau) := \frac{w}{\mu} - \beta_g, \tag{3.35}$$

(i) If $w \geqslant \frac{g}{r}$, then the following W_t and H_t solves the optimization (3.27),

$$W_t = w, \text{ and } H_t = \frac{w}{\mu} - w. \tag{3.36}$$

(ii) If $w < \frac{g}{r}$, the optimization problem (3.27) is infeasible, or, by convention, $G(w, \tau) = -\infty$.

We define the *discounted length* of the probation period to be

$$\bar{\tau} := \frac{1 - e^{-r\tau}}{r}. \tag{3.37}$$

Further define function

$$J(w, \bar{\tau}) := G\left(w, -\frac{\log(1 - r\bar{\tau})}{r}\right), \text{ for } \bar{\tau} \in \left[0, \frac{1}{r}\right], \ w \geqslant g\bar{\tau}. \tag{3.38}$$

Based on Lemma 3.4.2, we have the following result.

Proposition 3.4.3. *Function $J(w, \bar{\tau})$ is jointly concave in w and $\bar{\tau}$, and increasing in $\bar{\tau}$. Furthermore, we have*

$$Y = \max_{w_g, w_b, \bar{\tau}} p \cdot J(w_g, \bar{\tau}) + (1 - p) \min \left\{ F_b(w_b), \ \frac{w_g - w_b}{b - g}(\mu R - b)^+ - w_b \right\} \tag{3.39}$$

$$s.t. \ w_g \geqslant w_b \geqslant g \cdot \bar{\tau} \tag{3.40}$$

$$0 \leqslant \bar{\tau} \leqslant \frac{1}{r}. \tag{3.41}$$

Because the minimum of two concave functions is concave, the objective function in (3.39) is concave. Therefore, Proposition 3.4.3 implies that we can convert the non-convex optimization problem (3.21)-(3.25) into a convex optimization problem with linear constraints, which can be solved efficiently.

3.4.4 Optimal menu of contracts

Now we define a menu of contracts based on the upper bound optimization problem (3.39)-(3.41). Let $(w_g^*, w_b^*, \bar{\tau}^*)$ represent an optimal solution of the convex optimization (3.39)-(3.41). Define

$$\tau^* := -\frac{1}{r}\log(1 - r\bar{\tau}^*), \text{ and } z^* := \mathrm{z}(w_g^*, \tau^*), \tag{3.42}$$

in which the function $\mathrm{z}(w, \tau)$ is defined according to Lemma 3.4.2 and 3.4.3. We can then construct a probation contract $\gamma_{\mathtt{P}}^g(\tau^*, z^*)$ of Definition 3.3.2 for the good agent.

To construct the contract for the bad agent, we first present the following lemma.

Lemma 3.4.5. *We have $w_b^* \leqslant \bar{w}_b$. Furthermore, if $\mu R > b$, there exists a quantity $w \in [0, w_b^*]$ such that*

$$F_b(w) \leqslant \frac{\left(w_g^* - w_b^*\right)(\mu R - b)}{b - g} - w. \tag{3.43}$$

Lemma 3.4.5 implies that the following threshold is well-defined,

$$w_B := \begin{cases} \max\left\{ w \in [0, w_b^*] \;\middle|\; F_b(w) \leqslant \frac{\left(w_g^* - w_b^*\right)(\mu R - b)}{b-g} - w \right\}, & \text{if } \mu R > b, \\ 0, & \text{if } \mu R \leqslant b. \end{cases} \tag{3.44}$$

and the principal should give the bad agent a sign-on-bonus contract $\gamma_{\mathtt{B}}^b(w_B, w_b^* - w_B)$ of Definition 3.3.1.

To summarize, we define the following menu of contracts and show that it is optimal.

Definition 3.4.4. *Given the optimal solution $(w_g^*, w_b^*, \bar{\tau}^*)$ to the convex optimization (3.39)-(3.41), define a menu of contracts $\Gamma_{\{g,b\}}^* := \left\{\gamma_{\mathtt{P}}^g(\tau^*, z^*),\ \gamma_{\mathtt{B}}^b(w_B, w_b^* - w_B)\right\}$, in which τ^* and z^* are defined in (3.42), and w_B in (3.44).*

	$b \geqslant \bar{b}$	$b < \bar{b}$
Good agent	Probation contract $\gamma^g_{\mathtt{P}}(\tau^*, z^*)$	Probation contract $\gamma^g_{\mathtt{P}}(\tau^*, z^*)$
Bad agent	Pay-to-leave contract $\gamma^b_{\mathtt{B}}(0, w^*_b)$	Sign-on-bonus contract $\gamma^b_{\mathtt{B}}(w_B, w^*_b - w_B)$

Table 3.1: **Optimal Menu of Contracts**

Lemma 3.4.6. *There exists $\bar{b} \in [g, \mu R]$, such that $w_B = 0$ for $b \geqslant \bar{b}$ and $w_B > 0$ for $b < \bar{b}$.*

Lemma 3.4.6 shows that it is still possible that we have $w_B = 0$ even if $b < \mu R$.

Therefore, the good agent is always given a probation contract. The bad agent's contract depends on how high his operating cost is. If $b \geqslant \bar{b}$, or, the operating cost of the bad agent is too high to be worth hiring, then $w_B = 0$, and $\gamma^b_{\mathtt{B}}(0, w^*_b, 0)$ is a pay-to-leave contract. That is, the bad agent is paid an upfront payment w^*_b and asked to leave. If $b < \bar{b}$, on the other hand, the bad agent is still socially efficient, and would be hired by contract $\gamma^b_{\mathtt{B}}(w_B, w^*_b - w_B, 0)$, which allows the bad agent to work from an initial promised utility w_B.

We are now ready to present the main result of this section.

Theorem 3.4.5. *The menu of contracts $\Gamma^*_{\{g,b\}}$ satisfies* (LL), (PK), (IC), (IR), *and* (TT) *with $C = \{g, b\}$. Furthermore, we have $U(\Gamma^*_{\{g,b\}}) = Y$, in which Y is defined in* (3.21)-(3.25). *Therefore, we have $U(\Gamma^*_{\{g,b\}}) = Z(\{g, b\})$, or, the menu of contract $\Gamma^*_{\{g,b\}}$ solves the optimal contract design problem* (3.8) *with two types.*

We present in Table 3.1 a summary of the menu of optimal contracts in the two-type case. Although the good agent's contract is always a probation contract regardless of b, values τ^* and z^* are still functions of b.

It is worth reflecting incentives around the optimal menu of contracts $\Gamma^*_{\{g,b\}}$. In the case of $b \geqslant \bar{b}$, the initial bonus w^*_b to the bad agent equals the discounted total operating cost g that the agent can collect by pretending to be the good agent while shirking until the end of the probation period. This initial bonus mitigates the bad agent's incentive to lie about the high cost. It is also worth noting that the (TT) constraint for the good agent is not binding. That is, the good agent's promised utility w_g^* under the probation contract is strictly higher than the bonus w_b^*.

If $b < \bar{b}$, however, the principal may allow the bad agent to work following contract $\gamma_{\mathtt{B}}^b(w_B, w_b^* - w_B, 0)$, and provides sufficient information rent, w_b^*, for the bad agent to tell the truth. In order to discourage the good agent from pretending to be bad and exerting effort while collecting a higher operating cost reimbursement, b, the principal needs to lower the bad agent's contract's initial promised utility w_B. If this initial promised utility w_B is lower than the information rent w_b^*, however, the principal needs to pay the difference as an initial sign-on-bonus to the bad agent.

Before closing this section, we highlight the following properties of the optimal menu of contracts $\Gamma^*_{\{g,b\}}$.

Property 1:

$$\ell_t^g = g, \text{ and } \ell_t^b = b.$$

The optimal contracts never over compensates operating costs. That is, before contract termination, the good agent receives a flow payment g, and the bad agent receives a flow payment b.

Property 2:

$$\bar{\nu} \in \mathfrak{N}\Big(\gamma_{\mathtt{B}}^b(w_B, w_b^* - w_B),\ g\Big), \text{ and } N\Big(\gamma_{\mathtt{P}}^g(\tau^*, z^*),\ b\Big) = \{\nu^0\},$$

where $\nu^0 := \{\nu_t^0\}_{t\geqslant 0}$ is the *always shirking effort process.*

The first part of the property states that a good agent who pretends to be bad would exert full effort in response to the contact for the bad agent. It is rigorously proved in Lemma C.3.11 in the appendix. The second part of the property states that a bad agent who pretends to be good has to shirk until the end. This is because according to Property 1, the contract for the good agent only compensates the operating cost at rate g, which is too low to cover the bad agent's effort. As a result, under the probation contract $\gamma^g_{\text{P}}(\tau^*, z^*)$ for the good agent, the first arrival would confirm that the agent's type is indeed good, resolving the adverse selection issue. This property plays an important role in showing that $\Gamma^*_{\{g,b\}}$ is feasible to the (TT) constraint.

Property 3:

$$H^g_t = \beta_g, \ \forall t > \tau^N_1, \ \text{and} \ H^b_t = \beta_b, \ \forall t \geqslant 0.$$

This property indicates that under the menu $\Gamma^*_{\{g,b\}}$, the (IC) constraint is binding in the good agent's contract after the first arrival, and in the bad agent's contract the entire time. As mentioned earlier, the first arrival under the good agent's contract resolves adverse selection. Therefore the principal follows the most efficient contract, by setting the (IC) constraint binding. For the bad agent, on the other hand, arrivals do not resolve adverse selection because the good agent is able to mimick bad agent and generate arrivals. Therefore the principal always offers a dynamically efficient contract (binding (IC)) to the bad agent and adjusts other paramters in the menu to achieve optimality.

3.4.7 Welfare implications of unknown cost

In this section, we present how unknown cost affects the welfare of the principal and the agent, compared to the situations with known cost. We show that unknown cost always hurts the principal, but may hurt or benefit the agent, depending on whether or not the bad agent is efficient.

Denote $\bar{\mathcal{Y}}$ to represent the principal's expected payoff when cost is observable, and either takes value g with probability p, or b with probability $1-p$. That is,

$$\bar{\mathcal{Y}} := p\, Z(\{g\}) + (1-p)\, Z(\{b\}), \tag{3.45}$$

in which $Z(\{g\})$ and $Z(\{b\})$ are the principal's optimal utility earned from the good agent and the bad agent, respectively, following (3.8).[4] It is clear that the principal is

[4] In appendix C.2.2, we formally present the result when the agent's cost is known by the principal.

always better off knowing the cost of the agent before issuing the contract, or, $\mathcal{Y} \leqslant \bar{\mathcal{Y}}$. Intuitively, this conclusion follows from the basic idea of value of information. In particular, with known cost, the principal does not need to pay the information rent associated with unknown cost.

Now we consider the agent's utility in the following two different cases. Define w_*^g and w_*^b to be the maximizers of functions $F_g(w)$ and $F_b(w)$, respectively. Following Proposition 3.3.4, we know that they are the good and bad agents' utilities when the cost is observable under the respective optimal contracts. First, consider the situation that the bad agent is not efficient, or, $\beta_b \geqslant R$. In this case, the good agent is worse off and the bad agent is better off in the unknown cost situation, compared with the known cost one, as stated in the following result.

Proposition 3.4.6. *If $\beta_b \geqslant R$, we have*

$$w_g^* \leqslant w_*^g, \quad \textit{and} \quad w_b^* \geqslant w_*^b = 0, \tag{3.46}$$

where w_g^* and w_b^* are from the optimal solution of (3.39)-(3.41).

Apparently, the bad agent can earn the information rent if cost is unknown. Such an information rent does not exist if cost is known. Therefore the bad agent is better off with unknown cost. The good agent is worse off because with unknown cost, the bad agent could mimic the good agent, triggering the principal to curtail the good agent's payoff to prevent paying the bad agent too much information rent.

If the bad agent is efficient, or, $\beta_b < R$, then either agents can be better or worse off with unknown cost. We illustrate this with the following two examples.

Example 1. $p = 0.4$, $\mu = 0.8$, $r = 1 - \mu = 0.2$, $R = 20$, $g = 0.3$ and $b = 0.5$. In this case, we have

$$w_g^* = 2.76 > w_*^g = 1.16 \ \text{ and } \ w_b^* = 1.76 < w_*^b = 1.86.$$

Furthermore, in this example, $\bar{\tau}^* = 1/r$, which means that the probation period of the good agent's contract is infinite. That is, under the good agent's contract, the principal is willing to wait arbitrarily long for the first arrival, i.e. the good agent is never terminated.

Example 2. $p = 0.9$, $\mu = 0.8$, $r = 1 - \mu = 0.2$, $R = 3$, $g = 0.3$ and $b = 3.7$. In this case, we have

$$w_g^* = 0.93 < w_*^g = 0.941 \quad \text{and} \quad w_b^* = 0.93 > w_*^b = 0 \tag{3.47}$$

Furthermore, in this example, the bad agent is paid an amount $w_b^* = 0.93$ at time 0 to leave.

The reason that either agent can be better or worse off is because two competing forces influence agents' welfare when cost is unknown. First, as explained earlier, the bad agent benefits from mimicking the good agent, while this behavior hurts the good agent. This force is present no matter whether the bad agent is efficient. The second force is unique to the efficient bad agent case, where the good agent can potentially mimic the bad agent. This possibility can benefit the good agent while hurting the bad agent. Therefore, whether or not an agent is better off depends on which of the two forces dominates. In fact, in Example 1, the second force dominates the first, while in Example 2, the first dominates the second.

3.5 Continuous-type case

In this section we generalize the two-type case to a situation where the agent's operating cost may take value from an interval $C := [\underline{c}, \bar{c}]$ with $\underline{c} < \bar{c}$, following a commonly known cumulative distribution function $P(c)$ with probability density function $\rho(c)$. In this section we require the following assumption.

Assumption 3.5.1.

$$\int_{\underline{c}}^{\mu R} \rho(c) dc > 0.$$

This assumption is similar to the condition $\mu R \geqslant g$ in Assumption 3.4.1 for the two-type case, which guarantees that the agent is efficient with a positive probability, which excludes the trivial case where the agent is inefficient with probability 1. (If the agent is known to be inefficient, it is obviously a dominant strategy for the principal to immediately terminate the agent with no payment.) Note that we make no assumption on the worst cost $\bar{c}$. If $\bar{c} > \mu R$, an agent with cost $c > \mu R$ is not efficient, that is, not worth hiring.

In the contract design optimization (3.8) defined in Section 3.2, the objective function becomes

$$\mathcal{U}(\Gamma_{\mathcal{C}}) = \int_{\underline{c}}^{\bar{c}} \rho(c) U(\gamma^c, \nu^c) dc \tag{3.48}$$

in which the menu Γ_C offered by the principal contains a continuum of contracts γ^c for $c \in [\underline{c}, \bar{c}]$. Unlike for the two-type case, it appears hard to characterize the optimal solution for this infinite-dimensional optimization problem. Therefore, in this section, we focus on good approximations.

In Section 3.5.1, we first construct an optimization formulation similar to Section 3.4.1. However, this upper bound is hard to solve. Therefore, in Section 3.5.3, we provide a further relaxation that is easy to compute, using a dynamic programming approach. This upper bound calculation not only yields an upper bound for the optimal contract design, but also a way for us to design a menu of contracts. Therefore, in Section 3.5.4, we specify this menu of contracts, and compare its performance (a lower bound) with the upper bound. Furthermore, we show that if the solution in the upper bound calculation satisfies a simple condition, then the upper and lower bounds match, which implies that our contract design is in fact optimal. Numerical test illustrates that the condition is often satisfied with commonly used distributions.

3.5.1 Upper bound optimization

Similar to Section 3.4.1, we present a new optimization problem, which provides an upper bound to the contract design problem (3.48). First, we expand the definition of function J in (3.38) to include the cost variable as the following

$$J(w,\bar{\tau},c) := G\left(w, -\frac{\log(1-r\bar{\tau})}{r}, c\right), \text{ for } \bar{\tau} \in \left[0, \frac{1}{r}\right], \ c \in [\underline{c}, \min\{\bar{c}, \mu R\}] \tag{3.49}$$

and $w \geqslant c\bar{\tau}$, in which function function G is defined similar to function G in (3.26) and (3.27) as

$$G(w,\tau,c) := \max_{W_t, H_t} \int_0^\tau \mu e^{-t}[R + F_c(W_t + H_t)]dt - c(1-e^{-\tau}), \tag{3.50}$$

$$\text{s.t. } \frac{dW_t}{dt} = rW_{t-} - \mu H_t, \text{ for } t \in [0,\tau]; \ W_0 = w, \ W_\tau = 0,$$

$$H_t \geqslant \beta_c, \ \forall t \in [0,\tau].$$

Therefore, function $J(w,\bar{\tau},c)$ represents the principal's optimal utility when offering a type c agent a probation period with discounted length $\bar{\tau}$ and an initial promised utility level w. The relationship between the discounted length $\bar{\tau}$ and the real length τ of the probation period is defined in (3.37). Note that function J is well-defined only for $\bar{\tau} \in [0, 1/r]$, $c \in [\underline{c}, \ \min\{\bar{c}, \mu R\}]$, and $w \geqslant c\bar{\tau}$, when the corresponding optimal control problem (3.50) is admissible.

Lemma 3.5.2. *For $\bar{\tau} \in \left[0, \frac{1}{r}\right]$, $c \in [\underline{c}, \min\{\bar{c}, \mu R\}]$, and $w \geqslant c\bar{\tau}$, we have the following properties for function $J(w,\bar{\tau},c)$,*

(i) $J(w,\bar{\tau},c)$ is jointly concave in w and $\bar{\tau}$;

(ii) $J(w,\bar{\tau},c)$ is increasing in $\bar{\tau}$ and $J(w,0,c) = -w$;

(iii) $J(w,\bar{\tau},c) + w$ is non-decreasing in w;

(iv) $0 \leqslant J(w, \bar{\tau}, c) + w \leqslant \dfrac{\mu R - c}{r}$.

Based on the definition of the value function J we are ready to present the following upper bound optimization problem,

$$Y^C := \sup_{\mathsf{w}(\cdot)} \int_{\underline{c}}^{\min\{\bar{c},\mu R\}} \xi\big(c; \mathsf{w}(\cdot)\big)\rho(c)dc - \int_{\min\{\bar{c},\mu R\}}^{\bar{c}} \mathsf{w}(c)\rho(c)dc \tag{3.51}$$

$$s.t. \;\; \mathsf{w}(c) \text{ is non-increasing in } c \in [\underline{c}, \bar{c}], \tag{3.52}$$

in which for any $c \in [\underline{c}, \bar{c}]$ we define,

$$\xi\big(c; \mathsf{w}(\cdot)\big) := \min\left\{ J\left(\mathsf{w}(c),\;\; \min\left\{\frac{\mathsf{w}(\bar{c})}{c},\; \frac{1}{r}\right\},\; c\right),\right.$$

$$\left. \inf_{\tilde{c}\in[\underline{c},c)} \left[\frac{\mathsf{w}(\tilde{c}) - \mathsf{w}(c)}{c - \tilde{c}}\right] \cdot (\mu R - c) - \mathsf{w}(c) \right\}. \tag{3.53}$$

Theorem 3.5.2. *For any feasible menu of contracts $\Gamma_{\mathcal{C}}$ that satisfies* (LL), (PK), (IC), (IR), *and* (TT), *we have $Y^C \geqslant \mathcal{U}(\Gamma_{\mathcal{C}})$, where $\mathcal{U}(\Gamma_{\mathcal{C}})$ is defined in* (3.48).

The optimization problem (3.51)-(3.52) is a generalization of one defined in Proposition 3.4.3 for the two type case. First, the decision variables $\mathsf{w}(c)$ in the maximization problem (3.51)-(3.52) represent the initial promised utility assigned to the type c agent. If the agent is efficient ($c \leqslant \mu R$), the function $\xi(c, \mathsf{w}(\cdot))$ represents the principal's utility facing a type c agent when initial promised utility function is w, similar to the variable ξ in Proposition 3.4.2. If the agent is inefficient ($c > \mu R$), on the other hand, the objective function (3.51) implies that the principal's utility is $-\mathsf{w}(c)$, that is, the agent should be paid off and terminated immediately.

The constraint (3.52) states that the principal needs to offer a higher promised utility to the agent with a better type (lower cost) than to a worse type (higher cost). This monotonicity constraint partly mitigates the agent's incentive to mimic a *worse* type. The first term of the ξ function in (3.53) indicates that the principal's utility from type c agent is upper bounded by the function J, which calculates the principal's maximum expected utility when the type c agent always exert effort under a probation contract. (Functions J and G in this section corresponds to functions J and G in the last section. Remark 3.4.1 provides detailed explanations on G.) The second term in ξ ensures that the type c agent does not benefit by mimicking a type $\tilde{c}$ agent, where $\tilde{c} \in [\underline{c}, c)$. (The detailed explanations correspond to the discussion of the constraint (3.24) in Remark 3.4.1.)

To understand how to mitigate the agent's incentive to mimic a *better* type, we need to look at the term J inside (3.53). If an agent with a higher cost $\tilde{c}$ mimics the lower cost c and shirks through the probation period, the total discounted utility would be $c\bar{\tau}$, in which $\bar{\tau}$ represents the discounted probation period offered to type c. A constraint $c\bar{\tau} \leqslant \mathsf{w}(\tilde{c})$, or, equivalently, $\bar{\tau} \leqslant \mathsf{w}(\tilde{c})/c$, would mitigate such an incentive. Monotonicity of w following (3.52) implies that we only need to require $\bar{\tau} \leqslant \mathsf{w}(\bar{c})/c$. Following Lemma 3.5.2(ii), the function $J(w, \bar{\tau}, c)$ is increasing in $\bar{\tau}$. Therefore, in order to maximize J, it helps to to set $\bar{\tau}$ to the upper bound $\mathsf{w}(\bar{c})/c$. However, by definition, the discounted probation period $\bar{\tau}$ cannot be longer than $1/r$. This explains the second argument in function J.

Finally, given constraint (3.52), if $\mu R < \bar{c}$, it is clear that the optimal $\mathsf{w}(c)$ value for any $c > \mu R$ should be a constant, $\mathsf{w}(\bar{c})$.

3.5.3 Computing an upper bound of Y^C

The optimization problem (3.51)-(3.52) is infinite-dimensional, which is hard to solve. Therefore, we provide an efficient algorithm to compute an upper bound of Y^C based on a finite-dimensional approximation of. In the next subsection, we provide conditions to verify if the bound is tight.

Towards this goal, we divide the interval $[\underline{c},\ \min\{\bar{c}, \mu R\}]$ into a finite number of pieces. In particular, for a positive integer N, define $\delta := (\min\{\bar{c}, \mu R\} - \underline{c})/N$ and $c_i := \underline{c} + i\delta$ for $i \in \{0, \ldots, N\}$, such that $c_0 = \underline{c}$ and $c_N = \min\{\bar{c}, \mu R\}$.

Proposition 3.5.3. *Define*

$$\hat{y}(N) := \max_{w_0,\ldots,w_N} \sum_{i=1}^{N} [P(c_i) - P(c_{i-1})] \min\left\{ J\left(w_i, \min\left\{\frac{w_N}{c_i}, \frac{1}{r}\right\},\ c_i\right),\right.$$

$$\left.\frac{w_{i-1} - w_i}{\delta}(\mu R - c_i) - w_i\right\} \tag{3.54}$$

$$- w_N \int_{\min\{\mu R, \bar{c}\}}^{\bar{c}} \rho(c) dc \tag{3.55}$$

$$s.t.\ w_i \geqslant w_{i+1},\ \forall i \in \{0, \ldots, N-1\}.$$

We have

$$Y^C \leqslant \liminf_{N \to \infty} \hat{y}(N). \tag{3.56}$$

Therefore, we use w_i to approximate $\mathsf{w}(c_i)$ to obtain a finite-dimensional optimization (3.55). It is worth noting that the key difference between this upper bound optimization and the original problem. For a type $c = c_i$ and the corresponding $\mathsf{w}(c) = w_i$, the term $\inf_{\tilde{c} \in [\underline{c}, c)} \left[\frac{\mathsf{w}(\tilde{c}) - \mathsf{w}(c)}{c - \tilde{c}}\right]$ in the original formulation is replaced with a higher value $(w_{i-1} - w_i)/\delta$, which yields the upper bound. (In fact, if the optimal w function is concave, then this upper bound is tight. In the next subsection, we explore this further in the context of contract design.) The benefit of this change is computational efficiency. In fact, the optimization problem (3.55) can be solved using a dynamic programming approach.

Given a value $w_N \geqslant 0$, define the following deterministic dynamic programming recursion for any $i = \{1, \ldots, N\}$, starting from the boundary condition $\mathfrak{J}_0(w|w_N) = 0$

for all $w \geqslant w_N$,

$$\mathfrak{J}_i(w_i|w_N) = \max_{w_{i-1}\in[w_i,\infty)} \left[P(c_i) - P(c_{i-1})\right] \min\left\{\frac{w_{i-1} - w_i}{\delta}(\mu R - c_i) - w_i,\right.$$
$$\left. J\left(w_i, \min\left\{\frac{w_N}{c_i}, \frac{1}{r}\right\}, c_i\right)\right\}$$
$$+ \mathfrak{J}_{i-1}(w_{i-1}|w_N), \quad \forall w_i \geqslant w_N, \tag{3.57}$$

It is clear that

$$\hat{y}(N) = \max_{w_N\in[0,\infty)} \mathfrak{J}_N(w_N|w_N) - w_N(P(\bar{c}) - P(c_N)), \tag{3.58}$$

which implies that we can obtain the upper bound approximation $\hat{y}(N)$ by solving a sequence of the dynamic programming formulation (3.57) together with a one-dimensional search for the w_N value. Furthermore, we have the following result, which provides the closed-form solution to the maximization problem in (3.57).

Proposition 3.5.4. *For any given $i = 1, \ldots, N$ and $w_N \geqslant 0$, function $\mathfrak{J}_i(w|w_N)$ is concave in w. Use $\mathfrak{J}'_{i-1}(w|w_N)$ to represent the its left-derivative at w. Further fix a value $w_i \geqslant w_N$, and define*

$$\check{w} := \sup\left\{w \;\middle|\; w \geqslant w_i \text{ and } \mathfrak{J}'_{i-1}(w|w_N) \geqslant 0\right\},$$

$$\hat{w} := \inf\left\{w \;\middle|\; w \geqslant w_i \text{ and } \mathfrak{J}'_{i-1}(w|w_N) \leqslant -(\mu R - c_i)\frac{P(c_i) - P(c_{i-1})}{\delta}\right\}, \text{ and}$$

$$\bar{u} := \begin{cases} \dfrac{J\left(w_i, \min\left\{w_N/c_i, 1/r\right\}, c_i\right) + w_i}{\mu R - c_i}, & \text{if } \mu R - c_i > 0 \\ 0, & \text{if } \mu R - c_i = 0. \end{cases}$$

We have $\check{w} \leqslant \hat{w}$, and the following defined w^*_{i-1} solves the right-hand-side optimization problem in (3.57),

$$w^*_{i-1} := \begin{cases} \check{w}, & \text{if } w_i \leqslant \check{w} - \bar{u}\delta, \\ w_i + \bar{u}\delta, & \text{if } w_i \in (\check{w} - \bar{u}\delta,\ \hat{w} - \bar{u}\delta], \\ \hat{w}, & \text{if } w_i \in (\hat{w} - \bar{u}\delta,\ \hat{w}], \\ w_i, & \text{if } w_i > \hat{w}. \end{cases}$$

Concavity of $\mathfrak{J}_i(w|w_N)$ follows from an induction proof showing that the objective of the maximization in (3.57) is jointly concave in w_i and w_{i-1}. This concavity property is crucial for us to obtain the closed-form optimal solution u^*.

