Diagrammatics in Categorification and Compositionality

by

Dmitry Vagner

Department of Mathematics
Duke University

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Paul Bendich

Dissertation submitted in partial fulfillment of the
requirements for the degree of Doctor of Philosophy
in the Department of Mathematics
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ABSTRACT

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Abstract

In the present work, I explore the theme of diagrammatics and their capacity to shed insight on two trends—categorification and compositionality—in and around contemporary category theory. The work begins with an introduction of these meta-phenomena in the context of elementary sets and maps. Towards generalizing their study to more complicated domains, we provide a self-contained treatment—from a pedagogically novel perspective that introduces almost all concepts via diagrammatic language—of the categorical machinery with which we may express the broader notions found in the sequel. The work then branches into two seemingly unrelated disciplines: dynamical systems and knot theory. In particular, the former research defines what it means to compose dynamical systems in a manner analogous to how one composes simple maps. The latter work concerns the categorification of the $\mathfrak{sl}_N$ link invariant. In particular, we use a virtual filtration to give a more diagrammatic reconstruction of Khovanov-Rozansky homology via a smooth TQFT. Finally, the work culminates in a manifesto on the philosophical place of category theory.

Acknowledgements

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- Duke University: for being my intellectual incubator and patron.
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I am grateful to my loved ones

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- my grandpa Boris: for instilling in me an early love of doing mathematics and taking long walks
- my late grandmother Yelena, to whom I dedicate this thesis: for an influence on my life that is so profound I haven't the language to articulate it.

Contents

Chapter 1

Introduction: Aesthetics of Abstraction

The present work is obsessed with the visual language used to to render and manipulate mathematical notions. This yoga, which we call diagrammatics, was formally introduced in [JS91] by Joyal and Street to mechanize reasoning within monoidal categories. In departure from tradition, we develop from the ground up the entire theory of categories with diagrammatics as first principles. Diagrammatics are in some sense a higher-dimensional—typically, given the dimensionality of our communication media, planar—notation for formal symbolics in place of the traditional one-dimensional notation for mathematical expressions. Such visual language makes legible a wider class of abstract relationships, and hence, by extension, facilitates us to reason about them.

In particular, we apply this diagrammatic approach to explore compositionality and categorification, two emergent developments in and around category theory. The next two chapters of the thesis are pedagogical in nature, but nonetheless amount to a novel curation and narrative of concepts.

In Chapter 2, we introduce the core concepts—diagrammatics, categorification, and compositionality—in the setting of elementary sets and maps. Restricting our attention to this context has at least two advantages. The first is pedagogical, as the world of sets is sufficiently familiar so as to allow for the transfer of intuition. The second, and deeper, purpose is to illuminate the peculiarities of set theory, making more visible its limitations. The chapter is divided into two sections: the first, Section 2.1, uses diagrammatics to characterize the class of all compositions we may define on maps; while the second, Section 2.2, presents natural number arithmetic as a shadow of elementary set-theoretic operations.

In the subsequent chapter, we step back to provide the necessary categorical preliminaries so as to generalize the diagrammatics introduced in the prior chapter to a much broader setting. In the process, we reveal phenomena—such as coalgebras, duals, and non-degenerate braidings—that are invisible in the set theoretic context. Section 3.1 overviews the classical theory of categories, while Section 3.2 covers the theory of monoidal categories, namely those equipped with both sequential and parallel composition. Our presentation is novel in its use of diagrammatics as native to any notion that supports them.

The next two chapters—Chapter 4 and Chapter 5—may be read in either order as they cover independent and essentially unrelated incarnations of the conceptual cosmology introduced in the prior two chapters.

Chapter 4, based on joint work with David I. Spivak and Eugene Lerman [VL15] and mentorship from Patrick Schultz, defines what it means to compose, in much the same way as we did for set maps in 2, continuous-time dynamical systems. The approach taken is derived from Spivak's work on operads of wiring diagrams [Spi13, SR13], which casts various compositions as themselves arrows in some other categorical structure. This makes it possible for wiring diagrams to express holarchies [Koe67] of compositions—meaning each level can simultaneously be a whole of parts and a part of a whole.

The compositionality movement, to which Chapter 4 contributes, is inspired by the unifying role that category theory has played across diverse contexts such as algebra, topology, logic, physics, and computation [BS11, Coe13]. Category theory thus finds itself in a position of linguistic significance as it "far exceeds in its expressive power anything, even imaginable, say, before 1960. Any meaningful idea coming from science can be fully developed in this language" [Gro13]. More recently, the field has been emboldened with a manifest destiny towards what John Baez has heralded as "a foundation of applied mathematics" [Bae13]. Recent noteworthy contributions include compositional approaches to game theory [NGZ18] and machine learning [FT17].

Chapter 5 is situated in a research program that studies phenomena in low-dimensional topology and representation theory with the point of view that the structures at hand are mere "shadows" of deeper structures at the categorified level. In particular, this contribution, based on joint work with Michael Abel, provides a new construction for $\mathfrak{sl}_N$ link homology, which categorifies the $\mathfrak{sl}_N$ polynomial. This construction, a variant of the original by Khovanov-Rozansky [KR08], gives a more diagrammatic presentation of the theory, based on a smooth TQFT, that mirrors the playful nature of Khovanov's original categorification of the Jones Polynomial [Kho00], especially as it is presented in the excellent [BN02].

Finally, our conclusion in Chapter 6 is a self-indulgent speculation as to the philosophical spirit of category theory, not so much as a foundation of mathematics, but rather as an archetype for a freer and deeper—while still unassailable in rigor—style of reasoning than its predecessors.

Chapter 2

Compositionality, Categorification in Sets

Our odyssey begins in the mathematical womb of sets. We now fix some philosophy, terminology, and notation.

A *set* X is a collection of elements $x, y, \ldots$, which are, unless otherwise specified, indeterminate. For instance this implies, contrary to traditional philosophy, that given an element x of set X and an element y of set Y, it simply makes no sense to ask if $x = y$. We therefore cannot, without additional information, even construct the union $X \cup Y$ and intersection $X \cap Y$. This is the perspective of type theory rather than that of Zermelo-Fraenkel set theory, in which everything is a set (See discussion in [Shu13]). Although this choice may at first seem odd and overly restrictive, remember that there are far stranger well-defined questions, such as if $\mathbb{Z}$ is an element of π, that axiomatic set theory allows but that mathematicians have chosen to ignore. To emphasize our metaphysical choice, we write the *type declaration* $x \colon X$—pronounced "x is of type X"—in place of the *membership proposition* $x \in X$.

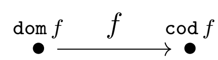
A *map* f is an assignment to each *argument* $x \colon \mathtt{dom}\, f$ of its *domain* type $\mathtt{dom}\, f$, a *value* $fx \colon \mathtt{cod}\, f$ of its *codomain* type $\mathtt{cod}\, f$. We write $X \to Y$ for the *mapping type*, defined as the collection of maps f with $\mathtt{dom}\, f = X$ and $\mathtt{cod}\, f = Y$. This then implies the traditional notation $f \colon X \to Y$, but perhaps with a different mindset. In particular, the arrow $\to$, a *type-level operator*, binds more tightly than the typing declaration $:$, meaning that $f \colon X \to Y$ is an equivalent expression to $f \colon [X \to Y]$.

A *specification* for a map f is an explicit description, typically as a symbolic expression, of the value fx assigned to an argument x. We denote the specification by $x \mapsto fx$ and write $f :: x \mapsto fx$ to declare that "f is specified by $x \mapsto fx$". When we wish to simultaneously declare both type and specification, we take the convention that the specification is written first as in

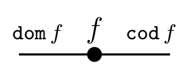
$$f :: x \mapsto fx \colon X \to Y.$$

Alternatively, we may wish to simply name a map by its specification. This is called an *anonymous declaration*, which we denote with lambda calculus notation in which the expression $\lambda x.fx$ denotes the map given by the specification $x \mapsto fx$. For instance, the rule $\lambda x.x$ specifies an identity map.

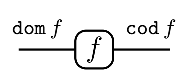
Traditionally, we use the abstract representation of a 1-dimensional directed edge for the map f and 0-dimensional source and target vertices respectively for its domain and codomain types.



There are illuminating consequences to dualizing this picture; that is, representing vertices by 1-dimensional *strings* or *wires* and maps by 0-dimensional *beads*:



So as to allow for maps to have internal structure, we replace the bead with a *box*.

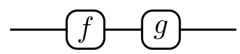


We call such pictures *string diagrams*. We typically omit labels from strings when they are either evident from context or simply irrelevant. We allow labels to imply orientation. We now use this visual language to explore the set-theoretic phenomena of interest to our agenda.

2.1 Compositionality of Maps

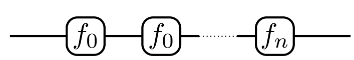
2.1.1 Composites

We depict the composed map—henceforth, the *composite*—$g \circ f$ of maps f and g for which $\texttt{cod}\, f = \texttt{dom}\, g$ by stringing the corresponding boxes together:



We lament the fact that the notation $g \circ f$ reads in the opposite direction from that of the string diagram. When we prefer to denote a composite in *diagrammatic order*, we use the symbol $\fatsemi$ as in $f \fatsemi g$ for the above. Just as one reads aloud $g \circ f$ as "g of f", we suggest reading aloud the equal $f \fatsemi g$ as "f then g".

We write $\texttt{map}$ for the collection of *all* maps. An n-tuple $(f_0, \ldots, f_{n-1})\colon \texttt{map}^n$ is *composable* when $\texttt{cod}\, f_i = \texttt{dom}\, f_{i+1}$ whenever this equation is defined. We call a composable n-tuple an *n-path* and write $n\,\texttt{path}$ for the subset of $\texttt{map}^n$ of n-paths. In string diagrams, we represent the composite map of a path with a chain of boxes, as such:



2.1.2 Compositions

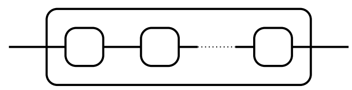
A *composition* map is a *higher-order* map—i.e. one whose arguments and values are themselves allowed to be maps—that takes arguments tuples of composable maps and outputs their composite map. In particular, for each $n\colon \mathbb{N}$, there is a composition

$n\,\texttt{chain}\colon n\,\texttt{path} \to \texttt{map}$ specified by $(f_0, \ldots, f_{n-1}) \mapsto f_0 \fatsemi \cdots \fatsemi f_{n-1}$.

We depict the composition $n\,\texttt{chain}$ as a so called *schematic*. A schematic is drawn similarly to a string diagram but with the following two formal distinctions:

- the boxes, called *slots*, of a schematic are left blank as arguments;
- a single outer box, called a *screen*, circumscribes the schematic.

We depict $n\,\texttt{chain}$ with the following schematic.



We can articulate the distinction between schematics, which represent compositions, and string diagrams, which represent composites, via analogy with the distinction between a map f and its evaluation fx on an argument $x\colon \texttt{dom}\, f$. A schematic $n\,\texttt{chain}$ is a higher-order map and a string diagram $n\,\texttt{chain}\,\mathbf{f}$ is the (lower-order) map resulting from evaluating the schematic $n\,\texttt{chain}$ on an argument $\mathbf{f}\colon n\,\texttt{path}$.

Define the map $\texttt{typing}\colon n\,\texttt{path} \to \texttt{set}^{n+1}$ taking $(f_0, \ldots, f_{n-1})\colon n\,\texttt{path}$ to

$$(\texttt{dom}\, f_0, \texttt{cod}\, f_0 = \texttt{dom}\, f_1, \texttt{cod}\, f_1 = \texttt{dom}\, f_2, \ldots, \texttt{cod}\, f_{n-1} = \texttt{dom}\, f_n)\colon\ \texttt{set}^{n+1}\,.$$

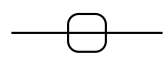
In turn, given $\mathbf{X}\colon\ \texttt{set}^{n+1}$, we define $\mathbf{X}\,\texttt{path}$ as the preimage of $\mathbf{X}$ under $\texttt{typing}$, i.e. the subset of $n\,\texttt{path}$ with typing $\mathbf{X}$. We can define $\mathbf{X}\colon\ \texttt{path}$ more concretely as the product of mapping sets

$$\mathbf{X}\,\texttt{path} \triangleq [X_0 \to X_1] \times [X_1 \to X_2] \times \cdots \times [X_{n-1} \to X_n],$$

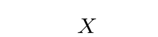
where we let $\mathbf{X}$ be the tuple $(X_0, \ldots, X_n)$. In turn, we can define $n\,\texttt{path}$ as the sum of $\mathbf{X}\colon\ \texttt{path}$ across $\mathbf{X}\colon\ \texttt{set}^{n+1}$. This gives us a satisfactory definition of $0\,\texttt{path}$, which we define as the sum across $X\colon$ `set` of the empty product—given by the generic singleton set $\star = \{\bullet\}$—and hence in bijection with `set`. This is in place of defining $0\,\mathtt{path}$ as a subset of the singleton $\mathtt{map}^0$.

We can then define $\mathbf{X}\colon$ `chain` as the restriction of $n\,\mathtt{chain}$ to the subdomain $\mathbf{X}\,\mathtt{path}$. This composition $\mathbf{X}\colon$ `chain` has type $\mathbf{X}\colon \mathtt{path} \to [i\mathbf{X} \to t\mathbf{X}]$, where i and t respectively extract the initial and terminal terms of a list. When we wish to depict $\mathbf{X}\colon$ `chain`, we replace the unlabelled wires of a schematic with labelled strings. We call a labelled schematic a *wiring diagram*.

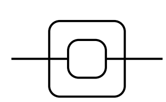
We give the schematic $0\,\mathtt{chain}$ the special name `unit`. Since this has domain $0\,\mathtt{path}$, or the bijectively equivalent `set`, it picks out for each set $X\colon$ `set` its *identity* map $\mathtt{id}_X :: x \mapsto x\colon X \to X$. This is represented by the following peculiar schematic.



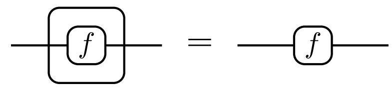
Said otherwise, there is a wiring diagram $X\,\mathtt{chain}$ for every singleton $X\colon$ `set`. Since the schematic has no slots, the composite of the composition $X\,\mathtt{chain}$ is merely the string



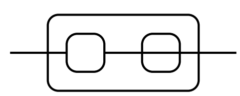
We give the schematic $1\,\mathtt{chain}$, depicted below, the special name `inert`. Since $1\,\mathtt{path}$ is simply `map`, `inert` has type $\mathtt{map} \to \mathtt{map}$. We give `inert` its name because it is simply the higher-order identity $\mathtt{inert} = \mathtt{id}_{\mathtt{map}}$. Alternatively, in the language of wiring diagrams, we have $(X, Y)\colon \mathtt{chain} = \mathtt{id}_{[X \to Y]}$ for every pair $(X, Y)\colon \mathtt{set}^2$.



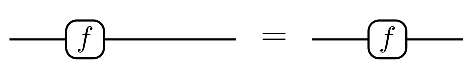
This fact implies the following relation.



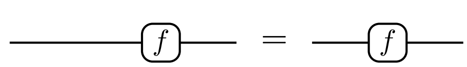
This relation is a special case of the more general *coherence* condition that entirely characterizes the behavior of chains and hence their string diagrams. Define a *nesting* of a schematic $\mathfrak{S}$ into another schematic $\mathfrak{S}'$ as follows. Substitute $\mathfrak{S}$ into some slot of $\mathfrak{S}'$ so that the $\mathfrak{S}$-screen aligns with this $\mathfrak{S}'$-slot. Erase this *intermediary* box given by the alligned $\mathfrak{S}$-screen and $\mathfrak{S}'$-slot. The coherence condition can then be stated as the fact that nesting any schematic into another schematic yields a schematic. In our current setting, which we will subsequently extend, the only schematics are n `chain`, so this merely states that nesting m `chain` into a slot of n `chain` yields another chain, in particular $(m + n - 1)$ `chain`. For example, consider the following schematic nesting.



This nested schematic is given by substituting the `unit` into the second slot of 2 `chain`. This composite schematic, which has a single argument, must thus be `inert`. Therefore, evaluating this composite wiring diagram on any map f simply returns the map f, yielding the following identity of string diagrams.



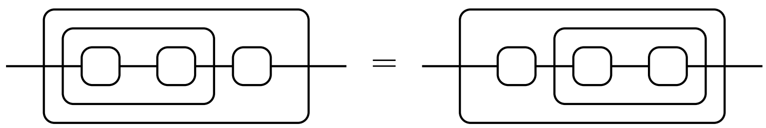
By symmetry, we also have:



These two relations are none other than the *unitality* of identity maps; i.e. the fact that for any $f\colon \texttt{map}$, we have that $\texttt{id}_{\texttt{dom}\, f} \fatsemi f = f$ and $f \fatsemi \texttt{id}_{\texttt{cod}\, f} = f$.

String diagrams thus allow algebraic relations to be implied by intuitive spatial reasoning—formalized with topology—since this law implies that our diagrams only rely on order of boxes as opposed to geometric data concerning length of string.

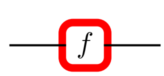
Similarly, the two nested schematics below are both equal to 3 `chain`, and hence imply the *associativity* of the binary composition operator.



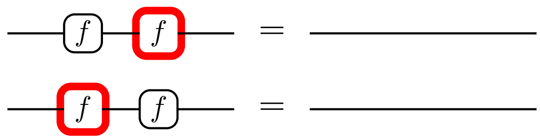
We remark that such identities of wiring diagrams provide a *tacit* or *point-free*—i.e. with no reference to arguments—way to express properties of map composition. This allows us to replace equalities of values universally quantified across arguments—e.g. in the *pointful* associativity equation in which we quantify—across all 3-paths (f, g, h)—the identity $(f \fatsemi g) \fatsemi h = f \fatsemi (g \fatsemi h)$. Instead we have direct equality between the processes that enact composition. This is merely a higher-order analog of stating $f = g$ in place of specifying that $f(x) = g(x)$ for all x.

2.1.3 Inverses

Recall that a map $f\colon X \to Y$ witnesses a bijection $X \cong Y$ if and only if there is an *inverse* map $f^{-1}\colon Y \to X$ for which $f \fatsemi f^{-1} = \texttt{id}_X$ and $f^{-1} \fatsemi f = \texttt{id}_Y$. We choose to represent f^{-1} as the box for f with its border replaced by a thick red band.

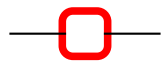


The inverse axioms can then be rewritten in string diagrams.

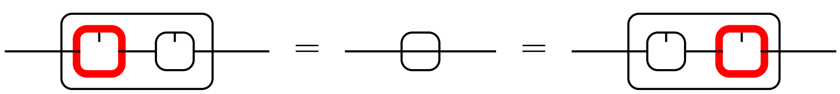


We call such maps $X \to Y$ *isomorphisms* and say that X and Y are *isomorphic*.

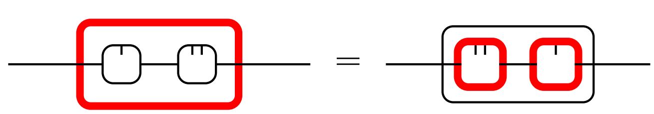
If we denote `bij` as the subset of `map` consisting of bijections, we have a schematic `inv`: `bij` $\to$ `bij`, which we render as follows.



This lets us to codify the inverse condition with the following schematic equalities.

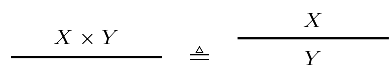


We introduce the tick mark decoration to indicate which slots have the same input—and not merely input type. This decoration is a throwback to those used in Euclidean geometry to indicate identity of angles. This is necessary since we must compose a map with its own inverse as opposed to the inverse of some other map. We will return to this point in depth in the following section. We also note that inversion reverses order of composition: $(f \fatsemi g)^{-1} = g^{-1} \fatsemi f^{-1}$. We can express this schematically by using different tick markings to keep track of which box is which.



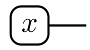
2.1.4 Products

In addition to this *sequential* map composition, we may wish to introduce a notion of *parallel* map composition. More precisely, the domain and codomain types of a given map may decompose into distinct pieces on which the map may act in interesting ways. Throughout the rest of this section, we take this to be defined as the *cartesian product*, or simply *product*, $X \times Y \triangleq \{(x, y) \mid x\colon X, y\colon Y\}$ of sets X, Y. We depict the product of sets by drawing parallel strings:

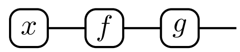


Before proceeding, we must address the associativity of products. The sets $(X\times Y)\times Z$ and $X \times (Y \times Z)$ are not exactly equal, since the former contains elements that look like $(x,(y, z))$ and the latter like $((x, y), z)$. These terms are of course in canonical bijection. We exploit the presence of this canonicity to *strictify*—a concept we will review in the next section—the product and simply write $X \times Y \times Z$ without concern for bracketing. Similarly, denote by $\star = \{\bullet\}$ the generic singleton set. This set plays the role of unit for the product; namely, given a set X, the sets $\star \times X$, and $X \times \star$ are all in canonical bijection via the cycle of mappings: $x \mapsto (\bullet, x) \mapsto (x, \bullet) \mapsto x$. We strictify these as well, considering the sets X, $\star\times X$, and $X\times\star$ as equal. Therefore, just as we depicted identity maps—the units for composition—with no box, we similarly depict the singleton $\star$ with no string. The fact fact that the presence of $\mathtt{id}_\star$ is completely compositionally inert is then reflected in it being depicted as an empty picture.

As we discussed before, maps of the form $x\colon \star \to X$ enjoy a special interpretation: they are determined by a single choice of element $x(\bullet)\colon X$, and can thus be identified with that element. Stated otherwise, this gives a canonical bijection between $[\star \to X]$ and X, which inspires us to abuse notation and simply write x for $x(\bullet)$. We may hence diagram the element $x\colon X$ as follows.

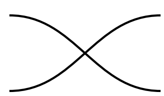


This notation allows us to diagram the evaluation of maps. For example, the following string diagram corresponds to the value $g(f(x))$, i.e. the map $x \fatsemi f \fatsemi g$.

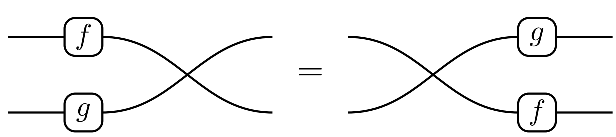


2.1.5 Swaps

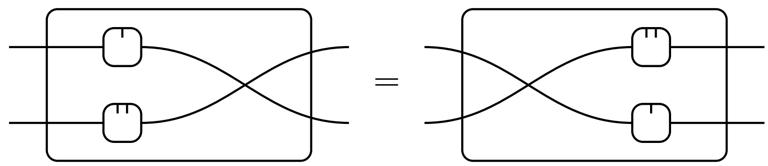
In addition to associativity and unitality, the product is, up to canonical bijection, commutative. In particular, the *swap* $(x, y) \mapsto (y, x)$ induces a bijection $X \times Y \to Y \times X$, which we denote generically by τ and in typed fashion by $\tau_{X,Y}$. Unlike associativity and unitality, however, we bestow a string diagram to the symmetry map instead of strictifying it away. In particular, we visualize τ with a so called *crossing*.



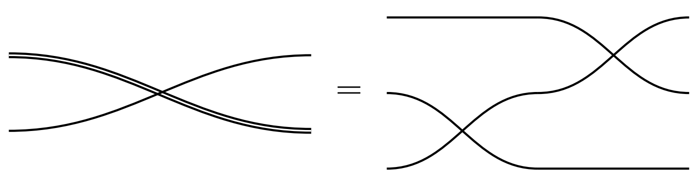
Swap maps enjoy the following *naturality* condition.



Such bead sliding codifies the equality of the composites $(x, y) \mapsto (y, x) \mapsto (gy, fx)$ and $(x, y) \mapsto (fx, gy) \mapsto (gy, fx)$. We can also use distinct tick markings to tacitly redraw the above condition.



In addition to naturality, swap maps enjoy the following condition, which describes how they interact with products. To neatly visualize this, we use a single double-stranded string to depict two strings that combine to form a product.

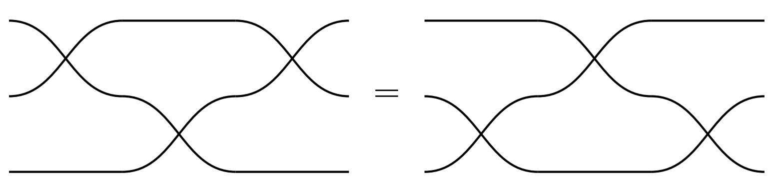


This condition merely says that swapping a pair of strings with a third string is equivalent to iteratively swapping each of the strings of the pair with the third string.

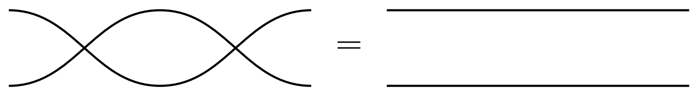
Perhaps the most important consequence of these conditions is that swap maps satisfy the so called *braid relation*

$$\tau_{1,2} \fatsemi \tau_{1,3} \fatsemi \tau_{2,3} = \tau_{2,3} \fatsemi \tau_{1,3} \fatsemi \tau_{1,2}$$

In string diagrams, this is drawn as follows.



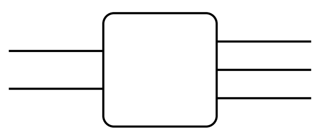
This implies a symmetry group action on products. Such topological interpretations breathe life into this *diagrammatics* we preach. Since transposition is involutive, we have that $\tau_{X,Y}$ and $\tau_{Y,X}$ are always mutually inverse. In string diagrams this is visualized as follows.



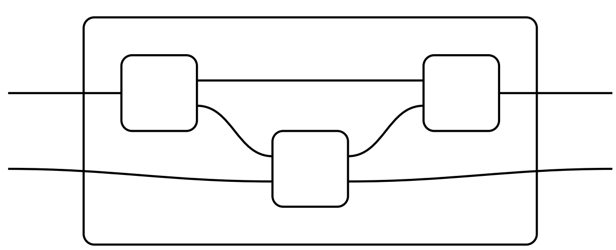
This means that our braid group action is in fact a permutation group action.

2.1.6 Flows

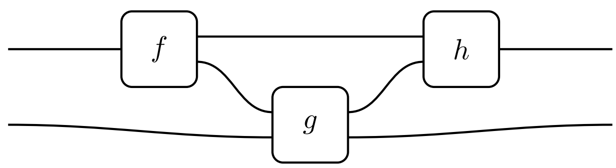
In general, multivariate maps—those with products in their domain and or codomain—are to be depicted as boxes with several input and output wires. For example, we render a generic map with three inputs and two outputs:



We consider the inputs and outputs as lists and take the convention that they are enumerated from top to bottom and starting at 0. The multivariate setting allows for a larger class of schematics, which we call *flows*, such as the one below.



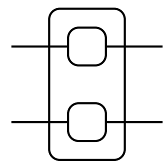
For a given schematic, a *filling* is a choice of map for every slot. We say that a filling *typechecks* when it induces a wiring diagram with well defined strings. In other words, only strings of the same type are wired together. We call a filling that typechecks, and hence constitutes a valid argument, an *animation*. In the case of the schematic n `chain`, a filling is precisely an n-tuple $\mathbf{f}\colon \mathtt{map}^n$, while an animation is precisely an n-path $\mathbf{f}\colon n\,\mathtt{path}$. When we animate the above flow with a triple (f, g, h), the resultant composite φ is given by the following string diagram.



How does the composite φ behave as a map? Although relatively simple to visualize with diagrams, expressing φ is annoying in classical notation with its limited one-dimensional medium. To give a *pointful* definition—i.e. a specification of map by its behavior on a generic argument—of φ on (s, t) requires using projection maps to isolate components of its multivariate composands. Formally, given $(x, y)\colon X \times Y$, the projection $\pi_0\colon X \times Y \to X$ has the rule $(x, y) \mapsto x$, and similarly for π_1. This all amounts to the following horrifying expression for $\varphi(s, t)$.

$$\varphi(s,t) = (h(\pi_0(f(s)), \pi_0(g(\pi_1 f(s), t))), \pi_1(g(\pi_1(f(s)), t)))$$

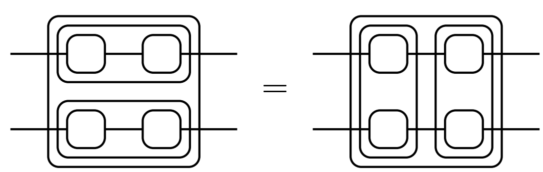
One flow of interest is that which enacts the product of maps. More precisely, given $f\colon X \to X'$ and $g\colon Y \to Y'$, we define the *map product* or *parallel composite* $f \times g$ via the rule $(x, y) \mapsto (fx, gy)$. We represent the process of parallel composition with the flow 2 `para`, with n `para` defined similarly.



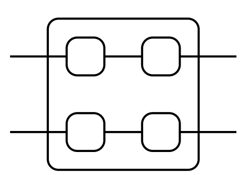
Although we save for the next section the coherence of multivariate wiring diagrams, we note perhaps its most significant instance—the commutativity of sequential and parallel map composition, i.e. 2 `para` ⨾ 2 `chain` = 2 `chain` ⨾ 2 `para`, which is traditionally written as the universally quantified equality of expressions

$$(f \times g) \fatsemi (f' \times g') = (f \fatsemi f') \times (g \fatsemi g'),$$

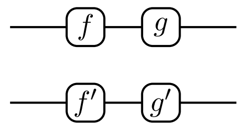
This identity holds since both composites are specified by $(x, y) \mapsto (f'fx, g'gy)$. In the language of schematics, this is the following *interchange law*.



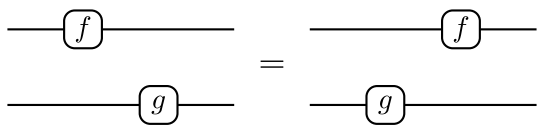
This means that the following is a well-defined flow



and hence that the following is an unambiguous string diagram.



In the special case where one of the diagonals consists of identity maps, we have the following "time-like" commutativity relation, instantiating the independence of the two strings.



2.1.7 Forks

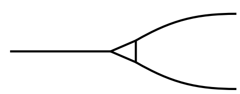
Of course, not all multivariate maps enjoy an internal structure that renders their strings independent. Such independence, however, does turn out to be half-true in our current setting. In particular, a map with multivariate output, such as $f\colon X \to Y_1 \times Y_2$ can be *decomposed* into a pair of *component maps* $f_j\colon X \to Y_j$ for $j = 1, 2$. Since f gives values of the form (y_1, y_2) to an argument x, we can define f_j by the rule $x \mapsto y_j$ for $j = 1, 2$. We can express this point-free in terms of projections.

$$f_j = f \fatsemi \pi_j$$

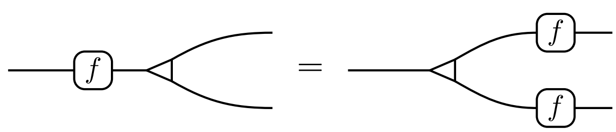
In turn, given a pair of maps $f\colon X \to Y$ and $g\colon X \to Z$, we can construct their *fork* $f \vartriangle g\colon X \to Y \times Z$ given point-wise by $x \mapsto (fx, gx)$. These two processes—decomposing and forking—turn out to be mutually inverse and hence instantiate a bijection of the following type.

$$[X \to Y] \times [X \to Z] \cong [X \to Y \times Z]$$

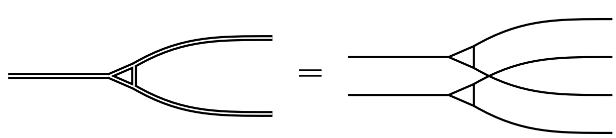
To diagram this correspondence, we must introduce some structure maps. We define the map `copy` by rule $x \mapsto (x, x)$. This map is defined for all sets X and hence has *polymorphic* type $X \to X \times X$, where X is a variable over which we can universally quantify. We type `copy` as $\mathtt{1} \to \mathtt{2}$ to denote that it has domain any set and codomain two copies of that set. This notation will be made more clear in the next section. We render `copy` with the following string diagram.



Copy maps possess three critical properties, the first of which is naturality.

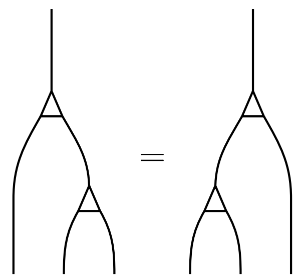


This property, instantiated as passing boxes through the copy map, holds since both composites have the rule $x \mapsto (fx, fx)$. The second property, *uniformity*, codifies the interaction between the copy map and product. We depict parallel strings passing together through a copy node as a single double-stranded string passing through a copy node.



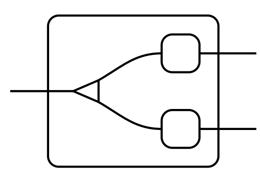
In particular, both string diagrams represent the mapping $(x, y) \mapsto (x, y, x, y)$. Note the use of the swap map in order to have outputs in the correct order.

We now introduce the third and final property of copy maps: *coassociativity*. It is visually suggestive to draw this equation vertically, rotating our diagrams by ninety degrees counter-clockwise so that they read from top to bottom.

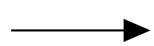


The left hand side maps x to $(x, (x, x))$ and the right to $((x, x), x)$, which via strictification we identify.

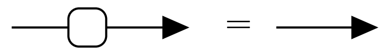
With copy map in hand, we can then define the fork $f \vartriangle g$ point-free as the composition $f \vartriangle g = \texttt{copy} \mathbin{⨾} [f \times g]$. We can hence represent the fork operation with the schematic `fork`.



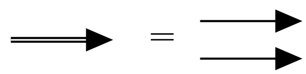
We now turn to making diagrammatic the process of extracting components, which requires encoding projections. We make the important observation that for any set X, there is precisely one map, which we call *discard*, of type $X \to \star$, given by the constant rule $x \mapsto \bullet$. Since `discard` is as polymorphic as `copy`, we type it as $\mathtt{1} \to \mathtt{0}$—taking any set as domain and having zero copies of it, i.e. $\star$, as codomain. We diagram `discard` as follows.



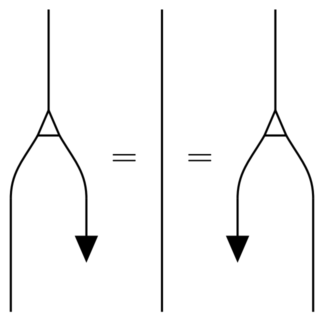
The map `discard` comes with its own naturality property.



Discard maps also satisfy uniformity.

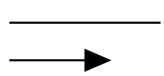


Finally, discard maps interact with copy maps through the *counitality* property.

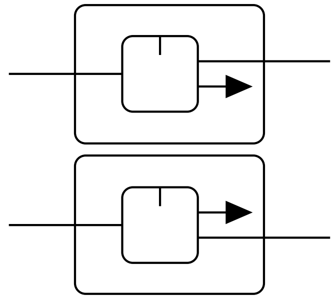


The diagrams respectively map x to $(x, \bullet)$, x, and $(\bullet, x)$, which we identify via strictification. In words, this is just the fact that copying something and then discarding one of the copies is equivalent to having done nothing at all. The string diagram aspires to visually imply this relation via the imagery of retracting the black discard triangle into the empty slot of the copy triangle, which, upon shrinking the induced bump, create a single unbroken identity string.

Via the discard map, we can define projections. For example, the following string diagram represents projection onto the first factor.

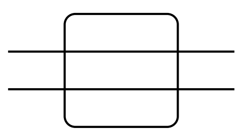


We can then diagram the extraction of components as the following generalized—allowing for multiple screens—schematic `unfork`.

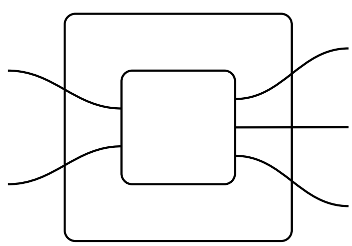


Recalling the meaning of tick marks, a filling for `unfork` amounts to a single map, inserted in both slots.

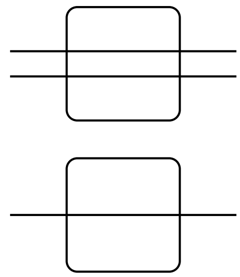
Recall the schematics `unit` and `inert` which respectively pick out the (lower-order) identity map and act as the higher-order identity map. These schematics take as argument what we would now retroactively call univariate maps. Since any multivariate map may be reconceptualized, via ignoring the product partitioning on domain and codomain, as univariate, we use the terms `unit` and `inert` to denote any of these higher-order schematics. For example, the following schematic is an instances of `unit`



and this next one of `inert`



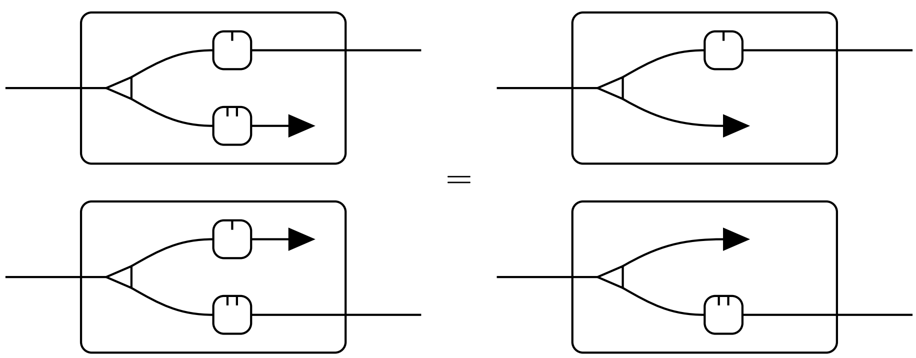
The same principle also goes for generalized schematics with multiple output boxes; e.g. the following is also an instance of `unit`.

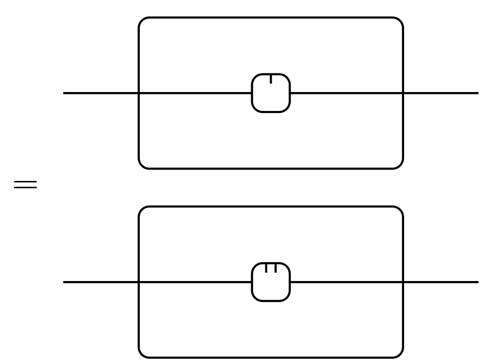


We now formally state and prove the inverse relation between `fork` and `unfork`.

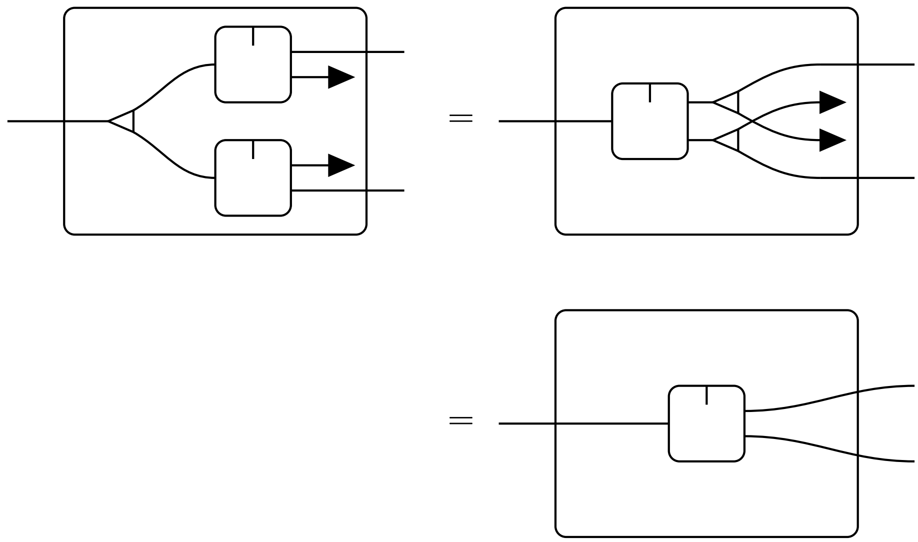
Proposition 2.1.1. *The schematics* `fork` *and* `unfork` *are mutually inverse, i.e.* `fork` ⨟ `unfork` = `inert` *and* `unfork` ⨟ `fork` = `inert`*.*

Proof. We simplify `fork` ⨟ `unfork` to `inert`.





We simplify `unfork` ⨾ `fork` to `inert`.

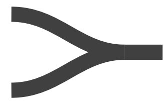


□

2.1.8 Algebras

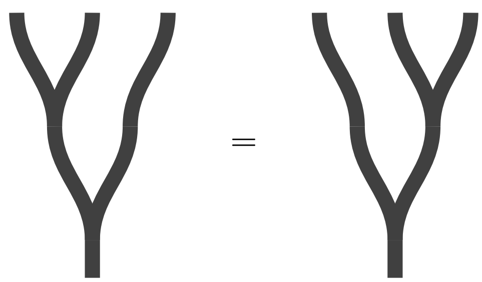
The field of algebra is concerned with well-behaved multivariate maps. Of particular significance perhaps are *binary operators* $\odot\colon X \times X \to X$. These are bestowed a special *infix* notation $x \odot y$ in place of the usual prefix function application $\odot\colon (x, y)$. A pair $(X, \odot)$, with X a set and $\odot$ a binary operator, is called a *magma*. Typically we are interested in magmas that enjoy certain properties. In the case that $\odot$ is associative and has a unit e, we say that the triple $(X, \odot, e)$ is a *monoid*. In this case, we choose to slightly interval-thicken the string into a *band*:

We then represent the monoid operation $\odot$ with the following image.

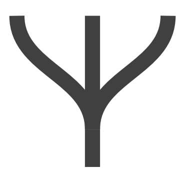


Similarly, we represent the unit e as a map $\star \to X$ via the *cap*.

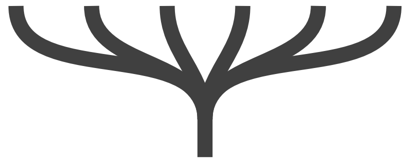
The associativity condition can be written pointfully as $(x \odot x') \odot x'' = x \odot (x' \odot x'')$ and hence point-free as $(\odot \times \mathtt{id}_X) \fatsemi \odot = (\mathtt{id}_X \times \odot) \fatsemi \odot$. This renders as the following identity of string diagrams, which we draw vertically, running downwards.



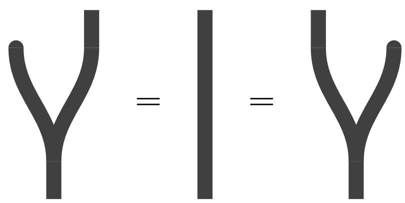
The diagrams are thus defined so as to have their relations implied by planar topology. Note that this planar transformation is further restricted by a *rel-boundary* condition. More precisely, if we consider these diagrams as living on an infinite strip $\mathbb{R} \times [0, 1]$, then we leave fixed the boundary lines, given by the set $\mathbb{R} \times \{0, 1\}$. If we like, this justifies drawing this ternary composite with the following "un-bracketed" diagram.



Associativity can be applied iteratively, as is made evident by the topology, to show that this induces a unique map for any number, e.g. six, of arguments.



The unitality axiom can also be given by a cycle of equalities; expressed pointfully as $e \odot x = x = x \odot e$ and point-free as $(e \times \mathtt{id}_X) \mathbin{⨟} \odot = \mathtt{id}_X = (\mathtt{id}_X \times e) \mathbin{⨟} \odot$. In diagrams, this is given as follows.

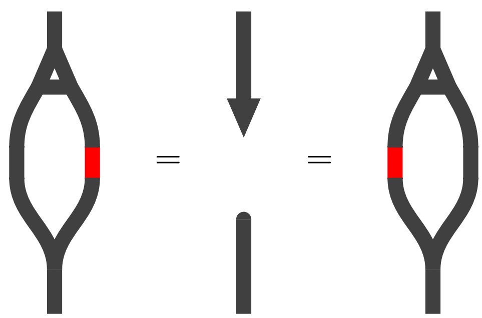


Again, the algebraic condition is reduced to planar topology. This all amounts to the fact that we can extend $\odot$ to its *unbiased* version—denoted by the enlarged symbol $\bigodot$—which takes arguments in *lists* $\mathbf{x}$ in X, and returns a single $\bigodot \mathbf{x} \colon X$, given by the unit in the case of an empty list.

When in addition our operation satisfies $x \odot y = y \odot x$, we call the monoid $(X, \odot, e)$ *commutative*. To describe this purely in terms of compositions, we need to make use of the swap map τ. The commutativity is equivalent to the identity $\tau \mathbin{⨟} \odot = \odot$. Since our bands have too planar an imagery to properly render crossings, we circle-thicken, rather than interval-thicken, the string into a *tube*.

What about the beloved group notion? A group is a monoid $(G, \odot, e)$ equipped with an extra unary *inverse* $(-)^{-1}\colon G \to G$, which we here depict as follows.

This operator satisfies *invertibility*, classically written $x \odot x^{-1} = e = x^{-1} \odot x$. Since the left and right-most expressions involve two instances of the same term x, we must use `copy` to render this relation. Furthermore, since the middle term e involves *no* instances of the term x, we must use `discard`. This lets us render this relation diagrammatically.



This relation, unfortunately, is not implied by planar topology. Because of this, it is= the last time we use this diagrammatic rendering for groups. We will return to this point in a subsequent section.

2.2 Categorification of Counting

Categorification is the act of replacing quantities with structures—and, more interestingly, relations between quantities with transformations of structures. Most famously, in (CITE), John Baez argued that our everyday arithmetic is itself a *decategorification* of naive set theory. In particular, the concept of a natural quantity n is an abstraction of the various instances of (finite) sets, e.g. of plants or animals, that contain n elements. Denoting the collection of finite sets as `fin`, we say that *cardinality* $\texttt{card}\colon \texttt{fin} \to \mathbb{N}$, which maps a finite set X to its number of elements $|X|$, is a decategorification of finite sets to natural numbers; and, dually, that sets categorify natural numbers.

2.2.1 Equality

We first categorify the notion of equality between numbers, i.e. we would like a relationship between two sets X and Y that implies the equality of their cardinalities $|X| = |Y|$. This is achieved by the notion of isomorphism $X \cong Y$, in that $|X| = |Y|$ if and only if there exists a bijection $f\colon X \to Y$.

For the sake of reference, we denote $\mathbf{n} \triangleq \{0, 1, \ldots, n-1\}$ as a generic n-element set. That is $|X| = n$ if and only if $X \cong \mathbf{n}$. We note that $\mathbf{0}$ is equal to the empty set $\varnothing$, while $\mathbf{1}$ is isomorphic to the terminal set $\star$.

This categorification in fact *generalizes* equality of numbers by allowing for a notion of sameness between any sets, including infinite ones. This extension of the equality of quantitites is precisely what Cantor used to present his hierarchy of infinities and articulate ideas like the fact that there are "more" real numbers than there are rational numbers—and, surprisingly, that there are equally many rationals as integers.

2.2.2 Multiplication

Not only would we like to categorify equality of terms, but furthermore we wish to lift operations. We spent much of the prior section discussing the product of sets. This operation is aptly named as the following identity holds.

$$|X \times Y| = |X| \times |Y|$$

This codifies the pedagogical explanation of multiplication of numbers n, m as the counting of points in an $n \times m$ grid, i.e. points in the product $\mathbf{n} \times \mathbf{m}$ of sets with cardinalities.

The beautiful thing about categorification is that we can recover *properties* of operations via constructing *structures* on their categorified level. For instance, multiplication of natural numbers is associative: $(k \cdot m) \cdot n = k \cdot (m \cdot n)$. Although this fact is "obvious," its formal proof relies on a tedious mathematical induction argument. Contrast this with the highly intuitive isomorphism

$$(X \times Y) \times Z \xrightarrow{\cong} X \times (Y \times Z)$$
$$((x, y), z) \mapsto (x, (y, z)).$$

This isomorphism both generalizes to infinitary quantities and also, by virtue of the fact that cardinality preserves products, implies the associativity of multiplication. Multiplication of natural numbers is also unital in the sense that the number 1 satisfies $1 \cdot n = n = n \cdot 1$. This too can be categorified, avoiding an induction argument, by the following chain of isomorphisms.

$$\mathbf{1} \times X \xrightarrow{\cong} X \xrightarrow{\cong} X \times \mathbf{1}$$
$$(0, x) \mapsto x \mapsto (x, 0)$$

Finally, multiplication enjoys commutativity $n \cdot m = m \cdot n$, which is categorified $\texttt{swap}\colon X \times Y \to Y \times X$.

2.2.3 Exponentiation

Perhaps more surprisingly than multiplication, sets have a natural exponentiation operator, given by the mapping set $[X \to Y]$, which we denote by exponential notation Y^X. In particular, we have the equality

$$|Y^X| = |Y|^{|X|}$$

by virtue of the fact that a function is determined by a choice of $y\colon Y$ for each $x\colon X$, i.e. an $|X|$'s worth of $|Y|$'s. We now hope to see that this operation interacts as expected with multiplication. Such a relationship is instantiated by the following higher-order *currying* map.

$$\begin{aligned} \texttt{curry}\colon [X \times Y \to Z] &\xrightarrow{\cong} [X \to [Y \to Z]] \\ \lambda(x, y).f(x, y) &\mapsto \lambda x.[\lambda y.f(x, y)] \end{aligned}$$

For the reader unfamiliar with lambda calculus, currying sends a bivariate map $f\colon X \times Y \to Z$ to a higher-order map which sends the element $x : X$ to the univariate map $f(x, -) :: y \mapsto f(x, y)\colon Y \to Z$. Using exponential notation, we can rewrite this isomorphism as follows.

$$Z^{X \times Y} \cong \left(Z^Y\right)^X$$

This is none other than a law of exponents! In the last section, we in fact constructed another isomorphism

$$\texttt{fork}\colon [X \to Y] \times [X \to Z] \xrightarrow{\cong} [X \to Y \times Z].$$

We can rewrite this in exponential notation, yielding another law of exponents:

$$Y^X \times Z^X \cong (Y \times Z)^X.$$

We also defined `discard`, which mapped any X to the singleton $\mathbf{1}$. This is in fact the unique map of this type, implying the isomorphism $\texttt{discard} \mapsto 0\colon [X \to \mathbf{1}] \xrightarrow{\cong} \mathbf{1}$, and hence the categorified law of exponents.

$$\mathbf{1}^X \cong \mathbf{1}$$

Finally, we discussed how an element $x\colon X$ was uniquely identifiable with the map $0 \mapsto x\colon \mathbf{1} \to X$, implying the isomorphism $[\mathbf{1} \to X] \xrightarrow{\cong} X$ and hence the law

$$X^{\mathbf{1}} \cong X.$$

2.2.4 Addition

We now climb down the recursive ladder to explore the simplest operation, addition.

We define the *cocartesian sum* or simply *sum* $X + Y$ of two sets X, Y as the *disjoint* union of X and Y, which by construction satisfies

$$|X + Y| = |X| + |Y|.$$

As per our discussion at the beginning of the chapter, the disjoint union may as well be the primitive union-like notion. For those insisting on a traditional set-theoretic approach, we may define the disjoint union by respectively tagging elements of X and Y with an ι_0 and ι_1. When done to each element, this act of tagging can be seen as a bijection $X \to \iota_0(X)$, and similarly for Y. This forces $\iota_0 X \cap \iota_1 Y = \varnothing$ and thus facilitates the definition

$$X + Y \triangleq \iota_0 X \cup \iota_1 Y.$$

From this definition, we readily derive associativity

$$(X + Y) + Z \cong X + (Y + Z)$$

defined point-wise by the mappings

$$\begin{aligned} \iota_0\iota_0 x &\mapsto \iota_0 x \\ \iota_0\iota_1 y &\mapsto \iota_1\iota_1 y \\ \iota_1 z &\mapsto \iota_1\iota_1 z. \end{aligned}$$

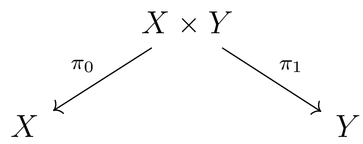
Noting that $\varnothing = \mathbf{0}$, we also have unitality

$$\varnothing + X \cong X \cong X + \varnothing$$

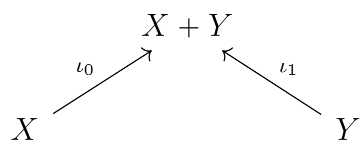
codifying the fact that including nothing to a set leaves it unchanged. Finally, we have a `swap` isomorphism

$$\begin{aligned} X + Y &\xrightarrow{\cong} Y + X \\ \iota_i a &\mapsto \iota_{1-i} a. \end{aligned}$$

Not only does the set sum categorify numerical addition, it ends up being dual to the set product. Recall that we can define a product term $a : X \times Y$ by the capacity to extract terms $a_0 \colon X$ and $a_1 \colon Y$. These derivations are given by the projection maps:

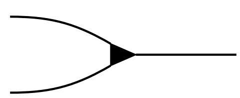


We can characterize the sum $X + Y$ as dual to this. More precisely, given a term $x \colon X$, we can construct a term $x_0 \colon X + Y$; or, similarly, given a term $y \colon Y$, we can construct a term $y_1 \colon X + Y$. This can be codified by *inclusion* maps:

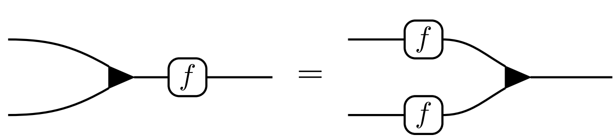


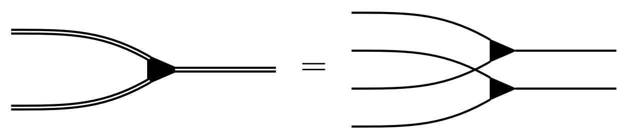
To demonstrate the duality between sum and product, we will now use parallel composition to denote sums, again strictify associativity and unitality. What is interesting is that, rather than possessing structure maps `copy` : $\mathtt{1} \to \mathtt{2}$ and `discard` : $\mathtt{1} \to \mathtt{0}$, duality gives us dual structure maps `fold` : $\mathtt{2} \to \mathtt{1}$ and `create` : $\mathtt{0} \to \mathtt{1}$.

The `fold` map is polymorphic of type $X + X \to X$ and is given by the mapping $\iota_i x \mapsto x$ for $i = 0, 1$. We depict it as an inverted `copy` diagram.

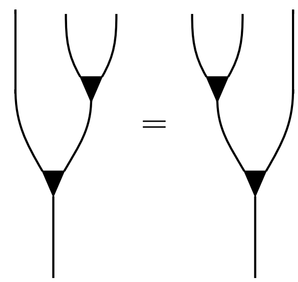


Just like `copy`, `fold` enjoys naturality and uniformity.

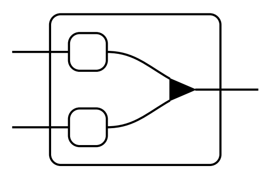




Duality leads to associativity rather than coassociativity for `fold`:



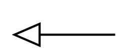
With `fold` in hand, we can define the *join* $f \blacktriangledown g : X + Y \to Z$ of two maps $f : X \to Z$ and $g : Y \to Z$. The join is enacted by the schematic `join`, drawn as inverted `fork`.



