Symmetrizing Black Hole Horizons and a Positive Mass Theorem for Spin Creased Initial Data

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Defense Date: April 2, 2024

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Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
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ABSTRACT

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Abstract

This thesis examines aspects of hypersurfaces inside initial data for the Einstein equations. We prove that 3-dimensional initial data with an apparent horizon boundary can be perturbed to data with a $H = 0$, $k = 0$ boundary while preserving the dominant energy condition. This yields a reduction of the spacetime Penrose conjecture to the case of $H = 0$, $k = 0$ boundaries. We also give an upper bound of the spacetime Bartnik mass of apparent horizon Bartnik data satisfying a stability condition in terms of the area.

Secondly, we define a new type of singularity, called a “DEC-crease”, across a hypersurface for initial data, modeled on two spacelike slices of a spacetime meeting at a hyperbolic dihedral angle. We prove that a positive mass theorem holds for spin initial data sets containing such singularities, in any dimension. Our proof is based on Dirac-Witten spinors. At the singularity, a transmission-type boundary condition for spinors is defined and we show its ellipticity.

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Chapter 1. Introduction

1.1 General Relativity, Energy Conditions

An *initial data set* is a triple (M, g, k), where (M, g) is an n-dimensional Riemannian manifold and k is a symmetric 2-tensor. Whenever convenient, we will simply refer to initial data for short by the space M. It has the interpretation as a spacelike slice of a Lorentzian spacetime $(N^{n+1}, \overline{g})$, with k as the second fundamental form of M in N, although we will work with general initial data sets defined in the absence of a spacetime N. The definition is important in the initial value formulation of General Relativity, where $(N, \overline{g})$ is determined from evolving a Cauchy surface (M, g, k) according to a set of evolution equations.

On a Lorentzian manifold $(N, \overline{g})$, define the *Einstein tensor*

$$G := \mathrm{Ric} - \frac{1}{2} R \overline{g}, \tag{1.1}$$

where Ric and R denote the Ricci and scalar curvatures of $\overline{g}$, respectively. The usage of the letter g to denote the spacetime metric corresponds to Einstein's conceptualization of $\overline{g}$ being a 10-component gravitational potential. The tensor G is given by a non-linear second order differential operator on $\overline{g}$; formally, this structure of General Relativity reminds one of the Laplacian acting on the electric potential in electrostatics. Given a physics model of matter on N, we get a divergence-free *stress-energy tensor* T on N, which models the matter distribution in the spacetime. The tensor T plays the role of the source of the gravitational potential $\overline{g}$. That is, in a universe with zero cosmological constant, $(N, \overline{g})$ is hypothesized to satisfy the *Einstein field equations* (in natural units where $c = G_0 = 1$),

$$G = (n-1)\omega_{n-1} T, \tag{1.2}$$

where ω_{n-1} is the area of the unit $n-1$ sphere. For $n=3$, our universe, the factor in front of T is 8π. For a future timelike unit vector τ, which we interpret as an observer, define the *energy and momentum density* observed by τ, by

$$\mu = (n-1)\omega_{n-1}T(\tau,\tau), \quad J(\cdot) = (n-1)\omega_{n-1}T(\tau,\cdot), \tag{1.3}$$

where $\cdot$ takes in spacelike vectors orthogonal to τ. We say $(N,\overline{g})$ satisfies the *spacetime dominant energy condition* if for any future non-spacelike vector Y, the covector $-T(Y,\cdot)$ is future non-spacelike. That is, every observer observes positive matter density flowing with speed at most the speed of light.

If (M,g,k) embeds into N, then there is a future timelike unit normal τ, and by the Gauss-Codazzi equations, we may compute μ and J with respect to τ just from (M,g,k):

$$\mu = \frac{1}{2}\left(R_g + \left(\operatorname{Tr}_g k\right)^2 - |k|_g^2\right), \tag{1.4}$$

$$J = \operatorname{div}_g k - d\left(\operatorname{Tr}_g k\right), \tag{1.5}$$

where R_g is the scalar curvature of (M,g). More generally, we define the energy and momentum densities μ and J on M as above without a reference to a spacetime. The *dominant energy condition* is the condition that $\mu \geqslant |J|_g$, and the strict dominant energy condition is the condition that $\mu > |J|_g$. If M embeds into M, the dominant energy condition on M is implied by the spacetime dominant energy condition on N.

If $k \equiv 0$, (M,g,k) is called *time-symmetric*, and the dominant energy condition is equivalent to $R_g \geqslant 0$. If $\operatorname{Tr}_g k = 0$, then the (M,g,k) is called *maximal* (the analogue of minimal hypersurface in the Riemannian case), and the dominant energy condition implies $R_g \geqslant 0$.

1.2 Asymptotically Flat Initial Data Sets

An initial data set (M,g,k) of dimension n is *asymptotically flat* if there is a compact set $K \subset M$ such that $M \backslash K$ (which we all an *end*) is diffeomorphic to $\mathbb{R}^n \backslash B_1$, and there is a chart on $M \backslash K$ such that

$$|g_{ij} - \delta ij| = O_2(|x|^{-q}), \quad k_{ij} = O_1(|x|^{-q-1}), \tag{1.6}$$

for some $q > \frac{n-2}{2}$, and $\mu, J \in L^1(M)$. Here $O_k(|x|^{-s})$ means some unspecified function such that $\sum_{i=0}^k |x|^i |\partial^i f| < C|x|^{-s}$. We refer to such charts as "asymptotically flat charts". This definition models the notion of an isolated gravitational system in General Relativity.

Under the asymptotically flat assumption, we can define a Lorentz vector (E, P), the *ADM energy-momentum*, by

$$E = \lim_{R\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_R} \sum_i \left(g_{ij,i} - g_{ii,j}\right) \nu^j dA \tag{1.7}$$

$$P_i = \lim_{R\to\infty} \frac{1}{(n-1)\omega_{n-1}} \int_{S_R} \left(k_{ij} - (\mathrm{Tr}_g\, k) g_{ij} \nu^j\right) dA. \tag{1.8}$$

These quantities, due to Arnowitt, Deser, and Misner, and are well-defined and independent of asymptotically flat chart. In the case that (E, P) is non-spacelike, the *ADM mass* m_{ADM} is defined by

$$m_{ADM} = \sqrt{E^2 - |P|^2}. \tag{1.9}$$

Note that in the time-symmetric case, $E = m_{ADM}$ and $P = 0$.

The positive mass conjecture asserts that for a complete asymptotically flat manifold satisfying the dominant energy condition $\mu \geqslant |J|_g$, (E, P) is non-spacelike, which, informally, is the assertion that an isolated system with positive local energy density has positive total mass, as measured from infinity. The assumption of completeness is important, as there exist negative mass initial data with a curvature singularity, an example being a negative mass Schwarzschild metric, which is vacuum and has negative mass. The conjecture has been proven for $n < 8$ [Eic+11] using the analogue of a minimal surface approach pioneered by Schoen and Yau [SY79,SY81], and for all spin manifolds by Witten using spinors [Wit81]. Furthermore, there is a closely related question of *rigidity*, the statement that if $m_{ADM} = 0$, then M arises as a slice of Minkowski space. This has been only proven to hold under additional assumptions on the decay rate q, with counter-examples being given by "pp-wave spacetimes" [HL20]. We refer the reader to the book by Lee [Lee19] to a survey of topics surrounding the positive mass theorem, or to the introduction of [HKK22] for a quick history of the proofs in various generalities.

In Chapter 4, we prove a positive mass theorem for spin initial data with a "DEC-crease" singularity on a hypersurface, modeled on two spacelike slices of a spacetime meeting at a hyperbolic dihedral angle. The study of positive mass theorem for similarly singular metrics was initialized by Miao [Mia03], and has applications to the study of quasi-local mass and compact manifolds with non-negative scalar curvature, see Shi-Tam [ST02]. We anticipate that the theorem will yield analogous applications. We also anticipate that a rigidity statement also holds under additional assumptions on the decay rate q; namely if $m_{ADM} = 0$ then the initial data arises as a $C^{0,1}$ graph of Minkowski space.

1.3 Horizons and the Penrose Conjecture

Black holes are a famous astrophysical feature predicted by general relativity. For a spacetime with a notion of *future null infinity* $\mathcal{I}^+$ (which we will not rigorously define here), the *black hole region* is defined to be set of points that cannot be connected to $\mathcal{I}^+$ via future-directed non-spacelike curves, and *domain of outer communication* is defined to be the complement of the black hole region. Informally, a black hole is the region of a spacetime where no observer can escape, even if they travel at the speed of light. The event horizon $\mathcal{H}$ of a black hole is defined to be the boundary of the black hole region, which is a null hypersurface of the spacetime whenever its tangent space exists. Thus a spacelike slice (M, g, k) of a spacetime intersects $\mathcal{H}$ at a codimension 2 spacelike surface Σ.

For an initial data set, as its future evolution is not determined or given, there is no computable notion black hole or event horizon. However, one can still alternatively consider the following class of quasi-locally defined surfaces:

For a hypersurface Σ and unit normal ν in an initial data set, (M, g, k), define the *outer* and *inner null expansions*,

$$\theta^+ = H + \mathrm{Tr}_\Sigma k, \quad \theta^- = \mathrm{Tr}_\Sigma k - H, \tag{1.10}$$

respectively, where H is the mean curvature of Σ in the direction ν. If $\theta^+ = 0$, then Σ is a *marginally outer trapped surface* with respect to ν or *MOTS* for short, and if $\theta^- = 0$, then Σ

is a *marginally inner trapped surface* or *MITS* for short. If Σ is the outermost (see Definitions 3.1) MOTS, then is called an *apparent horizon*. Changing $k \to -k$ swaps the MOTS and MITS in an initial data set. In the $k \equiv 0$ time-symmetric case, the condition of being a MOTS and a MITS reduces to being a minimal surface. If (M, g, k) is a spacelike slice of a spacetime, then a MOTS/MITS Σ is a codimension 2 spacelike surface. Finally, we note that there is a interpretation of MOTS/MITS intrinsic to the embedding of the surface into the spacetime: the MOTS (MITS) condition is that the mean curvature vector $\vec{H}$ of Σ is outward future or inward past (inward future, outward past) null.

Due to the Penrose incompleteness theorem, MOTS are null geodesically incomplete in a generic maximal globally hyperbolic development satisfying the dominant energy condition (Theorem 7.29 [Lee19]). The weak cosmic censorship hypothesis would imply that all MOTS are hidden behind event horizons from the domain of outer communication. More rigorously this is due to an argument of Hawking, we refer the reader to 7.4.3 of [Lee19] for a more detailed discussion. The upshot is that MOTS should physically indicate the presence of a black hole, in that $\mathcal{H} \cap M$ should enclose (see definitions 3.1) a MOTS from the asymptotically flat end. There is some evidence that the outermost MOTS becomes asymptotic to the event horizon as the spacetime evolves [Wil08].

Based on a heuristic argument that combines the final state conjecture, the Hawking area theorem, and the weak cosmic censorship hypothesis, R. Penrose proposed that the mass of an asymptotically flat slice M of a spacetime satisfying a non-negative energy condition should be bounded above by the area of the event horizon. For spacetime dimension is 4,

$$\sqrt{\frac{|M \cap \mathcal{H}|}{16\pi}} \leqslant m_{ADM}. \tag{1.11}$$

As discussed, the event horizon is not generally locateable in M given only initial data, but under the weak cosmic censorship, an apparent horizon Σ should be enclosed by $|M \cap \mathcal{H}|$. Let $A_{\min}(\Sigma)$ denote the area of the minimal area enclosure of Σ; in particular $A_{\min}(\Sigma) < |M \cap \mathcal{H}|$.

The *Penrose conjecture*, based on Penrose's heuristic argument, is then the following:

$$\sqrt{\frac{A_{\min}(\Sigma)}{16\pi}} \leqslant m_{ADM}, \tag{1.12}$$

where Σ is the outermost MOTS in a complete asymptotically flat initial data set satisfying the dominant energy condition. Due to using $A_{\min}$, this version of the conjecture is equivalent to the statement where the outermost requirement is dropped. We can also extend conjecture to allow the case that M has a compact boundary enclosed by a MOTS.

In the case $k = 0$ case, the inequality is termed the *Riemannian Penrose inequality*. The outermost MOTS becomes the outermost minimal surface, which is already outer-minimizing, and the dominant energy condition becomes the condition of non-negative scalar curvature. Huisken and Ilmanen proved the case of a single black hole in dimension 3 using the *weak inverse mean curvature flow* [HI01], showing that the Hawking mass is monotone under this flow and converges to the ADM mass at infinity. Bray proved the case of multiple black holes using the *conformal flow* [Bra01], and this work was generalized by Lee and Bray for dimensions less than 8 [BL09].

The general "spacetime" case of $k \not\equiv 0$ remains wide open outside of spherical symmetry. Unlike the positive mass theorem, in which one can solve the "Jang equation" effectively reducing to the $k = 0$ case [SY81], attempting to apply the same technique for Penrose, one loses control on the area of the horizon. Bray and Khuri proposed an approach using the "generalized Jang equation" [BK11], which in a sense is able to reduce the problem to known $k = 0$ techniques. However, the existence theory proves difficult, as the proposed equation is degenerate elliptic near the horizon, and the proposed additional equations needed are highly coupled to the generalized Jang equation. Still, bridging the gap from the spacetime case to the Riemannian case remains an intriguing route.

The simplest examples demonstrating the spacetime Penrose inequality are slices of the Schwarzschild spacetime, which represents a single uncharged non-rotating black hole in vacuum. Any spherically symmetric slice (M, g, k) of Schwarzschild thus can be visualized as a curve C in the Kruskal plane, whose slope does not exceed ± 45 degrees (Figure 1.1). Suppose that the curve has the right asymptotics at spacelike infinity of the I region such that M is asymptotically flat. The event horizon in the slice, $\Sigma = \{u = 0\} \cap M$, is in fact an apparent horizon with area $16\pi m^2$. If in addition, $v > 0$ at the apparent horizon, the slice is foliated by positive mean curvature spheres and hence Σ is outer area minimizing; these examples saturate the Penrose inequality.

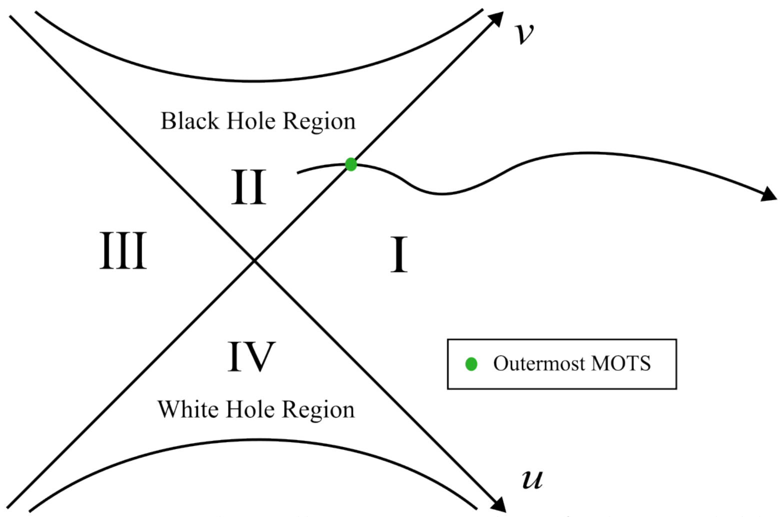

FIGURE 1.1: Spherically Symmetric slice of Schwarzschild

If we truncate the curve C close to the horizon and connect C with a line segment to the $(u, v) = 0$ minimal sphere, we get a modified initial data set with $k = 0$ on the line segment and minimal apparent horizon (Figure 1.2). Moreover, the closer the truncation point to the original apparent horizon, the more null the line segment becomes, corresponding to a short spacelike distance. This leads to the informal conjecture that the region near a MOTS boundary is a small perturbation away from being minimal and time symmetric near the boundary. In this thesis, we show that this heuristic is supported in dimension 3 in a specific sense.

In Chapter 2, we show that in dimension 3, for any MOTS Bartnik data (see Definition 2.1) satisfying a stability condition, there exists a collar initial data set of the form $S^2 \times [0, 1]$ satisfying the dominant energy condition such that one boundary component

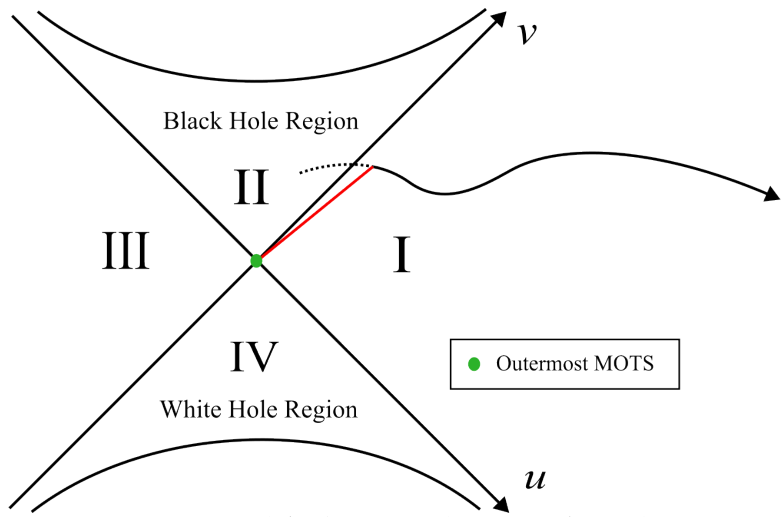

FIGURE 1.2: Modified slice with $H = 0$, $k = 0$ horizon

satisfies geometric constraints given by the MOTS Bartnik data, and the other boundary component is a minimal surface where $k = 0$. Moreover, the length of the collar in the $[0,1]$-direction is arbitrarily small and the slices have arbitrarily similar area. Applying this collar construction, in Section 3.1 we prove a reduction of the spacetime Penrose conjecture the the case where $k = 0$ near the horizon, which is a minimal surface. Hence we give a partial reduction of the spacetime Penrose conjecture to $k = 0$ case, and we discuss the possible advantages of this reduction.

Incidentally, the collar construction also yields an explicit class of spherically symmetric counterexamples to the *apparent horizon Penrose inequality*, which is unsupported by Penrose's original argument.

We use the same collar construction to study the spacetime Bartnik mass in Section 3.2. The problem of defining a suitable total energy for compact regions in an initial data set in GR is wide open. A good definition, called a *quasi-local mass*, typically only depends on the bounding surface of a region. Bartnik mass is a quasi-local mass defined (with many variants) as the infimum of the ADM mass over all asymptotically flat extensions with no horizons satisfying the dominant energy condition. We show that a spacetime Bartnik mass of a class of apparent horizons is bounded above by $\sqrt{\frac{A}{16\pi}}$ in Chapter 3, a spacetime generalization of the work of Mantoulidis-Schoen [MS15].

Chapter 2. MOTS-to-Minimal DEC Collars

The work in this and the following chapter is based on joint work with Demetre Kazaras. Here we work only in dimension 3.

2.1 Preliminaries

A smooth embedded surface Σ in an initial data set (M^3, h, k) inherits the following data: a Riemannian metric g, spatial mean curvature function H_0 with respect to a normal ν, the time mean curvature $\kappa = \mathrm{Tr}_\Sigma k$, and a 1-form $\omega(\cdot) = k(\nu, \cdot)$, where "$\cdot$" takes in tangent vectors to Σ.

We can also obtain the same data if Σ is a spacelike surface embedded within a spacetime. Given a choice of $O(1,1)$-framing $\{\tau, \nu\}$ for its normal bundle, where τ is timelike and ν is spacelike, Bartnik data is induced – a metric from the Lorentzian structure, H_0 and κ from the ν-and τ-components of its mean curvature vector, and a 1-form $\omega(\cdot) = \langle \nabla_\cdot \tau, \nu \rangle$ from the induced connection on its normal bundle.

Definition 2.1. [MM19] Given a Riemannian 2-surface (Σ, g), functions H_0, κ, and a 1-form ω, we call $(\Sigma, g, H_0, \kappa, \omega)$ *Bartnik data*. We say the tuple $(\Sigma, g, H_0, \kappa, \omega)$ defines *MOTS Bartnik data* (Σ, g, H_0, ω) if $H_0 + \kappa = 0$.

For MOTS in an initial data set, Anderson, Mars, and Simon introduced an important notion of stability in [AMS08], generalizing that of minimal surfaces. For MOTS Bartnik data, we define a related notion of stability through the following.

Definition 2.2. Given a one-form β and metric g on a closed surface Σ, we define the non-symmetric second order operator L_g^β by

$$L_g^\beta \varphi := -\Delta_g \varphi + 2\langle \beta, \nabla \varphi \rangle_g + \left(\operatorname{div}_g \beta - |\beta|_g^2 + K_g \right) \varphi, \tag{2.1}$$

and denote its real principal eigenvalue by $\lambda_1 \left(L_g^\beta \right)$. Let $\mathscr{M}_+^{MOTS}$ denote the class of MOTS Bartnik data (Σ, g, H, ω) such that $\lambda_1(L_g^\omega) > 0$.

We will see later that $\lambda_1 \left(L_g^\beta \right) < \lambda_1 \left(L_g^0 \right) = \lambda_1 \left(-\Delta_g + K_g \right)$, and hence the class $\mathscr{M}_+^{MOTS}$ contains only topological spheres. The definition of $\mathscr{M}_+^{MOTS}$ is a generalization of $\mathscr{M}_+$ defined in [MS15], the class of metrics on the 2-sphere with $\lambda_1 \left(-\Delta_g + K_g \right) > 0$. More precisely, if $g \in \mathscr{M}_+$, then $(\Sigma, g, 0, 0) \in \mathscr{M}_+^{MOTS}$.

The main result of this chapter is the following theorem:

Theorem 2.3. *Suppose $(\Sigma, g, H_0, \omega) \in \mathscr{M}_+^{MOTS}$ is given MOTS Bartnik data. There exists an initial data set $(C = [0,1] \times \Sigma, h, k)$ of the form*

$$\begin{aligned} h &= \eta(t,\cdot)^2 dt^2 + e^{2f(t,\cdot)} g \\ k|_\Sigma &= a(t,\cdot) e^{2f(t,\cdot)} g, \end{aligned} \tag{2.2}$$

with the following properties:

1. *The foliating spheres $\Sigma_t = \{t\} \times \Sigma$ are MOTS for the outward unit normal $\nu = \eta^{-1}\partial_t$ for $t \in [0,1]$,*
2. *(C, h, k) satisfies the strict dominant energy condition $\mu > |J|$ everywhere,*
3. *Σ_0 agrees with the given MOTS Bartnik data, that is, $f(0,\cdot) = 0$, $2a(0,\cdot) = -H_0$, $k_{\nu,\cdot}(0,\cdot) = \omega$,*
4. *Σ_1 is minimal and $k \equiv 0$ in a neighborhood there, that is, $a = 0$, $k_{\nu,\cdot} = 0$ and $k_{\nu,\nu} = 0$ in a neighborhood of Σ_1.*

Furthermore, if $H_0 > 0$, then the foliating spheres $\{\Sigma_t\}$ can be arranged to satisfy $H \geqslant 0$, and if $H_0 < 0$, the foliating spheres satisfy $H \leqslant 0$.

2.1.1 MOTS Stability and $L_g^{k_{\nu,\cdot}}$

We describe the notion of MOTS stability, introduced in [AMS08] by Andersson, Mars, and Simon. Let Σ be a MOTS with respect to a unit normal ν in an initial data set (M, g, k), and let φ be smooth function on Σ. For any normal variation Σ_t with velocity $\varphi\nu$, the variation in θ^+ is given by

$$\left.\frac{d}{dt}\right|_{t=0} \theta^+(\Sigma_t) = \mathcal{L}\varphi, \tag{2.3}$$

where $\mathcal{L}$ is the *MOTS stability operator*

$$\mathcal{L}\varphi = -\Delta_\Sigma \varphi + 2\langle k_{\nu,\cdot}, \nabla_\Sigma \varphi\rangle_\Sigma \tag{2.4}$$

$$+ \left(\operatorname{div}_\Sigma k_{\nu,\cdot} - |k_{\nu,\cdot}|^2_\Sigma + K_\Sigma - (\mu + J(\nu)) - \frac{1}{2}|k_\Sigma + A_\Sigma|^2\right)\varphi. \tag{2.5}$$

Here k_Σ denotes the two-tensor k restricted to vectors tangent to Σ and A_Σ denotes the second fundamental form of Σ. Though non-symmetric, $\mathcal{L}$ admits a real principal eigenvalue, which we denote as $\lambda_1(\mathcal{L})$. If $\lambda_1(\mathcal{L})$ is (positive) non-negative, then the we say Σ is *(strictly) stable.*

When the ambient initial data set is left unspecified for MOTS Bartnik data (Σ, g, H_0, ω), the stability operator $\mathcal{L}$ is not defined for Σ. Instead, we define the following

Definition 2.4 (The operator L_g^ω)**.** For a one form ω and metric g on a 2-dimensional closed manifold Σ, define the operator L_g^ω by

$$L_g^\omega \varphi := -\Delta_g \varphi + 2\langle \omega, \nabla\varphi\rangle_g + \left(\operatorname{div}_g \omega - |\omega|^2_g + K_g\right)\varphi. \tag{2.6}$$

When $(\Sigma, g, H_0, k_{\nu,\cdot})$ comes from an embedded surface inside initial data, one can compare the stability operator $\mathcal{L}$ to the operator $L_g^{k_{\nu,\cdot}}$ to find

$$\mathcal{L}\varphi = \left(L_g^{k_{\nu,\cdot}} - (\mu + J(\nu)) - \frac{1}{2}|k_\Sigma + A_\Sigma|^2\right)\varphi \tag{2.7}$$

If the dominant energy condition $\mu \geqslant |J|$ holds at Σ, we see that

$$\lambda_1(\mathcal{L}) \leqslant \lambda_1\left(L_g^{k_{\nu,\cdot}}\right), \tag{2.8}$$

and that $\lambda_1(\mathcal{L}) < \lambda_1\left(L_g^{k_{\nu,\cdot}}\right)$ whenever $\mu > |J|$ somewhere on Σ. Therefore the set $\mathscr{M}_+^{MOTS}$ contains Bartnik data arising from strictly stable MOTS inside DEC regions, or stable MOTS inside strict DEC regions.