Finally, we have the following result, which provides an upper bound for the optimal w_N.

Proposition 3.5.5. *Define* $\bar{\mathrm{w}} := \min\{\mu R - \underline{c}, \bar{c}\}/r$. *For any* $w_N \geqslant \mathrm{w}$, *we have* $\mathfrak{J}_N(w_N|w_N) \leqslant \mathfrak{J}_N(\bar{\mathrm{w}}|\bar{\mathrm{w}})$.

Proposition 3.5.5 implies that we can focus the search for the optimal w_N that solves (3.58) in the interval $[0, \bar{\mathrm{w}}]$.

In the following subsection, we construct a menu of contracts based on the optimal sequence of promised utilities obtained in Proposition 3.5.4 using the optimal w_N value that solves (3.58).

3.5.4 Contract design

Note that the upper bound computation from the last subsection generates a non-increasing sequence w_i of initial promised utilities. Now we construct a menu of contracts based on this sequence.

From a generic non-negative and non-increasing sequence $\mathsf{w} := \{w_i\}_{i=0,\ldots,N}$, we propose a menu of contracts. In particular, if the cost c is higher than μR, then the agent is paid w_N and terminated immediately, which corresponds to contract $\gamma_{\mathsf{B}}^{c}(0, w_N, 0)$ following Definition 3.3.1. If $c \in (c_{i-1}, c_i]$, on the other hand, the agent is given a probation contract $\gamma_{\mathsf{P}}^{c_i}(\tau, z)$, where we need to specify a probation time τ and a threshold z according to Definition 3.3.2. The following *discounted* probation period length is well-defined for $i = 1, \ldots, N$, according to Lemma 3.5.2,

$$\bar{\tau}_{\mathsf{w}}^{i} :=$$
$$\max\left\{\bar{\tau} \in \left[0, \min\left\{\frac{w_N}{\bar{c}}, \frac{1}{r}\right\}\right] \;\middle|\; J\left(w_i, \bar{\tau}, c_i\right) \leqslant \min_{j\in\{0,\ldots,i-1\}} \frac{w_j - w_i}{(i-j)\delta}(\mu R - c_i) - w_i\right\}. \tag{3.59}$$

That is, the discounted probation period $\bar{\tau}_{\mathsf{w}}^{i}$ allows the corresponding principal's utility to mimic $\xi(c_i; w_i)$ defined in (3.53).

Based on the definition of the discounted probation period, we can define the *actual* probation period length and the threshold as, respectively,

$$\tau_{\mathsf{w}}^{i} := -\frac{1}{r}\log(1 - r\bar{\tau}_{\mathsf{w}}^{i}), \quad \text{and} \quad z_{\mathsf{w}}^{i} := \frac{w_i}{\mu(1 - e^{-\tau_{\mathsf{w}}^{i}})} - \beta_{c_i}, \tag{3.60}$$

in which the calculation of the threshold z_{w}^{i} is consistent with the function z defined in (3.33), with c_i replacing g and w_i replacing w.

Lemma 3.5.5. *For any non-negative and non-increasing sequence* $\mathsf{w} := \{w_i\}_{i=0,\ldots,N}$*, define a menu of contracts* $\hat{\Gamma}_C^{\mathsf{w}} := \{\gamma_{\mathsf{w}}^{c}\}_{c\in C}$*, in which*

$$\gamma_{\mathsf{w}}^{c} = \begin{cases} \gamma_{\mathsf{P}}^{c_i}(\tau_{\mathsf{w}}^{i}, z_{\mathsf{w}}^{i}), & \text{if } c \in (c_{i-1}, c_i],\ \forall i = 1, \ldots, N, \\ \gamma_{\mathsf{B}}^{c}(0, w_N, 0), & \text{if } c \in (\mu R, \bar{c}], \end{cases}$$

and $\gamma_{\mathsf{w}}^{\underline{c}} = \gamma_{\mathsf{P}}^{c_0}(\tau_{\mathsf{w}}^{0}, z_{\mathsf{w}}^{0})$*. The menu of contracts* $\hat{\Gamma}_C^{\mathsf{w}}$ *satisfies* (LL)*,* (PK)*,* (IC)*,* (IR)*, and* (TT)*.*

In particular, following (3.58) and Proposition 3.5.4, we have a sequence of initial promised utilities $\boldsymbol{w}_N^* := \{w_i^*\}_{i=1,\ldots,N}$ that are optimal solutions to the upper bound problem $\hat{y}(N)$. Lemma 3.5.5 and Proposition 3.5.2 imply that

$$Y^C \geqslant U\left(\hat{\Gamma}_C^{\boldsymbol{w}_N^*}\right). \tag{3.61}$$

The following result provides a condition that one can use to verify when the inequality (3.61) holds as an equality.

Theorem 3.5.6. *If the sequence $\boldsymbol{w}_N^*$ is "convex," in the sense that $2w_i \leqslant w_{i-1}+w_{i+1}$ for all $i = 1, \ldots, N-1$, we have*

$$\hat{y}(N) = U\left(\hat{\Gamma}_C^{\boldsymbol{w}_N^*}\right).$$

If $\boldsymbol{w}_N^*$ is always convex when N is large enough, then Theorem 1, together with (3.56), implies that as N approaches infinity, (3.61) holds as an equality, or, the menu of contracts $\hat{\Gamma}_C^{\boldsymbol{w}_N^*}$ is optimal in the limit.

We conduct a numerical study to see the performance of $\hat{\Gamma}_C^{\boldsymbol{w}_N^*}$. In the numerical study, we keep the support of the cost as $[\underline{c}, \bar{c}]$ where $\underline{c} = 1$ and $\bar{c} = 5$. Further, we let the density function of the cost be uniform distribution, symmetric triangular distribution, truncated exponential distribution ($\lambda = 1$), truncated normal distribution ($\mu = 3, \sigma = 1$) and U-quadratic distribution, respectively. We take $\mu = \{0.1, 0.2, ..., 0.9\}$, $R = \{2, 4, ..., 20\}$, $r = 1 - \mu$ and $N = 200$. For each distribution, there are 80 cases that $\mu R \geqslant \underline{c}$. In each of the cases, we calculate the optimal solution to the upper bound problem $\hat{y}(N)$, i.e., $\boldsymbol{w}_N^*$. Numerical result shows that in all of these cases, $\boldsymbol{w}_N^*$ is "convex". Therefore, the menu of contracts $\hat{\Gamma}_C^{\boldsymbol{w}_N^*}$ is a good approximation of the optimal solution to the original contract design problem when N is large.

3.6 Conclusion

We study an optimal incentive design problem in continuous time over an infinite horizon with both moral hazard and adverse selection. Specifically, the principal hires an agent to exert effort to increase the arrival rate of a Poisson process where the agent's efforts are unobservable by the principal and the agent's capabilities, measured by the operating cost, are unknown by the principal. This type of problem is common in many businesses, R&D, and political environments.

Although combining dynamic moral hazard and adverse selection is generally hard, we can completely solve the problem when there are two types of agents. We show that the optimal contracts take simple and intuitive forms: the low-cost agent always takes the form of a probation contract, and the high-cost agent takes the form of a sign-on-bonus contract or a pay-to-leave contract, depending on how high his cost is. All of these forms of contracts are easy to compute and implement. Furthermore, based on the solution of the two-type problem, we can tackle the continuous-type problem. In the continuous-type problem, the principal designs a menu of contracts, each of which has a form of pay-to-leave, sign-on-bonus, or probation contract.

Our model serves as a foundation for dynamic contract design problems with both moral hazard and adverse selection. First, in our model, the agent's effort cost is unknown by the principal. It is also worthwhile to think about the model when the arrival rate is the agent's private information. Second, in our model, the agent exerts effort to increase the arrival rate of "good" events. Existing literature reveals that the optimal contract takes a very different structure when the arrivals are bad to the principal (Biais et al., 2010c). Hence, it is worth considering when the agent is hired to decrease the arrival rate of adverse events, and the agent's capabilities/costs are unknown by the principal. Another possible extension is to consider the replacement of agents. Each time the principal terminates an agent, she can find another agent as the replacement from a pool of agents with uncertain capabilities. We leave these extensions to future studies.

Appendix A

Concealing Losses in Dynamic Relationships

A.1 Proofs in Section 1.3 and 1.4

Definition A.1.1. *A (non-homogeneous) Poisson process N_t is called a **thinning** of another (non-homogeneous) Poisson process $\tilde{N}_t$ if*

$$dN_t = I_t d\tilde{N}_t \tag{A.1}$$

for some Binary stochastic process $I_t \in \{0, 1\}$.

Proposition A.1.2. *A public history h^t is consistent with a private history $\tilde{h}^t$ if and only if the publicly observed Process N_t is a thinning of the agent's privately observed Process $\tilde{N}_t$.*

Proof of Proposition A.1.2

Proof.

- If $\theta = 0$, then let $I_t = 1$ if nature discloses it and $I_t = a_t$ otherwise. An arrival is publicly observed if and only if the agent privately observes it and chooses to disclose it. Hence $dN_t = 1$ only when $d\tilde{N}_t = 1$ and $I_t = 1$; otherwise $dN_t = 0$. To sum up, $dN_t = I_t d\tilde{N}_t$.

- If $\theta = 1$, then let $I_t \equiv 1$. Obviously $dN_t = d\tilde{N}_t$ because the honest agent always reports all arrivals.

□

Proof of Lemma 1.3.1

Proof.

$$\begin{aligned}
\frac{\partial p_t'}{\partial \sigma} &= \lambda p_t (1 - p_t) \lambda \geqslant 0 \\
\frac{\partial j(p_t)}{\partial \sigma} &= -\frac{\lambda p (1-p)}{(p\lambda + (1-p)\lambda\sigma)^2} \leqslant 0
\end{aligned} \tag{A.2}$$

□

Lemma A.1.1. *$U(.)$ and $V(.)$ are both bounded above and below:*

$$0 \leqslant V(.) \leqslant \frac{B}{\rho}; \qquad 0 \leqslant U(.) \leqslant \frac{b}{r}. \tag{A.3}$$

The best-case scenario for both the principal and the agent is that no loss ever occurs, in which case both of them enjoy an eternal flow benefit. On the contrary, immediate termination yields their lower-bound continuation values of 0. As will be seen later, the upper bound coincides with the principal and agent's value at $p = 1$, when the public is certain that the agent is honest, while the lower bound coincides with their values at termination.

Proof of Lemma A.1.1

Proof. For $\forall p$, if the agent chooses $\sigma(p) \equiv 1$, immediately $U(p) \geqslant 0$. If the principal chooses to stop immediately, immediately she gets an outside of option of 0. Hence $V(p) \geqslant 0$. Furthermore, the maximal value the agent or the principal can get cannot exceed collecting flow benefit indefinitely.

$$U(p) \leqslant \int_0^\infty e^{-rt} b dt = \frac{b}{r}$$

$$V(p) \leqslant \int_0^\infty e^{-\rho t} B dt = \frac{B}{\rho} \tag{A.4}$$

□

Lemma A.1.2. *If $U(p) > 0$ then it must be that $\alpha(p_t) = 0$ whenever $dN_t = 0$.*

Lemma A.1.2 requires the principal not to implement randomization absent losses. Indeed, randomization as a method to penalize the agent is a harsh penalty. In Poisson processes, there is almost surely no arrival within a short duration. If the principal were to implement randomization even absent arrivals, such randomization is triggered so frequently that it is as if the principal terminates the agent immediately. It is worth noting that α is a function of the belief before a disclosed loss p_t rather than after the loss $j(p_t)$.

Proof of Lemma A.1.2

Proof. Suppose not. Then $U(p) = \alpha 0 + (1-\alpha)U(p)$, which yields $U = 0$. $\square$

A.2 Proofs in Section 1.5

Proof of Lemma 1.5.1

Proof. If the principal is myopic and $\sigma \equiv 1$, her problem (1.16) becomes

$$V^c(p_t) = Bdt - (\lambda_0 p + (1-p)\lambda)Kdt \tag{A.5}$$

This is strictly increasing in p with $V^c(p^m) = 0$. Hence the principal terminates if $V^c(p) < 0 = V^c(p^m)$, which is $p < p^m$. $\square$

Proof of Theorem 1.5.1

Proof. Letting $\sigma = 1$ and $V(j(p_t)) = 0$ in (1.16) yields

$$0 = Bdt + \bar{\lambda}(p_t)Kdt - \rho dt V(p_t) + \bar{\lambda}(p_t)dt\left[V(j(p_t)) - V(p_t)\right] + V'(p_t)p_t' dt + o(dt) \tag{A.6}$$

Omitting $o(dt)$ and plugging in $p_t' = p_t(1-p_t)(\lambda - \lambda_0)$ yields (1.22). Following the arguments in Keller and Rady (2015), $C_1 > 0$ is a constant of integration, as the value function is at least weakly concave in p and C_1 multiplies a concave term. Next I show $\bar{p} < p^m$. Notice that $V(p)$ is increasing in p and $V(p^m) = C_1 p^m \left(\frac{p^m}{1-p^m}\right)^{\frac{\lambda_0+\rho}{\lambda-\lambda_0}} > 0$. It has to be that $V(\bar{p})$ is solved by $\bar{p} < p^m$. $\square$

A.3 Proofs in Section 1.6

To analyze the equilibria, I start by considering all three possibilities in (BR): fully conceal, fully disclose and mix.

Lemma A.3.1. *In any MPE, $\sigma(p) \neq 0, \forall p$.*

Lemma A.3.2. *If $\sigma(p) = 1$ and $\alpha(p) = 0$, then the agent's value function $U(p)$ follows*

$$U'(p)p(1-p)(\lambda-\lambda_0) = rU(p) + \lambda\left[U(p) - U(j(p))\right] - b, \tag{A.7}$$

where $j(p) = \frac{p\lambda_0}{p\lambda_0+(1-p)\lambda\sigma} = \frac{p\lambda_0}{p\lambda_0+(1-p)\lambda}$ is the posterior belief after a loss is disclosed.

Notice that (A.7) is a delay differential equation (DDE) with a variable delay $j(p)$, which generally does not have a closed-form solution. Fortunately, a closed-form solution does exist in my model.

Proof of Lemma A.3.2

Proof. Notice $\alpha = 0$ implies $\underline{U}(p) = U(p)$. Letting $\sigma = 1$ and $\alpha = 0$ in (1.19) yields

$$\begin{aligned}0 &= \lambda dt U(j(p_t)) + (1-\lambda dt)e^{-rdt}U(p_{t+dt}) + bdt - U(p_t)\\ &= \left[U(p_{t+dt}) - U(p_t)\right] - rdtU(p_t) - \lambda dt\left[U(p_t) - U(j(p_t))\right] + o(dt).\end{aligned} \tag{A.8}$$

Omitting $o(dt)$, taking limit as dt tends to 0 yields

$$U'(p_t)p_t' = rU(p) + \lambda\left[U(p) - U(j(p))\right] - b. \tag{A.9}$$

The belief evolves by $p_t' = p(1-p)(\lambda-\lambda_0)$ following (1.13). Plugging p_t' into (A.9) yields (A.7). □

Proof of Proposition 1.6.1

Proof. First I show $U(p)$ in (1.24) solves (A.7).

$$\begin{aligned}LHS &= \frac{1}{p^2}\gamma C\left(\frac{1-p}{p}\right)^{\gamma-1}p(1-p)(\lambda-\lambda_0) = \gamma C\left(\frac{1-p}{p}\right)^{\gamma}(\lambda-\lambda_0)\\ RHS &= C\left(\frac{1-p}{p}\right)^{\gamma}\left[\lambda\left(\frac{\lambda}{\lambda_0}\right)^{\gamma} - \lambda + r\right]\end{aligned} \tag{A.10}$$

They are equal by definition of γ in (CE). By Lemma A.1.1, $U(p) \leqslant \frac{b}{r}$ and hence $C \geqslant 0$. Next, I show (CE) has a unique positive root. Consider the function $h(\gamma)$ as follows

$$h(\gamma) = \left(\frac{\lambda}{\lambda_0}\right)^{\gamma} - 1 - \frac{r}{\lambda} - \left(1 - \frac{\lambda_0}{\lambda}\right)\gamma \tag{A.11}$$

Obviously $h(\gamma)$ is continuous, with $h(0) = -\frac{r}{\lambda} < 0$ and $\lim_{\gamma\to\infty} h(\gamma) \to +\infty$. This guarantees the existence of positive root. Meanwhile, $h(\gamma)$ is strictly convex as $h''(\gamma) = \log^2\left(\frac{\lambda}{\lambda_0}\right)\left(\frac{\lambda}{\lambda_0}\right)^{\gamma} > 0$. This leads to the uniqueness of the positive root. □

Lemma A.3.3. *The agent's value function $U(p)$ when $\sigma(p) = 1$ and $\alpha(p) = 0$ satisfies the following properties*

- $U'(p) \geqslant 0$
- $\lim\limits_{p\to 1} U(p) = \frac{b}{r}$
- $\gamma = -\frac{d\left[\log\left(\frac{b}{r} - U(p)\right)\right]}{d\left[\log\left(\frac{p}{1-p}\right)\right]}$

Proof of Lemma A.3.3

Proof. First, $U'(p) = \gamma C\left(\frac{1-p}{p}\right)^{\gamma}\frac{1}{p^2} \geqslant 0$. Next, $\lim\limits_{p\to 1} U(p) = \frac{b}{r} - C\lim\limits_{p\to 1}\left(\frac{1-p}{p}\right)^{\gamma} = \frac{b}{r}$. Last, define $x = \frac{p}{1-p}$ then $-\frac{d\left[\log\left(\frac{b}{r} - U(p)\right)\right]}{d\left[\log\left(\frac{p}{1-p}\right)\right]} = -\frac{-\gamma d\log(x)}{d\log(x)} = \gamma$. □

Proof of Lemma 1.6.1

Proof. $U(p) - U(j(p)) = C\left(\frac{1-p}{p}\right)^{\gamma}\left[\left(\frac{\lambda}{\lambda_0}\right)^{\gamma} - 1\right]$, which is decreasing in p because $\frac{d[U(p)-U(j(p))]}{dp} = -C\left(\frac{1-p}{p}\right)^{\gamma}\frac{1}{p^2}\left[\left(\frac{\lambda}{\lambda_0}\right)^{\gamma} - 1\right] \leqslant 0$. □

Proof of Corollary A.3.2

Proof. Notice that by (BR), the agent will fully disclose if $U(p) - U(j(p)) \leqslant k$. Plugging in (1.24) yields (A.24).

$$k \geqslant U(p) - U(j(p)) = -C\left(\frac{1-p}{p}\right)^{\gamma} + C\left(\frac{1-j(p)}{j(p)}\right)^{\gamma}$$
$$= C\left(\frac{1-p}{p}\right)^{\gamma}\left(\left(\frac{\lambda}{\lambda_0}\right)^{\gamma} - 1\right) \tag{A.12}$$

□

Proof of Lemma 1.6.2

Proof. Suppose this is true for an interval, such that p and $j(p)$ are both the interior, then $V(p) = V^c(p) = 0$. By (1.25), $Bdt - \bar{\lambda}(p_t)Kdt$, which yields $\sigma(p) = \frac{B-\lambda_0 Kp}{\lambda K(1-p)}$. □

Proof of Proposition 1.6.2

Proof. First, $U'(p) = \eta D p^{\eta-1}$ and $\lambda\sigma(p) - \lambda_0 = \frac{B-\lambda_0 K}{K(1-p)} = \frac{r}{\eta(1-p)}$. Plugging (1.26) into (A.18) yields

$$LHS = \eta D p^{\eta-1} p(1-p)\frac{r}{\eta(1-p)} = b - \lambda k + rDp^{\eta} + \lambda k - b = RHS. \tag{A.13}$$

□

Lemma A.3.4. *The agent's mixed probability $\sigma(p)$ satisfies the following properties*

- $\sigma(p)$ *is strictly increasing in* p *and* $\sigma(\bar{p}) = 1$*;*
- $\frac{\lambda_0}{\lambda} < \sigma(p) \leqslant 1$*;*
- $j(p) = \frac{\lambda_0 K}{B} p < p$.

Proof of Lemma A.3.4

Proof. First, $\sigma(p)$ is strictly increasing because

$$\sigma'(p) = \frac{B - \lambda_0 K}{\lambda K(1-p)^2} > 0. \tag{A.14}$$

Next, evaluating $\sigma(p^m)$ yields

$$\sigma(p^m) = \frac{B - \lambda_0 K \frac{\lambda - \frac{B}{K}}{\lambda - \lambda_0}}{\lambda K (1 - \frac{\lambda - \frac{B}{K}}{\lambda - \lambda_0})} = 1. \tag{A.15}$$

Meanwhile, evaluating $\sigma(0)$ yields

$$\sigma(0) = \frac{B}{\lambda K} > \frac{\lambda_0 K}{\lambda K} = \frac{\lambda_0}{\lambda} \tag{A.16}$$

Finally,

$$j(p) = \frac{p\lambda_0}{p\lambda_0 + (1-p)\lambda\sigma(p)} = \frac{\lambda_0 K}{b} p < p \tag{A.17}$$

□

Lemma A.3.5. *If* $0 < \sigma(p) < 1$ *and* $\alpha(p) > 0$*, then the agent's value function follows*

$$U'(p)p(1-p)\left(\frac{B - \lambda_0 K p}{K(1-p)} - \lambda_0\right) = rU(p) + \lambda k - b \tag{A.18}$$

The differential equation (A.18) is a first order linear differential equation, whose general solutions are computed in (1.26).

Proof of Lemma A.3.5

Proof. First, notice that the agent playing $0 < \sigma(p) < 0$ requires $U(p) - (1 - \alpha(p))U(j(p)) = k$. This implies that I can rewrite (1.19) as

$$U(p_t) = bdt + \lambda dt \left(U(p_t) - k\right) + (1 - \lambda dt)e^{-rdt}U(p_{t+dt}). \tag{A.19}$$

This leads to the following differential equation

$$U'(p)p(1-p)\left(\lambda\sigma(p)-\lambda_0\right) = rU(p)+\lambda k - b \tag{A.20}$$

Plugging in $\sigma(p)$ from Lemma 1.6.2 yields (A.18). □

Lemma A.3.6. *The agent's value function $U(p)$ when $0 < \sigma(p) < 1$ and $\alpha(p) > 0$ satisfies the following properties*

- $U'(p) \geqslant 0$
- $\lim\limits_{p\to 0} U(p) = \frac{b-\lambda k}{r}$
- $\eta = \frac{d\left[\log\left(U(p)-\frac{b-\lambda k}{r}\right)\right]}{d[\log(p)]}$

Proof of Lemma A.3.6

Proof. First, $U'(p) = D\eta p^{\eta-1} > 0$. Next, $\lim\limits_{p\to 0} U(p) = \frac{b-\lambda k}{r} - D\lim\limits_{p\to 0} p^{\eta} = \frac{b-\lambda k}{r}$. Last, $\frac{d\left[\log\left(U(p)-\frac{b-\lambda k}{r}\right)\right]}{d[\log(p)]} = \frac{d\log(p^{\eta})}{d\log(p)} = \eta$. □

Lemma A.3.7. *The principal terminates the agent with probability*

$$\alpha(p) = 1 - \frac{b-(\lambda+r)k+rDp^{\eta}}{b-rk+rD\left(\frac{\lambda_0 K}{B}\right)^{\eta}p^{\eta}} \leqslant 1. \tag{A.21}$$

Furthermore, $\alpha'(p) < 0$.

The principal's termination probability is calibrated as in Lemma A.3.7 such that the agent is exactly indifferent between disclosing and concealing his losses. It is also intuitive that the principal's termination probability decreases as the belief p increases, which means she becomes less harsh over time as the belief drifts up in the absence of a disclosed loss. Notice that $0 \leqslant \alpha(p) \leqslant 1$ must hold. Therefore, to verify $\alpha(p) \geqslant 0$, it is sufficient to consider the maximum p.

Proof of Lemma A.3.7

Proof. In order for the agent to be indifferent, by (BR),

$$\alpha(p) = 1 - \frac{U(p) - k}{U(j(p))} = 1 - \frac{\frac{b-\lambda k}{r} + Dp^\eta - k}{\frac{b-\lambda k}{r} + Dj^\eta(p)} = 1 - \frac{b - (\lambda + r)k + rDp^\eta}{b - rk + rD\left(\frac{\lambda_0 K}{B}\right)^\eta p^\eta} \tag{A.22}$$

By Assumption 1.3.2, the numerator is positive and hence $\alpha(p) \leqslant 1$. Furthermore, $\alpha'(p) = \eta \frac{rD\left(\left(\frac{\lambda_0 K}{B}\right)^\eta - 1\right)p^{\eta-1}\left(b - rk + rD\left(\frac{\lambda_0 K}{B}\right)^\eta p^\eta\right) - \lambda k rD\left(\frac{\lambda_0 K}{B}\right)^\eta p^{\eta-1}}{\left(b - rk + rD\left(\frac{\lambda_0 K}{B}\right)^\eta p^\eta\right)^2} < 0.$ □

Corollary A.3.1. *$\alpha(p) > 0$ holds $\forall p' \leqslant p$ if*

$$Dp^\eta \left(1 - \left(\frac{\lambda_0 K}{B}\right)^\eta\right) \leqslant k \tag{A.23}$$

Proof of Corollary A.3.1

Proof. By Lemma A.3.7, $\alpha(p)$ is decreasing and hence it is sufficient to check $\alpha(p) > 0$ the maximum value of p. Checking $\alpha(p) > 0$ yields $b - (\lambda + r)k + rDp^\eta \leqslant b - rk + rD\left(\frac{\lambda_0 K}{B}\right)^\eta p^\eta$, which is equivalent to (A.23). □

Proof of Lemma 1.6.3

Proof. Given the agent's strategy in (1.27), I verify that the principal's value in (1.28) satisfies (1.16). First, for $p < \bar{p}$, as $V^c(p) = bdt - \bar{\lambda}(p_t)Kdt + (1 - \bar{\lambda}(p_t)dt)V(p_{t+dt}) + \bar{\lambda}(p_t)dte^{-rdt}V\left(j(p_t)\right) = 0$. Second, for $p > \bar{p}$, following Theorem 1.5.1 yields $V^c(p) > 0$. □

Corollary A.3.2. *There exists a unique threshold p satisfying*

$$C\left(\left(\frac{\lambda}{\lambda_0}\right)^\gamma - 1\right)\left(\frac{1-p}{p}\right)^\gamma = k \tag{A.24}$$

such that the agent is willing to fully disclose his losses for all $p' \geqslant p$.