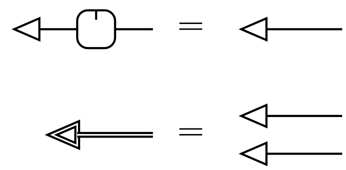
This map gives us a map of type $[X \to Z] \times [Y \to Z] \xrightarrow{\cong} [X + Y \to Z]$. To prove that this map is an isomorphism, we must define an inverse. We will define this dually to how we defined `discard`, defining the map `create`: $\mathtt{0} \to X$ as the vacuously unique map of its type. This uniqueness instantiates an isomorphism $[\mathtt{0} \to X] \cong \mathtt{1}$, thus instantiating another law

$$X^{\mathbf{0}} \cong \mathbf{1}.$$

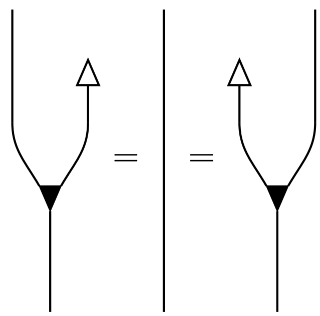
In accord with duality, we render `create` as an inverted `discard`.



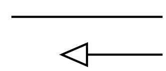
This map also enjoys naturality and uniformity.



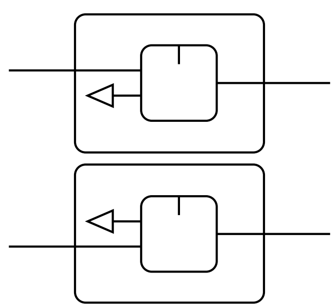
Continuing with the duality, `create` acts as unit for `fold`:



This makes any set X a monoid in the context of sums. Just as we defined projections via `discard`, we define inclusions via `create`. For instance, the following diagrams inclusion of the first summand.



We now depict the polymorphic map `unjoin` : $[X + Y \to Z] \to [X \to Z] \times [Y \to Z]$.



The beauty of duality is that, rather than proving that `join` and `unjoin` are mutually inverse, we may simply dualize the entire proof of Proposition 2.1.1, automatically deriving the isomorphism $[X + Y \to Z] \cong [X \to Z] \times [Y \to Z]$, which gives us our final law of exponents.

$$Z^{X+Y} \cong Z^X \times Z^Y$$

2.2.5 Combinatorics

We have already categorified formulae in which products show up in both arguments of exponentiation $(-)^-$, and in which sums show up in the exponents. The final combination, the binomial theorem, is also significant.

$$(n+m)^k = \sum_{j=0}^{k} \binom{k}{j} n^j m^{k-j}$$

The categorification of the left-hand side is the mapping set

$$[\mathbf{k} \to \mathbf{n} + \mathbf{m}].$$

Before categorifying the right-hand side, we must categorify the binomial coefficient

$$\binom{k}{j} \triangleq \frac{k!}{j!(k-j)!}$$

and hence of the factorial

$$n! \triangleq n \cdot (n-1) \cdots 2 \cdot 1.$$

Luckily, the factorial is already a decategorification—of the group $\mathbf{n}!$, classically written S_n, of permutations of the set $\mathbf{n}$, i.e. isomorphisms $\mathbf{n} \to \mathbf{n}$. The fact that this is a group lets us tackle the binomial coefficient. In fact, a permutation ρ_0 of $\mathbf{n}$ and ρ_1

of $\mathbf{m}$ combine to give a permutation $\rho_0 + \rho_1$ on $\mathbf{m} + \mathbf{n}$. This defines an inclusion

$$\mathbf{n}! \times \mathbf{m}! \to (\mathbf{n} + \mathbf{m})!$$

and hence a subgroup. We can thus apply Lagrange's theorem to give a proof by categorification of the integrality of the binomial coefficient. We can then define a free action of $\mathbf{n}! \times \mathbf{m}!$ on $(\mathbf{n} + \mathbf{m})!$. We can quotient by this action and, by Burnside's Lemma, see it as a categorification of the binomial coefficient.

$$\left| \frac{(\mathbf{n} + \mathbf{m})!}{\mathbf{n}! \times \mathbf{m}!} \right| = \frac{|(\mathbf{n} + \mathbf{m})!|}{|\mathbf{n}! \times \mathbf{m}!|} = \frac{(n+m)!}{n! \cdot m!}$$

Furthermore, it is known that the binomial coefficient $\binom{n}{k}$ measures the number of size-k subsets of a size-n set. This is precisely because the orbits of this action enact a partition of the set into two components, one of size n and one of size m.

Chapter 3

Categorical Technology

3.1 Classical Category Theory

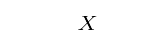
3.1.1 Categories

We abstract on the prior section with an overview of basic category theory principles.

Definition 3.1.1. A *category* $\mathcal{C}$ consists of the following data.

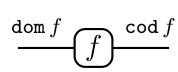
- a collection $\texttt{ob}\,\mathcal{C}$ of *objects*,

  generically depicted as *strings*



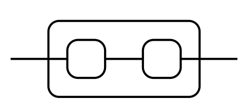
- a collection $\texttt{ar}\,\mathcal{C}$ of *arrows* and two maps $\texttt{dom}, \texttt{cod}\colon\ \texttt{ar}\,\mathcal{C} \to \texttt{ob}\,\mathcal{C}$,

  generically depicted as *boxes* attached to their $\texttt{dom}$ and $\texttt{cod}$ string



- a partial *composition* binary operator $\texttt{then}\colon\ \texttt{ar}\,\mathcal{C} \times \texttt{ar}\,\mathcal{C} \to \texttt{ar}\,\mathcal{C}$

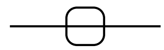
  depicted as a binary schematic



with domain given by the fiber product $2\,\mathtt{path} \triangleq \{(f,g) \mid \mathtt{cod}\, f = \mathtt{dom}\, g\}$

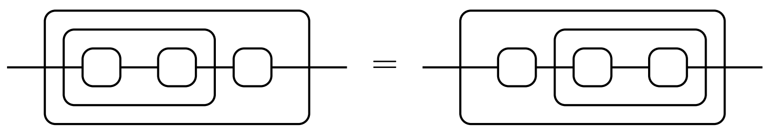
- a *unit* map $\mathtt{unit}\colon \mathtt{ob}\,\mathcal{C} \to \mathtt{ar}\,\mathcal{C}$

  depicted as a nullary schematic

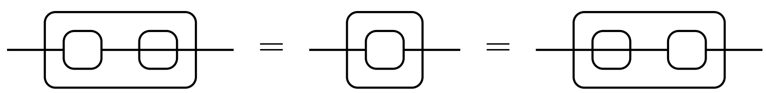


These data must satisfy the following two conditions.

- *associativity*:

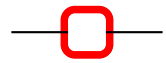


- *unitality*:

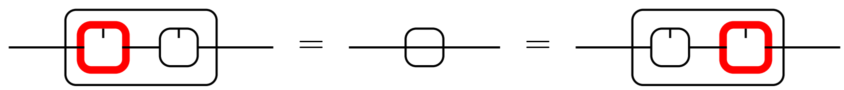


Typically, rather than writing $\mathtt{then}(f,g)$ and $\mathtt{unit}(X)$, we will respectively use the infix $f \mathbin{⨟} g$ and subscripted $\mathtt{id}_X$. An alternative to using $\mathtt{dom}$ and $\mathtt{cod}$ is to organize maps via $\mathtt{hom}\colon \mathtt{ob}\,\mathcal{C} \times \mathtt{ob}\,\mathcal{C} \to \mathbf{2}^{\mathtt{ar}\,\mathcal{C}}$, taking a pair (c,c') of objects to the set $\mathtt{hom}(c,c')$ of arrows f of type $c \to c'$, i.e. with $\mathtt{dom}\, f = c$ and $\mathtt{cod}\, f = c'$. In contexts in which we encounter several different categories, we will subscript the maps $\mathtt{dom}, \mathtt{cod}, \mathtt{hom}, \mathtt{then}$, and $\mathtt{unit}$, by the name of the respective category.

An *isomorphism* is an arrow $f\colon \mathtt{ar}\,\mathcal{C}$ for which there is another arrow $f^{-1}\colon \mathtt{ar}\,\mathcal{C}$ satisfying $f \mathbin{⨟} f^{-1} = \mathtt{id}_{\mathtt{dom}\, f}$ and $f^{-1} \mathbin{⨟} f = \mathtt{id}_{\mathtt{cod}\, f}$. We write $\mathtt{iso}\,\mathcal{C}$ for the subset of $\mathtt{ar}\,\mathcal{C}$ of isomorphisms. A *groupoid* $\mathcal{G}$ is a category for which $\mathtt{iso}\,\mathcal{G} = \mathtt{ar}\,\mathcal{G}$. Groupoids thus possess the following additional schematic $\mathtt{inv}\colon \mathtt{ar}\,\mathcal{G} \to \mathtt{ar}\,\mathcal{G}$



satisfying the following relation.



Examples of categories abound, first as generalizations of some common mathematical structures. A category $\mathcal{C}$ is *discrete* if $\mathtt{id}\colon \mathtt{ob}\,\mathcal{C} \to \mathtt{ar}\,\mathcal{C}$ is a bijection, i.e. it has no non-identity arrows. Any set can be thought of as a discrete category. For instance, it is worth individuating the discrete category on $\star$, which we call the *terminal category* and denote $\star$, which consists of but a single object and only its identity arrow. A category is *thin* if $\hom(c, c')$ has cardinality either 0 or 1, in which case the existence of such an arrow may be identified with a truth value. Such a category is precisely an *order*, where the composition $⨟ : 2\,\mathtt{path} \to \mathtt{ar}\,\mathcal{C}$ specializes to transitivity and unit $\mathtt{id}\colon \mathtt{ob}\,\mathcal{C} \to \mathtt{ar}\,\mathcal{C}$ to reflexivity. Furthermore, a category $\mathcal{C}$ with just a single object is exactly a monoid on $\mathtt{ar}\,\mathcal{C}$ since the composition becomes a total associative binary operator $⨟\colon \mathtt{ar}\,\mathcal{C} \times \mathtt{ar}\,\mathcal{C} \to \mathtt{ar}\,\mathcal{C}$ that is unital with respect to a choice of a single arrow $\mathtt{id}\colon \star \to \mathtt{ar}\,\mathcal{C}$. Alternatively, $\mathtt{end}(x) \triangleq \mathtt{hom}(x, x)$ is a monoid for all $x\colon \mathtt{ob}\,\mathcal{C}$.

Not only can categories be seen as mathematical objects but they may also play a role in organizing the theory of some other mathematical object. In the previous section, we worked with the category **Set**, defined by $\mathtt{ob}\,\mathbf{Set} = \mathtt{set}$ and $\mathtt{ar}\,\mathbf{Set} = \mathtt{map}$. We may also choose to restrict to finite sets and maps between them as in the *subcategory* **Fin**. We also draw attention to **Top** whose objects are topological spaces and arrows continuous maps. The isomorphisms in **Top** are precisely the homeomorphisms. For any algebraic structure—e.g. groups, rings, modules etc.—there is a category whose objects are the chosen algebraic structure and whose arrows are homomorphisms of those structures. We will shortly investigate such algebraic structures in full generality. An important family of these categories of algebraic structures is the category $\mathbf{Vect}_k$, whose objects are k-vector spaces and arrows are k-linear maps. Much of the classical discipline of linear algebra may be identified with the study of this family of categories.

3.1.2 Functors

Another, pleasingly self referential, category is that of categories themselves. More precisely, we define **Cat** to have objects small categories, and arrows functors, which we now define.

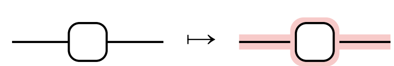
Definition 3.1.2. For categories $\mathcal{C}$ and $\mathcal{D}$, a *functor* $F \colon \mathcal{C} \to \mathcal{D}$ is given by the data

- an object map $\mathtt{ob}\, F \colon \mathtt{ob}\, \mathcal{C} \to \mathtt{ob}\, \mathcal{D}$.

  generically depicted by an *aura* around the string

- an arrow map $\mathtt{ar}\, F \colon \mathtt{ar}\, \mathcal{C} \to \mathtt{ar}\, \mathcal{D}$.

  generically depicted by an aura around the box

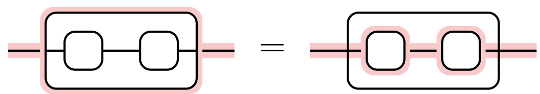


Note that the diagrams subtly imply that F preserves $\mathtt{dom}$ and $\mathtt{cod}$.

These data satisfy the *functoriality* conditions.

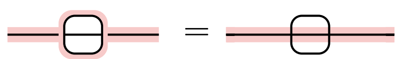
- composition preservation: $F(f \fatsemi g) = Ff \fatsemi Fg$

  depicted schematically as



- unit preservation: $F\,\mathtt{id}_X = \mathtt{id}_{FX}$

  depicted schematically as



Functors reduce to maps, monotone maps, and monoid homomorphisms respectively in the discrete, thin, and one-object special cases. When the involved categories are that of some species of mathematical object, functors between them codify relations between the corresponding mathematical theories, e.g. between naive set theory and linear algebra. In this context, there is a *forgetful* functor $U\colon \mathbf{Vect}_k \to \mathbf{Set}$ that takes a vector space to its underlying set, and a *free* functor $F\colon \mathbf{Set} \to \mathbf{Vect}_k$ which takes a set to the k-vector space generated by its k-linear combinations. Such free-forgetful pairings occur frequently in pairs of categories of a more and less structured mathematical object. Perhaps more interestingly, much of algebraic topology is the study of functors with domain category $\mathbf{Top}$, or a variant thereof, and codomain category consisting of algebraic structures.

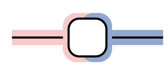
3.1.3 Naturality

The analogy that categories are to sets what functors are to maps is complicated by the fact that categories have one extra "level" relative to sets. This level affords another layer of structures that act like arrows between functors and are hence sometimes referred to as 2-arrows.

Definition 3.1.3. Given functors F and G both of type $\mathcal{C} \to \mathcal{D}$, a *natural transformation* $T\colon F \to G$ is given by the following data.

- for all $X\colon \mathtt{ob}\,\mathcal{C}$, a *component* arrow $T_X\colon FX \to GX$

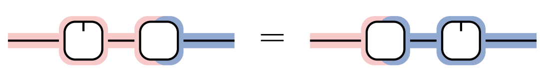
  depicted with a bicolored aura box



These data must satisfy the following *naturality* condition.

- for all $f\colon \mathtt{ar}\,\mathcal{C}$, the commutativity $Ff \mathbin{⨾} T_{\mathtt{cod}\,f} = T_{\mathtt{dom}\,f} \mathbin{⨾} Gf$

  depicted schematically as



When all of the components of T are isomorphisms, we call T a *natural isomorphism*.

The categorified setting introduces further novelty in the context of sameness. In particular, rather than isomorphism, it is more meaningful to consider equivalence.

Definition 3.1.4. Functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ form an *equivalence of categories* when there exist natural isomorphisms $S\colon F \mathbin{⨾} G \to \mathtt{id}_{\mathcal{C}}$ and $T\colon G \mathbin{⨾} F \to \mathtt{id}_{\mathcal{C}}$.

Polymorphism.

3.1.4 Universality

Unlike in typical treatments of the subject, the central notion of universality is not located at the heart of the present work. For completeness, we briefly review the relevant instances of universality.

Definition 3.1.5. We say a covariant functor $F\colon \mathcal{C} \to \mathbf{Set}$ is *representable* by $c\colon \mathtt{ob}\,\mathcal{C}$ when there is a natural isomorphism $F \cong \mathtt{hom}(c, -)$. Similarly, we say a contravariant functor $F'\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is *representable* by $c'\colon \mathtt{ob}\,\mathcal{C}$ when there is a natural isomorphism $F' \cong \mathtt{hom}(-, c)$.

Important instances of representable functors are given by the definition of product and coproduct in a generic category.

Definition 3.1.6. We say $c, c'\colon \mathtt{ob}\,\mathcal{C}$ have a *product* $c \times c'$ when there is a natural isomorphism

$$\mathtt{hom}(-, c \times c') \cong \mathtt{hom}(-, c) \times \mathtt{hom}(-, c')$$

i.e. when $c \times c'$ represents the functor mapping an object x to the cartesian product of the hom-sets from x to c and c'.

Similarly, we say $c, c'\colon \mathtt{ob}\,\mathcal{C}$ have a *coproduct* or *sum* $c + c'$ when there is a natural isomorphism

$$\mathtt{hom}(c + c', -) \cong \mathtt{hom}(c, -) \times \mathtt{hom}(c', -)$$

i.e. when $c + c'$ represents the functor mapping an object x to the cartesian product of the hom-sets from c and c' to x. When the product $c \times c'$ and coproduct $c + c'$ are the same object, we call it the *biproduct* or *direct sum* and write it as $c \oplus c'$.

This definition, in terms of representable functors, allows us to extend the up-to-natural-isomorphism associativity, unitality, and commutativity of the cartesian product to any product or sum.

We call a category for which every pair of objects has a product / sum / direct sum a *category with products / sums / direct sums.* In **Set**, the product is the cartesian product and coproduct is the cocartesian sum i.e. disjoint union. In $\mathbf{Vect}_k$, the aptly named direct sum $\oplus$ serves as a biproduct. In order to use these bifunctors as tensors, we need a notion of unit for each.

Definition 3.1.7. We say $\star\colon \mathtt{ob}\,\mathcal{C}$ is the *terminal object* when $\star$ represents the constant contravariant functor with value $\star$. Similarly, we say $\varnothing\colon \mathtt{ob}\,\mathcal{C}$ is the *initial object* of $\mathcal{C}$ when $\varnothing$ represents the constant covariant functor with value $\star$. When these are the same object $\star = \varnothing$, we call it a *zero object* and denote it as $\mathbf{0}$.

From this definition, it follows that $\star$ is the monoidal unit of $\times$, $\varnothing$ the monoidal unit of $+$, and 0 the monoidal unit of $\oplus$. We compute this directly in the case of the former, with duality implying the latter two.

$$\begin{aligned}\mathtt{hom}(-, c \times \star) &\cong \mathtt{hom}(-, c) \times \mathtt{hom}(-, \star)\\ &\cong \mathtt{hom}(-, c) \times \star\\ &\cong \mathtt{hom}(-, c)\end{aligned}$$

Representable functors are more well-understood than generic functors by virtue of the *Yoneda Lemma.*

Lemma 3.1.8. *Let $F\colon \mathcal{C} \to \mathbf{Set}$ and $c\colon \mathtt{ob}\,\mathcal{C}$. Then there is a natural isomorphism*

$$\mathtt{Nat}(\mathtt{hom}(c, -), F) \cong Fc$$

determined by mapping $[T\colon \mathtt{hom}(c, -) \to F]$ to its value on the identity $T(\mathtt{id}_c)\colon Fc$.

Similarly, let $F' \colon \mathcal{C}^{\mathtt{op}} \to \mathbf{Set}$ *and* $c' \colon \mathtt{ob}\,\mathcal{C}$*. Then there is a natural isomorphism*

$$\mathtt{Nat}(\mathtt{hom}(-, c'), F') \cong F'c'$$

determined by mapping $[T \colon \mathtt{hom}(-, c') \to F']$ *to its value on the identity* $T(\mathtt{id}_{c'}) \colon F'c'$*.*

The Yoneda Lemma implies that the Yoneda Embeddings

$$\mathcal{Y}_* :: x \mapsto \mathtt{hom}(x, -) \colon \mathcal{C}^{\mathtt{op}} \to [\mathcal{C} \to \mathbf{Set}]$$
$$\mathcal{Y}^* :: x \mapsto \mathtt{hom}(-, x) \colon \mathcal{C} \to [\mathcal{C}^{\mathtt{op}} \to \mathbf{Set}]$$

reflect isomorphisms, i.e. if there is a natural isomorphism $\mathtt{hom}(x, -) \cong \mathtt{hom}(x', -)$ or $\mathtt{hom}(-, x) \cong \mathtt{hom}(-, x')$, that implies that there is an isomorphism $x \cong x'$.

Our final universal construction of interest is that of *adjoint functors*.

Definition 3.1.9. The pair of functors $L \colon \mathcal{C} \to \mathcal{D}$ and $R \colon \mathcal{D} \to \mathcal{C}$ forms an *adjunction*, written $L \dashv R$, when there is a natural isomorphism

$$\mathtt{hom}_{\mathcal{D}}(L-, -) \cong \mathtt{hom}_{\mathcal{C}}(-, R-).$$

Note that both of these functors are of type $\mathcal{C} \times \mathcal{D}^{\mathtt{op}}$. We call L the *left adjoint* and R the *right adjoint*.

The most common instances of adjunctions are pairs of free and forgetful functors. For instance, we have the free functor $F \colon \mathbf{Set} \to \mathbf{Vect}_k$ mapping a set S to the vector space kS of its k-linear combinations and linearly extending maps $f \colon S \to S'$, and its right adjoint the underlying set functor $U \colon \mathbf{Vect}_k \to \mathbf{Set}$ which returns the set of vectors of a vector space and forgets that the linear maps preserve linear combinations. This is an adjunction because any map $kS \to V$ is *freely* determined (i.e. without restriction) by the mapping $S \to V$ on the basis. Similar adjunctions exist between **Set** and any category of algebras.

Interestingly, the underlying set functors **Top** → **Set** and **Cat** → **Set** have both a left and right adjoint. The right adjoint is given by the *discrete* functors **Set** → **Cat** and **Set** → **Top**, making the set a discrete category and space, respectively. These are left adjoint since functors and continuous maps out of discrete categories and spaces are freely determined by the object and point maps, respectively. Dually, the right adjoint is given by the *indiscrete* functors **Set** → **Cat** and **Set** → **Top**, respectively mapping a set to the thin category with no empty hom-sets and to the space whose only open sets are the empty and total set. This instantiates a right adjoint since functors and continuous maps into indiscrete categories and spaces are freely determined by object and point maps, respectively. This similarity between categories and spaces is among so many that one should conceive of a category as a hybrid gadget with spatial as well as algebraic features.

Another common family of adjunctions are various notions of currying. For instance, in **Set**, we have that taking the product with some set X is left adjoint to mapping out of X, i.e.

$$\mathtt{hom}(X \times -, -) \cong \mathtt{hom}(-, [X \to -]).$$

There is an analogous, so called *tensor-hom*, adjunction in the context of $\mathbf{Vect}_k$

$$\mathtt{hom}(V \otimes -, -) \cong \mathtt{hom}(-, [V \to -]).$$

The main difference between these two situations is that the linear tensor $\otimes$ is not the categorical product in $\mathbf{Vect}_k$. We will return to this point in a subsequent section.

We end the section with an important fact about how adjoint functors interact with universal structures.

Proposition 3.1.10. *Right adjoints preserve products and terminal objects, while left adjoints preserve sums and initial objects. I.e. there are natural isomorphisms*

$$R(d \times d') \cong Rd \times Rd' \qquad R\star = \star$$

$$L(c + c') \cong Lc + Lc' \qquad L\varnothing = \varnothing$$

We will use this fact in the sequel to explore the structural properties of various categories.

3.1.5 Exponentials

In the case of a cartesian category $(\mathcal{V}, \times, \star)$, we say $\mathcal{V}$ is *cartesian closed* and call the internal hom $[y \to z]$ an *exponential*, often denoting it via the notation z^y. Furthermore, we often write $\mathbf{1}$ for the terminal object and, if it exists, $\mathbf{0}$ for the initial object. This of course generalizes the case of set in which we canonically have $\star \cong \mathbf{1}$ and $\varnothing = \mathbf{0}$. As we discussed in the prior section, this notation allows us to codify the adjunction as the following categorified law of exponents.

$$z^{x \times y} \cong (z^y)^x$$

These notations are justified by the fact that it behaves just like exponentiation of natural numbers, and the behavior is provided by the adjunction properties. More precisely, since $(-)^y$ is a right adjoint, it preserves products and terminal objects, implying the following isomorphisms.

$$(a \times b)^y \cong a^y \times b^y$$

$$\mathbf{1}^y \cong \mathbf{1}$$

Since $- \times y$ is a left adjoint, it preserves sums and initial objects when they exist.

$$(a + b) \times y \cong a \times y + b \times y$$

$$\mathbf{0} \times y \cong \mathbf{0}$$

In addition, Proposition 3.2.20 implies that $z^{(-)}$ is a left adjoint, sending sums in $\mathcal{V}$ to sums in $\mathcal{V}^{\mathsf{op}}$, i.e. products in $\mathcal{V}$. This implies the isomorphisms.

$$z^{a+b} \cong z^a \times z^b$$

$$z^{\mathbf{0}} \cong \mathbf{1}$$

This amounts to the view that cartesian closed categories are the right setting for categorified natural number arithmetic.

3.1.6 Theories

Recall the classical definition of a group.

Definition 3.1.11. A group is a set G with operations

- binary *multiplication* $\cdot \colon G \times G \to G$
- nullary *unit* $e \colon \star \to G$

- unary *inverse* $(-)^{-1}\colon G \to G$

satisfying the equations

- *associativity* $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- *unitality* $a \cdot e = a = e \cdot a$
- *invertibility* $a \cdot a^{-1} = e = a^{-1} \cdot a$

This definition gives an arbitrary presentation of the group notion, in the sense that the following is an equivalent definition.

Definition 3.1.12. A group is a set G with

- a ternary *heap* operation $[-,-,-]\colon G^3 \to G$

subject to the heap equations

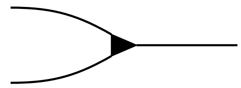
- $[[a,b,c],d,e]] = [a,[d,c,b],e] = [a,b,[c,d,e]]$
- $[a,a,x] = [x,a,a]$

A natural question for the category theorist is to ask of what structure the above definitions are presentations? This question also leads us to the desire of distinguishing between the essence of a group and its implemented setting, e.g. in **Set**. We therefore can generalize a group to any category with products.

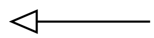
Definition 3.1.13. Let $\mathcal{C}$ be a category possessing finite products. A *group object* G in $\mathcal{C}$ consists of the following data.

- a $\mathcal{C}$-object G

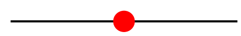
- a *multiplication* arrow $\mu\colon G \times G \to G$ depicted as



- a *unit* arrow $\eta\colon \star \to G$ depicted as

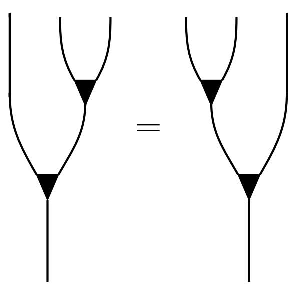


- an *inverse* arrow $(-)^{-1}\colon G \to G$ depicted as

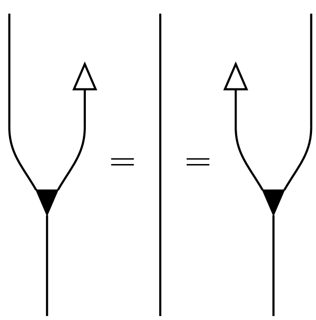


These data satisfy the following conditions.

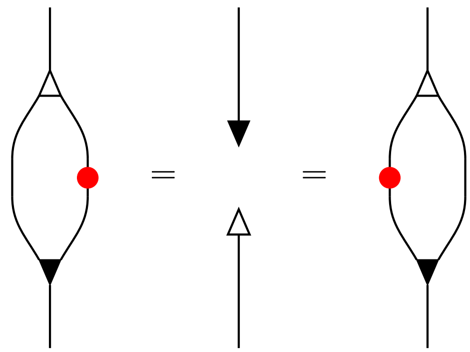
- *associativity*



- *unitality*



- *invertibility*



The fact that $\mathcal{C}$ has finite products is critical for defining the invertibility axiom, as `copy` and `discard` are needed to express it. This allows us to define groups internal to any finite product category $\mathcal{C}$, giving us topological groups when $\mathcal{C}$ is **Top** and Lie groups when $\mathcal{C}$ is the category **Diff** of smooth manifolds.

More generally, the field of univeresal algebra seeks to study generic algebraic structures, i.e. sets T endowed with collections of operations $\{\varphi\colon T^{\text{arity}\,\varphi} \to T\}$ satisfying a collection of equations. In his famed doctoral thesis, William Lawvere advanced universal algebra via the introduction of *algebraic theories*, now named *Lawvere theories*. Theories allow for the separation of the essence of an algebraic structure with any particular implementation of it, rendering the operations and equations of a universal algebra respectively as generating arrows and relations among them within a single meta-algebraic structure.

Definition 3.1.14. A *Lawvere Theory* is a finite product category $\mathfrak{T}$ generated by a single object. A *model* of $\mathfrak{T}$ in a finite product category $\mathcal{C}$ is a product-preserving functor $M\colon \mathfrak{T} \to \mathcal{C}$. The category $\mathfrak{T}(\mathcal{C})$ is defined as having objects models of $\mathfrak{T}$ in $\mathcal{C}$ and arrows natural transformations between them.

The Lawvere theory notion thus not only abstracts an algebraic structure from its presentation and from its ambient category but furthermore makes automatic the appropriate definition of a homomorphism for an algebraic structure. More concretely, the definition compactly packages the fact that homomorphisms are precisely those arrows which commute with the operations of the algebraic structure.

For instance, the Lawvere theory $\mathfrak{Grp}$ has objects $\mathbb{N}$ and arrows $n \to 1$ all of the n-ary operations for groups. A group in the classic sense is then a product-preserving functor $\mathfrak{Grp} \to \mathbf{Set}$, and hence the category **Grp** can be defined as $\mathfrak{Grp}(\mathbf{Set})$, while the categories of topological groups and Lie groups are respectively $\mathfrak{Grp}(\mathbf{Top})$ and $\mathfrak{Grp}(\mathbf{Diff})$.

3.2 Monoidal Category Theory

3.2.1 Tensors

As sets have a notion of product, so do categories.

Definition 3.2.1. For categories $\mathcal{C}$ and $\mathcal{D}$ the *product category* $\mathcal{C} \boxtimes \mathcal{D}$ is defined by

- $\mathtt{ob}[\mathcal{C} \boxtimes \mathcal{D}] = \mathtt{ob}\,\mathcal{C} \times \mathtt{ob}\,\mathcal{D}$
- $\mathtt{ar}[\mathcal{C} \boxtimes \mathcal{D}] = \mathtt{ar}\,\mathcal{C} \times \mathtt{ar}\,\mathcal{D}$
- $\mathtt{dom}_{\mathcal{C} \boxtimes \mathcal{D}} = \mathtt{dom}_{\mathcal{C}} \times \mathtt{dom}_{\mathcal{D}}$
- $\mathtt{cod}_{\mathcal{C} \boxtimes \mathcal{D}} = \mathtt{cod}_{\mathcal{C}} \times \mathtt{cod}_{\mathcal{D}}$
- $\mathtt{then}_{\mathcal{C} \boxtimes \mathcal{D}} = \mathtt{then}_{\mathcal{C}} \times \mathtt{then}_{\mathcal{D}}$
- $\mathtt{unit}_{\mathcal{C} \boxtimes \mathcal{D}} = \mathtt{unit}_{\mathcal{C}} \times \mathtt{unit}_{\mathcal{D}}$

In other words, objects of $\mathcal{C} \boxtimes \mathcal{D}$ are pairs $(c, d)\colon \mathtt{ob}\,\mathcal{C} \times \mathtt{ob}\,\mathcal{D}$ and arrows of type $(c, d) \to (c', d')$ are simply pairs of arrows $(f\colon c \to c', g\colon d \to d')$ whose compositions are independent. A *bifunctor* $\otimes\colon \mathcal{V} \boxtimes \mathcal{V} \to \mathcal{V}$ is a category-level analog—or *categorification*—of a binary operation. The bifunctoriality condition corresponds to the following two laws.

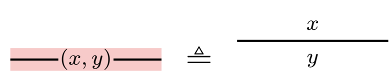
$$(f \mathbin{⨾} f') \otimes (g \mathbin{⨾} g') = (f \otimes g) \mathbin{⨾} (f' \otimes g')$$

$$\mathtt{id}_c \otimes \mathtt{id}_{c'} = \mathtt{id}_{c \otimes c'}$$

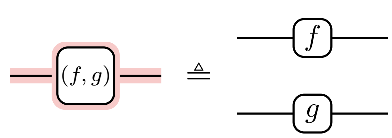
If $\otimes$ enjoys further properties, such as a unit I, the triple $(\mathcal{V}, \otimes, I)$ forms a kind of categorified monoid.

Definition 3.2.2. A *monoidal category* $(\mathcal{V}, \otimes, I)$ consists of the following data.

- a category $\mathcal{V}$
- a *tensor* bifunctor $\mathtt{tensor}\colon \mathcal{V} \boxtimes \mathcal{V} \to \mathcal{V}$ with infix notation $\otimes$
  - depicted on objects via parallelization

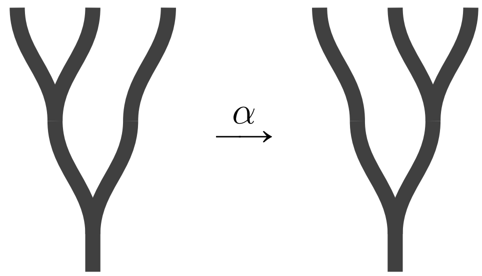


  - depicted on arrows via parallelization



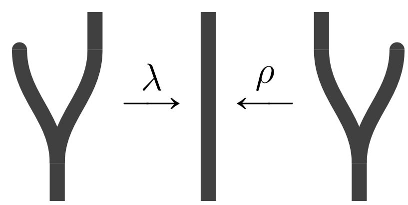
- a *monoidal unit* object $I\colon \mathtt{ob}\,\mathcal{V}$, depicted as an invisible string
- an *associator* natural transformation $\alpha\colon (- \otimes -) \otimes - \to - \otimes (- \otimes -)$

depicted at the object level with monoid diagrammatics as



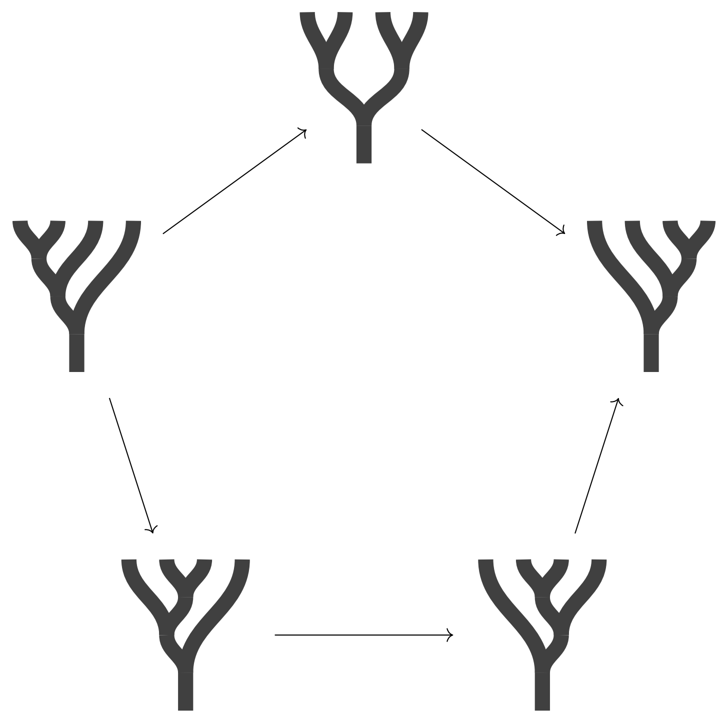
- left and right natural *unitor* arrows $\lambda\colon I \otimes - \to \mathtt{id}_{\mathcal{V}}$ and $\rho\colon -\otimes I \to \mathtt{id}_{\mathcal{V}}$

  depicted at the object level with monoid diagrammatics as

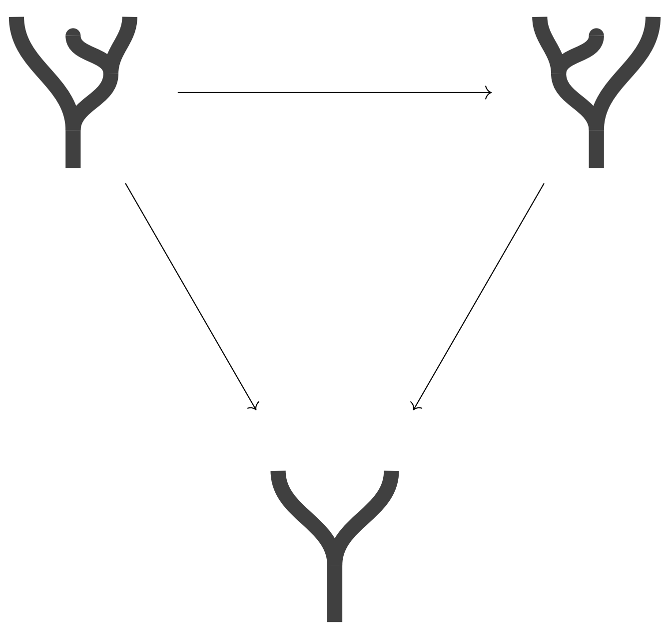


This data must satisfy the following conditions.

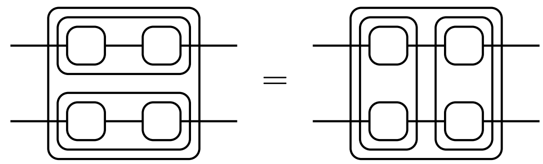
- the *pentagon* identity



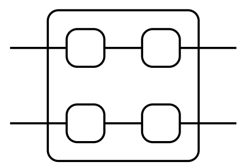
- the *triangle* identity



The interchange law can be rendered schematically, as in the prior section.



This means that the following is a valid schematic in a monoidal category.

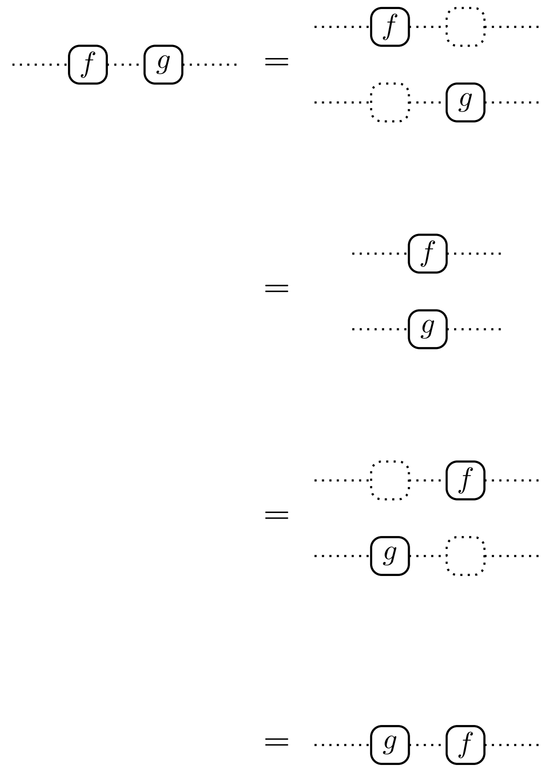


Recall that the arrows of a one-object category form a monoid. Said otherwise, the collection $\mathtt{end}(x)$ of endomorphisms of any object $x \colon \mathtt{ob}\,\mathcal{C}$ is a monoid under composition. In the context of a monoidal category $(\mathcal{V}, \otimes, I)$, we give a special name of *scalars* to $\mathtt{end}(I)$, which turns out to satisfy an important further property.

Proposition 3.2.3. *Let* $(\mathcal{V}, \otimes, I)$ *be a monoidal category. The scalars* $\mathtt{end}(I)$ *form a commutative monoid.*

Proof. For the sake of readability, we draw a dotted wire for I and a dotted box for

$\mathtt{id}_I$. Let $f, g\colon \mathtt{end}(I)$.



□

This method is called the Eckmann-Hilton argument. In the previous section, we encountered two monoidal structures on **Set**, namely $(\mathbf{Set}, \times, \star)$ and $(\mathbf{Set}, +, \varnothing)$. Similarly, we have two noteworthy monoidal structures on $\mathbf{Vect}_k$. The first is given by the *direct sum* $V \oplus W$, defined as the cartesian product on underlying sets with addition and scalar multiplication defined component-wise. The zero vector space 0 acts as unit for direct sum, giving us a monoidal category $(\mathbf{Vect}_k, \oplus, \mathbf{0})$. The other significant monoidal structure is given by the tensor product $V \otimes W$, defined as the vector space for which $\hom(V \otimes W, U)$ is canonically identifiable with the set of bilinear maps of type $V \times W \to U$. If we wish to disambiguate this specific bifunctor from the generic tensor, we will refer to it as a *linear tensor* or k-tensor.

Perhaps the most significant distinction between monoids and their categorification is *weakness*, i.e. the fact that associativity and unitality are, rather than satisfied as equalities, codified by pre-equipped natural isomorphisms. In contrast, a *strict* monoidal category is one for which α, λ, and ρ are all identities. Unfortunately, most monoidal categories found in nature are weak. Fortunately, the following theorem states that any monoidal category can be losslessly identified with a strict monoidal category called its *strictification*.

Theorem 3.2.4. *Every monoidal category is equivalent to a strict monoidal category.*

This theorem justifies working with strictified versions of every monoidal category we subsequently encounter. We in fact outright elide this issue and never again refer to associators and unitors. The fact that strict monoidal categories categorify monoids can be codified by the statement that, just as discrete categories are sets, discrete monoidal categories are monoids.

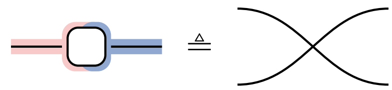
3.2.2 Braids

In the prior section, we discussed the weak commutativity of products and sums. Rather than strictifying this, we had to specify an isomorphism that instantiated it for each pair of sets. In the case of products, this was given by the swap $(x, y) \mapsto (y, x) \colon X \times Y \to Y \times X$. Using our new vocabulary, we can define the swap as a natural isomorphism. In particular, consider the categorified swap functor

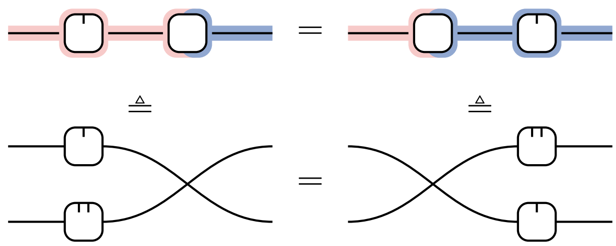
$$\mathtt{Swap} \colon \mathcal{C} \boxtimes \mathcal{D} \to \mathcal{D} \boxtimes \mathcal{C}$$

which specializes to $\mathtt{swap}$ on objects $\mathtt{ob}\,\mathcal{C} \times \mathtt{ob}\,\mathcal{D}$ and arrows $\mathtt{ar}\,\mathcal{C} \times \mathtt{ar}\,\mathcal{D}$. The swap map $\mathtt{swap}$ could then be defined as a natural isomorphism from $\mathtt{product} \colon \mathbf{Set} \boxtimes \mathbf{Set} \to$

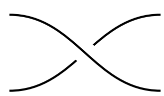
Set to Swap ⨾ product: **Set** ⊠ **Set** → **Set**, i.e. of type tensor → swap ⨾ product.



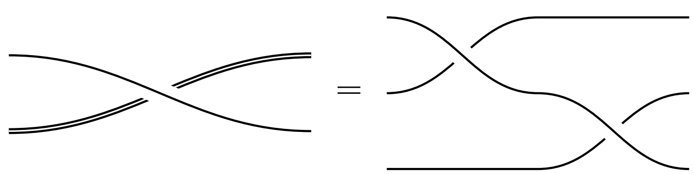
The naturality property can then be seen as an instance of the naturality condition.

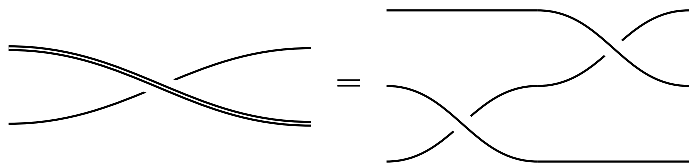


Returning to the context of a generic monoidal category $\mathcal{V}$, consider a natural isomorphism over of type tensor → Swap ⨾ tensor. We diagram this as a three-dimensionally embedded over-crossing:



We say that over is a *braiding* and $(\mathcal{V}, \otimes, I,$ over$)$ a *braided monoidal category* or just *braided category* when over respects $\otimes$ in the following two ways.

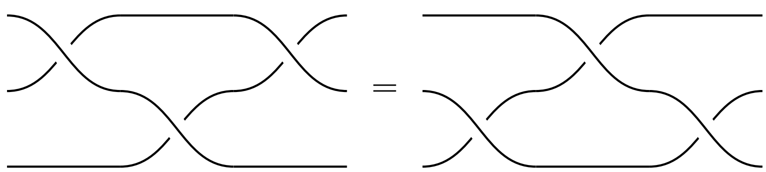




Along with the naturality condition, these imply the *braid relation*

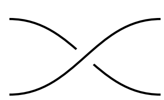
$$\texttt{over}_{1,2} \mathbin{⨾} \texttt{over}_{1,3} \mathbin{⨾} \texttt{over}_{2,3} = \texttt{over}_{2,3} \mathbin{⨾} \texttt{over}_{1,3} \mathbin{⨾} \texttt{over}_{1,2}$$

which is rendered graphically as follows.

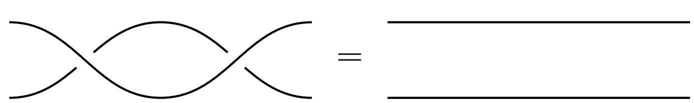


One can think of this relation as codifying the fact that if each of the strands sits in its own altitude—in the direction normal to the page—then, keeping each of their altitudes fixed, they can be arranged with no mutual interference.

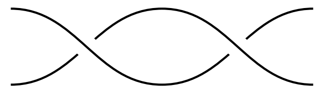
We depict the inverse transformation `under`: `tensor` → `Swap` ⨾ `tensor` as the opposite, i.e. under, crossing.



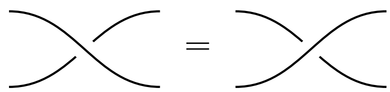
This gives invertibility `over` ⨾ `under` = `id` a graphical interpretation.



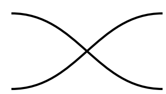
In contrast to swap maps, performing `over` twice in a row will just yield something even more twisted that cannot be simply pulled apart.



Many braidings enjoy the further *symmetry property* `over` = `under`, depicted as follows.



In this case, we write `swap` for `over` and draw the braiding without *crossing data*, as follows.



We call such braidings *symmetries* and their braided categories $(\mathcal{V}, \otimes, I, \texttt{swap})$ *symmetric monoidal categories*, sometimes simply denoting them simply by $(\mathcal{V}, \otimes, I)$. We note that most braided monoidal categories found "in nature" are symmetric. In the next chapter, we explore those which are not.

3.2.3 Strength

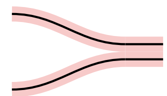
We typically want functors between monoidal categories to be equipped with features that respect the monoidal structure.

Definition 3.2.5. A *lax monoidal functor* $F\colon (\mathcal{V}, \otimes, I) \to (\mathcal{V}, \otimes', I')$ is

- a functor $F\colon \mathcal{V} \to \mathcal{V}'$

equipped with the following two *laxator* natural transformations.

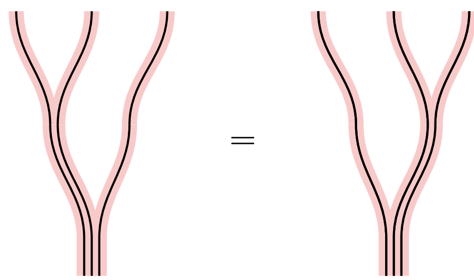
- `laxate`: $Fx \otimes' Fy \to F(x \otimes y)$ depicted as



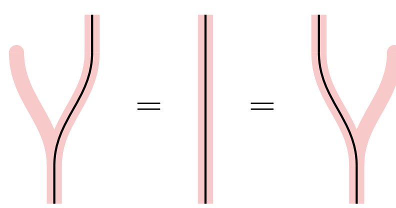
- `laxit`: $I' \to FI$ depicted as

These laxators satisfy the following conditions.

- *associativity* depicted as

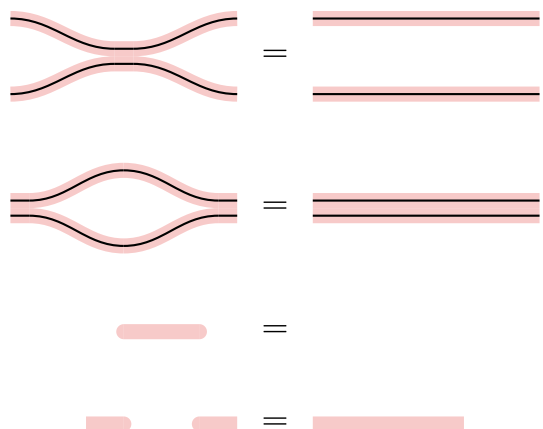


- *unitality* depicted as



An immediate and fascinating consequence of this definition is that a lax monoidal functor $\star \to (\mathcal{V}, \otimes, I)$, where $\star$ is the terminal category with the trivial monoidal structure, is precisely a monoid in $(\mathcal{V}, \otimes, I)$. This can be seen by simply erasing the black wires—since they now all represent the unit object—in the above associativity and unitality axioms.

An *oplax* monoidal functor is simply the dual of a lax monoidal functor. We call the duals of the laxators *colaxators* and depict them as their mirror image. A functor that is both lax and oplax is called *strong* when these are mutually inverse, i.e. when the laxator transformations are isomorphisms:

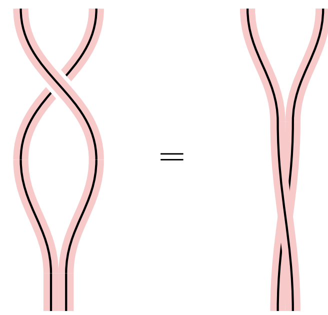


Furthermore, if our monoidal categories are braided, we may wish for a more structured functor notion.

Definition 3.2.6. A monoidal functor $F\colon (\mathcal{V}, \otimes, I, \texttt{swap}) \to (\mathcal{V}', \otimes', I', \texttt{swap}')$ is *braided monoidal* if `laxate` respects braidings, i.e. the following holds.

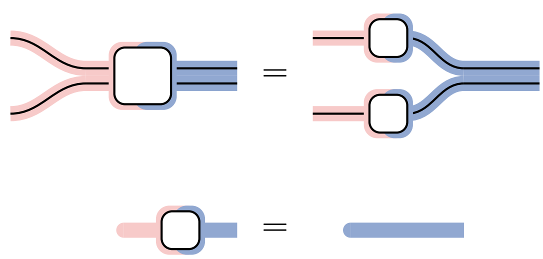
$$\texttt{swap}' \fatsemi \texttt{laxate} = \texttt{laxate} \fatsemi F(\texttt{swap})$$

We diagram this equation as follows.



Monoidal functors have a structure preserving notion of natural transformation.

Definition 3.2.7. A natural transformation $T\colon F \to F'$ is monoidal when the following two *uniformity* conditions are satisfied.



3.2.4 Monoids

Recall our definition of monoids. Its diagrammatic presentation assumed nothing about $(\mathbf{Set}, \times, \star)$ aside from the fact that it is a monoidal category. We can thus define a monoid in the exact same way in any monoidal category. Given this fact, there should be a way to abstract the essence of the monoid notion in a manner that does not refer to an ambient monoidal category. This leads us to the notion of *algebraic theories*, which allow context-independent formulations of algebraic structures.

Before providing a general definition, we investigate the example of monoids. In particular, we would like a monoidal category $\mathfrak{Mon}$ whose objects, arrows, and relations are the minimal structure needed for defining a monoid. First, we need an object, denote it $\underline{1}$, atop which to form a monoid structure. Next, since this is a monoidal category, we need to formally include all of its n-fold tensor powers, which we denote $\underline{n}$, where $\underline{0}$ acts as tensor unit. We depict $\underline{n}$ as a sequence of bands.

As for arrows, we must have the identity (twigs), units (bush), and all n-ary trees.

We define composition of maps as usual via stacking. Associativity and unitality guarantee that there is a unique arrow of type $\underline{n} \to \underline{1}$. We give the names `bush` and `tree` to the binary and nullary such arrows.

We define the tensor of two diagrams via placing the left tensorand to the left of the right tensorand. Generated arrows of type $\underline{n} \to \underline{m}$ are thus *gardens*, i.e. tensors of n_i-trees $n_i \to 1$, where $\sum_{i=1}^{m} n_i = n$. This completes our characterization of $\mathfrak{Mon}$, leading to our general definition of monoids.

Definition 3.2.8. A *monoid* $(M, \odot, e)$ in a monoidal category $(\mathcal{V}, \otimes, I)$ is a strong monoidal functor $M : \mathfrak{Mon} \to (\mathcal{V}, \otimes, I)$, where $\underline{1} \mapsto M$, `tree` $\mapsto \odot$, and `bush` $\mapsto e$. We call $\mathfrak{Mon}$ the *theory* of monoids, and M a *model* of monoids. We define $\mathfrak{Mon}(\mathcal{V})$ to be the category whose objects are models of monoids and arrows are monoidal natural transformations between them.

This definition is powerfully flexible. For instance, we can almost immediately define the famed monad concept.

Definition 3.2.9. A *monad* in a category $\mathcal{C}$ is a monoid in the category $\mathtt{End}\,\mathcal{C}$ of endofunctors on $\mathcal{C}$.

It turns out that $\mathfrak{Mon}$ is monoidally equivalent to a more familiar category—that of finite total orders and monotone maps between them.

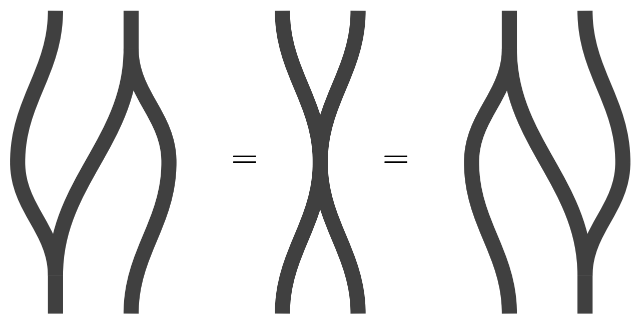
Proposition 3.2.10. *The monoidal category* $(\mathbf{FinTot}, \triangleright, \underline{0})$*—defined as having objects finite total orders* $\{(\underline{n}, \leqslant)\}$*, arrows monotone maps, and tensor as addition* $\underline{n} \triangleright \underline{m} = \underline{n+m}$ *with left-hand precedence—is monoidally equivalent to* $\mathfrak{Mon}$.

Proof. We give a sketch of proof. Any monotone map $\varphi\colon \underline{n} \to \underline{m}$ can be represented by the length-n increasing list $[\varphi(0), \varphi(1), \ldots, \varphi(n-1)]$ in $\underline{m}$. We make the following alterations to this list. First, whenever $\varphi(j+1) > \varphi(j) + 1$, we insert between $\varphi(j)$ and $\varphi(j+1)$ the sequence of unoccupied $\underline{m}$-values, but with an asterisk. We then partition this resultant list in terms of clusters of equal consecutive values, separating parts with the symbol $\triangleright$. For instance, the map given by $[0, 1, 1, 1, 3]$ yields the tensor product $[0] \triangleright [1, 1, 1] \triangleright [2^*] \triangleright [4]$. Tensorands are then exclusively of the forms $[a]$, $[b^*]$, and $[c, \ldots, c]$, which we respectively identify with $\mathtt{id}_{\underline{1}}$, $\mathtt{bush}$, and $\mathtt{tree}_n$. □

In general a *monoidal theory*, or sometimes *PRO* (short for "product"), $\mathfrak{M}$ is a monoidal category with underlying monoid isomorphic to $(\mathbb{N}, +, 0)$, where strong monoidal functors out of this category correspond to models of the theory. The arrows $n \to m$ of the theory correspond precisely to all of the possible n-coary m-ary operations combinable from the generating ones. As another example, duality gives us for free that the theory for comonoids is $\mathfrak{Mon}^{\mathrm{op}}$.

