Poisson Summation Conjecture on Braverman-Kazhdan Spaces

by

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Defense Date: March 25, 2024

Approved:

Jayce Getz, Supervisor

Samit Dasgupta

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Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
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ABSTRACT

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Abstract

Braverman and Kazhdan proposed a conjecture, later refined by Ngô and broadened to the framework of spherical varieties by Sakellaridis, that asserts that affine spherical varieties admit Schwartz spaces, Fourier transforms, and Poisson summation formulae. In the dissertation, we develop local Fourier theory and give explicit formulae for Fourier transforms on Braverman–Kazhdan spaces attached to maximal parabolic subgroups of split, almost simple, simply connected groups. In the nonarchimedean setting, we also give explicit representation theoretic descriptions of Schwartz spaces and verify several conjectural properties of Schwartz spaces.

Part of the thesis is based on joint work with Jayce Getz and Spencer Leslie.

Dedication

To Chanel and Tanya

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List of Symbols

$\widehat{K}_{\mathbb{G}_m}$	equivalence class of (unitary) characters of $F^\times$	(2.2)
X_P°	$P^{\mathrm{der}}\backslash G$	§3.1
β	simple root in Δ associated to P	§3.1.1
ω_β	fundamental weight associated to β	§3.1.1
ω_P	weight in $X^*(T)$ attached to P	(3.2)
X_P	affine closure of X_P°	§3.1.1
v_P	highest weight vector in $V_P(F)$	§3.1.1
$v^*_{P^{\mathrm{op}}}$	lowest weight vector dual to v_p in $V_P^\vee(F)$	§3.1.1
$\mathrm{Pl} = \mathrm{Pl}_{v_P}$	Plücker embedding $\mathrm{Pl} : X_P \longrightarrow V_P$	§3.1.1
V_P	right representation of G of highest weight $-\omega_P$	§3.1.1
$\langle \cdot, \cdot \rangle_{P \mid P^{\mathrm{op}}}$	pairing on $X_P^\circ \times X_{P^{\mathrm{op}}}^\circ$	(3.4)
$(\cdot)_{\chi,P}$, $(\cdot)^{\mathrm{op}}_{\chi,P^{\mathrm{op}}}$	Mellin transform along χ	(3.6)
$I_P(\chi), \overline{I}_{P^{\mathrm{op}}}(\chi)$	normalized parabolic induction	(3.5)
$V_{A,B}$	$\{s \in \mathbb{C} : A < \mathrm{Re}(s) < B\}$	(3.18)
$\lvert \cdot \rvert_{A,B,p}$	$\sup_{s \in V_{A,B}} \lvert p(s)\phi(s) \rvert$	(3.17)
$\lvert \cdot \rvert_{A,B,p_{P \mid Q},\Omega,D}$	seminorm	(3.45)
$U(\mathfrak{m}^{\mathrm{ab}} \oplus \mathfrak{g})$	universal enveloping algebra of $(\mathfrak{m}^{\mathrm{ab}} \oplus \mathfrak{g})_{\mathbb{C}}$	§3.2
$\lambda_!(\mu_s)$	normalized operator attached to (s, λ)	(3.9)
L	graded representation L of $\mathbb{G}_m$ with attached data $\{(s_i, \lambda_i)\}$	§3.2
$A(L), B(L)$	extended real numbers attached to L	(3.15)
$a_L(\chi)$	$\prod_{i \in I} L(-s_i, \chi^{\lambda_i})$	(3.14)
$a_{P \mid P}(\chi)$	$a_{\widetilde{L}}(\chi^{-1})$ with $L = \widehat{\mathfrak{n}}_P^e$	(3.42)
$a_{P \mid P^{\mathrm{op}}}(\chi)$	$a_L(\chi)$ with $L = \widehat{\mathfrak{n}}_P^e$	(3.42)
$\mathcal{S}_L$	Fréchet space attached to L	§3.2
$\mathcal{S}(X_P(F))$	Schwartz space on $X_P(F)$	Def. 3.3.5
$\{e, h, f\}$	principal $\mathfrak{sl}_2$ triple in $\widehat{\mathfrak{m}}$	§3.2.1
$\widehat{\mathfrak{n}}_P^e$	space of highest weight vectors in $\widehat{\mathfrak{n}}_P$ for a principal $\mathfrak{sl}_2$-triple	(3.36)
μ_L	normalized operator attached to L	(3.33)
$\mu_L(\chi)$	$\prod_{i=1}^k \gamma(-s_i, \chi^{\lambda_i}, \psi)$	(3.34)
μ_P	μ_L with $L = \widehat{\mathfrak{n}}_P^e$	(3.37)
$\mathcal{R}_{P \mid P}, \mathcal{R}_{P \mid P^{\mathrm{op}}}$	Radon transform	(3.41)
ι_{w_0}	isomorphism $\iota_{w_0} : \mathcal{S}(X_P(F)) \xrightarrow{\sim} \mathcal{S}(X_{P^{\mathrm{op}}}(F))$	§3.4
$\mathcal{F}_{P \mid P^{\mathrm{op}}}$	$\mu_P \circ \mathcal{R}_{P \mid P^{\mathrm{op}}}$	(3.47)
$\mathcal{F}^{\mathrm{geo}}_{P \mid P^{\mathrm{op}}}$	$\mu_P^{\mathrm{geo}} \circ \mathcal{R}_{P \mid P^{\mathrm{op}}}$	(3.70)
$\mathcal{F}_{X_P}$	$\iota_{w_0} \circ \mathcal{F}_{P \mid P^{\mathrm{op}}}$	(3.74)
μ_P^{aug}	$\lambda_{1!}(\mu_{s_1}) \circ \cdots \circ \lambda_{(k-1)!}(\mu_{s_{k-1}})$	(3.69)
μ_P^{geo}	$[1]_!(\mu_{s_k})$	(3.69)
$L(d), L_d$	asymptotics data attached to $\{(s_i, \lambda_i)\}$	(4.2)
$A_{r,n,\chi}, A_{r,\chi}$	asymptotics toward the origin of $\mathcal{S}(X_P(F))$	(4.7)
$\Phi_w^\vee$	$\{\beta^\vee \in (\Phi^\vee)^+ : w\beta^\vee \in (\Phi^\vee)^-\}$	§5.1
M'_w	normalized intertwining operator associated to $w \in W/W_M$	(5.3)

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Chapter 1. Introduction

The Riemann zeta function $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$, originally defined on $\mathrm{Re}(s) > 1$, is known to extend to a meromorphic function on the complex plane and satisfies a functional equation:

$$\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

The key ingredient in Riemann's proof (in 1859) is applying the Poisson summation formula to a specific choice of Schwartz function. Riemann's idea was later extended by Hecke in 1918-1920 to L-functions attached to Hecke characters over number fields.

In 1950, Tate in his thesis reinterpreted (completed) Hecke L-functions as certain adelic integrals and showed that functional equations and the analytic continuation of Hecke L-functions are roughly equivalent to adelic Fourier analysis on a one dimensional vector space.

1.1 A short overview of Tate's thesis

Let E be a global field and $\mathbb{A}_E$ be its ring of adeles. Let $|\cdot| := \prod_v |\cdot|_v : \mathbb{A}_E^\times \to \mathbb{R}_{>0}$ be the idelic norm. For a Schwartz function $f \in \mathcal{S}(\mathbb{A}_E) := \mathcal{S}(E_\infty) \otimes \bigotimes'_{v\nmid\infty} \mathcal{S}(E_v)$, an idelic character $\chi = \otimes_v \chi_v : \mathbb{A}_E^\times \to \mathbb{C}^\times$, and $s \in \mathbb{C}^\times$, Tate defined the zeta integral

$$Z(f,\chi,s) := \int_{\mathbb{A}_E^\times} f(x)\chi(x)|x|^s d^\times x$$

which converges absolutely for $\mathrm{Re}(s) > 1$. If $f = \otimes_v f_v$ is factorizable, the zeta integral can be written as an Euler product

$$Z(f,\chi,s) = \prod_{v\text{: places of } E} Z_v(f_v,\chi_v,s) := \prod_{v\text{: places of } E} \int_{E_v^\times} f_v(x)\chi_v(x)|x|_v^s d^\times x.$$

These local zeta integrals enjoy the following properties:

- The integral defining $Z_v(f_v,\chi_v,s)$ is absolutely convergent for $\mathrm{Re}(s) > 0$.
- $Z_v(f_v,\chi_v,s)$ has a meromorphic continuation in s to the whole $\mathbb{C}$. If E_v is nonarchimedean, it is a rational function in q_v^{-s}, where q_v is the size of the residue field of E_v.

- Upon normalization, there is a unique meromorphic function $L_v(\chi_v, s)$ such that
$$\frac{Z_v(f_v, \chi_v, s)}{L_v(\chi_v, s)}$$
is holomorphic for all Schwartz functions f_v. If E_v is nonarchimedean, it lies in $\mathbb{C}[q_v^{-s}, q_v^s]$.

Furthermore, each Hecke L-function is equal to the infinite product $\prod_{v\nmid\infty} L_v(\chi_v, s)$ realized by a particular Schwartz function and an idelic character. Functional equations of completed Hecke L-functions follow from Fourier analysis over each local field E_v, and the analytic continuation is a consequence of the Poisson summation formula:
$$\sum_{x\in E} f(x) = \sum_{x\in E} \mathcal{F}(f)(x).$$

Tate's thesis not only largely simplifies the original proof of Hecke but also opens up the study of automorphic forms in the adelic context. Tate's idea was later successfully generalized by Godement and Jacquet [GJ72] to L-functions attached to representations of general linear groups GL_n by replacing $\chi(x)$ with matrix coefficients. The main ingredient is Fourier analysis on the variety of n-by-n matrices, which is also a vector space.

Over the years, mathematicians have been able to come up with ad-hoc but ingenious approaches to construct integral representations of numerous L-functions without appealing to Fourier analysis. This leads to the following questions: Should every L-function possess an integral representation? Is there a uniform explanation behind different approaches?

1.2 An unifying theme

In the seminal paper [BK00], Braverman and Kazhdan realized that if one assumes the (local) Langlands correspondence and the Langlands functoriality conjecture, then there is a unifying theme in producing L-functions and establishing their analytic properties. Let us briefly explain their idea in the split case.

Let φ be a Langlands parameter of a split connected reductive group H. Given a finite dimensional representation of the complex dual group $\widehat{H}$, i.e., a morphism $\rho : \widehat{H} \to \widehat{\mathrm{GL}}_n = \mathrm{GL}_n(\mathbb{C})$, we obtain a Langlands parameter $\rho(\varphi) = \rho \circ \varphi$ of GL_n. Then by the local Langlands correspondence for general linear groups and the Godement-Jacquet theory, the Fourier theory on n-by-n matrices would induce a Fourier theory on certain reductive monoids attached to ρ under a mild hypothesis.

Thus [BK00] conjectured a notion of Schwartz spaces, Fourier transforms, and Poisson summation formulae on certain reductive monoids, which are referred to as L-monoids. If the conjecture is true, then it would prove the functorial property in the Langlands program in great generality by the converse theorem. The conjecture is later refined by Lafforgue [Laf14], Ngô [Ngô20], and many others, and broadened by Sakellaridis [Sak12] to the setting of affine spherical varieties. We will refer to this conjecture as the **Poisson summation conjecture**.

1.3 The Poisson summation conjecture

Let E be a global field and G be a split reductive group over E. Let X be an E-variety with a G-action. Then X is said to be **G-spherical** or simply **spherical** if X is normal and there is an open orbit of a Borel subgroup of G. The Poisson summation conjecture suggests that there should be a Fourier theory on an affine spherical variety. Precisely, it predicts (at least) the following.

Conjecture 1.3.1 (Poisson summation conjecture). *Let X be an affine G-spherical E-variety and X^{sm} be its smooth locus. Suppose $X^{\mathrm{sm}}(E) \neq \varnothing$. For every place v of E, there are*

- *a Schwartz space $\mathcal{S}(X(E_v))$ such that $\mathcal{S}(X^{\mathrm{sm}}(E_v)) \subseteq \mathcal{S}(X(E_v)) \subset C^\infty(X^{\mathrm{sm}}(E_v))$,*
- *a Fourier transform $\mathcal{F}_v : \mathcal{S}(X(E_v)) \to \mathcal{S}(X(E_v))$ that satisfies a twisted equivariance property,*
- *a basic function b_v such that $\mathcal{F}_v(b_v) = b_v$ at almost all places v.*

One defines the adelic Schwartz space

$$\mathcal{S}(X(\mathbb{A}_E)) := \bigotimes_v{}' \mathcal{S}(X(E_v)),$$

where the restricted tensor product is taken with respect to the basic function b_v, and an adelic

Fourier transform

$$\mathcal{F} := \otimes_v \mathcal{F}_v : \mathcal{S}(X(\mathbb{A}_E)) \to \mathcal{S}(X(\mathbb{A}_E)).$$

Furthermore, one has

- *a Poisson summation formula: for $f \in \mathcal{S}(X(\mathbb{A}_E))$*

$$\sum_{x \in X^{\mathrm{sm}}(E)} f(x) + (*) = \sum_{x \in X^{\mathrm{sm}}(E)} \mathcal{F}(f)(x) + (**)$$

where $(*), (**)$ are ***boundary terms*** *that vanish if f satisfies some local assumptions.*

The only examples where the Poisson summation conjecture is fully understood are vector spaces $\mathbb{G}_a^n$. Braverman and Kazhdan generalized the case of vector spaces to the affine closure X_P of the variety $X_P^\circ := P^{\mathrm{der}} \backslash G$ [BK02], where G is a split reductive group whose derived subgroup is simply connected and $P < G$ is a parabolic subgroup. We refer to spaces X_P as **Braverman-Kazhdan spaces**. Nevertheless, in [BK02] there are various conjectural properties of Schwartz spaces left unchecked and the discussion is incomplete in the archimedean case.

Aside from Braverman-Kazhdan spaces, there are few cases on which the Poisson summation conjecture is (partially) established. The first such cases are varieties built of triples of quadratic spaces. They are first studied in [GL19], and are completed in [GH20] and [GHL21]. Following the idea of [GL19], Gu [Gu21] established Poisson summation formulae on certain family of non-affine spherical varieties. In [CG21], Choie and Getz established the Poisson summation conjecture for certain Schubert varieties under mild assumptions on poles of degenerate Eisenstein series.

Remark 1.3.2. While the existence of local Fourier theory is established in many cases, only few cases have completely understood Poisson summation formulae, i.e., $(*)$ and $(**)$ are explicitly known. It is proved in [GL21] for the Lagrangian Grassmannian and in [Get22] for (split) odd dimensional cones. Both cases are proved via representation theory, and there is currently no geometric intuition for what boundary terms should be.

Remark 1.3.3. In many cases, including L-monoids, basic functions have been constructed via geometric approaches [BNS16, SW22]. In the case of L-monoids, they are realized as IC-functions on the arc spaces. This aligns with the expectation that Schwartz functions can be constructed via sheaf-function dictionary on appropriate spaces.

Remark 1.3.4. In the forthcoming work of Zhilin Luo and Ngô, they have constructed an analytically well behaved Fourier kernel for L-monoids of GL_2.

1.4 Main results

The dissertation is a combination of the first half of my joint work with Jayce Getz and Spencer Leslie [GHL21] and my other paper [Hsu21]. The goal is to complete the study of (local) Fourier theory on Braverman-Kazhdan spaces in the simplest cases, i.e., when P is maximal. We summarize our results roughly as follows.

Theorem 1.4.1. *Let G be a split, almost simple, simply connected group, and $P < G$ be a maximal parabolic subgroup. Over arbitrary local field F, there are an analytically defined explicit Schwartz space $\mathcal{S}(X_P(F))$ contained in $L^1(X_P(F)) \cap L^2(X_P(F))$, and a unitary operator $\mathcal{F}_{P|P^{\mathrm{op}}}$ that restricts to a $G(F)$-equivariant isomorphism*

$$\mathcal{F}_{P|P^{\mathrm{op}}} : \mathcal{S}(X_P(F)) \to \mathcal{S}(X_{P^{\mathrm{op}}}(F)).$$

The Fourier transform $\mathcal{F}_{P|P^{\mathrm{op}}}$ can be explicitly written down as some iterated integral.

When F is nonarchimedean and G is not of type E or F, one has

$$\mathcal{S}(X_P(F)) = C_c^\infty(X_P^\circ(F)) + \mathcal{F}_{P^{\mathrm{op}}|P}(C_c^\infty(X_{P^{\mathrm{op}}}^\circ(F))) \tag{1.1}$$

and $\mathcal{S}(X_P(F))$ is a $C^\infty(X_P(F))$-module under multiplication of functions. Furthermore, there is a (co)sheaf-theoretic way to describe $\mathcal{S}(X_P(F))$.

There are several reasons to develop harmonic analysis on X_P in such an explicit manner. First, one should think of these simple cases as building blocks for one to establish the Poisson summation conjecture for new cases via other methods. Indeed, a special case of Theorem 1.4.1 has been used in the second part of [GHL21] to study harmonic analysis on varieties built out of triples of quadratic spaces considered in [GL19].

Second, these are the cases where one has rather clear understanding on the spectral side of the Schwartz spaces, i.e., the degenerate principal series I_P^G have been studied thoroughly. They form precious examples where one can have less geometric intuition but can still prove the Poisson summation conjecture. It is then an interesting question to ask what geometric data can be recovered from spectral definition. This should be viewed as a reverse process of the Poisson summation conjecture, from which one could gain insight into general cases. See also the discussion in Chapter 6.

Finally, it is expected that Fourier theory on X_P should find applications in other fields, e.g., analytic number theory. For instance, the Poisson summation formulae on odd dimensional cones are used in [Get22] to count global points of quadrics with explicit asymptotics.

1.5 Outline of the thesis

We introduce notations and state our conventions on quasi-characters, measure, Schwartz spaces, and estimates in Chapter 2. We develop Fourier theory on Braverman-Kazhdan spaces over local fields in Chapter 3. We begin by recalling some basic facts on Braverman–Kazhdan spaces in §3.1. The definition of the Fourier transform on the Schwartz space of a Braverman–Kazhdan space relies on operators that correspond, under the Mellin transform, to multiplication by γ-factors. Only the nonarchimedean case appears in the literature. Even in this case the domain and range of these operators is never elucidated. This makes it problematic to define the composition of the operator on an explicit space of functions. We develop a new approach to these operators that works uniformly in the archimedean and nonarchimedean cases in §3.2. The new approach allows us to explicitly control the domain and range of the operators and to compose them. We expect these ideas will have applications to Fourier transforms beyond those constructed by Braverman and Kazhdan.

In §3.3 we give a refined definition of the Schwartz space of a Braverman–Kazhdan space whenever P is a maximal parabolic subgroup of a split, almost simple, and simply connected G, and prove in §3.4 that the Fourier transform preserves this space. In the special case where P is the Siegel parabolic of $G = \mathrm{Sp}_{2n}$, this definition is contained in [GL21]. This refinement goes beyond the work in [BK02], in which the Fourier transform is only defined via a transform defined on an inexplicit dense subspace of the L^2 space and then extended by continuity.

In §3.5 we prove an explicit formula of $\mathcal{F}_{P|P^{\mathrm{op}}}$ with geometric flavor in Theorem 3.5.5 below. The proof requires computations of various normalizing factors which are given in Appendix A. This is particularly important for readers without extensive background in representation theory who may want to apply our formula. This completes the proof of the first half of Theorem 1.4.1.

Starting from Chapter 4, we assume F is nonarchimedean. In Chapter 4, we assume the inclusion $C_c^\infty(X_P^\circ(F)) \leqslant \mathcal{S}(X_P(F))$ holds and study the quotient $\mathcal{S}(X_P(F))/C_c^\infty(X_P^\circ(F))$. Since $X_P - X_P^\circ = \{0\}$, motivated by the approach in [JLZ20, §4], we begin with rephrasing the definition of $\mathcal{S}(X_P(F))$ in terms of its asymptotic behavior towards the origin via the work of Igusa [Igu78, §1.5]. We also offer a list of asymptotics data $L(d) \cap \mathbb{R}$ and s_k (see (4.2)) in Appendix B. Then we restate the asymptotics in terms of representations in Theorem 4.2.2, which will justify the last assertion of Theorem 1.4.1.

In Chapter 5, we prove the inclusion $C_c^\infty(X_P^\circ(F)) \leqslant \mathcal{S}(X_P(F))$ when G is not of type E or F. Actually, we prove a stronger statement on poles of intertwining operators from which the rest of the assertion in Theorem 1.4.1 follows. Our proof is a case-by-case discussion.

In Chapter 6, we briefly explain our ongoing work to establish harmonic analysis on X_P over archimedean local fields and its connection to Weyl algebras on X_P.

Chapter 2. Preliminaries

Throughout G always denotes a split connected almost simple algebraic group over a field F, so $\mathfrak{g} := \mathrm{Lie}(G)$ is a simple Lie algebra. Fix a maximal split torus and a Borel subgroup $T < B < G$. For a standard parabolic subgroup $B \leqslant P$, let P^{op} denote its opposite and $N_P := R_u(P)$ be its unipotent radical. We often write $N := N_B$. Let $\Phi := \Phi_G \subset X^*(T)$ be the set of roots of (G, B, T) and $\Delta := \Delta_G \subset \Phi$ be its base.

Throughout F always denotes a local field. When F is nonarchimedean, we let $\mathcal{O}$ be its ring of integers. We fix a choice of uniformizer ϖ and let q be the cardinality of the residue field $\mathcal{O}/\varpi$. We denote by $|\cdot|$ the number theorist's norm on F. Thus $|\cdot|$ is the usual Euclidean norm if $F = \mathbb{R}$, $|z| = z\bar{z}$ if $F = \mathbb{C}$, and $|\varpi^{-1}| = q$ if F is nonarchimedean.

We also denote the usual norm on $\mathbb{C}$ by $|\cdot|$. This creates the possibility of confusion when we have chosen an identification $F = \mathbb{C}$. When F is denoted by $\mathbb{C}$, we use the standard norm, and when F is denoted simply F, we use the number-theorist's norm. Thus, for example, if X is a set and $f : X \to \mathbb{C}$ is a function, then $|f(x)| = (f(x)\overline{f(x)})^{1/2}$ for $x \in X$. This is a standard convention adopted to lighten notation.

2.1 Quasi-characters

Let $\widehat{F^\times}$ be the group of quasi-characters of $F^\times$. For $\chi \in \widehat{F^\times}$ and $s \in \mathbb{C}$, define $\chi_s := \chi|\cdot|^s$. Let $\mathrm{Re}(\chi)$ be the unique real number such that $\chi_{-\mathrm{Re}(\chi)}$ is a character (i.e., is unitary). Let $\widehat{K}_{\mathbb{G}_m}$ be a set of representatives for the characters of $F^\times$ modulo the equivalence relation

$$\chi_1 \sim \chi_2 \text{ if and only if } \chi_1 = \chi_2 |\cdot|^{it}$$

for some $t \in \mathbb{R}$. The set of equivalence classes can be identified with the set of characters of the maximal compact subgroup $K_{\mathbb{G}_m} < F^\times$, which explains the notation.

When F is nonarchimedean, we have a (noncanonical) group isomorphism

$$\begin{aligned} F^\times &\xrightarrow{\sim} \mathbb{Z} \times \mathcal{O}^\times \\ a &\mapsto (\mathrm{ord}(a), \tilde{a}), \end{aligned} \tag{2.1}$$

where

$$\mathrm{ord(a)} := -\log_q |a|, \quad \tilde{a} := a \cdot \varpi^{-\mathrm{ord(a)}}.$$

The isomorphism (2.1) induces a group isomorphism

$$\mathbb{C}/\tfrac{2\pi\sqrt{-1}}{\log q}\mathbb{Z} \times \widehat{\mathcal{O}^\times} \xrightarrow{\sim} \widehat{F^\times}$$

$$(s, \chi) \mapsto (a \mapsto |a|^s \chi(\tilde{a}))\,.$$

We henceforth identify $\widehat{\mathcal{O}^\times}$ as a subgroup of $\widehat{F^\times}$ under this isomorphism, and thus $\chi(a) = \chi(\tilde{a})$ for $\chi \in \widehat{\mathcal{O}^\times}$. Let $\mathrm{ord}(\chi)$ be the order of χ, i.e., the smallest positive integer d such that $\chi^d = 1$. We often write $\widehat{K}_{\mathbb{G}_m}$ and $\widehat{\mathcal{O}^\times}$ interchangeably.

When F is archimedean, let

$$\mu(z) := \frac{z}{(z\overline{z})^{1/2}},$$

where we use the positive square root. We always choose the representatives

$$\widehat{K}_{\mathbb{G}_m} := \left\{ \mu^\alpha : \begin{array}{c} \alpha \in \{0,1\} \text{ if } F \text{ is real} \\ \alpha \in \mathbb{Z} \text{ if } F \text{ is complex} \end{array} \right\}. \tag{2.2}$$

2.2 Measures

If dx denotes a Haar measure on F, then $d^\times x := \frac{\zeta(1)dx}{|x|}$ where ζ is the usual local zeta function. We often regard dx as a measure on the open dense subset $F^\times \subset F$. We fix once and for all a nontrivial additive character $\psi : F \to \mathbb{C}^\times$. The measure dx will always be normalized so that it is self-dual with respect to the Fourier transform on $\mathcal{S}(F)$ defined by ψ.

Fix a Chevalley basis of $\mathfrak{g}$ with respect to $\mathfrak{t} := \mathrm{Lie}(T)$. Let

$$\Theta : \mathfrak{g} \longrightarrow \mathfrak{g}$$

be the opposite involution attached to (G, B, T) (here we follow the conventions of [Mil17, §23.h]). For all roots α, this gives us vectors X_α in the root space of α that satisfy $X_{-\alpha} = -\Theta(X_\alpha)$ [Mil17, §23.h], and provides us with isomorphisms

$$\mathbb{G}_a \longrightarrow N_\alpha$$

where $N_\alpha < G$ is the root subgroup of α. We use this to endow each $N_\alpha(F)$ with the measure dx by transport of structure, which in turn induces measures on unipotent subgroups of $B(F)$ and its opposite $B^{\mathrm{op}}(F)$. This is the same normalization used in [Lan71].

2.3 Schwartz spaces

Let X be an irreducible smooth quasi-affine scheme of finite type over F. Suppose $X(F)$ is nonempty. We endow $X(F)$ with the analytic topology. If F is nonarchimedean, we define $\mathcal{S}(X(F)) := C_c^\infty(X(F))$ to be the space of locally constant functions on $X(F)$ with compact support. When F is archimedean, we define $\mathcal{S}(X(F)) = \mathcal{S}(\mathrm{Res}_{F/\mathbb{R}}X(\mathbb{R}))$ as in [ES18, Remark 3.2] (this is based on previous work in [AG08]). Briefly, one chooses an embedding $\mathrm{Res}_{F/\mathbb{R}}X(\mathbb{R}) \to \mathbb{R}^n$ in the category of real algebraic varieties with closed image and then defines $\mathcal{S}(X(F)) = \mathcal{S}(\mathbb{R}^n)/I$, where $I \leqslant \mathcal{S}(\mathbb{R}^n)$ is the (closed) ideal of functions that vanish identically on $X(F)$. The embedding $\mathrm{Res}_{F/\mathbb{R}}X(\mathbb{R}) \to \mathbb{R}^n$ always exists in the real algebraic category, even if X is merely quasi-affine (see [ES18, §2.1] for references). One endows $\mathcal{S}(X(F))$ with the quotient topology, which is Fréchet and nuclear. The space $\mathcal{S}(X(F))$ and its topology are independent of the choice of embedding [ES18, Lemma 3.6(i)].

2.4 Asymptotic notation

Let $g_1 : X \to \mathbb{R}_{\geqslant 0}$ and $g_2 : X \to \mathbb{R}_{\geqslant 0}$ be functions defined on a set X. We write

$$g_1(x) \ll_? g_2(x), \quad g_1(x) = O_?(g_2(x)) \tag{2.3}$$

if there is a constant $C_? > 0$ depending on the set ? such that $g_1(x) < C_? g_2(x)$ for all $x \in X$. We drop set symbols when denoting the set, e.g. we write $C_{a,b}$ instead of $C_{\{a,b\}}$. We will also say g_2 **dominates** g_1 in order to avoid repeating the phrase "is bounded by a constant times." If F is archimedean and ? contains an element of $\mathcal{S}(V(F))$ (or another topological vector space of functions) (2.3) will in addition mean that the implied constant can be chosen continuously as a function of f when the other elements in ? are fixed.

Chapter 3. Local Fourier theory on Braverman-Kazhdan spaces

In this chapter, we develop Fourier theory on Braverman-Kazhdan spaces over local fields. In §3.1 we introduce Braverman-Kazhdan spaces and study their geometry. Then we move on to introduce normalizing operators in §3.2 as a preparation to define Schwartz spaces and Fourier operators on Braverman-Kazhdan spaces in §3.3 and §3.4. Finally, in §3.5 we give formulae of Fourier operators with geometric flavor and work out several examples explicitly in §3.6, connecting this result to known formulae in the literature.

3.1 Braverman–Kazhdan spaces

Let $P < G$ be a standard maximal parabolic subgroup with Levi decomposition $P = MN_P$ such that $T \leqslant M$. Set

$$X_P^\circ := P^{\mathrm{der}} \backslash G.$$

It is known as a **pre-flag variety** since it is a $\mathbb{G}_m$-torsor over the generalized flag variety $P\backslash G$. This is a right $M^{\mathrm{ab}} \times G$-space, where the action is given on points in an F-algebra R by

$$\begin{aligned} X_P^\circ(R) \times M^{\mathrm{ab}}(R) \times G(R) &\longrightarrow X_P^\circ(R) \\ (x, m, g) &\longmapsto m^{-1}xg. \end{aligned} \tag{3.1}$$

3.1.1 Plücker embeddings

Suppose that $\beta \in \Delta$ is the simple root of (G, B, T) associated to P; that is, we have that $\Delta_M = \Delta - \{\beta\}$ is the set of simple roots for the based root system of $(M, M \cap B, T)$. Let $\omega_\beta \in X^*(T)_{\mathbb{Q}} := X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ be the fundamental weight of T determined by the relation

$$\langle \omega_\beta, \alpha^\vee \rangle = \delta_{\alpha,\beta} \quad \text{for all } \alpha \in \Delta,$$

where $\delta_{\alpha,\beta}$ is the Kronecker δ. It is not necessarily true that $\omega_\beta \in X^*(T)$. We let m_β be the least positive rational number such that $m_\beta \omega_\beta \in X^*(T)$ and define

$$\omega_P := m_\beta \omega_\beta. \tag{3.2}$$

We claim that $m_\beta \in \mathbb{Z}$. To see this, note that if Λ is the lattice in $X^*(T)_{\mathbb{Q}}$ spanned by the fundamental weights, one has

$$\lambda \in \Lambda \iff \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$$

for all simple roots $\alpha \in \Delta$. Since $X^*(T) \leqslant \Lambda$, the claim now follows by pairing ω_P with $\beta^\vee$.

Lemma 3.1.1. *If T is a maximal torus of a (connected) reductive F-group H, then $T \cap H^{\mathrm{der}}$ is a maximal torus of H^{der}. If T is split, then so are $T \cap H^{\mathrm{der}}$ and $T/T \cap H^{\mathrm{der}}$.* ☐

Proof. The first statement follows from the fact that $H = H^{\mathrm{der}}Z(H)$, where $Z(H)$ is the center of H. For the second statement, see e.g., [Mil17, Chapter 12]. ☐

Lemma 3.1.2. *The torus M^{ab} is split and isomorphic to $\mathbb{G}_m$. The map $M(F) \to M^{\mathrm{ab}}(F)$ is surjective.*

Proof. Since by our assumption G is split and almost simple, and P is maximal, the first assertion follows from Lemma 3.1.1. For the second assertion, by Lemma 3.1.1 we have an exact sequence of split tori

$$1 \longrightarrow T \cap M^{\mathrm{der}} \longrightarrow T \longrightarrow M^{\mathrm{ab}} \longrightarrow 1$$

obtained by restricting the map $M \to M^{\mathrm{ab}}$ to T. Therefore, $T(F) \to M^{\mathrm{ab}}(F)$ is surjective by Hilbert's theorem 90 and we deduce the lemma. ☐

Corollary 3.1.3. *The map $G(F) \to X_P^\circ(F)$ is surjective.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
G(F) \arrow[r, "q_1"] \arrow[dr, "q_3"'] & X_P^\circ(F) \arrow[d, "q_2"] \\
& (P\backslash G)(F),
\end{tikzcd}$$

where the q_i are the canonical quotient maps. The map q_3 is surjective [Mil17, Theorem 25.9]. For $y \in (P\backslash G)(F)$, choose $g \in G(F)$ such that $q_3(g) = y$. Set $x = q_1(g)$. Since M^{ab} is a split torus by Lemma 3.1.2, $q_2^{-1}(y)$ is a $M^{\mathrm{ab}}(F)$-torsor. In other words,

$$q_2^{-1}(y) = \{tx : t \in M^{\mathrm{ab}}(F)\} = \{q_1(mg) : m \in M(F)\}$$

since $M(F) \to M^{\mathrm{ab}}(F)$ is surjective by Lemma 3.1.2. Thus $q_2^{-1}(y)$ is in the image of q_1 for all $y \in (P\backslash G)(F)$. □

Let V_P be the right representation of G of highest weight $-\omega_P$. We remind the reader that for a right representation, the character of a highest weight vector is anti-dominant, explaining why the highest weight is $-\omega_P$. Fix a highest weight vector $v_P \in V_P(F)$.

Lemma 3.1.4. *The derived subgroup* P^{der} *is the stabilizer of* v_P, *so that the map* $\mathrm{Pl} := \mathrm{Pl}_{v_P} : X_P^\circ \to V_P$ *induced by*

$$\begin{aligned} G(R) &\longrightarrow V_P(R) \\ g &\longmapsto v_P g, \end{aligned}$$

maps X_P° *isomorphically onto to the orbit of* v_P *under* G. *The map* ω_P, *originally a character of* T, *extends to a character of* M, *and the induced map*

$$\omega_P : M^{\mathrm{ab}} \longrightarrow \mathbb{G}_m$$

is an isomorphism. For $m \in M^{\mathrm{ab}}(R)$, *one has*

$$\mathrm{Pl}(m^{-1}g) = \omega_P(m)\mathrm{Pl}(g). \tag{3.3}$$

Proof. It is well-known that P is the stabilizer of the line spanned by v_P (this follows from the discussion in [Bor91, §24.4]), and thus this line is a one-dimensional representation of P. We deduce that $-\omega_P$ extends from T to a character of P, and P acts via the character $-\omega_P$ on the line and hence the stabilizer of v_P contains P^{der}.

Since $P^{\mathrm{der}} = M^{\mathrm{der}}N_P$, to prove that P^{der} is the full stabilizer, it suffices to check that $\omega_P : M^{\mathrm{ab}} \to \mathbb{G}_m$ is an isomorphism. Upon choosing an isomorphism $M^{\mathrm{ab}} \cong \mathbb{G}_m$, we have that ω_P is given on points by $x \mapsto x^n$ for some non-zero $n \in \mathbb{Z}$. Then $\omega_P/n \in X^*(T)$. By our choice of ω_P, we deduce $n = \pm 1$ and ω_P is an isomorphism. The equivariance property (3.3) of Pl is now clear. □

Consider the affine closure

$$X_P := \overline{X_P^\circ}^{\mathrm{aff}} := \mathrm{Spec}(F[X_P^\circ]).$$

The affine F-scheme X_P is normal and of finite type, and the natural map $X_P^\circ \to X_P$ is an open immersion [BG02, Theorem 1.1.2]. We refer to X_P as a **Braverman-Kazhdan space** attached to G and P. We have the following explicit description of X_P [VP73, Theorem 1 and 2].

Theorem 3.1.5. *The embedding* $\mathrm{Pl} : X_P^\circ \to V_P$ *extends to a closed immersion* $\mathrm{Pl} : X_P \to V_P$. *The closed subscheme* $X_P - X_P^\circ$ *is a point and it is mapped under* Pl *to* 0. □

This implies the origin 0 is the only possible singularity of X_P. Moreover, one can easily check X_P is smooth if and only if $X_P = V_P$ is a vector space. Therefore, in general X_P is singular.

Let $V_P^\vee$ be the representation of G dual to V_P and let $v_{P^{\mathrm{op}}}^* \in V_P^\vee(F)$ be the lowest weight vector dual to v_P. We then have an embedding $\mathrm{Pl}_{v_{P^{\mathrm{op}}}^*} : X_{P^{\mathrm{op}}}^\circ \to V_P^\vee$ induced by

$$\begin{aligned} G(R) &\longrightarrow V_P^\vee(R) \\ g &\longmapsto v_{P^{\mathrm{op}}}^* g. \end{aligned}$$

Let $\langle \cdot, \cdot \rangle$ be the canonical pairing of V_P and $V_P^\vee$. Consider the G-equivariant pairing given on F-algebras R by

$$\begin{aligned} \langle \cdot, \cdot \rangle_{P|P^{\mathrm{op}}} : X_P^\circ(R) \times X_{P^{\mathrm{op}}}^\circ(R) &\longrightarrow R \\ (x, x^*) &\longmapsto \langle \mathrm{Pl}_{v_P}(x), \mathrm{Pl}_{v_{P^{\mathrm{op}}}^*}(x^*) \rangle. \end{aligned} \tag{3.4}$$

If we replace v_P by any other highest weight vector v_P', then $v_P' = t v_P$ for some $t \in F^\times$. Thus the dual vector of v_P' is $t^{-1} v_{P^{\mathrm{op}}}^*$. It follows that $\langle \cdot, \cdot \rangle_{P|P^{\mathrm{op}}}$ is independent of the choice of v_P.

3.1.2 Relation to induced representations

The space $\mathcal{S}(X_P^\circ(F))$, equipped with the $M^{\mathrm{ab}}(F)$-action induced by (3.1), can be thought of as a universal (degenerate) principal series representation. For a quasi-character χ :

$F^\times \to \mathbb{C}^\times$, let

$$I(\chi) := I_P(\chi) := \mathrm{Ind}_{P(F)}^{G(F)}(\chi \circ \omega_P), \quad \overline{I}(\chi) := \overline{I}_{P^{\mathrm{op}}}(\chi) := \mathrm{Ind}_{P^{\mathrm{op}}(F)}^{G(F)}(\chi \circ \omega_P) \tag{3.5}$$

be the normalized inductions in the category of smooth representations. Let δ_P be the modular character of P. We define Mellin transforms

$$\begin{aligned} \mathcal{S}(X_P^\circ(F)) &\longrightarrow I(\chi) \\ f &\longmapsto f_\chi(\cdot) := f_{\chi,P}(\cdot) := \int_{M^{\mathrm{ab}}(F)} \delta_P^{1/2}(m)\chi(\omega_P(m))f(m^{-1}\cdot)dm, \\ \mathcal{S}(X_{P^{\mathrm{op}}}^\circ(F)) &\longrightarrow \overline{I}(\chi) \\ f &\longmapsto f_\chi^{\mathrm{op}}(\cdot) := f_{\chi,P^{\mathrm{op}}}^{\mathrm{op}}(\cdot) := \int_{M^{\mathrm{ab}}(F)} \delta_{P^{\mathrm{op}}}^{1/2}(m)\chi(\omega_P(m))f(m^{-1}\cdot)dm. \end{aligned} \tag{3.6}$$

Here dm is the Haar measure on $M^{\mathrm{ab}}(F)$ obtained from the isomorphism $\omega_P : M^{\mathrm{ab}}(F) \to F^\times$ and the Haar measure $d^\times x$ on $F^\times$ by our convention in §2.2. In the notation $\overline{I}_{P^{\mathrm{op}}}(\chi)$ and $f_{\chi,P^{\mathrm{op}}}^{\mathrm{op}}$, the bar and the superscript op indicate that we are inducing from $\chi \circ \omega_P$ instead of $\chi \circ \omega_{P^{\mathrm{op}}}$.

We use the same notation for extensions of the Mellin transform to larger subsets of $C^\infty(X_P^\circ(F))$ and $C^\infty(X_{P^{\mathrm{op}}}^\circ(F))$, when in general the integrals defining $f_\chi, f_\chi^{\mathrm{op}}$ only exist for $\mathrm{Re}(\chi)$ in a proper subset of $\mathbb{R}$, and in some cases will be extended to larger complex domains by analytic continuation.

When F is archimedean (resp. nonarchimedean), we say that a section $f^{(s)} \in I(\chi_s)$ is **holomorphic** if for all $g \in G(F)$, the function

$$\begin{aligned} \mathbb{C} &\longrightarrow \mathbb{C} \\ s &\longmapsto f^{(s)}(g) \end{aligned} \tag{3.7}$$

is holomorphic (resp. lies in $\mathbb{C}[q^{-s}, q^s]$). We say $f^{(s)}$ is **meromorphic** if there is a nonzero holomorphic function $a(s)$ (resp. $a(s) \in \mathbb{C}[q^{-s}, q^s]$) such that $a(s)f^{(s)}$ is holomorphic.

3.2 Twisting by abelian γ-factors

As discussed in §3.4 below, the definition of the Fourier transform $\mathcal{F}_{P|P^{\mathrm{op}}}$ involves normalization operators $\lambda_!(\mu_s)$ which correspond, under the Mellin transform, to multiplication by $\gamma(-s, \chi^{\lambda}, \psi)$ (see Lemma 3.2.3). Here and below $\gamma(s, \chi, \psi)$ denotes the usual Tate γ-factor attached to a complex number s, a quasi-character $\chi : F^{\times} \to \mathbb{C}^{\times}$, and the additive character ψ. The operators $\lambda_!(\mu_s)$ were previously defined in [BK02] and an exposition is given in [Sha18]. The approach of [BK02] is inconvenient in the sense that each operator is only defined on an inexplicit subspace of $\mathcal{S}(X_P^{\circ}(F))$ that is dense in $L^2(X_P(F))$. Thus as one composes operators, one loses control of their domain and range. Moreover, the operators are only defined in the nonarchimedean case in [BK02].