For non-symmetric operators, we have the following characterization of λ_1.

Proposition 2.5. *(Andersson-Mars-Simon, Section 4 [AMS08]; the formula 2.11 stated in their paper is slightly incorrect) Let L be an operator on a closed manifold of the form*

$$L\varphi = -\Delta\varphi + 2\langle\beta, \nabla\varphi\rangle + (\operatorname{div}\beta - |\beta|^2 + Q)\varphi, \tag{2.9}$$

for a one-form β and smooth function Q. Then L has a real principal eigenvalue λ_1, characterized by $\Re(\lambda) \geqslant \lambda_1$ for all other eigenvalues λ, with one-dimensional eigenspace spanned by a real positive smooth eigenfunction η.

By the Hodge decomposition, write $\beta = dV + z$, where z is co-closed. Then λ_1 is characterized by

$$\lambda_1 = \inf_u \int |\nabla u|^2 + Qu^2 - |\nabla v[u]_z + z|^2 d\sigma, \tag{2.10}$$

where the infimum is taken over positive smooth functions u, and where $v[u]_z$ is the unique solution to

$$-\Delta v[u]_z - 2u^{-1}\langle\nabla v[u]_z, \nabla u\rangle = 2u^{-1}z(\nabla u), \quad \int u^2 v[u]_z d\sigma = 0. \tag{2.11}$$

This eigenvalue characterization leads to the following lemma, which states that the principal eigenvalue of L_g^β increases when shrinking β.

Lemma 2.6. *Fix a 1-form β on Σ. If $0 \leqslant s \leqslant t \leqslant 1$, then*

$$\lambda_1\left(L_g^{t\beta}\right) \leqslant \lambda_1\left(L_g^{s\beta}\right). \tag{2.12}$$

Proof. We may assume $t = 1$. Writing $\beta = dV + z$ with z co-closed, we can write $s\beta = d(sV) + sz$, with sz co-closed. For any smooth positive function u, we have that $sv[u]_z$ solves (2.11) when replacing z with sz, that is, $v[u]_{sz} = sv[u]_z$. Thus we have

$$\int |\nabla u|^2 + K_g u^2 - |\nabla v[u]_z + z|^2 d\sigma \leqslant \int |\nabla u|^2 + K_g u^2 - s^2|\nabla v[u]_z + z|^2 d\sigma \tag{2.13}$$

$$= \int |\nabla u|^2 + K_g u^2 - |\nabla v[u]_{sz} + (sz)|^2 d\sigma. \tag{2.14}$$

Taking the infimum over all positive smooth functions u yields the desired inequality in light of Proposition 2.5. □

Here we provide the following computation for a conformal change:

Proposition 2.7. *The operator L_g^β satisfies the following formula under a conformal change of metric:*

$$L_g^\beta = e^{2f} L^\beta_{e^{2f}g} + \Delta_g f. \tag{2.15}$$

Proof. In dimension 2, the Laplacian and Gauss curvature satisfy the well-known formulae

$$\Delta_g = e^{2f}\Delta_{e^{2f}g}, \tag{2.16}$$

$$K_g = e^{2f}K_{e^{2f}g} + \Delta_g f. \tag{2.17}$$

We also have

$$\langle \beta, \nabla^g \varphi\rangle = g^{ij}\beta_i\partial_j\varphi \tag{2.18}$$

$$= e^{2f}(e^{2f}g)^{ij}\beta_i\partial_j\varphi \tag{2.19}$$

$$= e^{2f}\langle \beta, \nabla^{e^{2f}g}\varphi\rangle, \tag{2.20}$$

and similarly, $|\beta|_g^2 = e^{2f}|\beta|^2_{e^{2f}g}$. For vector fields, we have the formula

$$\left(\operatorname{div}_g X\right)\mu_g = \mathcal{L}_X\mu_g \tag{2.21}$$

$$= \mathcal{L}_X\left(e^{-2f}\mu_{e^{2f}g}\right) \tag{2.22}$$

$$= e^{-2f}\left(\operatorname{div}_{e^{2f}g} X\right)\mu_{e^{2f}g} - 2e^{-2f}X(f)\mu_{e^{2f}g} \tag{2.23}$$

$$= \left(\operatorname{div}_{e^{2f}g} X - 2X(f)\right)\mu_g, \tag{2.24}$$

so $\operatorname{div}_g X = \operatorname{div}_{e^{2f}g} X - 2X(f)$. Transferring to 1-forms, we have

$$\operatorname{div}_g \beta = \operatorname{div}_g \beta^{\sharp_g} \tag{2.25}$$

$$= \operatorname{div}_{e^{2f}g} \beta^{\sharp_g} - 2\beta^{\sharp_g}(f) \tag{2.26}$$

$$= \operatorname{div}_{e^{2f}g}(e^{2f}\beta^{\sharp_{e^{2f}g}}) - 2\beta^{\sharp_g}(f) \tag{2.27}$$

$$= e^{2f} \operatorname{div}_{e^{2f}g}(\beta^{\sharp_{e^{2f}g}}) + 2e^{2f}\beta^{\sharp_{e^{2f}g}}(f) - 2\beta^{\sharp_g}(f) \tag{2.28}$$

$$= e^{2f} \operatorname{div}_{e^{2f}g} \beta, \tag{2.29}$$

where we used $\beta^{\sharp_{e^{2f}g}} = e^{-2f}g^{ij}\beta_i\partial_j = e^{-2f}\beta^{\sharp_g}$. □

We record a lemma on the smooth dependence of the principal eigenfunctions of a family non-self-adjoint elliptic operators whose coefficients smoothly depend on a parameter t. It is used in the proof of Theorem 2.3 to ensure the smoothness of the warping factor η. The proof is relegated to the Appendix A.1.

Lemma 2.8. *Let L_t be a family of second-order elliptic operators with smooth coefficients on a Riemannian manifold (M, g) of the form* (2.9)*, whose first and zeroth order coefficients depend smoothly on a parameter $t \in \mathbb{R}$. Then the principal eigenvalue and normalized eigenfunction pair $(\lambda(t), \varphi(t))$ depends smoothly on t.*

2.1.2 Computation of μ and J

Consider an initial data set $(C = [0,1] \times \Sigma, h, k)$ satisfying:

1. the metric has the form

$$h = \eta(t,\cdot)^2 dt^2 + e^{2f(t,\cdot)}g \tag{2.30}$$

   for functions $\eta, f \in C^\infty([0,1] \times \Sigma)$ and a fixed Riemannian metric g on Σ,

2. the components of k tangent to Σ have the form

$$k|_\Sigma = a(t,\cdot)e^{2f(t,\cdot)}g \tag{2.31}$$

   for some function $a \in C^\infty([0,1] \times \Sigma)$,

3. the remaining components $k_{\nu,\cdot}$ and $k_{\nu,\nu}$ are arbitrary, where $\nu = \eta^{-1}\partial_t$ is the unit normal to the slices $\{t_0\} \times \Sigma$, $t_0 \in [0,1]$.

From the product structure of C, we may regard a, η, f, and $k_{\nu,\cdot}$ as 1-parameter families of objects on Σ, and we denote their t-derivatives by $'$. Define a function $\bar{k}$ by $\bar{k} = \eta(k_{\nu,\nu} - a)$. We adopt the notation $\Sigma_t = \{t\} \times \Sigma$.

Lemma 2.9. *Suppose $(C = [0,1] \times \Sigma, h, k)$ is an initial data set satisfying conditions (1), (2), and (3) above. Then Σ_t is a MOTS with respect to ν if and only if*

$$f' = -\eta a. \tag{2.32}$$

When Σ_t is a MOTS for all t, we have the following formulae for energy and momentum density:

$$\mu = -\eta^{-1}\Delta_\Sigma \eta - |k_{\nu,\cdot}|^2 + K_\Sigma + 2\eta^{-1}(a\bar{k} + a') \tag{2.33}$$

$$J(\nu) = \operatorname{div}_\Sigma k_{\nu,\cdot} + 2\langle d^\Sigma \log \eta, k_{\nu,\cdot}\rangle_\Sigma - 2\eta^{-1}(a\bar{k} + a') \tag{2.34}$$

$$J^\top = \eta^{-1}k'_{\nu,\cdot} - 2ak_{\nu,\cdot} - 2d^\Sigma a - \eta^{-1}d^\Sigma \bar{k}, \tag{2.35}$$

where $J^\top$ denotes J restricted to vectors tangent to Σ_t, K_Σ denotes the Gauss curvature of $(\Sigma_t, e^{2f}g)$, d^Σ denotes the differential restricted to vectors tangent to Σ_t, and any other symbols with subscript Σ denote operations on $(\Sigma_t, e^{2f}g)$.

Proof. We first compute the extrinsic curvature of the surfaces Σ_t.

For the rest of the proof, let $\{x^\alpha\}_{\alpha=1,2}$ denote local coordinates on Σ and t the canonical coordinate on $[0,1]$; this gives coordinates on $U \times [0,1]$ for some open $U \subset \Sigma$. The second fundamental form of Σ_t is given by

$$A_{\alpha\beta} = \frac{1}{2}D_\nu(e^{2f}g_{\alpha\beta}) = \eta^{-1}f'e^{2f}g_{\alpha\beta}. \tag{2.36}$$

From this, it quickly follows that the norm and trace of A with respect to $h|_\Sigma = e^{2f}g$ are given by

$$|A|^2_\Sigma = A_{\alpha\beta}A^{\alpha\beta} = 2\eta^{-2}(f')^2, \tag{2.37}$$

$$H = \operatorname{Tr}_\Sigma A_{\alpha\beta} = 2\eta^{-1}f'. \tag{2.38}$$

Moreover, we have

$$\mathrm{Tr}_\Sigma\, k = \mathrm{Tr}_\Sigma\, ae^{2f} g = 2a, \tag{2.39}$$

$$\theta^+ = H + \mathrm{Tr}_\Sigma\, k = 2\eta^{-1} f' + 2a. \tag{2.40}$$

The MOTS condition, $\theta^+ = 0$, allows us to obtain f in terms of a and η. In particular,

$$f' = -\eta a. \tag{2.41}$$

Making use of this identity, we obtain

$$|A|^2_\Sigma = 2a^2, \qquad H = -2a. \tag{2.42}$$

Computation of μ: We first compute R_h. Recall the traced Gauss-Codazzi identity

$$R_h = 2K_\Sigma + 2\,\mathrm{Ric}(\nu,\nu) + |A|^2_\Sigma - H^2. \tag{2.43}$$

Using the first variation of mean curvature under $\eta\nu = \partial_t$, we have

$$H' = -\Delta_\Sigma \eta - \left(|A|^2_\Sigma + \mathrm{Ric}(\nu,\nu)\right)\eta \tag{2.44}$$

$$= -\Delta_\Sigma \eta + \left(-\frac{1}{2}|A|^2_\Sigma - \frac{1}{2}R_h + K_\Sigma - \frac{1}{2}H^2\right)\eta. \tag{2.45}$$

On the other hand, from (2.42), we have $H' = -2a'$ and $H^2 = 4a^2$. It follows that

$$\frac{1}{2}R_h = -\eta^{-1}\Delta_\Sigma \eta + K_\Sigma - 3a^2 + 2\eta^{-1}a'. \tag{2.46}$$

In an orthonormal frame including ν, we have the following form for k:

$$k = \begin{pmatrix} a & 0 & \multirow{2}{*}{$k_{\nu,\cdot}$} \\ 0 & a & \\ \multicolumn{2}{c}{k^T_{\nu,\cdot}} & k_{\nu,\nu} \end{pmatrix} \tag{2.47}$$

Also note that

$$\mathrm{Tr}_h\, k = \mathrm{Tr}_\Sigma\, k + k_{\nu,\nu} = 2a + k_{\nu,\nu}, \tag{2.48}$$

$$|k|^2_h = 2a^2 + 2|k_{\nu,\cdot}|^2_\Sigma + (k_{\nu,\nu})^2. \tag{2.49}$$

Now we are ready to compute the local energy density by combining (2.46)-(2.49):

$$\mu = \frac{1}{2}\left[R_h + (\operatorname{Tr}_h k)^2 - |k|_h^2\right] \tag{2.50}$$

$$= -\eta^{-1}\Delta_\Sigma \eta + K_\Sigma - 3a^2 + 2\eta^{-1}a' + \frac{1}{2}(2a + k_{\nu,\nu})^2 - \frac{1}{2}(2a^2 + 2|k_{\nu,\cdot}|_\Sigma^2 + k_{\nu,\nu}^2) \tag{2.51}$$

$$= -\eta^{-1}\Delta_\Sigma \eta + K_\Sigma - 2a^2 + 2\eta^{-1}a' + 2ak_{\nu,\nu} - |k_{\nu,\cdot}|_\Sigma^2 \tag{2.52}$$

$$= -\eta^{-1}\Delta_\Sigma \eta + K_\Sigma - 2a^2 + 2\eta^{-1}a' + 2a\left(\eta^{-1}\bar{k} + a\right) - |k_{\nu,\cdot}|_\Sigma^2 \tag{2.53}$$

$$= -\eta^{-1}\Delta_\Sigma \eta + K_\Sigma + 2\eta^{-1}\left(a\bar{k} + a'\right) - |k_{\nu,\cdot}|_\Sigma^2, \tag{2.54}$$

making the substitution $k_{\nu,\nu} = \eta^{-1}\bar{k} + a$ in the penultimate line. This yields the desired formula for μ.

Computation of J: From now on, we fix t_0 and work at a point $x \in \Sigma_{t_0}$. We further suppose that $\{x^\alpha\}_{\alpha=1,2}$ restrict to normal geodesic coordinates on Σ_{t_0} at x. In these coordinates, the second fundamental form satisfies $A_{\alpha\beta} = -ah_{\alpha\beta}$, k restricted to Σ_{t_0} satisfies $k_{\alpha\beta} = ah_{\alpha\beta}$, and, at x, the metric satisfies $h_{\alpha\beta}(x) = \delta_{\alpha\beta}$ and $\partial_\gamma h_{\alpha\beta}(x) = 0$. We compute the covariant derivatives of the frame at x. We have at x,

$$\nabla_\alpha \partial_\beta = (\nabla_\alpha \partial_\beta)^\top + (\nabla_\alpha \partial_\beta)^\perp = 0 - A_{\alpha\beta}\nu = a\delta_{\alpha\beta}\nu, \tag{2.55}$$

$$\nabla_\alpha \nu = (\nabla_\alpha \nu)^\top + (\nabla_\alpha \nu)^\perp = A_{\alpha\beta}\partial_\beta + 0 = -a\partial_\alpha. \tag{2.56}$$

By computing

$$\langle \partial_\alpha, \nabla_\nu \nu\rangle = -\langle \nabla_\nu \partial_\alpha, \nu\rangle + \nu\langle \partial_\alpha, \nu\rangle \tag{2.57}$$

$$= -\langle \eta^{-1}\nabla_t \partial_\alpha, \nu\rangle + 0 \tag{2.58}$$

$$= -\langle \eta^{-1}\nabla_\alpha \partial_t, \nu\rangle \tag{2.59}$$

$$= \eta\partial_\alpha(\eta^{-1})\langle \nu, \nu\rangle - \langle \nabla_\alpha \nu, \nu\rangle \tag{2.60}$$

$$= -\partial_\alpha \log \eta, \tag{2.61}$$

we obtain

$$\nabla_\nu \nu = -\nabla^\Sigma \log \eta, \tag{2.62}$$

$$\nabla_\nu \partial_\alpha = \partial_\alpha(\log \eta)\nu + A_{\alpha\beta}\partial_\beta = \partial_\alpha(\log \eta)\nu - a\partial_\alpha. \tag{2.63}$$

First, we compute the ν-component of J. We obtain, using the previously computed identities,

$$\operatorname{div}(k)(\nu) = \left(\sum_\alpha (\nabla_\alpha k)(\partial_\alpha, \nu)\right) + (\nabla_\nu k)(\nu,\nu) \tag{2.64}$$

$$= \left(\sum_\alpha \partial_\alpha(k_{\alpha\nu}) - k(\nabla_\alpha \partial_\alpha, \nu) - k(\partial_\alpha, \nabla_\alpha \nu)\right) + D_\nu k_{\nu,\nu} - 2k(\nabla_\nu \nu, \nu) \tag{2.65}$$

$$= \left(\sum_\alpha \partial_\alpha(k_{\alpha\nu}) - k(a\delta_{\alpha\alpha}\nu, \nu) - k(\partial_\alpha, -a\partial_\alpha)\right) + D_\nu k_{\nu,\nu} - 2k(-\nabla^\Sigma \log\eta, \nu) \tag{2.66}$$

$$= \left(\sum_\alpha \partial_\alpha(k_{\alpha\nu}) - a\delta_{\alpha\alpha} k_{\nu,\nu} + a^2\delta_{\alpha\alpha}\right) + \eta^{-1}k'_{\nu,\nu} + 2k(\nabla^\Sigma \log\eta, \nu) \tag{2.67}$$

$$= \operatorname{div}_\Sigma k_{\nu,\cdot} - 2ak_{\nu,\nu} + 2a^2 + \eta^{-1}k'_{\nu,\nu} + 2\langle d^\Sigma \log\eta, k_{\nu,\cdot}\rangle_\Sigma, \tag{2.68}$$

and

$$d(\operatorname{Tr}_h k)(\nu) = D_\nu(\operatorname{Tr}_h k) = 2\eta^{-1}a' + \eta^{-1}k'_{\nu,\nu}. \tag{2.69}$$

This yields

$$J(\nu) = \operatorname{div}(k)(\nu) - d(\operatorname{Tr}_h k)(\nu) \tag{2.70}$$

$$= \operatorname{div}_\Sigma k_{\nu,\cdot} - 2ak_{\nu,\nu} + 2a^2 + 2\langle d^\Sigma \log\eta, k_{\nu,\cdot}\rangle_\Sigma - 2\eta^{-1}a'. \tag{2.71}$$

Next we compute the tangential part of J. Again using the identities for the derivatives of the frame,

$$\operatorname{div}(k)(\partial_\alpha) = \left(\sum_\beta \nabla_\beta k(\partial_\beta, \partial_\alpha)\right) + \nabla_\nu k(\nu, \partial_\alpha) \tag{2.72}$$

$$= \left(\sum_\beta \partial_\beta k_{\beta\alpha} - k(\nabla_\beta \partial_\beta, \partial_\alpha) - k(\partial_\beta, \nabla_\beta \partial_\alpha)\right) \tag{2.73}$$

$$+ \eta^{-1} k'_{\nu\alpha} - k(\nabla_\nu \nu, \partial_\alpha) - k(\nu, \nabla_\nu \partial_\alpha) \tag{2.74}$$

$$= \left(\sum_\beta \partial_\beta (a g_{\alpha\beta}) - a\delta_{\beta\beta} k_{\nu\alpha} - a\delta_{\alpha\beta} k_{\nu\beta}\right) \tag{2.75}$$

$$+ \eta^{-1} k'_{\nu\alpha} + k(\nabla^\Sigma \log \eta, \partial_\alpha) - \partial_\alpha(\log \eta) k_{\nu,\nu} + a k_{\nu\alpha} \tag{2.76}$$

$$= \partial_\alpha a - 2a k_{\nu\alpha} + \eta^{-1} k'_{\nu\alpha} + (a - k_{\nu\nu}) \partial_\alpha (\log \eta). \tag{2.77}$$

Subtracting $d(\operatorname{Tr}_h k)(\partial_\alpha) = 2\partial_\alpha a + \partial_\alpha k_{\nu\nu}$, we arrive at

$$J(\partial_\alpha) = \eta^{-1} k'_{\nu\alpha} - 2a k_{\nu\alpha} + (a - k_{\nu\nu}) \partial_\alpha (\log \eta) - \partial_\alpha (a + k_{\nu,\nu}). \tag{2.78}$$

Note that $k'_{\nu,\cdot}(\partial_\alpha) = k'_{\nu\alpha}$ since the coordinates $\{x^\alpha\}$ are constant on $[0,1] \times \{p\}$ and thus ∂_α is constant in t as a t-dependent vector field. Again, the substitution $k_{\nu,\nu} = \eta^{-1}\bar{k} + a$ yields the desired formulae in the lemma. □

2.2 The MOTS Collar Construction for Theorem 2.3

Suppose we are given $a_0 := a(0,x) = -\frac{H_0}{2}, k_{\nu,\cdot}(0,x) = \beta$, and g, as in Theorem 2.3. Fix $\epsilon \in (0,1)$, $\delta \in (0,1/4)$, $\theta > 0$, to be chosen sufficiently small later on. We proceed by constructing a metric h and a symmetric 2-tensor k on $[0,1] \times \Sigma$ as described in (2.2), then verifying the desired properties.

2.2.1 Definition of $k_{\nu,\cdot}$

The first step is to define $k_{\nu,\cdot}$ by scaling it from β to 0. Let $\xi : [0,1] \to [0,1]$ be a fixed non-increasing smooth function such that

$$\xi(t) \equiv \begin{cases} 1 & t \in [0, \frac{1}{4}] \\ 0 & t \in [\frac{1}{2}, 1]. \end{cases} \tag{2.79}$$

Define $k_{\nu,\cdot}$ by

$$k_{\nu,\cdot}(t,x) = \xi(t)\beta(x), \tag{2.80}$$

and note $k_{\nu,\cdot} \equiv 0$ for $t \geqslant 1/2$, and $k'_{\nu,\cdot}$, $\|k_{\nu,\cdot}\|_{C^1(\Sigma_t,g)}$ is bounded independent of t, ϵ, δ, and θ.

2.2.2 Definition of η

Next, we construct the warping factor $\eta(t,x)$. For each $t \in [0,1]$, let $\widehat{\eta}(t) \in C^\infty(\Sigma^2)$ be the positive principal eigenfunction of $L_g^{\xi(t)\beta} = L_g^{k_{\nu,\cdot}(t)}$, normalized such that $\|\widehat{\eta}(t)\|_{L^2(\Sigma,g)} = 1$. By Lemma 2.6, $\lambda_1\left(L_g^{k_{\nu,\cdot}(t)}\right)$ is non-decreasing in t. Set

$$\eta(t,x) = \epsilon\widehat{\eta}(t,x). \tag{2.81}$$

2.2.3 Definition of a, and $\overline{k}$

The following lemma fixes our choice of a and $\overline{k}$.

Lemma 2.10. *There exists functions a, $\overline{k}$ satisfying*

1. *$a_0(x) = a(0,x)$, and $a \equiv 0$, $\overline{k} \equiv 0$ for $t \in [\frac{7}{8}, 1]$,*
2. *$a\overline{k} + a' \geqslant 1/\delta$ for $t \in [0, \frac{1}{2}]$,*
3. *$a\overline{k} + a' > -\theta$ and $d^\Sigma a = 0$ for $t \in [\frac{1}{2}, 1]$,*
4. *the norm $||a||_{C^2(\Sigma_t,g)}$ is bounded independent of t, ϵ, θ, and δ.*

If additionally $a_0 < 0$ (resp. $a_0 \geqslant 0$), a can be chosen to satisfy $a \leqslant 0$ (resp. $a \geqslant 0$).

Proof of Lemma 2.10. We proceed by first constructing $a, \overline{k}$ satisfying items (1)–(4) for general a_0. From this construction it will be clear that $a \geqslant 0$ in the special case where $a_0 \geqslant 0$. After this, we describe the alterations required to achieve $a \leqslant 0$ in the situation where $a_0 < 0$.

Case of general a_0: We construct a to smoothly interpolate from a_0 for $t = 0$, to a positive constant B for $t \in [\frac{1}{\delta}, \frac{3}{4}]$, and finally to 0 for $t = [\frac{7}{8}, 1]$. This construction preserves $a \geqslant 0$ whenever $a_0 \geqslant 0$.