Perhaps more interestingly, we may ask what could happen if some object is simultaneously equipped with both monoid and comonoid? There are two canonical ways in which these two structures can interact. We consider here the more diagrammatic of the two.

Definition 3.2.11. An object X in a monoidal category $(\mathcal{V}, \otimes, I)$ has the structure of a *Frobenius monoid* if it is both a monoid (X, μ, η) and a comonoid (X, δ, ϵ), and furthermore the two interact via the following *Frobenius law.*



The Frobenius law makes a Frobenius monoid an inherently topological object in a way made precise by the following theorem.

Theorem 3.2.12. *(Cite Lauda) The theory* $\mathfrak{Frob}$ *for Frobenius monoids can be described as follows.*

- *the object n is a sequence of n disjoint colinear bands.*
- *an arrow $n \to m$ is a planar cobordism between the dpomain and codomain bands.*

We need more structure than a monoidal category, however, to define a commutative monoid. In particular, we need to be in a symmetric monoidal category in order to define the commutativity relation $\texttt{swap} \mathbin{⨟} \odot = \odot$. The theory for commutative monoids must therefore be a symmetric monoidal category. We thus call it a *symmetric monoidal theory*, or sometimes *PROP* (short for "product and permutation"). In particular, we define $\mathfrak{CMon}$ analogously to $\mathfrak{Mon}$, except we replace bands with tubes and planar-embedded cobordisms with non-embedded two-dimensional cobordisms, thus allowing for permutations. Note that we could have defined an intermediate notion of braided monoids, for which these cobordisms would be spatially embedded. It turns out that $\mathfrak{CMon}$ is equivalent to a very familair category.

Proposition 3.2.13. *The monoidal category* $(\mathbf{Fin}, +, \varnothing)$ *of finite sets, maps, and sums is monoidally equivalent to* $\mathfrak{CMon}$*.*

Proof. The idea is similar to the case of monoids. Each finite map $\varphi\colon n \to m$ can be partitioned (without a notion of order) as the sum of fibers $(\varphi^*(k))_{k:\, m}$, each of which is tube-thickened un-embedded tree. □

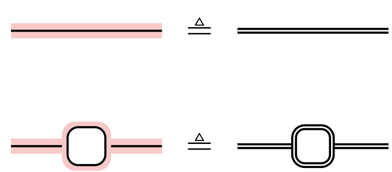
As before, $\mathfrak{CMon}^{\mathrm{op}}$ comes for free as the theory for cocommutative comonoids. The bicommutative notion of Frobenius monoid has a similar, and in fact more famous, characterization.

Theorem 3.2.14. *The theory* $\mathfrak{CFrob}$ *is given by the category* $\mathbf{Cob}_2$*, with objects disjoint unions of closed and compact* 1*-manifolds, i.e. circles, and arrows* 2*-dimensional cobordisms between these.*

3.2.5 Scarcity

Recall that **Set** enjoyed copy and discard maps. The situation is similar in **Cat**. Using capital letters to disambiguate this categorified setting, we can define, for any category $\mathcal{C}$, the functor $\mathtt{Copy}\colon \mathcal{C} \to \mathcal{C} \boxtimes \mathcal{C}$ given by $\mathtt{copy}$ on both $\mathtt{ob}\,\mathcal{C}$ and $\mathtt{ar}\,\mathcal{C}$. The terminal category $\star$, defined as having singleton object with only identity arrow, comes equipped, for any category $\mathcal{C}$, with a unique functor $\mathtt{Discard}\colon \mathcal{C} \to \star$ given by $\mathtt{discard}$ on both $\mathtt{ob}\,\mathcal{C}$ and $\mathtt{ar}\,\mathcal{C}$, i.e. by sending each $\mathcal{C}$-object to the unique $\star$-object and all $\mathcal{C}$-arrows to this object's identity arrow. The fact that the various properties—naturality and uniformity for each map, along with coassociativity and counitality—hold lifts immediately from the **Set** context. Just as in the case of $\times$, we strictify $\boxtimes$ as well.

In turn, we can use this categorified copy-discard structure to descend downwards and define what it would mean for *any* symmetric monoidal category $(\mathcal{V}, \otimes, I)$ to have copy and discard, along with their duals fold and create. In particular, we define a monoidal category $(\mathtt{tuples}\,\mathcal{V}, \otimes, \mathfrak{o})$ that has as its object set the *n-tupling* endofunctors $\mathfrak{n}\colon \mathcal{V} \to \mathcal{V}$, which act on both $\mathtt{ob}\,\mathcal{V}$ and $\mathtt{ar}\,\mathcal{V}$ via $v \mapsto v^{\otimes n}$. In the $n = 2$ and $n = 0$ cases, we respectively call tupling *doubling* and *nulling*. In accord, we depict $\mathfrak{n}$ by n-tupling wires; for instance, doubling is depicted as follows.



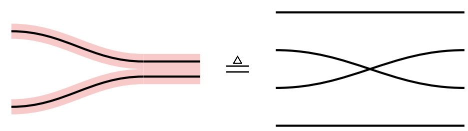
The tupling functors are in fact monoidal. Since, by strictness, we have $\mathfrak{n}(I) = I$, we can set $\mathtt{laxit}$ to $\mathtt{id}_I$ and need only to define $\mathtt{laxate}$, which is of type

$$\mathfrak{n}(x) \otimes \mathfrak{n}(y) \to \mathfrak{n}(x \otimes y).$$

We can define this map because $\mathcal{V}$ is symmetric, which facilitates the permutation

$$x \otimes \cdots \otimes x \otimes y \otimes \cdots \otimes y \to (x \otimes y) \otimes \cdots \otimes (x \otimes y)$$

For example, the laxator in the case of doubling is given by the following permutation.

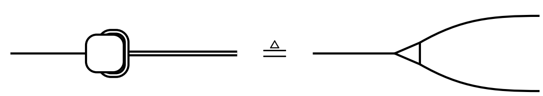


The tensor product is given by addition, i.e. acts by tensoring outputs $[\mathfrak{n} \otimes \mathfrak{m}]v = v^{\otimes(n+m)}$, for both objects and arrows. The nulling functor, which sends any object to I and arrow to $\mathtt{id}_I$, thus acts as monoidal unit, and is depicted by erasing whatever it highlights. More formally, $\mathfrak{n}$ is the composite functor

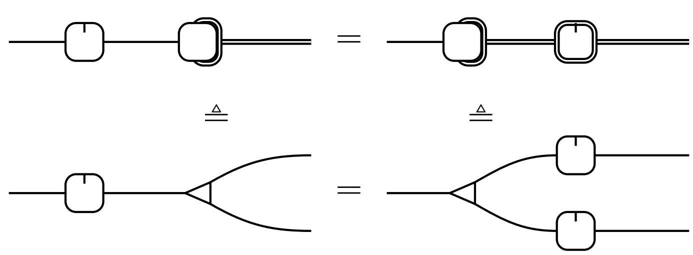
$$\mathcal{V} \xrightarrow{n\,\mathtt{Copy}} \mathcal{V}^{\boxtimes n} \xrightarrow{n\,\mathtt{tensor}} \mathcal{V}$$

We then define the arrows in $\mathtt{tuples}\,\mathcal{V}$ as monoidal natural transformations.

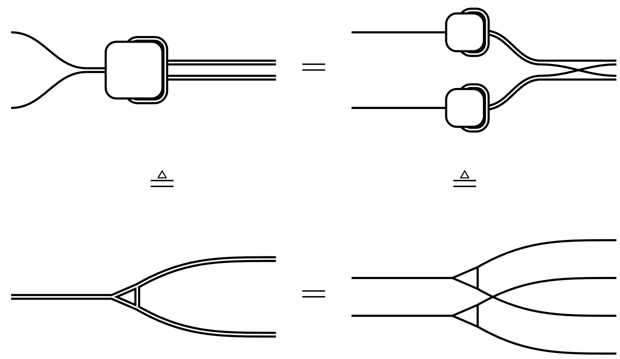
We say $\mathcal{V}$ is *cartesian monoidal* or simply *cartesian* when there is a comonoid in $\mathtt{tuples}\,\mathcal{V}$. We name, as in the case of **Set**, the comultiplication $\mathtt{copy}\colon \mathtt{1} \to \mathtt{2}$, and depict it as such.



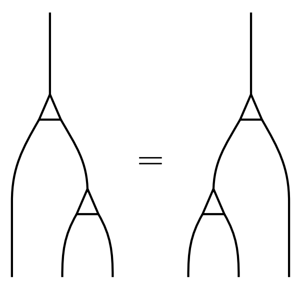
The naturality property instantiates the fact that copy is natural:



The uniformity property instantiates the uniformity of `copy`.

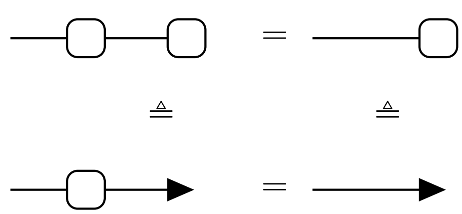


Finally, coassociativity is as before.

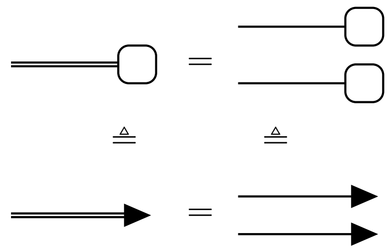


Similarly, we name the counit `discard`: $\mathtt{1} \to \mathtt{0}$ and depict it as follows.

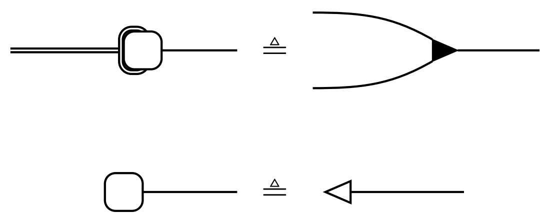
The naturality property instantiates the fact that `discard` is natural.



The uniformity property instantiates the uniformity of `discard`.



Dually, we say that $\mathcal{V}$ is *cocartesian* when there is a monoid in `tuples` $\mathcal{V}$. We name, as in the case of $(\mathbf{Set}, +, \varnothing)$, the multiplication `fold`: $\mathtt{2} \to \mathtt{1}$ and unit `create`: $\mathtt{0} \to \mathtt{1}$ and depict it as in that setting.



Such structures amount to an alternative characterization of the universal binary operators we introduced in the prior section.

Theorem 3.2.15. *A symmetric monoidal category* $(\mathcal{V}, \otimes, I)$ *is (co)cartesian if and only if* $\otimes$ *is the categorical (co)product in* $\mathcal{V}$.

Proof. By duality, we need only prove the cartesian case.

We have in fact already proven in Proposition 2.1.1, albeit with less general language, the rightward implication: that a cartesian category has its tensor as the categorical product. This was done by constructing—solely in terms of the cartesian structure of **Set**—the schematics `fork` and `unfork`, which instantiated the product-defining natural isomorphism

$$\texttt{hom}(v \times v', -) \cong \texttt{hom}(v, -) \times \texttt{hom}(v', -).$$

To prove the leftward implication, i.e. that $(\mathcal{V}, \times, \star)$ is cartesian, we must define $\texttt{copy}\colon \texttt{hom}(v, v \times v)$. By universality, this is equivalent to choosing an element of $\texttt{hom}(v, v) \times \texttt{hom}(v, v)$. There is only one generically available choice: $(\texttt{id}_v, \texttt{id}_v)$. It is straightforward to check that this map possesses all of the necessary properties. Defining `discard` is even easier: it comes for free by the terminality of $\star$. □

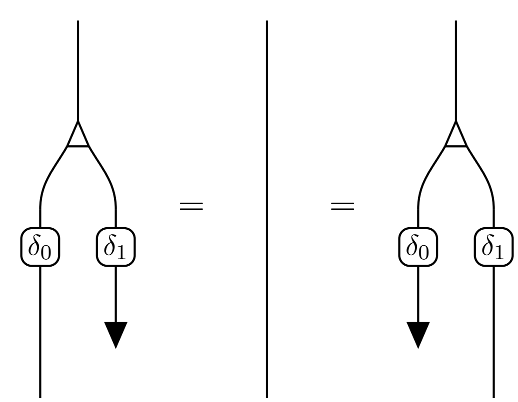
This theorem has an interesting corollary.

Corollary 3.2.16. *Each object in a cartesian (cocartesian) category* $(\mathcal{V}, \times, \star)$ *is a comonoid (monoid) in a unique way.*

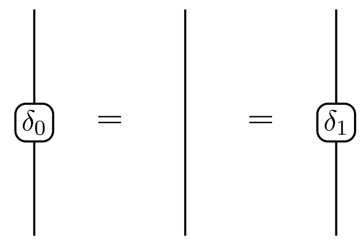
Proof. Suppose (X, δ, ϵ) is a comonoid in $(\mathcal{V}, \times, \star)$. Terminality forces

$$\epsilon = \texttt{discard}\colon X \to \star.$$

By universality, we can rewrite $\delta = \texttt{fork}(\delta_0, \delta_1)$. The counitality condition becomes



Applying the counitality of $(X, \mathtt{copy}, \mathtt{discard})$ yields



forcing $\delta = \mathtt{fork}(\mathtt{id}_X, \mathtt{id}_X) = \mathtt{copy}$ and thus $(X, \delta, \epsilon) = (X, \mathtt{copy}, \mathtt{discard})$. $\square$

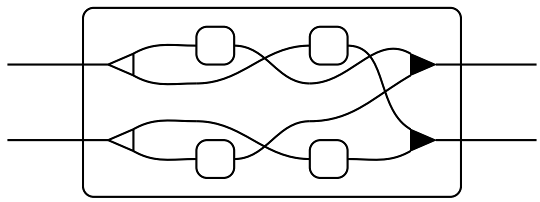
We say that $(\mathcal{V}, \otimes, I)$ is *bicartesian* when it is both cartesian and cocartesian. By the above theorem, this means that $\otimes$ is the direct sum $\oplus$ and I the zero object 0. The fact that both monoid and comonoid are natural means that the various operators interact as follows.

These categories are particularly well behaved; one could in fact conceive of much of linear algebra as the study bicartesian categories. This is by virtue of the fact that we have available both fork and join, thus instantiating the following isomorphism.

$$\mathtt{hom}(\bigoplus_{i=1}^{n} V_i, \bigoplus_{j=1}^{m} W_j) \cong \begin{bmatrix} \mathtt{hom}(V_1, W_1) & \cdots & \mathtt{hom}(V_n, W_1) \\ \vdots & \ddots & \vdots \\ \mathtt{hom}(V_1, W_m) & \cdots & \mathtt{hom}(V_n, W_m) \end{bmatrix}$$

where we suggestively use matrix notation to denote a cartesian product of factors arranged in a 2-dimensional grid, rather than a 1-dimensional line. Note that, when writing matrices, we will follow classical notation in arranging domain summands as columns and codomain summands as rows.

The backward map in this isomorphism, which we henceforth call $\mathtt{matrix}_{m\times n}$ is the composite $\mathtt{fork}_m \mathbin{⨾} \mathtt{join}_n = \mathtt{join}_n \mathbin{⨾} \mathtt{fork}_m$. We call the forward map $\mathtt{unmatrix}_{m\times n}$ and define it as the composite of $\mathtt{unfork}_m$ and $\mathtt{unjoin}_n$. When $m = n = 2$, we drop the subscripts since $\texttt{matrix} = \texttt{fork} \fatsemi \texttt{join} = \texttt{join} \fatsemi \texttt{fork}$. This implies the following unambiguous schematic for `matrix`, in which, contrasting with matrix notation, we will arrange domain summands as rows and codomain summands as columns.



To spare the reader the headache of visually processing this wiring, we suggest to merely mentally register that columns share outputs and rows share inputs.

In $(\mathbf{Vect}_k, \oplus, \mathbf{0})$, $\texttt{copy}\colon V \to V \oplus V$ is given as in **Set** by the rule $v \mapsto (v, v)$. More interestingly, the map $\texttt{fold}\colon V \oplus V \to V$ is given by addition $(v, v') \mapsto v + v'$. The duality of these two operations can be seen directly in their classical block-matrix representations.

$$\texttt{copy} = \begin{bmatrix} I \\ I \end{bmatrix} : V \to V \oplus V$$

$$\texttt{fold} = \begin{bmatrix} I & I \end{bmatrix} : V \oplus V \to V$$

This allows us to codify in purely compositional terms the fact that linear maps can be represented as matrices of k-elements. More precisely, a choice of basis $\mathcal{B} = \{b_i\}_{i=1}^n$ is an isomorphism $V \cong k\mathcal{B}$. The forward arrow maps a vector v to a tuple of *components*, i.e. projections onto $b_1, \ldots, b_n$. Equivalently, this map is the `fork` of all of the linear projections $V \to kb_i$. The backward arrow maps a tuple $(c_1, \ldots, c_n)\colon k^n$ to the sum $c_1 v_1 + \cdots + c_n v_n$. Equivalently, this map is the `join` of all of the maps $c \mapsto c \cdot b_i \colon k \to V$.

Since the free vector space functor is a left adjoint, it preserves coproducts, implying that these two are also isomorphic to the direct sum $\bigoplus_{i=1}^{n} kb_i$. Since kv is isomorphic—via mapping $c \cdot v$ to c—to the *line* k for all non-zero vectors v, these are isomorphic to k^n. Combining this fact and the above yields the following sequence of identifications for converting a linear map into a matrix.

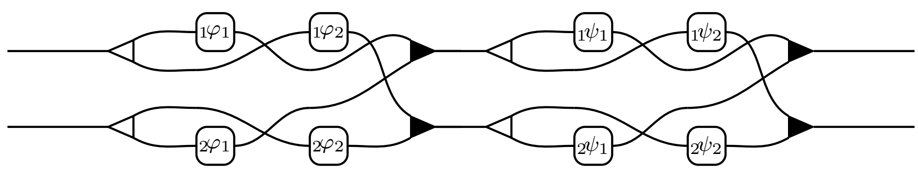
$$
\begin{aligned}
(L\colon V \to W) &\mapsto \left({}_{\mathcal{A}}[L]_{\mathcal{B}}\colon \bigoplus_{i=1}^{n} kb_i \to \bigoplus_{j=1}^{m} ka_j \right) \\
&\mapsto \begin{bmatrix} {}_{a_1}L_{b_1}\colon kb_1 \to ka_1 & \cdots & {}_{a_1}L_{b_n}\colon kb_n \to ka_1 \\ \vdots & \ddots & \vdots \\ {}_{a_m}L_{b_1}\colon kb_1 \to ka_m & \cdots & {}_{a_m}L_{b_n}\colon kb_n \to ka_m \end{bmatrix}
\end{aligned}
$$

In the highly specific context of classical linear algebra, we can use the free-forgetful adjunction to identify this matrix in $\mathbf{Vect}_k$ with the following matrix in $\mathbf{Set}$

$$
\begin{aligned}
(L\colon V \to W)_{\mathbf{Vect}_k} &\mapsto \begin{bmatrix} l_{11}\colon \star \to k & \cdots & l_{1n}\colon \star \to k \\ \vdots & \ddots & \vdots \\ l_{m1}\colon \star \to k & \cdots & l_{mn}\colon \star \to k \end{bmatrix}_{\mathbf{Set}} \\
&\mapsto \begin{bmatrix} \hat{l}_{11}\colon k & \cdots & \hat{l}_{1n}\colon k \\ \vdots & \ddots & \vdots \\ \hat{l}_{m1}\colon k & \cdots & \hat{l}_{mn}\colon k \end{bmatrix}_{\mathbf{Set}} \\
&\mapsto \hat{l}\colon \begin{bmatrix} k & \cdots & k \\ \vdots & \ddots & \vdots \\ k & \cdots & k \end{bmatrix}_{\mathbf{Set}}
\end{aligned}
$$

This leaves us with an element in a cartesian product taken over a two-dimensional grid, i.e. a two-dimensional array of numbers, i.e. a matrix.

Not only do we recover matrices, but we also recover matrix multiplication. For visual clarity, we perform this identification in the special case of multiplying two 2×2 matrices. The general case follows from the same principles. A matrix is simply the result of applying `matrix` to a two-dimensional array of arrows ${}_i\varphi_j \colon X_i \to X_j$ in some bicartesian category $(\mathcal{V}, \oplus, \mathbf{0})$. Since matrix multiplication is function composition, we have the following string diagram corresponding to the product of two matrices.



3.2.6 Bimonoids

Recall that the definition of group required use of the `copy` and `discard` maps. We hence require that $(\mathcal{V}, \otimes, I)$ be cartesian in order to support a group structure on an object. We name algebraic theories definable in cartesian categories *Lawvere Theories* after William Lawvere, whose doctoral thesis introduced the notion of an algebraic theory.

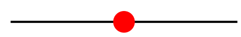
An alternative to building atop a comonoid native to the host category is to bring one's own comonoid. This is precisely the structure of a bimonoid.

By virtue of Corollary 3.2.16, a bimonoid in a cartesian category $(\mathcal{V}, \times, \star)$ is simply a monoid. We can use this principle to provide a more symmetric notion of a group that collapses to the classical notion in the setting of a cartesian category.

Definition 3.2.17. A *Hopf monoid* consists of

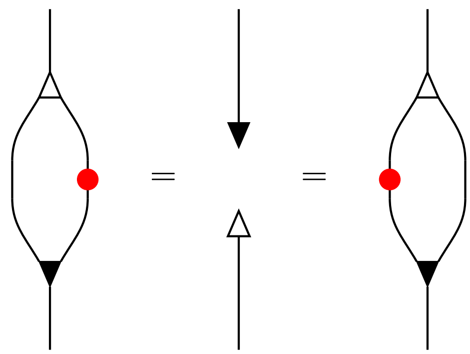
- a bimonoid $(X, \mu, \eta, \delta, \epsilon)$

- an *antipode* $\sigma\colon X \to X$, depicted as



This data must satisfy

- the *antipodality* condition



Since the comonoid collapses to the one instantiating cartesianness, a Hopf monoid in a cartesian category $(\mathcal{V}, \times, \star)$ collapses to a classical group.

3.2.7 Closure

In the context of **Set**, we were able to use currying to convert between multivariate maps and univariate higher-order maps via the adjunction $- \times X \dashv [X \to -]$. We extend this notion to the generic context of monoidal categories.

Definition 3.2.18. A monoidal category $(\mathcal{V}, \otimes, I)$ is *right closed* if for each $v\colon \texttt{ob}\,\mathcal{V}$, the endofunctor $- \otimes v\colon \mathcal{V} \to \mathcal{V}$ has a right adjoint $[v \to -]\colon \mathcal{V} \to \mathcal{V}$. It is *left closed* if the endofunctor $v \otimes -\colon \mathcal{V} \to \mathcal{V}$ has a right adjoint $[v \leftarrow -]\colon \mathcal{V} \to \mathcal{V}$. We say $\mathcal{V}$ is *biclosed* if it is both left and right closed.

In most contexts we will be interested in braided if not symmetric monoidal categories. In this setting, since there is an isomorphism $x \otimes y \to y \otimes x$, we have that a braided category $\mathcal{V}$ is left closed if and only if it is right closed. In this context, we simply say that $\mathcal{V}$ is *closed* and use the notation $[v \to -]$ for the right closure adjoint. This adjoint extends to an *internal hom* bifunctor $[- \to -]\colon \mathcal{V}^{\mathsf{op}} \boxtimes \mathcal{V} \to \mathcal{V}$, named because it behaves like an enhancement of the hom-set to a hom-$\mathcal{V}$-object. In particular, the adjunction-instantiating natural isomorphism in **Set**

$$\mathtt{hom}(x \otimes y, z) \cong \mathtt{hom}(x, [y \to z])$$

lifts to a natural isomorphism in $\mathcal{V}$.

Proposition 3.2.19. *For closed $(\mathcal{V}, \otimes, I)$, there is a natural isomorphism*

$$[x \otimes y \to z] \cong [x \to [y \to z]]$$

Proof. Let $a\colon \mathtt{ob}\,\mathcal{V}$ be any object.

$$\begin{aligned} \mathtt{hom}(a, [x \otimes y \to z]) &\cong \mathtt{hom}(a \otimes x \otimes y, z) \\ &\cong \mathtt{hom}(a, [x \to [y \to z]]) \end{aligned}$$

The Yoneda Lemma implies the result. □

Closed categories enjoy a further adjunction.

Proposition 3.2.20. *Given a braided closed monoidal category $(\mathcal{V}, \otimes, I)$ with internal hom $[- \to -]\colon \mathcal{V}^{\mathsf{op}} \boxtimes \mathcal{V} \to \mathcal{V}$, we have for any $v\colon \mathtt{ob}\,\mathcal{V}$ that the functor $[- \to v]\colon \mathcal{V}^{\mathsf{op}} \to \mathcal{V}$ has as left adjoint its opposite $[- \to v]\colon \mathcal{V} \to \mathcal{V}^{\mathsf{op}}$.*

Proof. The adjunction is given by the following natural isomorphism

$$\mathtt{hom}_{\mathcal{V}^{\mathrm{op}}}([a \to v], b) \cong \mathtt{hom}_{\mathcal{V}}(a, [b \to v])$$

By definition, the left-hand side is equal to

$$\mathtt{hom}_{\mathcal{V}}(b, [a \to v]).$$

Applying closedness reduces the task to constructing an isomorphism

$$\mathtt{hom}_{\mathcal{V}}(b \otimes a, v) \cong \mathtt{hom}_{\mathcal{V}}(a \otimes b, v)$$

which is given by applying $\mathtt{hom}(-, v)$ to $\mathtt{over}$. □

Monoidal closed categories $(\mathcal{V}, \otimes, I)$ with sums are an interesting structure in their own right. For one, the adjointness implies that the tensor distributes over sums:

$$x \otimes (a + b) \cong x \otimes a + x \otimes b$$

This feature allows for the definition of free monoids.

Proposition 3.2.21. *Let $(\mathcal{V}, \otimes, I)$ be symmetric closed with countable sums. The forgetful functor $\mathfrak{Mon}(\mathcal{V}) \to \mathcal{V}$ has left adjoint $F \colon \mathcal{V} \to \mathfrak{Mon}(\mathcal{V})$ given by*

$$Fx \triangleq \sum_{n \colon \mathbb{N}} x^{\otimes n} = I + x + x \otimes x + \cdots + x \otimes \cdots \otimes x + \cdots$$

with unit map $\eta \colon I \to Fx$ defined as the canonical inclusion

$$x^{\otimes 0} \to \sum_{n \colon \mathbb{N}} x^{\otimes n}$$

and multiplication $\mu\colon Fx \otimes Fx \to Fx$ *defined as the composition*

$$\begin{aligned} Fx \otimes Fx &\triangleq \left(\sum_{j\,:\,\mathbb{N}} x^{\otimes j}\right) \otimes \left(\sum_{k\,:\,\mathbb{N}} x^{\otimes k}\right) \\ &\cong \sum_{(j,k)\,:\,\mathbb{N}^2} x^{\otimes (j+k)} \\ &\to \sum_{n\,:\,\mathbb{N}} x^{\otimes n} \\ &\triangleq Fx \end{aligned}$$

where the arrow is given by the `join` *of canonical inclusions.*

Proof. We exclude the proof that (Fx, μ, η) satisfies the monoid conditions as it is straightforward but tredious. We now prove the adjunction isomorphism

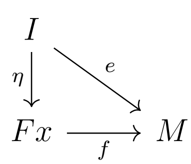
$$\mathtt{hom}_{\mathfrak{Mon}(\mathcal{V})}((Fx, \mu, \eta), (M, \odot, e)) \cong \mathtt{hom}_{\mathcal{V}}(x, M).$$

Recall that arrows on the left-hand side are natural transformations, i.e. $\mathcal{V}$-arrows $f\colon Fx \to M$ that commute with the monoid operations. By universality of sums

$$[f\colon Fx \to M] = \mathtt{join}(f_n\colon x^{\otimes n} \to M)_{n\,:\,\mathbb{N}}$$

We define the rightward map by sending $f\colon Fx \to M$ to $f_1\colon x \to M$.

We now define the leftward map by constructing an arrow $f\colon Fx \to M$ from an arrow $\tilde{f}\colon x \to M$, setting $f_1 = \tilde{f}$. The preservation of the unit



implies that $f_0 = e$. The preservation of multiplication

$$\begin{array}{ccc} Fx \otimes Fx & \xrightarrow{f \otimes f} & M \otimes M \\ {\scriptstyle \mu}\big\downarrow & & \big\downarrow{\scriptstyle \odot} \\ Fx & \xrightarrow[f]{\qquad} & M \end{array}$$

can be similarly analyzed by universality via the identification

$$[\varphi \colon Fx \otimes Fx \to M] = \mathtt{join}(\varphi_{j,k} \colon x^{\otimes(j+k)} \to M)_{(j,k) \colon \mathbb{N}^2}$$

Since μ is the join of canonical inclusions, $\mu \fatsemi f$ is given by

$$\mathtt{join}(f_{j+k} \colon x^{\otimes(j+k)} \to M)_{(j,k) \colon \mathbb{N}^2}.$$

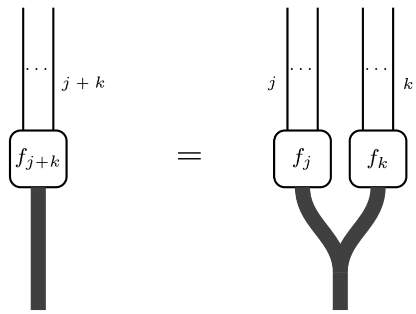
In turn, since $(f \otimes f) = \mathtt{join}(f_j \otimes f_k)_{(j,k) \colon \mathbb{N}^2}$, we have that $(f \otimes f) \fatsemi \odot$ is

$$\mathtt{join}\left([f_j \otimes f_k] \fatsemi \odot \colon x^{\otimes[j+k]} \to M\right)_{(j,k) \colon \mathbb{N}^2}.$$

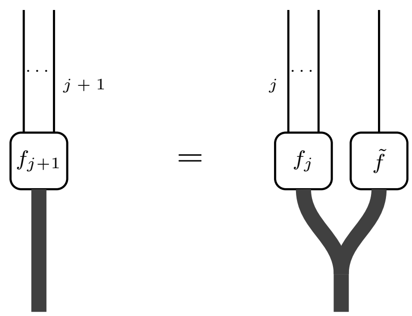
This means that for each pair $(j,k) \colon \mathbb{N}^2$, we must have the equality

$$f_{j+k} = (f_j \otimes f_k) \fatsemi \odot$$

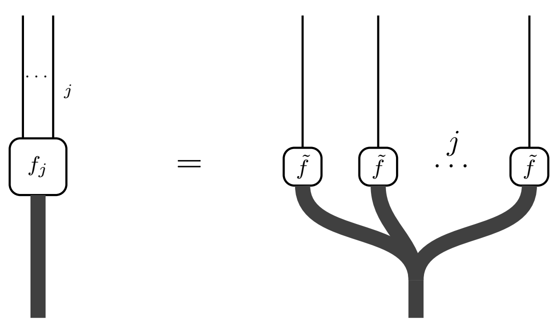
which, diagrammatically, is given by



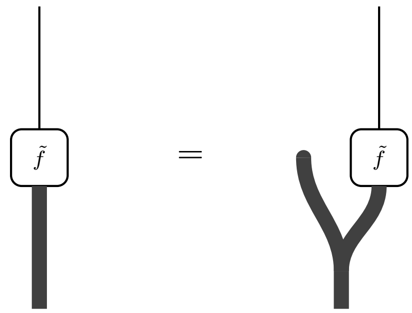
Setting $k = 1$ yields the recursion



Applying induction and associativity then determines $f_j\colon x^{\otimes j} \to M$ as



We thus get $f\colon Fx \to M$. Note that we could instead have set $j = 1$, which, despite defining the recursion in the opposite order, recurses to the same resultant f_j. We must now ascertain that $\tilde{f}$ plays nicely with $f_0 = e$. Setting $(j, k) = (0, 1)$, we require



which holds automatically by the unitality of $(M, \odot, e)$. $\square$

Not only can we define free monoid objects in closed categories with sums, but we can also define commutative monoid objects.

Proposition 3.2.22. *Let $(\mathcal{V}, \otimes, I)$ be symmetric closed with countable sums. The forgetful functor $\mathfrak{CMon}(\mathcal{V}) \to \mathcal{V}$ has left adjoint $\texttt{exp}\colon \mathcal{V} \to \mathfrak{CMon}(\mathcal{V})$ given by*

$$\texttt{exp}\, x \triangleq \sum_{n\colon \mathbb{N}} \frac{x^{\otimes n}}{\mathbf{n!}} = I + x + \frac{1}{\mathbf{2!}} x^{\otimes 2} + \cdots + \frac{1}{\mathbf{n!}} x^{\otimes n} + \cdots$$

where dividing by $\mathbf{n}!$ is categorifier's notation for quotienting by the canonical S_n-action on n tensor powers in a symmetric monoidal category. As before, we define the unit $\eta\colon I \to \texttt{exp}\, x$ by the canonical inclusion. We define multiplication as the *following composition.*

$$\begin{aligned}
\mathtt{exp}\, x \otimes \mathtt{exp}\, x &\triangleq \left(\sum_{j\,:\,\mathbb{N}} \frac{x^{\otimes j}}{\mathbf{j}!}\right)\left(\sum_{k\,:\,\mathbb{N}} \frac{x^{\otimes k}}{\mathbf{k}!}\right) \\
&\cong \sum_{(j,k)\,:\,\mathbb{N}^2} \frac{x^{\otimes (j+k)}}{\mathbf{j}! \otimes \mathbf{k}!} \\
&\to \sum_{(j,k)\,:\,\mathbb{N}^2} \frac{x^{\otimes (j+k)}}{(\mathbf{j}+\mathbf{k})!} \\
&\to \sum_{n\,:\,\mathbb{N}^2} \frac{x^{\otimes n}}{\mathbf{n}!} \\
&\triangleq \mathtt{exp}\, x
\end{aligned}$$

where the first arrow is the canonical quotient.

The free commutative monoid functor may be thought of as a categorified exponential function. In particular, simply by virtue of the fact that `exp` is a left adjoint, we have that it sends sums in $\mathcal{V}$ to sums in $\mathfrak{CMon}(\mathcal{V})$, i.e. tensors!

$$\mathtt{exp}(x+y) \cong \mathtt{exp}(x) \otimes \mathtt{exp}(y).$$

Before continuing onward, we provide a brief dictionary of some of the varied notations for entities in structured categories.

cartesian closed categories

hom	$y^x\colon \texttt{ob}\,\mathcal{V}$
tensor	$\times$
unit	$\mathbf{1}$

closed categories

hom	$[x \to y]\colon \texttt{ob}\,\mathcal{V}$
tensor	$\otimes$
unit	I

cartesian categories

hom	$\texttt{hom}(x, y)\colon \texttt{ob}\,\mathbf{Set}$
tensor	$\times$
unit	$\star$

monoidal categories

hom	$\texttt{hom}(x, y)\colon \texttt{ob}\,\mathbf{Set}$
tensor	$\otimes$
unit	I

3.2.8 Duals

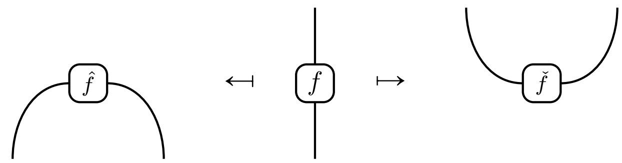
Suppose $(\mathcal{V}, \otimes, I)$ is a right closed category, except the internal hom, the left adjoint to $- \otimes x$, is naturally isomorphic to $x^* \otimes -$ for some other object x^*, i.e.

$$\texttt{hom}(a \otimes x, b) \cong \texttt{hom}(a, x^* \otimes b)$$

Applying this isomorphism to $\texttt{hom}(x, y)$ yields so-called *process-state duality*:

$$\texttt{hom}(I, x^* \otimes y) \cong \texttt{hom}(x, y) \cong \texttt{hom}(x \otimes y^*, I)$$

We visualize these identifications by bending wires:



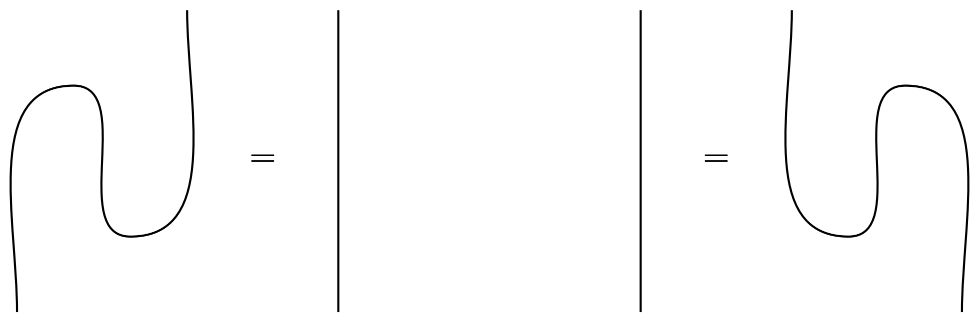
Applying the leftward identification to $\mathtt{id}_x$ yields $\mathtt{cap}_x \colon I \to x^* \otimes x$

Applying the rightward identification to $\mathtt{id}_{x*}$ yields $\mathtt{cup}_x \colon x \otimes x^* \to I$

We now define categories for which objects enjoy duals.

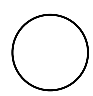
Definition 3.2.23. Let $(\mathcal{V}, \otimes, I)$ be a monoidal category. The pair $(l, r) \colon [\mathtt{ob}\,\mathcal{V}]^2$ is a *left-right dual pair* when there exist arrows $\mathtt{cap} \colon I \to r \otimes l$ and $\mathtt{cup} \colon l \otimes r \to I$

for which the following *zig-zag* equations are satisfied.

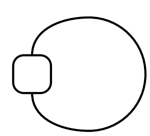


We say $\mathcal{V}$ is *right (left) autonomous* if each $x\colon \texttt{ob}\,\mathcal{V}$ has a right dual x^* (left dual $^*x$), and simply *autonomous* if it has both.

Curiously, by composing $\texttt{cap}_x$ and $\texttt{cup}_x$, an autonomous category $\mathcal{V}$ has for each $x\colon \texttt{ob}\,\mathcal{V}$ an arrow $\texttt{loop}_x\colon \texttt{end}(I)$, depicted as



More generally, these categories have the schematic $\texttt{trace}\colon \texttt{end}(-) \to \texttt{end}(I)$



where $\texttt{loop}_x = \texttt{trace}(\texttt{id}_x)$.

Chapter 4

Compositional Systems

The goal of this chapter is to define how to compose continuous-time dynamical systems in much the same way—using schematics—that we did for set maps in Section 2.1.

In Section 4.1, we define and characterize an appropriately composable notion of dynamical system, called an *open dynamical system*. In Section 4.2, we will review what we henceforth call the *organizational approach* to compositionality. In particular, an organization is given by defining a monoidal category of schematics, representing the type of allowable compositions, and providing it a semantics via a $(\mathbf{Set}, \times, \star)$-valued lax monoidal functor. Finally, in Section 4.3, we define a composition of open dynamical systems for each valid schematic.

4.1 Open Dynamical Systems Preliminaries

4.1.1 Motivation

More precisely, we will define a symmetric monoidal category $\mathbf{W}$ of black boxes and wiring diagrams. Its underlying operad $\mathcal{O}\mathbf{W}$ is a graphical language for building larger black boxes out of an interconnected set of smaller ones. We then define two $\mathbf{W}$-algebras, $\mathcal{G}$ and $\mathcal{L}$, which encode *open dynamical systems*, i.e., differential equations of the form

$$\begin{cases} \dot{Q} = f^-(Q, input) \\ output = f^+(Q) \end{cases} \tag{4.1}$$

where Q represents an internal state vector, $\dot{Q} = \frac{dQ}{dt}$ represents its time derivative, and *input* and *output* represent inputs to and outputs from the system. In $\mathcal{G}$, the functions f^- and f^+ are smooth, whereas in the subalgebra $\mathcal{L} \subseteq \mathcal{G}$, they are moreover linear. The fact that $\mathcal{G}$ and $\mathcal{L}$ are **W**-algebras captures the fact that these systems are closed under wiring diagram interconnection.

Our notion of interconnection is a generalization of that in Deville and Lerman [DL10], [DL15], [DL13]. Their version of interconnection produces a closed system from open ones, and can be understood in the present context as a morphism whose codomain is the closed box (see Definition 4.2.11). Graph fibrations between wiring diagrams form an important part of their formalism, though we do not discuss that aspect here.

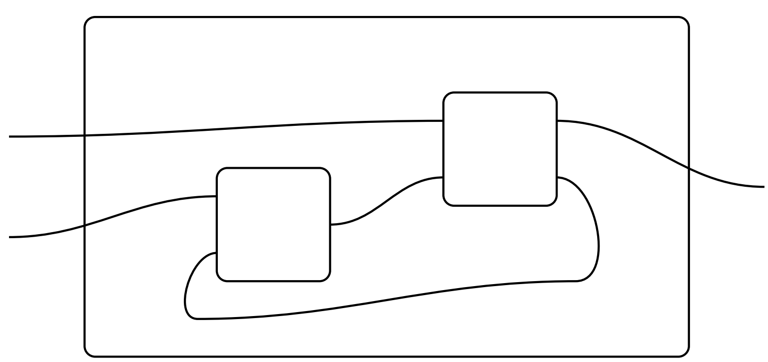
This chapter is the third work in a series, following [SR13] and [Spi13], on using wiring diagrams to model interactions. The algebra we present here, that of open systems, is distinct from the algebras of relations and of propagators studied in earlier works. Beyond the dichotomy of discrete vs. continuous, these algebras are markedly different in structure. For one thing, the internal wires in [SR13] themselves carry state, whereas here, a wire should be thought of as instantaneously transmitting its contents from an output site to an input site. Another difference between our algebra and those of previous works is that the algebras here involve *open systems* in which, as in (4.1), the instantaneous change of state is a function of the current state and the input, whereas the output depends only on the current state (see Definition 4.3.2).

The differences between these algebras is also reflected in a mild difference between the operad we use here and the one used in previous work.

4.1.2 Motivating example

The motivating example for the algebras in this chapter comes from classical differential equations pedagogy; namely, systems of tanks containing salt water concentrations, with pipes carrying fluid among them. The systems of ODEs produced by such applications constitute a subset of those our language can address; they are linear systems with a certain form (see Example 4.3.9). To ground the discussion, we consider a specific example.

Example 4.1.1. *Figure 4.1.1 below reimagines a problem from Boyce and DiPrima's differential equations text [BD65, Figure 7.1.6] as a dynamical system over a* wiring diagram.



In this diagram, the two boxes—hereby X_1 and X_2 from left to right—represent tanks consisting of salt water solution. The functions $Q_1(t)$ and $Q_2(t)$ represent the amount of salt (in ounces) found in 30 and 20 gallons of water, respectively. These tanks are interconnected with each other by pipes embedded within a total system Y. The prescription for how wires are attached among the boxes is formally encoded in *the wiring diagram $\Phi\colon X_1, X_2 \to Y$, as we will discuss in Definition 4.2.4.*

Both tanks are being fed salt water concentrations at constant rates from the outside world. Specifically, X_1 is fed a 1 ounce salt per gallon water solution at 1.5 gallons per minute and X_2 is fed a 3 ounce salt per gallon water solution at 1 gallon per minute. The tanks also both feed each other their solutions, with X_1 feeding X_2 at 3 gallons per minute and X_2 feeding X_1 at 1.5 gallons per minute. Finally, X_2 feeds the outside world its solution at 2.5 gallons per minute.

The dynamics of the salt water concentrations both within and leaving each tank X_i is encoded in a linear open system f_i, consisting of a differential equation for Q_i and a readout map for each X_i output (see Definition 4.1.10). Our algebra $\mathcal{L}$ allows one to assign a linear open system f_i to each tank X_i, and by functoriality the morphism $\Phi\colon X_1, X_2 \to Y$ produces a linear open system for the larger box Y. We will explore this construction in detail, in particular providing explicit formulas for it in the linear case, as well as for more general systems of ODEs.

4.1.3 Preliminary Notions

Throughout this chapter we use the language of monoidal categories and functors. Depending on the audience, appropriate background on basic category theory can be found in MacLane [ML98], Awodey [Awo10], or Spivak [Spi14]. Leinster [Lei04] is a good source for more specific information on monoidal categories and operads. We refer the reader to [KFA69] for an introduction to dynamical systems.

4.1.4 Monoidal categories and operads

In Section 4.2.3, we will construct the symmetric monoidal category $(\mathbf{W}, \oplus, 0)$ of boxes and wiring diagrams, which we often simply denote as $\mathbf{W}$. We will sometimes consider the underlying operad $\mathcal{O}\mathbf{W}$, obtained by applying the fully faithful functor

$$\mathcal{O}\colon \mathbf{SMC} \to \mathbf{Opd}$$

to $\mathbf{W}$. A brief description of this functor $\mathcal{O}$ is given below in Definition 4.1.2.

Definition 4.1.2. Let **SMC** denote the category of symmetric monoidal categories and lax monoidal functors; and **Opd** be the category of operads and operad functors. Given a symmetric monoidal category $(\mathcal{C}, \otimes, I_{\mathcal{C}})\colon \mathtt{ob}\,\mathbf{SMC}$, we define the operad $\mathcal{O}\mathcal{C}$ as follows:

$$\mathtt{ob}\,\mathcal{O}\mathcal{C} \triangleq \mathtt{ob}\,\mathcal{C}, \qquad \mathrm{Hom}_{\mathcal{O}\mathcal{C}}(X_1, \dots, X_n; Y) \triangleq \mathrm{Hom}_{\mathcal{C}}(X_1 \otimes \cdots \otimes X_n, Y)$$

for any $n\colon \mathbb{N}$ and objects $X_1, \dots, X_n, Y\colon \mathtt{ob}\,\mathcal{C}$.

Now suppose $F\colon (\mathcal{C}, \otimes, I_{\mathcal{C}}) \to (\mathcal{D}, \odot, I_{\mathcal{D}})$ is a lax monoidal functor in **SMC**. By definition such a functor is equipped with a morphism

$$\mu\colon FX_1 \odot \cdots \odot FX_n \to F(X_1 \otimes \cdots \otimes X_n),$$

natural in the X_i, called the *coherence map*. With this map in hand, we define the operad functor $\mathcal{O}F\colon \mathcal{O}\mathcal{C} \to \mathcal{O}\mathcal{D}$ by stating how it acts on objects X and morphisms $\Phi\colon X_1, \dots, X_n \to Y$ in $\mathcal{O}\mathcal{C}$:

$$\mathcal{O}F(X) \triangleq F(X), \quad \mathcal{O}F(\Phi\colon X_1, \dots, X_n \to Y) \triangleq F(\Phi) \circ \mu\colon FX_1 \odot \cdots \odot FX_n \to FY.$$

Example 4.1.3. *Consider the symmetric monoidal category* $(\mathbf{Set}, \times, \star)$*, where* $\times$ *is the cartesian product of sets and* $\star$ *a one element set. Define* $\mathbf{Sets} \triangleq \mathcal{O}\mathbf{Set}$ *as in Definition 4.1.2. Explicitly,* **Sets** *is the operad in which an object is a set and a* *morphism* $f\colon X_1, \ldots, X_n \to Y$ *is a function* $f\colon X_1 \times \cdots \times X_n \to Y$.

Definition 4.1.4. Let $\mathcal{C}$ be a symmetric monoidal category and let $\mathbf{Set} = (\mathbf{Set}, \times, \star)$ be as in Example 4.1.3. A $\mathcal{C}$*-algebra* is a lax monoidal functor $\mathcal{C} \to \mathbf{Set}$. Similarly, if $\mathcal{D}$ is an operad, a $\mathcal{D}$*-algebra* is defined as an operad functor $\mathcal{D} \to \mathbf{Sets}$.

To avoid subscripts, we will generally use the formalism of SMCs in this chapter. Definitions 4.1.2 and 4.1.4 can be applied throughout to recast everything we do in terms of operads. The primary reason operads may be preferable in applications is that they suggest more compelling pictures. Hence throughout this chapter, depictions of wiring diagrams will often be operadic, i.e., have many input boxes wired together into one output box.

4.1.5 Typed sets

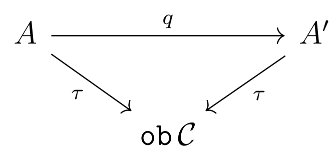
Each box in a wiring diagram will consist of finite sets of ports, each labelled by a type. To capture this idea precisely, we define the notion of typed finite sets. By a *finite product* category, we mean a category that is closed under taking finite products.

Definition 4.1.5. Let $\mathcal{C}$ be a small finite product category. The category of $\mathcal{C}$*-typed finite sets*, denoted $\mathbf{TFS}_{\mathcal{C}}$, is defined as follows. An object in $\mathbf{TFS}_{\mathcal{C}}$ is a map from a finite set to the objects of $\mathcal{C}$:

$$\mathtt{ob}\,\mathbf{TFS}_{\mathcal{C}} \triangleq \{(A, \tau) \mid A\colon \mathtt{ob}\,\mathbf{Fin}, \tau\colon A \to \mathtt{ob}\,\mathcal{C})\}.$$

Intuitively, one can think of a typed finite set as a finite unordered list of $\mathcal{C}$-objects. For any element $a\colon A$, we call the object $\tau(a)$ its *type*. If the typing function τ is clear from context, we may denote (A, τ) simply by A.

A morphism $q\colon (A,\tau) \to (A',\tau')$ in $\mathbf{TFS}_{\mathcal{C}}$ consists of a function $q\colon A \to A'$ that makes the following diagram of finite sets commute:



Note that $\mathbf{TFS}_{\mathcal{C}}$ is a cocartesian monoidal category.

We refer to the morphisms of $\mathbf{TFS}_{\mathcal{C}}$ as *$\mathcal{C}$-typed functions.* If a $\mathcal{C}$-typed function q is bijective, we call it a *$\mathcal{C}$-typed bijection.*

In other words, $\mathbf{TFS}_{\mathcal{C}}$ is the comma category for the diagram

$$\mathbf{Fin} \xrightarrow{i} \mathbf{Set} \xleftarrow{\mathtt{ob}\,\mathcal{C}} \{*\}$$

where i is the inclusion.

Definition 4.1.6. Let $\mathcal{C}$ be a finite product category, and let $(A,\tau)\colon \mathtt{ob}\,\mathbf{TFS}_{\mathcal{C}}$ be a $\mathcal{C}$-typed finite set. Its *dependent product* $\overline{(A,\tau)}\colon \mathtt{ob}\,\mathcal{C}$ is defined as

$$\overline{(A,\tau)} \triangleq \prod_{a\colon A} \tau(a).$$

Coordinate projections and diagonals are generalized as follows. Given a typed function $q\colon (A,\tau) \to (A',\tau')$ in $\mathbf{TFS}_{\mathcal{C}}$ we define

$$\overline{q}\colon \overline{(A',\tau')} \to \overline{(A,\tau)}$$

to be the unique morphism for which the following diagram commutes for all $a\colon A$:

$$\begin{array}{ccc} \prod_{a'\colon A'} \tau'(a') & \xrightarrow{\overline{q}} & \prod_{a\colon A} \tau(a) \\ \downarrow{\scriptstyle \pi_{q(a)}} & & \downarrow{\scriptstyle \pi_a} \\ \tau'(q(a)) & =\!=\!=\!= & \tau(a) \end{array}$$

By the universal property for products, this defines a functor,

$$\overline{\cdot}\colon \mathbf{TFS}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{C}.$$

Lemma 4.1.7. *The dependent product functor* $\mathbf{TFS}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{C}$ *is strong monoidal. In particular, for any finite set* I *whose elements index typed finite sets* (A_i, τ_i)*, there is a canonical isomorphism in* $\mathcal{C}$*,*

$$\overline{\coprod_{i\colon I}(A_i, \tau_i)} \cong \prod_{i\colon I} \overline{(A_i, \tau_i)}.$$

Remark 4.1.8. The category of second-countable smooth manifolds and smooth maps is essentially small (by the embedding theorem) so we choose a small representative and denote it **Man**. Note that **Man** is a finite product category. Manifolds will be our default typing, in the sense that we generally take $\mathcal{C} \triangleq \mathbf{Man}$ in Definition 4.1.5 and denote

$$\mathbf{TFS} \triangleq \mathbf{TFS}_{\mathbf{Man}}. \tag{4.2}$$

We thus refer to the objects, morphisms, and isomorphisms in **TFS** simply as *typed finite sets*, *typed functions*, and *typed bijections*, respectively.

Remark 4.1.9. The ports of each box in a wiring diagram will be labeled by manifolds because they are the natural setting for geometrically interpreting differential equations (see [Spi65]). For simplicity, one may wish to restrict attention to the full subcategory **Euc** of Euclidean spaces $\mathbb{R}^n$ for $n\colon \mathbb{N}$, because they are the usual domains for ODEs found in the literature; or to the (non-full) subcategory **Lin** of Euclidean spaces and linear maps between them, because they characterize linear systems of ODEs. We will return to $\mathbf{TFS}_{\mathbf{Lin}}$ in Section 4.3.2.

4.1.6 Open systems

As a final preliminary, we define our notion of open dynamical system. Recall that every manifold M has a *tangent bundle* manifold, denoted TM, and a smooth projection map $p\colon TM \to M$. For any point $m\colon M$, the preimage $T_mM \triangleq p^{-1}(m)$ has the structure of a vector space, called the *tangent space of M at m*. If $M \cong \mathbb{R}^n$ is a Euclidean space then also $T_mM \cong \mathbb{R}^n$ for every point $m\colon M$. A *vector field on M* is a smooth map $g\colon M \to TM$ such that $p \circ g = \mathtt{id}_M$. See [Spi65] or [War83] for more background.

For the purposes of this chapter we make the following definition of open systems; this may not be completely standard.

Definition 4.1.10. Let $M, U^-, U^+\colon \mathtt{ob}\,\mathbf{Man}$ be smooth manifolds and TM be the tangent bundle of M. Let $f = (f^-, f^+)$ denote a pair of smooth maps

$$\begin{cases} f^-\colon M \times U^- \to TM \\ f^+\colon M \to U^+ \end{cases}$$

where, for all $(m, u)\colon M \times U^-$ we have $f^-(m, u)\colon T_mM$; that is, the following diagram commutes:

$$\begin{array}{ccc} M \times U^- & \xrightarrow{\quad f^- \quad} & TM \\ & {\scriptstyle \pi_M} \searrow \quad \swarrow {\scriptstyle p} & \\ & M & \end{array}$$

We sometimes use f to denote the whole tuple,

$$f = (M, U^-, U^+, f),$$

which we refer to as an *open dynamical system* (or *open system* for short). We call M the *state space*, U^- the *input space*, U^+ the *output space*, f^- the *differential equation*, and f^+ the *readout map* of the open system.

