RANDOM SPLITTING OF FLUID MODELS: ERGODICITY AND CONVERGENCE

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Abstract. We introduce a family of stochastic models motivated by the study of nonequilibrium steady states of fluid equations. These models decompose the deterministic dynamics of interest into fundamental building blocks, i.e., minimal vector fields preserving some fundamental aspects of the original dynamics. Randomness is injected by sequentially following each vector field for a random amount of time. We show under general assumptions that these random dynamics possess a unique invariant measure and converge almost surely to the original, deterministic model in the small noise limit. We apply our construction to the Lorenz-96 equations, often used in studies of chaos and data assimilation, and Galerkin approximations of the 2D Euler and Navier-Stokes equations. An interesting feature of the models developed is that they apply directly to the conservative dynamics and not just those with excitation and dissipation.

1. Introduction

This paper studies the long time dynamics of fluid-like equations which are kept out of equilibrium. Among the simplest examples of fluid models displaying interesting out-of-equilibrium behavior (such as fluxes across scales) are the two-dimensional Euler and incompressible Navier-Stokes equations.

On the 2-dimensional torus $\mathbb{T}$, i.e., $\mathbb{T} \coloneqq [0, 2\pi]^2$ with periodic boundary conditions, the Navier-Stokes equations, which model the flow of an incompressible fluid, are

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p + F + \nu \Delta u \,, \\ \operatorname{div}(u) \coloneqq \nabla \cdot u = 0 \,, \end{cases} \tag{1.1}$$

where $u : \mathbb{T} \times \mathbb{R} \to \mathbb{R}^2$ is the fluid velocity, $p : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ the fluid pressure,

$$(u \cdot \nabla) u = (u_1 \partial_1 u_1 + u_2 \partial_2 u_1, u_1 \partial_1 u_2 + u_2 \partial_2 u_2) \qquad \text{and} \qquad \Delta u = \partial_1^2 u_1 + \partial_2^2 u_2 \,.$$

Here $u = (u_1, u_2)$ and $\partial_j \coloneqq \partial_{x_j}$. The viscosity $\nu > 0$ measures the strength of the dissipation introduced by the Laplacian Δ, and $F(x,t)$ is an external driving force whose role is to keep the system from relaxing to the trivial state $u \equiv 0$. By balancing the dissipative effect of Δu, the forcing term allows the system to establish an out-of-equilibrium steady state. Such statistical equilibria often develop fluxes across scales, a phenomenon whose study is an active area of research. Often F is taken to live on only a few scales so that the flux out of those scales can be studied [EM01, Mat03, HM06, KNS20a]. In practice, the forcing $F(x,t)$ is usually taken to be stochastic in space and time for some stationary distribution which is typically white in time [FM95, EM01, DPD02, HM06]. A common choice in the literature is $F(x,t) = \sum \psi_k(x) \dot{W}_k(t)$ where each $\psi_k(x)$ is a fixed spatial forcing and $\dot{W}_k(t)$ is a white in time noise term written here as the formal derivative of a Brownian motion. Stochastic forcing serves multiple purposes in these settings. On one hand, as already mentioned, it provides the energetic excitation which keeps the system out of equilibrium and allows for the establishment of a nontrivial statistical steady state. On the other hand, it provides local agitation which ensures the existence of a unique statistical steady state to

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which the system converges regardless of initial condition. In other words, it guarantees the forcing is sufficiently varied and generic to ensure convergence to a single long time statistical behavior of the system independent of the system's initial configuration.

This paper studies a class of processes, introduced in the next section, injecting randomness into the fluid models of interest while separating in a simple way the various roles served by noise in previous works. In particular, the randomness is used primarily to ensure that the dynamics is sufficiently generic that unique ergodicity follows for a broad class of initial conditions. This will free one to use a much less disruptive class of forcing to keep the system out of equilibrium. More specifically, the class of models introduced below have a number of desirable properties:

(1) They allow one to separate the effect of forcing, which keeps the system out of equilibrium, and stochastic agitation, which ensures the system has a unique long time statistical behavior.
(2) The stochastic agitation is strongly non-reversible being constructed from dynamics which only flow in the directions the original dynamics was heading.
(3) The stochastic agitation preserves the conserved quantities of the original dynamics. This allows the properties of the (stochastic) conservative dynamics to be studied directly.
(4) The model dynamics will be constructed as the composition of simple dynamics, isolating particular nonlinear interactions which are relatively intuitive and can be explicitly analyzed.

This decomposition is also partially motivated by the classical stylized models of dynamics studied in depth at the dawn of the theory of dynamical systems. Balancing between solvability and simplicity of its fundamental building blocks and preservation of the fundamental macroscopic properties of the original fluid equation, we expect that the stochastic models introduced in this paper will provide meaningful physical and dynamical insight in nonequilibrium steady states of models such as (1.1).

1.1. **A class of stochastic models.** We now introduce the general idea underlying the class of stochastic models, called *random splitting*, that we study in this paper. A more systematic definition of these models is deferred to Section 2. Consider an ordinary differential equation (ODE)

$$\dot{x} = V(x) = \sum_{k=1}^{n} V_k(x)\,, \tag{1.2}$$

where $n \in \mathbb{N}$ and V and $\{V_k\}_{k=1}^n$ are vector fields on $\mathbb{R}^d$. In what follows, we choose the V_k so that the dynamics

$$\dot{x} = V_k(x) \tag{1.3}$$

are in some sense simpler than the dynamics corresponding to (1.2). We then approximate the solution $\Psi_t \colon x(0) \mapsto x(t)$ of (1.2) with compositions of the solution maps $\varphi_t^{(k)} \colon x(0) \mapsto x(t)$ of (1.3). This procedure is known as operator splitting in the numerical analysis literature and is often used in numerical simulation of various ordinary, partial and stochastic differential equations [BNS96, AR96, GK96, MS16, Kus04, NV08, NN09, COS19, CST$^+$21]. Typically, the goal is to leverage the fact that each of the dynamics in (1.3) is more computationally tractable than (1.2) to construct an efficient and accurate numerical method. A variant of these models was also explored in the thesis [Wil19].

Here our goal is related but slightly different. Specifically, the random splitting model associated to (1.2) is constructed by composing each of the $\varphi^{(k)}$ for a random length of time with a common mean. In other words, instead of evolving each $\varphi^{(k)}$ for a fixed time h as in traditional operator splitting methods, we evolve each of the $\varphi^{(k)}$ for a random time with mean h. Repeated composition then produces dynamics on $\mathcal{O}(1)$ times. The evolution times for each $\varphi^{(k)}$, and over each cycle, will be identically distributed and mutually independent. As in the numerical analysis context, we hope to leverage the simplified nature of each $\varphi^{(k)}$, obtained from (1.3), to gain insight into the complex dynamics of the composition of maps. We will also see that as the mean evolution time $h \to 0$, the random splitting associated to (1.2) will converge to the deterministic dynamics Ψ_t on finite time intervals. Furthermore, we are most interested in studying the random splitting in its own right and not as an approximation of (1.2). We will be particularly interested in its long time dynamics and qualitative understanding of the stationary dynamics the random splitting produces when $h > 0$.

1.2. **Two motivating examples.** In this paper, we consider two motivating examples: A conservative version of the Lorenz-96 model and the Galerkin approximations of the vorticity formulation of the 2D Euler equations. We then use these analyses to study the full Lorenz-96 model and Galerkin approximations of the vorticity formulation of 2D Navier-Stokes.

Lorenz-96. Fix $n \geq 4$ and let $\{e_k\}_{k=1}^n$ denote the standard basis of $\mathbb{R}^n$. The Lorenz-96 model is

$$\dot{x} = \sum_{k=1}^{n} \big((x_{k+1} - x_{k-2})x_{k-1} - \nu x_k + F_k\big)e_k \tag{1.4}$$

for $x \in \mathbb{R}^n$, $\nu > 0$, and nonnegative constants F_k, where the indices are periodized via the identities $x_{-1} \coloneqq x_{n-1}$, $x_0 \coloneqq x_n$, and $x_{n+1} \coloneqq x_1$. The $-\nu x_k$ term in (1.4) represents dissipation in the kth coordinate and F_k is a forcing constant. Initially, we study a variant of Lorenz-96, called *conservative Lorenz-96*, obtained by removing the dissipation and forcing terms from Lorenz-96. That is,

$$\dot{x} = V(x) \coloneqq \sum_{k=1}^{n} (x_{k+1} - x_{k-2})x_{k-1}e_k\,. \tag{1.5}$$

We sometimes refer to the original Lorenz-96 model as the *forced Lorenz-96* model to emphasize the forcing (though the dissipation is equally important). For conservative Lorenz-96, we will decompose V into a collection of simple rotations by observing that

$$V(x) = \sum_{k=1}^{n} V_k(x) \tag{1.6}$$

where $V_k(x) \coloneqq (x_{k+1}e_k - x_k e_{k+1})x_{k-1}$. The dynamics given by $\dot{x} = V_k(x)$ are easy to understand on their own; any complex behavior comes from interactions of the rotations. Importantly, each V_k is chosen to conserve, like V, the system's *energy*, which is the square of the usual Euclidean norm of x, namely $\|x\|^2 \coloneqq \sum_{k=1}^n x_k^2$.

2D Euler. Returning to (1.1), we begin by defining the scalar vorticity $q(x,t) = \operatorname{curl} u(x,t)$ of the velocity field $u(x,t)$. Initially, we will consider the Euler equations which are obtained from (1.1) by taking $\nu = F = 0$. Writing the equation for the jth Fourier mode $q_j \in \mathbb{C}$, defined by $q(x,t) = \sum_j q_j(t)e_j(x)$ for $e_j(x) \coloneqq e^{ix\cdot j}$, and $j \in \{j \in \mathbb{Z}^2 : |j| < N, j \neq 0\}$, we have

$$\dot{q}_j = -\sum_{j+k+\ell=0} C_{k\ell}\bar{q}_k\bar{q}_\ell \tag{1.7}$$

for a constant $C_{k\ell}$ defined in Section 6.1. We will see that this system has two conserved quantities, the *enstrophy*, $\sum_j |q_j|^2$, and the *energy*, $\sum_j |j|^{-2}|q_j|^2$. Notice that the definition of energy differs between this equation and the Lorenz-96 model.

As in the Lorenz-96 model, we introduce the simpler dynamics $\dot{q} = V_{jk\ell}(q)$ where $V_{jk\ell}(q) = -C_{k\ell}\bar{q}_k\bar{q}_\ell e_j - C_{j\ell}\bar{q}_j\bar{q}_\ell e_k - C_{jk}\bar{q}_j\bar{q}_k e_\ell$ and observe that

$$V(q) = \sum_{j+k+\ell=0} V_{jk\ell}(q)\,.$$

We will see in Section 6 that with this choice of splitting the dynamics $\dot{q} = V_{jk\ell}(q)$, like the original system $V(q)$, preserve the important physical quantities of enstrophy and energy.

Remark 1.1. *In Section 6, we further simplify these complex-valued dynamics by projecting onto a real basis. The current choice is sufficient for an introductory discussion.*

Remark 1.2. *Our results do not focus on establishing minimal hypoellipticity assumptions for our systems; the stochastic agitation we use is more global than the minimal hypoellipticity forcing considered in* [EM01, HM06]. *We hope this will allow us to progress further than with previous models while still preserving much of the physically interesting dynamics.*

Remark 1.3. *It is important to emphasize that, with regard to unique ergodicity, the main role of the forcing* $F(x) = \sum F_k e_k(x)$ *is only to destroy the fixed points and other low-dimensional invariant structures of the original flows and not to provide the stochastic mixing which ensures ergodicity. This is provided by the random splitting and is in contrast to the results in* [EM01, KS00, EMS01, KS03, HM06, KNS20b, BBPS21].

Organization of paper. In Section 2, we introduce random splitting. In Section 3, we give conditions for random splitting to be uniquely ergodic, i.e., to have a unique invariant measure. In Section 4, we show that random splitting converges to its deterministic counterpart (1.2) on finite time intervals both in terms of its transition kernel and almost surely as the average time step goes to zero. In Sections 5 and 6, we construct random splittings of conservative Lorenz-96 and Galerkin approximations of 2D Euler and apply the preceding results to show that these splittings are uniquely ergodic and converge as above. In Section 7, we consider the Lorenz and Euler models when fixed forcing and dissipation are added. We again construct random splittings of these models, prove convergence, and show that if the forcing is not aligned with the equations' invariant structures (such as fixed points) then both randomly split Lorenz-96 and Galerkin approximations of 2D Navier-Stokes are uniquely ergodic.

2. Random splitting in a general setting

Let $\{V_k\}_{k=1}^n$ be $\mathcal{C}^2$ vector fields[1] on $\mathbb{R}^d$ and set

$$V \coloneqq \sum_{k=1}^{n} V_k\,. \tag{2.1}$$

Denote the flow of $\dot{x} = V(x)$ by Ψ and the flow of $\dot{x} = V_k(x)$ by $\varphi^{(k)}$. Ψ is the *true dynamics*. To construct a random dynamics approximating Ψ, fix $h > 0$, let $\tau = (\tau_k)_{k=1}^{\infty}$ be a sequence of independent exponential random variables with mean 1, and set $h\tau \coloneqq (h\tau_k)_{k=1}^{\infty}$. The approximating dynamics, henceforth referred to as the *random splitting associated to* (2.1) or just *random splitting* for short, is the Markov chain $\{\Phi_{h\tau}^m\}_{m=0}^{\infty}$ defined by $\Phi_{h\tau}^0 \coloneqq I$ and, for $m > 0$,

$$\Phi_{h\tau}^m \coloneqq \varphi_{h\tau_{mn}}^{(n)} \circ \cdots \circ \varphi_{h\tau_{(m-1)n+1}}^{(1)}(\Phi_{h\tau}^{m-1}), \tag{2.2}$$

where I is the identity on $\mathbb{R}^d$, $\Phi \coloneqq \varphi^{(n)} \circ \cdots \circ \varphi^{(1)}$, and Φ^m is the m-fold composition of Φ. Note $h\tau_k \overset{iid}{\sim} \mathrm{Exp}(1/h)$. Therefore, starting from the current step, the next step of the chain is obtained by following each V_k for $\mathrm{Exp}(1/h)$ time in order from $k = 1$ to n. The chain is Markovian because the random times are independent. Its transition kernel P_h acts on measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ via

$$P_h f(x) = \mathbb{E}\big(f(\Phi_{h\tau}(x))\big) = \int_{\mathbb{R}_+^n} f(\Phi_{ht}(x)) e^{-\sum_{k=1}^n t_k} dt \tag{2.3}$$

where $\mathbb{R}_+ \coloneqq (0, \infty)$, $t = (t_1, \dots, t_n)$, and $dt = dt_1 \cdots dt_n$.

[1] We use calligraphic $\mathcal{C}^k$ for k-times continuously differentiable maps throughout to avoid confusion with constants which are often denoted by normal script C (for example, the constants C_{jk} in 2D Euler).

Remark 2.1. *Throughout this paper the superscripts k in $\varphi^{(k)}$ and subscripts k in V_k are understood to be taken modulo n if $k \bmod n \neq 0$ and to be n otherwise. For example, if $n = 3$,*

$$\varphi^{(6)} \circ \varphi^{(5)} \circ \varphi^{(4)} \circ \varphi^{(3)} \circ \varphi^{(2)} \circ \varphi^{(1)} = \varphi^{(3)} \circ \varphi^{(2)} \circ \varphi^{(1)} \circ \varphi^{(3)} \circ \varphi^{(2)} \circ \varphi^{(1)}.$$

Also, the t in Φ_t^m is always a sequence $t = (t_1, \dots, t_{mn})$ or, more generally, $t = (t_k)_{k=1}^{\infty}$, so that

$$\Phi_t^m(x) = \varphi_{t_{mn}}^{(n)} \circ \cdots \circ \varphi_{t_1}^{(1)}(x)\,.$$

Note the above is a composition of mn flows, as in (2.2).

Remark 2.2. *All results in this paper remain true if at each step we randomly permute indices in the composition Φ. That is, given a current state x, the next step is $\varphi_{h\tau_n}^{(\sigma(n))} \circ \cdots \circ \varphi_{h\tau_1}^{(\sigma(1))}(x)$ where σ is a random permutation of $\{1, \dots, n\}$. This yields both additional randomness and an avenue to higher order approximations of the true dynamics* [Kus04, NV08, NN09, COS19, CST+21]. *We forgo this more general setting however to keep exposition more approachable and notationally light.*

Remark 2.3. *The times are assumed exponentially distributed for convenience. All results extend to any distribution on $[0, \infty)$ with positive density at zero and sufficiently fast decay at infinity.*

Though our setting is $\mathbb{R}^d$, it will be convenient in what follows to narrow our focus to

$$\mathcal{X}(x_0) \coloneqq \left\{\Phi_t^m(x_0) : m \geq 0, t \in \mathbb{R}_+^{mn}\right\}, \tag{2.4}$$

the set of points in $\mathbb{R}^d$ that can be reached by the split dynamics starting from x_0 in any finite number of steps and over arbitrary finite times. Note that if random splitting starts from any $x \in \mathcal{X}(x_0)$ it will stay in $\mathcal{X}(x_0)$ for all time. When x_0 is clear from context, we simply denote this space by $\mathcal{X}$. The following assumption is used at various points throughout the paper.

Assumption 1. *$\mathcal{X}(x_0)$ is bounded for each x_0 in $\mathbb{R}^d$.*

Since the vector fields V_k are $\mathcal{C}^2$, Assumption 1 further implies that for every $x_0 \in \mathbb{R}^d$ the V_k are bounded with bounded first and second derivatives on $\mathcal{X}$. In particular,

$$C_*(x_0) \coloneqq \sup_{x \in \mathcal{X}(x_0)} \left\{\|V_k(x)\|, \|DV_k(x)\|, \|D^2V_k(x)\| : 1 \leq k \leq n\right\} < \infty\,, \tag{2.5}$$

where $\|V_k(x)\|$ is the usual Euclidean norm, $\|DV_k(x)\|$ is the operator norm of the linear map $DV_k(x) : \mathbb{R}^d \to \mathbb{R}^d$, and $\|D^2V_k(x)\|$ is the operator norm of the bilinear map $D^2V_k(x) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$.

Recall that *Hausdorff measure* on a smooth submanifold $\mathcal{M}$ embedded in $\mathbb{R}^d$ is, up to multiplication by some constant, equal to the volume form on $\mathcal{M}$ induced by the Euclidean metric on $\mathbb{R}^d$; see for example [Fol13, Chapter 11]. With this terminology we state the following assumption, also used at various points throughout the paper, which ensures there is a natural reference measure on $\mathcal{X}$.

Assumption 2. *There is a smooth submanifold $\mathcal{M}$ embedded in $\mathbb{R}^d$ such that $\lambda(\mathcal{X}) > 0$, where λ is the Hausdorff measure on $\mathcal{M}$.*

In all examples in this paper, $\mathcal{X}$ will either lie on a sphere (conservative Lorenz-96), on the intersection of a sphere and an ellipsoid (2D Euler), or inside a ball of finite radius in $\mathbb{R}^d$ (Lorenz-96 and 2D Navier-Stokes). In the first two cases, λ is the Hausdorff measure on the sphere and on the intersection of the sphere and ellipsoid, respectively, and in the latter it is the usual Lebesgue measure on $\mathbb{R}^d$.

3. Ergodicity

Throughout this section $\mathcal{X}$ satisfies Assumption 2, $h > 0$ is fixed, and $P \coloneqq P_h$ is the transition kernel in (2.3). For notational convenience we make the change of variable $ht \mapsto t$ and write (2.3) as

$$Pf(x) \coloneqq P_h f(x) = \int_{\mathbb{R}^n_+} f(\Phi_t(x))\rho_h(t)dt$$

where $\rho_h(t)$ is the joint density of n independent exponential random variables with mean h. A measure μ on $\mathcal{X}$ is *P-invariant* if $\mu P = \mu$ where μP is defined by

$$\mu P f \coloneqq \int_{\mathcal{X}} Pf(x)\mu(dx)$$

for all bounded, measurable functions $f : \mathcal{X} \to \mathbb{R}$. The main result of this section is

Theorem 3.1. *If there exists $x_* \in \mathcal{X}$ such that for all $x \in \mathcal{X}$ there is an m and t with $\Phi^m(x,t) = x_*$ and $D_t\Phi^m(x,t) : T_t\mathbb{R}^{mn}_+ \to T_{x_*}\mathcal{M}$ surjective, then P has at most one invariant measure.*

Here $\mathcal{M}$ is the manifold from Assumption 2 and $T_{x_*}\mathcal{M}$ is the tangent space of $\mathcal{M}$ at x_*. The proof of Theorem 3.1 follows from the classical minorization condition [Num78, MT09, MSH02, HM11] given by the following result, which appears in [BBMZ15, Lemma 6.3].

Lemma 3.2. *Suppose U is an open subset of $\mathbb{R}^m$ and τ a U-valued random variable with continuous density ρ. Let $d \le m$ and let $F : \mathcal{X} \times U \to \mathcal{X}$ be $\mathcal{C}^1$. If for some $(x,t) \in \mathcal{X} \times U$ the map $D_tF(x,t)$ has rank d and ρ is bounded below by $c_0 > 0$ on a neighborhood of t, then there exists a constant $c > 0$ and neighborhoods U_x of x and U_* of $x_* \coloneqq F(x,t)$ such that*

$$\mathbb{P}\big(F(y,\tau) \in B\big) \ge c\lambda(B \cap U_*) \tag{3.1}$$

for all $y \in U_x$ and $B \in \mathcal{B}(\mathcal{X})$.

Remark 3.3. *In our setting, $U = \mathbb{R}^{mn}_+$, $F = \Phi^m : \mathcal{X} \times \mathbb{R}^{mn}_+ \to \mathcal{X}$, and $\tau = (\tau_1, \ldots, \tau_{mn})$ with the τ_k independent exponential random variables with mean h. In this case, if $x_* = \Phi^m(x,t)$ for some t with $D_t\Phi^m(x,t)$ surjective, then Lemma 3.2 guarantees the existence of a constant $c > 0$ and neighborhoods U_x of x and U_* of x_* such that, for all $y \in U_x$ and $B \in \mathcal{B}(\mathcal{X})$,*

$$P^m(y,B) \ge c\lambda(B \cap U_*)\,. \tag{3.2}$$

Proof of Theorem 3.1. The proof is by contradiction. Suppose μ_1 and μ_2 are distinct P-invariant probability measures. Without loss of generality we assume both μ_i are ergodic and therefore mutually singular. Therefore, there exist disjoint measurable sets A_1 and A_2 partitioning $\mathcal{X}$ such that $\mu_i(B) = \mu_i(B \cap A_i)$ for all $B \in \mathcal{B}(\mathcal{X})$. Fix x_i in the support of μ_i so, by definition, μ_i gives strictly positive measure to every neighborhood of x_i. By the hypothesis and Remark 3.3 there exist $c_i > 0$, $m_i \in \mathbb{N}$, and neighborhoods U_i of x_i and U_* of x_* such that $P^{m_i}(x,\cdot) \ge c_i\lambda(\cdot \cap U_*)$ for all $x \in U_i$. So,

$$\mu_i(B) = \mu_i P^{m_i}(B) \ge \int_{U_i} P^{m_i}(x,B)\mu_i(dx) \ge c_i\lambda(B \cap U_*)\mu_i(U_i)$$

for all $B \in \mathcal{B}(\mathcal{X})$. In particular, $\mu_i(B) = 0$ implies $\lambda(B \cap U_*) = 0$ since c_i and $\mu_i(U_i)$ are strictly positive. But $\mu_1(A_2 \cap U_*) = \mu_2(A_1 \cap U_*) = 0$ and hence

$$0 < \lambda(U_*) = \lambda(A_1 \cap U_*) + \lambda(A_2 \cap U_*) = 0,$$

a contradiction. ∎

$$
\begin{array}{ccccccccccccc}
V_1 & \xrightarrow{D_x\varphi^{(2)}} & V_2 & \xrightarrow{D_x\varphi^{(3)}} & \cdots & \xrightarrow{D_x\varphi^{(k)}} & V_k & \xrightarrow{D_x\varphi^{(k+1)}} & \cdots & \xrightarrow{D_x\varphi^{(N-1)}} & V_{N-1} & \xrightarrow{D_x\varphi^{(N)}} & V_N \\
\uparrow & & \uparrow & & & & \uparrow & & & & \uparrow & & \uparrow \\
x^{(1)} & \xrightarrow{\varphi^{(2)}} & x^{(2)} & \xrightarrow{\varphi^{(3)}} & \cdots & \xrightarrow{\varphi^{(k)}} & x^{(k)} & \xrightarrow{\varphi^{(k+1)}} & \cdots & \xrightarrow{\varphi^{(N-1)}} & x^{(N-1)} & \xrightarrow{\varphi^{(N)}} & x^{(N)}
\end{array}
$$

FIGURE 1. Columns of $D_t\Phi^m(x,t)$.