Set

$$B \geqslant \max\left(\max_{x\in\Sigma} a_0(x) + 1, -\min_{x\in\Sigma} a_0(x) + 2\right), \tag{2.82}$$

and one can verify that for all $x \in \Sigma$, B satisfies

$$B - a_0(x) \geqslant 1, \quad \frac{B + a_0(x)}{2} \geqslant 1, \quad B > 0. \tag{2.83}$$

Let $\rho : [0, \frac{1}{2}] \to [0, 1]$ be a smooth non-decreasing function such that $\rho(0) = 0$ and $\rho(t) \equiv 1$ for $t \in [\delta, \frac{1}{2}]$, additionally satisfying

$$\rho'(t) \geqslant \tfrac{1}{\delta} \ \text{ for } t \in [0, \tfrac{3\delta}{4}], \qquad \rho(\tfrac{\delta}{2}) \geqslant \tfrac{1}{2}. \tag{2.84}$$

Also fix a smooth non-increasing function $\zeta : [\frac{1}{2}, 1] \to [0, 1]$ such that $\zeta(t) = 1$ for $t \in [\frac{1}{2}, \frac{3}{4}]$ and $\zeta(t) = 0$ for $t \in [\frac{7}{8}, 1]$.

Define $a : [0, 1] \times \Sigma \to \mathbb{R}$ by

$$a(t, x) = \begin{cases} B\rho(t) + a_0(x)\,(1 - \rho(t)) & t < \frac{1}{2}, \\ B\zeta(t) & t \geqslant \frac{1}{2}. \end{cases} \tag{2.85}$$

so that $a(0, x) = a_0(x)$, and $a(t, x) \equiv B$ for $t \in [\delta, \frac{3}{4}]$. It is evident this method of interpolation ensures $||a||_{C^2(\Sigma_t, g)}$ depends only on a_0, establishing property (4). Note that the construction preserves $a \geqslant 0$ if $a_0 \geqslant 0$.

On the interval $t \in [0, \frac{3\delta}{4}]$, one may use (2.84) and (2.83) to find

$$a'(t, x) = (B - a_0(x))\rho'(t) \geqslant \frac{1}{\delta}. \tag{2.86}$$

Since ρ is non-decreasing and B satisfies (2.83), a is non-decreasing for $t < \frac{1}{2}$. As a consequence, for $t \in [\frac{\delta}{2}, \frac{1}{2}]$ we have

$$a(t, x) \geqslant a(\tfrac{\delta}{2}, x) \tag{2.87}$$

$$= \rho(\tfrac{\delta}{2})\,(B - a_0(x)) + a_0(x) \tag{2.88}$$

$$\geqslant \frac{B + a_0(x)}{2} \geqslant 1, \tag{2.89}$$

where we have used (2.84) and (2.83).

Now that the choice of a is fixed, we turn our attention to $\bar{k}$. The function $\bar{k} = \bar{k}(t)$ is chosen as a function of t only, so that $d^\Sigma \bar{k} = 0$. To define $\bar{k}$ and establish its properties, we first consider $t \in [0, \frac{1}{2}]$. On this interval, we choose $\overline{k}(t)$ to be an non-decreasing function satisfying $\overline{k} = 0$ for $t \in [0, \frac{\delta}{2}]$ and equal to $\frac{1}{\delta}$ for $t \in [\frac{3\delta}{4}, \frac{1}{2}]$. We claim item (2), namely

$$a\overline{k} + a' \geqslant 1/\delta \quad \text{on } [0, \tfrac{1}{2}] \times \Sigma^2. \tag{2.90}$$

We establish inequality (2.90) by considering three possible cases:

- for $t \in [0, \frac{\delta}{2}]$, we have $a\overline{k} = 0$, so the result follows from (2.86),
- for $t \in [\frac{\delta}{2}, \frac{3\delta}{4}]$, we have $\overline{k} \geqslant 0$ and (2.89), so the result follows from (2.86),
- for $t \in [\frac{3\delta}{4}, \frac{1}{2}]$, we have $a' \geqslant 0$ and (2.89), so the result follows from $\overline{k} = \frac{1}{\delta}$.

Next, we extend $\overline{k}$ over the interval $[\frac{1}{2}, 1]$ and verify property (3). Fix a choice of $\overline{k} : [0, 1] \to \mathbb{R}$ smoothly extending the above choice and such that $\overline{k}$ decreases from $\frac{1}{\delta}$ to 0 over $[\frac{1}{2}, \frac{5}{8}]$, and $\overline{k} = -\frac{a'}{a+\epsilon_1}$ on $[\frac{5}{8}, 1]$ for some $\epsilon_1 > 0$. These two conditions are compatible since $a' = 0$ on $t \in [\delta, \frac{3}{4}]$. Since $a \equiv 0$ on $[\frac{7}{8}, 1]$, this choice of $\overline{k}$ satisfies property (1). We seek to establish property (3). For $t \in [\frac{1}{2}, \frac{5}{8}]$, we have

$$a\overline{k} + a' = B\overline{k} + 0 \geqslant 0 > -\theta, \tag{2.91}$$

as desired. On the remaining interval $t \in [\frac{5}{8}, 1]$, we compute

$$a\overline{k} + a' = -\frac{a'a}{a + \epsilon_1} + a' = \frac{\epsilon_1 a'}{a + \epsilon_1} = \frac{a'}{\epsilon_1^{-1} a + 1}. \tag{2.92}$$

Note that $a \geqslant 0, a' \leqslant 0$ on $[\frac{5}{8}, 1]$ and $a' = 0$ when $a = 0$. As such, (2.92) monotonically converges to 0 on $[\frac{5}{8}, 1]$ from below as $\epsilon_1 \to 0$, hence uniformly converges to 0 by Dini's theorem. It follows that there exists a choice of ϵ_1 so that the desired bound in item (3) holds, finishing the proof of items (1)–(4) in the general case and the additional statement in the special case where $a_0 \geqslant 0$.

Case where $a_0 < 0$: In this case, we need not force a_0 to a positive constant. Instead of the choice (2.83), we define

$$B := \max_{x \in \Sigma} a_0(x) < 0, \tag{2.93}$$

and define a as in (2.85) with this new B. Thus as t varies from 0 to 1, a interpolates from a_0, to a negative constant B, and finally to 0. In particular, $a \leqslant 0, a' \geqslant 0$ for all t. Define

$\overline{k}(t) = \frac{1}{B\delta}\zeta(t)$, where $\zeta : [0,1] \to \mathbb{R}$ is a non-increasing function such that $\zeta(t) \equiv 1$ for $t \in [0, \frac{1}{2}]$ and $\zeta(t) \equiv 0$ for $t \in [\frac{7}{8}, 1]$. These choices of a and $\overline{k}$ satisfy items (1) and (4).

Next, we establish items (2) and (3) for these new choices. Since $a \leqslant B, \overline{k} \leqslant 0, a' \geqslant 0$ for $t \in [0, \frac{1}{2}]$, we have on this interval

$$a\overline{k} + a' \geqslant a\overline{k} \geqslant B\overline{k} = \frac{1}{\delta}; \tag{2.94}$$

moreover since $a\overline{k} \geqslant 0$ and $a' \geqslant 0$ by construction, we have for $t \in [\frac{1}{2}, 1]$,

$$a\overline{k} + a' \geqslant 0 > -\theta. \tag{2.95}$$

□

2.2.4 Definition of f

Set $\widehat{f}(x) = -\int_0^t \widehat{\eta}(x)a(x)dt$, where $\widehat{\eta}$ was defined in 2.2.2. The MOTS condition $f' = -\eta a$ is therefore satisfied by setting

$$f(t,x) = \epsilon\widehat{f}(t,x) = -\int_0^t \eta(t,x)a(t,x)dt. \tag{2.96}$$

Convention: In what follows, C_k, $k = 1, 2, \ldots$, will be used for constants which are independent of t, ϵ, θ, and δ.

Now that we have constructed the initial data set, is important to note that there is a constant C_1 independent of t, ϵ, θ, and δ so that

$$\text{the } C^2(\Sigma_t, g)\text{-norms of } a, \widehat{\eta}, \widehat{\eta}^{-1}, f, k_{\nu,\cdot}, k'_{\nu,\cdot}, \text{ and } \overline{k} \text{ are less than } C_1. \tag{2.97}$$

2.2.5 Estimate for $J(\nu)$ Using $a\overline{k} + a$

We proceed by estimating the expression for $J(\nu)$ given by Lemma 2.9. From item (2) of Lemma 2.10, for $t \in [0, \frac{1}{2}]$ we have $a\overline{k} + a' \geqslant 1/\delta$, so leveraging this, we have

$$\begin{aligned} J(\nu) &= \operatorname{div}_\Sigma k_{\nu,\cdot} + 2\langle d^\Sigma \log \eta, k_{\nu,\cdot}\rangle_\Sigma - 2\eta^{-1}(a\overline{k} + a') \\ &\leqslant e^{2f} \operatorname{div}_g k_{\nu,\cdot} + 2e^{-2f}|\nabla^g \log \widehat{\eta}|_g |k_{\nu,\cdot}|_g - 2\epsilon^{-1}\delta^{-1}\widehat{\eta}^{-1} \\ &\leqslant C_2 - C_3\epsilon^{-1}\delta^{-1}, \end{aligned} \tag{2.98}$$

where we have also used (2.97) and the conformal change formula for the divergence of 1-forms in 2 dimensions $\mathrm{div}_{e^{2f}g} = e^{2f}\,\mathrm{div}_g$. On the complimentary interval $t \in [\frac{1}{2}, 1]$, we have $k_{\nu,\cdot} = 0$, so $J(\nu) = -2\eta^{-1}(a\overline{k} + a')$. Combining this with item (3) of Lemma 2.10, the following holds for $t \in [\frac{1}{2}, 1]$:

$$J(\nu) = -2\eta^{-1}(a\overline{k} + a') < 2\epsilon^{-1}\widehat{\eta}^{-1}\theta < C_4\epsilon^{-1}\theta. \tag{2.99}$$

2.2.6 Estimate for $J^\top$

Inspecting the formula for $J^\top$ in Lemma 2.9, we find

$$\begin{aligned} |J^\top|_h &= |\eta^{-1}k'_{\nu,\cdot} - 2ak_{\nu,\cdot} - 2d^\Sigma a - \eta^{-1}d^\Sigma\overline{k}|_{e^{2f}g} \\ &= e^{-f}|\epsilon^{-1}\widehat{\eta}^{-1}k'_{\nu,\cdot} - 2ak_{\nu,\cdot} - 2d^\Sigma a|_g \\ &\leqslant C_5\epsilon^{-1} \end{aligned} \tag{2.100}$$

where we have used (2.97) and $d^\Sigma\overline{k} = 0$ from Lemma 2.10. Moreover, for $t \geqslant \frac{1}{2}$ we have $d^\Sigma a = 0$ and $k_{\nu,\cdot} = 0$, so

$$J^\top = 0 \quad \text{ for } t \in [\tfrac{1}{2}, 1]. \tag{2.101}$$

2.2.7 Verifying the Dominant Energy Condition

By construction, Σ_t is a flow of surfaces with velocity vector $\partial_t = \eta\nu$. According to our choice (2.96) of f, each Σ_t is a MOTS. Due to the form of the initial data (2.2), the surfaces are in fact umbillic and the MOTS condition additionally implies $k_\Sigma = -A_\Sigma$. We may apply this observation with equation (2.7) to compute the variation of θ^+ according to (2.3), finding

$$0 = \frac{d}{dt}\theta^+(\Sigma_t) = \mathcal{L}\eta = L^{k_{\nu,\cdot}}_{e^{2f}g}\eta - (\mu + J(\nu))\eta. \tag{2.102}$$

Next, we apply the conformal change formula Proposition 2.7 to deduce

$$\begin{aligned}(\mu + J(\nu))\eta &= L^{k_{\nu,\cdot}}_{e^{2f}g}\eta \\ &= e^{-2f}\left(L^{k_{\nu,\cdot}}_g\eta - \epsilon\Delta_g \widehat{f}\right)\eta \\ &\geqslant (e^{-2f_0})^{\epsilon}\left(\lambda_1\left(L^{k_{\nu,\cdot}(t)}_g\right) - C_6\epsilon\right)\eta \\ &\geqslant (e^{-2f_0})^{\epsilon}\left(\lambda_1\left(L^{k_{\nu,\cdot}(0)}_g\right) - C_6\epsilon\right)\eta,\end{aligned} \tag{2.103}$$

where we have used the fact that $\lambda_1\left(L^{k_{\nu,\cdot}(t)}_g\right)$ is non-decreasing by Lemma 2.6. Denote $\lambda_1\left(L^{k_{\nu,\cdot}(0)}_g\right)$ by λ, noting that $\lambda > 0$ by assumption. As a consequence of (2.103), we may choose ϵ sufficiently small so that

$$\mu + J(\nu) \geqslant \lambda/2 \tag{2.104}$$

holds throughout the initial data set.

Now that we have established (2.104), we will proceed to estimate $\mu - |J|$ using our previously established estimates on $J(\nu)$ and $J^{\top}$. First consider the case where $t \in [0, \frac{1}{2}]$. Inspecting inequality (2.98), we may find δ_0 to ensure $J(\nu) \leqslant C_2 - C_3\epsilon^{-1}\delta^{-1} < 0$ for any $\delta \in (0, \delta_0)$, so $|J(\nu)| = -J(\nu)$ and

$$\mu - |J| = \mu + J(\nu) - (|J| + J(\nu)) \tag{2.105}$$

$$\geqslant \lambda/2 - (|J| - |J(\nu)|)\,. \tag{2.106}$$

We bound the second term of 2.106:

$$|J| - |J(\nu)| = \sqrt{|J^{\top}|^2 + |J(\nu)|^2} - |J(\nu)| \tag{2.107}$$

$$= \left(\sqrt{1 + \frac{|J^{\top}|^2}{|J(\nu)|^2}} - 1\right)|J(\nu)| \tag{2.108}$$

$$\leqslant \frac{|J^{\top}|^2}{2|J(\nu)|}, \tag{2.109}$$

where we have used $\sqrt{1+x} \leqslant 1 + \frac{x}{2}$ for $x > 0$.

Using $|J(\nu)| \geqslant |C_2 - C_3\epsilon^{-1}\delta^{-1}|$ and 2.100, the estimate

$$\frac{|J^{\top}|^2}{2|J(\nu)|} \leqslant \frac{C_5^2\epsilon^{-2}}{2|C_2\delta - C_3\epsilon^{-1}|}\delta \tag{2.110}$$

implies that taking δ sufficiently small yields $\mu - |J| > 0$, for $t \in [0, \frac{1}{2}]$.

For $t \in [\frac{1}{2}, 1]$, $J(\nu) \leqslant C_4\epsilon^{-1}\theta$, and $J^T = 0$, so

$$\mu - |J| = \mu + J(\nu) - (|J| + J(\nu)) \tag{2.111}$$

$$\geqslant \lambda/2 - |J(\nu)| - J(\nu) \tag{2.112}$$

$$= \lambda/2 - 2\max(0, J(\nu)) \tag{2.113}$$

$$\geqslant \lambda/2 - 2C_4\epsilon^{-1}\theta, \tag{2.114}$$

and so taking θ sufficiently small yields $\mu - |J| > 0$ in this region.

2.3 Gluing Theorem and Perturbation for the Collar

The main application of the collar construction of Theorem 2.3 is to construct initial data sets, which requires a gluing theorem in order to attach the collar to other initial data sets. Moreover, we would like to make modifications in order to bump up or down the mean curvature at the gluing surface, while preserving properties of the foliation relating to mean curvature.

Definition 2.11. We say that (M, h, k) is *extendable allowable at* Σ (compare with Definition 8 of [Jau19]) if Σ is a boundary component of M and there exists an initial data set $(\widetilde{M}, \widetilde{h}, \widetilde{k})$ such that (M, h) embeds into $(\widetilde{M}, \widetilde{h})$ isometrically, $k = \widetilde{k}|_M$, and Σ has a collar neighborhood K inside $\widetilde{M}$, where the strict dominant energy condition holds on $K \backslash M$.

Remark 2.12. If (M, h, k) has a compact boundary component Σ and the strict dominant energy condition holds in a neighborhood of Σ, then M is automatically extendable allowable at Σ. Arbitrarily extend M, h, and k near Σ outwards, and the strict dominant energy holds in a collar neighborhood of Σ by continuity.

We have the following gluing theorem that preserves the strict dominant energy condition, which follows from Theorem A.3, proved in the Appendix. The initial idea appeared in the work of Jauregui [Jau19] and Brendle-Marques-Neves [BMN11], which we have generalized to the initial data set case.

Theorem 2.13. *Suppose (M_+, h_+, k_+) and (M_-, h_-, k_-) are n-dimensional initial data sets with boundary satisfying the dominant energy condition. Let Σ_+ be a compact component of ∂M_+ and let Σ_- be a compact component of ∂M_-. Let ν_+ be the inward unit normal to Σ_+ and ν_- be the outward unit normal to Σ_-. Suppose that M_+ satisfies the strict dominant energy condition in a collar neighborhood of Σ_+ and M_- is extendable allowable near Σ_-. If*

- $h_+|_{T\Sigma_+} \cong h_-|_{T\Sigma_-}$,
- $H_+ < H_-$, *the mean curvatures of Σ_+ and Σ_- with respect to ν_+ and ν_- in M_+ and M_-, respectively,*
- $\mathrm{Tr}_{\Sigma_+} k_+ = \mathrm{Tr}_{\Sigma_-} k_-$, *and* $k_+(\nu_+, \cdot) = k_-(\nu_-, \cdot)$,

then there exists a sequence of initial data sets $(M, h_\lambda, k_\lambda)$ such that

- $M = M_+ \cup_{\Sigma_\pm} M_-$, *and $(M, h_\lambda, k_\lambda)$ satisfies the strict dominant energy condition near the glued surface $\Sigma_\pm$*
- $(h_\lambda, k_\lambda) = (h_-, k_-)$ *restricted to M_-,*
- $\|h - h_+\|_{C^0(M_+)} < N\lambda^{-1}$, *and $(h_\lambda, k_\lambda)|_{M_+} = (h_+, k_+)$ outside a compact neighborhood of Σ_+.*

Proof. Extend h_-, k_- into an arbitrary metric and 2-tensor into $M_- \cup_{\Sigma_\pm} M_+$ satisfying the strict dominant energy condition in portion of a neighborhood of $\Sigma_\pm$ that lies in M_+, denoting this pair (h_-, k_-) as well. This is possible due to the extendable allowable property of M_-.

Note $\nu_+ = \nu_-$ in $M_- \cup_{\Sigma_\pm} M_+$, hence the mean curvatures H_+, H_- are equal to that of taken with respect to inward normal to ∂M_+, and so we may apply Theorem A.3 to $(h_\pm, k_\pm)$ on M_+. This yields the a sequence of initial data (h_λ, k_λ) on M_+, such that (h_λ, k_λ) smoothly extends to equal h_-, k_- on M_-; we also call this extension on all of $M_- \cup_{\Sigma_\pm} M_+$

(h_λ, k_λ) and we are done. $\square$

We state and prove following perturbation lemmas which allow us to prepare the $t = 0$ and $t = 1$ ends for gluing. These perturbations preserve strict dominant energy condition and therefore are extendable allowable at both boundary components by Remark 2.12.

Lemma 2.14. *Suppose $(C = \Sigma \times [0,1], h, k)$ is an initial data set given by Theorem 2.3. For small $z > 0$, the modified initial data set $(\widehat{C} = \Sigma \times [0,1], \widehat{h}, \widehat{k})$ of the form*

$$\widehat{h} = \eta^2 dt^2 + e^{2(f+zt^2)} g, \tag{2.115}$$

$$\widehat{k}|_{\Sigma_t} = a e^{2(f+zt^2)} g, \tag{2.116}$$

$$\widehat{k}_{\nu,\cdot} = k_{\nu,\cdot} \tag{2.117}$$

$$\widehat{k}_{\nu\nu} = k_{\nu,\nu}, \tag{2.118}$$

satisfies the following properties, where $\widehat{\cdot}$ denotes quantities associated to the modified initial data:

1. *Σ_0 is a MOTS, and the MOTS Bartnik data at Σ_0 is unchanged,*
2. *For $t \in (0,1]$, $\widehat{H} > H$ and $\widehat{\mathrm{Tr}}_{\Sigma_t} \widehat{k} = \mathrm{Tr}_{\Sigma_t} k$ on Σ_t. In particular, $\widehat{\theta}^+ > 0$.*
3. *$\widehat{C}$ satisfies the strict dominant energy condition everywhere.*
4. *Σ_1 is mean convex and satisfies $\lambda_1 \left(-\Delta_\Sigma + K_{\Sigma_1}\right) > 0$, and $\widehat{k} \equiv 0$ in a neighborhood there.*

Proof. From formulas (2.38) and (2.39) in the proof of Lemma 2.9, substituting f with $f + z^2 t$, we have

$$\widehat{H} = 2\eta^{-1} \left(f' + 2zt\right) = H + 4\eta^{-1} zt > H, \tag{2.119}$$

$$\widehat{\mathrm{Tr}}_{\Sigma_t} \widehat{k} = 2a = \mathrm{Tr}_{\Sigma_t} k. \tag{2.120}$$

for $t > 0$. Also recall that $\theta^+ = H + \mathrm{Tr}_{\Sigma_t} k = 0$ by the MOTS condition of Theorem 2.3, so

$$\widehat{\theta}^+ = \widehat{H} + \widehat{\mathrm{Tr}}_{\Sigma_t} \widehat{k} > H + \mathrm{Tr}_{\Sigma_t} k = 0 \tag{2.121}$$

for $t > 0$. This proves (2). By shrinking z, we can achieve strict DEC, an open condition, which proves (3). Observe that setting $t = 0$, we obtain $\widehat{H} = H$, $\widehat{h} = h$, and $\widehat{k} = k$ at Σ_0; this shows (1). We have $k \equiv 0$ in a neighborhood of Σ_1 in C, and Σ_1 satisfies $\widehat{H} = \widehat{\theta}^+ > 0$.

The spectral condition $\lambda_1(-\Delta_\Sigma + K_{\Sigma_1}) > 0$ holds since Σ_1 a stable minimal sphere in a region of positive scalar curvature (equivalent to strict DEC where $k \equiv 0$) of $(\Sigma \times [0,1], h)$, a property preserved by scaling the metric by e^{2z} for small z. This verifies (4). □

Lemma 2.15. *Suppose $(C = \Sigma \times [0,1], h, k)$ is an initial data set given by Theorem 2.3. For small $z > 0$, the modified initial data set $(\widehat{C} = \Sigma \times [0,1], \widehat{h}, \widehat{k})$ of the form*

$$\widehat{h} = \eta^2 dt^2 + e^{2(f - zt(2-t))} g \tag{2.122}$$

$$\widehat{k}|_{\Sigma_t} = a e^{2(f - zt(2-t))} g \tag{2.123}$$

$$\widehat{k}_{\nu,\cdot} = k_{\nu,\cdot}, \tag{2.124}$$

$$\widehat{k}_{\nu\nu} = k_{\nu,\nu}, \tag{2.125}$$

satisfies the following properties, where $\widehat{\cdot}$ denotes quantities associated to the modified initial data:

1. *Σ_1 is minimal and $\widehat{k} \equiv 0$ in a neighborhood there.*
2. *For $t \in [0,1)$, $\widehat{H} < H$ and $\widehat{\mathrm{Tr}}_{\Sigma_t} \widehat{k} = \mathrm{Tr}_{\Sigma_t} k$ on Σ_t. In particular, $\widehat{\theta}^+ < 0$.*
3. *$(\widehat{C}, \widehat{h}, \widehat{k})$ satisfies the strict dominant energy condition everywhere.*
4. *$\widehat{k_{\nu,\cdot}} = k_{\nu,\cdot}$ at $t = 0$.*

Proof. The proof is similar to the previous lemma, taking note that

$$\widehat{H} = 2\eta^{-1}(f' + z(2 - 2t)) = H - 4\eta^{-1} z(1 - t) < H \tag{2.126}$$

for $t \in [0,1)$. □

Finally, we have the following lemma useful for projections of cross-sections of $\widehat{C}$, which will be used to show that the minimal area enclosure of the boundary of a manifold does not change much when one attaches $\widehat{C}$ to its boundary.

Lemma 2.16. *Suppose $(\Sigma, g, H_0, \beta) \in \mathscr{M}_+^{MOTS}$ and we apply Theorem 2.3 and then Lemma 2.15. Then we can arrange the construction of $(\widehat{C}, \widehat{h}, \widehat{k})$ such that the map $p : (\Sigma \times [0,1], \widehat{h}) \to (\Sigma_0, g)$ defined by*

$$p(x,t) = (x,0) \tag{2.127}$$

is a $(1+\epsilon)$-Lipschitz map.