In this section we set up a general theory of the operators $\lambda_!(\mu_s)$ that is applicable uniformly in both archimedean and nonarchimedean settings. We also explain how to control their domain and range. This is quite delicate. In particular, to construct the Fourier transform, the normalizing operators $\lambda_!(\mu_s)$ have to be composed in a particular order. This motivates the definition of a **good ordering** in Definition 3.2.9 below. Essentially the situation is as follows: to compose the operators $\lambda_!(\mu_s)$, we require the domain of absolute convergence of certain Tate integrals to overlap. This is only possible if we arrange the operators in a particular order.

Remark 3.2.1. This difficulty was also encountered in the nonarchimedean setting in a special case in [JLZ20]. They overcame it by packaging all the normalizing operators together and relating them to transforms coming from prehomogeneous vector spaces. We do not know if their method can be used to obtain an explicit formula for the Fourier transform, or if it can be applied in the generality considered here.

For nonzero $\lambda \in \mathbb{Z}$ and arbitrary $s \in \mathbb{C}$, we define a linear map

$$\lambda_!(\mu_s) : \mathcal{S}(X_{P^{\mathrm{op}}}^{\circ}(F)) \longrightarrow C^{\infty}(X_{P^{\mathrm{op}}}^{\circ}(F)) \tag{3.8}$$

given by

$$\lambda_!(\mu_s)(f)(x) := \int_{M^{\mathrm{ab}}(F)} \psi(\omega_P(m))|\omega_P(m)|^{s+1}\delta_{P^{\mathrm{op}}}^{\lambda/2}(m) f(m^{-\lambda}x)\frac{dm}{\zeta(1)}. \tag{3.9}$$

This was denoted $\lambda_!(\eta_{\psi}^{s})$ in [BK02]. In loc. cit. a measure is incorporated into the distribution; this is why our formula looks different.

We work with $X_{P^{\mathrm{op}}}^{\circ}$ here to be consistent with our notation later on, when these operators are applied after the operator $\mathcal{R}_{P|P^{\mathrm{op}}}$ of (3.41). Of course in the formula for (3.9) we could write everything in terms of P or P^{op} by taking appropriate inverses. We have written it in the form above to remind the reader that f is a function on $X_{P^{\mathrm{op}}}^{\circ}(F)$, but the normalizing factors λ and s we will use in our case of primary interest are defined in terms of P (see §3.2.1 below).

To extend the domain of definition of $\lambda_!(\mu_s)$, choose $\Phi \in \mathcal{S}(F)$ such that $\Phi(0) = 1$ and $\widehat{\Phi} \in C_c^{\infty}(F)$. Here $\widehat{\Phi}(x) := \int_F \Phi(y)\psi(xy)dx$ is the Fourier transform of Φ. For continuous functions $f : X_{P^{\mathrm{op}}}^{\circ}(F) \to \mathbb{C}$ and $x \in X_{P^{\mathrm{op}}}^{\circ}(F)$, we define the regularized integral

$$\lambda_!(\mu_s)^{\mathrm{reg}}(f)(x) := \lim_{|b|\to\infty} \int_{M^{\mathrm{ab}}(F)} \Phi\left(\frac{\omega_P(m)}{b}\right)\psi(\omega_P(m))|\omega_P(m)|^{s+1}\delta_{P^{\mathrm{op}}}^{\lambda/2}(m)f(m^{-\lambda}x)\frac{dm}{\zeta(1)}. \tag{3.10}$$

We say this integral is well defined if

$$\int_{M^{\mathrm{ab}}(F)} |\Phi|\left(\frac{\omega_P(m)}{b}\right)|\omega_P(m)|^{\mathrm{Re}(s)+1}\delta_{P^{\mathrm{op}}}^{\lambda/2}(m)|f|(m^{-\lambda}x)dm \tag{3.11}$$

is finite for $|b|$ sufficiently large, and the limit in the definition of $\lambda_!(\mu_s)^{\mathrm{reg}}(f)(x)$ exists and is independent of Φ.

Lemma 3.2.2. *If the integral defining $\lambda_!(\mu_s)(f)(x)$ is absolutely convergent, then $\lambda_!(\mu_s)^{\mathrm{reg}}(f)(x) = \lambda_!(\mu_s)(f)(x)$. In particular, $\lambda_!(\mu_s)^{\mathrm{reg}}(f) = \lambda_!(\mu_s)(f)$ whenever $f \in \mathcal{S}(X_{P^{\mathrm{op}}}^{\circ}(F))$.* $\square$

To avoid more proliferation of notation, we will drop the reg from notation. Lemma 3.2.2 shows this is harmless, as it implies that the two integrals yield the same result when both are well defined.

Lemma 3.2.3. *Assume $f \in \overline{I}(\chi)$ and that $\mathrm{Re}(s) + 1 - \lambda\mathrm{Re}(\chi) > 0$. The function*

$$\lambda_!(\mu_s)(f)(x)$$

is well defined and equal to $\gamma(-s, \chi^{\lambda}, \psi)f(x)$.

Proof. Since $\mathrm{Re}(s) + 1 - \lambda \mathrm{Re}(\chi) > 0$, (3.11) is finite for all b. By the functional equation of Tate zeta functions, we have

$$\begin{aligned} &\int_{M^{\mathrm{ab}}(F)} \Phi\left(\frac{\omega_P(m)}{b}\right) \psi(\omega_P(m)) |\omega_P(m)|^{s+1} \chi^{-\lambda}(\omega_P(m)) f(x) \frac{dm}{\zeta(1)} \\ &= \gamma(-s, \chi^{\lambda}, \psi) f(x) \int_{F^\times} \left(\int_F \Phi\left(\frac{t}{b}\right) \psi(t) \overline{\psi}(yt) dt \right) |y|^{-s} \chi^{\lambda}(y) \frac{d^\times y}{\zeta(1)}. \end{aligned}$$

Using our assumption that $\Phi(0) = 1$, we have

$$\begin{aligned} &\lim_{|b| \to \infty} |b| \int_{F^\times} \left(\int_F \Phi(t)\, \psi(bt(1-y)) dt \right) |y|^{-s} \chi^{\lambda}(y) \frac{d^\times y}{\zeta(1)} \\ &= \lim_{|b| \to \infty} |b| \int_{F^\times} \widehat{\Phi}(b(1-y)) |y|^{-s} \chi^{\lambda}(y) \frac{d^\times y}{\zeta(1)} \\ &= \lim_{|b| \to \infty} \int_F \widehat{\Phi}(y) |1 - \tfrac{y}{b}|^{-s-1} \chi^{\lambda}(1 - \tfrac{y}{b}) dy \\ &= 1. \end{aligned} \tag{3.12}$$

Here for small $|b|$ the integral may diverge, but since $\widehat{\Phi} \in C_c^{\infty}(F)$, the integral converges for $|b|$ sufficiently large. $\square$

Now consider a graded $\mathbb{G}_m$-representation

$$L = \bigoplus_{i \in I} L_i$$

for some finite index set I. We assume that each L_i is 1-dimensional and that $\mathbb{G}_m$ acts via a character λ_i on L_i. We identify $X^*(\mathbb{G}_m)$ with $\mathbb{Z}$ by taking the identity character to 1, so we can speak of positive or negative characters. We assume that each character λ_i is non-zero and assign to each L_i a real number s_i.

We then have linear maps

$$\lambda_{i!}(\mu_{s_i}) : \mathcal{S}(X^{\circ}_{P^{\mathrm{op}}}(F)) \longrightarrow C^{\infty}(X^{\circ}_{P^{\mathrm{op}}}(F)) \tag{3.13}$$

for each $i \in I$. Following [BK02], we wish to compose these linear maps to give a single transform

$$\mu_L : \mathcal{S}(X^{\circ}_{P^{\mathrm{op}}}(F)) \longrightarrow C^{\infty}(X^{\circ}_{P^{\mathrm{op}}}(F))$$

associated to the $\mathbb{G}_m$-module L and the data $\{(s_i, \lambda_i) \in \mathbb{R} \times \mathbb{Z} : i \in I\}$.

It is convenient (and perhaps necessary) to extend the work in [BK02] by elucidating the domain and range of these operators. We proceed as in [GL21], which borrows from [Ike92]. Let

$$a_L(\chi) := \prod_{i \in I} L(-s_i, \chi^{\lambda_i}). \tag{3.14}$$

We introduce extended real numbers $A(L), B(L)$ as follows:

$$\begin{aligned} A(L) &:= \begin{cases} \max\left\{\frac{s_i}{\lambda_i} : i \in I, \lambda_i > 0\right\} & \text{if } \lambda_i > 0 \text{ for some } i, \\ -\infty & \text{otherwise,} \end{cases} \\ B(L) &:= \begin{cases} \min\left\{\frac{s_i}{\lambda_i} : i \in I, \lambda_i < 0\right\} & \text{if } \lambda_i < 0 \text{ for some } i, \\ \infty & \text{otherwise.} \end{cases} \end{aligned} \tag{3.15}$$

Assume that $A(L) < B(L)$.

Lemma 3.2.4. *The function $a_L(\chi)$ has no poles for $A(L) < \mathrm{Re}(\chi) < B(L)$.* □

We now define the space

$$\mathcal{S}_L := \mathcal{S}_L(X_{P^{\mathrm{op}}}(F)) < C^\infty(X^\circ_{P^{\mathrm{op}}}(F)). \tag{3.16}$$

When F is nonarchimedean, we define $\mathcal{S}_L$ to be the space of smooth functions $f : X^\circ_{P^{\mathrm{op}}}(F) \to \mathbb{C}$ that are finite under a maximal compact subgroup of $G(F)$ and satisfy the following additional condition: the integral defining $f^{\mathrm{op}}_{\chi_s}(x)$ is absolutely convergent for $A(L) < \mathrm{Re}(s) < B(L)$ and $x \in X^\circ_P(F)$, and

$$\frac{f^{\mathrm{op}}_{\chi_s}}{a_L(\chi_s)}$$

is a holomorphic section for all (unitary) characters $\chi : F^\times \to \mathbb{C}^\times$.

When F is archimedean, we require a bit more notation. For real numbers $A < B$, $p \in \mathbb{C}[s]$, and meromorphic functions $\phi : \mathbb{C} \to \mathbb{C}$, we let

$$|\phi|_{A,B,p} := \sup_{s \in V_{A,B}} |p(s)\phi(s)| \tag{3.17}$$

where

$$V_{A,B} := \{s \in \mathbb{C} : A < \mathrm{Re}(s) < B\}. \tag{3.18}$$

Consider the Lie algebra

$$\mathfrak{m}^{\mathrm{ab}} \oplus \mathfrak{g} := \mathrm{Lie}(M^{\mathrm{ab}} \times G). \tag{3.19}$$

It acts on $C^\infty(X^\circ_{P^{\mathrm{op}}}(F))$ via the differential of the action (3.1) and hence we obtain an action of $U(\mathfrak{m}^{\mathrm{ab}} \oplus \mathfrak{g})$, the universal enveloping algebra of $(\mathfrak{m}^{\mathrm{ab}} \oplus \mathfrak{g})_{\mathbb{C}}$ (here we view $\mathfrak{m}^{\mathrm{ab}} \oplus \mathfrak{g}$ as a real Lie algebra).

Recall our choice of $\widehat{K}_{\mathbb{G}_m}$ at archimedean places in (2.2). We let $\mathcal{S}_L$ be the space of smooth functions $f : X^\circ_{P^{\mathrm{op}}}(F) \to \mathbb{C}$ such that for all $\eta \in \widehat{K}_{\mathbb{G}_m}$ and all $D \in U(\mathfrak{m}^{\mathrm{ab}} \oplus \mathfrak{g})$, the integral defining

$$(D.f)^{\mathrm{op}}_{\eta_s}(x)$$

converges absolutely for all $A(L) < \mathrm{Re}(s) < B(L)$, and admits a meromorphic continuation to the plane such that

1. for all $A < B$,
2. all polynomials $p \in \mathbb{C}[s]$ such that $p(s)a_L(\eta_s)$ has no poles in $V_{A,B}$ for all $\eta \in \widehat{K}_{\mathbb{G}_m}$,
3. all compact subsets $\Omega \subset X^\circ_{P^{\mathrm{op}}}(F)$,
4. all $D \in U(\mathfrak{m}^{\mathrm{ab}} \oplus \mathfrak{g})$,

one has that

$$|f|_{A,B,p,\Omega,D} := \sum_{\eta \in \widehat{K}_{\mathbb{G}_m}} \sup_{x \in \Omega} |(D.f)^{\mathrm{op}}_{\eta_s}(x)|_{A,B,p} < \infty. \tag{3.20}$$

This collection of seminorms gives $\mathcal{S}_L$ the structure of a Fréchet space by the same argument as in [GH20, Lemma 3.2].

In all cases, this definition allows us to recover analytic properties of f from its Mellin transforms via Mellin inversion. More specifically, Let $\kappa \in \mathbb{R}_{>0}$ (depending on ψ) be chosen so that

$$\kappa dx$$

is the standard Haar measure on F. Here the standard Haar measure is the Lebesgue measure if $F = \mathbb{R}$, twice the Lebesgue measure if $F = \mathbb{C}$, and satisfies $\kappa dx(\mathcal{O}) = |\mathfrak{d}|^{1/2}$ where $\mathfrak{d}$ is a generator for the absolute different of $\mathcal{O}$ when F is nonarchimedean. We then let

$$I_F := \begin{cases} \left[-\frac{\pi}{\log q}, \frac{\pi}{\log q}\right] & \text{if } F \text{ is nonarchimedean,} \\ \mathbb{R} & \text{if } F \text{ is archimedean,} \end{cases} \tag{3.21}$$

and

$$c_F := \begin{cases} \kappa \log q & \text{if } F \text{ is nonarchimedean,} \\ \frac{\kappa}{2} & \text{if } F = \mathbb{R}, \\ \frac{\kappa}{2\pi} & \text{if } F = \mathbb{C}. \end{cases}$$

We fix now a maximal compact subgroup $K < G(F)$ such that the Iwasawa decomposition

$$P(F)K = G(F) \tag{3.22}$$

holds. The following is a version of Mellin inversion (see [GL21, Lemma 4.3], [Fol16, Theorem 4.32], [BB11, (2.2)]):

Lemma 3.2.5. *Let $f \in C^\infty(X^\circ_{P^{\mathrm{op}}}(F))$ and assume for all $\eta \in \widehat{K}_{\mathbb{G}_m}$ the integral defining $f^{\mathrm{op}}_{\eta_s}$ is absolutely convergent for $\mathrm{Re}(s) = \sigma$. Suppose moreover that for all $x \in X^\circ_{P^{\mathrm{op}}}(F)$ one has*

$$\sum_{\eta \in \widehat{K}_{\mathbb{G}_m}} \int_{\sigma + iI_F} |f^{\mathrm{op}}_{\eta_s}(x)| ds < \infty.$$

Then for all $x \in X^\circ_{P^{\mathrm{op}}}(F)$ one has

$$f(x) = \sum_{\eta \in \widehat{K}_{\mathbb{G}_m}} \int_{\sigma + iI_F} f^{\mathrm{op}}_{\eta_s}(x) \frac{c_F ds}{2\pi i}. \tag{3.23}$$

Moreover, f is K-finite if and only if the sum over η has support in a finite set independent of x.

Conversely, suppose that we are given continuous $f(\eta)^{(s)} \in \overline{I}(\eta_s)$ for all s with $\mathrm{Re}(s) = \sigma$ and all $\eta \in \widehat{K}_{\mathbb{G}_m}$ and that

$$\sum_{\eta \in \widehat{K}_{\mathbb{G}_m}} \int_{\sigma + iI_F} |f(\eta)^{(s)}(x)| ds < \infty$$

for all $x \in X^{\circ}_{P^{\mathrm{op}}}(F)$*. Assume moreover in the nonarchimedean case that* $f(\eta)^{(s+\frac{2\pi i}{\log q})} = f(\eta)^{(s)}$*. Define*

$$f(x) := \sum_{\eta \in \widehat{K}_{\mathbb{G}_m}} \int_{\sigma + iI_F} f(\eta)^{(s)}(x) \frac{c_F ds}{2\pi i}.$$

If the integral defining $f^{\mathrm{op}}_{\eta_s}$ *is absolutely convergent for all* $\eta \in \widehat{K}_{\mathbb{G}_m}$ *and* s *with* $\mathrm{Re}(s) = \sigma$ *then* $f^{\mathrm{op}}_{\eta_s} = f(\eta)^{(s)}$*.* $\square$

The lemma implies in particular that (3.23) holds for $f \in \mathcal{S}_L$ and $A(L) < \sigma < B(L)$.

As an immediate consequence of Mellin inversion (3.23), we deduce the following estimate for functions in $\mathcal{S}_L$:

Lemma 3.2.6. *Assume* $\varepsilon > 0$ *is chosen so that* $A(L) + \varepsilon < B(L) - \varepsilon$*, and let* $\Omega \subset X_{P^{\mathrm{op}}}(F)$ *be a compact subset. For each* $f \in \mathcal{S}_L$ *and* $(m, x) \in M^{\mathrm{ab}}(F) \times \Omega$*, one has an estimate*

$$|f(mx)| \ll_{\Omega, f, \varepsilon} \delta^{1/2}_{P^{\mathrm{op}}}(m) \min(|\omega_P(m)|^{A(L)+\varepsilon}, |\omega_P(m)|^{B(L)-\varepsilon}).$$

Here when $A(L) = -\infty$ *we interpret* $A(L)+\varepsilon$ *as any negative real number* A*, and when* $B(L) = \infty$ *we interpret* $B(L) - \varepsilon$ *as any positive real number* B*. In these cases, the implied constant depends on* A *and* B*.* $\square$

We now use this to give a criterion for when the regularized integral is the usual integral:

Lemma 3.2.7. *Assume* $\lambda > 0$ *and let* $s \in \mathbb{C}$*. If*

$$A(L) < \frac{\mathrm{Re}(s) + 1}{\lambda} < B(L),$$

then the integral defining $\lambda_!(\mu_s)(f)$ *is absolutely convergent for* $f \in \mathcal{S}_L$*.*

Proof. Substituting the bounds from Lemma 3.2.6, it suffices to observe that

$$\int_{F^\times} |t|^{\mathrm{Re}(s)+1} \min(|t|^{-\lambda(A(L)+\varepsilon)}, |t|^{-\lambda(B(L)-\varepsilon)}) d^\times t$$

is convergent for $\varepsilon > 0$ sufficiently small. Here when $A(L) = -\infty$ or $B(L) = \infty$, we interpret $A(L) + \varepsilon$ and $B(L) - \varepsilon$ as in Lemma 3.2.6. $\square$

For each i, let

$$\widetilde{L}_i \tag{3.24}$$

be $L_i^\vee$ (the one-dimensional vector space on which $\mathbb{G}_m$ acts via $-\lambda_i$) attached with the real number $-1-s_i$. If $-\infty < A(L)$, choose L_k such that $A(L) = \frac{s_k}{\lambda_k}$, and define

$$L' := \widetilde{L}_k \oplus \bigoplus_{i \neq k} L_i.$$

Since we have assumed $A(L) < B(L)$, we have that

$$A(L') \leqslant A(L) < B(L') \leqslant B(L), \tag{3.25}$$

so

$$(A(L), B(L)) \cap (A(L'), B(L')) = (A(L), B(L')) \neq \varnothing. \tag{3.26}$$

Using this observation, we prove the following proposition:

Proposition 3.2.8. *For* $-\infty < A(L) < \mathrm{Re}(\chi) < B(L')$, *there is a commutative diagram*

$$\begin{tikzcd}
\mathcal{S}_L \arrow[r, "\lambda_{k!}(\mu_{s_k})"] \arrow[d, "(\cdot)^{\mathrm{op}}_\chi"] & \mathcal{S}_{L'} \arrow[d, "(\cdot)^{\mathrm{op}}_\chi"] \\
\overline{I}(\chi) \arrow[r] & \overline{I}(\chi)
\end{tikzcd}$$

where the bottom arrow is multiplication by $\gamma(-s_k, \chi^{\lambda_k}, \psi)$ *and the vertical arrows are* $f \mapsto f^{\mathrm{op}}_\chi$. *In particular, the regularized integral* $\lambda_{k!}(\mu_{s_k})$ *is well-defined on* $\mathcal{S}_L$.

Proof. Let $f \in \mathcal{S}_L$ and $x \in X^\circ_{P^{\mathrm{op}}}(F)$. By Lemma 3.2.6, for any $\varepsilon > 0$ we have

$$\int_{M^{\mathrm{ab}}(F)} |\Phi| \left(\frac{\omega_P(m)}{b}\right) |\omega_P(m)|^{s_k+1} \delta_{P^{\mathrm{op}}}^{\lambda_k/2}(m) |f|(m^{-\lambda_k}x) \frac{dm}{\zeta(1)}$$

$$\ll_{f,\varepsilon,x} \int_{F^\times} |\Phi| \left(\frac{t}{b}\right) |t|^{s_k+1-\lambda_k(A(L)+\varepsilon)} d^\times t,$$

which is finite for any b when ε is sufficiently small.

We claim that

$$\lambda_{k!}(\mu_{s_k})(f)(x) = \lim_{|b|\to\infty}\int_{M^{\mathrm{ab}}(F)} \Phi\left(\frac{\omega_P(m)}{b}\right)\psi(\omega_P(m))|\omega_P(m)|^{s_k+1}\delta_{P^{\mathrm{op}}}^{\lambda_k/2}(m)f(m^{-\lambda_k}x)\frac{dm}{\zeta(1)}$$

converges and is equal to

$$h(x) := \sum_{\eta\in\widehat{K}_{\mathbb{G}_m}}\int_{\sigma+iI_F}\gamma(\lambda_k s - s_k,\eta^{\lambda_k},\psi)f^{\mathrm{op}}_{\eta_s}(x)\frac{c_F ds}{2\pi i} \tag{3.27}$$

for

$$A(L) < \sigma < B(L').$$

Before proving the claim, it is convenient to study $h(x)$. By standard properties of the Tate γ-factor, we have

$$\frac{\gamma(\lambda_k s - s_k,\eta^{\lambda_k},\psi)f^{\mathrm{op}}_{\eta_s}}{a_{L'}(\eta_s)} = \frac{g(s,\eta,\psi)f^{\mathrm{op}}_{\eta_s}}{a_L(\eta_s)} \tag{3.28}$$

where $g(s,\eta,\psi)$ lies in $\mathbb{C}[q^{-s},q^s]$ in the nonarchimedean case and is holomorphic and bounded in $V_{A,B}$ for all $-\infty < A < B < \infty$ by a constant independent of η when F is archimedean. Thus the expression defining $h(x)$ is absolutely convergent for $A(L) < \sigma < B(L')$ since $a_{L'}(\eta_s)$ has no poles in this range (see (3.26)). Here when F is nonarchimedean, we have used the fact that functions in $\mathcal{S}_L$ are finite under a maximal compact subgroup of $G(F)$ and hence the sum over η in (3.27) has finite support. In the archimedean case we have used the fact that $\gamma(\lambda_k s - s_k,\eta^{\lambda_k},\psi)$ is bounded by a polynomial in s independent of η for $A(L) < \mathrm{Re}(s) < B(L')$ (see the proof of [GL21, Lemma 3.3]).

Let $A(L) < \sigma < B(L')$ and $x \in X^{\circ}_{P^{\mathrm{op}}}(F)$. We claim the integral

$$\begin{aligned}&\int_{M^{\mathrm{ab}}(F)}\delta_{P^{\mathrm{op}}}^{1/2}(m)|\omega_P(m)|^{\sigma}|h|(m^{-1}x)dm\\ &=\int_{M^{\mathrm{ab}}(F)}\left|\sum_{\eta\in\widehat{K}_{\mathbb{G}_m}}\int_{iI_F}(\eta_s)^{-1}(\omega_P(m))\gamma(\lambda_k\sigma+\lambda_k s - s_k,\eta^{\lambda_k},\psi)f^{\mathrm{op}}_{\eta_{\sigma+s}}(x)\frac{c_F ds}{2\pi i}\right|dm\end{aligned} \tag{3.29}$$

is convergent. This implies in particular that $h^{\mathrm{op}}_{\chi_s}$ is well-defined for $A(L) < \sigma < B(L')$. If F is nonarchimedean, it suffices to fix $\eta \in \widehat{K}_{\mathbb{G}_m}$ and show

$$\int_{M^{\mathrm{ab}}(F)}\left|\int_{iI_F}(\eta_s)^{-1}(\omega_P(m))\gamma(\lambda_k\sigma+\lambda_k s - s_k,\eta^{\lambda_k},\psi)f^{\mathrm{op}}_{\eta_{\sigma+s}}(x)\frac{c_F ds}{2\pi i}\right|dm < \infty$$

By the smoothness of η, it suffices to show

$$\sum_{n\in\mathbb{Z}}\left|\int_{iI_F} q^{ns}\gamma(\lambda_k\sigma+\lambda_k s-s_k,\eta^{\lambda_k},\psi)f^{\mathrm{op}}_{\eta_{\sigma+s}}(x)\frac{c_F ds}{2\pi i}\right|<\infty \tag{3.30}$$

This is nothing but the ℓ^1-norm of the Fourier transform of the smooth function

$$\mathbb{R}/\tfrac{2\pi}{\log q}\mathbb{Z}\longrightarrow\mathbb{C}$$

$$s\longmapsto\gamma(\lambda_k\sigma+\lambda_k is-s_k,\eta^{\lambda_k},\psi)f^{\mathrm{op}}_{\eta_{\sigma+is}}(x).$$

Hence (3.30) is valid by a standard integration by parts argument. In the archimedean case the proof that (3.29) converges is similar. One uses the fact that $f\in\mathcal{S}_L$ and that $\gamma(\lambda_k s-s_k,\eta^{\lambda_k},\psi)$ is bounded by a polynomial in s independent of η for $A(L)<\mathrm{Re}(s)<B(L')$ as mentioned above.

We conclude that $h^{\mathrm{op}}_{\eta_s}=\gamma(\lambda_k s-s_k,\eta^{\lambda_k},\psi)f^{\mathrm{op}}_{\eta_s}$ by Mellin inversion, specifically the converse statement in Lemma 3.2.5. Using (3.28) (and its analogues with f and h replaced by various derivatives in the archimedean setting) we also deduce that $h\in\mathcal{S}_{L'}$. Thus we can conclude the commutativity of the diagram upon verifying our claim that $\lambda_{k!}(\mu_{s_k})(f)(x)$ is equal to $h(x)$.

Observe that the convergence in (3.12) is uniform in $\mathrm{Re}(s),\mathrm{Re}(\chi),\lambda$ in a compact set. Therefore, we can reverse the proof of Lemma 3.2.3 and deduce that (3.27) is equal to the limit as $|b|\to\infty$ of

$$\begin{aligned}&\sum_{\eta\in\widehat{K}_{\mathbb{G}_m}}\int_{\sigma+iI_F}\gamma(\lambda_k s-s_k,\eta^{\lambda_k},\psi)\\ &\times\left(\int_{F^\times}\left(\int_F\Phi\left(\frac{t}{b}\right)\psi(t)\overline{\psi}(yt)dt\right)|y|^{-s_k}\eta_s(y^{\lambda_k})d^\times y\right)f^{\mathrm{op}}_{\eta_s}(x)\frac{c_F ds}{2\pi i}\\ &=\sum_{\eta\in\widehat{K}_{\mathbb{G}_m}}\int_{\sigma+iI_F}\left(\int_{M^{\mathrm{ab}}(F)}\Phi\left(\frac{\omega_P(m)}{b}\right)\psi(\omega_P(m))\right.\\ &\left.\times|\omega_P(m)|^{s_k+1}\eta_s(\omega_P(m)^{-\lambda_k})f^{\mathrm{op}}_{\eta_s}(x)\frac{dm}{\zeta(1)}\right)\frac{c_F ds}{2\pi i}.\end{aligned}\tag{3.31}$$

Moreover, the expression

$$\sum_{\eta \in \widehat{K}_{\mathbb{G}_m}} \int_{\sigma + iI_F} \int_{M^{\mathrm{ab}}(F)} \left| \Phi\left(\frac{\omega_P(m)}{b}\right) |\omega_P(m)|^{s_k+1} \eta_s(\omega_P(m)^{-\lambda_k}) \right| \left| f_{\eta_s}^{\mathrm{op}}(x) \right| dm ds$$

is finite. Indeed, the inner integral is bounded independently of η and s since we have assumed $\sigma < B(L')$ and

$$\sum_{\eta \in \widehat{K}_{\mathbb{G}_m}} \int_{\sigma + iI_F} \left| f_{\eta_s}^{\mathrm{op}}(x) \right| ds$$

is finite by definition of $\mathcal{S}_L$ since $a_L(\eta_s)$ has no poles for $A(L) < \mathrm{Re}(s) < B(L')$.

Therefore, we can rearrange the order of the integral in (3.31) and arrive at

$$\begin{aligned} &\int_{M^{\mathrm{ab}}(F)} \Phi\left(\frac{\omega_P(m)}{b}\right) \psi(\omega_P(m)) |\omega_P(m)|^{s_k+1} \delta_{P^{\mathrm{op}}}^{\lambda_k/2}(m) \\ &\times \left(\sum_{\eta \in \widehat{K}_{\mathbb{G}_m}} \int_{\sigma + iI_F} f_{\eta_s}^{\mathrm{op}}(m^{-\lambda_k} x) \frac{c_F ds}{2\pi i} \right) \frac{dm}{\zeta(1)} \\ &= \int_{M^{\mathrm{ab}}(F)} \Phi\left(\frac{\omega_P(m)}{b}\right) \psi(\omega_P(m)) |\omega_P(m)|^{s_k+1} \delta_{P^{\mathrm{op}}}^{\lambda_k/2}(m) f(m^{-\lambda_k} x) \frac{dm}{\zeta(1)}. \end{aligned}$$

Here in the last step we have used Mellin inversion (Lemma 3.2.5), which is valid by definition of $\mathcal{S}_L$ because $A(L) < \sigma < B(L')$. This completes the proof of our claim that $\lambda_{k!}(\mu_{s_k})(f)(x)$ is equal to $h(x)$. □

Definition 3.2.9. Let $L = \bigoplus_{i \in I} L_i$ and $\{(s_i, \lambda_i)\}_{i \in I}$ be as above. Assume all $\lambda_i > 0$. A **good ordering** of $\{L_i\}$ is a bijection $I \tilde{\to} \{1, \dots, k\}$ for some k, such that after identifying I with $\{1, \dots, k\}$ via the bijection one has

$$\frac{s_{i+1}}{\lambda_{i+1}} \geqslant \frac{s_i}{\lambda_i} \tag{3.32}$$

for $1 \leqslant i \leqslant k-1$.

We also refer to a good ordering of $\{L_i\}$ as a good ordering of $\{(s_i, \lambda_i)\}_{i \in I}$. We henceforth assume that $\lambda_i > 0$ for all i and $\{L_i\}$ is equipped with a good ordering (it is easy to see it exists). In particular we use the good ordering to identify I and $\{1, \dots, k\}$.

For $0 \leqslant i \leqslant k$, we define

$$L(i) := \left(\bigoplus_{1 \leqslant j \leqslant k-i} L_j \right) \oplus \left(\bigoplus_{k-i<j \leqslant k} \widetilde{L}_j \right).$$

Note that $L = L(0)$, and set

$$\widetilde{L} := L(k).$$

Under assumption (3.32), for each $1 \leqslant i < k$ one has

$$A(L(i)) = \tfrac{s_{k-i}}{\lambda_{k-i}} < B(L(i+1)) = \min_{k-(i+1)<j \leqslant k} \left\{ \tfrac{1+s_j}{\lambda_j} \right\} \leqslant B(L(i)) = \min_{k-i<j \leqslant k} \left\{ \tfrac{1+s_j}{\lambda_j} \right\}$$

and

$$(A(L), B(L)) = \left(\tfrac{s_k}{\lambda_k}, \infty \right) \quad \text{and} \quad (A(\widetilde{L}), B(\widetilde{L})) = \left(-\infty, \min_{1 \leqslant j \leqslant k} \left\{ \tfrac{1+s_j}{\lambda_j} \right\} \right).$$

In particular, for each $0 \leqslant i < k$ we have $A(L(i)) < B(L(i))$, so Proposition 3.2.8 implies the map

$$\lambda_{(k-i)!}(\mu_{s_{k-i}}) : \mathcal{S}_{L(i)} \longrightarrow \mathcal{S}_{L(i+1)}$$

is well defined. Thus we define

$$\mu_L := \lambda_{1!}(\mu_{s_1}) \circ \cdots \circ \lambda_{k!}(\mu_{s_k}) : \mathcal{S}_L \longrightarrow \mathcal{S}_{\widetilde{L}} \tag{3.33}$$

as an iterated composition. Define

$$\mu_L(\chi) := \prod_{i=1}^{k} \gamma(-s_i, \chi^{\lambda_i}, \psi). \tag{3.34}$$

Corollary 3.2.10. *One has a commutative diagram*

$$\begin{tikzcd}
\mathcal{S}_L \arrow[r, "\mu_L"] \arrow[d, "(\cdot)^{\mathrm{op}}_{\chi}"] & \mathcal{S}_{\widetilde{L}} \arrow[d, "(\cdot)^{\mathrm{op}}_{\chi}"] \\
\overline{I}(\chi) \arrow[r] & \overline{I}(\chi)
\end{tikzcd}$$

where the bottom arrow is multiplication by $\mu_L(\chi)$.

Some care is needed in interpreting the commutativity of this diagram. Indeed, for general elements of $\mathcal{S}_L$, the half planes of absolute convergence of f_χ^{op} and $\mu_L(f)_\chi^{\mathrm{op}}$ may be disjoint. Thus, the identity $\mu_L(\chi)f_\chi^{\mathrm{op}} = \mu_L(f)_\chi^{\mathrm{op}}$ (for $f \in \mathcal{S}_L$) asserted by the corollary must be understood in the sense of meromorphic continuation.

Proof. Suppose that $A(L(i)) < \mathrm{Re}(\chi) < B(L(i+1))$ and consider the diagram in Proposition 3.2.8 in the special case $L = L(i)$. Using the string of inequalities (3.25) we see that both vertical arrows in Proposition 3.2.8 are given by absolutely convergent integrals. The diagram in Proposition 3.2.8 continues to commute for arbitrary $\mathrm{Re}(\chi)$ if interpreted in the sense of meromorphic continuation. In other words, for all $0 \leqslant i < k$ and arbitrary χ, we have an identity of meromorphic functions

$$\gamma(-s_{k-i}, \chi^{\lambda_{k-i}}, \psi) f_\chi^{\mathrm{op}} = \lambda_{(k-i)!}(\mu_{s_{k-i}})(f)_\chi^{\mathrm{op}}$$

for $f \in \mathcal{S}_{L(i)}$. The corollary follows. $\square$

3.2.1 Braverman and Kazhdan's graded representation

We now recall the graded representation L identified by Braverman and Kazhdan, restricting our attention to the case of a fixed maximal parabolic P containing M and its opposite P^{op}. We use fraktur letters to denote Lie algebras and $\widehat{\cdot}$ to denote the complex-algebraic dual groups and dual Lie algebras. We have embeddings of Lie algebras

$$\widehat{\mathfrak{n}}_P \longrightarrow \widehat{\mathfrak{p}} \longrightarrow \widehat{\mathfrak{g}}.$$

Let $\{e, h, f\}$ be a principal $\mathfrak{sl}_2$-triple in $\widehat{\mathfrak{m}}$; it defines a morphism $\mathfrak{sl}_2 \to \widehat{\mathfrak{m}}$. The adjoint action of $\widehat{\mathfrak{m}}$ on $\widehat{\mathfrak{n}}_P$ restricts to yield an action of $\mathfrak{sl}_2$ on $\widehat{\mathfrak{n}}_P$, and we let $\widehat{\mathfrak{n}}_P^e$ denote the space of highest weight vectors.

Recall our fixed isomorphism

$$\omega_P : M^{\mathrm{ab}} \xrightarrow{\sim} \mathbb{G}_m.$$

This induces a dual isomorphism

$$\widehat{\omega}_P : \mathbb{G}_m \xrightarrow{\sim} \widehat{M}^{\mathrm{ab}} = Z(\widehat{M}) \tag{3.35}$$

where $Z(\widehat{M})$ is the center of $\widehat{M}$. Thus we obtain a $\mathbb{G}_m$-action on $\widehat{\mathfrak{n}}_P^e$. Setting

$$L := \widehat{\mathfrak{n}}_P^e = \bigoplus_i L_i, \tag{3.36}$$

we let λ_i be the $\mathbb{G}_m$-character and s_i be $\frac{1}{2}$ times the h-eigenvalue on the line L_i.

Lemma 3.2.11. *For each L_i as above s_i is nonnegative and λ_i is positive.*

Proof. The s_i are all $\frac{1}{2}$ times the h-eigenvalue of a highest weight vector of a $\mathfrak{sl}_2$-representation and hence are nonnegative. The λ_i are all positive by Lemma A.0.1. $\square$

We define

$$\mu_P := \mu_L : \mathcal{S}_L \longrightarrow \mathcal{S}_{\widetilde{L}} \quad \text{and} \quad \mu_P(\chi) := \mu_L(\chi) \tag{3.37}$$

for the choice of L given in (3.36). Here $\mu_L(\chi)$ is defined as in (3.33).

Remark 3.2.12. One can compute that the dual group $\check{G}_{X_P}$ [GN10, SV17, KS17] of the spherical variety X_P is isomorphic to $\mathbb{G}_m$. The morphism induced from $\widehat{\omega}_P$ and the lifting from the principal $\mathfrak{sl}_2$-triple is a distinguished morphism

$$\check{G}_{X_P} \times \mathrm{SL}_2 \to \widehat{M} \hookrightarrow \widehat{G}$$

defined in [SV17], which depends only on the smooth locus X_P°. The specific choice of the graded representation L (or more precisely the adjoint action of $\check{G}_{X_P} \times \mathfrak{sl}_2$ on $\widehat{\mathfrak{n}}_P$) relates to the colored cone corresponding to the spherical embedding $X_P^\circ \to X_P$. For general affine spherical embedding, it is not clear how one should extract normalizing operators from the associated colored cone. In the unramified setting, this has been studied in various cases in [SW22] and the recent joint work of Ben-Zvi, Venkatesh, and Sakellaridis.

3.2.2 Switching to the opposite parabolic

In Corollary 3.5.6 we will switch between P and P^{op} for self-associate parabolic subgroups. This requires care regarding signs. We choose a principal $\mathfrak{sl}_2$-triple $\{e, h, f\}$ as above and consider $L^{\mathrm{op}} = (\widehat{\mathfrak{n}}_{P^{\mathrm{op}}})^e$. We claim that

$$L^{\mathrm{op}} = \bigoplus_{i \in I} L_i^{\mathrm{op}}, \tag{3.38}$$

where $\mathbb{G}_m$ and h act on L_i^{op} via λ_i and $2s_i$, respectively. Indeed, the $\mathfrak{sl}_2(\mathbb{C}) \times Z(\widehat{M})$-representations $\widehat{\mathfrak{n}}_P$ and $\widehat{\mathfrak{n}}_{P^{\mathrm{op}}}$ are dual. Thus as representations of $\mathfrak{sl}_2(\mathbb{C})$ they have the same highest weights. Since the parameters λ are defined using (3.35), we deduce the claim from the observation that

$$\omega_P = \omega_{P^{\mathrm{op}}}^{-1}.$$

3.2.3 The Lagrangian Grassmannian

As an example, let Sp_{2n} denote the symplectic group on a $2n$-dimensional vector space and let $P < \mathrm{Sp}_{2n}$, $M < P$ denote the Siegel parabolic and Levi subgroup, respectively. Specifically, for $\mathbb{Z}$-algebras R, set

$$\begin{aligned} \mathrm{Sp}_{2n}(R) &:= \left\{ g \in \mathrm{GL}_{2n}(R) : g^t \left(\begin{smallmatrix} & I_n \\ -I_n & \end{smallmatrix}\right) g = \left(\begin{smallmatrix} & I_n \\ -I_n & \end{smallmatrix}\right) \right\}, \\ M(R) &:= \{ \left(\begin{smallmatrix} A & \\ & A^{-t} \end{smallmatrix}\right) : A \in \mathrm{GL}_n(R) \}, \\ N(R) &:= \{ \left(\begin{smallmatrix} I_n & Z \\ & I_n \end{smallmatrix}\right) : Z \in \mathfrak{gl}_n(R), Z^t = Z \}, \end{aligned}$$

and $P = MN$. We have

$$\begin{aligned} \omega_P : M(R) &\longrightarrow R^\times \\ \left(\begin{smallmatrix} m & \\ & m^{-t} \end{smallmatrix}\right) &\longmapsto \det m, \end{aligned}$$

$\widehat{\mathfrak{g}} = \mathfrak{so}_{2n+1}$, and $\widehat{\mathfrak{m}} = \mathfrak{gl}_n$. Moreover, as a representation of $\widehat{\mathfrak{m}}$,

$$\widehat{\mathfrak{n}}_P \cong V_{\mathrm{st}} \oplus \wedge^2 V_{\mathrm{st}},$$

where V_{st} is the standard representation of $\mathfrak{gl}_n$. We use the standard principal $\mathfrak{sl}_2$-triple in $\mathfrak{gl}_n$. Concretely it is the image of $\mathfrak{sl}_2$ under Sym^{n-1}. The space $\widehat{\mathfrak{n}}_P^e$ is just the direct sum of the highest weight spaces of the $\mathfrak{sl}_2$-representation

$$\mathrm{Sym}^{n-1}(\mathbb{C}^2) \oplus \wedge^2 \mathrm{Sym}^{n-1}(\mathbb{C}^2) \cong \mathrm{Sym}^{n-1}(\mathbb{C}^2) \oplus \bigoplus_{j=0}^{\lfloor (n-2)/2 \rfloor} \mathrm{Sym}^{2(n-2)-4j}(\mathbb{C}^2).$$

Here we have used some well-known plethysms (see Lemma A.2.3 below). Then

$$(s_r, \lambda_r) = (n + 2r - 2\lfloor n/2 \rfloor - 2, 2) \text{ for } 1 \leqslant r \leqslant \lfloor n/2 \rfloor \text{ and } (s_{\lfloor n/2 \rfloor + 1}, \lambda_{\lfloor n/2 \rfloor + 1}) = \left(\tfrac{n-1}{2}, 1\right).$$

This is a good ordering.