The time derivative $D_t\Phi^m(x,t)$ of the composite flow Φ^m is an important object in the preceding discussion. Setting $N \coloneqq mn$ and recalling Remark 2.1 on sub-and-superscript conventions, $D_t\Phi^m(x,t)$ is expressed in local coordinates by the $d \times N$ matrix

$$
D_t\Phi^m(x,t) = \begin{pmatrix} | & & | & | \\ J_{2,N}(x^{(1)},t)V_1\big(x^{(1)}\big) & \cdots & J_{N,N}(x^{(N-1)},t)V_{N-1}\big(x^{(N-1)}\big) & V_N\big(x^{(N)}\big) \\ | & & | & | \end{pmatrix}. \tag{3.3}
$$

Here $x^{(k)}$ is defined recursively, starting from $x^{(1)} \coloneqq \varphi^{(1)}_{t_1}(x)$, by $x^{(k)} \coloneqq \varphi^{(k)}_{t_k}(x^{(k-1)})$, and

$$
J_{k,\ell}(x,t) \coloneqq D_x\big(\varphi^{(\ell)} \circ \cdots \circ \varphi^{(k)}\big)(x,t)
$$

for any $k \le \ell$. So, the kth column of $D_t\Phi^m(x,t)$ is obtained by evaluating V_k at the point $x^{(k)}$ then pushing forward the vector $V_k(x^{(k)}) \in T_{x^{(k)}}\mathcal{X}$ to $T_{x^{(N)}}\mathcal{X}$ via $J_{k+1,N}(x^{(k)},t)$.

Suppose we are at the point $x^{(N)} = \Phi^m(x,t)$ for $t = (t_1,\dots,t_{mn})$ some arbitrary vector of positive times. Adjoining n zeros to t, i.e., $t' \coloneqq (t,0,\dots,0)$, trivially gives $\Phi^{m+1}(x,t') = \Phi^m(x,t) = x^{(N)}$. And since $J_{k,\ell}(x,0)$ is the identity for all x, Equation (3.3) gives

$$
D_t\Phi^{m+1}(x,t') = \begin{pmatrix} | & & | & | & & | \\ J_{2,N}(x^{(1)},t)V_1\big(x^{(1)}\big) & \cdots & V_1\big(x^{(N)}\big) & V_2(x^{(N)}) & \cdots & V_N\big(x^{(N)}\big) \\ | & & | & | & & | \end{pmatrix}. \tag{3.4}
$$

In particular, the last n columns of $D_t^{(m+1)}(x,t')$ are the V_k evaluated at $x^{(N)}$. This observation gives a sufficient criterion for verifying the surjectivity condition in Theorem 3.1.

Corollary 3.4. *Suppose there exists an $x_* \in \mathcal{X}$ such that for every $x \in \mathcal{X}$ there is an $m \in \mathbb{N}$ and $t \in \mathbb{R}^{mn}_+$ satisfying $\Phi^m_t(x) = x_*$. If the vector fields V_k span the tangent space $T_{x_*}\mathcal{X}$ at x_* then P has at most one invariant measure.*

Proof. Fix $x \in \mathcal{X}$ and let m and $t \in \mathbb{R}^{mn}_+$ be such that $\Phi^m(x,t) = x_*$. Setting $t' \coloneqq (t,0,\dots,0)$ as above, the assumption that the V_k span $T_{x_*}\mathcal{X}$ together with (3.4) imply $D_t\Phi^{m+1}(x,t')$ is surjective. By continuity of Φ there exists $t'' = (t,\varepsilon)$ with $\varepsilon \in \mathbb{R}^n_+$ such that $D_t\Phi^{m+1}(x,t'')$ is surjective. In particular, ε is independent of x and hence $x'_* \coloneqq \Phi(x_*,\varepsilon)$ satisfies the conditions of Theorem 3.1. So, P has at most one invariant measure. □

4. Convergence as mean time step goes to zero

A well-known result in the operator splitting literature is that the error incurred in approximating Ψ by the deterministic splitting scheme $\Phi_h = \varphi^{(n)}_h \circ \cdots \circ \varphi^{(1)}_h$ is $\mathcal{O}(h)$ [MS16]. That is, the deterministic splitting converges to the true dynamics at worst linearly in h as $h \to 0$. In this section we give analogous results for random splitting; the pluralized "results" reflects that with randomness comes several different notions of convergence. Specifically, we give two main results. First, as in the deterministic case, the transition kernel P_h of random splitting converges to the true dynamics linearly in h as $h \to 0$. Second, random splitting converges almost-surely to the true dynamics as

$h \to 0$. These statements are made precise in Theorems 4.1 and 4.5, respectively, but to make sense of them we first introduce the appropriate setting.

Let $\mathcal{X}$ be as in (2.4). For a positive integer k let $\mathcal{C}^k(\mathbb{R}^d)$ denote the space of k-times continuously differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ and, viewing $\mathcal{X}$ as a subset of $\mathbb{R}^d$, let $\mathcal{C}^k(\mathcal{X})$ be the space of functions $f \in \mathcal{C}^k(\mathbb{R}^d)$ restricted to $\mathcal{X}$. For $f \in \mathcal{C}^k(\mathbb{R}^d)$ and $\ell \leq k$, the ℓth derivative $D^\ell f(x)$ of f at x is a multilinear operator from $\otimes_1^\ell \mathbb{R}^d$ to $\mathbb{R}$. The operator norm of $D^\ell f(x)$ is then

$$\|D^\ell f(x)\| \coloneqq \sup_{\|\eta\|=1} \left\{ |D^\ell f(x)\eta| \right\} ,$$

where $\eta \in \otimes_1^\ell \mathbb{R}^d$. Defining $D^0 f(x) \coloneqq f(x)$, this in turn induces a norm on $\mathcal{C}^k(\mathcal{X})$ given by

$$\|f\|_k \coloneqq \sup_{x \in \mathcal{X}} \left\{ \|D^\ell f(x)\| : 0 \leq \ell \leq k \right\} .$$

The corresponding operator norm is denoted $\|\cdot\|_{k\to k}$. More generally, for any k and ℓ define a norm $\|\cdot\|_{k\to\ell}$ on the space of linear operators $L : \mathcal{C}^k(\mathcal{X}) \to \mathcal{C}^\ell(\mathcal{X})$ by

$$\|L\|_{k\to\ell} \coloneqq \sup_{\|f\|_k=1} \|Lf\|_\ell \, .$$

We make frequent use of the *submultiplicity* of $\|\cdot\|_{k\to\ell}$. Namely, if A and B are bounded linear operators from $\mathcal{C}^j(\mathcal{X})$ to $\mathcal{C}^k(\mathcal{X})$ and from $\mathcal{C}^k(\mathcal{X})$ to $\mathcal{C}^\ell(\mathcal{X})$, respectively, then

$$\|BA\|_{j\to\ell} \leq \|B\|_{k\to\ell} \|A\|_{j\to k} \, .$$

The results below are stated in terms of semigroups of the flows Ψ and $\varphi^{(j)}$, which are $\mathcal{C}^2$ by assumption. Hence for all $k \leq 2$ the semigroup $\{S_t\}_{t\geq 0}$ corresponding to Ψ acts on $f \in \mathcal{C}^k(\mathcal{X})$ via

$$S_t f(x) = e^{tV} f(x) = f(\Psi_t(x)) \tag{4.1}$$

and, similarly, the semigroup $\{\widetilde{S}_t^{(j)}\}_{t\geq 0}$ corresponding to $\varphi^{(j)}$ is given by

$$\widetilde{S}_t^{(j)} f(x) = e^{tV_j} f(x) = f(\varphi_t^{(j)}(x)) \, . \tag{4.2}$$

In particular, m steps of random splitting corresponds to $\widetilde{S}_{h\tau}^m \coloneqq \widetilde{S}_{h\tau_1}^{(1)} \cdots \widetilde{S}_{h\tau_{mn}}^{(mn)}$ with superscripts taken as in Remark 2.2. The transition kernel P_h^m and semigroup composition $\widetilde{S}_{h\tau}^m$ are related via

$$P_h^m f = \mathbb{E}(f(\Phi_{h\tau}^m)) = \mathbb{E}(\widetilde{S}_{h\tau}^m f) \, . \tag{4.3}$$

With the above notation we now present the two main results of this section, Theorems 4.1 and 4.5, which follow from Lemmas 4.2 and 4.6, respectively. The full proofs of both lemmas are given in the Appendix, but we discuss the general idea behind each at the end of this section.

Theorem 4.1. *Suppose Assumption 1 holds and fix $t > 0$. For all h sufficiently small and satisfying $mh = t$ for some $m \in \mathbb{N}$, there exists a constant $C(t)$ depending on t but not on h such that*

$$\|P_h^m - S_t\|_{2\to 0} \leq C(t)h. \tag{4.4}$$

Lemma 4.2. *If Assumption 1 holds then there exists a constant C such that*

$$\|P_h - S_h\|_{2\to 0} \leq Ch^2 \tag{4.5}$$

for all h sufficiently small.

Recalling from (4.3) that $P_h = \mathbb{E}(\widetilde{S}_{h\tau}^1)$, Lemma 4.2 states that the average difference between one step of random splitting and the true dynamics is $\mathcal{O}(h^2)$ for sufficiently small h. For any finite time interval $[0, t]$ we can leverage this result to approximate S_t by successive steps of P_h. Specifically, choose h sufficiently small so that (4.5) holds and there exists an integer m with $mh = t$. Then the composition P_h^m corresponds to $\mathcal{O}(1/h)$ steps of P_h. Consequently, since the difference between P_h and S_h is $\mathcal{O}(h^2)$, the difference between P_h^m and S_t is $\mathcal{O}(h)$.

Remark 4.3. *As we have made minimal assumptions on the splitting, we will only be able to deduce that $P_h - S_h = \mathcal{O}(h^2)$. In specific examples, it is often possible to arrange the splitting so that $P_h - S_h = \mathcal{O}(h^p)$ with $p > 2$. An example of a higher order splitting is Strang splitting* [MS16]. *Alternatively higher order can also be obtained by fully randomizing the order* [COS19] *or randomly choosing between one ordering and its reverse* [Kus04, NV08, NN09].

Proof of Theorem 4.1. Let h be sufficiently small that (4.5) holds and such that $mh = t$ for some $m \in \mathbb{N}$. The quantity of interest can be written as the following telescoping sum:

$$P_h^m - S_t = \sum_{k=1}^{m} P_h^{k-1}(P_h - S_h)S_{h(m-k)}\,. \tag{4.6}$$

For any k and continuous function f with $\|f\|_0 = 1$,

$$\|P_h^k f\|_0 \leq \mathbb{E}\left(\left\|f(\Phi_{h\tau}^k)\right\|_0\right) = 1\,.$$

Hence $\|P_h^k\|_{0\to 0} = 1$. Similarly, since $mh = t$ implies $h(m-k) \leq t$ for $k \geq 0$, and since $\mathcal{X}$ is bounded by Assumption 1 (so Ψ and its first and second derivatives are bounded on $\mathcal{X}$, uniformly on $[0,t]$),

$$\|S_{h(m-k)}\|_{2\to 2} \leq K(t)$$

for some $K(t)$ depending on t but not h. Therefore, by submultiplicity together with (4.6) and Lemma 4.2, we have

$$\|P_h^m - S_t\|_{2\to 0} \leq \sum_{k=1}^{m} \|P_h^{k-1}\|_{0\to 0}\|P_h - S_h\|_{2\to 0}\|S_{h(m-k)}\|_{2\to 2} \leq K(t)\sum_{k=1}^{m} Ch^2 = C(t)h\,,$$

where $C(t) \coloneqq K(t)C$, with C the constant from (4.5) in Lemma 4.2. □

Remark 4.4. *Theorem 4.1 had the relation $h = t/m$, while in the almost-sure results below we will take $h = t/m^2$ (note we explicitly write t/m^2, making no reference to the variable h). The reason, loosely speaking, is that the transition kernel depends only on the expectation of the randomness, while the almost-sure results additionally depend on fluctuations of the randomness about its mean. For example, Lemma 4.6 prepares for an application of the Borel-Cantelli lemma by establishing the summability of probabilities of "large" fluctuations over sets of $\mathcal{O}(m) = \mathcal{O}(1/\sqrt{h})$ cycles. This is discussed in more detail at the end of this section and worked out in full in the Appendix.*

Theorem 4.5. *Suppose Assumption 1 holds and fix $t > 0$. Then for any $\varepsilon > 0$,*

$$\mathbb{P}\left(\limsup_{m\to\infty}\|\widetilde{S}_{t\tau/m^2}^{m^2} - S_t\|_{2\to 0} > \varepsilon\right) = 0\,. \tag{4.7}$$

Lemma 4.6. *Suppose Assumption 1 holds and fix $t > 0$. Then for any $\varepsilon > 0$,*

$$\sum_{m=1}^{\infty} \mathbb{P}\left(\|\widetilde{S}_{t\tau/m^2}^{m} - S_{t/m}\|_{2\to 0} > \tfrac{\varepsilon}{m}\right) < \infty\,. \tag{4.8}$$

Proof of Theorem 4.5. By the Borel-Cantelli Lemma it suffices to show

$$\sum_{m=1}^{\infty} \mathbb{P}\left(\|\widetilde{S}_{t\tau/m^2}^{m^2} - S_t\|_{2\to 0} > \varepsilon\right) < \infty\,.$$

Consider the telescoping sum

$$\widetilde{S}^{m^2}_{t\tau/m^2} - S_t = \sum_{k=1}^{m} \widetilde{S}^{(k-1)}_{t\tau/m^2} \left(\widetilde{S}^{m}_{t\tau/m^2} - S_{t/m} \right) S_{(m-k)t/m} \, . \tag{4.9}$$

For any k and continuous function f with $\|f\|_0 = 1$,

$$\big\| \widetilde{S}^{k}_{t\tau/m^2} f \big\|_0 = \big\| f(\Phi^k_{h\tau}) \big\|_0 = 1 \, .$$

Hence $\|\widetilde{S}^{(k-1)}_{t\tau/m^2}\|_{0\to 0} = 1$. Similarly, since $(m-k)t/m \le t$ for $k \ge 0$, and since $\mathcal{X}$ is bounded by Assumption 1 (so Ψ and its first and second derivatives are bounded on $\mathcal{X}$, uniformly on $[0,t]$),

$$\|S_{(m-k)t/m}\|_{2\to 2} \le K(t)$$

for some $K(t)$ depending on t but not h. Hence, by submultiplicity together with (4.9) and Lemma 4.6, we have

$$\big\|\widetilde{S}^{m^2}_{t\tau/m^2} - S_t\big\|_{2\to 0} \le K(t) \sum_{k=1}^{m} \big\|\widetilde{S}^{m}_{t\tau/m^2} - S_{t/m}\big\|_{2\to 0} = K(t) m \big\|\widetilde{S}^{m}_{t\tau/m^2} - S_{t/m}\big\|_{2\to 0} \, ,$$

and hence by Lemma 4.6,

$$\sum_{m=1}^{\infty} \mathbb{P}\left(\big\|\widetilde{S}^{m^2}_{t\tau/m^2} - S_t\big\|_{2\to 0} > \varepsilon \right) \le \sum_{m=1}^{\infty} \mathbb{P}\left(\big\|\widetilde{S}^{m}_{t\tau/m^2} - S_{t/m}\big\|_{2\to 0} > \tfrac{\varepsilon}{K(t)m} \right) < \infty. \qquad \square$$

We conclude this section by sketching the proofs of Lemmas 4.2 and 4.6, which are inspired by ideas from [COS19, CST+21] and given in full detail in the Appendix. Consider first Lemma 4.2. We begin by differentiating $\widetilde{S}_{h\tau}$ in h:

$$\partial_h \widetilde{S}_{h\tau} = \sum_{k=1}^{n} \tau_k e^{h\tau_1} \cdots e^{h\tau_{k-1}} V_k e^{h\tau_k} \cdots e^{h\tau_n} = \sum_{k=1}^{n} \tau_k \widetilde{S}^{(1,k-1)}_{h\tau} V_k \widetilde{S}^{(k,n)}_{h\tau}.$$

Next, commute $\widetilde{S}^{(1,k-1)}_{h\tau}$ and V_k via the Lie bracket $[\widetilde{S}^{(1,k-1)}_{h\tau}, V_k] \coloneqq \widetilde{S}^{(1,k-1)}_{h\tau} V_k - V_k \widetilde{S}^{(1,k-1)}_{h\tau}$ to get

$$\partial_h \widetilde{S}_{h\tau} = \sum_{k=1}^{n} \tau_k V_k \widetilde{S}_{h\tau} + \sum_{k=1}^{n} \tau_k [\widetilde{S}^{(1,k-1)}_{h\tau}, V_k] \widetilde{S}^{(k,n)}_{h\tau} = V \widetilde{S}_{h\tau} + (V_\tau - V)\widetilde{S}_{h\tau} + E_{h\tau}$$

where $V_\tau \coloneqq \sum_{k=1}^{n} \tau_k V_k$ and $E_{h\tau} \coloneqq \sum_{k=1}^{n} \tau_k [\widetilde{S}^{(1,k-1)}_{h\tau}, V_k] \widetilde{S}^{(k,n)}_{h\tau}$. So, by variation of constants,

$$\widetilde{S}_{h\tau} - S_h = \int_0^h S_{h-r}(V_\tau - V)\widetilde{S}_{r\tau} dr + \int_0^h S_{h-r} E_{r\tau} dr. \tag{4.10}$$

Loosely speaking, the first integrand is $\mathcal{O}(h)$ because

$$\mathbb{E}(V_\tau - V) = \sum_{k=1}^{n} \mathbb{E}(\tau_k - 1) V_k = 0 \tag{4.11}$$

cancels out first order terms from the full expression, $S_{h-r}(V_\tau - V)\widetilde{S}_{r\tau}$. On the other hand the second integrand is $\mathcal{O}(h)$ because the bracket terms in $E_{h\tau}$ also cancel out first order terms (most of the work of the proof in the Appendix is to make these two statements precise). Thus, integrating these $\mathcal{O}(h)$ terms over the interval $(0,h)$, we have that the difference on the right side of (4.10) is $\mathcal{O}(h^2)$, as claimed.

The proof of Lemma 4.6 is similar in that it again begins with an application of variation of constants. However, in this case one can no longer apply expectation directly to the randomness, as in (4.11), to estimate the analogue of the first term in (4.10). Instead we rely on a strong law of large numbers obtained by averaging the (random) vector fields over m cycles. This results in

$\mathcal{O}(m^2)$ commutators in the analogue of the second term in (4.10), each of which contributes $\mathcal{O}(h^2)$ as in the previous analysis. To balance these two summands we choose $m \sim \mathcal{O}(1/\sqrt{h})$, whence the relation $h = t/m^2$.

5. Conservative Lorenz-96

In this section, we apply results of the previous sections to the conservative Lorenz-96 model introduced in Section 1.2. There we noted that the vector field V in (1.5) splits as (1.6) where the flow of each V_k is a rotation; specifically, each flow $\varphi^{(k)}$ of the splitting vector fields

$$V_k(x) = (x_{k+1}e_k - x_k e_{k+1})x_{k-1} \tag{5.1}$$

is a rotation in the (x_k, x_{k+1})-plane with angular velocity x_{k-1}. Therefore the associated random splitting preserves Euclidean norm, which we refer to as the *energy* of the system, and remains on whichever sphere centered at the origin in $\mathbb{R}^n$ it starts on for all time. In particular, we have

Proposition 5.1. *All convergence results of Section 4 apply to the random splitting* (1.6) *of conservative Lorenz-96 starting from any initial condition.*

Proof. The splitting vector fields are smooth and Assumption 1 is satisfied since the dynamics stays on whichever sphere it starts on, so the conclusions of Theorems 4.1 and 4.5 both hold. □

5.1. Ergodicity.

A complicating feature of the dynamics (1.5) is that it has fixed points. Specifically, a point $x \in \mathbb{R}^n$ is a fixed point of (1.5) if and only if $\sum_{k=1}^n (x_k^2 + x_{k+1}^2)x_{k-1}^2 = 0$. For a 2-sphere embedded in $\mathbb{R}^3$ these are precisely the 6 points of intersection of the sphere with the standard coordinate axes. In higher dimensions, these fixed points lie on submanifolds that in general have dimension greater than 0 and in particular are no longer isolated. Nevertheless, nonfixed points cannot reach fixed points via the dynamics in finite time. For these reasons we study the splitting (1.6) on the space $\mathcal{X}$ consisting of a sphere in $\mathbb{R}^n$ minus the set of fixed points. That is,

$$\mathcal{X} := \left\{ x \in \mathbb{R}^n : \|x\| = R \text{ and } \sum_{k=1}^{n} (x_k^2 + x_{k+1}^2)x_{k-1}^2 \neq 0 \right\} \tag{5.2}$$

for some fixed but arbitrary radius $R > 0$. Note $\mathcal{X}$ is the complement of a set of Hausdorff measure zero on the sphere and therefore satisfies Assumption 2.