Lemma A.3.8. *The set $\mathcal{S}$ is nonempty.*

Proof of Lemma A.3.8

Proof. By the value matching condition (1.29), $C\left(\frac{1-\bar{p}}{\bar{p}}\right)^{\gamma} + D\bar{p}^{\eta} = \frac{\lambda k}{r}$. Meanwhile, (A.24) requires $C\left(\left(\frac{\lambda}{\lambda_0}\right)^{\gamma} - 1\right)\left(\frac{1-p}{p}\right)^{\gamma} \leqslant k$ and (A.23) requires $Dp^{\eta}\left(\left(\frac{\lambda_0 K}{B}\right)^{\eta} - 1\right) \leqslant k$. To show $\mathcal{S}$ is nonempty is equivalent to show

$$\frac{1}{1-\left(\frac{\lambda_0 K}{B}\right)^{\eta}} + \frac{1}{\left(\frac{\lambda}{\lambda_0}\right)^{\gamma} - 1} \geqslant \frac{\lambda}{r}. \tag{A.25}$$

First, I show $\left(\frac{\lambda_0 K}{B}\right)^{\eta} \geqslant e^{-\frac{r}{\lambda_0}}$. This comes from the fact that $\left(\frac{\lambda_0 K}{B}\right)^{\eta} = \left(\frac{\lambda_0 K}{B}\right)^{\frac{\frac{r}{\lambda_0}}{\frac{B}{\lambda_0 K}-1}}$. Define auxiliary function $h_a(t) = \left(\frac{1}{t}\right)^{\frac{1}{t-1}}$ for $t \geqslant 1$. Then $\log\left(h_a(t)\right) = -\frac{1}{t-1}\log(t)$, and $\frac{d\log(h_a(t))}{dt} = \frac{\log(t)-1+\frac{1}{t}}{(t-1)^2} \geqslant 0$. Hence $h_a(t) \geqslant \lim\limits_{t\to 1^+} h_a(t) = \lim\limits_{z\to+\infty}\left(1+\frac{1}{z}\right)^{-z} = \frac{1}{e}$. Consequently, $\left(\frac{\lambda_0 K}{B}\right)^{\frac{\frac{r}{\lambda_0}}{\frac{B}{\lambda_0 K}-1}} = h_a\left(\frac{B}{\lambda_0 K}\right)^{\frac{r}{\lambda_0}} \geqslant e^{-\frac{r}{\lambda_0}}$. It follows that $\frac{1}{1-\left(\frac{\lambda_0 K}{B}\right)^{\eta}} \geqslant \frac{\lambda_0}{r}$. Next, I show $\frac{1}{\left(\frac{\lambda}{\lambda_0}\right)^{\gamma}-1} \geqslant \frac{\lambda-\lambda_0}{r}$. Let $q = \left(\frac{\lambda}{\lambda_0}\right)^{\gamma} - 1$, then $\gamma = \frac{\log(q+1)}{\log\left(\frac{\lambda}{\lambda_0}\right)}$. By definition of γ, the equation that determines q is $q = \left(\frac{\lambda}{\lambda_0}\right)^{\gamma} - 1 = \frac{r}{\lambda} + \left(1 - \frac{\lambda_0}{\lambda}\right)\gamma = \frac{r}{\lambda} + \left(1-\frac{\lambda_0}{\lambda}\right)\frac{\log(q+1)}{\log\left(\frac{\lambda}{\lambda_0}\right)}$. Define auxiliary function $g_a(\xi) = \frac{r}{\lambda} + \left(1-\frac{\lambda_0}{\lambda}\right)\frac{\log(\xi+1)}{\log\left(\frac{\lambda}{\lambda_0}\right)} - \xi$. It is easy to show that $g_a(\xi)$ is monotone decreasing with $g_a(0) = \frac{r}{\lambda} > 0$ and $\lim\limits_{\xi\to+\infty} g_a(\xi) > 0$. Therefore, to show that $q \leqslant \frac{r}{\lambda-\lambda_0}$, it is sufficient to show that $g_a(\frac{r}{\lambda-\lambda_0}) \geqslant 0$. This is true because $\frac{r}{\lambda} > 0$ and $0 < \frac{\lambda_0}{\lambda} < 1$. □

Proof of Theorem 1.6.4

Proof. Given the agent's strategy $\sigma(p)$ and the principal's strategy $\alpha(p)$. Their value functions satisfy (1.24), (1.26) and (1.23). By Corollary A.3.2, Corollary A.3.1 and Theorem 1.5.1, both players' strategies $(\alpha(p), \sigma(p))$ satisfy their best response conditions (1.9) and (1.8). □

Proof of Theorem 1.6.5

Proof. This corresponds to Theorem 1.6.4 where $D = \underline{D} = 0$. Specifically, the agent's value function for $p < \bar{p}$ is $U(p) = \frac{b-\lambda k}{r} + Dp^{\eta} = \frac{b-\lambda k}{r}$. In the meantime, since $C = \bar{C} = \frac{\lambda k}{r}\left(\frac{\bar{p}}{1-\bar{p}}\right)^{\gamma}$, the agent's value for $p > \bar{p}$ is $U(p) = \frac{b}{r} - \frac{\lambda k}{r}\left(\frac{\bar{p}}{1-\bar{p}}\right)^{\gamma}\left(\frac{1-p}{p}\right)^{\gamma}$. Since $(\bar{C}, \underline{D})$ are feasible, the agent's value function can be supported in the equilibrium. Furthermore, because $\underline{D} = 0$ is the lower bound and therefore $\bar{C} = \frac{\lambda k}{r}\left(\frac{\bar{p}}{1-\bar{p}}\right)^{\gamma}$ is the upper bound in all feasible set of (C, D), it has to be that $\tilde{U}(p) \geqslant U(p)$ for all p for all $\tilde{U}(.)$ that can be supported by feasible (C, D). Hence this is the agent's worst equilibrium. □

Proof of Theorem 1.6.6

Proof. This corresponds to Theorem 1.6.4 where $D = \bar{D}$. Specifically, the agent's value function for $p < \bar{p}$ is $U(p) = \frac{b-\lambda k}{r} + \bar{D}p^{\eta}$. In the meantime, the agent's value for $p > \bar{p}$ is $U(p) = \frac{b}{r} - \underline{C}\left(\frac{\bar{p}}{1-\bar{p}}\right)^{\gamma}\left(\frac{1-p}{p}\right)^{\gamma}$. Since $(\underline{C}, \bar{D})$ are feasible, the agent's value function can be supported in the equilibrium. Furthermore, because $\bar{D}$ is the upper bound and therefore $\underline{C}$ is the lower bound in all feasible set of (C, D), it has to be that $\tilde{U}(p) \leqslant U(p)$ for all p for all $\tilde{U}(.)$ that can be supported by feasible (C, D). Hence this is the agent's best equilibrium. □

A.4 Proofs in Section 1.7

Proof of Lemma 1.7.1

Proof. For $\forall z > 0$,

$$F(\gamma; z\lambda_0, z\lambda, zr) = \left(\frac{z\lambda}{z\lambda_0}\right)^{\gamma} - 1 - \frac{zr}{z\lambda} - \left(1 - \frac{z\lambda_0}{z\lambda}\right)\gamma = F(\gamma; \lambda_0, \lambda, r) \quad \text{(A.26)}$$

□

Proof of Proposition 1.7.2

Proof. Notice that γ as a function of parameter r and λ_0 is defined by the implicit function (CE).

$$\begin{aligned}
\frac{\partial\gamma}{\partial r} &= -\frac{\partial F(\gamma;\lambda_0,\lambda,r)/\partial r}{\partial F(\gamma;\lambda_0,\lambda,r)/\partial\gamma} = -\frac{-1/\lambda}{\log(\lambda/\lambda_0)(\lambda/\lambda_0)^\gamma-(1-\lambda/\lambda_0)} > 0 \\
\frac{\partial\gamma}{\partial\lambda_0} &= -\frac{\partial F(\gamma;\lambda_0,\lambda,r)/\partial\lambda_0}{\partial F(\gamma;\lambda_0,\lambda,r)/\partial\gamma} = -\frac{\gamma(\lambda/\lambda_0)^{\gamma-1}-\gamma(\lambda/\lambda_0)^{-2}}{\log(\lambda/\lambda_0)(\lambda/\lambda_0)^\gamma-(1-\lambda/\lambda_0)}(-\frac{\lambda}{\lambda_0^2}) < 0
\end{aligned} \tag{A.27}$$

$\square$

Proof of Proposition 1.7.3

Proof. First, $\frac{\partial\eta}{\partial r} = \frac{K}{B-\lambda_0 K} > 0$. Second, $\frac{\partial\eta}{\partial\lambda_0} = \frac{rK^2}{(B-\lambda_0 K)^2} > 0$. $\square$

Lemma A.4.1. *If $\theta = 1$, x_t is a submartingale; if $\theta = 0$, x_t is a supmartingale.*

Proof of Lemma A.4.1

Proof. For $x > 1$, consider the following two functions: $h_1(x) = x - 1 - \log(x)$ and $h_2(x) = x - 1 - x\log(x)$. We have $h_1'(x) = 1 - 1/x > 0$ and $h_2'(x) = 1 - \log(x) - x\frac{1}{x} = -\log(x) < 0$. Hence $h_1(x) > h_1(1) = 1 - 1 - \log(1) = 0$ and $h_2(x) < h_2(1) = 1 - 1 + 1\log(1) = 0$. Now

$$\begin{aligned}
&\lambda - \lambda_0 + \lambda_0\log(\frac{\lambda_0}{\lambda}) = \lambda_0 h_1(\frac{\lambda}{\lambda_0}) > 0 \\
&\lambda - \lambda_0 + \lambda\log(\frac{\lambda_0}{\lambda}) = \lambda_0 h_2(\frac{\lambda}{\lambda_0}) < 0
\end{aligned} \tag{A.28}$$

here $\lambda/\lambda_0 > 1$. Furthermore, if $\theta = 1$ then $\mathbb{E}[dN_t - \lambda_0 dt|\theta = 1] = 0$, and thus $\mathbb{E}[dx_t|\theta = 1] > 0$; if $\theta = 0$ then $\mathbb{E}[dN_t - \lambda dt|\theta = 0] = 0$ and thus $\mathbb{E}[dx_t|\theta = 0] < 0$. $\square$

Proof of Proposition 1.7.4

Proof. I first prove $\mathbb{E}[\tau|\theta = 1] = \infty$ by contradiction. Suppose not, then $\mathbb{E}[\tau|\theta = 1] < \infty$. Notice that when $\theta = 1$, the stochastic process $\{x_t - (\lambda - \lambda_0 + \lambda_0 \log(\frac{\lambda_0}{\lambda}))t\}$ is a martingale because

$$\mathbb{E}\left[d\left(x_t - \left(\lambda - \lambda_0 + \lambda_0 \log(\frac{\lambda_0}{\lambda})\right)t\right)\middle|\theta = 1\right]$$

$$= \mathbb{E}\left[\log(\frac{\lambda_0}{\lambda})\,(dN_t - \lambda_0 dt)\middle|\,\theta = 1\right] = 0. \tag{A.29}$$

Moreover, its conditional expectations of the absolute value of the martingale increments are almost surely bounded since $|\log(\frac{\lambda_0}{\lambda})| < \infty$. Hence the conditions for applying optional stopping theorem hold. Therefore,

$$\mathbb{E}^{\tau}\left[x_\tau - \left(\lambda - \lambda_0 + \lambda_0 \log(\frac{\lambda_0}{\lambda})\right)\tau\middle|\,\theta = 1\right] = x_0. \tag{A.30}$$

Hence

$$\mathbb{E}[\tau|\theta = 1] = \frac{\mathbb{E}^{\tau}[x_\tau|\theta = 1] - x_0}{\lambda - \lambda_0 + \lambda_0 \log(\frac{\lambda_0}{\lambda})} \leqslant \frac{x^{\dagger} - x_0}{\lambda - \lambda_0 + \lambda_0 \log(\frac{\lambda_0}{\lambda})} < 0. \tag{A.31}$$

Contradiction. Next, when $\theta = 0$, the stochastic process $\{x_t - (\lambda - \lambda_0 + \lambda \log(\frac{\lambda_0}{\lambda}))t\}$ is a martingale because

$$\mathbb{E}\left[d\left(x_t - \left(\lambda - \lambda_0 + \lambda \log(\frac{\lambda_0}{\lambda})\right)t\right)\middle|\theta = 0\right]$$

$$= \mathbb{E}\left[\log(\frac{\lambda_0}{\lambda})\,(dN_t - \lambda_0 dt)\middle|\,\theta = 0\right] = 0. \tag{A.32}$$

In this case τ is almost surely bounded, because

$$\lim_{t\to\infty}\mathbb{P}(\tau > t) \leqslant \lim_{t\to\infty}\mathbb{P}(N_t < \frac{x_0+(\lambda-\lambda_0)t-x^\dagger}{\log(\frac{\lambda}{\lambda_0})})$$

$$\leqslant \lim_{t\to\infty}\mathbb{P}(N_t-\mathbb{E}[N_t] < \frac{x_0+(\lambda-\lambda_0)t-x^\dagger}{\log(\frac{\lambda}{\lambda_0})} - \lambda t < 0)$$

$$\leqslant \lim_{t\to\infty}\mathbb{P}(|N_t-\mathbb{E}[N_t]| > -\frac{x_0+(\lambda-\lambda_0)t-x^\dagger}{\log(\frac{\lambda}{\lambda_0})} + \lambda t)$$

$$\leqslant \lim_{t\to\infty}\frac{\mathrm{Var}(N_t)}{(-\frac{x_0+(\lambda-\lambda_0)t-x^\dagger}{\log(\frac{\lambda}{\lambda_0})}+\lambda t)^2} = \lim_{t\to\infty}\frac{\lambda t}{(-\frac{x_0+(\lambda-\lambda_0)t-x^\dagger}{\log(\frac{\lambda}{\lambda_0})}+\lambda t)^2} = 0, \tag{A.33}$$

where the last inequality follows from Chebyshev's inequality. Hence optional stopping theorem yields

$$\mathbb{E}^\tau\left[x_\tau - \left(\lambda-\lambda_0+\lambda\log(\frac{\lambda_0}{\lambda})\right)\tau\middle|\theta=0\right] = x_0. \tag{A.34}$$

Hence

$$\mathbb{E}[\tau|\theta=0] = \frac{\mathbb{E}^\tau[x_\tau|\theta=0]-x_0}{\lambda-\lambda_0+\lambda\log(\frac{\lambda_0}{\lambda})} = \frac{x_0-\mathbb{E}^\tau[x_\tau|\theta=0]}{-\big(\lambda-\lambda_0+\lambda\log(\frac{\lambda_0}{\lambda})\big)}. \tag{A.35}$$

By definition of τ, let x_τ^- be the value before the Poisson arrival at τ, we have $x_\tau^- \geqslant x^\dagger$. Therefore,

$$x^\dagger + \log(\frac{\lambda_0}{\lambda}) \leqslant \mathbb{E}^\tau[x_\tau|\theta=0] \leqslant x^\dagger. \tag{A.36}$$

which yields

$$\frac{x_0-x^\dagger}{-\big(\lambda-\lambda_0+\lambda\log(\frac{\lambda_0}{\lambda})\big)} \leqslant \mathbb{E}[\tau|\theta=0] \leqslant \frac{x_0-x^\dagger-\log(\frac{\lambda_0}{\lambda})}{-\big(\lambda-\lambda_0+\lambda\log(\frac{\lambda_0}{\lambda})\big)}. \tag{A.37}$$

□

A.5 Proofs in Section 1.8

Proof of Theorem 1.8.3

Proof.

$$U'(p)p(1-p)(\lambda-\lambda_0) = rU(p) + \lambda\left[U(p) - U(j(p))\right] - b, \tag{A.38}$$

where $j(p) = \frac{p\lambda_0}{p\lambda_0+(1-p)\lambda\sigma} = \frac{p\lambda_0}{p\lambda_0+(1-p)\lambda}$ is the posterior belief after a loss is disclosed. The delay differential equation (A.38) is solved by

$$U(p) = \frac{b}{r} - C\left(\frac{p}{1-p}\right)^{\gamma} \tag{A.39}$$

for undetermined constant $C \geqslant 0$. Here $\gamma > 0$ is the unique positive root of the following characteristic equation

$$\left(\frac{\lambda_0}{\lambda}\right)^{\gamma} - 1 = \frac{r}{\lambda} + \left(\frac{\lambda_0}{\lambda} - 1\right)\gamma. \tag{CE'}$$

Next, suppose the agent plays $\sigma(p) = 0$, then

$$U'(p)p(1-p)(-\lambda_0) = rU(p) + \lambda k - b \tag{A.40}$$

This is solved by

$$U(p) = \frac{b-\lambda k}{r} + C_4\left(\frac{1-p}{p}\right)^{\frac{r}{\lambda_0}} \tag{A.41}$$

Therefore, the value function of the agent is identical to the baseline model, which yield the identical equilibrium. □

Proof of Theorem 1.8.4

Proof. Maximization over $\sigma(p_t)$ yields following first order conditions:

$$\sigma(p) = \begin{cases} 1, & U(p) - \alpha(p)U\left(j(p)\right) \leqslant k \\ 0 < \sigma < 1, & U(p) - \alpha(p)U\left(j(p)\right) = k \\ 0, & U(p) - \alpha(p)U\left(j(p)\right) \geqslant k. \end{cases} \tag{BR'}$$

Letting $k = \frac{b}{\lambda}$ yields the same HJB equation for the agent as in (1.19). Because the principal's and the agent's problem are identical between the baseline model and preemptive action extension, their MPBE are identical. $\square$

Proof of Proposition 1.8.5

Proof. Let $\sigma_\theta(p)$ denote the mixed strategy. The off-equilibrium belief is vacuously defined. If the principal believes that the off-path belief is 0, she would terminate whenever a loss is disclosed, which triggers termination. Therefore, to disclose a loss at any belief p, an agent receives a payoff of 0. However, if an agent of type θ chooses to conceal all losses, he receives $(b - \lambda_\theta k)/r - k > 0$, which means he prefers to conceal losses. $\square$

Appendix B

Setbacks, Shutdowns and Overruns

B.1 Proof of Lemmas 2.4.2, 2.4.4, and 2.4.6

B.1.1 Proof of Lemma 2.4.2

First, the principal always induces the high action ($a_t = 1$). Imagine there is an interval of time in which the principal induces shirking. The project does not advance, nor is there a setback. The principal can award the agent intermediate consumption without paying the flow cost of the project during the shirking interval without changing the agent's continuation utility. This makes the agent indifferent and the principal better off, because $c > b$ implies that assigning the agent any positive amount of utility by allowing shirking is more costly for the principal than directly paying the agent.

Second, any contract with intermediate payment can be weakly improved by one without. Because the principal and the agent share the same discount rate (0), the principal can simply delay any intermediate payments until the end, leaving both participants indifferent.

Third, any contract with severance pay upon termination can be improved by one that pays only on the event of success. Notice that the principal can re-start any existing incentive compatible contract and both participants will have positive value going forward. Any contract that ends with a severance payment can be replaced with one that randomizes between re-starting the contract and termination with zero payment. The probability of re-starting the project can be set to make the agent indifferent to the randomization. The principal is better off because she receives zero (termination) or a positive value (re-start) instead of making a severance payment.

B.1.2 Proof of Lemma 2.4.4

First, by Lemma 2.4.2, $dC_t = 0$ for all $t < \tau$. Therefore (2.6) becomes

$$W_t^{a,X} = E_{a,X}\left[\int_0^{\tau} b(1 - a_s)ds + K_\tau \,|\mathcal{F}_t\right],$$

where $\mathcal{F}_t$ is the filtration generated by the agent's report $\{\hat{X}_t\}_{t\in[0,\tau]}$. Note that W_t is an $\mathcal{F}_t$-martingale. Thus, by the martingale representation theorem for jump processes, there exists a $\mathcal{F}_t$-predictable, integrable process J such that

$$dW_t^{a,X} = J_t(\lambda dt - d\hat{N}_t). \tag{B.1}$$

If the contract is incentive compatible, the agent will exert high effort and report the setback N_t truthfully. In this case, the analysis following the statement of Lemma 2.4.4 applies, and the (NPS) condition

$$-J_t \geqslant b\delta + \int_0^\delta \lambda J_{t+s} ds - J_{t+\delta}, \quad \forall \delta \in (0, \bar{X} - X) \tag{B.2}$$

must hold as a necessary condition. Finally , J_t must be weakly positive, otherwise the agent would gain from falsely reporting a setback. □

B.1.3 Proof of Lemma 2.4.6

Suppose $W_t < b\bar{X}$ following a reported setback. Then randomization between $W_{t'} = 0$ and $W_{T'} \geqslant b\bar{X}$ is necessary. To see why, suppose that immediately after randomization (including degenerate randomization) or at the beginning of a phase of Poisson termination, the agent's continuation utility is $W_{t'} \in (0, b\bar{X})$. Imagine the agent continues to work for $\delta \in (W_{t'}/b, \bar{X})$ and then suffers a setback which he truthfully reports. By the round-trip property of (NPS) we have

$$W_{t'+\delta} = W_{t'} + \int_0^\delta \lambda J_s ds - J_\delta \tag{B.3}$$

$$\leqslant W_{t'} - J_0 - b\delta = W_{t'} - b\delta < 0, \tag{B.4}$$

which violates limited liability. Finally, concavity of the principal's value function (established in Proposition 2.5.4) implies that randomization should involve minimal dispersion. Hence, to deliver W_t to the agent, he should receive $W_{t'} = 0$ with probability $(1 - p)$ and $W_{t'} = b\bar{X}$ with probability $p = \frac{W_t}{b\bar{X}}$. □

B.2 Proof of Proposition 2.5.2

The proof of Proposition 2.5.2 consists of four parts. Part 1 demonstrates that the (NPS) constraint can be re-written in a more tractable way. Part 2 shows that (NPS) binds with equality in an optimal contract. Part 3 shows that if (NPS) holds with equality, then the optimal contract can be implemented with a simple time budget. Part 4 shows that a simple time budget is sufficient to make truthful reporting optimal for the agent. We will also take as given that the principal's value function is concave with respect to W (or, S), which is verified in Proposition 2.5.3 and the analysis in Section 2.5.2.

We begin with the following lemma which is helpful several times:

Lemma B.2.1 (Lower bound for the marginal value of agent's utility)**.** *Let $F(W, X)$ be the principal's value function (2.7), then $F_W(W, X) \geqslant -1$.*

Proof: Imagine not: $F_W(W, X) < -1$. Then the principal would gain by giving the agent intermediate consumption. But this cannot be the case (Lemma 2.4.2).□

B.2.2 Re-writing (NPS)

An important fact is that J can only vary with time (or, the progress of X) between t and $t + \delta$. Because no setbacks are being reported, the passage of time is the only thing the principal can observe. Thus, we can write J as a function of current project progress and all of history prior to the previous setback, $J(X, \cdot)$. We will suppress the $(\cdot)$ notation for convenience. Then, (NPS) becomes

$$J(X_t) \leqslant -b\delta - \int_0^\delta \lambda J(X_t + s)ds + J(X_t + \delta),\ \forall X_t \in [0, \bar{X}) \text{ and } \delta > 0. \qquad \text{(B.5)}$$

First, we reformulate (NPS) so that J can be written as the sum of its minimum, binding value and a term capturing the excess. If (NPS) binds everywhere, (B.5) holds with equality and we can take the derivative with respect to δ to obtain

$$0 = -b - \lambda J(X + \delta) + J'(X + \delta). \tag{B.6}$$

This has the (general) solution $J(X) = C_1 e^{\lambda X} - \frac{b}{\lambda}$, where C_1 is a constant. Because $J(X) \geqslant 0$, the minimum value of C_1 is $\frac{\beta}{\lambda}$. Thus, the minimum, binding value of $J(X)$ is

$$J^{min}(X) = \frac{b}{\lambda} e^{\lambda X} - \frac{b}{\lambda}. \tag{B.7}$$

Next, we define the functions $f(X)$ and $g(X)$ such that

$$f(X) = \frac{1}{b}\left[J(X) - J^{min}(X)\right] \tag{B.8}$$

$$g(X) = f(X) - \int_0^X \lambda f(u) du. \tag{B.9}$$

So, $f \geqslant 0$ captures the difference between J and its minimum, and g is a convenient summarizing function. Note that $f(0) = g(0)$, so $f(X)$ can be fully recovered from $g(X)$.

Second, we show that (NPS) is satisfied if and only if $g(X) \geqslant 0$ and f and g are weakly increasing. Substituting the definition of $f(X)$ into (B.5) yields

$$f(X) \leqslant -\int_0^\delta \lambda f(X + u) du + f(X + \delta), \ \forall X_t \in [0, \bar{X}) \text{ and } \delta > 0. \tag{B.10}$$

Then, from the definition of g, we have

$$g(X + \delta) - g(X) = f(X + \delta) - \int_0^{X+\delta} \lambda f(u) du - f(X) + \int_0^X \lambda f(u) du \tag{B.11}$$

$$= f(X + \delta) - \int_X^{X+\delta} \lambda f(u) du - f(X) \tag{B.12}$$

$$\geqslant 0, \tag{B.13}$$

where (B.10) shows that (B.12) is non-negative, so $g(X)$ is weakly increasing. More, $f(0) = g(0)$ and g weakly increasing imply $g(X) \geqslant 0$. Then, using the definition of g

and $f \geqslant 0$, we see that we must also have f weakly increasing. Using the definitions of f and g, we see that f and g positive and weakly increasing are also sufficient to show (B.5). Thus, we can use f and g instead of (B.10) to characterize (NPS).

We can now write the change in the agent's continuation utility as X starts at $X_t = 0$ and progresses until a setback is experienced at $X = X_{t+s}$. Using $X_{t+u} = u$ and the f and g notation, we have

$$\begin{aligned}\int_0^s \lambda J(X_{t+u})du - J(X_{t+s}) &= \int_0^s be^{\lambda u}du - b\bar{X} - \frac{b}{\lambda}e^{\lambda s} + \frac{b}{\lambda} - bg(X_{t+s}) \\ &= \frac{b}{\lambda}\left(e^{\lambda s} - 1\right) - bs - \frac{b}{\lambda}e^{\lambda s} + \frac{b}{\lambda} - bg(X_{t+s}) \\ &= -bs - bg(X_{t+s}). \qquad \text{(B.14)}\end{aligned}$$

Similarly, we can write down the prize the agent receives, given that the agent starts at time t with W_t and $X_t = 0$ and progresses to project completion:

$$W_t + \int_0^{\bar{X}} \lambda J(x)dx = W_t + \int_0^{\bar{X}} be^{\lambda x}dx - b\bar{X} + b\int_0^{\bar{X}} \lambda f(x)dx. \qquad \text{(B.15)}$$

B.2.3 The Optimality of Binding the (NPS)

We show that the optimal contract has $f = g = 0$. To proceed, using (B.14 and B.15) and that the probability that any particular try at the project is successful is $e^{-\lambda\bar{X}}$, we have that $F(W, X = 0)$ can be written as

$$F(W, X = 0) = e^{-\lambda\bar{X}}\left[R - \frac{b}{\lambda}\left(e^{\lambda\bar{X}} - 1\right) + (b - c)\bar{X} - W - b\int_0^{\bar{X}} \lambda f(u)du\right] \qquad \text{(B.16)}$$

$$+ \int_0^{\bar{X}} \lambda e^{-\lambda u}F(W - bu - bg(u), X = 0)du, \qquad \text{(B.17)}$$

with constraints $g'(u) \geqslant 0$ and $g(u) \geqslant 0$. We will use the Hamiltonian maximization method with $\zeta(u) = g'(u)$ as the control variable, $f(u)$ and $g(u)$ as the state variables, and $f'(u) = \lambda f(u) + \zeta(u)$ and $g'(u) = \zeta(u)$ as the laws of motion. The constraints are $\zeta(u) \geqslant 0$ and $g(u) \geqslant 0$. The objective function, ignoring constant terms, is

$$\max \int_0^{\bar{X}} \left[\lambda e^{-\lambda u} F\left(W - bu - bg(u), X = 0\right) - b\lambda e^{-\lambda \bar{X}} f(u)\right] du. \tag{B.18}$$

Then, the Hamiltonian is

$$\begin{aligned} \mathcal{H} =& e^{-\lambda u}\lambda F\left(W - bu - bg, X = 0\right) - b\lambda e^{-\lambda \bar{X}} f \\ &+ \gamma_1(\lambda f + \zeta) + \gamma_2 \zeta + \eta_1(\zeta - 0) + \eta_2(g - 0). \end{aligned} \tag{B.19}$$

The optimality conditions are

$$0 = \frac{\partial \mathcal{H}}{\partial \zeta} = \gamma_1 + \gamma_2 + \eta_1 \tag{B.20}$$

$$-\gamma_1' = \frac{\partial \mathcal{H}}{\partial f} = -b\lambda e^{-\lambda \bar{X}} + \gamma_1 \lambda \tag{B.21}$$

$$-\gamma_2' = \frac{\partial \mathcal{H}}{\partial g} = -e^{-\lambda X} b\lambda F_W\left(W - bu - bg, X = 0\right) + \eta_2. \tag{B.22}$$

We can solve for $\gamma_1(u)$ directly:

$$\gamma_1(u) = k_1 e^{-\lambda u} + b e^{-\lambda \bar{X}}, \tag{B.23}$$

for some constant k_1.