Proof. By revisiting the proof of Theorem 2.3 we can see that for any $\epsilon > 0$ sufficiently small, there exists a construction where the metric $\widehat{h}$ has the form

$$\widehat{h} = \eta^2 dt^2 + e^{2(\epsilon \widehat{f} - zt(2-t))} g, \tag{2.128}$$

where the minimum of $\widehat{f}$ on $\widehat{C}$ is bounded by a constant K independent of ϵ. Note that in the construction of Theorem 2.3, $\widehat{f} = -\int_0^t \widehat{\eta} a$ will depend on ϵ, but the bound K does not. For a tangent vector $v = v_t \partial_t + v^T \in T_{(x,t)} \widehat{C}$, where v^T is tangent to Σ_t, we have

$$\widehat{h}(v, v) = \eta^2(x,t) v_0^2 + e^{2(\epsilon \widehat{f}(x,t) - zt(2-t))} \|v^T\|_g^2 \tag{2.129}$$

$$\geqslant \inf_{\widehat{C}} e^{2(\epsilon \widehat{f})} \|v^T\|_g^2 \tag{2.130}$$

$$= e^{2K\epsilon} \|dp(v)\|_g^2. \tag{2.131}$$

□

Chapter 3. Applications of the Collar

3.1 Applications to the Spacetime Penrose Conjecture

Definitions 3.1. We say a compact embedded surface S in an initial data set M is an *enclosing surface* if $S = \partial\Omega$ for an open set Ω containing the asymptotically flat end of M; in particular ∂M is an enclosing surface. We also say S_1 *encloses* S_2 if S_1 and S_2 are enclosing surfaces, $S_1 = \partial\Omega_1$, $S_2 = \partial\Omega_2$ and $\Omega_1 \subset \Omega_2$. Given a class of surfaces $\mathcal{C}$, $S \in \mathcal{C}$ is *outermost* if no other surfaces of $\mathcal{C}$ encloses S. Moreover, for dimensions less than 8, any C^2 enclosing surface S has a unique *outermost minimal area enclosure* S'. Importantly, S' is $C^{1,1}$ and smooth minimal away from S, and is the outermost among all surfaces which enclose S and has less than or equal area than any other surface enclosing S [HI01]. In particular, S' is itself *strictly outer minimizing*, that is, has strictly less area than any surface enclosing S'.

We restate a version of spacetime Penrose conjecture.

Conjecture 3.2. *Suppose (M, g, k) is a 3-dimensional asymptotically flat initial data set satisfying the dominant energy condition such that ∂M is a MOTS. Then*

$$\sqrt{\frac{A_{\min}(\partial M)}{16\pi}} \leqslant m_{ADM} \tag{3.1}$$

where m_{ADM} is the ADM mass of (M, g, k) and $A_{\min}(\partial M)$ is the area of the minimal area enclosure of ∂M.

We list two applications of the collar construction of Chapter 2 to the spacetime Penrose inequality. The first is a reduction to the case of $H = 0, k = 0$ boundary.

Theorem 3.3. *Suppose (M, h, k) is 3-dimensional asymptotically flat initial data satisfying the DEC such that ∂M is a MOTS. Then there exists an initial data set $(\widetilde{M}, \widetilde{h}, \widetilde{k})$ satisfying the following:*

1. *The ADM mass of $\widetilde{M}$ is ϵ-close to that of M.*
2. *$(\widetilde{M}, \widetilde{h}, \widetilde{k})$ satisfies the DEC,*
3. *$\widetilde{k} = 0$ and $H = 0$ on $\partial\widetilde{M}$,*
4. *$A_{\min}(\partial M) < A_{\min}(\partial\widetilde{M})(1+\epsilon)$.*

Proof. Let $A_{\min}(Z, P, \gamma)$ denote the area of the minimal area enclosure of Z in the Riemannian manifold (P, γ).

First, if $\gamma_k \to \gamma$ in C^0 on a manifold P, then there exists $\epsilon_k \to 0$ such that for all surfaces $S \subset P$,

$$|S|_\gamma < (1+\epsilon_k)|S|_{\gamma_k}, \quad |S|_{\gamma_k} < (1+\epsilon_k)|S|_\gamma. \tag{3.2}$$

So for any $\epsilon < 0$,

$$A_{\min}(Z, P, \gamma) < A_{\min}(Z, P, \gamma_k)(1+\epsilon) \tag{3.3}$$

for sufficiently large k.

First, we reduce to the case where M has strict dominant energy condition and stable MOTS boundary. By a density theorem of Lee-Lesourd-Unger (Theorem 1.1 of [LLU22]), there exists initial data (h', k') on M satisfying the strict dominant energy condition everywhere, such that ∂M remains a MOTS in (h', k') that is close to (h, k) in $W^{2,p}_{-q}(M) \times W^{1,p}_{-q-1}(M)$, and the energy and momentum densities of (h, k) and (h', k') are close in $L^1(M)$. Note that this implies that the ADM energy-momentum is ϵ-close by Lemma 8.4 of [Lee19]. Also as the density theorem works for all $p > n$, h is close to h' in C^0 by Sobolev embedding.

Let (M', h', k') be the initial data set that is the truncation of M containing the asymptotically flat end at the *outermost* MOTS of M. Since ∂M is a MOTS in (h', k'), the outermost MOTS of M exists and is smooth by a theorem of Andersson and Metzger [AM09].

Being the outermost MOTS, $\partial M'$ is a stable MOTS with respect to the inward unit normal ν by [AMS05]. Moreover, the strict dominant energy condition holds on M', hence $\partial M' \in \mathscr{M}_+^{MOTS}$.

We have

$$A_{\min}(\partial M, M, h) \leqslant A_{\min}(\partial M', M, h) \tag{3.4}$$

$$= A_{\min}(\partial M', M', h) \tag{3.5}$$

$$< A_{\min}(\partial M', M', h')(1+\epsilon), \tag{3.6}$$

since $\partial M'$ encloses ∂M, and h' is close to h in C^0.

As $A_{\min}$ does not decrease too much and m_{ADM} does not change much, without loss of generality, we may proceed by replacing M with M' constructed above, and assume that $\partial M \in \mathscr{M}_+^{MOTS}$.

We would like to construct an initial data collar C as in Theorem 2.3 and attach the $t = 0$ end of C to ∂M, such that $\partial(C \cup_{\Sigma_0} M) = \Sigma_1$, where $\tilde{k} = 0$ and $H = 0$. To do this, $\Sigma_0 \subset \partial C$ needs to satisfy the matching conditions on k and mean curvatures as required by Theorem 2.13. However, the unit normal $\eta^{-1}\partial_t$ used in the statement of Theorem 2.3 is inwards at the Σ_0 end, as opposed to outwards as required by Theorem 2.13, so we describe the correct procedure carefully.

Via Theorem 2.3, let C be the constructed collar starting with $(\partial M, g, -H, k_{\nu\cdot}) \in \mathscr{M}_+^{MOTS}$, and let $(\widehat{C} = \Sigma \times [0,1], \widehat{h}, \widehat{k})$ be the modified initial data given by Proposition 2.15. Let $\widehat{\nu}$ be inward normal to Σ_0, and let ν be the inward normal to ∂M.

At Σ_0, we have the following, given by properties 2 and 4 of Lemma 2.15

$$\widehat{H}_{\widehat{\nu}} < -H \tag{3.7}$$

$$\mathrm{Tr}_{\Sigma_0} \widehat{k} = H \tag{3.8}$$

$$\widehat{k}_{\widehat{\nu}\cdot} = k_{\nu\cdot}, \tag{3.9}$$

where $\widehat{H}_{\widehat{\nu}}$ is mean curvature of Σ_0 in $\widehat{C}$ with respect to $\widehat{\nu}$. Since ∂M is a MOTS, we have

$H = -\operatorname{Tr}_{\partial M} k$, and the outward normal to Σ_0 is $-\widehat{\nu}$, so

$$H < \widehat{H}_{-\widehat{\nu}} \tag{3.10}$$

$$\operatorname{Tr}_{\partial M} k = \operatorname{Tr}_{\Sigma_0}(-\widehat{k}) \tag{3.11}$$

$$k_{\nu \cdot} = -\widehat{k}_{-\widehat{\nu},\cdot}. \tag{3.12}$$

We now satisfy the hypotheses of Theorem 2.13, permitting us to glue $(\widehat{C}, \widehat{h}, -\widehat{k})$ and (M, h, k), identifying Σ_0 with ∂M, resulting in an initial data set $(\widetilde{M} = \widehat{C} \cup_{\Sigma_0} M, \widetilde{h}, \widetilde{k})$. The resulting $\widetilde{h}$ restricted to M is C^0-close to h. Furthermore, $\widetilde{M}$ satisfies the dominant energy condition, $\partial\widetilde{M} = \Sigma_1$ is minimal, and $\widetilde{k} = 0$ in a neighborhood there.

For property 4, we would like to bound $A_{\min}(\partial M, M, h)$ in terms of $A_{\min}(\partial\widetilde{M}, \widetilde{M}, \widetilde{h})$; note that ∂M encloses $\partial\widetilde{M}$. Let S be any surface enclosing ∂M. Define S' the modification of S that projects the subset of S that intersects $\widehat{C}$ onto ∂M while preserving the portion that intersects M. As the projection is a $(1+\epsilon)$-Lipschitz map by Lemma 2.16,

$$|S'|_{\widetilde{h}} < (1+\epsilon)^2 |S|_{\widetilde{h}}. \tag{3.13}$$

Moreover, as S' is a surface enclosing ∂M, we have

$$A_{\min}(\partial M, \widetilde{M}, \widetilde{h}) < |S'|_{\widetilde{h}}. \tag{3.14}$$

Hence $A_{\min}(\partial M, \widetilde{M}, \widetilde{h}) < (1+\epsilon)^2 |S|_{\widetilde{h}}$ for any surface S enclosing $\partial\widetilde{M}$, so we have

$$A_{\min}(\partial M, \widetilde{M}, \widetilde{h}) < (1+\epsilon)^2 A_{\min}(\partial\widetilde{M}, \widetilde{M}, \widetilde{h}). \tag{3.15}$$

Finally since $\widetilde{h}$ restricted to M is close to h in C^0, so

$$A_{\min}(\partial M, M, h) < (1+\epsilon) A_{\min}(\partial M, M, \widetilde{h}) = (1+\epsilon) A_{\min}(\partial M, \widetilde{M}, \widetilde{h}), \tag{3.16}$$

so all together we have

$$A_{\min}(\partial M, M, h) < A_{\min}(\partial M, \widetilde{M}, \widetilde{h})(1+\epsilon)^3, \tag{3.17}$$

as desired. □

This immediately implies the following reduction of the Spacetime Penrose inequality to initial data sets with $H = 0$ $k = 0$ boundary:

Corollary 3.4. *If for any asymptotically flat data set (M, g, k) satisfying the dominant energy condition with $H = 0$ and $k = 0$ on ∂M,*

$$\sqrt{\frac{A}{16\pi}} \leqslant m_{ADM}, \tag{3.18}$$

where A is the minimum area enclosure of ∂M, then the spacetime Penrose inequality is true.

This reduction of the Penrose inequality to the case where $H = 0$ and $k = 0$ has several interesting and potentially useful properties. If ∂M is minimal, then its minimal area enclosure is either equal to ∂M or a smooth minimal surface disjoint from ∂M, which gives more regularity on the surface whose area one is required to bound.

Moreover, in the generalized Jang equation approach proposed by Bray and Khuri, the boundary condition proposed for minimal apparent horizon boundary is that the Jang graph f stays bounded near ∂M, and the warping function $\phi = 0$ on ∂M. For a general MOTS, f is prescribed to blow up or blow down at a particular rate. This could possibly simplify the analysis required, as the PDE fails to elliptic where $\phi = 0$.

We also further remark that the minimal boundary condition would allow one to reflect the initial data across the boundary, resulting in an initial data set with two asymptotically flat ends with a $\mathbb{Z}/2$-symmetry. This doubling operation plays an crucial role in Bray's conformal flow approach to prove the Riemannian Penrose inequality for multiple black holes. In particular, the derivative of mass under the flow satisfies $m'(t) = -2\widetilde{m}$, where $\widetilde{m}$ is the mass of the double, with one end compactified by a global harmonic function. Since conformal change by a harmonic function preserves non-negative scalar curvature, $\widetilde{m}$ is positive due to the positive mass theorem. Also important is the fact that a harmonic function asymptotic to 1 at one end and asymptotic to -1 at the other end must equal zero at the minimal surface.

Our construction also yields a class of spherically symmetric counterexamples to the apparent horizon Penrose inequality. Already known to be false, an example in the literature where the the MOTS boundary satisfies $H < 0$ was constructed by Ben-Dov in [Ben04]. Our example works for any $H > 0$.

Theorem 3.5. *For any set of positive constants H, A, ϵ, there exists a spherically symmetric asymptotically flat initial data set (M, g, k) satisfying the dominant energy condition such that*

1. *∂M is isometric to a round sphere with area A,*
2. *∂M is an outermost MOTS,*
3. *∂M has mean curvature H,*
4. *$m_{ADM} < \epsilon$.*

Proof. In spherical symmetry we have $J^\top = 0$. Moreover, a and η are constant functions on Σ_t. Therefore, so is $f(t)$ and therefore $\Delta_g f(t) = 0$. Revisiting the computation (2.103), we see that strict dominant energy condition holds no matter what ε is, as long as δ is chosen appropriately small. Set $\varepsilon = 1$. When we construct a, it is sent to a large positive constant B, for $t \in [\frac{1}{\delta}, \frac{1}{2}]$ after which it is non-negative. Since $f(1) = -\int_0^t \eta a$, so by choosing the B large, $f(1)$ can be arbitrarily negative. Hence $(\Sigma_1, e^{2f(1)}g)$ is an arbitrarily small $H = 0$ sphere. Using the perturbation result, make Σ_1 mean convex, which allows us to glue to the time-symmetric slice of Schwarzschild with small mass as in [MS15]. This yields a spherically symmetric manifold with a $\theta^+ > 0$-foliation and dominant energy condition, with a $\theta^+ = 0$ boundary. □

3.2 Spacetime Bartnik Mass of Apparent Horizons

In [Bar97], R. Bartnik proposed a definition of quasi-local mass for a compact initial data sets with boundary (Ω, g_0, k_0), which we will call *spacetime Bartnik mass*. In this paper we will define more generally the spacetime Bartnik mass $m_B(\Sigma, g_0, H, \kappa, \beta)$ for any Bartnik data $(\Sigma, g_0, H, \kappa, \beta)$.

Let $\mathcal{B}$ denote the family of *admissible extensions* of $(\Sigma, g_0, H_0, \kappa, \beta)$, defined as the family of initial data sets (M, g, k) satisfying the the dominant energy condition and the *geometric* *boundary conditions*:

$$g_0 = g|_{T\partial M} \tag{3.19}$$

$$H_0 = H_{\partial M} \tag{3.20}$$

$$\kappa = \operatorname{Tr}_{\partial M} k \tag{3.21}$$

$$\beta = k(\nu, \cdot), \tag{3.22}$$

where ν is the unit normal of ∂M pointing into M and therefore towards infinity.

Define

$$m_B^{\mathcal{N}}(\Sigma, g_0, H, \kappa, \beta) = \left\{ \inf_{(M,g,k)\in\mathcal{B}} m_{ADM}(M,g,k) \,\middle|\, (M,g,k) \text{ satisfies condition } \mathcal{N} \right\}, \tag{3.23}$$

where $\mathcal{N}$ is a *no-horizon condition* to be chosen. As of recent, there has not been an extensive study in the literature on no-horizon conditions for the spacetime case except in Bartnik's original lecture [Bar97] and in [HL20]. This is in contrast to the better-studied time-symmetric notion of Bartnik mass, for example, see [Jau19] for relations between different notions of extensions and no-horizon conditions. We define a few candidates for $\mathcal{N}$.

1. $\mathcal{N}_1$ is the condition that ∂M is strictly outer minimizing. If Σ is a MOTS, then the spacetime Penrose conjecture would imply

$$\sqrt{\frac{|\Sigma|}{16\pi}} \leqslant m_B^{\mathcal{N}_1}(\Sigma, g_0, H_0, -H_0, \beta). \tag{3.24}$$

2. $\mathcal{N}_2$ is the condition that there does not exist an enclosing surface Σ with

$$H_\Sigma \leqslant -|\operatorname{Tr}_\Sigma k|, \tag{3.25}$$

that is, there does not exist an enclosing surface with inward space-like spacetime mean curvature, originally proposed in [Bar97]. Equivalently, there does not exist an enclosing surface Σ satisfying both $\theta^+ \leqslant 0$ and $\theta^- \geqslant 0$ everywhere on Σ.

3. $\mathcal{N}_3$ is the condition that M contains no enclosing MOTS, except the possibility of ∂M itself.

There is an example of Bartnik data that poses a problem for all of these conditions. For example if the data does not satisfy $H \geqslant 0$, no admissible extension satisfies $\mathcal{N} = \mathcal{N}_1$. Moreover, for spherically symmetric MOTS boundary data $(\Sigma, g_{\text{round}}, H \equiv C, 0)$, one can verify that the construction of Theorem 3.5 is an admissible extension satisfying $\mathcal{N}_2$ and $\mathcal{N}_3$, therefore $m_B(\Sigma) = 0$ holds for $\mathcal{N} = \mathcal{N}_2, \mathcal{N}_3$. This shows that a surface being an outermost MOTS is not a sufficient restriction to prevent being hidden behind a horizon, unlike the time-symmetric case.

Conjecture 3.6. *The Bartnik mass satisfies* $m_B^{\mathcal{N}}(\Sigma, g_0, H_0, \kappa, k_{\nu,\cdot}) = 0$ *for any Bartnik boundary data under conditions* $\mathcal{N} = \mathcal{N}_2$ *and* $\mathcal{N} = \mathcal{N}_3$.

Next, we give a generalization of the Mantoulidis-Schoen estimate of horizon Bartnik data to the spacetime case, assuming mean convex MOTS Bartnik data in $\mathscr{M}_+^{MOTS}$.

Theorem 3.7. *For any MOTS Bartnik data* $(\Sigma, g_0, H_0, -H_0, \omega) \in \mathscr{M}_+^{MOTS}$ *with* $H_0 > 0$, *the Bartnik mass satisfies*

$$m_B^{\mathcal{N}_1}(\Sigma, g_0, H_0, -H_0, \omega) \leqslant \sqrt{\frac{|\Sigma|}{16\pi}}. \tag{3.26}$$

Proof. The main idea is to glue the $t = 1$ end of the collar construction of Theorem 2.3 to the Mantoulidis-Schoen Bartnik extension of minimal Bartnik data satisfying $\lambda_1(-\Delta + K) > 0$. The mass of the extension will be $\sqrt{\frac{|\Sigma_1|}{16\pi}} + \epsilon$. Then we verify that Σ_0 is strictly outer minimizing in the glued extension, and Σ_1 has similar area to Σ_0, yielding an admissible extension satisfying $\mathcal{N}_1$ of mass less than

$$\sqrt{\frac{|\Sigma_0|}{16\pi}} + 2\epsilon.$$

This gives the requisite estimate of the spacetime Bartnik mass.

As $H_0 > 0$, by Theorem 2.3 there exists a collar $C = \Sigma \times [0,1]$ foliated by mean convex surfaces obtained where the Bartnik data at Σ_0 agrees with the given Bartnik data $(\Sigma, g_0, H_0, \omega)$. By Lemma 2.14 there exists a strict DEC modification $(C' = \Sigma \times [0,1], g', k)$

of C, perturbed so that C' is foliated by *strictly* mean convex surfaces for $t \in [0,1]$, preserves the Bartnik data at $t = 0$, and satisfies $\lambda_1(-\Delta_{\Sigma_1} + K_{\Sigma_1}) > 0$ at $t = 1$.

By the result of Mantoulidis-Schoen [MS15], there exists a time symmetric Bartnik extension (E, g_E) of Σ_1 with $g_E|_{\partial E} \cong g'|_{\Sigma_1}$ and ∂E minimal and outer minimizing in g_E. In a collar neighborhood U of ∂E, g_E has positive scalar curvature. Outside a compact neighborhood of ∂E, (E, g_E) is isometric to the $t = 0$ slice of Schwarzschild of mass $\sqrt{\frac{|\partial E|}{16\pi}} + \epsilon$.

Since $0 = H_{\partial E} < H_{\Sigma_1}$, where H_{Σ_1} is mean curvature with respect to the outward normal, and the strict dominant energy condition holds near the gluing surfaces, Theorem 2.13 gives a gluing of C' and E via identifying Σ_1 and ∂E. Let

$$(E' = C' \cup_{\Sigma_1} E, g_\lambda, k_\lambda) \tag{3.27}$$

be given by Theorem 2.13. View C' and E as domains of E' with a common boundary $\partial E \cong \Sigma_1$ in the interior of E'. The dominant energy condition holds on all of $(E', g_\lambda, k_\lambda)$ and is an admissible extension of Σ_0.

It remains to show that Σ_0 is strictly outer minimizing in E'. Let g be the Lipschitz metric on E' given by $g|_{C'} = g'$ and $g|_E = g_E$, which is smooth away from Σ_1. As stated in Theorem 2.13, in a compact neighborhood U with $\partial E \subset U \subset E$, we have that g_λ converges to g in C^0, and outside of U, we have that $g_\lambda = g$. Since Σ_0 has positive mean curvature, Lemma 13 of [Jau19] applies, and Σ_0 remains strictly outer minimizing in (E', g'_λ) for λ large, provided that $\Sigma_0 = \partial \overline{M}$ is strictly outer minimizing in g. The rest of the proof is devoted to showing this.

We construct a minimal area enclosure of S of Σ_0 in the metric g, where Ω is an open set containing the AF end. The existence of Ω follows from a standard argument as follows. In particular, take a sequence Ω_i of open sets containing the AF end, such that $|\partial\Omega_i|$ approaches to the minimal value. As Σ_1 is outer-minimizing, we can ensure S is contained in C' by taking the modified sequence $\Omega'_i = E \cup \Omega_i$. By standard properties of the perimeter,

$$|\partial\Omega'_i| \leqslant |\partial E| + |\partial\Omega_i| - |\partial(E \cap \Omega_i)| < |\partial\Omega_i|, \tag{3.28}$$

so Ω_i' is also a minimizing sequence. Let W be a 3-ball that fills in Σ_0, and extend the metric g into W. Now $\Omega_i'^c$ is a set of finite perimeter in the bounded region of $C' \cup W$, hence by the compactness theorem (see, for example, Theorem 12.26 of [Mag12]) for such sets, $\Omega_i'^c$ converges to a set of finite perimeter Z, after possibly passing to a subsequence. Up to modifying by a set of measure zero, $W \subset Z \subset (C' \cup W)$. The perimeter is lower semicontinuous under this convergence, hence let $S := \partial^* Z$ is a minimizer of area for all enclosing surfaces contained in C'. We can view S as an integral rectifiable varifold, by De Giorgi's theorem for reduced boundaries.

We would like to use the mean convex foliation of C' to conclude that $S = \Sigma_0$. As S is a-priori only known to be $C^{1,1}$, care beyond the usual theory of minimal surfaces is needed to rule out nontrivial intersection of S with the boundary component Σ_1. We invoke a one-sided maximum principle for varifolds [SW89] which shows that S cannot touch Σ_1 on the inside. As Σ_1 is mean convex with respect to the outward normal, and the derivative of area of S is non-negative with respect to inward variations near Σ_1, the conclusion follows from combining the main theorem with Additional Remarks 1 and 2 of [SW89] (also see Theorem 4 of [Whi09]). The same argument shows that at the maximum value of t on S cannot be greater than 0, since Σ_t is mean convex. Hence $S = \Sigma_0$.

Now we have shown that Σ_0 is outer minimizing. If Σ_0 were not strictly outer minimizing, then there exists a surface S' enclosing Σ_0 with the same area. But S' would also be a minimal area enclosure of Σ_0. As before, since Σ_1 is strictly outer minimizing, the modification argument using (3.28) shows that S' must lie inside C', and the generalized maximum principal shows that $S' = \Sigma_0$. □

Chapter 4. Positive Mass Theorem with Creases

4.1 Preliminaries

4.1.1 Intrinsic Bartnik data

A spacelike co-dimension two submanifold Σ embedded in a Lorentzian spacetime (N, h) inherits the following data:

1. a Riemannian metric γ,
2. an oriented $SO(1,1)$-normal bundle $\mathcal{N}$, and a compatible connection $\nabla^{\mathcal{N}}$,
3. a symmetric 2-tensor χ with values in $\mathcal{N}$, the second fundamental form of Σ.

The spacetime mean curvature vector is denoted by $\vec{H} = \mathrm{Tr}_\gamma \chi$. This motivates the following definition:

Definition 4.1. Given a closed orientable manifold Σ with an oriented $SO(1,1)$-bundle $\mathcal{N}$, a triple of Riemannian metric γ, section $\vec{H}$ of $\mathcal{N}$, and connection on $\mathcal{N}$, $\left(\gamma, \mathcal{N}, \vec{H}, \nabla^{\mathcal{N}}\right)$ is called *intrinsic Bartnik data* for Σ.