Note that the pair $f = (f^-, f^+)$ is determined by a single smooth map

$$f \colon M \times U^- \to TM \times U^+,$$

which, by a minor abuse of notation, we also denote by f.

In the special case that $M, U^{\text{in}}, U^{\text{out}} \colon \texttt{ob}\,\mathbf{Lin}$ are Euclidean spaces and f is a linear map (or equivalently f^- and f^+ are linear), we call f a *linear open system*.

Remark 4.1.11. Let M be a smooth manifold, and let $U^- = U^+ = \mathbb{R}^0$ be trivial. Then an open system in the sense of Definition 4.1.10 is a smooth map $f \colon M \to TM$ over M, in other words, a vector field on M. From the geometric point of view, vector fields are autonomous (i.e., closed!) dynamical systems; see [Tes12].

Remark 4.1.12. For an arbitrary manifold U^-, a map $M \times U^- \to TM$ can be considered as a function $U^- \to \mathbf{VF}(M)$, where $\mathbf{VF}(M)$ is the set of vector fields on M. Hence, U^- *controls* the behavior of the system in the usual sense.

Remark 4.1.13. Given an open system f we can form a new open system by feeding the readout of f into the inputs of f. For example suppose the open system is of the form

$$\begin{cases} M \times A \times B \xrightarrow{F} TM \\ g = (g_A, g_B) \colon M \to C \times B, \end{cases}$$

where A, B, C and M are manifolds. Define $F' \colon M \times A \to TM$ by

$$F'(m, a) \triangleq F(m, a, g_B(m)) \qquad \text{for all} \quad (m, a) \colon M \times A.$$

Then

$$\begin{cases} M \times A \xrightarrow{F'} TM \\ g_A \colon M \to C \end{cases}$$

is a new open system obtained by plugging a readout of f into the space of inputs B.

This becomes more interesting when we start with several open systems, take their product and then plug (some of the) outputs into inputs. For example suppose we start with two open systems

$$\begin{cases} M_1 \times A \times B \xrightarrow{F_1} TM_1 \\ g_1 \colon M_1 \to C \end{cases}$$

and

$$\begin{cases} M_2 \times C \xrightarrow{F_2} TM_2 \\ g_2 = (g_B, g_D) \colon M_2 \to B \times D \end{cases} .$$

Their product is given by

$$\begin{cases} M_1 \times A \times B \times M_2 \times C \xrightarrow{(F_1, F_2)} TM_1 \times TM_2 \\ (g_1, g_2) \colon M_1 \times M_2 \to C \times B \times D \end{cases}$$

Now plug in the functions g_B and g_1 into inputs. We get a new system

$$\begin{cases} M_1 \times M_2 \times A \xrightarrow{F'} TM_1 \times TM_2 \\ g' \colon M_1 \times M_2 \to D \end{cases}$$

where

$$F'(m_1, m_2, a) \triangleq (F_1(m_1, a, g_B(m_2)), F_2(m_2, g_1(m_1)).$$

Compare with Figure 4.2.15.

Making these kinds of operations on open systems precise for an arbitrary number of interacting systems is the point of our work.

By defining the appropriate morphisms, we can consider open dynamical systems as being objects in a category. We are not aware of this notion being defined previously in the literature, but it is convenient for our purposes.

Definition 4.1.14. Suppose that $M_i, U_i^-, U_i^+ \colon \texttt{ob}\,\mathbf{Man}$ and (M_i, U_i^-, U_i^+, f_i) is an open system for $i \colon \{1, 2\}$. A *morphism of open systems*

$$\zeta \colon (M_1, U_1^-, U_1^+, f_1) \to (M_2, U_2^-, U_2^+, f_2)$$

is a triple $(\zeta_M, \zeta_{U^-}, \zeta_{U^+})$ of smooth maps $\zeta_X \colon X_1 \to X_2$ for $X \colon \{M, U^-, U^+\}$ such that the following diagram commutes:

$$\begin{array}{ccc} M_1 \times U_1^- & \xrightarrow{f_1} & TM_1 \times U_1^+ \\ {\scriptstyle \zeta_M \times \zeta_{U^-}} \downarrow & & \downarrow {\scriptstyle T\zeta_M \times \zeta_{U^+}} \\ M_2 \times U_2^- & \xrightarrow[f_2]{} & TM_2 \times U_2^+ \end{array}$$

This defines the category **ODS** of open dynamical systems. We define the subcategory $\mathbf{ODS_{Lin}} \subseteq \mathbf{ODS}$ by restricting our objects to linear open systems, as in

Definition 4.1.10, and imposing that the three maps in ζ are linear.

As in Remark 4.1.13, we will often want to combine two or more interconnected open systems into one larger one. As we shall see in Section 4.3, this will involve taking a product of the smaller open systems. Before we define this formally, we first remind the reader that the tangent space functor T is strong monoidal, i.e., it canonically preserves products,

$$T(M_1 \times M_2) \cong TM_1 \times TM_2.$$

Lemma 4.1.15. *The category* **ODS** *of open systems has all finite products. That is, if I is a finite set and $f_i = (M_i, U_i^-, U_i^+, f_i)\colon \mathtt{ob}\,\mathbf{ODS}$ is an open system for each $i\colon I$, then their product is*

$$\prod_{i\colon I} f_i = \left(\prod_{i\colon I} M_i, \prod_{i\colon I} U_i^-, \prod_{i\colon I} U_i^+, \prod_{i\colon I} f_i\right)$$

with the obvious projection maps.

4.2 Compositionality with Wiring Diagrams

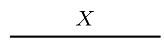
4.2.1 Chains

We now abstract the prior section by defining the general structures that admit string diagrams. This will let us use our visual language in settings not limited to sets, maps, and Cartesian products. To this end, we introduce the *category* notion with the following unorthodox unbiased definition.

Definition 4.2.1. A *category* $\mathcal{C}$ consists of the following data.

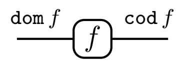
- a collection $\mathtt{ob}\,\mathcal{C}$ of *objects*.

  These are depicted as labelled *strings*, e.g.



- a collection $\mathtt{ar}\,\mathcal{C}$ of *arrows* and two maps $\mathtt{dom}, \mathtt{cod}\colon \mathtt{ar}\,\mathcal{C} \to \mathtt{ob}\,\mathcal{C}$.

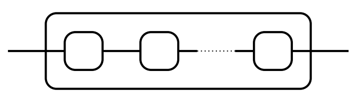
  These are depicted as *boxes* on strings, e.g.



  We write $n\,\mathtt{path}\,\mathcal{C}$ for the collection of n-paths of $\mathcal{C}$-arrows.

- for each $n\colon \mathbb{N}$, a *composition* map $n\,\mathtt{chain}\colon n\,\mathtt{path}\,\mathcal{C} \to \mathtt{ar}\,\mathcal{C}$.

  These are depicted as *schematics*:



  We write $\mathtt{chain}$ for the set of schematics $\{n\,\mathtt{chain}\}_{n\colon \mathbb{N}}$.

This data satisfies the following *coherence* condition.

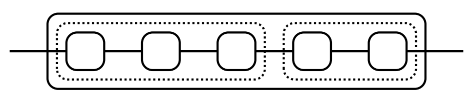
- $\mathtt{chain}$ is closed under nesting

An alternative to using $\mathtt{dom}$ and $\mathtt{cod}$ is the map $\mathtt{hom}\colon \mathtt{ob}\,\mathcal{C} \times \mathtt{ob}\,\mathcal{C} \to \mathbf{2}^{\mathtt{ar}\,\mathcal{C}}$, taking a pair (c, c') of objects to the set $\mathtt{hom}(c, c')$ of arrows f of type $c \to c'$, i.e. with $\mathtt{dom}\,f = c$ and $\mathtt{cod}\,f = c'$. In contexts in which we encounter several different categories, we will subscript the maps $\mathtt{dom}, \mathtt{cod}, \mathtt{hom}, \mathtt{chain}$ by the name of the respective category.

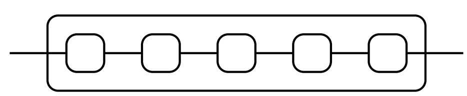
Instead of using schematics, the classic definition of category specifies a binary composition $\fatsemi\colon 2\,\mathtt{path}\,\mathcal{C} \to \mathtt{ar}\,\mathcal{C}$ and nullary composition $\mathtt{id}\colon 0\,\mathtt{path} \to \mathtt{ar}\,\mathcal{C}$. Since

$0\,\texttt{path}$ is in bijection with $\texttt{ob}\,\mathcal{C}$, the nullary composition is a map of type $\texttt{ob}\,\mathcal{C} \to \texttt{ar}\,\mathcal{C}$ that assigns to each $c\colon \texttt{ob}\,\mathcal{C}$ an arrow $\texttt{id}_c$, called its *identity*. We say the classic definition is *biased* towards 0 and 2, whereas our definition does not privilege any input arrangement.

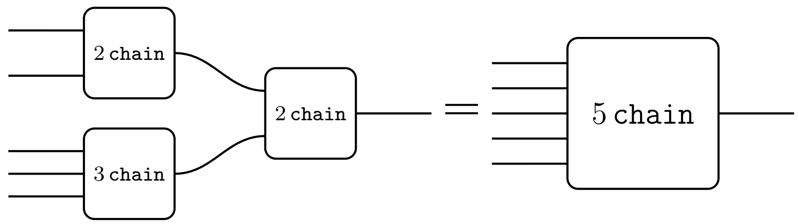
As far as conditions, the classic definition stipulates that binary composition is *associative*— given by the *associativity* equation $(f \mathbin{⨾} g) \mathbin{⨾} h = f \mathbin{⨾} (g \mathbin{⨾} h)$—and *unital*—given by the unitality equations $\texttt{id}_{\texttt{dom}\,f} \mathbin{⨾} f = f = f \mathbin{⨾} \texttt{id}_{\texttt{cod}\,f}$. These conditions can be combined together to derive a unique n-ary composition. Our definition, in contrast, takes n-ary composition as native to its definition. The cost of doing so is the coherence condition. Visually, all this means is that nesting schematics into a schematic so that the cases of the former align with slots of the latter—calling these aligned boundaries *intermediaries*—and then erasing these intermediaries yields a schematic. For a concrete example, consider the following nested schematic, respectively nesting $3\,\texttt{chain}$ and $2\,\texttt{chain}$ in the first and second slots of $2\,\texttt{chain}$.



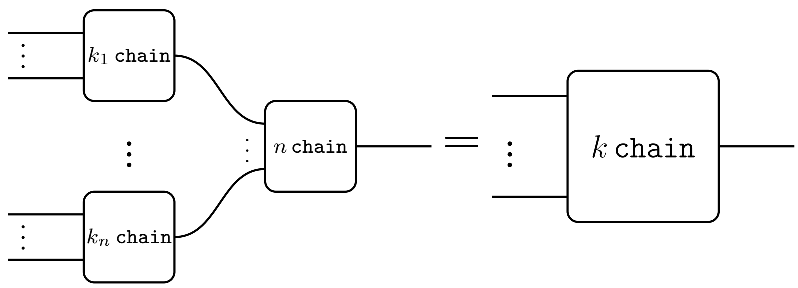
Erasing the intermediary dotted boxes, we simply get $5\,\texttt{chain}$.



Recalling that schematics are just set maps, we can render this equality of schematic nestings using the string diagram language of the prior section.



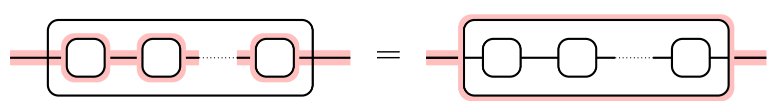
In general, letting $k = \sum_j k_j$, we represent the coherence condition for `chain` via the following string diagram of set maps.



This extends to iterated nesting via the associativity of addition.

- schematic preservation: $[\texttt{ar}\, F]^n \fatsemi n\,\texttt{chain} = n\,\texttt{chain} \fatsemi \texttt{ar}\, F$.

  This is depicted as the following schematic equation.



4.2.2 Flows

With this in mind, we now take a schematic approach to characterizing symmetric monoidal categories. Just as we named the schematics for categories "chains" and wrote `chain` for the collection of them, we name the schematics for symmetric monoidal categories "flows" and write `flow` for the collection of them. Before we define flows, we introduces potential complications that are worth defending against via greater formality.

Definition 4.2.2. A `flow` *box* s is a pair $(s_-, s_+)\colon \mathtt{ob}\,\mathbf{Fin} \times \mathtt{ob}\,\mathbf{Fin}$. We respectively call s^- and s^+ the *negative* and *positive ports* of s.

We now consider what it means to be a `flow` schematic. A schematic is essentially a collection of wires connecting ports of slots and ports of the case. Since both the slots and the case are boxes, we can think of this as a way to pair the total set of ports together. We note that although two connected slot ports have opposite sign, a case port connected to a slot port have the same sign. Thus, letting $S_\pm = \bigsqcup_{s\colon S} s_\pm$, a schematic with slots S and case T can be codified as a map

$$\Phi\colon S_+ + T_- \to S_- + T_+$$

This map must satisfy certain properties. First of all, it must be a bijection since every port must be wired to some other port. In addition, we must consider the fact that we cannot create any cycles among the slots, as this would render their composite ill-defined. Letting `of` be a map that takes a port to its box, we proceed as follows. We take the slots S to come with a partial ordering $\preceq\colon S \times S \to \mathbf{2}$, and write $s \prec s'$, when both $s \preceq s'$ and $s' \npreceq s$. From there, we can derive a partial ordering—which we also denote by $\preceq$—on $S_+ + S_- + T_+ + T_-$ as follows.

$$p \preceq q = \begin{cases} \mathtt{of}\,p \preceq \mathtt{of}\,q & p, q\colon S_- + S_+ \\ 1 & p\colon T^- \text{ or } q\colon T^+ \\ 0 & \text{otherwise} \end{cases}$$

We then use this ordering to define a flow as follows.

Definition 4.2.3. A *flow* with slots S and case T is given by the data

- a partial order $\preceq : S \times S \to \mathbf{2}$
- a bijection $\Phi \colon S_+ + T_- \to S_- + T_+$

subject to the following *progress* condition

- for all $p \colon \mathtt{dom}\,\Phi$, $p \prec \Phi p$.

Clearly chains are a special case of flows. In fact, we did not even bother to define a `chain` box since the notion is essentially degenerate, as each box s has precisely one positive port s_+ and one negative port s_-. At the schematic level, $n\,$`chain` has slot set $\mathbf{n}$ and case t. Then, slyly labeling the case ports as $t_- \triangleq -1_+$ and $t_+ \triangleq n_-$, the corresponding flow would have domain $\{-1_+, 0_+, \ldots, [n-1]_+\}$, codomain $\{0_-, \ldots, [n-1]_-, n_-\}$. We can thus define an embedding $\iota \colon$ `chain` $\to$ `flow` via sending $n\,$`chain` to the pair $(\leqslant, \mathtt{nxt})$, where $\leqslant \colon \mathbf{n} \times \mathbf{n} \to \mathbf{2}$ is the classic inequality and `nxt` is given by the rule $k_- \mapsto [k+1]_+$.

We are now interested in nesting flows. In particular, given the flows

$$\Phi \colon S_+ + T_- \to S_- + T_+$$

$$\Psi \colon T_+ + U_- \to T_- + U_+$$

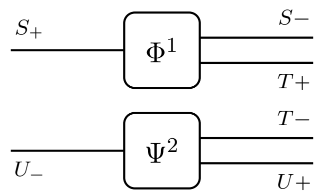
we wish to construct the flow $\Omega \colon S_+ + U_- \to S_- + U_+$ that corresponds to nesting Φ within the slots of Ψ and erasing intermediaries. We can attempt to trace the wires of Ω by inspecting each domain summand. A Φ-wire connecting $p \colon S_+$ to $p' \colon S_-$ will be an Ω-wire since it does not interact with the intermediaries. Likewise, a Ψ-wire connecting $r \colon U_-$ to $r' \colon U_+$ will be an Ω-wire. The more interesting cases are wires that pass through intermediaries. For instance, consider the following sample wiring trajectory that begins at port $p\colon U_-$

$$(r : U_-) \rightsquigarrow (q_1 : T_-) \rightsquigarrow (q_1' : T_+) \rightsquigarrow (q_2 : T_-) \rightsquigarrow \cdots \rightsquigarrow (q_n' : T_+) \rightsquigarrow (r' : U_+)$$

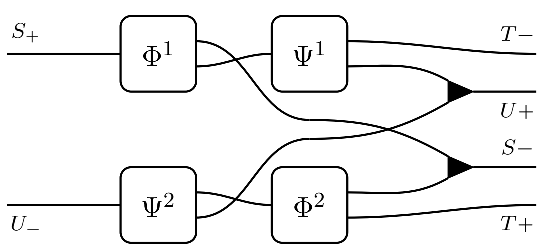
In turn, this trajectory is given by applying to $r : U_-$ the following composite.

$$\Psi \fatsemi \Phi \fatsemi \Psi \fatsemi \cdots \fatsemi \Psi$$

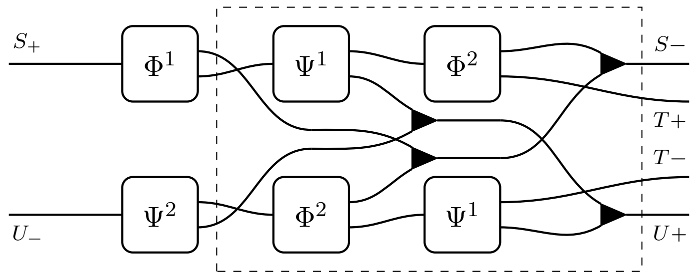
In contrast, there may be an $r : U_-$ that immediately gets mapped to U_+ by Ψ. Thus Ω must apply Φ and Ψ in varying quantity to different arguments. We observe the general pattern by considering the first application $\mathsf{go} \triangleq \Phi^1 + \Psi^2$ to the domain $S_+ + U_-$ of Ω.



The arguments that get sent to either S_- or U_+ have reached their final destination, but those that land in T_+ or T_- require further application of Φ and Ψ. This in turn maps more terms to their final destinations S_- and U_+, which we fold into their pre-existing replicas, along with remainder terms in T_+ and T_-.



This may seem to bring us to back where we were at the end of the previous stage since we now must apply Φ^2 and Ψ^1 respectively to the remaining T_- and T_+ ports:



Now, at least as types are concerned, we are *exactly* where we were after we applied `go`. We thus have for $S_- + T_+ + T_- + U_+$ an endomorphism—delimited above by a dashed box—which we name `step`. Since we have not gotten rid of T-boxes, it may seem as though we need to keep `step`, threatening us with an infinite regress. Luckily, the progress condition guarantees that we need only `step` a finite number of times until the image of the composite map is the desired $S_- + U_+$.

More precisely, let $T^0 \subseteq T$ be the set of T-boxes that have occupied ports after applying `go` and $T^i \subseteq T$ the set of T-boxes with occupied ports after the i^{th} application of `step`. Define the set $T^{i\triangleright} = \{t : T \mid \exists_{(t^i:T^i)} \,.\, t^i \prec t\}$ of boxes that may be accessed in the future. The progress condition implies that for those $t : T^i$ that `step` sends to some $t' : T^{i+1}$, we have $t \prec t'$. This then implies the strict containment $T^{i\triangleright} \supsetneq T^{[i+1]\triangleright}$. Since finite powersets satisfy the descending chain condition, we have the existence of an N for which $T^{N\triangleright} = \varnothing$. We then have that $S_- + U_+$ is the image of the following composite.

$$\texttt{go} \mathbin{⨾} \texttt{step} \mathbin{⨾} \cdots \mathbin{⨾} \texttt{step}$$

4.2.3 The Operad of Wiring Diagrams

In this section, we define the symmetric monoidal category $(\mathbf{W}, \boxplus, 0)$ of wiring diagrams. We then use Definition 4.1.2 to define the wiring diagram operad $\mathcal{O}\mathbf{W}$, which situates our pictorial setting. We begin by formally defining the underlying category $\mathbf{W}$ and continue with some concrete examples to explicate this definition.

Definition 4.2.4. The category $\mathbf{W}$ has objects *boxes* and morphisms *wiring diagrams*. A box X is an ordered pair of $\mathbf{Man}$-typed finite sets (Definition 4.1.5),

$$X = (X^-, X^+) \colon \mathtt{ob}\,\mathbf{TFS} \times \mathtt{ob}\,\mathbf{TFS}.$$

Let $X^- = (A, \tau)$ and $X^+ = (A', \tau')$. Then we refer to elements $a \colon A$ and $a' \colon A'$ as *input ports* and *output ports*, respectively. We call $\tau(a) \colon \mathtt{ob}\,\mathbf{Man}$ the *type* of port a, and similarly for $\tau'(a')$.

A wiring diagram $\Phi \colon X \to Y$ in $\mathbf{W}$ is a triple (X, Y, φ), where φ is a typed bijection (see Definition 4.1.5)

$$\varphi \colon X^- + Y^+ \xrightarrow{\cong} X^+ + Y^-, \tag{4.3}$$

satisfying the following condition:

no passing wires $\varphi(Y^+) \cap Y^- = \varnothing$, or equivalently $\varphi(Y^+) \subseteq X^+$.

This condition allows us to decompose φ into a pair $\varphi = (\varphi^-, \varphi^+)$:

$$\begin{cases} \varphi^- \colon X^- \to X^+ + Y^- \\ \varphi^+ \colon Y^+ \to X^+ \end{cases} \tag{4.4}$$

We often identify the wiring diagram $\Phi = (X, Y, \varphi)$ with the typed bijection φ, or equivalently its corresponding pair (φ^-, φ^+).

By a *wire* in Φ, we mean a pair (a, b), where $a\colon X^- + Y^+$, $b\colon X^+ + Y^-$, and $\varphi(a) = b$. In other words a wire in Φ is a pair of ports connected by ϕ.

The *identity* wiring diagram $\iota\colon X \to X$ is given by the identity morphism $X^- + X^+ \to X^- + X^+$ in **TFS**.

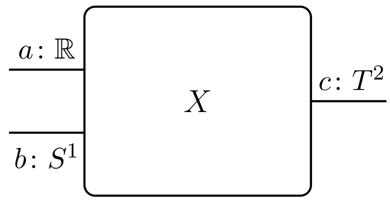
Now suppose $\Phi = (X, Y, \varphi)$ and $\Psi = (Y, Z, \psi)$ are wiring diagrams. We define their *composition* as $\Psi \circ \Phi = (X, Z, \omega)$, where $\omega = (\omega^-, \omega^+)$ is given by the pair of dashed arrows making the following diagrams commute.

$$\begin{array}{ccc} X^- & \overset{\omega^-}{\dashrightarrow} & X^+ + Z^- \\ {\scriptstyle \varphi^-}\downarrow & & \uparrow{\scriptstyle (\mathtt{id} \,\blacktriangledown\, \varphi^+) + \mathtt{id}} \\ X^+ + Y^- & \underset{\mathtt{id} + \psi^-}{\longrightarrow} & X^+ + Y^+ + Z^- \end{array} \qquad \begin{array}{ccccc} Z^+ & & \overset{\omega^+}{\dashrightarrow} & & X^+ \\ & {\scriptstyle \psi^+}\searrow & & \nearrow{\scriptstyle \varphi^+} & \\ & & Y^+ & & \end{array} \tag{4.5}$$

Remark 4.2.5. For any finite product category $\mathcal{C}$, we may define the category $\mathbf{W}_{\mathcal{C}}$ by replacing **Man** with $\mathcal{C}$, and **TFS** with $\mathbf{TFS}_{\mathcal{C}}$, in Definition 4.2.4. In particular, as in Remark 4.1.9, we have the symmetric monoidal category $\mathbf{W}_{\mathbf{Lin}}$ of linearly typed wiring diagrams.

What we are calling a box is nothing more than an interface; at this stage it has no semantics, e.g., in terms of differential equations. Each box can be given a pictorial representation, as in Example 4.2.6 below.

Example 4.2.6. *As a convention, we depict a box $X = (\{a, b\}, \{c\})$ with input ports connecting on the left and output ports connecting on the right, as in Figure 4.2.6 below. When types are displayed, we label ports on the exterior of their box and their types adjacently on the interior of the box with a ':' symbol in between to designate typing. Reading types off of this figure, we see that the type of input port a is the* *manifold* $\mathbb{R}$*, that of input port* b *is the circle* S^1*, and that of output port* c *is the torus* T^2*.*



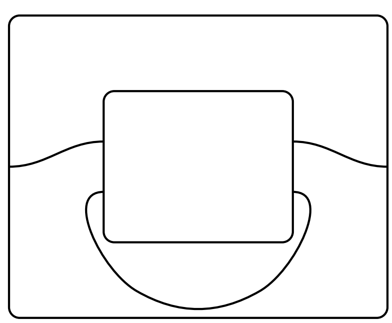
A morphism in **W** is a wiring diagram $\Phi = (X, Y, \varphi)$, the idea being that a smaller box X (the domain) is nested inside of a larger box Y (the codomain). The ports of X and Y are then interconnected by wires, as specified by the typed bijection φ. We will now see an example of a wiring diagram, accompanied by a picture.

Example 4.2.7. *Reading off the wiring diagram* $\Phi = (X, Y, \varphi)$ *drawn below in Figure 4.2.7, we have the following data for boxes:*

$$X^- = \{a, b\} \quad X^+ = \{c, d\}$$
$$Y^- = \{m\} \quad Y^+ = \{n\}$$

Table 4.2.7 makes φ *explicit via a list of its wires, i.e., pairs* $(\gamma, \varphi(\gamma))$*.*

$\gamma\colon X^- + Y^+$	a	b	n
$\varphi(\gamma)\colon X^+ + Y^-$	m	d	c



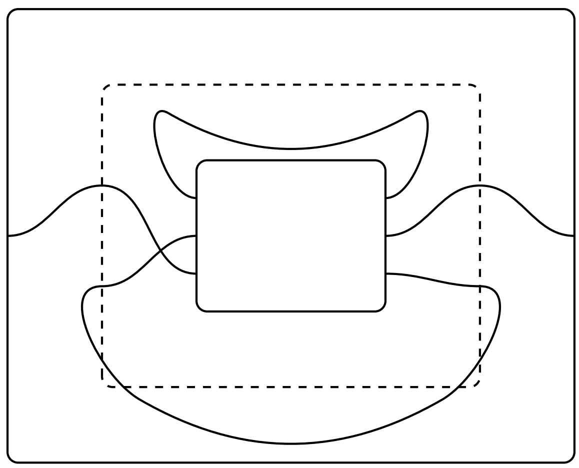
Remark 4.2.8. The condition that φ be typed, as in Definition 4.1.5, ensures that if two ports are connected by a wire then the associated types are the same. In particular, in Example 4.2.7 above, (a, b, n) must be the same type tuple as (m, d, c).

Now that we have made wiring diagrams concrete and visual, we can do the same for their composition.

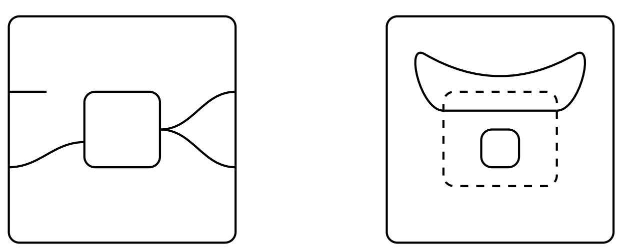
Example 4.2.9. *In Figure 4.2.9, we visualize the composition of two wiring diagrams* $\Phi = (X, Y, \varphi)$ *and* $\Psi = (Y, Z, \psi)$ *to form* $\Psi \circ \Phi = (X, Z, \omega)$. *Composition is depicted by drawing the wiring diagram for* Ψ *and then, inside of the* Y *box, drawing in the wiring diagram for* Φ. *Finally, to depict the composition* $\Psi \circ \Phi$ *as one single wiring diagram, one simply "erases" the* Y *box, leaving the* X *and* Z *boxes interconnected among themselves. Figure 4.2.9 represents such a procedure by depicting the* Y *box with a dashed arrow.*

It's important to note that the wires also connect, e.g. if a wire in Ψ *connects a* Z *port to some* Y *port, and that* Y *port attaches via a* Φ *wire to some* X *port, then these wires "link together" to a total wire in* $\Psi \circ \Phi$, *connecting a* Z *port with an* X *port. Table 4.2.9 below traces the wires of* $\Psi \circ \Phi$ *through the* ω^- *and* ω^+ *composition diagrams in (4.5) on its left and right side, respectively. The left portion of the table starts with* $\gamma\colon X^-$ *and ends at* $\omega^-(\gamma)\colon X^+ + Z^-$, *with intermediary steps of the composition denoted with superscripts* γ^n. *The right portion of the table starts with* $\gamma\colon Z^+$ *then goes through the intermediary of* $\gamma'\colon Y^+$ *and finally reaches* $\omega^+(\gamma)\colon Z^+$. *We skip lines on the right portion to match the spacing on the left.*

$\gamma\colon X^-$	a	b	c	v	$\gamma\colon Z^+$
$\gamma^1\colon X^+ + Y^-$	d	k	l		
$\gamma^2\colon X^+ + Y^+ + Z^-$	d	u	n	m	$\gamma'\colon Y^+$
$\gamma^3\colon X^+ + X^+ + Z^-$	d	u	f		
$\omega^-(\gamma)\colon X^+ + Z^-$	d	u	f	e	$\omega^+(\gamma)\colon X^+$



Remark 4.2.10. The condition that φ be both injective and surjective prohibits *exposed* ports and *split* ports, respectively, as depicted in Figure 4.2.3**a**. The *no passing wires* condition on $\varphi(Y^+)$ prohibits wires that go straight across the Y box, as seen in the intermediate box of Figure 4.2.3**b**.



Now that we have formally defined the category **W**, we will make it into a monoidal category by defining its tensor product.

Definition 4.2.11. Let $X_1, X_2, , Y_1, Y_2 \colon \mathtt{ob}\,\mathbf{W}$ be boxes and $\Phi_1 \colon X_1 \to Y_2$ and $\Phi_2 \colon X_2 \to Y_2$ be wiring diagrams. The *monoidal product* $\boxplus$ is given by

$$X_1 \boxplus X_2 \triangleq \left(X_1^- + X_2^-\ ,\ X_1^+ + X_2^+\right), \qquad \Phi_1 \boxplus \Phi_2 \triangleq \Phi_1 + \Phi_2.$$

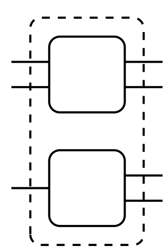
The *closed box* $0 = \{\varnothing, \varnothing\}$ is the monoidal unit.

Remark 4.2.12. Once we add semantics in Section 4.3, closed boxes will correspond to *autonomous systems*, which do not interact with any outside environment (see Remark 4.1.11).

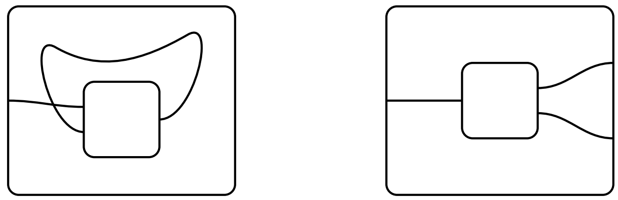
We now make this monoidal product explicit with an example.

Example 4.2.13. *Consider boxes* $X = (\{x_1, x_2\}, \{x_3, x_4\})$ *and* $Y = (\{y_1\}, \{y_2, y_3\})$ *depicted below.*

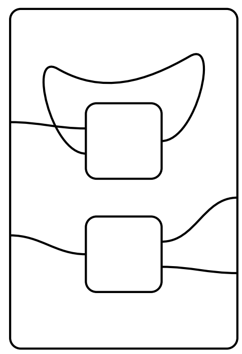
Their tensor product is given by stacking.



Similarly, consider the following wiring diagrams.



We can depict their composition via stacking.



We now prove that the above data characterizing $(\mathbf{W}, \boxplus, 0)$ indeed constitutes a symmetric monoidal category, at which point we can, as advertised, invoke Definition 4.1.2 to define the operad $\mathcal{O}\mathbf{W}$.

Proposition 4.2.14. *The category* $\mathbf{W}$ *in Definition 4.2.4 and the monoidal product* $\boxplus$ *with unit* 0 *in Definition 4.2.11 form a symmetric monoidal category* $(\mathbf{W}, \boxplus, 0)$.

Proof. We begin by establishing that $\mathbf{W}$ is indeed a category. We first show that our class of wiring diagrams is closed under composition. Let $\Phi = (X, Y, \varphi)$, $\Psi = (Y, Z, \psi)$, and $\Psi \circ \Phi = (X, Z, \omega)$.

To show that ω is a typed bijection, we replace the pair of maps (φ^-, φ^+) with a pair of bijections $(\widetilde{\varphi^-}, \widetilde{\varphi^+})$ as follows. Let $X^{\text{exp}}_{\varphi} \subseteq X^+$ (for *exports*) denote the image of φ^+, and X^{loc}_{φ} (for *local ports*) be its complement. Then we can identify φ with the following pair of typed bijections

$$\begin{cases} \widetilde{\varphi^-}\colon X^- \xrightarrow{\cong} X_\varphi^{\text{loc}} + Y^- \\ \widetilde{\varphi^+}\colon Y^+ \xrightarrow{\cong} X_\varphi^{\text{exp}} \end{cases}$$

Similarly, identify ψ with $(\widetilde{\psi^-}, \widetilde{\psi^+})$. We can then rewrite the diagram defining ω in (4.5) as one single commutative diagram of typed finite sets.

$$\begin{array}{ccc}
X^- + Z^+ & \overset{\omega}{\dashrightarrow} & X^+ + Z^- \\
\Big\downarrow{\scriptstyle \widetilde{\varphi^-}+\widetilde{\psi^+}} & & \Big\uparrow{\scriptstyle \cong} \\
X_\varphi^{\text{loc}} + Y^- + Y_\psi^{\text{exp}} & & X_\varphi^{\text{loc}} + X_\varphi^{\text{exp}} + Z^- \\
\Big\downarrow{\scriptstyle \mathtt{id}+\widetilde{\psi^-}+\mathtt{id}} & & \Big\uparrow{\scriptstyle \mathtt{id}+\widetilde{\varphi^+}+\mathtt{id}} \\
X_\varphi^{\text{loc}} + Y_\psi^{\text{loc}} + Z^- + Y_\psi^{\text{exp}} & \xrightarrow[\cong]{} & X_\varphi^{\text{loc}} + Y^+ + Z^-
\end{array}$$

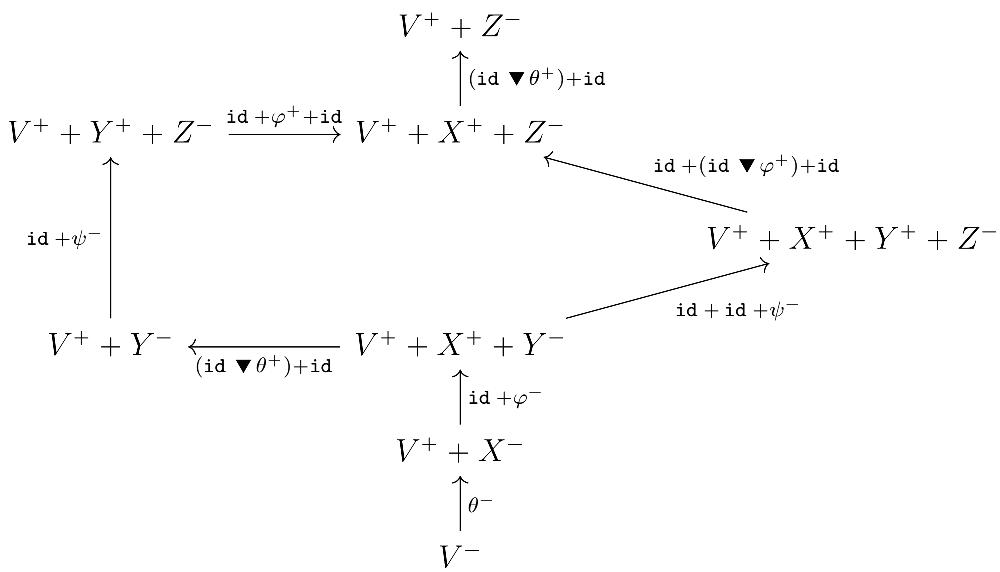
As a composition of typed bijections, ω is also a typed bijection.

The following computation proves that ω has no passing wires:

$$\omega(Z^+) = \varphi\big(\psi(Z^+)\big) \subseteq \varphi(Y^+) \subseteq X^+.$$

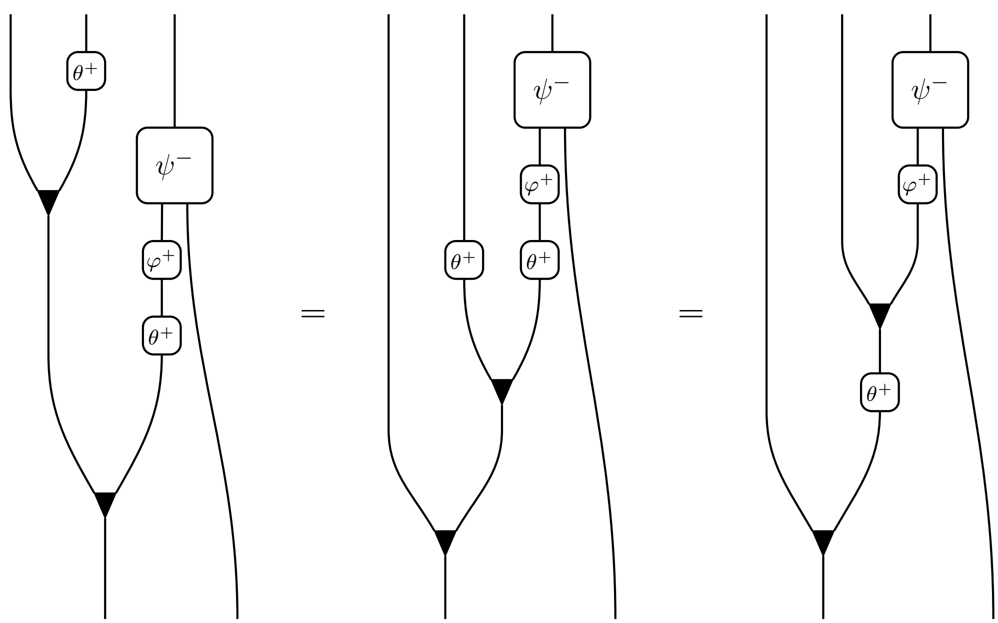
Therefore **W** is closed under wiring diagram composition. To show that **W** is a category, it remains to prove that composition of wiring diagrams satisfies the unit and associativity axioms. The former is straightforward and will be omitted. We now establish the latter.

Consider the wiring diagrams $\Theta = (V, X, \theta), \Phi = (X, Y, \varphi), \Psi = (Y, Z, \psi)$; and let $(\Psi \circ \Phi) \circ \Theta = (V, Z, \kappa)$ and $\Psi \circ (\Phi \circ \Theta) = (V, Z, \lambda)$. We readily see that $\kappa^+ = \lambda^+$ by the associativity of composition in **TFS**. Proving that $\kappa^- = \lambda^-$ is equivalent to establishing the commutativity of the following diagram:



(4.6)

This diagram commutes in any cocartesian category as shown below.



The first step of the proof follows from the topological nature of string diagrams, which mirror the axioms of monoidal categories. The second step invokes the associativity of codiagonal maps while the third invokes their naturality.

Now that we have shown that $\mathbf{W}$ is a category, we show that $(\boxplus, 0)$ is a monoidal structure on $\mathbf{W}$. Let $X, X', X''\colon \texttt{ob}\,\mathbf{W}$ be boxes. We readily observe the following canonical isomorphisms.

$$X \boxplus 0 = X = 0 \boxplus X \qquad \textit{(unity)}$$

$$(X \boxplus X') \boxplus X'' = X \boxplus (X' \boxplus X'') \qquad \textit{(associativity)}$$

$$X \boxplus X' = X' \boxplus X \qquad \textit{(commutativity)}$$

Hence the monoidal product $\boxplus$ is well behaved on objects. It is similarly easy, and hence will be omitted, to show that $\boxplus$ is functorial. This completes the proof that $(\mathbf{W}, \boxplus, 0)$ is a symmetric monoidal category. □

Having established that $(\mathbf{W}, \boxplus, 0)$ is an SMC, we can now speak about the operad $\mathcal{O}\mathbf{W}$ of wiring diagrams. In particular, we can draw operadic pictures, such as the one in our motivating example in Figure 4.1.1, to which we now return.

Example 4.2.15. *Figure 4.2.15 depicts an $\mathcal{O}\mathbf{W}$ wiring diagram $\Phi\colon X_1, X_2 \to Y$, which we may formally denote by the tuple $\Phi = (X_1, X_2; Y; \varphi)$. Reading directly from Figure 4.2.15, we have the boxes:*

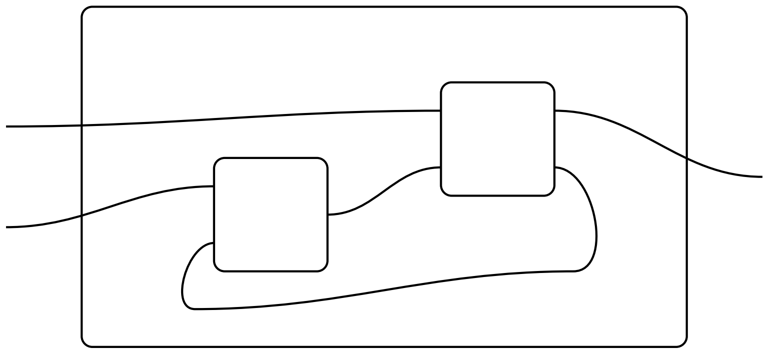
$$X_1 = \big(\{X_{1a}^-, X_{1b}^-\}, \{X_{1a}^+\}\big)$$

$$X_2 = \big(\{X_{2a}^-, X_{2b}^-\}, \{X_{2a}^+, X_{2b}^+\}\big)$$

$$Y = \big(\{Y_a^-, Y_b^-\}, \{Y_a^+\}\big)$$

The wiring diagram Φ is visualized by nesting the domain boxes X_1, X_2 within *the codomain box Y, and drawing the wires prescribed by φ, as recorded below in Table 4.2.15.*

$w\colon X^- + Y^+$	X^-_{1a}	X^-_{1b}	X^-_{2a}	X^-_{2b}	Y^+_a
$\varphi(w)\colon X^+ + Y^-$	Y^-_b	X^+_{2b}	Y^-_a	X^+_{1a}	X^+_{2a}



The following remark explains that our pictures of wiring diagrams are not completely ad hoc—they are depictions of 1-dimensional oriented manifolds with boundary. The boxes in our diagrams simply tie together the positively and negatively oriented components of an individual oriented 0-manifold.

Remark 4.2.16. For any set S, let $\mathbf{Cob}_1/S$ denote the symmetric monoidal category of oriented 0-manifolds over S and the 1-dimensional cobordisms between them. We call its objects *oriented S-typed 0-manifolds.* Recall that $\mathbf{W} = \mathbf{W}_{\mathbf{Man}}$ is our category of **Man**-typed wiring diagrams; let $\mathbf{M} \triangleq \mathtt{ob}\,\mathbf{Man}$ denote the set of manifolds (see Remark 4.1.8). There is a faithful, essentially surjective, strong monoidal functor

$$\mathbf{W} \to \mathbf{Cob}_1/\mathbf{M},$$

sending a box (X^-, X^+) to the oriented $\mathbf{M}$-typed 0-manifold $X^- + X^+$ where X^- is oriented positively and X^+ negatively. Under this functor, a wiring diagram $\Phi =$

(X, Y, φ) is sent to a 1-dimensional cobordism that has no closed loops. A connected component of such a cobordism can be identified with either its left or right endpoint, which correspond to the domain or codomain of the bijection $\varphi\colon X^- + Y^+ \xrightarrow{\cong} X^+ + Y^-$. See [SSR15].

In fact, with the *no passing wires* condition on morphisms (cobordisms) $X \to Y$ (see Definition 4.2.4), the subcategory $\mathbf{W} \subseteq \mathbf{Cob}_1 / \mathbf{M}$ is the left class of an orthogonal factorization system. See [Aba15].

Let $\Phi = (X, Y, \varphi)$ be a wiring diagram. Applying the dependent product functor (see Definition 4.1.6) to φ, we obtain a diffeomorphism of manifolds

$$\overline{\varphi}\colon \overline{X^+} \times \overline{Y^-} \to \overline{X^-} \times \overline{Y^+}. \tag{4.7}$$

Equivalently, if φ is represented by the pair (φ^-, φ^+), as in Definition 4.2.4, we can express $\overline{\varphi}$ in terms of its pair of component maps:

$$\begin{cases} \overline{\varphi^-}\colon \overline{X^+} \times \overline{Y^-} \to \overline{X^-} \\ \overline{\varphi^+}\colon \overline{X^+} \to \overline{Y^+} \end{cases}$$

It will also be useful to apply the dependent product functor to the commutative diagrams in (4.5), which define wiring diagram composition. Note that, by the contravariance of the dependent product, the codiagonal $\nabla\colon X^+ + X^+ \to X^+$ gets sent to the diagonal map $\Delta\colon \overline{X^+} \to \overline{X^+} \times \overline{X^+}$. Thus we have the following commutative diagrams:

$$\begin{array}{ccc} \overline{X^+} \times \overline{Z^-} & \xrightarrow{\overline{\omega^-}} & \overline{X^-} \\ {\scriptstyle (\mathtt{id}\,\Delta\,\overline{\varphi^+})}\downarrow & & \uparrow{\scriptstyle \overline{\varphi^-}} \\ \overline{X^+} \times \overline{Y^+} \times \overline{Z^-} & \xrightarrow[\mathtt{id}\,\times\overline{\psi^-}]{} & \overline{X^+} \times \overline{Y^-} \end{array} \qquad \begin{array}{ccccc} \overline{X^+} & & \xrightarrow{\overline{\omega^+}} & & \overline{Z^+} \\ & {\scriptstyle \overline{\varphi^+}}\searrow & & \nearrow{\scriptstyle \overline{\psi^+}} & \\ & & \overline{Y^+} & & \end{array} \tag{4.8}$$

4.3 Compositional Dynamical Systems

4.3.1 The Algebra of Open Dynamical Systems

In this section we define an algebra $\mathcal{G}\colon (\mathbf{W}, \oplus, 0) \to (\mathbf{Set}, \times, \star)$ (see Definition 4.1.4) of general open dynamical systems. A $\mathbf{W}$-algebra can be thought of as a choice of semantics for the syntax of $\mathbf{W}$, i.e., a set of possible meanings for boxes and wiring diagrams. As in Definition 4.1.2, we may use this to construct the corresponding operad algebra $\mathcal{O}\mathcal{G}\colon \mathcal{O}\mathbf{W} \to \mathbf{Sets}$. Before we define $\mathcal{G}$, we revisit Example 4.1.1 for inspiration.

Example 4.3.1. *As the textbook exercise [BD65, Problem 7.21] prompts, let's begin by writing down the system of equations that governs the amount of salt Q_i within the tanks X_i. This can be done by using dimensional analysis for each port of X_i to find the the rate of salt being carried in ounces per minute, and then equating the rate $\dot{Q}_i$ to the sum across these rates for X_i^- ports minus X_i^+ ports.*

$$\dot{Q}_1 \frac{oz}{min} = -\left(\frac{Q_1 oz}{30 gal} \cdot \frac{3 gal}{min}\right) + \left(\frac{Q_2 oz}{20 gal} \cdot \frac{1.5 gal}{min}\right) + \left(\frac{1 oz}{gal} \cdot \frac{1.5 gal}{min}\right)$$
$$\dot{Q}_2 \frac{oz}{min} = -\left(\frac{Q_2 oz}{20 gal} \cdot \frac{(1.5 + 2.5) gal}{min}\right) + \left(\frac{Q_1 oz}{30 gal} \cdot \frac{3 gal}{min}\right) + \left(\frac{3 oz}{gal} \cdot \frac{1 gal}{min}\right)$$

Dropping the physical units, we are left with the following system of ODEs:

$$\begin{cases} \dot{Q}_1 = -.1Q_1 + .075Q_2 + 1.5 \\ \dot{Q}_2 = .1Q_1 - .2Q_2 + 3 \end{cases} \tag{4.9}$$

The derivations for the equations in (4.9) involved a hidden step in which the connection pattern in Figure 4.1.1, or equivalently Figure 4.2.15, was used. Our wiring diagram approach explains this step and makes it explicit. Each box in a wiring diagram should only "know" about its own inputs and outputs, and not how they are connected to others. That is, we can only define a system on X_i by expressing $\dot{Q}_i$ just in terms of Q_i and X_i^-—this is precisely the data of an open system (see Definition 4.1.10). We now define our algebra $\mathcal{G}$, which assigns a set of open systems to a box. Given a wiring diagram and an open system on its domain box, it also gives a functorial procedure for assigning an open system to the codomain box. We will then use this new machinery to further revisit Example 4.3.1 in Example 4.3.9.

Definition 4.3.2. We define $\mathcal{G}\colon (\mathbf{W}, \oplus, 0) \to (\mathbf{Set}, \times, \star)$ as follows. Let $X\colon \mathtt{ob}\,\mathbf{W}$. The *set of open systems on* X, denoted $\mathcal{G}(X)$, is defined as

$$\mathcal{G}(X) = \{(S, f) \mid S\colon \mathtt{ob}\,\mathbf{TFS}, (\overline{S}, \overline{X^-}, \overline{X^+}, f)\colon \mathtt{ob}\,\mathbf{ODS}\}.$$

We call S the set of *state variables* and its dependent product $\overline{S}$ the *state space.*

Let $\Phi = (X, Y, \varphi)$ be a wiring diagram. Then $\mathcal{G}(\Phi)\colon \mathcal{G}(X) \to \mathcal{G}(Y)$ is given by $(S, f) \mapsto (\mathcal{G}(\Phi)S, \mathcal{G}(\Phi)f)$, where $\mathcal{G}(\Phi)S = S$ and $g = \mathcal{G}(\Phi)f\colon \overline{S} \times \overline{Y^-} \to T\overline{S} \times \overline{Y^+}$ is defined by the dashed arrows (g^-, g^+) (see Definition 4.1.10) that make the diagrams below commute:

$$
\begin{array}{ccc}
\overline{S}\times\overline{Y^-} & \overset{g^-}{\dashrightarrow} & T\overline{S} \\
{\scriptstyle (\mathtt{id}\,\triangle f^+)\times\mathtt{id}}\downarrow & & \uparrow{\scriptstyle f^-} \\
\overline{S}\times\overline{X^+}\times\overline{Y^-} & \xrightarrow[\mathtt{id}\times\varphi^-]{} & \overline{S}\times\overline{X^-}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\overline{S} & \overset{g^+}{\dashrightarrow} & \overline{Y^+}\\
{\scriptstyle f^+}\searrow & & \nearrow{\scriptstyle \overline{\varphi^+}} \\
& \overline{X^+} &
\end{array}
\tag{4.10}
$$

One may note strong resemblance between the diagrams in (4.10) and those in (4.5).

We give $\mathcal{G}$ a lax monoidal structure: for any pair $X, X'\colon \mathbf{W}$ we have a coherence map $\mu_{X,X'} \colon \mathcal{G}(X) \times \mathcal{G}(X') \to \mathcal{G}(X \oplus X')$ given by

$$\big((S,f),(S',f')\big) \mapsto (S+S', f \times f'),$$

where $f \times f'$ is as in Lemma 4.1.15.

Remark 4.3.3. Recall from Remark 4.1.8 that **Man** is small, so the collection $\mathcal{G}(X)$ of open systems on X is indeed a set.

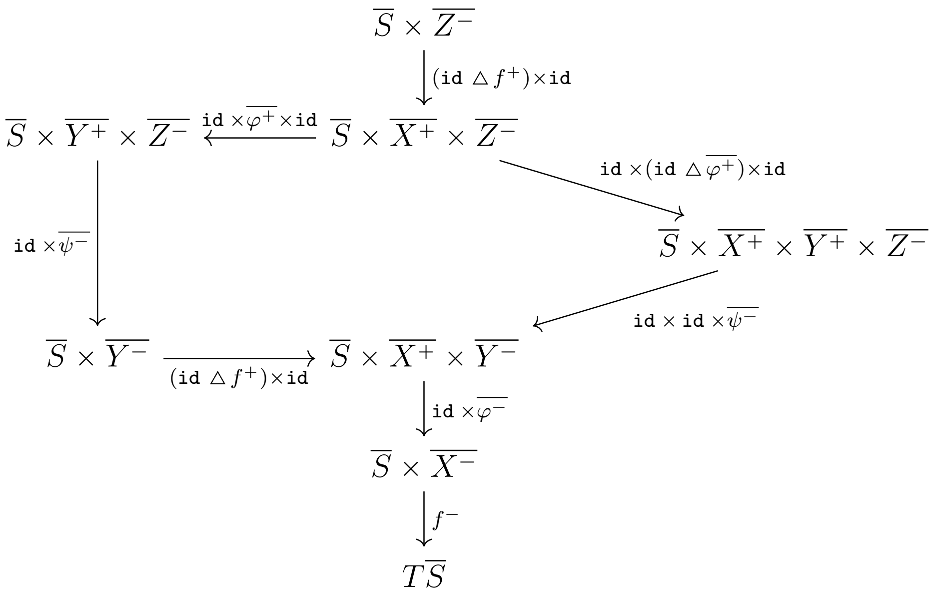
Remark 4.3.4. One may also encode an initial condition in $\mathcal{G}$ by using $\mathbf{Man}_*$ instead of **Man** in Remark 4.1.8 as the default choice of finite product category, where $\mathbf{Man}_*$ is the category of pointed smooth manifolds and base point preserving smooth maps. The base point represents the initialization of the state variables.

We now establish that $\mathcal{G}$ is indeed an algebra.

Proposition 4.3.5. *The pair $(\mathcal{G}, \mu)$ of Definition 4.3.2 is a lax monoidal functor, i.e., $\mathcal{G}$ is a* **W***-algebra.*

Proof. Let $\Phi = (X, Y, \varphi)$ and $\Psi = (Y, Z, \psi)$ be wiring diagrams in $\mathbf{W}$. To show that $\mathcal{G}$ is a functor, we must have that $\mathcal{G}(\Psi \circ \Phi) = \mathcal{G}(\Psi) \circ \mathcal{G}(\Phi)$. Immediately we have $\mathcal{G}(\Psi \circ \Phi)S = S = \mathcal{G}(\Psi)(\mathcal{G}(\Phi)S)$.

Now let $h \triangleq \mathcal{G}(\Psi \circ \Phi)f$ and $k \triangleq \mathcal{G}(\Psi)(\mathcal{G}(\Phi)f)$. It suffices to show $h = k$, or equivalently $(h^-, h^+) = (k^-, k^+)$. One readily sees that $h^+ = k^+$. We use (4.8) and (4.10) to produce the following diagram; showing it commutes is equivalent to proving that that $h^- = k^-$.



(4.11)

The commutativity of this diagram, which is dual to the one for associativity in (4.6), holds in an arbitrary category with products. Although the middle square fails to commute by itself, the composite of the first two maps equalizes it; that is, the two composite morphisms $\overline{S} \times \overline{Z^-} \to \overline{S} \times \overline{X^+} \times \overline{Y^-}$ agree.

