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NOISE-INDUCED STABILIZATION OF PLANAR FLOWS I

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Abstract. We show that the complex-valued ODE

$$\dot{z}_t = a_{n+1} z^{n+1} + a_n z^n + \cdots + a_0,$$

which necessarily has trajectories along which the dynamics blows up in finite time, can be stabilized by the addition of an arbitrarily small elliptic, additive Brownian stochastic term. We also show that the stochastic perturbation has a unique invariant measure which is heavy-tailed yet is uniformly, exponentially attracting. The methods turn on the construction of Lyapunov functions. The techniques used in the construction are general and can likely be used in other settings where a Lyapunov function is needed. This is a two-part paper. This paper, Part I, focuses on general Lyapunov methods as applied to a special, simplified version of the problem. Part II [8] extends the main results to the general setting.

1. Introduction

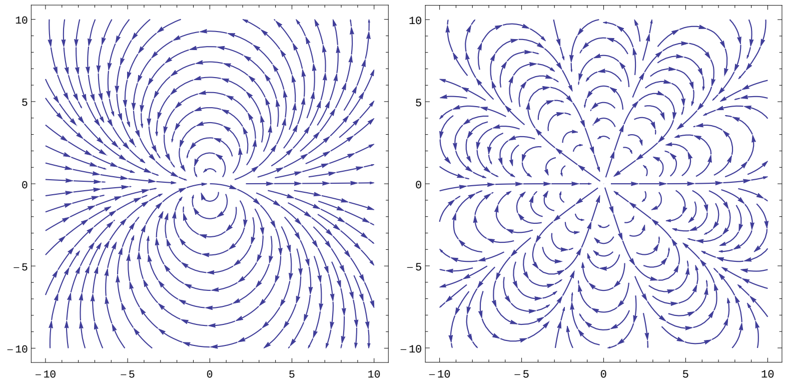
We study the following complex-valued system

$$\begin{cases} dz_t = (a_{n+1} z_t^{n+1} + a_n z_t^n + \cdots + a_0)\, dt + \sigma\, dB_t \\ z_0 \in \mathbf{C} \end{cases} \tag{1.1}$$

where $n \geq 1$ is an integer, $\sigma \geq 0$ is constant, and $B_t = B_t^{(1)} + iB_t^{(2)}$ is a complex Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. (We in fact prove many result for the slightly more general form given in (3.1).)

When $\sigma = 0$ in equation (1.1), the resulting deterministic system has solutions which blow up in finite time[1]. Interestingly, however, we will see that with the addition of noise (i.e. when $\sigma > 0$ in (1.1)), not only does the system stabilize so that solutions to (1.1) exist for all initial conditions and all finite times, but the dynamics settles down into a unique statistical steady state with corresponding invariant measure.

[1]This can be intuited by noting that the asymptotic equation $\dot{z}_t = a_{n+1} z_t^{n+1}$ at infinity has n explosive trajectories (see Figure 1)



(A) $\dot{z} = z^2$ (case: $n = 1$) (B) $\dot{z} = z^6$ (case: $n = 5$)

FIGURE 1. The orbits of $\dot{z} = z^{n+1}$. Trajectories with initial condition $z_0 = re^{i\theta}$ satisfying $r > 0$ and $\theta = \frac{2\pi k}{n}$, $k \in \mathbf{Z}$, explode to infinity in finite time.

Strikingly, too, we will find that this invariant measure has the following two properties. First, it possesses an everywhere positive density (with respect to Lebesgue measure on $\mathbf{C}$) which decays polynomially in $|z|$ at infinity. Second, it attracts all initial distributions exponentially fast in time. Note that these two properties cannot be simultaneously realized in a gradient system with polynomial potential, but it is possible here because the system is strongly non-reversible with a nontrivial probability flux in equilibrium.

It will be noted that the polynomial decay follows from a delicate balance between the noise and unstable dynamics resulting in a global circulation in equilibrium. Such circulation produces what might be called "intermittent" behavior. Namely, the system spends long periods of time in an order one region about the origin but at approximately exponential times the system "spikes", making rapid exclusions to large values followed by an equally rapid returns to order one values (see Figure 2). The same balancing of noise and explosion which produces the polynomial decay at infinity also implies a specific scaling between the level of the spike and the parameter of the interspike-time distribution. This and its relationship to the equilibrium flux are highlighted in the heuristic discussions in Section 9.

Although the results obtained here are specific to the equation (1.1) (and a mild generalization (3.1)), the methods used should be applicable to a wide range of problems. That is, to establish the main results, a sequence of "optimal" Lyapunov functions is constructed, and because of the delicate interplay between the noise and the instabilities of the underlying deterministic system, it forces one to know how to build such functions well. Although we do not claim to have a step-by-step algorithmic procedure which would produce a Lyapunov function for a given stochastic differential equation, we at least give a framework that could be molded to handle a variety of situations. We believe that many of the core ideas can be applied quite broadly.

It is important to remark that the system (1.1) and other similar planar systems have been studied before [1, 2, 4, 7, 13]. Most relevant to this work are the references [1, 4, 7] where (1.1) was investigated in special cases of the polynomial drift term. In addition to proving results for the general system (1.1), we will also improve upon the existing ones in these special cases. Specifically, our results will be seen to be optimal in the following sense: If μ denotes the unique invariant measure for (1.1), then for any $\gamma \in (0, 2n)$ we will succeed in constructing a Lyapunov function for (1.1) whose growth at infinity will imply that

$$\int_{\mathbf{C}} (1 + |z|^{\gamma}) d\mu(z) < \infty,$$

yet we will see that for any $\gamma \geq 2n$

$$\int_{\mathbf{C}} (1 + |z|^{\gamma}) d\mu(z) = \infty.$$

Hence it would be impossible to build a Lyapunov function with power growth at infinity which grows faster than those we construct.

The layout of this paper is as follows. In Section 2, we give a non-rigorous, formal asymptotic matching argument which suggests a possible rate of decay at infinity for the invariant probability density function. Section 9.1 gives a second heuristic explanation of the decay using arguments based on stochastic dynamics. Section 3 contains the statements of the main results of this paper, Part I, and its continuation, Part II [8], which henceforth will be referred to as the sequel. In Section 4, we give the precise definition of a Lyapunov function we will use and list some of the consequences of its existence. It is interesting to note that, although our notion of a Lyapunov function is similar to that of Meyn and Tweedie [10] or Khasminskii [9], we only require Lyapunov functions to be piecewise C^2 and globally continuous rather than globally C^2. While this is useful in constructing the Lyapunov functions, it forces us to employ a generalized Itô-Tanaka formula due to Peskir [12] to estimate contributions along curves where our Lyapunov function is not C^2. This allows us to avoid smoothing or mollifying along the boundaries which leads to a substantial reduction in the complexity of

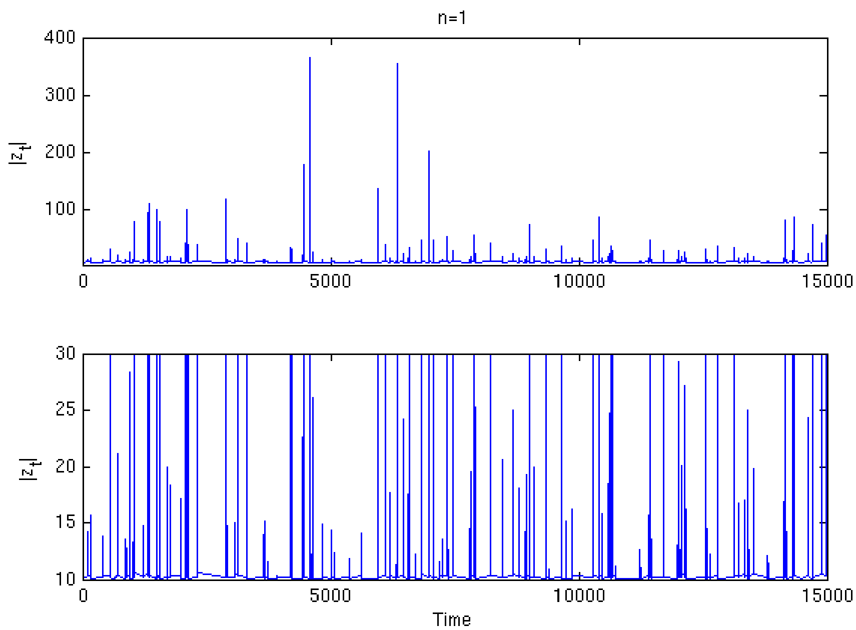


FIGURE 2. A realization of the process $|z_t|$ plotted on the time interval $[0, 15000]$ where z_t solves equation (1.1) with $n = 1$, $a_2 = 1$, $a_1 = a_0 = 0$ and $\sigma = 1$. See Section 9.2 for a discussion of the spacing between the "spikes."

the argument when compared to previous works [1, 4, 7, 2]. In Section 5, we state the precise results we will actually prove which, when combined with the results in Section 4, will imply the main results as stated in Section 3. In Section 6, the key initial steps of the construction procedure that will produce the required Lyapunov functions are discussed. In particular, we show how we plan to apply Peskir's result [12], allowing us to work with less regular Lyapunov functions. To illustrate our general methods, in Section 7 we build our Lyapunov functions corresponding to the system (1.1) assuming that there are no "significant" lower-order terms in the drift of (1.1). Having introduced the main construction of the Lyapunov functions, in Section 7.4 we explain the critical connection between the maximal growth rate at infinity of $2n$ and the local, stochastically dominated behavior in a neighborhood of the deterministically unstable directions. Section 8 finishes the remaining details in this special case. In Section 9, we discuss heuristically the behavior of the system when the noise coefficient

σ approaches zero. In particular, we give heuristics explaining the decay rate of the invariant measure and the inter-spike spacing exhibited in Figure 2.

In the sequel (Part II of this paper) [8], we prove the results given in Section 5 pertaining to (1.1) but in the general setting without this simplifying assumption. Extending the result is subtly intricate as the presence of higher-order polynomial terms with degree $\leq n$ drastically alters the nature of the process at infinity. Moreover, we will also give a short proof of a weaker version of Peskir's result [12] suitable for our needs. In Section 10, we summarize what has been accomplished in Part I of the paper with the advantage of hindsight. We postpone discussions of possible future directions to Part II of the work.

Acknowledgements

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2. Formal Derivation of the Decay at Infinity of the Invariant Measure

In this section, we present some heuristic, non-rigorous arguments which give information about the possible structure of the invariant measure at infinity. We will concentrate on the simplified equation

$$dz_t = z^{n+1}dt + \sigma dB_t \tag{2.1}$$

where $z_t \in \mathbf{C}$, $n \geq 1$, $\sigma > 0$, and B_t is a complex Brownian motion. Although matters could be complicated by the presence of lower order terms in the drift, studying the equation above is a good starting point and reveals much of the structure of the general equation (1.1).

We begin by writing the generator $\mathcal{L}$ of the process defined by (2.1) in polar coordinates (r, θ) producing

$$\mathcal{L} = r^{n+1}\cos(n\theta)\partial_r + r^n \sin(n\theta)\partial_\theta + \frac{\sigma^2}{r}\partial_r + \frac{\sigma^2}{2}\partial_r^2 + \frac{\sigma^2}{2r^2}\partial_\theta^2 \,.$$

Since the Euclidean volume element in (r,θ) is $r\,dr\,d\theta$, the action of the adjoint of $\mathcal{L}$ on a smooth test function $f\colon \mathbf{R}^2 \to \mathbf{R}$ is given by

$$\mathcal{L}^* f = -\cos(n\theta)\frac{1}{r}\partial_r(r^{n+2}f) - r^n\partial_\theta(\sin(n\theta)f) \\ - \frac{\sigma^2}{r}\partial_r f + \frac{\sigma^2}{2r}\partial_r^2(rf) + \frac{\sigma^2}{2r^2}\partial_\theta^2 f\,.$$

If we assume that any invariant measure has a C^2 density ρ with respect to Lebesgue measure, then it must satisfy $\mathcal{L}^*\rho = 0$. Since the diffusion is uniformly elliptic, it is relatively easy to justify this and further deduce that $\rho(r,\theta) > 0$ for all (r,θ) with $r > 0$. To begin to study the the behavior at infinity, we consider the effect on the equation $\mathcal{L}^*\rho = 0$ under the scaling transformation $(r,\theta) \mapsto (\lambda r, \theta)$ which relives the asymptotic behavior along a fixed radial direction. Considering $\mathcal{L}^*$ under the effect of this transformation produces

$$\lambda^n r^n\big[(2n+2)\cos(n\theta)\rho + r\cos(n\theta)\partial_r\rho + \sin(n\theta)\partial_\theta\rho\big] \\ - \frac{\sigma^2}{\lambda^2}\big[\frac{1}{2}\partial_r^2\rho + \frac{1}{2r^2}\partial_\theta^2\rho\big] = 0\,.$$

Extracting the leading order λ^n term and assuming that $\partial_\theta\rho = 0$ and $\cos(n\theta) \neq 0$, we obtain the leading order equation

$$(2n+2)\rho + r\partial_r\rho = 0$$

Solving this equation produces

$$\rho(r,\theta) \sim \frac{1}{r^{2n+2}}\,. \tag{2.2}$$

We will see in fact that this scaling is essentially correct. A heuristic discussion from a probabilistic point of view and highlighting the equilibrium flux is given in Section 9.1.

3. Main Results: Ergodicity, Mixing, and the Behavior of the Stationary Measure at Infinity

Although we have thus far only discussed equation (1.1), we will see that our main results, to be stated in this section, hold for more general complex-valued SDEs. In particular, our analysis can tolerate more general lower-order terms in the drift. Therefore, throughout the remainder of this paper and the sequel [8] we assume that the complex-valued process z_t satisfies more generally the following SDE

$$dz_t = [a_{n+1}z_t^{n+1} + F(z_t,\bar{z}_t)]\,dt + \sigma\,dB_t \tag{3.1}$$

where a_{n+1}, $n \geq 1$, σ and B_t are as before and $F(z,\bar{z})$ is a complex polynomial in $(z,\bar{z})$ with $F(z,\bar{z}) = \mathcal{O}(|z|^n)$ as $|z| \to \infty$. Notice that

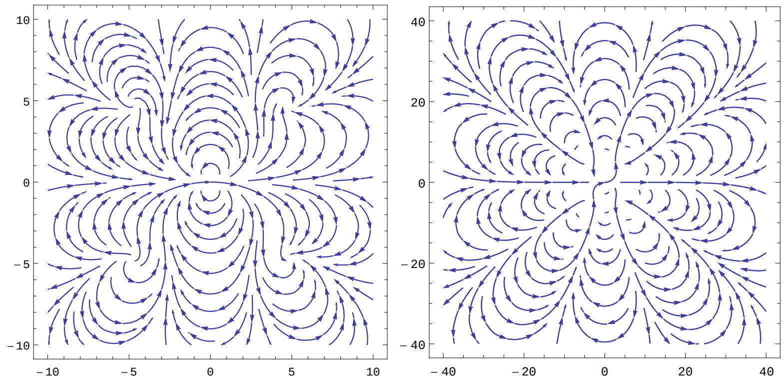


FIGURE 3. The trajectories of $\dot{z} = z^6 + 2000z^2$ plotted on $[-10, 10]^2$ (left) and $[-40, 40]^2$ (right). For $|z|$ small, the dynamics qualitatively resembles that of $\dot{z} = z^2$ (see Figure 1a). As $|z|$ becomes larger, the dynamics starts to resemble that of $\dot{z} = z^6$ (see Figure 1b).

equation (1.1) is the special case of equation (3.1) where $F(z, \bar{z}) \equiv F(z)$ is a complex polynomial in the variable z only with degree$(F) \leq n$.

The global-in-time existence of the Markov process induced by (3.1) is neither obvious nor certain given the unstable nature of the underlying deterministic dynamics. Consequently, even if the process is shown to be non-explosive in finite time, the existence of an invariant measure is still in question. Assuming, however, both issues can be settled, the formal asymptotic calculations of Section 2 suggest that the probability density function of the invariant measure has a certain polynomial decay rate at infinity. The following result, one of the principal rigorous results of this article and the sequel [8], shows that these formal computations are correct.