This definition of Bartnik data captures the geometric quantities on Σ without reference to an initial data set. Suppose we have an initial data set (M, g, k) and embedded hypersurface $(\Sigma, \gamma) \subset M$ and choice of unit normal ν. Recall that the *spacetime Bartnik data* on Σ is the tuple

$$(\gamma, H, \mathrm{Tr}_\Sigma k, k_{\nu,\cdot}) \tag{4.1}$$

where H is the mean curvature of Σ in (M, g) with respect to ν. This definition has been important in the study of stationary vacuum constraint equations and for defining quasilocal mass in initial data. As spacetime Bartnik data encodes intrinsic Bartnik data in the particular frame given by the tangent space of M, we can obtain intrinsic Bartnik data on Σ as follows:

Definition 4.2. An embedded hypersurface (Σ, γ) in an initial data set (M, g, k) with choice of unit normal ν yields intrinsic Bartnik data $\left(\gamma, \mathcal{N}, \vec{H}, \nabla^{\mathcal{N}}\right)$ as follows. Let $\mathcal{N} := \mathbb{R} \oplus N\Sigma$, equipped with a global orthonormal $SO(1,1)$ frame $\{\tau = (1,0), \nu := (0,\nu)\}$, oriented with τ future timelike. Define the mean curvature vector and connection on $\mathcal{N}$ by

$$\vec{H} = H\nu - (\mathrm{Tr}_\Sigma k)\tau, \quad \left\langle \nabla^{\mathcal{N}}_{\cdot} \nu, \tau \right\rangle = -k_{\nu,\cdot}. \tag{4.2}$$

Note that, in our convention, we have $H = \left\langle \vec{H}, \nu \right\rangle$ and $\mathrm{Tr}_\Sigma k = \left\langle \vec{H}, \tau \right\rangle$. If (M, g, k) arises as a slice of spacetime (N, h), and $\Sigma \subset M$, then the intrinsic Bartnik data on Σ with choice of ν determined from (M, g, k) is equivalent to the data induced from the spacetime (N, h).

4.1.2 Creased Data

For the rest of the paper, we adhere to the following setup. Suppose (M_+, g_+, k_+) and (M_-, g_-, k_+) are n-dimensional initial data sets, with M_+ asymptotically flat and M_- compact. Suppose the boundaries $\partial M_+, \partial M_-$ are isometric to a closed manifold, denoted Σ. Let ν_+ be the inward unit normal to ∂M_+ pointing towards infinity, and ν_- be the outward unit normal to ∂M_- (see Figure 4.1). Let $(\mathcal{N}_\pm, \vec{H}_\pm, \nabla^\pm)$, be the intrinsic Bartnik data on Σ constructed from $(M_\pm, g\pm, k_\pm, \nu_\pm)$ as in the previous subsection. Let $\tau_\pm$ be the future timelike unit vectors in $\mathcal{N}_\pm$ orthogonal to $\nu_\pm$.

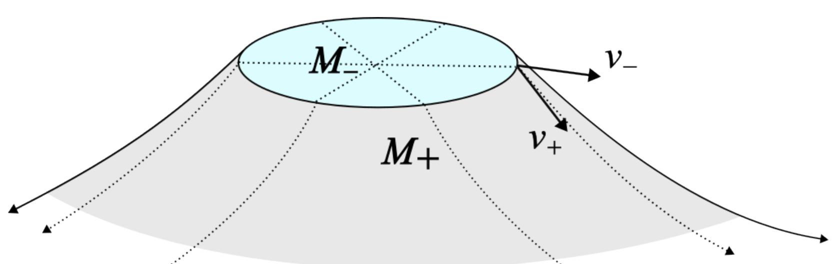

FIGURE 4.1: Creased initial data

For any smooth function f on Σ, we can isometrically identify $\mathcal{N}_\pm$ with a normal bundle

$$\mathcal{N}_f \cong \mathcal{N}_\pm \tag{4.3}$$

such that

$$\tau_+ = \cosh(f)\tau_- + \sinh(f)\nu_- \tag{4.4}$$

$$\nu_+ = \sinh(f)\tau_- + \cosh(f)\nu_- \tag{4.5}$$

inside $\mathcal{N}_f$.

Under this identification, the connection one form of ∇^- with respect to ν_+ equals

$$\begin{aligned} \left\langle \nabla^-_X \nu_+, \tau_+ \right\rangle &= \left\langle \nabla^-_X \left(\sinh(f)\tau_- + \cosh(f)\nu_-\right), \cosh(f)\tau_- + \sinh(f)\nu_- \right\rangle \\ &= -k^-_{\nu_-,X} - df(X). \end{aligned} \tag{4.6}$$

The difference of connection one forms of $\nabla^\pm$ is frame-independent and equals

$$\alpha^\Delta(\cdot) := \left\langle (\nabla^-_\cdot - \nabla^+_\cdot)\nu_+,\ \tau_+ \right\rangle = k^+_{\nu_+,\cdot} - k^-_{\nu_-,\cdot} - df. \tag{4.7}$$

We will displace the $\pm$-subscript on the one forms $k^\pm_{\nu_\pm,\cdot}$ for readability.

The components of $\vec{H}_\pm$ in the τ_+, ν_+ frame are

$$\begin{aligned} \left\langle \vec{H}_+, \nu_+ \right\rangle &= H_+, & \left\langle \vec{H}_-, \nu_+ \right\rangle &= \cosh(f)H_- + \sinh(f)\operatorname{Tr}_\Sigma k_-, \\ \left\langle \vec{H}_+, \tau_+ \right\rangle &= \operatorname{Tr}_\Sigma k_+, & \left\langle \vec{H}_-, \tau_+ \right\rangle &= \sinh(f)H_- + \cosh(f)\operatorname{Tr}_\Sigma k_-, \end{aligned} \tag{4.8}$$

where $H_\pm$ denotes the mean curvature of Σ in $(M_\pm, g_\pm)$ with respect to $\nu_\pm$.

We call f the *hyperbolic angle of rotation* and the identification of $\mathcal{N}_\pm$ with $\mathcal{N}_f$ allows us to compare the Bartnik data $(\vec{H}_\pm, \nabla^\pm)$.

The natural setting where creased data arises is when $(M_+, g_+, k_+), (M_-, g_-, k_-) \subset (N, h)$ are spacelike hypersurfaces whose common boundary is a closed spacelike codimension-two surface $\Sigma = \partial M_+ = \partial M_- \subset N$. Suppose M_+ and M_- meet at Σ with hyperbolic angle f, that is, equations of the form (4.4)-(4.5) hold in the spacetime at Σ. Then the normal bundle of Σ from N is isomorphic to $\mathcal{N}_f$, and we naturally have $\vec{H}_+ = \vec{H}_-$

and $\nabla^+ = \nabla^-$ on the normal bundle $\mathcal{N}_f$. We observe that as long as the dominant energy condition is satisfied in N, the positive mass theorem should hold for such initial data.

More generally, a positive mass theorem should still hold with a mismatch of intrinsic Bartnik data as long as the boundary metrics agree and a certain quantity has the right sign. Our condition generalizes the $H_- > H_+$ condition of P. Miao [Mia03] in the Riemannian case and the jump conditions of T.Y. Tsang [Tsa22], allowing for a gauge freedom given by f.

Definition 4.3. Fix a smooth function f on Σ. We say $(M_\pm, g_\pm, k_\pm)$ is *"DEC-creased"* with (hyperbolic) angle f if on $\mathcal{N}_f$, $\vec{H}_- - \vec{H}_+$ is outward spacelike and

$$\left|\vec{H}_- - \vec{H}_+\right| \geqslant \left|\alpha^\Delta\right|_\Sigma. \tag{4.9}$$

Equivalently,

$$\left\langle \vec{H}_- - \vec{H}_+, \nu_+ \right\rangle - \sqrt{\left\langle \vec{H}_- - \vec{H}_+, \tau_+ \right\rangle^2 + |\alpha^\Delta|^2_\Sigma} \geqslant 0. \tag{4.10}$$

We give a proof of the positivity of ADM mass for spin DEC-creased initial data sets with angle f satisfying the dominant energy condition, modeled on the spinor proof argument discovered by Witten [Wit81]. We define transmission-type boundary conditions for spinors that depend on the angle f. Under these boundary conditions, we apply the Lichnerowicz-Weitzenböck formula to show coercivity, which yields the following existence theorem for the Dirac-Witten harmonic spinor equation:

Theorem 4.4. *Suppose $(M_\pm, g_\pm, k_\pm)$ are n-dimensional spin initial data sets satisfying the following: (M_+, g_+, k_+) asymptotically flat with compact boundary and (M_-, g_-, k_-) compact, $(\partial M_\pm, g_\pm)$ are isometric, and there exists a choice of spin structure on $M_- \cup_\partial M_+$. Let $S(M_\pm)$ be the spacetime spinor bundle on $M_\pm$ and let $\Phi : S(M_-)|_{\partial M_-} \to S(M_+)|_{\partial M_+}$ be the induced isomorphism from the unified spin structure. Assume $(M_\pm, g_\pm, k_\pm)$ satisfy the dominant energy condition and there exists a smooth function f such that $(M_\pm, g_\pm, k_\pm)$ is DEC-creased with angle f. Set $A = \cosh f/2$, $B = \sinh f/2$ and $\epsilon = \nu_+\tau_+$.*

Fix a constant spinor ψ_∞ at infinity. Then there exist Dirac-Witten-harmonic spinors $\psi_\pm$ on $M_\pm$ satisfying the boundary condition

$$\Phi(\psi_-) = (A + B\epsilon)\psi_+ \tag{4.11}$$

on ∂M_+ and of regularity $C^\infty(M_\pm)$ with ψ_+ asymptotic to ψ_∞ at infinity.

Consequently, we obtain a positive mass theorem for DEC-creased initial data satisfying the dominant energy condition:

Theorem 4.5. *Assume the hypotheses on $(M_\pm, g_\pm, k_\pm)$ as in Theorem 4.4. If $\psi_\pm$ are the solutions given by Theorem 4.4, then*

$$\begin{aligned} &\frac{(n-1)\omega_{n-1}}{2}\left(E|\psi_\infty|^2 - \langle\psi_\infty, P\tau\psi_\infty\rangle\right) \\ &\geqslant \int_{M_-}\left(|\overline{\nabla}\psi_-|^2 + \frac{1}{2}(\mu_-|\psi_-|^2 - \langle\psi_-, J_-\tau_-\psi_-\rangle)\right)dV \\ &+ \int_{M_+}\left(|\overline{\nabla}\psi_+|^2 + \frac{1}{2}(\mu_+|\psi_+|^2 - \langle\psi_+, J_+\tau_+\psi_+\rangle)\right)dV, \end{aligned} \tag{4.12}$$

where ω_k denotes the volume of the unit sphere in $\mathbb{R}^k$, (E, P) is the ADM energy-momentum of M_+, $(\mu_\pm, J_\pm)$ are the energy-momentum densities on $M_\pm$, and $\overline{\nabla}$ is the spacetime connection on $S(M_\pm)$, defined by (4.28).

By choosing ψ_∞ to satisfy $P\tau\psi_\infty = |P|\psi_\infty$, we have $E - |P| \geqslant 0$ which shows that (E, P) is non-spacelike.

4.2 Spin Geometry Background

4.2.1 Spin Structures and the Map Φ

As the metrics g_+ and g_- in general define a C^0-metric on $M_+ \cup_\Sigma M_-$ that is smooth away from Σ, we must define the notion of a spin structure for $M_+ \cup_\Sigma M_-$ in this setting. Let P_+^{SO} and P_-^{SO} be the orthonormal frame bundles over (M_+, g_+) and (M_-, g_-), respectively. Let $\phi : \partial M_- \to \partial M_+$ be the isometry of the boundaries, and let us freely identify $T\Sigma :\cong T\partial M_- \cong T\partial M_+$. Then define the map

$$\phi_* : TM_-|_{\partial M_-} \to TM_+|_{\partial M_+} \tag{4.13}$$

such that

$$\phi_*(v) = d\phi(v) = v, \quad v \in T\Sigma, \tag{4.14}$$

$$\phi_*(\nu_-) = \nu_+; \tag{4.15}$$

we also regard ϕ_* as a map $P_-^{SO}|_{\partial M_-} \to P_+^{SO}|_{\partial M_+}$. A *spin structure* on $M_+ \cup_\Sigma M_-$ consists of lifts $\pi_- : P_-^{Spin} \to P_-^{SO}$ and $\pi_+ : P_+^{Spin} \to P_+^{SO}$ such that π_- and π_+ define spin structures over M_- and M_+ respectively, and a map of principal bundles $\Phi_0 : P_-^{Spin}|_{\partial M_-} \to P_+^{Spin}|_{\partial M_+}$ such that the following diagram commutes

$$\begin{array}{ccc} P_-^{Spin}|_{\partial M_-} & \xrightarrow{\Phi_0} & P_+^{Spin}|_{\partial M_+} \\ \pi_- \big\downarrow & & \big\downarrow \pi_+ \\ P_-^{SO}|_{\partial M_-} & \xrightarrow{\phi_*} & P_+^{SO}|_{\partial M_+} \end{array} . \tag{4.16}$$

We remark that the set of such spin structures is in 1-1 correspondence to the set of ones defined with any smooth metric on $M_+ \cup_\Sigma M_-$ (see 3.1.1 of [LL15]), so an equivalent condition for existence is that $M_+ \cup_\Sigma M_-$ is spin as a smooth manifold.

Given a spin structure on $M_+ \cup_\Sigma M_-$, we obtain two associated spinor bundles $S_0(M_+)$ and $S_0(M_-)$ over M_+ and M_-. Each spinor bundle is equipped with Clifford multiplication, a hermitian metric $\langle \cdot, \cdot \rangle_0$, and a compatible spin connection ∇_0. Interpreting $P_\pm^{Spin}$ as the bundle of orthonormal frames of $S_0(M_\pm)$, Φ_0 induces an isometry $S_0(M_-)|_{\partial M_-} \to S_0(M_+)|_{\partial M_+}$ which we also denote by Φ_0. By (4.16), Φ_0 is a lift of the map of tangent bundle frames given by ϕ_* to a map of spin frames, so naturally we have for all $v \in TM_-|_{\partial M_-}$ and $\psi \in S_0(M_-)|_{\partial M_-}$, $\Phi_0(v\psi) = \phi_*(v)\Phi_0(\psi)$. In particular, $\Phi_0(\nu_-\psi) = \nu_+\Phi_0$, and $\Phi_0(v\psi) = v\Phi_0(\psi)$ for $v \in T\Sigma$.

Fix a local spin frame $\{\psi_I\}$ of $S_0(M_-)|_{\partial M_-}$ corresponding to the frame $\{e_1, ..., e_{n-1}, \nu_-\}$, where $\{e_i\}$ is a frame on ∂M_-. Let $\psi = \sum_I c_I \psi_I$ be any section of $S_0(M_-)|_{\partial M_-}$. The *boundary spin connection* [CB03] on ∂M_- is given by

$$\nabla_X^{\partial M_-} \psi = \sum_I X(c_I)\psi_I - \frac{1}{4} \sum_{I, 1 \leqslant i,j \leqslant n-1} c_I \omega_{ij}(X) e_i e_j \cdot \psi_I \tag{4.17}$$

for any $X \in T\partial M_-$, where $\omega_{ij}(X) = \langle e_i, \nabla_X e_j \rangle$. The boundary spin connection $\nabla_X^{\partial M_+}$ over ∂M_+ is given by the analogous formula for a spin frame of $S_0(M_+)|_{\partial M_+}$ corresponding to the tangent bundle frame $\{e_1, ..., e_{n-1}, \nu_+\}$. The diagram (4.16) implies that $\Phi_0(\psi_I)$ gives such a frame. Hence we have

$$\Phi_0\left(\nabla_X^{\partial M_-}\psi\right) = \Phi_0\left(\sum_I X(c_I)\psi_I - \frac{1}{4}\sum_{I,1\leqslant i,j\leqslant n-1} c_I\omega_{ij}(X)e_ie_j \cdot \psi_I\right) \tag{4.18}$$

$$= \sum_I X(c_I)\Phi_0(\psi_I) - \frac{1}{4}\sum_{I,1\leqslant i,j\leqslant n-1} c_I\omega_{ij}(X)e_ie_j \cdot \Phi_0(\psi_I) \tag{4.19}$$

$$= \nabla_X^{\partial M_+}\Phi_0(\psi). \tag{4.20}$$

Construct the spacetime spinor bundle $S(M_\pm) := S_0(M_\pm) \oplus S_0(M_\pm)$ and equip it with a $Cl(n,1)$-Clifford module structure by the following: for $v \in TM_\pm$ and $\psi = \psi_1 \oplus \psi_2$, define

$$v\psi = v\psi_1 \oplus -v\psi_2, \tag{4.21}$$

and define the endomorphism $\tau_\pm$ of $S(M_\pm)$ by

$$\tau_\pm\psi = \psi_2 \oplus \psi_1. \tag{4.22}$$

Note $\tau_\pm$ satisfies $\tau_\pm^2 = 1$ and anticommutes with Clifford multiplication of $TM_\pm$, so $\tau_\pm$ can be interpreted as Clifford multiplication by the timelike normal to $M_\pm$. The connection ∇ on $S(M_\pm)$ is given by applying the connection from S_0 on each summand, which commutes with multiplication by $\tau_\pm$. Define the isometry

$$\Phi : S(M_-)|_{\partial M_-} \to S(M_+)|_{\partial M_+} \tag{4.23}$$

by applying Φ_0 on each summand, and define the boundary connection on $S(M_\pm)|_{\partial M_\pm}$, which we now denote by ∇^Σ, by applying (4.17) on each factor. It's easy to verify that Φ satisfies

$$\Phi(v\psi) = v\Phi(\psi) \text{ for } v \in T\Sigma \tag{4.24}$$

$$\Phi(\nu_-\psi) = \nu_+\Phi(\psi) \tag{4.25}$$

$$\Phi(\tau_-\psi) = \tau_+\Phi(\psi) \tag{4.26}$$

$$\nabla^\Sigma \circ \Phi = \Phi \circ \nabla^\Sigma. \tag{4.27}$$

Remark 4.6. Identities (4.25) and (4.26) may initially cause alarm, as different spacetime vectors appear to act the same on the spacetime spinor under the identification Φ. In our presentation, we defined Φ as above for calculational convenience: it arises naturally from the Riemannian spin structure of $M_+ \cup M_-$, it is an isometry for the hermitian inner product on $S(M_\pm)$, and it commutes with the boundary Dirac operator. Later, we will see that "discontinuous" boundary conditions across Σ for ψ lead to cancellation of the boundary integrals. One concludes that Φ should not be viewed as identifying continuous spinors on the spacetime.

4.2.2 Lichnerowicz-Weitzenböck Formula

Recall the definition of spacetime connection on $S(M_\pm)$

$$\overline{\nabla}_i = \nabla_i + \frac{1}{2} k_{ij} e^j \tau \tag{4.28}$$

and corresponding Dirac-Witten operator

$$D_W = \sum_{i=1}^{n} e^i \overline{\nabla}_i. \tag{4.29}$$

The Lichnerowicz-Weitzenböck formula (Equation 11.13 of [CB03]) reads for any smooth spacetime spinor ψ on compact initial data (Ω, g, k),

$$\begin{aligned} &\int_\Omega |\overline{\nabla}\psi|^2 - |D_W \psi|^2 + \frac{1}{2}\langle \psi,\ (\mu + J\tau)\psi \rangle \, dV \\ &= \int_{\partial\Omega} \left\langle \psi,\ \mathcal{D}_\partial \psi + \frac{1}{2} H_{in} \psi + \frac{1}{2}\left[(\mathrm{Tr}_{\partial\Omega}\, k)\, \nu_{in} - k_{\nu_{in},\cdot} \right] \tau \psi \right\rangle dA \\ &= \int_{\partial\Omega} \left\langle \psi,\ \mathcal{D}_\partial \psi - \frac{1}{2} H_{out} \psi - \frac{1}{2}\left[(\mathrm{Tr}_{\partial\Omega}\, k)\, \nu_{out} - k_{\nu_{out},\cdot} \right] \tau \psi \right\rangle dA \end{aligned} \tag{4.30}$$

where H_{in} (H_{out}) is the inward (outward) mean curvature, ν_{in} (ν_{out}) is the inward (outward) pointing unit normal to $\partial\Omega$, and $\mathcal{D}_\partial$ denotes the *boundary Dirac operator*,

$$\mathcal{D}_\partial \psi = \nu_{out} e^\alpha \nabla^\Sigma_\alpha \psi \tag{4.31}$$

where ∇^Σ is the boundary spin connection arising from $(\partial\Omega, g)$, and e^α for $\alpha = 1, \ldots, n-1$ form an orthonormal basis for the boundary.

4.2.3 Constant Spinors at Infinity

Throughout the paper, let $q = \frac{n-2}{2}$. For an asymptotically flat initial data set M, the weighted space $W^{1,2}_{-q}(M)$ is the completion of $C^\infty_c(M)$ under the norm

$$\|\psi\|^2_{W^{1,2}_{-q}(M)} := \|\nabla\psi\|^2_{L^2(M)} + \|\psi/r\|^2_{L^2(M)}, \tag{4.32}$$

where r is a positive function such that $r = |x|$ in the asymptotically flat chart.

We denote the subset of M diffeomorphic to $\mathbb{R}^n \backslash B_R(0)$ in the asymptotically flat chart by M_R. Let $\{e_i\}$ be the Graham-Schmidt orthonormalization of the frame $\{\partial_i\}$ in M_R (where R is chosen large such that ∂_i is linearly independent over any point of M_R). Since the spin structure M_R is trivial, $\{e_i\}$ lifts to a spin frame $\{\psi_I\}$ for $S(M_R)$. We say ψ_∞ is a *constant spinor at infinity* if $\psi_\infty = \sum_I c_I\psi_I$ for constant functions c_I. We say a spinor ψ is asymptotic to ψ_∞ if $\psi - \psi_\infty \in W^{1,2}_{-q}(M_R)$.

By a standard computation, the boundary term limits to an ADM energy momentum term for a constant spinor ψ_∞:

$$\lim_{R\to\infty}\int_{\partial M_R}\left\langle \psi_\infty,\ \mathcal{D}_\partial\psi_\infty - \frac{1}{2}H_{out}\psi_\infty - \frac{1}{2}\left[(\mathrm{Tr}_{\partial\Omega}\, k)\,\nu_{out} - k_{\nu_{out},\cdot}\right]\tau\psi_\infty\right\rangle dA \tag{4.33}$$

$$= \frac{(n-1)\omega_{n-1}}{2}\left(E|\psi_\infty|^2 - \langle\psi_\infty, P\tau\psi_\infty\rangle\right). \tag{4.34}$$

4.3 Spinor Calculations

4.3.1 The Boundary Value Problem

With the hyperbolic angle of rotation f, let

$$a = \cosh(f), \quad b = \sinh(f) \tag{4.35}$$

$$A = \cosh(f/2) \quad B = \sinh(f/2). \tag{4.36}$$

Let $\epsilon = \nu_+\tau_+$, an isometry of $S(M_+)$ that satisfies

$$\epsilon^2 = 1, \qquad \epsilon\nu_+ = -\nu_+\epsilon = \tau_+, \qquad \epsilon\tau_+ = -\tau_+\epsilon, \qquad \epsilon\nabla^\Sigma = \nabla^\Sigma\epsilon. \tag{4.37}$$

The boundary value problem we would like to consider is

$$\begin{cases} D_W\psi_+ = 0 & \text{in } M_+ \\ D_W\psi_- = 0 & \text{in } M_- \\ (A - B\epsilon)\Phi(\psi_-) = \psi_+ & \text{on } \Sigma = \partial M_+ \\ \psi_+ \to \psi_\infty & \text{as } |x| \to \infty. \end{cases} \tag{4.38}$$

Remark 4.7. To explain the boundary condition, note that $(A - B\epsilon)(A + B\epsilon) = 1$ and so the condition is equivalent to

$$\Phi(\psi_-) = (A + B\epsilon)\psi_+. \tag{4.39}$$

In the picture where $M_\pm$ are embedded in a spacetime N, note that hyperbolic rotation by f on TN induces the"rotation" $(A + B\epsilon)$ on the associated spinor bundle. With this in mind, the boundary condition reads "the identification Φ of the $\pm$ spinor bundles at Σ takes ψ_- to the the rotation of ψ_+ by f," which is a convoluted way of saying that (ψ_-, ψ_+) come from restricting a continuous spinor on the spacetime.

4.3.2 Boundary Term Computation

Let $\mathcal{D}^\pm = \mp\nu_\pm\mathcal{D}$ denote the boundary Dirac operators where $\mathcal{D} = e^\alpha\nabla^\Sigma_\alpha$ is the standard Dirac operator of (Σ, g). The boundary contributions from the Lichnerowicz-Weitzenböck formula (4.30) applied to $\psi_\pm$ are (taking into account the direction of $\nu_\pm$)

$$\mathbb{I}_- := \int_{\partial M_-} \left\langle \psi_-,\, \mathcal{D}^-\psi_+ - \frac{1}{2}H_-\psi_- - \frac{1}{2}\left[(\operatorname{Tr}_\Sigma k_-)\,\nu_- - k^-_{\nu_-,\cdot}\right]\tau_-\psi_- \right\rangle dA, \tag{4.40}$$

$$\mathbb{I}_+ := \int_{\partial M_+} \left\langle \psi_+,\, \mathcal{D}^+\psi_+ + \frac{1}{2}H_+\psi_+ + \frac{1}{2}\left[(\operatorname{Tr}_\Sigma k_+)\,\nu_+ - k^+_{\nu_+,\cdot}\right]\tau_+\psi_+ \right\rangle dA. \tag{4.41}$$

We would like to show that $\mathbb{I}_- + \mathbb{I}_+$ has a favorable sign under appropriate boundary conditions for $\psi_\pm$ and the geometric situation of DEC-creased data.