Fix $h > 0$ and let P_h be the Markov transition kernel of the random splitting associated to (1.6) as defined in (2.3). The main result of this section is

Proposition 5.2. *Hausdorff measure, normalized to be probability measure, is the unique P_h-invariant probability measure on $\mathcal{X}$.*

Proof. We first show Lebesgue in $\mathbb{R}^n$ is P_h-invariant. Let $\lambda_t^{(k)} := (\varphi_t^{(k)})_\# \lambda$ be the pushforward of λ by $\varphi_t^{(k)}$. Since the V_k defined in (5.1) are divergence free, the continuity equation becomes

$$0 = \partial_t \lambda_t^{(k)} + \operatorname{div}\left(V_k \lambda_t^{(k)}\right) = \partial_t \lambda_t^{(k)} + \nabla \lambda_t^{(k)} \cdot V_k \,.$$

The latter is a transport equation with constant initial condition $\lambda_0^{(k)} \equiv 1$ and hence $\lambda_t^{(k)} = \lambda$ for all t. Because the trajectories of all $\{V_k\}$ conserve the energy $\|x\|$, we fiber $\mathbb{R}^n$ using spherical coordinates $(r, \vartheta) \in \mathbb{R}_+ \times S^{n-1}$. In these coordinates, we have that $V_k(r, \vartheta) = 0\, \partial_r + r v_k(\vartheta) \nabla_\vartheta$ and by a change of coordinates of the divergence operator the stationarity equation becomes

$$0 = \operatorname{div}\left(V_k(x)\lambda(x)\right) = u(r) w(\vartheta) \operatorname{div}_\vartheta (\lambda(r, \vartheta) v_k(\vartheta)) \,,$$

where $\operatorname{div}_\vartheta$ denotes the angular terms of the divergence in spherical coordinates, and $u(r), w(\vartheta)$ result from the change of variables. Hence, we can factor the solution $\lambda(r, \vartheta) = \bar{\lambda}(\vartheta|r) \cdot \mu_R(dr) =$

$\bar\lambda(\vartheta)\cdot\mu_R(dr)$, where $\bar\lambda(\vartheta|r)$ is the conditional density of Lebesgue measure on a fiber. The measure $\bar\lambda$ solves $w(\vartheta)\,\mathrm{div}_\vartheta(\bar\lambda(\vartheta)v_k(\vartheta))=0$ and is therefore invariant under the flows $\varphi_t^{(k)}$. By rotational symmetry of $\lambda(x)$, we must have that $\bar\lambda(\vartheta)$ is the Hausdorff measure on S^{n-1}.

P_h-invariance of λ, and hence of $\bar\lambda$, follows immediately from the above: letting $\lambda_t := (\Phi_t)_\#\lambda$ be the pushforward of λ by Φ_t we have

$$\lambda_t = (\Phi_t)_\#\lambda = \big(\varphi_{t_n}^{(n)}\big)_\# \circ\cdots\circ \big(\varphi_{t_1}^{(1)}\big)_\#\lambda = \lambda.$$

Therefore, for any Borel subset B of $\mathcal{X}$,

$$\lambda P_h(B) = \int_{\mathcal{X}} P_h(x,B)\lambda(dx) = \mathbb{E}\bigg(\int_{\mathcal{X}} \mathbb{1}_B\left(\Phi(x,t)\right)\lambda(dx)\bigg) = \mathbb{E}\big(\lambda_\tau(B)\big) = \lambda(B)$$

and so λ is P_h-invariant.

Next we prove uniqueness by verifying the hypotheses of Corollary 3.4. By direct computation

$$\begin{pmatrix} | & | & & | \\ V_1(x) & V_2(x) & \cdots & V_{n-1}(x) \\ | & | & & | \end{pmatrix} = \begin{pmatrix} x_2x_n & 0 & \dots & 0 \\ -x_1x_n & x_3x_1 & \dots & 0 \\ 0 & -x_2x_1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & x_nx_{n-2} \\ 0 & 0 & & -x_{n-1}x_{n-2} \end{pmatrix}$$

has rank $n-1$ whenever all x_k are nonzero. So, the V_k span the tangent space at every point in $\mathcal{X}$ with nonzero coordinates. It suffices then to show the existence of an $x_*\in\mathcal{X}$ with nonzero coordinates such that for every $x\in\mathcal{X}$ there is an m and a t satisfying $\Phi^m(x,t)=x_*$. To that end, we first show any $x\in\mathcal{X}$ can be evolved to have nonzero coordinates. Recall $\mathcal{X}$ is a sphere minus points satisfying $\sum_{k=1}^n(x_k^2+x_{k+1}^2)x_{k-1}^2=0$. So, since $x\in\mathcal{X}$ there exists k such that $x_{k-1}\neq 0$ and x_k or x_{k+1} is nonzero. Further, since $\varphi^{(k)}$ is a rotation in the (x_k,x_{k+1})-plane with angular velocity x_{k-1}, there is a time $t_k>0$ such that the k and $k+1$ coordinates of $\varphi^{(k)}(x,t_k)$ are both nonzero. By the same argument there is a time $t_{k+1}>0$ such that the k, $k+1$, and $k+2$ coordinates of $x^{(k+1)}=\varphi^{(k+1)}(\varphi^{(k)}(x,t_k),t_{k+1})$ are nonzero. Continuing this way, we see x can be made to have nonzero coordinates in a finite number of steps.

We complete the proof by showing any x can be evolved via random switching to $x_*=(R/\sqrt{n},\dots,R/\sqrt{n})$. By the preceding argument, we assume without loss of generality that x has nonzero coordinates. Since $\|x\|=R$, there exists an index k such that $|x_k|\geq R/\sqrt{n}$. If $k=n$, rotate in the $(n-1,n)$-plane so that the nth coordinate of x becomes $R/\sqrt{n}$. If $k<n$, rotate in the $(k,k+1)$-plane so that the $k+1$ coordinate of x becomes $R/\sqrt{n}$, then rotate in the $(k+1,k+2)$-plane so that the $k+2$ coordinate of x becomes $R/\sqrt{n}$, and so on until the nth-coordinate of x becomes $R/\sqrt{n}$. Such rotations are always possible because all coordinates are nonzero. Thus, whether $k=n$ or $k<n$ we can evolve x via random switching so that its last coordinate, x_n, is $R/\sqrt{n}$. In particular, there now must exist an index $k<n$ such that $|x_k|\geq R/\sqrt{n}$. By the same procedure, and without disturbing the last coordinate, we can use rotations to make the $n-1$ coordinate of x equal $R/\sqrt{n}$. Iterating this process we obtain $x=x_*$ in a finite number of steps. □

6. Galerkin approximations of 2D Euler

The 2D Euler equations on the torus $\mathbb{T}$ are obtained from the 2D Navier-Stokes equations (1.1) by dropping the dissipative and forcing terms:

$$\begin{cases}\partial_t u + (u\cdot\nabla)u = -\nabla p \\ \mathrm{div}(u) := \nabla\cdot u = 0\end{cases} \tag{6.1}$$

where, as before, $u : \mathbb{T} \times \mathbb{R} \to \mathbb{R}^2$ is the fluid velocity, $p : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ the fluid pressure, and

$$(u \cdot \nabla)u = (u_1\partial_1 u_1 + u_2\partial_2 u_1, u_1\partial_1 u_2 + u_2\partial_2 u_2)\,.$$

In this section we construct a convenient random splitting of (6.1). To do so we first write (6.1) in vorticity form and apply the Fourier transform. This yields an infinite system of ODEs which we truncate to systems of arbitrary finite size, referred to throughout as Galerkin approximations. Finally, we split these Galerkin approximations and apply the results of Sections 3 and 4 to the associated random splitting.

6.1. **Constructing the splitting.** The vorticity formulation of (6.1) is obtained by taking the curl of velocity. Specifically, setting $q \coloneqq \mathrm{curl}(u) \coloneqq \partial_2 u_1 - \partial_1 u_2$, equation (6.1) becomes

$$\begin{cases} \partial_t q + (\mathcal{K}q \cdot \nabla)q = 0\,, \\ \mathrm{div}(q) = 0\,, \end{cases} \tag{6.2}$$

where $\mathcal{K} \coloneqq \nabla^{\perp}(-\Delta)^{-1}$ with $\nabla^{\perp} \coloneqq (\partial_2, -\partial_1)$. To express (6.2) in Fourier space, set $\mathbb{Z}^2_{\infty} \coloneqq \mathbb{Z}^2 \setminus \{(0,0)\}$ and let $\{e_j\}_{j\in\mathbb{Z}^2_\infty}$ be the orthonormal basis of $L^2(\mathbb{T}, \mathbb{R})$ given by $e_j(x) \coloneqq (2\pi)^{-1}\exp(ix\cdot j)$. Then $q(x,t) = \sum_{j\in\mathbb{Z}^2_\infty} q_j(t)e_j(x)$ where

$$q_j(t) \coloneqq \langle q, e_j\rangle_{L^2} = \int_{\mathbb{T}} q(x,t)\overline{e}_j(x)dx$$

is the jth Fourier mode of q. Here $\langle\cdot,\cdot\rangle_{L^2}$ is the standard inner product on $L^2(\mathbb{T}, \mathbb{R})$ with $\overline{e}_j$ denoting the complex conjugate of e_j. The jth Fourier mode of $(\mathcal{K}q \cdot \nabla)q$ is

$$\langle(\mathcal{K}q \cdot \nabla)q, e_j\rangle_{L^2} = \sum_{k+\ell=j} C_{k\ell}q_k q_\ell$$

where

$$C_{k\ell} \coloneqq \frac{\langle k, \ell^{\perp}\rangle}{4\pi}\left(\frac{1}{|k|^2} - \frac{1}{|\ell|^2}\right) \tag{6.3}$$

with $\langle\cdot,\cdot\rangle$ the standard inner product in $\mathbb{R}^2$, $\ell^{\perp} \coloneqq (\ell_2, -\ell_1)$, and $|\ell|^2 \coloneqq \ell_1^2 + \ell_2^2$. Therefore

$$\sum_j \dot{q}_j e_j = \partial_t q = -(\mathcal{K}q \cdot \nabla)q = -\sum_j \left(\sum_{k+\ell=j} C_{k\ell}q_k q_\ell\right)e_j$$

and hence $\dot{q}_j = -\sum_{k+\ell=j} C_{k\ell}q_k q_\ell$. Moreover, since q is real-valued,

$$\sum_j q_j e_j = q = \overline{q} = \sum_j \overline{q}_j e_{-j}$$

which gives $q_j = \overline{q}_{-j}$. In particular,

$$\dot{q}_j = \dot{\overline{q}}_{-j} = -\sum_{j+k+\ell=0} C_{k\ell}\overline{q}_k\overline{q}_\ell\,.$$

Writing each Fourier mode $q_j = a_j + ib_j$ in terms of real and imaginary parts then gives

$$\begin{aligned} \dot{a}_j + i\dot{b}_j = \dot{q}_j &= -\sum_{j+k+\ell=0} C_{k\ell}(a_k - ib_k)(a_\ell - ib_\ell) \\ &= \sum_{j+k+\ell=0} C_{k\ell}(b_k b_\ell - a_k a_\ell) + i\sum_{j+k+\ell=0} C_{k\ell}(a_k b_\ell + a_\ell b_k)\,. \end{aligned}$$

Thus the Fourier modes of solutions to the Euler equation in vorticity form satisfy

$$\begin{cases} \dot a_j = \displaystyle\sum_{j+k+\ell=0} C_{k\ell}(b_k b_\ell - a_k a_\ell) \\ \dot b_j = \displaystyle\sum_{j+k+\ell=0} C_{k\ell}(a_k b_\ell + a_\ell b_k) \end{cases} \tag{6.4}$$

for all $j \in \mathbb{Z}^2_\infty$. While (6.4) could be studied as is, notice the constraint $q_{-j} = \overline{q}_j$ implies $a_{-j} = a_j$ and $b_{-j} = -b_j$, which introduces redundancy in (6.4). Therefore we restrict to the subset

$$\mathbb{Z}^2_+ \coloneqq \{j \in \mathbb{Z}^2 : j_2 > 0\} \cup \{j \in \mathbb{Z}^2 : j_2 = 0 \text{ and } j_1 > 0\}\,.$$

Specifically, by straightforward computation together with the identities $a_{-j} = a_j$, $b_{-j} = -b_j$, and $C_{k\ell} = C_{-k,-\ell} = -C_{-k,\ell} = -C_{k,-\ell}$, the system (6.4) can be re-expressed as

$$\begin{cases} \dot a_j = \displaystyle\sum_{j+k-\ell=0} C_{k\ell}(a_k a_\ell + b_k b_\ell) + \sum_{j-k-\ell=0} C_{k\ell}(b_k b_\ell - a_k a_\ell) \\ \dot b_j = \displaystyle\sum_{j+k-\ell=0} C_{k\ell}(a_k b_\ell - b_k a_\ell) - \sum_{j-k-\ell=0} C_{k\ell}(a_k b_\ell + b_k a_\ell) \end{cases} \tag{6.5}$$

for all $j \in \mathbb{Z}^2_+$ with each sum running over all pairs $k, \ell \in \mathbb{Z}^2_+$ satisfying the specified identity. To split (6.5) note that for any $j, k, \ell \in \mathbb{Z}^2_+$ satisfying $j + k - \ell = 0$ (and hence $\ell - j - k = 0$) we can isolate from the above sums exactly 6 equations involving only these indices:

$$\begin{aligned} &\dot a_j = C_{k\ell}(a_k a_\ell + b_k b_\ell)\,, && \dot a_k = C_{j\ell}(a_j a_\ell + b_j b_\ell)\,, && \dot a_\ell = C_{jk}(b_j b_k - a_j a_k)\,, \\ &\dot b_j = C_{k\ell}(a_k b_\ell - b_k a_\ell)\,, && \dot b_k = C_{j\ell}(a_j b_\ell - b_j a_\ell)\,, && \dot b_\ell = -C_{jk}(a_j b_k + b_j a_k)\,. \end{aligned} \tag{6.6}$$

For reasons to be made clear shortly, we recombine (6.6) into 4 groups of 3 equations:

$$\begin{cases} \dot a_j = C_{k\ell} a_k a_\ell \\ \dot a_k = C_{j\ell} a_j a_\ell \\ \dot a_\ell = -C_{jk} a_j a_k \end{cases} \quad \begin{cases} \dot a_j = C_{k\ell} b_k b_\ell \\ \dot b_k = C_{j\ell} a_j b_\ell \\ \dot b_\ell = -C_{jk} a_j b_k \end{cases} \quad \begin{cases} \dot b_j = C_{k\ell} a_k b_\ell \\ \dot a_k = C_{j\ell} b_j b_\ell \\ \dot b_\ell = -C_{jk} b_j a_k \end{cases} \quad \begin{cases} \dot b_j = -C_{k\ell} b_k a_\ell \\ \dot b_k = -C_{j\ell} b_j a_\ell \\ \dot a_\ell = C_{jk} b_j b_k \end{cases}\,. \tag{6.7}$$

Let $V_{a_j a_k a_\ell}$, $V_{a_j b_k b_\ell}$, $V_{b_j a_k b_\ell}$, and $V_{b_j b_k a_\ell}$ be the vector fields associated to the equations of (6.7) from left to right. For example, $V_{a_j a_k a_\ell}$ is the vector field on $\mathbb{R}^\infty$ mapping the a_j coordinate to $-C_{k\ell} a_k a_\ell$, the a_k coordinate to $-C_{j\ell} a_j a_\ell$, the a_ℓ coordinate to $-C_{jk} a_j a_k$, and all other coordinates to 0. These are the *splitting vector fields*. Our sought-after splitting is

$$V = \sum_{j+k-\ell=0} V_{a_j a_k a_\ell} + V_{a_j b_k b_\ell} + V_{b_j a_k b_\ell} + V_{b_j b_k a_\ell}\,, \tag{6.8}$$

where V is the vector field associated to (6.5). As noted earlier, our focus will be on finite truncations of the infinite-dimensional system (6.5). Thus we fix an integer $N \geq 2$ and define the *Nth Galerkin approximation* of (6.5) to be (6.5) with indices restricted to the set

$$\mathbb{Z}^2_N \coloneqq \left\{ j \in \mathbb{Z}^2_+ : \max\{|j_1|, |j_2|\} \leq N \right\}.$$

The splitting (6.8) remains valid in this finite-dimensional setting, bearing in mind that now all indices are restricted to lie in $\mathbb{Z}^2_N$. By a slight abuse of notation, we denote the finite-dimensional counterpart of V by V as well and similarly for the splitting vector fields. Since $\mathbb{Z}^2_N$ has cardinality $2N(N+1)$ and each index $j \in \mathbb{Z}^2_N$ has an associated a_j and b_j coordinate, these are all vector fields on $\mathbb{R}^n$, where throughout this section we set $n \coloneqq 4N(N+1)$. We also abuse notation by conflating elements $j \in \mathbb{Z}^2_N$ with elements $j \in \{1, \ldots, n/2\}$, which can be formalized via any bijection between the two sets. Moreover, we denote elements of $\mathbb{R}^n$ by $q = (a_j, b_j)_{j=1}^{n/2}$. This reflects that the a_j and

b_j coordinates of $q \in \mathbb{R}^n$ are in one-to-one correspondence with the real and imaginary parts of the jth mode of q.

Remark 6.1. *There are many possible splittings of a given equation. For the Euler equations, we made the particular choice we have so that both energy and enstrophy are conserved but the dynamics of each splitting are still relatively easily understood. We could have further decomposed the three-dimensional dynamics in the above splitting into a number of two-dimensional dynamics, similar in spirit to the decomposition into rotations used in Lorenz-96. However, that would have necessitated only conserving either the energy or the enstrophy.*

6.2. **Conservation and convergence.** The Lorenz system presented in Section 5 conserves Euclidean norm (energy in that case) and therefore remains on whichever sphere it started on. Furthermore, the corresponding splitting consists of rotations which also conserve energy. Returning to Galerkin approximations of 2D Euler, define the *energy* and *enstrophy* of $q = (a_j, b_j)_{j=1}^{n/2} \in \mathbb{R}^n$ by

$$E(q) \coloneqq \sum_{j\in\mathbb{Z}_N^2} \frac{a_j^2 + b_j^2}{|j|^2} \qquad \text{and} \qquad \mathcal{E}(q) \coloneqq \sum_{j\in\mathbb{Z}_N^2} a_j^2 + b_j^2\,, \tag{6.9}$$

respectively (note the aforementioned conflation of $j \in \mathbb{Z}_N^2$ and $j \in \{1, \dots, n/2\}$ in the summations). Straightforward computation shows that for all $j, k, \ell \in \mathbb{Z}_N^2$ with $j + k - \ell = 0$,

$$C_{k\ell} + C_{j\ell} - C_{jk} = \frac{C_{k\ell}}{|j|^2} + \frac{C_{j\ell}}{|k|^2} - \frac{C_{jk}}{|\ell|^2} = 0\,,$$

which in turn implies that under the dynamics (6.5),

$$\partial_t E(q) = \partial_t \mathcal{E}(q) = 0$$

for all $q \in \mathbb{R}^n$. That is, both energy and enstrophy are conserved by the true dynamics and the set

$$\mathcal{Q}_0(E, \mathcal{E}) \coloneqq \left\{ q \in \mathbb{R}^n : E(q) = E,\ \mathcal{E}(q) = \mathcal{E} \right\}. \tag{6.10}$$

is invariant under (6.5), a well-established property of the 2D Euler equations. Moreover, if we flow by $V_{a_j a_k a_\ell}$ starting from q for any $j, k, \ell \in \mathbb{Z}_N^2$ with $j + k - \ell = 0$, then

$$\tfrac{1}{2}\partial_t E(q) = \frac{a_j \dot{a}_j}{|j|^2} + \frac{a_k \dot{a}_k}{|k|^2} + \frac{a_\ell \dot{a}_\ell}{|\ell|^2} = \left(\frac{C_{k\ell}}{|j|^2} + \frac{C_{j\ell}}{|k|^2} - \frac{C_{jk}}{|\ell|^2} \right) a_j a_k a_\ell = 0\,,$$

and similarly $\partial_t \mathcal{E}(q) = 0$. The same computation shows energy and enstrophy are conserved by *all* of the splitting vector fields corresponding to (6.7), which provides the motivation for recombining (6.6) as (6.7) in the first place. In particular, we have

Proposition 6.2. *All convergence results of Section 4 apply to the random splitting* (6.8) *of every Galerkin approximation of 2D Euler starting from any initial condition.*

Proof. The splitting vector fields are smooth and Assumption 1 is satisfied since the dynamics stays on whichever sphere it starts on, so the conclusions of Theorems 4.1 and 4.5 both hold. ☐

6.3. **Ergodicity.** Throughout this subsection we fix energy and enstrophy values E and $\mathcal{E}$ and set $\mathcal{Q}_0 \coloneqq \mathcal{Q}_0(E, \mathcal{E})$. As with conservative Lorenz-96, Galerkin approximations of 2D Euler have degenerate sets, i.e., points $q \in \mathcal{Q}_0$ such that $\mathcal{X}(q)$ as defined in (2.4) is not dense in $\mathcal{Q}_0$. For example, any vector with exactly one nonzero coordinate is a fixed point of (6.5) and of all the equations (6.7). In this subsection we characterize these degenerate sets and prove that the random splitting associated to (6.8) has exactly one invariant measure on the complement of such sets. To that end, we enumerate the coordinates of $q \in \mathbb{R}^n$ by extending the indices $j \in \mathbb{Z}_N^2$ with an element $\chi \in \{+, -\}$ which denotes the real $(+)$ or imaginary $(-)$ part of the corresponding mode. Then, for

$\mathbf{j} = (j, \chi) \in \mathbb{Z}_N^2 \times \{+, -\}$ we define the *type* of such coordinates via the function $\mathrm{T}(\mathbf{j}) = \chi$ so that $q_{\mathbf{j}}$ is identified with a_j if $\mathrm{T}(\mathbf{j}) = +$ and with b_j if $\mathrm{T}(\mathbf{j}) = -$. For $q \in \mathbb{R}^n$, we denote throughout by

$$\mathcal{A}(q) := \left\{ \mathbf{j} \in \mathbb{Z}_N^2 \times \{+, -\} \ : \ q_{\mathbf{j}} \neq 0 \right\}$$

the set of "active" coordinates. To streamline our analysis, we define the following operation to expand the set $\mathcal{A}$:

$$\mathcal{A} \oplus \boldsymbol{\ell} := \begin{cases} \mathcal{A} \cup \{\boldsymbol{\ell}\} & \text{if } \ell \in \{j+k, j-k\} \cap \mathbb{Z}_N^2 \text{ for } \mathbf{j}, \mathbf{k} \in \mathcal{A}, C_{jk} \neq 0, \mathrm{T}(\mathbf{j}) \cdot \mathrm{T}(\mathbf{k}) = \mathrm{T}(\boldsymbol{\ell}), \\ \mathcal{A} & \text{else}, \end{cases} \tag{6.11}$$

where $\mathrm{T}(\mathbf{j}) \cdot \mathrm{T}(\mathbf{k})$ is $+$ if $\mathrm{T}(\mathbf{j}) = \mathrm{T}(\mathbf{k})$ and $-$ if $\mathrm{T}(\mathbf{j}) \neq \mathrm{T}(\mathbf{k})$. This operation corresponds to extending the nonzero coordinates of q from $\boldsymbol{j}, \boldsymbol{k}$ to $\boldsymbol{\ell}$ by letting a triple $\iota = \boldsymbol{jk\ell}$ interact.

We assume that the initial condition is sufficiently nondegenerate, as stated in the following assumption similar to the one made in [HM06, Thm. 2.1].