Theorem 3.2. *The Markov process defined by* (3.1) *is non-explosive and possesses a unique stationary measure* μ. *In addition,* μ *satisfies:*

$$\int_{\mathbf{C}} (1 + |z|)^\gamma \, d\mu(z) < \infty \quad \text{if and only if} \quad \gamma < 2n.$$

Furthermore, μ *is ergodic and has a probability density function* ρ *with respect to Lebesgue measure on* $\mathbf{R}^2$ *which is smooth and everywhere positive.*

Given the existence and uniqueness of the stationary measure, it is also natural to explore if arbitrary initial distributions converge to it and, if so, to determine the rate of convergence. To see what happens in the present context, given any function $w\colon \mathbf{C} \to [1, \infty)$, define the weighted total variation metric d_w by

$$d_w(\nu_1,\nu_2) = \sup_{\substack{\phi:\mathbf{C}\to\mathbf{R}\\ |\phi(z)|\le w(z)}} \left[\int \phi(z)\,\nu_1(dz) - \int \phi(z)\,\nu_2(dz)\right]$$

where ν_1 and ν_2 are arbitrary probability measures on $\mathbf{C}$.

Theorem 3.3. *Let P_t denote the Markov semi-group corresponding to (3.1) and let $\alpha \in (0,n)$ be arbitrary. Then there exists a function $\Psi\colon \mathbf{C} \to (0,\infty)$ satisfying the global bound*

$$c|z|^{\alpha} \le \Psi(z) \le d|z|^{\alpha+\frac{n}{2}+1}$$

for some positive constants c,d so that if $w(z) = 1+\beta\Psi(z)$ for some $\beta > 0$, then there exist constants $C>0$ and $\gamma>0$ such that for any two probability measures ν_1 and ν_2 on $\mathbf{C}$ and any $t \ge 0$

$$d_w(\nu_1 P_t, \nu_2 P_t) \le Ce^{-\gamma t}\|\nu_1-\nu_2\|_{TV}.$$

Remark 3.4. In Theorem 3.3, $\|\nu_1-\nu_2\|_{TV}$ denotes the total variation distance between the measures ν_1 and ν_2. Note that $\|\nu_1-\nu_2\|_{TV} = d_w(\nu_1,\nu_2)$ when $w \equiv 1$.

4. Consequences of Lyapunov Structure and Implications for the Invariant Measure

Most of the results in Section 3 turn on the existence of a certain type of Lyapunov function corresponding to the Markov semi-group P_t. Because we require the additional flexibility, we make use of a slightly more general notion of a Lyapunov function than is usually employed for diffusion processes. In this section, therefore, we will define what we mean by a Lyapunov function and give some results that follow from its existence. Because it is simpler, we will work in this section in the general context of an Itô diffusion ξ_t on $\mathbf{R}^m$ whose Markov generator $\mathscr{L}$ has smooth (C^∞) coefficients. We first introduce some notation.

For $n \in \mathbf{N}$, let $\tau_n = \inf\{t>0 : |\xi_t| \ge n\}$ and $\mathscr{P}_t$ denote the Markov semi-group corresponding to ξ_t.

Definition 4.1. Let $\Psi,\Phi : \mathbf{R}^m \to [0,\infty)$. We call (Ψ,Φ) a Lyapunov pair corresponding to ξ_t if:

a) Ψ and Φ are locally bounded on $\mathbf{R}^m$ and $\Psi(\xi)\wedge\Phi(\xi) \to \infty$ as $|\xi| \to \infty$;
b) There exist constants $m,b>0$ such that

$$\mathbf{E}_{\xi_0}\Psi(\xi_{t\wedge\tau_n}) - \Psi(\xi_0) \le \mathbf{E}_{\xi_0}\int_0^{t\wedge\tau_n} (-m\Phi(\xi_s)+b)\,ds \tag{4.2}$$

for all $t \ge 0$, $n \in \mathbf{N}$, and $\xi_0 \in \mathbf{R}^m$.

The function Ψ in a Lyapunov pair (Ψ, Φ) is called a LYAPUNOV FUNCTION.

Remark 4.3. One usually requires Ψ above to be globally C^2, in which case the bound in condition b) is often replaced by the global inequality

$$\mathscr{L}\Psi \leq -m\Phi + b.$$

Since the Ψ we construct will only be globally continuous, we employ the slightly more general formulation above.

The first result we state and prove gives the basic consequences of the existence of a Lyapunov pair (Ψ, Φ).

Lemma 4.4. *If the Markov process ξ_t possesses a Lyapunov pair (Ψ, Φ), then the following conclusions hold:*

a) ξ_t is well-defined for all $t \geq 0$ and all initial conditions $\xi_0 \in \mathbf{R}^m$ almost surely;
b) ξ_t possesses at least one invariant measure π;
c) Such an invariant measure π satisfies

$$\int_{\mathbf{R}^m} \Phi(\xi)\pi(d\xi) < \infty.$$

Proof. Parts a) and b) are straightforward extensions of the proofs of, respectively, Theorem 3.5 and Theorem 3.7 of [9]. Part c) is a straightforward extension of the proof of Theorem 4.3 (ii) of [10] or of the proof of Proposition 5.1 of [5]. For the interested reader, we have provided a full proof of this result in Appendix A. □

We also have the following results giving uniqueness of the invariant measure and characterizing the convergence rate of the process ξ_t to this equilibrium.

Theorem 4.5. *If $\mathscr{L}$ is uniformly elliptic and ξ_t has a Lyapunov pair (Ψ, Φ), then the following conclusions also hold:*

a) ξ_t possesses a unique invariant measure π. Furthermore, π is ergodic and has a smooth and everywhere positive density with respect to Lebesgue measure on $\mathbf{R}^m$.
b) If $\Phi = \Psi$ and $w(\xi) = 1 + \beta\Psi(\xi)$ for some $\beta > 0$, then there exist constants $C > 0$ and $\eta > 0$ so that

$$d_w(\nu_1\mathscr{P}_t, \nu_2\mathscr{P}_t) \leq Ce^{-\eta t} d_w(\nu_1, \nu_2)$$

for all probability measures ν_1 and ν_2 and all times $t \geq 0$.

c) If $\Phi = \Psi^{1+\delta}$ for some $\delta > 0$, then the bound in part b) can be improved to

$$d_w(\nu_1\mathscr{P}_t, \nu_2\mathscr{P}_t) \leq Ce^{-\eta t}\|\nu_1 - \nu_2\|_{TV}.$$

Remark 4.6. The strong assumption that $\mathscr{L}$ is uniformly elliptic is not needed for many of the results to hold. However it will simplify our discourse significantly. See [1, 4] for examples considering degenerate noise in a similar setting to this paper.

Proof of Theorem 4.5. The existence in part a) follows from the previous theorem. Uniqueness of π and the existence of a smooth and everywhere positive density are well-known consequences of uniform ellipticity and smoothness of the coefficients of $\mathscr{L}$. Part b) follows easily from the work [6] and part c) is an immediate consequence of Section 6 in [1]. □

As we saw in the previous lemma, if ξ_t possesses a Lyapunov pair of the form $(\Psi, \Psi^{1+\delta})$ for some $\delta > 0$, then the standard geometric ergodicity bound given in part b) can be improved in the sense that the right-hand side no longer depends on the initial state. This is also reflected in the following lemma, as return times to large compact sets are small and independent of where the process starts.

Proposition 4.7. *Suppose that ξ_t has a Lyapunov pair $(\Psi, \Psi^{1+\delta})$ for some $\delta > 0$ and define $\upsilon_\gamma = \inf\{t > 0 : |\xi_t| \leq \gamma\}$ for $\gamma > 0$. Then for each $\gamma > 0$ sufficiently large*

$$\inf_{\xi_0 \in \mathbf{R}^m} \mathbf{P}_{\xi_0}[\upsilon_\gamma < \infty] = 1.$$

Moreover, for each $t, \epsilon > 0$ there exists $\gamma > 0$ large enough so that

$$\sup_{\xi_0 \in \mathbf{R}^m} \mathbf{P}_{\xi_0}[\upsilon_\gamma \geq t] \leq \epsilon.$$

Proof of Proposition 4.7. By assumption, we have the bound

$$\begin{aligned} \mathbf{E}_{\xi_0} \Psi(\xi_{t\wedge\tau_n}) &\leq \Psi(\xi_0) + \mathbf{E}_{\xi_0} \int_0^{t\wedge\tau_n} -m\Psi(\xi_s)^{1+\delta} + b\, ds \\ &\leq \Psi(\xi_0) + \mathbf{E}_{\xi_0} \int_0^{t\wedge\tau_n} -m\Psi(\xi_s) + b\, ds \end{aligned}$$

for all $n \in \mathbf{N}$, $t \geq 0$. By Fatou's lemma, monotone convergence and non-explosivity of ξ_t we obtain the bound

$$\begin{aligned} \mathbf{E}_{\xi_0} \Psi(\xi_t) &\leq \Psi(\xi_0) + \mathbf{E}_{\xi_0} \int_0^t -m\Psi(\xi_s) + b\, ds \\ &= \Psi(\xi_0) + \int_0^t -m\mathbf{E}_{\xi_0} \Psi(\xi_s) + b\, ds \end{aligned}$$

where the last line follows from Tonelli's theorem. Gronwall's inequality then gives

$$\mathbf{E}_{\xi_0}\Psi(\xi_t) \le (\Psi(\xi_0) + bt)e^{-mt}.$$

But note also that

$$\mathbf{E}_{\xi_0}\Psi(\xi_t) \ge \mathbf{E}_{\xi_0}1_{\{\upsilon_\gamma \ge t\}}\Psi(\xi_t) \ge \inf_{|\xi|\ge\gamma}\Psi(\xi)\cdot\mathbf{P}_{\xi_0}[\upsilon_\gamma \ge t].$$

Hence for $\gamma > 0$ large enough we have

$$\mathbf{P}_{\xi_0}[\upsilon_\gamma \ge t] \le \frac{(\Psi(\xi_0)+bt)}{\inf_{|\xi|\ge\gamma}\Psi(\xi)}e^{-mt}.$$

Taking $t \to \infty$ in the above we obtain the first result. To obtain the second conclusion, apply Lemma 6.1 of [1] to see that there exists a positive function K_t independent of ξ_0 which is continuous and monotone decreasing on $(0,\infty)$ and satisfies

$$\mathbf{E}_{\xi_0}\Psi(\xi_t) \le K_t$$

for all $t \ge 0$, $\xi_0 \in \mathbf{R}^m$. In particular, we have that

$$\begin{aligned} K_t \ge \mathbf{E}_{\xi_0}\Psi(\xi_t) &\ge \mathbf{E}_{\xi_0}1_{\{\upsilon_\gamma \ge t\}}\Psi(\xi_t) \\ &\ge \inf_{|x|\ge\gamma}\Psi(x)\cdot\mathbf{P}_{\xi_0}[\upsilon_\gamma \ge t]. \end{aligned}$$

Thus for $\gamma > 0$ large enough

$$\mathbf{P}_{\xi_0}[\upsilon_\gamma \ge t] \le \frac{K_t}{\inf_{|x|\ge\gamma}\Psi(x)}.$$

Since $\Psi(x) \to \infty$ as $|x| \to \infty$, it follows that for each $\epsilon, t > 0$ there exists a $\gamma > 0$ such that

$$\sup_{\xi_0\in\mathbb{R}^d}\mathbf{P}_{\xi_0}[\upsilon_\gamma \ge t] \le \epsilon$$

which finishes the proof. □

5. Proving the Main Results: An Outline

We now give two results which, when combined with the results of the previous section, will yield all of main results of this paper and the sequel [8]. Theorem 5.2 below will provide the needed Lyapunov pair to make use of the results of Section 4. And, Theorem 5.5 below will give the required lower bound on the density of the invariant measure whose existence will be now ensured by Theorem 5.2.

First notice we may assume, without loss of generality, that the coefficient a_{n+1} of z^{n+1} in (3.1) is identically one. Indeed, the system can be rescaled and rotated so that it is one and, since any rotation of B_t is also a complex Brownian motion, the resulting system will be of the form (3.1) but with $a_{n+1} = 1$. Hence for the remainder of the paper we will assume that (3.1) takes the from

$$dz_t = [z_t^{n+1} + F(z_t, \bar{z}_t)]\, dt + \sigma\, dB_t \tag{5.1}$$

where $n \geq 1$, F, σ and B_t are as in equation (3.1). By a simple change of variables, all results translate easily to the general system (3.1).

Theorem 5.2. *For each $\gamma \in (n, 2n)$ and $\delta = \delta_\gamma > 0$ sufficiently small, there exist a function Ψ so that $(\Psi, \Psi^{1+\delta})$ and $(\Psi, |z|^\gamma)$ are Lyapunov pairs corresponding to the dynamics (5.1). Moreover, Ψ satisfies the global bounds*

$$c|z|^{\gamma-n} \leq \Psi(z) \leq d|z|^{\gamma-n+\frac{n}{2}+1}$$

for some positive constants c, d.

Remark 5.3. We will see that for a "very large" set $X \subset \mathbf{C}$, the bound

$$C|z|^{\gamma-n} \leq \Psi(z) \leq D|z|^{\gamma-n}$$

holds for all $z \in X$ for some $C, D > 0$. The only region Y in which Ψ grows faster than a constant times $|z|^{\gamma-n}$ satisfies the property

$$\lim_{R\to\infty} \lambda(Y \cap \{|z| > R\}) = 0$$

where λ denotes Lebesgue measure on $\mathbf{C}$. As will be apparent later, any increase in growth in Ψ is exactly compensated by the decrease in the measure of the set $Y \cap \{|z| > R\}$ as $R \to \infty$. Although we did not state it this way in Theorem 5.2, we could have also chosen the second function in the pair $(\Psi, |z|^\gamma)$ to have this property.

Translating back to the general system (3.1), notice by combining Lemma 4.4 and Theorem 4.5, we see that Theorem 5.2 implies Theorem 3.3 as well as all results of Theorem 3.2 except

$$\int_{\mathbf{C}} (1 + |z|)^\gamma\, d\mu(z) = \infty \quad \text{whenever} \quad \gamma \geq 2n. \tag{5.4}$$

To prove this last point, we will show the following stronger result.

Theorem 5.5. *Let ρ denote the invariant probability density function of (5.1) with respect to Lebesgue measure $\frac{i}{2}(dz \wedge d\bar{z})$ on $\mathbf{R}^2$. Then there exist positive constants c, K such that*

$$|z|^{2n+2}\rho(z, \bar{z}) \geq c \quad \textit{for} \quad |z| \geq K. \tag{5.6}$$

The focus of this paper is to prove Theorem 5.2 under the following simplifying assumption while the sequel [8] will remove this restriction.

Assumption 5.7. *In equation (5.1), either F is a constant function or*

$$F(z,\bar{z}) = \mathcal{O}(|z|^{\lfloor \frac{n}{2} \rfloor - 2}) \text{ as } |z| \to \infty.$$

This is done to highlight the general procedure used to yield our Lyapunov pairs. Moreover, it allows us to avoid significant complexities created by the presence of large lower-order terms. The full proof of Theorem 5.2 and the proof of Theorem 5.5 are given in Part II [8] by building on the structures and techniques developed in this work.

6. Building Lyapunov Functions: The Key Initial Steps

In this section, we make some beginning observations which will help us get started with constructing a Lyapunov function Ψ for the system (5.1). Everything done in this section applies to equation (5.1) even if we do not employ Assumption 5.7.

6.1. **The Coordinate and Time Changes.**

When building Ψ, it is paramount that one first pick a convenient coordinate system in which to work. For equation (5.1), there are at least three choices: standard Euclidean coordinates (x, y), the two-dimensional complex system $(z, \bar{z})$, and polar coordinates (r, θ). Notice, however, since stability of the process (5.1) (or any $\mathbf{R}^m$-valued process for that matter) is completely determined by the distribution of the radial component r, the polar system (r, θ) is arguably most natural.

In light of this remark, observe that the generator of the Markov process defined by (5.1) has the following form when written in the variables (r, θ):

$$\begin{aligned} \mathcal{L} = r^{n+1}\cos(n\theta)\partial_r + r^n \sin(n\theta)\partial_\theta + \frac{\sigma^2}{2}\partial_r^2 + \frac{\sigma^2}{2r^2}\partial_\theta^2 \\ + r^n P(r,\theta)\partial_r + r^n Q(r,\theta)\partial_\theta \end{aligned} \tag{6.1}$$

where $P(r,\theta) = \sum_{k=0}^n r^{k-n} f_k(\theta)$ and $Q(r,\theta) = \sum_{k=0}^n r^{k-n-1} g_k(\theta)$ for some collection of bounded, smooth real-valued functions f_k and g_k.