Since we proved the analogous result via string diagrams in the proof of Proposition 4.2.14, we show it concretely using elements this time. Let $(s, z)\colon \overline{S} \times \overline{Z^-}$ be an arbitrary element. Composing six morphisms $\overline{S} \times \overline{Z^-} \longrightarrow \overline{S} \times \overline{X^+} \times \overline{Y^-}$ through the left of the diagram gives the same answer as composing through the right; namely,

$$\Big(s, f^+(s), \psi^-\big(\varphi^+ \circ f^+(s), z\big)\Big)\colon \overline{S} \times \overline{X^+} \times \overline{Y^-}.$$

Since the diagram commutes, we have shown that $\mathcal{G}$ is a functor. To prove that the pair $(\mathcal{G}, \mu)$ constitutes a lax monoidal functor $\mathbf{W} \to \mathbf{Set}$, i.e., a $\mathbf{W}$-algebra, we must establish coherence. Since μ simply consists of a coproduct and a product, this is straightforward and will be omitted. □

As established in Definition 4.1.2, the coherence map μ allows us to define the operad algebra $\mathcal{O}\mathcal{G}$ from $\mathcal{G}$. This finally provides the formal setting to consider open dynamical systems over operadic wiring diagrams, such as our motivating one in Figure 4.1.1. We note that, in contrast to the trivial equality $\mathcal{G}(\Phi)S = S$ found in Definition 4.3.2, in the operadic setting we have

$$\mathcal{O}\mathcal{G}(\Phi)(S_1, \ldots, S_n) = \amalg_{i=1}^{n} S_i.$$

This simply means that the set of state variables of the larger box Y is the disjoint union of the state variables of its constituent boxes X_i. Now that we have the tools to revisit Example 4.3.1, we do so in the following section, but first we will define the subalgebra $\mathcal{L}$ to which it belongs—that of linear open systems.

4.3.2 The Subalgebra of Open Linear Systems

In this section, we define the algebra $\mathcal{L}\colon \mathbf{W_{Lin}} \to \mathbf{Set}$, which encodes linear open systems. Here $\mathbf{W_{Lin}}$ is the category of **Lin**-typed wiring diagrams, as in Remark 4.2.5. Of course, one can use Definition 4.1.2 to construct an operad algebra $\mathcal{O}\mathcal{L}\colon \mathcal{O}\mathbf{W_{Lin}} \to$ **Sets**.

Before we give a formal definition for $\mathcal{L}$, we first provide an alternative description for linear open systems and wiring diagrams in $\mathbf{W_{Lin}}$. The category **Lin** enjoys special properties—in particular it is an additive category, as seen by the fact that there is an equivalence of categories $\mathbf{Lin} \cong \mathbf{Vect}_{\mathbb{R}}$. Specifically, finite products and finite coproducts are isomorphic. Hence a morphism $f\colon A_1 \times A_2 \to B_1 \times B_2$ in **Lin** canonically decomposes into a matrix equation

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \mapsto \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} f^{1,1} & f^{1,2} \\ f^{2,1} & f^{2,2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

This matrix is naturally equivalent to the whole map f by universal properties. We use these to rewrite our relevant **Lin** maps as follows.

Suppose that (M, U^-, U^+, f) is a linear open system. Hence we have a linear map $f\colon M \times U^- \to TM \times U^+$, which decomposes into four linear maps:

$$f^{M,M}\colon M \to TM \qquad f^{M,U}\colon U^- \to TM$$

$$f^{U,M}\colon M \to U^+ \qquad f^{U,U}\colon U^- \to U^+$$

By Definition 4.1.10, we know $f^{U,U} = 0$. If we let $(m, u^-, u^+)\colon M \times U^- \times U^+$, these equations can be organized into a single matrix equation

$$\begin{bmatrix} \dot{m} \\ u^+ \end{bmatrix} = \begin{bmatrix} f^{M,M} & f^{M,U} \\ f^{U,M} & 0 \end{bmatrix} \begin{bmatrix} m \\ u^- \end{bmatrix} \tag{4.12}$$

We will exploit this form to define how $\mathcal{L}$ acts on wiring diagrams in terms of one single matrix equation, in place of the seemingly complicated commutative diagrams in (4.10). To do so, we recast wiring diagrams in matrix format.

Suppose $\Phi = (X, Y, \varphi)$ is a wiring diagram in $\mathbf{W_{Lin}}$. Recalling (4.7), we apply the dependent product functor to φ:

$$\overline{\varphi}\colon \overline{X^+} \times \overline{Y^-} \to \overline{X^-} \times \overline{Y^+}$$

Since this is a morphism in **Lin**, it can be decomposed into four linear maps

$$\overline{\varphi}^{X,X} \colon \overline{X^+} \to \overline{X^-} \qquad \overline{\varphi}^{X,Y} \colon \overline{X^+} \to \overline{Y^+}$$

$$\overline{\varphi}^{Y,X} \colon \overline{Y^-} \to \overline{X^+} \qquad \overline{\varphi}^{Y,Y} \colon \overline{Y^-} \to \overline{Y^+}$$

By virtue of the no passing wires condition in Definition 4.2.4, we must have $\overline{\varphi}^{Y,Y} = 0$. We can then, as in (4.12), organize this information in one single matrix:

$$\overline{\varphi} = \begin{bmatrix} \overline{\varphi^{X,X}} & \overline{\varphi^{X,Y}} \\ \overline{\varphi^{Y,X}} & 0 \end{bmatrix}$$

Remark 4.3.6. The bijectivity condition in Definition 4.2.4 implies that $\overline{\varphi}$ is a permutation matrix.

We now employ these matrix characterizations to define the algebra $\mathcal{L}$ of linear open systems.

We define the algebra $\mathcal{L} \colon (\mathbf{W_{Lin}}, \oplus, 0) \to (\mathbf{Set}, \times, \star)$ as follows. Let $X \colon \mathtt{ob}\, \mathbf{W_{Lin}}$. Then the *set of linear open systems* $\mathcal{L}(X)$ *on* X is defined as

$$\mathcal{L}(X) \triangleq \left\{(S, f) \mid S \colon \mathtt{ob}\, \mathbf{TFS_{Lin}}, (\overline{S}, \overline{X^-}, \overline{X^+}, f) \colon \mathtt{ob}\, \mathbf{ODS_{Lin}}\right\}.$$

Let $\Phi = (X, Y, \varphi)$ be a wiring diagram. Then, as in Definition 4.3.2, $\mathcal{L}(\Phi)(S, f) \triangleq$

(S, g), where we define g as follows.

$$
\begin{aligned}
g = \begin{bmatrix} g^{S,S} & g^{S,X} \\ g^{X,S} & g^{X,X} \end{bmatrix} &= \begin{bmatrix} f^{S,X} & 0 \\ 0 & I \end{bmatrix} \overline{\varphi} \begin{bmatrix} f^{X,S} & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} f^{S,S} & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} f^{S,X} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \overline{\varphi}^{X,X} & \overline{\varphi}^{X,Y} \\ \overline{\varphi}^{Y,X} & \overline{\varphi}^{Y,Y} \end{bmatrix} \begin{bmatrix} f^{X,S} & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} f^{S,S} & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} f^{S,X}\overline{\varphi}^{X,X} f^{X,S} + f^{S,S} & f^{S,X}\overline{\varphi}^{X,Y} \\ \overline{\varphi}^{Y,X} f^{X,S} & 0 \end{bmatrix}
\end{aligned}
\tag{4.13}
$$

This is really just a linear version of the commutative diagrams in (4.10). For example, the equation $g^{S,S} = f^{S,X}\overline{\varphi}^{X,X} f^{X,S} + f^{S,S}$ can be read off the diagram for g^- in (4.10), using the additivity of **Lin**.

Finally, The coherence map $\mu_{\mathbf{Lin}_{X,X'}} \colon \mathcal{L}(X) \times \mathcal{L}(X') \to \mathcal{L}(X \oplus X')$ is given, as in Definition 4.3.2, by $\big((S, f), (S', f')\big) \mapsto (S + S', f \times f')$.

We now establish that this constitutes an algebra.

Proposition 4.3.7. *The pair $(\mathcal{L}, \mu_{\mathbf{Lin}})$ is a lax monoidal functor, i.e. a $\mathbf{W}_{\mathbf{Lin}}$-algebra.*

Proof. Since coherence is identical to that in Proposition 4.3.5, it will suffice to show functoriality. Let $\Phi = (X, Y, \varphi)$ and $\Psi = (Y, Z, \psi)$ be wiring diagrams with composition $\Psi \circ \Phi = (X, Z, \omega)$. We now rewrite $\overline{\omega}$ using a matrix equation in terms of $\overline{\varphi}$ and $\overline{\psi}$ by recasting (4.5) in matrix form below.

$$\overline{\omega} = \begin{bmatrix} \overline{\omega^{X,X}} & \overline{\omega^{X,Z}} \\ \overline{\omega^{Z,X}} & \overline{\omega^{Z,Z}} \end{bmatrix} = \begin{bmatrix} \overline{\varphi}^{X,Y} & 0 \\ 0 & I \end{bmatrix} \overline{\psi} \begin{bmatrix} \overline{\varphi}^{Y,X} & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} \overline{\varphi}^{X,X} & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \overline{\varphi}^{X,Y}\overline{\psi}^{Y,Y}\overline{\varphi}^{Y,X} + \overline{\varphi}^{X,X} & \overline{\varphi}^{X,Y}\overline{\psi}^{Y,Z} \\ \overline{\psi}^{Z,Y}\overline{\varphi}^{Y,X} & 0 \end{bmatrix} \tag{4.14}$$

We now prove that $\mathcal{L}(\Psi \circ \Phi) = \mathcal{L}(\Psi) \circ \mathcal{L}(\Phi)$. We immediately have $\mathcal{L}(\Psi \circ \Phi)S = S = \mathcal{L}(\Psi)(\mathcal{L}(\Phi)S)$. Let $h \triangleq \mathcal{L}(\Psi \circ \Phi)f$ and $k \triangleq \mathcal{L}(\Psi)(\mathcal{L}(\Phi)f)$. We must show $h = k$. Let $g = \mathcal{L}(\Phi)f$ and $\Psi \circ \Phi = (X, Z, \omega)$. It is then straightforward matrix arithmetic to see that

$$k = \mathcal{L}(\Psi)g = \begin{bmatrix} g^{S,Y} & 0 \\ 0 & I \end{bmatrix} \overline{\psi} \begin{bmatrix} g^{Y,S} & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} g^{S,S} & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} f^{S,X}(\overline{\varphi}^{X,Y}\overline{\psi}^{Y,Y}\overline{\varphi}^{Y,X} + \overline{\varphi}^{X,X})f^{X,S} + f^{S,S} & f^{S,X}\overline{\varphi}^{X,Y}\overline{\psi}^{Y,Z} \\ \overline{\psi}^{Z,Y}\overline{\varphi}^{Y,X}f^{X,S} & 0 \end{bmatrix} \tag{4.15}$$

$$= \begin{bmatrix} f^{S,X} & 0 \\ 0 & I \end{bmatrix} \overline{\omega} \begin{bmatrix} f^{X,S} & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} f^{S,S} & 0 \\ 0 & 0 \end{bmatrix} = \mathcal{L}(\Psi \circ \Phi)f = h$$

Therefore, the pair $(\mathcal{L}, \mu_{\mathbf{Lin}})$ constitutes a lax monoidal functor $\mathbf{W_{Lin}} \to \mathbf{Set}$, i.e., a $\mathbf{W_{Lin}}$-algebra. □

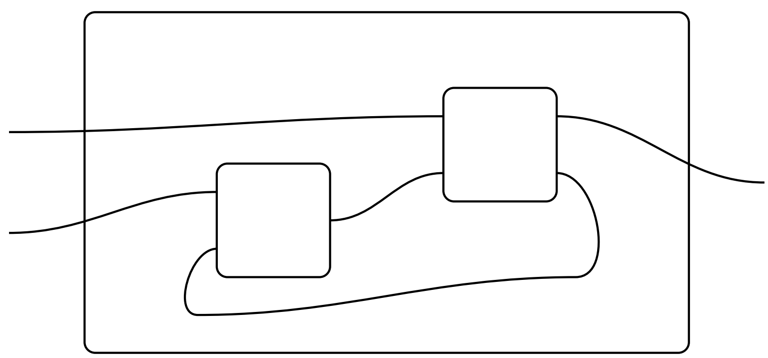
Remark 4.3.8. Although we've been referring to $\mathcal{L}$ as a subalgebra of $\mathcal{G}$, this is technically not the case since they have different source categories. The following diagram illustrates precisely the relationship between the $\mathbf{W_{Lin}}$-algebra $\mathcal{L}$, defined above, and the $\mathbf{W}$-algebra $\mathcal{G}$.

$$
\begin{array}{ccc}
\mathbf{W_{Lin}} & \xhookrightarrow{\mathbf{W}_i} & \mathbf{W} \\
& \mathcal{L} \searrow \quad \overset{\epsilon}{\Longrightarrow} \quad \swarrow \mathcal{G} & \\
& \mathbf{Set} &
\end{array}
\tag{4.16}
$$

Here, the natural inclusion $\mathbf{W}_i \colon \mathbf{W_{Lin}} \hookrightarrow \mathbf{W}$ corresponds to $i \colon \mathbf{Lin} \hookrightarrow \mathbf{Man}$, and we have a natural transformation $\epsilon \colon \mathcal{L} \to \mathcal{G} \circ i$. Hence for each $X \colon \mathtt{ob}\,\mathbf{W_{Lin}}$, we have a function $\epsilon_X \colon \mathcal{L}(X) \to \mathcal{G}(i(X)) = \mathcal{G}(X)$ that sends the linear open system $(S, f) \colon \mathcal{L}(X)$ to the open system $(\mathbf{TFS}_i(S), i(f)) = (S, f) \colon \mathcal{G}(X)$.

As promised, we now reformulate Example 4.1.1 in terms of our language.

Example 4.3.9. *For the reader's convenience, we reproduce Figure 4.1.1 and Table 4.2.15.*



$w \colon X^- + Y^+$	X^-_{1a}	X^-_{1b}	X^-_{2a}	X^-_{2b}	Y^+_a
$\varphi(w) \colon X^+ + Y^-$	Y^-_b	X^+_{2b}	Y^-_a	X^+_{1a}	X^+_{2a}

We can write $\overline{\varphi}$ as a matrix below:

$$
\begin{bmatrix} \overline{X_{1a}^+} \\ \overline{X_{2a}^+} \\ \overline{X_{2b}^+} \\ \overline{Y_a^-} \\ \overline{Y_b^-} \end{bmatrix} = \begin{bmatrix} 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & I & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} \overline{X_{1a}^-} \\ \overline{X_{1b}^-} \\ \overline{X_{2a}^-} \\ \overline{X_{2b}^-} \\ \overline{Y_a^+} \end{bmatrix} \tag{4.17}
$$

One can think of $\overline{\varphi}$ *as a block permutation matrix consisting of identity and zero matrix blocks. An identity matrix in block entry* (i, j) *represents the fact that the port whose state space corresponds to row* i *and the one whose state space corresponds to column* j *get linked by* Φ. *In general, the dimension of each* I *is equal to the dimension of the corresponding state space and hence the formula in (4.17) is true, independent of the typing. In the specific example of this system, however, all of these ports are typed in* $\mathbb{R}$, *and so we have* $I = 1$ *in (4.17).*

As promised in Example 4.3.1, we now write the open systems for the X_i *in Figure 4.1.1 as elements of* $\mathcal{L}(X_i)$. *The linear open systems below in (4.18) represent* f_1 *and* f_2, *respectively.*

$$
\begin{bmatrix} \dot{Q}_1 \\ X_{1a}^+ \end{bmatrix} = \begin{bmatrix} -.1 & 1 & 1 \\ .1 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1 \\ X_{1a}^- \\ X_{1b}^- \end{bmatrix}, \begin{bmatrix} \dot{Q}_2 \\ X_{2a}^+ \\ X_{2b}^+ \end{bmatrix} = \begin{bmatrix} -.2 & 1 & 1 \\ .125 & 0 & 0 \\ .075 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_2 \\ X_{2a}^- \\ X_{2b}^- \end{bmatrix} \tag{4.18}
$$

Note the proportion of zeros and ones in the f*-matrices of (4.18)—this is perhaps why the making explicit of these details was an afterthought in (4.9). Because we may have arbitrary nonconstant coefficients, our formalism can capture more intricate systems.*

We then use (4.17) to establish that $X_{1b}^- = X_{2b}^+$ *and* $X_{2b}^- = X_{1a}^+$. *This allows us to recover the equations in (4.9):*

$$\begin{cases} \dot{Q}_1 = -.1Q_1 + X_{1a}^- + X_{1b}^- = -.1Q_1 + 1.5 + X_{2b}^+ = -.1Q_1 + .075Q_2 + 1.5 \\ \dot{Q}_2 = -.2Q_2 + X_{2a}^- + X_{2b}^- = -.2Q_2 + 3 + X_{1a}^+ = -.2Q_2 + .1Q_1 + 3 \end{cases}$$

The coherence map gives us the combined tank system:

$$(Q, f) \triangleq \mu_{\mathbf{Lin}}((\{Q_1\}, f_1), (\{Q_2\}, f_2)) = (\{Q_1, Q_2\}, f_1 \times f_2) \colon \mathcal{L}(X).$$

This system can then be written out as a matrix below

$$\begin{bmatrix} \dot{Q}_1 \\ \dot{Q}_2 \\ X_{1a}^+ \\ X_{2a}^+ \\ X_{2b}^+ \end{bmatrix} = \begin{bmatrix} -.1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -.2 & 0 & 0 & 1 & 1 \\ .1 & 0 & 0 & 0 & 0 & 0 \\ 0 & .125 & 0 & 0 & 0 & 0 \\ 0 & .075 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ X_{1a}^- \\ X_{1b}^- \\ X_{2a}^- \\ X_{2ba}^- \end{bmatrix} \tag{4.19}$$

Finally, we can apply formula (4.13) to (4.19) above to express as a matrix the open system $(Q, g) = (\Phi)f \colon \mathcal{L}(Y)$ *for the outer box* Y*.*

$$\begin{bmatrix} \dot{Q}_1 \\ \dot{Q}_2 \\ Y^+ \end{bmatrix} = \begin{bmatrix} -.1 & .075 & 0 & 1 \\ .1 & -.2 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Y_a^- \\ Y_b^- \end{bmatrix}$$

Chapter 5

Categorified Invariants

In the present chapter, we work up to a new construction—via a smooth TQFT—for the categorification of the $\mathfrak{sl}_N$ link polynomial. In Section 5.1, we give an overview of the relevant knot theory, leading up to their quantum invariants in general and the $\mathfrak{sl}_N$ polynomial in particular. In Section 5.2, we give a self contained treatment of categorifying various algebraic structures in categories built up from linear structures. Finally, in Section 5.3, we combine these to define our reconstruction of Khovanov-Rozansky $\mathfrak{sl}_N$-homology.

5.1 Quantum Link Invariants Preliminaries

5.1.1 Isotopy Types

Knot theory concerns itself with the topological structure of embedded spaces. This means that we are interested in restricting the usually allowable topological deformations to those inherited by deformations of the ambient space.

Definition 5.1.1. An *embedding* $f : S \to X$ is a map that restricts to a homeomorphism onto its image. An *isotopy* is a continuous family of homeomorphisms $\{F_t : X \to Y\}_{t\colon I}$. Two embeddings $f_0, f_1 : S \to X$ are *ambient isotopic* when there is an isotopy $\{F_t : X \to X\}_{t\colon I}$ of the ambient space for which $F_0 \circ f_0 = F_1 \circ f_1$. An *embedding type* is an equivalence class of embeddings up to ambient isotopy.

This provides the general topological setting for knot theory.

Definition 5.1.2. A *link* is an embedding $l\colon x \to S^3$ of a closed compact 1-manifold—i.e. a sum of circles—x into the 3-sphere. When there is only one component, we call l a *knot*.

Remark 5.1.3. To avoid degenerate cases, we typically stipulate that the embedding is smooth or piece-wise linear, but we elide this point in the present text.

We denote by **link** the set of link types. This can be seen as the set of isomorphism classes of a category.

Definition 5.1.4. The category **Link** has objects oriented links. An arrow between oriented links $l_0\colon x_0 \to S^3$ and $l_1\colon x_1 \to S^3$ is an embedding $\varphi\colon M \to S^3 \times I$, where M is an oriented cobordism from x_0 to x_1 and $\varphi \circ \iota_j = x_j$ for $j = 0, 1$.

Links L are often studied in terms of their diagrams—generic projections onto a plane—D. Two diagrams D and D' represent the same link if and only if one can transform D into D' via a combination of *planar* isotopies and sequences of so called Reidemeister moves.

(RI)

(RII)

(RIII)

We may interpret the latter two of these as equations between morphisms in a braided category. In particular, (RII) corresponds to the fact that `over` and `under` are mutually inverse, while (RIII) corresponds to what we have called the braid relation.

In contrast, (RI) has a less direct interpretation in terms of string diagrams.

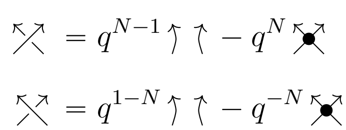
5.1.2 Resolution Cubes

Since the category **Tang** is the free ribbon category on a single object, a ribbon functor **Tang** $\to \mathcal{B}$ is uniquely determined by where one sends the point $1\colon$ `ob` **Tang**. This is sometimes called the "primacy of the point."

It has historically been difficult to come about ribbon categories that are both non-degenerate but actually computationally tractable. Such categories were discovered in the 1980's in the context of quantum groups (CITE: KASSLE). In particular, the $\mathfrak{sl}_N$ link invariant is extracted from the category $\mathbf{Fund}(\mathcal{U}_q\mathfrak{sl}_N)$, consisting of fundamental representations of quantum deformations of the universal enveloping algebra of the lie algebras $\mathfrak{sl}_N$ (CITE: LAUDA).

The associated $\mathfrak{sl}_N$ link invariant takes values in $\mathbf{Fund}(\mathcal{U}_q\mathfrak{sl}_N)$-scalars, i.e. endomorphisms of the monoidal unit $\mathbb{C}[q^{\mathbb{Z}}]$. Since these endomorphisms must be $\mathbb{C}[q^{\mathbb{Z}}]$-linear, there are, by universality, canonically identifiable with $\mathbb{C}[q^{\mathbb{Z}}]$ itself. This explains why the invariants are called the $\mathfrak{sl}_N$ polynomials, which in the $N = 2$ case specialize to the Jones polynomial.

We compute the $\mathfrak{sl}_N$ polynomials by first mapping `over` and `under` in the formal link diagram to their implementations, called *resolutions*, in $\mathbf{Fund}(\mathcal{U}_q\mathfrak{sl}_N)$.



The latter summand of these resolutions is a so called *wide edge* or *singular crossing*. Representation theoretically, this string diagram corresponds to the idempotent projector $V \otimes V \to \bigwedge^2 V \to V \otimes V$ onto the irreducible sub-representation $\bigwedge^2 V$ of $V \otimes V$. In the case that $N = 2$, the module $\bigwedge^2 V$ reduces to the monoidal unit, i.e. the base field. Hence, since V is self dual, the singular crossing becomes a composition of the cup and cup structure morphisms. This fact allows for the following simplified *smooth* resolutions.

Given a link diagram D with n crossings, we may resolve the entire diagram by resolving each crossing, yielding 2^n resolution summands. Fixing an ordering on the crossings, we may label each of these resolutions by a binary string corresponding the the choice of resolution for each crossing. These are typically arranged in a *resolution cube*.

To compute the invariant, we must then have a way of evaluating each of these resolutions. Since resolutions may consist of disjoint components, we may simplify further by merely specifying how to evaluate each component. This is by virtue of the fact that these components are scalars and their adjacency is evaluated as the composition of scalars, namely multiplication of polynomials. The most important component to evaluate is the circle. In particular, the $\mathfrak{sl}_N$ invariant assigns to it the *quantum integer*

$$[N] \triangleq \frac{q^N - q^{-N}}{q - q^{-1}}.$$

In the $N = 2$ case, the cube consists solely of smooth resolutions, namely disjoint copies of the circle. The above hence suffices to characterize the Jones polynomial. In the general case, however, our resolutions are *singular*. In particular, they have the structure of a so called *MOY Graph*, defined to be a closed graph in which each vertex has degree 4. The evaluation of such graphs is a more intricate affair dubbed the *MOY Calculus*, described in full detail in [MOY98].

5.2 Categorification with Linear Structures

5.2.1 Categorifying Natural Numbers

In Section 2.2, we discussed how **Fin** categorifies $\mathbb{N}$ with decategorification given by `card`. In this section, we will categorify $\mathbb{N}$—and several extensions of it—with $\mathbf{FdV}_k$, taking `dim` as our decategorification. As discussed throughout Section 3.2, the monoidal category $(\mathbf{FdV}_k, \otimes, k)$ is more flexible than $(\mathbf{Fin}, \times, \star)$. Namely, it is compact closed. Furthermore, since it is neither cartesian nor cocartesian, it can support nontrivial monoids and comonoids.

Just as the sum $+$ in **Fin**, the direct sum $\oplus$ in $\mathbf{FdV}_k$ categorifies numerical addition. The most direct way to see this is by virtue of the fact that the free k-vector space functor $\mathbf{Fin} \to \mathbf{FdV}_k$ is a left adjoint and hence preserves coproducts:

$$kS \oplus kS' \cong k[S + S']$$

Since $\mathtt{dim}\, kS \triangleq \mathtt{card}\, S$, we have that

$$\begin{aligned}
\mathtt{dim}[kS \oplus kS'] &= \mathtt{dim}\, k[S + S'] \\
&= \mathtt{card}[S + S'] \\
&= \mathtt{card}\, S + \mathtt{card}\, S' \\
&= \mathtt{dim}\, kS + \mathtt{dim}\, kS'
\end{aligned}$$

Since every k-vector space V is isomorphic to kS for any V-basis S, we can work directly with this form. The fact that the forgetful functor $\mathbf{FdV}_k \to \mathbf{Fin}$ is a right adjoint explains why the sum of vector spaces is constructed by taking the product of their underlying sets.

Analogously, just as the product in **Fin**, the tensor $\otimes$ in $\mathbf{FdV}_k$ categorifies multiplication. We can show this in the same manner as above by proving that $kS \otimes kS' \cong k[S \times S']$. This can be shown by recalling that the endofunctors $V \otimes -$ and $- \otimes V$ are both left adjoints and hence preserve sums.

$$\begin{aligned}
kS \otimes kS' &\cong (\bigoplus_{s\colon S} ks) \otimes (\bigoplus_{s'\colon S'} ks') \\
&\cong \bigoplus_{(s,s')\colon S\times S'} (ks \otimes ks') \\
&\cong \bigoplus_{(s,s')\colon S\times S'} k(s \otimes s') \\
&\cong k\left(\coprod_{(s,s')\colon S\times S'} s \otimes s'\right) \\
&\cong k[S \times S']
\end{aligned}$$

We can then exploit `card` just as above.

$$\begin{aligned}
\mathtt{dim}[kS \otimes kS'] &= \mathtt{dim}\, k[S \times S'] \\
&= \mathtt{card}[S \times S'] \\
&= \mathtt{card}\, S \times \mathtt{card}\, S' \\
&= \mathtt{dim}\, kS \times \mathtt{dim}\, kS'
\end{aligned}$$

The distributivity of tensor over direct sum thus establishes that $(\mathbf{FdV}_k, \oplus, \mathbf{0}, \otimes, k)$ categorifies the rig $(\mathbb{N}, +, 0, \times, 1)$. In fact, the free vector space functor establishes a morphism of categorifications.

$$\begin{array}{ccc} (\mathbf{Fin}, +, \varnothing, \times, \star) & \xrightarrow{\quad k(-) \quad} & (\mathbf{FdV}_k, \oplus, \mathbf{0}, \otimes, k) \\ & {\scriptstyle \texttt{card}} \searrow \qquad \swarrow {\scriptstyle \texttt{dim}} & \\ & (\mathbb{N}, +, 0, \cdot, 1) & \end{array}$$

This free functor is faithful and essentially wide—but it is not full. The image of the finite maps are simply the basis mappings, i.e. linear maps arising from mapping the domain basis to the codomain basis. We may thus consider this categorification morphism an embedding into a larger ambient categorification. Thus, in addition to enjoying the newly afforded structure of $(\mathbf{FdV}_k, \otimes, k)$, we have that this *linear categorification* is strictly richer, containing far more arrows to play with. One such arrow is in fact the decategorification itself! In particular, recall that the dimension of a space V is simply the endomorphism $\texttt{loop}\colon V \to V$.

5.2.2 Categorifying Polynomials

We may extend the construction in the prior section to a categorification of the quantum numbers $\mathbb{N}[q^{\pm}]$, i.e. the rig of Laurent polynomials over natural number coefficients. We do so via the notion of a graded vector space.

Definition 5.2.1. For $(C, +, 0)$ a commutative monoid, we define the category $\mathrm{gr}_C\, \mathbf{Vect}_k$ of *C-graded k-vector spaces* as follows.

- an object is a k-vector space V equipped with a *grading* map $\mathrm{gr}\colon V \to C$, such that for any $c\colon C$, the fiber $V_c \triangleq \mathrm{gr}^*(c) \subseteq V$ is a V-subspace called the *degree-c homogeneous subspace*. This can also be phrased as a direct sum decomposition

$$V = \bigoplus_{c\colon C} V_c$$

- We give the hom space the grading

$$\texttt{hom}(V, W) = \bigoplus_{d\colon C} \texttt{hom}_d(V, W)$$

where $\texttt{hom}_d(V, W)$ is the set of maps $f\colon V \to W$ for which $f(V_c) \subseteq W_{c+d}$. Said otherwise, $\texttt{hom}_d(V, W)$ is the set of maps $f\colon V \to W$ that can be written as a direct sum of maps $f_c\colon V_c \to W_{c+d}$.

These objects enjoy a graded notion of dimension.

Definition 5.2.2. Let $\mathbb{N}[q^C]$ be the $\mathbb{N}$-linear extension of C, writing q^c rather than c so as to disambiguate the addition inherited from $\mathbb{N}$ and that from C, which now operates on the exponents. Then, for a C-graded vector space V, its q-dimension is the following formal polynomial in $\mathbb{N}[q^C]$.

$$q\,\texttt{dim}\,V \triangleq \sum_{j\colon C} (\texttt{dim}\,V_j) q^j$$

Just as we write k^n in linear algebra for an n-dimensional k-vector space, we write k^p, for $p\colon \mathbb{N}[q^C]$, for a graded vector space of q-dimension p.

With this as our notion of decategorification, we define the grading on the direct sum $V \oplus W$ to be the join of the gradings of the summands. This immediately satisfies $q\,\texttt{dim}(V \oplus W) = q\,\texttt{dim}\,V + q\,\texttt{dim}\,W$.

We can also define a tensor product so as to make this a monoidal category.

Proposition 5.2.3. *The monoidal category* $(\mathbf{Vect}_k, \otimes, k_0)$ *given as follows is closed.*

- *the tensor* $V \otimes W$ *is given the grading.*

$$(V \otimes W)_c \triangleq \bigoplus_{c' + c'' = c} (V_{c'} \otimes V_{c''})$$

- *the unit k_0 is the line purely graded by the C-unit 0.*

Proof. We construct an isomorphism of degree-d summands.

$$\begin{aligned}
\mathtt{hom}_d(U\otimes V,W) &= \bigoplus_{c\colon C}\mathtt{hom}((U\otimes V)_c,W_{c+d})\\
&= \bigoplus_{c\colon C}\mathtt{hom}\left(\bigoplus_{c'+c''=c}(U_{c'}\otimes V_{c''}),W_{c+d}\right)\\
&\cong \bigoplus_{c\colon C}\bigoplus_{c'+c''=c}\mathtt{hom}(U_{c'}\otimes V_{c''},W_{c+d})\\
&\cong \bigoplus_{c'\colon C}\bigoplus_{c''\colon C}\mathtt{hom}(U_{c'}\otimes V_{c''},W_{c'+c''+d})\\
&\cong \bigoplus_{c'\colon C}\bigoplus_{c''\colon C}\mathtt{hom}(U_{c'},[V_{c''}\to W_{c'+c''+d}])\\
&\cong \bigoplus_{c'\colon C}\mathtt{hom}\left(U_{c'},\bigoplus_{c''\colon C}[V_{c''}\to W_{c''+c'+d}]\right)\\
&= \bigoplus_{c'\colon C}\mathtt{hom}(U_{c'},\mathtt{hom}_{c'+d}(V\to W))\\
&= \mathtt{hom}_d(U,\mathtt{hom}(V,W))
\end{aligned}$$

□

Such closure justifies the moral correctness of our definition of tensor product. We note that the tensor product decategorifies into the product of formal polynomials.

$$q\,\mathtt{dim}(V\otimes W) = q\,\mathtt{dim}\,V\cdot q\,\mathtt{dim}\,W$$

We henceforth specialize to finite dimensional spaces graded in $\mathbb{Z}$.

Although it may seem perverse to do so, we may envision the integers $\mathbb{Z}$ as the polynomial ring $\mathbb{N}[(-1)^{\mathbb{Z}}]$. We can therefore alter the above graded vector space formalism to categorify the integers, except now we set $q=-1$ in our q-dimension decategorification and call it the *Euler characteristic*.

This allows us to state the seemingly vacuous claim that the additive inverse of

k^p is simply k^{-p}, defined as tensoring with k_1 and hence multiplying the polynomial by $q = -1$. But what good is this when it is clearly not the case that $k^p \oplus k^{-p} \cong \mathbf{0}$?

This is where the idea of homotopy equivalence returns as a softer notion of equality that can support a notion of subtraction. If a direct sum becomes contractible, then the summands cancel each other out like puzzle pieces. To use such a scheme, we need a notion of homotopy between some kind of graded spaces. This is in fact the theory of homological algebra, which we henceforth develop in the setting of dg-categories.

5.2.3 DG Categories and Twisted Complexes

Differential graded—or henceforth: dg—categories are a canonical setting for working with iterated mapping cones, and hence give a categorification for subtraction. Recall that an **Ab**-category is a category enriched in $(\mathbf{Ab}, \otimes, \mathbb{Z})$. A dg category is then just a refined version of an Ab-category.

Definition 5.2.4. A *differential graded category* is a category enriched in a monoidal category $(\operatorname{Ch}\mathcal{A}, \otimes, I)$ of chain complexes over a monoidal additive category $(\mathcal{A}, \otimes, I)$.

In other words, for every pair A, B of $\mathcal{D}$-objects, there exists a chain complex $\hom^\bullet(A, B)$ of morphisms $A \to B$, such that the composition of morphisms is a chain map. One can give any additive category $\mathcal{A}$ the *trivial* dg structure by assigning to all morphisms the grading 0 and defining the differential to be 0. For a more interesting example, consider the category $\operatorname{Ch}\mathcal{A}$ of chain complexes over some additive category $\mathcal{A}$. This is a dg category, where $\hom^k(A, B)$ is given by the homogeneous homological-degree-k maps, and the boundary map ∂ of a morphism $f\colon \hom^k(A, B)$ is given by

$$\partial(f) = d^B f - (-1)^k f d^A.$$

In the following subsection, we will use this framework to introduce matrix factorizations. We now explore how dg categories provide a natural setting in which to do linear homotopy theory.

Definition 5.2.5. We say two morphisms $f, g : A \to B$ in a dg category $\mathcal{D}$ are *homotopic*—and write $f \sim g$—when there exists a morphism $h : A \to B$ for which

$$f - g = \partial h.$$

A morphism $f : A \to B$ is *closed* when $\partial f = 0$ and *exact* when $f = \partial h$. We respectively write $Z^k \hom^\bullet(A, B)$ and $B^k \hom^\bullet(A, B)$ for the degree k closed and exact morphisms, and $H^k \hom^\bullet(A, B)$ for the quotient Z^k/B^k. We define the *homology category* $H(\mathcal{D})$ as having the same objects as $\mathcal{D}$ and hom spaces $H(\mathcal{D})(A, B)$ given by $H \hom^\bullet(A, B)$, and the *homotopy category* $\mathcal{K}(\mathcal{D})$ as the restriction of $H(\mathcal{D})$ to degree 0 morphisms $H^0 \hom^\bullet(A, B)$.

We say two objects A, A' in $\mathcal{D}$ are *homotopy equivalent*, and write $A \simeq A'$, when they are isomorphic in $H(\mathcal{D})$, i.e. when there exist closed morphisms $F : A \to A'$ and $F' : A' \to A$ for which $F \circ F' - \mathtt{id}_{A'} = \partial f'$ and $F' \circ F - \mathtt{id}_A = \partial f$. In the case that $F' \circ F = \mathtt{id}_B$, we say A' is a *deformation retract* of A with *projection* F, *inclusion* F', and *homotopy* f. Furthermore, we call the deformation retract *strong* when $fF' = Ff = ff = 0$, and refer to the triple (F, F', f) as the SDR data for this deformation retract.

In the case of chain complexes A, B, the closed degree zero maps $f \colon Z^0 \hom^\bullet(A, B)$ are precisely the chain maps since

$$d^B f - f d^A = \partial(f) = 0.$$

Hence $\mathcal{K}(\mathrm{Ch}\,\mathcal{A})$ recovers the homotopy category of chain complexes, typically denoted $\mathcal{K}(\mathcal{A})$.

Just as defining the category of chain complexes aids in the study of an additive category, defining the category of twisted complexes illuminates a dg category. We henceforth assume that $\mathcal{D}$ is additive—i.e. has direct sums and zero object. This is harmless since, were this not the case, we could replace $\mathcal{D}$ with its additive closure $\mathcal{D}^{\oplus}$, which produces a dg category equivalent to $\mathcal{D}$ in the case that $\mathcal{D}$ was already additive.

Definition 5.2.6. For $\mathcal{D}$ a dg category, a *twisted complex in* $\mathcal{D}$ is a tuple $A = (A_i, d^A_{ji})_{i<j\colon \mathbb{Z}}$, where

- A_i is a $\mathcal{D}$-object
- $d_{ji}\colon \hom^1(A_i, A_j)$

satisfying the conditions

1. the *support* of A—the set of indices for which A_i is nonzero—$\operatorname{supp} A \subset \mathbb{Z}$ is finite
2. $\partial(d^A_{ji}) + \sum_{k:[i<k<j]} d^A_{jk} d^A_{ki} = 0$ for each pair $i < j$.

We define the *shift* $A[n]$ of a twisted complex A as follows.

$$A[n] = (A[n]_i, d^{A[n]}_{ji})_{i<j\colon \mathbb{Z}} = (A_{i+n}, (-1)^n d^A_{j+n,i+n})_{i<j\colon \mathbb{Z}}.$$

We call the d^A_{ij} *differentials*. We define the *length* of differential d_{ji} as $j - i$. We say differentials of length 1 are *short* and differentials of length $\geqslant 2$ are *long*. Condition (2) implies that the composition $d^A_{j+1,j} d^A_{j,j-1}$ of short differentials is null-homotopic with homotopy given by the long differential $d^A_{j+1,j-1}$. In consequence, a twisted complex for which all long differentials vanish is then just a chain complex.

Definition 5.2.7. The dg category $\operatorname{Tw}\mathcal{D}$ is defined as having objects twisted complexes in $\mathcal{D}$. Given a pair A, B of twisted complexes in $\mathcal{D}$, we define the graded hom spaces as

$$\hom^k(A,B) = \bigoplus_{l:[k-l=j-i]} \hom^l(A_i, B_j).$$

Given a morphism $f_{ji}\colon \hom^l(A_i, B_j)$, we define its differential $\partial(f_{ji})$ to be

$$\partial(f_{ji}) = \partial_{\mathcal{D}}(f_{ji}) + \sum_{m\colon \mathbb{Z}} (d^B_{mj} f_{ji} + (-1)^{l(m-i+1)} f_{ji} d^A_{im}).$$

A *twisted morphism* $f : A \to B$ is a closed morphism of degree zero. We write $\operatorname{tw}\mathcal{D}$ for the restriction of $\operatorname{Tw}\mathcal{D}$ to twisted morphisms.

There is an inclusion $\epsilon : \mathcal{D} \to \operatorname{Tw}\mathcal{D}$ which maps an object A to the twisted complex (A_i, d^A_{ji}) for which $A_0 = A$, $A_i = 0$ for $i \neq 0$, and $d_{ji} = 0$ for all i.

It is often useful to conceive of a dg category $\mathcal{D}$ as the dg category $\operatorname{tw}\mathcal{D}'$ of twisted complexes and twisted morphisms in some other dg category $\mathcal{D}'$. In particular, when we give an additive category $\mathcal{A}$ the trivial dg structure, we can see $\operatorname{Ch}\mathcal{A}$ as the subcategory of $\operatorname{tw}\mathcal{A}$ whose objects have vanishing long differentials. We now define the mapping cone of a twisted morphism, which specializes to the classic definition in the context of a chain map.

Definition 5.2.8. Let $\mathcal{D}$ be a dg category and $f : A \to B$ a morphism in $\operatorname{tw}\mathcal{D}$. The *mapping cone* $\boxed{f : A \to B}$, or simply $\boxed{f}$, is the twisted complex (C, d^C) given by

$$C = A[1] \oplus B \qquad\qquad d^C = \begin{bmatrix} d^{A[1]} & 0 \\ f[1] & d^B \end{bmatrix}$$

where the shift $f[n]$ is given by $f[n]_{ji} = f_{j,i+n}$.

Just as in the familiar case of chain maps, one can easily show that the canonical inclusion $\iota : B \to \boxed{f}$ and projection $\pi : \boxed{f} \to A[1]$ are twisted morphisms.

Proposition 5.2.9. *[BK90] Let $\mathcal{D}$ be a dg category. The category $\mathcal{K}(\operatorname{Tw}\mathcal{D})$ can be given the structure of a triangulated category with the shift as defined above and distinguished triangles of the form*

$$A \xrightarrow{f} B \xrightarrow{\iota} \boxed{f} \xrightarrow{\pi} A[1].$$

The mapping cone is a special case of a more general construction.

Definition 5.2.10. Let $\mathcal{D}$ be a dg category and $A = \left(A_{(i,m)}, d^A_{(j,n),(i,m)}\right)_{(i,m)\leqslant(j,n)}$ a *doubly twisted complex*, i.e. an object in $\operatorname{Tw}^2\mathcal{D}$. The *convolution* $\operatorname{Conv}(A)$, or simply $\boxed{A}$, is an object in $\operatorname{Tw}\mathcal{D}$ defined as follows.

$$\boxed{A}_k = \bigoplus_{i+m=k} A_{(i,j)} \qquad\qquad d^{\boxed{A}}_{lk} = \sum_{(j+n,i+m)=(l,k)} d^A_{(j,n),(i,m)}$$

The convolution specializes to the total complex of a bicomplex when restricting to the subcategory $\operatorname{Ch}^2\mathcal{A}$ of $\operatorname{Tw}^2\mathcal{A}$. The convolution also restricts to the mapping cone, by conceiving of a twisted morphism as a certain twisted complex. More precisely, a twisted morphism $f_{10} : A_0 \to A_1$ can be identified with the doubly twisted complex $(A_{(i,j)}, d^A)$ for which $A_{(i,\bullet)} = A_i[1-i]$ for $i = 0, 1$ with the following non-vanishing differentials

$$d^A_{(i,\bullet),(i,\bullet)} = d^{A_i[1-i]} \qquad\qquad d^A_{(1,\bullet),(0,\bullet)} = f_{10}[1]$$

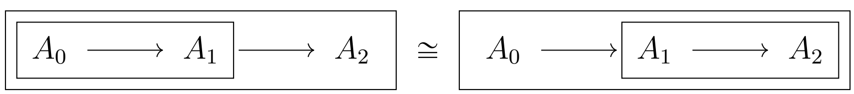
In this case, the mapping cone $\boxed{f_{10} : A_0 \to A_1}$ is homotopy equivalent to $\boxed{A}$.

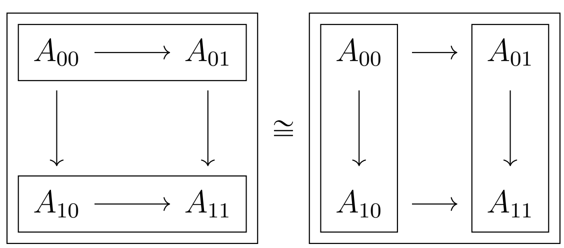
The convolution satisfies many desirable properties, which may be neatly organized in the following terms.

Proposition 5.2.11. *The triple* (Tw, Conv, ϵ) *forms a commutative monad.*

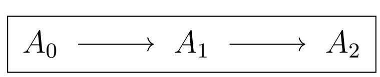
The fact that (Tw, Conv, ϵ) is a monad was proven in [BK90]. We define the braiding to be the index-swap map given by $A_{(i,m)} \mapsto A_{(m,i)}$ and $d_{(j,n),(i,m)} \mapsto d_{(n,j),(m,i)}$. Since the definition of the convolution is symmetric in these two arguments, the commutativity is immediate.

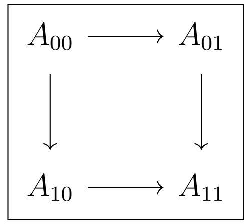
Concretely, the commutative monad structure means that, given an n-fold twisted complex, i.e. an object in $\mathrm{Tw}^n \mathcal{D}$, there is a unique *total convolution* in $\mathrm{Tw}\,\mathcal{D}$. The uniqueness implies that one may arrive at this total convolution via any sequence of intermediary convolutions. Two visually suggestive consequences of this are given by the following isomorphisms of iterated mapping cones.





We can thus unambiguously depict the above iterated mapping cones as total convolutions:



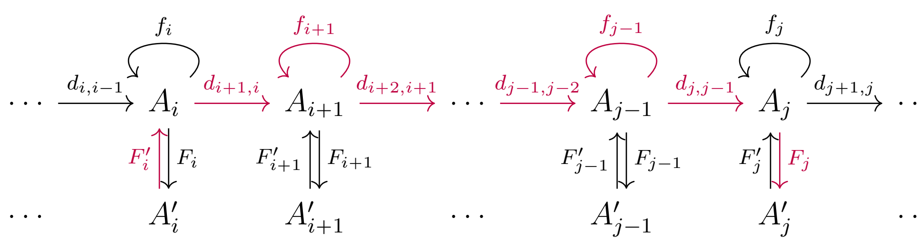


We finish the section with a useful lemma for simplifying convolutions.

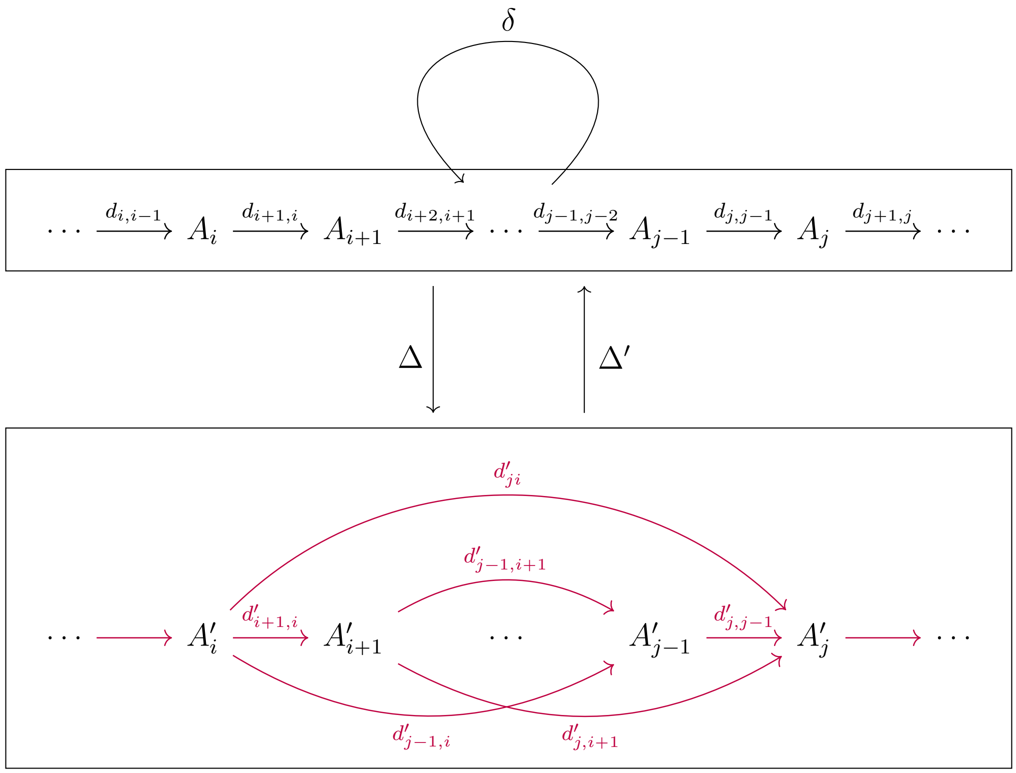
Lemma 5.2.12. *Let $A = (A_i, d_{ji})$ be a twisted complex, with vanishing long differentials, of twisted complexes A_i in some dg category $\mathcal{D}$. Let $A_i \to A'_i$ be strong deformation retracts with SDR data (F_i, F'_i, f_i). Let $A' = (A'_i, d'_{ji})$ be a twisted complex with differentials*

$$d'_{ji} = F_j d_{j,j-1} f_{j-1} d_{j-1,j-2} f_{j-2} \cdots f_{i+2} d_{i+2,i+1} f_{i+1} d_{i+1,i} F'_i.$$

Then there is a deformation retract $\Delta : \boxed{A} \to \boxed{A'}$. In diagrams, the strong deformation retracts



imply—with d'_{ji} given by the above purple composition—a deformation retract of convolutions



Proof. We proceed by induction on n. In the case that $n = 1$ then there is nothing to prove as d_{ji} and thus d'_{ji} is zero for all i, j and hence (F_1, F'_1, f_1) is already SDR data.

Now suppose $(\Delta, \Delta', \delta)$ constitute a strong deformation retract whenever the maximum support index is n. We proceed with the induction step to show that this is also the case when the maximum support index is $n + 1$.

Now suppose $n = 2$. Consider the following square of arrows.

$$\begin{array}{ccc} A_1 & \xrightarrow{d_{21}} & A_2 \\ {\scriptstyle F_1}\downarrow & & \downarrow{\scriptstyle F_2} \\ A'_1 & \xrightarrow[F_2 d_{21} F'_1]{} & A'_2 \end{array}$$

Since the F_i are deformation retracts, this square commutes up to homotopy. That is

$$F_2 d_{21} - F_2 d_{21} F_1' F_1 = F_2 d_{21}(\mathtt{id}_{A_1} - F_1' F_1) = F_2 d_{21} \partial(-f_1) = \partial(-F_2 d_{21} f_1)$$

It is a standard fact that homotopy communtative squares induce chain maps between mapping cones [Wei94]. Let $d_{21}' = -F_2 d_{21} f_1$. Then this homotopy commutative square gives a map of mapping cones

$$\nabla = \begin{pmatrix} F_1 & 0 \\ d_{21}' & F_2 \end{pmatrix}.$$

Now we want to construct a candidate for the reverse map $\nabla' : \boxed{F_2 d_{21} F_1'} \to \boxed{d_{21}}$. Consider the square, Now we need to find a homotopy. Let

$$\delta = \begin{pmatrix} \psi_1 & 0 \\ \psi_2 f_{21} \psi_1 & \psi_2 \end{pmatrix}.$$

First we prove condition (2). Note that

$$\nabla\nabla' = \begin{pmatrix} \Psi_1 \Psi_1' & 0 \\ h\Psi_1' + \Psi_2 h' & \Psi_2 \Psi_2'. \end{pmatrix}.$$

Since Ψ_1 and Ψ_2 are deformation retracts, then $\Psi_i' \Psi_i = \mathtt{id}_{M_i}$, $\Psi_2 h' = \Psi_2 \psi_2 f_{21} \Psi_1' = 0$ and $h\Psi_1 = \Psi_2 f_{21} \psi_1 \Psi_1 = 0$. Therefore $\nabla\nabla' = \mathtt{id}_{\mathrm{Con}(\Psi_2 f_{21} \Psi_1')}$.

Next we will prove condition (1). Note that

$$\nabla'\nabla = \begin{pmatrix} \Psi_1'\Psi_1 & 0 \\ h'\Psi_1' + \Psi_2 h & \Psi_2'\Psi_2. \end{pmatrix} = \begin{pmatrix} \mathtt{id}_{M_1} + \partial(\psi_1) & 0 \\ \partial(\psi_2 f_{21}\psi_1) & \mathtt{id}_{M_2} + \partial(\psi_2) \end{pmatrix}$$
$$= \begin{pmatrix} \mathtt{id}_{M_1} & 0 \\ 0 & \mathtt{id}_{M_2} \end{pmatrix} + \partial \begin{pmatrix} \psi_1 & 0 \\ \psi_2 f_{21}\psi_1 & \psi_2 \end{pmatrix}.$$

Therefore $\nabla'\nabla = \mathtt{id}_{\mathrm{Con}(f_{21})}$ as desired. Condition (3) follows from similar computations.

Now suppose that $M_i\colon \mathbf{MF}(R,w)$ for $i = 0, 1, \ldots, n$, and suppose that $\Psi_i : M_i \to M_i'$ is a deformation retract. Suppose that $M_\bullet = (M_i, f_{ij})$ is a twisted complex. Then there exist maps $f_{ij}' : M_j' \to M_i'$ such that $M_\bullet' = (M_i', f_{ij}')$ is a twisted complex and there exists a map $\nabla : \mathrm{Con}(M_\bullet) \to \mathrm{Con}(M_\bullet')$ such that ∇ is a deformation retract.

We prove this by induction on the length of the twisted complex. The case that the twisted complex has length 2 was proven in Lemma 4.4.7. Suppose that the twisted complex $M_\bullet$ has length $k+1$. Then we may consider $\mathrm{Con}(M_\bullet)$ as a mapping cone,

$$\mathrm{Con}(M_0 \xrightarrow{\sum_{i=1}^k f_{i1}} \mathrm{Con}((M_\ell, f_{ij})_{\ell\geqslant 1})).$$

By the induction hypothesis, since $\mathrm{Con}((M_\ell, f_{ij})_{\ell\geqslant 1})$ is a convolution of a twisted complex of length k, then there exists a deformation retract $\nabla_1 : \mathrm{Con}((M_\ell, f_{ij})_{\ell\geqslant 1}) \to \mathrm{Con}((M_\ell', f_{ij}')_{\ell\geqslant 1})$. Therefore by Lemma 4.4.7 there exists a deformation retract,

$$\nabla : \mathrm{Con}(M_0 \xrightarrow{\sum_{i=1}^k f_{i0}} \mathrm{Con}((M_\ell, f_{ij})_{\ell\geqslant 1})) \to \mathrm{Con}(M_0' \xrightarrow{\sum_{i=1}^k f_{i0}'} \mathrm{Con}((M_\ell', f_{ij}')_{\ell\geqslant 1})).$$

But, $\mathrm{Con}(M_0' \xrightarrow{\sum_{i=1}^k f_{i0}'} \mathrm{Con}((M_\ell', f_{ij}')_{\ell\geqslant 1}) = \mathrm{Con}(M_\bullet'))$.