Assumption 3. *There exists $M \in \mathbb{N}$, $j^* \in \mathbb{Z}_N^2$ with $|j^*|^2 > 1$, and an ordered set of indices $(\boldsymbol{\ell}_i)_{i=1}^M$ in $\mathbb{Z}_N^2 \times \{+, -\}$ such that*

$$\left\{(1,0,+), (0,1,+), (j^*, -)\right\} \subseteq \left((\mathcal{A}(q) \oplus \boldsymbol{\ell}_1) \oplus \boldsymbol{\ell}_2\right) \cdots \oplus \boldsymbol{\ell}_M.$$

Assumption 3 gives sufficient conditions for the controllability of the dynamics induced by (6.7) on $\mathcal{Q}_0$. The set of nondegenerate points on $\mathcal{Q}_0$ is defined as

$$\mathcal{Q} := \left\{ q \in \mathcal{Q}_0 : q \text{ satisfies Assumption 3} \right\}. \tag{6.12}$$

Fix $h > 0$ and let P_h be the Markov transition kernel of the random splitting associated to (6.8) as defined in (2.3). The main result of this subsection is

Proposition 6.3. *Hausdorff is the unique P_h-invariant probability measure on $\mathcal{Q}$.*

The proof follows three steps, detailed in the subsections below. Existence is proved in Lemma 6.6. For uniqueness, we first show in Lemma 6.7 that the splitting vector fields span the tangent space of $\mathcal{Q}$ at every point with all nonzero coordinates. Then in Proposition 6.8 we prove the existence of a q_* in $\mathcal{Q}$ with all nonzero coordinates that can be reached from any other point in $\mathcal{Q}$. The Proposition then follows from Corollary 3.4.

Remark 6.4. *Comparing* (6.12) *with* (5.2)*, we see that the conditions defining the nondegenerate subset of phase space are more complicated for the randomly split Euler equations than for the conservative Lorenz-96 model. The difference is that there are proper subspaces of the surface* (6.10) *which are invariant for our splitting of the Euler dynamics but are not fixed points. This is not true for the conservative Lorenz-96 model. One such subspace is the collection of purely real points; another is the purely imaginary points.*

Remark 6.5. *Continuing in the spirit of Remark 6.1, we observe the splitting in* (6.7) *splits q_j into its real and imaginary parts. We could have chosen another basis of $\mathbb{C}$ and even randomized over this choice for each evolution of an interacting triple (j, k, ℓ). More explicitly, if we define $e(\vartheta) = cos(\vartheta) + i \sin(\vartheta)$ then $e(\vartheta)$ and $e(\vartheta + \frac{\pi}{2})$ form an orthonormal basis of $\mathbb{C}$ for any ϑ. Then we can drive a system analogous to* (6.7) *by setting $q_\ell = a_\ell^\vartheta e(\vartheta) + b_\ell^\vartheta e(\vartheta + \frac{\pi}{2})$. As the form is similar to* (6.7)*, the results of the paper extend to this system. In particular, by randomizing the choice of ϑ for each such triple (j, k, ℓ), we can relax Assumption 3 by destroying some of the invariant structures discussed in Remark 6.4 which obstruct controllability starting from some initial conditions.*

6.3.1. *Existence of invariant measure.* As with conservative Lorenz-96, each vector field of the 2D Euler splitting is divergence free and so Lebesgue measure in $\mathbb{R}^n$ is invariant. Consequently, we have

Lemma 6.6. *Let λ denote the Lebesgue measure on $\mathbb{R}^n$. The measure obtained by conditioning λ to lie on $\mathcal{Q}$ is P_h-invariant.*

Proof. As in the proof of Proposition 5.2 we have that Lebesgue measure in $\mathbb{R}^n$ is P_h-invariant. Since the vector fields V_k defined in (5.1) are divergence free, the continuity equation reads

$$\partial_t \lambda + \operatorname{div}(V_k \lambda) = \partial_t \lambda + \nabla \lambda \cdot V_k = 0\,.$$

Because each flow $\varphi^{(k)}$ conserves energy E and enstrophy $\mathcal{E}$, we locally fiber $\mathbb{R}^n$ using coordinates $(E, \mathcal{E}, \vartheta) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^{n-2}$. In these coordinates, we have $V_k(E, \mathcal{E}, \vartheta) = 0\,\partial_E + 0\,\partial_{\mathcal{E}} + v_k(E, \mathcal{E}, \vartheta)\nabla_\vartheta$ so by a change of coordinates of the divergence operator the stationary equation becomes

$$0 = \operatorname{div}(V_k(x)\lambda(x)) = u(E, \mathcal{E}, \vartheta)\operatorname{div}_\vartheta(\lambda(E, \mathcal{E}, \vartheta) v_k(E, \mathcal{E}, \vartheta))\,,$$

where $\operatorname{div}_\vartheta$ denotes the "angular" terms of the divergence in $(E, \mathcal{E}, \vartheta)$-coordinates, and $u(E, \mathcal{E}, \vartheta)$ result from the change of variables. Hence, we can factor the solution $\lambda(E, \mathcal{E}, \vartheta) = \bar{\lambda}(\vartheta|E, \mathcal{E}) \cdot \lambda^\perp(E, \mathcal{E})$, where $\bar{\lambda}(\vartheta|E, \mathcal{E})$ is the conditional density of Lebesgue measure on a fiber, solving $u(E, \mathcal{E}, \vartheta)\operatorname{div}_\vartheta(\bar{\lambda}(\vartheta|E, \mathcal{E}) v_k(E, \mathcal{E}, \vartheta)) = 0$ for any choice of $E/(2N^2) < \mathcal{E} < E$. This proves the invariance of $\bar{\lambda}(\vartheta|E, \mathcal{E})$ under the flow map for any value of the flow times τ. The stationarity of $\bar{\lambda}(\vartheta)$ under P follows immediately as in Proposition 5.2 ◻

6.3.2. *Spanning.* For $j, k, \ell \in \mathbb{Z}_N^2$ with $j + k - \ell = 0$ define $M_{jk\ell}$ to be the matrix

$$M_{jk\ell} := \begin{pmatrix} | & | & | & | \\ V_{a_j a_k a_\ell} & V_{a_j b_k b_\ell} & V_{b_j a_k b_\ell} & V_{b_j b_k a_\ell} \\ | & | & | & | \end{pmatrix} = \begin{pmatrix} C_{k\ell} a_k a_\ell & C_{k\ell} b_k b_\ell & 0 & 0 \\ 0 & 0 & C_{k\ell} a_k b_\ell & -C_{k\ell} b_k a_\ell \\ C_{j\ell} a_j a_\ell & 0 & C_{j\ell} b_j b_\ell & 0 \\ 0 & C_{j\ell} a_j b_\ell & 0 & -C_{j\ell} b_j a_\ell \\ -C_{jk} a_j a_k & 0 & 0 & C_{jk} b_j b_k \\ 0 & -C_{jk} a_j b_k & -C_{jk} b_j a_k & 0 \end{pmatrix} \tag{6.13}$$

and let $M'_{jk\ell}$ and $M''_{jk\ell}$ be the 4-by-4 and 2-by-4 matrices consisting of the bottom four and bottom two rows of $M_{jk\ell}$, respectively. For notational convenience we often suppress indices and simply write M, M', and M''. Straightforward Gaussian elimination shows that M, M', and M'' have ranks 4, 3, and 2 whenever C_{jk}, $C_{j\ell}$, $C_{k\ell}$, a_j, b_j, a_k, b_k, a_ℓ, and b_ℓ are nonzero.

Denoting throughout a point $q \in \mathbb{R}^n$ as generic if all its coordinates are nonzero, we have

Lemma 6.7. *The family of vector fields*

$$\mathcal{V} := \left\{V_{a_j a_k a_\ell}, V_{a_j b_k b_\ell}, V_{b_j a_k b_\ell}, V_{b_j b_k a_\ell} : j, k, \ell \in \mathbb{Z}_N^2 \text{ and } j + k - \ell = 0\right\}$$

span $T_q\mathcal{Q}$ at every generic point $q \in \mathcal{Q}$.

Proof. Fix a generic point q. The main idea of the proof is to choose an enumeration of $\mathbb{Z}_N^2$ and a subset of vector fields from $\mathcal{V}$ so that the matrix made up of these vector fields evaluated at q is in a convenient form whose rank is readily deduced. Formally, the enumeration is the bijection $F : \mathbb{Z}_N^2 \to \{1, \ldots, 2N(N+1)\}$ given by

$$F(j) := \begin{cases} 1 & j = (1,0)\,, \\ 5 + N & j = (2,0)\,, \\ j_1 + N(2N+1) & j = (j_1, 0) \text{ with } j_1 > 2\,, \\ j_1 + 2 + N & j = (j_1, 1) \text{ with } j_1 < 3\,, \\ j_1 + 3 + N & j = (j_1, 1) \text{ with } j_1 \geq 3\,, \\ j_1 + 2 - N + (2N+1)j_2 & j = (j_1, j_2) \text{ with } j_2 > 1\,. \end{cases}$$

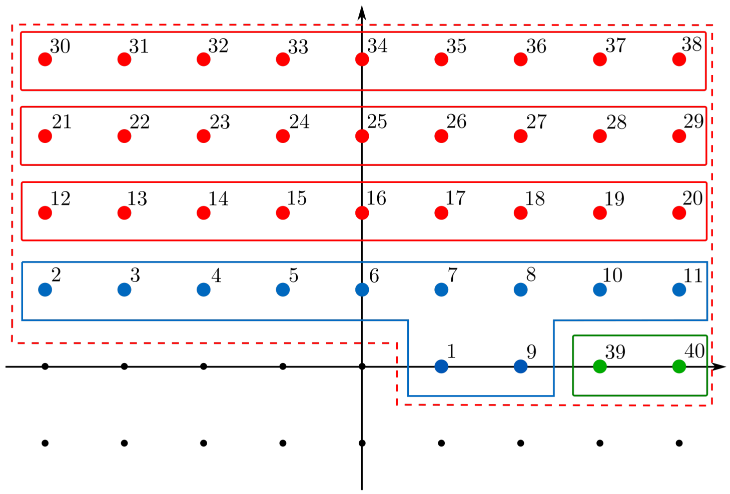


FIGURE 2. Ordering of $\mathbb{Z}_N^2$ when $N = 4$.

Figure 2 gives this enumeration in the case $N = 4$. Informally, F starts at $(1, 0)$, then counts lattice points from left to right along the horizontal line $y = 1$ until the point $(2, 1)$, which corresponds to $4 + N$. It then assigns $5 + N$ to $(2, 0)$ and continues counting along the line $y = 1$. From there it moves up to the lines $y = 2$, $y = 3$, and so on, counting from left to right along each. Finally, it goes back down to the line $y = 0$ and counts the remaining indices from left to right.

The motivation for F is that all horizontally-adjacent indices (j_1, j_2) and $(j_1 + 1, j_2)$ form an interacting triple together with $(1, 0)$. Fix for the moment an integer $y > 1$ and consider the yth horizontal line of $\mathbb{Z}_N^2$; that is, the points with second coordinate y. These are outlined by red blocks in Figure 2. By the preceding remarks we can choose the vector fields corresponding to the horizontally-adjacent indicies to get the block matrix

$$B_y \coloneqq \left(\begin{array}{c|c|c|c} \widetilde{M} & * & * & * \\ \hline 0 & M'' & * & * \\ \hline 0 & 0 & \ddots & * \\ \hline 0 & 0 & 0 & M'' \end{array}\right).$$

Here each M'' is the 2-by-4 matrix consisting of the bottom two rows of (6.13) for the relevant indices and

$$\widetilde{M} \coloneqq \begin{pmatrix} C_{j\ell}a_j a_\ell & 0 & C_{j\ell}b_j b_\ell & 0 & 0 & 0 \\ 0 & C_{j\ell}a_j b_\ell & 0 & -C_{j\ell}b_j a_\ell & 0 & 0 \\ -C_{jk}a_j a_k & 0 & 0 & C_{jk}b_j b_k & -C_{j'k'}a_{j'}a_{k'} & \\ 0 & -C_{jk}a_j b_k & -C_{jk}b_j a_k & 0 & 0 & -C_{j'k'}a_{j'}b_{k'} \end{pmatrix}$$

where $j = (1, 0), k = (-N, y), \ell = (-N + 1, y)$ and $j' = (0, 1)$ and $k' = (-N + 1, y - 1)$. This is M' with two columns from the interacting triple $(0, 1), (-N + 1, y - 1), (-N + 1, y)$ adjoined to the end. Note that these adjoined columns contribute entries in the coordinates corresponding to $(0, 1)$ and $(-N + 1, y - 1)$, but these come before all indices in the yth row for our ordering. By adding the latter two columns, $\widetilde{M}$ has rank 4 at any generic point. Further, since each M'' has rank 2, each B_y has rank $4 + 2(2N - 1) = 4N + 2$. This establishes spanning of the red blocks in Figure 2.

For the blue block we perform a similar procedure to the one above to get

$$B_1 := \left(\begin{array}{c|c|c|c|c|c} M_{123} & * & * & * & * & * \\ \hline 0 & M'' & * & * & * & * \\ \hline 0 & 0 & \ddots & * & * & * \\ \hline 0 & 0 & 0 & \widehat{M} & * & * \\ \hline 0 & 0 & 0 & 0 & \ddots & * \\ \hline 0 & 0 & 0 & 0 & 0 & M'' \end{array}\right)$$

where M_{123} is the matrix from (6.13) for the interacting triple $(1,0),(-N,1),(-N+1,1)$, each M'' is as before, and $\widehat{M}$ is the 6-by-8 matrix

$$\widehat{M} := \left(\begin{array}{cccc|cccc} & & & & 0 & 0 & 0 & 0 \\ & M'_{1,N+3,N+4} & & & 0 & 0 & 0 & 0 \\ & & & & --- & --- & --- & --- \\ --- & --- & --- & --- & & & & \\ 0 & 0 & 0 & 0 & & M'_{N+2,N+4,N+5} & & \\ 0 & 0 & 0 & 0 & & & & \end{array}\right)$$

located at the rows corresponding to $N+3, N+4$, and $N+5$. The reason for $\widehat{M}$, and for considering the blue block separately, is that $C_{jk} = 0$ when $j = (1,0)$ and $k = (0,1)$. The matrix M has rank 6 at a generic point. Since M_{123} has rank 4, $\widehat{M}$ has rank 6, and each of the $2N-3$ remaining M'' blocks has rank 2, the matrix B_y has rank $4 + 6 + 2(2N-3) = 4N+4$.

Finally, none of the indices of the green block interact with $(1,0)$ since the C_{jk} are all 0 in this case. However, by an entirely similar procedure to above, we can use the interactions between $(0,1), (x,0)$, and $(x,1)$ for $x > 1$ to get a rank $2(N-2)$ block matrix for the last $N-2$ coordinates of the form

$$B_{N+1} := \left(\begin{array}{c|c|c|c} M'' & * & * & * \\ \hline 0 & M'' & * & * \\ \hline 0 & 0 & \ddots & * \\ \hline 0 & 0 & 0 & M'' \end{array}\right).$$

Combining the above results we observe that there is an ordering of indices and vector fields such that the matrix whose columns consist of these vector fields has the form

$$B := \left(\begin{array}{c|c|c|c} B_1 & * & * & * \\ \hline 0 & B_2 & * & * \\ \hline 0 & 0 & \ddots & * \\ \hline 0 & 0 & 0 & B_{N+1} \end{array}\right).$$

Moreover, B has rank

$$\operatorname{rank}(B) = \operatorname{rank}(B_1) + \operatorname{rank}(B_{N+1}) + \sum_{y=2}^{N} \operatorname{rank}(B_y) = 4N(N+1) - 2 = n - 2$$

at every generic point $q \in \mathcal{Q}$. Now since the dynamics conserve energy and enstrophy, every tangent vector to $\mathcal{Q}$ is perpendicular to the normal vectors for these two quantities which are linearly independent at every generic point. Therefore the maximum dimension of $T_q\mathcal{Q}$ is $n-2$, and by the above argument we have shown the vector fields $\mathcal{V}$ span $T_q\mathcal{Q}$ at every generic point q. $\square$

6.3.3. *Controllability.* In this section, we prove controllability of the dynamics (6.7). By (6.9), the orbit of an initial condition $q^{(0)} \in \mathcal{Q}$ is confined to the space $\mathcal{Q}$ defined in (6.12). Recalling the definition of extended indices in Section 6.3, we define the set of *interacting coordinate triples*

$$\mathcal{I} := \left\{ (\mathbf{j}, \mathbf{k}, \boldsymbol{\ell}) \in (\mathbb{Z}_N^2 \times \{+,-\})^3 \ : \ j + k = \ell,\ (C_{jk}, C_{j\ell}, C_{k\ell}) \neq (0,0,0),\ \mathrm{T}(\mathbf{j}) \cdot \mathrm{T}(\mathbf{k}) = \mathrm{T}(\boldsymbol{\ell}) \right\}.$$

Then, for any such triple of interacting indices $\iota \in \mathcal{I}$ we denote by $\varphi_t^\iota \ : \ \mathcal{Q} \to \mathcal{Q}$ the flow of the ODEs (6.7) evolving the corresponding coordinates. The dynamics we consider is then obtained by cycling through the set $\mathcal{I}$ in a fixed or random order. For any $\iota \in \mathcal{I}$ we denote by $\Phi_t^\iota \ : \ \mathcal{Q} \to \mathcal{Q}$ the flow of (6.7) after one such full cycle where the flow times are chosen as

$$\tau^\xi = \begin{cases} t & \text{if } \xi = \iota\,, \\ 0 & \text{else}\,, \end{cases}$$

so that for any $q \in \mathcal{Q}$, $\Phi_t^\iota(q) = \varphi_t^\iota(q)$.

Given a shell $\mathcal{Q}$ from (6.12) we can uniquely define a point $q^* = (a_j^*, b_j^*)_{j=1}^{n/2} \in \mathcal{Q}$ as follows:

$$q^*_{(1,0)} = q^*_{(0,1)} = (a^*, 0)\,, \qquad q^*_{(N,N)} = (0, b^*)\,, \tag{6.14}$$

for $a^*, b^* \geq 0$ and $q_j^* = (0,0)$ for all other $j \in \mathbb{Z}_N^2$. We show below that for any initial condition $q^{(0)} \in \mathcal{Q}$ satisfying Assumption 3 the system can be driven to the unique point $q^* \in \mathcal{Q}$.

Proposition 6.8. *For any initial* $q^{(0)} = (a_j^{(0)}, b_j^{(0)})_{j=1}^{n/2} \in \mathcal{Q}$ *satisfying Assumption 3 there exists* $M \in \mathbb{N}$ *and a joint sequence of transition times and coordinate triples* $\{(\iota(m), \tau(m))\}_{m=1}^M$ *such that*

$$\Phi_{\tau(M)}^{\iota(M)} \circ \cdots \circ \Phi_{\tau(1)}^{\iota(1)}(q^{(0)}) = q^*\,. \tag{6.15}$$

Proof of Proposition 6.8. We prove the above result by first evolving the initial condition $q^{(0)}$ into a sufficiently nondegenerate state $q^{(1)}$, and then by sequentially shrinking the set of active components of the coordinate vector q to the ones listed in (6.14). We realize this program by following, in order, the sequence of steps described below, represented schematically in Fig. 3:

(0) If it is not the case at initialization, Lemma 6.12 shows that we can "prepare" our state by evolving $q^{(0)}$ into $q^{(1)}$ such that

$$a^{(1)}_{(1,0)}, b^{(1)}_{(1,0)}, a^{(1)}_{(0,1)}, b^{(1)}_{(0,1)}, a^{(1)}_{(1,1)}, b^{(1)}_{(1,1)} \neq 0\,, \tag{6.16}$$

as represented in Fig. 3a.

(1) As shown in Lemma 6.13, we can then transform $q^{(1)}$ into $q^{(2)}$ with the property

$$q_j^{(2)} = (0,0) \qquad \text{for all } j \in \mathbb{Z}_N^2 \setminus \{(0,1),(1,0),(1,1),(N,N),(-N,N)\}\,, \tag{6.17}$$

as represented in Fig. 3b, and

$$a^{(2)}_{(1,0)}, b^{(2)}_{(1,0)}, a^{(2)}_{(0,1)}, b^{(2)}_{(0,1)}, a^{(2)}_{(1,1)}, b^{(2)}_{(1,1)} \neq 0\,. \tag{6.18}$$

(2) Lemma 6.14 shows that we can then "transfer" the amplitude from modes $a_{(-N,N)}$, $b_{(-N,N)}$, $a_{(N,N)}$ to mode $b_{(N,N)}$ i.e., we can reach a state $q^{(3)}$ that satisfies

$$q_j^{(3)} = (0,0) \qquad \text{for all } j \in \mathbb{Z}_N^2 \setminus \{(0,1),(1,0),(1,1),(N,N)\}\,, \tag{6.19}$$

$$q^{(3)}_{(N,N)} = (0, b^{(3)}_{(N,N)}) \quad \text{with } b^{(3)}_{(N,N)} \geq 0\,. \tag{6.20}$$

This state is represented in Fig. 3c.

(3) Finally, Lemma 6.16 shows that we can "transfer" the amplitude from modes $a_{(1,1)}$, $b_{(1,1)}$, $b_{(0,1)}$ and $b_{(1,0)}$ to modes $a_{(0,1)}, a_{(1,0)}, b_{(N,N)}$ so that, after the transfer, $a_{(0,1)} = a_{(1,0)}$ and $a_{(0,1)}, a_{(1,0)}, b_{(N,N)} > 0$ i.e., we reach the unique state q^* from (6.14) (represented in Fig. 3d).

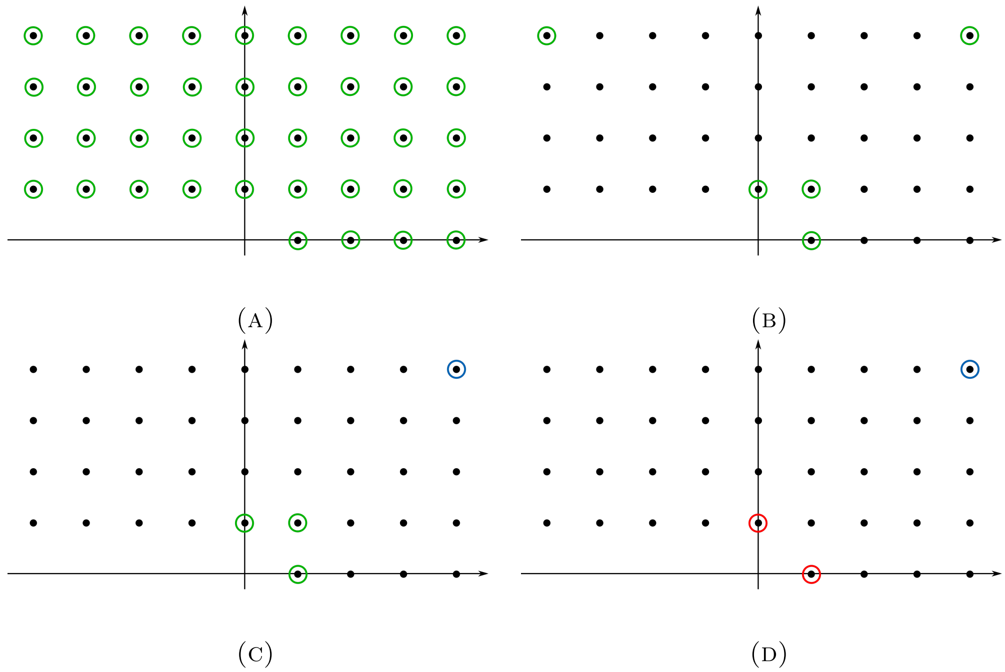


Figure 3. Representation of the state of the network in a generic initial state (a), after step 1 of the procedure in the proof of Proposition 6.8 (b), and after step 2 (c) and after step 3 (d) of the same procedure. In the above pictures, each point corresponds to a mode, i.e., an element of $\mathbb{Z}_N^2$ while the color of each circle represents the real/complex value of the corresponding mode: zero (white, no circle), purely imaginary (red), purely real (blue) or having both nonvanishing real and imaginary parts (green).