When proving Theorem 5.5, we will need certain stability properties of a diffusion process related to the formal adjoint $\mathcal{L}^*$. Because there is one additional term in the generator of this diffusion, we will construct the appropriate Lyapunov pairs assuming the slightly more general form of P and Q given by

$$\begin{aligned} P(r,\theta) &= \sum_{k=0}^{n} r^{k-n} f_k(\theta) + \left(h(r)\cos(\theta) + \frac{\sigma^2}{2}\right) r^{-n-1} \\ Q(r,\theta) &= \sum_{k=0}^{n} r^{k-n-1} g_k(\theta) + h(r) r^{-n-2}\sin(\theta) \end{aligned} \tag{6.2}$$

where f_k, g_k are arbitrary bounded smooth functions on $\mathbf{R}$ and h is a smooth cutoff function satisfying

$$h(r) = \begin{cases} c & \text{for } r \geq 2 \\ 0 & \text{for } r \leq 1 \end{cases}$$

for some constant $c \in \mathbf{R}$.

As suggested by the appearances of r^n in (6.1), it is helpful to pull out a number of factors of r so that the resulting underlying dynamics is stabilized at infinity. More precisely, write

$$\mathcal{L} = r^n L \tag{6.3}$$

where

$$\begin{aligned} L = r\cos(n\theta)\partial_r + \sin(n\theta)\partial_\theta + \frac{\sigma^2}{2r^n}\partial_r^2 + \frac{\sigma^2}{2r^{n+2}}\partial_\theta^2 \\ + P(r,\theta)\partial_r + Q(r,\theta)\partial_\theta. \end{aligned} \tag{6.4}$$

We will see that using the operator L instead of $\mathcal{L}$ itself to define Ψ results in a number of simplifications, the most notable of which is that the asymptotic flow along L is much more straightforward than that of $\mathcal{L}$. Notice that this is expected since the stochastic dynamics (r_t, θ_t) defined by L moves according to a slower clock at infinity than the process (R_t, Θ_t) determined by $\mathcal{L}$. Indeed, observe that $(r_{T_t}, \theta_{T_t}) = (R_t, \Theta_t)$ where T_t is the time change

$$T_t = \int_0^t R_s^n ds.$$

Considering, too, the nature of Lyapunov functions, all results obtained in terms of L will translate easily back to the original operator $\mathcal{L}$ since $\mathcal{L} = r^n L$ and $r^n > 0$.

6.2. **The General Structure of Ψ.** We now take a look at some of the characteristics that our Ψ will exhibit. As dictated by the dynamics and suggested by previous works [1, 4], it is easiest construct Lyapunov functions piecewise. More precise reasons for why this is the case are given in the following section and in the sequel [8], but here we focus on, at least abstractly, how Ψ will look in our context.

We begin by partitioning $\mathbf{R}^2$ into the open ball of radius $r^* > 0$ about zero, denoted by $B_{r^*}(0)$, and a collection of closed regions $\{\mathcal{S}_i : i = 0, \dots, l\}$, the union of which captures all routes to infinity.

Definition 6.5. We say that a collection of subsets $\mathcal{S} = \{\mathcal{S}_i : i = 0, \dots, l\}$ is a GOOD RADIAL PARTITION OF $\mathcal{U} \subset \mathbf{R}^2$ if the following conditions hold:

a) Each $\mathcal{S}_i$ is closed and there exists an $r^* > 0$ so that

$$\bigcup_{j=1}^{l} \mathcal{S}_j = \mathcal{U} \cap \{(r,\theta) : r \geq r^*\}$$

b) For any $i \neq j$, interior$(\mathcal{S}_i) \cap$ interior$(\mathcal{S}_j) = \emptyset$.
c) For all distinct i,j, and k, then $\mathcal{S}_i \cap \mathcal{S}_j \cap \mathcal{S}_k = \emptyset$
d) For any $i \neq j$, $\mathcal{S}_i \cap \mathcal{S}_j$ is either empty or a collection of disjoint curves, each of which can be written as $\{(r, f(r)) : r \geq r^*\}$ for some smooth function f.

Definition 6.6. Given a good radial partition $\mathcal{S} = \{\mathcal{S}_i : i = 0, \dots, l\}$ of $\mathcal{U}$ and functions $f_* \colon B_{r^*}(0) \to \mathbf{R}$ and $f_i \colon \mathcal{S}_i \to \mathbf{R}$, we define the NATURAL EXTENSION TO $\mathcal{U}$ by

$$F(r,\theta) = \begin{cases} f_*(r,\theta) & \text{if } (r,\theta) \in B_{r^*}(0) \cap \mathcal{U} \\ f_i(r,\theta) & \text{if } (r,\theta) \in \text{interior}(\mathcal{S}_i) \\ \frac{1}{2}(f_i(r,\theta) + f_j(r,\theta)) & \text{if } (r,\theta) \in \mathcal{S}_i \cap \mathcal{S}_j. \end{cases}$$

In Section 7, we will succeed in constructing a good radial partition $\mathcal{S} = \{\mathcal{S}_i : i = 0, \dots, l\}$ of $\mathbf{R}^2$ and two collections of functions $\{\psi_i \colon \mathcal{S}_i \to (0,\infty) : i = 0, \dots, l\}$ and $\{\varphi_i \colon \mathcal{S}_i \to (0,\infty) : i = 0, \dots, l\}$ such that all functions are locally bounded and continuous and the ψ_i are C^2 on their domains, which we recall were assumed to be closed. Additionally, the (ψ_i, φ_i) pairs will satisfy the following bound with respect to $L = r^{-n}\mathcal{L}$ on $\mathcal{S}_i$

$$(L\psi_i)(r,\theta) \leq -m_i\varphi_i(r,\theta) + b_i \tag{6.7}$$

for some constants $m_i, b_i > 0$. The fact that each ψ_i is C^2 on the closed set $\mathcal{S}_i$ implies that $L\psi_i$ is continuous on $\mathcal{S}_i$ up to and including its boundary. Undoing the time change and using the fact that the number of inequalities is finite, it follows easily that on $\mathcal{S}_i$

$$(\mathcal{L}\psi_i)(r,\theta) \leq -m[r^n\varphi_i(r,\theta)] + b \tag{6.8}$$

for some global choice of constants $m, b > 0$.

Let Ψ and Φ be the natural extensions to $\mathbf{R}^2$ of the ψ_i's and φ_i's respectively. By equation (6.8) and Remark 4.3, it is clear that Ψ and

Φ are locally a Lyapunov pair on the interior of $\mathcal{S}_i$ for each $i = 1, \dots, l$. Unfortunately, we will see that this approach does not naturally produce a Ψ which is C^2. Rather, Ψ will only be globally continuous as it is possible that the first and second derivatives may not match along the boundaries between the regions $\mathcal{S}_i$ and $\mathcal{S}_j$. This prevents us from applying Itô's formula in a straightforward way to show that Ψ is a Lyapunov function in the sense of this paper.

A typical way around this difficulty is to smooth the function Ψ along these interfaces rendering it C^2. However when doing this, special care must be taken to preserve the Lyapunov property expressed in (6.8). This often leads to long and, at least to us, less than intuitive calculations. This was the approach taken in [1, 4]. Here we take a different path.

To deal with the issue at hand, we employ a generalization of Itô's Formula due to Peskir [12]. This result allows us to apply the Itô differential to functions which are not C^2 along a collection of curves expressing θ as a function of r. We now state a corollary of Peskir's formula in the context of our problem. A more general and detailed treatment is given Section 8 of Part II [8] along with a repackaged proof of a slightly weaker result suiting the needs of this paper. The proof of the following corollary is a direct consequence of Theorem 8.1 of that section provided one establishes the key jump conditions (6.10) along the curves $\mathcal{S}_i \cap \mathcal{S}_j$ of non-differentiability.

Corollary 6.9. *Let $\mathcal{S} = \{\mathcal{S}_i : i = 0, \dots, l\}$ be a good radial partition of $\mathbf{R}^2$ and suppose that $\{\psi_i \colon \mathcal{S}_i \to (0,\infty) : i = 0, \dots, l\}$ is a collection of C^2 functions and $\{\varphi_i \colon \mathcal{S}_i \to (0,\infty) : i = 0, \dots, l\}$ is a collection of locally bounded functions such that for each $i \in \{0, \dots, l\}$ the estimate in (6.7) holds. Furthermore, assume that for some ψ_* the natural extension Ψ of the ψ_i's is everywhere continuous and satisfies the flux condition*

$$\lim_{\substack{(R,\Theta)\to(r,\theta)\\ \Theta>\theta}} \partial_\Theta \Psi(R,\Theta) - \lim_{\substack{(R,\Theta)\to(r,\theta)\\ \Theta<\theta}} \partial_\Theta \Psi(R,\Theta) \leq 0 \tag{6.10}$$

for all (r,θ) with $r \geq r^*$. Then there exist some bounded function φ_* so that if Φ is the natural extension of the φ_i's then (Ψ, Φ) is a Lyapunov pair on $\mathbf{R}^2$.

Remark 6.11. The condition (6.10) speaks to the convexity along the curves where Ψ is not differentiable. Hence it is related to the classical generalization of Itô's formula to functions which are the difference of two convex functions.

Remark 6.12. As we will see, Ψ will be defined in such a way so that it is 2π-periodic in θ. In particular, since Ψ will be built so that it is C^2 at $\theta = \pm\pi$ for $r \geq r^*$, we will only need to check that the boundary-flux condition (6.10) is satisfied along those curves of non-differentiability, which are finite in number, on the set $\{r \geq r^*,\ -\pi \leq \theta \leq \pi\}$.

6.3. **Reduction of the Construction to the Principal Wedge.** First observe that any system which can be described by (6.1) remains a system which can be described by (6.1) (with perhaps different f_k's and g_k's) after being rotated by $\theta \mapsto \theta + \frac{2k\pi}{n}$ for any integer k . In particular, we now describe how we can use this fact to reduce the construction of our Lyapunov pair from $\mathbf{R}^2 \setminus B_{r^*}(0)$ to the *principal wedge*

$$\mathcal{R} = \{r \geq r^*,\ -\tfrac{\pi}{n} \leq \theta \leq \tfrac{\pi}{n}\}. \tag{6.13}$$

Defining the remaining wedges by

$$\mathcal{R}_k = \{(r,\theta) : (r, \theta - \tfrac{2k\pi}{n}) \in \mathcal{R}\},$$

we will now see that our construction on $\mathcal{R}$ will allow us to also build a Lyapunov Function on all of the of the other $\mathcal{R}_k$'s. This is the content of the following proposition, which is a straightforward consequence of the above observation.

Proposition 6.14. *Fix $n \geq 1$ in (5.1). Assume that there exists positive constants γ, δ and p so that for any system of the form (5.1), there exists a good radial partition $\{\mathcal{S}_i : i = 0,\dots,l\}$ of $\mathcal{R}$ and a collection of C^2-functions $\{\psi_i \colon \mathcal{S}_i \to (0,\infty) : i = 0,\dots,l\}$ satisfying the bound*

$$\mathcal{L}\psi_i(r,\theta) \leq -m\big[r^\gamma \vee \psi_i^{1+\delta}(r,\theta)\big] + b$$

on $\mathcal{S}_i$ for $i = 0,\dots,l$ and some positive constants m and b. Furthermore, assume that there exists a function ψ_* so that the natural extension Ψ of the ψ_i's on $\mathcal{R}$ is globally continuous, satisfies the flux condition (6.10) for all $(r,\theta) \in \mathcal{R}$, and is such that $\Psi(r,\theta) = r^p$ for all $(r,\theta) \in \mathcal{R}$ with $|\theta - \frac{\pi}{n}| \wedge |\theta + \frac{\pi}{n}| \leq \epsilon$ for some $\epsilon > 0$. Then there exists a function Ψ defined on all of $\mathbf{R}^2$ so that $(\Psi, \Psi^{1+\delta})$ and $(\Psi, |z|^\gamma)$ are Lyapunov pairs corresponding to the dynamics (5.1).

Remark 6.15. To prove Theorem 5.2, we construct Ψ on $\mathcal{R}$ satisfying the properties above, the hypotheses of Proposition 6.14 with $\gamma \in (n, 2n)$ arbitrary, and the following bound on $\mathcal{R}$

$$cr^{\gamma - n} \leq \Psi(r,\theta) \leq dr^{\gamma - n + \frac{n}{2} + 1} \tag{6.16}$$

for some positive constants c, d.

Remark 6.17. Since $\Psi(r,\theta)=r^p$ in a neighborhood of $\theta=\pm\frac{\pi}{n}$, this allows us rotate Ψ, initially defined only on the principle wedge $\mathcal{R}$, by integer multiples of $\frac{2\pi}{n}$ to produce the desired globally-defined Lyapunov pairs. Moreover, after such rotations the flux condition (6.10) will be satisfied globally.

7. The Construction of Ψ on the Principal Wedge in a Simple Case

In this section, we will build Ψ on $\mathcal{R}$ under Assumption 5.7. We will see that this assumption assures that the lower-order terms collected in F in the drift part of (5.1) play no role in the arguments.

The layout of this section is as follows. First in Section 7.1, we study the asmyptotic behavior of L as $r\to\infty$. This will help yield the fundamental building blocks of the construction procedure: the so-called asymptotic operators and their associated regions. In Section 7.2, we explain the intuition behind how the local functions ψ_i will be defined in Section 7.3 as solutions to certain PDEs involving these operators. In Section 7.3, we will also see that each ψ_i is smooth and non-negative on $\mathcal{S}_i$ and that $\psi_i(r,\theta)\to\infty$ as $r\to\infty$ with $(r,\theta)\in\mathcal{S}_i$.

In Section 8, we will finish proving Theorem 5.2 under Assumption 5.7 by checking the details outlined in Remark 6.15.

7.1. The Asymptotic Operators and Their Associated Regions.

Intuitively it is clear that certain terms in the operator L are asymptotically dominant over other terms as $r\to\infty$ and such dominance can change from region to region in the plane. Here our goal is to elucidate these ideas by studying more carefully L along various paths to infinity. Doing such analysis is indispensable, as the dominant balances of terms in L yielded from it will be used to construct the local Lyapunov functions ψ_i in subsequent sections.

Throughout, we will make use of the scaling transformations

$$S_\alpha^\lambda\colon (r,\theta)\mapsto(\lambda r,\lambda^{-\alpha}\theta).$$

for any $\lambda\geq 1$ and $\alpha\geq 0$ in order to parameterize the various routes to infinity. In particular, we will determine, heuristically, the behavior as $\lambda\to\infty$ of

$$\begin{aligned}L\circ S_\alpha^\lambda(r,\theta)&=r\cos(n\theta\lambda^{-\alpha})\partial_r+\lambda^\alpha\sin(n\theta\lambda^{-\alpha})\partial_\theta+\lambda^{-2-n}\frac{\sigma^2}{2r^n}\partial_r^2\\&\quad+\lambda^{2\alpha-n-2}\frac{\sigma^2}{2r^{n+2}}\partial_\theta^2+\lambda^{-1}P(\lambda r,\lambda^{-\alpha}\theta)\partial_r+\lambda^\alpha Q(\lambda r,\lambda^{-\alpha}\theta)\partial_\theta.\end{aligned}$$

Because we have restricted the construction to the principal wedge $\mathcal{R}$, we only consider routes to infinity contained in $\mathcal{R}$. The two cases $\alpha = 0$ and $\alpha > 0$ are qualitatively different, so they are handled separately.

Suppose first that $\alpha = 0$. Provided $\theta \neq 0$, we see that the fist two terms in $L \circ S_0^\lambda(r,\theta)$ are unchanged and all other terms go to zero as $\lambda \to \infty$. More precisely,

$$L \circ S_0^\lambda(r,\theta) = r\cos(n\theta)\partial_r + \sin(n\theta)\partial_\theta + O(\lambda^{-1}) \text{ as } \lambda \to \infty.$$

We thus conclude that the leading order behavior of L as $r \to \infty$ in $\mathcal{R}$ along the rays traced out by $\lambda \mapsto S_0^\lambda(r,\theta)$ (with $\theta \neq 0$) is given by

$$T_1 = r\cos(n\theta)\partial_r + \sin(n\theta)\partial_\theta\,.$$

Of course one is not simply restricted to such radial paths. So long as one does not asymptote to the line $\theta = 0$ then the same dominate balances hold. More precisely, $L \approx T_1$ as $r \to \infty$ when the paths to infinity are restricted to a region $\mathcal{S}_1$ of the form

$$\mathcal{S}_1 = \{(r,\theta) \in \mathcal{R} \,:\, 0 < \theta_1^* \leq |\theta| \leq \theta_0^* \leq \tfrac{\pi}{n}\} \tag{7.1}$$

for any fixed positive constants $\theta_1^* > \theta_0^* > 0$.