The formula

$$f_{ji}' = \Psi_j f_{j,j-1}\psi_{j-1} f_{j-1,j-2}\psi_{j-2}\cdots f_{i+1,i}\Psi_i'.$$

follows from repeatedly applying the computations from the proof in the $n = 2$ case. $\square$

5.3 Categorified Link Invariants

5.3.1 Khovanov Homology

Before discussing our categorification of the $\mathfrak{sl}_N$ polynomial, we briefly review the construction of Khovanov Homology, which categorifies the $\mathfrak{sl}_2$, i.e. Jones, polynomial. We will try to mirror this construction for the more general case we cover.

Khovanov Homology is, up to grading shift, the homology of a chain complex called the Khovanov Bracket. The bracket itself is a link invariant when evaluated in the homotopy category. This bracket categorifies the Kauffman Bracket $\langle - \rangle$ in the sense of the following commutative diagram.

$$\begin{array}{ccc} \mathbf{Link} & \overset{[\![-]\!]}{\dashrightarrow} & \mathcal{K}\,\mathrm{Ch}\,\mathrm{gr}_{\mathbb{Z}}\,\mathbf{FdV}_k \\ {\scriptstyle /\cong}\big\downarrow & & \big\downarrow{\scriptstyle \chi_q} \\ \mathbf{link} & \underset{\langle - \rangle}{\longrightarrow} & \mathbb{Z}[q^{\mathbb{Z}}] \end{array}$$

We define this bracket by categorifying the entire procedure for computing the Jones polynomial. In particular, let D be an n-crossing link diagram with writhe $w(D)$.

1. define a resolution cube functor $\rho\colon \underline{\mathbf{2}}^n \to \mathbf{Cob}_2$ via

   - mapping the binary sequence $a\colon \mathbf{2}^n$ to its corresponding smooth resolution $D_a\colon \texttt{ob}\,\mathbf{Cob}_2$
   - mapping $s_{a,a'}\colon a \to a'$, where a and a' differ only in the j^{th} component, to a cobordism that enacts, respectively via pant or copant cobordisms, the merging or splitting of circles.

2. apply a TQFT or, equivalently, define a Frobenius object $\mathcal{F}\colon \mathbf{Cob_2} \to \mathbf{FdV}_k$

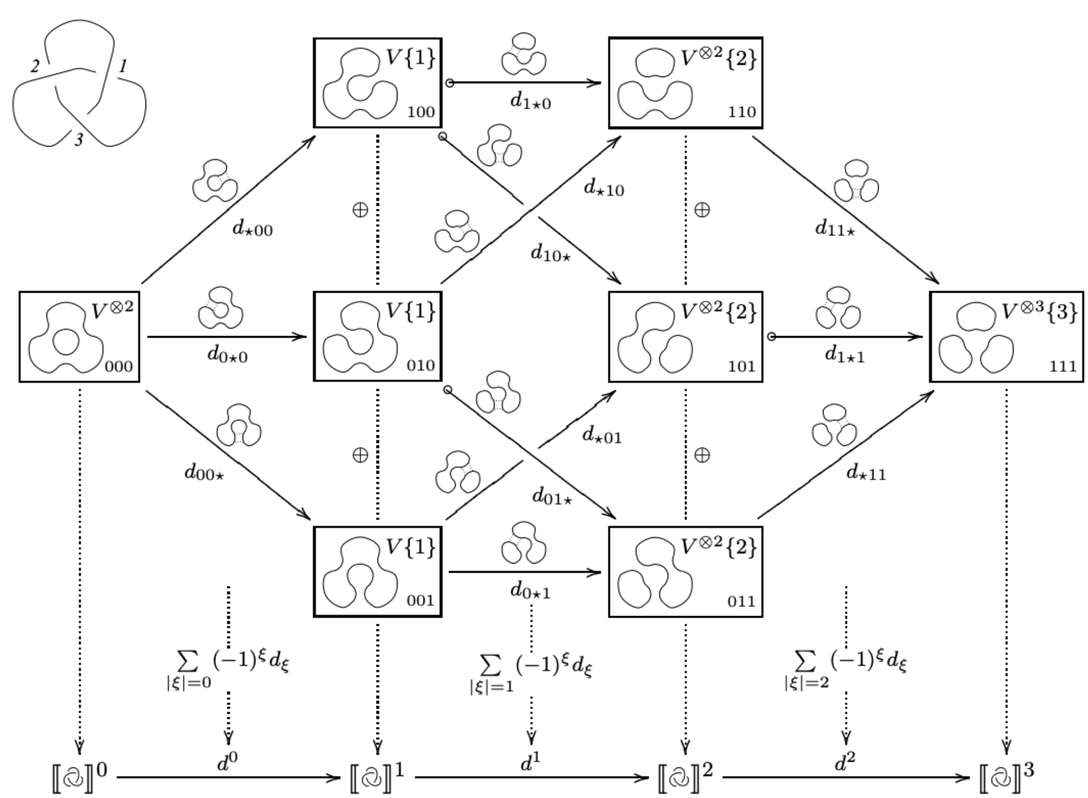
3. define the i^{th} graded piece $[\![D]\!]^i$ as the direct sum

$$[\![D]\!]^i \triangleq \bigoplus_{|a|=i} \mathcal{F}\rho(a)$$

and the i^{th} differential $d^i\colon [\![D]\!]^i \to [\![D]\!]^{i+1}$ as the direct sum

$$d^i = \sum_{|a|=i} \mathcal{F}(-1)^{|a|}\rho(s_{a,a'})$$

This procedure is neatly visualized in the following diagram, borrowed from [BN02], which performs it in the case of a trefoil knot.



Just as we mapped an m-component resolution to the power $(q+q^{-1})^m$, we define the TQFT $\mathcal{F}\colon \mathbf{Cob}_2 \to \mathbf{Vect}_k$ by assigning to it the vector space $V^{\otimes m}$, where V is a Frobenius algebra isomorphic to $k^{q+q^{-1}}$ in $\operatorname{gr}_{\mathbb{Z}} \mathbf{FdV}_k$. We then assign to each cobordism the corresponding Frobenius monoid operation. In particular, as a monoid, V is the polynomial algebra $k[x]/\langle x^2\rangle$. In turn, the comultiplication $\delta\colon V \mapsto V \otimes V$ is defined, using polynomial notation, as follows.

$$
\begin{aligned}
1 &\mapsto x \otimes 1 + 1 \otimes x \\
x &\mapsto x \otimes x
\end{aligned}
$$

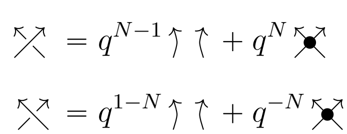
To properly set the gradings, we define $\deg_q x = 2$ and tensor the polynomial algebra by k^{-q}. This is typically written in the literature as

$$V \triangleq \frac{k[x]}{\langle x^2\rangle}\{1\}.$$

By virtue of the properties of $\mathcal{F}$, the complex $[\![D]\!]$ amounts to a chain complex in $\operatorname{Ch} \operatorname{gr}_{\mathbb{Z}} \mathbf{FdV}_k$, which we conceive of in the homotopy category $\mathcal{K}(\operatorname{Ch} \operatorname{gr}_{\mathbb{Z}} \mathbf{FdV}_k)$.

5.3.2 Khovanov-Rozansky Homology

Recall that the $\mathfrak{sl}_N$ invariant assigns the following resolutions to crossings.



One approach, taken by Queffelec-Rose in [QR16], that mirrors the one presented for Khovanov Homology, is to define $\mathfrak{sl}_N$-homology by constructing a *seamed TQFT*, which assigns linear structures to singular resolutions and coherently defined morphisms thereof to singular cobordisms called *foams*. This seamed TQFT is far more complicated than its smooth analog since it must respect all the relations of singular graphs and their transformations implied by the representation theory.

In contrast, the original definition of $\mathfrak{sl}_N$ homology, given by Khovanov and Rozansky in [KR08], assigns to each singular resolution a certain kind of graded vector spaced called a matrix factorization. This approach, however, does not involve any TQFT-like construction. Thus, despite evading the difficult task of constructing a seamed TQFT, it fails to provide a topological interpretation for the edge maps within the resolution complex.

The present construction takes a hybrid approach: rather than a cube of singular resolutions, we in some sense define a *meta-cube* of smooth resolutions. More precisely, we replace each singular resolution with an equivalent convolution of an n-fold twisted complex of smooth resolutions. This equivalence is achieved by iteratively applying the following equivalence, introduced in [KR07], of the singular crossing to a mapping cone of virtual crossings.

Applying this *virtual filtration* to each singular crossing within a singular resolution yields an iterated mapping cone, which, as discussed in Section 5.2.3, is simply a convolution of a twisted complex. Each term in this twisted complex is thus a virtual link diagram and hence, by the virtual Reidemeister moves, a disjoint union of circles. Not only does this give us smooth resolutions, but it makes the *outer* morphisms—those at the level of the cube of resolutions—into canonical inclusions and projections of mapping cones.

While our construction indeed leads to simpler shapes than in the seamed approach, the complexity gets pushed to the intricacy of combination of our shapes. In particular, the convolutions yield twisted complexes that admit long differentials.

Although we do not give a closed form expression for these long differentials, we provide an explicit algorithm for computing them.

Given that the resolution cube flattens to a complex of singular crossings, our virtual filtration yields a complex of convolutions and hence a single bicomplex, whose horizontal differentials are given by the outer complex derived from the resolution cube and whose vertical differentials arise from those in the resultant twisted complexes of the convolutions. The second page of the corresponding spectral sequence then exactly matches the original definition of Khovanov-Rozansky homology.

5.3.3 Matrix factorizations

Let R be a $\mathbb{Z}$-graded, commutative, unital ring. We refer to this grading as the *quantum grading* or the *q-grading* and denote it by $\deg_q$ and its (upward) grading shift functors by $\bullet(i)$.

Definition 5.3.1. A *matrix factorization* (M, d) *of potential* $w\colon R$ consists of a $\mathbb{Z}_2$-graded R-module M equipped with a q-homogeneous R-endomorphism $d : M \to M$ such that

1. d is homogeneous of $\mathbb{Z}_2$-degree 1.

2. $d^2 = w \cdot \mathtt{id}_M$.

We call d the *differential* of the matrix factorization M.

We will use $\deg_h$ to denote the $\mathbb{Z}_2$-grading, and call it the h-grading. We will often express the matrix factorization $M = M_0 \oplus M_1$ as a 2-periodic sequence

$$\cdots \xrightarrow{d} M_1 \xrightarrow{d} M_0 \xrightarrow{d} M_1 \xrightarrow{d} M_0 \xrightarrow{d} \cdots$$

with subscripts indicating h-degree. If $w = 0$, this is just a 2-periodic chain complex. If $w \neq 0$ and M is finite rank, it is straightforward to show that M_0 and M_1 have equal rank. We define the h-grading shift $\bullet[i]$ by setting $M[i] = M_i \oplus M_{i+1}$ (with indices in $\mathbb{Z}_2$) and $d[1] = -d$.

Matrix factorizations of potential w assemble into a dg category $\mathcal{MF}(R, w)$ with hom complex $\hom^\bullet(M, N)$ whose k^{th} graded piece $\hom^k(M, N)$ is given by $\mathrm{Hom}_R(M, N[k])$ of homogeneous R-linear maps of h-degree k. Given $f\colon \hom^k(M, N)$, its differential is given by

$$\partial(f) = d_N f - (-1)^k f d_M.$$

We say degree zero closed maps, i.e. those in $Z^0 \hom^\bullet(M, M')$, are *matrix factorization morphisms*, and define these as the arrows in the category $\mathbf{MF}(R, w)$, whose objects are matrix factorizations of potential w. We write $\mathbf{HMF}(R, w)$ for the homology category $H(\mathcal{MF}(R, w))$. We denote the graded pieces of the hom complexes $\mathrm{Hom}^\bullet(M, N)$ of $\mathbf{HMF}(R, w)$ with the notation $\mathrm{Hom}(M, N) = \mathrm{Hom}^0(M, N)$ and $\mathrm{Ext}(M, N) = \mathrm{Hom}^1(M, N)$.

Now consider the tensor product bifunctor $\otimes_R$ on R-modules. We can extend this bifunctor to matrix factorizations. In particular we define a bifunctor

$$\otimes_R : \mathbf{MF}(R, w_1) \times \mathbf{MF}(R, w_2) \to \mathbf{MF}(R, w_1 + w_2).$$

On objects, $M \otimes_R N$ is the matrix factorization with the underlying graded R-module given by usual tensor product and $d_{M\otimes N}(m \otimes n) = d_M(m) \otimes n + (-1)^{\deg_h(m)} m \otimes d_N(n)$ for any homogeneous $m \otimes n$. Direct computation shows that $d^2_{M\otimes N} = (w_1 + w_2) \cdot \mathtt{id}_{M\otimes N}$. The bifunctor applied to morphisms is defined the same as in the case of

$\mathbb{Z}_2$-graded modules.

It will be useful to compute $\hom^\bullet(M,N)$ and $\mathrm{Hom}^\bullet(M,N)$ via the homology of certain objects in $\mathbf{MF}(R,0)$. Suppose $(M,d)\colon \mathbf{MF}(R,w)$ and suppose that M is a free R-module. Then we define the matrix factorization $(M^*,d^*)\colon \mathbf{MF}(R,-w)$ by

$$\mathrm{Hom}_R(M_0,R) \xrightarrow{-d^*} \mathrm{Hom}_R(M_0,R) \xrightarrow{d^*} \mathrm{Hom}_R(M_0,R).$$

The following proposition, whose proof we omit, will be useful in the sequel. The reader is encouraged to consult [KR08] for further detail.

Proposition 5.3.2. *Suppose M and N are finite rank objects of $\mathbf{MF}(R,w)$. Then, in $\mathbf{MF}(R,0)$, there is a natural isomorphism*

$$\mathrm{Hom}_R(M,N) \cong M^* \otimes_R N.$$

More generally, if M and N are homotopy equivalent to finite rank objects of $\mathbf{MF}(R,w)$, there is a natural isomorphism of homology groups

$$\mathrm{Hom}^\bullet(M,N) \cong H(M^* \otimes_R N).$$

Not only do potential-w matrix factorizations assemble into a dg category $\mathcal{MF}(R,w)$, but furthermore this is equivalent to a dg category $\mathrm{Tw}\,\mathcal{D}$ of twisted complexes over another dg category $\mathcal{D}$. Then $\mathrm{tw}\,\mathcal{D}$ recovers $\mathbf{MF}(R,w)$. The description of $\mathcal{D}$ is outside the scope of this paper, but we invite the reader to consult for more detail. For our purposes, the meaningful implication of this is that we can define a convolution of a twisted complex of matrix factorizations, i.e. an object in $\mathrm{Tw}\,\mathbf{MF}(R,w)$, as an object in $\mathbf{MF}(R,w)$. This is just a rerendering of the convolution map $\mathrm{Tw}^2\,\mathcal{D} \to \mathrm{Tw}\,\mathcal{D}$. Since we suppressed the nature of $\mathcal{D}$, we provide the concrete form, with only reference to matrix factorizations, for what such convolutions amount to.

Definition 5.3.3. Let $A = (A_i, d_{ji})_{i<j}$ be a twisted complex of matrix factorizations for which $\operatorname{supp} A = \{0, 1, \ldots, n\}$. Let d_{A_i} be the matrix factorization differential for A_i and define new diagonal maps $d_{ii} = d_{A_i}$. Then the convolution $\boxed{A}$ is the matrix factorization with underlying module $\bigoplus_i A_i$ with differential $d = \sum d_{ji}$, i.e. the lower triangular matrix whose ij^{th} entry is given by d_{ji}.

We may check that $\boxed{A}$ is a matrix factorization by directly computing the ij^{th} entry of d^2. When $j = i$ this is simply $d_{ii}^2 = d_{A_i}^2 = w \cdot \mathtt{id}$. When $j > i$, we can use condition (2) of Definition 5.2.6 to compute

$$\begin{aligned}(d^2)_{ji} &= \textstyle\sum_{k:[i \leqslant k \leqslant j]} d_{jk} d_{ki} \\ &= -\partial(d_{ji}) + d_{jj} d_{ji} + d_{ji} d_{ii} \\ &= 0.\end{aligned}$$

We conclude this subsection by establishing a form of Gaussian Elimination for matrix factorizations.

Proposition 5.3.4. *Let $A, B, C, D : \mathtt{ob}\,\mathbf{Mod}_R$ and $\alpha : A \xrightarrow{\cong} C$. Consider the following matrix factorization M in $\mathbf{MF}(R, w)$.*

$$A \oplus B \xrightarrow{\left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right]} C \oplus D \xrightarrow{\left[\begin{smallmatrix} \varphi & \lambda \\ \mu & \nu \end{smallmatrix}\right]} A \oplus B$$

Then the Gaussian elimination *morphism $\Upsilon \colon M \to M'$ in $\mathbf{MF}(R, w)$ is a deformation* *retract with inclusion* $\Upsilon' \colon M' \to M$

$$
\begin{array}{ccccc}
A \oplus B & \xrightarrow{\left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right]} & C \oplus D & \xrightarrow{\left[\begin{smallmatrix} \varphi & \lambda \\ \mu & \nu \end{smallmatrix}\right]} & A \oplus B \\
\left[\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right] \uparrow \downarrow \left[\begin{smallmatrix} -\alpha^{-1}\beta \\ 1 \end{smallmatrix}\right] & & \left[\begin{smallmatrix} -\gamma\alpha^{-1} & 1 \end{smallmatrix}\right] \uparrow \downarrow \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] & & \left[\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right] \uparrow \downarrow \left[\begin{smallmatrix} -\alpha^{-1}\beta \\ 1 \end{smallmatrix}\right] \\
B & \xrightarrow[\delta - \gamma\alpha^{-1}\upsilon\beta]{} & D & \xrightarrow[\nu]{} & B
\end{array}
\tag{5.1}
$$

Proof. Via direct computation, this diagram commutes and $\Upsilon\Upsilon' = \texttt{id}$.

We define a homotopy map $\upsilon\colon \hom^1(M, M)$ as follows.

$$
\begin{array}{ccccc}
A \oplus B & \xrightarrow{\left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right]} & C \oplus D & \xrightarrow{\left[\begin{smallmatrix} \varphi & \lambda \\ \mu & \nu \end{smallmatrix}\right]} & A \oplus B \\
 & \swarrow \left[\begin{smallmatrix} -\alpha^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right] & & \swarrow 0 & \\
A \oplus B & \xrightarrow[\left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right]]{} & C \oplus D & \xrightarrow[\left[\begin{smallmatrix} \varphi & \lambda \\ \mu & \nu \end{smallmatrix}\right]]{} & A \oplus B
\end{array}
\tag{5.2}
$$

We now want to show $\Upsilon'\Upsilon = \partial\upsilon + \texttt{id}$. $\Upsilon'\Upsilon$ is the map

$$
\begin{array}{ccccc}
A \oplus B & \xrightarrow{\left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right]} & C \oplus D & \xrightarrow{\left[\begin{smallmatrix} \varphi & \lambda \\ \mu & \nu \end{smallmatrix}\right]} & A \oplus B \\
\left[\begin{smallmatrix} 0 & -\alpha^{-1}\beta \\ 0 & 1 \end{smallmatrix}\right] \downarrow & & \left[\begin{smallmatrix} 0 & 0 \\ -\gamma\alpha^{-1} & 1 \end{smallmatrix}\right] \downarrow & & \left[\begin{smallmatrix} 0 & -\alpha^{-1}\beta \\ 0 & 1 \end{smallmatrix}\right] \downarrow \\
A \oplus B & \xrightarrow[\left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right]]{} & C \oplus D & \xrightarrow[\left[\begin{smallmatrix} \varphi & \lambda \\ \mu & \nu \end{smallmatrix}\right]]{} & A \oplus B
\end{array}
\tag{5.3}
$$

and $\partial \upsilon$ is given by

$$\begin{array}{ccccc}
A \oplus B & \xrightarrow{\left[\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right]} & C \oplus D & \xrightarrow{\left[\begin{smallmatrix}\varphi & \lambda \\ \mu & \nu\end{smallmatrix}\right]} & A \oplus B \\
{\scriptstyle\left[\begin{smallmatrix}-1 & -\alpha^{-1}\beta \\ 0 & 0\end{smallmatrix}\right]}\downarrow & & {\scriptstyle\left[\begin{smallmatrix}-1 & 0 \\ -\gamma\alpha^{-1} & 0\end{smallmatrix}\right]}\downarrow & & {\scriptstyle\left[\begin{smallmatrix}-1 & -\alpha^{-1}\beta \\ 0 & 0\end{smallmatrix}\right]}\downarrow \\
A \oplus B & \xrightarrow[\left[\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right]]{} & C \oplus D & \xrightarrow[\left[\begin{smallmatrix}\varphi & \lambda \\ \mu & \nu\end{smallmatrix}\right]]{} & A \oplus B
\end{array} \tag{5.4}$$

Thus, $\Upsilon'\Upsilon = \mathtt{id} + \partial \upsilon$ as claimed. We leave verifying condition (3) for $(\Upsilon, \Upsilon', \upsilon)$ being SDR data as a simple calculation for the reader. □

We now specialize to the specific type of matrix factorization, related to Koszul complexes, that is relevant to our construction. We henceforth let $R = \mathbb{C}[\mathbf{x}]$ for some list of variables $\mathbf{x} = x_1, \ldots, x_n$, and set $\deg_q(x_i) = 2$ for $i = 1, \ldots, n$. Let $a, b \colon R$ be two homogeneous elements of q-degrees k and k' respectively. We define the *Koszul matrix factorization* $(a\ b)_R$ to be

$$R_1(\ell) \xrightarrow{a} R_0 \xrightarrow{b} R_1(\ell),$$

where $R_0 = R_1$ as modules, with subscript indicating h-degree, and $\ell = \frac{1}{2}(k - k')$. Note that the matrix factorization potential is $w = ab$ and that $\deg(ab) = k + k'$. The grading shift ℓ is chosen so that the matrix factorization differential is homogeneous of q-degree $\frac{1}{2}(k + k')$.

We will perform calculations of Koszul matrix factorization using the language of exterior algebra, also known as *Grassmann calculus*. In particular, given the Koszul matrix factorization $(a\ b)_R$, we identify its underlying module with $R \otimes \bigwedge \mathbb{C}$, where the tensor is over $\mathbb{C}$, and $\bigwedge$ is the functor taking a $\mathbb{C}$-vector space to its exterior algebra. Letting $\theta \colon \bigwedge^1 \mathbb{C}$ be the generator of the exterior algebra $\bigwedge \mathbb{C}$, we can express this equivalence more concretely by relabelling direct summands as follows.

$$R_0 \mapsto R \qquad\qquad R_1 \mapsto R\theta.$$

Given this correspondence, we have that $\deg_h \theta = 1$. Furthermore, we set $\deg_q \theta = \ell$, for the sake of recording both degrees in a single variable. We then define two linear endomorphisms $\partial, \hat{\theta} : \bigwedge \mathbb{C} \to \bigwedge \mathbb{C}$ via their behavior on generators:

$$\begin{aligned} \partial(1) &= 0 & \hat{\theta}(1) &= \theta \\ \partial(\theta) &= 1 & \hat{\theta}(\theta) &= 0. \end{aligned}$$

It follows that $\partial^2 = 0 = \hat{\theta}^2$. We can then express the matrix factorization differential

$$d = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} : R_0 \oplus R_1 \to R_0 \oplus R_1$$

in terms of these endomorphisms as

$$d = a \otimes \partial + b \otimes \hat{\theta} : R \otimes \bigwedge \mathbb{C} \to R \otimes \bigwedge \mathbb{C}.$$

We will henceforth omit the $\otimes$ symbols in the differential and simply write

$$d = a\partial + b\hat{\theta}.$$

As an algebra, this space is generated by ∂ and $\hat{\theta}$, but as a vector space, $\mathrm{Hom}_{\mathbb{C}}(\bigwedge \mathbb{C}, \bigwedge \mathbb{C})$ is 4-dimensional. In particular, writing $\mathtt{id}$ for $\mathtt{id}_{\wedge \mathbb{C}}$, the following gives a basis.

$$\mathtt{id}, \partial, \hat{\theta}, \partial\hat{\theta}.$$

One can easily check that these are all linearly independent. This leaves us with the question of what exactly happens to $\hat{\theta}\partial$ and all other alternating words in ∂ and $\hat{\theta}$. We note the following *anti*-commutator relation:

$$[\partial, \hat{\theta}]_- \triangleq \partial\hat{\theta} + \hat{\theta}\partial = \mathtt{id}\,.$$

We now have a description of $\mathrm{Hom}_{\mathbb{C}}(\bigwedge\mathbb{C}, \bigwedge\mathbb{C})$ as both an algebra and vector space. For example, we check that $d^2 = (ab) \cdot \mathtt{id}$.

$$\begin{aligned} d^2 &= (a\partial + b\hat{\theta})^2 \\ &= a^2\partial^2 + ab\partial\hat{\theta} + ba\hat{\theta}\partial + b^2\hat{\theta}^2 \\ &= 0 + ab[\partial, \hat{\theta}]_- + 0 \\ &= (ab) \cdot \mathtt{id}\,. \end{aligned}$$

Note that since $\deg_h \theta = 1$, we have that ∂ and $\hat{\theta}$ are homogeneous of h-degree 1, while $\mathtt{id}, \partial\hat{\theta}$ are homogenous of h-degree 0. Hence the differential of such a Koszul matrix factorization could only have consisted of ∂ and $\hat{\theta}$ terms.

More generally, let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ be vectors with components in R of q-degree $(k_1, \ldots, k_n)$ and $(k_1', \ldots, k_n')$, respectively. We define the Koszul factorization $(\mathbf{a}\ \mathbf{b})_R$ as the R-tensor product

$$\bigotimes_{i=1}^{n} \begin{pmatrix} a_i & b_i \end{pmatrix}_R$$

We say $\mathbf{a}$ and $\mathbf{b}$ have length n, and denote the component corresponding to $R_{s_1} \otimes \cdots \otimes R_{s_n}$ by R_s, with $s\colon \{0,1\}^n$ a binary finite sequence of length n. This matrix factorization has potential

$$\mathbf{a} \cdot \mathbf{b} \triangleq \sum_{i=1}^{n} a_i b_i.$$

We can codify Koszul complexes—those made by tensoring smaller complexes—in the language of exterior algebra. The same is also true for matrix factorizations. In particular, we let $\bigwedge \mathbb{C}^n$ be the $\mathbb{C}$-algebra generated by $(\theta_i)_{i=1}^n$ with relations $[\theta_i, \theta_j]_- = 0$ for all i, j. We will use the basis given by expressions $\theta^s \triangleq \theta_1^{s_1} \cdots \theta_n^{s_n}$ with s again a length n binary sequence. When s is the 0 sequence, we simply write 1 for the corresponding basis element. This basis has 2^n elements—one for each subset of the n generators. As before $\deg_h(\theta_i) = 1$ for all i, and $\deg_q(\theta_i) = \ell_i$, with $\ell_i = \frac{1}{2}(k_i - k_i')$. Then, at the level of graded modules, we have the isomorphism

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix}_R \xrightarrow{\cong} R \otimes \bigwedge \mathbb{C}^n$$
$$R_s \ni 1 \mapsto \theta^s.$$

We omit subscripts in the case where $n = 1$. We will often implicitly use the isomorphism $\bigwedge \mathbb{C}^n \otimes \bigwedge \mathbb{C}^{n'} \xrightarrow{\cong} \bigwedge \mathbb{C}^{n+n'}$ of $\mathbb{C}$-vector spces given by $\theta^s \otimes \theta^{s'} \mapsto \theta^{s''}$, with s'' the length $n + n'$ concatenation of s and s'. This isomorphism could be used in the induction step for formally proving the identification of Koszul complexes with exterior algebras. In the case when we have Koszul R-matrix factorizations M and M' of length n and n' respectively, both given in exterior form, we can use this isomorphism to express the tensor product $M \otimes M'$ as a single Koszul matrix factorization of length $n + n'$ in exterior form.

Before we extend this isomorphism to the matrix factorization differentials, we will also fix a basis for $\mathrm{Hom}_{\mathbb{C}}(\bigwedge \mathbb{C}^m, \bigwedge \mathbb{C}^n)$. To do so, we define generalized versions of ∂ and $\hat{\theta}$ given by $(\partial_i)_{i=1}^m$ and $(\hat{\theta}_i)_{i=1}^n$. Note the indices: we want m ∂_i's to be able to vanish any domain generators and n $\hat{\theta}_i$'s to be able to produce any codomain generators. More concretely, these act on generators via partial differentiation and multiplication respectively:

$$\partial_i(\theta_j) = \delta_{ij} \qquad \hat{\theta}_i(\theta_j) = \theta_i\theta_j,$$

where, since $\theta_i^2 = 0$, this forces $\hat{\theta}_i(\theta_i) = 0$. This can be extended to an action on the entire basis via algebra relations, e.g.

$$\partial_2(\theta_1\theta_2\theta_3) = -\partial_2(\theta_2\theta_1\theta_3) = -\theta_1\theta_3.$$

Via pre- and post-composition, $\mathrm{Hom}_{\mathbb{C}}(\bigwedge\mathbb{C}^m, \bigwedge\mathbb{C}^n)$ respectively inherits a left and right module structure over the endomorphism $\mathbb{C}$-algebras $\mathrm{End}_{\mathbb{C}}(\bigwedge\mathbb{C}^m)$ and $\mathrm{End}_{\mathbb{C}}(\bigwedge\mathbb{C}^n)$. Then $\mathrm{Hom}_{\mathbb{C}}(\bigwedge\mathbb{C}^m, \bigwedge\mathbb{C}^n)$ has as generators the maps $(\partial_i)_{i=1}^m$ and $(\hat{\theta}_i)_{i=1}^n$, which satisfy the following relations.

$$[\partial_i, \hat{\theta}_j]_- = \delta_{ij} \qquad [\partial_i, \partial_j]_- = 0 = [\hat{\theta}_i, \hat{\theta}_j]_-.$$

We will commonly use the basis

$$\partial_1^{s_1} \cdots \partial_m^{s_m} \hat{\theta}_1^{s_{m+1}} \cdots \hat{\theta}_n^{s_{m+n}},$$

with s a length $m+n$ binary sequence, writing `id` when s is the 0 sequence. This set has the correct cardinality of 2^{m+n}. The reader may wish to check it for linear independence, thus confirming it as a basis. Then the length n Koszul matrix factorization $(\mathbf{a}\ \mathbf{b})_R$ is isomorphic to $R \otimes \bigwedge \mathbb{C}^n$ with the differential

$$\sum_{i=1}^{n} (a_i \partial_i + b_i \hat{\theta}_i).$$

A simple calculation confirms that $d^2 = (\mathbf{a} \cdot \mathbf{b}) \cdot \mathtt{id}$:

$$\begin{aligned}
d^2 &= [\textstyle\sum_i (a_i \partial_i + b_i \hat{\theta}_i)]^2 \\
&= \textstyle\sum_{i,j} (a_i a_j \partial_i \partial_j + a_i b_j \partial_i \hat{\theta}_j + b_i a_j \hat{\theta}_i \partial_j + b_i b_j \hat{\theta}_i \hat{\theta}_j) \\
&= \textstyle\sum_{i,j} a_i b_j [\partial_i, \hat{\theta}_j]_- + \sum_{i \neq j} (a_i a_j \partial_i \partial_j + b_i b_j \hat{\theta}_i \hat{\theta}_j) \\
&= \textstyle\sum_{i,j} a_i b_i \delta_{ij}\, \mathtt{id} + \sum_{i>j} (a_i a_j \partial_i \partial_j + b_i b_j \hat{\theta}_i \hat{\theta}_j) + \sum_{j>i} (a_i a_j \partial_i \partial_j + b_i b_j \hat{\theta}_i \hat{\theta}_j) \\
&= \textstyle\sum_i a_i b_i\, \mathtt{id} + \sum_{i>j} (a_i a_j \partial_i \partial_j + b_i b_j \hat{\theta}_i \hat{\theta}_j) - \sum_{i>j} (a_i a_j \partial_i \partial_j + b_i b_j \hat{\theta}_i \hat{\theta}_j) \\
&= (\mathbf{a} \cdot \mathbf{b}) \cdot \mathtt{id}
\end{aligned}$$

Koszul matrix factorizations enjoy some well known identities, which we include here for completeness. The first result is known as "change of basis".

Lemma 5.3.5. *Let* $\mathbf{a} = (a_1, \ldots, a_n)$ *and* $\mathbf{b} = (b_1, \ldots, b_n)$ *be sequences of elements in* $\mathbb{C}[\mathbf{x}, \mathbf{y}]$. *Let* $(\mathbf{a}, \mathbf{b})$ *be the associated Koszul matrix factorization. The change of basis transformation*

$$\begin{pmatrix} a_i & b_i \\ a_j & b_j \end{pmatrix} \to \begin{pmatrix} a_i & b_i + \lambda b_j \\ a_j - \lambda a_i & b_j \end{pmatrix},$$

where all other rows are fixed, yields isomorphic matrix factorizations for all $\lambda \colon \mathbb{C}[\mathbf{x}, \mathbf{y}]$. *Similarly, if* $\lambda \colon S$ *is invertible, then* $(a_i\ b_i) \cong (\lambda a_i\ \lambda \colon v b_i)$.

Proof. The first isomorphism is given by the maps $f = \mathtt{id} - \lambda \partial_j \hat{\theta}_i$ and $g = \mathtt{id} + \lambda \partial_j \hat{\theta}_i$, which can be checked directly to be matrix factorization morphisms of correct type. The second isomorphism is given by the matrix factorization morphims $\lambda \partial_i \hat{\theta}_i + \hat{\theta}_i \partial_i$

and $\lambda^{-1}\partial_i\hat{\theta}_i + \hat{\theta}_i\partial_i$. $\square$

The next result is Theorem 2.1 in [KR07].

Theorem 5.3.6. *Suppose $a_1, \ldots, a_n \colon R$ forms a regular sequence and suppose that $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{b}'$ for some $\mathbf{b}, \mathbf{b}' \colon R^{\oplus n}$. Then the following two Koszul matrix factorizations are isomorphic.*

$$(\mathbf{a}\ \mathbf{b})_R \cong (\mathbf{a}\ \mathbf{b}')_R.$$

The proof of this result follows from the fact that, since $\mathbf{a}$ is a regular sequence, $b_n - b'_n \colon (a_1, \ldots, a_n)R$. One then proceeds with a sequence of change of bases transformations.

Finally, we introduce the notion of "excluding a variable". When R is a polynomial ring, we can simplify Koszul matrix factorizations by canceling rows of the form $(0\ b_i)$ or $(a_i\ 0)$, subject to certain restrictions.

Lemma 5.3.7. *Suppose $(\mathbf{a}\ \mathbf{b})$ is a Koszul matrix factorization with potential $w = \mathbf{a}\cdot\mathbf{b}$. Let y be a generator of S and write $S = S'[y]$. Assume $z \colon S'$ and that one of the rows of $(\mathbf{a}\ \mathbf{b})$ has the form $(0\ y - p)$ for $p \colon S'$. Then $(\mathbf{a}\ \mathbf{b})_S$ is homotopy equivalent to the matrix factorization of potential $\mathbf{a}\cdot\mathbf{b}$ over S', where the row $(0\ y - p)$ is removed, and p is substituted everywhere for y in all other rows.*

We omit the proof of this lemma in this text and refer the reader to [Ras15] for a detailed proof. However, we will prove a special case of this lemma (Lemma 5.3.27) and compute the explicit maps in the homotopy equivalence for later use.

5.3.4 Smooth Resolutions for MOY Graphs

In this section we recall facts about the specific matrix factorizations used in the construction of $\mathfrak{sl}_N$ link homology by Khovanov and Rozansky in [KR08]. We will then introduce the virtual crossing expansion of the MOY wide edge graph which was introduced in [KR07]. After introducing a useful calculus for working with constituent maps of matrix factorizations, we prove results about certain homotopy equivalences and the differentials appearing in our construction.

Definition 5.3.8. A *MOY graph* is an oriented 4-regular planar graph, possibly with incoming and outgoing edges, such that each vertex has the local orientation as shown below in Figure 5.1 (up to rotation in the plane). We consider the endpoints of the graph as being degree 1 vertices, though we will often not include a vertex in the diagram for the endpoints.

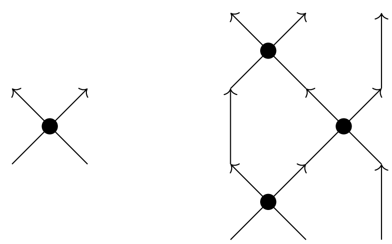


Figure 5.1: Local orientation of vertices in a MOY graph and an example of a MOY graph.

We also define a *braid-graph* to be a MOY graph formed by replacing every crossing in a braid diagram with a degree 4 vertex, such that the orientation of the graph is consistent with the original orientation of the braid. The MOY graph on the right-hand side of Figure 5.1 is an example of a braid-graph.

A *marked MOY graph*is a MOY graph Γ with markings (degree 2 vertices) such that the marks partition the graph into some combination of *elementary MOY graphs* as shown in Figure 5.2. We label the marks and the endpoints of the graph with variables. Typically, though not necessarily, we will label outgoing edges by variables y_i, incoming edges by variables x_i, and internal marks by variables t_i. An example of this process is given in Figure 5.2 for the braid-graph from Figure 5.1.

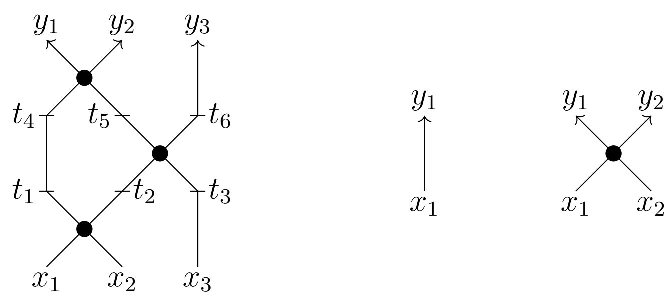


Figure 5.2: An example of a marked braid-graph and the elementary braid-graphs

Definition 5.3.9. Let Γ be a marked MOY graph, and suppose the outgoing endpoints are labeled by variables $\mathbf{y} = y_1, \ldots, y_n$, the incoming endpoints are labeled by $\mathbf{x} = x_1, \ldots, x_n$, and the internal marks are labeled by $\mathbf{t} = t_1, \ldots, t_k$. The *edge ring* $E(\Gamma)$ of Γ is defined as the polynomial ring $\mathbb{C}[\mathbf{x}, \mathbf{y}]$; whereas its *total ring* is defined as the polynomial ring $E^t(\Gamma) = \mathbb{C}[\mathbf{x}, \mathbf{y}, \mathbf{t}]$. We make these rings graded by assigning the q-grading $\deg_q(z_i) = 2$ for $z = x, y, t$ and all relevant i. A variable z is *internal* if it corresponds to a mark. More precisely, a variable z is internal if it appears in $E^t(\Gamma)$ but not $E(\Gamma)$.

For the rest of this section, we fix $N \geqslant 1$ and let $w(x) = x^{N+1}$, which we often shorthand by $w_x = x^{N+1}$. We then define some special polynomials which will appear in the sequel.

$$\Delta(y, x) = y - x \qquad \pi(y, x) = \frac{\Delta(w_y, w_x)}{\Delta(y, x)} \tag{5.5}$$

Note that we conceive of $\pi(x, y)$ as the resultant polynomial of the fraction as opposed to the fraction itself. This is important for the case of $\pi(x, x)$, which then equals $\dot{w}_x = (N+1)x^N$, the derivative of w_x with respect to x. In addition, we often employ the shorthand notations $\Delta_{yx} = \Delta(y, x)$ and $\pi_{yx} = \pi(y, x)$, or, alternatively, $\Delta_{ij} = \Delta(y_i, x_j)$ and $\pi_{ij} = \pi(y_i, x_j)$, choosing whichever is convenient and evident from context. Similarly, we occasionally use w_i and $\dot{w}_i$ for w_{x_i} and $\dot{w}_{x_i}$, respectively, in contexts where variation only occurs on the level of this subscript. We can generalize our various polynomial constructions via the following definition.

Definition 5.3.10. Given $\mathbf{x} = x_0, \ldots, x_n$, we define $S(\mathbf{x})$ to be the complete degree $N - n + 1$ homogeneous symmetric polynomial on $\mathbf{x}$.

Then $S(x) = w(x)$ and $S(x, y) = \pi(x, y)$, but we will continue to use the w and π notations to distinguish the roles each polynomial plays. The three variable case $S(x, y, z)$ will feature prominently in the sequel. We also note the following relation. Letting $\mathbf{x}_m^n = x_m, x_{m+1}, \ldots, x_{n-1}, x_n$, we have that

$$\frac{\Delta(S(\mathbf{x}_1^n), S(\mathbf{x}_0^{n-1}))}{\Delta(x_n, x_0)} = S(\mathbf{x}_0^n). \tag{5.6}$$

Suppose Γ is a marked MOY graph with outgoing and incoming endpoints labeled by the variables $\mathbf{y} = y_1, \ldots, y_n$ and $\mathbf{x} = x_1, \ldots, x_n$, respectively. We will define the matrix factorization $[\![\Gamma]\!]$ as an object in $HMF(E(\Gamma), w(\Gamma))$, where

$$w(\Gamma) = \sum_{i=1}^{n} (w_{y_i} - w_{x_i}). \tag{5.7}$$

We will determine these matrix factorizations explicitly by first defining the matrix factorizations $[\![\uparrow]\!]$ and $[\![\times]\!]$ for basic braid-graphs and then applying rules for juxtaposing and gluing these graphs together. First we consider $\Gamma = \uparrow$, as labeled in Figure 5.2. In this case, $E(\uparrow) = \mathbb{C}[x_1, y_1]$ and $w(\uparrow) = w_{y_1} - w_{x_1}$. To this graph we associate the following Koszul matrix factorization.

$$[\![\uparrow]\!] = (\Delta_{11} \ \pi_{11})_{E^t(\uparrow)} \tag{5.8}$$

Since $\Delta_{11}\pi_{11} = w_{y_1} - w_{x_1} = \pi_{11}\Delta_{11}$, the potential of $[\![\uparrow]\!]$ is $w(\uparrow)$ as hoped. Note that in this scenario we have $E^t(\uparrow) = E(\uparrow)$. This will not always be the case, and the distinction is important: for a graph Γ, the a-homogeneous components of its associated matrix factorization are each isomorphic to shifted direct sums of $E^t(\Gamma)$—but are viewed as modules over $E(\Gamma)$!

Now consider the case of $\Gamma = \bullet\!\!\!\!\times$. In this case, $w(\Gamma) = w_{y_1} + w_{y_2} - w_{x_1} - w_{x_2}$ and $E(\Gamma) = \mathbb{C}[x_1, x_2, y_1, y_2]$. We define $[\![\bullet\!\!\!\!\times]\!]$ as follows.

$$[\![\bullet\!\!\!\!\times]\!](1) = \begin{pmatrix} y_1 + y_2 - x_1 - x_2 & u_1 \\ y_1 y_2 - x_1 x_2 & u_2 \end{pmatrix}_{E^t(\bullet\!\!\!\!\times)} \tag{5.9}$$

Where u_1 and u_2 are polynomials in $E(\Gamma)$ such that

$$(y_1 + y_2 - x_1 - x_2)u_1 + (y_1 y_2 - x_1 x_2)u_2 = w(\bullet\!\!\!\!\times).$$

Remark 5.3.11. We note that as long as such polynomials u_1 and u_2 exist, then $[\![\bullet\!\!\!\!\times]\!]$ is well-defined up to homotopy equivalence by Theorem 5.3.6. To see why such polynomials exist it is enough to note that $w_v + w_z$ is a polynomial in $v + z$ and vz, and we can use this to explicitly write polynomials u_1 and u_2. In this text, we will never need explicit expressions for u_1 and u_2 so we shall omit this process. We welcome the interested reader to do this calculation on their own or see [KR08] for more details.

We can then rewrite the above two matrix factorizations in terms of the exterior algebra framework.

$$[\![\uparrow]\!] = (E^t(\uparrow) \otimes \bigwedge \mathbb{C}, \Delta\partial + \pi\hat{\theta})$$

$$[\![\times]\!](1) = (E^t(\times) \otimes \bigwedge \mathbb{C}^2, (y_1 + y_2 - x_1 - x_2)\partial_1 + (y_1 y_2 - x_1 x_2)\partial_2 + u_1\hat{\theta}_1 + u_2\hat{\theta}_2) \tag{5.10}$$

Now suppose Γ and Γ' are marked MOY graphs with respective edge rings E and E' and total rings E^t and E'^t. The disjoint union $\Gamma \sqcup \Gamma'$ of these graphs then has edge ring $E'' \cong E \otimes_{\mathbb{C}} E'$ and total ring $E''^t \cong E^t \otimes_{\mathbb{C}} E'^t$. To the marked MOY graph $\Gamma \sqcup \Gamma'$ we then associate the E''-matrix factorization

$$[\![\Gamma \sqcup \Gamma']\!] \triangleq [\![\Gamma]\!] \otimes_{\mathbb{C}} [\![\Gamma']\!]. \tag{5.11}$$

Note that $w(\Gamma \sqcup \Gamma') = w(\Gamma) \otimes 1 + 1 \otimes w(\Gamma')$ by our discussion in Section 5.3.3. A picture of the corresponding diagram is shown in Figure 5.3

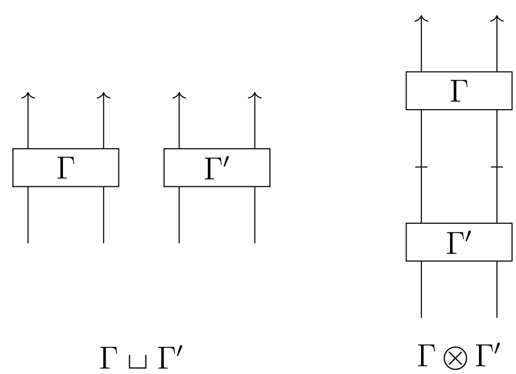


Figure 5.3: Examples of disjoint union and gluing of marked MOY graphs

Finally, we define a matrix factorization for when we glue two marked MOY graphs together. Let Γ and Γ' be two marked MOY graphs. We can glue outgoing edges of Γ to incoming edges of Γ' (or vice versa) to get a new marked MOY graph. First suppose only one endpoint from each graph are being glued together. Suppose thatendpoint in both Γ and Γ' is labeled by the variable$z$ (that is, $z\colon E^i(\Gamma) \cap E^o(\Gamma')$

or $z\colon E^i(\Gamma') \cap E^o(\Gamma)$). Then we define the new graph $\Gamma \cup_z \Gamma'$ by identifying the endpoints labeled by z and associate to $\Gamma \cup_z \Gamma'$ the matrix factorization

$$[\![\Gamma \cup_z \Gamma']\!] \triangleq [\![\Gamma]\!] \otimes_{\mathbb{C}[z]} [\![\Gamma']\!] \tag{5.12}$$

The edge ring of $\Gamma \cup_z \Gamma'$ is $E(\Gamma \cup_z \Gamma') = (E(\Gamma) \otimes_{\mathbb{C}[z]} E(\Gamma'))/(z)$ and the total ring is $E^t(\Gamma \cup_z \Gamma') = E^t(\Gamma) \otimes_{\mathbb{C}[z]} E^t(\Gamma')$. Note that after gluing, z is no longer in the edge ring as it is an internal variable. We may glue multiple edges at once in a similar manner. If $\mathbf{z} = z_1, \ldots, z_n$ are the variables at the marked endpoints being identified, then we define

$$[\![\Gamma \cup_{\mathbf{z}} \Gamma']\!] \triangleq [\![\Gamma]\!] \otimes_{\mathbb{C}[\mathbf{z}]} [\![\Gamma']\!]. \tag{5.13}$$

Similar to the case where we only identified one pair of edges, the edge ring of $\Gamma \cup_{\mathbf{z}} \Gamma'$ is $E(\Gamma \cup_{\mathbf{z}} \Gamma') = (E(\Gamma) \otimes_{\mathbb{C}[\mathbf{z}]} E(\Gamma'))/(z_1, \ldots, z_n)$ and the total ring is $E^t(\Gamma \cup_{\mathbf{z}} \Gamma') = E^t(\Gamma) \otimes_{\mathbb{C}[\mathbf{z}]} E^t(\Gamma')$.

Note that $w(\Gamma \otimes \Gamma') = w(\Gamma) \otimes 1 + 1 \otimes w(\Gamma')$ by the discussion at the end of Section 5.3.3.

It will be useful to describe general disjoint unions and gluings of marked MOY graphs in terms of Koszul matrix factorizations. Suppose $[\![\Gamma]\!]$ and $[\![\Gamma']\!]$ are given by the Koszul complexes

$$[\![\Gamma]\!] = \begin{pmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_m & b_m \end{pmatrix}_{E^t(\Gamma)} \qquad [\![\Gamma']\!] = \begin{pmatrix} a'_1 & b'_1 \\ \vdots & \vdots \\ a'_m & b'_m \end{pmatrix}_{E^t(\Gamma')}$$

We can then present $[\![\Gamma \sqcup \Gamma']\!]$ and $[\![\Gamma \otimes \Gamma']\!]$ as the following Koszul complexes:

$$[\![\Gamma \sqcup \Gamma']\!] = \begin{pmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_m & b_m \\ a'_1 & b'_1 \\ \vdots & \vdots \\ a'_n & b'_n \end{pmatrix}_{E^t(\Gamma \sqcup \Gamma')} \qquad [\![\Gamma \otimes \Gamma']\!] = \begin{pmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_m & b_m \\ a'_1 & b'_1 \\ \vdots & \vdots \\ a'_n & b'_n \end{pmatrix}_{E^t(\Gamma \otimes \Gamma')} \tag{5.14}$$

Here the distinction comes from the difference in total and edge rings. $[\![\Gamma \sqcup \Gamma']\!]$ is a matrix factorization over $E(\Gamma \sqcup \Gamma')$, but $[\![\Gamma \otimes \Gamma']\!]$ is a matrix factorization over $E(\Gamma \otimes \Gamma')$. As an immediate application, we can define the matrix factorization associated to two arcs $\uparrow\,\uparrow$ as

$$[\![\uparrow\,\uparrow]\!] = \begin{pmatrix} \Delta_{11} & \pi_{11} \\ \Delta_{22} & \pi_{22} \end{pmatrix}_{E^t(\uparrow\,\uparrow)} \tag{5.15}$$

Or equivalently, in exterior algebra notation, as

$$[\![\uparrow\,\uparrow]\!] = (E^t(\uparrow\,\uparrow) \otimes \bigwedge \mathbb{C}^2, \Delta_{11}\partial_1 + \Delta_{22}\partial_2 + \pi_{11}\hat{\theta}_1 + \pi_{22}\hat{\theta}_2) \tag{5.16}$$

Lemma 5.3.7 allows us to freely add or remove marks without changing the homotopy type of the matrix factorization (as a $E(\Gamma)$-matrix factorization). This implies the following very useful statement which we now record.

Corollary 5.3.12. *Let Γ and Γ' be two marked MOY graphs whose underlying (unmarked) MOY graphs are the same (isomorphic as oriented graphs rel boundary). Then $[\![\Gamma]\!] \simeq [\![\Gamma']\!]$ as matrix factorizations over the ring $E(\Gamma) = E(\Gamma')$.*

Example 5.3.13. *Consider the marked MOY graph (braid-graph) from Figure 5.4. The marks partition the MOY graph into four elementary MOY graphs (two degree 4 vertices and two arcs) which are drawn below in Figure 5.4.*

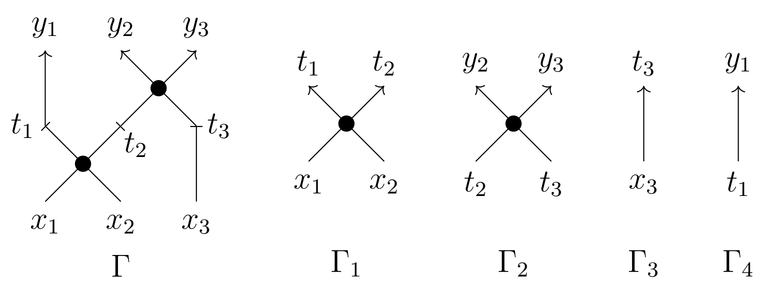


Figure 5.4: The marked braid-graph in Example 5.3.13 and its elementary braid–graphs

We can write Γ *as* $(\Gamma_1 \sqcup \Gamma_3) \otimes (\Gamma_2 \sqcup \Gamma_4)$ *and therefore*

$$[\![\Gamma]\!] = ([\![\Gamma_1]\!] \otimes_{\mathbb{C}} [\![\Gamma_3]\!]) \otimes_{\mathbb{C}[t_1,t_2,t_3]} ([\![\Gamma_2]\!] \otimes_{\mathbb{C}} [\![\Gamma_4]\!]).$$

We can write $[\![\Gamma]\!]$*, after some applications of mark removal to remove* t_1 *and* t_3*, as*

$$[\![\Gamma]\!] \simeq \begin{pmatrix} y_1 + t_2 - x_1 - x_2 & * \\ y_1 t_2 - x_1 x_2 & * \\ y_2 + y_3 - t_2 - x_3 & * \\ y_2 y_3 - t_2 x_3 & * \end{pmatrix}_{\mathbb{C}[x_1,x_2,x_3,y_1,y_2,y_3,t_2]}$$

Note the elements $*$ *are determined as discussed in Remark 5.3.11. We invite the reader to finish the process of removing the internal variable* t_2*.*

Many of the examples we will work with in detail involve braid-graphs which arise from resolving the crossings in braid closure presentations of links. Along these lines, we introduce the notion of "closure" of a braid-graph. The diagrammatic form of this process is shown in Figure 5.5.

Definition 5.3.14. Let Γ be a marked braid-graph with outgoing variables $y_1, \ldots, y_n$ and incoming variables $x_1, \ldots, x_n$. The *closure* of Γ, denoted by $\overline{\Gamma}$, is the graph formed by identifying the vertex for x_i with y_i for all i. We leave a mark on the corresponding edge for the variable x_i.

Let $\phi : E(\Gamma) \to E(\Gamma)/(\Delta_{ii})$ be the quotient map. The matrix factorization associated to $\overline{\Gamma}$ is defined to be the $\mathbb{C}$-matrix factorization $[\![\overline{\Gamma}]\!] = \phi([\![G]\!])$.

We can also define the notation of a *partial closure* of Γ, which we will denote by $\overline{\Gamma}^{(i)}$. $\overline{\Gamma}^{(i)}$ is the graph formed by identifying the vertex for x_i with the vertex for y_i. As before we will leave a mark on the corresponding edge.

Note that the edge ring of $\overline{\Gamma}$ is simply $\mathbb{C}$ and all remaining variables are now internal. Furthermore, since $w(\Gamma) = \sum_i (w_{y_i} - w_{x_i})$, then the potential of $\overline{\Gamma}$ is $\sum_i (w_{x_i} - w_{x_i}) = 0$. Therefore, $[\![\overline{\Gamma}]\!] \colon HMF(\mathbb{C}, 0)$, or in other words $[\![\overline{\Gamma}]\!]$ is a $\mathbb{Z}_2$-graded (or 2-periodic) chain complex of $\mathbb{C}$-vector spaces. In the next section, we will discuss how closures can be used to study morphisms between matrix factorizations of braid-graphs.