□

Defining similarly to (6.11) the operation of removing a coordinate from the set $\mathcal{A}$

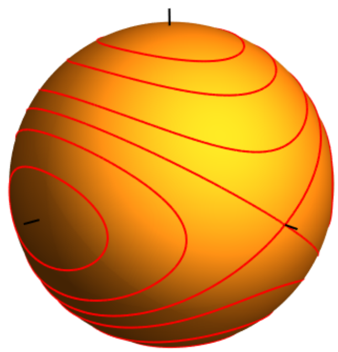
$$\mathcal{A} \ominus \boldsymbol{\ell} = \begin{cases} \mathcal{A} \setminus \{\boldsymbol{\ell}\} & \text{if } \ell \in \{j+k, j-k\} \cap \mathbb{Z}_N^2 \text{ for } \mathbf{j}, \mathbf{k} \in \mathcal{A}, C_{jk} \neq 0, \mathrm{T}(\mathbf{j}) \cdot \mathrm{T}(\mathbf{k}) = \mathrm{T}(\boldsymbol{\ell}) \,, \\ \mathcal{A} & \text{else} \,, \end{cases} \tag{6.21}$$

we now proceed to *construct* (sequences of) times τ and interacting triples ι such that the transformations $\Phi_\tau^{(\iota)}$ of q implement the operations $\oplus, \ominus$ from (6.11), (6.21) through the flow of (6.7), i.e., such that $\mathcal{A}(q) \oplus \boldsymbol{\ell} = \mathcal{A}(\Phi_\tau^\iota(q))$ or $\mathcal{A}(q) \ominus \boldsymbol{\ell} = \mathcal{A}(\Phi_\tau^\iota(q))$ respectively. To do so we separate the possible interactions between the modes in two types:

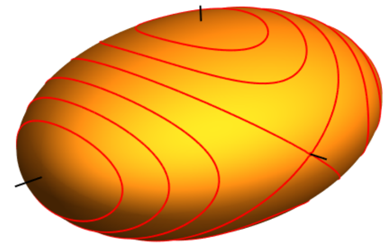
$$\begin{aligned} &a) \quad \iota = \boldsymbol{jk\ell} \in \mathcal{I} \;:\; |j| \neq |k| \neq |\ell| \,, \\ &b) \quad \iota = \boldsymbol{jk\ell} \in \mathcal{I} \;:\; |j| = |k| \neq |\ell| \,. \end{aligned} \tag{6.22}$$

Note that these two types of interactions are exhaustive, since if $|j| = |k| = |\ell|$, $C_{j\ell} = C_{jk} = C_{k\ell} = 0$.

The following preparatory lemmas describe the properties of these two types of interactions that we will leverage throughout our proof. The first one shows that for interactions of type a), ordering the indices so that $|j| < |k| < |\ell|$, it is always possible to activate all modes $\boldsymbol{j}, \boldsymbol{k}, \boldsymbol{\ell}$ or to distribute the amplitude of the k-mode to the j and ℓ-modes reaching, in finite time, a state with $q_{\boldsymbol{k}} = 0$. As we show in the proof below, while such a point with $q_{\boldsymbol{k}} = 0$ always exists on the orbits of (6.7), this



(A)



(B)

FIGURE 4. Orbits $\mathcal{Q}_\iota$ of (6.7) (in red) corresponding in (A) to various values of the energy $\mathcal{E}_\iota(q)$ on the sphere of constant enstrophy $E_\iota(q)$ and in (B) to various values of the enstrophy $E_\iota(q)$ on the ellipsoid of constant energy $\mathcal{E}_\iota(q)$. The axes are, sequentially, $q_{\boldsymbol{k}}, q_{\boldsymbol{j}}, q_{\boldsymbol{\ell}}$. The orbit with a degenerate point at the pole of the sphere or ellipsoid corresponds to values of $E_\iota, \mathcal{E}_\iota$ violating (6.23).

point is reachable in finite time for $\iota = \boldsymbol{jk\ell} \in \mathcal{I}$ with $|j| < |k| < |\ell|$ only if

$$E_\iota(q) \neq |k|^2 \mathcal{E}_\iota(q)\,, \tag{6.23}$$

where $E_\iota(q)$ and $\mathcal{E}_\iota(q)$ denote the energy and enstrophy of the coordinates in $\iota \in \mathcal{I}$:

$$E_\iota(q) \coloneqq \sum_{\boldsymbol{\ell}\in\iota} |q_{\boldsymbol{\ell}}|^2\,, \qquad \mathcal{E}_\iota(q) \coloneqq \sum_{\boldsymbol{\ell}\in\iota} \frac{|q_{\boldsymbol{\ell}}|^2}{|\ell|^2}\,. \tag{6.24}$$

In the following lemma and throughout the section, we abuse notation slightly by defining $\mathrm{sign}(x) = +1$ for $x \in [0, \infty)$ and -1 otherwise.

Lemma 6.9. *Fix $\iota = \boldsymbol{jk\ell} \in \mathcal{I}$ with $|j| < |k| < |\ell|$. Let $q \in \mathcal{Q}$ satisfy (6.23) and let $q_{\boldsymbol{l}} = 0$ for at most an index $\boldsymbol{l} \in \{\boldsymbol{j}, \boldsymbol{k}, \boldsymbol{\ell}\}$. Then there exist $\tau_-^\iota, \tau_+^\iota \geq 0$ such that*

(a) $\varphi^\iota_{\tau^\iota_-}(q) = q'$ with $q'_{\boldsymbol{k}} = 0$, $\mathrm{sign}(q_{\boldsymbol{j}}) = \mathrm{sign}(q'_{\boldsymbol{j}})$ and $\mathrm{sign}(q_{\boldsymbol{\ell}}) = \mathrm{sign}(q'_{\boldsymbol{\ell}})$,

(b) $\varphi^\iota_{\tau^\iota_+}(q) = q''$ with $q''_{\boldsymbol{j}}, q''_{\boldsymbol{k}}, q''_{\boldsymbol{\ell}} \neq 0$, $\mathrm{sign}(q_{\boldsymbol{j}}) = \mathrm{sign}(q''_{\boldsymbol{j}})$ and $\mathrm{sign}(q_{\boldsymbol{\ell}}) = \mathrm{sign}(q''_{\boldsymbol{\ell}})$.

Furthermore, if $|j|^2\mathcal{E}_\iota(q) < E_\iota(q) < |k|^2\mathcal{E}_\iota(q)$, there exists $\tau_=^\iota \geq 0$ such that

(c) $\varphi^\iota_{\tau^\iota_=}(q) = q'''$ with $q'''_{\boldsymbol{\ell}} = 0$, $\mathrm{sign}(q_{\boldsymbol{j}}) = \mathrm{sign}(q'''_{\boldsymbol{j}})$ and $\mathrm{sign}(q_{\boldsymbol{k}}) = \mathrm{sign}(q'''_{\boldsymbol{k}})$.

Proof. We consider the intersection between the sphere and the ellipse corresponding to the enstrophy and the energy in the coordinates $\iota = \boldsymbol{jk\ell} \in \mathcal{I}$ of interest, resulting in the set

$$\mathcal{Q}_\iota \coloneqq \left\{ (q'_{\boldsymbol{j}}, q'_{\boldsymbol{k}}, q'_{\boldsymbol{\ell}}) \in \mathbb{R}^3 \;:\; |q'_{\boldsymbol{j}}|^2 + |q'_{\boldsymbol{k}}|^2 + |q'_{\boldsymbol{\ell}}|^2 = E_\iota(q), \frac{|q'_{\boldsymbol{j}}|^2}{|j|^2} + \frac{|q'_{\boldsymbol{k}}|^2}{|k|^2} + \frac{|q'_{\boldsymbol{\ell}}|^2}{|\ell|^2} = \mathcal{E}_\iota(q) \right\}.$$

This set is represented in Fig. 4. We observe that this set has exactly 2 disjoint simply connected components when $|j|^2\mathcal{E}_\iota(q) < E_\iota(q) < |k|^2\mathcal{E}_\iota(q)$ and $|k|^2\mathcal{E}_\iota(q) < E_\iota(q) < |\ell|^2\mathcal{E}_\iota(q)$. These components are diffeomorphic to S^1. By continuity the dynamics are limited to one such component of $\mathcal{Q}_\iota$. Furthermore, $|\dot{q}|^2$ is uniformly bounded away from 0 on each such component: the fixed points of (6.7) must have at least two coordinates vanishing, which cannot be realized on the curves of interest. Therefore the dynamics on these sets are periodic.

We start by proving part (b) of the lemma. If $q_{\boldsymbol{j}}, q_{\boldsymbol{k}}, q_{\boldsymbol{\ell}} \neq 0$ the result follows by choosing $\tau_+^\iota = 0$. Else, if $q_{\boldsymbol{l}} = 0$ for $\boldsymbol{l} \in \iota$ the result follows immediately choosing τ_+^ι small enough by combining the continuity of the flow Φ_t^ι and the fact that $\dot{q}_{\boldsymbol{l}} = C_{l'l''} q_{\boldsymbol{l}'} q_{\boldsymbol{l}''} \neq 0$ for $\{\boldsymbol{l}', \boldsymbol{l}''\} = \iota \setminus \{\boldsymbol{l}\}$.

To prove part (a) we consider the cases where $|j|^2 \mathcal{E}_\iota(q) < E_\iota(q) < |k|^2 \mathcal{E}_\iota(q)$ and $|k|^2 \mathcal{E}_\iota(q) < E_\iota(q) < |\ell|^2 \mathcal{E}_\iota(q)$ separately. In the first case, we see that there is no point $q \in \mathcal{Q}_\iota$ with $q_{\boldsymbol{j}} = 0$: if that were the case we would have

$$E_\iota(q) = q_{\boldsymbol{k}}^2 + q_{\boldsymbol{\ell}}^2 = |k|^2 \left(\frac{q_{\boldsymbol{k}}^2}{|k|^2} + \frac{q_{\boldsymbol{\ell}}^2}{|k|^2} \right) > |k|^2 \mathcal{E}_\iota(q) \,,$$

contradicting our assumption. Consequently the points $(p_{\boldsymbol{j}}, 0, p_{\boldsymbol{\ell}}), (p_{\boldsymbol{j}}, 0, -p_{\boldsymbol{\ell}})$ with $p_{\boldsymbol{\ell}} > 0$, $\mathrm{sign}(p_{\boldsymbol{j}}) = \mathrm{sign}(q_{\boldsymbol{j}})$ and

$$p_{\boldsymbol{j}}^2 + p_{\boldsymbol{\ell}}^2 = E_\iota(q) \,, \qquad \frac{p_{\boldsymbol{j}}^2}{|j|^2} + \frac{p_{\boldsymbol{\ell}}^2}{|\ell|^2} = \mathcal{E}_\iota(q) \,,$$

belong to the same connected component as q and by the lower bound on the velocity on this connected component both these points are reachable in finite time from q. This also proves part (c) by continuity of the dynamics. The second case where $|k|^2 \mathcal{E}_\iota(q) < E_\iota(q) < |\ell|^2 \mathcal{E}_\iota(q)$ can be handled analogously: in this case we have $\mathcal{Q}_\iota \cap \{q_{\boldsymbol{\ell}} = 0\} = \emptyset$ and we can reach $(p_{\boldsymbol{j}}, 0, p_{\boldsymbol{\ell}}), (-p_{\boldsymbol{j}}, 0, p_{\boldsymbol{\ell}})$ with $p_{\boldsymbol{j}} > 0$, $\mathrm{sign}(p_{\boldsymbol{\ell}}) = \mathrm{sign}(q_{\boldsymbol{\ell}})$ in finite time. □

The following lemma considers interactions of type b) in (6.22). Recalling the definition $j^\perp := (j_2, -j_1)$ we show that interactions with $|j| = |k| \neq |\ell|$ leave component $\boldsymbol{\ell}$ fixed and move $\boldsymbol{j}, \boldsymbol{k}$ in a circle at constant angular speed.

Lemma 6.10. *Fix an (unordered) interacting triple $\iota = \boldsymbol{jk\ell}$ with $|k| = |j|$ and let $q_{\boldsymbol{\ell}} \neq 0$. Then for every $\vartheta \in [0, 2\pi)$ there exists $t \geq 0$ such that $\varphi_t^\iota(q) = q'$ with $(q'_{\boldsymbol{j}}, q'_{\boldsymbol{k}}) = \sqrt{q_{\boldsymbol{j}}^2 + q_{\boldsymbol{k}}^2}(\cos(\vartheta), \sin(\vartheta))$ and $q'_{\boldsymbol{\ell}} = q_{\boldsymbol{\ell}}$.*

Corollary 6.11. *Fix an (unordered) interacting triple $\iota = \boldsymbol{jk\ell} \in \mathcal{I}$ with $|k| = |j|$ and let $q_{\boldsymbol{\ell}}, q_{\boldsymbol{k}} \neq 0$. Then there exist $\tau_+^\iota, \tau_-^\iota \geq 0$ such that $(\varphi_{\tau_+^\iota}^\iota(q))_{\mathbf{j}} > 0$ and $(\varphi_{\tau_-^\iota}^\iota(q))_{\mathbf{j}} = 0$.*

Proof of Lemma 6.10. Recall from (6.3) that if $|j| = |k| \neq |\ell|$ we have $C_{jk} = 0$. This implies that, by our choice of $|k| = |j|$, $\dot{q}_{\boldsymbol{\ell}} = 0$ and $q'_{\boldsymbol{\ell}} = q_{\boldsymbol{\ell}}$. Again by (6.3) and since to have an interacting triple $\ell = j + k$ we must have

$$\langle k^\perp, \ell \rangle = \langle k^\perp, k + j \rangle = \langle k^\perp, j \rangle = \langle (k+j)^\perp - j^\perp, j \rangle = \langle \ell^\perp, j \rangle = -\langle j^\perp, \ell \rangle \,,$$

so that

$$C_{k\ell} = \frac{\langle k, \ell^\perp \rangle}{4\pi} \left(\frac{1}{|k|^2} - \frac{1}{|\ell|^2} \right) = -\frac{\langle j, \ell^\perp \rangle}{4\pi} \left(\frac{1}{|j|^2} - \frac{1}{|\ell|^2} \right) = -C_{j\ell} \,.$$

This implies that the dynamics of the vector $\tilde{q} := (q_{\boldsymbol{j}}, q_{\boldsymbol{k}})$ can be written as $\dot{\tilde{q}} = \tilde{C} \tilde{q}^\perp$ for $\tilde{C} := C_{j\ell} q_{\boldsymbol{\ell}} \neq 0$, proving the claim. □

Combining the partial results obtained above we show the existence of transformations implementing the steps listed at the beginning of the section:

Lemma 6.12. *Let $q^{(0)} \in \mathcal{Q}$ satisfy Assumption 3, then there exists M_1 and a sequence of transition times and interaction triples $\{\iota(m), \tau(m)\}_{m=1}^{M_1}$ such that $\Phi_{\tau(M_1)}^{\iota(M_1)} \circ \cdots \circ \Phi_{\tau(1)}^{\iota(1)}(q^{(0)}) = q^{(1)}$ as in (6.16).*

Proof. If the condition (6.16) is satisfied by $q^{(0)}$ we simply set $M_1 = 0$, $q^{(1)} = q^{(0)}$. If this is not the case, by Assumption 3 there exists a sequence of triples $\{\iota(m)\}_{m=1}^M$ with $\iota(m) = \boldsymbol{j}(m)\boldsymbol{k}(m)\boldsymbol{\ell}(m)$ such that $\mathcal{A}_0 := \mathcal{A}(q^{(0)})$ and $\mathcal{A}_m = \mathcal{A}_{m-1} \oplus \boldsymbol{\ell}(m)$ with $\{(0,1,+), (1,0,+), (j^*, -)\} \subset \mathcal{A}_M$. We notice that all steps of this procedure satisfy, upon possibly reordering the indices within each triple, either the conditions of Lemma 6.9 (b) or of Lemma 6.10, so we sequentially choose $\tau(m) = \tau_+^{\iota(m)}$ from those lemmas.

To activate coordinate $(1, 1, -)$ – if this was not already done in the previous procedure – we start with component $b_{j^*} \neq 0$ for $|j^*| \neq 1$ and consider a nearest neighbors path $\{\ell(n)\}_{n=1}^{M'}$ in $\mathbb{Z}_N^2$ connecting j^* to $(1, 1)$ without performing any step on the axes. It is easy to see that such path can be realized through repeated application of Lemma 6.9 (b) by choosing for the n-th step the triples $\iota(n) = (0, 1, +)(\ell(n), -)(\ell(n) \pm (0, 1), -)$ or $\iota(n) = (1, 0, +)(\ell(n), -)(\ell(n) \pm (1, 0), -)$ for vertical and horizontal steps respectively.

Finally, coordinates $(1, 0, -)$ and $(0, 1, -)$ can be activated by applying Lemma 6.11 to the triples $(1, 0, -)(0, 1, +)(1, 1, -)$ and $(1, 0, +)(0, 1, -)(1, 1, -)$ respectively, while $(1, 1, +)$ is activated by (b) by interchanging the type of modes $(1, 1, -)$ and $(1, 0, +)$ (or $(0, 1, +)$) in $\iota(M')$ from the previous paragraph to $(1, 1, +)$ and $(1, 0, -)$ (or $(0, 1, -)$). ☐

Lemma 6.13. *Let $q^{(1)} \in \mathcal{Q}$ satisfy (6.16). Then there exists M_2 and a sequence of interacting triples and transition times $\{\iota(m), \tau(m)\}_{m=1}^{M_2}$ such that $\Phi_{\tau(M_2)}^{\iota(M_2)} \circ \cdots \circ \Phi_{\tau(1)}^{\iota(1)}(q^{(1)}) = q^{(2)} \in \mathcal{Q}$ satisfying (6.17) and (6.18).*

Proof. In this part of the proof, we only consider interactions involving triples $\iota(m)$ of the form

$$\Big\{(0,1)(l,h)(l,h\pm 1) \text{ or } (1,0)(l,h)(l\pm 1,h) : \ |l|,|h| \le N,\ |(l,h)| \neq 1\Big\}. \tag{6.25}$$

By Lemma 6.9 (a), if $|j| < |k| < |\ell|$ and $(0, 1), (l, h) \in \mathcal{A}(q)$ there exists $\tau(m) = \tau_-^{\iota(m)}$ such that defining $\mathcal{A}_m = \mathcal{A}(\varphi_{\tau(m)}^{\iota(m)}(q))$ we have $(l, h) \notin \mathcal{A}_m$ and $(0, 1) \in \mathcal{A}_m$ (and similarly for $(1, 0)$) [2]. Note that while a triple as above satisfies by assumption that $|j| < |k| < |\ell|$ and at least two of its coordinates are nonvanishing, it does not, in general, satisfy (6.23). However, assuming that q does not satisfy (6.23), by Lemma 6.10 and setting $\iota' = (1, 0)(0, 1)(1, 1)$, there exists $\tau^{\iota'}$ such that $|q_{(1,0)}| \neq |(\Phi_{\tau^{\iota'}}^{\iota'}(q))_{(1,0)}| > 0$. Since none of the coordinates in $\mathbb{Z}_N^2 \setminus \{(1, 0)(0, 1)(1, 1)\}$ are affected by this operation, $(\Phi_{\tau^{\iota'}}^{\iota'}(q))$ satisfies (6.23) and Lemma 6.9 can be applied to this state.

To conclude the proof we identify a sequence of triples $\iota(m) = (j(m), k(m), \ell(m)) \in \mathcal{I}$ of the form (6.25) such that for $\mathcal{A}_0 = \mathcal{A}(q^{(1)}) \subseteq \mathbb{Z}_N^2 \times \{+, -\}$

$$(((\mathcal{A}_0 \ominus k(1)) \ominus k(2)) \ominus \dots) \ominus k(M_2) = \{(1,0,\chi),(0,1,\chi),(1,1\chi),(N,N\chi),(-N,N\chi)\,,\chi\in\{+,-\}\}\,.$$

A possible such sequence is given by triples of the form

$$\Big\{(1,0,+)(l,h,\chi)(l+1,h,\chi) \ : \ (l,h)\in\{(0,2),\dots,(0,N)\}\,,\chi\in\{+,-\}\Big\}$$

to remove the vertical column of $\mathbb{Z}_N^2$ (which cannot interact with $(0, 1)$), followed by

$$\Big\{\Big((0,1,+)(l,h,\chi)(l,h+1,\chi) \ : \ (l,h)\in\big\{(l,0),\dots,(l,N):|l|\in(1,\dots,N-1)\big\}\setminus\{(1,1)\}\Big)\,,\chi\in\{+,-\}\Big\}\,,$$

where importantly the set of transitions for each l is ordered. The above transformation zeroes all coefficients except those in the set $\{(1, 1), (0, 1), (1, 0)\} \cup \{(l, N) \ : \ l \in (-N, \dots, N)\}$. We further remove the coefficients from $\{(l, N) \ : \ l \in (-N+1, \dots, N-1)\}$ by sequentially applying Lemma 6.9 to the ordered sequence of interacting triples

$$\Big((1,0,+)(l,h,\chi)(l+1,h,\chi) \ : \ (l,h)\in\{(0,N),\dots,(N-1,N)\,,\chi\in\{+,-\}\}\Big)\,,$$

and then

$$\Big((1,0,+)(l,h,\chi)(l-1,h,\chi) \ : \ (l,h)\in\{(-1,N),\dots,(-N+1,N)\}\,,\chi\in\{+,-\}\Big)\,.$$

[2] Note that the same result can trivially be obtained if $(l, h) \notin \mathcal{A}(q)$ setting $\tau_-^{\iota(m)} = 0$

It is easy to check that each transition in the above construction sequentially satisfies the assumptions of Lemma 6.9 (a), and that once a mode has been removed from $\mathcal{A}$ it will not interact again in this procedure. The fact that (6.18) holds follows from (6.16) and that in an interacting triple $\iota = \boldsymbol{jk\ell}$ with $|j| < |k| < |l|$ both modes $\boldsymbol{j}$ and $\boldsymbol{\ell}$ are in $\mathcal{A}$ at the end of the interaction by τ_-^ι. □

Lemma 6.14. *Let $q^{(2)} \in \mathcal{Q}$ satisfy (6.17) and (6.18). Then there exists M_3 and a sequence of interacting triples and transition times $\{\iota(m), \tau(m)\}_{m=1}^{M_3}$ such that $\Phi_{\tau(M_3)}^{\iota(M_3)} \circ \cdots \circ \Phi_{\tau(1)}^{\iota(1)}(q^{(2)}) = q^{(3)} \in \mathcal{Q}$ satisfying (6.19) and (6.20).*

Since it may not be possible to "transfer" the content of *e.g.*, mode $(-N, N)$ to $(-N+1, N)$ through one single interaction with mode $(1, 0)$ – and therefore it won't be possible to transfer the amplitude of mode $(-N, N)$ to (N, N) in one single "pass" – we proceed to prove that, through a sequence of interactions, we can transfer a *finite* and $q_{(-N,N)}$-independent amount of energy from mode $(-N, N)$ to (N, N). Therefore, the transfer of amplitude from mode $(-N, N)$ to (N, N) may be accomplished by repeating this sequence of interactions sufficiently many times.