The situation becomes more complicated if $|\theta| \to 0$ as $r \to \infty$. To see what happens, we begin by considering $L \circ S_\alpha^\lambda(r,\theta)$ as $\lambda \to \infty$ for $\alpha > 0$. In this setting as $\lambda \to \infty$,

$$\begin{aligned} L \circ S_\alpha^\lambda(r,\theta) = r\partial_r + n\theta\partial_\theta + \lambda^{d+\alpha-(n+1)} r^{d-(n+1)} g_d(0)\partial_\theta + \lambda^{2\alpha-(n+2)}\frac{\sigma^2}{2r^{n+2}}\partial_\theta^2 \\ + O(\lambda^{-1}) + o(\lambda^{d+\alpha-(n+1)}) \end{aligned}$$

where $d \in \{0, 1, \dots, n\}$ is the largest index for which $g_d(0) \neq 0$ where the g_k are the coefficient functions of $Q(r,\theta)$ introduced in (6.2). If no such index exists, then

$$\lambda^{d+\alpha-(n+1)} g_d(0)\partial_\theta + o(\lambda^{d+\alpha-(n+1)})$$

is simply absent from the expression above and the following analysis still holds regardless.

First realize that if $\alpha > 0$ is sufficiently small, the linearization of T_1

$$T_2 = r\partial_r + n\theta\partial_\theta$$

gives the leading order asymptotic behavior as $r \to \infty$. Recalling the discussion of T_1 above, we see that even when $\alpha = 0$, we have that L is asymptotically well approximated by T_2 provided $|\theta|$ is small since

$$L \circ S_0^\lambda(r,\theta) = r\partial_r + n\theta\partial_\theta + O(\lambda^{-1}) + O(\theta^2)\,.$$

In particular, one has $L \approx T_2$ as $r \to \infty$ provided the paths to infinity are restricted to a region of the form

$$\mathcal{S}_2 = \{(r,\theta) \in \mathcal{R} \,:\, b(r) \leq |\theta| \leq \theta_1^*\}$$

were $\theta_1^* > 0$ is small and the boundary curve $b(r)$ has the property that $b(r) \to 0$ sufficiently slowly as $r \to \infty$. To define b explicitly and also discover what happens to L when $|\theta| \leq b(r)$, we must see for what powers of α other terms in the expansion $L \circ S_\alpha^\lambda(r,\theta)$ become asymptotically relevant as $\lambda \to \infty$.

We now claim that Assumption 5.7 allows to disregard

$$\lambda^{d+\alpha-(n+1)} r^{d-(n+1)} g_d(0)\partial_\theta + o(\lambda^{d+\alpha-(n+1)})$$

in $L \circ S_\alpha^\lambda(r,\theta)$ as $\lambda \to \infty$ for all choices of $\alpha \geq 0$. Indeed, the value of $\alpha \geq 0$ where

$$\lambda^{d+\alpha-(n+1)} r^{d-(n+1)} g_d(0)\partial_\theta + \lambda^{2\alpha-(n+2)} \frac{\sigma^2}{2r^{n+2}}\partial_\theta^2 + O(\lambda^{-1}) + o(\lambda^{d+\alpha-(n+1)}) \tag{7.2}$$

is $O(1)$ as $\lambda \to \infty$ is precisely

$$\alpha = \frac{n+2}{2}.$$

But note that for $\alpha \geq \frac{n+2}{2}$, by Assumption 5.7 the term

$$\lambda^{2\alpha-(n+2)} \frac{\sigma^2}{2r^{n+2}}\partial_\theta^2$$

dominates all the remaining contributions in (7.2) in λ as $\lambda \to \infty$.

If Assumption 5.7 is not made, then the term

$$\lambda^{d+\alpha-(n+1)} r^{d-(n+1)} g_d(0)\partial_\theta \tag{7.3}$$

initially dominates the remaining terms in (7.2) as $\lambda \to \infty$. However, at a certain threshold in α, (7.3) can cancel with $n\theta\partial_\theta$ implying that we must expand $L \circ S_\alpha^\lambda(r,\theta)$ further asymptotically in λ to uncover the next lower-order term. The total analysis in the general case is quite involved and requires another novel idea. This, in addition to our desire to focus first on the general elements of the construction, is why we save it for Part II [8].

Again operating under Assumption 5.7, observe that the above analysis suggests that the operator

$$A = r\partial_r + n\theta\partial_\theta + \frac{\sigma^2}{2r^{n+2}}\partial_\theta^2 \tag{7.4}$$

satisfies $L \approx A$ as $r \to \infty$ in the remaining portion of $\mathcal{R}$, namely $\{(r,\theta) \in \mathcal{R} : |\theta| \leq b(r)\}$. To determine the correct choice of $b(r)$ note that

$$2\alpha - (n+2) \geq 0 \iff \alpha \geq \frac{n+2}{2}.$$

Specifically, the threshold $\alpha = \frac{n+2}{2}$ is precisely where

$$r\partial_r + n\theta\partial_\theta + \lambda^{2\alpha-(n+2)}\frac{\sigma^2}{2r^{n+2}}\partial_\theta^2 = O(1)$$

as $\lambda \to \infty$. Therefore, we now definitively set

$$\mathcal{S}_2 = \{(r,\theta) \in \mathcal{R} \,:\, r \geq r^*,\, \eta^* r^{-\frac{n+2}{2}} \leq |\theta| \leq \theta_1^*\}$$
$$\mathcal{S}_3 = \{(r,\theta) \in \mathcal{R} \,:\, r \geq r^*,\, |\theta| \leq \eta^* r^{-\frac{n+2}{2}}, |\theta| \leq \theta_1^*\}$$

where $\eta^* > 0$ is a constant which will be chosen later. Intuitively though, η^* should be thought of as large so that in $\mathcal{S}_2$ the term $\frac{\sigma^2}{2r^{n+2}}\partial_\theta^2$ is small in comparison to the rest of A. Hence we still expect the approximation $L \approx T_2$ as $r \to \infty$ to hold when paths to infinity are restricted to $\mathcal{S}_2$ even though $\frac{\sigma^2}{2r^{n+2}}\partial_\theta^2$ does not vanish on the lower boundary curve. This term will be negligible on this curve not because $r \to \infty$ but rather because η^* was chosen large which makes the coefficient in front of $\frac{\sigma^2}{2r^{n+2}}\partial_\theta^2$ small.

Remark 7.5. Notice that for any choice of $\eta^*, \theta_1^* > 0$, we may always pick $r^* > 0$ large enough so that the bound $|\theta| \leq \theta_1^*$ can be removed from the definition of $\mathcal{S}_3$. In particular after making this choice, $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ are a elements of a good radial partition of $\mathcal{R}$ as discussed in Section 6.2 and Section 6.3.

In summary, under Assumption 5.7 we have found the *asymptotic operators* T_1, T_2 and A which "approximate" L well for $r > 0$ large in the regions $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$ respectively.

7.2. **Overview of Local Lyapunov Function Construction.** Recall that for a sufficiently smooth function Ψ, Itô's formula states that the time-changed process (r_t, θ_t) with generator L as in (6.7) satisfies

$$\mathbf{E}\Psi(r_{t\wedge\tau}, \theta_{t\wedge\tau}) = \Psi(r_0,\theta_0) + \mathbf{E}\int_0^{t\wedge\tau}(L\Psi)(r_s,\theta_s)ds \tag{7.6}$$

for any stopping time τ such that $\Psi(r_{t\wedge\tau}, \theta_{t\wedge\tau})$ is almost surely bounded for all $t \geq 0$. Comparing this equality with the inequality satisfied by a Lyapunov pair in (4.2), we conclude that it would be enough to find a non-negative function Ψ such that $\Psi(r,\theta) \to \infty$ as $r \to \infty$ and

$$(L\Psi)(r,\theta) \approx -\Phi(r,\theta)$$

for some nonnegative continuous function Φ satisfying $\Phi(r,\theta)\to\infty$ as $r\to\infty$.

As mentioned in Section 6.2, most notably in Corollary 6.9, we will construct our Lyapunov function in piecewise fashion. The regions used in the construction are precisely the $\{\mathcal{S}_i : i=1,2,3\}$ from Section 7.1. On each $\mathcal{S}_i$ we will construct a ψ_i such that

$$(L\psi_i)(r,\theta)\approx -c\,\varphi_i(r,\theta) \tag{7.7}$$

for r large, $(r,\theta)\in\mathcal{S}_i$, and such that $\psi_i(r,\theta)\to\infty$ as $r\to\infty$ in $\mathcal{S}_i$. In the expression above, $c>0$ is a constant and φ_i is a nonnegative function satisfying $\varphi_i(r,\theta)\to\infty$ as $r\to\infty$, $(r,\theta)\in\mathcal{S}_i$. We will then use the natural extensions of the ψ_i's and φ_i's, as defined in Section 6.2, to yield a Lyapunov pair. To do so, we will have to ensure that our construction produces functions which satisfy the hypotheses of Proposition 6.14.

The advantage of this methodology is that on each $\mathcal{S}_i$, L is well approximated for large r by a simplified differential operator as identified in Section 7.1. Hence we will aim to construct ψ_i satisfying (7.7) but with L replaced by this simplified operator.

Since every simplified operator contains only terms which dominate at infinity under the associated scaling transformation of that region, each of the simplified operators necessarily scales homogeneously under this transformation. This will be critical in showing that the terms neglected when using the simplified operators are in fact negligible. We now illustrate this by considering each of the regions in order. In this overview, we still omit some of the details leaving them for Section 7.3 where we will give the precise constructions of the local Lyapunov functions.

Beginning in $\mathcal{S}_1$, since $L\approx T_1$ for large r this suggests defining ψ_1 as the solution of the equation

$$(T_1\psi_1)(r,\theta)=-c\,\varphi_1(r,\theta) \tag{7.8}$$

on $\mathcal{S}_1$ with the appropriate boundary conditions. For (7.8) to imply that (7.7) holds for large r in $\mathcal{S}_1$, we need to know that the terms in $(L-T_1)\psi_1$ are negligible asymptotically as $r\to\infty$, $(r,\theta)\in\mathcal{S}_1$, when compared to those in $T_1\psi_1$. By the analysis of Section 7.1, we expect the terms $(L-T_1)\psi_1$ to be negligible if ψ_1 scales homogeneously of degree $p>0$ under S_0^λ, for then $\psi_1(r,\theta)=r^p\frac{1}{(r^*)^p}\psi_1(r^*,\theta)$. In other words, $\psi_1(r,\theta)=r^pC(\theta)$ for some function $C(\theta)$, so the action of L on ψ_1 will mimic the action of S_0^λ on L.

To see why we are able to construct ψ_1 so as to have this homogeneously scaling property, first observe that T_1 scales homogeneously

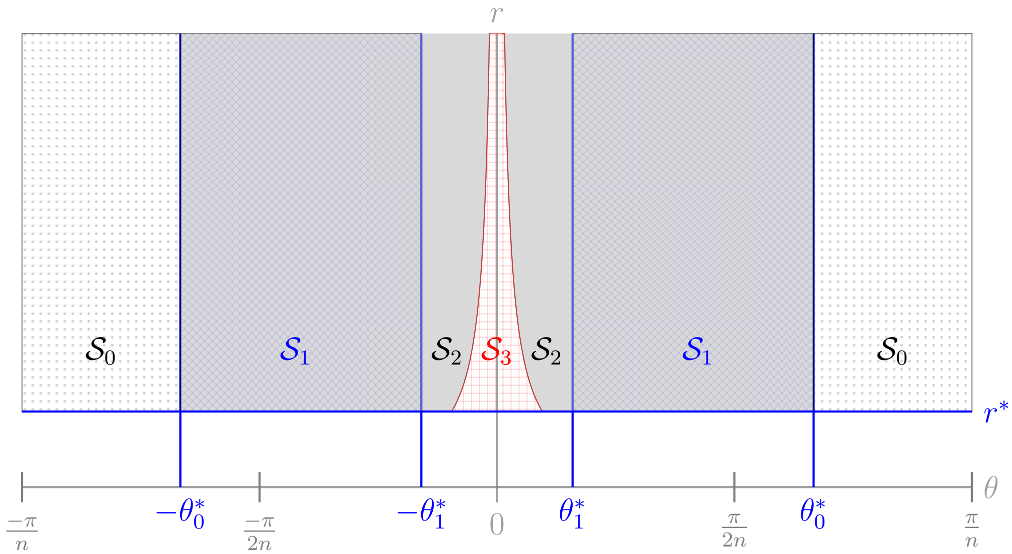


FIGURE 4. The regions $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$. In the diagram, $\theta_1^* > 0$ is chosen much larger than in reality to make visualization easier. The regions $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$ are discussed in Section 7.1–7.3 while $\mathcal{S}_0$ is only introduced in Section 7.3.

under S_0^λ. Therefore if one chooses φ_1 in (7.8) to scale homogeneously under S_0^λ, then as the solution to (7.8), ψ_1 will, with the appropriate boundary data, also scale homogeneously under S_0^λ with the same scaling exponent as φ_1. If we chose the scaling exponent to be positive and φ_1 to be continuous, then it will be relatively easy to see that both $\psi_1(r,\theta), \varphi_1(r,\theta) \to \infty$ as $r \to \infty$, $(r,\theta) \in \mathcal{S}_1$, and

$$(L\psi_1)(r,\theta) \le -m\,\varphi_1(r,\theta) + b,$$

for some $m, b > 0$.

Jumping ahead to region $\mathcal{S}_3$, since $L \approx A$ for large r, it makes sense to choose ψ_3 as the solution of the equation

$$(A\psi_3)(r,\theta) = -c\,\varphi_3(r,\theta) \tag{7.9}$$

on $\mathcal{S}_3$ where $c > 0$ and $\varphi_3(r,\theta) \to \infty$ as $r \to \infty$, $(r,\theta) \in \mathcal{S}_3$. Since $\lambda \mapsto S^\lambda_{\frac{n+2}{2}}(r^*,\theta)$ covers $\mathcal{S}_3$ as θ varies in $\mathcal{S}_3$ and A is invariant under $S^\lambda_{\frac{n+2}{2}}$, the same reasoning used above suggests that we should choose φ_3 to be homogeneous of positive degree under $S^\lambda_{\frac{n+2}{2}}$. Again, it will then follow that, with the appropriate boundary data, ψ_3 and all of its derivatives are asymptotically homogeneous under $S^\lambda_{\frac{n+2}{2}}$ and that

$(L-A)\psi_3$ is negligible relative to $A\psi_3$. Thus by the results of Section 7.1, we anticipate the following bound

$$(L\psi_3)(r,\theta) \leq -m\varphi_3(r,\theta) + b,$$

on $\mathcal{S}_3$ for some positive constants m, b.

The set $\mathcal{S}_2$ serves as a transition region between the two other sets $\mathcal{S}_1$ and $\mathcal{S}_3$. Hence ψ_2 must connect ψ_1, which scales homogeneously under S_0^λ in $\mathcal{S}_1$, to ψ_3, which scales homogeneously under $S_{\frac{n+2}{2}}^\lambda$ in $\mathcal{S}_3$. Thus we should setup the equation so that ψ_2 and its derivatives asymptotically scale homogeneously under both mappings, otherwise the ψ_i together could not be extended to a continuous function as required in Corollary 6.9. Requiring this duel scaling property is further suggested by the fact that both paths of the form $\lambda \mapsto S_0^\lambda(r_*, \theta)$ and $\lambda \mapsto S_{\frac{n+2}{2}}^\lambda(r_*, \theta)$ are required to cover $\mathcal{S}_2$.

Since $L \approx T_2$ for $r > 0$ large in $\mathcal{S}_2$, we take ψ_2 as the solution of

$$(T_2\psi_2)(r,\theta) = -c\,\varphi_2(r,\theta) \tag{7.10}$$

on $\mathcal{S}_2$. Since T_2 is homogeneous of degree zero under S_α^λ for any $\alpha \geq 0$, choosing φ_2 to scale homogeneously under S_α^λ for all $\alpha \geq 0$ with positive degree will, with the right choice of boundary data, lead us to a ψ_2 which will asymptotically scale homogeneously under S_α^λ for all $\alpha \geq 0$ with the same positive degree.