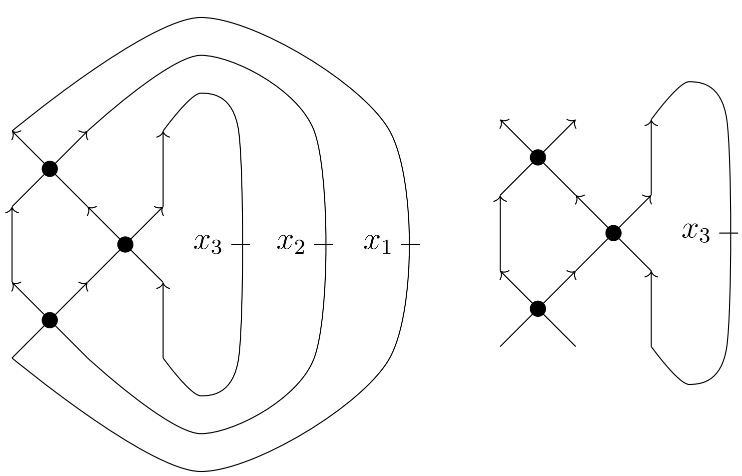


Figure 5.5: The closure of a braid-graph $\overline{\Gamma}$ and a partial closure $\overline{\Gamma}^{(3)}$.

In the following proposition we recall some relations on MOY graphs. We will not use all of these relations in this text, but include them for completeness.

Proposition 5.3.15. *Let* $\Gamma_0, \Gamma_{1a}, \Gamma_{1b}, \Gamma_{2a}, \Gamma_{2b}, \Gamma_{3a}, \Gamma_{3b}, \Gamma_{3c}, \Gamma_{3d}, \Gamma_{4a}, \Gamma_{4b},$ *and* Γ_{4c} *be the MOY graphs as in Figure 5.6 below. Also consider* $\mathbb{C}$: $HMF(\mathbb{C}, 0)$ *as the matrix factorization with a single copy of* $\mathbb{C}$ *in degree* 0 *and zero differential. Then*

$$
\begin{aligned}
[\![\Gamma_0]\!] &\simeq \bigoplus_{i=0}^{N-1} \mathbb{C}(2i+1-N)[1] \\
[\![\Gamma_{1a}]\!] &\simeq \bigoplus_{i=0}^{N-2} [\![\Gamma_{1b}]\!](2i-N) \\
[\![\Gamma_{2a}]\!] &\simeq [\![\Gamma_{2b}]\!](-1) \oplus [\![\Gamma_{2b}]\!](1) \\
[\![\Gamma_{3a}]\!] \oplus [\![\Gamma_{3b}]\!] &\simeq [\![\Gamma_{3c}]\!] \oplus [\![\Gamma_{3d}]\!] \\
[\![\Gamma_{4a}]\!] &\simeq [\![\Gamma_{4b}]\!] \oplus \bigoplus_{i=0}^{N-3} [\![\Gamma_{4c}]\!](2i-N+1)
\end{aligned}
$$

where $\simeq$ *denotes homotopy equivalence in the appropriate category of matrix factorizations.*

We refer the reader to [KR08] for a precise proof of the isomorphisms. We will present an alternate proof of the first isomorphism in Section 5.3.5

In this section we introduce the notion of a "virtual crossing", which we denote by $\boxtimes$ in this framework. We then recall a result from [KR07] on the relationship among $\uparrow\uparrow$, $\boxtimes$, and $\times$.

To a virtual crossing whose incoming edges are labeled by x_1 and x_2 and outgoing edges are labeled by y_1 and y_2 (see Figure 5.7) we associate the matrix factorization

$$
[\![\boxtimes]\!] = \begin{pmatrix} \Delta_{12} & \pi_{12} \\ \Delta_{21} & \pi_{21} \end{pmatrix}_{E^t(\boxtimes)} = (E^t(\boxtimes) \otimes \textstyle\bigwedge \mathbb{C}^2, \Delta_{12}\partial_1 + \Delta_{21}\partial_2 + \pi_{12}\hat{\theta}_1 + \pi_{21}\hat{\theta}_2). \quad (5.17)
$$

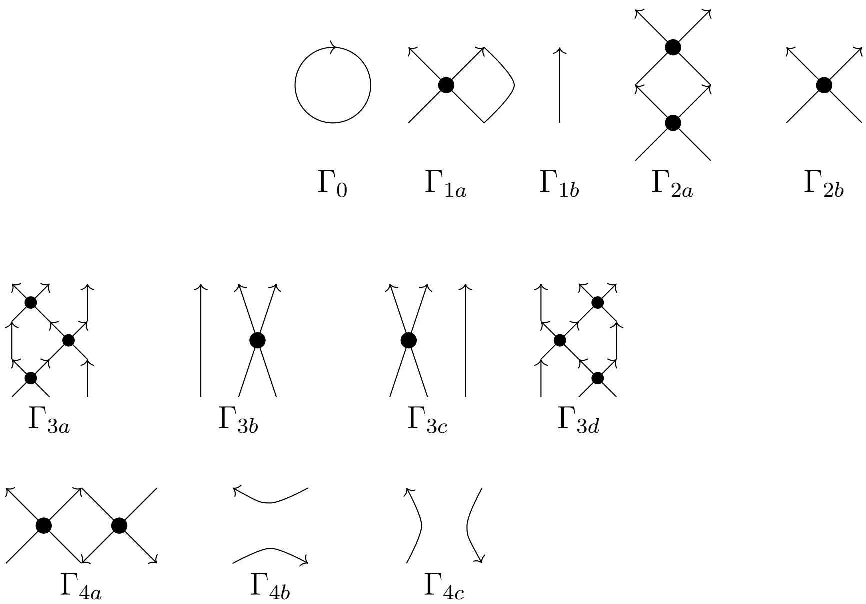


Figure 5.6: MOY graphs for Proposition 5.3.15

Note that the total ring $E^t(\otimes)$ is defined in the same manner as it was for braid-graphs before. We can combine virtual crossings and MOY graphs to form *virtual MOY graphs* in the obvious manner. Equivalently, a virtual MOY graph is a MOY graph where we relax the planarity condition and assume the only self-intersections of the projection onto the plane are transverse double points. These points are the virtual crossings. We define a *virtual braid-graph* to be a braid-graph where the vertices are possibly replaced by virtual crossings.

The definition of $\llbracket \otimes \rrbracket$ is the same as the definition for $\llbracket \uparrow\ \uparrow \rrbracket$, except for a permutation $x_1 \leftrightarrow x_2$. This hints at a relation between virtual crossings and permutations which we now expand upon.

Definition 5.3.16. Let $\sigma\colon S_n$ be a permutation, $\mathbf{x} = x_1, \ldots, x_n$ and $\mathbf{y} = y_1, \ldots, y_n$.

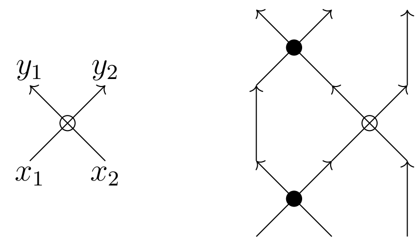


Figure 5.7: A (marked) virtual crossing and an example of a virtual MOY graph

We define the matrix factorization

$$[\![\sigma]\!] = \begin{pmatrix} \Delta_{1,\sigma(1)} & \pi_{1,\sigma(1)} \\ \vdots & \vdots \\ \Delta_{n,\sigma(n)} & \pi_{n,\sigma(n)} \end{pmatrix}_{\mathbb{C}[\mathbf{x},\mathbf{y}]} .$$

When we wish to record the sets of variables being used, we will use the notation ${}_{\mathbf{y}}[\![\sigma]\!]_{\mathbf{x}}$.

Letting s_i denote the simple transposition $(i, i+1)$, $[\![s_i]\!]$ is the matrix factorization of the virtual braid graph with a virtual crossing between the ith and $(i+1)$st strands and vertical strands elsewhere. The following lemma is a direct application of change of basis and mark removal.

Lemma 5.3.17. *Let s_i and s_j be two simple transpositions in S_n, then $[\![s_i]\!] \otimes [\![s_j]\!] \simeq [\![s_i s_j]\!]$.*

We do not include a proof of Lemma 5.3.17 for brevity, however we do give an illuminating example. Suppose $s_1, s_2 \colon S_3$. Then consider ${}_{\mathbf{y}}[\![s_1]\!]_{\mathbf{t}} \otimes_{\mathbb{C}[\mathbf{t}]} [\![s_2]\!]_{\mathbf{x}}$. This is the $\mathbb{C}[\mathbf{x}, \mathbf{y}]$-matrix factorization given by

$$\begin{pmatrix} \Delta_{y_1,t_2} & \pi_{y_1,t_2} \\ \Delta_{y_2,t_1} & \pi_{y_2,t_1} \\ \Delta_{y_3,t_3} & \pi_{y_3,t_3} \\ \Delta_{t_1,x_1} & \pi_{t_1,x_1} \\ \Delta_{t_2,x_3} & \pi_{t_2,x_3} \\ \Delta_{t_3,x_2} & \pi_{t_3,x_2} \end{pmatrix}_{\mathbb{C}[\mathbf{x},\mathbf{y},\mathbf{t}]}.$$

We can perform mark removal (Lemma 5.3.7) on the bottom three rows, setting $t_1 = x_1, t_2 = x_3, t_3 = x_2$ and removing the bottom three rows to get the equivalent matrix factorization

$$\begin{pmatrix} \Delta_{y_1,x_3} & \pi_{y_1,x_3} \\ \Delta_{y_2,x_1} & \pi_{y_2,x_1} \\ \Delta_{y_3,x_2} & \pi_{y_3,x_2} \end{pmatrix}_{\mathbb{C}[\mathbf{x},\mathbf{y}]}.$$

This matrix factorization is by definition equal to $[\![s_1 s_2]\!]$ as expected.

Proposition 5.3.18. *The following homotopy equivalences hold:*

1. $[\![\overline{\sigma s_n}^{(n+1)}]\!] \simeq [\![\sigma]\!]$.

2. $[\![s_i]\!] \otimes [\![s_i]\!] \simeq [\![1_n]\!] = \upharpoonleft \sqcup \cdots \sqcup \upharpoonleft$.

3. $[\![s_i]\!] \otimes [\![s_{i+1}]\!] \otimes [\![s_i]\!] \simeq [\![s_i s_{i+1} s_i]\!]$.

Here $\sigma \colon S_n$ and σs_n is considered as a permutation in S_{n+1}.

Proof. Isomorphisms (2) and (3) are immediate corollaries of Lemma 5.3.17. For the first isomorphism, apply Lemma 5.3.7 to the mark associated to the variable x_{n+1}. □

A graphical version of Proposition 5.3.18 is given in Figure 5.8.

Remark 5.3.19. We can interpret (2) and (3) of Proposition 5.3.18 in two ways. First is that virtual crossings give a categorical S_n-action on $HMF(\mathbb{C}[\mathbf{x},\mathbf{y}],w)$, where w is defined as in (5.7), given by $s_i \cdot M = [\![s_i]\!] \otimes M$.

Second, the full subcategory of $HMF(\mathbb{C}[\mathbf{x},\mathbf{y}],w)$ generated by $[\![s_i]\!]$ up to $\otimes, \oplus$ and grading shifts is a categorification of $\mathbb{Z}[q^{\pm 1}][S_n]$. More precisely the Groethendieck ring of this tensor category is $\mathbb{Z}[q^{\pm 1}][S_n]$.

Remark 5.3.20. The isomorphisms in Proposition 5.3.18 for any allowed choice of orientations on the underlying graphs. From the virtual knot theory point of view, these are three of the *virtual Reidemeister moves*. We omit the proofs for these extra cases, but they follow easily from definitions and via mark removal.

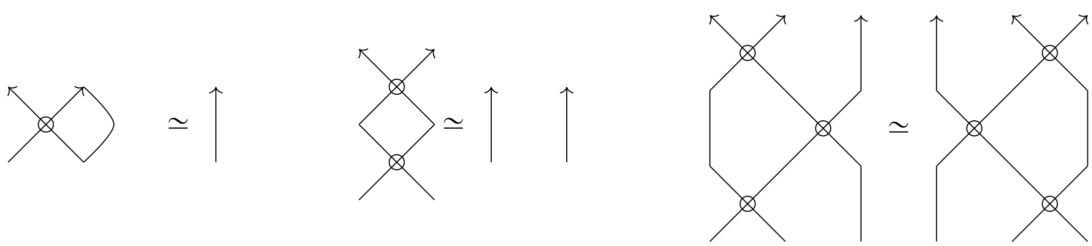


Figure 5.8: Graphical version of Proposition 5.3.18

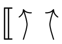
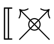
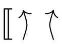
We now work toward describing the relationship among the three graphs , , and . First we need to define a new method to combine matrix factorizations.

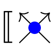
Definition 5.3.21. Let $M, N\colon HMF(R,w)$ and $F\colon Hom(M,N)$. The mapping cone of F, denoted by Cone(F), is the matrix factorization whose underlying module is $M \oplus N[1]$ and differential given by

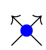
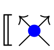
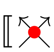
$$d_{\mathrm{Cone}(F)} = \begin{bmatrix} d_M & 0 \\ F & -d_N \end{bmatrix}.$$

Likewise, we can define the mapping cone of a map $F\colon \mathrm{Ext}(M, N)$. The underlying module is $M \oplus N$ and the differential is

$$d_{\mathrm{Cone}(F)} = \begin{bmatrix} d_M & 0 \\ F & d_N \end{bmatrix}.$$

Proposition 5.3.22. *There exists a unique map* $F\colon \mathrm{Ext}(\llbracket$  $\rrbracket, \llbracket$  $\rrbracket)$ *of degree* $1 - N$*, up to rescaling, such that* $\mathrm{Cone}(F) \simeq \llbracket$  $\rrbracket$*. Likewise there exists a unique map, up to rescaling,* $G\colon \mathrm{Ext}(\llbracket$  $\rrbracket, \llbracket$  $\rrbracket)$ *such that* $\mathrm{Cone}(G)$ *is homotopy equivalent to* $\llbracket$  $\rrbracket$.

We will call maps of the form F and G *virtual saddle* maps. We will distinguish these mapping cone presentations in the following manner. We will write $\llbracket$  $\rrbracket$ for $\mathrm{Cone}(G)$ and $\llbracket$  $\rrbracket$ for $\mathrm{Cone}(F)$ from Proposition 5.3.22. Along these lines we can define "signed virtual MOY graphs". A *signed virtual MOY graph* is a virtual MOY graph where each vertex is colored red or blue.

Suppose Γ is a signed virtual MOY graph marked so that it is partitioned into graphs of the form ,  and , then we can write $\llbracket\Gamma\rrbracket$ as an iterated mapping cone or a convolution. To each vertex colored blue, we associate $\llbracket$  $\rrbracket$, and to each MOY vertex colored red, we associate $\llbracket$  $\rrbracket$. By Proposition 5.3.22 this gives a homotopy equivalent presentation for $\llbracket\Gamma\rrbracket$. The choices of colors on vertices will be important in §5.3.6 in defining the complex for a virtual link diagram.

Proposition 5.3.22 is proven in [KR07]. We will not repeat the proof of Proposition 5.3.22 here, but only mention some relevant details to our discussion in the sequel. In particular, the fact that the maps F and G are unique up to rescaling follows from the following lemma.

Lemma 5.3.23. $\mathrm{Ext}(\llbracket \uparrow\uparrow \rrbracket, \llbracket \times \rrbracket) \cong \llbracket \overline{\times} \rrbracket \cong \mathbb{C}[x]/\langle x^N \rangle$. *In particular,*

$$\dim_{\mathbb{C}} \mathrm{Ext}(\llbracket \uparrow\uparrow \rrbracket, \llbracket \times \rrbracket)_{N-1} = 1.$$

The analogous statement holds if the roles of $\uparrow\uparrow$ *and* $\times$ *are switched.*

Proof. First recall from Proposition 5.3.2 that $\mathrm{Ext}(\llbracket \uparrow\uparrow \rrbracket, \llbracket \times \rrbracket) \cong H^1(\llbracket \times \rrbracket \otimes \llbracket \uparrow\uparrow \rrbracket^*)$ where for any Koszul matrix factorization $(\mathbf{a}, \mathbf{b})^* = (\mathbf{a}, \mathbf{b})[|\mathbf{a}|]$. One can directly compute (or use Proposition 5.3.15 and 5.3.18) to show that

$$\llbracket \times \rrbracket \otimes \llbracket \uparrow\uparrow \rrbracket^* \simeq \llbracket \overline{\times} \rrbracket \simeq \llbracket \overline{\uparrow} \rrbracket \simeq \mathbb{C}[x]/\langle x^N \rangle (N-1).$$

□

We now give an explicit presentation of the virtual saddle map $F : \llbracket \uparrow\uparrow \rrbracket \to \llbracket \times \rrbracket$ in terms of the exterior algebra. It will suffice to find an element of $\mathrm{Ext}(\llbracket \uparrow\uparrow \rrbracket, \llbracket \times \rrbracket)$ of the right degree which is not null-homotopic. We set

$$F = \partial_1 - \partial_2 + P_{12}\hat{\theta}_1 - P_{21}\hat{\theta}_2, \tag{5.18}$$

where $P_{ij} = (\pi_{ii} - \pi_{ij})(\Delta_{ii} - \Delta_{ij})^{-1}$. One can check that F is not null-homotopic and of the proper q-degree. Therefore, up to a scalar, it is homotopic to the virtual saddle introduced in [KR07] and $\mathrm{Cone}(F) \simeq \llbracket \times_\bullet \rrbracket$ as claimed.

The presentation of $G\colon \mathrm{Ext}(\llbracket \times \rrbracket, \llbracket \uparrow\uparrow \rrbracket)$ is actually the same as the presentation of F in the exterior algebra. That is $G = \partial_1 - \partial_2 + P_{12}\hat{\theta}_1 - P_{21}\hat{\theta}_2$. With this in mind we define the following map.

Definition 5.3.24. Let $\sigma\colon S_n$ and define $P^{\sigma}_{ij} = (\pi_{i\sigma(i)} - \pi_{i\sigma(j)})(\Delta_{i\sigma(i)} - \Delta_{i\sigma(j)})^{-1}$. The map $\Sigma_{ij}\colon \mathrm{Ext}(\llbracket \sigma \rrbracket, \llbracket (i\,j)\sigma \rrbracket)$ is the degree $N-1$ morphism presented in the exterior algebra by

$$\Sigma_{ij} = \partial_i - \partial_j + P^{\sigma}_{ij}\hat{\theta}_i - P^{\sigma}_{ji}\hat{\theta}_j. \tag{5.19}$$

We call the map Σ_{ij} a *virtual saddle map.* In the case we need to specify the domain of Σ_{ij} we write Σ^{σ}_{ij}

We can view the map Σ_{ij} as a map in $\mathrm{Ext}([\![1]\!] \otimes [\![\sigma]\!], [\![(i\,j)]\!] \otimes [\![\sigma]\!])$. However, we could have also defined $\Sigma_{ij}\colon \mathrm{Ext}([\![1]\!], [\![(i\,j)]\!])$ in a similar manner and then considered the map $\Sigma'_{ij} = \Sigma_{ij} \otimes \mathbf{1}\colon \mathrm{Ext}([\![1]\!] \otimes [\![\sigma]\!], [\![(i\,j)]\!] \otimes [\![\sigma]\!])$. Fortunately, this choice does not matter up to homotopy.

Lemma 5.3.25. *Consider the Koszul matrix factorization $M = (a\ b)$, and let $Hatp$ denote the endomorphism of M given by multiplication by $p\colon \mathbb{C}[\mathbf{x}, \mathbf{y}]$. Then $Hata \sim Hatb \sim 0$.*

Proof. Recall the differential of $(a\ b)$ has the form $d = a\partial + b\hat{\theta}$. Direct computation shows that $[d, \hat{\theta}] = a$ and $[d, \partial] = b$, proving the lemma. □

Proposition 5.3.26. *The maps Σ_{ij} and Σ'_{ij} are homotopy equivalent as elements of* $\mathrm{Ext}([\![\sigma]\!], [\![(i\,j)\sigma]\!])$.

Proof. Consider ${}_{\mathbf{y}}[\![\sigma]\!]_{\mathbf{x}} \simeq {}_{\mathbf{y}}[\![1]\!]_{\mathbf{t}} \otimes {}_{\mathbf{t}}[\![\sigma]\!]_{\mathbf{x}}$ and ${}_{\mathbf{y}}[\![(i\,j)\sigma]\!]_{\mathbf{x}} \simeq {}_{\mathbf{y}}[\![(i\,j)]\!]_{\mathbf{t}} \otimes {}_{\mathbf{t}}[\![\sigma]\!]_{\mathbf{x}}$. We first write Σ'_{ij} as $\partial_i - \partial_j + P'_{ij}\hat{\theta}_i - P'_{ji}\hat{\theta}_j$, where

$$P'_{ij} = (\pi(y_i, t_i) - \pi(y_i, t_j))(\Delta(y_i, t_i) - \Delta(y_i, t_j))^{-1}.$$

However, $Hatt_i \sim Hatx_{\sigma(i)}$ as elements of $Hom([\![\sigma]\!], [\![\sigma]\!])$ by Lemma 5.3.25. Therefore the result follows. □

5.3.5 Smooth Resolution Meta-Cube

We now turn our attention to the problem of mark removal in the context of matrix factorizations assigned to braid graphs and their closures. We will use the exterior algebra formalism to show that mark removal amounts to a strong deformation retract. We will orient our analysis around the two basic cases: that of a simple arc $\uparrow$ with one incoming variable x, one outgoing variable y, and a single internal variable t; and that of a marked circle $\circlearrowleft$ with one internal variable x. The latter case is the closure of a simple arc $\uparrow$ with one incoming variable x and one outgoing variable y, which become identified, without loss of generality, to a single internal variable x. In practice, the former case will help us reduce the number of marks on a component from n to 1, while the latter will help us eliminate the sole remaining mark. We henceforth refer to the first case as *standard mark removal* and second case as *final mark removal*.

We begin our analysis by writing the Koszul matrix factorizations $[\![\uparrow]\!]$ and $[\![\circlearrowleft]\!]$.

$$\begin{aligned}
[\![\uparrow]\!] &= \begin{pmatrix} \Delta_{yt} & \pi_{yt} \\ \Delta_{tx} & \pi_{tx} \end{pmatrix}_{E^t(\uparrow)} \\
&\cong (E^t(\uparrow) \otimes \bigwedge \mathbb{C}^2, \Delta_{yt}\partial_1 + \Delta_{tx}\partial_2 + \pi_{yt}\hat{\theta}_1 + \pi_{tx}\hat{\theta}_2)
\end{aligned}$$

$$\begin{aligned}
[\![\circlearrowleft]\!] &= \begin{pmatrix} 0 & \dot{w}_x \end{pmatrix}_{E^t(\circlearrowleft)} \\
&\cong (E^t(\circlearrowleft) \otimes \bigwedge \mathbb{C}, \dot{w}_x\hat{\theta})
\end{aligned}$$

We remind the reader that although these are matrix factorizations whose a-graded pieces are shifted direct sums of the *total rings* $E^t(\Gamma)$, that these are considered as modules over the *edge rings* $E(\Gamma)$. This is a crucial fact since our deformation retracts will involve morphisms that are $E(\Gamma)$-linear but *not* $E^t(\Gamma)$-linear.

In both cases, we have a ring isomorphism $E^t(\Gamma) \cong E(\Gamma)[z]$ for a variable z corresponding to the situation's internal mark. Given a polynomial $p(z)\colon E(\Gamma)[z]$, we then have a decomposition of $E(\Gamma)$-modules $E^t(\Gamma) \cong E^t(\Gamma)/\langle p\rangle \oplus E(\Gamma)\langle p\rangle$. We denote the projection onto the first component, which precisely kills the ideal $\langle p\rangle$, by ϵ_p and call it the *evaluation* map. The name is due to the fact that, if $p = \Delta_{zv}$, for an edge variable v, the map is given by evaluating z to v. Note that in this case we also have $E^t(\Gamma)/\langle\Delta_{zv}\rangle \cong E(\Gamma)$.

We will also need another morphism, which arises from the map $E(\Gamma)\langle p\rangle \to E^t(\Gamma)$ given by dividing by p. We define the *division map* ρ_p as its precomposition with the projection $1 - \epsilon_p$ onto $E^t(\Gamma)\langle p\rangle$.

$$E^t(\Gamma) \xrightarrow{1-\epsilon_p} E(\Gamma)\langle p\rangle \xrightarrow{\cdot(p)^{-1}} E^t(\Gamma).$$

This map divides polynomials in the ideal $E(\Gamma)\langle p\rangle$ by p and sends all other polynomials to zero. It has the algebraic expression $\rho_p = p^{-1}(1 - \epsilon_p)$, which can be reformulated as the commutator relation $\epsilon_p = [\rho_p, p]$. When evident from context, we will suppress the Δ from the subscripts for ϵ and q; i.e. when $p = \Delta_s$, for some subscript s, we write ϵ_s and ρ_s instead of ϵ_{Δ_s} and ρ_{Δ_s}.

We now specialize to the first case, that of ⤉, for which we have $E^t(\text{⤉}) = \mathbb{C}[x, t, y]$ and $E(\text{⤉}) = \mathbb{C}[x, y]$. Letting $p = \Delta_{tx}$, we have the decomposition of $E(\text{⤉})$-modules:

$$E^t(\text{⤉}) \cong E(\text{⤉}) \oplus E(\text{⤉})\langle\Delta_{tx}\rangle,$$

with evaluation map ϵ_{tx} the projection $E^t(\text{⤉}) \to E(\text{⤉})$ and division map ρ_{tx} :

$E^t(\uparrow\!\!\!\!-) \to E^t(\uparrow\!\!\!\!-)$.

Recalling Definition 5.3.10, we now define the deformation retract taking $[\![\uparrow\!\!\!\!-]\!]$ to $[\![\uparrow]\!]$.

Lemma 5.3.27. *The triple* $(\Psi : [\![\uparrow\!\!\!\!-]\!] \to [\![\uparrow]\!], \Psi' : [\![\uparrow]\!] \to [\![\uparrow\!\!\!\!-]\!], \psi : [\![\uparrow\!\!\!\!-]\!] \to [\![\uparrow\!\!\!\!-]\!])$ *given by*

- $\Psi = \epsilon_{tx}\partial_2\hat{\theta}_2$
- $\Psi' = \mathbf{1} - \partial_1\hat{\theta}_2 + S(y,t,x)\hat{\theta}_1\hat{\theta}_2$
- $\psi = -\rho_{tx}\hat{\theta}_2$

constitutes data for a strong deformation retract.

Proof. This amounts to showing that various expressions must vanish. In particular,

1. Ψ, Ψ' must be chain maps:
   (a) $d_{\uparrow}\Psi - \Psi d_{\uparrow\!\!\!\!-}$
   (b) $d_{\uparrow\!\!\!\!-}\Psi' - \Psi' d_{\uparrow}$
2. they must constitute a deformation retract:
   (c) $\Psi\Psi' - \mathbf{1}$
   (d) $\Psi'\Psi - \mathbf{1} - d_{\uparrow\!\!\!\!-}\varphi - \varphi d_{\uparrow\!\!\!\!-}$
3. this deformation retract must be strong:
   (e) $\Psi\psi$
   (f) $\psi\Psi'$
   (g) ψ^2

These all amount to straightforward calculations, so we omit all but (d), which is the most intricate. Direct calculation and formulation in terms of our basis yields:

$$\begin{aligned}
\Psi'\Psi - \mathbf{1} &= \epsilon_{tx}\partial_2\hat{\theta}_2 + S_{ytx}\epsilon_{tx}\hat{\theta}_1\hat{\theta}_2 - \epsilon_{tx}\partial_1\hat{\theta}_2 - \mathbf{1} \\
d_{\curvearrowleft}\psi + \psi d_{\curvearrowleft} &= \epsilon_{tx}\partial_2\hat{\theta}_2 + (\rho_{tx}\pi_{yt} + \pi_{yt}\rho_{tx})\hat{\theta}_1\hat{\theta}_2 + (\rho_{tx}\Delta_{yt} + \Delta_{yt}\rho_{tx})\partial_1\hat{\theta}_2 - \mathbf{1}
\end{aligned}$$

We now use Equation 5.6 to show that the $\hat{\theta}_1\hat{\theta}_2$ and $\partial_1\hat{\theta}_2$ terms match.

$$\begin{aligned}
\rho_{tx}\pi_{yt} + \pi_{yt}\rho_{tx} &= \Delta_{tx}^{-1}(1-\epsilon_{tx})\pi_{yt} - \pi_{yt}\Delta_{tx}^{-1}(1-\epsilon_{tx}) \\
&= \Delta_{tx}^{-1}(\pi_{yt} - \epsilon_{tx}\pi_{yt}) - \Delta_{tx}^{-1}\pi_{yt}(1-\epsilon_{tx}) \\
&= \Delta_{tx}^{-1}(\pi_{yt} - \pi_{yx}\epsilon_{tx}) - \Delta_{tx}^{-1}\pi_{yt}(1-\epsilon_{tx}) \\
&= \Delta_{tx}^{-1}[(\pi_{yt} - \pi_{yt}) + (-\pi_{yx} + \pi_{yt})\epsilon_{tx}] \\
&= S_{ytx}\epsilon_{tx}
\end{aligned}$$

$$\begin{aligned}
\rho_{tx}\Delta_{yt} + \Delta_{yt}\rho_{tx} &= \Delta_{tx}^{-1}(1-\epsilon_{tx})\Delta_{yt} - \Delta_{yt}\Delta_{tx}^{-1}(1-\epsilon_{tx}) \\
&= \Delta_{tx}^{-1}(\Delta_{yt} - \epsilon_{tx}\Delta_{yt}) - \Delta_{tx}^{-1}\Delta_{yt}(1-\epsilon_{tx}) \\
&= \Delta_{tx}^{-1}(\Delta_{yt} - \Delta_{yx}\epsilon_{tx}) - \Delta_{tx}^{-1}\Delta_{yt}(1-\epsilon_{tx}) \\
&= \Delta_{tx}^{-1}[(\Delta_{yt} - \Delta_{yt}) + (-\Delta_{yx} + \Delta_{yt})\epsilon_{tx}] \\
&= -\Delta_{tx}^{-1}\Delta_{tx}\epsilon_{tx} \\
&= -\epsilon_{tx}
\end{aligned}$$

□

In practice, we will deal with the closure of the above situation; i.e. the strong deformation retract $[\![\circlearrowright]\!] \to [\![\circlearrowright]\!]$. In this case, our edge variables x, y will be identified, without loss of generality, to the internal variable x. So as to make these symbols line up with those in the exterior algebra, we will write x_1 for x and x_2 for our original internal mark t. Formally, this means that $E^t(\;) = \mathbb{C}[x_1, x_2]$. Its Koszul matrix factorization is then given as follows.

$$\llbracket\;\rrbracket = \begin{pmatrix} \Delta_{12} & \pi_{12} \\ -\Delta_{12} & \pi_{12} \end{pmatrix}_{E^t(\;)}$$

$$\cong (E^t(\;) \otimes \bigwedge \mathbb{C}^2, \Delta_{12}(\partial_1 - \partial_2) + \pi_{12}(\hat{\theta}_1 + \hat{\theta}_2))$$

Then, for consistency, we write x_1 for the internal variable in . Since this situation just involved relabelling and identifying variables, an amended version of the above SDR data still holds for this closed up version. For completeness, we write below the SDR data (Ψ, Ψ', ψ) for the deformation retract $\llbracket\;\rrbracket \to \llbracket\;\rrbracket$.

- $\Psi = \epsilon_{21}\partial_2\hat{\theta}_2$
- $\Psi' = \mathbf{1} - \partial_1\hat{\theta}_2 + S_{121}\hat{\theta}_1\hat{\theta}_2$
- $\psi = -\rho_{21}\hat{\theta}_2$

We now consider the second case, that of , for which $E^t(\;) = \mathbb{C}[x]$ and $E(\;) = \mathbb{C}$. We will begin by expressing this matrix factorization in chain complex notation. We briefly ignore the q-grading shifts.

$$\mathbb{C}[x]\theta \xrightarrow{0} \mathbb{C}[x] \xrightarrow{\dot{w}_x} \mathbb{C}[x]\theta$$

Because we are dealing with a closed braid graph. we have $d^2 = 0$, thus making this an actual 2-periodic chain complex. Recalling our algebra $\mathcal{A} = \mathbb{C}[x]/\langle x^N \rangle$, which we here conceive of as simply a $\mathbb{C}$-vector space, we take homology with respect to the matrix factorization differential.

$$\mathcal{A}\theta \to 0 \to \mathcal{A}\theta$$

In the category of vector spaces, chain complexes are homotopy equivalent to their homologies. This is precisely what final mark removal is: taking homology with respect to the closed matrix factorization differential. This entails constructing a deformation retract $[\![\circlearrowright]\!] \to [\![\circlearrowright]\!]$, where, returning to exterior notation and hence recovering q-grading, we have $\circlearrowright = \mathcal{A}\theta$ with a 0 differential. As in the prior case, we will use the evaluation and division maps defined above. Letting $p = \dot{w}_x$, we have the following decomposition *of complex vector spaces.*

$$E^t(\circlearrowright) = \mathbb{C}[x] \cong \mathcal{A} \oplus \mathbb{C}\langle \dot{w}_x \rangle = \mathcal{A} \oplus E(\circlearrowright)\langle \dot{w}_x \rangle.$$

As before this yields evaluation $\epsilon_{\dot{w}_x} : E^t(\circlearrowright) \to \mathcal{A}$ and division $\rho_{\dot{w}_x} : E^t(\circlearrowright) \to E^t(\circlearrowright)$ maps. We are now ready to define the final mark removal.

Lemma 5.3.28. *The triple* $(\Phi : [\![\circlearrowright]\!] \to [\![\circlearrowright]\!], \Phi' : [\![\circlearrowright]\!] \to [\![\circlearrowright]\!], \varphi : [\![\circlearrowright]\!] \to [\![\circlearrowright]\!])$ *given by*

- $\Phi = \epsilon_{\dot{w}_x} \hat{\theta} \partial$
- $\Phi' = \mathbf{1}$
- $\varphi = -\rho_{\dot{w}_x} \partial$

constitutes data for a strong deformation retract.

The proof is straightforward and hence we leave it to the reader. Note that this gives an alternate proof of the first isomorphism of Proposition 5.3.15. Now that we have established the base cases, we now turn to the hybrid case that will be of primary relevance: the deformation retract $[\![\Gamma]\!] \to [\![\dot{\bigcirc}]\!]$ that removes all the marks on an n-marked circle Γ. We call this *complete mark removal*.

We fix the convention that a circle with n marks has internal variables $x_0, x_1, \ldots, x_n$, labeled in a counterclockwise manner. We will remove the highest subscript mark at each step, always absorbing it into the x_0 variable, which will be the final mark. We give the 0 subscript a privileged status by suppressing it from the notation. In particular, since our subsequent ϵ_{ij} and ρ_{ij} maps will always have $i = 0$, we will simply write ϵ_j and ρ_j for $\epsilon_{j,0}$ and $\rho_{j,0}$ respectively. We will also simply write ϵ and ρ for $\epsilon_{\dot{w}_0}$ and $\rho_{\dot{w}_0}$.

Remark 5.3.29. We will later encounter marks that move across circles throughout a single diagram. In this case, we fix a total set J of marks with a choice of ordering. Then each circle possesses some subset of J with the induced ordering.

In the following definition, we set some notation for sequences.

Definition 5.3.30. Given a length n sequence f and a symbol X, we let X_f denote the product $X_{f(1)} \cdots X_{f(n)}$. We will only use strictly monotonic sequences, which will be increasing or decreasing depending on the function composition at hand: increasing for projection maps and decreasing for inclusion maps. Let $|f|$ denote the length of f. We write $f \| x$ to denote the sequence f with the term x inserted so as to maintain monotonicity, and $f \backslash x$ to denote the sequence f with the term x removed, setting the convention that this removal is idempotent. For $k \leqslant n$, define the sequences, $\mathbf{k} : \mathbf{n} = (k+1, k+2, \ldots, n)$ and $\mathbf{n} : \mathbf{k} = (n, n-1, \ldots, k+1)$ with the special cases $\mathbf{0} : \mathbf{n}$ written as $\mathbf{n}$ and $\mathbf{n} : \mathbf{0}$ written as $\overleftarrow{\mathbf{n}}$. We set the convention that $\mathbf{n} : \mathbf{n}$ is the empty sequence $\varnothing$ and that $X_\varnothing = 1$. Given a sub-sequence f of s, we denote its monotonic complement in s by $\check{f}$. Let $\mathcal{P}_\alpha(s)$ denote the set of all (even, odd) length non-empty sub-sequences f of s and $\iota_\alpha f$ denote the sub-sequence of f (and hence sub-sub-sequence of s) consisting of its (even, odd) place terms for $\alpha = (e, o)$ respectively. Given a set S consisting of sequences, we denote by $S \| x$ and $S \backslash x$ images of S under the application of the specified sequence operation.

We then have the projection map $[\![\Gamma]\!] \to [\![\circlearrowright]\!]$ given by $\Psi_{\mathbf{n}}$, which directly computes to:

$$\Psi_{\mathbf{n}} = \epsilon_1 \cdots \epsilon_n (\partial_1 \hat{\theta}_1)(\partial_2 \hat{\theta}_2) \cdots (\partial_n \hat{\theta}_n)$$

If we denote $\partial_k \hat{\theta}_k$ by $(\partial \hat{\theta})_k$, we may express this more succinctly as

$$\Psi_{\mathbf{n}} = \epsilon_{\mathbf{n}} (\partial \hat{\theta})_{\mathbf{n}}.$$

In turn, $\Psi'_{\overleftarrow{\mathbf{n}}}$ gives the inclusion map $[\![\circlearrowright]\!] \to \Gamma$, with the choice $\Psi'_j = \mathbf{1} + \hat{\theta}_j \partial - \mathbb{S}_j \hat{\theta}_j \hat{\theta}$, where $\mathbb{S}_j$ is shorthand for $S_{j-1,j,0}$, whose three subscripts correspond to the mark x_{j-1} directly above the one being removed, the mark x_j being removed, and the mark x_0 which due to closure is always directly below the one being removed. In the following proposition, we give a closed form expression for $\Psi'_{\overleftarrow{\mathbf{n}}}$.

Proposition 5.3.31. *For a sequence f, define $\zeta(f) = \sin(\frac{\pi}{2}|f|)$ when $f \colon \mathcal{P}_o(\mathbf{n})$ and $\zeta(f) = \cos(\frac{\pi}{2}|f|)$ when $f \colon \mathcal{P}_e$. More concretely, this means we have the values $\zeta(f) = 1, 1, -1, -1, 1, 1, -1, -1, \ldots$ when f has length $0, 1, 2, \ldots$. We then have the formula*

$$\Psi'_{\overleftarrow{\mathbf{n}}} = \mathbf{1} - \left(\sum_{f \colon \mathcal{P}_o(\mathbf{n})} \zeta(f) \mathbb{S}_{\iota_o(f)} \hat{\theta}_f \right) \hat{\theta} + \left(\sum_{f \colon \mathcal{P}_o(\mathbf{n})} \zeta(f) \mathbb{S}_{\iota_e(f)} \hat{\theta}_f \right) \partial$$
$$+ \left(\sum_{f \colon \mathcal{P}_e(\mathbf{n})} \zeta(f) \mathbb{S}_{\iota_o(f)} \hat{\theta}_f \right) \partial \hat{\theta} + \left(\sum_{f \colon \mathcal{P}_e(\mathbf{n})} \zeta(f) \mathbb{S}_{\iota_e(f)} \hat{\theta}_f \right) \hat{\theta} \partial.$$

Proof. We proceed by induction. The base case $\Psi'_{\overleftarrow{\mathbf{1}}} = \Psi'_1 = \mathbf{1} + \hat{\theta}_1\partial - \mathbb{S}_1\hat{\theta}_1\hat{\theta}$ holds since $\mathbb{S}_{\iota_e(\mathbf{1})} = \mathbb{S}_\varnothing = 1$. Now suppose $\Psi'_{\overleftarrow{\mathbf{n}}}$ holds. We attempt to show this implies $\Psi'_{\overleftarrow{\mathbf{n+1}}}$ follows.

$$
\begin{aligned}
\Psi'_{\overleftarrow{\mathbf{n+1}}} &= \Psi'_{n+1}\Psi'_{\overleftarrow{\mathbf{n}}} \\
&= (\mathbf{1} + \hat{\theta}_{n+1}\partial - \mathbb{S}_{n+1}\hat{\theta}_{n+1}\hat{\theta})\Psi'_{\overleftarrow{\mathbf{n}}} \\
&= \mathbf{1} \; - \left(\sum_{f\colon \mathcal{P}_o(\mathbf{n})} \zeta(f)\mathbb{S}_{\iota_o(f)}\hat{\theta}_f\right)\hat{\theta} + \left(\sum_{f\colon \mathcal{P}_o(\mathbf{n})} \zeta(f)\mathbb{S}_{\iota_e(f)}\hat{\theta}_f\right)\partial \\
&\quad + \left(\sum_{f\colon \mathcal{P}_e(\mathbf{n})} \zeta(f)\mathbb{S}_{\iota_o(f)}\hat{\theta}_f\right)\partial\hat{\theta} + \left(\sum_{f\colon \mathcal{P}_e(\mathbf{n})} \zeta(f)\mathbb{S}_{\iota_e(f)}\hat{\theta}_f\right)\hat{\theta}\partial \\
&\quad + \hat{\theta}_{n+1}\partial - \left(\sum_{f\colon \mathcal{P}_o(\mathbf{n})} \zeta(f)\mathbb{S}_{\iota_o(f)}\hat{\theta}_{f\|n+1}\right)\partial\hat{\theta} + \left(\sum_{f\colon \mathcal{P}_e(\mathbf{n})} \zeta(f)\mathbb{S}_{\iota_e(f)}\hat{\theta}_{f\|n+1}\right)\partial \\
&\quad - \mathbb{S}_{n+1}\hat{\theta}_{n+1}\hat{\theta} - \left(\sum_{f\colon \mathcal{P}_o(\mathbf{n})} \zeta(f)\mathbb{S}_{\iota_e(f)\|n+1}\hat{\theta}_{f\|n+1}\right)\hat{\theta}\partial - \left(\sum_{f\colon \mathcal{P}_e(\mathbf{n})} \zeta(f)\mathbb{S}_{\iota_o(f)\|n+1}\hat{\theta}_{f\|n+1}\right)\hat{\theta} \\
&= \mathbf{1} \; - \left[\left(\sum_{f\colon \mathcal{P}_o(\mathbf{n})} \zeta(f)\mathbb{S}_{\iota_o(f)}\hat{\theta}_f\right) + \left(\sum_{f\colon \mathcal{P}_e(\mathbf{n})} \zeta(f)\mathbb{S}_{\iota_o(f)\|n+1}\hat{\theta}_{f\|n+1}\right) + \mathbb{S}_{n+1}\hat{\theta}_{n+1}\right]\hat{\theta} \\
&\quad + \left[\left(\sum_{f\colon \mathcal{P}_o(\mathbf{n})} \zeta(f)\mathbb{S}_{\iota_e(f)}\hat{\theta}_f\right) + \left(\sum_{f\colon \mathcal{P}_e(\mathbf{n})} \zeta(f)\mathbb{S}_{\iota_e(f)}\hat{\theta}_{f\|n+1}\right) + \hat{\theta}_{n+1}\right]\partial \\
&\quad + \left[\left(\sum_{f\colon \mathcal{P}_e(\mathbf{n})} \zeta(f)\mathbb{S}_{\iota_o(f)}\hat{\theta}_f\right) - \left(\sum_{f\colon \mathcal{P}_o(\mathbf{n})} \zeta(f)\mathbb{S}_{\iota_o(f)}\hat{\theta}_{f\|n+1}\right)\right]\partial\hat{\theta} \\
&\quad + \left[\left(\sum_{f\colon \mathcal{P}_e(\mathbf{n})} \zeta(f)\mathbb{S}_{\iota_e(f)}\hat{\theta}_f\right) - \left(\sum_{f\colon \mathcal{P}_o(\mathbf{n})} \zeta(f)\mathbb{S}_{\iota_e(f)\|n+1}\hat{\theta}_{f\|n+1}\right)\right]\hat{\theta}\partial
\end{aligned}
$$

Note that we bind $+$ tighter than $\|$. Observe that when $f\colon \mathcal{P}_\alpha$, we have that $\iota_\alpha(f\|n+1) = \iota_\alpha(f)$, but when $\beta \neq \alpha$, we have $\iota_\beta(f\|n+1) = \iota_\beta(f)\|n+1$. Also, for a singleton x, we have that $\iota_o(x) = x$ and $\iota_e(x) = \varnothing$. We denote by $\mathcal{P}_\alpha(\mathbf{n})\|x$ the set of sequences, each appended by x, in $\mathcal{P}_\alpha(\mathbf{n})$. Furthermore, $\zeta(f\|x) = \zeta(f)$ for $f\colon \mathcal{P}_e(\mathbf{n})$, $\zeta(f\|x) = -\zeta(f)$ for $f\colon \mathcal{P}_o(\mathbf{n})$, and $\zeta(x) = 1$. This yields the following simplification.

$$\begin{aligned}
\Psi'_{\overleftarrow{\mathbf{n+1}}} = \mathbf{1} &- \left[\left(\sum_{f\colon \mathcal{P}_o(\mathbf{n})} \zeta(f)\mathbb{S}_{\iota_o(f)}\hat{\theta}_f\right) + \left(\sum_{f\colon \mathcal{P}_e(\mathbf{n})\|n+1} \zeta(f)\mathbb{S}_{\iota_o(f)}\hat{\theta}_f\right) + \zeta((n+1))\mathbb{S}_{\iota_o((n+1))}\hat{\theta}_{n+1}\right]\hat{\theta} \\
&+ \left[\left(\sum_{f\colon \mathcal{P}_o(\mathbf{n})} \zeta(f)\mathbb{S}_{\iota_e(f)}\hat{\theta}_f\right) + \left(\sum_{f\colon \mathcal{P}_e(\mathbf{n})\|n+1} \zeta(f)\mathbb{S}_{\iota_e(f)}\hat{\theta}_f\right) + \zeta((n+1))\mathbb{S}_{\iota_e((n+1))}\hat{\theta}_{n+1}\right]\partial \\
&+ \left[\left(\sum_{f\colon \mathcal{P}_e(\mathbf{n})} \zeta(f)\mathbb{S}_{\iota_o(f)}\hat{\theta}_f\right) + \left(\sum_{f\colon \mathcal{P}_o(\mathbf{n})\|n+1} \zeta(f)\mathbb{S}_{\iota_o(f)}\hat{\theta}_f\right)\right]\partial\hat{\theta} \\
&+ \left[\left(\sum_{f\colon \mathcal{P}_e(\mathbf{n})} \zeta(f)\mathbb{S}_{\iota_e(f)}\hat{\theta}_f\right) + \left(\sum_{f\colon \mathcal{P}_o(\mathbf{n})\|n+1} \zeta(f)\mathbb{S}_{\iota_e(f)}\hat{\theta}_f\right)\right]\hat{\theta}\partial
\end{aligned}$$

Now that all summands are in uniform format, we can combine the appropriate sums. We do this by noting that if $f\colon \mathcal{P}_\alpha(\mathbf{n})$, then, for a singleton x, $f\|x\colon \mathcal{P}_\beta(\mathbf{n+1})$ with $\alpha \neq \beta$. Now consider the inductive structure of $\mathcal{P}_\alpha(\mathbf{n+1})$. A sequence in this set is either a sequence in $\mathcal{P}_\alpha(\mathbf{n})$ or it is a sequence that ends with $n+1$. Aside from the singleton $(n+1)$ itself, any sequence ending with $n+1$ can be seen as a sequence in $\mathcal{P}_\beta(\mathbf{n})$ appended by $(n+1)$. This singleton $(n+1)$ belongs to $\mathcal{P}_o(\mathbf{n+1})$. Hence $\mathcal{P}_o(\mathbf{n+1}) = \mathcal{P}_o(\mathbf{n}) \sqcup (\mathcal{P}_e(\mathbf{n})\|n+1) \sqcup (n+1)$ and $\mathcal{P}_e(\mathbf{n+1}) = \mathcal{P}_e(\mathbf{n}) \sqcup (\mathcal{P}_o(\mathbf{n})\|n+1)$. This allows us to complete the induction.

$$\Psi'_{\overleftarrow{\mathbf{n+1}}} = \mathbf{1} - \left(\sum_{f\colon \mathcal{P}_o(\mathbf{n+1})} \zeta(f)\mathbb{S}_{\iota_o(f)}\hat{\theta}_f \right)\hat{\theta} + \left(\sum_{f\colon \mathcal{P}_o(\mathbf{n+1})} \zeta(f)\mathbb{S}_{\iota_e(f)}\hat{\theta}_f \right)\partial$$

$$+ \left(\sum_{f\colon \mathcal{P}_e(\mathbf{n+1})} \zeta(f)\mathbb{S}_{\iota_o(f)}\hat{\theta}_f \right)\partial\hat{\theta} + \left(\sum_{f\colon \mathcal{P}_e(\mathbf{n+1})} \zeta(f)\mathbb{S}_{\iota_e(f)}\hat{\theta}_f \right)\hat{\theta}\partial$$

□

We next turn our attention to the homotopy map, which we denote by $\widetilde{\psi}_n$, for the deformation retract $[\![\Gamma]\!] \to [\![\circlearrowleft]\!]$. Recall that, given a sequence of deformation retracts $C_k \to C_{k-1}$ with SDR data (r_k, i_k, h_k), the SDR data for the composed deformation retract $C_n \to C_1$ is given by $(r_{\mathbf{n}}, i_{\overleftarrow{\mathbf{n}}}, \sum_k i_{\mathbf{n:k}} h_k r_{\mathbf{k:n}})$. We will hence write a formula for $\widetilde{\psi}_n$ in terms of its summands $\psi_n^k = \Psi'_{\mathbf{n:k}}\psi_k\Psi_{\mathbf{k:n}}$, which we sum across $k = n, \ldots, 1$. Recall that $\psi_k = -\rho_k\hat{\theta}_k$, and note that, by mere substitution, $\Psi_{\mathbf{k:n}} = \epsilon_{\mathbf{k:n}}(\partial\hat{\theta})_{\mathbf{k:n}}$. Then, we can write

$$\psi_k\Psi_{\mathbf{k:n}} = -\rho_k\epsilon_{\mathbf{k:n}}\hat{\theta}_k(\partial\hat{\theta})_{\mathbf{k:n}}.$$

The more complicated $\Psi'_{\mathbf{n:k}}$ is, similarly, expressed via mere relabelling:

$$\Psi'_{\mathbf{n:k}} = \mathbf{1} - \left(\sum_{f\colon \mathcal{P}_o(\mathbf{k:n})} \zeta(f)\mathbb{S}_{\iota_o(f)}\hat{\theta}_f \right)\hat{\theta} + \left(\sum_{f\colon \mathcal{P}_o(\mathbf{k:n})} \zeta(f)\mathbb{S}_{\iota_e(f)}\hat{\theta}_f \right)\partial$$

$$+ \left(\sum_{f\colon \mathcal{P}_e(\mathbf{k:n})} \zeta(f)\mathbb{S}_{\iota_o(f)}\hat{\theta}_f \right)\partial\hat{\theta} + \left(\sum_{f\colon \mathcal{P}_e(\mathbf{k:n})} \zeta(f)\mathbb{S}_{\iota_e(f)}\hat{\theta}_f \right)\hat{\theta}\partial.$$

To compute the composition of these two terms, we must analyze the product $\hat{\theta}_f(\partial\hat{\theta})_{\mathbf{k:n}}$ for $f\colon \mathcal{P}(\mathbf{k}:\mathbf{n})$. Via commutativity, we have $\hat{\theta}_f(\partial\hat{\theta})_f(\partial\hat{\theta})_{\check{f}}$, which can be rearranged to $(\hat{\theta}\partial\hat{\theta})_f(\partial\hat{\theta})_{\check{f}} = \hat{\theta}_f(\partial\hat{\theta})_{\check{f}}$. Then, upon invoking parity, we have

$$
\begin{aligned}
\psi_n^k =& \Psi'_{\mathbf{n:k}}\psi_k\Psi_{\mathbf{k:n}} \\
=& -\rho_k\epsilon_{\mathbf{k:n}}\hat{\theta}_k(\partial\hat{\theta})_{\mathbf{k:n}} \\
&+ \left(\sum_{f\colon \mathcal{P}_o(\mathbf{k:n})} \zeta(f)\mathbb{S}_{\iota_o(f)}\rho_k\epsilon_{\mathbf{k:n}}\hat{\theta}_{f\|k}(\partial\hat{\theta})_{\check{f}}\right)\hat{\theta} \\
&- \left(\sum_{f\colon \mathcal{P}_o(\mathbf{k:n})} \zeta(f)\mathbb{S}_{\iota_e(f)}\rho_k\epsilon_{\mathbf{k:n}}\hat{\theta}_{f\|k}(\partial\hat{\theta})_{\check{f}}\right)\partial \\
&- \left(\sum_{f\colon \mathcal{P}_e(\mathbf{k:n})} \zeta(f)\mathbb{S}_{\iota_o(f)}\rho_k\epsilon_{\mathbf{k:n}}\hat{\theta}_{f\|k}(\partial\hat{\theta})_{\check{f}}\right)\partial\hat{\theta} \\
&- \left(\sum_{f\colon \mathcal{P}_e(\mathbf{k:n})} \zeta(f)\mathbb{S}_{\iota_e(f)}\rho_k\epsilon_{\mathbf{k:n}}\hat{\theta}_{f\|k}(\partial\hat{\theta})_{\check{f}}\right)\hat{\theta}\partial.
\end{aligned}
$$

We are now in a position to write down a closed form for complete mark removal.

Proposition 5.3.32. *The SDR data $(\Upsilon_n, \Upsilon'_n, \upsilon_n)$ for the deformation retract $[\![\Gamma]\!] \to [\![\circlearrowright]\!]$ consists of maps with the following expressions.*

$$
\begin{aligned}
\Upsilon_n &= \epsilon\epsilon_{\mathbf{n}}(\hat{\theta}\partial)(\partial\hat{\theta})_{\mathbf{n}} \\
\Upsilon'_n &= \Psi'_{\overleftarrow{\mathbf{n}}} \\
\upsilon_n &= \widetilde{\psi}_n + \widetilde{\varphi}_n,
\end{aligned}
$$

where $\widetilde{\varphi}_n$ is given by $\Psi'_{\overleftarrow{\mathbf{n}}}\varphi\Psi_{\mathbf{n}}$, which can be evaluated as follows.