We observe that

$$a_{I_{2n}}(s,\chi) = a_{\widetilde{L}}((\chi_s)^{-1}) \quad \text{and} \quad a_{w_0}(s,\chi) = a_L(\chi_s) \tag{3.39}$$

in the notation of [GL21, §3].

3.3 The Schwartz space of a Braverman–Kazhdan space

From now on we assume that G is simply connected so that we can apply the results of [BK02]. Braverman and Kazhdan originally defined operators $\mathcal{F}_{P^{\mathrm{op}}|P}$ via a series of integral operators on an inexplicit subspace of $\mathcal{S}(X^{\circ}_{P^{\mathrm{op}}}(F))$, proved that the operators extended to unitary operators on $L^2(X_{P^{\mathrm{op}}}(F))$, and then proposed the following definition:

Definition 3.3.1. The **BK-Schwartz space** $\mathcal{S}_{BK}(X_P(F))$ is defined as the sum

$$\mathcal{S}_{BK}(X_P(F)) := \mathcal{S}(X^{\circ}_P(F)) + \mathcal{F}_{P^{\mathrm{op}}|P}(\mathcal{S}(X^{\circ}_{P^{\mathrm{op}}}(F))).$$

Here the sum is taken in $L^2(X_P(F))$. We point out that the expression $\mathcal{F}_{P^{\mathrm{op}}|P}(\mathcal{S}(X^{\circ}_{P^{\mathrm{op}}}(F)))$ means that we apply the L^2-extension of $\mathcal{F}_{P^{\mathrm{op}}|P}$ to $\mathcal{S}(X^{\circ}_{P^{\mathrm{op}}}(F))$. It is far from obvious that the integral operators defining $\mathcal{F}_{P^{\mathrm{op}}|P}$ converge when applied to elements of $\mathcal{S}(X^{\circ}_{P^{\mathrm{op}}}(F))$. Indeed, this was in general unknown before our work is done. We postpone the discussion until Chapter 4.

Remark 3.3.2. Braverman and Kazhdan only stated this definition in the nonarchimedean case, but the extension to the archimedean case is natural and was suggested to Getz by Kazhdan.

In [GL21] Getz and Liu refined Braverman and Kazhdan's definition when $G = \mathrm{Sp}_{2n}$ and P is the Siegel parabolic, and gave explicit spaces of functions that are mapped to each other under the Fourier transform. We do the same for Braverman–Kazhdan spaces attached to general G and maximal parabolic subgroups $P < G$ in this section. This goes beyond the work of Braverman and Kazhdan in that it allows us to isolate an explicit subspace on which our formulae for the Fourier transforms given in §3.5 are valid.

3.3.1 Measures redux

Thus far we have only made use of Haar measures dx and $d^\times x$ on F and $F^\times$ related as in §2.2. In order to study the Schwartz space and the Fourier transform, we require right $G(F)$-invariant measures on $X_P^\circ(F)$, $X_{P^{\mathrm{op}}}^\circ(F)$.

Recall that we have chosen Haar measures on $N_{P^{\mathrm{op}}}(F)$ and $N_P(F)$ in §2.2, and a Haar measure on $M^{\mathrm{ab}}(F)$ from the isomorphism $\omega_P : M^{\mathrm{ab}}(F) \to F^\times$. By the Bruhat decomposition, one has an injection

$$\begin{aligned} M^{\mathrm{ab}}(F) \times N_{P^{\mathrm{op}}}(F) &\longrightarrow X_P^\circ(F) \\ (m,u) &\longmapsto P^{\mathrm{der}}(F)mu \end{aligned}$$

with Zariski open and dense (hence, of full measure) image. We can and do normalize the right $G(F)$-invariant nonnegative Radon measure dx on $X_P^\circ(F)$ such that

$$d(mu) = \frac{\delta_{P^{\mathrm{op}}}(m)dmdu}{\zeta(1)}. \tag{3.40}$$

Similarly, we normalize the right $G(F)$-invariant non-negative Radon measure dx on $X_{P^{\mathrm{op}}}^\circ(F)$ so that

$$d(mu) = \frac{\delta_P(m)dmdu}{\zeta(1)}$$

for $(m,u) \in M^{\mathrm{ab}}(F) \times N_P(F)$.

3.3.2 The Schwartz space

For functions $f \in C^\infty(X_P^\circ(F))$ and $x = P^{\mathrm{op,der}}(F)g \in X_{P^{\mathrm{op}}}^\circ(F)$, we define the unnormalized intertwining operator

$$\mathcal{R}_{P|P^{\mathrm{op}}}(f)(x) := \int_{N_{P^{\mathrm{op}}}(F)} f\left(P^{\mathrm{der}}(F)ug\right)du = \int_{N_{P^{\mathrm{op}}}(F)} f(ug)du \tag{3.41}$$

whenever this integral is absolutely convergent (or obtained via some regularization procedure). We refer to $\mathcal{R}_{P|P^{\mathrm{op}}}$ as a **Radon transform**, as it is a generalization of the classical Radon transform [BK02, §2.9]. That this agrees with the operator defined by Braverman and Kazhdan is proved in [Sha18, §5]. For example, we have maps

$$\mathcal{R}_{P|P^{\mathrm{op}}} : \mathcal{S}(X_P^\circ(F)) \longrightarrow C^\infty(X_{P^{\mathrm{op}}}^\circ(F))$$

and

$$\mathcal{R}_{P|P^{\mathrm{op}}} : I(\chi_s) \longrightarrow \overline{I}(\chi_s)$$

for $\mathrm{Re}(s)$ sufficiently large that may be extended meromorphically to $\mathbb{C}$ [Wal92, §10.1.2, §10.1.6] [Wal03, Theorem IV.1.1]. For notational convenience, we write

$$\mathcal{R}_{P|P} : C^\infty(X_P^\circ(F)) \longrightarrow C^\infty(X_P^\circ(F))$$

for the identity operator.

Let L be the graded $\mathbb{G}_m$-representation associated to P in §3.2.1 and let $\{(s_i, \lambda_i)\}$ be a good ordering of L. For quasi-characters $\chi : F^\times \to \mathbb{C}^\times$, we set

$$a_{P|P}(\chi) := a_{\widetilde{L}}(\chi^{-1}) \quad \text{and} \quad a_{P|P^{\mathrm{op}}}(\chi) := a_L(\chi). \tag{3.42}$$

Bearing in mind the discussion in §3.2.2, the definition (3.42) implies

$$a_{P|P}(\chi) = a_{P^{\mathrm{op}}|P^{\mathrm{op}}}(\chi) \quad \text{and} \quad a_{P|P^{\mathrm{op}}}(\chi) = a_{P^{\mathrm{op}}|P}(\chi). \tag{3.43}$$

Lemma 3.3.3. *The function $a_{P|P}(\chi)$ is holomorphic for $\mathrm{Re}(\chi) \geqslant 0$.*

Proof. It suffices to show that $s_i + 1 > 0$ and $\lambda_i > 0$ for all L_i. This follows from Lemma 3.2.11. □

For a character χ, we say a section $f(\chi)^{(s)}$ of $I(\chi_s)$ is **good** if it is meromorphic, and if for $Q \in \{P, P^{\mathrm{op}}\}$ the sections

$$\frac{\mathcal{R}_{P|Q} f(\chi)^{(s)}}{a_{P|Q}(\chi_s)} \tag{3.44}$$

of $I(\chi_s)$ and $\overline{I}(\chi_s)$ are holomorphic.

Definition 3.3.4. Assume F is nonarchimedean. The Schwartz space $\mathcal{S}(X_P(F))$ is defined to be the space of right K-finite functions $f \in C^\infty(X_P^\circ(F))$ such that for each $g \in G(F)$ and character χ of $F^\times$, the integral (3.6) defining $f_{\chi_s}(g)$ is absolutely convergent for $\mathrm{Re}(s)$ large enough and defines a good section.

For F archimedean, recall we have an action of $U(\mathfrak{m}^{\mathrm{ab}} \oplus \mathfrak{g})$ on $C^\infty(X_P^\circ(F))$ via the differential of (3.1).

Definition 3.3.5. Assume F is archimedean. The Schwartz space $\mathcal{S}(X_P(F))$ is defined to be the space of functions $f \in C^\infty(X_P^\circ(F))$ such that for all $D \in U(\mathfrak{m}^{\mathrm{ab}} \oplus \mathfrak{g})$, $g \in G(F)$, and each character χ of $F^\times$, the integral (3.6) defining $(D.f)_{\chi_s}(g)$ is absolutely convergent for $\mathrm{Re}(s)$ large enough, defines a good section, and satisfies the following condition: For all real numbers $A < B$, $Q \in \{P, P^{\mathrm{op}}\}$, any polynomial $p_{P|Q} \in \mathbb{C}[s]$ such that $p_{P|Q}(s)a_{P|Q}(\eta_s)$ has no poles for all $(s, \eta) \in V_{A,B} \times \widehat{K}_{\mathbb{G}_m}$, and compact subsets $\Omega \subset X_P^\circ(F)$ one has that

$$|f|_{A,B,p_{P|Q},\Omega,D} := \sum_{\eta \in \widehat{K}_{\mathbb{G}_m}} \sup_{g \in \Omega} |\mathcal{R}_{P|Q}(D.f)_{\eta_s}(g)|_{A,B,p_{P|Q}} < \infty. \tag{3.45}$$

To understand this definition, it is useful to point out that it is indeed possible to choose $p_{P|Q}$ (independently of η) that satisfy the given assumptions. This follows directly from the definition of the $a_{P|Q}(\eta_s)$. We also observe that the $|\cdot|_{A,B,p_{P|Q},\Omega,D}$ are seminorms and they give $\mathcal{S}(X_P(F))$ the structure of a Fréchet space by essentially the same argument proving [GH20, Lemma 3.2].

Remark 3.3.6. Note that $f_{\chi,P}^{\mathrm{op}} = f_{\chi^{-1},P}$. Using this observation and the discussion in §3.2.2 we see that $\mathcal{S}(X_P(F)) \leqslant \mathcal{S}_{\widetilde{L}}(X_P(F))$.

For any F, the action of $M^{\mathrm{ab}}(F) \times G(F)$ on $X_P^\circ(F)$ induces a smooth action on $\mathcal{S}(X_P(F))$. In the archimedean setting, this action is continuous in the Fréchet topology of $\mathcal{S}(X_P(F))$.

The elements of the Schwartz space are well-behaved analytically. They can be bounded in an intuitive manner using the Plücker embedding. Let

$$\mathrm{Pl} : X_P \longrightarrow V_P$$

be the Plücker embedding defined by a choice of highest weight vector v_P as in Lemma 3.1.4. Choose a norm $|\cdot|$ on $V_P(F)$ that is invariant under K and let

$$\begin{aligned} |\cdot| : X_P^\circ(F) &\longrightarrow \mathbb{R}_{>0} \\ x &\longmapsto |\mathrm{Pl}(x)|; \end{aligned}$$

here, K is chosen as in (3.22). Replacing v_P by tv_P for $t \in F^\times$ multiplies this norm by $|t|$. We normalize the norm so that

$$|mk| = |\omega_P(m)|^{-1} \quad \text{for } m \in P(F), k \in K.$$

Let $r \in \mathbb{Q}_{>0}$ be such that

$$|\omega_P|^r = \delta_P.$$

Note that our assumption that G is simply connected implies that $r \in \mathbb{Z}_{>0}$; indeed, we compute this value in Proposition 3.5.2 below.

Lemma 3.3.7. *Assume $\alpha > 0$ is sufficiently small. Let $f \in \mathcal{S}(X_P(F))$. When F is nonarchimedean, $f(x)$ vanishes for $|x|$ sufficiently large and*

$$|f(x)| \ll_\alpha |x|^{-r/2+\alpha}.$$

When F is archimedean, for all $N \in \mathbb{Z}_{\geqslant 0}$ one has

$$|f(x)| \leqslant \nu_{N,\alpha}(f)|x|^{-r/2+\alpha} \max(1,|x|)^{-N}$$

where $\nu_{N,\alpha}$ is a continuous seminorm on $\mathcal{S}(X_P(F))$.

Proof. Write $x = P^{\mathrm{der}}(F)mk$ with $m \in M^{\mathrm{ab}}(F)$ and $k \in K$. By definition of $\mathcal{S}(X_P(F))$ and Mellin inversion (3.23), we have

$$f(x) = \delta_P^{1/2}(m) \sum_{\eta \in \widehat{K}_{\mathbb{G}_m}} \int_{\sigma+iI_F} \eta_s(\omega_P(m)) f_{\eta_s}(k) \frac{c_F ds}{2\pi i} \tag{3.46}$$

provided that there are no poles of $a_{P|P}(\eta_s)$ for $\mathrm{Re}(s) \geqslant \sigma$. Moreover the sum and integral converge absolutely. Therefore to prove the bounds for $|x| \leqslant 1$ in the archimedean case and for all x in the nonarchimedean case it suffices to recall $a_{P|P}(\eta_s)$ has no poles for $\mathrm{Re}(s) \geqslant 0$ by Lemma 3.3.3.

The support assertion in the nonarchimedean case follows as in [GL21, Lemma 5.1]. The bound for $|x| \gg 1$ in the archimedean case follows as in [GH20, Lemma 3.5]. □

As $X_P^\circ(F)$ is open and dense in $X_P(F)$, we can and do extend the right $G(F)$-invariant Radon measure on $X_P^\circ(F)$ by zero to $X_P(F)$.

Corollary 3.3.8. *One has* $\mathcal{S}(X_P(F)) < L^2(X_P(F)) \cap L^1(X_P(F))$.

Proof. This follows from Lemma 3.3.7 and the Iwasawa decomposition. $\square$

3.4 The Fourier transform

Braverman and Kazhdan [BK02] proved that the Fourier transform

$$\mathcal{F}_{P|P^{\mathrm{op}}} := \mu_P \circ \mathcal{R}_{P|P^{\mathrm{op}}} \tag{3.47}$$

is well defined on a subspace of $\mathcal{S}(X_P^\circ(F))$ that is dense in $L^2(X_P(F))$ and defines an isometry

$$\mathcal{F}_{P|P^{\mathrm{op}}} : L^2(X_P(F)) \longrightarrow L^2(X_{P^{\mathrm{op}}}(F)). \tag{3.48}$$

They also proved that

$$\mathcal{F}_{P|P^{\mathrm{op}}} \circ \mathcal{F}_{P^{\mathrm{op}}|P} = \mathrm{Id}. \tag{3.49}$$

We use the results of the previous sections to refine $\mathcal{F}_{P|P^{\mathrm{op}}}$ to an isomorphism between $\mathcal{S}(X_P(F))$ and $\mathcal{S}(X_{P^{\mathrm{op}}}(F))$ in this section.

Lemma 3.4.1. *One has a commutative diagram*

$$\begin{tikzcd}
\mathcal{S}(X_P(F)) \arrow[r, "\mathcal{R}_{P|P^{\mathrm{op}}}"] \arrow[d, "(\cdot)_\chi"] & \mathcal{S}_L \arrow[d, "(\cdot)_\chi^{\mathrm{op}}"] \\
I(\chi) \arrow[r, "\mathcal{R}_{P|P^{\mathrm{op}}}"] & \bar{I}(\chi)
\end{tikzcd}$$

for $\mathrm{Re}(\chi)$ *sufficiently large.*

Proof. For $g \in G(F)$ consider the integral

$$\int_{N_{P^{\mathrm{op}}}(F)} \int_{M^{\mathrm{ab}}(F)} \delta_P^{1/2}(m)|\chi|(\omega_P(m))|f|(m^{-1}ug)dmdu. \tag{3.50}$$

The inner integral converges and defines an element of $I(|\chi|)$ for $\mathrm{Re}(\chi)$ sufficiently large by definition of $\mathcal{S}(X_P(F))$, and the outer integral converges for $\mathrm{Re}(\chi)$ sufficiently large [Wal92, Lemma 10.1.2] [Wal03, Theorem IV.1.1]. Thus by Fubini's theorem, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{S}(X_P(F)) \arrow[r, "\mathcal{R}_{P|P^{\mathrm{op}}}"] \arrow[d, "(\cdot)_\chi"] & \mathcal{R}_{P|P^{\mathrm{op}}}(\mathcal{S}(X_P(F))) \arrow[d, "(\cdot)_\chi^{\mathrm{op}}"] \\
I(\chi) \arrow[r, "\mathcal{R}_{P|P^{\mathrm{op}}}"] & \overline{I}(\chi)
\end{tikzcd}$$

for $\mathrm{Re}(\chi)$ sufficiently large. We are left with proving that $\mathcal{R}_{P|P^{\mathrm{op}}}(\mathcal{S}(X_P(F))) \leqslant \mathcal{S}_L$. By the definitions of $\mathcal{S}(X_P(F))$ and $\mathcal{S}_L$ it suffices to check that

$$\mathcal{R}_{P|P^{\mathrm{op}}}(\mathcal{S}(X_P(F)) \leqslant C^\infty(X_P^\circ(F)). \tag{3.51}$$

Let $f \in \mathcal{S}(X_P(F))$. By Fubini's theorem and the argument above for almost every m with respect to dm we have that $\int_{N_{P^{\mathrm{op}}}(F)} f(m^{-1}ug)du$ converges. When F is nonarchimedean we can use the fact that f is K-finite and Lemma 3.2.5 to deduce that f is fixed by a compact open subgroup of $M^{\mathrm{ab}}(F)$. This implies that $\int_{N_{P^{\mathrm{op}}}(F)} f(ug)du$ converges absolutely. Since $\mathcal{R}_{P|P^{\mathrm{op}}}$ is a $G(F)$-intertwining map this implies that $\mathcal{R}_{P|P^{\mathrm{op}}}(f)$ is smooth. Now assume that F is archimedean. In this case we can view the integral (3.50), as g varies, as valued in the Fréchet space $C^\infty(G(F))$ (with the usual Fréchet topology). Using the Fubini theorem in this setting [Tho75, Theorem 8], we deduce that for almost all m with respect to dm, $\int_{N_{P^{\mathrm{op}}}(F)} f(m^{-1}ug)du$ converges absolutely and defines a smooth function of g. For such an m, we change variables $u \mapsto mum^{-1}$ and replacing g by mg, we deduce that $\mathcal{R}_{P|P^{\mathrm{op}}}(f)$ is smooth. ☐

To proceed, we recall the subspaces $\mathcal{C}_Q < \mathcal{S}(X_Q^\circ(F))$ for $Q \in \{P, P^{\mathrm{op}}\}$ considered in [BK02, Proposition 4.2] that are used to proved the unitarity of the operator $\mathcal{F}_{Q|Q^{\mathrm{op}}}$ on $L^2(X_Q(F))$. In the following, we will use the notation in (3.5) and (3.6) to keep track of the domain of our Mellin transforms.

Lemma 3.4.2. *For each $\chi \in \widehat{K}_{\mathbb{G}_m}$ we can choose holomorphic functions $h_Q(\chi_s)$ that lie in $\mathbb{C}[q^{-s}, q^s]$ in the nonarchimedean case and are bounded in vertical strips in the archimedean case such that*

$$h_Q(\chi_s)\mathcal{R}_{Q|Q^{\mathrm{op}}} : I_Q(\chi_s) \longrightarrow \overline{I}_{Q^{\mathrm{op}}}(\chi_s) \tag{3.52}$$

is holomorphic when evaluated on a holomorphic section $f(\chi)^{(s)} \in I_Q(\chi_s)$ *and an isomorphism for* s *outside a discrete countable subset of* $\mathbb{C}$.

Proof. Assume first that F is nonarchimedean. Then one can use the usual normalizing factors for intertwining operators [Art89, §2-4] to construct $h_Q(\chi_s)$ satisfying the requirements in the lemma. If F is archimedean, loc. cit. implies the existence of a set $\{a_i, b_i\}_{i=1}^n$ of complex numbers such that

$$\left(\prod_{i=1}^{n} \frac{1}{\Gamma(a_i s + b_i)}\right) \mathcal{R}_{Q|Q^{\mathrm{op}}} \tag{3.53}$$

is holomorphic when evaluated on a holomorphic section $f(\chi)^{(s)} \in I_Q(\chi_s)$ and is an isomorphism for s outside a discrete countable subset of $\mathbb{C}$. The factors of Γ here are archimedean L-functions of quasi-characters of $F^\times$ up to irrelevant factors. To obtain $h(\chi_s)$ we take the product of reciprocals of Γ-functions and multiply by e^{s^2} to make the result rapidly decreasing in vertical strips. □

We henceforth assume the $h_Q(\chi_s)$ are chosen as in Lemma 3.4.2. Let $\mathcal{S}(X_Q^\circ(F), K)$ be the space of K-finite functions in $\mathcal{S}(X_Q^\circ(F))$, and let

$$\mathcal{C}_Q := \left\{ f \in \mathcal{S}(X_Q^\circ(F), K) : \begin{array}{c} \text{There exists an } f' \in \mathcal{S}(X_Q^\circ(F), K) \text{ such that} \\ f_{\chi_s,Q} = h_{Q^{\mathrm{op}}}((\chi_s)^{-1}) h_Q(\chi_s) f'_{\chi_s,Q} \\ \text{for all characters } \chi : F^\times \to \mathbb{C}^\times \text{ and all } s \in \mathbb{C} \end{array} \right\}.$$

For a subspace $W \leqslant \mathcal{S}(X_Q^\circ(F))$, let $W_{\chi_s,Q}$ denote its image in $I(\chi_s)$ under the Mellin transform (3.6). We also use the notation $\mathcal{S}(X_Q(F))_{\chi_s,Q}$ for the image of $\mathcal{S}(X_Q(F))$ in $I(\chi_s)$ under the Mellin transform, which must be understood in the following sense: For $\mathrm{Re}(s)$ sufficiently large these Mellin transforms are absolutely convergent by definition of the Schwartz space. Again by definition of the Schwartz space the Mellin transforms are defined by meromorphic continuation for s outside a countable subset of $\mathbb{C}$ independent of $\chi \in \widehat{K}_{\mathbb{G}_m}$.

Lemma 3.4.3. *For* $f \in \mathcal{C}_Q$ *the functions* $\mathcal{R}_{Q|Q^{\mathrm{op}}}(f_{\chi_s,Q})$ *and* $\mathcal{R}_{Q^{\mathrm{op}}|Q}(\mathcal{R}_{Q|Q^{\mathrm{op}}}(f_{\chi_s,Q}))$ *are holomorphic for all* $\chi \in \widehat{K}_{\mathbb{G}_m}$. *One has* $\mathcal{C}_Q < \mathcal{S}(X_Q(F))$. *For* s *outside a countable subset of* $\mathbb{C}$

(independent of χ) one has

$$(\mathcal{C}_Q)_{\chi_s,Q} = \mathcal{S}(X_Q^\circ(F), K)_{\chi_s,Q},$$

which is dense in $\mathcal{S}(X_Q(F))_{\chi_s,Q}$ in the usual Fréchet topology if F is archimedean and equal to $\mathcal{S}(X_Q(F))_{\chi_s,Q}$ if F is nonarchimedean.

Proof. The first assertion is immediate from the definition of $\mathcal{C}_Q$. The inclusion $\mathcal{C}_Q < \mathcal{S}(X_Q(F))$ follows from the fact that $a_{Q|Q^{\mathrm{op}}}(\chi_s)$ and $a_{Q|Q}(\chi_s)$ have no zeros. As the function $h_{Q^{\mathrm{op}}}((\chi_s)^{-1})h_Q(\chi_s)$ is nonzero outside a discrete countable set we have $(\mathcal{C}_Q)_{\chi_s,Q} = \mathcal{S}(X_Q^\circ(F), K)_{\chi_s,Q}$ outside a discrete countable set. The union of these sets is again countable. Since $\mathcal{S}(X_Q^\circ(F), K)_{\chi_s,Q}$ is the space of K-finite vectors in $I(\chi_s)$ (which is all of $I(\chi_s)$ in the nonarchimedean case) the last assertion of the lemma follows. $\square$

We remark that here the definition of $\mathcal{C}_Q$ depends on the choice of $h_Q(\chi_s)$ and $h_{Q^{\mathrm{op}}}((\chi_s)^{-1})$. Using Corollary 3.2.10, and a minor variant of the proof of Lemma 3.4.2 above we fix a choice of $h_Q(\chi_s)$ and $h_{Q^{\mathrm{op}}}((\chi_s)^{-1})$ such that $\mathcal{F}_{P|P^{\mathrm{op}}}(\mathcal{C}_P) < \mathcal{S}(X_{P^{\mathrm{op}}}(F))$.

Theorem 3.4.4. *We have a well defined isomorphism*

$$\mathcal{F}_{P|P^{\mathrm{op}}} : \mathcal{S}(X_P(F)) \longrightarrow \mathcal{S}(X_{P^{\mathrm{op}}}(F)),$$

that is continuous with respect to the Fréchet topologies in the archimedean case. The diagram

$$\begin{tikzcd}
\mathcal{S}(X_P(F)) \arrow[r, "\mathcal{F}_{P|P^{\mathrm{op}}}"] \arrow[d, "(\cdot)_\chi"] & \mathcal{S}(X_{P^{\mathrm{op}}}(F))) \arrow[d, "(\cdot)_\chi^{\mathrm{op}}"] \\
I(\chi) \arrow[r, "\mu_P(\chi)\mathcal{R}_{P|P^{\mathrm{op}}}"] & \overline{I}(\chi)
\end{tikzcd} \tag{3.54}$$

commutes.

As in Corollary 3.2.10, some care is required in interpreting the statement that the diagram commutes. The Mellin transform $(\cdot)_\chi$ converges absolutely for $\mathrm{Re}(\chi)$ large and the Mellin transform $(\cdot)_\chi^{\mathrm{op}}$ converges absolutely for $\mathrm{Re}(\chi)$ small. The factor $\mu_P(\chi)$ is meromorphic, and the operator $\mathcal{R}_{P|Q} : I(\chi) \to \overline{I}(\chi)$, originally defined for $\mathrm{Re}(\chi)$ large, extends to an operator sending meromorphic sections to meromorphic sections. The definition of $\mathcal{S}(X_P(F))$ is designed to control the poles of all of these objects in terms of the functions $a_{P|Q}(\chi)$.

Proof. By Corollary 3.3.8, $\mathcal{S}(X_Q(F)) < L^2(X_Q(F))$ for $Q \in \{P, P^{\mathrm{op}}\}$. Combining this with (3.48) and (3.49), we see that to prove $\mathcal{F}_{P|P^{\mathrm{op}}}$ is an isomorphism, it suffices to check that

$$\mathcal{F}_{P|P^{\mathrm{op}}}(\mathcal{S}(X_P(F))) \leqslant \mathcal{S}(X_{P^{\mathrm{op}}}(F)). \tag{3.55}$$

On the other hand, Corollary 3.2.10 and Lemma 3.4.1 imply that $\mathcal{F}_{P|P^{\mathrm{op}}}(\mathcal{S}(X_P(F))) \leqslant \mathcal{S}_{\widetilde{L}}$ and that if we replace $\mathcal{S}(X_{P^{\mathrm{op}}}(F))$ by $\mathcal{S}_{\widetilde{L}}$ in (3.54) we obtain a commutative diagram. Thus proving (3.55) implies everything in the theorem besides the continuity assertion.

Since $a_{P|P}(\chi) = a_{\widetilde{L}}(\chi^{-1})$ for all quasi-characters χ, by (3.43) we deduce that for $f \in \mathcal{S}(X_P(F))$

$$\frac{\mathcal{F}_{P|P^{\mathrm{op}}}(f)_{\chi_s,P^{\mathrm{op}}}(x)}{a_{P^{\mathrm{op}}|P^{\mathrm{op}}}(\chi_s)} = \frac{\mathcal{F}_{P|P^{\mathrm{op}}}(f)^{\mathrm{op}}_{(\chi_s)^{-1},P^{\mathrm{op}}}(x)}{a_{\widetilde{L}}((\chi_s)^{-1})} \in \mathbb{C}[q^s, q^{-s}]$$

in the nonarchimedean case, and

$$|\mathcal{F}_{P|P^{\mathrm{op}}}(f)|_{A,B,p_{P^{\mathrm{op}}|P^{\mathrm{op}}},\Omega,D} < \infty$$

for all data as in Definition 3.3.5 in the archimedean case since $\mathcal{F}_{P|P^{\mathrm{op}}}(f) \in \mathcal{S}_{\widetilde{L}}$. Hence we are left with checking that

$$\frac{\mathcal{R}_{P^{\mathrm{op}}|P}(\mathcal{F}_{P|P^{\mathrm{op}}}(f)_{\chi_s,P^{\mathrm{op}}})}{a_{P^{\mathrm{op}}|P}(\chi_s)} \in \mathbb{C}[q^s, q^{-s}] \tag{3.56}$$

in the nonarchimedean case and

$$|\mathcal{F}_{P|P^{\mathrm{op}}}(f)|_{A,B,p_{P^{\mathrm{op}}|P},\Omega,D} < \infty \tag{3.57}$$

in the archimedean case for all data as in Definition 3.3.5.

For any $f \in \mathcal{C}_P$ and any $\chi \in \widehat{K}_{\mathbb{G}_m}$, by Corollary 3.2.10, Lemma 3.4.1, (3.49), and our choice of $\mathcal{C}_P$, we have the identities

$$\begin{aligned}
\frac{\mu_L((\chi_s)^{-1})\mathcal{R}_{P^{\mathrm{op}}|P}(\mathcal{R}_{P|P^{\mathrm{op}}}(f_{(\chi_s)^{-1},P}))}{a_{P^{\mathrm{op}}|P}(\chi_s)} &= \frac{\mathcal{R}_{P^{\mathrm{op}}|P}(\mathcal{F}_{P|P^{\mathrm{op}}}(f)_{\chi_s,P^{\mathrm{op}}})}{a_{P^{\mathrm{op}}|P}(\chi_s)} \\
&= \frac{\left(\mathcal{R}_{P^{\mathrm{op}}|P}\mathcal{F}_{P|P^{\mathrm{op}}}(f)\right)^{\mathrm{op}}_{\chi_s,P}}{a_{P^{\mathrm{op}}|P}(\chi_s)} \\
&= \frac{\left(\mathcal{F}_{P^{\mathrm{op}}|P}\mathcal{F}_{P|P^{\mathrm{op}}}(f)\right)^{\mathrm{op}}_{\chi_s,P}}{\mu_{L^{\mathrm{op}}}(\chi_s)a_{P^{\mathrm{op}}|P}(\chi_s)} \\
&= \frac{f^{\mathrm{op}}_{\chi_s,P}}{\mu_{L^{\mathrm{op}}}(\chi_s)a_{L^{\mathrm{op}}}(\chi_s)}. \qquad (3.58)
\end{aligned}$$

Since $f_{(\chi_s)^{-1},P} = f^{\mathrm{op}}_{\chi_s,P}$, the first and last quantities in (3.58) depend only on the image of f under the map to $I((\chi_s)^{-1})$. Let $\mathcal{S}(X_P(F),K)$ be the space of K-finite functions in $\mathcal{S}(X_P(F))$; it is all of $\mathcal{S}(X_P(F))$ when F is nonarchimedean. By Lemma 3.4.3, the equality of the first and last terms in (3.58) holds for all $f \in \mathcal{S}(X_P(F),K)$ and all $\chi \in \widehat{K}_{\mathbb{G}_m}$ for s in a dense subset of $\mathbb{C}$.

Since the first equality in the previous calculation is valid for all $f \in \mathcal{S}(X_P(F),K)$ by Corollary 3.2.10, we deduce that

$$\frac{\mathcal{R}_{P^{\mathrm{op}}|P}(\mathcal{F}_{P|P^{\mathrm{op}}}(f)_{\chi_s,P^{\mathrm{op}}})}{a_{P^{\mathrm{op}}|P}(\chi_s)} = \frac{f^{\mathrm{op}}_{\chi_s,P}}{\mu_{L^{\mathrm{op}}}(\chi_s)a_{L^{\mathrm{op}}}(\chi_s)} \qquad (3.59)$$

for all $f \in \mathcal{S}(X_P(F),K)$ and $\chi \in \widehat{K}_{\mathbb{G}_m}$, at least for all s in a dense subset of $\mathbb{C}$. But then (3.59) is valid as an identity of meromorphic functions for all s. As discussed in §3.2.2, we have $\mu_{L^{\mathrm{op}}}(\chi_s)a_{L^{\mathrm{op}}}(\chi_s) = \mu_L(\chi_s)a_L(\chi_s)$. Moreover, with L_i defined as in (3.36)

$$\begin{aligned}
\mu_{L_i}(\chi_s)a_{L_i}(\chi_s) = \gamma(-s_i,(\chi_s)^{\lambda_i},\psi)L(-s_i,(\chi_s)^{\lambda_i}) &= \varepsilon(-s_i,(\chi_s)^{\lambda_i},\psi)L(1+s_i,(\chi_s)^{-\lambda_i}) \\
&= \varepsilon(-s_i,(\chi_s)^{\lambda_i},\psi)a_{\widetilde{L}_i}(\chi_s)
\end{aligned}$$

Here $\varepsilon(-s_i,(\chi_s)^{\lambda_i},\psi)$ denotes the usual Tate ε-factor. Therefore,

$$g(s,\chi,\psi)\frac{\mathcal{R}_{P^{\mathrm{op}}|P}(\mathcal{F}_{P|P^{\mathrm{op}}}(f)_{\chi_s,P^{\mathrm{op}}})}{a_{P^{\mathrm{op}}|P}(\chi_s)} = \frac{f_{(\chi_s)^{-1},P}}{a_{\widetilde{L}}(\chi_s)} = \frac{f_{(\chi_s)^{-1},P}}{a_{P|P}((\chi_s)^{-1})} \qquad (3.60)$$

where $g(s,\chi,\psi) = \prod_i \varepsilon(-s_i,(\chi_s)^{\lambda_i},\psi)$.

In the remainder of the proof we use some basic facts on ε-factors that are nicely summarized in [Tat79, §3.2]. Assume F is nonarchimedean. In this case $g(s,\chi,\psi)$ is equal to $cq^{p(s)}$ for some polynomial p and some $c \in \mathbb{C}^\times$. Thus (3.60) and the fact that $f \in \mathcal{S}(X_P(F),K)$ implies (3.56). Now assume that F is archimedean. Then

$$g(s,\chi,\psi) = \prod_i \epsilon_{i,\chi} r \frac{(\chi_s)^{\lambda_i}(a)}{|a|^{1+s_i}}$$

where a and r depend only on ψ (which determines the normalization of the Haar measure) and $\epsilon_{i,\chi}$ is a fourth root of unity. By an analogue of [GH20, Lemma 3.6], (3.60) and the fact that $f \in \mathcal{S}(X_P(F),K)$ implies (3.57), at least in the special case where D is the identity operator. It also follows for general D once we note that $\mathcal{R}_{P|P^{\mathrm{op}}} \circ R(m,g) = \delta_{P^{\mathrm{op}}}(m)R(m,g) \circ \mathcal{R}_{P|P^{\mathrm{op}}}$. Here we have used R to denote the right action of $M(F) \times G(F)$ on $C^\infty(X_P^\circ(F))$ and $C^\infty(X_{P^{\mathrm{op}}}^\circ(F))$.

To deduce (3.57) without the condition of K-finiteness, we point out that the same argument proving [GH20, Proposition 3.7] implies that

$$\mathcal{F}_{P|P^{\mathrm{op}}} : \mathcal{S}(X_P(F),K) \longrightarrow \mathcal{S}(X_{P^{\mathrm{op}}}(F),K)$$

is continuous in the Fréchet topology. Since $\mathcal{S}(X_P(F),K)$ is dense in $\mathcal{S}(X_P(F))$ [War72, §§4.4.3.1] it extends to a topological isomorphism

$$\mathcal{F}_{P|P^{\mathrm{op}}} : \mathcal{S}(X_P(F)) \longrightarrow \mathcal{S}(X_{P^{\mathrm{op}}}(F)).$$

This already implies the first assertion of the theorem, and additionally (3.57). ☐

As usual, we say that a parabolic subgroup of a reductive group is self-associate if it is conjugate to its opposite. Assume P is self-associate. Choose

$$w_0 \in G(F) \tag{3.61}$$

normalizing M such that $w_0^{-1}Pw_0 = P^{\mathrm{op}}$. Then conjugation by w_0 acts as inversion on M^{ab}.

Lemma 3.4.5. *Let w be a representative for the long Weyl element of the Weyl group of T in G. Then one has $w_0 \in M(F)w = wM(F)$.*

Proof. The normalizer of P in G is P and the normalizer of M in P is M. Therefore, $w_0 \in P(F)w$ as $w\left(w_0^{-1}Pw_0\right)w^{-1} = P$. As w normalizes M, for w_0 to normalize M, one must have $w_0 = mnw$ for some $n \in N(F)$ such that $n^{-1}Mn = M$. This is only possible when n is the identity. ☐

We observe that $w_0Pw_0^{-1} = P^{\mathrm{op}}$. Indeed, by Lemma 3.4.5 it suffices to check this in the special case where w_0 is the long Weyl element of $T(F)$, in which case w_0 and w_0^{-1} differ by an element in $T(F)$.

We assume that w_0 is chosen so that measures on $N_P(F)$ and $N_{P^{\mathrm{op}}}(F)$ correspond under

$$\begin{aligned} N_P(F) &\longrightarrow N_{P^{\mathrm{op}}}(F) \\ n &\longmapsto w_0^{-1}nw_0. \end{aligned} \tag{3.62}$$

Lemma 3.4.6. *One has an isomorphism*

$$\begin{aligned} \iota_{w_0} : \mathcal{S}(X_{P^{\mathrm{op}}}(F)) &\longrightarrow \mathcal{S}(X_P(F)) \\ f &\longmapsto (x \mapsto f(w_0^{-1}x)). \end{aligned}$$

Proof. We have an isomorphism

$$\begin{aligned} \iota_{w_0} : C^\infty(X^\circ_{P^{\mathrm{op}}}(F)) &\longrightarrow C^\infty(X^\circ_P(F)) \\ f &\longmapsto (x \mapsto f(w_0^{-1}x)). \end{aligned}$$

For $f \in \mathcal{S}(X_{P^{\mathrm{op}}}(F))$ and $Q \in \{P, P^{\mathrm{op}}\}$, one has

$$\frac{\mathcal{R}_{P|Q}(\iota_{w_0}(f)_{\chi_s,P})}{a_{P|Q}(\chi_s)} = \frac{\mathcal{R}_{P|Q}(\iota_{w_0}(f_{\chi_s,P^{\mathrm{op}}}))}{a_{P|Q}(\chi_s)} = \frac{\iota_{w_0} \circ \mathcal{R}_{P^{\mathrm{op}}|Q^{\mathrm{op}}}(f_{\chi_s,P^{\mathrm{op}}})}{a_{P^{\mathrm{op}}|Q^{\mathrm{op}}}(\chi_s)},$$

where we have used (3.43). Assume F is nonarchimedean. Then since $f_{\chi_s,P^{\mathrm{op}}}$ is a good section, we deduce that $\iota_{w_0}(f)_{\chi_s,P}$ is a good section. Thus the lemma follows from the definition of the Schwartz space. A similar argument proves the lemma in the archimedean case. ☐

Thus when P is self-associate, we have an isomorphism

$$\mathcal{F}_{X_P} := \mathcal{F}_{X_P,w_0} := \iota_{w_0} \circ \mathcal{F}_{P|P^{\mathrm{op}}} : \mathcal{S}(X_P(F)) \longrightarrow \mathcal{S}(X_P(F)). \tag{3.63}$$

By Theorem 3.4.4 and (3.39), we see that the Fourier transform $\mathcal{F}_{X_P}$ agrees with the Fourier transform used in [GL21, GL19, GH20] when X_P is as in §3.2.3 and w_0 is chosen as in loc. cit.

For use in §3.5.2, we also consider how ι_{w_0} interacts with the operators $\lambda_!(\mu_s)$. Suppose that L and L', etc., are as in the discussion prior to Proposition 3.2.8. Recall L^{op} and its associated data $\{(s_i, \lambda_i)\}$ from (3.38). Arguing as in the proof of Lemma 3.4.6, we have an isomorphism

$$\iota_{w_0} : \mathcal{S}_L(X_{P^{\mathrm{op}}}(F)) \longrightarrow \mathcal{S}_{L^{\mathrm{op}}}(X_P(F)).$$

Lemma 3.4.7. *We have a commutative diagram*

$$\begin{tikzcd}
\mathcal{S}_L(X_{P^{\mathrm{op}}}(F)) \arrow[r, "\lambda_{k!}(\mu_{s_k})"] \arrow[d, "\iota_{w_0}"] & \mathcal{S}_{L'}(X_{P^{\mathrm{op}}}(F)) \arrow[d, "\iota_{w_0}"] \\
\mathcal{S}_{L^{\mathrm{op}}}(X_P(F)) \arrow[r, "\lambda_{k!}(\mu_{s_k})"] & \mathcal{S}_{L'^{\mathrm{op}}}(X_P(F)).
\end{tikzcd}$$

We caution the reader that the bottom row in the diagram is given by the same definition as (3.10), but the roles of P and P^{op} switched as we are applying the operator $\lambda_{k!}(\mu_{s_k})$ to functions on $X_P(F)$.