7.3. **Defining the Local Lyapunov Functions.** So far we have subdivided the principal wedge $\mathcal{R}$ into the following three regions:

$$\mathcal{S}_1 = \{r \geq r^*,\ 0 < \theta_1^* \leq |\theta| \leq \theta_0^* \leq \tfrac{\pi}{n}\}$$
$$\mathcal{S}_2 = \{r \geq r^*,\ |\theta| \leq \theta_1^*,\ r^{\frac{n+2}{2}}|\theta| \geq \eta^*\}$$
$$\mathcal{S}_3 = \{r \geq r^*,\ |\theta| \leq \theta_1^*,\ r^{\frac{n+2}{2}}|\theta| \leq \eta^*\}$$

where r^*, $\eta^* > 0$ (see Figure 4). To initialize the construction procedure, we will in fact need an additional region $\mathcal{S}_0$ given by

$$\mathcal{S}_0 = \{(r,\theta) \in \mathcal{R} \,:\, r \geq r^*,\ \theta_0^* \leq |\theta| \leq \tfrac{\pi}{n}\}$$

where we fix $\theta_0^* \in (\frac{\pi}{2n}, \frac{\pi}{n})$. Because the vector field induced by T_1 points radially inward in $\mathcal{S}_0$, defining

$$\psi_0(r,\theta) = r^p,\ p > 0, \tag{7.11}$$

and noting that ψ_0 clearly scales homogeneous under S_0^λ, it follows easily that

$$L\psi_0(r,\theta) \leq -mr^p + b,$$

on $\mathcal{S}_0$ for some positive constants m, b. The function ψ_0 will now serve as the boundary condition for the equation satisfied by ψ_1 which is defined on the neighboring region $\mathcal{S}_1$.

Remark 7.12. Each of the regions comes with a number of parameters such as $\theta_1^* > 0$, $\eta^* > 0$ and $r^* > 0$. Instead of giving these constants specific values at the start, it is much easier to leave them as parameters because they will need to be adjusted many times. To assure that each of these adjustments is consistent, we note that throughout we will always pick $\theta_1^* > 0$ sufficiently small, then $\eta^* = \eta^*(\theta_1^*) > 0$ sufficiently large, and then $r^* = r^*(\theta_1^*, \eta^*) > 0$ sufficiently large.

This multitude of parameters is not central to the construction. Mainly, they exist in order to quantify the idea of being "close enough" to ∞ or 0 so that certain asymptotic properties hold up to some error. The parameter p from the definition of ψ_0 is different as it measures growth at infinity of our Lyapunov function, thus having implications for the decay at infinity of any invariant measure. Hence the maximal allowed p is a fundamental property of the system whose value we will see is linked to the exit distribution from the region $\mathcal{S}_3$ of the diffusion process defined by L.

The Construction in $\mathcal{S}_1$. Choosing $p \in (0, n)$, the function ψ_1 is defined as the solution to the following equation

$$\begin{cases} (T_1\psi_1)(r,\theta) & = -h_1 r^p |\theta|^{-q} \\ \psi_1(r, \pm\theta_0^*) & = \psi_0(r, \pm\theta_0^*) \end{cases} \tag{7.13}$$

on $\mathcal{S}_1$ where $h_1 > 0$ and $q \in (p/n, 1)$. We mention the restrictions on p and q now for simplicity. They will not truly be needed until the construction in the region $\mathcal{S}_3$. However, putting them in place now will prevent us from considering generalities now which, in the end, are not needed given the restrictions stemming from the dynamics in $\mathcal{S}_3$. Notice that the form of the righthand side of (7.13) scales homogeneously under S_0^λ as suggested by the considerations in Section 7.2. The dependence on θ is introduced to facilitate matching with ψ_2 along the boundary with $\mathcal{S}_2$.

Since we have picked $\theta_0^* > \frac{\pi}{2n}$, one notices that the PDE given in (7.13) is not well-defined with the given boundary data since some of the characteristics along T_1 cross $r = r^*$ before reaching the lines $|\theta| = \theta_0^*$. Because it is convenient to only give data on the lines $|\theta| = \theta_0^*$, we slightly modify the domain of definition of the PDE to be

$$\widetilde{\mathcal{S}}_1 = \Big\{(r,\theta) \in \mathcal{R} \,:\, 0 < \theta_1^* \le |\theta| \le \theta_0^*,\, r|\sin(n\theta_0^*)|^{\frac{1}{n}} \ge r^*\Big\}.$$

With this modification, all characteristics now exit the domain through the boundary $r \geq r^*$, $|\theta| = \theta_0^*$. Thus, solving (7.13), we see that for $(r,\theta) \in \widetilde{\mathcal{S}}_1$

(7.14)
$$\psi_1(r,\theta) = \begin{cases} \dfrac{r^p}{|\sin(n\theta)|^{\frac{p}{n}}}\left(\big|\sin(n\theta_0^*)\big|^{\frac{p}{n}} + h_1\displaystyle\int_\theta^{\theta_0^*} \frac{|\sin(n\alpha)|^{\frac{p}{n}}}{|\alpha|^q \sin(n\alpha)}d\alpha\right) & \text{if } \theta > 0 \\ \dfrac{r^p}{|\sin(n\theta)|^{\frac{p}{n}}}\left(\big|\sin(n\theta_0^*)\big|^{\frac{p}{n}} + h_1\displaystyle\int_\theta^{-\theta_0^*} \frac{|\sin(n\alpha)|^{\frac{p}{n}}}{|\alpha|^q \sin(n\alpha)}d\alpha\right) & \text{if } \theta < 0. \end{cases}$$

In particular, we note that ψ_1 can be extended smoothly to all of $\mathcal{S}_1$ and is a homogeneous function of degree p under S_0^λ. Moreover, $\psi_1(r,\theta) \geq 0$ on $\mathcal{S}_1$ and $\psi_1(r,\theta) \to \infty$ as $r \to \infty$ with $(r,\theta) \in \mathcal{S}_1$.

The Construction in $\mathcal{S}_2$. Let ψ_2 be defined on $\mathcal{S}_2$ by

$$\begin{cases} (T_2\psi_2)(r,\theta) = -h_2 r^p |\theta|^{-q} \\ \psi_2(r,\pm\theta_1^*) = \psi_1(r,\pm\theta_1^*) \end{cases} \tag{7.15}$$

where $h_2 > 0$. This time, the PDE above is clearly well-defined. Observe that the righthand side of (7.15) scales homogeneously under S_α^λ for all $\alpha \geq 0$ as the considerations in Section 7.2 suggested.

Using the method of characteristics we see that

$$\psi_2(r,\theta) = \left(|\theta_1^*|^{\frac{p}{n}}\psi_1(1,\theta_1^*) - h_2\frac{|\theta_1^*|^{\frac{p}{n}-q}}{qn-p}\right)\frac{r^p}{|\theta|^{\frac{p}{n}}} + \frac{h_2}{qn-p}\frac{r^p}{|\theta|^q} \tag{7.16}$$

In particular, we notice that ψ_2 is homogeneous under S_0^λ of degree p and is the sum of two terms which are each homogeneous under S_α^λ for every $\alpha \geq 0$ (though each term has a different degree). Moreover, on $\mathcal{S}_2$

$$\psi_2(r,\theta) \geq c r^p |\theta|^{-\frac{p}{n}}$$

for some $c > 0$. Hence $\psi_2(r,\theta) \geq 0$ on $\mathcal{S}_2$ and $\psi_2(r,\theta) \to \infty$ as $r \to \infty$ with $(r,\theta) \in \mathcal{S}_2$.

The Construction in $\mathcal{S}_3$. Finally, define ψ_3 on $\mathcal{S}_3$ as the solution of the equation

$$\begin{cases} (A\psi_3)(r,\theta) = -h_3 r^{p_3} \\ \psi_3(r,\theta) = \psi_2(r,\theta), \qquad r^{\frac{n+2}{2}}|\theta| = \eta^* \end{cases} \tag{7.17}$$

where $h_3 > 0$ and $p_3 = p + q\frac{n+2}{2}$. Again, our choice of righthand side is consistant with the considerations of Section 7.2 since it scales homogeneously under S_α^λ for all $\alpha \geq 0$.

Recycling notation, we let (r_t, θ_t) denote the stochastic flow generated by A and define

$$\tau = \inf\Big\{t > 0 \,:\, r_t^{\frac{n+2}{2}} |\theta_t| = \eta^*\Big\}.$$

Setting $p_2 = p + \frac{p}{n}\frac{n+2}{2} = p(\frac{3n+2}{2n})$, notice we may express ψ_3 formally as

$$\psi_3(r,\theta) = c_1 r^{p_3} \mathbf{E}_{(r,\theta)} e^{p_3 \tau} + c_2 r^{p_2} \mathbf{E}_{(r,\theta)} e^{p_2\tau} - c_3 r^{p_3} \tag{7.18}$$

where

$$\begin{aligned} c_1 &= \frac{h_3}{p_3} + \frac{1}{(\eta^*)^q}\frac{h_2}{qn-p}, \\ c_2 &= \frac{1}{(\eta^*)^{p/n}}\left[|\theta_1^*|^{p/n}\psi_1(1,\theta_1^*) - h_2 \frac{|\theta_1^*|^{p/n-q}}{qn-p}\right], \\ c_3 &= \frac{h_3}{p_3}. \end{aligned}$$

The main question in the validity of this expression arises from the possibility that $\mathbf{E}_{(r,\theta)} e^{p_3\tau}$ or $\mathbf{E}_{(r,\theta)} e^{p_2\tau}$ might be infinite. The restriction that $p \in (0,n)$ and $q \in (p/n, 1)$, implies that $p_2, p_3 \in (0, \frac{3}{2}n + 1)$. Because of this restriction, Proposition 7.20 below will imply that $\mathbf{E}_{(r,\theta)} e^{p_i\tau} < \infty$ for $i = 2, 3$. Thus, the expression (7.17) is well defined.

To setup the statement of the result, first observe that the process η_t defined by

$$\eta_t := r_t^{\frac{n+2}{2}} \theta_t$$

solves the Gaussian SDE

$$d\eta_t = \big(\tfrac{3}{2}n + 1\big)\eta_t \, dt + \sigma \, dW_t. \tag{7.19}$$

In particular, we may write $\tau = \inf\{t > 0 \,:\, |\eta_t| = \eta^*\}$ and

$$\mathbf{E}_{(r,\theta)} e^{p_i\tau} = \mathbf{E}_\eta e^{p_i \tau}$$

where $\eta = r^{\frac{n+2}{2}}\theta$ for $i = 2, 3$.

Proposition 7.20. *Fix a constant $c \in \mathbb{R}$, let $\eta^* > |c|$, and define*

$$\tau_c = \inf_{t>0}\{t > 0 \,:\, \eta_t \notin [-\eta^* + c, \eta^* + c]\}.$$

Then for all $\eta \in [-\eta^* + c, \eta^* + c]$ and $a < \frac{3}{2}n + 1$

$$\mathbf{E}_\eta e^{a\tau_c} < \infty.$$

In particular, $\tau_0 = \tau$ almost surely and the mappings

$$(r,\theta) \mapsto \mathbf{E}_{(r,\theta)} e^{p_i\tau}$$

for $i = 2, 3$ *are well-defined and homogeneous of degree 0 under* $S^\lambda_{\frac{n+2}{2}}$.

Remark 7.21. In this article we only need the case $c = 0$. However in the sequel [8], we will need the full strength of Proposition 7.20; that is, all results above when $c \neq 0$.

Proof of Proposition 7.20. All we need to show is that $\mathbf{E}_\eta e^{a\tau_c} < \infty$ whenever $a < \frac{3}{2}n + 1$. The case when $a \leq 0$ is clear, thus suppose $0 < a < \frac{3}{2}n + 1$. Since η_t with $\eta_0 = \eta$ is normally distributed with mean $e^{(\frac{3}{2}n+1)t}\eta$ and variance

$$\frac{\sigma^2}{3n+2}(e^{(3n+2)t} - 1),$$

we obtain

$$\begin{aligned}
\mathbf{E}_\eta e^{a\tau_c} &= \int_0^\infty \mathbf{P}_\eta\{e^{a\tau_c} > t\}\, dt \\
&\leq 2 + \int_2^\infty \mathbf{P}_\eta\{\tau_c > a^{-1}\log(t)\}\, dt \\
&\leq 2 + \int_2^\infty \mathbf{P}_\eta\{\eta_{a^{-1}\log(t)} \in [-\eta^* + c, \eta^* + c]\}\, dt \\
&\leq 2 + K\int_2^\infty \frac{1}{\sqrt{t^{\frac{3n+2}{a}} - 1}}\, dt
\end{aligned}$$

for some constant $K > 0$. Notice that the last integral above is finite provided $0 < a < \frac{3}{2}n + 1$, giving the claimed result. □

Notice by the proof of Proposition 7.20 we may write

$$\psi_3(r, \theta) = c_1 r^{p_3}\mathbf{E}_\eta e^{p_3\tau} + c_2 r^{p_2}\mathbf{E}_\eta e^{p_2\tau} - c_3 r^{p_3} \tag{7.22}$$

where $\eta = r^{\frac{n+2}{2}}\theta$. This implies that ψ_3 is the sum of three terms, each of which is homogeneous under the scaling transformation $S^\lambda_{\frac{n+2}{2}}$. In Appendix A, we will see that functions of the form

$$\eta \mapsto \mathbf{E}_\eta e^{a\tau}, \quad a < \tfrac{3}{2}n + 1$$

are smooth up to the boundary $|\eta| = \eta^*$, allowing us to conclude, moreover, that $\psi_3 \in C^\infty(\mathcal{S}_3)$. Also, it is not hard to see that on $\mathcal{S}_3$

$$\psi_3(r, \theta) \geq c r^{p_2}\mathbf{E}_{(r,\theta)} e^{p_2\tau}$$

for some $c > 0$. Hence, $\psi_3(r, \theta) \geq 0$ on $\mathcal{S}_3$ and $\psi_3(r, \theta) \to \infty$ as $r \to \infty$ with $(r, \theta) \in \mathcal{S}_3$.

7.4. **The Relationship Between the Scaling of $\mathcal{S}_1$ and $\mathcal{S}_3$ and the Origin of the Restriction on p.** Now that the basic construction is finished, let us take a moment to elucidate the relationship between the scaling exponents p and p_2. We will show that the shape of region $\mathcal{S}_2$ dictates the relationship between the two. This is fundamental to understanding the problem since we saw that the equation for ψ_3 places a restriction of the exponent p_2 which in turn cascades through the remaining dependencies to place a restriction on p.

The function ψ_2 consists of two terms: one comes from the boundary data propagated along the flow and the other from integrating the right-hand side along the characteristics. As a function of θ, the characteristics corresponding to T_2 stopped at the boundary $r \geq r^*$, $|\theta| = \theta_1^*$ are given by

$$r : \theta \mapsto r^* \left|\frac{\theta_1^*}{\theta}\right|^{\frac{1}{n}} \tag{7.23}$$

and hence, denoting the part of solution ψ_2 which comes from the boundary data by $\hat{\psi}_2$, one has

$$\begin{aligned}\hat{\psi}_2 \circ S_\alpha^\lambda(r,\theta) &= \psi_1\Big(\lambda r \Big|\frac{\theta_1^*}{\lambda^{-\alpha}\theta}\Big|^{\frac{1}{n}}, \theta_1^*\Big) = \lambda^{p(1+\frac{\alpha}{n})} r^p \Big|\frac{\theta_1^*}{\theta}\Big|^{\frac{p}{n}} \psi_1(1,\theta_1^*) \\ &= \lambda^{p(1+\frac{\alpha}{n})}\psi_1\Big(r\Big|\frac{\theta_1^*}{\theta}\Big|^{\frac{1}{n}}, \theta_1^*\Big) = \lambda^{p(1+\frac{\alpha}{n})}\hat{\psi}_2(r,\theta)\end{aligned}$$

Hence, since the boundary data scales homogeneously under S_0^λ with degree p then the part of the solution generated from the boundary data scales under S_α^λ with degree $p(1+\frac{\alpha}{n})$. Since the lower boundary of $\mathcal{S}_2$ is homogeneous under S_α^λ with $\alpha = \frac{n+2}{2}$, we see that $\hat{\psi}_2$ must be homogeneous under $S_{\frac{n+2}{2}}^\lambda$ of degree

$$p_2 = p\big(\tfrac{3n+2}{2n}\big).$$

Since we saw that p_2 was required to be less than $\frac{3n+2}{2}$, we conclude that p has to be less than

$$\big(\tfrac{2n}{3n+2}\big)\big(\tfrac{3n+2}{2}\big) = n$$

which was the restriction placed on p when just after it was introduced when we defined ψ_1 in Section 7.3.