We now work through the various cases with respect to η_1 and η_2:

- Imagine that neither constraint binds for some X, so $\eta_1 = \eta_2 = 0$. Then, $\gamma_2 = -\gamma_1 = -k_1 e^{-\lambda u} + b e^{-\lambda \bar{X}}$ and so $-\gamma_2' = -\lambda k_1 e^{-\lambda u}$. Plugging that back in (B.22), we obtain $bF_W\left(W - bu - bg, X = 0\right) = k_1$ which implies $u + g$ is a constant, which violates g weakly increasing.

- Imagine that $g' \geqslant 0$ binds but $g \geqslant 0$ does not, so $\eta_1 > 0$ and $\eta_2 = 0$. Then, $-\gamma_2' = \gamma_1' + \eta_1' = -\lambda k_1 e^{-\lambda u} + \eta_1'$. This is a valid differential equation solution.

- Imagine that $g' \geqslant 0$ does not bind but $g \geqslant 0$ does, so $\eta_2 > 0$ and $\eta_1 = 0$. This is the solution in which $g(u) = 0$ and is valid.

Thus, we need only to consider the case in which $g(u) = g(0)$ is constant and optimize over that constant.

To proceed, we differentiate the definition of g and solve for f to obtain

$$f(u) = e^{\lambda u}\left[g(0) + \int_0^u e^{-\lambda v} g'(v)dv\right]. \tag{B.24}$$

If g is constant, we have $f(u) = e^{\lambda u} g(0)$. Using the definition of $g(u)$ to show that $\int_0^u \lambda f(v)dv = g(\bar{X}) - f(\bar{X})$, the objective function for the principal can be written as

$$\max \int_0^{\bar{X}} \left[\lambda e^{-\lambda u} F\left(W - bu - bg(0), X = 0\right) du\right] + b(g(\bar{X}) - f(\bar{X})) \tag{B.25}$$

$$= \max \int_0^{\bar{X}} \left[\lambda e^{-\lambda u} F\left(W - bu - bg(0), X = 0\right) du\right] + bg(0)(1 - e^{\lambda \bar{X}}). \tag{B.26}$$

Given the concavity of F in W, the first-order condition for $g(0)$ is

$$-\int_0^{\bar{X}} \left[\lambda e^{-\lambda u} b F_W\left(W - bu - bg(0), X = 0\right) du\right] - b(e^{\lambda \bar{X}} - 1). \tag{B.27}$$

By Lemma B.2.1, $F_W(\cdot, X = 0) > -1$, so this expression is negative and the optimal value of $g(0)$ is 0. Since g is constant, we also have $f = g = 0$, so

$$J(X) = J^{min} = \frac{b}{\lambda}\left(e^{\lambda X} - 1\right). \tag{B.28}$$

B.2.4 Implementation

The optimal contract can be implemented with a time budget with the following properties:

1. $S_0 = W_0/b$.

2. $dS_t = -dt - M_t dN_t$. $M_t = 0$ if $S_t \geqslant \bar{X}$. If $S_t < \bar{X}$, M_t is a binary random variable that takes value $\bar{X} - S_t$ with probability $p = S_t/\bar{X}$ and $-S_t$ with probability $1 - p$.

3. The contract ends at $t = \tau$ if $S_\tau = 0$.

Part I and II of the proof shows that if a setback is reported at any time t, the agent's continuation utility is $W_t = W_0 - bt$. Therefore, $W_t = b(S_0 - t) = bS_t$, where the first equality comes from Properties 1 above and the second equality utilizes Property 2, respectively. Randomization occurs when a setback is reported and $W_t < b\bar{X}$ where W_t jumps to $b\bar{X}$ with probability $p = W_t/b\bar{X}$ or 0 with probability $1 - p$. Because $W_t = bS_t$ following any setback, randomization occurs if $S_t < \bar{X}$, and S_t jumps to $\bar{X}$ with probability $p = bS_{t-}/b\bar{X} = S_{t-}/\bar{X}$ or 0 with probability $1 - p$ (Property 2 and 3). Finally, combining (B.14) and (B.15) and using the fact that $f = g = 0$, the prize for project completion is

$$K_\tau = W_0 - b\tau + \frac{b}{\lambda}\left(e^{\lambda\bar{X}} - 1\right) \tag{B.29}$$

$$= bS_\tau + \frac{b}{\lambda}\left(e^{\lambda\bar{X}} - 1\right). \tag{B.30}$$

B.2.5 Truthful Reporting

Next we show that the linear time-budget is sufficient to induce the agent to report truthfully and take the high action. The agent's objective is to solve (2.6), subject to the evolution of state variables:

$$dX_t = a_t(dt - X_t dN_t) \tag{B.31}$$

$$dS_t = -dt - M_t d\hat{N}_t, \tag{B.32}$$

where

$$M_t = \begin{cases} 0 & \text{if } S_{t-} \geqslant \bar{X} \\ \bar{X} - S_{t-} & \text{if } S_{t-} < \bar{X} \text{ , with probability } \frac{S_{t-}}{\bar{X}} \\ -S_{t-} & \text{if } S_{t-} < \bar{X} \text{ , with probability } 1 - \frac{S_{t-}}{\bar{X}}. \end{cases} \tag{B.33}$$

Let $U(X,S)$ denote the agent's value function. Then $U(X,S)$ satisfies

$$\begin{aligned} 0 = \max \Bigg\{ & \mathrm{E}U(X, S-M) - U(X,S), \\ & U_X a + (1-a)b - U_S \\ & + \lambda \max \left\{ \mathrm{E}U(0, S-M) - U(X,S), U(0,S) - U(X,S) \right\} \Bigg\} \end{aligned} \tag{B.34}$$

with the following boundary conditions,

$$U(\bar{X}, S) = bS + \frac{b}{\lambda}\left(e^{\lambda \bar{X}} - 1\right) \tag{B.35}$$

$$U(X, 0) = 0. \tag{B.36}$$

The first line of (B.34) represents the change of utility from reporting a false setback or not. The second line represents the flow utility from working or shirking, and the third line represents the change of utility from postponing the report of a true setback or not. The two boundary conditions come from project completion and termination, respectively, both of which are verifiable events for the principal.

The idea behind (B.34) is that the agent is making a branched choice. The first max is over whether to announce a false setback or not to do that and instead to see what happens over dt. If there is no false setback, the agent chooses to work or shirk, and whether to postpone the report of a setback when one occurs.

Outside the randomization region, $M = 0$ which implies $U(X, S-M) - U(X,S) = 0$. That is, the agent does not benefit from falsely reporting a setback. Furthermore, when $M = 0$, (B.34) implies

$$0 \geqslant \max_a \left[U_X a + (1-a)b - U_S + \lambda \max \left(U(0,S) - U(X,S) \right) \right], \tag{B.37}$$

where the equality is achieved when $a = 1$ and

$$U(X,S) = bS + \frac{b}{\lambda}\left(e^{\lambda X} - 1\right). \tag{B.38}$$

Moreover, (B.38) satisfies the boundary conditions (B.35) and (B.36). Therefore, the agent achieves the highest utility from exerting the high effort and reporting actual setbacks immediately outside the randomization region.

Inside the randomization region, M_t equals $\bar{X} - S_{t-}$ with probability $S_{t-}/\bar{X}$ and equals $-S_{t-}$ with probability $1 - S_{t-}/\bar{X}$. Then equation (B.34) becomes

$$0 = \max \left\{ \left(1 - \frac{S}{\bar{X}}\right) U(X,0) + \frac{S}{\bar{X}} U(X,\bar{X}) - U(X,S), \right. \tag{B.39}$$

$$U_X a + (1-a)b - U_S$$

$$\left. + \lambda \max \left\{ \left(1 - \frac{S}{\bar{X}}\right) U(0,0) + \frac{S}{\bar{X}} U(0,\bar{X}) - U(X,S), U(0,S) - U(X,S) \right\} \right\}.$$

With boundary conditions $U(X,0) = 0$ and $U(X,\bar{X}) = b\bar{X} + \frac{b}{\lambda}\left(e^{\lambda X} - 1\right)$ (from (B.38) at $S = \bar{X}$). Replacing $U(X,0)$ and $U(X,\bar{X})$ in (B.39) with these boundary conditions and substituting in $U(X,S) = bS + \frac{b}{\lambda}\left(e^{\lambda X} - 1\right)$ yields

$$0 = \max \left\{ \left(\frac{S}{\bar{X}} - 1\right) \frac{b}{\lambda} \left(e^{\lambda X} - 1\right), b\left(e^{\lambda X} - 1\right)(a-1) \right\}, \tag{B.40}$$

where we have used the fact that $S/\bar{X} < 1$ inside the randomization region. Equation (B.40) implies that the agent prefers working if $X > 0$ and is indifferent if $X = 0$. The agent strictly prefers not announcing a false setback, and is indifferent between postponing the report of a setback or not when one actually occurs. $\square$

B.3 Proof of Proposition 2.5.4

Recall the definition of $\sigma(S)$ from (2.17). From the randomization for $S \in [0, \bar{X}]$, we have $\sigma(S) = \frac{S}{\bar{X}}\sigma(\bar{X})$ for $S \in [0, \bar{X}]$.

We will use an iterative procedure to find $\sigma(S) = \sigma_n(S)$ on interval $S \in (\bar{X} + n\bar{X}, \bar{X} + (n+1)\bar{X}]$ for $n \geqslant 0$. We start by observing that for any $S \geqslant \bar{X}$,

$$\sigma(S) = \int_0^{\bar{X}} \lambda e^{-\lambda t}(t + \sigma(S-t))dt + e^{-\lambda \bar{X}}\bar{X}. \tag{B.41}$$

Further define

$$\phi_n(\nu) = \int_{\nu - n\bar{X}}^{\bar{X}} \lambda e^{-\lambda t} \sigma_{n-1}(\nu - t) dt \tag{B.42}$$

$$\xi = \int_0^{\bar{X}} \lambda e^{-\lambda t} t dt + e^{-\lambda \bar{X}} \bar{X} = \frac{1}{\lambda} \left(1 - e^{-\lambda \bar{X}}\right). \tag{B.43}$$

To continue,

$$\begin{aligned}
\sigma(S) &= \int_0^{S-n\bar{X}} \lambda e^{-\lambda t}(t + \sigma(S-t)) dt + \int_{S-n\bar{X}}^{\bar{X}} \lambda e^{-\lambda t}(t + \sigma(S-t)) dt + e^{-\lambda \bar{X}} \bar{X} \\
&= \int_0^{S-n\bar{X}} \lambda e^{-\lambda t} \sigma(S-t) dt + \int_0^{S-n\bar{X}} \lambda e^{-\lambda t} t dt \\
&+ \int_{S-n\bar{X}}^{\bar{X}} \lambda e^{-\lambda t}(t + \sigma(S-t)) dt + e^{-\lambda \bar{X}} \bar{X} \\
&= \int_0^{S-n\bar{X}} \lambda e^{-\lambda t} \sigma(S-t) dt + \int_0^{\bar{X}} \lambda e^{-\lambda t} t dt \\
&+ \int_{S-n\bar{X}}^{\bar{X}} \lambda e^{-\lambda t} \sigma(S-t) dt + e^{-\lambda \bar{X}} \bar{X} \\
&= \int_0^{S-n\bar{X}} \lambda e^{-\lambda t} \sigma(S-t) dt + \phi_n(S) + \xi.
\end{aligned} \tag{B.44}$$

Differentiating with respect to S and then integrating by parts yields

$$\sigma'(S) = \lambda \phi_n(S) + \phi_n'(S) + \lambda \xi. \tag{B.45}$$

So,

$$\sigma(S) = \sigma(n\bar{X}) + \int_{n\bar{X}}^{S} \sigma'(\nu) d\nu = \sigma(n\bar{X}) + \int_{n\bar{X}}^{S} (\lambda \phi_n(\nu) + \phi_n'(\nu) + \lambda \xi) d\nu, \tag{B.46}$$

where $\nu - t \leqslant n\bar{X}$ and $\sigma(\nu), \nu \leqslant \bar{X}$ is known from iteration in the previous round.

B.3.1 Part (i)

First, $\sigma(S)$ is increasing. For $S \leqslant \bar{X}$, we have $\sigma(S) = \frac{\sigma_1(\bar{X})}{\bar{X}} S$, and hence $\sigma'(S) > 0$. For $S > \bar{X}$,

$$\sigma(S) = \int_0^{\bar{X}} \lambda e^{-\lambda t}(t + \sigma(S-t))dt + e^{-\lambda \bar{X}} \bar{X}, \tag{B.47}$$

and taking the derivative yields

$$\sigma'(S) = \int_0^{\bar{X}} \lambda e^{-\lambda t} \sigma'(S-t) dt. \tag{B.48}$$

Hence $\sigma'(S) > 0$ because $\sigma'(S-t) < 0$ for $\forall 0 < t < S$.

Second, $\sigma(S)$ is concave for $S \in (n\bar{X}, (n+1)\bar{X}]$. When $n = 1$, we have $\sigma_1(S) = \frac{\sigma_1(\bar{X})}{\bar{X}} S$, which is weakly concave in S. Then, suppose concavity holds for $n-1$, we show that it holds for n as well. Since $\sigma_n'(S) = \lambda \phi_n(S) + \phi_n'(S) + \lambda \xi$, differentiating yields $\sigma_n''(S) = \lambda \phi_n'(S) + \phi_n''(S)$. Differentiating ϕ yields

$$\phi_n'(S) = -\lambda e^{-\lambda(S-n\bar{X})} \sigma_{n-1}(n\bar{X}) + \int_{S-n\bar{X}}^{\bar{X}} \lambda e^{-\lambda t} \sigma_{n-1}'(S-t) dt \tag{B.49}$$

$$\phi_n''(S) = \lambda^2 e^{-\lambda(S-n\bar{X})} \sigma_{n-1}(n\bar{X}) - \lambda e^{-\lambda(S-n\bar{X})} \sigma_{n-1}'(n\bar{X}) \tag{B.50}$$

$$- \lambda e^{-\lambda(S-n\bar{X})} \sigma_{n-1}'(n\bar{X}) + \int_{S-n\bar{X}}^{\bar{X}} \lambda e^{-\lambda t} \sigma_{n-1}''(S-t) dt \tag{B.51}$$

and

$$\begin{aligned}
\sigma_n''(S) &= \lambda \phi_n'(S) + \phi_n''(S) \\
&= \lambda \int_{S-n\bar{X}}^{\bar{X}} \lambda e^{-\lambda t} \sigma_{n-1}'(S-t) dt - 2\lambda e^{-\lambda(S-n\bar{X})} \sigma_{n-1}'(n\bar{X}) \\
&+ \lambda e^{-\lambda t} \sigma_{n-1}''(S-t) dt \\
&\leqslant \lambda \int_{S-n\bar{X}}^{\bar{X}} \lambda e^{-\lambda t} \sigma_{n-1}' \left(S - (S - n\bar{X})\right) dt - 2\lambda e^{-\lambda(S-n\bar{X})} \sigma_{n-1}'(n\bar{X}) + 0 \\
&= -\lambda e^{-\lambda \bar{X}} \sigma_{n-1}'(n\bar{X}) - \lambda e^{-\lambda(S-n\bar{X})} \sigma_{n-1}'(n\bar{X}) < 0.
\end{aligned} \tag{B.52}$$

The last step is from the fact that σ_{n-1} is a increasing function. So, by induction we have $\sigma_n(S)$ is concave on interval $S \in (n\bar{X}, (n+1)\bar{X}]$ for every n.

Third, we show that $\sigma_n(S)$ is a polynomial of order n. We have

$$\begin{aligned}
\lambda\phi_n(S) + \phi_n'(S) &= \lambda \int_{S-n\bar{X}}^{\bar{X}} \lambda e^{-\lambda t} \sigma_{n-1}(S-t)dt - \lambda e^{-\lambda(S-n\bar{X})}\sigma_{n-1}(n\bar{X}) \\
&\quad + \int_{S-n\bar{X}}^{\bar{X}} \lambda e^{-\lambda t} \sigma_{n-1}'(S-t)dt \\
&= \lambda \int_{S-n\bar{X}}^{\bar{X}} \lambda e^{-\lambda t} \sigma_{n-1}(S-t)dt - \lambda e^{-\lambda(S-n\bar{X})}\sigma_{n-1}(n\bar{X}) \\
&\quad + \lambda e^{-\lambda(S-n\bar{X})}\sigma_{n-1}(n\bar{X}) - \lambda e^{-\lambda\bar{X}}\sigma_{n-1}(S-\bar{X}) \\
&\quad - \lambda \int_{S-n\bar{X}}^{\bar{X}} \lambda e^{-\lambda t} \sigma_{n-1}(S-t)dt \\
&= -\lambda e^{-\lambda\bar{X}}\sigma_{n-1}(S-\bar{X}),
\end{aligned} \tag{B.53}$$

and therefore (B.46) implies

$$\sigma_n(S) = \sigma(n\bar{X}) + \int_{n\bar{X}}^{S} \left(\lambda\xi - \lambda e^{-\lambda\bar{X}}\sigma_{n-1}(\nu-\bar{X})\right) d\nu. \tag{B.54}$$

Since σ_1 is linear and hence polynomial of order 1, by (B.54) σ_2 is quadratic and σ_n is polynomial of order n.

B.3.2 Parts (ii) and (iii)

Because

$$\sigma(S) = \frac{1}{\lambda}\left[1 - \pi(S)e^{\lambda\bar{X}} - (1-\pi(S))\right] < \infty, \tag{B.55}$$

$\sigma(S)$ is a bounded function. Since every monotone and bounded function in $\mathbb{R}$ converges, let the limit be

$$\sigma(\infty) = \lim_{S\to\infty}\sigma(S). \tag{B.56}$$

Since

$$\sigma(S) = \int_0^{\bar{X}} \lambda e^{-\lambda t}(t + \sigma(S-t))dt + e^{-\lambda \bar{X}} \bar{X}, \tag{B.57}$$

taking $S \to \infty$ yields

$$\sigma(\infty) = \int_0^{\bar{X}} \lambda e^{-\lambda t}(t + \sigma(\infty))dt + e^{-\lambda \bar{X}} \bar{X} = \frac{1}{\lambda}(1 - e^{-\lambda \bar{X}}) + \sigma(\infty)(1 - e^{-\lambda \bar{X}}), \tag{B.58}$$

and hence we have

$$\sigma(\infty) = \frac{e^{\lambda \bar{X}} - 1}{\lambda}. \tag{B.59}$$

Moreover, by the convergence of $\sigma(S)$, we have

$$\lim_{S\to\infty} \sigma'(S) = \lim_{S\to\infty} \lim_{\Delta S\to 0} \frac{\sigma(S+\Delta S) - \sigma(S)}{\Delta S} = 0. \tag{B.60}$$

Then, at $S = n\bar{X}$, the left slope is

$$\begin{aligned}
\sigma'_{n-1}(n\bar{X}) &= \lambda\phi_{n-1}(n\bar{X}) + \phi'_{n-1}(n\bar{X}) + \lambda\xi \\
&= \lambda \int_{\bar{X}}^{\bar{X}} \lambda e^{-\lambda t}\sigma_{n-1}(n\bar{X} - t)dt - \lambda e^{-\lambda \bar{X}}\sigma_{n-1}(n\bar{X}) \\
&\quad + \int_{\bar{X}}^{\bar{X}} \lambda e^{-\lambda t}\sigma'_{n-1}(n\bar{X} - t)dt + \lambda\xi \\
&= -\lambda e^{-\lambda \bar{X}}\sigma_{n-1}(n\bar{X}) + \lambda\xi.
\end{aligned} \tag{B.61}$$

The right slope is

$$\begin{aligned}
\sigma'_n(n\bar{X}) &= \lambda\phi_n(n\bar{X}) + \phi'_n(n\bar{X}) + \lambda\xi \\
&= \lambda \int_0^{\bar{X}} \lambda e^{-\lambda t}\sigma_{n-1}(n\bar{X} - t)dt - \lambda\sigma_{n-1}(n\bar{X}) + \lambda\xi.
\end{aligned} \tag{B.62}$$

Hence

$$\sigma'_{n-1}(n\bar{X}) - \sigma'_n(n\bar{X})$$

$$= -\lambda e^{-\lambda \bar{X}} \sigma_{n-1}(n\bar{X}) - \lambda \int_0^{\bar{X}} \lambda e^{-\lambda t} \sigma_{n-1}(n\bar{X} - t) dt + \lambda \sigma_{n-1}(n\bar{X})$$

$$> -\lambda e^{-\lambda \bar{X}} \sigma_{n-1}(n\bar{X}) - \lambda \int_0^{\bar{X}} \lambda e^{-\lambda t} \sigma_{n-1}(n\bar{X}) dt + \lambda \sigma_{n-1}(n\bar{X})$$

$$= -\lambda e^{-\lambda \bar{X}} \sigma_{n-1}(n\bar{X}) - \lambda \sigma_{n-1}(n\bar{X}) + \lambda \sigma_{n-1}(n\bar{X}) - \lambda \sigma_{n-1}(n\bar{X})$$

$$= 0,$$

where the inequality comes from the fact that σ_{n-1} is a (strictly) increasing function.

□

B.4 Proof of Proposition 2.6.1 and Proposition 2.6.2

B.4.1 Value Function Without Commitment to Randomization

The same steps used to prove Proposition 2.5.2 apply in this setting, except that upon receiving an extension the agent must randomize between working and shirking such that the principal's expected payoff from granting the extension is the same as from canceling the project, namely 0.

First observe that the agent's expected utility from working immediately following an extension is $b\bar{X}$, which is the same as his utility from shirking out the clock – so he is willing to randomize at this point. Now, suppose the agent receives an extension and works. Let $s \in [0, \bar{X}]$ be the amount of time since the extension was granted. Any setback during the extension will reset the principal's expected utility to 0. Therefore her expected utility when the agent works as a function of s is

$$\left(\mathcal{R} - c(\bar{X} - s)\right) e^{-\lambda(\bar{X}-s)} - \int_0^{\bar{X}-s} ct\lambda e^{-\lambda t} dt \tag{B.63}$$

$$= \mathcal{R} e^{-\lambda(\bar{X}-s)} - \frac{c}{\lambda}\left(1 - e^{-\lambda(\bar{X}-s)}\right) \equiv \hat{F}(\bar{X} - s, s), \tag{B.64}$$

where $\mathcal{R} \equiv R - \frac{b}{\lambda}\left(e^{\lambda\bar{X}} - 1\right)$ is the principal's surplus at the completion of the project.

Let q be the probability the agent shirks out the clock following an extension. Then randomization between cancelation and granting an extension is incentive compatible for the principal iff

$$(1-q)\hat{F}(\bar{X},0) - qc\bar{X} = 0. \tag{B.65}$$

Solving for q yields the formula given in the claim.

There is one further item that must be checked: the principal must be willing to continue the contract if the agent does not report a setback. This must be checked because the principal does not know if the agent is working or shirking out the clock and shirking does not generate setbacks. The probability that the principal believes the agent is shirking conditional on having reported no setbacks by s is

$$q(s) = \frac{\hat{F}(\bar{X},0)}{\hat{F}(\bar{X},0) + c\bar{X}e^{-\lambda s}}.$$

Then, the principal is willing to let the clock run during the extension iff for all $s \in [0, \bar{X}]$,

$$\begin{aligned}
&(1-q(s))\hat{F}(\bar{X}-s,s) - q(s)c(\bar{X}-s) \geqslant 0 \\
\iff\ & \bar{X}e^{-\lambda s}\hat{F}(\bar{X}-s,s) \geqslant (\bar{X}-s)\hat{F}(\bar{X},0) \\
\iff\ & \bar{X}e^{-\lambda\bar{X}}\left(\mathcal{R}e^{-\lambda(\bar{X}-s)} - \frac{c}{\lambda}\left(1-e^{-\lambda(\bar{X}-s)}\right)\right) \\
&\geqslant (\bar{X}-s)\left(\mathcal{R}e^{-\lambda\bar{X}} - \frac{c}{\lambda}\left(1-e^{-\lambda\bar{X}}\right)\right) \\
\iff\ & \frac{c}{\lambda}\left(\bar{X}-s+se^{-\lambda\bar{X}} - \bar{X}e^{-\lambda s}\right) \geqslant -s\mathcal{R}e^{-\lambda\bar{X}}.
\end{aligned}$$

The right side is non-positive, so the claim will hold if the left side is non-negative for $s \in [0, \bar{X}]$:

$$\bar{X} - s + se^{-\lambda\bar{X}} - \bar{X}e^{-\lambda s} \geqslant 0.$$

At $s \in \{0, \bar{X}\}$ the inequality above evidently binds. For $s \in (0, \bar{X})$, differentiating the left side w.r.t. s yields $-1 + e^{-\lambda \bar{X}} + \lambda \bar{X} e^{-\lambda s}$. This is positive at $s = 0$, negative at $s = \bar{X}$, and 0 at a single point between 0 and $\bar{X}$.□

B.4.2 Optimal Initial Time Budget

Define $\hat{\pi}(S_0)$ to be the probability that the project succeeds without commitment given an initial time budget S_0 and initial progress $X_0 = 0$. Then, the principal's value function can be written

$$\hat{F}(S_0, X = 0) = \hat{\pi}(S_0) E\left[R - bS_{\hat{\tau}^{proj}} - \frac{b}{\lambda}(e^{\lambda \bar{X}} - 1) \middle| S_0, X_{\hat{\tau}^{proj}} = \bar{X} \right]$$
$$- cE[\hat{\tau}^{proj} | S_0]$$

The first term is the expected reward minus payment, the second term is expected running cost.

Define $\hat{\tau}^{work}$ as the random time the agent stops working (either by completion of project, or by the agent shirking) and $\hat{\tau}^{proj}$ as the random time the project stops (either by completion of project, or by termination of project). Both $\hat{\tau}^{work}$ and $\hat{\tau}^{proj}$ are $\mathcal{F}_t^N-$ stopping times, and $\hat{\tau}^{work} \leqslant \hat{\tau}^{proj}$.