Proof. Let $f \in \mathcal{S}_L(X_{P^{\mathrm{op}}}(F))$. Then by Proposition 3.2.8, $\lambda_{k!}(\mu_{s_k})(f) \in \mathcal{S}_{L'}(X_{P^{\mathrm{op}}}(F))$, and for $A(L) < \mathrm{Re}(\chi) < B(L')$, traversing the top of the diagram and applying a Mellin transform yields

$$\begin{aligned}
(\iota_{w_0}(\lambda_{k!}(\mu_{s_k})(f)))^{\mathrm{op}}_{\chi,P} &= \iota_{w_0}\left(\lambda_{k!}(\mu_{s_k})(f)^{\mathrm{op}}_{\chi,P^{\mathrm{op}}}\right) \\
&= \gamma(-s_k, \chi^{\lambda_k}, \psi)\iota_{w_0}\left(f^{\mathrm{op}}_{\chi,P^{\mathrm{op}}}\right) \\
&= \gamma(-s_k, \chi^{\lambda_k}, \psi)\left(\iota_{w_0}(f)\right)^{\mathrm{op}}_{\chi,P}.
\end{aligned}$$

Noting that

$$A(L^{\mathrm{op}}) = A(L) \quad \text{and} \quad B(L^{\mathrm{op}}) = B(L),$$

we may apply Proposition 3.2.8 again to see that this equals

$$(\lambda_{k!}(\mu_{s_k})(\iota_{w_0}(f)))^{\mathrm{op}}_{\chi,P}.$$

This is the result of traversing the bottom of the diagram and applying a Mellin transform. Thus applying Mellin inversion yields the lemma. □

3.5 A formula for the Fourier transform

In this section, we combine our analytic results with the geometric pairing between opposite Braverman–Kazhdan spaces to give a formula for the Fourier transform.

3.5.1 Preliminary calculations

Recall that ω_β is the fundamental weight attached to P as in (3.2). Since G is simply connected $\omega_P = \omega_\beta$ in the notation of (3.2). As above V_P is the associated highest weight representation. By Lemma 3.1.4, ω_P may be extended to a character of P (trivial on P^{der}) and defines an isomorphism

$$\omega_P = \omega_\beta : M^{\mathrm{ab}} \xrightarrow{\sim} \mathbb{G}_m.$$

Recall the graded representation $L = \widehat{\mathfrak{n}}_P^e$ of §3.2.1. We fix a good ordering

$$\{(s_i, \lambda_i) : 1 \leqslant i \leqslant k\},$$

so

$$\frac{s_{i+1}}{\lambda_{i+1}} \geqslant \frac{s_i}{\lambda_i}$$

for $1 \leqslant i < k$ and $k = \dim L$. In particular, we have the highest datum (s_k, λ_k), which is unique in the following sense:

Proposition 3.5.1. *Any good ordering of $L = \widehat{\mathfrak{n}}_P^e$ satisfies $\lambda_k = 1$. Furthermore, $s_k > \frac{s_i}{\lambda_i}$ for all $i < k$.*

Proof. Our proof is a case-by-case analysis. As this is a computation on the Langlands dual group, we defer the details to Appendix A. In fact we compute all of the parameters

$\{(s_i, \lambda_i)\}$ for all simple Cartan types. The results required to observe the current proposition are lemmas A.1.2 and A.2.4 and the tables at the end of Appendix A. We alert the reader that we work entirely on the Langlands dual side in the appendix. One must use the following well-known computations of Langlands dual groups:

$$\widehat{\mathrm{Sp}}_{2n} = \mathrm{SO}_{2n+1}(\mathbb{C}), \quad \widehat{\mathrm{Spin}}_{2n} = \mathrm{PSO}_{2n}(\mathbb{C}), \quad \widehat{\mathrm{Spin}}_{2n+1} = \mathrm{PSp}_{2n}(\mathbb{C})$$

together with the fact that the dual group of a simply connected semisimple group is adjoint. □

Proposition 3.5.2. *One has*

$$\delta_P = |\omega_P|^{2s_k+2}.$$

Remark 3.5.3. The proof shows that the proposition is still valid if we weaken the assumption that G is simply connected to the assumption that $\omega_\beta \in X^*(T)$.

Proof. Let $\Phi^+ \subset \Phi$ denote the set of positive roots and $\Phi_M^+ \subset \Phi^+$ denote the subset of positive roots of $(M, M \cap B, T)$. For $t \in T(F)$, we have

$$\delta_P(t) = \left| t^{\sum_{\gamma \in \Phi^+ - \Phi_M^+} \gamma} \right|.$$

On the other hand $X^*(M) = \mathbb{Z}\omega_P$, so there is an integer $r > 0$ such that

$$\sum_{\gamma \in \Phi^+ - \Phi_M^+} \gamma = r\omega_P.$$

We are to show that $r = 2s_k + 2$. This can be directly verified for $G = \mathrm{SL}_2$.

Suppose $G \neq \mathrm{SL}_2$. Let $\{e, h, f\} \subset \widehat{\mathfrak{m}}$ be a principal $\mathfrak{sl}_2$-triple. The copy of $\mathfrak{sl}_2$ it spans acts on $\widehat{\mathfrak{n}}_P$ by the adjoint action. The root systems of M and $\widehat{M}$ are in Langlands duality. We use this to identify

$$\widehat{\mathfrak{t}} = X_*(\widehat{T}) \otimes_{\mathbb{Z}} \mathbb{C} = X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}. \tag{3.64}$$

Under this identification, $h \in \widehat{\mathfrak{t}}$ may be chosen so that it is sent to the sum of positive coroots of $\widehat{M}$ [Gro97, Section 2]:

$$2\rho_M^\vee := \sum_{\gamma \in \Phi^+_{\widehat{M}}} \gamma^\vee \in X_*(\widehat{T}),$$

which corresponds under the second equality of (3.64) to

$$2\rho_M := \sum_{\gamma \in \Phi^+_M} \gamma = 2 \sum_{\alpha \in \Delta_M} \widetilde{\omega}_\alpha \in X^*(T) \tag{3.65}$$

where $\widetilde{\omega}_\alpha$ is the weight of the fundamental representation of M associated to $\alpha \in \Delta_M$. Thus

$$h + r\omega_P = \sum_{\gamma \in \Phi^+} \gamma = 2 \sum_{\alpha \in \Delta} \omega_\alpha, \tag{3.66}$$

where ω_α is the fundamental weight of G associated to $\alpha \in \Delta$. Note that in general $\widetilde{\omega}_\alpha \neq \omega_\alpha$ for $\alpha \in \Delta_M$ since $\widetilde{\omega}_\alpha \in X^*(T \cap M^{\mathrm{der}})$.

Consider now the h-eigenvalues on the space of highest weight vectors $L = \widehat{\mathfrak{n}}_P^e$. By Proposition 3.5.1,

$$L_k = \mathbb{C}v_k \leqslant \widehat{\mathfrak{n}}_P(1)^e$$

where the 1 indicates the subspace on which $Z(\widehat{M})$ acts via 1. As mentioned in [Man13, §5.2], the space $\widehat{\mathfrak{n}}_P(1)$ is the irreducible representation of $\widehat{M}$ with lowest weight space corresponding to $\beta^\vee$, the coroot of β.

By the definition of a good ordering, the h-eigenvalue $2s_k$ is largest among all h-eigenvalues occurring in the $\widehat{M}$ representation $\widehat{\mathfrak{n}}_P(1)$. It follows that v_k is a highest weight vector for $\widehat{\mathfrak{n}}_P(1)$. Let

$$\gamma_0^\vee = \beta^\vee + \sum_{\alpha \in \Delta_M} c_\alpha(\gamma_0^\vee)\alpha^\vee$$

be the weight of v_k. We claim that $2s_k = \sum_{\alpha \in \Delta_M} c_\alpha(\gamma_0^\vee)$.

Since this is the largest h-eigenvalue in L, it follows that the lowest weight space $\widehat{\mathfrak{n}}_{\beta^\vee} < \widehat{\mathfrak{n}}_P(1)$ is the lowest-weight space for the irreducible $\mathfrak{sl}_2$-representation containing

v_k, and thus has the eigenvalue

$$\langle h, \beta^\vee \rangle = -2s_k. \tag{3.67}$$

Here $\langle \cdot, \cdot \rangle$ is the pairing on $X^*(T) \otimes X_*(T)$. Therefore, since (3.65) implies $\langle h, \alpha^\vee \rangle = 2$ for all $\alpha \in \Delta_M$,

$$\begin{aligned} 2s_k = \langle h, \gamma_0^\vee \rangle &= \sum_{\alpha \in \Delta_M} c_\alpha(\gamma_0^\vee) \langle h, \alpha^\vee \rangle + \langle h, \beta^\vee \rangle \\ &= 2 \sum_{\alpha \in \Delta_M} c_\alpha(\gamma_0^\vee) - 2s_k, \end{aligned}$$

proving the claim that $2s_k = \sum_{\alpha \in \Delta_M} c_\alpha(\gamma_0^\vee)$.

Since $\omega_P = \omega_\beta$, we see that for any root $\gamma^\vee$ occurring in $\hat{\mathfrak{n}}_P(1)$, $\langle \omega_P, \gamma^\vee \rangle = 1$. Evaluating both sides of (3.66) on $\gamma_0^\vee$ thus implies

$$2s_k + r = \left\langle 2 \sum_{\alpha \in \Delta} \omega_\alpha, \gamma_0^\vee \right\rangle = 2 + 2 \sum_{\alpha \in \Delta_M} c_\alpha(\gamma_0^\vee) = 2 + 4s_k.$$

We deduce that $r = 2s_k + 2$, and the proposition follows. □

3.5.2 The general formula

For integers n, let

$$[n] : \mathbb{G}_m \longrightarrow \mathbb{G}_m \tag{3.68}$$

be the map $x \mapsto x^n$. We define

$$\mu_P^{\mathrm{aug}} := \lambda_{1!}(\mu_{s_1}) \circ \cdots \circ \lambda_{(k-1)!}(\mu_{s_{k-1}}) \quad \text{and} \quad \mu_P^{\mathrm{geo}} := [1]_!(\mu_{s_k}), \tag{3.69}$$

where the aug stands for "augmented" and consider the factorization

$$\mu_P = \mu_P^{\mathrm{aug}} \circ \mu_P^{\mathrm{geo}}.$$

Remark 3.5.4. In light of our formula for the Fourier transform below, it would be interesting to illuminate the relationship between the operator μ_P^{aug} and the singularity of X_P at 0. In the archimedean case, it is my ongoing work to relate μ_P^{aug} to a correct candidate of Weyl algebra on X_P that incorporates the singularity of X_P. See Chapter 6 for a short discussion.

Set

$$\mathcal{F}^{\mathrm{geo}}_{P|P^{\mathrm{op}}} := \mu^{\mathrm{geo}}_P \circ \mathcal{R}_{P|P^{\mathrm{op}}} : \mathcal{S}(X_P(F)) \longrightarrow \mathcal{S}_{L(1)}. \tag{3.70}$$

Theorem 3.5.5. *For $f \in \mathcal{S}(X_P(F))$ and $x^* \in X^{\circ}_{P^{\mathrm{op}}}(F)$, we have $\mathcal{F}_{P|P^{\mathrm{op}}} = \mu^{\mathrm{aug}}_P \circ \mathcal{F}^{\mathrm{geo}}_{P|P^{\mathrm{op}}}$ where*

$$\mathcal{F}^{\mathrm{geo}}_{P|P^{\mathrm{op}}}(f)(x^*) = \int_{X^{\circ}_P(F)} f\,(x)\,\psi\left(\langle x, x^*\rangle_{P|P^{\mathrm{op}}}\right) dx.$$

Here $\langle \cdot, \cdot \rangle_{P|P^{\mathrm{op}}}$ is as in (3.4) *and the measure on $X^{\circ}_P(F)$ is normalized as in §3.3.1.*

Proof. For $x^* \in X^{\circ}_{P^{\mathrm{op}}}(F)$, we have

$$\begin{aligned}\mathcal{F}^{\mathrm{geo}}_{P|P^{\mathrm{op}}}(f)(x^*) &= \frac{1}{\zeta(1)} \int_{M^{\mathrm{ab}}(F)} \psi(\omega_P(m))|\omega_P(m)|^{s_k+1}\delta^{1/2}_{P^{\mathrm{op}}}(m)\mathcal{R}_{P|P^{\mathrm{op}}}(f)(m^{-1}x^*)dm \\ &= \frac{1}{\zeta(1)} \int_{M^{\mathrm{ab}}(F)} \psi(\omega_P(m))\mathcal{R}_{P|P^{\mathrm{op}}}(f)(m^{-1}x^*)dm.\end{aligned}$$

Here we have used Proposition 3.5.2. We note that there is no need to regularize the outer integral: the absolute convergence of $[1]_!(\mu_{s_k})$ on $\mathcal{R}_{P|P^{\mathrm{op}}}(\mathcal{S}(X_P(F)))$ follows from our use of a good ordering and from lemmas 3.2.7 and 3.4.1. If we write $x^* = P^{\mathrm{op,der}}(F)g$,

$$\begin{aligned}\mathcal{R}_{P|P^{\mathrm{op}}}(f)(m^{-1} \cdot x^*) &= \int_{N_{P^{\mathrm{op}}}(F)} f\left(um^{-1}g\right) du \\ &= \delta_P(m) \int_{N_{P^{\mathrm{op}}}(F)} f\left(m^{-1}ug\right) du. \end{aligned} \tag{3.71}$$

We have an injection

$$\begin{aligned}\Phi_g : M^{\mathrm{ab}}(F) \times N_{P^{\mathrm{op}}}(F) &\longrightarrow X^{\circ}_P(F) \\ (m, u) &\longmapsto P^{\mathrm{der}}(F)m^{-1}ug\end{aligned}$$

with dense image denoted by $X^{\circ}_{P,g}(F)$. Moreover, we have

$$d(m^{-1}ug) = \frac{\delta_{P^{\mathrm{op}}}(m^{-1})dmdu}{\zeta(1)} = \frac{\delta_P(m)dmdu}{\zeta(1)} \tag{3.72}$$

by (3.40).

For $x \in X^\circ_{P,g}(F)$, let

$$(m(x), u(x)) := \Phi_g^{-1}(x).$$

By (3.71) and (3.72), we have

$$\mathcal{F}^{\mathrm{geo}}_{P|P^{\mathrm{op}}}(f)(x^*) = \int_{X^\circ_{P,g}(F)} \psi(\omega_P(m(x)))f(x)dx. \tag{3.73}$$

Now for $(m, u) \in M^{\mathrm{ab}}(F) \times N_{P^{\mathrm{op}}}(F)$ and g chosen as above (so $P^{\mathrm{op,der}}(F)g = x^*$), we have

$$\langle v_P m^{-1} ug, v^*_{P^{\mathrm{op}}} g\rangle = \langle v_P m^{-1}, v^*_{P^{\mathrm{op}}}\rangle = \omega_P(m);$$

here we have used (3.3). Thus (3.73) is

$$\mathcal{F}^{\mathrm{geo}}_{P|P^{\mathrm{op}}}(f)(x^*) = \int_{X^\circ_{P,g}(F)} \psi(\langle x, x^*\rangle_{P|P^{\mathrm{op}}})f(x)dx = \int_{X^\circ_P(F)} \psi(\langle x, x^*\rangle_{P|P^{\mathrm{op}}})f(x)dx$$

since $X^\circ_{P,g}(F)$ is open and of full measure in $X^\circ_P(F)$. □

Assume for the moment that P is self-associate. In this special case, fix a $w_0 \in G(F)$ normalizing M such that $w_0^{-1}Pw_0 = P^{\mathrm{op}}$ and such that (3.62) is measure preserving. We saw in (3.63) that this allows us to define a Fourier transform

$$\mathcal{F}_{X_P} := \mathcal{F}_{X_P, w_0} := \iota_{w_0} \circ \mathcal{F}_{P|P^{\mathrm{op}}} : \mathcal{S}(X_P(F)) \longrightarrow \mathcal{S}(X_P(F)). \tag{3.74}$$

Corollary 3.5.6. *Assume that $P = w_0 P^{\mathrm{op}} w_0^{-1}$ is self-associate. Then for $f \in \mathcal{S}(X_P(F))$, we have $\mathcal{F}_{X_P}(f) = \mu^{\mathrm{aug}}_{P^{\mathrm{op}}} \circ \mathcal{F}^{\mathrm{geo}}_{X_P}(f)$ where*

$$\mathcal{F}^{\mathrm{geo}}_{X_P}(f)(x') = \int_{X^\circ_P(F)} f(x)\, \psi\left(\langle x, w_0^{-1}x'\rangle_{P|P^{\mathrm{op}}}\right) dx$$

for $x' \in X^\circ_P(F)$. Here the measure on $X^\circ_P(F)$ is normalized as in §3.3.1.

Proof. By the discussion in §3.2.2, we have

$$\iota_{w_0} \circ \mu^{\mathrm{aug}}_P = \mu^{\mathrm{aug}}_{P^{\mathrm{op}}} \circ \iota_{w_0}$$

and it is clear that

$$\mathcal{F}^{\mathrm{geo}}_{X_P} = \iota_{w_0} \circ \mathcal{F}^{\mathrm{geo}}_{P|P^{\mathrm{op}}}.$$ □

Remark 3.5.7. By Corollary 3.3.8, $\mathcal{S}(X_P(F)) < L^1(X_P(F))$. Thus the integrals in the definition of $\mathcal{F}^{\mathrm{geo}}_{P|P^{\mathrm{op}}}$ and $\mathcal{F}^{\mathrm{geo}}_{X_P}$ converge absolutely.

3.6 Examples

In this section, we explicate the objects appearing in Theorem 3.5.5 in several cases of interest.

3.6.1 Line bundles over Grassmannians

The maximal parabolic subgroups of SL_n are stabilizers of planes. Concretely, fix $1 \leqslant \ell < n$ and let P be the stabilizer of the ℓ-plane $\{e_{n-\ell+1}, \ldots, e_n\}$. Here we use the standard basis of F^n, viewed as row vectors with a right action of G. Then $P\backslash G$ is a classical Grassmannian, and $X_P^\circ(F)$ can be viewed as the space of ℓ-planes $W \subset F^n$ together with an associated non-zero vector in $\wedge^\ell W$.

For F-algebras R, we have

$$P(R) = \left\{ \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix} \begin{pmatrix} I_{n-\ell} & w \\ & I_\ell \end{pmatrix} \in \mathrm{SL}_n(R) : (m_1, m_2, w) \in \mathrm{GL}_{n-\ell}(R) \times \mathrm{GL}_\ell(R) \times M_{n-\ell,\ell}(R) \right\}.$$

Then

$$P^{\mathrm{op}}(R) = \left\{ \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix} \begin{pmatrix} I_{n-\ell} & \\ w^t & I_\ell \end{pmatrix} \in \mathrm{SL}_n(R) : (m_1, m_2, w) \in \mathrm{GL}_{n-\ell}(R) \times \mathrm{GL}_\ell(R) \times M_{n-\ell,\ell}(R) \right\}.$$

In this setting,

$$\begin{aligned} \omega_P : M(F) &\longrightarrow F^\times \\ \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix} &\longmapsto \det(m_1) = \det(m_2)^{-1}. \end{aligned}$$

Our representation V_P is just $\wedge^\ell \mathbb{G}_a^n$. We realize the dual as the space $\wedge^{n-\ell}\mathbb{G}_a^n$ equipped with the pairing

$$\begin{aligned} \wedge^\ell R^n \times \wedge^{n-\ell} R^n &\longrightarrow R \\ (w_1, w_2) &\longmapsto e_1^\vee \wedge \cdots \wedge e_n^\vee (w_2 \wedge w_1) \end{aligned}$$

We choose the highest weight vector $v_P := e_{n-\ell+1} \wedge \cdots \wedge e_n$ and dual lowest weight vector $v_{P^{\mathrm{op}}}^* := e_1 \wedge \cdots \wedge e_{n-\ell}$. With these choices,

$$\mathrm{Pl}_{v_P} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \wedge^\ell \begin{pmatrix} c & d \end{pmatrix}$$

where we are taking the (ordered) wedge product of the row vectors from top to bottom. Similarly,

$$\mathrm{Pl}_{v^*_{P^{\mathrm{op}}}} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \longmapsto \wedge^{n-\ell} \begin{pmatrix} a & b \end{pmatrix}$$

where the wedge product is taken from top to bottom.

3.6.2 Orthogonal groups and the transform on the isotropic cone

Assume the characteristic of F is not 2. Consider the split orthogonal group SO_n for $n > 4$, defined with respect to the matrix

$$J_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

Denote the corresponding pairing by $\langle \cdot, \cdot \rangle$, and let

$$Q_n(v) := \tfrac{1}{2}\langle v, v \rangle.$$

Let T be the split maximal torus of diagonal matrices and let B be the Borel subgroup of upper triangular matrices of SO_n. There is a natural right action of SO_n on $V_n = \mathbb{G}_a^n$. We let $P < \mathrm{SO}_n$ be the parabolic subgroup fixing the line spanned by $e_n = (0, \ldots, 0, 1)$. Then $V_P = V_n$, and we choose the highest weight vector $v_P := e_n$.

Consider the split spin group $G = \mathrm{Spin}_n$ over SO_n and let $p : G \to \mathrm{SO}_n$ be the double cover. Then $\widetilde{P} := p^{-1}(P)$ is a maximal parabolic subgroup of G. Moreover, the representation $V_{\widetilde{P}}$ of G descends to the representation V_n of SO_n via p. It therefore follows from Lemma 3.1.4 that p induces an isomorphism

$$p : X^\circ_{\widetilde{P}} = \widetilde{P}^{\mathrm{der}} \backslash G \xrightarrow{\sim} P^{\mathrm{der}} \backslash \mathrm{SO}_n.$$

Let $\widetilde{M}$ be a Levi subgroup of $\widetilde{P}$ and $M := p(\widetilde{M})$. Since $V_{\widetilde{P}}$ descends to V_n, it also follows from Lemma 3.1.4 that the map $\widetilde{M}^{\mathrm{ab}} \to M^{\mathrm{ab}}$ induced by p is an isomorphism and the diagram

$$\begin{tikzcd} \widetilde{M} \times X^\circ_{\widetilde{P}} \arrow[r] \arrow[d, "p"] & X^\circ_{\widetilde{P}} \arrow[d, "p"] \\ M \times X^\circ_P \arrow[r] & X^\circ_P \end{tikzcd} \tag{3.75}$$

commutes. Here the horizontal arrows are the action maps. Thus we can and do work with $P^{\mathrm{der}}\backslash G$ in place of $\widetilde{P}^{\mathrm{der}}\backslash\widetilde{G}$ below.

The Plücker embedding

$$\mathrm{Pl}_{e_n} : X_P \longrightarrow V_n$$

maps X_P isomorphically onto the affine scheme whose points in an F-algebra R are

$$C(R) := \{v \in V_n(R) : Q_n(v) = 0\}.$$

This is the isotropic cone of Q_n.

We define the Schwartz space $\mathcal{S}(C(F))$ to be

$$(\mathrm{Pl}_{e_n}^{-1})^*(\mathcal{S}(X_P(F))) < C^\infty(C(F) - \{0\}).$$

The parabolic P is self-associate. Thus the Schwartz space comes equipped with a Fourier transform

$$\mathcal{F}_C := (\mathrm{Pl}_{e_n}^{-1})^* \circ \mathcal{F}_{X_P,w_0} \circ \mathrm{Pl}_{e_n}^* : \mathcal{S}(C(F)) \longrightarrow \mathcal{S}(C(F)).$$

Here w_0 is chosen as in Lemma 3.6.1 below. There is a natural measure on $C(F)$ as we now explain. Let dv_i be the standard 1-form on $\mathbb{G}_a$, viewed as the ith coordinate of $V_n = \mathbb{G}_a^n$. Recall [GS16, §III.1.2] that to give a measure on $C(F)$, we may choose any $(n-1)$-form ω_{Q_n} such that

$$dv_1 \wedge \cdots \wedge dv_n = dQ_n \wedge \omega_{Q_n} \tag{3.76}$$

and then consider the measure $|\omega_{Q_n}|$. If we write

$$Q_n(v_1, \ldots, v_n) = \begin{cases} \frac{1}{2}v_{r+1}^2 + \sum_{i=1}^r v_i v_{n+1-i} & \text{if } n = 2r+1, \\ \sum_{i=1}^r v_i v_{n+1-i} & \text{if } n = 2r, \end{cases}$$

with respect to the standard basis of F^n, then on $\mathbb{G}_a^{n-1} \times \mathbb{G}_m$ we choose $\omega_{Q_n} = \frac{1}{v_n} dv_2 \wedge \cdots \wedge dv_n$.

Lemma 3.6.1. *We can choose $w_0 \in \mathrm{SO}_n(F)$ normalizing M such that $w_0^{-1} P w_0 = P^{\mathrm{op}}$ and such that for $x, x' \in X_P^\circ(F)$ one has*

$$\langle x, w_0^{-1} x' \rangle_{P|P^{\mathrm{op}}} = \langle \mathrm{Pl}_{e_n}(x), \mathrm{Pl}_{e_n}(x') \rangle.$$

Moreover, $\mathrm{Pl}^*_{e_n}(|\omega_{Q_n}|) = cdx$ *for some* $c \in \mathbb{R}_{>0}$.

Proof. We identify the dual of V_n with V_n itself via the form $\langle\cdot,\cdot\rangle$. Then the vector dual to e_n is e_1. Let

$$w_0 := \begin{cases} J_n & \text{if } n \equiv 0 \pmod 4 \text{ or } n \equiv 1 \pmod 4, \\ \begin{pmatrix} & & J_{(n-1)/2} \\ & -1 & \\ J_{(n-1)/2} & & \end{pmatrix} & \text{if } n \equiv 3 \pmod 4, \\ \begin{pmatrix} & & J_{(n-2)/2} \\ & I_2 & \\ J_{(n-2)/2} & & \end{pmatrix} & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Thus $w_0 \in \mathrm{SO}_n(F)$, $\mathrm{Pl}_{e_n}(g) = e_n g$ and $\mathrm{Pl}_{e_1}(w_0^{-1}g) = e_n g$. This yields the first assertion. For the second assertion, it suffices to observe that (3.76) implies that ω_{Q_n} is SO_n-invariant. □

Corollary 3.6.2. *If the measure* $|\omega_{Q_n}(v)|$ *is normalized so that* $\mathrm{Pl}^*_{e_n}(|\omega_{Q_n}|) = dx$, *then for* $f \in \mathcal{S}(C(F))$ *one has*

$$\mathcal{F}_C(f)(v') = \int_{F^\times} \psi(t^{-1})|t|^{(n-4)/2} \left(\int_{C(F)-\{0\}} f(v)\psi\left(\langle v, tv'\rangle\right)|\omega_{Q_n}(v)| \right) \frac{d^\times t}{\zeta(1)} \tag{3.77}$$

if $n > 4$ *is even and*

$$\mathcal{F}_C(f)(v') = \int_{F^\times} \psi(t^{-1})|t|^{n-3} \left(\int_{C(F)-\{0\}} f(v)\psi\left(\langle v, t^2v'\rangle\right)|\omega_{Q_n}(v)| \right) \frac{d^\times t}{\zeta(1)} \tag{3.78}$$

if $n > 3$ *is odd.*

Proof. This is a consequence of Corollary 3.5.6 and Lemma 3.6.1 as we now explain. Using (3.75), we are free to work with the action of M^{ab} instead of $\widetilde{M}^{\mathrm{ab}}$ in applying Corollary 3.5.6.

For $(t, g) \in R^\times \times \mathrm{SO}_{n-2}(R)$ write

$$m(t,g) := \begin{pmatrix} t & & \\ & g & \\ & & t^{-1} \end{pmatrix} : t \in R^\times, g \in \mathrm{SO}_{n-2}(R)$$

The character ω_P is given by $\omega_P(m(t,g)) = t$. Note that for $x, x' \in X_P^\circ(F)$ and $\lambda \in \mathbb{Z}$,

$$\langle x, w_0^{-1}m(t,g)^{-\lambda}x'\rangle_{P|P^{\mathrm{op}}} = \omega_P(m(t,g))^\lambda \langle x, w_0^{-1}x'\rangle_{P|P^{\mathrm{op}}} = t^\lambda \langle x, w_0^{-1}x'\rangle_{P|P^{\mathrm{op}}}.$$

Applying Lemma 3.6.1 now shows that if $v = \mathrm{Pl}_{e_n}(x)$ and $v' = \mathrm{Pl}_{e_n}(x')$, then

$$\langle x, w_0^{-1}m(t,g)^{-\lambda}x'\rangle_{P|P^{\mathrm{op}}} = \langle v, t^{\lambda}v'\rangle.$$

By Lemma A.2.4, we have $\mu^{\mathrm{geo}}_{\mathrm{P^{op}}} = [1]_!(\mu_{\frac{n-4}{2}})$ for all n, and

$$\mu^{\mathrm{aug}}_{P^{\mathrm{op}}} = \begin{cases} [1]_!(\mu_0) & \text{if } n > 4 \text{ is even,} \\ [2]_!(\mu_0) & \text{if } n > 3 \text{ is odd.} \end{cases}$$

The regularized operators are equal to the unregularized operators by Lemma 3.2.7. □

When F is nonarchimedean with odd or zero characteristic, Corollary 3.6.2 implies that when n is even $\mathcal{F}_C$ agrees with the operator $\Pi(r) = \Phi$ of [GK23, (1.5)]. Gurevich and Kazhdan also treat nonsplit isotropic quadratic forms. When $F = \mathbb{R}$ and n is even, a Fourier transform on $L^2(C(F), |\omega_Q|)$ was investigated in [KM11] (they also treated arbitrary isotropic quadratic forms in an even number of variables). It agrees with $\mathcal{F}_C$ up to a constant when the form is split, but we will not verify the claim here.

3.6.3 The Lagrangian Grassmannian

Define Sp_{2n} and P as in §3.2.3. We let Sp_{2n} act on $V = \mathbb{G}_a^{2n}$ on the right. The representation V_P may be realized as an irreducible subrepresentation of $\wedge^n V$, and we choose the highest weight vector to be $v_P := e_{n+1} \wedge \cdots \wedge e_{2n}$. Thus

$$\mathrm{Pl}_{v_P}\begin{pmatrix} * \\ a_{n+1} \\ \vdots \\ a_{2n} \end{pmatrix} = a_{n+1} \wedge \cdots \wedge a_{2n} \tag{3.79}$$

is the (ordered) wedge product of the last n rows.

There is a perfect pairing

$$\langle \cdot, \cdot \rangle : \wedge^n \mathbb{G}_a^{2n} \times \wedge^n \mathbb{G}_a^{2n} \longrightarrow \wedge^{2n}\mathbb{G}_a^{2n} \xrightarrow{\sim} \mathbb{G}_a, \tag{3.80}$$

where the first map is canonical and the second is obtained by specifying that $e_1 \wedge \cdots \wedge e_{2n}$ is sent to 1. We use this pairing to identify the dual of V_P with V_P. Thus

$$\langle x, x^* \rangle_{P|P^{\mathrm{op}}} = \langle \mathrm{Pl}_{v_P}(x), \mathrm{Pl}_{v^*_{P^{\mathrm{op}}}}(x^*)\rangle$$

where $v^*_{P^{\mathrm{op}}} = (-1)^n e_1 \wedge \cdots \wedge e_n$ is the lowest weight vector dual to v_P.

The parabolic subgroup P is self-associate. More precisely $w_0^{-1} P w_0 = P^{\mathrm{op}}$ for

$$w_0 = \left(\begin{smallmatrix} & -I_n \\ I_n & \end{smallmatrix}\right). \tag{3.81}$$

Corollary 3.6.3. *For $f \in \mathcal{S}(X_P(F))$ we have that $\mathcal{F}_{X_P}(f)$ is*

$$[2]_!(\mu_{n-2\lfloor n/2 \rfloor}) \circ [2]_!(\mu_{n-2\lfloor n/2\rfloor+2}) \cdots \circ [2]_!(\mu_{n-2}) \circ \int_{X_P^\circ(F)} f(x)\psi((-1)^n\langle \mathrm{Pl}_{v_P}(x), \mathrm{Pl}_{v_P}(\cdot)\rangle)dx.$$

Here $[2]_!(\mu_s)$ is defined as in (3.9) but with P replaced with P^{op}. See Lemma 3.4.7.

Proof. Since $v^*_{P^{\mathrm{op}}} w_0^{-1} = (-1)^n v_P$, we have

$$\langle x, w_0^{-1} x' \rangle_{P|P^{\mathrm{op}}} = (-1)^n \langle \mathrm{Pl}_{v_P}(x), \mathrm{Pl}_{v_P}(x')\rangle.$$

Thus the result follows from Corollary 3.5.6 and the computation in §3.2.3. □

Corollary 3.6.4. *When $n = 3$, one has*

$$\mathcal{F}_{X_P}(f)(x') = \int_{F^\times} \psi(t^{-1})|t|^2 \left(\int_{X_P^\circ(F)} f(x)\overline{\psi}(t^2 \langle \mathrm{Pl}(x), \mathrm{Pl}(x')\rangle)dx \right) \frac{d^\times t}{\zeta(1)} \tag{3.82}$$

for all $f \in \mathcal{S}(X_P(F))$. In particular, the integral over t is absolutely convergent.

Proof. Only the last claim is not clear from Corollary 3.6.3. By Lemma 3.2.7 the regularized operator $[2]_!(\mu_1)$ is equal to the unregularized operator in the case at hand as

$$A(L(1)) = \tfrac{1}{2}, \quad B(L(1)) = 2, \quad \tfrac{s_1+1}{\lambda_1} = 1.$$

This implies that the integral over t converges absolutely. □

Chapter 4. Characterization of Schwartz spaces

Braverman and Kazhdan's definition of $\mathcal{S}_{\mathrm{BK}}(X_P(F))$ is beautifully succinct. However, it is difficult to extract analytic information about elements of the Schwartz space from the definition. The definition of $\mathcal{S}(X_P(F))$ is more involved, but it seems to be the correct definition. For example, in the nonarchimedean case we certainly want the image of $\mathcal{S}(X_P(F))$ under various Mellin transforms to consist exactly of good sections, and we have defined $\mathcal{S}(X_P(F))$ so that this is true. Moreover, analytic information is relatively straightforward to extract from the definition of $\mathcal{S}(X_P(F))$. However, it is not clear at all what the relations are between two spaces $\mathcal{S}_{\mathrm{BK}}(X_P(F))$ and $\mathcal{S}(X_P(F))$. In particular, it is not clear if we have the inclusion

$$C_c^\infty(X_P^\circ(F)) = \mathcal{S}(X_P^\circ(F)) < \mathcal{S}(X_P(F)) \tag{4.1}$$

which is crucial for applications. It has been justified in few cases that (4.1) holds by ad-hoc approaches. In Chapter 5, we will prove the inclusion holds when F is nonarchimedean and G is not of type E or F. Indeed, we will show in these cases $\mathcal{S}_{\mathrm{BK}}(X_P(F)) = \mathcal{S}(X_P(F))$ (see Corollary 5.1.8).

Throughout this chapter, we will assume F is nonarchimedean and the inclusion (4.1) holds. The goal of this chapter is to give an alternative representation theoretic description of $\mathcal{S}(X_P(F))$, which should find applications in study boundary terms of the Poisson summation conjecture on X_P (see §4.3.2). Since functions in $\mathcal{S}(X_P(F))$ have compact support in $X_P(F)$, under our assumption, to study $\mathcal{S}(X_P(F))$ amounts to study asymptotics of functions toward the origin. Therefore, we will rewrite the definition of $\mathcal{S}(X_P(F))$ in terms of asymptotics condition.

By the definition of $\mathcal{S}(X_P(F))$ and Lemma 3.4.1, a smooth function $f \in C^\infty(X_P^\circ(F))$ lies in $\mathcal{S}(X_P(F))$ if and only if it satisfies the following two conditions.

Condition 4.0.1. (1) The function f is right K-finite.

(2) For each $g \in G(F)$ and $\chi \in \widehat{\mathcal{O}^\times}$, the integral defining $f_{\chi_s}(g)$ is absolutely convergent for $\mathrm{Re}(s)$ sufficiently large, and

(3) the section

$$\frac{f_{\chi_s}}{\prod_{i=1}^{k} L(s_i+1,\chi_s^{\lambda_i})}$$

is holomorphic.

Condition 4.0.2. The section

$$\frac{(\mathcal{R}_{P|P^{\mathrm{op}}}(f))^{\mathrm{op}}_{\chi_s}}{\prod_{i=1}^{k} L(-s_i,\chi_s^{\lambda_i})}$$

is holomorphic for each $\chi \in \widehat{\mathcal{O}^{\times}}$.

By the proof of Lemma 3.3.7 and Corollary 3.3.8 if f satisfies Condition 4.0.1, then $f \in (L^1 \cap L^2)(X_P(F))$ and has compact support in $X_P(F)$, i.e., $f(x) = 0$ for $|x|$ sufficiently large (depending on f).

We will reformulate Conditions 4.0.1 and 4.0.2 in terms of asymptotics of $f(x)$ as $|x| \to 0$ in §4.1. Then restate these asymptotic conditions in terms of representations of $G(F)$ in §4.2. The main result of this chapter is Theorem 4.2.2 that gives a detailed description of the $G(F)$-module $\mathcal{S}(X_P(F))/\mathcal{S}(X_P^{\circ}(F))$. Finally, we apply Theorem 4.2 in §4.3 to examples considered in §3.6.

4.1 Asymptotics toward the origin

We start by introducing our convention on multisets. A multiset L consists of a set U and a function $m(L) : U \to \mathbb{Z}_{>0}$. Equivalently, we can understand L as the set

$$\{(r, m_r) : r \in U\}$$

where $m_r := m_r(L) \in \mathbb{Z}_{>0}$ is the multiplicity of r in L. We will write $\mathrm{Supp}(L) := U$ as the underlying set of L. Given two multisets L_1, L_2, their sum $L_1 + L_2$ is the multiset such that

$$m_r(L_1 + L_2) = m_r(L_1) + m_r(L_2),$$

where the domain of $m(L_i)$ is extended to $\mathrm{Supp}(L_1) \cup \mathrm{Supp}(L_2)$ by setting $m_r(L_i) = 0$ if $r \notin \mathrm{Supp}(L_i)$.

Recall that we have fixed a good ordering of $\{(\lambda_i, s_i)\}_{1\leqslant i\leqslant k}$ and thus $\lambda_k = 1$. For each $d \in \mathbb{Z}_{>0}$, define multisets, whose underlying sets are subsets of $\mathbb{C}/\frac{2\pi\sqrt{-1}}{\log q}\mathbb{Z}$,

$$\begin{aligned} L(d) &:= \sum_{i:\lambda_i=d} \left\{ \tfrac{s_i+1}{\lambda_i} - (s_k+1) + \tfrac{\mu}{\lambda_i}\tfrac{2\pi\sqrt{-1}}{\log q}\mathbb{Z} : 0 \leqslant \mu < \lambda_i \right\}, \\ L_d &:= \sum_{d|\lambda} L(\lambda), \end{aligned} \tag{4.2}$$

where sums are taken in the sense of multisets. From Proposition 3.5.1 we deduce

Lemma 4.1.1. *For* $r \in \mathrm{Supp}(L_1)$, $0 \geqslant \mathrm{Re}(r) > -(s_k+1)$. *Moreover,* $\mathrm{Re}(r) = 0$ *if and only if* $r = 0$, *and* $m_0(L_1) = 1$. □

Let $m : \mathbb{G}_m \to M$ be a section of ω_P and $X_P^1 := \{x \in X_P(F) : |x| = 1\}$.

Proposition 4.1.2. *Let* $f \in C^\infty(X_P^\circ(F))$. *Then* f *satisfies Condition 4.0.1 if and only if*

(1) f *has compact support in* $X_P(F)$, *and*

(2) *for* $n \gg_f 0$ *and* $x \in X_P^1$, *we have*

$$f(m(\varpi^n)^{-1}x) = \sum_{d=1}^{\infty} \sum_{\substack{\chi\in\widehat{\mathcal{O}^\times} \\ \mathrm{ord}(\chi)=d}} \sum_{r\in\mathrm{Supp}(L_d)} \sum_{j=1}^{m_r(L_d)} c_{r,j,\chi}(x) q^{-nr} n^{j-1}, \tag{4.3}$$

for some $c_{r,j,\chi} \in C^\infty(X_P^1)$ *satisfying* $c_{r,j,\chi}(m(a)^{-1}x) = \chi(a)c_{r,j,\chi}(x)$ *for* $a \in \mathcal{O}^\times$.

Remark 4.1.3. Since $\{(\lambda_i, s_i)\}$ is a finite multiset, the sum in (4.3) is actually finite. Indeed, $L(d)$ and L_d are empty for $d > 6$ (see charts in Appendix B).

The key ingredient to establish this equivalence is the work of Igusa in [Igu78, §1.5], which we now review. Let L be a finite multiset whose underlying set is a subset of $\mathbb{C}/\frac{2\pi\sqrt{-1}}{\log q}\mathbb{Z}$. We define $C_L^\infty(F^\times)$ to be the subspace of $C^\infty(F^\times)$ that consists of functions f satisfying

(1) f has compact support in F, and

(2) for $|a| \ll_f 1$, we have

$$f(a) = \sum_{r\in\mathrm{Supp}(L)} \sum_{j=1}^{m_r(L)} c_{r,j}(\tilde{a})|a|^r \mathrm{ord}(a)^{j-1}$$

for some $c_{r,j} \in C^\infty(\mathcal{O}^\times)$.