In summary, the solution of the exit problem associated to $\mathcal{S}_3$ is only well defined if $p_2 < \frac{3n+2}{2}$ by Proposition 7.20. The interaction of the scaling relations associated with $\mathcal{S}_3$ and $\mathcal{S}_1$ translates this restriction on p_2 the restriction $p < n$ where p was power in the definition of $\psi_0(r,\theta) = r^p$.

8. Proof of Theorem 5.2

We now prove that under Assumption 5.7, the functions $\{\psi_i : i = 0, 1, 2, 3\}$ together with their corresponding domains of definition $\{\mathcal{S}_i : i = 0, \ldots, 3\}$ satisfy the hypotheses of Proposition 6.14 with the appropriate choice of the parameters $\theta_1^*, \eta^*, r^*, h_1, h_2, h_3$. Having done so, we will have proven Theorem 5.2, as the bound (6.16) will follow almost immediately.

The layout of this section is as follows. First, we will deduce the large r asymptotics of the functions ψ_1, ψ_2, and ψ_3, allowing us to validate the bound (6.16). Second, we will show that for all r^*, η^* sufficiently large and all θ_1^* sufficiently small, the boundary-flux conditions given in (6.10) are satisfied for some choice of the positive parameters h_1, h_2, h_3. Recall that θ_1^*, η^*, r^* are to be chosen according to Remark 7.12. Lastly, we will verify the local Lyapunov property from (6.8). The second and third items in the agenda will check the hypotheses of Proposition 6.14.

Beginning with the large r asymptotics, the following proposition derives them quickly from the construction of the ψ_i's and the accompanying discussions.

Proposition 8.1. *There exist positive constants l_i, u_i such that*

$$\begin{aligned} 2l_1 r^p &\leq \psi_1(r,\theta) \leq u_1 r^p && (r,\theta) \in \mathcal{S}_1 \\ l_2 \frac{r^p}{|\theta|^{\frac{p}{n}}} &\leq \psi_2(r,\theta) \leq u_2 \frac{r^p}{|\theta|^q} && (r,\theta) \in \mathcal{S}_2 \\ l_3 r^{p_2} &\leq \psi_3(r,\theta) \leq u_3 r^{p_3} && (r,\theta) \in \mathcal{S}_3. \end{aligned} \tag{8.2}$$

Proof of Proposition 8.1. We begin with ψ_1. Since ψ_1 scales homogeneously under S_0^λ with degree p, for any $(r,\theta) \in \mathcal{S}_1$ we have that

$$\psi_1(r,\theta) = \left(\frac{r}{r_*}\right)^p \psi_1(r_*, \theta)$$

and hence

$$\frac{m}{r_*^p} r^p \leq \psi_1(r,\theta) \leq \frac{M}{r_*^p} r^p$$

where $M = \sup\{\psi_1(r_*, \theta) : \theta \in [\theta_0^*, \theta_1^*]\}$ and $m = \inf\{\psi_1(r_*, \theta) : \theta \in [\theta_0^*, \theta_1^*]\}$. Since $\psi_1(r_*, \theta)$ is continuous $M \geq m > 0$. The bounds on ψ_3 are handled in a completely analogous way only using the scaling generated by $S^\lambda_{\frac{n+2}{2}}$ rather than S_0^λ. From (7.18), we see that the terms which make up ψ_3 do not all scale with the same degree. Hence we obtain a upper bound of r^{p_3} and a lower bound of r^{p_2}.

Since the region $\mathcal{S}_2$ requires both scalings to reach all points, we would need a slightly more complicated construction which mixed the two scaling to obtain the bounds on ψ_2 using just the abstract scaling. While this is not difficult, in light of the explicit representation of ψ_2 given in (7.16), we see that the quoted bounds follow by inspection. □

With these estimates in hand, we turn to the more techincal of the two remaining topics.

8.1. Boundary-flux conditions.

Boundary between $\mathcal{S}_0$ and $\mathcal{S}_1$. Because $\psi_1(r,\theta) = r^p\psi_1(1,\theta)$ and

$$-h_1 r^p|\theta|^{-q} = \frac{\partial\psi_1}{\partial r} r\cos(n\theta) + \frac{\partial\psi_1}{\partial\theta}\sin(n\theta)$$

one has

$$\frac{\partial\psi_1}{\partial\theta} = -r^p\Big(\frac{p\cos(n\theta)\psi_1(1,\theta) + h_1|\theta|^{-q}}{\sin(n\theta)}\Big). \tag{8.3}$$

Therefore combining $\frac{\partial\psi_0}{\partial\theta} = 0$ with (8.3) produces

$$\Big[\frac{\partial\psi_0}{\partial\theta} - \frac{\partial\psi_1}{\partial\theta}\Big]_{\theta=\theta_0^*} = r^p\Big(\frac{p\cos(n\theta_0^*)\psi_1(1,\theta_0^*) + h_1|\theta_0^*|^{-q}}{\sin(n\theta_0^*)}\Big).$$

Since $\psi_j(r,\theta) = \psi_j(r,-\theta)$ on $\mathcal{S}_j$ for $j = 0,1$, we note also that

$$\Big[\frac{\partial\psi_1}{\partial\theta} - \frac{\partial\psi_0}{\partial\theta}\Big]_{\theta=-\theta_0^*} = \Big[\frac{\partial\psi_0}{\partial\theta} - \frac{\partial\psi_1}{\partial\theta}\Big]_{\theta=\theta_0^*}. \tag{8.4}$$

Since $\psi_1(1,\theta_0^*) = 1$, $\sin(n\theta_0^*) > 0$ and $\cos(n\theta_0^*) < 0$, picking

$$0 < h_1 < p|\theta_0^*|^q|\cos(n\theta_0^*)| \tag{8.5}$$

implies that the quantity (8.4) is negative.

Boundary between $\mathcal{S}_1$ and $\mathcal{S}_2$. Similar to the previous computations, observe that $\psi_2(r,\theta) = r^p\psi_2(1,\theta)$ implies

$$\frac{\partial\psi_2}{\partial\theta} = -r^p\Big[\frac{p\psi_2(1,\theta) + h_2|\theta|^{-q}}{n\theta}\Big].$$

Since $\psi_1(1,\theta_1^*) = \psi_2(1,\theta_1^*)$, we then obtain

$$\begin{aligned}
&\Big[\frac{\partial\psi_1}{\partial\theta} - \frac{\partial\psi_2}{\partial\theta}\Big]_{\theta=\theta_1^*}\\
&= -r^p\Big[\frac{p\cos(n\theta_1^*)\psi_1(1,\theta_1^*) + h_1|\theta_1^*|^{-q}}{\sin(n\theta_1^*)} - \frac{p\psi_1(1,\theta_1^*) + h_2|\theta_1^*|^{-q}}{n\theta_1^*}\Big]\\
&= -\frac{r^p}{|\theta_1^*|^{q+1}}\Big[\Big(\frac{p\cos(n\theta_1^*)}{\sin(n\theta_1^*)} - \frac{p}{n\theta_1^*}\Big)\psi_1(1,\theta_1^*)|\theta_1^*|^{q+1} + \Big(\frac{h_1}{\sin(n\theta_1^*)} - \frac{h_2}{n\theta_1^*}\Big)|\theta_1^*|\Big].
\end{aligned}$$

Using the expression (7.14), it is not hard to check that $\psi_1(1,\theta_1^*)|\theta_1^*|^q$ is bounded as $\theta_1^* \downarrow 0$. Employing the Taylor expansions for $\sin(n\theta_1^*)$ and $\cos(n\theta_1^*)$ about $\theta_1^*=0$, we thus obtain

$$\left[\frac{\partial \psi_1}{\partial \theta} - \frac{\partial \psi_2}{\partial \theta}\right]_{\theta=\theta_1^*} \simeq -\frac{r^p}{|\theta_1^*|^{q+1}}\left(\frac{h_1}{n} - \frac{h_2}{n}\right) \tag{8.6}$$

as $\theta_1^* \downarrow 0$ where $\simeq$ denotes asymptotic equivalence. Thus for all $\theta_1^*>0$ sufficiently small, we may choose $h_2>0$ small enough so that the quantity on the left-hand side of (8.6) is negative for $r \geq r^*$. Since $\psi_j(r,-\theta) = \psi_j(r,\theta)$ on $\mathcal{S}_j$ for $j=1,2$, notice that we also have the equality

$$\left[\frac{\partial \psi_2}{\partial \theta} - \frac{\partial \psi_1}{\partial \theta}\right]_{\theta=-\theta_1^*} = \left[\frac{\partial \psi_1}{\partial \theta} - \frac{\partial \psi_2}{\partial \theta}\right]_{\theta=\theta_1^*}$$

Hence, the same choice of θ_1^* and $h_2>0$ results in a negative sign for the flux across the boundary $\theta=-\theta_1^*$ as well.

Boundary between $\mathcal{S}_2$ and $\mathcal{S}_3$. Thus far it has been fairly straightforward to compute and analyze fluxes across boundaries where noise plays no role. In such cases, we saw that we could find convenient expressions for $\partial_\theta \psi_i$, $i=0,1,2$, simply by using the first-order PDEs these functions satisfy. A similar approach, however, does not work when studying the flux across the boundaries between $\mathcal{S}_2$ and $\mathcal{S}_3$ since the operator A contains second-order partial derivatives in θ. Therefore, to study study $\partial_\theta\psi_3$ along these interfaces, we opt to employ the somewhat explicit expression (7.22) derived in the previous section. Because there is no closed form expression for the functions $G_{p_i}(\eta) := \mathbf{E}_\eta e^{p_i\tau}$, $i=2,3$, we still have some work to do, but we expect analysis of $G_{p_i}'(\eta^*)$ to be possible for large $\eta^*>0$. This is because, by the computations of Section 7.1, the noise term in A formally scales away as $r^{\frac{n+2}{2}}|\theta| \to \infty$. Indeed, the following lemma, which is a slight generalization of what we need here, says precisely this.

Lemma 8.7. *Fix $c\in\mathbb{R}$, let $\eta^*>|c|$, and recall that*

$$\tau_c = \inf\{t>0 \,:\, \eta_t \notin [-\eta^*+c, \eta^*+c]\}$$

where η_t solves (7.19). Let $0<a<\frac{3}{2}n+1$ and $G_{a,c}(\eta) := \mathbf{E}_\eta e^{a\tau_c}$. Then for all $\eta^*>|c|$ large enough, $G_{a,c}\in C^\infty([-\eta^*+c,\eta^*+c])$ and

$$G_{a,c}'(\pm\eta^*+c) = \mp\frac{2a}{3n+2}(\eta^*)^{-1} + o((\eta^*)^{-1}) \text{ as } \eta^*\to\infty. \tag{8.8}$$

Moreover, $G_{a,0}=G_a$ and $G_a(-\eta)=G_a(\eta)$ for all $\eta\in[-\eta^*,\eta^*]$.

Remark 8.9. We will need the lemma above in this generality in the sequel [8]. In this paper, all we need are the results when $c = 0$.

Proof of Lemma 8.7. See Appendix A. □

We now apply the lemma to control the flux terms across the boundaries $r^{\frac{n+2}{2}}\theta = \pm\eta^*$. By the symmetry $G_{p_i}(\eta) = G_{p_i}(-\eta)$ for $\eta \in [-\eta^*, \eta^*]$, we must only show that for $\eta^* > 0$ sufficiently large, $h_3 > 0$ can be chosen so that the flux across the boundary $r^{\frac{n+2}{2}}\theta = \eta^* > 0$ is negative for $r^* > 0$ sufficiently large. Therefore, observe that

$$\begin{aligned}&\Big[\frac{\partial\psi_2}{\partial\theta} - \frac{\partial\psi_3}{\partial\theta}\Big]_{r^{\frac{n+2}{2}}\theta=\eta^*}\\ &= -\Big(\frac{qh_2}{qn-p}\frac{1}{(\eta^*)^{q+1}} + \frac{1}{(\eta^*)^q}\frac{h_2}{qn-p}G'_{p_3}(\eta^*) + \frac{h_3}{p_3}G'_{p_3}(\eta^*)\Big)r^{p+\frac{n+2}{2}(q+1)}\\ &\quad + o(r^{p+\frac{n+2}{2}(q+1)}) \text{ as } r \to \infty.\end{aligned}$$

Recalling the assumption that $q > p/n$, observe that (8.8) implies that for $\eta^* > 0$ large enough and $h_3 > 0$ small enough the righthand sisde of the above expression above is negative for all sufficiently large r^*.

8.2. **Checking the local Lyapunov property.** We now turn to verifying the local Lyapunov property given in (6.8). To do so, we will not need to change the values of the h_i, $i = 1, 2, 3$, set in the previous section. We will need to, however, increase r^*, η^* as well as decrease θ_1^*, but this will consistent with all previous choices, including those made in the preceding section to assure that each boundary-flux term had the appropriate sign.

Letting B denote the asymptotic operator corresponding to L in $\mathcal{S}_i$, this involves first writing

$$L\psi_i(r,\theta) = B\psi_i(r,\theta) + (L - B)\psi_i(r,\theta)$$

on $\mathcal{S}_i$. Since $B\psi_i$ is of the desired form, all we must do is estimate the remainder term $(L - B)\psi_i$ to see that

$$|(L - B)\psi_i| \ll |B\psi_i|$$

as $r \to \infty$, $(r,\theta) \in \mathcal{S}_i$. We proceed region by region starting with:

Region $\mathcal{S}_0$. Since $\psi_0(r,\theta) = r^p$, it is not hard to see that as $r \to \infty$, $(r,\theta) \in \mathcal{S}_0$,

$$L\psi_0(r,\theta) = pr^p\cos(n\theta) + o(r^p). \tag{8.10}$$

Since $\cos(n\theta) \leq -c < 0$ for $(r,\theta) \in \mathcal{S}_0$ and some $c > 0$, the relation (8.10) implies that there exist positive constants c_0, d_0 such that

$$L\psi_0(r,\theta) \leq -c_0 r^p + d_0$$

for all $(r,\theta) \in \mathcal{S}_0$. Undoing the time change, we find easily that on $\mathcal{S}_0$

$$\mathcal{L}\psi_0(r,\theta) \leq -C_0 r^{p+n} + D_0 \tag{8.11}$$

for some $C_0, D_0 > 0$.

Region $\mathcal{S}_1$. First observe that by definition of ψ_1

$$\begin{aligned} L\psi_1(r,\theta) &= T_1\psi_1(r,\theta) + (L - T_1)\psi_1(r,\theta) \\ &= -h_1 \frac{r^p}{|\theta|^q} + (L - T_1)\psi_1(r,\theta) \end{aligned}$$

on $\mathcal{S}_1$. To bound the remainder term $(L - T_1)\psi_1(r,\theta)$, notice by (7.14) we may write $\psi_1(r,\theta) = r^p g(\theta)$ where g is a smooth and positive function in θ for all $0 < \theta_1^* \leq |\theta| \leq \theta_0^*$. In particular, since $0 < \theta_1^* \leq |\theta| \leq \theta_0^*$ for $(r,\theta) \in \mathcal{S}_1$, we see that as $r \to \infty$ with $(r,\theta) \in \mathcal{S}_1$

$$L\psi_1(r,\theta) = -h_1 \frac{r^p}{|\theta|^q} + o(r^p). \tag{8.12}$$

Using the asymptotic formula above as well as positivity and smoothness of g on the domain for θ in $\mathcal{S}_1$, we obtain the inequality

$$L\psi_1(r,\theta) \leq -c_1 \frac{r^p}{|\theta|^q} + d_1$$

on $\mathcal{S}_1$ for some constants $c_1, d_1 > 0$. Undoing the time change, we also find that

$$\mathcal{L}\psi_1(r,\theta) \leq -C_1 \frac{r^{p+n}}{|\theta|^q} + D_1 \tag{8.13}$$

on $\mathcal{S}_1$ for some constants $C_1, D_1 > 0$.