The following corollary of Lemma 6.10 will be instrumental for the proof of Lemma 6.14:

Corollary 6.15. *Let $q_{(1,1)}, b_{(1,1)} \neq 0$ then for any q, q' with $q_{\boldsymbol{j}} = q'_{\boldsymbol{j}}$ for all $|j| > 1$ there exist a sequence $\{\iota(m), \tau(m)\}_{m=1}^{4}$ such that $\Phi_{\tau(4)}^{\iota(4)} \circ \cdots \circ \Phi_{\tau(1)}^{\iota(1)}(q) = q'$.*

Proof of Lemma 6.14. The desired result follows upon showing that for any $i \in \{-N, \ldots, N\}$, setting $\boldsymbol{\ell} = (-i, N, \chi), \boldsymbol{\ell}' = (i, N, \chi')$ for $\chi, \chi' \in \{-, +\}$ there exists $M_{\boldsymbol{\ell},\boldsymbol{\ell}'}$ and a sequence of triples and interaction times $\{\iota(m), \tau(m)\}_{m=1}^{M_{\boldsymbol{\ell},\boldsymbol{\ell}'}}$ such that for any q satisfying $\bigcup_{|i'|<i}\{(i', N, +), (i', N, -)\} \cap \mathcal{A}(q) = \emptyset$ and $q' = \Phi_{\tau(M_{\boldsymbol{\ell},\boldsymbol{\ell}'})}^{\iota(M_{\boldsymbol{\ell},\boldsymbol{\ell}'})} \circ \cdots \circ \Phi_{\tau(1)}^{\iota(1)}(q)$ we have

$$q'_{\boldsymbol{j}} = \begin{cases} q_{\boldsymbol{j}} & \text{for } \boldsymbol{j} \in \mathbb{Z}_N^2 \setminus \{\boldsymbol{\ell}, \boldsymbol{\ell}'\}, \\ 0 & \text{for } \boldsymbol{j} = \boldsymbol{\ell} \text{ if } \boldsymbol{\ell} \neq \boldsymbol{\ell}', \end{cases} \tag{6.26}$$

and for $\boldsymbol{k} \in \{\boldsymbol{\ell}, \boldsymbol{\ell}'\}$, $\mathrm{sign}(q_{\boldsymbol{k}}) = \mathrm{sign}(q'_{\boldsymbol{k}})$ holds if $q'_{\boldsymbol{k}} \neq 0$ (recalling our choice of notation $\mathrm{sign}(0) = +1$). Indeed, if $\mathrm{sign}(b_{(N,N)}) \geq 0$ we sequentially apply the above result to the pairs

$$(\boldsymbol{\ell}, \boldsymbol{\ell}') = ((N, N, +), (-N, N, +)), ((-N, N, +), (N, N, -)), ((-N, N, -), (N, N, -)).$$

Otherwise, when $\mathrm{sign}(b_{(N,N)}) = -1$ we first apply the above result to $\boldsymbol{\ell} = (N, N, -)$, $\boldsymbol{\ell}' = (-N, N, -)$ and then proceed as in the previous case.

We prove the result above by induction on $i \in \{0, \ldots, N\}$. The proof for $i \leq 0$ is analogous.

Base case ($i = 0 : (0, N, \chi) \to (0, N, \chi')$): If $\boldsymbol{\ell} = \boldsymbol{\ell}'$ there is nothing to show. We proceed to consider the case $\boldsymbol{\ell} = (0, N, +)$, $\boldsymbol{\ell}' = (0, N, -)$, as the converse follows by analogous arguments. In this case, for a sufficiently small $\varepsilon > 0$ we consider the interactions $\iota = (1, 0, +)(0, N, +)(1, N, +)$ and $\iota' = (1, 0, -)(0, N, -)(1, N, +)$, running the corresponding flow maps by a small amount of time $\tau(\varepsilon)$, $\tau'(\varepsilon)$ such that $(\Phi_{\tau'(\varepsilon)}^{\iota'} \circ \Phi_{\tau(\varepsilon)}^{\iota}(q)_{(0,N,-)})^2 = b_{(0,N)}^2 + \varepsilon$. We then apply Corollary 6.15 to the coordinates $(1, 0, +), (1, 0, -)$ to return them in the initial configuration. Note that the existence of a uniform $\varepsilon > 0$ such that the transitions above can be performed in a single pair of interactions (and therefore the finiteness of the total number of interactions required to perform the desired transformation) follows from the fact that $b_{(0,N)}$ is nondecreasing and the continuity of the dynamics together with Lemma 6.9.

Induction step ($i > 0 : (-i, N, \chi) \to (i, N, \chi')$): We consider two possibilities for q: a) there exists q'' with $|a''_{(1,0)}| \in [|a_{(1,0)}|/2, |a_{(1,0)}|]$, $q''_{(-i,N,\chi)} = 0$ and for $\iota'' = (1, 0, +)(-i+1, N, \chi)(-i, N, \chi)$

$$E_{\iota''}(q) = E_{\iota''}(q''), \qquad \mathcal{E}_{\iota''}(q) = \mathcal{E}_{\iota''}(q''),$$

or b) such q'' does not exist.

In case a) the state q'' can be reached by letting $\iota = (1,0,+)(-i+1,N,\chi)(-i,N,\chi)$ interact for a finite amount of time τ from Lemma 6.9 (c). Then, by the induction assumption there is a sequence of triples and interaction times allowing to reach a state q''' with $q'''_{(-i+1,N,\chi)} = 0$, $q'''_{(i-1,N,\chi')} = q''_{(-i+1,N,\chi)}$ and $q'''_{\boldsymbol{j}} = q''_{\boldsymbol{j}}$ for all other $j \in \mathbb{Z}_N^2$. The desired state can then be reached by application of Lemma 6.9 (a) to the triple $\iota = (1,0,+)(i-1,N,\chi')(i,N,\chi')$. We proceed to check that the final state satisfies (6.26). Because modes $j \notin \{(-i,N),\dots,(i,N),(1,0)\}$ did not interact in the procedure above for such $\boldsymbol{j}$ we must have that $q_{\boldsymbol{j}} = q'_{\boldsymbol{j}}$. The fact that for $j \in \{(-i,N),\dots,(i-1,N)$ $q'_{\boldsymbol{j}} = 0$ follows by construction and the induction assumption. It remains to check that $|a'_{(1,0)}| = |a_{(1,0)}|$. Since the only modes affected by the above transformation are $(-i,N,\chi),(i,N\chi'),(1,0,+)$, this follows directly by conservation of energy and enstrophy:

$$(q_{(-i,N,\chi)})^2 + (q_{(i,N,\chi')})^2 + (q_{(1,0,+)})^2 = (q'_{(i,N,\chi')})^2 + (q'_{(1,0,+)})^2\,,$$
$$\frac{(q_{(-i,N,\chi)})^2}{N^2+i^2} + \frac{(q_{(i,N,\chi')})^2}{N^2+i^2} + (q_{(1,0,+)})^2 = \frac{(q'_{(-i,N,\chi)})^2}{N^2+i^2} + (q'_{(1,0,+)})^2\,.$$

In case b) we proceed to show that case a) can be reached with a finite number of interactions. More specifically if condition a) is not satisfied we let the triple $\iota'' = (-i,N,\chi)(-i+1,N,\chi)(1,0,+)$ for $\chi \in \{+,-\}$ interact as described by Lemma 6.9 for a time τ'' to reach $q'' \in \mathcal{Q}$ with $q''_{\boldsymbol{j}} = q_{\boldsymbol{j}}$ for $\boldsymbol{j} \notin \{(-i,N,\chi),(-i+1,N,\chi),(1,0,+)\}$, $a''_{(1,0)} = a_{(1,0)}/2$ and $q''_{(-i,N,\chi)}, q''_{(-i+1,N,\chi)}$ satisfying the conservation laws

$$(q_{(-i,N,\chi)})^2 + (q_{(1,0,+)})^2 = (q''_{(-i,N,\chi)})^2 + (q''_{(-i+1,N,\chi)})^2 + (q_{(1,0,+)}/2)^2\,,$$
$$\frac{(q_{(-i,N,\chi)})^2}{N^2+i^2} + (q_{(1,0,+)})^2 = \frac{(q''_{(-i,N,\chi)})^2}{N^2+i^2} + \frac{(q''_{(-i+1,N,\chi)})^2}{N^2+(i-1)^2} + (q_{(1,0,+)}/2)^2\,,$$

so that $(q''_{(-i,N,\chi)})^2 = (q_{(-i,N,\chi)})^2 - C_{N,i}(q_{(1,0)})^2$ for $C_{N,i} = \frac{3}{4}\frac{N^2+i^2}{i^2-(i-1)^2}(N^2+(i-1)^2-1)$. We see that a positive, $q_{(1,0,+)}$-dependent amplitude is removed from $(q_{(-i,N,\chi)})^2$. Again applying the induction step and Lemma 6.9 (a) to transfer, respectively, the amplitude from $(-i+1,N,\chi)$ to $(i-1,N,\chi')$ and from $(i-1,N,\chi')$ to (i,N,χ') we reach the state q' with $q_{\boldsymbol{j}} = q'_{\boldsymbol{j}}$ for modes $j \notin \{(-i,N),\dots,(i,N),(1,0)\}$ (since these modes either vanish in both cases or they did not interact). Further, by conservation of energy and enstrophy, we have that

$$(q_{(-i,N,\chi)})^2 + (q_{(i,N,\chi')})^2 + (q_{(1,0,+)})^2 = (q''_{(-i,N,\chi)})^2 + (q''_{(i,N,\chi')})^2 + (q''_{(1,0,+)})^2\,,$$
$$\frac{(q_{(-i,N,\chi)})^2}{N^2+i^2} + \frac{(q_{(i,N,\chi')})^2}{N^2+i^2} + (q_{(1,0,+)})^2 = \frac{(q''_{(-i,N,\chi)})^2}{N^2+i^2} + \frac{(q''_{(i,N,\chi')})^2}{N^2+i^2} + (q''_{(1,0,+)})^2\,,$$

so that $|q''_{(1,0,+)}| = |q_{(1,0,+)}|$. This shows that the amplitude $C_{N,i}(q_{(1,0,+)})^2$ subtracted to $q_{(-i,N,\chi)}$ is constant at each cycle, showing by boundedness of $q_{(-i,N,\chi)}$ that with a finite number of iterations as the one described above we can reach state a), concluding the proof. □

Lemma 6.16. *Let $q^{(3)} \in \mathcal{Q}$ satisfy (6.19) and (6.20). Then there exists M_4 and a sequence of interacting triples and transition times $\{\iota(m),\tau(m)\}_{m=1}^{M_4}$ such that $\Phi_{\tau(M_4)}^{\iota(M_4)} \circ \cdots \circ \Phi_{\tau(1)}^{\iota(1)}(q^{(3)}) = q^* \in \mathcal{Q}$ satisfying (6.14).*

Proof. We start the proof by applying Corollary 6.15 to transform the state $q^{(3)}$ into $q = \Phi_{\tau(1)}(q^{(3)})$ satisfying $q^{(3)}_{\boldsymbol{j}} = q_{\boldsymbol{j}}$ for all $|j| > 1$ and $a_{(0,1)} = b_{(0,1)} = b_{(1,0)} = a_{(1,0)} > 0$. Throughout this proof, we refer to states q such that $q_{(i,i',\chi)} = q_{(i',i,\chi)}$ for all $i,i' \in (0,\dots,N)$, $\chi \in \{+,-\}$ as *symmetric*.

We then proceed to transfer the amplitude from $a_{(1,1)}$ to $b_{(2,1)}, b_{(1,2)}$ by transforming q into another symmetric state q' with $(2,1,-),(1,2,-) \in \mathcal{A}(q')$ and $(1,1,+) \notin \mathcal{A}(q')$. This can be done by letting triples $\iota(2) = (1,0,-)(1,1,+)(2,1,-) \in \mathcal{I}$ and $\iota(3) = (0,1,-)(1,1,+)(1,2,-) \in \mathcal{I}$ interact, and choosing the interaction times $\tau, \tau'(\tau)$ such that $\Phi^{\iota(3)}_{\tau'(\tau)} \circ \Phi^{\iota(2)}_{\tau}(q)_{(1,1,+)} = 0$. Further, we note that the difference $b'_{(1,2)} - b'_{(2,1)}$ is negative for $\tau = 0$, positive for $\tau'(\tau) = 0$ and is continuous in τ, so there must exist τ^* such that $b'_{(1,2)} = b'_{(2,1)}$. To show that q' is symmetric it only remains to show that $b'_{(1,0)} = b'_{(0,1)}$. This follows from the conservation laws:

$$B_{(1,0)(1,1)} \left((b'_{(1,0)})^2 - (b_{(1,0)})^2 \right) = B_{(2,1)(1,1)} (b'_{(2,1)})^2 = B_{(1,2)(1,1)} (b'_{(1,2)})^2 = B_{(0,1)(1,1)} \left((b'_{(0,1)})^2 - (b_{(0,1)})^2 \right)$$

where

$$B_{jk} \coloneqq \frac{1}{|j|^2} - \frac{1}{|k|^2} \,.$$

Next, we let the triples $\iota(4) = (1,0,-)(0,1,+)(1,1,-)$ and $\iota(5) = (0,1,-)(1,0,+)(1,1,-)$ interact. By Lemma 6.10 there exists an interaction time such that the initial state q' is mapped to q'' with $b''_{(1,0)} = b''_{(0,1)} = 0$ and $a''_{(1,0)} = a''_{(0,1)} > 0$, so that $(1,0,-),(0,1,-) \notin \mathcal{A}(q'')$.

We then proceed to transfer the amplitude from modes $(1,2,-)$ and $(2,1,-)$ to $(2,2,-)$. This is done letting triples $\iota(6) = (1,0,+)(1,2,-)(2,2,-)$ and $\iota(7) = (0,1,+)(2,1,-)(2,2,-)$ interact until the modes $(2,1,-),(1,2,-)$ are depleted, as proved in Lemma 6.9. The symmetry of the final state q''' is again a consequence of the conservation laws:

$$B_{(1,0)(2,2)} \left((a'''_{(1,0)})^2 - (a''_{(1,0)})^2 \right) = B_{(2,1)(2,2)} (b''_{(2,1)})^2 = B_{(1,2)(2,2)} (b''_{(1,2)})^2 = B_{(0,1)(2,2)} \left((a'''_{(0,1)})^2 - (a''_{(0,1)})^2 \right) .$$

Summarizing, we have reached a symmetric state $q''' = \Phi^{\iota(7)}_{\tau(7)} \circ \cdots \circ \Phi^{\iota(2)}_{\tau(2)}(q)$ with

$$\mathcal{A}(q''') = \{(1,0,+),(0,1,+),(2,2,-),(1,1,-),(N,N,-)\} \,.$$

The desired result then follows immediately if we can show that we can transfer the amplitude of mode $(i-1,i-1,-)$ to $(i,i,-)$ for $i \in (2,\ldots,N)$ while preserving the fact that $a'_{(1,0)} = a'_{(0,1)}$. We show this by considering, sequentially, the interaction triples

$$\iota(4i) = (1,0,+)(i-1,i-1,-)(i,i-1,-) \,, \qquad \iota(4i+1) = (0,1,+)(i-1,i-1,-)(i-1,i,-) \,,$$
$$\iota(4i+2) = (0,1,+)(i,i-1,-)(i,i,-) \,, \qquad \iota(4i+3) = (1,0,+)(i-1,i,-)(i,i,-) \,.$$

More specifically, we consider the family of endpoints

$$q''(t) = \Phi^{\iota(4i+3)}_{\tau_-^{\iota(4i+3)}} \circ \Phi^{\iota(4i+2)}_{\tau_-^{\iota(4i+2)}} \circ \Phi^{\iota(4i+1)}_{\tau_-^{\iota(4i+1)}} \circ \Phi^{\iota(4i)}_{t}(q') \,,$$

where τ_-^{ι} is defined in Lemma 6.9 (a). By construction, this sequence implies that $a''_{(i-1,i-1)} = a''_{(i-1,i)} = a''_{(i,i-1)} = 0$ and $a''_{(i,i)} \neq 0$. It remains to prove that $a''_{(1,0)} = a''_{(0,1)}$. As a composition of continuous functions, $q''(t)$ is continuous in t and therefore so is $\Delta q(t) = a''_{(1,0)}(t) - a''_{(0,1)}(t)$. Further, since by symmetry $a''_{(1,0)}(0) = a''_{(0,1)}(\tau_-^{\iota(4i)})$, we must have $\mathrm{sign}(\Delta q(0)) = -\mathrm{sign}(\Delta q(\tau_-^{\iota_1}))$. This implies the existence of $\tau(4i) \in [0, \tau_-^{\iota_1}]$ with $\Delta q(0) = 0$, concluding the proof. □

7. Adding forcing and dissipation: Lorenz-96 and 2D Navier-Stokes

In this section, we add dissipation and fixed body forcing to both conservative Lorenz-96 and Galerkin approximations of 2D Euler by introducing a new vector field

$$V_0(x) = -\nu \Lambda x + F \tag{7.1}$$

to the splittings constructed in Sections 5 and 6, where $\nu > 0$ is an arbitrary constant, F a fixed nonzero vector with nonnegative entries, and Λ a linear operator satisfying

$$\Lambda x \cdot x \geq \alpha \|x\|^2 \tag{7.2}$$

for some $\alpha > 0$. For the remainder of this section we consider random splittings associated to collections of continuous vector fields $\{V_k\}_{k=0}^n$ on $\mathbb{R}^d$ satisfying

Assumption 4. *V_0 is as in (7.1) and the flows of the other V_k conserve Euclidean norm.*

Fix $h > 0$ and let P_h be the transition kernel of a random splitting satisfying Assumption 4. When Λ is the identity matrix, the addition of V_0 to the splitting of conservative Lorenz-96 gives a splitting of the full Lorenz-96 model, (1.4), while for 2D Euler the resulting V_0 corresponds to a friction or drag term sometimes called *Ekman damping*. When Λ is diagonal with diagonal entry $|k|^2$ In the spots associated to[3] a_k and b_k, which corresponds to a Laplacian written in Fourier space, the addition of V_0 to the splitting of 2D Euler gives a splitting of 2D Navier-Stokes, (1.1).

Note the dissipative part of V_0 in (7.1) depends linearly on x whereas the forcing is constant. Thus dissipation dominates forcing for sufficiently large x and, since the remaining vector fields are conservative, the splitting dynamics cannot grow too large. Specifically, letting $\Phi_{h\tau}$ be as in (2.2) but with the solution $\varphi^{(0)}$ of $\dot{x} = V_0(x)$ appended to the beginning of each cycle, we have

Lemma 7.1. *Under Assumption 4 for any initial x and $m > 0$,*

$$\|\Phi_{h\tau}^m(x)\|^2 \leq \|x\|^2 e^{-\nu\alpha h \sum_{k=0}^m \tau_{k(n+1)}} + \frac{1}{\nu^2\alpha^2}\|F\|^2 \left(1 - e^{-\nu\alpha h \sum_{k=0}^m \tau_{k(n+1)}}\right). \tag{7.3}$$

Proof. Letting $\varphi = \varphi^{(0)}$, we have

$$\partial_t \|\varphi_t\|^2 = 2\langle F, \varphi_t\rangle - 2\nu\langle \Lambda\varphi_t, \varphi_t\rangle \leq \frac{1}{\nu\alpha}\|F\|^2 + \nu\alpha\|\varphi_t\|^2 - 2\nu\alpha\|\varphi_t\|^2 = \frac{1}{\nu\alpha}\|F\|^2 - \nu\alpha\|\varphi_t\|^2,$$

where the inequality follows from (7.2) and $2\langle F, \varphi_t\rangle \leq (\nu\alpha)^{-1}\|F\|^2 + \nu\alpha\|\varphi_t\|^2$. Solving

$$\dot{y} = \frac{1}{\nu\alpha}\|F\|^2 - \nu\alpha y$$

from $y(0) = \|x\|$ together with the Comparison Theorem for ODEs [McN86] then gives

$$\|\varphi_t(x)\|^2 \leq \|x\|^2 e^{-\nu\alpha t} + \frac{1}{\nu^2\alpha^2}\|F\|^2 \left(1 - e^{-\nu\alpha t}\right)$$

for all time. And since $\varphi^{(k)}$ conserves norm for $1 \leq k \leq n$, the above implies

$$\|\Phi_{h\tau}(x)\|^2 = \|\varphi_{h\tau_n}^{(n)} \circ \cdots \circ \varphi_{h\tau_0}^{(0)}(x)\|^2 = \|\varphi_{h\tau_0}^{(0)}(x)\|^2 \leq \|x\|^2 e^{-\nu\alpha\tau_0} + \frac{1}{\nu^2\alpha^2}\|F\|^2 \left(1 - e^{-\nu\alpha\tau_0}\right).$$

The result then follows by straightforward induction on the number of cycles, namely m. □

Remark 7.2. *The above implies that for both Lorenz-96 and Galerkin approximations of 2D Navier-Stokes, the random splitting starting from any point x will remain inside the ball of radius $\|x\|^2 + (\nu\alpha)^{-2}\|F\|^2$ centered at the origin for all time. In particular, for both models the random splitting dynamics are confined to a compact subset of Euclidean space. And since the splitting vector fields are smooth, all convergence results of Section 4 hold for these random splittings.*

Corollary 7.3. *The Euclidean norm is a Lyapunov function for P_h. That is, there exist constants $K \geq 0$ and $\gamma \in (0, 1)$ such that for all $x \in \mathbb{R}^d$,*

$$(P_h\|\cdot\|)(x) \leq \gamma\|x\| + K.$$

[3] Recall that for each index $k \in \mathbb{Z}_N^2$, we have two real coordinates a_k and b_k.

Proof. By Lemma 7.1, specifically $\|\Phi_{ht}(x)\| \le \|x\| e^{-\frac{1}{2}\nu\alpha t_0} + (\nu\alpha)^{-1}\|F\|$, we have

$$(P_h\|\cdot\|)(x) = \int_{\mathbb{R}_+^{n+1}} \|\Phi_{ht}(x)\| e^{-\sum t_k} dt \le \frac{1}{1+\frac{1}{2}\nu\alpha h}\|x\| + \frac{1}{\nu\alpha}\|F\|$$

for any x. The result follows with $K = (\nu\alpha)^{-1}\|F\|$ and $\gamma = (1+\frac{1}{2}\nu\alpha h)^{-1}$. □

7.1. **Ergodicity.** We now present a variation of Theorem 3.1, namely Theorem 7.8, which simplifies verification of ergodicity in the present setting. This involves two observations. First, recall from Sections 5 and 6 that one of the difficulties in verifying Theorem 3.1 was proving controllability, i.e., the existence of a distinguished point x_* that could be reached by the splitting dynamics in finite time from any other point. With the addition of dissipation, the fixed point $\nu^{-1}\Lambda^{-1}F$ of $\dot{x} = V_0(x)$ is a natural candidate for x_* and, as we will see, the fact that it is globally attracting obviates several technicalities associated with controllability in the conservative cases discussed above.