Note that we can further write

$$c_{r,j} = \sum_{\chi \in \widehat{\mathcal{O}^\times}} c_{r,j,\chi}\chi$$

for some constants $c_{r,j,\chi} \in \mathbb{C}$ that are zero for all but finitely many χ. On the other hand, let $\mathcal{Z}_L(\widehat{F^\times})$ be the set of complex-valued functions Z on $\widehat{F^\times}$ for which

(1) for every $\chi \in \widehat{\mathcal{O}^\times}$, there exist constants $b_{r,j,\chi} \in \mathbb{C}$ such that

$$Z(\chi_s) - \sum_{r \in \mathrm{Supp}(L)} \sum_{j=1}^{m_r(L)} b_{r,j,\chi}\zeta(s+r)^j$$

is a function in $\mathbb{C}[q^{-s}, q^s]$, and

(2) for all but finitely many $\chi \in \widehat{\mathcal{O}^\times}$, $Z(\chi_s) = 0$ for all s.

Igusa showed in [Igu78, Theorem 1.5.3] the following:

Theorem 4.1.4. *For* $f \in C_L^\infty(F^\times)$, *the Mellin transform*

$$M : f \mapsto \left(\chi \mapsto \int_{F^\times} f(a)\chi(a)d^\times a\right),$$

originally defined on $\chi \in \widehat{F^\times}$ *with* $\mathrm{Re}(\chi) \gg_L 0$, *extends analytically to whole* $\widehat{F^\times}$ *and gives rise to an isomorphism between* $C_L^\infty(F^\times)$ *and* $\mathcal{Z}_L(\widehat{F^\times})$. *Moreover, the Mellin inversion* M^{-1} *is given as follows. Given* $Z \in \mathcal{Z}_L(\widehat{F^\times})$, *for each* $\chi \in \widehat{\mathcal{O}^\times}$, *let* $Z_\chi(z)$ *be the complex function obtained from* $Z(\chi_s)$ *by substituting* z *for* q^{-s}. *Then for* $a \in F^\times$,

$$M^{-1}(Z)(a) = \frac{1}{d^\times x(\mathcal{O}^\times)} \sum_{\chi \in \widehat{\mathcal{O}^\times}} \mathrm{Res}_{z=0}(Z_\chi(z)z^{-\mathrm{ord}(a)-1})\chi^{-1}(a). \tag{4.4}$$

□

Proof of Proposition 4.1.2. Clearly, both conditions imply f is right K-finite and has compact support in $X_P(F)$, which we assume henceforth. Let $K' \leqslant K$ be a compact open subgroup such that f is right K'-invariant. Let $\{g_i\}$ be a set of representatives of left cosets of K' in

K. By Iwasawa decomposition, (2)(3) in Condition 4.0.1 are satisfied if and only if they are satisfied for all g_i. On the other hand, since the set $\{g_i\}$ is finite, asymptotics (4.3) can be checked at each g_i separately.

Therefore, it suffices to fix $g \in G(F)$ and study the smooth function

$$f_g : F^\times \longrightarrow \mathbb{C}$$

$$a \mapsto f(m(a)^{-1}g),$$

Note that f_g has compact support in F and is $\mathcal{O}^\times$-finite by our assumption. Moreover, for $\chi \in \widehat{\mathcal{O}^\times}$ by Proposition 3.5.2 we can write

$$f_{\chi_s}(g) = M(f_g)(\chi_{s+s_k+1}).$$

Then Condition 4.0.1 is reduced to

- $(2')$ The integral defining $M(f_g)(\chi_s)$ is absolutely convergent for $\mathrm{Re}(s)$ sufficiently large, and
- $(3')$ the complex function

  $$\frac{M(f_g)(\chi_s)}{\prod_{i=1}^k L(s_i + 1 - \lambda_i(s_k+1), \chi_s^{\lambda_i})}$$

  is a function in $\mathbb{C}[q^{-s}, q^s]$.

To prove the proposition, we need to show f_g satisfies conditions $(2')$ and $(3')$ if and only if there exist constants $c_{r,j,\chi} \in \mathbb{C}$ such that for $|a| \ll_{f_g} 1$ we have

$$f_g(a) = \sum_{d=1}^{\infty} \sum_{\substack{\chi \in \widehat{\mathcal{O}^\times} \\ \mathrm{ord}(\chi)=d}} \sum_{r \in \mathrm{Supp}(L_d)} \sum_{j=1}^{m_r(L_d)} c_{r,j,\chi}\chi(a)|a|^r \mathrm{ord}(a)^{j-1}.$$

Observe that for $\chi \in \widehat{\mathcal{O}^\times}$ of order d, we have

$$\prod_{i=1}^k L(s_i + 1 - \lambda_i(s_k+1), \chi_s^{\lambda_i}) = \prod_{r \in \mathrm{Supp}(L_d)} \zeta(s+r)^{m_r(L_d)}.$$

Therefore, the set

$$\{M(f_g) : f_g \text{ satisfying conditions } (2') \text{ and } (3')\}$$

is the subset of functions in $\mathcal{Z}_{L_1}(\widehat{F^\times})$ such that the associated constants $b_{r,j,\chi} = 0$ unless there is some $d \in \mathbb{Z}_{>0}$ such that $r \in \mathrm{Supp}(L_d)$, $j \leqslant m_r(L_d)$, and $\mathrm{ord}(\chi) = d$. By Theorem 4.1.4 and (4.4), this is equivalent to saying the associated constants $c_{r,j,\chi^{-1}}$ of $f_g \in C^\infty_{L_1}(F^\times)$ are zero unless there is some $d \in \mathbb{Z}_{>0}$ such that $r \in \mathrm{Supp}(L_d)$, $j \leqslant m_r(L_d)$, and $\mathrm{ord}(\chi) = d$. Thus the assertion follows as $\mathrm{ord}(\chi) = \mathrm{ord}(\chi^{-1})$. □

For $j \in \mathbb{Z}_{>0}$, let $h_j \in \mathbb{Z}[z]$ be the monic polynomial of degree $j-1$ such that

$$\frac{h_j(z)}{(1-z)^j} = \sum_{i=0}^{\infty} i^{j-1} z^i.$$

Proposition 4.1.5. *Let $f \in C^\infty(X_P^\circ(F))$ satisfy equivalent conditions in Proposition 4.1.2. Then $f \in \mathcal{S}(X_P(F))$ if and only if for every $d \in \mathbb{Z}_{>0}$, $r \in \mathrm{Supp}(L_d), \chi \in \widehat{\mathcal{O}^\times}$ of order d, and $x^* \in X_{P^{\mathrm{op}}}^\circ(F)$, the meromorphic function*

$$\sum_{j=1}^{m_r(L_d)} h_j(q^{-s})\zeta(s)^{j-m_r(L_d)} \int_{F^\times} |t|^{s-r-2s_k-2} \int_{X_P^1} c_{r,j,\chi}(x)\psi(\varpi^{-\mathrm{ord}(t)}\langle x, x^*\rangle) dx d^\times t \qquad (4.5)$$

has zeros at $s = 0$ of order at least $m_r(L_d)$.

To prove the proposition, we first give an alternative definition of $\mathcal{S}(X_P(F))$ in terms of $\mathcal{F}^{\mathrm{geo}}_{P|P^{\mathrm{op}}}$ instead of $\mathcal{R}_{P|P^{\mathrm{op}}}$.

Lemma 4.1.6. *Suppose $f \in C^\infty(X_P^\circ(F))$ satisfies Condition 4.0.1. Then Condition 4.0.2 is equivalent to*

1. *For each $\chi \in \widehat{\mathcal{O}^\times}$ and $g \in G(F)$, the integral defining $(\mathcal{F}^{\mathrm{geo}}_{P|P^{\mathrm{op}}}(f))^{\mathrm{op}}_{\chi_s}(g)$ is absolutely convergent for $\frac{s_{k-1}}{\lambda_{k-1}} < \mathrm{Re}(s) < s_k + 1$, and*
2. *the section*

$$\frac{(\mathcal{F}^{\mathrm{geo}}_{P|P^{\mathrm{op}}}(f))^{\mathrm{op}}_{\chi_s}}{L(s_k+1, \chi_s^{-1}) \prod_{i=1}^{k-1} L(-s_i, \chi_s^{\lambda_i})}$$

is holomorphic.

Proof. Recall by definition $\mathcal{F}^{\mathrm{geo}}_{P|P^{\mathrm{op}}} = 1_!(\mu_{s_k}) \circ \mathcal{R}_{P|P^{\mathrm{op}}}$. The sufficiency follows from the work in Chapter 3. For the converse, in view of Proposition 3.2.8 and its proof, it suffices to check the integral defining $(\mathcal{R}_{P|P^{\mathrm{op}}}(f))^{\mathrm{op}}_{\chi_s}$ is holomorphic for $\mathrm{Re}(s) > s_k + 1 - \epsilon$ for some $\epsilon > 0$ for all $\chi \in \widehat{\mathcal{O}^\times}$. Since $f \in L^1(X_P(F))$, by Fubini-Tonelli theorem one may reverse the proof of Theorem 3.5.5 to see the integral defining $(\mathcal{R}_{P|P^{\mathrm{op}}}(f))^{\mathrm{op}}_{\chi_s}$ is absolutely convergent for $\mathrm{Re}(s) = s_k + 1$. Since it is also absolutely convergent for $\mathrm{Re}(s) \gg 0$, it is absolutely convergent for $\mathrm{Re}(s) \geqslant s_k + 1$. As $(\mathcal{R}_{P|P^{\mathrm{op}}}(f))^{\mathrm{op}}_{\chi_s}$ is meromorphic and vanishes for all but finitely many χ, there exists $\epsilon > 0$ such that it is holomorphic for $\mathrm{Re}(s) > s_k + 1 - \epsilon$ for all χ. $\square$

Proof of Proposition 4.1.5. Since $\mathcal{S}(X_P^\circ(F)) < \mathcal{S}(X_P(F))$, we may assume

$$f(m(\varpi^n)^{-1}x) = \sum_{d=1}^{\infty} \sum_{\substack{\chi \in \widehat{\mathcal{O}^\times} \\ \mathrm{ord}(\chi)=d}} \sum_{r \in \mathrm{Supp}(L_d)} \sum_{j=1}^{m_r(L_d)} c_{r,j,\chi}(x) q^{-nr} n^{j-1}$$

for $n \geqslant 0$ and vanishes for $n < 0$ for all $x \in X_P^1$. We fix (d, χ) such that $\mathrm{ord}(\chi) = d$ and further assume for $n \geqslant 0$ and $x \in X_P^1$

$$f(m(\varpi^n)^{-1}x) = \sum_{r \in \mathrm{Supp}(L_d)} \sum_{j=1}^{m_r(L_d)} c_{r,j,\chi}(x) q^{-nr} n^{j-1}.$$

Let $x^* = P^{\mathrm{op},\mathrm{der}}(F)g \in X^\circ_{P^{\mathrm{op}}}(F)$. Viewing $c_{r,j,\chi}$ as a function in $\mathcal{S}(X_P^\circ(F))$ supported on X_P^1, we have by Theorem 3.5.5

$$\mathcal{F}^{\mathrm{geo}}_{P|P^{\mathrm{op}}}(c_{r,j,\chi})(x^*) = \int_{X_P^1} c_{r,j,\chi}(x)\psi(\langle x, x^* \rangle)dx.$$

By Lemma 4.1.1, $\mathrm{Re}(r + s_k + 1) > 0$ for all $r \in \mathrm{Supp}(L_d)$. Therefore, taking a change of variables $t \mapsto at$, the integral

$$\int_{F^\times} \int_{|a| \leqslant 1} |t|^{s-s_k-1} |a|^{r+2s_k+2} \mathrm{ord}(a)^{j-1} \left| \mathcal{F}^{\mathrm{geo}}_{P|P^{\mathrm{op}}}(c_{r,j,\chi})(m(a^{-1}t)^{-1}x^*) \right| d^\times a d^\times t$$

$$= \int_{|a| \leqslant 1} |a|^{s+r+s_k+1} \mathrm{ord}(a)^{j-1} \int_{F^\times} |t|^{s-s_k-1} \left| \mathcal{F}^{\mathrm{geo}}_{P|P^{\mathrm{op}}}(c_{r,j,\chi})(m(t)^{-1}x^*) \right| d^\times t d^\times a$$

converges for $\frac{s_{k-1}}{\lambda_{k-1}} < \mathrm{Re}(s) < s_k + 1$ by Lemma 4.1.6.

Thus by Fubini-Tonelli theorem and Iwasawa decomposition, for $\eta \in \widehat{\mathcal{O}^\times}$

$$(\mathcal{F}^{\mathrm{geo}}_{P|P^{\mathrm{op}}}(f))^{\mathrm{op}}_{\eta_s}(g)$$

$$= \sum_{r \in \mathrm{Supp}(L_d)} \sum_{j=1}^{m_r(L_d)} \int_{F^\times} \eta(t)|t|^{s-s_k-1}$$

$$\times \int_{|a| \leqslant 1} \chi(a)|a|^{r+2s_k+2} \mathrm{ord}(a)^{j-1} \mathcal{F}^{\mathrm{geo}}_{P|P^{\mathrm{op}}}(c_{r,j,\chi})(m(a^{-1}t)^{-1}x^*) d^\times a d^\times t$$

$$= \sum_{r \in \mathrm{Supp}(L_d)} \sum_{j=1}^{m_r(L_d)} \int_{|a| \leqslant 1} (\eta\chi)(a)|a|^{s+r+s_k+1} \mathrm{ord}(a)^{j-1}$$

$$\times \int_{F^\times} \eta(t)|t|^{s-s_k-1} \mathcal{F}^{\mathrm{geo}}_{P|P^{\mathrm{op}}}(c_{r,j,\chi})(m(t)^{-1}x^*) d^\times t d^\times a,$$

which is zero unless $\eta\chi = 1$; if $\eta = \overline{\chi}$, we have

$$(d^\times a(\mathcal{O}^\times))^{-1} (\mathcal{F}^{\mathrm{geo}}_{P|P^{\mathrm{op}}}(f))^{\mathrm{op}}_{\overline{\chi}_s}(g)$$

$$= \sum_{r \in \mathrm{Supp}(L_d)} \sum_{j=1}^{m_r(L_d)} h_j(q^{-s-r-s_k-1}) \zeta(s+r+s_k+1)^j (\mathcal{F}^{\mathrm{geo}}_{P|P^{\mathrm{op}}}(c_{r,j,\chi}))^{\mathrm{op}}_{\overline{\chi}_s}(g).$$

By Lemmas 4.1.1 and 4.1.6,

$$\frac{(\mathcal{F}^{\mathrm{geo}}_{P|P^{\mathrm{op}}}(f))^{\mathrm{op}}_{\overline{\chi}_s}(g)}{L(s_k+1, \overline{\chi}_s^{-1}) \prod_{i=1}^{k-1} L(-s_i, \overline{\chi}_s^{\lambda_i})}$$

lies in $\mathbb{C}[q^{-s}, q^s]$ if and only if

$$\sum_{r \in \mathrm{Supp}(L_d)} \sum_{j=1}^{m_r(L_d)} h_j(q^{-s-r-s_k-1}) \zeta(s+r+s_k+1)^j$$

$$\times \int_{F^\times} |t|^{s-s_k-1} \int_{X_P^1} c_{r,j,\chi}(x) \psi(\varpi^{-\mathrm{ord}(t)} \langle x, x^* \rangle) dx d^\times t$$

has no poles at $s = -r - s_k - 1$ for all $r \in \operatorname{Supp}(L_d)$, which is equivalent to

$$\sum_{j=1}^{m_r(L_d)} h_j(q^{-s-r-s_k-1})\zeta(s+r+s_k+1)^{j-m_r(L_d)}$$

$$\times \int_{F^\times} |t|^{s-s_k-1} \int_{X_P^1} c_{r,j,\chi}(x)\psi(\varpi^{-\operatorname{ord}(t)}\langle x, x^*\rangle) dx d^\times t$$

has zeros at $s = -r - s_k - 1$ of order at least $m_r(L_d)$ for every $r \in \operatorname{Supp}(L_d)$. Observe that conditions among distinct (d, χ) are independent. The assertion then follows from linearity and Lemma 4.1.6. □

4.2 An explicit characterization

Fix $d \in \mathbb{Z}_{>0}$ with $L_d \neq \varnothing$, $r \in \operatorname{Supp}(L_d)$ and $\chi \in \widehat{\mathcal{O}^\times}$ of order d. For $n \in \mathbb{Z}_{\geqslant 0}$, consider smooth $G(F)$-modules $I_{r,n,\chi}$ defined by the quotient of

$$\left\{ f \in C^\infty(X_P^\circ(F)) : \begin{array}{c} f(x) = 0 \text{ for } |x| \gg_f 1, \\ f(x) = \sum_{j=1}^n |x|^r \log_{1/q}^{j-1}(|x|) c_{r,j,\chi}(m(\varpi^{\log_q |x|})^{-1}x) \text{ for } |x| \ll_f 1 \end{array} \right\}$$

by $\mathcal{S}(X_P^\circ(F))$. For ease of notation, we will write interchangeably f as the vector $(c_{r,j,\chi})_{1 \leqslant j \leqslant n}$.

Lemma 4.2.1. *For $n \in \mathbb{Z}_{>0}$, we have canonical $G(F)$-equivariant isomorphisms*

$$\varphi_{r,n,\chi} : I_{r,n,\chi}/I_{r,n-1,\chi} \xrightarrow{\sim} I_P(\overline{\chi}_{-r-s_k-1}).$$

by sending f to the unique function in $I_P(\overline{\chi}_{-r-s_k-1})$ that equals $c_{r,n,\chi}$ on X_P^1.

Proof. The only statement that is unclear is the $G(F)$-equivariance. Given $f = (c_{r,j,\chi})_{1 \leqslant j \leqslant n} \in I_{r,n,\chi}$ and $g \in G(F)$, we have

$$\begin{aligned} R(g)f(x) &:= f(xg) \\ &= \sum_{j=1}^n |x|^r \left(\log_{1/q}(|x|) + \log_{1/q}\left(\frac{|xg|}{|x|}\right)\right)^{j-1} \left(\frac{|xg|}{|x|}\right)^r c_{r,j,\chi}(m(\varpi^{\log_q |xg|})^{-1}xg) \\ &= \sum_{j=1}^n |x|^r \log_{1/q}^{j-1}(|x|) \sum_{i=j}^n \binom{i-1}{j-1} \log_{1/q}^{i-j}\left(\frac{|xg|}{|x|}\right) \left(\frac{|xg|}{|x|}\right)^r c_{r,i,\chi}(m(\varpi^{\log_q |xg|})^{-1}xg) \end{aligned}$$

for $|x|$ sufficiently small. In other words, $R(g)f = (\tilde{c}_{r,j,\chi})_{1\leqslant j\leqslant n}$, where

$$\tilde{c}_{r,j,\chi}(x) := \sum_{i=j}^{n} \binom{i-1}{j-1} \log_{1/q}^{i-j}\left(\frac{|xg|}{|x|}\right)\left(\frac{|xg|}{|x|}\right)^{r} c_{r,i,\chi}(m(\varpi^{\log_q |xg|})^{-1}xg). \tag{4.6}$$

Therefore for $x \in X_P^{\circ}(F)$,

$$\varphi_{r,n,\chi}(R(g)f)(x) = |xg|^{r} c_{r,n,\chi}(m(\varpi^{\log_q |xg|})^{-1}xg) = R(g)\varphi_{r,n,\chi}(f)(x).$$

□

For $n \leqslant m_r(L_d)$, define $G(F)$-submodules

$$\begin{aligned} A_{r,n,\chi} &:= \left\{f \in I_{r,n,\chi} : (c_{r,j,\chi})_{1\leqslant j\leqslant n} \text{ satisfies } \mathrm{ord}_{s=0}(4.5) \geqslant m_r(L_d)\right\}, \\ A_{r,\chi} &:= A_{r,m_r(L_d),\chi}. \end{aligned} \tag{4.7}$$

Now we are ready to state our main result.

Theorem 4.2.2. *We have an exact sequence of smooth $G(F)$-modules*

$$0 \longrightarrow \mathcal{S}(X_P^{\circ}(F)) \longrightarrow \mathcal{S}(X_P(F)) \longrightarrow \bigoplus_{d\geqslant 1, L_d\neq\varnothing} \bigoplus_{\substack{\chi\in\widehat{\mathcal{O}^{\times}} \\ \mathrm{ord}(\chi)=d}} \bigoplus_{r\in\mathrm{Supp}(L_d)} A_{r,\chi} \longrightarrow 0.$$

Moreover, $A_{r,\chi}$ admits a natural filtration of $G(F)$-submodules

$$0 = A_{r,0,\chi} < A_{r,1,\chi} < \cdots < A_{r,m_r(L_d),\chi} = A_{r,\chi}$$

together with canonical $G(F)$-equivariant injections

$$\begin{aligned} 0 \neq A_{r,m_r(L_d),\chi}/A_{r,m_r(L_d)-1,\chi} &\hookrightarrow \cdots \hookrightarrow A_{r,2,\chi}/A_{r,1,\chi} \\ &\hookrightarrow A_{r,1,\chi} \cong \mathrm{Ker}(\mathcal{R}_{P|P^{\mathrm{op}}}(\overline{\chi}_{-r-s_k-1})) \subsetneq I_P(\overline{\chi}_{-r-s_k-1}). \end{aligned}$$

Remark 4.2.3. Theorem 4.2.2 allows one to construct a candidate of (co)sheaf on $X_P(F)$ whose global sections with compact support are $\mathcal{S}(X_P(F))$, and hence justifies an expectation in [BK00, Ngô20].

The exact sequence in Theorem 4.2.2 follows directly from Proposition 4.1.5 and the definition of $A_{r,\chi}$. Therefore, we are left to inspect the filtration $0 = A_{r,0,\chi} \leqslant A_{r,1,\chi} \leqslant \cdots \leqslant A_{r,m_r(L_d),\chi} = A_{r,\chi}$.

Define the map

$$\begin{aligned} I_{r,1,\chi} &\xrightarrow{\Phi} I_{P^{\mathrm{op}}}(\chi_{r+s_k+1}) \\ (c_{r,1,\chi}) &\mapsto \int_{F^\times} |t|^{s-r-2s_k-2} \int_{X_P^1} c_{r,1,\chi}(x)\psi(\varpi^{-\mathrm{ord}(t)}\langle x,\cdot\rangle)dxd^\times t \Bigg|_{s=0} . \end{aligned}$$

Here the integral is viewed as a meromorphic section (see the proof of Proposition 4.1.5), and Φ is well defined by Lemma 4.1.6.

Lemma 4.2.4. *We have a commutative diagram*

$$\begin{tikzcd} I_{r,1,\chi} \arrow[d, "\varphi_{r,1,\chi}"] \arrow[dr, "c\cdot\Phi"] & \\ I_P(\overline{\chi}_{-r-s_k-1}) \arrow[r, "\mathcal{R}_{P|P^{\mathrm{op}}}"] & I_{P^{\mathrm{op}}}(\chi_{r+s_k+1}), \end{tikzcd}$$

where c is a nonzero constant. In particular, Φ is $G(F)$-equivariant and $\varphi_{r,1,\chi}$ restricts to an isomorphism

$$A_{r,1,\chi} = A_{r,\chi} \cap I_{r,1,\chi} \xrightarrow{\sim} \mathrm{Ker}\left(\mathcal{R}_{P|P^{\mathrm{op}}}(\overline{\chi}_{-r-s_k-1})\right).$$

Proof. By Proposition 3.2.8 and Lemma 3.4.1, we have an identity of meromorphic sections

$$\begin{aligned} &\int_{F^\times} \overline{\chi}(t)|t|^{s-r-2s_k-2} \int_{X_P^1} c_{r,1,\chi}(x)\psi(t^{-1}\langle x,\cdot\rangle)dxd^\times t \\ &= \left([1]!(\mu_{s_k}) \circ \mathcal{R}_{P|P^{\mathrm{op}}}\left(c_{r,1,\chi}\mathbf{1}_{X_P^1}\right)\right)^{\mathrm{op}}_{\overline{\chi}_{s-r-s_k-1}} \\ &= \gamma(-s_k, \overline{\chi}_{s-r-s_k-1}, \psi)^{-1} \mathcal{R}_{P|P^{\mathrm{op}}}\left(\left(c_{r,1,\chi}\mathbf{1}_{X_P^1}\right)_{\overline{\chi}_{s-r-s_k-1}}\right) \end{aligned}$$

where $\gamma(-s_k, \overline{\chi}_{s-r-s_k-1}, \psi)$ is the Tate γ-factor. By Lemma 4.1.1, $\gamma(-s_k, \overline{\chi}_{-r-s_k-1}, \psi)$ is a nonzero constant, and evaluating at $s = 0$ we have

$$\Phi((c_{r,1,\chi})) = \gamma(-s_k, \overline{\chi}_{-r-s_k-1}, \psi)^{-1} \mathcal{R}_{P|P^{\mathrm{op}}} \circ \varphi_{r,1,\chi}((c_{r,1,\chi})).$$

□

Lemma 4.2.5. *The map Φ is nonzero.*

Proof. We can and do assume the conductor of ψ is $\mathcal{O}$. It suffices to show there exists $(c_{r,1,\chi}) \in I_{r,1,\chi}$ such that $\Phi((c_{r,1,\chi}))(\mathrm{Id}) \neq 0$. Choose a compact open subgroup $K' \leqslant K$ sufficiently small such that $\omega_P(K' \cap P(F)) \leqslant (\mathcal{O}^\times)^d$ and $\langle v_P g, v^*_{P^{\mathrm{op}}}\rangle \in \ker(\chi) \cap \mathcal{O}^\times$ for all $g \in K'$. Let $c_{r,1,\chi}$ be the unique function supported on $P^{\mathrm{der}}(F)m(\mathcal{O}^\times)K'$ such that for $a \in \mathcal{O}^\times$ and $g \in K'$,

$$c_{r,1,\chi}(m(a)^{-1}g) = \chi(a).$$

Then we have

$$\int_{X_P^1} c_{r,1,\chi}(x)\psi(\varpi^{-\mathrm{ord}(t)}\langle x, \mathrm{Id}\rangle)dx$$

is up to a nonzero constant

$$\int_{\mathcal{O}^\times} \chi(a)\psi(\varpi^{-\mathrm{ord}(t)}a)d^\times a$$

which is the well-known Gauss sum

$$\mathfrak{G}(\varpi^{-\mathrm{ord}(t)}, \chi) = \begin{cases} 1 & \text{if } \chi = 1, |t| \geqslant 1, \\ \zeta(-1) & \text{if } \chi = 1, |t| = q^{-1}, \\ \neq 0 & \text{if } c(\chi) = \mathrm{ord}(t) > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $c(\chi)$ is the conductor of χ. Therefore, $\Phi((c_{r,1,\chi}))(\mathrm{Id})$ is up to a nonzero constant

$$\begin{cases} \zeta(r+2s_k+2) + \zeta(-1)q^{r+2s_k+2} & \text{if } \chi = 1, \\ q^{-c(\chi)(-r-2s_k-2)}\mathfrak{G}(\varpi^{-c(\chi)}, \chi) & \text{if } \chi \neq 1, \end{cases}$$

which is nonzero since $r + 2s_k + 2 > 1$. □

Proof of Theorem 4.2.2. The nontriviality statement

$$A_{r,m_r(L_d),\chi}/A_{r,m_r(L_d)-1,\chi} \neq 0$$

will be proved in Corollary 5.1.8. By Lemmas 4.2.4 and 4.2.5, it remains to show for $m_r(L_d) \geqslant n \geqslant 2$ there is a (canonical) $G(F)$-equivariant injection

$$A_{r,n,\chi}/A_{r,n-1,\chi} \hookrightarrow A_{r,n-1,\chi}/A_{r,n-2,\chi}.$$

Suppose $(c_{r,j,\chi})_{1\leqslant j\leqslant n}\in A_{r,n,\chi}$, i.e.,

$$\sum_{j=1}^{n} h_j(q^{-s})\zeta(s)^{j-n}\int_{F^\times}|t|^{s-r-2s_k-2}\int_{X_P^1} c_{r,j,\chi}(x)\psi(\varpi^{-\mathrm{ord}(t)}\langle x,x^*\rangle)dxd^\times t$$

has at least order n at $s=0$ for all $x^*\in X^\circ_{P^{\mathrm{op}}}(F)$. Therefore,

$$\sum_{j=2}^{n} h_j(q^{-s})\zeta(s)^{j-n}\int_{F^\times}|t|^{s-r-2s_k-2}\int_{X_P^1} c_{r,j,\chi}(x)\psi(\varpi^{-\mathrm{ord}(t)}\langle x,x^*\rangle)dxd^\times t \tag{4.8}$$

has at least order $n-1$ at $s=0$ for all $x^*\in X^\circ_{P^{\mathrm{op}}}(F)$. Note that for $j\geqslant 2$, we have the identity

$$h_j(z)=z(1-z)\frac{dh_{j-1}}{dz}(z)+(j-1)zh_{j-1}(z). \tag{4.9}$$

Thus, (4.8) can be written as

$$\sum_{j=1}^{n-1}\left(1+\frac{h_j'(q^{-s})\zeta(s)^{-1}}{jh_j(q^{-s})}\right)q^{-s}h_j(q^{-s})\zeta(s)^{j-(n-1)}$$

$$\times\int_{F^\times}|t|^{s-r-2s_k-2}\int_{X_P^1} jc_{r,j+1,\chi}(x)\psi(\varpi^{-\mathrm{ord}(t)}\langle x,x^*\rangle)dxd^\times t$$

As h_j has positive coefficients (which follows from (4.9) and induction),

$$\frac{h_j'(q^{-s})\zeta(s)^{-1}}{jh_j(q^{-s})}\in\mathbb{C}(q^{-s}) \tag{4.10}$$

has a simple zero at $s=0$. Therefore, by considering the expansion of the function (4.10) in $\zeta(s)^{-1}$ at $s=0$, we can find inductively (unique) constants $b_{mj}\in\mathbb{C}$ such that

$$\begin{aligned} c'_{r,n-1,\chi} &:= (n-1)c_{r,n,\chi} \quad \text{and} \\ c'_{r,m,\chi} &:= mc_{r,m+1,\chi}+\sum_{j=m+1}^{n-1} jb_{mj}c_{r,j+1,\chi} \quad \text{for } 1\leqslant m\leqslant n-2 \end{aligned} \tag{4.11}$$

satisfies $(c'_{r,m,\chi})_{1\leqslant m\leqslant n-1}\in A_{r,n-1,\chi}$.

Therefore, we have a linear map

$$\Psi: A_{r,n,\chi}\longrightarrow A_{r,n-1,\chi}$$

$$(c_{r,j,\chi})_{1\leqslant j\leqslant n}\mapsto (c'_{r,j,\chi})_{1\leqslant j\leqslant n-1}.$$

that induces an injection $A_{r,n,\chi}/A_{r,n-1,\chi} \hookrightarrow A_{r,n-1,\chi}/A_{r,n-2,\chi}$. We claim the map is $G(F)$-equivariant. Indeed, by (4.6) and (4.11) we have for $g \in G(F)$,

$$\Psi(R(g)(c_{r,j,\chi})_{1\leqslant j\leqslant n}) - R(g)\Psi((c_{r,j,\chi})_{1\leqslant j\leqslant n}) \in A_{r,n-2,\chi}.$$

Thus the theorem follows. □

Let us study in more details what representations should occur in $\mathcal{S}(X_P(F))/\mathcal{S}(X_P^\circ(F))$. Define

$$R(G,P) := \left\{(s,\chi) \in \mathbb{C}/\tfrac{2\pi\sqrt{-1}}{\log q}\mathbb{Z} \times \widehat{\mathcal{O}^\times} : I(\chi_s) \text{ is reducible and } \mathrm{Re}(s) < 0\right\}.$$

Let $P_\ell = M_\ell N_\ell < G$ denote the maximal parabolic associated to the ℓ-th node of the Dynkin diagram of G, using the Bourbaki numbering.

Theorem 4.2.6. *Suppose* $\mathrm{char}\, F = 0$ *and* $\ell < n-1$ *if* G *is of type* D_n *with* $n \geqslant 5$. *Let* $(s,\chi) \in \mathbb{C}/\frac{2\pi\sqrt{-1}}{\log q}\mathbb{Z} \times \widehat{\mathcal{O}^\times}$ *with* $\mathrm{Re}(s) < 0$ *and* $d := \mathrm{ord}(\chi)$. *Then*

$$(s,\chi) \in R(G,P) \iff L_d \neq \varnothing \text{ and } -(s + s_k + 1) \in \mathrm{Supp}(L_d).$$

Consequently,

$$\bigoplus_{d\geqslant 1, L_d\neq\varnothing} \bigoplus_{\substack{\chi\in\widehat{\mathcal{O}^\times}\\ \mathrm{ord}(\chi)=d}} \bigoplus_{r\in\mathrm{Supp}(L_d)} A_{r,1,\chi} = \bigoplus_{(s,\chi)\in R(G,P)} \ker(\mathcal{R}_{P|P^{\mathrm{op}}}(\chi_s)). \tag{4.12}$$

Furthermore, if G is of type A, then $A_{r,1,1}$ is irreducible for all $r \in L_1$. In general if G is a classical group, $A_{r,1,\chi}$ has length at most 3 for all χ (of order at most 2) and $r \in \mathrm{Supp}(L_{\mathrm{ord}(\chi)})$.

Our proof relies on general results on degenerate principal series. It would be more satisfying if one can prove (4.12) directly without resorting to representation theory.

Proof. **Type A:** We can extend the natural $\mathrm{SL}_n(F)$-action on $\mathcal{S}(X_P(F))$ to an action of $\mathrm{GL}_n(F)$ by stipulating that the similitude acts trivially. Then induced representations in this case is of the form $\chi \times 1$ in [Gou17] and thus the assertion follows from [Gou17, Theorem 1.1(i) and Lemma 4.2].

Type B_n: Let $\varphi : G = \mathrm{Spin}_{2n+1} \to \mathrm{SO}_{2n+1}$ be the degree 2 isogeny. Note that $\varphi(P_\ell)$ is a parabolic subgroup of SO_{2n+1} corresponding to the same node. Consider first $\ell < n$. In this case the fundamental weight corresponding to the ℓth node lies inside the root lattice of SO_{2n+1}, i.e., the $G(F)$-representation $V_P(F)$ descends to a representation of $\mathrm{SO}_{2n+1}(F)$. Therefore, we can assume G to be SO_{2n+1}. The assertion then follows from [Jan96, Theorem 4.1 and 7.1].

Suppose $\ell = n$. Since φ is of degree 2, the assertion follows from [Jan96, Proposition 3.6] if $\chi_{\mathrm{Im}(s)}$ is of odd order. Note that the points of reducibility one obtained in loc. cit. are a half of the claimed values; the factor 2 comes from the fact that the twice of the fundamental weight attached to P_n lies in the root lattice of SO_{2n+1}. Therefore, we remain to show $I_{P_n}(\chi_s)$ is irreducible if $\chi_{\mathrm{Im}(s)}$ is of even order. As explained in the case of type A, we can extend the action of G to GSpin_{2n+1}. Then in this case following the notation in [KLM20], we are to show $\zeta([\nu^{-b}\rho_0, \nu^{-a}\rho_0]) \rtimes \sigma$ is irreducible, where $a, b \in \mathbb{R}$ whose difference is an integer, and ρ_0, σ are unitary characters of $F^\times$ with at least one of ρ_0, σ has even order as $\chi_{\mathrm{Im}(s)}$ has even order. By the discussions in [KLM20, §2 and §7], $\zeta([\nu^{-b}\rho_0, \nu^{-a}\rho_0]) \rtimes \sigma$ is irreducible unless $\rho_0^2\sigma = 1$ and there is a half-integer β such that $\nu^{-\beta}\rho_0 \rtimes \sigma$ is reducible. By the representation theory of GL_2, it happens only when $\rho_0\sigma^{-1} = 1$. Together, we have $\rho_0^3 = 1$, and thus both ρ_0 and σ are of odd order, which contradicts our assumption. This completes our proof for the case $\ell = n$.

Type C_n: This follows from [Jan96, Theorem 4.3 and 7.2].

Type $D_n, \ell < n-1$: For the same reason as in the case of type B, for $\ell < n-1$ we can take G to be SO_{2n}, and the assertion in this case follows from [BJ03, Theorem 5.3 and 5.5, Remark 5.6].

Exceptional types: Each case is treated separately in [Cho06, CJ10, HS20, HS21, HS22].

□

Remark 4.2.7. The assertion for the only excluded cases, i.e., $G = \mathrm{Spin}_{2n}$ and $\ell = n-1$ (or n), is likely to be true. For $\mathrm{ord}(\chi_{\mathrm{Im}(s)})$ odd, this follows again from [BJ03] as in the case of type B. Therefore, it remains to show $I_{P_\ell}(\chi_s)$ is irreducible if $\chi_{\mathrm{Im}(s)}$ is of even order. In this case, the assertion could possibly be proved directly using e.g., the Geometric Lemma [BZ77, Lemma 2.12].

4.3 Examples revisited

We illustrate Theorem 4.2.2 (and Theorem 4.2.6) with examples considered in §3.6.

4.3.1 Line bundles over Grassmannians

Suppose $G = \mathrm{SL}_n$ and P is the stabilizer of an ℓ-plane for some $1 \leqslant \ell < n$. By Appendix B we have

$$\begin{aligned} L(1) &= \{0, -1, \dots, -\min(\ell-1, n-\ell-1)\}, \\ s_k &= \tfrac{n-2}{2}. \end{aligned}$$

Therefore, L_1 is indeed a set. By Theorem 4.2.6, which remains valid for general F by [Gou17], we have

$$\mathcal{S}(X_P(F))/\mathcal{S}(X_P^\circ(F)) \cong \bigoplus_{(s,\chi)\in R(G,P)} \ker(\mathcal{R}_{P|P^{\mathrm{op}}}(\chi_s)).$$

4.3.2 Odd dimensional Isotropic cones

Suppose $\mathrm{char}F \neq 2$. Let $G_n = \mathrm{SO}_{2n}$ for $n \geqslant 2$ and P_n be the stabilizer of the line spanned by an isotropic vector, so that X_{P_n} is a cone. By Appendix B we have for $n \geqslant 3$

$$\begin{aligned} L(1) &= \{0, 2-n\}, \\ s_k &= n-2. \end{aligned}$$

In this case the exact sequence is also obtained in [GK23]:

$$\mathcal{S}(X_{P_n}(F))/\mathcal{S}(X_{P_n}^\circ(F)) \cong \mathbb{C} \oplus \mathcal{S}(X_{P_{n-1}}(F)). \tag{4.13}$$

They also have similar results for nonsplit cases. However, their approach relies on the fact that $\mathcal{S}(X_{P_n}(F))$ is actually a representation of G_{n+1} and their exact sequence is a consequence of the theory of Jacquet functors.

In [Get22], the isomorphism (4.13) is used to give a geometrization of boundary terms in Poisson summation formulae on X_{P_n}. We expect this to be true for general; that is, global boundary terms should come from “compatible” restricted tensor products of local boundary terms of Schwartz spaces. This is part of the motivation to prove Theorem 4.2.2 and Theorem 4.2.6.

4.3.3 The Lagrangian Grassmannian

Suppose $G = \mathrm{Sp}_{2n}$ and P is the Siegel parabolic. By Appendix B we have

$$\begin{aligned} L(1) &= \{0\}, \\ L(2) &= \{-i + j\tfrac{\pi\sqrt{-1}}{\log q}\mathbb{Z} : 1 \leqslant i \leqslant \lfloor \tfrac{n}{2} \rfloor, 0 \leqslant j \leqslant 1\}, \\ s_k &= \tfrac{n-1}{2}. \end{aligned}$$

Therefore, L_1 and L_2 are indeed sets. By Theorem 4.2.2, we have

$$\mathcal{S}(X_P(F))/\mathcal{S}(X_P^{\circ}(F)) \cong \bigoplus_{d=1}^{2} \bigoplus_{\substack{\chi \in \widehat{\mathcal{O}^{\times}} \\ \mathrm{ord}(\chi)=d}} \bigoplus_{r \in L_d} \mathrm{Ker}(\mathcal{R}_{P|P^{\mathrm{op}}}(\overline{\chi}_{-r-\frac{n+1}{2}})).$$

When F is of characteristic zero, by the work of Kudla and Rallis [KR92], we have

$$\bigoplus_{d=1}^{2} \bigoplus_{\substack{\chi \in \widehat{\mathcal{O}^{\times}} \\ \mathrm{ord}(\chi)=d}} \bigoplus_{r \in L_d} \mathrm{Ker}(\mathcal{R}_{P|P^{\mathrm{op}}}(\overline{\chi}_{-r-\frac{n+1}{2}})) \cong \bigoplus_{V} \mathcal{S}(V^n(F))_{O(V)(F)}$$

where V ranges over all equivalence classes of nondegenerate even dimensional quadratic spaces over F of dimension not larger than n, and $\mathcal{S}(V^n(F))_{O(V)(F)}$ is the space of $O(V)(F)$-coinvariants of the Weil representation of $G = \mathrm{Sp}_{2n}$ realized on $\mathcal{S}(V^n(F))$.

Chapter 5. Poles of intertwining operators

In this chapter we assume F is nonarchimedean and prove the inclusion $\mathcal{S}(X_P^\circ(F)) < \mathcal{S}(X_P(F))$ for G not of type E or F. We proceed by studying more generally poles of intertwining operators to prove Theorem 5.1.3, which implies the desired inclusion. We start with a general setup in §5.1, regardless of the type of G, to prove Theorem 5.1.3 and discuss its consequences. Our proof follows the approach in [Ike92, §1] and requires case-by-case discussion. We will prove the theorem for classical groups and $G = G_2$ in §5.2-§5.6.