Region $\mathcal{S}_2$. By definition of ψ_2, first observe that on $\mathcal{S}_2$

$$\begin{aligned} L\psi_2(r,\theta) &= T_2\psi_2(r,\theta) + (L - T_2)\psi_2(r,\theta) \\ &= -h_2 \frac{r^p}{|\theta|^q} + (T_1 - T_2)\psi_2(r,\theta) + (L - T_1)\psi_2(r,\theta). \end{aligned}$$

Using the Taylor expansions for $\sin(n\theta)$ and $\cos(n\theta)$ about $\theta = 0$ notice that there exists a constant $C > 0$ which is independent of $\theta_1^* > 0$ such that

$$\begin{aligned}
&(T_1 - T_2)\psi_2(r,\theta)\\
&\leq C\theta^2\bigg[\Big(|\theta_1^*|^{\frac{p}{n}}\psi_1(1,\theta_1^*) + h_2\frac{|\theta_1^*|^{\frac{p}{n}-q}}{qn-p}\Big)\frac{r^p}{|\theta|^{p/n}} + \frac{h_2}{qn-p}\frac{r^p}{|\theta|^q}\bigg]\\
&\leq C(\theta_1^*)^2\bigg[\Big(|\theta_1^*|^{\frac{p}{n}}\psi_1(1,\theta_1^*) + h_2\frac{|\theta_1^*|^{\frac{p}{n}-q}}{qn-p}\Big)\frac{r^p}{|\theta|^{p/n}} + \frac{h_2}{qn-p}\frac{r^p}{|\theta|^q}\bigg]
\end{aligned}$$

for all $(r,\theta) \in \mathcal{S}_2$. In particular, since $\psi_1(1,\theta_1^*) = O((\theta_1^*)^{-1})$ as $\theta_1^* \downarrow 0$, it follows that for all $\epsilon > 0$, there exists $\theta_1^* > 0$ small enough so that

$$(T_1 - T_2)\psi_2(r,\theta) \leq \epsilon\frac{r^p}{|\theta|^q}$$

for all $(r,\theta) \in \mathcal{S}_2$. Therefore, in particular, we may choose $\theta_1^* > 0$ small enough so that

$$L\psi_2(r,\theta) \leq -\frac{h_2}{2}\frac{r^p}{|\theta|^q} + (L - T_1)\psi_2(r,\theta)$$

on $\mathcal{S}_2$. To control the remaining term $(L - T_1)\psi_2(r,\theta)$, recall that we are operating under Assumption 5.7. Therefore, we find that there exists positive constants C, D independent of η^*, r^* such that on $\mathcal{S}_2$

$$(L - T_1)\psi_2(r,\theta) \leq \Big(\frac{C}{\eta^*} + \frac{D}{r^*}\Big)\frac{r^p}{|\theta|^q}.$$

Picking $\eta^*, r^* > 0$ sufficiently large we see that there exist constants $c_2, d_2 > 0$ such that

$$L\psi_2(r,\theta) \leq -c_2\frac{r^p}{|\theta|^q} + d_2$$

for all $(r,\theta) \in \mathcal{S}_2$. Undoing the time change, we obtain the bound

$$\mathcal{L}\psi_2(r,\theta) \leq -C_2\frac{r^{p+n}}{|\theta|^q} + D_2 \tag{8.14}$$

on $\mathcal{S}_2$ for some constants $C_2, D_2 > 0$.

Region $\mathcal{S}_3$. First decompose $L\psi_3$ on $\mathcal{S}_3$ as follows

$$L\psi_3(r,\theta) = A\psi_3(r,\theta) + (T - A)\psi_3(r,\theta) + (L - T)\psi_3(r,\theta)$$

where

$$T = r\cos(n\theta)\partial_r + \sin(n\theta)\partial_\theta + \frac{\sigma^2}{2r^{n+2}}\partial_\theta^2.$$

Using the definition of ψ_3 we find that

$$\begin{aligned} L\psi_3(r,\theta) &= -h_3 r^{p_3} + (T-A)\psi_3(r,\theta) + (L-T)\psi_3(r,\theta) \\ &= -h_3 r^{p_3} + (T_1 - T_2)\psi_3(r,\theta) + (L-T)\psi_3(r,\theta). \end{aligned}$$

Let us first see how to bound $(T_1 - T_2)\psi_3$. Again, making use of the Taylor expansions for $\sin(n\theta)$ and $\cos(n\theta)$, we see that there exists a constant $C > 0$ which is independent of η^* such that

$$(T_1 - T_2)\psi_3(r,\theta) \leq C\theta^2\big(|r\partial_r\psi_3(r,\theta)| + |\partial_\theta\psi_3(r,\theta)|\big).$$

To control derivatives of ψ_3, recall the expression (7.18). Applying Lemma 8.7, we deduce the existence of a constant $C = C(\eta^*) > 0$ such that

$$\theta^2(|r\partial_r\psi_3(r,\theta)| + |\partial_\theta\psi_3(r,\theta)|) \leq C(\eta^*) r^{p_3 - \frac{(n+2)}{2}}$$

on $\mathcal{S}_3$. In particular, we have thus far obtained

$$L\psi_3(r,\theta) \leq -h_3 r^{p_3} + C(\eta^*) r^{p_3 - \frac{n+2}{2}} + (L-T)\psi_3(r,\theta) \tag{8.15}$$

for all $(r,\theta) \in \mathcal{S}_3$. To estimate the remaining term $(L-T)\psi_3$, proceed in a similar fashion using Assumption 5.7 to see that

$$(L-T)\psi_3(r,\theta) \leq D(\eta^*) r^{p_3 - 1} \tag{8.16}$$

on $\mathcal{S}_3$. Putting (8.15) together with (8.16), there exist constants $c_3, d_3 > 0$ such that on $\mathcal{S}_3$

$$L\psi_3(r,\theta) \leq -c_3 r^{p_3} + d_3. \tag{8.17}$$

Undoing the time change, we also see that on $\mathcal{S}_3$

$$\mathcal{L}\psi_3(r,\theta) \leq -C_3 r^{p_3+n} + D_3. \tag{8.18}$$

for some constants $C_3, D_3 > 0$.

Remark 8.19. Using the bounds obtained in Proposition 8.1 and the inequalities (8.11), (8.13), (8.14), and (8.18) we can easily see that the bound for $\mathcal{L}\psi_i$, $i = 0, 1, 2, 3$, on $\mathcal{S}_i$ required by Proposition 6.14 is satisfied. Because the boundary flux terms have the appropriate sign by the arguments of Section 8.1, we have now finished proving Theorem 5.2 under Assumption 5.7.

9. Heuristic Discussions and Small Noise Asymptotics

We now revisit the discussions of Section 7.1 with an eye to more heuristic calculations rather than constructing a provably correct Lyapunov function. For simplicity, we restrict ourselves to the monomial case

$$dz_t = z_t^{n+1}\, dt + \sigma\, dB_t \tag{9.1}$$

where σ and B_t are as before in (1.1). As discussed in Section 6.3, the dynamics of this system breaks into n wedge-shaped regions which can all be treated similarly. Hence, we will consider the wedge saddling the non-negative real numbers, namely

$$\widetilde{\mathcal{R}} = \left\{(r,\theta) \,:\, r \geq 0,\, |\theta| \leq \frac{\pi}{n}\right\}.$$

In what follows, we will shift between the complex coordinate z and polar coordinates (r,θ), $z = re^{i\theta}$, as it is convenient.

Fixing $\eta^* > 0$ large, to leading order in $\widetilde{\mathcal{R}}$, the noise only has an effect inside of the region

$$\mathcal{S}(\eta^*) = \{(r,\theta) \in \mathcal{R} \,:\, |\theta| r^{\frac{n+2}{2}} \leq \eta^*\}$$

for all $r \geq r^*$ if r^* is sufficiently large. Outside of $\mathcal{S}(\eta^*)$ when r is large, the dynamics approximately follows the deterministic equation $\dot{z} = z^{n+1}$. The orbits of this system are concentric loops which share a single common point $z = 0$. These loops are the locus of points

$$\mathcal{J}(K) = \left\{(r,\theta) \,:\, \frac{r}{|\sin(n\theta)|^{\frac{1}{n}}} = K,\, 0 < |\theta| < \frac{\pi}{n}\right\} \tag{9.2}$$

for $K > 0$.

Let $\zeta \in \widetilde{\mathcal{R}}$ be some point with $|\zeta|$ large. Since $\mathcal{S}(\eta^*)$ becomes vanishingly thin for $r^* > 0$ large we assume that $\zeta \notin \mathcal{S}(\eta^*)$. With high probability, any trajectory which reaches such a point ζ first transitioned to a relatively large radial value inside of $\mathcal{S}(\eta^*)$, then exited $\mathcal{S}(\eta^*)$, and then approximately followed the deterministic trajectory of the form given in (9.2) passing through ζ. Since each orbit $\mathcal{J}(K)$ is indexed by the constant K, the chance of being on given orbit is determined by the chance of leaving the region $\mathcal{S}(\eta^*)$ at the point where the orbit with the given value of K intersects the boundary of $\mathcal{S}(\eta^*)$. As before, define the stopping time $\tau = \inf\{t > 0 : |\eta_t| > \eta_*\} = \inf\{t > 0 : z_t \notin \mathcal{S}(\eta^*)\}$ and we assume that $r(0) = r^*$. Define the K value upon exit from $\mathcal{S}(\eta^*)$ as

$$K_\tau = \frac{r_\tau}{|\sin(n\theta_\tau)|^{\frac{1}{n}}}.$$

Since after the time change, $\dot{r} = r$ to leading order we see that $r_\tau = r^* e^\tau$. Since $|\eta_\tau| = \eta^*$ implies that

$$r_\tau^{\frac{n+2}{2}} |\theta_\tau| = \eta^*,$$

we conclude that

$$|\theta_\tau| = \eta^* (r^*)^{-\frac{n+2}{2}} e^{-\frac{n+2}{2}\tau} \tag{9.3}$$

and that as a function of τ

$$K_\tau \approx \frac{r_\tau}{((n\theta_\tau))^{\frac{1}{n}}} = \frac{(r^*)^{\frac{3n+2}{2n}}}{(n\eta^*)^{\frac{1}{n}}} e^{\frac{3n+2}{2n}\tau} \tag{9.4}$$

As discussed in Section 7.1, after the time change $t \to \int_0^t r_s^n ds$, the dynamics in $\mathcal{S}(\eta^*)$ as essentially given by $\eta_t = r_t^{\frac{n+2}{2}}\theta_t$ which satisfies (7.19). Moreover, this equation is coupled with the leading order r dynamics: $\dot{r} = r$.

From (7.19) we see that η_t is a Gaussian process and that as $t \to \infty$

$$\mathbf{P}(\tau > t) \approx e^{-\lambda_1 t} \tag{9.5}$$

where $\lambda_1 > 0$ is the smallest eigenvalue associated with the operator

$$(\mathcal{Q}f)(\eta) = -\frac{\sigma^2}{2}\partial_\eta^2 f - \left(\tfrac{3n+2}{2}\right)\partial_\eta(\eta f)$$

on the domain $[-\eta^*, \eta^*]$ with Dirichlet boundary conditions. Note that $\mathcal{Q}$ is the negative of the L^2-adjoint of the generator of η_t. Clearly, λ_1 is a function of η^*. However, it is not hard to see that as $\eta^* \to \infty$, $\lambda_1 \to \frac{3n+2}{2}$.

9.1. **Tails of the invariant measure.** Observe that the maximal distance from the origin on a given orbit $\mathcal{J}(K)$ occurs when $\theta = \pm\pi/(2n)$ since $|\sin(n\theta)|^{1/n} = 1$ when $\theta = \pm\pi/(2n)$. From this we see that the maximum distance from the origin on the orbit $\mathcal{J}(K)$ is simply K. Let $\tilde{\mu}$ denote the stationary measure (after the time change from Section 6.1), $\mathbf{P}_{\tilde{\mu}}$ the probability measure of the time-changed Markov process with initial distribution $\tilde{\mu}$, and $\mathbf{P}_{(r,\theta)}$ the probability measure of the time-changed Markov process with initial condition (r,θ). Recalling the preceding discussion, for $R > 0$ sufficiently large we have that with high probability, the path to any $\zeta \in \widetilde{R} \setminus \mathcal{S}(\eta^*)$ with $|\zeta| = R$ is through $\mathcal{S}(\eta^*)$. Hence for $R > 0$ large

$$\mathbf{P}_{\tilde{\mu}}(|z| \geq R) \approx c\mathbf{P}_{(r^*,0)}(K_\tau > R)$$

for some constant $c > 0$, as the flux in $\mathcal{S}(\eta^*)$ around r^* is some positive constant at equilibrium. Using the relation (9.4), we find that for $R > 0$ large

$$\mathbf{P}_{\tilde{\mu}}(|z| \geq R) \approx c\mathbf{P}_{(r^*,0)}\left(\tau > \tfrac{2n}{3n+2}\log\tfrac{R}{R_0}\right)$$

for some positive constant R_0. Now pick $\eta^* > 0$ large enough so that $\lambda_1 \approx \frac{3n+2}{2}$. Thus for $R > 0$ large enough, (9.5) gives

$$\mathbf{P}_{\tilde{\mu}}(|z| \geq R) \approx cR^{-\frac{2n\lambda_1}{3n+2}} \approx cR^{-n}$$

Letting $\rho(r,\theta)$ denote density of $\tilde{\mu}$, for $\theta \neq 0$ we have

$$\frac{c}{R^{n+1}} \approx \frac{\partial}{\partial R}\mathbf{P}_{\tilde{\mu}}(|z| \geq R) = \frac{\partial}{\partial R}\int_0^R \int_0^{2\pi} \rho(r,\theta)\, d\theta\, r\, dr = R\int_0^{2\pi} \rho(R,\theta)\, d\theta\, .$$

From this we conclude that

$$\mathbf{P}_{\tilde{\mu}}(|z| \in dR) \approx \frac{c}{R^{n+2}}\, dR$$

for large R. Hence if μ denotes the stationary measure from Section 6.1 without the time change we see that

$$\mathbf{P}_{\mu}(|z| \in dR) \approx \frac{c}{R^{2n+2}}\, dR$$

which agrees with the rigorous results of Theorem 3.2

9.2. **Spacing of the excursions.** We now investigate the distribution of the time between "spikes" of size $R > 0$, as illustrated in Figure 2.

It is reasonable to assume that for a large but fixed value of $r^* > 0$, trajectories with high probability spend a random amount of time before exiting through $\mathcal{S}(\eta^*)$ to larger radial values. For any $R \gg r^*$, define the following sequence of stopping times: $S_0 = 0$ and for $i \geq 1$ set

$$T_i = \inf\{t \geq S_{i-1} : r_t \geq R\} \qquad \text{and} \qquad S_i = \inf\{t \geq T_i : r_t \geq r_*\}\, .$$

We are interested in the distribution of the time between "spikes" $T_{i+1} - T_i$. In what follows, we will see that $T_{i+1} - T_i$ is distributed geometrically with parameter that scales like R^{-n}. Hence we expect

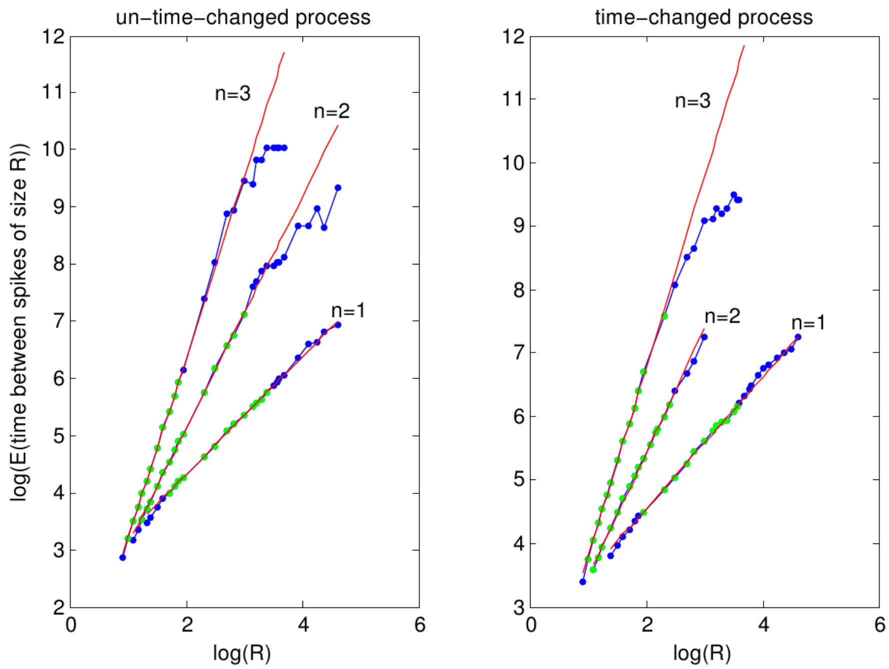
$$\mathbf{E}\big(T_{i+1} - T_i\big) = cR^n$$

for some positive constant c. We begin by defining successive exit times from the set $\{r \geq 2r^*\}$ with an intervening return to the set $\{r \geq r^*\}$: Let $s_0 = 0$ and for $i \geq 1$

$$t_i = \inf\{t \geq s_{i-1} : r_t \geq 2r^*\} \qquad \text{and} \qquad s_i = \inf\{t \geq t_i : r_t \geq r^*\}\, .$$

With high probability, each exit from $\{r \leq 2r^*\}$ happens through the region $\{r \leq 2r^*\} \cap \mathcal{S}(\eta^*)$. In turn, the location of exits through the boundary of $\{r \geq 2r^*\} \cap \mathcal{S}(\eta^*)$ determines which of the orbits $\mathcal{J}(K)$ the dynamics follows. This is because for large $r^* > 0$, the dynamics will be approximately deterministic upon exit from $\{r \geq 2r^*\} \cap \mathcal{S}(\eta^*)$. As in the discussion at the start of this section, let τ denote the exit time from $\{r \geq 2r^*\} \cap \mathcal{S}(\eta^*)$ and let K_τ be the value of the constant K used to index the orbit $\mathcal{J}(K)$ when the dynamics leaves $\{r \geq 2r^*\} \cap \mathcal{S}(\eta^*)$.