Next, we continue with the martingale analysis, following the main text. For $t \leqslant \hat{\tau}^{work}$, we have that $dX_t = dt - X_t dN_t$ implies $d(e^{\lambda X_t} - \lambda t) = (1 - \lambda e^{\lambda X_t})(dN_t - \lambda dt)$, which is a martingale. Applying the optional stopping theorem regarding $e^{\lambda X_t} - \lambda t$, we have

$$\begin{aligned}
1 &= \mathrm{E}[e^{\lambda X_t} - \lambda t | S_0, t = \hat{\tau}^{work}] \\
&= \hat{\pi}(S_0) \mathrm{E}[e^{\lambda X_{\hat{\tau}^{work}}} | S_0, X_{\hat{\tau}^{work}} = \bar{X}] \\
&+ (1 - \hat{\pi}(S_0)) \mathrm{E}[e^{\lambda X_{\hat{\tau}^{work}}} | S_0, X_{\hat{\tau}^{work}} = 0] - \lambda \mathrm{E}[\hat{\tau}^{work} | S_0] \\
&= \hat{\pi}(S_0) e^{\lambda \bar{X}} + (1 - \hat{\pi}(S_0)) - \lambda E[\hat{\tau}^{work} | S_0] \qquad \text{(B.66)}
\end{aligned}$$

Notice that we have used the fact that success or failure of the project is fully determined at $t = \hat{\tau}^{work}$; one does not have to wait until $t = \hat{\tau}^{proj}$. Solving, we have

$$\hat{\pi}(S_0) = \frac{\lambda}{e^{\lambda\bar{X}} - 1}\mathrm{E}[\hat{\tau}^{work}|S_0] \tag{B.67}$$

Next, apply the optional stopping theorem regarding $S_t + t$

$$\begin{aligned}
S_0 &= \mathrm{E}[S_t + t|S_0, t = \hat{\tau}^{proj}] \\
&= \hat{\pi}(S_0)\mathrm{E}[S_{\hat{\tau}^{proj}}|S_0, X_{\hat{\tau}^{proj}} = \bar{X}] + (1 - \hat{\pi}(S_0))\mathrm{E}[S_{\hat{\tau}^{proj}}|S_0, X_{\hat{\tau}^{proj}} = 0] \\
&+ \mathrm{E}[\hat{\tau}^{proj}|S_0] \\
&= \hat{\pi}(S_0)\mathrm{E}[S_{\hat{\tau}^{proj}}|S_0, X_{\hat{\tau}^{proj}} = \bar{X}] + \mathrm{E}[\hat{\tau}^{proj}|S_0]
\end{aligned} \tag{B.68}$$

The third line follows from that $\mathrm{E}[S_{\hat{\tau}^{proj}}|S_0, X_{\hat{\tau}^{proj}} = 0] = 0$. If the project has ended at $X_t = 0$, there must not be any time remaining.

Examining the terms in the principal's value function one at a time, we have

$$\hat{\pi}(S_0)R = \frac{R\lambda}{e^{\lambda\bar{X}} - 1}\mathrm{E}[\hat{\tau}^{work}|S_0] \tag{B.69}$$

$$-\hat{\pi}(S_0)E[bS_{\hat{\tau}^{proj}}|S, X_{\hat{\tau}^{proj}} = \bar{X}] = b\mathrm{E}[\hat{\tau}^{proj}|S_0] - bS_0 \tag{B.70}$$

$$-\hat{\pi}(S_0)\frac{b}{\lambda}(e^{\lambda\bar{X}} - 1) = -b\mathrm{E}[\hat{\tau}^{work}|S_0] \tag{B.71}$$

$$-cE[\hat{\tau}^{proj}|S_0] = -cE[\hat{\tau}^{proj}|S_0] \tag{B.72}$$

Adding these up and re-arranging, we obtain

$$\begin{aligned}
\hat{F}(S_0, X = 0) &= R\frac{\lambda\mathrm{E}[\hat{\tau}^{work}|S_0]}{e^{\lambda\bar{X}} - 1} - cE[\hat{\tau}^{proj}|S_0] - bS_0 \\
&+ b\left(\mathrm{E}[\hat{\tau}^{proj}|S_0] - \mathrm{E}[\hat{\tau}^{work}|S_0]\right)
\end{aligned}$$

which implies

$$\begin{aligned}
\hat{F}(S_0, X = 0) &= \left(\frac{R\lambda}{e^{\lambda\bar{X}} - 1} - c\right)\mathrm{E}[\hat{\tau}^{proj}|S_0] - bS_0 \\
&+ \left(\frac{R\lambda}{e^{\lambda\bar{X}} - 1} + b\right)\left(\mathrm{E}[\hat{\tau}^{work}|S_0] - \mathrm{E}[\hat{\tau}^{proj}|S_0]\right)
\end{aligned} \tag{B.73}$$

Next, we observe that upon entering the short-leash region, we have $E[\hat{\tau}^{proj}|S = \bar{X}, X = 0] = E[\tau|S = \bar{X}, X = 0] = \bar{X}$. This means that commitment does not change the average project duration. In addition, outside the short-leash region, the agent is not shirking, and the evolution of X and S are the same with and without commitment. Thus, we have $E[\hat{\tau}^{proj}|S_0] = E[\tau|S_0]$ (where τ is the with-commitment stopping time), and we can write the principal's value function without commitment as

$$\hat{F}(S_0, X = 0) = F(S_0, X = 0) - \left(\frac{R\lambda}{e^{\lambda\bar{X}} - 1} + b\right)\left(\mathrm{E}[\hat{\tau}^{proj}|S_0] - \mathrm{E}[\hat{\tau}^{work}|S_0]\right) \tag{B.74}$$

As a reminder, F is the principal's value function with commitment. We want to show that $\left(\mathrm{E}[\hat{\tau}^{proj}|S_0] - \mathrm{E}[\hat{\tau}^{work}|S_0]\right)$ is decreasing in S_0.

We will use the fact that outside of the short-leash region, the agent never shirks. Thus, if the project succeeds before entering the short-leash region, we have $\hat{\tau}^{work} = \hat{\tau}^{proj}$. Then, we can write the difference between $E[\hat{\tau}^{proj}|S_0]$ and $E[\hat{\tau}^{work}|S_0]$ entirely as a function of the probability of entering the short-leash region and the expected time spent there:

$$\begin{aligned} E[\hat{\tau}^{proj}|S_0] - E[\hat{\tau}^{work}|S_0] &= (1 - \Pr(X_{\hat{\tau}^{proj}} = \bar{X}, S_{\hat{\tau}^{proj}} > \bar{X})) \\ &\quad \times \left(\mathrm{E}[\hat{\tau}^{proj}|S_{\hat{\tau}^{proj}} \leqslant \bar{X}] - \mathrm{E}[\hat{\tau}^{work}|S_{\hat{\tau}^{work}} \leqslant \bar{X}]\right) \end{aligned} \tag{B.75}$$

where $(1 - \Pr(X_{\hat{\tau}^{proj}} = \bar{X}, S_{\hat{\tau}^{proj}} > \bar{X})) = (1 - \Pr(X_{\hat{\tau}^{work}} = \bar{X}, S_{\hat{\tau}^{work}} > \bar{X}))$ is the probability of entering the short leash region. Then we notice: $\mathrm{E}[\hat{\tau}^{proj}|S_{\hat{\tau}^{proj}} \leqslant \bar{X}] - \mathrm{E}[\hat{\tau}^{work}|S_{\hat{\tau}^{work}} \leqslant \bar{X}]$, the difference in the expected time taken by the end of work and the end of the project in the short leash region, does not depend on the starting value of S_0.

Since the probability that the agent does not succeed before the short-leash region is decreasing in S_0, we must have $E[\hat{\tau}^{proj}|S_0] - \mathrm{E}[\hat{\tau}^{work}|S_0]$ is decreasing in S_0.

Thus, from (B.74), the principal's value function without commitment is equal to the value function with commitment minus a term that is decreasing in S_0; without-commitment and with-commitment have increasing differences in S_0. Both value functions are bounded from above and are $-\infty$ for $S_0 \to \infty$. By the standard logic of increasing differences, the optimal S_0 without commitment must be weakly larger than the optimal S_0 with commitment. □

B.5 Optimal Contract with Discounting

This section presents the key results of the baseline model when both contracting parties have the same discount rate $r > 0$. Details of the derivations are mostly omitted in the interest of space, but are available upon request.

First, suppose the project is operated until completed. Compared to (2.3), F^{FB}, the first-best value to the principal at inception, is now given by

$$F^{FB} = R\left(\frac{e^{-(r+\lambda)\bar{X}}}{\lambda e^{-(r+\lambda)\bar{X}} + r}\right)(\lambda + r) - c\left(\frac{1 - e^{-(r+\lambda)\bar{X}}}{\lambda e^{-(r+\lambda)\bar{X}} + r}\right) \tag{B.76}$$

In the baseline model, F^{FB} is given by the final reward R minus the expected cost of operating the project. The same interpretation applies here, except that both costs and rewards are discounted. Taking the $r \to 0$ limit, wee see that (B.76) goes to (2.3).

We first consider the case that the agent's initial continuation utility (W_0) is less than b/r. In this case, the contract must terminate, otherwise the agent would be better off obtaining the perpetuity value of shirking b/r by adopting the report strategy that yields no termination. Conditional on the project being feasible, Lemma 2.4.2 still applies: high action is always optimal and no payment except for project completion. The No-Postponed-Setback (NPS) constraint can be written as

$$W_t - J_t \geqslant \int_0^\delta e^{-rt} b dt + e^{-r\delta}\left[W_{t+\delta} - J_{t+\delta}\right] \tag{B.77}$$

The law of motion for the agent's continuation utility W_t is

$$dW_t = rW_t dt + J_t(\lambda dt - dN_t) \tag{B.78}$$

Like in the baseline model without discounting, the agent's value drifts up over time in the absence of a setback. However, the drift is composed of two parts: the interest over time (captured by rW_t) and the reward for having no setback (captured by λJ_t).

We can show that binding NPS is still optimal. This, combined with (B.77) and (B.78), implies the utility penalty for reporting a setback is

$$J(X) = \frac{b}{\lambda + r}\left[e^{(\lambda+r)X} - 1\right] \tag{B.79}$$

Clearly, (B.79) becomes (2.12) when $r \to 0$.[1]

The optimal contract has the same structure as characterized in Proposition 2.5.2 in the baseline model: a time budget S with a soft deadline that involves randomization. Under this contract, the agent's utility when $X = 0$ (either at the outset of the contract or immediately after experiencing a setback) is

$$W(S, X = 0) = \frac{b}{r}\left(1 - e^{-rS}\right) \tag{B.80}$$

Randomization occurs when $S < \bar{X}$ and a setback is reported. Upon randomization, either $S = 0$, in which case the contract is terminated with no payment, or S is extended to $\bar{X}$. The probability of the extension is,

$$p(S) = \frac{1 - e^{-rS}}{1 - e^{-r\bar{X}}} \tag{B.81}$$

Finally, the agent is rewarded with a prize K for completion, where

$$K(S) = \frac{b}{r}\left(\frac{\lambda}{r+\lambda} - e^{-rS}\right) + \frac{b}{r+\lambda}e^{(r+\lambda)\bar{X}} \tag{B.82}$$

[1] Without discounting, $J(X)$ also equals b times the project's expected duration. However, this is a coincidence and does not hold with $r > 0$.

Unlike the baseline model, the probability of extension and the prize for completion with discounting are no longer linear functions of the remaining balance of the time budget. However, they are both still increasing functions in S, meaning that later reports of the setback result in a lower probability of extension, and later completion of the project results in a smaller prize. Moreover, straightforward algebra shows that they both converge to their baseline model counterparts; i.e., (B.81) becomes (2.13), and (B.82) becomes (2.15), when $r \to 0$.

As in the baseline model, the principal's initial value function at the outset of the contract $F(S)$, has kinks at the integer multiples of $\bar{X}$. We begin by defining the auxiliary functions,

$$\pi(S) = \mathrm{E}_t\left[e^{-r(\tau-t)}1_{X_\tau=\bar{X}}|X_t = 0\right] \tag{B.83}$$

$$\sigma(S) = \mathrm{E}_t\left[\int_t^\tau e^{-r(s-t)}ds|X_t = 0\right] \tag{B.84}$$

which are the discounted probability of success and expected duration of the project, analogous to (2.16) and (2.17). This allows us to write the principal's value function as

$$F(S) = \pi(S)R - \frac{b}{r}\left(1 - e^{-rS}\right) - c\sigma(S) \tag{B.85}$$

which is analogous to (2.19).

We proceed using the same martingale methods as in the baseline model. We observe that $e^{-rt}W_t$ and $1 - e^{-r(S_t+t)}$ are martingales, and thus we can repeat the steps beginning with (2.20) to obtain an analogy to (2.24):

$$\pi(S) = \frac{(r+\lambda)\sigma(S)}{e^{(r+\lambda)\bar{X}} - 1}. \tag{B.86}$$

The intuition is the same as in the baseline model: the agent's continuation utility has two terms, one that counts down as the time budget is used and one that counts up as the project accumulates progress. These two changes must cancel out on average to maintain incentives. Thus the passage of time is matched by an increase in the probability of eventual success. Then, combining (B.85) and (B.86), we obtain an analogy to (2.25):

$$F(S, X=0) = \left(\frac{(r+\lambda)}{e^{(r+\lambda)\bar{X}}-1}R - c\right)\sigma(S) - \frac{b}{r}\left(1-e^{-rS}\right). \tag{B.87}$$

In the "short-leash" region ($S \leqslant \bar{X}$), every setback triggers randomization, and the probability of retention is given by (B.81) above. Then, direct calculation shows that $\sigma(S) = \frac{1}{r}\left(1-e^{-rS}\right)$, so (B.87) becomes

$$F(S \leqslant \bar{X}, X=0) = \frac{1-e^{-rS}}{r}\left(\frac{(r+\lambda)}{e^{(r+\lambda)\bar{X}}-1}R - (b+c)\right) \tag{B.88}$$

$F(S)$ in any region $[n\bar{X}, (n+1)\bar{X}]$ (where $n \in \mathbb{N}$) can be obtained from the same iterative procedure used to derive $F(S)$ in the baseline model in Propositions 2.5.3 and 2.5.4, albeit with more cumbersome algebra. One can show through direct calculation that the limit as $r \to 0$ of $\sigma(S)$ in (B.84) converges to the value with $r = 0$ in (B.41). Then, using (B.87), one can show that the principal's value $F(S)$ for $r > 0$ converges to that in the baseline model (i.e. equation 2.26) when $r \to 0$ for all $S > 0$.

The above analysis requires $W_0 < \frac{b}{r}$. If $W_0 \geqslant \frac{b}{r}$ and $F^{\mathrm{FB}} - W_0 > 0$, then the optimal contract is to offer a fixed prize of $K = W_0\left(\frac{\lambda+re^{(\lambda+r)\bar{X}}}{\lambda+r}\right)$ upon completion, request no intermediate reports, and never fire the agent. This contract implements the first best policy, but is not feasible for small r, e.g., $r < b/R$.

Appendix C

Dynamic Moral Hazard with Adverse Selection

C.1 Proof in Section 2

C.1.1 Proof of Lemma 3.2.1

To characterize how the agent's continuation utility evolves over time, it is useful to consider her lifetime expected utility, evaluated conditionally upon the information available at time t

$$
\begin{aligned}
u_t(\gamma,\nu;c) &= \mathrm{E}^{\nu}\left[\int_0^{\tau} e^{-rs}\left(dL_s - c1_{\nu_s=\mu}ds\right)\middle|\,\mathcal{F}_t^N\right] \\
&= \int_0^{t\wedge\tau-} e^{-rs}(dL_s - c1_{\nu_s=\mu}ds) + e^{-rt}W_t(\gamma,\nu;c) \qquad \text{(C.1)}
\end{aligned}
$$

Since $u_t(\gamma,\nu;c)$ is the expectation of a given random variable conditional on $\mathcal{F}_t^N$, the process $\mathbf{u}(\gamma,\nu;c) = \{u_t(\gamma,\nu;c)\}_{t\geqslant 0}$ is an martingale under the probability measure $\mathbf{P}^{\nu}$. Relying on this martingale property, we now offer an alternative representation of $\mathbf{u}(\gamma,\nu;c)$. Consider the process $M^{\nu} = \{M_t^{\nu}\}_{t=\geqslant 0}$ defined by

$$
M_t^{\nu} = N_t - \int_0^t \nu_s ds \qquad \text{(C.2)}
$$

for all $t \geqslant 0$. The martingale representation theorem for point processes implies that the martingale $\mathbf{u}(\gamma,\nu;c)$ satisfies

$$
u_t(\gamma,\nu;c) = u_0(\gamma,\nu;c) + \int_0^{t\wedge\tau} e^{-rs}H_s(\gamma,\nu;c)dM_s^{\nu} \qquad \text{(C.3)}
$$

for all $t \geqslant 0$, $\mathbf{P}^{\nu}$-almost surely, for some $\mathcal{F}^N$-predictable process
$H(\gamma,\nu;c) = \{H_t(\gamma,\nu;c)\}_{t\geqslant 0}$. Then, (C.1) and (C.3) imply (PK). Next, we show that $\{\nu_t\}_{t\in[0,\tau]}$ defined in (IC) is a best response to contract γ.

Let u_t' denote the agent's lifetime expected payoff, given the information available at date t, when he acts according to $\nu' = \{\nu_t'\}_{t\geqslant 0}$ until date t and then reverts to $\nu = \{\nu_t\}_{t\geqslant 0}$:

$$
u_t' = \int_0^{t\wedge\tau^-} e^{-rs}\left(dL_s - 1_{\nu_s'=\mu}\cdot cds\right) + e^{-rt}W_t(\gamma,\nu;c) \qquad \text{(C.4)}
$$

Following Sannikov (2008) (Proposition 2), the proof now proceeds as follows. First, we show that if $u' = \{u'_t\}_{t \geqslant 0}$ is an $\mathcal{F}^N$-submartingale under $\mathbf{P}^{\nu}$ that is not a martingale, then ν is suboptimal for the agent. Indeed, in that case there exists some $t > 0$ such that

$$u_{0-}(\gamma, \nu; c) = u'_{0-} < \mathrm{E}^{\nu'}[u'_t] \tag{C.5}$$

where $u_{0-}(\gamma, \nu; c)$ and u'_{0-} correspond to unconditional expected payoffs at date 0. By (C.4), the agent is then strictly better off acting according to ν' until date t and then reverting to ν. The claim follows. Next, we show that if u' is a $\mathcal{F}^N$-supermartingale under $\mathbf{P}^{\nu'}$, then ν is at least as good as ν' for the agent. From (C.1) and (C.4),

$$u'_t = u_t(\gamma, \nu; c) + \int_0^{t \wedge \tau} e^{-rs}(1_{\nu'_s=0} - 1_{\nu_s=0})cds \tag{C.6}$$

for all $t \geqslant 0$. Hence, since $u_t(\gamma, \nu; c)$ is right-continuous with left-hand limits, so is u'. Moreover, since u' is non-negative, it has a last element. Hence, by the optional sampling theorem (Dellacherie and Meyer (2011), Chapter VI, Theorem 10)),

$$u'_0 \geqslant \mathrm{E}^{\nu'}[u'_\tau] = u_0(\gamma, \nu'; c) \tag{C.7}$$

where again $u_{0-}(\gamma, \nu')$ is an unconditional expected payoff at date 0. Since $u'_0 =$

$u_0(\gamma, \nu)$ by (C.4), the claim follows. Now, for each $t \geqslant 0$,

$$
\begin{aligned}
u'_t &= u_t(\gamma, \nu; c) + \int_0^{t\wedge\tau} e^{-rs}(1_{\nu'_s=0} - 1_{\nu_s=0}) c ds \\
&= u_0(\gamma, \nu; c) + \int_0^{t\wedge\tau} e^{-rs} H_s(\gamma, \nu; c) dM_s^{\nu} + \int_0^{t\wedge\tau} e^{-rs}(1_{\nu'=0} - 1_{\nu=0}) c ds \\
&= u_0(\gamma, \nu; c) + \int_0^{t\wedge\tau} e^{-rs} H_s(\gamma, \nu; c) dM_s^{\nu'} + \int_0^{t\wedge\tau} e^{-rs} H_s(\gamma, \nu; c)(\nu'_s - \nu_s) ds \\
&+ \int_0^{t\wedge\tau} e^{-rs}(1_{\nu'_s=0} - 1_{\nu_s=0}) c ds \\
&= u_0(\gamma, \nu; c) + \int_0^{t\wedge\tau} e^{-rs} H_s(\gamma, \nu; c) dM_s^{\nu'} \\
&+ \int_0^{t\wedge\tau} e^{-rs} \mu (1_{\nu'_s=0} - 1_{\nu_s=0}) \left[\frac{c}{\mu} - H_s(\gamma, \nu; c) \right] ds
\end{aligned} \tag{C.8}
$$

Since $H(\gamma, \nu; c)$ is $\mathcal{F}^N$-predictable and $M^{\nu'}$ is an $\mathcal{F}^N$-martingale under $P^{\nu'}$, the drift of u' has the same sign as

$$
(1_{\nu'_s=0} - 1_{\nu_s=0}) \left[\frac{c}{\mu} - H_s(\gamma, \nu; c) \right]
$$

for all $t \in [0, \tau)$. If (IC) holds, then this drift remains non-positive for all $t \in [0, \tau)$ and all choices of ν'. This implies that for any effort process ν', u' is an $\mathcal{F}^N$-supermartingale under $P^{\nu'}$ and, thus, that ν is at least as good as ν' for the agent. If (IC) does not hold for the effort process ν, then choose ν' such that for each $t \in [0, \tau)$, $\nu'_t = \mu$ if $H_t \geqslant \beta_c$ and $\nu'_t = 0$ if $H_t < \beta_c$. The drift of u' is then everywhere non-negative and strictly positive over a set of $P^{\nu'}$-strictly positive measure. As a result of this, u' is an $\mathcal{F}^N$-submartingale under $P^{\nu'}$ that is not a martingale and, thus, ν is suboptimal for the agent. This concludes the proof.

C.2 Proofs in Section 3.3

C.2.1 Proof of Proposition 3.3.3

(i) Following contract $\gamma^c_{\mathtt{B}}(w,B)$, since the promised utility process W^c_t follows (3.9) and the payment process follows (3.10), then $H^c_t = \beta_c \forall t$. Hence, $\bar{\nu} \in \mathfrak{N}(\gamma^c_{\mathtt{B}}(w,B),c)$. Meanwhile,

$$u\Big(\gamma^c_{\mathtt{B}}(w,B),\bar{\nu};c\Big) = \mathrm{E}^{\bar{\nu}}\left[\int_{0^-}^{\tau} e^{-rs}(dL_s - c1_{\bar{\nu}_s=\mu}ds)\middle|\,\mathcal{F}^N_t\right]$$

$$= B + \mathrm{E}^{\nu}\left[\int_{0^+}^{\tau} e^{-rs}(dL_s - c1_{\bar{\nu}_s=\mu}ds)\middle|\,\mathcal{F}^N_t\right] = B + w \tag{C.9}$$

where the third equality follows from the definition of $\gamma^c_{\mathtt{B}}(w,B)$.

(ii) By definition 3.3.2, we have that the promised utility of $\gamma^c_{\mathtt{P}}(\tau,z)$ follows (PK), where $H^c_t = \max\{\beta_c, z+\beta_c - W^c_t\}1_{t\leqslant\tau^N_1} + \beta_c 1_{t>\tau^N_1}$, $dL^c_t = cdt + (W^c_t + H^c_t - \bar{w}_c)^+ dN_t$, and $\tau^c = \min\{t : W^c_t = 0\}$. Hence, following (IC), we have $\bar{\nu} \in \mathfrak{N}(\gamma^c_{\mathtt{P}}(\tau,z),c)$. Further, by definition of W^c_t, we have

$$u\Big(\gamma^c_{\mathtt{P}}(\tau,z),\bar{\nu};c\Big) = W^c_0.$$

in which W^c_0 follows (3.17) if $z < \bar{w}_c$ and $\tau > \tau_z$, and (3.15) otherwise.

C.2.2 Proof of Proposition 3.3.4

The goal of this proposition is to solve the case $Z(\{c\})$. This benchmark case differs from (3.8) with the omission of the (TT) constraint, and contributes to an important building block to our general adverse selection problem. This benchmark setting is very similar, although not identical, to the model in Sun and Tian (2018). In their setting the principal does not have to reimburse the operating cost rate c in real time. That is, the constrain (LL) reduces to L^c_t monotonically non-decreasing in time t. In the following, we show the claims.

If $R \geqslant \beta_c$, the differential equation (3.18) with boundary condition (3.19) has a unique solution, $F_c(w)$, which is strictly concave on $[0, \bar{w}_c)$ and $F_c'(w) \geqslant -1$. The proof of this technical result can be directly adapted from the proof of Lemma 3 in Sun and Tian (2018), hence, is omitted here. Furthermore, the proof of $F_c(w) = U(\gamma_{\mathtt{B}}^c(w,0), \bar{\nu})$ can be directly adapted from the proof of Proposition 1 in Sun and Tian (2018), hence, is omitted here.

Next, we show that it is optimal for the principal to always induce effort from the agent before contract termination.

Lemma C.2.3. *For any contract γ^c, define a probation period*

$$\tau^0(\gamma^c) := \inf\{t : W_t(\gamma^c, \nu^0; c) = 0\}. \tag{C.10}$$

Then, for any effort process $\nu^c \in \mathfrak{N}(\gamma^c, c)$ that satisfies (IC)*, there exists a contract $\hat{\gamma}^c$ such that $\bar{\nu} \in \mathfrak{N}(\hat{\gamma}^c, c)$, $u(\gamma^c, \nu^c; c) = u(\hat{\gamma}^c, \bar{\nu}; c)$, $\tau^0(\gamma^c) = \tau^0(\hat{\gamma}^c)$ and*

$$U(\hat{\gamma}^c, \bar{\nu}) \geqslant U(\gamma^c, \nu^c),$$

as long as $R \geqslant \beta_c$.

Proof. Consider the contract $\gamma^c = \{L^c, \tau^c\}$ and the best response effort process $\nu^c \in \mathfrak{N}(\gamma^c, c)$ such that $H_t < \beta_c$ for $t \in T \subset [0, \tau^c]$. Define $\hat{\gamma}^c = \{\hat{L}^c, \tau^c\}$ such that $\hat{H}_t^c = H_t^c$, $\hat{\ell}_t^c = \ell_t^c$ and $\hat{I}_t^c = I_t^c$ except $\hat{H}_t^c = \beta_c$, $\hat{I}_t^c = \beta_c dN_t$ for $t \in T$. Following (PK), we know that $W_t(\gamma^c, \nu; c) = W_t(\hat{\gamma}^c, \bar{\nu}; c)$ for $t \in [0, \tau^c]$. Hence, $\bar{\tau}(\hat{\gamma}^c) = \bar{\tau}(\gamma^c)$. Furthermore,

$$U(\hat{\gamma}^c, \bar{\nu}) - U(\gamma^c, \nu^c) = \mathrm{E}^{\bar{\nu}}\left[\int_{t \in T} e^{-rt}(R - \beta_c) dN_t\right] \geqslant 0,$$

in which the inequality follows from $R \geqslant \beta_c$. $\square$

The quantity $\tau^0(\gamma^c)$ represents the time when contract γ^c terminates if there is no arrival. Lemma C.2.3 implies that it is without loss of generality to focus on the contracts that induce full effort from the agent. It is worth noting here that Sun and Tian (2018) claimed optimality of full effort contracts in their equal time discount case without providing a proof. Next, we show that $Z(\{c\}) = \max_{w \geqslant 0} F_c(w)$.

Following Lemma C.2.3, we know that it is without loss of generality to focus on contracts that induce full effort from the agent. If $R \geqslant \beta_c$, then the rest of proof can be easily adapted from the proof of Proposition 2 in Sun and Tian (2018).

If $R < \beta_c$, then for any γ^c that satisfies (LL), (PK), and (IR), and $\nu^c \in \mathfrak{N}(\gamma^c, c)$,

$$\begin{aligned} U(\gamma^c, \nu^c) + w = U(\gamma^c, \nu^c) + u(\gamma^c, \nu^c; c) &= \mathrm{E}^{\nu^c}\left[\int_0^\tau e^{-rt}(RdN_t - c1_{\nu_t^c=\mu}dt)\right] \\ &= \mathrm{E}^{\nu^c}\left[\int_0^\tau e^{-rt}(R\mu 1_{\nu_t^c=\mu}dt - c1_{\nu_t^c=\mu}dt)\right] \\ &= \mu(R-\beta_c)\mathrm{E}^{\nu^c}\left[\int_0^\tau e^{-rt}1_{\nu_t^c=\mu}dt\right] \leqslant 0, \end{aligned}$$

which verifies $Z(\{c\}) = 0$. This completes the proof.