The second observation is that, while the surjectivity assumption on $D_t\Phi^m$ is satisfied for Lorenz-96, it does not hold for Galerkin approximations of 2D Navier-Stokes. Indeed, as a result of the two conserved quantities, energy and enstrophy, we saw the splitting vector fields $\{V_k\}_{k=1}^n$ of 2D Euler can span at most $n-2$ dimensions. Thus the splitting vector fields $\{V_k\}_{k=0}^n$ of 2D Navier-Stokes can span at most $n-1$ dimensions and, in particular, cannot span the full n-dimensional tangent space of the space in which the dynamics live. To address this issue we introduce the following Hörmander-type hypoellipticity condition, discussed at length in [BH12, BBMZ15].

Definition 7.4. *Let $\mathcal{V}_0 = \{V_k\}_{k=0}^n$ be a family of vector fields on a manifold $\mathcal{M}$ and recursively set*

$$\mathcal{V}_{m+1} = \{[V_k, W] : W \in \mathcal{V}_m, 0 \le k \le n\} \cup \mathcal{V}_m.$$

The weak bracket condition *holds at $x \in \mathcal{M}$ if there exists m such that $\mathcal{V}_m$ spans $T_x\mathcal{M}$.*

Remark 7.5. *Note that if the vector fields $\{V_k\}_{k=0}^n$ span $T_x\mathcal{M}$ then the weak bracket condition holds at x. Thus the spanning criteria of Section 3 imply the weak bracket condition.*

The proof of Theorem 7.8 uses the following lemmas. The first, due to Krylov-Bogolubov, is a standard result from the theory Markov processes [Hai10]. The second, which can be regarded as a weakening of the surjectivity assumption in Lemma 3.2, is from [BBMZ15, Theorem 4.4]. For the statement of Lemma 7.6, recall a transition kernel P on $\mathbb{R}^d$ is *Feller* if Pf is continuous whenever $f : \mathbb{R}^d \to \mathbb{R}$ is continuous and bounded. Also, a sequence of probability measures $\{\mu_m\}$ on $\mathbb{R}^d$ is *tight* if for every $\varepsilon > 0$ there exists a compact subset K of $\mathbb{R}^d$ such that $\mu_m(K) \ge 1 - \varepsilon$ for all m.

Lemma 7.6. *Let P be a Feller probability transition kernel on $\mathbb{R}^d$. If there exists $x \in \mathbb{R}^d$ such that $\{P^m(x,\cdot)\}_{m=0}^\infty$ is tight, then P has an invariant probability measure.*

Lemma 7.7. *Suppose $\Phi_{ht}^m(x) = \widetilde{x}$ and the weak bracket condition holds at $\widetilde{x}$. Then there exists a $c > 0$, an $\widetilde{m}$, and neighborhoods U_x of x and $\widetilde{U}$ of $\widetilde{x}$ such that for all $y \in U_x$ and $B \in \mathcal{B}(\mathcal{X})$,*

$$P_h^{\widetilde{m}}(y, B) \ge c\lambda\left(B \cap \widetilde{U}\right).$$

Now to the main result of this subsection.

Theorem 7.8. *Suppose Assumption 4 holds and set $x_* = \nu^{-1}\Lambda^{-1}F$. If there exist $m \ge 0$ and t such that the weak bracket condition holds at $\widetilde{x} := \Phi_{ht}^m(x_*)$, then P_h has a unique invariant measure, μ. Furthermore, there exist constants $C > 0$ and $\gamma \in (0,1)$ such that for all $x \in \mathbb{R}^d$,*

$$\|P_h^m(x,\cdot) - \mu\| \le C\gamma^m \tag{7.4}$$

where $\|\cdot\|$ is the norm on probability measures induced by the weighted supremum norm $\|f\| := \sup_x|f(x)|/(1+\|x\|)$ on bounded measurable functions $f : \mathbb{R}^d \to \mathbb{R}$.

Proof. We first prove existence. Continuity of Φ_{ht} immediately implies P_h is Feller. Furthermore, Lemma 7.1 implies that random splitting starting from any x is constrained to lie in a compact subset of $\mathbb{R}^d$, namely the closed ball of radius $\|x\|^2 + (\nu\alpha)^{-2}\|F\|^2$ centered at the origin. Thus, for any x, the sequence $\{P_h^m(x,\cdot)\}_{m=0}^\infty$ is tight and existence follows from Lemma 7.6.

Next we prove uniqueness. The hypothesis and Lemma 7.7 together imply the existence of $c > 0$, $\widetilde{m}$, and neighborhoods U_* of x_* and $\widetilde{U}$ of $\widetilde{x}$ such that

$$P_h^{\widetilde{m}}(x,B) \geq c\lambda\left(B \cap \widetilde{U}\right) \tag{7.5}$$

for all $x \in U_*$ and Borel sets B. Also, positive-definiteness of Λ implies

$$\|\varphi_t^{(0)}(x) - x_*\| \leq e^{-\alpha t}\|x - x_*\|$$

for any $x \in \mathbb{R}^d$ and $t \geq 0$. In particular, for any open ball B_r of radius r centered at the origin, there exists $T_0 > 0$ such that $\varphi_{ht}^{(0)}(B_r)$ is properly contained in U_* whenever $ht > T_0$. And since $\varphi_{ht}^{(0)}(B_r)$ is properly contained in U_* and the $\varphi^{(k)}$ are continuous, there exist $T_k > 0$ such that $\Phi_{ht} = \varphi_{ht_n}^{(n)} \circ \cdots \circ \varphi_{ht_0}^{(0)}(x) \in U_*$ for all $x \in B_r$ and $ht_k \in (0, T_k)$. So, for any $x \in B_r$,

$$P_h(x,U_*) \geq \int_0^{T_n} \cdots \int_0^{T_1} \int_{T_0}^\infty \mathbb{1}_{U_*}\left(\Phi_{ht}(x)\right) e^{-\sum t_k} dt = \frac{1}{T_0}\prod_{k=1}^n \left(1 - e^{-T_k}\right) > 0$$

and hence $\inf_{x\in B_r} P_h(x,U_*) > 0$.

As in the proof of Theorem 3.1, suppose toward a contradiction that μ_1 and μ_2 are distinct P_h-ergodic probability measures and that A_1 and A_2 are disjoint measurable sets partitioning $\mathbb{R}^d$ with $\mu_i(B) = \mu_i(B \cap A_i)$ for all Borel sets B. Fix x_i in the support of μ_i, let r be sufficiently large that $x_1, x_2 \in B_r$, and set $\kappa := \inf_{x\in B_r} P_h(x,U_*) > 0$. Then by (7.5) for any Borel set B,

$$\begin{aligned}\mu_i(B) = \mu_i P_h^{\widetilde{m}+1}(B) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} P_h^{\widetilde{m}}(y,B)P_h(x,dy)\mu_i(dx) \\ &\geq \int_{B_r}\int_{U_*} P_h^{\widetilde{m}}(y,B)P_h(x,dy)\mu_i(dx) \geq \kappa c\lambda\left(B\cap\widetilde{U}\right)\mu_i\left(B_r\right).\end{aligned} \tag{7.6}$$

In particular, $\mu_i(B) = 0$ implies $\lambda(B \cap \widetilde{U}) = 0$ since c, κ, and $\mu_i(B_r)$ are all strictly positive (the latter because B_r is an open set containing both x_1 and x_2 which were chosen to be in the supports of μ_1 and μ_2, respectively). But $\mu_1(A_2 \cap \widetilde{U}) = \mu_2(A_1 \cap \widetilde{U}) = 0$ and so we obtain the contradiction

$$0 < \lambda\left(\widetilde{U}\right) = \lambda\left(A_1 \cap \widetilde{U}\right) + \lambda\left(A_2 \cap \widetilde{U}\right) = 0,$$

which concludes the proof of uniqueness.

Finally, for the exponential convergence statement (7.4), we have from (7.6) that for any $r > 0$,

$$\inf_{x\in B_r} P_h^{\widetilde{m}+1}(x,B) \geq \kappa c\lambda\left(B\cap\widetilde{U}\right)$$

for all Borel sets B. That is, the transition probabilities $P_h^{\widetilde{m}+1}(x,\cdot)$ are minorized uniformly over B_r by the probability measure $\widetilde{\lambda} := \lambda(\widetilde{U})^{-1}\lambda(\cdot\cap\widetilde{U})$. Exponential convergence then follows from Corollary 7.3 upon taking $r > 2K/(1-\gamma)$. See for example Theorem 1.2 in [HM11]. □

Corollary 7.9. *Consider the random splitting of n-dimensional Lorenz-96 associated to the vector fields* $\{V_k\}_{k=0}^n$*, where* $V_0(x) = -\nu x + F$ *and* $\{V_k\}_{k=1}^n$ *are the splitting vector fields of conservative Lorenz-96 from Section 5. If* $x_* := -\nu F$ *is not a fixed point of conservative Lorenz-96, i.e.,* $\nu^2\sum_{k=1}^n(F_k^2 + F_{k+1}^2)F_{k-1}^2 \neq 0$*, then the random splitting has a unique invariant measure on* $\mathbb{R}^n$ *and the dynamics converge to this measure at an exponential rate in the sense of* (7.4).

Proof. The determinant of the n-by-n matrix

$$\begin{pmatrix} | & | & & | \\ V_0(x) & V_1(x) & \cdots & V_{n-1}(x) \\ | & | & & | \end{pmatrix} = \begin{pmatrix} -\nu x_1 + F_1 & x_2 x_n & 0 & \dots & 0 \\ -\nu x_2 + F_2 & -x_1 x_n & x_3 x_1 & \dots & 0 \\ -\nu x_3 + F_3 & 0 & -x_2 x_1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & x_n x_{n-2} \\ -\nu x_n + F_n & 0 & 0 & & -x_{n-1}x_{n-2} \end{pmatrix}$$

is

$$x_1 x_{n-1} x_n \left(\prod_{x=2}^{n-2} x_k^2 \right) \left(\nu^2 \|x\|^2 - \langle F, x \rangle \right).$$

So the $\{V_k\}_{k=0}^n$ span $\mathbb{R}^n$ at every x with nonzero coordinates and satisfying $\nu^2\|x\|^2 \neq \langle F, x\rangle$. In particular, since x_* is not a fixed point of conservative Lorenz-96, we showed in the proof of Proposition 5.2 that x_* can be moved via the splitting dynamics to some $\widetilde{x}$ with nonzero coordinates. And by rotating slightly more on the last step if necessary, we can also guarantee that $\nu^2\|\widetilde{x}\|^2 \neq \langle F, \widetilde{x}\rangle$. Thus the weak bracket condition holds at $\widetilde{x}$ and the result follows by Theorem 7.8. □

Corollary 7.10. *Fix $N \geq 2$ and set $n = 4N(N+1)$. Consider the random splitting of the Nth Galerkin approximation of 2D Navier-Stokes associated to the vector fields $\{V_k\}_{k=0}^n$, where $V_0(x) = -\nu\Lambda x + F$ with Λ the n-by-n diagonal matrix corresponding to the Laplacian discussed at the beginning of this section, and $\{V_k\}_{k=1}^n$ are the splitting vector fields of 2D Euler from Section 6. If the forcing vector F satisfies Assumption 3, then the random splitting has a unique invariant measure and the dynamics converge to this measure at an exponential rate in the sense of* (7.4).

Proof. Recall in this case $V_0(x) = -\nu\Lambda x + F$ where Λ is the diagonal matrix with diagonal entry $|k|^2$ in the slots corresponding to the coordinates a_k and b_k. Fix $j, k, \ell \in \mathbb{Z}_N^2$ with $j + k - \ell = 0$ and let W be one of the vector fields $V_{a_j a_k a_\ell}$, $V_{a_j b_k b_\ell}$, $V_{b_j a_k b_\ell}$, or $V_{b_j b_k a_\ell}$. Letting e.g. $(x_j, x_k, x_\ell) = (a_j, a_k, a_\ell)$ when $W = V_{a_j a_k a_\ell}$ and similarly for the other cases, direct computation yields

$$\begin{aligned} [V_0, W]_j(x) &= C_{k\ell}\left(F_k x_\ell + F_\ell x_k + \nu(|j|^2 - |k|^2 - |\ell|^2) x_k x_\ell\right), \\ [V_0, W]_k(x) &= C_{j\ell}\left(F_j x_\ell + F_\ell x_j + \nu(|k|^2 - |j|^2 - |\ell|^2) x_j x_\ell\right), \\ [V_0, W]_\ell(x) &= -C_{jk}\left(F_j x_k + F_k x_j + \nu(|\ell|^2 - |j|^2 - |k|^2) x_j x_k\right), \end{aligned} \tag{7.7}$$

where $[V_0, W]_j(x)$ is the component of $[V_0, W]$ corresponding to the component x_j of x, and similarly for $[V_0, W]_k$ and $[V_0, W]_\ell$. As in the 2D Euler case, Gaussian elimination shows that the 6-by-6 matrix (see (6.13) for an explicit form of the middle 4 columns)

$$\begin{pmatrix} | & | & | & | & | & | \\ V_0 & V_{a_j a_k a_\ell} & V_{a_j b_k b_\ell} & V_{b_j a_k b_\ell} & V_{b_j b_k a_\ell} & [V_0, W] \\ | & | & | & | & | & | \end{pmatrix} \tag{7.8}$$

is rank 6 at every generic point $q \in \mathbb{R}^n$. Thus V_0 and $[V_0, W]$ add two new directions to the splitting vector fields of 2D Euler and by an entirely similar argument to the spanning argument in Section 6.3.2 we have that the weak bracket condition holds at every generic $q \in \mathbb{R}^n$. Furthermore, since F satisfies Assumption 3 the controllability argument of Section 6.3.3 implies x_* can be evolved via the random splitting to a generic point. The result then follows by Theorem 7.8. □

Remark 7.11. *A very similar argument to the one above proves unique ergodicity for Ekman damping as well, i.e., when Λ is the identity matrix on $\mathbb{R}^n$. In this case* (7.7) *becomes*

$$
\begin{aligned}
[V_0, W]_j(x) &= C_{k\ell}\left(F_k x_\ell + F_\ell x_k - \nu x_k x_\ell\right),\\
[V_0, W]_k(x) &= C_{j\ell}\left(F_j x_\ell + F_\ell x_j - \nu x_j x_\ell\right),\\
[V_0, W]_\ell(x) &= -C_{jk}\left(F_j x_k + F_k x_j - \nu x_j x_k\right),
\end{aligned}
$$

and the rest of the argument goes through unchanged.

Acknowledgments: All authors thank the NSF grant NSF-DMS-1613337 for partial support during this project. AA also acknowledges the partial support of NSF-CCF-1934964, and OM also thanks NSF-DMS-2038056 for partial support during this project. JCM thanks David Herzog and Brendan Williamson for discussions at the start of these investigations. JCM thanks the hospitality and support of the Institute for Advanced Study, where this manuscript was completed.

References

[AR96] Anton Arnold and Christian Ringhofer. An operator splitting method for the Wigner-Poisson problem. *SIAM J. Numer. Anal.*, 33(4):1622–1643, 1996.

[BBMZ15] Michel Benaïm, Stéphane Le Borgne, Florent Malrieu, and Pierre-André Zitt. Qualitative properties of certain piecewise deterministic Markov processes. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 51(3):1040 – 1075, 2015.

[BBPS21] Jacob Bedrossian, Alex Blumenthal, and Sam Punshon-Smith. Almost-sure enhanced dissipation and uniform-in-diffusivity exponential mixing for advection-diffusion by stochastic Navier-Stokes. *Probab. Theory Related Fields*, 179(3-4):777–834, 2021.

[BH12] Yuri Bakhtin and T. Hurth. Invariant densities for dynamical systems with random switching. *Nonlinearity*, 25:2937–2952, 2012.

[BNS96] B. Bermúdez, A. Nikolás, and F. J. Sánchez. On operator splitting methods with upwinding for the unsteady Navier-Stokes equations. *East-West J. Numer. Math.*, 4(2):83–98, 1996.

[COS19] Andrew M. Childs, Aaron Ostrander, and Yuan Su. Faster quantum simulation by randomization. *Quantum*, 3:182, Sep 2019.

[CST$^+$21] Andrew M. Childs, Yuan Su, Minh C. Tran, Nathan Wiebe, and Shuchen Zhu. Theory of trotter error with commutator scaling. *Physical Review X*, 11(1):011020, Feb 2021.

[DPD02] Giuseppe Da Prato and Arnaud Debussche. Two-dimensional Navier-Stokes equations driven by a space-time white noise. *J. Funct. Anal.*, 196(1):180–210, 2002.

[EM01] Weinan E and Jonathan C. Mattingly. Ergodicity for the Navier-Stokes equation with degenerate random forcing: finite-dimensional approximation. *Comm. Pure Appl. Math.*, 54(11):1386–1402, 2001.

[EMS01] Weinan E, J. C. Mattingly, and Ya. Sinai. Gibbsian dynamics and ergodicity for the stochastically forced Navier-Stokes equation. *Comm. Math. Phys.*, 224(1):83–106, 2001. Dedicated to Joel L. Lebowitz.

[FM95] Franco Flandoli and Bohdan Maslowski. Ergodicity of the 2-D Navier-Stokes equation under random perturbations. *Comm. Math. Phys.*, 172(1):119–141, 1995.

[Fol13] G.B. Folland. *Real Analysis: Modern Techniques and Their Applications*. Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts. Wiley, 2013.

[GK96] Daniel Goldman and Tasso J. Kaper. Nth-order operator splitting schemes and nonreversible systems. *SIAM J. Numer. Anal.*, 33(1):349–367, 1996.

[Hai10] Martin Hairer. Convergence of markov processes. *Lecture notes*, 2010.

[HM06] Martin Hairer and Jonathan C Mattingly. Ergodicity of the 2d navier-stokes equations with degenerate stochastic forcing. *Annals of Mathematics*, pages 993–1032, 2006.

[HM11] Martin Hairer and Jonathan C. Mattingly. Yet another look at Harris' ergodic theorem for Markov chains. In *Seminar on Stochastic Analysis, Random Fields and Applications VI*, volume 63 of *Progr. Probab.*, pages 109–117. Birkhäuser/Springer Basel AG, Basel, 2011.

[KNS20a] Sergei Kuksin, Vahagn Nersesyan, and Armen Shirikyan. Exponential mixing for a class of dissipative PDEs with bounded degenerate noise. *Geom. Funct. Anal.*, 30(1):126–187, 2020.

[KNS20b] Sergei Kuksin, Vahagn Nersesyan, and Armen Shirikyan. Exponential mixing for a class of dissipative PDEs with bounded degenerate noise. *Geom. Funct. Anal.*, 30(1):126–187, 2020.

[KS00] Sergei Kuksin and Armen Shirikyan. Stochastic dissipative PDEs and Gibbs measures. *Comm. Math. Phys.*, 213(2):291–330, 2000.

[KS03] Sergei Kuksin and Armen Shirikyan. Some limiting properties of randomly forced two-dimensional Navier-Stokes equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 133(4):875–891, 2003.

[Kus04] Shigeo Kusuoka. *Approximation of expectation of diffusion processes based on Lie algebra and Malliavin calculus*, volume 6 of *Adv. Math. Econ.*, page 69–83. Springer, 2004.

[Mat03] Jonathan C. Mattingly. On recent progress for the stochastic Navier Stokes equations. In *Journées "Équations aux Dérivées Partielles"*, pages Exp. No. XI, 52. Univ. Nantes, Nantes, 2003.

[McN86] Alex McNabb. Comparison theorems for differential equations. *Journal of mathematical analysis and applications*, 119(1-2):417–428, 1986.

[MS16] Shev MacNamara and Gilbert Strang. Operator splitting. In *Splitting methods in communication, imaging, science, and engineering*, Sci. Comput., pages 95–114. Springer, Cham, 2016.

[MSH02] J. C. Mattingly, A. M. Stuart, and D. J. Higham. Ergodicity for SDEs and approximations: locally Lipschitz vector fields and degenerate noise. *Stochastic Process. Appl.*, 101(2):185–232, 2002.

[MT09] Sean Meyn and Richard L. Tweedie. *Markov chains and stochastic stability.* Cambridge University Press, Cambridge, second edition, 2009. With a prologue by Peter W. Glynn.

[NN09] Mariko Ninomiya and Syoiti Ninomiya. A new higher-order weak approximation scheme for stochastic differential equations and the runge-kutta method. *Finance and Stochastics*, 13(3):415–443, 2009.

[Num78] E. Nummelin. The discrete skeleton method and a total variation limit theorem for continuous-time Markov processes. *Math. Scand.*, 42(1):150–160, 1978.

[NV08] S. Ninomiya and N. Victoir. Weak approximation of stochastic differential equations and application to derivative pricing. *Applied Mathematical Finance*, 15(2):107–121, 2008.

[Wil19] Brendan Williamson. *On SDEs with Partial Damping Inspired by the Navier-Stokes Equations.* PhD thesis, Duke University, May 2019. https://hdl.handle.net/10161/18773.

Appendix

A.1. **Semigroups, norms, and bounds.** In this subsection we elaborate on the semigroup framework of Section 4. The notation and results are used extensively in the proofs of Lemmas 4.2 and 4.6, which are given in subsections A.2 and A.4, respectively.