Proposition 4.8. *For any spinors $\psi_\pm$ on $S(M_\pm)$, if $\Phi(\psi_-) = (A + B\epsilon)\psi_+$, then*

$$\mathbb{I}_- + \mathbb{I}_+ \tag{4.42}$$

$$= \frac{1}{2}\int_{\partial M_+} |\psi_+|^2 \left\langle \vec{H}_+ - \vec{H}_-, \nu_+ \right\rangle + \left\langle \psi_+, \left(\left\langle \vec{H}_+ - \vec{H}_-, \tau_+ \right\rangle \nu_+ - \alpha^\Delta \right) \tau_+ \psi_+ \right\rangle dA \tag{4.43}$$

$$\leqslant \frac{1}{2}\int_{\partial M_+} |\psi_+|^2 \left[\left\langle \vec{H}_+ - \vec{H}_-, \nu_+ \right\rangle + \sqrt{\left\langle \vec{H}_- - \vec{H}_+, \tau_+ \right\rangle^2 + |\alpha^\Delta|^2_\Sigma} \right] dA. \tag{4.44}$$

In particular, if $M_\pm$ is DEC-creased with hyperbolic angle f (4.10), then $\mathbb{I}_- + \mathbb{I}_+ \leqslant 0$.

Note that under the matching condition $\vec{H}_+ = \vec{H}_-$ and $\nabla^+ = \nabla^-$, then $\mathbb{I}_- + \mathbb{I}_+ = 0$.

Proof. Note that the inequality follows from Cauchy-Schwarz and the fact that Clifford multiplication with a vector X has norm $|X|$.

To show that the boundary terms simplify under the condition $\Phi(\psi_-) = (A + B\epsilon)\psi_+$, we will compute all the $(-)$ side data in terms of $(+)$ side information.

For the the boundary Dirac operator term, note that

$$\Phi(\mathcal{D}^-\psi_-) = \Phi(\nu_- e^\alpha \nabla^\Sigma_\alpha \psi_-) \tag{4.45}$$

$$= \nu_+ e^\alpha \nabla^\Sigma_\alpha \Phi(\psi_-) \tag{4.46}$$

$$= -\mathcal{D}^+ \Phi(\psi_-), \tag{4.47}$$

where we have used (4.24), (4.25), and (4.27). It follows that

$$\left\langle \psi_-, \mathcal{D}^-\psi_- \right\rangle = -\left\langle \Phi(\psi_-), \mathcal{D}^+\Phi(\psi_-) \right\rangle. \tag{4.48}$$

It will be helpful to note that $dA = \frac{1}{2}Bdf$ and $dB = \frac{1}{2}Adf$, which is just an elementary calculation. Also useful: $\langle \phi, (s + t\epsilon)\psi \rangle = \langle (s + t\epsilon)\phi, \psi \rangle$ for any s, t, which just requires the fact that $\epsilon^2 = 1$. Now we continue computing

$$\left\langle \Phi(\psi_-),\, \mathcal{D}^+\Phi(\psi_-) \right\rangle \tag{4.49}$$

$$= \left\langle [A+B\epsilon]\psi_+,\, \mathcal{D}^+[A+B\epsilon]\psi_+ \right\rangle \tag{4.50}$$

$$= -\left\langle [A+B\epsilon]\psi_+, \nu_+\mathcal{D}_\partial[A+B\epsilon]\psi_+ \right\rangle \tag{4.51}$$

$$= -\left\langle [A+B\epsilon]\psi_+,\, \nu_+[dA+dB\epsilon]\psi_+ \right\rangle - \left\langle [A+B\epsilon]\psi_+,\, [A-B\epsilon]\nu_+\mathcal{D}_\partial\psi_+ \right\rangle \tag{4.52}$$

$$= -\frac{1}{2}\left\langle [A+B\epsilon]\psi_+,\, \nu_+[B+A\epsilon]df\psi_+ \right\rangle + \left\langle \psi_+, \mathcal{D}^+\psi_+ \right\rangle \tag{4.53}$$

$$= -\frac{1}{2}\left\langle [A+B\epsilon]\psi_+,\, [B-A\epsilon]\nu_+df\psi_+ \right\rangle + \left\langle \psi_+,\, \mathcal{D}^+\psi_+ \right\rangle \tag{4.54}$$

$$= \frac{1}{2}\left\langle [A^2-B^2]\epsilon\psi_+,\, \nu_+df\psi_+ \right\rangle + \left\langle \psi_+,\, \mathcal{D}^+\psi_+ \right\rangle \tag{4.55}$$

$$= \frac{1}{2}\left\langle \psi_+,\, \epsilon\nu_+df\psi_+ \right\rangle + \left\langle \psi_+,\, \mathcal{D}^+\psi_+ \right\rangle \tag{4.56}$$

$$= \frac{1}{2}\left\langle \psi_+,\, \tau_+df\psi_+ \right\rangle + \left\langle \psi_+,\, \mathcal{D}^+\psi_+ \right\rangle. \tag{4.57}$$

It follows that

$$\left\langle \psi_-,\, \mathcal{D}^-\psi_- \right\rangle + \left\langle \psi_+,\, \mathcal{D}^+\psi_+ \right\rangle = \frac{1}{2}\left\langle \psi_+,\, df\tau_+\psi_+ \right\rangle. \tag{4.58}$$

Next, let's look at the boundary term involving the connection 1-form:

$$\left\langle \psi_-,\, k^-_{\nu_-,\cdot}\tau_-\psi_- \right\rangle = \left\langle \Phi(\psi_-),\, \Phi\left(k^-_{\nu_-,\cdot}\tau_-\psi_-\right) \right\rangle \tag{4.59}$$

$$= \left\langle \Phi(\psi_-),\, k^-_{\nu_-,\cdot}\tau_+\Phi(\psi_-) \right\rangle \tag{4.60}$$

$$= \left\langle [A+B\epsilon]\psi_+,\, k^-_{\nu_-,\cdot}\tau_+[A+B\epsilon]\psi_+ \right\rangle \tag{4.61}$$

$$= \left\langle [A+B\epsilon]\psi_+,\, [A-B\epsilon]k^-_{\nu_-,\cdot}\tau_+\psi_+ \right\rangle \tag{4.62}$$

$$= \left\langle \psi_+,\, k^-_{\nu_-,\cdot}\tau_+\psi_+ \right\rangle \tag{4.63}$$

where we've used the fact that τ_+ anti-commutes with ϵ. It follows that

$$\frac{1}{2}\left\langle \psi_-,\, k^-_{\nu_-,\cdot}\tau_-\psi_- \right\rangle - \frac{1}{2}\left\langle \psi_+,\, k^+_{\nu_+,\cdot}\tau_+\psi_+ \right\rangle \tag{4.64}$$

$$= \frac{1}{2}\left\langle \psi_+,\, \left(k^-_{\nu_-,\cdot} - k^+_{\nu_+,\cdot}\right)\tau_+\psi_+ \right\rangle \tag{4.65}$$

$$= -\frac{1}{2}\left\langle \psi_+,\, \alpha^\Delta\tau_+\psi_+ \right\rangle - \frac{1}{2}\left\langle \psi_+,\, df\tau_+\psi_+ \right\rangle, \tag{4.66}$$

where we have used (4.7).

Now we should compute the terms involving the mean curvature of $\partial M_\pm$

$$-\left\langle \psi_-,\, \frac{1}{2}H_-\psi_- + \frac{1}{2}\left(\mathrm{Tr}_\Sigma\, k_-\right)\nu_-\tau_-\psi_- \right\rangle \tag{4.67}$$

$$= -\frac{1}{2}\langle \Phi(\psi_-),\, \Phi((H_- + \left(\mathrm{Tr}_\Sigma\, k_-\right)\nu_-\tau_-)\psi_-)\rangle \tag{4.68}$$

$$= -\frac{1}{2}\langle \Phi(\psi_-),\, (H_- + \left(\mathrm{Tr}_\Sigma\, k_-\right)\nu_+\tau_+)\Phi(\psi_-)\rangle \tag{4.69}$$

$$= -\frac{1}{2}\langle [A+B\epsilon]\psi_+,\, (H_- + \left(\mathrm{Tr}_\Sigma\, k_-\right)\epsilon)[A+B\epsilon]\psi_+\rangle \tag{4.70}$$

$$= -\frac{1}{2}\langle [A+B\epsilon]\psi_+,\, [A+B\epsilon](H_- + \left(\mathrm{Tr}_\Sigma\, k_-\right)\epsilon)\psi_+\rangle \tag{4.71}$$

$$= -\frac{1}{2}\left\langle \psi_+,\, [A+B\epsilon]^2(H_- + \left(\mathrm{Tr}_\Sigma\, k_-\right)\epsilon)\psi_+\right\rangle \tag{4.72}$$

$$= -\frac{1}{2}\langle \psi_+,\, [a+b\epsilon](H_- + \left(\mathrm{Tr}_\Sigma\, k_-\right)\epsilon)\psi_+\rangle \tag{4.73}$$

$$= -\frac{1}{2}\langle \psi_+,\, \left(aH_- + b\,\mathrm{Tr}_\Sigma\, k_- + \left(bH_- + a\,\mathrm{Tr}_\Sigma\, k_-\right)\epsilon\right)\psi_+\rangle \tag{4.74}$$

where we've used the fact that

$$[A+B\epsilon]^2 = (A^2+B^2) + 2AB\epsilon = a + b\epsilon$$

(i.e. rotating by $f/2$ twice is the same as rotating by f once). It follows that

$$-\left\langle \psi_-,\ \frac{1}{2}\left(H_- + (\mathrm{Tr}_\Sigma\, k_-)\,\nu_-\tau_-\right)\psi_- \right\rangle + \left\langle \psi_+,\ \frac{1}{2}\left(H_+ + (\mathrm{Tr}_\Sigma\, k_+)\,\nu_+\tau_+\right)\psi_+ \right\rangle \tag{4.75}$$

$$= \frac{1}{2}|\psi_+|^2\left[H_+ - (aH_- + b\,(\mathrm{Tr}_\Sigma\, k_-))\right] + \frac{1}{2}\langle\psi_+, \epsilon\psi_+\rangle\left[\mathrm{Tr}_\Sigma\, k_+ - (bH_- + a\,(\mathrm{Tr}_\Sigma\, k_-))\right] \tag{4.76}$$

$$= \frac{1}{2}|\psi_+|^2\left\langle \vec{H}_+ - \vec{H}_-, \nu_+\right\rangle + \frac{1}{2}\langle\psi_+,\ \epsilon\psi_+\rangle\left\langle \vec{H}_+ - \vec{H}_-, \tau_+\right\rangle \tag{4.77}$$

$$= \frac{1}{2}|\psi_+|^2\left\langle \vec{H}_+ - \vec{H}_-, \nu_+\right\rangle + \frac{1}{2}\left\langle \psi_+,\ \left\langle \vec{H}_+ - \vec{H}_-, \tau_+\right\rangle \nu_+\tau_+\psi_+\right\rangle. \tag{4.78}$$

Finally, summing (4.58), (4.66), and (4.78) yields the desired result.

□

4.4 Existence of the Required Spinor

In this section we solve for a Dirac-Witten-harmonic spinor satisfying the boundary conditions. Weighted spaces and associated Poincaré inequalities play an important role in the existence theory of Dirac spinors in the asymptotically flat setting [PT82], [CB03]. For convenience in defining our function spaces, define $M = M_+ \sqcup M_-$, as the *disjoint* union of M_+ and M_-, so that $\partial M = \partial M_+ \sqcup \partial M_-$. We denote the the spacetime connection by $\overline{\nabla}_i := \nabla_i + \frac{1}{2}k_i^j e_j\tau$.

To incorporate the boundary conditions, define the Hilbert space

$$H := \left\{\psi \in W^{1,2}_{-q}(M) \mid \psi|_{\partial M_+} = (A + \epsilon B)\Phi(\psi|_{\partial M_-}) \text{ on } \partial M_+\right\}. \tag{4.79}$$

Since the trace is continuous on $W^{1,2}_{-q}(M)$, H is indeed is a closed subset of $W^{1,2}_{-q}(M)$. Utilizing standard smooth approximation theorems yields the following approximation lemma:

Lemma 4.9. *$C^\infty_c(M) \cap H$ is dense in H.*

Proof. Let $\psi \in H$, and let ψ' be the $C^\infty_c(M)$-approximation of ψ that is close in $W^{1,2}_{-q}(M)$. The boundary values of ψ' may not satisfy the boundary conditions, so ψ' may not lie in H. Let $\mathcal{E} : H^{\frac{1}{2}}(\partial M) \to W^{1,2}_{-q}(M)$ be a bounded right inverse to the trace operator $\mathcal{T} : W^{1,2}_{-q}(M) \to H^{\frac{1}{2}}(\partial M)$; $\mathcal{E}$ can be constructed to send smooth spinors on ∂M to smooth spinors on M with compact support. Define $K : H^{\frac{1}{2}}(\partial M) \to H^{\frac{1}{2}}(\partial M)$ by

$$(\varphi_+, \varphi_-) \mapsto \frac{1}{2}\left(\varphi_+ - (A + \epsilon B)\Phi(\varphi_-), \varphi_- - (A - \epsilon B)\Phi^{-1}(\varphi_+)\right), \tag{4.80}$$

where $\psi_\pm$ are the respective restrictions of a spinor on ∂M to $\partial M_\pm$. Note that $K\mathcal{T}\varphi = 0$ if and only if $\varphi \in H$, and that $K^2 = K$.

Let $\psi'' = \psi' - \mathcal{E}K\mathcal{T}\psi'$, which is smooth, compactly supported, and constructed to be contained in H, as

$$K\mathcal{T}\psi'' = K\mathcal{T}\psi' - K^2\mathcal{T}\psi' = 0. \tag{4.81}$$

Since $\psi \in H$, we have $\mathcal{E}K\mathcal{T}\psi = 0$, so

$$\|\mathcal{E}K\mathcal{T}\psi'\|_{W^{1,2}_{-q}(M)} = \|\mathcal{E}K\mathcal{T}(\psi' - \psi)\|_{W^{1,2}_{-q}(M)} \leqslant C\|\psi' - \psi\|_{W^{1,2}_{-q}(M)}, \tag{4.82}$$

and thus

$$\|\psi - \psi''\|_{W^{1,2}_{-q}(M)} \leqslant (1 + C)\|\psi' - \psi\|_{W^{1,2}_{-q}(M)}. \tag{4.83}$$

□

The argument in this section is as follows. We first state the mass formula for spinors of regularity $W^{1,2}_{-q}(M)$. On each piece M_+ and M_-, we obtain Poincaré inequalities. This yields a Poincaré-type estimate for functions $\psi \in H$ using the L^2-norm of $\overline{\nabla}\psi$. Then we use the mass formula to show that $\mathcal{D} : H \to L^2(M)$ is an isomorphism; finally we solve $\mathcal{D}\psi = -\mathcal{D}\psi_0$ for a smooth spinor ψ_0 constant at infinity.

Proposition 4.10. *Let ψ_0 be a smooth spinor on M that is is equal to a constant spinor ψ_∞ over M_{2R}, and satisfying $\psi_0 \equiv 0$ on $M \backslash M_R$. Any spinor ψ such that $\psi - \psi_0 \in H$ satisfies:*

$$\begin{aligned} &\int_M |\overline{\nabla}\psi|^2 - |D_W\psi|^2 + \frac{1}{2}\langle \psi, (\mu + J\tau)\psi\rangle\, dV \\ &= \int_{\partial M_+} \langle \psi, \mathcal{B}\psi\rangle\, d\mu_{\partial M_+} + \frac{(n-1)\omega_{n-1}}{2}\left(E|\psi_\infty|^2 - \langle \psi_\infty, P\tau\psi_\infty\rangle\right), \end{aligned} \tag{4.84}$$

where

$$\mathcal{B} = \left\langle \vec{H}_+ - \vec{H}_-, \nu_+ \right\rangle + \left(\left\langle \vec{H}_+ - \vec{H}_-, \tau_+ \right\rangle \nu_+ - \alpha^\Delta\right)\tau_+. \tag{4.85}$$

Proof. If $\psi - \psi_0 \in H \cap C_c^\infty(M)$, then ψ is a smooth spinor satisfying the boundary conditions and equal to ψ_∞ outside a compact set. By Proposition 4.8, as well as a standard computation for the ADM energy momentum limit, the formula holds for such ψ. The first two terms of the LHS of (4.84) are continuous on $\{H + \psi_0\}$ in the $W^{1,2}_{-q}(M)$-topology due to the $O(|x|^{-q-1})$ fall off of k. For the third term on the LHS, $\mu + J\tau = O(|x|^{-q-2}) = O(|x|^{-n/2-1}) = O(|x|^{-2})$ (as $n \geqslant 3$), so this term is continuous as well. Finally the term $\int_{\partial M_+} \langle \psi, \mathcal{B}\psi \rangle \, d\mu_{\partial M_+}$ is a continuous functional on $\{H + \psi_0\}$ by the trace theorem. Noting that $H \cap C_c^\infty(M)$ is dense in H by 4.9, the mass formula holds for all spinors in $\{H + \psi_0\}$ by continuity. □

Lemma 4.11. *We have the following Poincaré-type inequalities:*

1. *For* $\psi \in W^{1,2}_{-q}(M_+)$,

$$\|\psi\|^2_{W^{1,2}_{-q}(M_+)} \leqslant C \|\overline{\nabla}\psi\|^2_{L^2(M_+)}, \tag{4.86}$$

and,

2. *for* $\psi \in W^{1,2}(M_-)$,

$$\|\psi\|^2_{W^{1,2}(M_-)} \leqslant C \left(\|\overline{\nabla}\psi\|^2_{L^2(M_-)} + \|\psi\|^2_{L^2(\partial M_-)} \right). \tag{4.87}$$

Proof. Since M_+ is connected and contains an asymptotically flat end, Theorem 9.5 of [CB03] implies there exists C such that the weighted Poincaré inequality

$$\|\psi/r\|^2_{L^2(M_+)} \leqslant C \|\overline{\nabla}\psi\|^2_{L^2(M_+)}, \tag{4.88}$$

holds. Now

$$\|\psi\|_{W^{1,2}_{-q}(M_+)} \leqslant C(\|\nabla\psi\|_{L^2(M_+)} + \|\psi/r\|_{L^2(M_+)}) \tag{4.89}$$

$$\leqslant C(\|\overline{\nabla}\psi\|_{L^2(M_+)} + \||k|\psi\|_{L^2(M_+)} + \|\psi/r\|_{L^2(M_+)}) \tag{4.90}$$

$$\leqslant C\left(\|\overline{\nabla}\psi\|_{L^2(M_+)} + \left(\sup_{M_+} r|k| + 1\right)\|\psi/r\|_{L^2(M_+)}\right) \tag{4.91}$$

$$\leqslant C\|\overline{\nabla}\psi\|_{L^2(M_+)} \tag{4.92}$$

with the last inequality due to the $O(r^{-\frac{n}{2}})$ asymptotic decay of k and (4.88).

For part (2) of the lemma, we first demonstrate the following Poincaré-type inequality:

$$\|\psi\|^2_{L^2(M_-)} \leqslant C\left(\|\overline{\nabla}\psi\|^2_{L^2(M_-)} + \|\psi\|^2_{L^2(\partial M_-)}\right). \tag{4.93}$$

Suppose the inequality does not hold, then there exists a sequence of spinors ψ_i such that $\|\psi_i\|_{L^2(M_-)} = 1$ and $\|\overline{\nabla}\psi_i\|^2_{L^2(M_-)} + \|\psi_i\|^2_{L^2(\partial M_-)} \to 0$. By the triangle inequality, (for a constant C depending on k)

$$\|\psi_i\|_{W^{1,2}(M_-)} \leqslant C\left(\|\overline{\nabla}\psi_i\|_{L^2(M_-)} + \|\psi_i\|_{L^2(M_-)}\right), \tag{4.94}$$

hence the sequence is bounded in $W^{1,2}$. By Rellich's theorem, after passing to a subsequence, ψ_i strongly converges in $L^2(M_-)$ and weakly converges in $W^{1,2}(M_-)$. Let $\psi = \lim_{i\to\infty}\psi_i$. We have $\overline{\nabla}\psi = 0$ and $\psi|_{\partial M_-} = 0$ by continuity under weak convergence and assumption on ψ_i. Then $\psi = 0$, contradicting $\|\psi\|_{L^2(\partial M_-)} = 1$. More specifically, ψ satisfies

$$\nabla_i\psi = -\frac{1}{2}k_i^j e_j n\psi, \tag{4.95}$$

hence if $\psi \in W^{k,2}$, then $\nabla\psi \in W^{k,2}$ so $\psi \in W^{k+1,2}$. Since $\psi \in W^{1,2}$, $\psi \in W^{k,2}$ for all $k \geqslant 1$, and by Sobolev embedding, ψ is smooth. Integrating along curves connecting interior points to the boundary where $\psi = 0$, $\psi = 0$ everywhere by the uniqueness of solutions to first order ODE.

Combining (4.93) and (4.94) yields the second inequality. □

Now we glue together the two Poincaré inequalities using the boundary condition, yielding a global weighted Poincaré inequality for spinors in H.

Proposition 4.12. *The inner product $\langle f, g\rangle = \int_M \langle\overline{\nabla}f, \overline{\nabla}g\rangle$ is equivalent to the $W^{1,2}_{-q}(M)$ inner product on H. In particular, $\|\psi\|_{W^{1,2}_{-q}(M)} \leqslant C\|\overline{\nabla}\psi\|_{L^2(M)}$.*

Proof. Noting $(A + \epsilon B)\circ\Phi$ is a bounded invertible map on $L^2(\partial M_-) \to L^2(\partial M_+)$, we obtain for $\psi \in H$,

$$\|\psi\|^2_{W^{1,2}(M_-)} \leqslant C\left(\|\overline{\nabla}\psi\|^2_{L^2(M_-)} + \|\psi\|^2_{L^2(\partial M_+)}\right) \tag{4.96}$$

$$\leqslant C\left(\|\overline{\nabla}\psi\|^2_{L^2(M_-)} + \|\psi\|^2_{W^{1,2}_{-q}(M_+)}\right) \tag{4.97}$$

$$\leqslant C\left(\|\overline{\nabla}\psi\|^2_{L^2(M_-)} + \|\overline{\nabla}\psi\|^2_{L^2(M_+)}\right), \tag{4.98}$$

where we used the two inequalities of Lemma 4.11 and the continuity of trace. Applying (1) of the Lemma again, we obtain for $\psi \in H$:

$$\|\psi\|^2_{W^{1,2}_{-q}(M)} \leqslant C\|\overline{\nabla}\psi\|^2_{L^2(M)}. \tag{4.99}$$

The reverse inequality holds by the decay of k. □

Theorem 4.13. *Assume the dominant energy condition holds on each component of M and M is DEC-creased with hyperbolic angle f. Then $D_W : H \to L^2(M)$ is an isomorphism.*

Proof. Due to the $O\left(|x|^{-q-1}\right)$ fall-off of k note that D_W is a bounded linear operator from H to $L^2(M)$. We first show injectivity, then show surjectivity. Applying the mass formula of Proposition 4.10 to a spinor $\psi \in H$ (i.e. $\psi_0 = 0$), the mass term vanishes and we have

$$\|\overline{\nabla}\psi\|^2_{L^2(M)} - \|D_W\psi\|^2_{L^2(M)} \leqslant \int_{\partial M_+} \langle \psi, \mathcal{B}\psi\rangle \, d\mu_{\partial M_+} \leqslant 0, \tag{4.100}$$

by the DEC-creased condition and dominant energy condition. Combining this with the weighted Poincaré inequality of Proposition 4.12, we obtain

$$\|\psi\|^2_{W^{1,2}_{-q}} \leqslant C\|\overline{\nabla}\psi\|^2_{L^2(M)} \leqslant C\|D_W\psi\|^2_{L^2(M)}. \tag{4.101}$$

This shows injectivity of D_W.

Now for surjectivity. Let $\eta \in L^2(M)$. As D_W maps H to $L^2(M)$, the mapping $\cdot \mapsto \langle \eta, D_W \cdot \rangle$ is a bounded linear functional on H. By (4.101), $\langle D_W \cdot, D_W \cdot \rangle$ is equivalent to our inner product on H. By the Riesz representation theorem, there exists $\omega \in H$ such that

$$\langle D_W\omega, D_W\psi\rangle = \langle \eta, D_W\psi\rangle, \tag{4.102}$$

for every ψ in H. Substituting $\varphi = D_W\omega - \eta \in L^2(M)$, we have

$$\langle \varphi, D_W\psi \rangle = 0, \tag{4.103}$$

for all $\psi \in H$. In particular, $D_W\varphi = 0$ weakly, but we do not yet know its regularity at the boundary. Proposition 4.14 below implies $\varphi \in W^{1,2}_{loc}(M)$, where the subscript *loc* means $W^{1,2}$ on compact subsets containing the boundary, and satisfies the boundary conditions that define H.