Finally, Proposition 3.3.4 implies that for any contract γ^c that satisfies (LL), (PK), and (IR), $\nu^c \in \mathfrak{N}(\gamma^c, c)$, and $u(\gamma^c, \nu^c; c) = w$, we have

$$U(\gamma^c, \nu^c) \leqslant F_c(w). \tag{C.11}$$

C.3 Proof in Section 3.4

C.3.1 Useful definitions and Technical results

Working Duration:

$$\bar{T}(\gamma, \nu) := \mathrm{E}^{\nu}\left[\int_0^\tau e^{-rt}1_{\nu_t=\mu}dt\right], \tag{C.12}$$

which measures the agent's expected working time under contract γ when the agent chooses the effort process ν.

Societal Value:

$$S(\gamma, \nu; c) = \mathrm{E}^{\nu}\left[\int_0^\tau e^{-rt}(RdN_t - c1_{\nu_t=\mu}dt)\right], \tag{C.13}$$

which measures the expected total value net of cost produced with effort ν when the agent's cost is c.

Lemma C.3.2. *The societal value produced is fractional to the working duration, i.e.,* $S(\gamma, \nu; c) = (\mu R - c)\bar{T}(\gamma, \nu)$.

Proof:

$$\begin{aligned}
S(\gamma, \nu; c) &= E^{\nu}\left[\int_0^{\tau} e^{-rt}(RdN_t - c1_{\nu_t=\mu}dt)\right] \\
&= E^{\nu}\left[\int_0^{\tau} e^{-rt}(R\mu 1_{\nu_t=\mu}dt - c1_{\nu_t=\mu}dt)\right] \\
&= (\mu R - c)\bar{T}(\gamma, \nu) \qquad \text{(C.14)}
\end{aligned}$$

Hence, for each moment the agent exerts effort, he produces an expected revenue of μR with a cost c. □

C.3.3 Proof of Proposition 3.4.2

For any contract pair (γ^g, γ^b), denote $w_g := u(\gamma^g, \nu^g; g)$, $w_b := u(\gamma^b, \nu^b; b)$ and $\tau := \tau^0(\gamma^g)$ where $\tau^0(.)$ is defined in (C.10). In the following, we establish the proof in two steps.

Step 1: Constraint is more relaxed: We prove that the constraint of $Z(\{g, b\})$ implies the constraint of Y. First, $\tau \geqslant 0$ which implies constraint (3.23). Second, (TT) implies that

$$w_g \geqslant \max_{\nu} u(\gamma^b, \nu; g) \geqslant u(\gamma^b, \nu^b; g) = w_b + (b - g)\bar{T}(\gamma^b, \nu^b) \geqslant w_b, \qquad \text{(C.15)}$$

where $\nu^b \in \mathfrak{N}(\gamma^b, b)$ and $\bar{T}(., .)$ is defined in (C.12).

$$w_b \geqslant \max_{\nu} u(\gamma^g, \nu; b) \geqslant u(\gamma^g, \nu^0; b) = g\int_0^{\tau} e^{-rt}dt = g/r \cdot (1 - e^{-r\tau}). \qquad \text{(C.16)}$$

Hence, constraint (TT) implies constraint (3.22).

Step 2: Objective is higher We prove that the objective of Y is greater or equal to the objective of $Z(\{g,b\})$.

Step 2.1: If $R > \beta_b$, then following (C.15), we have

$$w_g \geqslant w_b + (b-g)\bar{T}(\gamma^b,\nu^b) = w_b + \frac{(b-g)S(\gamma^b,\nu^b;b)}{\mu R - b}$$

$$= w_b + \frac{(b-g)(U(\gamma^b,\nu^b)+w_b)}{\mu R - b} \tag{C.17}$$

where the first equality follows from Lemma C.3.2 and the last equality follows from $S(\gamma^b,\nu^b;b) = U(\gamma^b,\nu^b) + u(\gamma^b,\nu^b;b)$. Rearrange (C.17), we have

$$U(\gamma^b,\nu^b) \leqslant \frac{(w_g - w_b)(\mu R - b)}{b-g} - w_b, \text{ if } R > \beta_b. \tag{C.18}$$

Further, following (C.11), we have

$$U(\gamma^b,\nu^b) \leqslant F_b(w_b). \tag{C.19}$$

On the other hand, if $R \leqslant \beta_b$, then

$$U(\gamma^b,\nu^b) \leqslant F_b(w_b) = -w_b, \text{ if } R \leqslant \beta_b \tag{C.20}$$

Hence, following (C.18) - (C.20), we have

$$U(\gamma^b,\nu^b) \leqslant \min\left\{\frac{(w_g - w_b)}{b-g}\max\{\mu R - b, 0\} - w_b, F_b(w_b)\right\}, \tag{C.21}$$

which corresponds to constraints (3.24) and (3.25).

Step 2.2: Next, following Lemma C.2.3, there exists $\hat{\gamma}^g = (\hat{L},\hat{\tau})$ such that $\bar{\nu} \in \mathfrak{N}(\hat{\gamma}^g, g)$, $u(\hat{\gamma}^g,\bar{\nu};g) = w_g$ and $\tau^0(\hat{\gamma}^g) = \tau$. Denote $\hat{W}_t$ as the agent's continuation utility under contract $\hat{\gamma}^g$ and τ_1^g as the time of the first arrival. Then, following Lemma 3.2.1, we have

$$d\hat{W}_t = [r\hat{W}_{t-} - \mu\hat{H}_t + g]dt - d\hat{L}_t, \hat{H}_t \geqslant \beta_g, \quad \text{for} \quad t < \min\{\tau_1^g, \tau\}. \tag{C.22}$$

Furthermore, denote $\hat{I}_{\tau_1^g}$ as the payment upon the first arrival. Thus,

$$
\begin{aligned}
U(\hat{\gamma}^g, \bar{\nu}) &= \mathrm{E}^{\bar{\nu}}\left[\int_0^\tau e^{-rt}(RdN_t - d\hat{L}_t)\right] \\
&\leqslant \left\{\mathrm{E}_{\tau_1^g}\left[e^{-r\tau_1^g}\left(R - I_{\tau_1^g} + U_{\tau_1^g}(\hat{\gamma}^g, \bar{\nu})\right)1_{\tau_1^g<\tau} - \int_0^{\min\{\tau_1^g,\tau\}} e^{-rt} g dt\right]\right\} \\
&= \left\{\int_0^\tau \left[e^{-r\tau_1^g}\left(R - I_{\tau_1^g} + U_{\tau_1^g}(\hat{\gamma}^g, \bar{\nu})\right) - \int_0^{\tau_1^g} e^{-rt} g dt\right] \mu e^{-\mu\tau_1^g} d\tau_1^g \right. \\
&\left. - \int_\tau^\infty \int_0^\tau e^{-rt} g dt \cdot \mu e^{-\mu\tau_1^g} d\tau_1^g\right\} \\
&= \left[\int_0^\tau \mu e^{-t}(R - I_t + U_t(\hat{\gamma}^g, \bar{\nu}))dt - \int_0^\tau g e^{-t} dt\right]
\end{aligned}
\tag{C.23}
$$

where the first inequality follows from that $d\hat{L}_t \geqslant 0$ and $\hat{\ell}_t \geqslant g$ for $t < \tau$ and the inequality is binding if and only if $d\hat{L}_t = gdt$. Finally, following Proposition 3.3.4 (since W_t is the state variable of the optimal control problem, we can easily generalize (C.11) to it at time t), we have

$$
-I_t + U_t(\hat{\gamma}^g, \bar{\nu}) \leqslant -I_t + F_g(\hat{W}_{t-} + \hat{H}_t - I_t) \leqslant F_g(\hat{W}_{t-} + \hat{H}_t), \tag{C.24}
$$

where the first inequality follows from that the agent's continuation utility $\hat{W}_t = \hat{W}_{t-} + H_t - I_t$ and the second inequality follows from $F_g' \geqslant -1$. Therefore, (C.22) - (C.24) imply that

$$
U(\hat{\gamma}^g, \bar{\nu}) \leqslant G(w_g, \tau), \tag{C.25}
$$

which further implies that

$$
U(\gamma^g, \nu^g) \leqslant U(\hat{\gamma}^g, \bar{\nu}) \leqslant G(w_g, \tau). \tag{C.26}
$$

Hence, following step 1 and 2, for any contract pair (γ^g, γ^b) that satisfy the constraints of the optimization problem $Z(\{g, b\})$, we are able to find w_g, w_b, τ, ξ that satisfy the constraints of the optimization problem Y such that the corresponding objective of Y is higher than it in $Z(\{g, b\})$. Therefore, we conclude that $Z(\{g, b\}) \leqslant Y$.

C.3.4 Proof of Lemma 3.4.2

First, we verify (iii). If $H_t = \beta_g$, $\forall t \in [0, \tau]$, then $W_0 = \breve{w}(\tau)$. Further since $H_t \geqslant \beta_g$, $\forall t \in [0, \tau]$, we have $W_0 \geqslant \breve{w}(\tau)$. Hence, if $w < \breve{w}(\tau)$, then the optimization problem (3.26) is infeasible, or, by convention, $G(w, \tau) = -\infty$.

Next we verify (i) and (ii) by solving the optimiation problem (3.26). Since $g(1-e^{-\tau})$ is fixed when τ is given, we only need to maximize the integral $\int_0^\tau \mu e^{-t}[R + F_g(W_t + H_t)]dt$. To solve the optimization problem, we can write down the Hamiltonian:

$$\mathcal{H} = e^{-t}\{\mu[R + F_g(W_t + H_t)]\} + \lambda(t)(rW_t - \mu H_t) + \eta(t)(H_t - \beta_g). \tag{C.27}$$

The optimality conditions are

$$\frac{\partial \mathcal{H}}{\partial H} = \mu e^{-t} F_g'(W_t + H_t) - \lambda(t)\mu + \eta(t) = 0, \tag{C.28}$$

$$\eta(t)(H_t - \beta_g) = 0; \quad \eta(t) \geqslant 0, \tag{C.29}$$

$$\frac{\partial \mathcal{H}}{\partial W} = \mu e^{-t} F_g'(W_t + H_t) + \lambda(t) r = -\lambda'(t). \tag{C.30}$$

Since the objective of the optimal control problem is jointly concave in (W_t, H_t), it is sufficient to verify the above optimality conditions.

Next, we verify (ii). If $W_0 = w \geqslant \hat{w}(\tau)$, then $W_t + H_t = \mathrm{z} + \beta_g, \forall t \in [0, \tau]$, where $\mathrm{z} + \beta_g = w/(\mu(1 - e^{-\tau}))$. We can easily verify the optimality conditions (C.28) - (C.30) by letting

$$\lambda(t) = F_g'(\mathrm{z} + \beta_g)e^{-t} \quad \text{and} \quad \eta(t) = 0.$$

Furthermore, we can verify that

$$W_t = \mu(\mathrm{z} + \beta_g) - \mu(\mathrm{z} + \beta_g)e^{t-\tau},$$

$$H_t = \mathrm{z} + \beta_g - W_t \geqslant \mathrm{z} + \beta_g - W_0 \geqslant w\left(\frac{1}{\mu(1 - e^{-\tau})} - 1\right) \geqslant \beta_g.$$

where the last inequality follows from $w = W_0 \geqslant \hat{w}(\tau)$.

Finally, we verify (i). If $w = W_0 \in [\check{w}(\tau), \hat{w}(\tau))$, we firstly prove that there exists a unique $z \in [0, g(1-e^{-\tau})/(r+\mu e^{-\tau}))$ such that

$$\bar{w}_g - (\bar{w}_g - z)e^{r(\tau_z - \tau)} = w.$$

Define

$$h(z) := \bar{w}_g - (\bar{w}_g - z)e^{r(\tau_z - \tau)},$$

then we can easily obtain that

$$h(0) = \bar{w}_g - \bar{w}_g \cdot e^{-r\tau} = g/r \cdot (1 - e^{-r\tau}),$$

where the first equality follows from $\tau_0 = 0$ and

$$\lim_{z \to g(1-e^{-\tau})/(r+\mu e^{-\tau})} h(z) = \bar{w}_g - (\bar{w}_g - z) = g(1-e^{-\tau})/(r+\mu e^{-\tau}),$$

where the first equality follows from that $\lim_{z \to g(1-e^{-\tau})/(r+\mu e^{-\tau})} \tau_z = \tau$. Furthermore, we have

$$\begin{aligned} h'(z) &= e^{-r\tau} e^{r\tau_z} \left(1 + (z - \bar{w}_g) r \cdot \frac{\partial \tau_z}{\partial z_1}\right) \\ &= e^{r(\tau_z - \tau)} \left(1 + (z - \bar{w}_g)\frac{r(\bar{w}_g + \beta_g)}{(z_1 + \beta_g)(\bar{w}_g - z_1)}\right) \\ &= e^{r(\tau_z - \tau)} \left(1 - \frac{r(\bar{w}_g + \beta_g)}{z + \beta_g}\right) = e^{r(\tau_z - \tau)} \frac{z}{z + \beta_g} > 0. \end{aligned}$$

Since $w \in [h(0), \lim_{z \to g(1-e^{-\tau})/(r+\mu e^{-\tau})}) h(z))$ and h is continuous, we have that there exists a unique $z \in [0, g(1-e^{-\tau})/(r+\mu e^{-\tau}))$ such that $w = h(z)$, denoted by $\mathrm{z}(w, \tau)$. It is easy to verify that $\tau_{\mathrm{z}} < \tau$ since $\mathrm{z} < g(1-e^{-\tau})/(r+\mu e^{-\tau})$. Hence, if we let H_t follows (3.32), then W_t follows (3.31).

We can easily verify the optimality conditions (C.28) - (C.30) by letting

$$\lambda(t) = \begin{cases} \left[\int_t^{\tau - \tau_{\mathrm{z}}} \mu e^{-\mu\xi} F_g'(W_\xi + \beta) d\xi + F_g'(\mathrm{z} + \beta_g) e^{-\mu(\tau - \tau_{\mathrm{z}})}\right] e^{-rt}, & t \in [0, \tau - \tau_{\mathrm{z}}], \\ F_g'(\mathrm{z} + \beta_g) e^{-t}, & t \in [\tau - \tau_{\mathrm{z}}, \tau], \end{cases}$$

$$\tag{C.31}$$

and

$$\eta(t) = \begin{cases} \mu e^{-rt}\gamma(t), & t \in [0, \tau - \tau_{\mathrm{z}}], \\ 0, & t \in [\tau - \tau_{\mathrm{z}}, \tau], \end{cases} \tag{C.32}$$

where

$$\gamma(t) := \left[\int_t^{\tau-\tau_{\mathrm{z}}} \mu e^{-\mu\xi} F_g'(W_\xi + \beta_g) d\xi + F_g'(\mathrm{z} + \beta_g) e^{-\mu s} - e^{-\mu t} F_g'(W_t + \beta_g)\right] \geqslant 0,$$

for $t \in [0, \tau - \tau_{\mathrm{z}}]$ and the inequality follows from that $\gamma(t)$ is decreasing in t and $\gamma(\tau - \tau_{\mathrm{z}}) = 0$. Furthermore, we have

$$H_t = \beta_g, t \in [0, \tau - \tau_{\mathrm{z}}],$$

$$H_t = \mathrm{z} + \beta_g - W_t \geqslant \mathrm{z} + \beta_g - W_{\tau-\tau_{\mathrm{z}}} = \beta_g, t \in [\tau - \tau_{\mathrm{z}}, \tau].$$

C.3.5 Proof of Lemma 3.4.3

Similar to the proof of Lemma 3.4.2, since the objective of the optimal control problem is jointly concave in (W_t, H_t), it is sufficient to verify the optimality conditions (C.28) - (C.30). We can verify (C.28) - (C.30) by simply letting

$$\lambda(t) = F_g'\left(\frac{w}{\mu}\right) e^{-t}, \eta(t) = 0,$$

and

$$W_t = w > 0, H_t = \frac{w}{\mu} - w \geqslant \frac{r}{\mu} \cdot \frac{g}{r} = \beta_g.$$

C.3.6 Proof of Proposition 3.4.3

First, we look at the function when $\bar{\tau} \in [0, 1/r)$. If $w \geqslant \hat{w}(\tau)$, according to the optimal solution in (3.34), we have

$$G(w, \tau) = \int_0^\tau \mu e^{-t} F_g(y^*) dt + \int_0^\tau (\mu R - c) e^{-t} dt, \tag{C.33}$$

where $y^* = \text{z} + \beta_g = \dfrac{w}{\mu(1-e^{-\tau})}$. Hence,

$$\begin{aligned}
\frac{\partial G(w,\tau)}{\partial \tau} &= \mu e^{-\tau} F_g(y^*) + \int_0^T \mu e^{-t} F_g'(y^*) \frac{\partial y^*}{\partial \tau} dt + (\mu R - g) e^{-\tau} \\
&= \mu e^{-\tau} F_g(y^*) + (\mu R - g) e^{-\tau} + F_g'(y^*) \frac{-we^{-\tau}}{\mu(1-e^{-\tau})^2} \int_0^T \mu e^{-t} dt \\
&= \mu e^{-\tau} (F_g(y^*) - y^* F_g'(y^*)) + (\mu R - c) e^{-\tau} \\
&= \mu e^{-\tau} y^* \left(\frac{F_g(y^*) - F_g(0)}{y^*} - F_g'(y^*) \right) + (\mu R - c) e^{-\tau} > 0, \qquad \text{(C.34)}
\end{aligned}$$

where the inequality follows from the concavity of F. As a result, we have

$$\frac{\partial J(w,\bar{\tau})}{\partial \bar{\tau}} = \frac{\partial \tau}{\partial \bar{\tau}} \frac{\partial G(w,\tau)}{\partial \tau} > 0, \qquad \text{(C.35)}$$

where the inequality follows from (C.34) and $\frac{\partial \tau}{\partial \bar{\tau}} > 0$, and

$$\frac{\partial J(w,\bar{\tau})}{\partial w} = F_g'(y^*). \qquad \text{(C.36)}$$

Hence, J is increasing in $\bar{\tau}$ when $w \geqslant \hat{w}(\tau)$. Next, we verify the concavity of J. Following (C.35) and (C.36), we have that the Hessian matrix of J is

$$\begin{bmatrix} \dfrac{\partial^2 J(w,\bar{\tau})}{\partial^2 w} & \dfrac{\partial^2 J(w,\bar{\tau})}{\partial w \partial \bar{\tau}} \\ \dfrac{\partial^2 J(w,\bar{T})}{\partial w \partial \bar{T}} & \dfrac{\partial^2 J(w,\bar{T})}{\partial^2 \bar{T}} \end{bmatrix}, \qquad \text{(C.37)}$$

where

$$\frac{\partial^2 J(w,\bar{\tau})}{\partial^2 w} = \frac{F_g''(y^*)}{\mu(1-e^{-\tau})} < 0, \qquad \text{(C.38)}$$

where the inequality follows from the concavity of F_g,

$$\frac{\partial^2 J(w,\bar{\tau})}{\partial w \partial \bar{\tau}} = \frac{\partial y^*}{\partial \bar{\tau}} F_g''(y^*) = \frac{\partial y^*}{\partial \tau} \frac{\partial \tau}{\partial \bar{\tau}} F_g''(y^*) > 0, \qquad \text{(C.39)}$$

where the inequality follows from $\frac{\partial y^*}{\partial \tau} < 0$, $\frac{\partial \tau}{\partial \bar{\tau}} > 0$, and the concavity of F_g, and

$$\begin{aligned}
\frac{\partial^2 J(w,\bar{\tau})}{\partial^2 \bar{\tau}} &= -\mu (1 - r\bar{\tau})^{\frac{\mu}{r}} \cdot y^* \cdot F_g''(y^*) \cdot \frac{\partial y^*}{\partial \bar{\tau}} - \mu^2 (1 - r\bar{\tau})^{\frac{\mu - r}{r}} [F(y^*) - y^* F_g'(y^*)] \\
&\quad - (\mu R - g) \mu (1 - r\bar{\tau})^{\frac{\mu - r}{r}} < 0. \qquad \text{(C.40)}
\end{aligned}$$

where the inequality follows from $\frac{\partial y^*}{\partial \tau} < 0$ and $F_g(y^*) - y^* F_g'(y^*) = y^*[(F_g(y^*) - F_g(0))/y^* - F_g'(y^*)] \geqslant 0$ (implied by the concavity of F_g).

Further, with (C.38) - (C.40), we can show that the Hessian matrix of J is negative definite which implies that J is jointly concave when $w \geqslant \hat{w}(\tau)$.

Second, we look at the function if $w \in [\check{w}(\tau), \hat{w}(\tau))$. According to (3.31) and (3.32), we have

$$W_{\tau_1} = \mathrm{z} = \mu(\mathrm{z} + \beta_g)(1 - e^{\tau_1 - \tau}) = \bar{w}_g + (w - \bar{w}_g)e^{r\tau_1},$$

by denoting $\tau_1(w,\tau) := \tau - \tau_{\mathrm{z}}$ and simplifying $\tau_1(w,\tau)$ with τ_1. Further, we denote $y_1^* = \mathrm{z} + \beta_g$. Hence,

$$y_1^* = \frac{g}{\mu(r + \mu e^{\tau_1 - \tau})},$$

and $\tau_1(w,\tau)$ is the solution of

$$\frac{g}{r + \mu e^{\tau_1 - \tau}}(1 - e^{\tau_1 - \tau}) = \bar{w}_g + (w - \bar{w}_g)e^{r\tau_1}, \tag{C.41}$$

where we again simplify $\tau_1(w,\tau)$ with τ_1. Therefore,

$$\begin{aligned} G(w,\tau) = &\int_0^{\tau_1(w,\tau)} \mu e^{-t} F_g\left(\bar{w}_g + (w - \bar{w}_g)e^{rt} + \beta_g\right) dt \\ &+ \int_{\tau_1(w,\tau)}^{\tau} \mu e^{-t} F_g(y_1^*) dt + \int_0^{\tau} (\mu R - g)e^{-t} dt. \end{aligned} \tag{C.42}$$

Then,

$$\begin{aligned} \frac{\partial G(w,\tau)}{\partial \tau} &= \left[\mu e^{-\tau_1} F_g\left(\bar{w}_g + (w - \bar{w}_g)e^{r\tau_1} + \beta_g\right) - \mu e^{-\tau_1} F_g(y_1^*)\right] \frac{\partial \tau_1(w,\tau)}{\partial \tau} \\ &+ \int_{\tau_1(w,\tau)}^{\tau} \mu e^{-t} F'(y_1^*) \frac{\partial y_1^*}{\partial \tau} dt + \mu e^{-\tau} F_g(y_1^*) \\ &= \int_{\tau_1(w,\tau)}^{\tau} \mu e^{-t} dt \cdot F_g'(y_1^*) \frac{-g\mu e^{\tau_1(w,\tau) - \tau} \cdot (\frac{\partial \tau_1(w,\tau)}{\partial \tau} - 1)}{\mu(r + \mu e^{\tau_1(w,\tau) - \tau})^2} \\ &+ \mu e^{-\tau} F_g(y_1^*) + (\mu R - g)e^{-\tau}, \end{aligned} \tag{C.43}$$

Since (C.41) implies that

$$\frac{\partial \tau_1(w,\tau)}{\partial \tau} - 1 = \frac{r + \mu e^{\tau_1 - \tau}}{\mu(1 - e^{\tau_1 - \tau})}, \tag{C.44}$$

we have

$$\frac{\partial G(w,\tau)}{\partial \tau} = \mu e^{-\tau}(F_g(y_1^*) - y_1^* F_g'(y_1^*)) + (\mu R - g)e^{-\tau} > 0, \tag{C.45}$$

where the inequality follows from $F_g(y_1^*) - y_1^* F_g'(y_1^*) = y_1^*[(F_g(y_1^*) - F_g(0))/y_1^* - F_g'(y_1^*)] \geqslant 0$. As a result, we have

$$\frac{\partial J(w,\bar{\tau})}{\partial \bar{\tau}} = \frac{\partial \tau}{\partial \bar{\tau}} \frac{\partial G(w,\tau)}{\partial \tau} > 0, \tag{C.46}$$

where the inequality follows from (C.45) and $\frac{\partial \tau}{\partial \bar{\tau}} > 0$, and

$$\frac{\partial J(w,\bar{\tau})}{\partial w} = \mu \int_0^{\tau_1} e^{-\mu t} F_g'(\bar{w}_g + (w - \bar{w}_g)e^{rt} + \beta_g)dt + F_g'(y_1^*) \cdot e^{-\mu \tau_1}. \tag{C.47}$$

Hence, J is increasing in $\bar{\tau}$ when $w \in [\check{w}(\tau), \hat{w}(\tau))$. Next, we verify the concavity of J. Following (C.46) and (C.47), we have that the Hessian matrix of J is

$$\begin{bmatrix} \frac{\partial^2 J(w,\bar{T})}{\partial^2 w} & \frac{\partial^2 J(w,\bar{T})}{\partial w \partial \bar{T}} \\ \frac{\partial^2 J(w,\bar{T})}{\partial w \partial \bar{T}} & \frac{\partial^2 J(w,\bar{T})}{\partial^2 \bar{T}} \end{bmatrix}, \tag{C.48}$$

where

$$\frac{\partial^2 J(w,\bar{\tau})}{\partial^2 w} = \mu \int_0^{\tau_1} e^{-\mu t} F''(\bar{w}_g + (w - \bar{w}_g)e^{rt} + \beta_g)e^{rt}dt + F''(y_1^*) \cdot e^{-\mu \tau_1} \cdot \frac{\partial y_1^*}{\partial w}, \tag{C.49}$$

and

$$\begin{aligned} \frac{\partial^2 J(w,\bar{\tau})}{\partial^2 \bar{\tau}} &= -\mu(1 - r\bar{\tau})^{\frac{\mu}{r}} \cdot y_1^* \cdot F_g''(y_1^*) \cdot \frac{\partial y_1^*}{\partial \bar{\tau}} \\ &\quad - \mu^2 (1 - r\bar{\tau})^{\frac{\mu - r}{r}} \cdot [F_g(y_1^*) - y_1^* F_g'(y_1^*)] \\ &\quad - (\mu R - g) \cdot \mu \cdot (1 - r\bar{\tau})^{\frac{\mu - r}{r}} < 0, \end{aligned} \tag{C.50}$$

where the inequality follows from $\frac{\partial y^*}{\partial \bar{\tau}} < 0$ and $F_g(y^*) - y^* F_g'(y^*) = y^*[(F_g(y^*) - F_g(0))/y^* - F_g'(y^*)] \geqslant 0$ (implied by the concavity of F_g), and

$$\frac{\partial^2 J(w, \bar{\tau})}{\partial w \partial \bar{\tau}} = -\mu \cdot \frac{\partial \tau}{\partial \bar{\tau}} \cdot e^{-\tau} \cdot y_1^* \cdot F''(y_1^*) \cdot \frac{\partial y_1^*}{\partial w}. \tag{C.51}$$

Further, the concavity of F_g and

$$\frac{\partial y_1^*}{\partial w} = \frac{\partial y_1^*}{\partial \tau_1} \cdot \frac{\partial \tau_1}{\partial w} = -e^{r\tau_1} \frac{(r + \mu e^{-(\tau-\tau_1)})^2}{g e^{-(\tau-\tau_1)} \mu (1 - e^{-(\tau-\tau_1)})} \cdot \frac{\partial y_1^*}{\partial \tau_1} > 0.$$

implies that

$$\frac{\partial^2 J(w, \bar{\tau})}{\partial^2 w} < 0, \frac{\partial^2 J(w, \bar{\tau})}{\partial w \partial \bar{\tau}} > 0. \tag{C.52}$$

Furthermore, following (C.49)-(C.51) we can show that the Hessian matrix of J is negative definite which implies that J is jointly concave when $w \in [\check{w}(\tau), \hat{w}(\tau))$. And following (C.33) and (C.42), we can show that $G(w, \tau)$ is continuously differentiable when $w = \hat{w}(\tau)$.