5.1 Poles of Intertwining operators

We first recall some general results on intertwining operators discussed in [Ike92, §1.2] Our main references are [Sha81, Sha10].

Let $W := W_G$ be the Weyl group of (G, B, T). For each $w \in W$, we choose a representative in K, which we continue to denote by w. For a quasi-character $\tilde{\chi}$ of $T(F)$, let $I_B(\tilde{\chi}) := \mathrm{Ind}_{B(F)}^{G(F)}(\tilde{\chi})$ be the normalized induced representation in the category of smooth representations. Here we have identified $\tilde{\chi}$ as a quasi-character of $B(F)$. For $w \in W$, let $\tilde{\chi}^w(t) := \tilde{\chi}(w^{-1}tw)$ and $N_w := N \cap wN^{\mathrm{op}}w^{-1}$. The (unnormalized) intertwining operator M_w is defined as

$$\begin{aligned} M_w(\tilde{\chi}) : I_B(\tilde{\chi}) &\longrightarrow I_B(\tilde{\chi}^w) \\ f &\mapsto \left(g \mapsto \int_{N_w(F)} f(w^{-1}ug)du \right). \end{aligned}$$

Here the measure du on N_w is defined as in §2.2. The integral defining $M_w(\tilde{\chi})f(g)$ converges absolutely for all $g \in G(F)$ for $\tilde{\chi}$ in some open cone, and extends meromorphically to all $\tilde{\chi}$.

For a simple root $\alpha \in \Delta$, let $\iota_\alpha : \mathrm{SL}_2 \to G$ be the homomorphism determined by our choice of Chevalley basis. Let $\tilde{w}_0$ be the nonidentity element in W_{SL_2}. We can and do assume $\iota_\alpha(\tilde{w}_0) = s_\alpha$ is the simple reflection defined by α. Then for $f \in I_B(\tilde{\chi})$, we have

$$\iota_\alpha^*(M_{s_\alpha}(\tilde{\chi})f) = M_{\tilde{w}_0}(\iota_\alpha^*\tilde{\chi})(\iota_\alpha^* f). \tag{5.1}$$

For $w \in W$, let $\ell(w)$ be the length of w. If $w_1, w_2 \in W$ satisfies $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$, then $M_{w_1w_2}(\tilde{\chi}) = M_{w_1}(\tilde{\chi}^{w_2}) \circ M_{w_2}(\tilde{\chi})$. Note that by definition M_w commutes with right translations, and thus to study poles of M_w, it suffices to have a good understanding for the case $G = \mathrm{SL}_2$, which we now recall.

Theorem 5.1.1. *Let $G = \mathrm{SL}_2$ and w_0 be the long Weyl element. Then the intertwining operator $M_{w_0}(\chi_s) : I_B(\chi_s) \to I_B(\chi_s^{-1})$ has following properties.*

(i) $L(0, \chi_s)^{-1}M_{w_0}(\chi_s)$ *is holomorphic.*

(ii) The kernel of $M_{w_0}(1_{-1})$ is the trivial representation.

(iii) $\mathrm{Res}_{s=0}M_{w_0}(1_s)$ *is a nonzero scalar multiplication.*

(iv) The image of $M_{w_0}(1_1)$ is the trivial representation.

(v) $\gamma(s, \tilde{\chi}, \psi)\gamma(-s, \chi^{-1}, \psi)M_{w_0}(\chi_s^{-1})M_{w_0}(\chi_s) = \mathrm{Id}$.

(vi) Suppose the conductor of ψ is $\mathcal{O}$. Let ϕ_s be the unique right $\mathrm{SL}_2(\mathcal{O})$-invariant section in $I_B(1_s)$ such that $\phi_s|_{\mathrm{SL}_2(\mathcal{O})} = 1$. Then

$$M_{w_0}(1_s)\phi_s = \frac{L(0, 1_s)}{L(1, 1_s)}\phi_{-s}.$$

Proof. See [Ike92, §1.2]. Note that there is a typo in (1.2.3) in *loc. cit.*. We refer one to [GH20, Lemma 3.11] for a corrected statement. □

Remark 5.1.2. Since $\gamma(s, \chi, \psi)$ equals $L(1, \chi_s^{-1})/L(0, \chi_s)$ up to a nowhere vanishing function, (v) implies

$$\frac{L(1, \chi_s^{-1})L(1, \chi_s)}{L(0, \chi_s)L(0, \chi_s^{-1})}M_{w_0}(\chi_s^{-1})M_{w_0}(\chi_s)$$

is an $\mathrm{SL}_2(F)$-equivariant isomorphism at all s.

When considering the quotient W/W_M, we always choose representatives of minimal length in each left coset of W_M. Let $w_0 \in W/W_M$ be (the representative of) the long Weyl element. Let $\Phi^\vee$ be the set of coroots and $(\Phi^\vee)^+$ (resp. $(\Phi^\vee)^-$) be the set of positive (resp.

negative) coroots. For $w \in W/W_M$, $\chi \in \widehat{\mathcal{O}^\times}$, and $s \in \mathbb{C}/\frac{2\pi\sqrt{-1}}{\log q}\mathbb{Z}$, define

$$\Phi_w^\vee := \{\beta^\vee \in (\Phi^\vee)^+ : w\beta^\vee \in (\Phi^\vee)^-\} \subset (\Phi^\vee)^+ - (\Phi_M^\vee)^+,$$

$$m_w(h,\lambda) := \left| \left\{ \beta^\vee \in \Phi_w^\vee : \sum_{\alpha\in\Delta} \langle \omega_\alpha, \beta^\vee\rangle = h, \langle \omega_P, \beta^\vee\rangle = \lambda \right\} \right| \text{ for } h, \lambda \in \mathbb{Z},$$

$$c_w(\chi_s) := \prod_{\beta^\vee \in \Phi_w^\vee} \frac{L\left(-\sum_{\alpha\in\Delta}\langle\omega_\alpha,\beta^\vee\rangle, \chi_{s+s_k+1}^{\langle\omega_P,\beta^\vee\rangle}\right)}{L\left(1-\sum_{\alpha\in\Delta}\langle\omega_\alpha,\beta^\vee\rangle, \chi_{s+s_k+1}^{\langle\omega_P,\beta^\vee\rangle}\right)}$$

$$= \prod_{\lambda=1}^{\infty}\prod_{h=1}^{\infty} \frac{L\left(-h, \chi_{s+s_k+1}^{\lambda}\right)^{\max(0, m_w(h,\lambda)-m_w(h+1,\lambda))}}{L\left(1-h, \chi_{s+s_k+1}^{\lambda}\right)^{\max(0, m_w(h,\lambda)-m_w(h-1,\lambda))}},$$

$$a_w(\chi_s) := \prod_{\lambda=1}^{\infty}\prod_{h=1}^{\infty} L\left(-h, \chi_{s+s_k+1}^{\lambda}\right)^{\max(0, m_w(h,\lambda)-m_w(h+1,\lambda))}.$$

Note that $I_P(\chi_s) < I_B(\tilde{\chi}_s)$, where

$$\tilde{\chi}_s := \delta_P^{1/2}\delta_B^{-1/2} \cdot \chi \circ \omega_P \cdot |\omega_P|^s = \left(\prod_{\alpha\in\Delta} |\omega_\alpha|^{-1}\right) \cdot \chi \circ \omega_P \cdot |\omega_P|^{s+s_k+1}, \tag{5.2}$$

and the restriction of $M_w(\tilde{\chi}_s)$ to $I_P(\chi_s)$, denoted by $M_w(\chi_s)$, is well defined outside a finite set of s. Also observe that, up to a conjugation by an element of K, M_{w_0} is essentially equal to $\mathcal{R}_{P|P^{\mathrm{op}}}$. In the case P is self-associate, one may choose the element to be w_0.

The goal of this chapter is to prove

Theorem 5.1.3. *Suppose G is not of type E or F. The intertwining operator*

$$M'_w(\chi_s) := a_w(\chi_s)^{-1} M_w(\chi_s). \tag{5.3}$$

is holomorphic. Moreover, for all $h, \lambda \in \mathbb{Z}_{>0}$,

$$\max(0, m_w(h,\lambda) - m_w(h-1,\lambda)) \leqslant \max(0, m_{s_\alpha w}(h,\lambda) - m_{s_\alpha w}(h-1,\lambda)) \tag{5.4}$$

for every $w \in W/W_M$ and $\alpha \in \Delta$ such that $\ell(s_\alpha w) = \ell(w) + 1$. Consequently,

$$d(\chi_s) := \prod_{\lambda=1}^{\infty}\prod_{h=1}^{\infty} L\left(1-h, \chi_{s+s_k+1}^{\lambda}\right)^{\max(0, m_{w_0}(h,\lambda)-m_{w_0}(h-1,\lambda))}$$

is the least common denominator of $c_w(\chi_s)$ for all $w \in W/W_M$.

Remark 5.1.4. Our proof should also work for G of type E, F. However, as the proof essentially boils down to certain combinatorics statements on the set of coroots in $(\Phi^\vee)^+ - (\Phi_M^\vee)^+$, we do not carry it out due to the complexity of W/W_M .

The rest of the paper is a case-by-case proof of Theorem 5.1.3. The case $G = G_2$ is proved in §5.2, and classical groups are discussed in §5.3-§5.6. Theorem 5.1.3 together with the following lemma has several consequences.

Lemma 5.1.5. *We have*

$$a_{w_0}(\chi_s) = a_{P|P^{\mathrm{op}}}(\chi_s) = \prod_{i=1}^{k} L(-s_i, \chi_s^{\lambda_i})$$

$$d(\chi_s) = a_{P|P}(\chi_s) = \prod_{i=1}^{k} L(1 + s_i, \chi_s^{\lambda_i}).$$

Proof. Both identities can be easily verified which we leave to reader. For a list of positive (co)roots see e.g., [Bou02] for exceptional groups, and see §5.3-§5.6 for classical groups; for a list of multiset $\{(s_i, \lambda_i)\}$, see Appendix A. We alert the reader that the result there is stated in the dual side. □

In the rest of the section, we assume G is not of type E or F, or more generally (5.3) and (5.4) hold.

Corollary 5.1.6. *We have* $\mathcal{S}(X_P^\circ(F)) < \mathcal{S}(X_P(F))$.

Proof. This follows from Theorem 5.1.3, Lemma 5.1.5, and the definition of $\mathcal{S}(X_P(F))$. □

We can rephrase our definition of good sections in the sense of [Yam14, Definition 3.1].

Corollary 5.1.7. *Let* $\chi \in \widehat{\mathcal{O}^\times}$. *A meromorphic section* $f^{(s)} \in I_P(\chi_s)$ *is good if and only if* $f^{(s)}$ *has no poles for* $\mathrm{Re}(s) > -\frac{1}{6}$ *and* $d(\chi_s^{-1}) a_{w_0}(\chi_s)^{-1} M_{w_0}(\chi_s) f^{(s)}$ *has no poles for* $\mathrm{Re}(s) < 0$.

Proof. Sufficiency follows from Remark 4.1.3. For the converse, by the assumption of $f^{(s)}$ and Theorem 5.1.3, the meromorphic section

$$\frac{M_{w_0}(\chi_s) f^{(s)}}{a_{w_0}(\chi_s)}$$

has no poles for $\mathrm{Re}(s) > -\frac{1}{6}$ or $\mathrm{Re}(s) < 0$. Thus it is holomorphic. On the other hand, by (5.1) and (v) we can write

$$f^{(s)} = h(s)\frac{d(\chi_s)d(\chi_s^{-1})}{a_{w_0}(\chi_s^{-1})a_{w_0}(\chi_s)}M_{w_0}(\chi_s^{-1})M_{w_0}(\chi_s)f^{(s)}$$

for some nowhere vanishing function $h(s) \in \mathbb{C}[q^{-s}, q^s]$. Then by assumption of $f^{(s)}$ and Theorem 5.1.3

$$\frac{f^{(s)}}{d(\chi_s)} = h(s)\frac{M_{w_0}(\chi_s^{-1})}{a_{w_0}(\chi_s^{-1})}\frac{d(\chi_s^{-1})}{a_{w_0}(\chi_s)}M_{w_0}(\chi_s)f^{(s)}$$

has no poles for $\mathrm{Re}(s) < 0$. Thus $d(\chi_s)^{-1}f^{(s)}$ is holomorphic, and $f^{(s)}$ is a good section. □

By the equivalent definitions of good sections, we obtain from [Yam14, Proposition 3.1] the following.

Corollary 5.1.8. *(a)* $\mathcal{S}(X_P(F)) = \mathcal{S}_{\mathrm{BK}}(X_P(F))$.

(b) Let $d \in \mathbb{Z}_{>0}$ *with* $L_d \neq \varnothing$. *The module* $A_{r,m_r(L_d),\chi}/A_{r,m_r(L_d)-1,\chi}$ *is nonzero for all* $\mathrm{ord}(\chi) = d$ *and* $r \in \mathrm{Supp}(L_d)$.

(c) $\mathbb{C} \leqslant A_{0,1,1} = A_{0,1}$ *and thus* $\mathcal{S} := \mathcal{S}(V_P(F))\big|_{X_P^\circ(F)} \leqslant \mathcal{S}(X_P(F))$.

Proof. By the Mellin inversion, sections that are holomorphic for all $\chi \in \widehat{\mathcal{O}^\times}$ correspond bijectively to functions in $\mathcal{S}(X_P^\circ(F))$. Therefore by Theorem 3.4.4 the first statement follows from the proof of [Yam14, Proposition 3.1 (4)]. The second statement is [Yam14, Proposition 3.1 (3)]. For the last statement, since the Schwartz space $\mathcal{S}(X_P(F))$ is independent of ψ, we may assume the conductor of ψ is $\mathcal{O}$. Let f be the unique right $G(\mathcal{O})$-invariant function in $C^\infty(X_P^\circ(F))$ such that $(f)_{1_s}|_{G(\mathcal{O})} = d(1_s)$. Then by the Gindikin-Karpelevič formula [Lai80, Proposition 4.6] , we have $f \in \mathcal{S}(X_P(F))$. Since $d(1_0) \neq 0$, the image of f in $A_{0,1}$ under the exact sequence in Theorem 4.2.2 is nonzero. As f is right $G(\mathcal{O})$-invariant, the image is a nonzero constant. □

The following corollary generalizes [Ike92, Lemma 1.2].

Corollary 5.1.9. *For $f \in \mathcal{S}(X_P(F))$, the section*

$$(d(\chi_s)c_w(\chi_s))^{-1} M_w(\chi_s)f_{\chi_s} \tag{5.5}$$

is holomorphic for any $w \in W/W_M$ and $\chi \in \widehat{\mathcal{O}^\times}$.

Proof. Replace χ_s with χ_s^{-1}. By Theorem 3.4.4, Theorem 5.1.3, and Corollary 5.1.8(a), we may replace $f_{\chi_s^{-1}}$ with

$$\frac{d(\chi_s^{-1})}{a_{w_0}(\chi_s)} M_{w_0}(\chi_s)f_{\chi_s} \text{ where } f \in C_c^\infty(X^\circ_{w_0Pw_0^{-1}}(F)).$$

Write $w_0 = ww'$ such that $\ell(w_0) = \ell(w) + \ell(w')$. Then we can rewrite (5.5) as

$$\begin{aligned}&\frac{1}{c_w(\chi_s^{-1})a_{w_0}(\chi_s)} M_w(\chi_s^{-1})M_{w_0}(\chi_s)f_{\chi_s}\\ &= \frac{1}{d(\chi_s)c_w(\chi_s^{-1})c_{w_0}(\chi_s)} M_w(\chi_s^{-1})M_w(\chi_s^{w'})M_{w'}(\chi_s)f_{\chi_s}\end{aligned}$$

By (5.1) and (v), this section has the same poles (counting multiplicities) as

$$\frac{1}{d(\chi_s)c_{w'}(\chi_s)} M_{w'}(\chi_s)f_{\chi_s},$$

which is holomorphic by Theorem 5.1.3. □

Remark 5.1.10. In [GL21] their notion of a Schwartz function f is defined so that

$$(d(\chi_s)c_w(\chi_s))^{-1} M_w(\chi_s)f_{\chi_s}$$

is holomorphic for all $w \in W/W_M$ and χ. By Corollary 5.1.9 it is enough to check for $w = \mathrm{Id}, w_0$, and thus the Schwartz spaces in [GL21] are equal to $\mathcal{S}(X_P(F))$ in the nonarchimedean case.

Corollary 5.1.11. *The Schwartz space $\mathcal{S}(X_P(F))$ is **local**, i.e., it is an $C^\infty(X_P(F))$-module under multiplication of functions.*

Proof. Let $\mathcal{S} := \mathrm{Im}(\mathcal{S}(V_P(F)) \longrightarrow C^\infty(X_P^\circ(F)))$, where the implicit map is the restriction. Note that $\mathcal{S}(X_P^\circ(F)) < \mathcal{S}$ and $\mathcal{S}/\mathcal{S}(X_P^\circ(F)) = \mathbb{C}$. Since each function in $\mathcal{S}(X_P(F))$ is compactly supported in $X_P(F)$, we have $C^\infty(X_P(F)) \cdot \mathcal{S}(X_P(F)) = \mathcal{S} \cdot \mathcal{S}(X_P(F))$. As $\mathcal{S}(X_P^\circ(F)) \cdot \mathcal{S}(X_P(F)) = \mathcal{S}(X_P^\circ(F)) < \mathcal{S}(X_P(F))$, it suffices to show $\mathcal{S}(X_P(F))/\mathcal{S}(X_P^\circ(F))$ is a $\mathcal{S}/\mathcal{S}(X_P^\circ(F))$-module, i.e., a $\mathbb{C}$-vector space, which is clear. $\square$

Remark 5.1.12. The notion of locality of a function space was introduced in [BK00, §5]. When X_P is a vector space, locality is equivalent to $\mathcal{S}(X_P(F))$ being an algebra which plays an important role in harmonic analysis.

5.2 Proof outline of Theorem 5.1.3

In this section, we discuss in detail the proof of Theorem 5.1.3 for the case $G = G_2$ and explain how to adapt it for classical groups. We record the following lemma and notations for later use (see e.g., [Bou02]).

Lemma 5.2.1. *Let $w \in W/W_M$ and $w = w_m \cdots w_2 w_1$ be a reduced expression in W. For each i, let $\alpha_{(i)} \in \Delta$ be the simple root such that w_i is the corresponding reflection. Then*

$$\Phi_w^\vee = \{\tilde{\alpha}_{(i)}^\vee := w_1 \cdots w_{i-2} w_{i-1} \alpha_{(i)}^\vee : 1 \leqslant i \leqslant m\}.$$

$\square$

We say the coroot $\tilde{\alpha}_{(i)}^\vee$ corresponds to the simple reflection w_i (under the reduced expression).

Recall $M'_w(\chi_s)$ defined in (5.3).

Lemma 5.2.2. *Let $w \in W/W_M$. Let α be a simple root such that $\ell(s_\alpha w) = \ell(w) + 1$, and (h, λ) be the unique pair such that $m_{s_\alpha w}(h, \lambda) - m_w(h, \lambda) = 1$. Suppose $M'_w(\chi_s)$ is holomorphic.*

Assume

$$m_w(h, \lambda) \geqslant m_w(h+1, \lambda). \tag{5.6}$$

Then $M'_{s_\alpha w}(\chi_s)$ is holomorphic if either $\chi^\lambda \neq 1$ or $\chi^\lambda = 1$ and one of the following holds.

(I) $m_w(h, \lambda) \geqslant m_w(h-1, \lambda)$.

(II) $M'_w(\chi_s) f^{(s)}|_{\lambda(s+s_k+1)=h-1}$ *is left* $\iota_\alpha(\mathrm{SL}_2)$*-invariant for all holomorphic sections* $f^{(s)}$ *of* $I_P(\chi_s)$.

Proof. By (5.1), (5.2), and (i),

$$L\left(-\sum_{\beta\in\Delta}\langle w\omega_\beta,\alpha^\vee\rangle,\chi_{s+s_k+1}^{\langle w\omega_P,\alpha^\vee\rangle}\right)^{-1} M_{s_\alpha}(\chi_s^w)\left(M'_w(\chi_s)f^{(s)}\right)$$

is holomorphic. By Lemma 5.2.1

$$L\left(-\sum_{\beta\in\Delta}\langle w\omega_\beta,\alpha^\vee\rangle,\chi_{s+s_k+1}^{\langle w\omega_P,\alpha^\vee\rangle}\right)=L\left(-\sum_{\beta\in\Delta}\langle \omega_\beta,w^{-1}\alpha^\vee\rangle,\chi_{s+s_k+1}^{\langle \omega_P,w^{-1}\alpha^\vee\rangle}\right)=L(-h,\chi_{s+s_k+1}^{\lambda}).$$

Thus by definition of $a_{s_\alpha w}(\chi_s)$ we may assume $\chi^\lambda=1$ and (I) fails, so that by assumption (5.6)

$$a_{s_\alpha w}(\chi_s)=\frac{\zeta\left(\lambda(s+s_k+1)-(h-1)\right)}{\zeta\left(\lambda(s+s_k+1)-h\right)}a_w(\chi_s).$$

By (II) and (ii),

$$M_{s_\alpha}(\chi_s^w)M'_w(\chi_s)f^{(s)}\big|_{\lambda(s+s_k+1)=h-1}=0,$$

and thus $M'_{s_\alpha w}(\chi_s)f^{(s)}$ is holomorphic. ☐

Note that for every $w\in W/W_M$ there is a reduced expression $w_m\cdots w_2w_1$ of w_0 such that $w=w_r\cdots w_2w_1$ for some $r\leqslant m$, and a reduced expression can be transformed to another expression by performing a sequence of defining relations of the Weyl group W. Explicitly, for two distinct simple (co)roots α and α', let $s_\alpha, s_{\alpha'}$ be the corresponding simple reflections and let $n_{\alpha\alpha'}$ be the number of edges between the corresponding nodes in the Dynkin diagram.

(a) If $n_{\alpha\alpha'}=0$, then replace $s_\alpha s_{\alpha'}$ with $s_{\alpha'}s_\alpha$;

(b) if $n_{\alpha\alpha'}=1$, then replace $s_\alpha s_{\alpha'}s_\alpha$ with $s_{\alpha'}s_\alpha s_{\alpha'}$;

(c) if $n_{\alpha\alpha'}=2$, then replace $(s_\alpha s_{\alpha'})^2$ with $(s_{\alpha'}s_\alpha)^2$;

(d) if $n_{\alpha\alpha'}=3$, then replace $(s_\alpha s_{\alpha'})^3$ with $(s_{\alpha'}s_\alpha)^3$.

The strategy to prove Theorem 5.1.3 is as follows. We will choose a reduced expression $w_m\cdots w_1$ of w_0 and list corresponding coroots $\tilde{\alpha}_{(i)}^\vee$ (see Lemma 5.2.1). For each case, we will first verify the combinatorics inequalities (5.4) and (5.6) for any $w \in W/W_M$ by studying the effects of operations above on the order of $\tilde{\alpha}_{(i)}^\vee$. Then we apply Lemma 5.2.2 repeatedly to show, inductively on the length of w, that the invariance property (II) holds whenever (I) fails.

In the rest of the section, we will use the Bourbaki numbering of the Dynkin diagram and (co)roots. Let $P := P_\ell$ be the maximal parabolic subgroup associated to the ℓth node of the Dynkin diagram of (G, B, T). Let α_i denote the simple root attached to the ith node and let s_i be the corresponding simple reflection. However, note that we also write s_k for the highest data. This should lead to little or no confusion as the subscript k is only used as the size of the multiset $\{(s_i, \lambda_i)\}$ throughout.

Proof of Theorem 5.1.3 for $G = G_2$. Observe that $M^{\mathrm{der}} \cong \mathrm{SL}_2$ for both cases and $|W/W_M| = 6 = \ell(w_0) + 1$, so that $w_0 \in W/W_M$ has a unique reduced expression. Inequalities (5.4) and (5.6) will be clear from the order of coroots below.

For $\ell = 1$, the reduced expression of w_0 is

$$s_1 s_2 s_1 s_2 s_1;$$

corresponding coroots $\tilde{\alpha}_{(i)}^\vee$ are

$$\alpha_1^\vee, \alpha_1^\vee + \alpha_2^\vee, 2\alpha_1^\vee + 3\alpha_2^\vee, \alpha_1^\vee + 2\alpha_2^\vee, \alpha_1^\vee + 3\alpha_2^\vee.$$

Since (I) holds for $w = \mathrm{Id}, s_2 s_1$, we may assume $\chi = 1$, and we are left to verify (II) holds for $w = s_1, s_1 s_2 s_1, s_2 s_1 s_2 s_1$.

A holomorphic section $f^{(s)}$ of $I_{P_1}(\chi_s)$ is left $\iota_{\alpha_2}(\mathrm{SL}_2)$-invariant, and so is

$$M'_{s_1}(1_s) f^{(s)}|_{s+s_k+1=1}$$

by (iii), which justifies the case $w = s_1$. For $w = s_1 s_2 s_1$, by (iv)

$$M'_{s_1}(\chi_s) f^{(s)}|_{s+s_k+1=2}$$

is left $\iota_{\alpha_1}(\mathrm{SL}_2)$-invariant, and so is

$$M'_{s_2 s_1}(\chi_s) f^{(s)}|_{s+s_k+1=2}$$

by (iii). As (I) holds for s_2s_1, by (ii)

$$M'_{s_1s_2s_1}(1_s)f^{(s)}|_{s+s_k+1=2} = 0.$$

This proves the case $w = s_1s_2s_1$. Finally for $w = s_2s_1s_2s_1$, by (iv)

$$M'_{s_1s_2s_1}(1_s)f^{(s)}|_{s+s_k+1=3}$$

is left $\iota_{\alpha_1}(\mathrm{SL}_2)$-invariant, and thus so is

$$M'_{s_2s_1s_2s_1}(1_s)f^{(s)}|_{s+s_k+1=3}$$

by (iii).

For $\ell = 2$, the reduced expression of w_0 is

$$s_2s_1s_2s_1s_2;$$

corresponding coroots $\tilde{\alpha}^\vee_{(i)}$ are

$$\alpha_2^\vee, \alpha_1^\vee + 3\alpha_2^\vee, \alpha_1^\vee + 2\alpha_2^\vee, 2\alpha_1^\vee + 3\alpha_2^\vee, \alpha_1^\vee + \alpha_2^\vee.$$

Since (I) holds for $w = \mathrm{Id}, s_2, s_1s_2$, we may assume $\chi^3 = 1$, and we are left to verify (II) holds for $w = s_2s_1s_2, s_1s_2s_1s_2$.

A holomorphic section $f^{(s)}$ of $I_{P_2}(\chi_s)$ is left $\iota_{\alpha_1}(\mathrm{SL}_2)$-invariant. Since

$$L(-4, \chi^3_{s+s_k+1})^{-1}M_{s_1}(\chi_s^{s_2})|_{3(s+s_k+1)=4}$$

is a nonzero scalar multiplication by (iii) and $s_2s_2 = \mathrm{Id}$, we have

$$M'_{s_2s_1s_2}(\chi_s)f^{(s)}|_{3(s+s_k+1)=4}$$

is left $\iota_{\alpha_1}(\mathrm{SL}_2)$-invariant, which justifies the case $w = s_2s_1s_2$. For $w = s_1s_2s_1s_2$, it suffices to consider $\chi = 1$. By (iii)

$$M'_{s_2}(\chi_s)f^{(s)}|_{s+s_k+1=1}$$

is left $\iota_{\alpha_1}(\mathrm{SL}_2)$-invariant. As (I) holds for s_2 and s_1s_2, by (ii)

$$M'_{s_1s_2}(\chi_s)f^{(s)}|_{s+s_k+1=1} = 0,$$

and hence

$$M'_{s_1s_2s_1s_2}(\chi_s)f^{(s)}|_{s+s_k+1=1} = 0.$$

□

Arguments are more complicated for classical groups (also for groups of type E, F) as in general there are more than one reduced expression of w_0. We close this subsection by studying the effects of (a), (b) and (c) on the order of coroots $\tilde{\alpha}^\vee_{(i)}$.

Let $w_0 = w_m \cdots w_1$ be a reduced expression.

(a′) If $n_{\alpha_{(i+1)}\alpha_{(i)}} = 0$ and $w_{i+1}w_i$ is replaced with w_iw_{i+1}, then

$$\text{the position of the coroots } \tilde{\alpha}^\vee_{(i)} \text{ and } \tilde{\alpha}^\vee_{(i+1)} \text{ are swapped.}$$

(b′) If $n_{\alpha_{(i+1)}\alpha_{(i)}} = 1$, $\alpha_{(i)} = \alpha_{(i+2)}$, and $w_{i+2}w_{i+1}w_i$ is replaced with $w_{i+1}w_iw_{i+1}$, then

$$\text{the position of the coroots } \tilde{\alpha}^\vee_{(i)} \text{ and } \tilde{\alpha}^\vee_{(i+2)} \text{ are swapped.}$$

Moreover, $\alpha^\vee_{(i+1)} = \alpha^\vee_{(i+2)} + \alpha^\vee_{(i)}$, so

$$\langle \omega_P, \alpha^\vee_{(i+1)} \rangle \geqslant 2$$

and thus such operations will not occur if G is of type A.

(c′) If $n_{\alpha_{(i+1)}\alpha_{(i)}} = 2$, $\alpha_{(i)} = \alpha_{(i+2)}$ and $\alpha_{(i+1)} = \alpha_{(i+3)}$, then

$$\Phi^\vee_{w_{i+3}\cdots w_1} - \Phi^\vee_{w_{i-1}\cdots w_1} = \{\beta^\vee, \beta^\vee + \gamma^\vee, \beta^\vee + 2\gamma^\vee, \gamma^\vee\}$$

for some coroots $\beta^\vee, \gamma^\vee \in \Phi^+ - \Phi^+_M$. In particular, we have $\langle \omega_P, \beta^\vee + 2\gamma^\vee \rangle \geqslant 3$, which is impossible for classical groups.

5.3 Type A_n

A reduced expression of w_0 is

$$(s_{n-\ell+1} \cdots s_{n-1}s_n) \cdots (s_{r-\ell+1} \cdots s_{r-1}s_r) \cdots (s_1 \cdots s_{\ell-1}s_\ell) \tag{5.7}$$

Here the parentheses are only present in this expression so that the reader can follow the pattern. The corresponding order of coroots $\tilde{\alpha}^\vee_{(i)}$ is

$$\begin{array}{ccccc}
\alpha^\vee_\ell, & \cdots & \sum_{j=t}^{\ell} \alpha^\vee_j, & \cdots & \sum_{j=1}^{\ell} \alpha^\vee_j, \\
\vdots & & \vdots & & \vdots \\
\sum_{j=\ell}^{r} \alpha^\vee_j, & \cdots & \sum_{j=t}^{r} \alpha^\vee_j, & \cdots & \sum_{j=1}^{r} \alpha^\vee_j, \\
\vdots & & \vdots & & \vdots \\
\sum_{j=\ell}^{n} \alpha^\vee_j, & \cdots & \sum_{j=t}^{n} \alpha^\vee_j, & \cdots & \sum_{j=1}^{n} \alpha^\vee_j.
\end{array}$$

The ith row of coroots (read from the top left) corresponds to the ith parenthesis of (5.7) (read from the right). Each positive coroot in $\Phi_{w_0}^\vee$ appears exactly once and there are no other coroots in $\Phi^\vee - \Phi_{w_0}^\vee$ in the above list, so (5.7) is indeed a reduced expression of w_0. The same justification of reduced expressions will be used in the rest of the section without mention.

We denote

$$\beta^\vee \longleftrightarrow \beta'^\vee$$

if the order of two coroots $\beta^\vee$ and $\beta'^\vee$ can be reversed under a series of operations (a$'$) and (b$'$). Denote coroots

$$\sum_{j=t}^{r} \alpha_j^\vee \quad \text{for } 1 \leqslant t \leqslant \ell \leqslant r \leqslant n$$

by (r, t). Since operations (b$'$) cannot be applied if G is of type A, via operations (a$'$) we have

$$\text{for } r \leqslant r', (r,t) \longleftrightarrow (r', t') \text{ if and only if } r < r' \text{ and } t < t'. \tag{$R1$}$$

Note that coroots share the same h-value if they lie on the same 45° line in the table above. Consequently for $w \in W/W_M$, $m_w(h, 1) \geqslant c$ only if $m_w(h-1, 1) \geqslant c$ unless

$$m_w(h,1) = m_{w_0}(h,1) = m_{w_0}(h-1,1) + 1 = m_w(h-1,1) + 1.$$

This verifies inequalities (5.4) and (5.6).

We prove the holomorphy of $M'_w(\chi_s)$ by induction on both n and length of w. It suffices to prove the holomorphy when $\chi = 1$. For $\ell = 1$, the reduced expression (5.7) is unique. As (I) only holds for Id, we need to check

$$M'_{s_r \cdots s_1}(1_s) f^{(s)}\big|_{s+s_k+1=r} \text{ is left } \iota_{\alpha_{r+1}}(\mathrm{SL}_2)\text{-invariant.}$$

for $1 \leqslant r < n$. Since the coroot $(r, 1)$ has h-value r and $f^{(s)}$ is left $\iota_{\alpha_{r+1}}(\mathrm{SL}_2)$-invariant, the assertion follows from (iii).

For the general case, by symmetry we may assume $n \geqslant 3$ and $\lceil n/2 \rceil \geqslant \ell \geqslant 2$. By induction hypothesis, we may assume $(r, 1) \in \Phi_w^\vee$ for some r. Choose r to be maximal. By

$(R1)$ we can write $w = s_{r-\ell+1}w_1$ such that $s_{r-\ell+1}$ corresponds to the coroot $(r,1)$. Assume first $r > \ell$. We need to show

$$M'_{w_1}(1_s)f^{(s)}\big|_{s+s_k+1=r-1} \text{ is left } \iota_{\alpha_{r-\ell+1}}(\mathrm{SL}_2)\text{-invariant.} \tag{5.8}$$

Write $w_1 = s_{r-\ell}w'$ such that $s_{r-\ell}$ corresponds to the coroot $(r-1,1)$. We have

$$w'^{-1}\alpha_{r-\ell+1} = w_1^{-1}\alpha_{r-\ell+1} - w'^{-1}\alpha_{r-\ell} = \sum_{j=1}^{r}\alpha_j - \sum_{j=1}^{r-1}\alpha_j = \alpha_r.$$

Since $f^{(s)}$ is left $\iota_{\alpha_r}(\mathrm{SL}_2)$-invariant, $M'_{w'}(1_s)f^{(s)}$ is left $\iota_{\alpha_{r-\ell+1}}(\mathrm{SL}_2)$-invariant. As the coroot $(r-1,1)$ has h-value $r-1$, (5.8) follows by (iii).

Suppose $r = \ell$. By induction hypothesis, we may assume $(n,t) \in \Phi_w^\vee$ for some t. Choose t to be minimal. A similar argument as above justifies the holomorphy of $M'_w(1_s)$ if $t < \ell$, and thus we can assume $t = \ell$. By $(R1)$ we can also write $w = s_n w_1$ such that s_n corresponds to the coroot (n,ℓ), which has h-value $n-\ell+1$. Therefore by induction hypothesis and (i), if $n-\ell \neq \ell-1$ then $M'_w(1_s)f^{(s)} = M'_{s_n w_1}(1_s)f^{(s)}$ is holomorphic at $s+s_k+1 = \ell-1$ and hence is holomorphic at all s. For the same reason, to justify $M'_w(1_s)f^{(s)}$ is holomorphic at $s+s_k+1 = n-\ell = \ell-1$, it suffices to consider the case where no coroot in $\Phi_w^\vee$ has h-value greater than $n-\ell+1 = \ell$.

Note that when $n = 2\ell-1$, (I) fails except when $m_w(\ell,1) = \ell$. Assume $m_w(\ell,1) < \ell$ and thus there exists an integer $0 \leqslant i < \ell-2$ such that $(n-i,\ell-i) \in \Phi_w^\vee$ but $(n-i-1,\ell-i-1) \notin \Phi_w^\vee$. By $(R1)$ we can rewrite $w = s_{n-2i}s_{n-2i-1}w'$ such that s_{n-2i-1} and s_{n-2i} correspond to coroots $(n-i-1,\ell-i)$ and $(n-i,\ell-i)$ respectively. We need to show

$$M'_{s_{n-2i-1}w'}(1_s)f^{(s)}\big|_{s+s_k+1=n-\ell} \text{ is left } \iota_{\alpha_{n-2i}}(\mathrm{SL}_2)\text{-invariant.}$$

We have

$$w'^{-1}\alpha_{n-2i} = (s_{n-2i-1}w')^{-1}\alpha_{n-2i} - w'^{-1}\alpha_{n-2i-1} = \sum_{j=\ell-i}^{n-i}\alpha_j - \sum_{j=\ell-i}^{n-i-1}\alpha_j = \alpha_{n-i}.$$

Since $f^{(s)}$ is left $\iota_{\alpha_{n-i}}(\mathrm{SL}_2)$-invariant and the coroot $(n-i-1,\ell-i)$ has h-value $n-\ell$, the assertion follows by (iii).

□

5.4 Type B_n $(n \geqslant 2)$

We break down the discussion into two cases: $n = \ell$ and $n > \ell$. Consider first $1 \leqslant \ell < n$. A reduced expression of w_0 is

$$\begin{aligned}&\big((s_\ell \cdots s_{n-1})(s_n)\big) \cdots \big((s_{\ell-r+1} \cdots s_{n-r})(s_{n-r+1} \cdots s_n)\big) \cdots \big((s_1 \cdots s_{n-\ell})(s_{n-\ell+1} \cdots s_n)\big) \cdot \\ &(s_{n-\ell} \cdots s_{n-2}s_{n-1}) \cdots (s_{r-\ell+1} \cdots s_{r-1}s_r) \cdots (s_1 \cdots s_{\ell-1}s_\ell);\end{aligned} \tag{5.9}$$

corresponding coroots $\tilde{\alpha}^\vee_{(i)}$ are

$$\begin{array}{ccccc} \alpha^\vee_\ell, & \cdots & \sum_{j=t}^{\ell} \alpha^\vee_j, & \cdots & \sum_{j=1}^{\ell} \alpha^\vee_j, \\ \vdots & & \vdots & & \vdots \\ \sum_{j=\ell}^{r} \alpha^\vee_j, & \cdots & \sum_{j=t}^{r} \alpha^\vee_j, & \cdots & \sum_{j=1}^{r} \alpha^\vee_j, \\ \vdots & & \vdots & & \vdots \\ \sum_{j=\ell}^{n-1} \alpha^\vee_j, & \cdots & \sum_{j=t}^{n-1} \alpha^\vee_j, & \cdots & \sum_{j=1}^{n-1} \alpha^\vee_j, \end{array} \tag{$*$}$$

$$\begin{array}{ccccc} \alpha^\vee_n + 2\sum_{j=\ell}^{n-1} \alpha^\vee_j, & \cdots & \alpha^\vee_n + 2\sum_{j=\ell}^{n-1} \alpha^\vee_j + \sum_{j=t}^{\ell-1} \alpha^\vee_j, & \cdots & \alpha^\vee_n + 2\sum_{j=\ell}^{n-1} \alpha^\vee_j + \sum_{j=1}^{\ell-1} \alpha^\vee_j, \\ \alpha^\vee_n + \sum_{j=\ell}^{n-1} \alpha^\vee_j, & \cdots & \alpha^\vee_n + 2\sum_{j=m}^{n-1} \alpha^\vee_j + \sum_{j=\ell}^{m-1} \alpha^\vee_j, & \cdots & \alpha^\vee_n + 2\sum_{j=\ell+1}^{n-1} \alpha^\vee_j + \alpha^\vee_\ell, \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ & & \alpha^\vee_n + 2\sum_{j=r}^{n-1} \alpha^\vee_j + \sum_{j=t}^{r-1} \alpha^\vee_j, & \cdots & \alpha^\vee_n + 2\sum_{j=r}^{n-1} \alpha^\vee_j + \sum_{j=1}^{r-1} \alpha^\vee_j, \\ \alpha^\vee_n + \sum_{j=r}^{n-1} \alpha^\vee_j, & \cdots & \alpha^\vee_n + 2\sum_{j=m}^{n-1} \alpha^\vee_j + \sum_{j=r}^{m-1} \alpha^\vee_j, & \cdots & \alpha^\vee_n + 2\sum_{j=\ell+1}^{n-1} \alpha^\vee_j + \sum_{j=r}^{\ell} \alpha^\vee_j, \\ \vdots & & \vdots & \ddots & \vdots \\ & & & & \alpha^\vee_n + 2\sum_{j=1}^{n-1} \alpha^\vee_j, \\ \alpha^\vee_n + \sum_{j=1}^{n-1} \alpha^\vee_j, & \cdots & \alpha^\vee_n + 2\sum_{j=m}^{n-1} \alpha^\vee_j + \sum_{j=1}^{m-1} \alpha^\vee_j, & \cdots & \alpha^\vee_n + 2\sum_{j=\ell+1}^{n-1} \alpha^\vee_j + \sum_{j=1}^{\ell} \alpha^\vee_j. \end{array}$$

Here $(*)$ corresponds to the second row of (5.9) (see §5.3). For rows of coroots below $(*)$, the ith odd (resp. even) row corresponds to the former (resp. latter) parenthesis in the ith big parenthesis (reading from the right).