As just discussed in Section 9.1, the maximum radius reached by the deterministic orbit $\mathcal{J}(K)$ (as parameterized by its exit point K) is in



n	Slope (no time change)	Slope (time change)
1	1.031	1.035
2	2.034	1.960
3	3.175	3.001

FIGURE 5. Simulation results of $\log \mathbf{E}[T_{j+1} - T_j]$ versus $\log(R)$ without the time change (left) and with the time change (right). Points sufficiently far from 0 but with enough data (in green) are fitted with a least squares approximation. Slopes of each line in either case are given in the table above and are as predicted.

fact also K. Hence we recall that if $p_R = \mathbf{P}(\text{height of spike is} \geq \text{R})$ on a given entry into $\mathcal{S}(\eta^*)$, then

$$p_R = \mathbf{P}(K_\tau > R) \approx cR^{-\frac{2n\lambda_1}{3n+2}} \approx cR^{-n}.$$

It is reasonable to assume that the $K_\tau^{(i)}$ associated to ith entry into $\mathcal{S}(\eta^*) \cap \{r \geq r^*\}$ is independent of the $K_\tau^{(j)}$ with $j \neq i$. Letting $n_R^{(i)}$ denote the number of excursions to level $2r^* < R$ needed to have an spike greater than R after time T_{i-1}, we observe that under the independence assumption $n_R^{(i)}$ is geometric with parameter p_R. Hence, $\mathbf{E}n_R^{(i)} = 1/p_R$.

Since

$$T_{j+1} - T_j = \sum_{i=N_j}^{N_{j+1}} (t_i - t_{i-1}) \qquad \text{where} \qquad N_j = N_{j-1} + n_R^{(j)}$$

we have

$$\mathbf{E}\big[T_{j+1} - T_j\big] \approx \mathbf{E}\big[n_R^{(j)}\big]\mathbf{E}\big[t_2 - t_1\big] \approx \frac{c}{p_R} \approx cR^n$$

Hence we expect the average spacing between peaks of spikes greater than R should grow like R^n. Consult Figure 5 for numerical results.

All of the analysis above holds equally well even if the dynamics has not been time-changed. The critical quantity is the location at which the process exits from $\mathcal{S}(\eta^*)$. This location is the same in either case, as the time-change only affects the time between entrances into $\mathcal{S}(\eta^*) \cap \{r \geq r^*\}$ and the time it takes to traverse the spike excursion out to level R (see Figure 5).

10. Conclusion

We have given a general methodology for constructing Lyapunov functions and applied it to study a family of equations in which the underlying deterministic dynamics is stabilized under the addition of noise. The method incorporates global information of the flow and hence is well suited in the setting where stability results from global rewiring of trajectories due to the addition of a small amount of noise. The use of auxiliary PDEs to define our Lyapunov functions was central to the construction as it allowed us to obtain radially optimal results. There are a number of points which, though technical, allow for a successful completion of the argument. We always use homogeneous operators in our local constructions, as this allows us to create local Lyapunov functions through the use of auxiliary PDEs which are the sum of homogeneously scaling terms. This greatly simplifies the general analysis. The homogeneous operators are also drastically simplified from the original generator. This makes many points of the analysis easier, often allowing explicit representations of solutions. We also employ a extension of Itô theorem which allows us to avoid smoothing the patched functions along interfaces.

Our construction of Lyapunov functions is closely related to the construction of sub/super solutions to certain PDEs associated to the SDEs considered. In particular, all of our results can be translated to the existence of a normalizable solutions with polynomial decay at infinity to the PDE $\mathcal{L}^*\rho = 0$ where $\mathcal{L}$ is the generator of the SDE (5.1).

In Part II [8] of this paper, we will consider the same class of problems but in a more general setting where Assumption 5.7 does not hold. In the conclusion of that paper we will give a number directions of possible future work.

Appendix A. Proofs of Lemma 4.4 and Lemma 8.7

We now give the proof of Lemma 4.4. After that we will give the proof of Lemma 8.7.

Proof of Lemma 4.4. To see part a), let $\tau_\infty = \lim_{n\uparrow\infty}\tau_n$ to be the explosion time of ξ_t. We must show that $\mathbf{P}_{\xi_0}[\tau_\infty < \infty] = 0$ for all $\xi_0 \in \mathbf{R}^m$. By Definition 4.1, there exists constants $m, b > 0$ such that

$$\mathbf{E}_{\xi_0}\Psi(\xi_{t\wedge\tau_n}) - \Psi(\xi_0) \leq \mathbf{E}_{\xi_0}\int_0^{t\wedge\tau_n} -m\Phi(\xi_s) + b\,ds \tag{A.1}$$

for all $t \geq 0$, $n \in \mathbf{N}$. Since $\Phi \geq 0$, we obtain the bound

$$\mathbf{E}_{\xi_0}\Psi(\xi_{t\wedge\tau_n}) \leq \Psi(\xi_0) + bt$$

for all $t \geq 0$, $n \in \mathbf{N}$. Since $\Psi(\xi) \to \infty$ as $|\xi| \to \infty$, for $n \in \mathbf{N}$ large enough the inequality above implies

$$\begin{aligned}\mathbf{P}_{\xi_0}[\tau_n \leq t] = \frac{\inf_{|x|\geq n}\Psi(x)\cdot\mathbf{P}_{\xi_0}[\tau_n \leq t]}{\inf_{|x|\geq n}\Psi(x)} &\leq \frac{\mathbf{E}_{\xi_0}[\Psi(\xi_{\tau_n})1_{\{\tau_n\leq t\}}]}{\inf_{|x|\geq n}\Psi(x)}\\ &\leq \frac{\Psi(\xi_0) + bt}{\inf_{|x|\geq n}\Psi(x)}\end{aligned}$$

for all $t \geq 0$. Using the fact that $\Psi(\xi) \to \infty$ as $|\xi| \to \infty$, Fatou's lemma gives

$$\mathbf{P}_{\xi_0}[\tau_\infty \leq t] = 0 \qquad \forall t \geq 0,$$

finishing the proof of a)

To see part b) and setup the proof of c), we follow the argument in Proposition 5.1 of [5]. Since $\Psi \geq 0$, relation (A.1) implies

$$\mathbf{E}_{\xi_0}\int_0^{t\wedge\tau_n}\Phi(\xi_s)\,ds \leq \Psi(\xi_0) + bt$$

for all $t \geq 0$, $n \in \mathbf{N}$. Using the Monotone convergence theorem and part a), we obtain

$$\mathbf{E}_{\xi_0}\int_0^t \Phi(\xi_s)\,ds \leq \Psi(\xi_0) + bt \tag{A.2}$$

for all $t \geq 0$. Letting $A_R = \{\xi \in \mathbf{R}^m : |\Phi(\xi)| \leq R\}$, we note that the bound above implies

$$\frac{1}{t}\int_0^t \mathbf{P}_{\xi_0}[\xi_s \in A_R^c]\, ds \leq \frac{\Psi(\xi_0) + bt}{Rt}.$$

In particular, it follows by the standard reference [9] that the sequence of measures

$$\pi_t^{\xi_0}(\,\cdot\,) = \frac{1}{t}\int_0^t P_{\xi_0}[\xi_s \in \,\cdot\,]\, ds$$

is compact in the weak topology. Part b) now follows easily. For the remaining assertion, in light of the proof of Theorem 4.3 of [10] observe that

$$\frac{1}{t}\mathbf{E}_{\xi_0}\int_0^t [\Phi(\xi_s) \wedge R]\, ds \leq \frac{\Psi(\xi_0)}{t} + b$$

which is valid for any $R > 0$. In particular, we obtain the inequality

$$\int_{\mathbf{R}^m} [\Phi(\xi) \wedge R]\, \pi^{\xi_0}(d\xi) \leq b$$

where π^{ξ_0} is an invariant measure defined as weak limit point as $t \to \infty$ of $\pi_t^{\xi_0}$ above. Applying the Monotone converge as $R \to \infty$ finishes c). □

Proof of Lemma 8.7. Note that the function $G_{a,c}$ solves the following boundary-value problem

$$\text{(A.3)} \qquad \frac{\sigma^2}{2}G''_{a,c}(\eta) + \Big(\frac{3}{2}n + 1\Big)\eta G'_{a,c}(\eta) + aG_{a,c}(\eta) = 0$$

$$\text{(A.4)} \qquad G_{a,c}(-\eta^* + c) = G_{a,c}(\eta^* + c) = 1.$$

To further understand solutions of (A.3), we transform the equation to Weber's equation. To this end, we write

$$\text{(A.5)} \qquad G_{a,c}(\eta) = e^{-\beta\eta^2/4} H(\sqrt{\beta}\eta)$$

where $\beta = (3n+2)/\sigma^2$ and note that H satisfies

$$\text{(A.6)} \qquad H''(v) - \Big(\frac{v^2}{4} + \frac{1}{2} - \frac{2a}{\sigma^2\beta}\Big)H(v) = 0.$$

The two linearly independent general solutions of (A.6), denoted by

$$U\Big(\frac{2a}{\sigma^2\beta} - 1/2, \pm iv\Big),$$

have the following integral representations (cf. Chapter 12.5 of [3])

$$U\Big(\kappa - 1/2, \pm iv\Big) = \frac{e^{\frac{v^2}{4}}}{\Gamma\big(\kappa\big)} \int_0^\infty t^{-1+\kappa} e^{-t^2/2 \mp ivt}\, dt, \quad v \in \mathbb{R}.$$

where we have introduced $\kappa = \frac{2a}{\sigma^2 \beta}$ in the interest of brevity. Using these expressions, the boundary conditions given in (A.3) and the assumption $0 < a < \frac{3}{2}n + 1$ we may formally write

$$\text{(A.7)} \quad G_{a,c}(\eta) = \frac{D - B}{AD - BC} \int_0^\infty f(t) \cos(\sqrt{\beta}\eta t)\, dt + \frac{A - C}{AD - BC} \int_0^\infty f(t) \sin(\sqrt{\beta}\eta t)\, dt$$

where $f(t) = t^{-1+\kappa} e^{-\frac{t^2}{2}}$ and

$$A = \int_0^\infty f(t) \cos(\sqrt{\beta}(\eta^* + c)t)\, dt, \quad B = \int_0^\infty f(t) \sin(\sqrt{\beta}(\eta^* + c)t)\, dt$$
$$C = \int_0^\infty f(t) \cos(\sqrt{\beta}(\eta^* - c)t)\, dt, \quad D = -\int_0^\infty f(t) \sin(\sqrt{\beta}(\eta^* - c)t)\, dt.$$

The only question pertaining to the validity of (A.7) is that $AD - BC$ could possibly be zero. We will now show that this is not the case for $\eta^* > |c|$ sufficiently large. It will then follow easily that $G_{a,c} \in C^\infty([-\eta^* + c, \eta^* + c])$ for all $\eta^* > |c|$ large enough by (A.7). Write

$$A = \Gamma(\kappa) e^{-\beta \frac{(\eta^* + c)^2}{4}} \operatorname{Re} U\big(\kappa - \frac{1}{2}, i\sqrt{\beta}(\eta^* + c)\big)$$
$$B = -\Gamma(\kappa) e^{-\beta \frac{(\eta^* + c)^2}{4}} \operatorname{Im} U\big(\kappa - \frac{1}{2}, i\sqrt{\beta}(\eta^* + c)\big)$$
$$C = \Gamma(\kappa) e^{-\beta \frac{(\eta^* - c)^2}{4}} \operatorname{Re} U\big(\kappa - \frac{1}{2}, i\sqrt{\beta}(\eta^* - c)\big)$$
$$D = \Gamma(\kappa) e^{-\beta \frac{(\eta^* - c)^2}{4}} \operatorname{Im} U\big(\kappa - \frac{1}{2}, i\sqrt{\beta}(\eta^* - c)\big).$$

One can then use the asymptotic formula for $U(a, z)$ as $z \to \infty$ in Section 12.9 of [11] to deduce that as $\eta^* \to \infty$ one has

$$A = \Gamma(\kappa) \frac{\cos(\frac{\pi}{2}\kappa)}{(\sqrt{\beta}\eta^*)^\kappa} \Big\{1 + O\big(\tfrac{1}{\eta^*}\big)\Big\}, \quad B = \Gamma(\kappa) \frac{\sin(\frac{\pi}{2}\kappa)}{(\sqrt{\beta}\eta^*)^\kappa} \Big\{1 + O\big(\tfrac{1}{\eta^*}\big)\Big\},$$
$$C = \Gamma(\kappa) \frac{\cos(\frac{\pi}{2}\kappa)}{(\sqrt{\beta}\eta^*)^\kappa} \Big\{1 + O\big(\tfrac{1}{\eta^*}\big)\Big\}, \quad D = -\Gamma(\kappa) \frac{\sin(\frac{\pi}{2}\kappa)}{(\sqrt{\beta}\eta^*)^\kappa} \Big\{1 + O\big(\tfrac{1}{\eta^*}\big)\Big\}.$$

From these formulas and the fact that $0 < \kappa < 1$, we can easily conclude that for η^* large enough $AD - BC \neq 0$. To see the claimed asymptotic formula for $G'_{a,c}$, we may differentiate under the integrals in (A.7) to obtain

$$G'_{a,c}(\pm\eta^*+c) = -\sqrt{\beta}\frac{D-B}{AD-BC}\int_0^\infty tf(t)\sin(\sqrt{\beta}(\pm\eta^*+c)t)\,dt + \sqrt{\beta}\frac{A-C}{AD-BC}\int_0^\infty tf(t)\cos(\sqrt{\beta}(\pm\eta^*+c)t)\,dt.$$

Using a similar trick, we may write the functions

$$\int_0^\infty tf(t)\sin(\sqrt{\beta}(\pm\eta^*+c)t)\,dt \quad \text{and} \quad \int_0^\infty tf(t)\cos(\sqrt{\beta}(\pm\eta^*+c)t)\,dt$$

in terms of the function U as

$$\int_0^\infty tf(t)\sin(\sqrt{\beta}(\pm\eta^*+c)t)\,dt = -\kappa\Gamma(\kappa)e^{-\frac{\beta}{4}(\pm\eta^*+c)^2}\operatorname{Im}U\big(\kappa+\frac{1}{2}, i\sqrt{\beta}(\pm\eta^*+c)\big),$$

and

$$\int_0^\infty tf(t)\cos(\sqrt{\beta}(\pm\eta^*+c)t)\,dt = \kappa\Gamma(\kappa)e^{-\frac{\beta}{4}(\pm\eta^*+c)^2}\operatorname{Re}U\big(\kappa+\frac{1}{2}, i\sqrt{\beta}(\pm\eta^*+c)\big).$$

Again, applying the asymptotic formula for $U(a,z)$ as $z\to\infty$ in Section 12.9 of [3] with those derived for A, B, C, D, we can obtain the claimed asymptotic formulas for $G'_{a,c}$.

To see the symmetry $G_{a,0}(-\eta) = G_{a,0}(\eta)$ for $\eta\in[-\eta^*+c, \eta^*+c]$, set $c=0$ in (A.7) to see that

$$G_{a,0} = \frac{\int_0^\infty f(t)\cos(\sqrt{\beta}\eta t)\,dt}{\int_0^\infty f(t)\cos(\sqrt{\beta}\eta^* t)\,dt}.$$

□

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