Finally, we show that $\lim_{\bar{\tau} \to 1/r} J(w, \bar{\tau}) = J(w, 1/r)$.

$$\lim_{\bar{\tau} \to 1/r} J(w, \bar{\tau}) = \lim_{\tau \to \infty} G\left(w, -\frac{\log(1 - r\bar{\tau})}{r}\right) \tag{C.53}$$

Since $\lim_{\tau \to \infty} \hat{w}(\tau) = g/r$, if $w < g/r$, we have $\lim_{\bar{\tau} \to 1/r} J(w, \bar{\tau}) = -\infty = J(w, \bar{\tau})$. If $w \geqslant g/r$, then since

$$\lim_{\tau \to \infty} z(w, \tau) = \lim_{\tau \to \infty} \frac{w}{\mu(1 - e^{-\tau})} - \beta_g = \frac{w}{\mu} - \beta_g$$

and

$$\lim_{\tau \to \infty} W_t = \lim_{\tau \to \infty} \mu(z + \beta_g)(1 - e^{t-\tau}) = w; \lim_{\tau \to \infty} H_t = \lim_{\tau \to \infty} z + \beta_g - W_t = w/\mu - w$$

Hence, following (3.36), we have $\lim_{\bar{\tau} \to \infty} J(w, \bar{\tau}) = J(w, 1/r)$ if $w \geqslant g/r$. This concludes the proof.

C.3.7 Proof of Lemma 3.4.5

Define

$$V_b(w) := F_b(w) + w. \tag{C.54}$$

Following Proposition 3.3.4, we have $F_b(0) = 0$. Further, since $(w_g^*, w_b^*, \bar{\tau}^*)$ are the optimal solution of (3.39)-(3.41), we have $w_g^* \geqslant w_b^*$. Hence, $F_b(0) \leqslant \dfrac{(w_g^* - w_b^*)(\mu R - b)}{b - g}$ which completes the proof.

C.3.8 Proof of Lemma 3.4.6

First, we present a technical lemma.

Lemma C.3.9. *For any $k \geqslant 0$, we have*

$$\frac{V_{b_1}(k \cdot b_1)}{\mu R - b_1} = \frac{V_{b_2}(k \cdot b_2)}{\mu R - b_2}, \forall b_1, b_2 < \mu R, \tag{C.55}$$

where V_b is defined in (C.54).

Proof. Following Proposition 3.3.4, we have

$$\begin{aligned} V_b(w) &= F_b(w) + w = U(\gamma_{\mathsf{B}}^b(w,0), \bar{\nu}) + w \\ &= S(\gamma_{\mathsf{B}}^b(w,0), \bar{\nu}; b) = (\mu R - b)\bar{T}(\gamma_{\mathsf{B}}^b(w,0), \bar{\nu}) \\ &= (\mu R - b) E\left[\int_0^{\tau_B^b} e^{-rt} dt\right], \end{aligned} \tag{C.56}$$

where the fourth equality follows from Lemma 9, and the fifth equality follows from (C.12). Following (3.9) and (3.11), we have for any b,

$$dW_t^b = [r(W_{t-}^b - b) + \min\{b/r - W_{t-}^b, b/\mu\} dN_t] 1_{W_{t-}^b \geqslant 0}, \tau_B^b = \min\{t : W_{t-}^b = 0\}.$$

Define process $w_t := W_t^b / b$, then we have

$$dw_t = [r(w_t - 1) + \min\{1/r - w_{t-}, 1/\mu\} dN_t] 1_{w_{t-} \geqslant 0}, \tau_B^b = \min\{t : w_{t-} = 0\},$$

Hence, for any b_1, b_2, $\tau_B^{b_1}$ and $\tau_B^{b_2}$ follows the same distribution if $W_0^{b_1}/b_1 = W_0^{b_2}/b_2$. Therefore,

$$\frac{V_{b_1}(k \cdot b_1)}{\mu R - b_1} = \mathrm{E}\left[\int_0^{\tau_B^{b_1}} e^{-rt} dt\right] = \mathrm{E}\left[\int_0^{\tau_B^{b_2}} e^{-rt} dt\right] = \frac{V_{b_2}(k \cdot b_2)}{\mu R - b_2},$$

which verifies (C.55). □

Lemma C.3.9 implies that for any $w > 0$ and $b_1 < b_2$, we have

$$V_{b_1}(w) = V_{b_2}\left(w\frac{b_2}{b_1}\right)\frac{\mu R - b_1}{\mu R - b_2} > V_{b_2}(w)\frac{\mu R - b_1}{\mu R - b_2} > V_{b_2}(w) \tag{C.57}$$

Next, we prove the desired results. For any given b, we denote the solution of optimization problem (3.39)-(3.41) as $(w_g^*(b), w_b^*(b), \bar{\tau}^*(b))$ and w_B defined in (3.44) as $w_B(b)$. If there exists $\check{b} \in [b, \mu R]$ such that $w_B(\check{b}) = 0$, then we prove by contradcition that for any $\hat{b} > \check{b}$, we have $w_B(\hat{b}) = 0$. It implies that there exists $\bar{b} \in [g, \mu R]$ such that $w_B = 0$ if and only if $w_B > 0$.

If $w_B(\hat{b}) > 0$, following (3.44), we have $\mu R > b$ and $w_g^*(\hat{b}) > w_b^*(\hat{b})$. Following (3.40) and (3.41), $(w_g^*(\hat{b}), w_b^*(\hat{b}), \bar{\tau}^*(\hat{b}))$ and $(w_g^*(\check{b}), w_b^*(\check{b}), \bar{\tau}^*(\check{b}))$ are feasible for both $b = \hat{b}$ and $b = \check{b}$. Therefore,

$$\begin{aligned}
&p \cdot J(w_g^*(\hat{b}), \bar{\tau}^*(\hat{b})) + (1-p)\min\left\{F_{\hat{b}}(w_b^*(\hat{b})), \frac{w_g^*(\hat{b}) - w_b^*(\hat{b})}{\hat{b} - g} - w_b^*(\hat{b})\right\} \\
&> p \cdot J(w_g^*(\check{b}), \bar{\tau}^*(\check{b})) + (1-p) w_b^*(\check{b}) \\
&\geqslant p \cdot J(w_g^*(\hat{b}), \bar{\tau}^*(\hat{b})) + (1-p)\min\left\{F_{\check{b}}(w_b^*(\hat{b})), \frac{w_g^*(\hat{b}) - w_b^*(\hat{b})}{\check{b} - g} - w_b^*(\hat{b})\right\}
\end{aligned}$$

which implies that

$$\begin{aligned}
&\min\left\{F_{\hat{b}}(w_b^*(\hat{b})), \frac{w_g^*(\hat{b}) - w_b^*(\hat{b})}{\hat{b} - g} - w_b^*(\hat{b})\right\} \\
&> \min\left\{F_{\check{b}}(w_b^*(\hat{b})), \frac{w_g^*(\hat{b}) - w_b^*(\hat{b})}{\check{b} - g} - w_b^*(\hat{b})\right\}
\end{aligned}$$

which is equivalent to

$$\min\left\{V_{\hat{b}}(w_b^*(\hat{b})), \frac{w_g^*(\hat{b}) - w_b^*(\hat{b})}{\hat{b} - g}\right\} \geqslant \min\left\{V_{\check{b}}(w_b^*(\hat{b})), \frac{w_g^*(\hat{b}) - w_b^*(\hat{b})}{\check{b} - g}\right\}$$

which contradicts with

$$V_{\hat{b}}(w_b^*(\hat{b})) < V_{\check{b}}(w_b^*(\hat{b})), \text{and}, \frac{w_g^*(\hat{b}) - w_b^*(\hat{b})}{\hat{b} - g} < \frac{w_g^*(\hat{b}) - w_b^*(\hat{b})}{\check{b} - g}.$$

where the first inequality follows from (C.57).

C.3.10 Proof of Theorem 3.4.5

Following proposition 3.3.3, contract pair $(\gamma_{\mathtt{P}}^g(\tau^*, z^*), \gamma_{\mathtt{B}}^b(w_B, w_b^* - w_B))$ satisfy constraints (LL), (PK), (IC), and (IR). In the following, we verify (TT).

$$u(\gamma_{\mathtt{B}}^b(w_B, w_b^* - w_B), \bar{\nu}; b) = w_b^* \geqslant g/r \cdot (1 - e^{-r\tau^*}) = u(\gamma_{\mathtt{P}}^g(\tau^*, z^*), \nu^0; b)$$
$$= \max_{\nu \in \mathcal{N}} u(\gamma_{\mathtt{P}}^g(\tau^*, z^*), \nu; b),$$

where the first equality follows from Proposition 3.3.3, the first inequality follows from constraint (3.22), and the last inequality follows from $\{\nu^0\} = \mathfrak{N}(\gamma_{\mathtt{P}}^g(\tau^*, z^*), b)$. If $\mu R \leqslant b$,

$$u(\gamma_{\mathtt{P}}^g(\tau^*, z^*), \bar{\nu}; g) = w_g^* \geqslant w_b^* = \max_{\nu \in \mathcal{N}} u(\gamma_{\mathtt{B}}^b(0, w_b^*), \nu; g),$$

where the first equality follows from proposition 3.3.3 and the first inequality follows from constraint (3.22). On the other hand, if $\mu R > b$,

$$\begin{aligned} u(\gamma_{\mathtt{P}}^g(\tau^*, z^*), \bar{\nu}; g) &= w_g^* \geqslant w_b^* + (b - g)\frac{\xi^*}{(\mu R - b)} = w_b^* + (b - g)\frac{V_b(w_B)}{(\mu R - b)} \\ &= w_b^* + (b - g)\bar{T}(\gamma_{\mathtt{B}}^b(w_B, w_b^* - w_B), \bar{\nu}) \\ &= u(\gamma_{\mathtt{B}}^b(w_B, w_b^* - w_B), \bar{\nu}; g) = \max_{\nu \in \mathcal{N}} u(\gamma_{\mathtt{B}}^b(w_B, w_b^* - w_B), \nu; g), \end{aligned}$$

where the first equality follows from proposition 3.3.3, the first inequality follows from constraint (3.25), the last equality follows from the following lemma.

Lemma C.3.11. *For any γ^b that $\bar{\nu} \in \mathfrak{N}(\gamma^b, b)$, we have $\bar{\nu} \in \mathfrak{N}(\gamma^b, g)$.*

Define bad agent's lifetime expected utility, evaluated conditionally upon the information available at time t under contract γ^b and effort process $\bar{\nu}$ as u_t^b, then

$$u_t^b = \mathrm{E}^{\bar{\nu}} \left[\int_0^{\tau^b} e^{-rs}(dL_s^b - bds) \middle| \mathcal{F}_t^N \right] = u_0^b + \int_0^t H_s^b dM_s^{\bar{\nu}},$$

where $M_t^{\bar{\nu}} = N_t - \mu t$ and $H_s^b \geqslant \beta_b$ for any s. Define good agent's lifetime expected utility, evaluated conditionally upon the information available at time t under contract γ^b and effort process $\bar{\nu}$ as u_t^g, then

$$\begin{aligned} u_t^g &= \mathrm{E}^{\bar{\nu}} \left[\int_0^{\tau^b} e^{-rs}(dL_s^b + (b-g)ds) \middle| \mathcal{F}_t^N \right] = u_t^b + \mathrm{E}^{\bar{\nu}} \left[\int_0^{\tau^b} e^{-rs}(b-g)ds \middle| \mathcal{F}_t^N \right] \\ &= u_0^b + \int_0^t H_s^b dM_s^{\bar{\nu}} + \mathrm{E}^{\bar{\nu}} \left[\int_0^{\tau^b} e^{-rs}(b-g)ds \middle| \mathcal{F}_t^N \right]. \end{aligned}$$

Next, we denote $u_t^{g'}$ as good agent's lifetime expected payoff, given the information available at time t, when he acts according to $\nu' = \{\nu_t'\}_{t \geqslant 0}$ until time t and then reverts to $\bar{\nu}$, then

$$\begin{aligned} u_t^{g'} &= u_t^g + \int_0^{t \wedge \tau -} e^{-rs}(1 - 1_{\nu_s'=\mu})gds \\ &= u_0^b + \int_0^{t \wedge \tau -} H_s^b dM_s^{\bar{\nu}} + \mathrm{E}^{\bar{\nu}} \left[\int_0^{\tau^b} e^{-rs}(b-g)ds \middle| \mathcal{F}_t^N \right] \\ &+ \int_0^{t \wedge \tau -} e^{-rs}(1 - 1_{\nu_s'=\mu})gds \\ &= u_0^b + \int_0^{t \wedge \tau^b -} H_s^b dM_s^{\nu'} + \mathrm{E}^{\bar{\nu}} \left[\int_0^{\tau^b} e^{-rs}(b-g)ds \middle| \mathcal{F}_t^N \right] \\ &+ \int_0^{t \wedge \tau^b -} e^{-rs}\mu(1 - 1_{\nu_s'=\mu})(\beta_g - H_s^b)ds, \end{aligned}$$

Then, for any $t' > t$,

$$\begin{aligned}
\mathrm{E}^{\nu'}[u_{t'}^{g'}|\mathcal{F}_t^N] = \mathrm{E}^{\nu'}\Bigg[& u_0^b + \int_0^{t'\wedge\tau^b-} H_s^b dM_s^{\nu'} + \mathrm{E}^{\bar{\nu}}\left[\int_0^{\tau^b} e^{-rs}(b-g)ds\bigg|\mathcal{F}_{t'}^N\right] \\
& + \int_0^{t'\wedge\tau^b-} e^{-rs}\mu(1-1_{\nu_s'=\mu})(\beta_g - H_s^b)ds\bigg|\mathcal{F}_t^N\Bigg] \\
& = u_0^b + \int_0^{t\wedge\tau^b-} H_s^b dM_s^{\nu'} \\
& + \mathrm{E}^{\bar{\nu}}\left[\int_0^{\tau^b} e^{-rs}(b-g)ds\bigg|\mathcal{F}_t^N\right] \\
& + \mathrm{E}^{\nu'}\left[\int_0^{t'\wedge\tau^b-} e^{-rs}\mu(1-1_{\nu_s'=\mu})(\beta_g - H_s^b)ds\bigg|\mathcal{F}_t^N\right] \\
& \leqslant u_0^b + \int_0^{t\wedge\tau^b-} H_s^b dM_s^{\nu'} + \mathrm{E}^{\bar{\nu}}\left[\int_0^{\tau^b} e^{-rs}(b-g)ds\bigg|\mathcal{F}_t^N\right] \\
& + \int_0^{t\wedge\tau^b-} e^{-rs}\mu(1-1_{\nu_s'=\mu})(\beta_g - H_s^b)ds = u_t^{g'},
\end{aligned}$$

where the second equality follows from law of iterated expectation and the first inequality follows from that $\mu(1-1_{\nu_s'=\mu})(\beta_g - H_s^b) \leqslant 0, \forall t$. Hence, $u_t^{g'}$ is $\mathcal{F}^N$-supermartingale under $P^{\nu'}$. Therefore, by the optional sampling theorem (Dellacherie and Meyer (2011), Chapter VI, Theorem 10),

$$u(\gamma^b, \bar{\nu}; g) = u_0^{g'} \geqslant \mathrm{E}^{\nu'}[u_\tau^{g'}] = u(\gamma^b, \nu'; g).$$

which implies that $\bar{\nu}$ is at least as good as ν' for the agent.

C.3.12 Proof of Proposition 3.4.6

Following Proposition 3.3.4, we have if $\beta_b \geqslant R$, then $U(\gamma^b, \nu^b) \leqslant -w$ which implies that $w_*^b = 0$. As a result, $w_b^* \geqslant w_*^b = 0$.

If $\beta_b \geqslant R$, then in the optimization problem (3.4.3), $\xi \leqslant 0$. Hence, the optimization problem (3.4.3) becomes

$$\max_{w_g, w_b, \bar{\tau}} \quad p \cdot J(w_g, \bar{\tau}) - (1-p)w_b$$

$$w_g \geqslant w_b \geqslant g \cdot \bar{\tau},$$

$$\bar{\tau} \geqslant 0, \bar{\tau} \leqslant 1/r.$$

Since the objective function is decreasing in w_b, we can let $w_b = g \cdot \bar{\tau}$. Hence, optimization problem (3.4.3) becomes

$$\max_{w_g, \bar{\tau}} \quad p \cdot J(w_g, \bar{\tau}) - (1-p)g \cdot \bar{\tau}$$

$$w_g \geqslant g \cdot \bar{\tau},$$

$$\bar{\tau} \geqslant 0, \bar{\tau} \leqslant 1/r.$$

We firstly consider the following optimization problem by adding the third constraint (corresponding dual variable defined after each constraint):

$$\max_{w_g, \bar{\tau}} \quad p \cdot J(w_g, \bar{\tau}) - (1-p)g \cdot \bar{\tau} \tag{C.58}$$

$$\bar{\tau} \geqslant 0; \eta_1,$$

$$w_g \geqslant g \cdot \bar{\tau}; \eta_2,$$

$$w_g \leqslant \bar{w}_g = g/r; \eta_3.$$

The Lagrangian of the above optimization problem is given as

$$\mathcal{L} = pJ(w_g, \bar{\tau}) - (1-p)g\bar{\tau} + \eta_1\bar{\tau} + \eta_2(w_g - g\bar{\tau}) + \eta_3(\bar{w}_g - w_g)$$

Optimality condition requires

$$\eta_1 \geqslant 0, \eta_1 \bar{\tau} = 0,$$

$$\eta_2 \geqslant 0, \eta_2(w_g - g\bar{\tau}) = 0,$$

$$\eta_3 \geqslant 0, \eta_3(\bar{w}_g - w_g) = 0,$$

$$\frac{\partial \mathcal{L}}{w_g} = pJ_1 + \eta_2 - \eta_3 = 0,$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\tau}} = pJ_2 - (1-p)g + \eta_1 - \eta_2 g = 0.$$

If $\eta_3 > 0$, then $w_g = \bar{w}_g$. Hence, $J_1 = F_g'(y^*) = -1$ (following (C.36)). Hence, $\eta_3 = -\eta_2 - pJ_1 < 0$ which leads to a contradiction. Hence, $\eta_3 = 0$. We further consider the following two cases:

1. If $\eta_2 > 0$, then $w = g\bar{\tau}$. Hence, $\eta_2 = -pJ_1(w, w/g) > 0$ and $\eta_1 = -pJ_1(w, w/g)g - pJ_2(w, w/g) + (1-p)g = -pgF_g'(w) + (1-p)g \geqslant 0$. Hence, in this case, the optimal solution will be either $w_g^* = \bar{\tau}^* = 0$ or $w_g^* = w^{**}$ and $\bar{\tau}^* = w^{**}/g$ where $F_g'(w^{**}) = (1-p)/p$. In both cases, $w_g^* < w_*^g$

2. If $\eta_2 = 0$, then we need to find $w_g^*, \bar{\tau}^*$ such that

$$\eta_2 = -pJ_1(w_g^*, \bar{\tau}^*) = 0,$$

$$\eta_1 = (1-p)g - pJ_2,$$

$$\eta_1 \bar{\tau} = 0.$$

Since $J_1(w_g^*, \bar{\tau}^*) = 0$, we have $\bar{\tau}^* > 0$ and $\eta_1 = 0$. (If $\eta_1 > 0$ and $\bar{\tau}^* = 0$, then $J_1(w_g^*, 0) = -1$) Hence, we require $J_1(w_g^*, \bar{\tau}^*) = 0$ and $J_2(w_g^*, \bar{\tau}^*) = (1-p)g/p$. Finally, we show that $w_g^* < w_*^g$. If $w_*^g > 0$, then $w_*^g = \{w : F_g'(w) = 0\}$. For any $w \geqslant w_*^g$, $F_g'(w) = J_1(w, w/g) + 1/g \cdot J_2(w, w/g) \leqslant 0$. Further, since $J_2(w, w/g) > 0$ (following (C.46)), we have $J_1(w, w/g) < 0$. Furthermore, since

$J_{12} > 0$ (following (C.52)), we have $J_1(w, \bar{\tau}) < J_1(w, w/g) < 0$ for any $w \geqslant w_g^*$ and $\bar{\tau} \leqslant w/g$. In this case, $w_g^* < w_*^g$.

On the other hand, if $w_*^g = 0$, then $F_g'(0) \leqslant 0$ which further implies that $F_g'(w) = J_1(w, w/g) + 1/g \cdot J_2(w, w/g) \leqslant 0$ for any $w \geqslant 0$. Again, since $J_2(w, w/g) > 0$(following (C.46)), we have $J_1(w, w/g) < 0$. Furthermore, since $J_{12} > 0$ (following (C.52)), we have $J_1(w, \bar{\tau}) < J_1(w, w/g) < 0$ for any $w \geqslant 0$ and $\bar{\tau} \leqslant w/g$. In this case, $(w_g^*, \bar{\tau}^*)$ does not exist.

Hence, $w_g^* \leqslant w_*^g$.

Since $w \leqslant w_*^g < \bar{w}_g$ in the optimization problem (C.58), the constraint $w \leqslant \bar{w}_g$ is redundant. Hence, in the optimization problem (3.4.3), we have if $\beta_b \geqslant R$, $w_g^* \leqslant w_*^g$. This concludes the proof.

C.4 Proofs in Section 5

C.4.1 Proof of Lemma 3.5.2

(i) Following Proposition 3.4.3, $J(w, \bar{\tau}, c)$ is jointly concave in $(w, \bar{\tau})$.

(ii) Following Proposition 3.4.3, $J(w, \bar{\tau}, c)$ is is increasing in $\bar{\tau}$. If $\bar{\tau} = 0$, then $\tau = -\log(1 - r\bar{\tau})/r = 0$. Hence, $\hat{w}(0) = 0$. Hence, for any $w \geqslant 0$, following (C.33), we have

$$
\begin{aligned}
J(w, 0, c) &= \lim_{\tau \to 0} \int_0^\tau \mu e^{-t} F_g \left(\frac{w}{\mu(1 - e^{-\tau})} \right) dt \\
&= \lim_{\tau \to 0} \int_0^\tau \mu e^{-t} \left[\frac{\mu R - c}{r} - \frac{w}{\mu(1 - e^{-\tau})} \right] dt \\
&= \lim_{\tau \to 0} -\mu(1 - e^{-\tau}) \frac{w}{\mu(1 - e^{-\tau})} = -w.
\end{aligned}
$$

(iii) It is equivalent to show that $J_1'(w, \bar{\tau}, c) = J_1'(w, \bar{\tau}) \geqslant -1$. First, J is concave in w. Hence, we only need to show $J_1'(w, \bar{\tau}) \geqslant -1$ when w is large enough.

Denote $\tau := \dfrac{\log(1-r\bar{\tau})}{r}$. Following (C.36), we have, for $w \geqslant \hat{w}(\tau)$, $J_1'(w,\bar{\tau}) = F_c'\left(\dfrac{w}{\mu(1-e^{-\tau})}\right) \geqslant -1$.

(iv) Following (ii) and (iii), we have

$$J(w,\bar{\tau},c) + w \geqslant J(w,0,c) + w = 0.$$

Again, following (ii) and definition of J, we have

$$J(w,\bar{\tau},c) + w \leqslant J\left(w,\frac{w}{c},c\right) + w = U(\gamma_{\mathsf{P}}^c(\tau,0),\bar{\nu})$$

$$= U(\gamma_{\mathsf{B}}^c(w,0),\bar{\nu}) + w = F_c(w) + w = V_c(w) \leqslant \frac{\mu R - c}{r},$$

where $\tau := \dfrac{\log(1-r\bar{\tau})}{r}$ and $\bar{\tau} = \dfrac{w}{c}$, the last inequality follows from 3.3.4.

C.4.2 Proof of Theorem 3.5.2

For any contract menu $\Gamma_C, C = [\underline{c},\bar{c}]$, denote $\mathsf{w}(c) := u(\gamma^c,\nu^c;c)$, and $\tau(c) := \tau^0(\gamma^c)$ for any $c \in [\underline{c},\bar{c}]$ where $\tau^0(.)$ is defined in (C.10).

Step 1: Constraint is more relaxed: We prove that the constraint of $Z(C)$ implies the constraint of Y^C. First,

$$\tau(c) \in [0,1/r] \tag{C.59}$$

Second, (TT) implies that for any $c_1 < c_2 \in C$,

$$\mathsf{w}(c_1) \geqslant \max_{\nu} u(\gamma^{c_2},\nu;c_1) \geqslant u(\gamma^{c_2},\nu^{c_2};c_1)$$

$$= \mathsf{w}(c_2) + (c_2 - c_1)\bar{T}(\gamma^{c_2},\nu^{c_2}) \geqslant \mathsf{w}(c_2), \tag{C.60}$$

where $\nu^{c_2} \in \mathfrak{N}(\gamma^{c_2},c_2)$ and $\bar{T}$ is defined in (C.12). Hence, (TT) implies that $\mathsf{w}(c)$ should be non-increasing.

Furthermore, for any $c < \bar{c}$, we have

$$\mathsf{w}(\bar{c}) \geqslant \max_{\nu} u(\gamma^c,\nu;\bar{c}) \geqslant u(\gamma^c,\nu^0;\bar{c}) = c\cdot\tau(c) = c\int_0^{\tau} e^{-rt}dt = c/r\cdot(1-e^{-r\tau}). \tag{C.61}$$

Step 2: Objective is higher We prove that the objective of Y^C is greater or equal to the objective of $Z(C)$.

If $R > \beta_c$, then for any $\tilde{c} < c$, following (C.60), we have

$$\begin{aligned} \mathsf{w}(\tilde{c}) &\geqslant \mathsf{w}(c) + (c-\tilde{c})\bar{T}(\gamma^c, \nu^c) = \mathsf{w}(c) + \frac{(c-\tilde{c})S(\gamma^c, \nu^c; c)}{\mu R - c} \\ &= \mathsf{w}(\tilde{c}) + \frac{(c-\tilde{c})(U(\gamma^c, \nu^c) + w_c)}{\mu R - c}, \end{aligned} \tag{C.62}$$

where the first equality follows from Lemma C.3.2 and the last equality follows from $S(\gamma^c, \nu^c; c) = U(\gamma^c, \nu^c) + u(\gamma^c, \nu^c; c)$. Rearrange (C.62), we have, for any $\tilde{c} < c$,

$$U(\gamma^c, \nu^c) \leqslant \frac{(\mathsf{w}(\tilde{c}) - \mathsf{w}(c))(\mu R - c)}{c - \tilde{c}} - \mathsf{w}(c), \text{ if } R > \beta_c. \tag{C.63}$$

Hence, for any $c < \mu R$, we have

$$U(\gamma^c, \nu^c) \leqslant \inf_{\tilde{c}<c} \left[\frac{\mathsf{w}(\tilde{c}) - \mathsf{w}(c)}{c - \tilde{c}} \right] (\mu R - c) - \mathsf{w}(c). \tag{C.64}$$

On the other hand, if $R \leqslant \beta_c$ $(c \geqslant \mu R)$, then

$$U(\gamma^c, \nu^c) \leqslant F_c(\mathsf{w}(c)) = -\mathsf{w}(c), \text{ if } R \leqslant \beta_c. \tag{C.65}$$

Finally, following (C.11) and definition of J, we have for any $c < \mu R$,

$$U(\gamma^c, \nu^c) \leqslant J(\mathsf{w}(\mathsf{c}), \tau(\mathsf{c}), \mathsf{c}) \leqslant \mathsf{J}\left(\mathsf{w}(\mathsf{c}), \min\left\{\frac{\mathsf{w}(\bar{\mathsf{c}})}{\mathsf{c}}, \frac{1}{\mathsf{r}}\right\}, \mathsf{c}\right), \tag{C.66}$$

where the second inequality follows from that the function $J(w, \bar{\tau}, c)$ is increasing in $\bar{\tau}$ (Lemma 3.5.2(ii)), (C.59) and (C.61). Therefore, (C.64) and (C.66) imply that for any $c < \mu R$,

$$U(\gamma^c, \nu^c) \leqslant \xi\big(c; \mathsf{w}(\cdot)\big). \tag{C.67}$$

With (C.65), we established that the objective of Y^C is higher than the objective of $Z(C)$. Hence, following step 1 and 2, for any contract menu Γ_C that satisfy the constraints of the optimization problem $Z(C)$, we are able to find $\mathsf{w}(.)$ that satisfy the constraints of the optimization problem Y^C such that the corresponding objective of Y^C is higher than it in $Z(C)$. Therefore, we conclude that, $Z(C) \leqslant Y^C$.