We retain the terminology in §5.3. Denote coroots

$$\alpha^\vee_n + 2\sum_{j=r}^{n-1} \alpha^\vee_j + \sum_{j=t}^{r-1} \alpha^\vee_j \qquad \text{for } 1 \leqslant t \leqslant r \leqslant \ell, \text{ and}$$

$$\alpha^\vee_n + 2\sum_{j=m}^{n-1} \alpha^\vee_j + \sum_{j=r}^{m-1} \alpha^\vee_j \qquad \text{for } 1 \leqslant r \leqslant \ell < m \leqslant n$$

by $(2, r, t)$ and $(1, m, r)$ respectively. Note that neither the sum nor the difference of two coroots of the same type is a coroot in $\Phi^\vee_{w_0}$. Therefore, (b′) has no effects on the order of two coroots of the same type, so by (a$'$) additional to ($R1$) we have

$$\text{for } r \geqslant r', (2,r,t) \longleftrightarrow (2,r',t') \text{ if and only if } r > r' \text{ and } t' > t; \tag{$R2$}$$

$$\text{for } r \geqslant r', (1,m,r) \longleftrightarrow (1,m',r') \text{ if and only if } r > r' \text{ and } m' > m. \tag{$R3$}$$

Arguing similarly as in §5.3, we have ($R2$) implies (5.4) and (5.6) for $\lambda = 2$. We also remark that since (c$'$) is not applicable, the coroot $(2,r,r)$ always corresponds to the $(\ell - r + 1)$th s_n in a reduced expression of w_0.

Observe that $(2,r,t)$ can be written as the sum

$$(1,m',r') + (r'',t'')$$

if and only if either $r' = r$ and $t'' = t$ or $r' = t$ and $t'' = r$. We claim the operation (b$'$) can only be applied to triples $(2,r,t), (1,m,r), (m-1,t)$ for $t < r$ due to the occurrence of s_n. Indeed, if $r = t = r'$, then $(2,r,r)$ corresponds to s_n; if $r > t = r'$, then by ($R2$) the coroot $(2,t,t)$, which corresponds to s_n, must come after $(2,r,t)$, while the original position of $(1,m,t)$ is behind $(2,t,t)$.

Consequently, together with operations (a$'$),

$$(2,r,t) \longleftrightarrow (r',t') \text{ if and only if } t' < r; \tag{$B1$}$$

$$(1,m,r) \longleftrightarrow (r',t') \text{ if and only if } t' < r; \tag{$B2$}$$

$$(2,r,t) \longleftrightarrow (1,m',r') \text{ if and only if } r < r' \text{ or } t < r = r'. \tag{$B3$}$$

Inequalities (5.4) and (5.6) for $\lambda = 1$ follow from ($R1$), ($R3$), and ($B2$) by an argument similar to that in §5.3. To see this, flip the above list of coroots of type $(1,m,r)$ along the diagonal and attach it below ($*$), so one obtains a table of coroots (with λ-value equal to 1) of size $2(n-\ell)$ by ℓ. Then rules ($R1$), ($R3$), and ($B2$) combined are essentially the same as the rule ($R1$) for G of type $A_{2n-\ell-1}$ with node ℓ.

We prove the holomorphy of $M'_w(\chi_s)$ by induction on both n and length of w. Only holomorphy of $M'_w(\chi_s)$ for $\chi^2 = 1$ requires a proof. Suppose $\ell = 1$ and $n \geqslant 2$. Then the reduced expression (5.9) of w_0 is unique. Note that (I) only holds for Id and $s_{n-1}s_{n-2}\cdots s_1$. As type A is justified, we may assume $w = s_{m-1}\cdots s_{n-1}s_n s_{n-1}\cdots s_1$ for some $2 \leqslant m \leqslant n$.

If $m < n$, then we need to show

$$M'_{s_m \cdots s_{n-1} s_n \cdots s_2 s_1}(1_s) f^{(s)}|_{s+s_k+1=2n-m-1} \text{ is left } \iota_{\alpha_{m-1}}(\mathrm{SL}_2)\text{-invariant.}$$

Since $M'_{s_{m+1} \cdots s_n s_{n-1} \cdots s_1} f^{(s)}$ is left $\iota_{\alpha_{m-1}}(\mathrm{SL}_2)$-invariant and the coroot $(1, m+1, 1)$ has h-value $2n-m-1$, this follows from (iii). For $m = n$, note that $f^{(s)}$ is left $\iota_{\alpha_n}(\mathrm{SL}_2)$-invariant. Since the coroot $(n-1, 1)$ has h-value $n-1$, by (iii) $M'_{s_{n-1} \cdots s_2 s_1}(1_s) f^{(s)}|_{s+s_k+1=n-1}$ is both left $s_{n-1}\iota_{\alpha_n}(\mathrm{SL}_2)$-invariant and left $\iota_{\alpha_n}(\mathrm{SL}_2)$-invariant. As

$$s_n \alpha_{n-1} = (\alpha_n + \alpha_{n-1}) + \alpha_n = s_{n-1}\alpha_n + \alpha_n,$$

we have $s_n \iota_{\alpha_{n-1}}(\mathrm{SL}_2)$ is contained in the group generated by $s_{n-1}\iota_{\alpha_n}(\mathrm{SL}_2)$ and $\iota_{\alpha_n}(\mathrm{SL}_2)$, and thus

$$M'_{s_n s_{n-1} \cdots s_2 s_1}(1_s) f^{(s)}|_{s+s_k+1=n-1} \text{ is left } \iota_{\alpha_{n-1}}(\mathrm{SL}_2)\text{-invariant.}$$

Now consider general $n > \ell \geqslant 2$. We will use rules $(R1)$-$(R3)$ and $(B1)$-$(B3)$ without further mention below. As type A is justified, by induction hypothesis it suffices to justify the assertion for w such that

$$\Phi_w^\vee \cap \{(r,1), (1,m,1), (2,r',1) : \ell \leqslant r \leqslant n-1, \ell < m \leqslant n, 1 \leqslant r' \leqslant \ell\}$$

is nonempty and $\Phi_w^\vee$ does not only consist of coroots of type (r,t). Write $w = s_\alpha w_1$ and let $\beta^\vee$ be the coroot in $\Phi_w^\vee - \Phi_{w_1}^\vee$.

Lemma 5.4.1. *There exists a reduced expression* $w = s_\alpha w_1$ *such that* $\beta^\vee$ *is either* $(r,1), (1,m,1)$, *or* $(2,r',1)$.

Proof. Observe that the coroot $(1,m,1)$ is not a part of any triple of coroots on which an operation (b′) can be applied. Therefore, if $(1,m,1) \in \Phi_w^\vee$ for some m, then by performing a series of (a′), one can take $\beta^\vee$ to be $(1,m',1) \in \Phi_w^\vee$ with m' minimal.

Thus we assume no coroots $(1,m,1)$ are contained in $\Phi_w^\vee$. Consider first the case that $(r,1) \in \Phi_w^\vee$ for some r. Choose r to be maximal. Let c_1 (resp. c_2) be the number of coroots of the form $(2,r',1)$ (resp. $(1,r+1,r')$) contained in $\Phi_w^\vee$. Note that coroots $(2,r',1), (1,r+1,r'), (r,1)$ form a triple on which the operation (b′) can be applied. Therefore, if $c_1 \leqslant c_2$

(resp. $c_1 > c_2$), then $\beta^\vee$ can be taken as $(r, 1)$ (resp. $(2, \ell - c_2 + 1, 1)$. The case $(2, r', 1) \in \Phi_w^\vee$ for some r' can be argued similarly.

$\square$

Suppose $\beta^\vee = (2, r, 1)$. Note that (I) fails for w_1 unless

$$\ell = r \text{ is odd and } (2, \ell - c, 1 + c) \in \Phi_w^\vee \text{ for all } 0 \leqslant c \leqslant (\ell - 1)/2. \tag{5.10}$$

We need to show

$$M'_{w_1}(\chi_s) f^{(s)}\big|_{2(s+s_k+1)=2(n-1)+1-r} \text{ is left } \iota_\alpha(\mathrm{SL}_2)\text{-invariant} \tag{5.11}$$

except when (5.10) occurs. By (i) and ($R2$) we can assume coroots in $\Phi_w^\vee$ with λ-value 2 have h-value at most $2(n-1) + 2 - r$.

Case $r = 1$: We have $\alpha = \alpha_n$ and we can write $w_1 = w'' s_{n-1} s_n w'$ where s_n and s_{n-1} correspond to coroots $(2, 2, 2)$ and $(2, 2, 1)$ respectively, and $\Phi_{w_1}^\vee - \Phi_{s_{n-1}s_n w'}^\vee$ consists of coroots of the form $(1, m, 2)$. Since coroots $(2, 2, 2)$ and $(2, 2, 1)$ have (h, λ)-value $(2(n-2)+1, 2)$ and $(2(n-1), 2)$ respectively, we have

$$M'_{s_{n-1}s_n w'}(\chi_s) f^{(s)}\big|_{2(s+s_k+1)=2(n-1)} \text{ is left } \iota_{\alpha_n}(\mathrm{SL}_2)\text{-invariant}$$

by (iii) and (iv). As coroots in $\Phi_{w_1}^\vee - \Phi_{s_{n-1}s_n w'}^\vee$ correspond to reflections s_{α_i} where $i < n-1$, to prove (5.11) by (i) it suffices to show

$$\frac{a_{s_{n-1}s_n w'}(\chi_s)}{a_{w_1}(\chi_s)} M_{w''}(\chi_s^{s_{n-1}s_n w'})$$

is holomorphic at $s + s_k + 1 = n - 1$. Observe that the coroot $(1, n-1, 2)$ has h-value n, and the coroot $(1, n, 2)$ must come before $(1, n-1, 2)$. Thus the assertion follows from the proof of type A with node 1.

Case r is even, $\ell > r$: Since we are to check holomorphy of $M'_w(\chi_s)$ at $2(s + s_k + 1) = 2(n-1) + 1 - r$, which is an odd integer, by (i) and (v) we may add to or remove from $\Phi_w^\vee$ coroots with λ-value 1 as long as rules ($R1$)-($R3$) and ($B1$)-($B3$) are obeyed. In particular, we can take w such that $\alpha = \alpha_{n-r+1}$ and $w_1 = s_{n-r} w'$ where s_{n-r} corresponds to the coroot $(2, r+1, 1)$. Then we have

$$\begin{aligned} w'^{-1}\alpha_{n-r+1} &= w_1^{-1}\alpha_{n-r+1} - w'^{-1}\alpha_{n-r} \\ &= (2\sum_{j=r}^{n}\alpha_j + \sum_{j=1}^{r-1}\alpha_j) - (2\sum_{j=r+1}^{n}\alpha_j + \sum_{j=1}^{r}\alpha_j) = \alpha_r. \end{aligned}$$

As the coroot $(2, r+1, 1)$ has h-value $2(n-1)+1-r$ and $f^{(s)}$ is left $\iota_{\alpha_r}(\mathrm{SL}_2)$-invariant, we deduce (5.11) from (iii).

Case $r = \ell$ **is even:** Let $0 \leqslant c \leqslant \ell/2 - 1$ be the largest integer such that $(2, \ell - c, 1+c) \in \Phi_w^\vee$. As in the previous case, we can assume w can be rewritten as $w = s_{n-\ell+1+2c}s_{n-\ell+2+2c}w'$, where $s_{n-\ell+2+2c}$ and $s_{n-\ell+1+2c}$ correspond to coroots $(2, \ell-c, c+2)$ and $(2, \ell-c, c+1)$ respectively. We claim

$$M'_{s_{n-\ell+2+2c}w'}(\chi_s)f^{(s)}|_{2(s+s_k+1)=2(n-1)+1-\ell} \text{ is left } \iota_{\alpha_{n-\ell+1+2c}}(\mathrm{SL}_2)\text{-invariant}$$

We have

$$\begin{aligned} w'^{-1}\alpha_{n-\ell+1+2c} &= (s_{n-\ell+2+2c}w')^{-1}\alpha_{n-\ell+1+2c} - (1+\delta_{c,\frac{\ell}{2}-1})w'^{-1}\alpha_{n-\ell+2+2c} \\ &= (2\sum_{j=\ell-c}^{n}\alpha_j + \sum_{j=c+1}^{\ell-c-1}\alpha_j) - (2\sum_{j=\ell-c}^{n}\alpha_j + \sum_{j=c+2}^{\ell-c-1}\alpha_j) = \alpha_{c+1} \end{aligned}$$

Since $(2, \ell-c, c+2)$ has h-value $2(n-1)+1-\ell$ and $f^{(s)}$ is left $\iota_{\alpha_{c+1}}(\mathrm{SL}_2)$-invariant, our claim follows from (iii).

Case r **is odd,** $\ell > r > 1$**:** Say $\alpha = \alpha_{n-r+1-i}$ where $n-\ell \geqslant i \geqslant 0$ is the number of coroots $(m, 1)$ not in $\Phi_w^\vee$. By our assumption on coroots, we have $(2, t, t) \notin \Phi_w^\vee$ for $2t < r+1$ and thus $(1, m, t) \notin \Phi_w^\vee$ for $t < (r+1)/2$. Write $w_1 = w''s_{n-r-i}w'$ where s_{n-r-i} corresponds to the coroot $(2, r+1, 1)$, and $\Phi_{w_1}^\vee - \Phi_{s_{n-r-i}w'}^\vee$ consists of coroots of the form $(1, m, t)$ where $t \geqslant (r+1)/2$. Choose a reduced expression such that reflections corresponding to coroots in $\Phi^\vee_{w_1} - \Phi^\vee_{s_{n-r-i}w'}$ fixes $\alpha_{n-r+1-i}$. Then we have

$$\begin{aligned} w'^{-1}\alpha_{n-r+1-i} &= (s_{n-r-i}w')^{-1}\alpha_{n-r+1-i} - w'^{-1}\alpha_{n-r-i} \\ &= w_1^{-1}\alpha_{n-r+1-i} - w'^{-1}\alpha_{n-r-i} \\ &= (2\sum_{j=r}^{n}\alpha_j + \sum_{j=1}^{r-1}\alpha_j) - (2\sum_{j=r+1}^{n}\alpha_j + \sum_{j=1}^{r}\alpha_j) = \alpha_r. \end{aligned}$$

As the coroot $(2, r+1, 1)$ has h-value $2(n-1)+1-r$ and $f^{(s)}$ is left $\iota_{\alpha_r}(\mathrm{SL}_2)$-invariant, we conclude by (iii)

$$M'_{s_{n-r-i}w'}(\chi_s)f^{(s)}|_{2(s+s_k+1)=2(n-1)+1-r} \text{ is left } \iota_{\alpha_{n-r+1-i}}(\mathrm{SL}_2)\text{-invariant.}$$

Therefore (5.11) follows if

$$\frac{a_{s_{n-r-i}w'}(\chi_s)}{a_{w_1}(\chi_s)} M_{w''}(\chi_s^{s_{n-r-i}w'})$$

is holomorphic at $2(s+s_k+1) = 2(n-1)+1-r$. By the proof of type A, this is true if $\chi \neq 1$ or $\chi = 1$ and the number of coroots of the form $(1, m, t)$ in $\Phi^\vee_w$ with h-value $\frac{2(n-1)+1-r}{2} + 1$ is at most that with h-value $\frac{2(n-1)+1-r}{2}$.

We assume this is not the case, so $\chi = 1$, $2(n-\ell) \geqslant \frac{2(n-1)+1-r}{2} + 1$, and $(1, n-c, \frac{r+1}{2} + c) \in \Phi^\vee_w$ for all c. Since $(2, \frac{r+1}{2}, \frac{r-1}{2}) \notin \Phi^\vee_w$ by assumption and $(1, n, \frac{r+1}{2}) \in \Phi^\vee_w$, we have $(n-1, \frac{r-1}{2}) \notin \Phi^\vee_w$ but $(n-1, \frac{r+1}{2}) \in \Phi^\vee_w$. Rewrite $w = s_{n-1}v's_{n-\ell+\frac{(r-1)}{2}}v$ such that s_{n-1} and $s_{n-\ell+\frac{(r-1)}{2}}$ correspond to coroots $(1, n, \frac{r+1}{2})$ and $(n-1, \frac{r+1}{2})$ respectively, and coroots of type (m, t) in $\Phi^\vee_w$ all lie in $\Phi^\vee_v$. We need to show

$$M'_{v's_{n-\ell+\frac{(r-1)}{2}}v}(1_s)f^{(s)}|_{s+s_k+1=\frac{2(n-1)+1-r}{2}} \text{ is left } \iota_{\alpha_{n-1}}(\mathrm{SL}_2)\text{-invariant.} \tag{5.12}$$

Lemma 5.4.2. *We have*

$$M'_{v's_{n-\ell+\frac{(r-1)}{2}}v}(1_s)f^{(s)}|_{s+s_k+1=\frac{2(n-1)+1-r}{2}}$$

is left $v'v\iota_{\alpha_n}(\mathrm{SL}_2)$*-invariant.*

Proof. Since the coroot $(n-1, \frac{r+1}{2})$ has h-value $\frac{2(n-1)+1-r}{2}$ and $f^{(s)}$ is left $\iota_{\alpha_n}(\mathrm{SL}_2)$-invariant, by (iii) to justify the lemma it suffices to show

$$\frac{a_{s_{n-\ell+\frac{(r-1)}{2}}v}(1_s)}{a_{v's_{n-\ell+\frac{(r-1)}{2}}v}(1_s)} M_{v'}(1_s^{s_{n-\ell+\frac{(r-1)}{2}}v})$$

is holomorphic at $s + s_k + 1 = \frac{2(n-1)+1-r}{2}$. By $(R2)$ and $(R3)$, the number of coroots in $\Phi^\vee_{v's_{n-\ell+\frac{(r-1)}{2}}v} - \Phi^\vee_{s_{n-\ell+\frac{(r-1)}{2}}v}$ with (h,λ)-value $(2(n-1)+2-r, 2)$ (resp. $(\frac{2(n-1)+1-r}{2}+1, 1)$) is at most that with (h,λ)-value $(2(n-1)+1-r, 2)$ (resp. $(\frac{2(n-1)+1-r}{2}, 1)$). Therefore, the holomorphy follows by the induction on length (and proofs). $\square$

Clearly, $M'_{v's_{n-\ell+\frac{(r-1)}{2}}v}(1_s)f^{(s)}$ is left $v's_{n-\ell+\frac{(r-1)}{2}}v\iota_{\alpha_n}(\mathrm{SL}_2)$-invariant. We have

$$v\alpha_n = \alpha_n + \alpha_{n-1} + \ldots + \alpha_{n-\ell+\frac{(r+1)}{2}},$$

and

$$v'v\alpha_n = s_n\alpha_n, \quad v's_{n-\ell+\frac{(r-1)}{2}}v\alpha_n = s_n(\alpha_{n-1}+\alpha_n).$$

Since $s_n\alpha_n + s_n(\alpha_{n-1}+\alpha_n) = s_ns_n\alpha_{n-1} = \alpha_{n-1}$, we deduce (5.12).

Case $\ell = r$ is odd: Since (5.10) does not hold, there exists smallest $0 \leqslant c \leqslant (\ell-1)/2 - 1$ such that $(2, \ell-c-1, 2+c) \notin \Phi^\vee_w$. We can rewrite $w = s_{n-\ell+1+2c-i}s_{n-\ell+2+2c-i}w'$ for some $i \geqslant 0$, where $s_{n-\ell+2+2c-i}$ and $s_{n-\ell+1+2c-i}$ correspond to coroots $(2, \ell-c, c+2)$ and $(2, \ell-c, c+1)$ respectively. A similar argument as in the case $r = \ell$ even justifies the holomorphy.

For the rest of the proof we may assume $\chi = 1$. Consider $\beta^\vee = (1, m, 1)$ and thus $\alpha = \alpha_{m-1}$. By $(B2)$ $\Phi^\vee_w$ contains all coroots of type (r, t). Therefore, we can choose a reduced expression of w with no operations (b$'$) carried out. Note that (I) fails for w_1 in this case. Suppose $\ell < m < n$. We can write $w_1 = s_mw'$ where s_m corresponds to the coroot $(1, m+1, 1)$. We have

$$\begin{aligned} w'^{-1}\alpha_{m-1} &= w_1^{-1}\alpha_{m-1} - w'^{-1}\alpha_m \\ &= (2\sum_{j=m}^{n}\alpha_j + \sum_{j=1}^{m-1}\alpha_j) - (2\sum_{j=m+1}^{n}\alpha_j + \sum_{j=1}^{m}\alpha_j) = \alpha_m. \end{aligned}$$

Since $f^{(s)}$ is left $\iota_{\alpha_m}(\mathrm{SL}_2)$-invariant and the coroot $(1, m+1, 1)$ has h-value $2n-m-1$, by (iii)

$$M'_{w_1}(1_s)f^{(s)}|_{s+s_k+1=2n-m-1} \text{ is left } \iota_{\alpha_{m-1}}(\mathrm{SL}_2)\text{-invariant.}$$

Suppose $m = n$. We break down the discussion into two cases.

Case $n < 2\ell$**:** Let $0 \leqslant c < n - \ell$ be the largest integer such that $(1, n-c, 1+c) \in \Phi_w^\vee$. Rewrite $w = s_{n-1-2c}s_{n-2-2c}w'$ where s_{n-2-2c} and s_{n-1-2c} correspond to coroots $(1, n-c, 2+c)$ and $(1, n-c, 1+c)$ respectively. We claim

$$M'_{s_{n-2-2c}w'}(1_s)f^{(s)}|_{s+s_k+1=n-1} \text{ is left } \iota_{\alpha_{n-1-2c}}(\mathrm{SL}_2)\text{-invariant}$$

We have

$$\begin{aligned} w'^{-1}\alpha_{n-1-2c} &= (s_{n-2-2c}w')^{-1}\alpha_{n-1-2c} - w'^{-1}\alpha_{n-2-2c} \\ &= (2\sum_{j=n-c}^{n}\alpha_j + \sum_{j=1+c}^{n-c-1}\alpha_j) - (2\sum_{j=n-c}^{n}\alpha_j + \sum_{j=2+c}^{n-c-1}\alpha_j) = \alpha_{c+1} \end{aligned}$$

Since the coroot $(1, n-c, 2+c)$ has h-value $n-1$ and $f^{(s)}$ is left $\iota_{\alpha_{c+1}}(\mathrm{SL}_2)$-invariant, our claim follows from (iii).

Case $n \geqslant 2\ell$**:** Suppose there exists $0 \leqslant c < \ell$ such that $(1, n-c-1, 2+c) \notin \Phi_w^\vee$. Choose c to be minimal, and rewrite $w = s_{n-1-2c}s_{n-2-2c}w'$ where s_{n-2-2c} and s_{n-1-2c} correspond to coroots $(1, n-c, 2+c)$ and $(1, n-c, 1+c)$ respectively. The holomorphy of $M'_w(1_s)$ can be justified similarly as the previous case. Therefore, we assume $(1, n-c, 1+c) \in \Phi_w^\vee$ for all c. We claim

$$M'_{w_1}(1_s)f^{(s)}|_{s+s_k+1=n-1} \text{ is left } \iota_{\alpha_{n-1}}(\mathrm{SL}_2)\text{-invariant.}$$

Lemma 5.4.3. $M'_{w_1}(1_s)f^{(s)}|_{s+s_k+1=n-1}$ *is left* $w''w'\iota_{\alpha_n}(\mathrm{SL}_2)$*-invariant.*

Proof. Write $w_1 = w'' s_{n-\ell} w'$ where $s_{n-\ell}$ corresponds to the coroot $(n-1,1)$ and $\Phi^\vee_{s_{n-\ell}w'}$ consists of all coroots of type (r,t). Since the coroot $(n-1,1)$ has h-value $n-1$, by (iii)

$$M'_{s_{n-\ell}w'}(1_s)f^{(s)}\big|_{s+s_k+1=n-1} \text{ is left } w'\iota_{\alpha_n}(\mathrm{SL}_2)\text{-invariant.}$$

Write $w'' = v'v$ where $\Phi^\vee_{vs_{n-\ell}w'} - \Phi^\vee_{s_{n-\ell}w'}$ consists of all coroots of type $(2,r,t)$. Since $(2,1,1)$ is the only coroot of type $(2,r,t)$ with h-value $2(n-1)+1$ and coroots $(2,2,2)$ and $(2,2,1)$ are in $\Phi^\vee_{vs_{n-\ell}w'} - \Phi^\vee_{s_{n-\ell}w'}$, it follows from the proof of the case $\beta^\vee = (2,1,1)$ that

$$\frac{a_{s_{n-\ell}w'}(1_s)}{a_{vs_{n-\ell}w'}(1_s)} M_v(1_s^{s_{n-\ell}w'})$$

is holomorphic at $s + s_k + 1 = n-1$. Therefore, the lemma follows once we show

$$\frac{a_{vs_{n-\ell}w'}(1_s)}{a_{w_1}(1_s)} M_{v'}(1_s^{vs_{n-\ell}w'})$$

is holomorphic at $s + s_k + 1 = n-1$. Note that $\Phi^\vee_{w_1} - \Phi^\vee_{vs_{n-\ell}w'}$ consists of coroots of type $(1,r,t)$. By $(R3)$ the number of coroots in $\Phi^\vee_{w_1} - \Phi^\vee_{vs_{n-\ell}w'}$ with h-value n is at most that with h-value $n-1$. Therefore, the holomorphy follows from the proof of type A. ☐

Clearly, $M'_{w_1}(1_s)f^{(s)}$ is left $w_1\iota_{\alpha_n}(\mathrm{SL}_2)$-invariant. Since $(1, n-c, 1+c) \in \Phi^\vee_w$ for all c, we have $(1,n,r) \in \Phi^\vee_w$ for all r, and thus a direct computation gives

$$w_1\alpha_n = \alpha_n + \alpha_{n-1}, \quad w''w'\alpha_n = -\alpha_n.$$

As $(\alpha_n + \alpha_{n-1}) + (-\alpha_n) = \alpha_{n-1}$, our claim follows from the above lemma.

Consider now $\beta^\vee = (r,1)$. By (i), $(R1)$, $(R3)$ and $(B2)$, we may assume every coroot with λ-value 1 in $\Phi^\vee_w$ has h-value at most r.

Case $n-1 \geqslant r > \ell$: Note that (I) fails for w_1 in this case. If $(1, r+1, \ell) \notin \Phi^\vee_w$, then $\alpha = \alpha_{r-\ell+1}$ and we can write $w_1 = s_{r-\ell}w'$ where $s_{r-\ell}$ corresponds to the coroot $(r-1,1)$. We have

$$w'^{-1}\alpha_{r-\ell+1} = w_1^{-1}\alpha_{r-\ell+1} - w'^{-1}\alpha_{r-\ell} = \sum_{j=1}^{r}\alpha_j - \sum_{j=1}^{r-1}\alpha_j = \alpha_r.$$

Since the coroot $(r-1,1)$ has h-value $r-1$ and $f^{(s)}$ is left $\iota_{\alpha_r}(\mathrm{SL}_2)$-invariant, by (iii)

$$M'_{w_1}(1_s)f^{(s)}\big|_{s+s_k+1=r-1} \text{ is left } \iota_{\alpha_r}(\mathrm{SL}_2)\text{-invariant.}$$

Therefore, we assume $(1,r+1,\ell)\in\Phi_w^\vee$. Let $\ell > c \geqslant 1$ be the number of coroots $(1,r+1,t)$ in $\Phi_w^\vee$. Suppose the coroot $(1,r+1,\ell+1-c)$ has h-value less than r. If $(1,r,\ell+1-c)\notin\Phi_w^\vee$, then we can rewrite $w=s'w'$ such that s' corresponds to the coroot $(1,r+1,\ell+1-c)$. The holomorphy of $M'_w(1_s)=M'_{s'w'}(1_s)$ follows by induction hypothesis and (i). If $(1,r,\ell+1-c)\in\Phi_w^\vee$, let $m\leqslant r$ be the smallest integer such that $(1,m,\ell+1-c)\in\Phi_w^\vee$. Rewrite $w=s_is_{i+1}w'$ $(i<n-1)$ such that s_{i+1} and s_i correspond to coroots $(1,m+1,\ell+1-c)$ and $(1,m,\ell+1-c)$ respectively. The holomorphy of $M'_w(1_s)$ in this case follows by a similar argument as in the case $\beta^\vee=(1,m,1)$.

Consequently, we assume the coroot $(1,r+1,\ell+1-c)$ has h-value r, i.e., $2n-r-\ell-1+c=r$. If $c\geqslant 2$ and $(1,r,\ell+2-c)\notin\Phi_w^\vee$, then we rewrite $w=s_{i-1}s_iw'$ $(i<n)$ where s_i and s_{i-1} corresponds to coroots $(1,r+1,\ell+2-c)$ and $(1,r+1,\ell+1-c)$ respectively. We have

$$\begin{aligned} w'^{-1}\alpha_{i-1} &= (s_iw_1)^{-1}\alpha_{i-1}-w'^{-1}\alpha_i \\ &= \Big(2\sum_{j=r+1}^{n}\alpha_j+\sum_{j=\ell+1-c}^{r}\alpha_j\Big)-\Big(2\sum_{j=r+1}^{n}\alpha_j+\sum_{j=\ell+2-c}^{r}\alpha_j\Big)=\alpha_{\ell+1-c}. \end{aligned}$$

Since the coroot $(1,r+1,\ell+1-c)$ has h-value $r-1$ and $f^{(s)}$ is left $\iota_{\alpha_{\ell+1-c}}(\mathrm{SL}_2)$-invariant, we have

$$M'_{s_iw'}(1_s)f^{(s)}\big|_{s+s_k+1=r-1} \text{ is left } \iota_{\alpha_{i-1}}(\mathrm{SL}_2)\text{-invariant.}$$

Suppose either $c=1$ or $c\geqslant 2$ and $(1,r,\ell+2-c)\in\Phi_w^\vee$. Let s_i be a reflection such that $\ell(s_iw)=1+\ell(w)$ and the coroot corresponding to s_i is $(1,r,\ell+1-c)$. Since the coroot $(1,r,\ell+1-c)$ has h-value $r+1$, by (i) and (v) $M'_w(1_s)$ is holomorphic at $s+s_k+1=r-1$ iff $M'_{s_iw}(1_s)$ is. To see $M'_{s_iw}(1_s)$ is holomorphic at $s+s_k+1=r-1$, rewrite $s_iw=s_{r-\ell+c+1}s_{r-\ell+c}w'$ where $s_{r-\ell+c}$ and $s_{r-\ell+c+1}$ correspond to coroots $(r,1)$ and $(r-1,1)$ respectively. The holomorphy follows from a similar argument as the case $(1,r+1,\ell)\notin\Phi_w^\vee$.

Case $r=\ell, 2(n-\ell)<\ell$**:** Note that in this case (I) fails for w_1. For the case that $(1,\ell+1+t,2n-2\ell-t)\in\Phi_w^\vee$ for some t, the holomorphy in this case follows from an argument analogous to that of the previous case when $(1, r+1, \ell+1-c) \in \Phi_w^\vee$ with $c \geqslant 2$ but $(1, r, \ell+2-c) \notin \Phi_w^\vee$. Therefore, suppose $(1, \ell+1+t, 2n-2\ell-t) \notin \Phi_w^\vee$ for any t. If there is no coroot of type $(1, m, t)$ in $\Phi_w^\vee$, then we can write $w = w''w'$, where $\Phi_{w'}^\vee$ consists of all coroots of type (m, t) in $\Phi_w^\vee$. The holomorphy then follows from the proof of the case $\beta^\vee = (2, m, 1)$. If $\Phi_w^\vee$ contains some coroot of type $(1, m, t)$, we can rewrite $w = s_i w'$ where s_i corresponds to a coroot with (h, λ)-value $(c, 1)$ where $c < \ell$. Then by induction hypothesis and (i) $M'_w(1_s) = M'_{s_i w'}(1_s)$ is holomorphic at $s + s_k + 1 = \ell - 1$.

Case $r = \ell, 2(n-\ell) \geqslant \ell$**:** By the same argument as the previous case, it suffices to consider when $n - \ell < \ell$ and $(1, 2(n-\ell)+1+t, \ell-t) \in \Phi_w^\vee$ for all t. Since $2(n-\ell) \geqslant \ell$, for (I) to fail for w_1, there exists smallest $n - \ell > c \geqslant 0$ such that $(\ell+c+1, 2+c) \notin \Phi_w^\vee$ and $(\ell+c, 1+c) \in \Phi_w^\vee$. Since $n < 2\ell$, we can rewrite $w = s_i s_{i+1} w'$ $(i < n-1)$ where s_{i+1} and s_i correspond to coroots $(\ell+c, 2+c)$ and $(\ell+c, 1+c)$. We have

$$w'^{-1}\alpha_i = (s_{i+1}w')^{-1}\alpha_i - w'^{-1}\alpha_{i+1} = \sum_{j=1+c}^{\ell+c} \alpha_j - \sum_{j=2+c}^{\ell+c} \alpha_j = \alpha_{c+1}.$$

Since the coroot $(\ell+c, 2+c)$ has h-value $\ell - 1$ and $f^{(s)}$ is left $\iota_{\alpha_{c+1}}(\mathrm{SL}_2)$-invariant, by (iii)

$$M'_{s_{i+1}w'}(1_s) f^{(s)}\big|_{s+s_k+1=\ell-1} \text{ is left } \iota_{\alpha_i}(\mathrm{SL}_2)\text{-invariant.}$$

This completes the proof for the case $\ell < n$.

For $\ell = n \geqslant 2$, a reduced expression of w_0 is

$$(s_n) \cdots (s_r \cdots s_n) \cdots (s_1 \cdots s_n); \tag{5.13}$$

corresponding coroots $\tilde{\alpha}_{(i)}^\vee$ are

$$\begin{array}{ccccc}
\alpha_n^\vee + 2\sum_{j=n}^{n-1} \alpha_j^\vee, & \cdots & \alpha_n^\vee + 2\sum_{j=n}^{n-1} \alpha_j^\vee + \sum_{j=t}^{n-1} \alpha_j^\vee, & \cdots & \alpha_n^\vee + 2\sum_{j=n}^{n-1} \alpha_j^\vee + \sum_{j=1}^{n-1} \alpha_j^\vee, \\
 & \ddots & \vdots & \ddots & \vdots \\
 & & \alpha_n^\vee + 2\sum_{j=r}^{n-1} \alpha_j^\vee + \sum_{j=t}^{r-1} \alpha_j^\vee, & \cdots & \alpha_n^\vee + 2\sum_{j=r}^{n-1} \alpha_j^\vee + \sum_{j=1}^{r-1} \alpha_j^\vee, \\
 & & & \ddots & \vdots \\
 & & & & \alpha_n^\vee + 2\sum_{j=1}^{n-1} \alpha_j^\vee.
\end{array}$$

The inequalities (5.4) and (5.6) and the holomorphy of $M'_w(\chi_s)$ (especially $\chi = 1$) follow from the same (and actually simpler) argument as the previous case for type $(2, r, t)$ coroots. $\square$

5.5 Type C_n $(n \geqslant 2)$

For $1 \leqslant \ell < n$, coroots $\tilde{\alpha}^{\vee}_{(i)}$ corresponding to the reduced expression (5.9) are

$$\begin{array}{ccccc}
\alpha^{\vee}_{\ell}, & \cdots & \sum_{j=t}^{\ell} \alpha^{\vee}_j, & \cdots & \sum_{j=1}^{\ell} \alpha^{\vee}_j, \\
\vdots & & \vdots & & \vdots \\
\sum_{j=\ell}^{r} \alpha^{\vee}_j, & \cdots & \sum_{j=t}^{r} \alpha^{\vee}_j, & \cdots & \sum_{j=1}^{r} \alpha^{\vee}_j, \\
\vdots & & \vdots & & \vdots \\
\sum_{j=\ell}^{n-1} \alpha^{\vee}_j, & \cdots & \sum_{j=t}^{n-1} \alpha^{\vee}_j, & \cdots & \sum_{j=1}^{n-1} \alpha^{\vee}_j,
\end{array}$$

$$\begin{array}{cccccc}
\sum_{j=\ell}^{n} \alpha^{\vee}_j & 2\sum_{j=\ell}^{n} \alpha^{\vee}_j + \alpha^{\vee}_{\ell-1}, & \cdots & 2\sum_{j=\ell}^{n} \alpha^{\vee}_j + \sum_{j=t}^{\ell-1} \alpha^{\vee}_j, & \cdots & 2\sum_{j=\ell}^{n} \alpha^{\vee}_j + \sum_{j=1}^{\ell-1} \alpha^{\vee}_j, \\
 & 2\alpha^{\vee}_n + \sum_{j=\ell}^{n-1} \alpha^{\vee}_j, & \cdots & 2\sum_{j=m}^{n} \alpha^{\vee}_j + \sum_{j=\ell}^{m-1} \alpha^{\vee}_j, & \cdots & 2\sum_{j=\ell+1}^{n} \alpha^{\vee}_j + \alpha^{\vee}_{\ell}, \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\sum_{j=r}^{n} \alpha^{\vee}_j & & & 2\sum_{j=r}^{n} \alpha^{\vee}_j + \sum_{j=t}^{r-1} \alpha^{\vee}_j, & \cdots & 2\sum_{j=r}^{n} \alpha^{\vee}_j + \sum_{j=1}^{r-1} \alpha^{\vee}_j, \\
 & 2\alpha^{\vee}_n + \sum_{j=r}^{n-1} \alpha^{\vee}_j, & \cdots & 2\sum_{j=m}^{n} \alpha^{\vee}_j + \sum_{j=r}^{m-1} \alpha^{\vee}_j, & \cdots & 2\sum_{j=\ell+1}^{n} \alpha^{\vee}_j + \sum_{j=r}^{\ell} \alpha^{\vee}_j, \\
\vdots & \vdots & & \vdots & \ddots & \vdots \\
\sum_{j=2}^{n} \alpha^{\vee}_j & & & & & 2\sum_{j=2}^{n} \alpha^{\vee}_j + \alpha^{\vee}_1, \\
 & 2\alpha^{\vee}_n + \sum_{j=2}^{n-1} \alpha^{\vee}_j, & \cdots & 2\sum_{j=m}^{n} \alpha^{\vee}_j + \sum_{j=2}^{m-1} \alpha^{\vee}_j, & \cdots & 2\sum_{j=\ell+1}^{n} \alpha^{\vee}_j + \sum_{j=2}^{\ell} \alpha^{\vee}_j, \\
\sum_{j=1}^{n} \alpha^{\vee}_j & & & & & \\
 & 2\alpha^{\vee}_n + \sum_{j=1}^{n-1} \alpha^{\vee}_j, & \cdots & 2\sum_{j=m}^{n} \alpha^{\vee}_j + \sum_{j=1}^{m-1} \alpha^{\vee}_j, & \cdots & 2\sum_{j=\ell+1}^{n} \alpha^{\vee}_j + \sum_{j=1}^{\ell} \alpha^{\vee}_j .
\end{array}$$

By duality, the proof for type B_n carries over with minor modification. The major difference is that ith s_n in any reduced expression of w_0 corresponds to the coroot $\sum_{j=\ell-i+1}^{n} \alpha^{\vee}_j$, which has λ-value 1 instead of 2. If the last coroot is $\sum_{j=1}^{n} \alpha^{\vee}_j$, one modifies the argument of the case $\beta^{\vee} = (1, n, 1)$ in §5.4. Some explanation is given in the case $\ell = n$ below. If the last coroot is $2\alpha^{\vee}_n + \sum_{j=1}^{n-1} \alpha^{\vee}_j$ one applies the argument of the case $\beta^{\vee} = (1, r, 1)$ with $r < n$ in §5.4. We leave the details of the other cases to the reader.

For $\ell = n$, coroots $\tilde{\alpha}^{\vee}_{(i)}$ corresponding to the reduced expression (5.13) are

$$\begin{array}{cccccc}
\alpha^{\vee}_n, & 2\alpha^{\vee}_n + \alpha^{\vee}_{n-1}, & \cdots & 2\sum_{j=n}^{n} \alpha^{\vee}_j + \sum_{j=t}^{n-1} \alpha^{\vee}_j, & \cdots & 2\sum_{j=n}^{n} \alpha^{\vee}_j + \sum_{j=1}^{n-1} \alpha^{\vee}_j, \\
\vdots & & \ddots & \vdots & \ddots & \vdots \\
\sum_{j=r}^{n} \alpha^{\vee}_j, & & & 2\sum_{j=r}^{n} \alpha^{\vee}_j + \sum_{j=t}^{r-1} \alpha^{\vee}_j, & \cdots & 2\sum_{j=r}^{n} \alpha^{\vee}_j + \sum_{j=1}^{r-1} \alpha^{\vee}_j, \\
\vdots & & & & \ddots & \vdots \\
\sum_{j=2}^{n} \alpha^{\vee}_j, & & & & & 2\sum_{j=2}^{n} \alpha^{\vee}_j + \alpha^{\vee}_1, \\
\sum_{j=1}^{n} \alpha^{\vee}_j . & & & & &
\end{array}$$

As in the case $\ell < n$, except for the case where the last coroot is $\sum_{j=1}^{n} \alpha_j^\vee$, a similar inductive proof as in §5.4 for type $(2, r, t)$ coroots justifies the holomorphy in this case. Therefore, we only explain how to prove the holomorphy of $M'_{w_0}(1_s)$ assuming $M'_w(\chi_s)$ is holomorphic for any $w \neq w_0$ and χ.