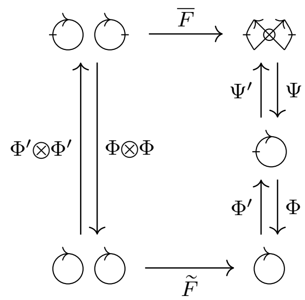
$$
\begin{aligned}
\widetilde{\varphi}_n =& \Psi'_{\overleftarrow{\mathbf{n}}}\varphi\Psi_{\mathbf{n}} \\
=& \Psi'_{\overleftarrow{\mathbf{n}}}(-\rho\partial)\epsilon_{\mathbf{n}}(\partial\hat{\theta})_{\mathbf{n}} \\
=& -\Psi'_{\overleftarrow{\mathbf{n}}}(\rho\epsilon_{\mathbf{n}})\partial(\partial\hat{\theta})_{\mathbf{n}} \\
=& -\rho\epsilon_{\mathbf{n}}(\partial\hat{\theta})_{\mathbf{n}}\partial \\
& + \left(\sum_{f\,:\,\mathcal{P}_o(\mathbf{n})} \zeta(f)\mathbb{S}_{\iota_o(f)}\rho\epsilon_{\mathbf{n}}\hat{\theta}_f(\partial\hat{\theta})_{\check{f}}\right)\hat{\theta}\partial \\
& - \left(\sum_{f\,:\,\mathcal{P}_e(\mathbf{n})} \zeta(f)\mathbb{S}_{\iota_o(f)}\rho\epsilon_{\mathbf{n}}\hat{\theta}_f(\partial\hat{\theta})_{\check{f}}\right)\partial
\end{aligned}
$$

The proof is a consequence of the homological algebra of composing deformation retracts.

Given the MOY braid graph Γ, we have a twisted complex corresponding to virtual resolutions of wide edges in Γ. We will now consider the *short*, or length 1, differentials of this complex and save the description of the *long*, or length > 1, differentials for the next section. The short differentials arise from mapping cones of the virtual saddle morphisms $F : [\![\uparrow\;\uparrow]\!] \to [\![\otimes]\!]$ and $G : [\![\otimes]\!] \to [\![\uparrow\;\uparrow]\!]$. We will attempt to demonstrate the following.

Proposition 5.3.33. *Up to homotopy equivalence and scalar multiple, the short differentials in the twisted complex associated to a braid graph Γ are either the multiplication map $\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$ or the comultiplication map $\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ of the Frobenius algebra $\mathcal{A}=\mathbb{C}[x]/\langle x^N\rangle$.*

To show this, consider the following commutative diagram, which, by Proposition 5.2.12, corresponds to the short differentials. In particular, this diagram consists of the closure of the virtual saddle and mark removal maps.



The bottom entries are $= \mathcal{A} = \mathbb{C}[x_1]/\langle x_1^N \rangle$ and $= \mathcal{A} \otimes \mathcal{A} \cong \mathbb{C}[x_1, x_2]/\langle x_1^N, x_2^N \rangle$, where the former is concentrated in the exterior basis θ_1 and the latter in $\theta_1\theta_2$. These disjoint circles arise upon taking the closure and removing all marks. We now stipulate that $\widetilde{F}$ is equivalent to the natural multiplication map $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ of the algebra $\mathcal{A}$. For self containment, we recall the definitions of the relevant maps in the above diagram whose composition we wish to show is equivalent to $\widetilde{F}$.

$$\Phi' \otimes \Phi' = \mathbf{1}$$

$$\overline{F} = \partial_1 - \partial_2 + \overline{P_{12}}\hat{\theta}_1 - \overline{P_{21}}\hat{\theta}_2$$

$$\Psi = \epsilon_{21}\partial_2\hat{\theta}_2$$

$$\Phi = \epsilon_{\dot{w}_1}\hat{\theta}_1\partial_1$$

It will be easiest to characterize $\widetilde{F}$ via how it acts on the generic basis element $x_1^{k_1}x_2^{k_2} \otimes \theta_1\theta_2$ of :

$$
\begin{aligned}
x_1^{k_1} x_2^{k_2} \otimes \theta_1\theta_2 &\xmapsto{\Phi'\otimes\Phi'} x_1^{k_1} x_2^{k_2} \otimes \theta_1\theta_2 \\
&\overset{\overline{F}}{\longmapsto} \quad x_1^{k_1} x_2^{k_2} \otimes (\theta_1 + \theta_2) \\
&\overset{\Psi}{\longmapsto} \quad x_1^{k_1+k_2} \otimes \theta_1 \\
&\overset{\Phi}{\longmapsto} \quad x_1^{k_1+k_2} \otimes \theta_1,
\end{aligned}
$$

where the last mapping ensures that $x_1^{k_1+k_2}$ is an element of $\mathcal{A}$ via vanishing it in the case when $k_1 + k_2 \geqslant N$. Then the following composition

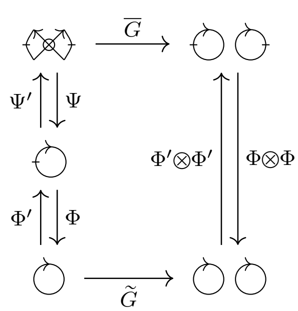
$$\mathcal{A} \otimes \mathcal{A} \xrightarrow{\cong} \mathbb{C}[x_1, x_2]/\langle x_1^N, x_2^N \rangle \otimes \theta_1\theta_2 \xrightarrow{\tilde{F}} \mathbb{C}[x_1]/\langle x_1^N \rangle \otimes \theta_1 \xrightarrow{\cong} \mathcal{A}$$

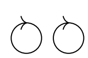
given by

$$x^{k_1} \otimes x^{k_2} \mapsto x_1^{k_1} x_2^{k_2} \otimes \theta_1\theta_2 \mapsto x_1^{k_1+k_2} \otimes \theta_1 \mapsto x^{k_1+k_2}$$

is precisely the multiplication map for the algebra $\mathcal{A}$.

We now consider the flipped case via writing its corresponding commutative diagram.



The same remains true for  and  just as above. We now enumerate the relevant maps again, in the order that they will be composed.

$$\begin{aligned}
\Phi' &= \mathbf{1} \\
\Psi' &= \mathbf{1} - \partial_1 \hat{\theta}_2 + S_{121} \hat{\theta}_1 \hat{\theta}_2 \\
\overline{G} &= \partial_1 - \partial_2 + \overline{P_{12}} \hat{\theta}_1 - \overline{P_{21}} \hat{\theta}_2 \\
\Phi \otimes \Phi &= \epsilon_{\dot{w}_1} \epsilon_{\dot{w}_2} \hat{\theta}_1 \partial_1 \hat{\theta}_2 \partial_2
\end{aligned}$$

This time, however, we will need to work more explicitly with $\overline{P_{ij}}$. In particular, we will need to calculate the following.

$$\begin{aligned}
\overline{P_{ij}} + \overline{P_{ji}} &= \overline{\left(\frac{\pi_{ii} - \pi_{ij}}{\Delta_{ii} - \Delta_{ij}} \right)} + \overline{\left(\frac{\pi_{jj} - \pi_{ji}}{\Delta_{jj} - \Delta_{ji}} \right)} \\
&= \frac{\overline{\pi_{ii}} - \overline{\pi_{jj}}}{\Delta_{ji}} \\
&= (-N-1) \frac{x_i^N - x_j^N}{x_i - x_j} \\
&= (-N-1) \sum_{l+m=N-1} x_i^l x_j^m
\end{aligned}$$

Since this quantity is a scalar multiple of the complete degree $N-1$ homogeneous polynomials in two variables, we write it suggestively as S_{ij}. Now, as in the prior case, we characterize $\tilde{G}$ in terms of how it acts on a generic basis element $x_1^{k_1} \otimes \theta_1$.

$$\begin{aligned}
x_1^{k_1} \otimes \theta_1 &\xmapsto{\Phi'} x_1^{k_1} \otimes \theta_1 \\
&\xmapsto{\Psi'} x_1^{k_1} \otimes (\theta_1 + \theta_2) \\
&\xmapsto{\overline{G}} S_{ij} x_1^{k_1} \otimes \theta_1 \theta_2 \\
&\xmapsto{\Phi \otimes \Phi} (-N-1) \sum_{l+m=k_1+N-1} x_1^l x_2^m \otimes \theta_1 \theta_2,
\end{aligned}$$

where, as in last time, the final mapping ensures that all the summands are in $\mathcal{A}\otimes\mathcal{A}$ by killing all terms with x_1^N or x_2^N. Thus the following composition

$$\mathcal{A} \xrightarrow{\cong} \mathbb{C}[x_1]/\langle x_1^N \rangle \otimes \theta_1 \xrightarrow{\tilde{G}} \mathbb{C}[x_1, x_2]/\langle x_1^N, x_2^N \rangle \otimes \theta_1\theta_2 \xrightarrow{\cong} \mathcal{A}\otimes\mathcal{A}$$

given by

$$\frac{-x^k}{N+1} \mapsto \frac{-x_1^k}{N+1} \otimes \theta_1 \mapsto \sum_{l+m=k+N-1} x_1^l x_2^m \otimes \theta_1\theta_2 \mapsto \sum_{l+m=k+N-1} x^l \otimes x^m$$

is a scalar multiple of the comultiplication map for the Frobenius algebra $\mathcal{A}$.

By Proposition 5.2.12, we calculate our long differentials as composition paths in a diagram of the following form.

Although the general form for such a long differential may be intractable to express in a succinct closed form, we are equipped with closed formulae for all of the pieces that constitute these compositions. We do, however, derive a nice combinatorial description for the top left corner of any such composition.

More precisely, suppose we have the situation of beginning with a single $\circlearrowright$ component, and thus with everything concentrated in the generator θ. We can then compute the exact expression for the result of applying to this generator our complete inclusion map.

$$\begin{aligned}
\Psi'_{\overleftarrow{\mathbf{n}}} = \mathbf{1} &- \left(\sum_{f\,:\,\mathcal{P}_o(\mathbf{n})} \zeta(f)\mathbb{S}_{\iota_o(f)}\hat{\theta}_f\right)\hat{\theta} + \left(\sum_{f\,:\,\mathcal{P}_o(\mathbf{n})} \zeta(f)\mathbb{S}_{\iota_e(f)}\hat{\theta}_f\right)\partial \\
&+ \left(\sum_{f\,:\,\mathcal{P}_e(\mathbf{n})} \zeta(f)\mathbb{S}_{\iota_o(f)}\hat{\theta}_f\right)\partial\hat{\theta} + \left(\sum_{f\,:\,\mathcal{P}_e(\mathbf{n})} \zeta(f)\mathbb{S}_{\iota_e(f)}\hat{\theta}_f\right)\hat{\theta}\partial.
\end{aligned}$$

Using f' as shorthand for $f\backslash 0$, we can simplify matters by applying this complete inclusion map to θ as follows:

$$\begin{aligned}
\Psi'_{\overleftarrow{\mathbf{n}}}(\theta) &= \theta + \left(\sum_{f \colon \mathcal{P}_o(\mathbf{n})} \zeta(f) \mathbb{S}_{\iota_e(f)} \theta_f \right) + \left(\sum_{f \colon \mathcal{P}_e(\mathbf{n})} \zeta(f) \mathbb{S}_{\iota_e(f)} \theta_f \right) \theta \\
&= \zeta((0)) \mathbb{S}_{\iota_e((0))} \theta + \left(\sum_{f \colon \mathcal{P}_o(\mathbf{n})} \zeta(f) \mathbb{S}_{\iota_e(f)} \theta_f \right) + \left(\sum_{f \colon \mathcal{P}_e(\mathbf{n})} \zeta(f) \mathbb{S}_{\iota_e(f)} \theta_{f \| 0} \right) \\
&= \zeta((0)') \mathbb{S}_{\iota_e((0)')} \theta + \left(\sum_{f \colon \mathcal{P}_o(\mathbf{n})} \zeta(f') \mathbb{S}_{\iota_e(f')} \theta_f \right) + \left(\sum_{f \colon \mathcal{P}_o(\mathbf{n}) \| 0} \zeta(f') \mathbb{S}_{\iota_e(f')} \theta_f \right) \\
&= \sum_{f \colon \mathcal{P}_o(\mathbf{n} \| 0)} \zeta(f') \mathbb{S}_{\iota_e(f')} \theta_f
\end{aligned}$$

We now consider the general case of beginning with several components. More formally, suppose we have a closed braid of index n, and hence we in total, before removing them, have n marks, recorded in the sequence $\mathbf{n}$. Suppose upon closing we have a set K of k components, each of which has a sequence $\mathbf{s}^m$ of marks. Note that the $\{\mathbf{s}^m\}_{m \colon K}$ must partition $\mathbf{n}$. We denote by $\mathbf{s}_0^m$ the initial term of $\mathbf{s}^m$ and hence conceive of it as being in the 0th place of $\mathbf{s}^m$. This $\mathbf{s}_0^m$ term corresponds to the final mark that was removed from the circle corresponding to $m \colon K$—upon removing marks, this circle is concentrated in exterior grading $\theta_{\mathbf{s}_0^m}$. We also generalize the meaning of f' to denote the sub-sequence f of $\mathbf{s}^m$ with instances of $\mathbf{s}_0^m$ deleted; i.e. for $f \colon \mathcal{P}(\mathbf{s}^m)$, $f' = f\backslash \mathbf{s}_0^m$. Then, the image of the complete inclusion map can be expressed as the following product of the above single component expression.

$$\prod_{m \colon K} \sum_{f \colon \mathcal{P}_o(\mathbf{s}^m)} \zeta(f') \mathbb{S}_{\iota_e(f')} \theta_f$$

Note that, since the parity of the θ_f terms is always odd, this product is skew commutative. At a higher level, this is due to the fact that the inclusion map was applied to a tensor of circles, and hence an exterior product of the $\theta_{\mathbf{s}_0^m}$ terms in which they are concentrated.

We now consider the opposite end of our diagram. Supposing that we end at a set L of  components, we know that, given the structure of our projection maps, that the final arrow's behavior on exterior generators amounts to a projection in the following sense. Given that the final complex is concentrated entirely in the degree $\prod_{l\colon L} \theta_{\mathbf{s}_0^l}$, the projection map eliminates all terms to which it is applied that are not concentrated entirely in this degree. Formally, this is because for every $j \notin \mathbf{s}_0^L$, we apply a $(\partial\hat{\theta})_j$ to the penultimate term. This effectively kills off any terms that include such j's. What remains is then, for each component l in the penultimate diagram, a term for $\theta_{\mathbf{s}_0^l}$.

5.3.6 Recovering Khovanov-Rozansky Homology

In this section we recover the $\mathfrak{sl}_N$ link homology theory constructed by Khovanov and Rozansky in [KR08]. We first recall some results from [KR07] which characterize the virtual filtration of $\mathfrak{sl}_N$ homology and its corresponding description of differentials between MOY resolutions as natural morphisms of mapping cones. We use this to construct a spectral sequence whose E^2-page is the Khovanov-Rozansky $\mathfrak{sl}_N$ homology of a given link and E^∞-page is the $\mathfrak{sl}_N$ homology of its corresponding unlink.

First, given a link diagram D, we recall the definition of $[\![D]\!]$. Suppose D has n components. Then the edge ring is given by

Suppose $\beta\colon \mathfrak{B}_n$. Then as in the case of MOY braid-graphs, we define $E(\beta) = \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ and $w(\beta) = \sum_{i=1}^{n}(w_{y_i} - w_{x_i})$. To each braid β we associate a chain complex of matrix factorizations in $\mathbf{MF}(E(\beta), w(\beta))$. In particular, for $[\![$  $]\!]$

and $\llbracket \nwarrow\!\!\nearrow \rrbracket$ we define

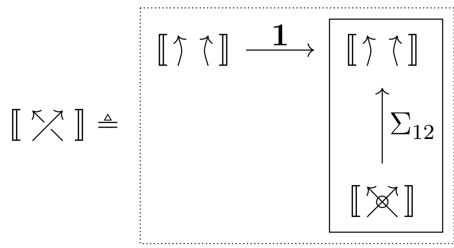
$$\llbracket \nwarrow\!\!\nearrow \rrbracket = (0 \to \underline{\llbracket \uparrow \uparrow \rrbracket} \xrightarrow{\chi_i} \llbracket \nwarrow\!\!\bullet\!\!\nearrow \rrbracket \to 0), \tag{5.20}$$

$$\llbracket \nwarrow\!\!\nearrow \rrbracket = (0 \to \llbracket \nwarrow\!\!\bullet\!\!\nearrow \rrbracket \xrightarrow{\chi_o} \underline{\llbracket \uparrow \uparrow \rrbracket} \to 0). \tag{5.21}$$

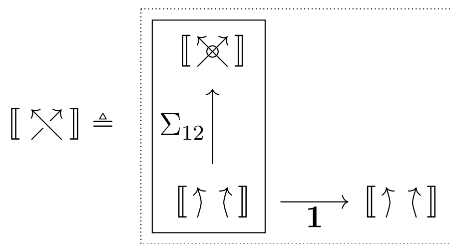
In the chain complexes above the underlined terms are in homological degree 0 and we assume that χ_i and χ_o have homological degree 1. We define the complex $\llbracket\beta\rrbracket$ for any braid β using the same gluing relations as described in (5.11) and (5.12). The maps χ_i and χ_o are explicitly given in [KR08]. However, these maps are naturally defined in terms of virtual filtrations as well. The following result is proven in [KR07].

Proposition 5.3.34. *Consider* $\llbracket \nwarrow\!\!\bullet\!\!\nearrow \rrbracket$ *as the mapping cone of* $F\colon \mathrm{Ext}(\llbracket \uparrow \uparrow \rrbracket, \llbracket \nwarrow\!\!\otimes\!\!\nearrow \rrbracket)$ *as described in Proposition 5.3.22. The map* $\chi_o : \llbracket \nwarrow\!\!\nearrow \rrbracket \to \llbracket \uparrow \uparrow \rrbracket$ *in (5.21) is homotopy equivalent to the natural projection of mapping cone of* F*. Likewise, consider* $\llbracket \nwarrow\!\!\bullet\!\!\nearrow \rrbracket$ *as the mapping cone of* $G\colon \mathrm{Ext}(\llbracket \nwarrow\!\!\otimes\!\!\nearrow \rrbracket, \llbracket \uparrow \uparrow \rrbracket)$ *as described in Proposition 5.3.22. The map* $\chi_i : \llbracket \uparrow \uparrow \rrbracket \to \llbracket \nwarrow\!\!\nearrow \rrbracket$ *in (5.20) is homotopy equivalent to the natural inclusion of the mapping cone of* G

We use solid boxes around chain complexes to represent their convolution. In particular, the convolutions of the complexes of matrix factorizations $\llbracket \nwarrow\!\!\otimes\!\!\nearrow \rrbracket \xrightarrow{\Sigma_{12}} \llbracket \uparrow \uparrow \rrbracket$ and $\llbracket \uparrow \uparrow \rrbracket \xrightarrow{\Sigma_{12}} \llbracket \nwarrow\!\!\otimes\!\!\nearrow \rrbracket$ represent $\llbracket \nwarrow\!\!\bullet\!\!\nearrow \rrbracket^+$ and $\llbracket \nwarrow\!\!\bullet\!\!\nearrow \rrbracket^-$ respectively. Given this, we will often represent $\llbracket \nwarrow\!\!\nearrow \rrbracket$ and $\llbracket \nwarrow\!\!\nearrow \rrbracket$ via the arrow diagrams, delimited by dotted boxes for readability.



(5.22)



(5.23)

Finally we give the definition for Khovanov-Rozansky homology. As before, N is a fixed positive integer. Let L be a link and suppose D is a link diagram for L. Then we consider the complex of matrix factorizations $[\![D]\!]$. This is actually a bicomplex. We denote the "horizontal" differential coming from the canonical maps described above as d_{KR} and the differential on the closed matrix factorizations as d_{MOY}.

Definition 5.3.35. The $\mathfrak{sl}_N$ *Khovanov-Rozansky homology* of a link diagram D is given by

$$H_N(D) = H_{d_{KR}}(H_{d_{MOY}}([\![D]\!])). \tag{5.24}$$

Theorem 5.3.36 (Khovanov-Rozansky [KR08])**.** *Let D be a link diagram of writhe $w(D)$ with underlying link L. Then the doubly-graded vector space*

$$H_N(D)(-w(D), (1-N)w(D), 0)$$

is a link invariant of L and furthermore categorifies the $\mathfrak{sl}_N$ link polynomial.

We will ignore the grading shift needed to make $H_N(D)$ a link invariant (instead of a framed link invariant) for the sake of simplicity. However since this shift only depends on the writhe and N, it will always be easy to compute quickly.

We now consider $[\![\beta]\!]$ as a bicomplex of matrix factorizations motivated by the diagrams (5.22) and (5.23). This was first explored by Becker in his thesis [Bec15] using stable Hochschild homology and a formal diagrammatic definition of the following complex of diagrams. Our approach here shares many similarities with his construction, but using the original matrix factorization construction and virtual filtrations instead.

First, we recall a definition from homological algebra regarding bicomplexes and their tensor products.

Definition 5.3.37. Let $C = (C_{\bullet\bullet}, d_h, d_v)$ and $C' = (C'_{\bullet\bullet}, d'_h, d'_v)$ be two bicomplexes of matrix factorizations. We define the *tensor product bicomplex* $C \otimes C' = ((C \otimes C')_{\bullet\bullet}, d_h^{\otimes}, d_v^{\otimes})$ as follows.

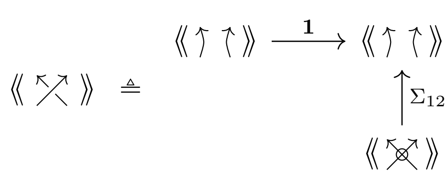
$$(C \otimes C')_{mn} = \bigoplus_{i+k=m, j+\ell=n} (C_{ij} \otimes C_{k\ell}),$$

$$d_h^{\otimes}(x \otimes y) = d_h(x) \otimes y + (-1)^i x \otimes d'_h(y),$$

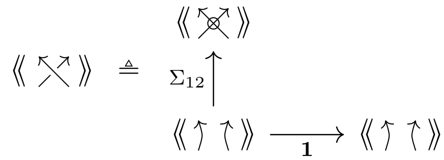
$$d_v^{\otimes}(x \otimes y) = d_v(x) \otimes y + (-1)^k x \otimes d'_v(y),$$

where above $x \colon C_{ij}$ and $y \colon C'_{k\ell}$.

We now can define our bicomplex for $[\![\beta]\!]$. For the sake of avoiding confusion, let us denote this bicomplex as $\langle\!\langle\beta\rangle\!\rangle$. First we define $\langle\!\langle\sigma\rangle\!\rangle = [\![\sigma]\!]$ for any $\sigma \colon S_n$. Next we define $\langle\!\langle\bullet\rangle\!\rangle$ for positive and negative crossings by



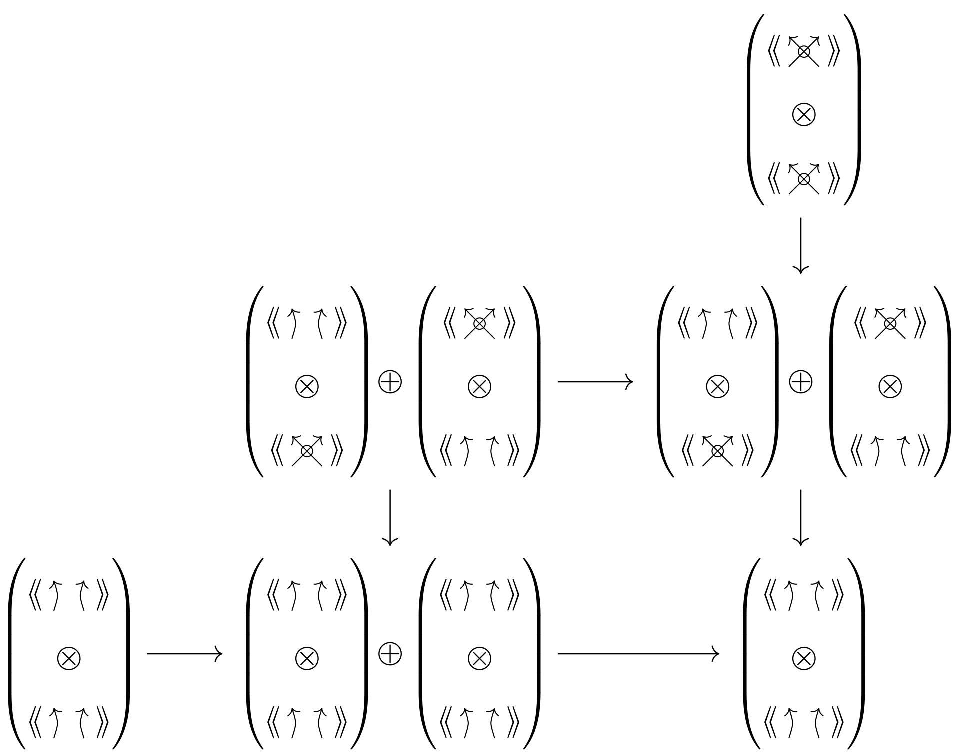
(5.25)



(5.26)

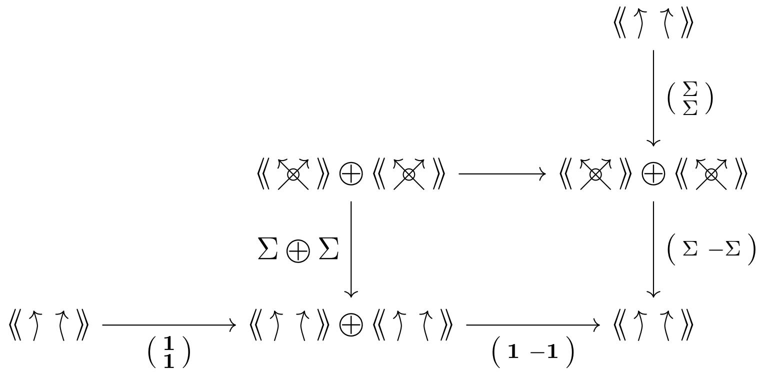
We then define $\langle\!\langle \beta \rangle\!\rangle$ via the gluing relations in (5.11) and (5.12), where the tensor product $\otimes$ now represents tensor products of bicomplexes. Likewise, we define $\langle\!\langle \overline{\beta} \rangle\!\rangle$ by applying the quotient map $E(\beta) \to E(\beta)/\langle \Delta_{ii} \rangle$.

Example 5.3.38. *We introduce a running example we will revisit in the following sections. Let* $\beta = (⤫)^2$*, then* $\overline{\beta}$ *is the (positive) Hopf link. Then* $\langle\!\langle \beta \rangle\!\rangle = \langle\!\langle ⤫ \rangle\!\rangle \otimes \langle\!\langle ⤫ \rangle\!\rangle$*. This is the bicomplex of matrix factorizations*



(5.27)

Above the maps in the bicomplex are given by Definition 5.3.37. Using Corollary 5.3.12 and Proposition 5.3.26, we simplify the complex above to



(5.28)

Note that $\langle\!\langle \overline{\beta} \rangle\!\rangle$ is actually a *tricomplex* of vector spaces.

Definition 5.3.39. A *tricomplex* over an abelian category $\mathcal{C}$ is a triply-graded object $(C_{i,j,k})_{i,j,k \colon \mathbb{Z}}$, where $C_{i,j,k} \colon \mathcal{C}$, equipped with three mutually anticommuting maps

$$d_1 : C_{i,j,k} \to C_{i+1,j,k},\ \ d_2 : C_{i,j,k} \to C_{i,j+1,k},\ \ d_3 : C_{i,j,k} \to C_{i,j,k+1}$$

such that $d_i^2 = 0$.

We will often write $C_{\bullet\bullet\bullet}$ or $(C_{i,j,k}, d_1, d_2, d_3)$ for the data comprising a tricomplex. From a tricomplex of vector spaces we have three natural ways to collapse to a bicomplex of vector spaces. We define define $\mathbf{Tot}(C)$ to be the *total complex* of C where

$$\mathbf{Tot}(C)_\ell = \bigoplus_{i+j+k=\ell} C_{i,j,k},\ \ d_{\mathbf{Tot}(C)} = d_1 + d_2 + d_3.$$

Now reconsider $\langle\!\langle \overline{\beta} \rangle\!\rangle$ as a tricomplex. We have three differentials: the first d_{KR} are the horizontal differentials from (5.25) and (5.26) coming from the canonical maps of (5.22) and (5.23). The second d_V are the virtual saddles (Σ_{ij}) or the vertical maps in the diagrams for (5.25) and (5.26). Finally, the last differential d_{MF} is the internal differential coming from the matrix factorization differentials on $\langle\!\langle \sigma \rangle\!\rangle$.

We can recover the bicomplex construction of $\langle\!\langle \overline{\beta} \rangle\!\rangle$ by considering the collapsed tricomplex with differentials d_{KR} and $d_V + d_{MF}$. We will specify when there could be any confusion if we are considering $\langle\!\langle \overline{\beta} \rangle\!\rangle$ as a bicomplex or a tricomplex.

Theorem 5.3.40. *Let L be an n-component link with braid representative β. Suppose β has writhe (braid exponent) $e(\beta)$. There exists a spectral sequence from $\langle\!\langle \overline{\beta} \rangle\!\rangle$ converging to $\mathcal{A}^{\otimes n}(e(\beta), (N-1)e(\beta), 0)$ whose E^2-page is isomorphic to $H_N(L)$.*

Remark 5.3.41. The convergence of such a spectral sequence, as in Theorem 5.3.40, was first proven by [Bec15] in a different framework. We will give an explicit proof here, but many of the same ideas appear as in the proof of his "cancelling spectral sequence".

Proof. We consider $\langle\!\langle \overline{\beta} \rangle\!\rangle$ as a bicomplex with differentials d_{KR} and $d_V + d_{MF}$. First we consider the convergence of the spectral sequence by computing the homology with respect to d_{KR} then $d_{MF} + d_V$. Note that all of the degree 0 rows in $\langle\!\langle \times \rangle\!\rangle$ and $\langle\!\langle \times \rangle\!\rangle$ are both contractible. Therefore, each row of $\langle\!\langle \beta \rangle\!\rangle$ will also be contractible except the row containing $\langle\!\langle \beta_{vir} \rangle\!\rangle(e(\beta), (N-1)e(\beta), 0)$, where β_{vir} is the virtual braid where we replace every crossing with a virtual crossing. Therefore, $H_{d_{KR}}(\langle\!\langle \overline{\beta} \rangle\!\rangle) \simeq \langle\!\langle \overline{\beta}_{vir} \rangle\!\rangle(e(\beta), (N-1)e(\beta), 0) \simeq \mathcal{A}^{\otimes n}(e(\beta), (N-1)e(\beta), 0)$. Therefore, the spectral sequence associated to the bicomplex collapses at the E^1-page to $\mathcal{A}^{\otimes n}(e(\beta), (N-1)e(\beta), 0)$ as desired.

We now compute the E^2-page by taking homology with respect to $d_{MF} + d_V$ and then d_{KR}. Let Γ be a signed MOY braid-graph. By Proposition 5.3.22 we know that $([\![\Gamma]\!], d_{MOY})$ is homotopy equivalent to $Tot(\langle\!\langle \Gamma \rangle\!\rangle)$ as matrix factorizations. Therefore, since closing the graphs will preserve this relation, we see that

$$H_{d_{MOY}}(\Gamma) \cong H(Tot(\langle\!\langle \Gamma \rangle\!\rangle), d_{MF} + d_V)$$

This proves that the E^1-page of our spectral sequence is equivalent to $H_{d_{MOY}}([\![\overline{\beta}]\!])$. Finally, since the differential d_{KR} is the same as before (up to homotopy equivalence), we get that the E^2-page is $H_N(L)$ as claimed before. □

We now discuss the connection with our construction in the last section. To compute

$$H(Tot(\langle\!\langle\overline{\Gamma}\rangle\!\rangle), d_{MF} + d_V)$$

we can apply a spectral sequence. Consider the E^1-page of this spectral sequence via computing homology with respect to d_{MF}. Each $\langle\!\langle\overline{\sigma}\rangle\!\rangle$ is reduced to a tensor power of the Frobenius algebra $\mathcal{A}$. Using Gaussian elimination to reduce $\langle\!\langle\overline{\sigma}\rangle\!\rangle$, induces the extra differentials introduced in the introduction and described more completely in the last section. Stopping at the E^1-page recovered the construction described in the introduction.

Remark 5.3.42. Note that in our construction we leave out the differentials on pages past the E^2-pages of the spectral sequence described in Theorem 5.3.40. It is not clear if the entire spectral sequence is a link invariant. In [AR14], the invariance of the virtual filtration for HOMFLY-PT homology is left as an open question due to the Reidemeister III isotopy. Though not explicitly discussed in that text, the same issue arises for $\mathfrak{sl}_N$ Khovanov-Rozansky homology. The possible noninvariance of the virtual filtration gives an obstruction to the spectral sequence being invariant under Reidemeister III as well. For this reason, we do not address the computation of these differentials, though the process in which we would do so is similar to the discussion.

Chapter 6

Conclusion: Broader Cosmologies of Structure

6.1 The Notionographic Perspective

This is not, unlike many philosophical texts on the subject, a work on the implications of category theory on the foundations of mathematics. Rather, we are interested in investigating what the internal conceptual organization of category theory has to say about what sorts of arrangements abstract ideas—from any context—may form. Put simply, what would it look like to have a philosophy whose shape mirrored that of category theory?

Our goals are less so that of a logical or ontological project—i.e. one concerned with the grounding basis for things—but more of a zoological one: in some sense we hope to catalogue the various species of creature that has and may form from the aggregate configuration of concepts. We introduce the term *notionography* for this inquiry of the variety and evolution of the various structures that arise in the medium of theory. As is often in philosophical contexts, we will use this term not only to refer to a field of study, but to any prevailing perspective within this discipline. Said otherwise, the term can signify both the question and an answer. This is in analogy with the term *epistemology* which can either mean the inquiry as to the nature of knowledge, or a particular approach—*an* epistemology—proposed as a solution to this question.

Epistemology, however, plays a role for us far beyond the linguistic homology that it shares with our proposed field. We will cast the field of notionography as an alternative approach to the field of epistemology. We are indeed interested in in the constitution of knowledge, but with an emphasis on the shape of this constitution as opposed to its boundary, e.g. in the form of a rule system for what is and is not allowed to be included within it. Our approach does not avoid this question, but rather seeks to demote it in relative significance.

If the reader considers our philosophical posture too eccentric, we hope that at the least the language we provide offers the thinker and communicator of abstract ideas sufficient expressive capacity so as to convey, or perhaps even imagine, something new. For example, in our language, we would say that analytic philosophy can be *defined* as the application of the notionography of early twentieth century foundations of mathematics—i.e. logic and set theory—to the broader medium of philosophy. We can then recast our opening paragraph's closing question as the application of the notionography of category theory to this broader philosophical medium. We hereby introduce the term *prismatic* for this notionography.

In which of these projects then are we interested: that of promoting the prismatic notionography or that of promoting the inquiry of notionography in general as a reassessment of epistemology? In some sense, we see these as one and the same. We do not mix levels unintentionally; in fact, the breaking down of this distinction of levels is paramount to our perspective. We aspire towards a *flat epistemology*, one in which "how the body of knowledge sits within totality" is insides of this very "body of knowledge." This is the prismatic notionography.

In Section 6.2, we discuss the relationship between (meta-)mathematics and philosophy. We then articulate more deeply both notionography in general and the prismatic notionography in particular—along with meditating on the non-hierarchical nature of these two currents.

6.2 The Role of Mathematics in Notionography

Mathematics enjoys a stature as the paragon of rigor. Much of philosophy and scientific theory aspire to its discursive structure of precise articulation of terms and subsequent logical deduction of conclusions.

There are two particularly noteworthy instances of a mode of practicing mathematics being emulated in philosophy. First, there is Euclid's *Elements* and its introduction of the axiomatic method. Even two millenia after its publication, Spinoza employed its methodology in his *Ethics* to present arguments about ideas as seemingly divorced from mathematics as the existence of God. Second, there is the early twentieth century emergence of formal logic, set theory, and the axiomatic foundations of mathematics, as seen—perhaps most iconically in *Principia Mathematica*—in the works of Russell, Whitehead, Frege, Cantor, Hilbert, and others. This second, in Kuhnian terms, revolution directly inspired the ideology of the analytic style of philosophy.

These examples imply a relationship in which one direction is characterized by mathematics' production of the most sophisticated methods for correct reasoning, and philosophy's subsequent attempts to apply these techniques to better probe its significantly broader, yet far more ambiguous and speculative, domain.

The transport in this direction, however, is not limited to *methodology*, but also includes *content*, in the sense that mathematical theories address existing or inspire new philosophical questions. For example, Zeno's paradoxes articulate the difficulty in explaining, at the least via our informal language, the infinity of the continuum. Such matters are addressed—although, as is common in philosophy, by no means closed once and for all—by the development of the calculus and its treatment of infinite series, and then more definitively with the rigor of mathematical analysis and point-set topology.

We now discuss a distinct—and, to our awareness, overlooked—potentially transferable material from mathematics to philosophy. As mathematics progresses in increasing abstraction, we encounter ever more ways in which mathematical concepts can intertwine. Such entanglements, when described in the collapsed terms of our informal language—including that of non-symbolic philosophy—are difficult to present without them appearing logically contradictory or circular.

For example, in algebraic topology we extract an algebraic object, e.g. the cohomology algebra, of a topological space in order to study it. The cohomology algebra itself though may be described as its own kind of geometric object, e.g. when it is given by a polynomial algebra—a structure which is often best understood through the methods of algebraic geometry, which see such objects as themselves instantiating a geometric space in their own right. These spaces we get from cohomology algebras, however, are themselves amenable to being studied by extracting from them a cohomology algebra. This example of course need not be—and indeed definitively is not—circular since the *specific* geometric and algebraic gadgets within each level are distinct. Imagine, however, that we'd used a language whose granularity bottomed out in just the words "space" and "algebra" It may well be tempting in this case for our minds to register a circularity. In the context of discussing highly abstract mathematical objects, the artificial limitation to a language with such poor expressivity seems preposterous if not downright masochistic. In the context of formal mathematics, we can typically increase the resolution of our terms until any sense of logical instability unravels.

The freedoms afforded to us in the realm of philosophy, however, do not necessarily offer as clear a way out. In particular, there is always an infinitude of possible resolutions, only limited in scope by the concepts we've charted within our imaginations—themselves bounded or at the least highly influenced by our biological and cultural contexts. Not only does this mean that we in general lack a principled way to navigate ourselves, but that our current map may lack routes altogether. This *resolution problem* can perhaps be seen as an explanation of the phenomena of paraodoxes.

The most salient separation between mathematics and philosophy is typically regarded as the former's capacity to reach unambiguous consensus. The latter is notorious for ceaselessly failing to do so. In our estimation, this state of affairs' most significant consequence is the lack, articulated in the preceding paragraphs, of definitive linguistic and conceptual resolutions. In particular, much like an impressionist painting, the body of philosophy can necessarily—by inherent lack in grounding pixelation—only be observed as an image out of focus.

It may therefore be worthwhile to stare at mathematical bodies until they too go out of focus. The blurred formations we register may very well hold deep insight into the often structurally bizarre the ways in which philosophical realms appear. In turn, this parallel implies that we need not panic about what looks to be paradoxical—if not felt outright deserving of prohibition—configurations of philosophical ideas.

We propose that this framing of mathematics and philosophy marks a distinct and, to our knowledge, newly articulated material that can be transported from mathematics to philosophy. This is precisely the context in which we introduce notionography. In the terms of this section, a notionography is the cosmology of allowed kinds of "meta-conceptual formations"—i.e. structural arrangements constituted by abstract concepts. Since, by some token, mathematics is the study of such formations, of abstractions, in the abstract, it is sensible to see it as a source from which to draw entries in our bestiary of them.

In particular, the present work is interested in the *prismatic* notionography, much of whose affordance has been endowed by category theory. In the following section, we discuss the prismatic notionography and the various phenomena that inhabit it. We aim to show that a prismatic notionography renders admissible hitherto excluded or outright unfathomable ideas, marking nothing short of an earnest expansion of our philosophical imagination.

6.3 The Prismatic Notionography of Categories

The data of a category is a collection of noun-like *objects*, often standing for mathematical structures, and verb-like *arrows*, in such contexts standing for their corresponding transformations. Arrows, which have a source and target object, and point from the former to the latter, may be composed into paths traversing some (potentially zero!) quantity of arrows across a sequence of objects. Any mathematical structure equipped with some notion of structure-respecting transformation—e.g. general sets and maps, orders (sets endowed with some, not necessarily complete, ordering relation) and order-preserving maps, topological spaces and continuous maps, etc.—can be assembled into a category. This category, in turn, often serves as an encoding into a single structure the *theory* of that given mathematical kind. For example, much of the theory of linear algebra can be seen as an inspection of the structure and properties of the category we call **Vect**, whose objects are vector spaces and arrows are linear maps.

The fun starts when you realize that certain familiar mathematical structures, e.g. sets, orders, and monoids (sets endowed with a notion of multiplication, allowing for the stringing together of elements into *words*) may be seen as special cases of a category. Sets are essentially categories without the arrows (and hence have objects reconsidered as elements), orderings categories in which there is either one or no arrow from any object to another (the existence of an arrow now instantiates the presence of an ordering relation, i.e. $x \to y$ standing for $x \preceq y$), and finally monoids are those categories consisting of only a single object so that the arrows pointing from said object to itself become the set of monoid elements which we cannow string together into words via arrow composition.

We must phenomenologically reconcile these two types of category, one *in* which we may store the entire theory of a mathematical kind, say that of monoids, and another which itself *is* a mathematical kind, say a monoid. There is, of course, a category of categories in which the category of monoids sits. The reader is to be assured that the necessary formalities are taken of so as to avoid a Russell-like paradox. In this sense, a category can simultaneously be the entity also the ambient frame within which the entities sit, thus eroding the distinction between the two.

We call such meta-conceptual events *pariefractures*. The term's etymology is given by the Latin *paries* for "wall, partition" and *fracture* for "(to) break." As its name suggests, the term's inspiration is the concept of "breaking the fourth wall," albeit a significant generalization. In a theatrical or literary text, a breaking of the fourth wall incites a renegotiation between the internal fiction of the text and the physical medium—occupying the same space as the audience—atop which it is rendered. Thus by pariefracture we mean any event marking the erosion of the boundary between frame and entity, between medium and message, between idea and paradigm, between ecology and organism, etc.

We hold that pariefracture is the primary meta-conceptual technology with which category theory terraforms the notionographic landscape. Most concepts that one encounters in category theory are scaffolded so as to afford its user a straightforward path to pariefracture. In fact, in the mental laboratory that is category theory, perhaps every single component of a mathematical gadget is amenable to a variation across an expansive possibility space in which, as one traverses across it, one encounters tiers of generality so varied that one it is common for one to see the original concept and the ambient framework in which that concept sits within during a single trip. One can see this in most of the articles on n-lab—the category theory wiki—in which any definition is almost, like clockwork, immediately followed by vast generalizations of it to contexts so distinct that the concept in its original incarnation becomes unrecognizable.

In this sense, the act of doing category theory is an invitation to a certain mode of thought which, although still buttressed by the mathematical hallmarks of formal definition and logical rigor, resembles in character an exercise in free-association, in which the usual rules of safely traversing concepts seem to break down.

The actual rules of course need not break down—although we personally take little exception with such happenings, as long as they are undertaken in a sufficiently careful manner! Rather, the way in which the elementary conceptual pieces may logically assemble—in the manner, as is classically depicted, like bricks with which we build, foundation-up, some architecture—into emergent forms within which one can simulate structures whose existence would be verboten were we to try to articulate them at the beginning.

This is reminiscent of the fact that, by virtue of its Turing completeness, Conway's Game of Life can at large scales render the image of a computer screen playing Conway's Game of Life. Given this, it could instead instead very well render some altered version of the game in which formerly disallowed local dynamics become available. And this of course is no contradiction! It is in this sense that we imagine a *prismatic notionography* in which concepts no longer feel like bricks to be stacked but become closer to permeable membranes that can tangle, enclose, thread or pass through, etc. each other. Except far *more* free than this, as we hope not to limit the expressivity of what we can imagine solely by the events allowable for permeable spatial membranes! In fact, we wish to articulate such great freedom of imagination that any entity from any context can very well serve as the structure encoding the formal relation itself among some other entities. One formal entities can very well enclose, compete with, simulate, seep in to, fall in love with, frame etc. any other within their shared frame, which itself is mere entity with which the relation can be renegotiated within some higher frame.

We now come back down to earth and return to describing some more explicit mathematical scenarios. The above situation—that of a category both *of* and *as* mathematical kind—is the first, but far from last or most imagination-bending, instance of level-mixing that one encounters as they travel further down the categorical rabbit hole, within which we now proceed further. One soon learns of functors, the arrow notion for the category-as-object. When the functors map between categories of some mathematical kind, and hence in some sense full theory, their aspects encode semantic relationships between the two theories. In particular, given a category $\mathcal{C}$ of one species of object and another $\mathcal{C}'$ of another species, which is in some sense less restricted than the first, we can define a *forgetful functor* that takes an object of $\mathcal{C}$ and simply discards the information—without editing the actual substance—that an object of $\mathcal{C}$ possesses that is not necessary to mention to see it as an instance of $\mathcal{C}'$.

For example, consider the relationship between monoids in general and commutative monoids—i.e. those in which $ab = ba$. There is a forgetful functor from the category of commutative monoids to that of all monoids that simply omits—but, again, does not actually change—the fact that its input monoids satisfy this extra *property*. In a somewhat similar yet slightly different vein, now consider the relationship between monoids and sets. By definition a monoids consists in part of an underlying set of elements. This allows us to define a forgetful functor that takes a monoid and simply discards the added *structure* with which we equipped the set to make it a monoid. We note that this procedure is somewhat distinct in that we may no longer recover that which we "forgot." In the first case, had we been given a monoid, without mention of its commutativity, we could just *test* to see if it was commutative. This is not so with the discarding of structure. For example, the natural numbers $\mathbb{N} = \{0, 1, \dots\}$ can be made into a monoid in at least two different ways—one in which the operation is given by addition and the other in which it is given by multiplication. Applying the forgetful functor to both of these, however, yields the set $\mathbb{N}$, without trace of the operational structure with which it was promoted to a monoid. The distinction between these philosophically distinct forms of forgetting—that of a binary condition to be satisfied vs. extra associated information in terms of which wholly new features can be defined—can in such a way be formulated purely in terms of the algebraic properties enjoyed by the relevant forgetful functor.

We in theory could choose to enter a metaphysically charged discussion on such a distinction—that between property and structure—but with category theory we are no longer required to do so—although we are of course still welcome!—in order to systematically refine the delineation of these two lofty concepts. In the language of forgetful functors, the act of "preserving property" and that of "preserving structure" are themselves, in an intriguingly self similar manner, *properties* that a forgetful functor may or may not possess. Thus, at least within this limited context, such lofty notions as "property" and "structure" are demoted to the status of mere symbolic expressions, only differing in degree and not in kind from the likes of the binomial formula. Thus, not only does category theory allow for a mathematically stable mixing of different levels of abstraction in mathematical objects, but it also facilitates the entanglement of different tiers of philosophical generality into this very mixture.

Sometimes a forgetful functor enjoys a sort of conceptual inverse, called a free functor. This is not an inverse in the sense that going one way and then going the other gets you back to where you started, but in the sense that they represent the least arbitrary and in some sense most efficient methods for transforming between the two species of mathematical object. For example, consider again the forgetful functor from monoids to sets—how do we go the other way; that is, given a set build a monoid from it? A naive attempt would be to somehow impose some multiplication table on the set's elements. Although we in principle could do such a thing, the choice is highly arbitrary. A more clever approach is to recall that in a monoid one can define the value of any word of elements. Well, if we need to be able to allow all strings, why not simply *freely* include all strings written in the alphabet of set's elements. For example, if the set is given by $S = \{a, b, c\}$, then its free monoid has as elements all of the strings of these letters, e.g. $abacba, c, \ , aabccc$—note the *non*-typo: there really is a blank there as we must as well include the empty word. Then, the binary operation on two such strings is their concatenation. We called this the free monoid on the set S.

The construction of the free monoid is what we mean by "conceptual inverse" to the forgetful functor that forgets all but a monoid's unerlying set of elements. This is *the* conceptual inverse in the vague sense that it is the least arbitrary way to transform a set into a monoid. In a formal sense, however, it is the "most efficient" way in which we can build a monoid from a set in that—and this is a theorem—*any other* monoid on the elements of S can be constructed by first forming the free monoiod and then subsequently imposing extra relations—e.g. of the form $b^3a^2 = ac^2$. We name call such characterizations *universal properties*, i.e. those possessed by constructions that one must always first "factor through" en route to constructing other instance of its kind. The Wikipedia article on adjoint functors provides a particularly compelling conceptual remark.

> The notion that F is the most efficient solution to the problem posed by G is, in a certain rigorous sense, equivalent to the notion that G poses the most difficult problem that F solves.

Such a relationship is again a deeply philosophical one. Just as in the previous case of structure and property, though, an abstraction into a more general mathematical framework—formally as dualizable 2-arrows in a 2-category—entirely reconfigures the concept's status. Ironically, this movement *upwards* in abstraction leaves *behind* the philosophy, towards a realm that is in this case not only symbolic but actually taken to the world of children's drawings, in which one plays with curved lines on the page as though they are toy strings that one can manipulate, free to forget the whole-theory's-worth of meaning being manipulated with every playful bend. In particular, although we won't go into formal detail, an adjunction given by functors F and G is a context in which it makes sense to draw the following pictures—taken from n-Lab, which obey the diagrammatic rules depicted below them.

In fact, applied category theorist Bob Coecke has given a complete formulation of finite dimensional quantum mechanics [Coe05]—in its own right a carrier of extremely disruptive philosophical thrust—purely in this mundane pictorial language of strings (which we carefully point out are not the strings of string theory, but exist at a wholly different conceptual location). He is currently setting up experiments in which he teaches school children to perform quantum mechanics calculations free of their physical meaning, in competition against Oxford University physics majors, equipped with the subject's classical Hilbert space formulation.

We finish the section by remarking on yet further escalations to meta-conceptual events. The *sets* of objects and *set* of arrows that constitute a category's data become generalized—in so called *internal categories*—to *objects of objects* and *objects of arrows*. Why not? Since sets, no longer enjoying a status of being some necessarily primitive form of plurality, are just objects in some category—that of sets and functions. Nothing—beyond the not-overly-restrictive stipulation of the presence of certain technically necessary features—stops us from varying this category. Do you want a space of objects? or would you like to go "upwards" and have a category of objects? Perhaps you instead prefer to go "downwards" and would like to select our object of objects from some category-as-structure, as in a set with indivisible elements? In the mathematically distinct but similar-in-spirit generalization of *enriched categories*, one can, with only minor cleverness, do something to this effect to yield a metric spaces—a set endowed with a notion of distance between its points—in which the entire collection of arrows from x to y is replaced by a single real number, encoding the "distance" from x to y. If we, instead of a real number, replace this set with a single Boolean value, we recover propositional logic [Law73]. Through such seemingly inert variation of some component within a theory-gadget—not unlike varying a parameter that dictates the curvature of some shape—category theory has the remarkable ability to flow across ideas which by their very essence seem incomparable.

The wavelength of thought one enters when pondering such concepts can be felt as a kind of defamiliarization—an experience induced upon repeating or stares at something until it takes on an unrecognizable form, stripped of the scaffolding of meaning we've built around it. A flat-screen television hung up on a wall can become a mere brushstroke of black paint across a canvas. This is, with more vivid mathematical detail, what in the prior section we meant by blurring our image of mathematics and philosophy so as to lose sight of the pixels but perhaps experience more directly the movements they simulate. A particularly Platonist stance—which we do not take but nonetheless feel compelled to mention—would posit that these movements are in fact the *real* object of study, forming the undercurrent that animates and perhaps directs the assembly of component conceptual formations, independent of the rigidity of our building blocks and assembly rules. This kind of perspective, which centers the "high-level" notions and hopes to see the "low-level" ones—what some may call the *foundations*—as a formality that is merely there to serve as a choice—one among many—of pixelation with which to render the actual image of interest. Such a posture—that of the "high-level" as the thing in itself—has been ascribed, albeit in the more constrained scope of just the mathematics, before, as in the following remark by David Corfield.

> To give an analogy in the field of architecture, when studying Notre Dame cathedral in Paris, you try to understand how the building relates to other cathedrals of the day, and then to earlier and later cathedrals, and other kinds of ecclesiastical building. What you dont do is begin by imagining it reduced to a pile of mineral fragments. Similarly, when category theorists think about those very important entities known as Lie groups, smooth manifolds that behave like groups, they do so by looking for group objects in the category of smooth manifolds, rather than by looking for sets which happen to be manifolds with compatible group structure. The idea here is to work out a theory of groups and then see which categories can support it, rather than just start out with sets. ([Cor03], 239).

If we take the—seemingly uncontroversial—presupposition that the dynamics of mathematical thought are strictly more constrained than those of philosophical thought, then a philosophical system is even more susceptible to be taken along the kind of journey that leads from a realm like property vs. structure to one children's drawings. It then becomes sensible to ask wholly new questions as to the underlying and overarching currents of that philosophical system, as its different transform in relative emphasis so as to place in relief distinct—and newly registrable—facets of its formation and potential variations there of.

Returning to the realm of visual images, if we conceive—and, again, not merely in ad hoc application of a metaphor but as earnest repositioning at which we may arrive through some tame variation of a formal component—a philosophy as a two dimensional picture, visible to our mind's eye, then what of the possibility of optical illusion? For example, when a human looks at a sufficiently human-face-like image—one which, however, definitively does depicts no face—then the mind still processes the image as a face. Phenomenological tendencies of this kind—ones in which, perhaps in the Kantian sense, involve the processing of inputs in terms of pre-installed (or, in the Kuhnian sense, installable) mechanisms that anticipate, and thus manipulate, the corresponding conceptual rendering—are at play in the interpretation of any conceptual system. If then, we can vary these perceptual encodings to unrecognizable shapes and levels, then we can make them available to markedly different perceptual tendencies that may very well lead us to what may seem as contradictory, absurd, or entirely defamiliarized impressions as to what the concepts *are*.

These are precisely the kinds of thought experiments that capture the spirit of our epistemology. We approach the problem of the typically hallowed category of knowledge by repositioning it via this flavor of definitional variation into something mundane and hence vulnerable to wholly new classes of phenomena—ones typically not conceived of as having any traceable along within the realm of abstract thought. The reconceptualizations afforded by such thought experiments introduce a potential infinitude of novel currents along which any theoretical system can be carried. This is the prismatic notionography.

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Biography

Dmitry Vagner attended Brown University from Fall 2008 until Spring 2013. He acquired a triple concentration in mathematics, applied mathematics, and economics along with a masters degree in applied mathematics. He subsequently enrolled in the Graduate School at Duke University in Fall 2013. At Duke, Dmitry pursued a doctorate in mathematics, a masters degree in computer science, and two graduate certificates, one in the "history and philosophy of science, technology, and medicine" and another in college teaching. During the 2014 and 2015 summers, Dmitry was a visiting student at MIT, where he pursued research in applied category theory under the tutelage of David Spivak. Their collaboration during this period resulted in a paper, *The Algebra of Open Dynamical Systems on the Operad of Wiring Diagrams*, that was published in the journal *Theory and Applications of Categories*. Furthermore, Dmitry was selected to participate in two competitive research schools: the MRC workshop on Homotopy Type Theory and the Applied Category Theory Adjoint School. In addition to his research accolades, Dmitry was also awarded the Bass Fellowship, which allowed him to design and teach a full-credit undergraduate course, Math 490: Introduction to Category Theory and Mathematized Thinking. In early 2019, Dmitry began working as a Simulation Scientist at the AI Research Division of Imbellus, an educational technology company.