C.4.3 Proof of Proposition 3.5.3

Define

$$y(N, \mathsf{w}(.)) := \sum_{i=1}^{N} [P(c_i) - P(c_{i-1})] \min \left\{ J\left(\mathsf{w}(c_i), \min\left\{ \frac{\mathsf{w}(\bar{c})}{c_i}, \frac{1}{r} \right\},\ c_i \right), \right.$$

$$\left. \inf_{\tilde{c} < c_i} \left[\frac{\mathsf{w}(\tilde{c}) - \mathsf{w}(c_i)}{\delta} \right] (\mu R - c_i) - \mathsf{w}(c_i) \right\}$$

$$- \mathsf{w}(c_N) \int_{\min\{\mu R, \bar{c}\}}^{\bar{c}} \rho(c) dc, \tag{C.68}$$

Following (3.51), we have

$$Y^c = \sup_{\mathsf{w}(.)} \quad \lim_{N \to \infty} y(N, \mathsf{w}(.)),$$

$$s.t\ \mathsf{w}(c) \text{ is non-increasing in } c. \tag{C.69}$$

As a result, we have for any $\epsilon > 0$, there exists non-increasing $\tilde{\mathsf{w}}(c)$ such that

$$Y^C \leqslant \lim_{N \to \infty} y(N, \tilde{\mathsf{w}}(.)) + \epsilon,$$

Since for any N,

$$y(N, \tilde{\mathsf{w}}(.)) \leqslant \hat{y}(N),$$

we have

$$\lim_{N \to \infty} y(N, \tilde{\mathsf{w}}(.)) = \liminf_{N \to \infty} y(N, \tilde{\mathsf{w}}(.)) \leqslant \liminf_{N \to \infty} \hat{y}(N).$$

Hence, for any $\epsilon > 0$, we have

$$Y^C \leqslant \liminf_{N \to \infty} \hat{y}(N) + \epsilon,$$

which finally implies (3.56).

C.4.4 Proof of Proposition 3.5.4

First, we show that $\mathfrak{J}_i(w|w_N)$ is concave in w by induction. $\mathfrak{J}_0(w|w_N) = 0$ is clearly concave in w. Next, if $\mathfrak{J}_{i-1}(w|w_N)$ is concave in w, we verify that $\mathfrak{J}_i(w|w_N)$ is also concave in w. Denote

$$f(w_{i-1}, w_i) := [P(c_i) - P(c_{i-1})] \min \left\{ \frac{w_{i-1} - w_i}{\delta}(\mu R - c_i) - w_i, \right.$$
$$\left. J\left(w_i, \min\left\{\frac{w_N}{c_i}, \frac{1}{r}\right\}, c_i\right)\right\} + \mathfrak{J}_{i-1}(w_{i-1}|w_N).$$

Since $\frac{w_{i-1} - w_i}{\delta}(\mu R - c_i) - w_i$ is linear in (w_{i-1}, w_i) and $J\left(w_i, \min\left\{\frac{w_N}{c_i}, \frac{1}{r}\right\}, c_i\right)$ is concave in w_i (follows Lemma 3.5.2), then $G(w_{i-1}, w_i)$ is jointly concave in (w_{i-1}, w_i). Hence, $\mathfrak{J}_i(w_i|w_N)$ is concave in w_i.

Since $\mathfrak{J}_{i-1}(w_{i-1}|w_N)$ is concave in w_{i-1}, $\breve{w}$ and $\hat{w}$ are well-defined and $\breve{w} \leqslant \hat{w}$. Next, we verify the optiaml solution in the following 3 cases.

Case 1. If $w_i \leqslant \breve{w} - \bar{u}\delta$, then we verify that $w^*_{i-1} = \breve{w}$. If $w_{i-1} \geqslant \breve{w}$, then $w_{i-1} \geqslant \breve{w} \geqslant w_i + \bar{u}\delta$. Hence

$$f(w_{i-1}, w_i) = [P(c_i) - P(c_{i-1})] J\left(w_i, \min\left\{\frac{w_N}{c_i}, \frac{1}{r}\right\}, c_i\right) + \mathfrak{J}_{i-1}(w_{i-1}|w_N),$$

and

$$f'_1(w_{i-1}, w_i) = \mathfrak{J}'_{i-1}(w_{i-1}|w_N) \leqslant 0,$$

for $w_{i-1} \geqslant \breve{w}$, where the last inequality follows from the definition of $\breve{w}$. If $w_{i-1} < \breve{w}$, then

$$f'_1(w_{i-1}, w_i) = \begin{cases} \frac{[P(c_i) - P(c_{i-1})](\mu R - c_i)}{\delta} + \mathfrak{J}'_{i-1}(w_{i-1}|w_N) > 0, & w_{i-1} \leqslant w_i + \bar{u}\delta \\ \mathfrak{J}'_{i-1}(w_{i-1}|w_N) \geqslant 0, & w_i \in (w_i + \bar{u}\delta,\ \breve{w}], \end{cases}$$

where the second inequality follows from the definition of $\breve{w}$. Hence, $f(w_{i-1}, w_i)$ is increasing in w_{i-1} if $w_{i-1} < \breve{w}$ and decreasing in w_{i-1} if $w_{i-1} \geqslant \breve{w}$, which imply that $w^*_{i-1} = \breve{w}$.

Case 2. If $w_i \in (\breve{w} - \bar{u}\delta, \hat{w} - \bar{u}\delta]$, then we verify that $w^*_{i-1} = w_i + \bar{u}\delta$. For $w_{i-1} < w_i + \bar{u}\delta$,

$$f'_1(w_{i-1}, w_i) = \frac{[P(c_i) - P(c_{i-1})](\mu R - c_i)}{\delta} + \mathfrak{J}'_{i-1}(w_{i-1}|w_N) \geqslant 0,$$

where the inequality follows from $w_{i-1} < w_i + \bar{u}\delta \leqslant \hat{w}$. Further, for $w_{i-1} > \breve{w} + \bar{u}\delta$,

$$f_1'(w_{i-1}, w_i) = \mathfrak{J}_{i-1}'(w_{i-1}|w_N) < 0,$$

where the inequality follows from $w_{i-1} > w_i + \bar{u}\delta > \breve{w}$.

Case 3. If $w_i \in (\hat{w} - \bar{u}\delta, \hat{w}]$, then we verify that $w_{i-1}^* = \hat{w}$. For $w_{i-1} < \hat{w} < w_i + \bar{u}\delta$, we have

$$f_1'(w_{i-1}, w_i) = \frac{[P(c_i) - P(c_{i-1})](\mu R - c_i)}{\delta} + \mathfrak{J}_{i-1}'(w_{i-1}|w_N) \geqslant 0,$$

where the inequality follows from $w_{i-1} < \hat{w}$. And for $w_{i-1} > \hat{w}$, then

$$f_1'(w_{i-1}, w_i) = \begin{cases} \frac{[P(c_i)-P(c_{i-1})](\mu R - c_i)}{\delta} + \mathfrak{J}_{i-1}'(w_{i-1}|w_N) \leqslant 0, & \text{if } w_{i-1} \in (\hat{w}, w_i + \bar{u}\delta], \\ \mathfrak{J}_{i-1}'(w_{i-1}|w_N) \leqslant 0, & \text{if } w_{i-1} > w_i + \bar{u}\delta, \end{cases} \tag{C.70}$$

where the first inequality follows from $w_{i-1} > \hat{w}$. Hence, $f(w_{i-1}, w_i)$ is increasing in w_{i-1} if $w_{i-1} < \hat{w}$ and decreasing in w_{i-1} if $w_{i-1} \geqslant \hat{w}$ which imply that $w_{i-1}^* = \hat{w}$.

Case 4. If $w_i > \hat{w}$, we verify that $w_{i-1}^* = w_i$. Folloing (C.70), we have $f(w_{i-1}, w_i)$ is decreasing in w_{i-1} for $w_{i-1} \geqslant w_i$. Hence, $w_{i-1}^* = w_i$.

C.4.5 Proof of Proposition 3.5.5

First, if $w_N \geqslant \dfrac{\mu R - \underline{c}}{r}$, then

$$\begin{aligned} \mathfrak{J}_N(w_N|w_N) &= \sum_{i=1}^{N} [P(c_i) - P(c_{i-1})] \\ &\quad \min\left\{\frac{w_{i-1} - w_i}{\delta}(\mu R - c_i) - w_i,\ J\left(w_i, \min\left\{\frac{w_N}{c_i}, \frac{1}{r}\right\}, c_i\right)\right\} \\ &\leqslant \sum_{i=1}^{N} [P(c_i) - P(c_{i-1})] J\left(w_i, \min\left\{\frac{w_N}{c_i}, \frac{1}{r}\right\}, c_i\right) \\ &\leqslant \sum_{i=1}^{N} [P(c_i) - P(c_{i-1})] \left[\frac{\mu R - c_i}{r} - w_i\right] \leqslant 0 \leqslant J_N(0|0), \end{aligned}$$

where the third inequality follows from $w_i \geqslant w_N \geqslant \dfrac{\mu R - \underline{c}}{r}$.

On the other hand, for any $w_N > \bar{c}/r$, then denote the corresponding optimal solution as $\{w_i^*\}_{i=0,...,N}$. Since $w_N > \bar{c}/r$, we have $w_i^* > \bar{c}/r \geqslant c_i/r$ for $i = 0, ..., N$. Hence, $\min\{w_N/c_i, 1/r\} = 1/r$ for $i = 1, ..., N$.

Following (C.36), we have

$$\frac{\partial J(w, 1/r, c)}{\partial w} = -1, \tag{C.71}$$

if $w \geqslant \hat{w}(\infty) = c/r$ and $w \geqslant \mu c/r$. Hence,

$$J(w, 1/r, c_i) \leqslant J(\bar{c}/r, 1/r, c), \tag{C.72}$$

for any $i \in \{0, ..., N\}$ and $w > \bar{c}/r$. Define $\{\tilde{w}_i\}_{i=0,...,N}$ as

$$\tilde{w}_i = w_i^* - (w_N^* - \bar{c}/r).$$

Hence, if $w_N \geqslant \bar{c}/r$,

$$\begin{aligned}
&\mathfrak{J}_N(w_N|w_N) = \sum_{i=1}^{N}[P(c_i) - P(c_{i-1})] \\
&\min\left\{\frac{w_{i-1}^* - w_i^*}{\delta}(\mu R - c_i) - w_i^*,\ J\left(w_i^*, \min\left\{\frac{w_N^*}{c_i}, \frac{1}{r}\right\}, c_i\right)\right\} \\
&\leqslant \sum_{i=1}^{N}[P(c_i) - P(c_{i-1})] \\
&\min\left\{\frac{\tilde{w}_{i-1} - \tilde{w}_i}{\delta}(\mu R - c_i) - \tilde{w}_i,\ J\left(\tilde{w}_i, \min\left\{\frac{\tilde{w}_N}{c_i}, \frac{1}{r}\right\}, c_i\right)\right\} \\
&\leqslant \sum_{i=1}^{N}[P(c_i) - P(c_{i-1})]\left[\frac{\mu R - c_i}{r} - w_i\right] \leqslant 0 \leqslant J_N(0|0) \leqslant \mathfrak{J}_N(\bar{c}/r|\bar{c}/r),
\end{aligned}$$

where the first inequality follows from (C.72). Therefore,

$$\mathfrak{J}_N(w_N|w_N) \leqslant \mathfrak{J}_N(\bar{\mathrm{w}}|\bar{\mathrm{w}}).$$

Since $c \leqslant c_i$, $\gamma_{\mathsf{P}}^{c_i}(\tau_{\mathsf{w}}^i, z_{\mathsf{w}}^i)$ satisfies (LL) for $c \in (c_{i-1}, c_i]$. Then, following the defition of $\gamma_{\mathsf{P}}^{c_i}(\tau_{\mathsf{w}}^i, z_{\mathsf{w}}^i)$ and γ_{B}^{c}, the menu of contracts $\hat{\Gamma}_C^{\mathsf{w}}$ satisfies (LL), (PK), (IC) and (IR). Hence, what is left is to verify (TT).

First, we present a technical lemma to show that for any $c \leqslant c_i$, $\bar{\nu}$ is type c agent's best response if he takes contract $\gamma_{\mathsf{P}}^{c_i}(\tau_{\mathsf{w}}^i, z_{\mathsf{w}}^i)$.

Lemma C.4.7. *For any $c \leqslant c_i$, we have $\bar{\nu} \in \mathfrak{N}(\gamma_{\mathsf{P}}^{c_i}(\tau_{\mathsf{w}}^i, z_{\mathsf{w}}^i), c)$.*

The proof of this lemma can be adapted from the proof of Lemma C.3.11, which is omitted here. Next, we verify (TT) for type c_i, $i = 0, ..., N$. Given any c_j such that $\underline{c} \leqslant c_i < c_j \leqslant c_N$. We have

$$\begin{aligned}
\max_{\nu} u\left(\gamma_{\mathsf{P}}^{c_j}(\tau_{\mathsf{w}}^j, z_{\mathsf{w}}^j), \nu; c_i\right) &= u\left(\gamma_{\mathsf{P}}^{c_j}(\tau_{\mathsf{w}}^j, z_{\mathsf{w}}^j), \bar{\nu}; c_i\right) \\
&= u\left(\gamma_{\mathsf{P}}^{c_j}(\tau_{\mathsf{w}}^j, z_{\mathsf{w}}^j), \bar{\nu}; c_j\right) + (c_j - c_i)\bar{\tau}(\gamma_{\mathsf{P}}^{c_j}(\tau_{\mathsf{w}}^j, z_{\mathsf{w}}^j), \bar{\nu}) \\
&= u\left(\gamma_{\mathsf{P}}^{c_j}(\tau_{\mathsf{w}}^j, z_{\mathsf{w}}^j), \bar{\nu}; c_j\right) + (c_j - c_i)\bar{\tau}_{\mathsf{w}}^i \leqslant \mathsf{w}_j + \mathsf{w}_i - \mathsf{w}_j \\
&= u\left(\gamma_{\mathsf{P}}^{c_i}(\tau_{\mathsf{w}}^i, z_{\mathsf{w}}^i), \bar{\nu}; c_i\right), \qquad \text{(C.73)}
\end{aligned}$$

where the first equality follows from Lemma C.4.7, the second equality follows from (C.12), the third equality follows from Lemma C.3.2, and the first inequality follows from the definition of $\bar{\tau}_{\mathsf{w}}^j$ in (3.59). On the other hand, given any type c_j such that $\underline{c} \leqslant c_j < c_i \leqslant c_N$,

$$\begin{aligned}
\max_{\nu} u\left(\gamma_{\mathsf{P}}^{c_j}(\tau_{\mathsf{w}}^j, z_{\mathsf{w}}^j), \nu; c_i\right) &= u\left(\gamma_{\mathsf{P}}^{c_j}(\tau_{\mathsf{w}}^j, z_{\mathsf{w}}^j), \nu^0; c_i\right) \\
&= c_j \bar{\tau}_{\mathsf{w}}^j \leqslant c_j \min\left\{\frac{\mathsf{w}_N}{c_j}, \frac{1}{r}\right\} \leqslant \mathsf{w}_N \\
&\leqslant \mathsf{w}_i = u\left(\gamma_{\mathsf{P}}^{c_i}(\tau_{\mathsf{w}}^i, z_{\mathsf{w}}^i), \bar{\nu}; c_i\right), \qquad \text{(C.74)}
\end{aligned}$$

where the first inequality follows from the definition of $\bar{\tau}_{\mathsf{w}}^j$ in (3.59), and the third inequality follows from that w is non-increasing. Furthermore,

$$\max_{\nu} u\left(\gamma_{\mathsf{B}}^{c}(0, w_N), \nu; c_i\right) = \mathsf{w}_N \leqslant \mathsf{w}_i = u\left(\gamma_{\mathsf{P}}^{c_i}(\tau_{\mathsf{w}}^i, z_{\mathsf{w}}^i), \bar{\nu}; c_i\right), \qquad \text{(C.75)}$$

where the inequality follows from that w is non-increasing. Hence, (C.73)-(C.75) imply that type c_i would not mimic any other type.

Before we consider a general type $c \in (c_{i-1}, c_i]$, we present a technical lemma.

Lemma C.4.8. *For any $k_0 \geqslant \bar{\tau} \geqslant 0$,*

$$\frac{V(k_0 c_1, \bar{\tau}; c_1)}{\mu R - c_1} = \frac{V(k_0 c_2, \bar{\tau}; c_2)}{\mu R - c_2}; \quad \forall c_1, c_2 < \mu R. \tag{C.76}$$

Proof: Let $\tau := -\log(1 - r\bar{\tau})/r$. First, $k_0 c_1 \geqslant \hat{w}_{c_1}(\tau)$ is equivalent to $k_0 c_2 \geqslant \hat{w}_{c_2}(\tau)$. Further,

$$\frac{z_{c_1}(k_0 c_1, \tau)}{z_{c_2}(k_0 c_2, \tau)} = \frac{c_1}{c_2}.$$

Similarly, $k_0 c_1 \in [\check{w}_{c_1}(\tau), \hat{w}_{c_1}(\tau))$ is equivalent to $k_0 c_2 \in [\check{w}_{c_2}(\tau), \hat{w}_{c_2}(\tau))$. Further, $\tau_z(c_1) = \tau_z(c_2)$ and

$$\frac{z_{c_1}(k_0 c_1, \tau)}{z_{c_2}(k_0 c_2, \tau)} = \frac{c_1}{c_2}.$$

Hence, for any agent c with $w = k_0 c$, $\bar{\tau}$, and let $\tau^P(w, \bar{\tau}; c)$ be the stochastic stopping time that an agent with cost c and initial promised utility $k_0 c$ and probation length $\bar{\tau}$ is terminated, when exerting full effort in a probation contract. This implies that $\tau^P(w, \bar{\tau}; c)$ are identically distributed for any c. Therefore,

$$\frac{V(k_0 c, \bar{\tau}; c)}{\mu R - c}; \quad \forall c < \mu R,$$

is a constant. □

Now consider an agent with type $c \in (c_{i-1}.c_i]$, given $c_j > c_i$, then

$$
\begin{aligned}
&\max_{\nu} u\left(\gamma_{\mathsf{P}}^{c_j}(\tau_{\mathsf{w}}^j, z_{\mathsf{w}}^j), \nu; c\right) = u\left(\gamma_{\mathsf{P}}^{c_j}(\tau_{\mathsf{w}}^j, z_{\mathsf{w}}^j), \bar{\nu}; c\right) \\
&= u\left(\gamma_{\mathsf{P}}^{c_j}(\tau_{\mathsf{w}}^j, z_{\mathsf{w}}^j), \bar{\nu}; c_j\right) + (c_j - c)\bar{\tau}_{\mathsf{w}}^j \\
&= \mathsf{w}_j + (c_j - c)\bar{\tau}_{\mathsf{w}}^j \leqslant \mathsf{w}_j + \mathsf{w}_i - \mathsf{w}_j + (c_i - c)\bar{\tau}_{\mathsf{w}}^j \\
&= \mathsf{w}_i + (c_i - c)\min\left\{\frac{V\left(\mathsf{w}_j, \min\{\mathsf{w}_N/c_j, 1/r\}, c_j\right)}{\mu R - c_j}, \inf_{c_k : k<j, k\in 1:N}\left[\frac{\mathsf{w}_k - \mathsf{w}_j}{c_j - c_k}\right]\right\} \\
&= \mathsf{w}_i + (c_i - c)\min\left\{\frac{V(\mathsf{w}_j \cdot c_i/c_j, \min\{\mathsf{w}_N/c_j, 1/r\}, c_i)}{\mu R - c_i}, \inf_{c_k : k<i, k\in 1:N}\left[\frac{\mathsf{w}_k - \mathsf{w}_i}{c_i - c_k}\right]\right\} \\
&\leqslant \mathsf{w}_i + (c_i - c)\min\left\{\frac{V(\mathsf{w}_i, \min\{\mathsf{w}_N/c_i, 1/r\}, c_i)}{\mu R - c_i}, \inf_{c_k : k<i, k\in 1:N}\left[\frac{\mathsf{w}_k - \mathsf{w}_i}{c_i - c_k}\right]\right\} \\
&= \mathsf{w}_i + (c_i - c)\bar{\tau}_{\mathsf{w}}^i = u\left(\gamma_{\mathsf{P}}^{c_i}(\tau_{\mathsf{w}}^i, z_{\mathsf{w}}^i), \bar{\nu}; c\right) \leqslant \max_{\nu} u\left(\gamma_{\mathsf{P}}^{c_i}(\tau_{\mathsf{w}}^i, z_{\mathsf{w}}^i), \nu; c\right),
\end{aligned}
\tag{C.77}
$$

where the first equality follows from Lemma C.4.7, the second equality follows from (C.12), the third equality follows from Lemma C.3.2, the first inequality and the forth equality follows from the definition of $\bar{\tau}_{\mathsf{w}}^j$ in (3.59), the fifth equality follows from Lemma C.4.8, the second inequality follows from that V is increasing in w and $\bar{\tau}$, and the sixth equality follows from (C.12). Meanwhile, given $c_j < c_i$,

$$
\begin{aligned}
\max_{\nu} u\left(\gamma_{\mathsf{P}}^{c_j}(\tau_{\mathsf{w}}^j, z_{\mathsf{w}}^j), \nu; c\right) &= u\left(\gamma_{\mathsf{P}}^{c_j}(\tau_{\mathsf{w}}^j, z_{\mathsf{w}}^j), \nu^0; c\right) = c_j\bar{\tau}_{\mathsf{w}}^j \leqslant c_j \min\left\{\frac{\mathsf{w}_N}{c_j}, \frac{1}{r}\right\} \\
&\leqslant \mathsf{w}_N \leqslant \mathsf{w}_i \leqslant \mathsf{w}_i + (c_i - c)\bar{\tau}_{\mathsf{w}}^i = u\left(\gamma_{\mathsf{P}}^{c_i}(\tau_{\mathsf{w}}^i, z_{\mathsf{w}}^i), \bar{\nu}; c\right),
\end{aligned}
\tag{C.78}
$$

where the first inequality follows from the definition of $\bar{\tau}_{\mathsf{w}}^j$ in (3.59), and the third inequality follows from that w is non-increasing. Furthermore,

$$
\max_{\nu} u\left(\gamma_{\mathsf{B}}^{c}(0, w_N), \nu; c\right) = \mathsf{w}_N \leqslant \mathsf{w}_i = u\left(\gamma_{\mathsf{P}}^{c_i}(\tau_{\mathsf{w}}^i, z_{\mathsf{w}}^i), \bar{\nu}; c\right), \tag{C.79}
$$

where inequality follows from that w is non-increasing. Hence, (C.77)-(C.79) imply that $c \in [\underline{c}, c_N]$ would not mimic any other type. Finally, for any type $c \in$

$[\min\{\mu R, \bar{c}\}, \bar{c}]$ and $j \in \{1, ..., N\}$,

$$\max_{\nu} u\left(\gamma_{\mathsf{P}}^{c_j}(\tau_{\mathsf{w}}^j, z_{\mathsf{w}}^j), \nu; c\right) = u\left(\gamma_{\mathsf{P}}^{c_j}(\tau_{\mathsf{w}}^j, z_{\mathsf{w}}^j), \nu^0; c\right)$$

$$= c_j \bar{\tau}_{\mathsf{w}}^j \leqslant c_j \min\left\{\frac{1}{r}, \frac{\mathsf{w}_N}{c_j}\right\} \leqslant w_N = u\left(\gamma_{\mathsf{B}}^c(0, w_N), \nu; c\right), \tag{C.80}$$

which implies that type $c \in [\min\{\mu R, \bar{c}\}, \bar{c}]$ does not mimic any other type. This concludes the proof.

C.4.9 Proof of Theorem 3.5.6

If $2w_i^* \leqslant w_{i-1}^* + w_{i+1}^*$, then $w_i^* - w_{i+1}^* \leqslant w_{i-1}^* - w_i^*$. Hence, for any $j < i$,

$$\frac{w_j^* - w_i^*}{i - j} = \frac{\sum_{k=j+1}^{i}(w_{k-1}^* - w_k^*)}{i - j} \geqslant \frac{(i - j)(w_{i-1}^* - w_i^*)}{i - j} = w_{i-1}^* - w_i^*. \tag{C.81}$$

Therefore,

$$U\left(\hat{\Gamma}_C^{\boldsymbol{w}_N^*}\right) = \int_{\underline{c}}^{\bar{c}} U(\gamma^c_{\boldsymbol{w}_N^*}, \nu^c)\rho(c)dc$$

$$= \sum_{i=1}^{N} \int_{c_{i-1}}^{c_i} U(\gamma_{\mathsf{P}}^{c_i}(\tau_{\mathsf{w}}^i, z_{\mathsf{w}}^i), \nu^c)\rho(c)dc + \int_{\mu R}^{\bar{c}} U(\gamma_{\mathsf{B}}^c(0, w_N), \nu^c)\rho(c)dc$$

$$= \sum_{i=1}^{N} \int_{c_{i-1}}^{c_i} U(\gamma_{\mathsf{P}}^{c_i}(\tau_{\mathsf{w}}^i, z_{\mathsf{w}}^i), \bar{\nu})\rho(c)dc - w_N \int_{\mu R}^{\bar{c}} \rho(c)dc$$

$$= \sum_{i=1}^{N} (P(c_i) - P(c_{i-1}))J(w_i^*, \bar{\tau}^i_{\boldsymbol{w}_N^*}, c_i) - w_N(P(\bar{c}) - P(c_N))$$

$$= \sum_{i=1}^{N} (P(c_i) - P(c_{i-1}))$$

$$\min\left\{ J\left(w_i^*, \min\left\{\frac{w_N^*}{c_i}, \frac{1}{r}\right\}, c_i\right), \min_{j\in\{0,\dots,i-1\}} \frac{w_j - w_i}{(i-j)\delta}(\mu R - c_i) - w_i \right\}$$

$$- w_N^*(P(\bar{c}) - P(c_N))$$

$$= \sum_{i=1}^{N} (P(c_i) - P(c_{i-1}))$$

$$\min\left\{ J\left(w_i^*, \min\left\{\frac{w_N^*}{c_i}, \frac{1}{r}\right\}, c_i\right), \frac{w_{i-1} - w_i}{\delta}(\mu R - c_i) - w_i \right\}$$

$$- w_N^*(P(\bar{c}) - P(c_N)) = \hat{y}(N) \tag{C.82}$$

where the third equality follows from Lemma C.4.7, the fifth equality follows from the definition of $\bar{\tau}_{\mathsf{w}}^j$ in (3.59) and sixth equality follows from (C.81).

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