Rewrite $w_0 = s_n s_{n-1} s_{n-2} s_n w'$ by switching the order of coroots $\sum_{j=2}^{n} \alpha_j^\vee$ and $2\sum_{j=3}^{n} \alpha_j^\vee + \sum_{j=1}^{2} \alpha_j^\vee$. We need to show

$$M'_{s_{n-1}s_{n-2}s_n w'}(1_s) f^{(s)}\big|_{s+s_k+1=n-1} \text{ is left } \iota_{\alpha_n}(\mathrm{SL}_2)\text{-invariant.}$$

We have $w'\alpha_{n-1} = s_{n-1}\alpha_{n-2}$ and

$$s_n s_{n-1}\alpha_{n-2} + s_{n-1}\alpha_{n-2} = \alpha_n + 2\alpha_n + 2\alpha_{n-2} = s_{n-2}s_{n-1}\alpha_n.$$

Since $\sum_{j=2}^{n} \alpha_j^\vee$ has h-value $n-1$ and $f^{(s)}$ is left $\iota_{\alpha_{n-1}}(\mathrm{SL}_2)$-invariant, by (iii)

$$M'_{s_n w'}(1_s) f^{(s)}\big|_{s+s_k+1=n-1} \text{ is left } s_{n-2}s_{n-1}\iota_{\alpha_n}(\mathrm{SL}_2)\text{-invariant.}$$

Therefore, to justify the holomorphy it suffices to show

$$\frac{a_{s_n w'}(1_s)}{a_{s_{n-1}s_{n-2}s_n w'}(1_s)} M_{s_{n-1}s_{n-2}}(1_s^{s_n w'})$$

is holomorphic at $s + s_k + 1 = n - 1$. Since coroots corresponding to s_{n-2} and s_{n-1} have (h, λ)-value $(2(n-1), 2)$ and $(2(n-1)+1, 2)$ respectively, the holomorphy follows again from (iii). □

5.6 Type D_n $(n \geqslant 4)$

For $1 \leqslant \ell < n-1$, a reduced expression of w_0 is

$$\begin{aligned}
&(s_\ell \cdots s_{n-3} s_{n-2} s_{n-(\ell+1 \mod 2)})\cdot \\
&\big((s_{\ell-1} \cdots s_{n-2})(s_{n-(\ell \mod 2)})\big)\cdot \\
&\vdots \\
&\big((s_4 \cdots s_{n-\ell+3})(s_{n-\ell+4} \cdots s_{n-3} s_{n-2} s_{n-1})\big)\cdot \\
&\big((s_3 \cdots s_{n-\ell+2})(s_{n-\ell+3} \cdots s_{n-3} s_{n-2} s_{n})\big)\cdot \\
&\big((s_2 \cdots s_{n-\ell+1})(s_{n-\ell+2} \cdots s_{n-3} s_{n-2} s_{n-1})\big)\cdot \\
&\big((s_1 \cdots s_{n-\ell})(s_{n-\ell+1} \cdots s_{n-3} s_{n-2} s_{n})\big)\cdot \\
&(s_{n-\ell} \cdots s_{n-1}) \cdots (s_2 \cdots s_\ell s_{\ell+1})(s_1 \cdots s_{\ell-1} s_\ell);
\end{aligned} \tag{5.14}$$

corresponding coroots $\tilde{\alpha}^\vee_{(i)}$ are

$$\begin{array}{ccccc}
\alpha^\vee_\ell, & \cdots & \sum_{j=t}^{\ell} \alpha^\vee_j, & \cdots & \sum_{j=1}^{\ell} \alpha^\vee_j, \\
\vdots & & \vdots & & \vdots \\
\sum_{j=\ell}^{r} \alpha^\vee_j, & \cdots & \sum_{j=t}^{r} \alpha^\vee_j, & \cdots & \sum_{j=1}^{r} \alpha^\vee_j, \\
\vdots & & \vdots & & \vdots \\
\sum_{j=\ell}^{n-1} \alpha^\vee_j, & \cdots & \sum_{j=t}^{n-1} \alpha^\vee_j, & \cdots & \sum_{j=1}^{n-1} \alpha^\vee_j,
\end{array} \tag{$*'$}$$

$$\begin{array}{ccccc}
\sum_{j=\ell}^{n-2} \alpha^\vee_j + \sum_{j=\ell-1}^{n} \alpha^\vee_j, & \cdots & \sum_{j=\ell}^{n-2} \alpha^\vee_j + \sum_{j=t}^{n} \alpha^\vee_j, & \cdots & \sum_{j=\ell}^{n-2} \alpha^\vee_j + \sum_{j=1}^{n} \alpha^\vee_j, \\
\alpha^\vee_n + \sum_{j=\ell}^{n-2} \alpha^\vee_j, & \cdots & \sum_{j=m}^{n} \alpha^\vee_j + \sum_{j=\ell}^{n-2} \alpha^\vee_j, & \cdots & \sum_{j=\ell+1}^{n} \alpha^\vee_j + \sum_{j=\ell}^{n-2} \alpha^\vee_j, \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
 & & \sum_{j=r}^{n-2} \alpha^\vee_j + \sum_{j=t}^{n} \alpha^\vee_j, & \cdots & \sum_{j=r}^{n-2} \alpha^\vee_j + \sum_{j=1}^{n} \alpha^\vee_j, \\
\alpha^\vee_n + \sum_{j=r}^{n-2} \alpha^\vee_j, & \cdots & \sum_{j=m}^{n} \alpha^\vee_j + \sum_{j=r}^{n-2} \alpha^\vee_j, & \cdots & \sum_{j=\ell+1}^{n} \alpha^\vee_j + \sum_{j=r}^{n-2} \alpha^\vee_j, \\
\vdots & & \vdots & \ddots & \vdots \\
 & & & & \sum_{j=2}^{n-2} \alpha^\vee_j + \sum_{j=1}^{n} \alpha^\vee_j, \\
\alpha^\vee_n + \sum_{j=2}^{n-2} \alpha^\vee_j, & \cdots & \sum_{j=m}^{n} \alpha^\vee_j + \sum_{j=2}^{n-2} \alpha^\vee_j, & \cdots & \sum_{j=\ell+1}^{n} \alpha^\vee_j + \sum_{j=2}^{n-2} \alpha^\vee_j, \\
 & & & & \\
\alpha^\vee_n + \sum_{j=1}^{n-2} \alpha^\vee_j, & \cdots & \sum_{j=m}^{n} \alpha^\vee_j + \sum_{j=1}^{n-2} \alpha^\vee_j, & \cdots & \sum_{j=\ell+1}^{n} \alpha^\vee_j + \sum_{j=1}^{n-2} \alpha^\vee_j,
\end{array}$$

Here $(*')$ corresponds to the last row of (5.14). For rows of coroots below $(*')$, the ith odd (resp. even) rows correspond to the former (resp. latter) parenthesis in the $(i+1)$th big parenthesis (counting from bottom to top and from right to left).

We retain the terminology in §5.3 and follow the idea in §5.4. Denote coroots

$$\sum_{j=r}^{n-2} \alpha_j^\vee + \sum_{j=t}^{n} \alpha_j^\vee = \alpha_n^\vee + \alpha_{n-1}^\vee + 2\sum_{j=r}^{n-2} \alpha_j^\vee + \sum_{j=t}^{r-1} \alpha_j^\vee \quad \text{for } 1 \leqslant t < r \leqslant \ell, \text{ and}$$

$$\sum_{j=m}^{n} \alpha_j^\vee + \sum_{j=r}^{n-2} \alpha_j^\vee \quad \text{for } 1 \leqslant r \leqslant \ell < m \leqslant n$$

by $(2, r, t)$ and $(1, m, r)$ respectively. As the sum of two coroots of the same type is not a coroot, so we still have rules ($R1$), ($R2$), ($R3$). The rule ($R2$) implies (5.4) and (5.6) for $\lambda = 2$ similarly.

Remark 5.6.1. Rules of coroots are stable under the symmetry of the Dynkin diagram D_n. For instance, ($R1$) implies the coroot $(1, n, t)$ must come after the coroot $(n-2, t)$ for any $1 \leqslant t \leqslant \ell$.

Observe that the coroot $(2, r, t)$ can be written as a sum of coroots

$$(1, m', r') + (r'', t'')$$

if and only if either $r' = r$ and $t'' = t$ or $r' = t$ and $t'' = r$. In both cases, $m' = r'' + 1$.

Lemma 5.6.2. *The operation (b′) cannot be applied to the triple of coroots* $(2, r, t), (1, m, t), (m - 1, r)$ *if (and only if)* $m < n$.

Proof. By ($R2$) and ($R3$), it suffices to show for $r = t + 1, m = n - 1$. By ($R1$), an operation (b′) that reverses the order of coroots $(n-1, t+1)$ and $(2, t+1, t)$ needs to be carried out first. However, in this case the coroot $(1, n, t+1)$ comes before $(2, t+1, t)$, but comes after $(n-2, t+1)$ by Remark 5.6.1. □

Consequently, together with operations (a′), we have

$$(2, r, t) \longleftrightarrow (r', t') \text{ if and only if } t' < r \text{ or } r' = n - 1; \tag{D1}$$

$$(1, m, r) \longleftrightarrow (r', t') \text{ if and only if } t' < r \text{ or } m = r' + 1 = n; \tag{D2}$$

$$(2, r, t) \longleftrightarrow (1, m', r') \text{ if and only if } r < r' \text{ or } t < r = r' \text{ or } m' = n. \tag{D3}$$

Inequalities (5.4) and (5.6) for $\lambda = 1$ follow similarly from ($R1$), ($R3$), and ($D2$).

As mentioned in Remark 5.6.1, the relaxation of rules, compared to rules of type B, arises from the symmetry of D_n. More precisely, the Dynkin diagram of D_n folds into that of B_{n-1}. This is the reason why we have named our coroots the same way as in §5.4. We explain how one can modify the inductive proof of holomorphy of $M'_w(\chi_s)$ of type B to that of type D.

By induction hypothesis we have an analogue of Lemma 5.4.1 that asserts that w can be written as $s_\alpha w_1$ where the corresponding coroot $\beta^\vee$ of s_α is either $(r,1)$, $(1,m,1)$, or $(2,r',1)$. For the case $\beta^\vee = (r,1)$ for $r \neq n-1$, $\beta^\vee = (1,m,1)$ for $m \neq n$, or $\beta^\vee = (2,r',1)$ for $1 < r'$, a similar proof for the same type of roots considered in §5.4 proves the holomorphy. Therefore, by the symmetry of D_n, it suffices to consider the case where $\beta^\vee = (n-1,1)$ and $\chi = 1$. We can assume any coroot in $\Phi_w^\vee$ with λ-value 1 has h-value at most $n-1$. Note that (I) fails if $\ell \geqslant 2$ or $\ell = 1$ and $(1,n,1) \notin \Phi_w^\vee$. If $(1,n,1) \notin \Phi_w^\vee$, then the holomorphy follows from the argument in §5.4 of the case $\beta^\vee = (n-1,1)$. Therefore, we assume $(1,n,1) \in \Phi_w^\vee$. Since (I) holds for w_1 if $\ell = 1$, it suffices to consider $\ell \geqslant 2$. In this case, a similar argument as in §5.4 of the case $\beta^\vee = (1,n,1)$ justifies the holomorphy.

For $\ell = n, n-1$, by symmetry it suffices to deal with either case. Let $\ell = n$. A reduced expression of w_0 is

$$\begin{aligned}
&(s_{n-(\ell+1 \mod 2)})\cdot\\
&(s_{n-2}s_{n-(\ell \mod 2)})\cdot\\
&\vdots\\
&(s_4 \cdots s_{n-3}s_{n-2}s_{n-1})\cdot\\
&(s_3 \cdots s_{n-3}s_{n-2}s_n)\cdot\\
&(s_2 \cdots s_{n-3}s_{n-2}s_{n-1})\cdot\\
&(s_1 \cdots s_{n-3}s_{n-2}s_n);
\end{aligned}$$

corresponding coroots $\tilde{\alpha}_{(i)}^\vee$ are

$$\begin{array}{cccccc}
\alpha_n^\vee & \cdots & \cdots & \alpha_n^\vee + \sum_{j=t}^{n-2}\alpha_j^\vee\,, & \cdots & \alpha_n^\vee + \sum_{j=1}^{n-2}\alpha_j^\vee \\
 & \sum_{j=n-2}^{n}\alpha_j^\vee\,, & \cdots & \sum_{j=t}^{n}\alpha_j^\vee\,, & \cdots & \sum_{j=1}^{n}\alpha_j^\vee\,, \\
 & & \ddots & \vdots & \ddots & \vdots \\
 & & & \sum_{j=r}^{n-2}\alpha_j^\vee + \sum_{j=t}^{n}\alpha_j^\vee\,, & \cdots & \sum_{j=r}^{n-2}\alpha_j^\vee + \sum_{j=1}^{n}\alpha_j^\vee\,, \\
 & & & & \ddots & \vdots \\
 & & & & & \sum_{j=2}^{n-2}\alpha_j^\vee + \sum_{j=1}^{n}\alpha_j^\vee\,.
\end{array}$$

As mentioned above, the proof of holomorphy for coroots of type $(2, r, t)$ with $r > t$ in §5.4 can be modified to justify the holomorphy. We leave the details to the reader. $\square$

Chapter 6. Conclusions

We have fully developed the Fourier theory on X_P over nonarchimedean local fields. It is my ongoing work to prove analogous results in Chapter 4 and Chapter 5 (in a non ad-hoc way) over archimedean local fields and develop harmonic analysis. Surprisingly, through harmonic analysis on X_P, we discover a candidate of the "correct" Weyl algebra on X_P. We briefly explain the idea below.

Recall that when $X_P = \mathbb{G}_a^n$ is a vector space, one has by integration by parts

$$\frac{\partial}{\partial x_i} \circ \mathcal{F}_{\mathbb{R}^n} = 2\pi\sqrt{-1} \cdot \mathcal{F}_{\mathbb{R}^n} \circ x_i,$$

$$-2\pi\sqrt{-1} \cdot x_i \circ \mathcal{F}_{\mathbb{R}^n} = \mathcal{F}_{\mathbb{R}^n} \circ \frac{\partial}{\partial x_i},$$

if $\psi(x) = e^{2\pi\sqrt{-1}x}$. Furthermore, this property uniquely characterizes $\mathcal{F}_{\mathbb{R}^n}$ up to a nonzero constant. This gives a canonical set of generators of the classical Weyl algebra on $\mathbb{A}^n$:

$$K\left[x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right]$$

where K is any field of characteristic zero. We can show that analogous statements hold for $\mathcal{F}_{P|P^{\mathrm{op}}}$ when X_P is singular, which relies heavily on the fact that $\mathcal{S}(X_P(F))$ can be shown to be local (cf. Remark 5.1.12). However, in this case the corresponding differential operators are no longer of first order, but is of order $\sum_{i=1}^{k} \lambda_i > 1$. This relates to the fact that one needs the additional normalization operator μ_P^{aug} to normalize the naive geometric Fourier transform $\mathcal{F}_{P|P^{\mathrm{op}}}^{\mathrm{geo}}$ (cf. Remark 3.5.4).

We note that in general one can take coordinate functions and derivations on X_P as generators to define a Weyl algebra on X_P :

$$\Delta(X_P) := K[X_P]\left[\Gamma(X_P, \mathcal{T}_{X_P})\right]$$

where $\mathcal{T}_{X_P}$ is the tangent sheaf on X_P. However, $\Delta(X_P)$ is a simple ring iff X_P is smooth. On the other hand, consider the Weyl algebra W_{X_P} defined via harmonic analysis as discussed above. We can show W_{X_P} is a simple domain and an analogue of the Bernstein's inequality holds. This opens up the possibility for studying the Riemann-Hilbert correspondence, Bernstein-Sato polynomials, and zeta functions on Braverman-Kazhdan spaces.

In conclusion, what harmonic analysis on affine spherical varieties would buy us is a bridge between analysis and geometry (in the spirit of the Riemann-Hilbert correspondence). We expect that through studying the Riemann-Hilbert correspondence on X_P, we can gain insights into how one should extract analytic data from geometric inputs in the Poisson summation conjecture.

Appendix A. Computation of normalizing factors

In this appendix we compute the normalizing factors $\{(s_i, \lambda_i)\}$. The parameters depend only on the action of the dual group $\widehat{M}$ acting on the dual Lie algebra $\widehat{\mathfrak{n}}_P$ and our fixed isomorphism

$$\omega_P : M^{\mathrm{ab}} \xrightarrow{\sim} \mathbb{G}_m \tag{A.1}$$

where ω_P is defined as in (3.1.1).

To avoid proliferation of duals, we work directly in the dual picture in this section. Thus now G denotes an adjoint simple group over $\mathbb{C}$ with maximal parabolic subgroup P, Levi subgroup M, and we are studying the action of M on $\mathfrak{n}_P$, the complex Lie algebra of the unipotent radical N_P of P. We define the parameters (s_i, λ_i) as in §3.2.1 but with $\widehat{M}$, $\widehat{\mathfrak{n}}_P$ in that section replaced by M and $\mathfrak{n}_P$, respectively. We let T be a maximal torus in M, $T < B \leqslant P$ a Borel subgroup, and Δ the corresponding set of simple roots. We let β be the simple root such that $\Delta - \{\beta\}$ is the set of simple roots of $(M, M \cap B, T)$. The dual of (A.1) is an isomorphism

$$\varphi : \mathbb{G}_m \xrightarrow{\sim} Z(M). \tag{A.2}$$

For any representation W of M and any integer λ, we write $W(\lambda)$ for the subspace on which $Z(M) = \mathbb{G}_m$ acts via $x \mapsto x^{\lambda}$.

Lemma A.0.1. *If* $\lambda \leqslant 0$, *then* $\mathfrak{n}_P(\lambda) = 0$.

Proof. Let γ be a positive root of (G, B, T). Note that the root space $(\mathfrak{n}_P)_\gamma$ is non-zero if and only if writing $\gamma = \sum_{\alpha \in \Delta} c_\alpha \alpha$ we have $c_\beta > 0$. It follows from (3.1.1) that

$$\langle \gamma, \varphi \rangle = c_\beta \langle \beta, \varphi \rangle = c_\beta m_{\beta^\vee} > 0.$$

We deduce the lemma. □

In each of the cases given below, the isomorphism $\varphi : \mathbb{G}_m \to Z(M)$ will be the "obvious one", so we will not record it. In fact, there are only two choices of isomorphism

$\mathbb{G}_m \xrightarrow{\sim} Z(M)$, and there is only one of them so that Lemma A.0.1 is true, so the reader can easily check which isomorphism is φ.

In the following computations, we interpret $\mathrm{Sym}^0(\mathbb{C}^2)$ as the trivial 1-dimensional representation of $\mathfrak{sl}_2$.

A.1 Projective general linear groups

The following is the classical Clebsch–Gordan rule [FH91, Exercise 11.11]:

Lemma A.1.1. *We have an isomorphism of $\mathfrak{sl}_2$-representations*

$$\mathrm{Sym}^n(\mathbb{C}^2) \otimes \mathrm{Sym}^m(\mathbb{C}^2) \cong \mathrm{Sym}^{n+m}(\mathbb{C}^2) \oplus \mathrm{Sym}^{n+m-2}(\mathbb{C}^2) \oplus \cdots \oplus \mathrm{Sym}^{|n-m|}(\mathbb{C}^2).$$

□

Lemma A.1.2. *Let $P \leqslant \mathrm{PGL}_n$ be the parabolic stabilizing an ℓ-plane. Then*

$$\left\{ \left(\tfrac{|n-2\ell|}{2}, 1\right), \left(\tfrac{|n-2\ell|+2}{2}, 1\right), \ldots, \left(\tfrac{n-2}{2}, 1\right) \right\}$$

is a good ordering for $\mathfrak{n}_P$.

Proof. It is not hard to see $\mathfrak{m}^{\mathrm{der}} \cong \mathfrak{sl}_{n-\ell} \times \mathfrak{sl}_\ell$ and $\mathfrak{n}_P$ is isomorphic as a representation of $\mathfrak{m}$ to

$$\mathrm{Hom}(\mathbb{C}^{n-\ell}, \mathbb{C}^\ell)$$

with the natural action. The induced representation of a principal $\mathfrak{sl}_2$-triple is

$$\begin{aligned}\mathrm{Sym}^{n-\ell-1}(\mathbb{C}^2)^\vee \otimes \mathrm{Sym}^{\ell-1}(\mathbb{C}^2) &\cong \mathrm{Sym}^{n-\ell-1}(\mathbb{C}^2) \otimes \mathrm{Sym}^{\ell-1}(\mathbb{C}^2) \\ &\cong \mathrm{Sym}^{n-2}(\mathbb{C}^2) \oplus \mathrm{Sym}^{n-4}(\mathbb{C}^2) \oplus \cdots \oplus \mathrm{Sym}^{|n-2\ell|}(\mathbb{C}^2)\end{aligned}$$

by Lemma A.1.1. The lemma follows. □

A.2 The classical groups

Let V be a complex vector space equipped with a nondegenerate ϵ-symmetric form $\langle \cdot, \cdot \rangle$, that is,

$$\langle v, w \rangle = \epsilon \langle w, v \rangle$$

for $v, w \in V$. We assume $\epsilon \in \{1, -1\}$. For $\mathbb{C}$-algebras R, let

$$G_V(R) := \{g \in \mathrm{SL}_V(R) : \langle gv, gw \rangle = \langle v, w \rangle\}.$$

We refer to G_V as a classical group. The corresponding Lie algebra is

$$\mathfrak{g}_V = \{X \in \mathfrak{sl}(V) : \langle Xv, w \rangle + \langle v, Xw \rangle = 0 \text{ for } v, w \in V\}.$$

Let PG_V be the associated projective group. Concretely,

$$PG_V \cong \begin{cases} \mathrm{PSO}_{\dim V} & \text{if } \epsilon = 1, \\ \mathrm{PSp}_{\dim V} & \text{if } \epsilon = -1. \end{cases}$$

We assume that PG_V is simple and not isomorphic to a projective general linear group. Thus $\dim V \notin \{2, 4\}$ if $\epsilon = 1$ and $\dim V \neq 2$ if $\epsilon = -1$. We also observe that $\mathrm{PSO}_{2r+1} = \mathrm{SO}_{2r+1}$.

The maximal parabolic subgroups of PG_V are precisely the stabilizers of isotropic subspaces. For a parabolic $P = MN_P$, we denote by W_P the corresponding isotropic subspace. We let $W_P^\vee$ be the linear dual of W_P with respect to $\langle \cdot, \cdot \rangle$. Then there exists a subspace $V_P < V$ such that the pair $(V_P, \langle \cdot, \cdot \rangle|_{V_P})$ is of the same symmetric type as V, and such that there is a direct sum decomposition $V = W_P \oplus V_P \oplus W_P^\vee$. Note that the pair $(W_P \oplus W_P^\vee, \langle \cdot, \cdot \rangle|_{W_P \oplus W_P^\vee})$ is also non-degenerate and of the same symmetric type as V. We have

$$\mathfrak{m} \cong \mathfrak{gl}_{W_P} \oplus \mathfrak{g}_{V_P}.$$

We refer to $\ell := \dim W_P$ as the **linear rank** of M or $\mathfrak{m}$.

The following lemma is well known. See for instance [Wol76, Theorems 8.6 and 12.6].

Lemma A.2.1. *As a representation of* $\mathfrak{m}$,

$$\mathfrak{n}_P \cong \mathrm{Hom}_{\mathbb{C}}(V_P, W_P) \oplus \mathrm{Sym}_V(W_P)$$

where

$$\mathrm{Sym}_V(W_P) := \begin{cases} \mathrm{Sym}^2(W_P) & \text{if } \epsilon = -1, \\ \mathrm{Alt}^2(W_P) & \text{if } \epsilon = 1. \end{cases}$$

We have

$$\mathfrak{n}_P(1) \cong \mathrm{Hom}_{\mathbb{C}}(V_P, W_P) \quad \text{and} \quad \mathfrak{n}_P(2) \cong \mathrm{Sym}_V(W_P) \tag{A.3}$$

unless PG_V is PSO_{2r} or PSp_{2r} and $\mathfrak{m} \cong \mathfrak{gl}_r$, in which case

$$\mathfrak{n}_P = \mathfrak{n}_P(1) \cong \mathrm{Sym}_V(W_P). \tag{A.4}$$

The following lemma explicates the principal $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}_{V_P}$.

Lemma A.2.2. *As a representation of a principal $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}_V$, the standard representation V of $\mathfrak{g}_V$ is isomorphic to $\mathrm{Sym}^{\dim V-1}(\mathbb{C}^2)$ unless $G \cong \mathrm{PSO}_{2r}$, in which case it is $\mathrm{Sym}^{\dim V-2}(\mathbb{C}^2) \oplus \mathbb{C}$.*

Proof. The n-th tensor power of the standard symplectic form on $\mathbb{C}^2$ is $(-1)^n$-symmetric, and the n-th symmetric power of the standard representation $\mathbb{C}^2$ of $\mathfrak{sl}_2$ is a subrepresentation of the nth tensor power. Thus the principal $\mathfrak{sl}_2 \to \mathfrak{sl}_V$ may be chosen to factor through the standard representation $\mathfrak{g}_V \to \mathfrak{sl}_V$. This implies the lemma unless $G \cong \mathrm{PSO}_{2r}$. For this last case see [Gro00, Section 7]. ☐

For the following lemma, see [FH91, Exercise 11.31 and 11.35]:

Lemma A.2.3. *For any $n \geqslant 1$, we have the following equivalences of $\mathfrak{sl}_2$-representations:*

$$\wedge^2(\mathrm{Sym}^n(\mathbb{C}^2)) \cong \mathrm{Sym}^2(\mathrm{Sym}^{n-1}(\mathbb{C}^2)) = \bigoplus_{j=0}^{\lfloor (n-1)/2 \rfloor} \mathrm{Sym}^{2(n-1)-4j}(\mathbb{C}^2).$$

☐

Let

$$p(V) := \dim V \pmod 2$$

be the parity of $\dim V$, viewed as an element of the set $\{0, 1\}$. Note that if $G \cong \mathrm{PSO}_{2r}$, linear ranks are either r or $\leqslant r-2$.

Lemma A.2.4. *Assume $r > 1$. Assume that either G is PSp_{2r} and that the linear rank ℓ of M is not r or $G \cong \mathrm{SO}_{2r+1}$. For $\ell > 1$, the parameters $\{(s_i, \lambda_i)\}$ are*

$$\left\{(\tfrac{|2r+p(V)-3\ell|}{2}, 1), (\tfrac{|2r+p(V)-3\ell|+2}{2}, 1), \dots, (\tfrac{2r+p(V)-\ell-2}{2}, 1)\right\}$$

$$\bigsqcup \{(\ell - 1 - p(V) - 2j, 2) : 0 \leqslant j \leqslant \lfloor(\ell - 1 - p(V))/2\rfloor\}.$$

If $\ell = 1$, the parameters are

$$\left\{(\tfrac{2r-2}{2}, 1)\right\} \text{ if } G \cong \mathrm{SO}_{2r+1}, \quad \text{and} \quad \left\{(\tfrac{2r-3}{2}, 1), (0, 2)\right\} \text{ if } G \cong \mathrm{PSp}_{2r}.$$

Suppose $G \cong \mathrm{PSO}_{2r}$ with $r \geqslant 3$ and that $\ell \leqslant r - 2$. If $\ell > 1$, then the parameters are

$$\left\{(\tfrac{|2r-1-3\ell|}{2}, 1), (\tfrac{|2r-1-3\ell|+2}{2}, 1), \dots, (\tfrac{2r-\ell-3}{2}, 1)\right\} \bigsqcup \{(\tfrac{\ell-1}{2}, 1)\}$$

$$\bigsqcup \{(\ell - 2 - 2j, 2) : 0 \leqslant j \leqslant \lfloor(\ell - 2)/2\rfloor\}.$$

If $\ell = 1$, the parameters are $\{(0, 1), (r - 2, 1)\}$. If $\ell = r$ and G is isomorphic to PSp_{2r} or PSO_{2r} then the parameters $\{(s_i, \lambda_i)\}$ are

$$\{(r - 1 - 2j, 1) : 0 \leqslant j \leqslant \lfloor(r - 1)/2\rfloor\} \text{ if } G \cong \mathrm{PSp}_{2r},$$

$$\{(r - 2 - 2j, 1) : 0 \leqslant j \leqslant \lfloor(r - 2)/2\rfloor\} \text{ if } G \cong \mathrm{PSO}_{2r} \text{ and } r \geqslant 3.$$

In all cases, every good ordering has the largest parameter (s_k, λ_k) with $\lambda_k = 1$.

Proof. We use Lemmas A.1.1, A.2.2, and A.2.3 freely in the following. If $G \cong \mathrm{PSp}_{2r}$ or $G \cong \mathrm{SO}_{2r+1}$, then as a representation under a principal $\mathfrak{sl}_2$-triple,

$$\begin{aligned}\mathrm{Hom}_{\mathbb{C}}(V_P, W_P) &\cong \mathrm{Sym}^{\ell-1}(\mathbb{C}^2) \otimes \mathrm{Sym}^{2r+p(V)-2\ell-1}(\mathbb{C}^2) \\ &\cong \mathrm{Sym}^{2r+p(V)-\ell-2}(\mathbb{C}^2) \oplus \mathrm{Sym}^{2r+p(V)-\ell-4}(\mathbb{C}^2) \oplus \cdots \oplus \mathrm{Sym}^{|2r+p(V)-3\ell|}(\mathbb{C}^2).\end{aligned}$$

This space is understood to be zero if $r = \ell$ and $G \cong \mathrm{PSp}_{2r}$. If $G \cong \mathrm{PSO}_{2r}$,

$$\begin{aligned}\mathrm{Hom}_{\mathbb{C}}(V_P, W_P) &\cong \mathrm{Sym}^{\ell-1}(\mathbb{C}^2) \otimes \left(\mathbb{C} \oplus \mathrm{Sym}^{2r-2\ell-2}(\mathbb{C}^2)\right) \\ &\cong \mathrm{Sym}^{\ell-1}(\mathbb{C}^2) \oplus \mathrm{Sym}^{2r-\ell-3}(\mathbb{C}^2) \oplus \cdots \oplus \mathrm{Sym}^{|2r-3\ell-1|}(\mathbb{C}^2).\end{aligned}$$

If $G \cong \mathrm{PSp}_{2r}$, we have

$$\mathrm{Sym}_V(W_P) \cong \mathrm{Sym}^2(\mathrm{Sym}^{\ell-1}(\mathbb{C}^2)) \cong \bigoplus_{j=0}^{\lfloor(\ell-1)/2\rfloor} \mathrm{Sym}^{2(\ell-1)-4j}(\mathbb{C}^2).$$

If G is PSO_{2r} or SO_{2r+1}, we have

$$\mathrm{Sym}_V(W_P) \cong \wedge^2(\mathrm{Sym}^{\ell-1}(\mathbb{C}^2)) \cong \bigoplus_{j=0}^{\lfloor(\ell-2)/2\rfloor} \mathrm{Sym}^{2(\ell-2)-4j}(\mathbb{C}^2).$$

Here by convention this space is zero if $\ell = 1$. The lemma now follows from Lemma A.2.1, (A.3) and (A.4). □

A.3 Exceptional cases

For exceptional types, we compute the decomposition of $\mathfrak{n}_P$ using LieART 2.0.2 (a Mathematica application) based on the tables of [MS12].

Assume G is adjoint of type E, F, or G. Let $P_\ell = M_\ell N_\ell \leqslant G$ denote the maximal parabolic associated to the ℓ-th node of the Dynkin diagram of G, using the Bourbaki numbering. For a given parabolic subgroup, consider the grading

$$\mathfrak{n}_{P_\ell} = \bigoplus_{i\geqslant 1} \mathfrak{n}_{P_\ell}(i),$$

associated to the action of $Z(M)$. The columns of the tables correspond to the graded piece we consider.

We list the resulting $\mathfrak{sl}_2$-representations by the highest weight. For example, the representation $\mathrm{Sym}^n(\mathbb{C}^2)$ will be denoted $\mathbf{n}$. In particular, under the assumption G is adjoint, the data (s, λ) associated to the representation $\mathbf{n}$ appearing in $\mathfrak{n}_{P_\ell}(i)$ is $(\frac{\mathbf{n}}{2}, i)$.

Table A.1: Normalizing factors for $\widehat{G}$ of type E_6

	E_6		
Node	$i = 1$	$i = 2$	$i = 3$
1	$\mathbf{10}, \mathbf{4}$		
2	$\mathbf{9}, \mathbf{5}, \mathbf{3}$	$\mathbf{0}$	
3	$\mathbf{7}, \mathbf{5}, \mathbf{3}, \mathbf{1}$	$\mathbf{4}$	
4	$\mathbf{5}, \mathbf{3}, \mathbf{3}, \mathbf{1}, \mathbf{1}$	$\mathbf{4}, \mathbf{2}, \mathbf{0}$	$\mathbf{1}$
5	$\mathbf{7}, \mathbf{5}, \mathbf{3}, \mathbf{1}$	$\mathbf{4}$	
6	$\mathbf{10}, \mathbf{4}$		

Table A.2: Normalizing factors for $\widehat{G}$ of type E_7, E_8, F_4, G_2

E_7

Node	$i = 1$	$i = 2$	$i = 3$	$i = 4$
1	$\mathbf{15, 9, 5}$	$\mathbf{0}$		
2	$\mathbf{12, 8, 6, 4, 0}$	$\mathbf{6}$		
3	$\mathbf{9, 7, 5, 3, 1}$	$\mathbf{8, 4, 0}$	$\mathbf{1}$	
4	$\mathbf{6, 4, 4, 2, 2, 0}$	$\mathbf{6, 4, 2, 2}$	$\mathbf{4, 2}$	$\mathbf{2}$
5	$\mathbf{8, 6, 4, 4, 2, 0}$	$\mathbf{6, 4, 2}$	$\mathbf{4}$	
6	$\mathbf{11, 9, 5, 3}$	$\mathbf{8, 0}$		
7	$\mathbf{16, 8, 0}$			

E_8

Node	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$
1	$\mathbf{21, 15, 11, 9, 3}$	$\mathbf{12, 0}$				
2	$\mathbf{15, 11, 9, 7, 5, 3}$	$\mathbf{12, 8, 4, 0}$	$\mathbf{7}$			
3	$\mathbf{11, 9, 7, 5, 3, 1}$	$\mathbf{12, 8, 6, 4, 0}$	$\mathbf{7, 5}$	$\mathbf{6}$		
4	$\mathbf{7, 5, 5, 3, 3, 1}$	$\mathbf{8, 6, 4, 4, 2, 0}$	$\mathbf{7, 5, 3, 1}$	$\mathbf{6, 4, 2}$	$\mathbf{3, 1}$	$\mathbf{4}$
5	$\mathbf{9, 7, 5, 5, 3, 3, 1}$	$\mathbf{8, 6, 4, 4, 2, 0}$	$\mathbf{7, 5, 3, 1}$	$\mathbf{6, 2}$	$\mathbf{3}$	
6	$\mathbf{12, 10, 8, 6, 4, 2}$	$\mathbf{10, 8, 6, 2}$	$\mathbf{10, 4}$	$\mathbf{2}$		
7	$\mathbf{17, 15, 9, 7, 1}$	$\mathbf{16, 8, 0}$	$\mathbf{1}$			
8	$\mathbf{27, 17, 9}$	$\mathbf{0}$				

F_4

Node	$i = 1$	$i = 2$	$i = 3$	$i = 4$
1	$\mathbf{9, 3}$	$\mathbf{0}$		
2	$\mathbf{5, 3, 1}$	$\mathbf{4, 0}$	$\mathbf{1}$	
3	$\mathbf{3, 1}$	$\mathbf{4, 2, 0}$	$\mathbf{1}$	$\mathbf{2}$
4	$\mathbf{6, 0}$	$\mathbf{6}$		

G_2

Node	$i = 1$	$i = 2$	$i = 3$
1	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{1}$
2	$\mathbf{3}$	$\mathbf{0}$	

Appendix B. Data for asymptotics

For readers' convenience, we list s_k and the multisets $L(d) \cap \mathbb{R}$ considered in Chapter 4, which we denote by $L(d)$ for simplicity, in each case. We again adopt the Bourbaki numbering as in Appendix A.3.

Table B.1: Asymptotics data for G of type A, B, C

A_n

Node	$L(1)$	s_k
ℓ	$\{-i : 0 \leqslant i \leqslant \min(\ell-1, n-\ell)\}$	$\frac{n-1}{2}$

B_n $(n \geqslant 2)$

Node	$L(1)$	$L(2)$	s_k
1	$\{0\}$	$\{1-n\}$	$\frac{2n-3}{2}$
$1 < \ell < n$	$\{-i : 0 \leqslant i \leqslant \min(\ell-1, 2n-2\ell-1)\}$	$\{\ell-n-i : 0 \leqslant i \leqslant \lfloor(\ell-1)/2\rfloor\}$	$\frac{2n-\ell-2}{2}$
n	$\{-2i : 0 \leqslant i \leqslant \lfloor(n-1)/2\rfloor\}$		$n-1$

C_n $(n \geqslant 2)$

Node	$L(1)$	$L(2)$	s_k
1	$\{0\}$		$n-1$
$1 < \ell$	$\{-i : 0 \leqslant i \leqslant \min(\ell-1, 2n-2\ell)\}$	$\{\ell-n-i : 1 \leqslant i \leqslant \lfloor\ell/2\rfloor\}$	$\frac{2n-\ell-1}{2}$

Table B.2: Asymptotics data for G of type D, E_6

D_n $(n \geqslant 3)$

Node	$L(1)$	$L(2)$	s_k
1	$\{0, 2-n\}$		$n-2$
$1 < \ell < n-1$	$\{-i, \ell-n+1 : 0 \leqslant i \leqslant \min(\ell-1, 2n-2\ell-2)\}$	$\{\ell-n-i : 0 \leqslant i \leqslant \lfloor(\ell-2)/2\rfloor\}$	$\frac{2n-\ell-3}{2}$
$n-1, n$	$\{-2i : 0 \leqslant i \leqslant \lfloor(n-2)/2\rfloor\}$		$n-2$

E_6

Node	$L(1)$	$L(2)$	$L(3)$	s_k
$1, 6$	$\{0, -3\}$			5
2	$\{0, -2, -3\}$	$\{-5\}$		$\frac{9}{2}$
$3, 5$	$\{0, -1, -2, -3\}$	$\{-3\}$		$\frac{7}{2}$
4	$\{0, -1, -1, -2, -2\}$	$\{-2, -\frac{5}{2}, -3\}$	$\{-3\}$	$\frac{5}{2}$

Table B.3: Asymptotics data for G of type E_7

E_7

Node	$L(1)$	$L(2)$	$L(3)$	$L(4)$	s_k
1	$\{0, -3, -5\}$	$\{-8\}$			$\frac{15}{2}$
2	$\{0, -2, -3, -4, -6\}$	$\{-5\}$			6
3	$\{0, -1, -2, -3, -4\}$	$\{-3, -4, -5\}$	$\{-5\}$		$\frac{9}{2}$
4	$\{0, -1, -1, -2, -2, -3\}$	$\{-2, -\frac{5}{2}, -3, -3\}$	$\{-3, -\frac{10}{3}\}$	$\{-\frac{7}{2}\}$	3
5	$\{0, -1, -2, -2, -3, -4\}$	$\{-3, -\frac{7}{2}, -4\}$	$\{-4\}$		4
6	$\{0, -1, -3, -4\}$	$\{-4, -6\}$			$\frac{11}{2}$
7	$\{0, -4, -8\}$				8

Table B.4: Asymptotics data for G of type E_8

E_8							
Node	$L(1)$	$L(2)$	$L(3)$	$L(4)$	$L(5)$	$L(6)$	s_k
1	$\{0, -3, -5, -6, -9\}$	$\{-8, -11\}$					$\frac{21}{2}$
2	$\{0, -2, -3, -4, -5, -6\}$	$\{-5, -6, -7, -8\}$	$\{-7\}$				$\frac{15}{2}$
3	$\{0, -1, -2, -3, -4, -5\}$	$\{-3, -4, -\frac{9}{2}, -5, -6\}$	$\{-5, -\frac{16}{3}\}$	$\{-\frac{11}{2}\}$			$\frac{11}{2}$
4	$\{0, -1, -1, -2, -2, -3\}$	$\{-2, -\frac{5}{2}, -3, -3, -\frac{7}{2}, -4\}$	$\{-3, -\frac{10}{3}, -\frac{11}{3}, -4\}$	$\{-\frac{7}{2}, -\frac{15}{4}, -4\}$	$\{-4, -\frac{21}{5}\}$	$\{-4\}$	$\frac{7}{2}$
5	$\{0, -1, -2, -2, -3, -3, -4\}$	$\{-3, -\frac{7}{2}, -4, -4, -\frac{9}{2}, -5\}$	$\{-4, -\frac{13}{3}, -\frac{14}{3}, -5\}$	$\{-\frac{9}{2}, -5\}$	$\{-5\}$		$\frac{9}{2}$
6	$\{0, -1, -2, -3, -4, -5\}$	$\{-4, -\frac{9}{2}, -5, -6\}$	$\{-5, -6\}$	$\{-\frac{13}{2}\}$			6
7	$\{0, -1, -4, -5, -8\}$	$\{-5, -7, -9\}$	$\{-9\}$				$\frac{17}{2}$
8	$\{0, -5, -9\}$	$\{-14\}$					$\frac{27}{2}$

Table B.5: Asymptotics data for G of type F_4, G_2

F_4

Node	$L(1)$	$L(2)$	$L(3)$	$L(4)$	s_k
1	$\{0, -3\}$	$\{-2\}$			3
2	$\{0, -1\}$	$\{-1, -\frac{3}{2}, -2\}$	$\{-2\}$	$\{-2\}$	$\frac{3}{2}$
3	$\{0, -1, -2\}$	$\{-2, -3\}$	$\{-3\}$		$\frac{5}{2}$
4	$\{0, -3\}$	$\{-5\}$			$\frac{9}{2}$

G_2

Node	$L(1)$	$L(2)$	$L(3)$	s_k
1	$\{0\}$	$\{-2\}$		$\frac{3}{2}$
2	$\{0\}$	$\{-1\}$	$\{-1\}$	$\frac{1}{2}$

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