To show $-q$-weighted Sobolev space decay of φ, we argue via approximation. Let χ_k be a nondecreasing sequence of cut-off functions, such that for large k, $\chi_k \equiv 1$ on $M \backslash M_{2^k}$ and $\chi_k = 0$ on $M_{2^{k+1}}$ and $|\nabla \chi_k| \leqslant C2^{-k}$. We have

$$D_W(\chi_k \varphi) = \chi_k D_W(\varphi) + (\nabla \chi_k)\varphi = (\nabla \chi_k)\varphi, \tag{4.104}$$

where $(\nabla \chi_k)\varphi$ is understood to be Clifford multiplication of $\nabla \chi_k$ on φ. Using (4.101),

$$\|\chi_k\varphi - \chi_{k+1}\varphi\|_{W^{1,2}_{-q}(M)} \leqslant C\|D_W(\chi_k\varphi - \chi_{k+1})\varphi\|_{L^2}(M) \tag{4.105}$$

$$\leqslant C\|\nabla(\chi_k - \chi_{k+1})\varphi\|_{L^2} \tag{4.106}$$

$$\leqslant C2^{-k}\|\varphi\|_{L^2(M)}, \tag{4.107}$$

so $\chi_k\varphi$ converges to φ in $W^{1,2}_{-q}(M)$. Therefore φ is in H and a solution to $D_W\varphi = 0$. Therefore $\varphi = 0$ by injectivity of D_W, so $D_W\omega = \eta$.

□

We now arrive at the main existence theorem.

Proof of Theorem 4.4. Fix any constant spinor ψ_∞ on M_R, and fix a smooth spinor ψ_0 with $\psi_0 \equiv 0$ on $M \backslash M_R$ and $\psi_0 = \psi_\infty$ on M_{2R}.

We have that $-D_W(\psi_0) = O\left(|\overline{\nabla}\psi_0|\right)$. Since ψ_0 is constant but not covariantly so, $-D_W(\psi_0)$ decays like the connection coefficients of g_+ and like k_+, which is $O(|x|^{-q-1-\epsilon}) = O(|x|^{-n/2-\epsilon})$. So $-D_W(\psi_0) \in L^2(M) \cap C^\infty(M)$. By Theorem 4.13, we obtain a solution $\omega \in H$ to

$$D_W\omega = -D_W\psi_0, \tag{4.108}$$

and $\omega \in C^{\infty}(M)$ by Proposition 4.15. Then $\psi = \omega + \psi_0$ is the desired spinor in the theorem. □

The existence of a Dirac-Witten harmonic spinor satisfying the boundary conditions yields the positive mass formula:

Proof. 4.5 Let ψ be the spinor given Theorem 4.4 satisfying $D_W\psi = 0$. Apply Proposition 4.10 to ψ and note that $\mathcal{B} \leqslant 0$ by the DEC-creased condition. □

4.5 Ellipticity of the Boundary Conditions

In [BB11], elliptic boundary conditions are defined for first-order operators and, in particular, Dirac-type operators, which we review here. Let S be the spacetime spinor bundle on a manifold M with compact boundary. Let $D_W : S \to S$ be the hypersurface Dirac operator, which we note is formally self-adjoint. Let $D_{\max}$ be the extension of D_W to $\mathrm{dom}(D_{\max}) \subset L^2(M)$, defined by the following: $\varphi \in \mathrm{dom}(D_{\max})$ whenever there exists $\eta \in L^2(M)$ such that

$$\langle \varphi, D_W\psi \rangle = \langle \eta, \psi \rangle \tag{4.109}$$

for all smooth spinors ψ compactly supported in the interior of M. In this case, we define $D_{\max}\varphi = \eta$; so φ is an L^2-weak solution to $D_W\varphi = \eta$, with no boundary conditions imposed. Note that $\mathrm{dom}(D_{\max})$ is complete with the graph norm:

$$\|\varphi\|_{\mathrm{dom}(D_{\max})} = \|\varphi\|_{L^2(M)} + \|D_{\max}\varphi\|_{L^2(M)}. \tag{4.110}$$

Standard elliptic regularity implies φ is H^1 in the interior, but we need to impose boundary conditions on φ to obtain regularity up to ∂M.

We say that a first-order operator A on $S|_{\partial M}$ is an *adapted operator* if the principal symbol σ_A of A satisfies

$$\sigma_A(\xi, x) = \nu\sigma_{D_W}(\xi, x) \tag{4.111}$$

for all $x \in \partial M$, $\xi \in T^*_x(\partial M)$, where ν is Clifford multiplication by the outward unit normal to ∂M. In particular, the boundary Dirac operator $\mathcal{D}_\partial = \sum_i \nu e_i \nabla^\partial_i$ is an adapted operator.

Now let us return to the setting where $M = M_+ \sqcup M_-$. We have that $\partial M = \partial M_+ \sqcup \partial M_-$ and $\Phi : S|_{\partial M_-} \to S|_{\partial M_+}$ induces an isomorphism $H^s(\partial M_-) \to H^s(\partial M_+)$ for all $s \in \mathbb{R}$. Let $H^s_{\geqslant 0}(\mathcal{D}_\partial)$ and $H^s_{<0}(\mathcal{D}_\partial)$ be subspaces of $H^s(\partial M)$ spanned by eigenspaces of $\mathcal{D}_\partial$ of non-negative and negative eigenvalues respectively.

Define the hybrid Sobolev space

$$\check{H}(D_\partial) := H^{1/2}_{<0}(\mathcal{D}_\partial) \oplus H^{-1/2}_{\geqslant 0}(\mathcal{D}_\partial). \tag{4.112}$$

As M is a complete Riemannian manifold with boundary, Theorem 6.7 of [BB11] asserts that the trace map uniquely extends to a surjective bounded linear map $\mathcal{T} : \mathrm{dom}(D_{\max}) \to \check{H}(\mathcal{D}_\partial)$, and for $\varphi \in \mathrm{dom}(D_{\max})$, $\varphi \in H^1_{loc}(M)$ if and only if $\mathcal{T}(\varphi) \in H^{1/2}(\partial M)$. Moreover we have the integration-by-parts formula,

$$\langle D_{\max}\varphi, \psi\rangle_{L^2(M)} - \langle \varphi, D_{\max}\psi\rangle_{L^2(M)} = \langle \nu\mathcal{T}\varphi, \mathcal{T}\psi\rangle_{L^2(\partial M)}. \tag{4.113}$$

The paring in the RHS is well-defined as Clifford multiplication by ν swaps the positive eigenspaces with the negative eigenspaces of $\mathcal{D}_\partial$.

We have the following Proposition on elliptic regularity of weak solutions satisfying the boundary condition in a weak way:

Proposition 4.14. *If $\varphi, \eta \in L^2(M)$ such that*

$$\langle \varphi, D_W\psi\rangle = \langle \eta, \psi\rangle \tag{4.114}$$

for all $\psi \in H$, then $\varphi \in H^1_{loc}(M)$ and φ satisfies the boundary conditions defining H.

Proof. Since, in particular, H contains all compactly supported spinors in the interior of M, $D_{\max}\varphi = \eta$. Using (4.113), for all $\psi \in H$, we have that

$$0 = \langle \nu\mathcal{T}\varphi, \mathcal{T}\psi\rangle_{L^2(\partial M)} \tag{4.115}$$

$$= \langle \nu_-\mathcal{T}\varphi, \mathcal{T}\psi\rangle_{L^2(\partial M_-)} - \langle \nu_+\mathcal{T}\varphi, \mathcal{T}\psi\rangle_{L^2(\partial M_+)} \tag{4.116}$$

$$= \langle \nu_+\Phi(\mathcal{T}\varphi), \Phi(\mathcal{T}\psi)\rangle_{L^2(\partial M_+)} - \langle \nu_+\mathcal{T}\varphi, \mathcal{T}\psi\rangle_{L^2(\partial M_+)} \tag{4.117}$$

$$= \langle \nu_+\Phi(\mathcal{T}\varphi), (A - \epsilon B)\mathcal{T}\psi)\rangle_{L^2(\partial M_+)} - \langle \nu_+\mathcal{T}\varphi, \mathcal{T}\psi\rangle_{L^2(\partial M_+)} \tag{4.118}$$

$$= \langle \nu_+[(A + \epsilon B)\Phi(\mathcal{T}\varphi) - \mathcal{T}\varphi], \mathcal{T}\psi)\rangle_{L^2(\partial M_+)}. \tag{4.119}$$

Since $\mathcal{T}\psi$ is an arbitrary function in $H^{1/2}(\partial M_+)$,

$$(A + \epsilon B)\Phi(\mathcal{T}\varphi) = \mathcal{T}\varphi \tag{4.120}$$

on ∂M_+ (as distributions).

We know that $\mathcal{T}\varphi$ is in $\check{H}(\mathcal{D}_\partial)$ by (ii) of Theorem 6.7 of [BB11]. It remains to show that $\mathcal{T}\varphi$ is in $H^{\frac{1}{2}}(\partial M)$, which will imply $\varphi \in H^{1/2}_{loc}(M)$ by (iii) of the same theorem, and (4.120) shows that φ also satisfies the boundary conditions pointwise.

We revisit the proof of Theorem 7.20 in [BB11], which equates well-studied pseudolocal elliptic boundary conditions to the notion of elliptic boundary conditions defined in the paper. Although the current boundary conditions are a form of *transmission conditions*, making them neither local nor pseudolocal, they can be recast as pointwise boundary conditions in a constructed bundle over the boundary.

The space of sections on the boundary is $H^s(\partial M, S) \cong H^s(\partial M_+, S) \oplus H^s(\partial M_-, S)$, and we define an isometry with $H^s(\partial M_+, S \oplus S)$ via

$$(\varphi_+, \varphi_-) \mapsto (\varphi_+, \Phi(\varphi_-)). \tag{4.121}$$

The boundary Dirac operator on $H^s(\partial M_+, S) \oplus H^s(\partial M_-, S)$ is given by

$$\mathcal{D}_\partial = (\mathcal{D}_+, \mathcal{D}_-). \tag{4.122}$$

We also have the identity from (4.45):

$$(\mathcal{D}_+\varphi_+, \Phi(\mathcal{D}_-\varphi_-)) = (\mathcal{D}_+\varphi_+, -\mathcal{D}_+\Phi(\varphi_-)), \tag{4.123}$$

which implies that $\mathcal{D}_\partial$ acts as $(\mathcal{D}_+, -\mathcal{D}_-)$ on $H^s(\partial M_+, S \oplus S)$.

From now on, we work on $H^s(\partial M_+, S \oplus S)$. Define $K : H^s(\partial M_+, S \oplus S) \to H^s(\partial M_+, S \oplus S)$ via

$$K(\varphi_+, \varphi_-) = \frac{1}{2}(\varphi_+ - (A + \epsilon B)\varphi_-, \varphi_- - (A - \epsilon B)\varphi_+), \tag{4.124}$$

whose kernel defines the boundary conditions. Let $Q_{<0}$ be the L^2-projection onto the negative eigenspace of $(\mathcal{D}_+, -\mathcal{D}_+)$. We check that $K - Q_{<0}$ is an elliptic pseudo-differential operator of order 0. It is known that the principal symbol $\sigma_{Q_{<0}}(\xi)$ of $Q_{<0}$ is orthogonal projection on the the negative eigenspace of $i(\sigma_{\mathcal{D}_+}(\xi), \sigma_{-\mathcal{D}_+}(\xi)) = i(-\nu_+\xi, \nu_+\xi)$, see proof of Theorem 7.20 of [BB11]. WLOG assume $|\xi| = 1$. It's easy to see that $i\nu_+\xi$ has eigenvalues ± 1, with eigenspaces of equal dimension. Hence

$$\sigma_{Q_{<0}}(\xi) = \frac{1}{2}\left(1 - i\nu_+\xi, 1 + i\nu_+\xi\right). \tag{4.125}$$

We check that the principal symbol of $K - Q_{<0}$, $\sigma_K(\xi) - \sigma_Q(\xi)$, is injective hence is an isomorphism. Suppose $(\sigma_K(\xi) - \sigma_Q(\xi))(\varphi_+, \varphi_-) = 0$. Then we have

$$(A + \epsilon B)\varphi_- = i\nu_+\xi\varphi_+ \tag{4.126}$$

$$(A - \epsilon B)\varphi_+ = -i\nu_+\xi\varphi_-. \tag{4.127}$$

Solving for φ_+ in the second equation, then inserting into the first equation, we have

$$(A + \epsilon B)\varphi_- = i\nu_+\xi(A + \epsilon B)(-i\nu_+\xi)\varphi_-, \tag{4.128}$$

so,

$$(a + \epsilon b)\varphi_- = -\varphi_-; \tag{4.129}$$

$$(a + 1)\varphi_- = -\epsilon b\varphi_-. \tag{4.130}$$

Since Clifford multiplication by ϵ is an isometry, we have $|\cosh(f) + 1| = |a + 1| = |b| = |\sinh(f)|$, which is never true, unless $\varphi_- = 0$. So $\varphi_- = 0$ and $\varphi_+ = 0$. This verifies that $\sigma_{K-Q}(\xi)$ is an isomorphism for $\xi \neq 0$, hence $K - Q$ is an elliptic pseudo-differential operator. and there is a parametrix R, a pseudo-differential operator of order 0, such that $R(K - Q) = I + S$ where S is a smoothing operator. We have $K\varphi = 0$, so

$$\varphi + S\varphi = R(K - Q_{<0})\varphi = RQ_{<0}\varphi. \tag{4.131}$$

Furthermore, observe that $\varphi \in \check{H}(\mathcal{D}_\partial)$ implies $Q_{<0}\varphi \in H^{1/2}(\partial M_1, S \oplus S)$, and therefore $RQ_{<0} \in H^{1/2}(\partial M_1, S \oplus S)$.

Since S is a smoothing operator, $S\varphi \in C^\infty(\partial M_1, S \oplus S)$, hence (4.131) implies that $\varphi \in H^{1/2}(\partial M_1, S \oplus S)$ and $\varphi \in H^{1/2}(\partial M, S)$. □

We have the following proposition about higher elliptic regularity:

Proposition 4.15. *If $\eta \in H^s_{loc}(M)$ and $\omega \in H$ satisfies*

$$D_W \omega = \eta, \tag{4.132}$$

then $\omega \in H^{s+1}_{loc}(M)$.

Proof. Essentially Proposition 4.14 shows that the boundary conditions defining H are self-adjoint and *elliptic* in the sense of [BB11], that is, $\mathcal{T}(H) = \mathcal{T}(H)^{ad} \subset H^{\frac{1}{2}}(\partial M)$. Our desired proposition is given by Theorem 7.17 of [BB11], provided that $\mathcal{T}(H)$ satisfies a property called ∞-*regular* (Definition 7.15 of [BB11]). Examining the proof of Proposition 7.24 of [BB11], the ∞-regular condition that reduces to ellipticity of the pseudodifferential operator $K - Q_{<0}$, which was shown in the the proof of Proposition 4.14. $\square$

Remark 4.16. We note that locally, the spinor PDE system can be locally translated to a boundary value problem on a half ball, and once can verify that the equations satisfy Agmon-Douglis-Nirenberg complementing conditions/are of Shapiro-Lopatinski type, and one gets elliptic estimates at the boundary and the Fredholm property. This gives an alternative route to proving the above regularity results.

Chapter 5. Conclusion

The main results in this thesis are

- the collar construction Theorem 2.3,
- Theorem 3.3, and the reduction of Penrose inequality Corollary 3.4,
- Bartnik mass estimate Theorem 3.7,
- and the creased positive mass theorem Theorem 4.5.

Future work remains in studying the implications of the reduction of the Penrose inequality, applying the Bartnik mass estimate to Kerr horizons and its ramifications for the Bartnik stationary conjecture, and the case of rigidity for creased positive mass.

Appendix A. MOTS Collar Calculations

A.1 Smooth Dependence of Eigenfunctions for a Family of Second Order Elliptic Operators on a Compact Manifold

We present the proof of Lemma 2.8.

Proof of Lemma 2.8. We will use the implicit function theorem for Banach spaces. Consider the map $F : H^2(M) \times \mathbb{R} \times (-\epsilon, \epsilon) \to L^2(M) \times \mathbb{R}$ defined by

$$F(\varphi, \lambda, t) = \left(L_t\varphi(t) - \lambda\varphi(t), \int_M \varphi^2 d\mu_M - 1 \right), \tag{A.1}$$

whose zero set characterizes normalized eigenfunctions with eigenvalues in $\mathbb{R}$. Let (φ_0, λ_0) be the principal normalized eigenfunction-eigenvalue pair for L_0 with $\varphi_0 > 0$; i.e.

$$F(\varphi_0, \lambda_0, 0) = 0. \tag{A.2}$$

We first show that F is Fréchet smooth. A simple computation shows that linear and quadratic terms of F are smooth, so it remains to show $(t, \varphi) \mapsto L_t\varphi$ is smooth. As $L_t\varphi$ is linear and bounded in φ, it suffices to check the smoothness in t, after verifying continuity. By assumption, there exist a smooth family of smooth vector fields X_t and a smooth family of smooth functions Q_t such that

$$L_t\varphi = \Delta_g\varphi + X_t(\varphi) + Q_t\varphi, \tag{A.3}$$

so it suffices to check the smoothness of the lower-order parts. For any first-order differential operator $A = X + Q$, where X is a vector field and Q is a function,

$$\|A\varphi\|_{H^k} \leqslant \sum_{j=0}^{k} \|\nabla^j (X \cdot \nabla\varphi + Q\varphi)\|_{L^2(M)} \tag{A.4}$$

$$\leqslant C(\|X\|_{C^k(M)} + \|Q\|_{C^k(M)})\|\varphi\|_{H^{k+2}(M)}. \tag{A.5}$$

In particular, if $A = L_{t+\delta} - L_t = X_{t+\delta} - X_t + Q_{t+\delta} - Q_t$, this shows that $L_t\varphi : \mathbb{R} \times H^{k+2}$ is continuous. Define the first-order operator L_t' by $L_t' = X_t' + Q_t'$. Now we have

$$\lim_{\delta\to 0}\left\|\frac{1}{\delta}(L_{t+\delta} - L_t)\varphi - L_t'\varphi\right\|_{H^k(M)} \tag{A.6}$$

$$\leqslant C\lim_{\delta\to 0}\left(\left\|\frac{1}{\delta}(X_{t+\delta} - X_t) - X_t'\right\|_{C^k} + \left\|\frac{1}{\delta}(Q_{t+\delta} - Q_t) - Q_t'\right\|_{C^k}\right)\|\varphi\|_{H^2(M)}. \tag{A.7}$$

By the differentiability of X_t and Q_t, the right hand side equals 0, which shows that $D_t(L_t) = L_t'$. This shows that L_t is C^1 in t, with derivative equal to L_t', a first-order operator. Repeating the argument shows L_t' is also C^1, and hence L_t is smooth in t by induction.

By the implicit function theorem, if the Fréchet derivative of F with respect to the first two variables at $(\varphi_0, \lambda_0, 0)$

$$D_{(\varphi,\lambda)}F[\varphi_0, \lambda_0, 0] : H^2(M) \times \mathbb{R} \to L^2(M) \times \mathbb{R} \tag{A.8}$$

is an isomorphism, then there exists a smooth map $t \mapsto (\varphi(t), \lambda(t))$ such that

$$F(\varphi(t), \lambda(t), t) = 0 \tag{A.9}$$

for $t \in (-\epsilon, \epsilon)$. To show that $\varphi(t)$ is a principal eigenfunction, note that $\varphi(t) > 0$ by continuity, hence,

$$\lambda_1(L_t) = \sup_{\psi>0}\inf_{x\in M}\frac{L_t\psi}{\psi} \geqslant \inf_{x\in M}\frac{L_t\varphi}{\varphi} = \lambda(t), \tag{A.10}$$

so equality holds by the minimality of $\lambda_1(L_t)$. Standard arguments show that $\varphi(t)$ is unique, hence independent of k. Having shown the path $t \mapsto \varphi(t)$ is smooth in $H^k(M)$ for all k, the path is smooth in C^l for all l by Sobolev embedding. This implies the weaker condition that $\varphi(t)$ is in $C^l(\mathbb{R} \times M)$ for all l and hence smooth on $\mathbb{R} \times M$.

Now we compute the Fréchet derivative of F at with respect to the first two variables at $(\varphi_0, \lambda_0, 0)$:

$$D_{(\varphi,\lambda)}F[\varphi_0, \lambda_0, 0](\varphi', \lambda') \tag{A.11}$$

$$= \frac{d}{ds}\bigg|_{s=0} F(\varphi_0 + s\varphi', \lambda_0 + s\lambda', 0) \tag{A.12}$$

$$= \frac{d}{ds}\bigg|_{s=0} \left(L_0(\varphi_0 + s\varphi') - (\lambda_0 + s\lambda')(\varphi_0 + s\varphi'), \int_M (\varphi_0 + s\varphi')^2 d\mu_M \right) \tag{A.13}$$

$$= \left(L_0\varphi' - \lambda_0\varphi' - \lambda'\varphi_0, 2\int_M \varphi_0\varphi' d\mu_M \right). \tag{A.14}$$

Clearly $F' := D_{(\varphi,\lambda)}F : H^2(M) \times \mathbb{R} \to L^2(M) \times \mathbb{R}$ is a bounded linear operator. It remains to check that F' is invertible.

For any $(V, \alpha) \in L^2(M) \times \mathbb{R}$, we would like to solve $F'(\varphi', \lambda') = (V, \alpha)$, that is,

$$(L_0 - \lambda_0)\varphi' = V + \lambda'\varphi_0, \tag{A.15}$$

$$2\int_M \varphi_0\varphi' = \alpha, \tag{A.16}$$

and show that φ', λ' is unique.

By Lemma 4.1 of [AMS08], the formal adjoint L_0^* has the same principal eigenvalue λ_0 as L_0, so $\ker(L_0^* + \lambda_0) = \langle \varphi_0^* \rangle$ for some positive function φ_0^*. Assume $\int_M (\varphi_0^*)^2 d\mu_M = 1$. Substituting the following expansions of V and φ_0:

$$V = a\varphi_0^* + \overline{V} \tag{A.17}$$

$$\varphi_0 = b\varphi_0^* + \overline{\varphi_0}, \tag{A.18}$$

where $\overline{V}, \overline{\varphi_0}$ are L^2-orthogonal to φ_0^*, and $b = \langle \varphi_0, \varphi_0^* \rangle_{L^2(M)} \neq 0$, since both functions are positive, we get

$$(L_0 - \lambda_0)\varphi' = a\varphi_0^* + \overline{V} + \lambda'(b\varphi_0^* + \overline{\varphi_0}) \tag{A.19}$$

$$2\int_M \varphi_0\varphi' = \alpha. \tag{A.20}$$

By elliptic theory, we can solve A.19 for φ' if and only if the RHS is orthogonal to $\ker(L_0^* - \lambda_0) = \langle \varphi_0^* \rangle$. This implies that $\lambda' = -\frac{a}{b}$. Furthermore, the solution φ' is unique up to an element of $\ker(L_0 - \lambda_0) = \langle \varphi_0 \rangle$, so applying A.20 yields a unique φ'. □

A.2 Gluing Initial Data Preserving the Dominant Energy Scalar

In this section we give a useful gluing lemma for initial data that locally preserves the strict dominant energy condition provided that certain Bartnik data quantities associated with k match on both sides, and a strict mean curvature difference holds. This is based on a theorem used in Brendle-Marques-Neves's counterexample to the Min-Oo conjecture [BMN11], that was observed to be highly applicable in constructing gluings for Bartnik extensions in [Jau19].

In the proof of their gluing theorem, Brendle, Marques, and Neves compute the second-order error of scalar curvature expanded around a metric g. Note that the last 3 terms in (A.21) is the familiar linearization of scalar curvature.

Proposition A.1. *(Proposition 16 of [BMN11]) If $|h|_g \leqslant \frac{1}{2}$, then $g + h$ is a Riemannian metric and*

$$\left| R_{g+h} - R_g - \sum_{i,j=1}^{n} (\nabla^2_{e_i,e_j} h)(e_i, e_j) + \Delta_g(\mathrm{tr}_g(h)) + \langle \mathrm{Ric}_g, h \rangle \right| \tag{A.21}$$

$$\leqslant C|h|^2 + C|\nabla h|^2 + C|h||\nabla^2 h|. \tag{A.22}$$

This yields the following gluing result, first given as Theorem 5 in [BMN11], and refined in the Appendix of [Jau19]. We revisit the construction in [BMN11] and present a slightly simplified proof, while showing that the C^1-norm of $\hat{g}_\lambda$ remains bounded as $\lambda \to \infty$.

Proposition A.2. *Let M be a smooth manifold with compact boundary ∂M, and let g and $\tilde{g}$ be two smooth Riemannian metrics on M such that $g - \tilde{g} = 0$ on ∂M. Assume $H_g - H_{\tilde{g}} < 0$ at each point on ∂M with respect to the inward normal.*

Let ρ denote the g-distance to ∂M. Given any $\epsilon > 0$ and any precompact neighborhood U of ∂M, there exists $(s_0, s_1) \subset (\frac{1}{2}, 1)$ and a sequence of smooth metrics $\{\hat{g}_\lambda\}_\lambda$ with the the following properties for large λ:

1. *$R_{\hat{g}_\lambda}(x) \geqslant \min\{R_g(x), R_{\tilde{g}}(x)\} - \epsilon$ for all $x \in M$,*

2. $\inf\left\{R_{\hat{g}_\lambda}(x) \mid s_0\lambda^{-1} < \rho(x) < s_1\lambda^{-1}\right\} \to \infty$,

3. *the* C^1*-norm with respect to* g *of* $\hat{g}_\lambda$ *is bounded on* U *as* $\lambda \to \infty$,

4. $\|\hat{g}_\lambda - \widetilde{g}\|_{C^1(\rho < s_0\lambda^{-1}, g)} < \epsilon$ *and* $\|\hat{g}_\lambda - g\|_{C^1(\rho > s_1\lambda^{-1}, g)} < \epsilon$,

5. $\|\hat{g}_\lambda - g\|_{C^0(M,g)} \leqslant N\lambda^{-1}$ *for a constant* N *independent of* λ,

6. $\hat{g}_\lambda = g$ *in* $M \backslash U$ *and* $\hat{g}_\lambda(x) = \widetilde{g}(x)$ *on* $\left\{\rho < \frac{1}{2}\lambda^{-1}\right\}$.

Note that the construction is local to U*, and the mean curvature is computed with respect to the inward normal, in the opposite convention of [BMN11].*

Proof. Let $\rho : M \to [0, \infty)$ be a smooth function such that $\rho^{-1}(0) = \partial M$ and $|\nabla \rho|_g = 1$ in a neighborhood $V \subset\subset U$ of ∂M such that $\rho(x) = \mathrm{dist}_g(x, \partial M)$ on V. Let $\eta : M \to [0, 1]$ be a cutoff function supported on U, and identically equal to 1 on V. Define T by

$$T = \frac{\widetilde{g} - g}{\rho}\eta.$$

As ρ and $\widetilde{g} - g$ vanish on ∂M, we use L'Hopital's rule along curves to deduce for any point $x \in \partial M$ and $X \in T_x M$ not tangent to ∂M,

$$T_{ij}(x) = \frac{X(\widetilde{g}_{ij} - g_{ij})}{X\rho} = \frac{X(\widetilde{g}_{ij} - g_{ij})}{\langle X, \nu \rangle_g} = \nabla_\nu(\widetilde{g}_{ij} - g_{ij}), \tag{A.23}$$

where ν is the inward unit normal on ∂M. So T is a smooth symmetric 2-tensor supported on U. Furthermore, giving ∂M local coordinates x^α and computing with the $\widetilde{g}$- and g-distance functions as the n-th coordinate, we have the identity

$$T_{\alpha\beta}(x) = 2(\widetilde{A}_{\alpha\beta} - A_{\alpha\beta})(x), \tag{A.24}$$

where $\widetilde{A}$ and A are the second fundamental forms of ∂M in $\widetilde{g}$ and g respectively. Hence $H_g - H_{\widetilde{g}} < 0$ shows that $\mathrm{Tr}_{\partial M} T > 0$.

Define $\hat{g}_\lambda = g + \lambda^{-1}\chi(\lambda\rho)T$, for a smooth bounded cut off function $\chi : \mathbb{R}^+ \to [0, 1]$ satisfying:

$$\chi(s) = s \text{ for } s < \tfrac{1}{2}, \quad \chi' \in [0, 1], \quad \chi''(s) < 0 \text{ for } s \in (\tfrac{1}{2}, 1), \quad \chi(s) \equiv C \text{ for } s \geqslant 1. \tag{A.25}$$

for some $0 < C < 1$. The existence of χ can easily be shown by integrating a suitable bump function. Suppose λ is large enough such that $\{\rho < \lambda^{-1}\}$ lies inside of V. Then

$$\hat{g}_\lambda = g + \lambda^{-1}\chi(\lambda\rho)T = \begin{cases} \tilde{g} & \text{for } \rho < \frac{1}{2}\lambda^{-1}, \\ \tilde{g} - (\rho - \lambda^{-1}\chi(\lambda\rho))T & \text{on } \left\{\frac{1}{2}\lambda^{-1} < \rho < \lambda^{-1}\right\}, \\ g & \text{on } M \backslash U \end{cases}$$

hence item 6 is satisfied.

As $\lambda \to \infty$, the tensor $\lambda^{-1}\chi(\lambda\rho)T$ uniformly converges to 0, and in particular $\|\hat{g}_\lambda - g\|_{C^0(M)} \leqslant N\lambda^{-1}$, which shows item 5. Moreover,

$$\nabla(\lambda^{-1}\chi(\lambda\rho)T) = \chi'(\lambda\rho)\nabla\rho \otimes T + \lambda^{-1}\chi(\lambda^{-1}\rho)\nabla T, \tag{A.26}$$

which remains bounded as $\lambda \to \infty$. Hence $\hat{g}_\lambda$ remains uniformly bounded in $C^1(U)$ as $\lambda \to \infty$, which shows item 3.

Skipping the details, which can be found in the proof of Proposition 18 of [BMN11], one computes using Proposition A.1 to obtain the estimate

$$R_{\hat{g}_\lambda} - R_g + \lambda\chi''(\lambda\rho)\left(|\nabla\rho|_g^2 \operatorname{Tr}_g T - T(\nabla\rho, \nabla\rho)\right) \tag{A.27}$$

$$\geqslant -N\left[\lambda^{-1} + \chi'(\lambda\rho) + \chi(\lambda\rho)(-\chi''(\lambda\rho))\right], \tag{A.28}$$

for some positive constant N independent of λ.

In the region $\{\rho \geqslant \lambda^{-1}\}$, $\chi''(\lambda\rho) = \chi'(\lambda\rho) = 0$, hence we have

$$R_{\hat{g}_\lambda} - R_g \geqslant -N\lambda^{-1}. \tag{A.29}$$

By examining (A.26), we in this region $\|\hat{g}_\lambda - g\|_{C^1} < N\lambda^{-1}$.

Since $\operatorname{Tr}_{\partial M} T > 0$, there exists $a > 0$ such that $|\nabla\rho|_g^2 \operatorname{Tr}_g(T) - T(\nabla\rho, \nabla\rho) \geqslant a$ in the region $\{\rho < \lambda^{-1}\}$ for λ large by continuity. Hence on $\{\rho < \lambda^{-1}\}$ we have

$$R_{\hat{g}_\lambda} - R_g \geqslant -N\left[\lambda^{-1} + \chi'(\lambda\rho)\right] + (\lambda a - N)(-\chi''(\lambda\rho)), \tag{A.30}$$

and we enlarge λ further to ensure $\lambda a - N > 0$. Since $\chi'(s) = 0$ for $s \geqslant 1$, there exists $s_1 \in (\frac{1}{2}, 1)$ such that $N\chi'(s) < \epsilon$ for all $s \geqslant s_1$. Then for $\rho \in [s_1\lambda^{-1}, \lambda^{-1})$,

$$R_{\hat{g}_\lambda} - R_g \geqslant -N\lambda^{-1} - \epsilon. \tag{A.31}$$

The choice of s_1 also ensures that $\|\hat{g}_\lambda - g\|_{C^1} < N\lambda^{-1} + \epsilon$ in this region, by examining (A.26), and we have verified the second part of item 4.

Now for any $s_0 \in (\frac{1}{2}, s_1)$, we have $\sup_{s\in(s_0,s_1)} \chi''(s) < 0$. Then from (A.30),

$$\inf R_{\hat{g}_\lambda} - R_g \to \infty \tag{A.32}$$

in the region $\{s_0\lambda^{-1} < \rho < s_1\lambda^{-1}\}$ as $\lambda \to \infty$, verifying item 2.

Consider $\widetilde{\chi}(s) = s - \chi(s)$; we have $\hat{g}_\lambda = \widetilde{g} - \lambda^{-1}\widetilde{\chi}(\lambda\rho)T$ on $\{\rho < s_0\lambda^{-1}\}$. Through a similar calculation as before, we apply Proposition A.1 to obtain

$$R_{\hat{g}_\lambda} - R_{\widetilde{g}} - \lambda\widetilde{\chi}''(\lambda\rho)\left(|\widetilde{\nabla}\rho|^2_{\widetilde{g}} \operatorname{Tr}_{\widetilde{g}} T - T(\widetilde{\nabla}\rho, \widetilde{\nabla}\rho)\right) \tag{A.33}$$

$$\geqslant -N\left[\lambda^{-1} + \widetilde{\chi}'(\lambda\rho) + \widetilde{\chi}(\lambda\rho)(\widetilde{\chi}''(\lambda\rho))\right]. \tag{A.34}$$

We have $\widetilde{\chi}'' \geqslant 0$ everywhere and $\widetilde{\chi}(s) \equiv 0$ for $s \leqslant \frac{1}{2}$. As before, by continuity, we enlarge λ such that $|\widetilde{\nabla}\rho|^2_{\widetilde{g}} \operatorname{Tr}_{\widetilde{g}} T - T(\widetilde{\nabla}\rho, \widetilde{\nabla}\rho) \geqslant a$ and such that $\lambda a > N$. Choose $s_0 \in (\frac{1}{2}, s_1)$ such that $N\widetilde{\chi}'(s) < \epsilon$ for all $s \leqslant s_0$. So on the region $\{\rho < s_0\lambda^{-1}\}$, we have

$$R_{\hat{g}_\lambda} - R_{\widetilde{g}} \geqslant -N\lambda^{-1} - \epsilon, \tag{A.35}$$

and this choice of s_0 ensures that $\|\hat{g}_\lambda - \widetilde{g}\|_{C^1} < N\lambda^{-1} + \epsilon$ in this region, by the analog of (A.26) for $-\lambda^{-1}\widetilde{\chi}(\lambda\rho)T$, verifying the first part of item 4.

Finally, combining equations (A.29), (A.31), (A.32), and (A.35) shows item 1. □

Taking advantage of the blowup of $R_{\hat{g}_\lambda}$ in the region $\rho \in (s_0\lambda^{-1}, s_1\lambda^{-1})$ and the λ^{-1}-decay of $\|\hat{g}_\lambda - g\|_{C^1}$ and $\|\hat{g}_\lambda - \widetilde{g}\|_{C^1}$ in the two components of the complement of this region, we obtain a gluing theorem for initial data sets. Recall that the *momentum tensor* π of (M, g, k) is defined by $\pi := k - (\operatorname{tr}_g k)g$. Let $(M, \widetilde{g}, \widetilde{k})$ be an initial data set such that $\widetilde{g} = g$ on ∂M, and let ν be the inward unit normal to ∂M. Then the condition that

$$k(\nu, X) = \widetilde{k}(\nu, X), \quad \operatorname{Tr}_{\partial M} k = \operatorname{Tr}_{\partial M} \widetilde{k} \tag{A.36}$$

for all tangent vectors X on ∂M is equivalent to the condition that

$$\pi(\nu, Z) = \widetilde{\pi}(\nu, Z) \tag{A.37}$$

for *all* vectors Z on ∂M, where $\widetilde{\pi}$ is the momentum tensor for $(M, \widetilde{g}, \widetilde{k})$.

Theorem A.3. *Let M be a smooth manifold with compact boundary ∂M, and let g and $\widetilde{g}$ be two smooth Riemannian metrics on M such that $g - \widetilde{g} = 0$ on ∂M. Assume $H_g - H_{\widetilde{g}} < 0$ at each point on ∂M with respect to the inward normal ν. Let $\pi, \widetilde{\pi}$ be two smooth symmetric 2-tensors on M such that $\pi(\nu, \cdot) = \widetilde{\pi}(\nu, \cdot)$ on ∂M.*

Given any $\epsilon > 0$ and any precompact neighborhood U of ∂M, there exists $(s_0, s_1) \subset (\frac{1}{2}, 1)$ and a sequence of smooth initial data $\{(\hat{g}_\lambda, \hat{\pi}_\lambda)\}_\lambda$ with the the following properties for large λ:

- $(\hat{\mu}_\lambda - |\hat{J}_\lambda|_{\hat{g}_\lambda}) \geqslant \min\left\{ \left(\mu - |J|_g\right), \left(\widetilde{\mu} - |\widetilde{J}|_{\widetilde{g}}\right) \right\} - \epsilon$ *on all of M, where*
$$(\hat{\mu}_\lambda, \hat{J}_\lambda), (\mu, J), (\widetilde{\mu}, \widetilde{J}) \tag{A.38}$$
are the constraints of $\hat{g}_\lambda, g, \widetilde{g}$, respectively,
- $(\hat{g}_\lambda, \hat{\pi}_\lambda) = (g, \pi)$ *in $M \backslash U$,*
- $(\hat{g}_\lambda, \hat{\pi}_\lambda) = (\widetilde{g}, \widetilde{\pi})$ *in a neighborhood of ∂M,*
- $\|\hat{g}_\lambda - g\|_{C^0(M)} \leqslant N\lambda^{-1}$.

Proof. Define the *constraint operator*
$$\Phi : \mathcal{M}(M) \times C^\infty(TM \odot TM) \to C^\infty(M) \times C^\infty(TM), \tag{A.39}$$
where $\mathcal{M}(M)$ is the set of Riemannian metrics on M, by the formula
$$\Phi(g, \pi) := (2\mu, J) = \left(R_g + \frac{1}{n-1}(\operatorname{tr}_g \pi)^2 - |\pi|_g^2, \operatorname{div}_g \pi \right). \tag{A.40}$$

We compute the change in Φ under a perturbation. Let $(h, \omega) \in C^\infty(TM \odot TM) \times C^\infty(TM \odot TM)$ with $|h|_g \leqslant \frac{1}{2}, |\omega|_g \leqslant C$. Then, applying Lemmas A.4 and A.5, we have
$$\Phi(g + h, \pi + \omega) \tag{A.41}$$
$$= \left(R_{g+h} + \frac{1}{n-1}(\operatorname{tr}_{g+h}(\pi + \omega))^2 - |\pi + \omega|_{g+h}^2, \operatorname{div}_{g+h}(\pi + \omega) \right) \tag{A.42}$$
$$= \Bigg(R_{g+h} + \frac{1}{n-1}(\operatorname{tr}_g \pi)^2 - |\pi|_g^2 + O(|h| + |\omega|), \tag{A.43}$$
$$\operatorname{div}_g \pi + \operatorname{div}_g \omega + O\left(|h| + |\nabla h| + |\nabla \omega||h| + |\nabla h||\omega|\right) \Bigg) \tag{A.44}$$
$$= \Phi(g, \pi) + \left(R_{g+h} - R_g + O(|h| + |\omega|), \operatorname{div}_g \omega + O\left(|h| + |\nabla h| + |\nabla \omega||h| + |\nabla h||\omega|\right)\right) \tag{A.45}$$

where $O(X)$ denotes terms that are whose g-norms are bounded by X times a constant determined by g and π.

Now we are ready to define the sequence of initial data. Let $\hat{g}_\lambda$ be the sequence of metrics given in Proposition A.2. Fix a function ρ in a neighborhood of ∂M such that $\rho^{-1}(0) = \partial M$, $\rho = 1$ outside a compact neighborhood of ∂M, and $|\nabla\rho|_g = 1$; ρ will be a g-distance function near ∂M. Let $\nu = \nabla\rho$, an extension of the inward pointing unit normal. Let $\gamma : [0,\infty) \to \mathbb{R}$ be a smooth cutoff function such that $\gamma(s) \equiv 1$ for $s < s_0$ and $\gamma(s) \equiv 0$ for $s > s_1$, where s_0 and s_1 are given by Proposition A.2. Define $\omega_0 = \widetilde{\pi} - \pi$ and define $\omega_\lambda = \gamma(\rho\lambda)\omega_0$. Let $\hat{\pi}_\lambda = \pi + \omega_\lambda$. Note that

$$\hat{\pi}_\lambda = \pi + \omega_\lambda \equiv \begin{cases} \widetilde{\pi} & \rho \leqslant s_0\lambda^{-1} \\ \pi & \rho > s_1\lambda^{-1}. \end{cases} \tag{A.46}$$

We focus on the region $\{s_0\lambda^{-1} < \rho < s_1\lambda^{-1}\}$. We can write $\omega_0(\nu,\cdot) = \rho\alpha$ for some smooth 1-form α, since $\pi'(\nu,\cdot) = \pi(\nu,\cdot)$ and thus $\omega_0(\nu,\cdot) = 0$ on ∂M.

Putting this together, recalling that $\nabla^g\rho = \nu$, we have that

$$\begin{aligned} \operatorname{div}_g \omega_\lambda &= \operatorname{div}_g (\gamma(\rho\lambda)\omega_0) && \text{(A.47)} \\ &= \lambda\gamma'(\rho\lambda)\omega_0(\nu,\cdot) + \gamma(\rho\lambda)\operatorname{div}_g \omega_0 && \text{(A.48)} \\ &= \rho\lambda\gamma'(\rho\lambda)\alpha + \gamma(\rho\lambda)\operatorname{div}_g \omega_0 && \text{(A.49)} \\ &= O(1) && \text{(A.50)} \end{aligned}$$

in the region $\rho < s_1\lambda^{-1}$. Set $h_\lambda = \hat{g}_\lambda - g$. We have

$$|h_\lambda| = O(\lambda^{-1}) \qquad |\nabla h_\lambda| = O(1) \tag{A.51}$$

$$|\omega_\lambda| = O(1) \qquad |\nabla\omega_\lambda| = O(\lambda). \tag{A.52}$$

So in the region $\{s_0\lambda^{-1} < \rho < s_1\lambda^{-1}\}$, we have

$$(2\hat{\mu}_\lambda, \hat{J}_\lambda) \tag{A.53}$$

$$= \Phi(\hat{g}_\lambda, \hat{\pi}_\lambda) \tag{A.54}$$

$$= \Phi(g + h_\lambda, \pi + \omega_\lambda), \tag{A.55}$$

$$= \Phi(g, \pi) + \Big(R_{\hat{g}_\lambda} - R_g + O(|h_\lambda| + |\omega_\lambda|), \tag{A.56}$$

$$\operatorname{div}_g \omega_\lambda + O\left(|h_\lambda| + |\nabla h_\lambda| + |\nabla \omega_\lambda||h_\lambda| + |\nabla h_\lambda||\omega_\lambda|\right)\Big) \tag{A.57}$$

$$= \left(2\mu + R_{\hat{g}_\lambda} - R_g + O(\lambda^{-1} + 1), J + O\left(1 + \lambda^{-1} + 1 + \lambda \cdot \lambda^{-1} + 1\right)\right) \tag{A.58}$$

$$= \left(2\mu + R_{\hat{g}_\lambda} - R_g + O(1), J + O(1)\right). \tag{A.59}$$

Since $\inf R_{\hat{g}_\lambda} - R_g \to \infty$ in this region, $\hat{\mu}_\lambda \to \infty$ while $|\hat{J}_\lambda|_{\hat{g}_\lambda}$ stays bounded.

In the region $\{\rho > s_1\lambda^{-1}\}$, we have $\hat{\pi}_\lambda = \pi$ and $|\nabla h_\lambda| = O(\epsilon)$ by Proposition A.2, so

$$(2\hat{\mu}_\lambda, \hat{J}_\lambda) = \Phi(\hat{g}_\lambda, \hat{\pi}_\lambda) \tag{A.60}$$

$$= \Phi(g + h_\lambda, \pi + 0), \tag{A.61}$$

$$= \Phi(g, \pi) + \left(R_{\hat{g}_\lambda} - R_g + O(|h_\lambda|), O\left(|h_\lambda| + |\nabla h_\lambda|\right)\right) \tag{A.62}$$

$$= \left(2\mu + R_{\hat{g}_\lambda} - R_g + O(\lambda^{-1}), J + O(\lambda^{-1} + \epsilon)\right). \tag{A.63}$$

Since $R_{\hat{g}_\lambda} - R_g > -\epsilon$, we have $\hat{\mu}_\lambda > \mu - \epsilon$, and $|J|_g > |\hat{J}_\lambda|_{\hat{g}_\lambda} - C\epsilon$, where C is independent of λ. This yields $\hat{\mu}_\lambda - |\hat{J}_\lambda|_{\hat{g}_\lambda} > \mu - |J|_g - C\epsilon$.

Set $\widetilde{h}_\lambda = \hat{g}_\lambda - \widetilde{g}$. Then in the region $\{\rho < s_0\lambda^{-1}\}$, we have $\hat{\pi}_\lambda = \widetilde{\pi}$ and $|\widetilde{\nabla}\widetilde{h}_\lambda| = O(|\nabla\widetilde{h}_\lambda|) = O(\epsilon)$, so

$$(2\hat{\mu}_\lambda, \hat{J}_\lambda) = \Phi(\hat{g}_\lambda, \hat{\pi}_\lambda) \tag{A.64}$$

$$= \Phi(\widetilde{g} + \widetilde{h}_\lambda, \widetilde{\pi} + 0), \tag{A.65}$$

$$= \Phi(\widetilde{g}, \widetilde{\pi}) + \left(R_{\hat{g}_\lambda} - R_{\widetilde{g}} + O(|\widetilde{h}_\lambda|), O\left(|\widetilde{h}_\lambda| + |\widetilde{\nabla}\widetilde{h}_\lambda|\right)\right) \tag{A.66}$$

$$= \left(2\widetilde{\mu} + R_{\hat{g}_\lambda} - R_{\widetilde{g}} + O(\lambda^{-1}), \widetilde{J} + O(\lambda^{-1} + \epsilon)\right). \tag{A.67}$$

Since $R_{\hat{g}_\lambda} - R_{\widetilde{g}} > -\epsilon$, we have $\hat{\mu}_\lambda > \widetilde{\mu} - \epsilon$, and $|\widetilde{J}|_{\widetilde{g}} > |\hat{J}_\lambda|_{\hat{g}_\lambda} - C\epsilon$, where C is independent of λ. This yields $\hat{\mu}_\lambda - |\hat{J}_\lambda|_{\hat{g}_\lambda} > \widetilde{\mu} - |\widetilde{J}|_{\widetilde{g}} - C\epsilon$.

□

We had to use the following two lemmas about metrics g and their perturbation $\hat{g} = g + h$:

Lemma A.4. *Let $\hat{g} = g + h$, with $|h|_g \leqslant \frac{1}{2}$. Then,*

$$\hat{g}^{ij} = g^{ij} - g^{ik}h_{kj} + O(|h|_g^2). \tag{A.68}$$

In particular, we have the following identities for any 2-tensor A:

$$\operatorname{Tr}_{g+h} A = \operatorname{Tr}_g A + O(|h|_g|A|_g) \tag{A.69}$$

$$|A|_{g+h}^2 = A_{ij}A_{kl}\hat{g}^{ik}\hat{g}^{jl} = |A|_g^2 + O(|h|_g|A|_g^2). \tag{A.70}$$

Proof. These identities come from the identity $(I+P)^{-1} = I - P + P^2(I+P)^{-1} = I - P + O(|P|^2)$ for any matrix with Frobenius norm less than $\frac{1}{2}$. □

Lemma A.5. *Let $\hat{g} = g + h$, with $|h|_g \leqslant \frac{1}{2}$. Then for any 2-tensor A, the following identity holds for divergence:*

$$\operatorname{div}_{g+h} A = \operatorname{div}_g A + O(|A||\nabla h|) + O(|\nabla A||h|). \tag{A.71}$$

Proof. The difference of the connections $\nabla^{\hat{g}} - \nabla^g$ is a tensor

$$W_{ij}^k = \frac{1}{2}\hat{g}^{kl}\left(\nabla_i^g h_{lj} + \nabla_j^g h_{il} - \nabla_l^g h_{ij}\right) = O(|\nabla h|_g). \tag{A.72}$$

We have

$$\operatorname{div}_{g+h} A = \hat{g}^{ij}(\nabla_i^{\hat{g}} A)(\partial_j, \partial_k)dx^k \tag{A.73}$$

$$= \hat{g}^{ij}\left(\partial_i A_{jk} - A(\nabla_i^{\hat{g}}(\partial_j), \partial_k) - A(\partial_j, \nabla_i^{\hat{g}}(\partial_k)\right) dx^k \tag{A.74}$$

$$= \hat{g}^{ij}\left((\nabla_i^g A)(\partial_j, \partial_k) - A(W_i(\partial_j), \partial_k) - A(\partial_j, W_i(\partial_k))\right) dx^k \tag{A.75}$$

$$= \operatorname{div}_g A + O(|A||\nabla h|) + O(|\nabla A||h|). \tag{A.76}$$

□

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