The $\mathcal{C}^2$ assumption implies the V_k, which act on functions f via $V_k f(x) = Df(x)V_k(x)$, are linear operators from $\mathcal{C}^2(\mathcal{X})$ to $\mathcal{C}^1(\mathcal{X})$ and from $\mathcal{C}^1(\mathcal{X})$ to $\mathcal{C}(\mathcal{X})$. It also implies the semigroups $\{S_t\}_{t\geq 0}$ and $\{\widetilde{S}_t^{(k)}\}_{t\geq 0}$ defined in (4.1) and (4.2) are linear operators on $\mathcal{C}^k(\mathcal{X})$ for all $k \leq 2$. Our aim now is to obtain bounds on norms of compositions of these random semigroups. To this end, for $i \leq j$ define $\Phi_{h\tau}^{(i,j)} \coloneqq \varphi_{h\tau_j}^{(j)} \circ \cdots \circ \varphi_{h\tau_i}^{(i)}$ and $\widetilde{S}_{h\tau}^{(i,j)} \coloneqq \widetilde{S}_{h\tau}^{(i)} \cdots \widetilde{S}_{h\tau}^{(j)}$. Note $\widetilde{S}_{h\tau}^{(i,j)}$ acts on functions f via

$$\widetilde{S}_{h\tau}^{(i,j)} f(x) = f\left(\Phi_{h\tau}^{(i,j)}(x)\right) = f\left(\varphi_{h\tau_j}^{(j)} \circ \cdots \circ \varphi_{h\tau_i}^{(i)}(x)\right).$$

So for any $f \in \mathcal{C}(\mathcal{X})$ with $\|f\|_\infty = 1$, we have

$$\|\widetilde{S}_{h\tau}^{(i,j)} f\|_\infty = \|f(\Phi_{h\tau}^{(i,j)})\|_\infty = 1$$

and hence $\|\widetilde{S}_{h\tau}^m\|_{0\to 0} = 1$. Next, let $\varphi = \varphi^{(k)}$ for arbitrary k. Then

$$\varphi_t(x) = x + \int_0^t V(\varphi_s(x))ds$$

and so

$$D\varphi_t(x) = I + \int_0^t DV(\varphi_s(x))D\varphi_s(x)ds$$

and

$$D^2\varphi_t(x) = \int_0^t D^2V(\varphi_s(x))\left(D\varphi_s(x), D\varphi_s(x)\right) + DV(\varphi_s(x))D^2\varphi_s(x)ds.$$

In particular, $\|D\varphi_t(x)\| \leq 1 + C_* \int_0^t \|D\varphi_s(x)\|ds$ for all $x \in \mathcal{X}$ and Grönwall's inequality implies

$$\sup_{x\in\mathcal{X}} \|D\varphi_t(x)\| \leq e^{C_* t}, \tag{A.1}$$

where C_* is as in (2.5). Similarly, since $\|D^2V(D\varphi, D\varphi)\| \leq \|D^2V\|\|D\varphi\|^2 \leq C_*\|D\varphi\|^2$,

$$\|D^2\varphi_t(x)\| \leq C_* \int_0^t \|D\varphi_s(x)\|^2 + \|D^2\varphi_s(x)\| ds \leq C_* t e^{2C_*t} + C_* \int_0^t \|D^2\varphi_s(x)\| ds$$

and Grönwall implies

$$\sup_{x\in\mathcal{X}} \|D^2\varphi_t(x)\| \leq C_* t e^{3C_*t}. \tag{A.2}$$

Note (A.1) and (A.2) hold uniformly over all $\varphi^{(k)}$. Thus, for $f \in C^1(\mathcal{X})$ with $\|f\|_1 = 1$,

$$\left\|D\left(\widetilde{S}_{h\tau}^{(i,j)} f\right)\right\| = \left\|Df\left(\Phi_{h\tau}^{(i,j)}\right) D\Phi_{h\tau}^{(i,j)}\right\| \leq \prod_{k=i}^{j} \|D\varphi_{h\tau_k}^{(k)}\| \leq e^{C_*h\sum_{k=i}^j \tau_k},$$

where the first inequality follows from submultiplicity and the second from (A.1). Similarly,

$$D^2\Phi_{h\tau}^{(i,j)} = \sum_{k=i}^{j} D\varphi_{h\tau_j}^{(j)} \cdots D\varphi_{h\tau_{k+1}}^{(k+1)} D^2\varphi_{h\tau_k}^{(k)} \left(D\Phi_{h\tau}^{(i,k-1)}, D\Phi_{h\tau}^{(i,k-1)}\right)$$

together with (A.1) and (A.2) gives

$$\begin{aligned}
\left\|D^2\Phi_{h\tau}^{(i,j)}\right\| &\leq \sum_{k=i}^{j} \left\|D\varphi_{h\tau_j}^{(j)}\right\| \cdots \left\|D\varphi_{h\tau_{k+1}}^{(k+1)}\right\| \left\|D^2\varphi^{(k)}\right\| \left\|D\Phi_{h\tau}^{(i,k-1)}\right\|^2 \\
&\leq C_* \sum_{k=i}^{j} h\tau_k e^{C_*h\sum_{k+1}^{j}\tau_\ell} e^{3C_*h\tau_k} e^{2C_*h\sum_{1}^{k-1}\tau_\ell} \leq C_* h e^{3C_*h\sum_{k=i}^j \tau_k} \sum_{k=i}^{j} \tau_k.
\end{aligned}$$

Therefore

$$\begin{aligned}
\left\|D^2\left(\widetilde{S}_{h\tau}^{(i,j)} f\right)\right\| &= \left\|D^2 f\left(\Phi_{h\tau}^{(i,j)}\right)\left(D\Phi_{h\tau}^{(i,j)}, D\Phi_{h\tau}^{(i,j)}\right) + Df\left(\Phi_{h\tau}^{(i,j)}\right) D^2\Phi_{h\tau}^{(i,j)}\right\| \\
&\leq \left\|D\Phi_{h\tau}^{(i,j)}\right\|^2 + \left\|D^2\Phi_{h\tau}^{(i,j)}\right\| \leq e^{2C_*h\sum_{k=i}^j \tau_k} + \left\|D^2\Phi_{h\tau}^{(i,j)}\right\| \\
&\leq e^{2C_*h\sum_{k=i}^j \tau_k} + C_* h e^{3C_*h\sum_{k=i}^j \tau_k} \sum_{k=i}^{j} \tau_k \\
&\leq \left(1 + C_* h \sum_{k=i}^{j} \tau_k\right) e^{3C_*h\sum_{k=i}^j \tau_k}.
\end{aligned}$$

The above computations prove

Lemma A.12. *For any $h > 0$ and $i \leq j$, we have $\|\widetilde{S}_{h\tau}^{(i,j)}\|_{0\to 0} = 1$ as well as*

$$\left\|\widetilde{S}_{h\tau}^{(i,j)}\right\|_{1\to 1} \leq e^{C_*h\sum_{k=i}^j \tau_k} \quad \textit{and} \quad \left\|\widetilde{S}_{h\tau}^{(i,j)}\right\|_{2\to 2} \leq \left(1 + C_* h \sum_{k=i}^{j} \tau_k\right) e^{3C_*h\sum_{k=i}^j \tau_k}.$$

In particular, $\left\|\widetilde{S}_{h\tau}^{(i,j)}\right\|_{\ell\to\ell} \leq \left(1 + C_*h\sum_{k=i}^j \tau_k\right) e^{3C_*h\sum_{k=i}^j \tau_k}$ for all $\ell \leq 2$.

Note that under the $\mathcal{C}^2$ assumption $\widetilde{S}_{h\tau}^{(i,j)}$ can also be regarded as a linear operator from $\mathcal{C}^2(\mathcal{X})$ to $\mathcal{C}^1(\mathcal{X})$. And since $\{f \in \mathcal{C}^2(\mathcal{X}) : \|f\|_2 = 1\}$ is a subset of $\{f \in \mathcal{C}^1(\mathcal{X}) : \|f\|_1 = 1\}$, we have

$$\left\|\widetilde{S}_{h\tau}^{(i,j)}\right\|_{2\to 1} = \sup_{\|f\|_2=1} \left\|\widetilde{S}_{h\tau}^{(i,j)} f\right\|_1 \leq \sup_{\|f\|_1=1} \left\|\widetilde{S}_{h\tau}^{(i,j)} f\right\|_1 = \left\|\widetilde{S}_{h\tau}^{(i,j)}\right\|_{1\to 1} \leq e^{C_*h\sum_{k=i}^j \tau_k}. \tag{A.3}$$

We also have the following corollary of Lemma A.12.

Corollary A.13. *Fix $i \leq j$ and set $m := j - i + 1$. For all $\ell \leq 2$ and polynomial $p : \mathbb{R}_+^m \to \mathbb{R}$ there exists $h_* > 0$ such that for all $h < h_*$,*

$$\mathbb{E}\|p(\tau_i, \dots, \tau_j)\widetilde{S}_{h\tau}^{(i,j)}\|_{k\to k} < \infty. \tag{A.4}$$

Proof. Writing $t = (t_i, \dots, t_j)$ and $dt = dt_i \cdots dt_j$, we have

$$\begin{aligned} \mathbb{E}\|p(\tau_i, \dots, \tau_j)\widetilde{S}_{h\tau}^{(i,j)}\|_{\ell\to\ell} &= \int_{\mathbb{R}_+^m} |p(t)| \left\|\widetilde{S}_{ht}^{(i,j)}\right\|_{\ell\to\ell} e^{-\sum t_k} dt \\ &\leq \int_{\mathbb{R}_+^m} |p(t)| \left(1 + C_* h \sum_{k=i}^{j} t_k\right) e^{(3C_*h-1)\sum_{k=i}^{j} t_k} dt \end{aligned}$$

which is finite for all $h < h_* := (3C_*)^{-1}$. □

A.2. **Proof of Lemma 4.2.** We highlight the steps of the proof with italicized font.

Variation of constants. We begin by differentiating $\widetilde{S}_{h\tau}$ in h:

$$\partial_h \widetilde{S}_{h\tau} = \sum_{k=1}^{n} \tau_k e^{h\tau_1} \cdots e^{h\tau_{k-1}} V_k e^{h\tau_k} \cdots e^{h\tau_n} = \sum_{k=1}^{n} \tau_k \widetilde{S}_{h\tau}^{(1,k-1)} V_k \widetilde{S}_{h\tau}^{(k,n)}.$$

Next, commute $\widetilde{S}_{h\tau}^{(1,k-1)}$ and V_k via $[\widetilde{S}_{h\tau}^{(1,k-1)}, V_k] := \widetilde{S}_{h\tau}^{(1,k-1)} V_k - V_k \widetilde{S}_{h\tau}^{(1,k-1)}$ to get

$$\partial_h \widetilde{S}_{h\tau} = \sum_{k=1}^{n} \tau_k V_k \widetilde{S}_{h\tau} + \sum_{k=1}^{n} \tau_k [\widetilde{S}_{h\tau}^{(1,k-1)}, V_k] \widetilde{S}_{h\tau}^{(k,n)} = V \widetilde{S}_{h\tau} + (V_\tau - V)\widetilde{S}_{h\tau} + E_{h\tau}$$

where $V_\tau := \sum_{k=1}^{n} \tau_k V_k$ and $E_{h\tau} := \sum_{k=1}^{n} \tau_k [\widetilde{S}_{h\tau}^{(1,k-1)}, V_k] \widetilde{S}_{h\tau}^{(k,n)}$. So, by variation of constants,

$$\widetilde{S}_{h\tau} - S_h = \int_0^h S_{h-r}(V_\tau - V)\widetilde{S}_{r\tau} dr + \int_0^h S_{h-r} E_{r\tau} dr. \tag{A.5}$$

Call $S_{h-r}(V_\tau - V)\widetilde{S}_{r\tau}$ *error term 1* and $S_{h-r}E_{r\tau}$ *error term 2*. These terms will be treated separately in what follows. First however, we invoke variation of constants again to get an expression for $[\widetilde{S}_{r\tau}^{(1,k-1)}, V_k]$ that will be used to control error term 2. Differentiating in r gives

$$\begin{aligned} \partial_r [\widetilde{S}_{r\tau}^{(1,k-1)}, V_k] &= \sum_{j=1}^{k-1} \tau_j [\widetilde{S}_{r\tau}^{(1,j-1)} V_j \widetilde{S}_{r\tau}^{(j,k-1)}, V_k] \\ &= \sum_{j=1}^{k-1} \tau_j \left([V_j \widetilde{S}_{r\tau}^{(1,k-1)}, V_k] + \big[[\widetilde{S}_{r\tau}^{(1,j-1)}, V_j]\widetilde{S}_{r\tau}^{(j,k-1)}, V_k\big] \right) \\ &= \sum_{j=1}^{k-1} \tau_j V_j [\widetilde{S}_{r\tau}^{(1,k-1)}, V_k] + \sum_{j=1}^{k-1} \tau_j \left([V_j, V_k] \widetilde{S}_{r\tau}^{(1,k-1)} + \big[[\widetilde{S}_{r\tau}^{(1,j-1)}, V_j]\widetilde{S}_{r\tau}^{(j,k-1)}, V_k\big] \right). \end{aligned}$$

The second equality follows from commuting $\widetilde{S}_{h\tau}^{(1,j-1)}$ and V_j as before, and the third follows from the identity $[XY, Z] = X[Y, Z] + [X, Z]Y$. So, by variation of constants,

$$\begin{aligned} [\widetilde{S}_{r\tau}^{(1,k-1)}, V_k] &= \sum_{j=1}^{k-1} \int_0^r \tau_j e^{(r-s)\sum_{j=1}^{k-1} \tau_j V_j} [V_j, V_k] \widetilde{S}_{s\tau}^{(1,k-1)} ds \\ &\quad + \sum_{j=1}^{k-1} \int_0^r \tau_j e^{(r-s)\sum_{j=1}^{k-1} \tau_j V_j} \big[[\widetilde{S}_{s\tau}^{(1,j-1)}, V_j]\widetilde{S}_{s\tau}^{(j,k-1)}, V_k\big] ds. \end{aligned} \tag{A.6}$$

Note $\|e^{(r-s)\sum_{j=1}^{k-1}\tau_j V_j}\|_{0\to 0} = 1$. So, by Corollary A.13 the integrands above satisfy

$$\mathbb{E}\|\tau_j e^{(r-s)\sum_{j=1}^{k-1}\tau_j V_j}[V_j, V_k]\widetilde{S}_{s\tau}^{(1,k-1)}\|_{2\to 0} \le \|[V_j, V_k]\|_{2\to 0}\mathbb{E}\|\tau_j \widetilde{S}_{s\tau}^{(1,k-1)}\|_{2\to 2} < C$$

and

$$\mathbb{E}\big\|\tau_j e^{(r-s)\sum_{j=1}^{k-1}\tau_j V_j}\big[[\widetilde{S}_{s\tau}^{(1,j-1)}, V_j]\widetilde{S}_{s\tau}^{(j,k-1)}, V_k\big]\big\|_{2\to 0} \le \mathbb{E}\big\|\tau_j\big[[\widetilde{S}_{s\tau}^{(1,j-1)}, V_j]\widetilde{S}_{s\tau}^{(j,k-1)}, V_k\big]\big\|_{2\to 0} < C$$

for some C. Therefore

$$\mathbb{E}\|[\widetilde{S}_{r\tau}^{(1,k-1)}, V_k]\|_{2\to 0} \le 2\sum_{j=1}^{k-1}\int_0^r C ds \le Cr \tag{A.7}$$

for some new constant C (we will often absorb arbitrary constants into existing ones).

Error term 1. Rewrite error term 1 as

$$\begin{aligned} S_{h-r}(V_\tau - V)\widetilde{S}_{r\tau} &= \sum_{k=1}^n (\tau_k - 1)S_{h-r}V_k\widetilde{S}_{r\tau} \\ &= \sum_{k=1}^n (\tau_k - 1)S_{h-r}V_k\widetilde{S}_{r\tau}^{(1,k-1)}\widetilde{S}_{r\tau}^{(k+1,n)} \\ &\qquad + \sum_{k=1}^n (\tau_k - 1)S_{h-r}V_k\widetilde{S}_{r\tau}^{(1,k-1)}(e^{r\tau_k V_k} - I)\widetilde{S}_{r\tau}^{(k+1,n)} \\ &=: \mathcal{A}_1 + \mathcal{A}_2 \end{aligned} \tag{A.8}$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are the first and second sums in the preceding expression. The second equality is obtained by adding and subtracting the identity I as follows:

$$\widetilde{S}_{r\tau} = \widetilde{S}_{r\tau}^{(1,k-1)}\big(e^{r\tau_k V_k} - I + I\big)\widetilde{S}_{r\tau}^{(k+1,n)} = \widetilde{S}_{r\tau}^{(1,k-1)}\widetilde{S}_{r\tau}^{(k+1,n)} + \widetilde{S}_{r\tau}^{(1,k-1)}(e^{r\tau_k V_k} - I)\widetilde{S}_{r\tau}^{(k+1,n)}.$$

Notice $\widetilde{S}_{r\tau}^{(1,k-1)}\widetilde{S}_{r\tau}^{(k+1,n)}$ does not depend on τ_k. So, since the τ_i are independent with mean 1,

$$\mathbb{E}(\mathcal{A}_1) = \sum_{k=1}^n S_{t-r}V_k\mathbb{E}(\tau_k - 1)\mathbb{E}\big(\widetilde{S}_{r\tau}^{(1,k-1)}\widetilde{S}_{r\tau}^{(k+1,n)}\big) = 0. \tag{A.9}$$

For the second sum, Taylor expanding $r \mapsto e^{r\tau_k V_k}$ about $r = 0$ with remainder gives

$$e^{r\tau_k V_k} - I = r\tau_k V_k e^{r_*\tau_k V_k}$$

for some $r_* \in [0, r]$. Therefore

$$\mathcal{A}_2 = r\sum_{k=1}^n \tau_k(\tau_k - 1)S_{h-r}V_k\widetilde{S}_{r\tau}^{(1,k-1)}V_k e^{r_*\tau_k V_k}\widetilde{S}_{r\tau}^{(k+1,n)}$$

and by Lemma A.12 and Corollary A.13,

$$\|\mathbb{E}(\mathcal{A}_2)\|_{2\to 0} \le Cr\sum_{k=1}^n \mathbb{E}\|\widetilde{S}_{r\tau}^{(1,k-1)}\|_{1\to 1}\mathbb{E}\|\tau_k(\tau_k - 1)\widetilde{S}_{r\tau}^{(k,n)}\|_{2\to 2} \le Cr \tag{A.10}$$

for some $C > 0$. Combining Equations (A.8), (A.9), and (A.10) gives

$$\|\mathbb{E}(S_{h-r}(V_\tau - V)\widetilde{S}_{r\tau})\|_{2\to 0} \le Cr. \tag{A.11}$$

Error term 2. Recall error term 2 is $S_{h-r}E_{r\tau} := \sum_{k=1}^{n} \tau_k S_{h-r}[\widetilde{S}_{r\tau}^{(1,k-1)}, V_k]\widetilde{S}_{r\tau}^{(k,n)}$. So, we have that

$$\|S_{h-r}E_{r\tau}\|_{2\to 0} \le \sum_{k=1}^{n} \tau_k \|S_{h-r}\|_{0\to 0} \|[\widetilde{S}_{r\tau}^{(1,k-1)}, V_k]\|_{2\to 0} \|\tau_k \widetilde{S}_{r\tau}^{(k,n)}\|_{2\to 2}\,.$$

Note $[\widetilde{S}_{r\tau}^{(1,k-1)}, V_k]$ is independent of τ_k. So, by (A.7) Corollary A.13,

$$\|\mathbb{E}(S_{h-r}E_{r\tau})\|_{2\to 0} \le Cr \tag{A.12}$$

for some $C > 0$.

Final step. Combining (A.5), (A.11), and (A.12) and absorbing constants into C, we have

$$\begin{aligned}\|P_h - S_h\|_{2\to 0} &= \|\mathbb{E}(\widetilde{S}_{h\tau} - S_h)\|_{2\to 0} \\ &\le \int_0^h \|\mathbb{E}(S_{h-r}(V_\tau - V)\widetilde{S}_{r\tau})\|_{2\to 0} dr + \int_0^h \|\mathbb{E}(S_{h-r}E_{r\tau})\|_{2\to 0} dr \\ &\le C\int_0^h r dr = \tfrac{1}{2}Ch^2.\end{aligned}$$

□

A.3. **Concentration of the sum of exponential random variables.** The proof of Lemma 4.6 will itself use two lemmas.

Lemma A.14. *Let $\{\tau_k\}_{k=1}^{\infty}$ be iid exponential with mean 1. For any $m \in \mathbb{N}$, $K > 0$ and $\beta > 1$,*

$$\mathbb{P}\left(\sum_{k=1}^{m} \tau_k > Km^\beta\right) \le 2^m e^{-\frac{1}{2}Km^\beta}. \tag{A.13}$$

Proof. Note if $\tau \sim \text{Exp}(1)$ then $\mathbb{E}(e^{\tau/2}) = 2$. So, by Markov's inequality and independence,

$$\mathbb{P}\left(\sum_{k=1}^{m} \tau_k > Km^\beta\right) = \mathbb{P}\left(e^{\frac{1}{2}\sum_{k=1}^{m}\tau_k} > e^{\frac{1}{2}Km^\beta}\right) \le e^{-\frac{1}{2}Km^\beta}\left(\mathbb{E}\left[e^{\frac{1}{2}\tau}\right]\right)^m = 2^m e^{-\frac{1}{2}Km^\beta}.$$

□

Lemma A.15. *Let $\{\tau_k\}_{k=1}^{\infty}$ be iid exponential with mean 1. For any $m \in \mathbb{N}$ and $K \in (0,1)$,*

$$\mathbb{P}\left(\left|\sum_{k=1}^{m} \tau_k - 1\right| > Km\right) < 2e^{-\frac{1}{2}K^2 m}. \tag{A.14}$$

Proof. Fix m. For any $\gamma \in (0,1)$,

$$\begin{aligned}\mathbb{P}\left(\left|\sum_{k=1}^{m} \tau_k - 1\right| > Km\right) &= \mathbb{P}\left(\sum_{k=1}^{m} \tau_k > (1+K)m\right) + \mathbb{P}\left(-\sum_{k=1}^{m} \tau_k > -(1-K)m\right) \\ &= \mathbb{P}\left(e^{\gamma\sum_{k=1}^{m}\tau_k} > e^{(1+K)\gamma m}\right) + \mathbb{P}\left(e^{-\gamma\sum_{k=1}^{m}\tau_k} > e^{-(1-K)\gamma m}\right) \\ &\le e^{-(1+K)\gamma m}\left(\mathbb{E}\left[e^{\gamma\tau}\right]\right)^m + e^{(1-K)\gamma m}\left(\mathbb{E}\left[e^{-\gamma\tau}\right]\right)^m \\ &= e^{-(1+K)\gamma m}(1-\gamma)^{-m} + e^{(1-K)\gamma m}(1+\gamma)^{-m} \\ &= \exp\left(-\gamma m\left[1 + K + \frac{\log(1-\gamma)}{\gamma}\right]\right) + \exp\left(\gamma m\left[1 - K - \frac{\log(1+\gamma)}{\gamma}\right]\right).\end{aligned}$$

The inequality is Markov's inequality and the equality immediately after the inequality follows from independence together with $\mathbb{E}[\exp(\alpha\tau)] = (1-\alpha)^{-1}$ for any $\alpha \in (-1,1)$. The other steps are all algebraic manipulations. By Taylor's theorem with remainder there exists $\gamma_1 \in (-\gamma, 0)$ such that

$$\frac{1}{\gamma}\log(1-\gamma) = -1 - \frac{\gamma}{2(1-\gamma_1)^2} > -1 - \frac{\gamma}{2},$$

where the inequality follows since $\gamma_1 < 0$. Therefore

$$\exp\left(-\gamma m\left[1 + K + \frac{\log(1-\gamma)}{\gamma}\right]\right) \leq \exp\left(-\gamma m\left[K - \frac{\gamma}{2}\right]\right).$$

Similarly,

$$\exp\left(\gamma m\left[1 - K - \frac{\log(1+\gamma)}{\gamma}\right]\right) \leq \exp\left(-\gamma m\left[K - \frac{\gamma}{2}\right]\right).$$

So combining with the first computation of this proof and taking $\gamma = K$ gives

$$\mathbb{P}\left(\left|\sum_{k=1}^{m}\tau_k - 1\right| > Km\right) \leq 2\exp\left(-\gamma m\left[K - \frac{\gamma}{2}\right]\right) = 2e^{-\frac{1}{2}K^2 m}. \qquad \square$$

A.4. **Proof of Lemma 4.6.** Fix $t > 0$. The argument is similar to that of Lemma 4.2.

Variation of constants. Fix $m \in \mathbb{N}$. Since $\widetilde{S}^m_{h\tau} = \exp(h\tau_1 V_1)\cdots\exp(h\tau_{mn}V_{mn})$,

$$\begin{aligned}\partial_h \widetilde{S}^m_{h\tau} &= \sum_{k=1}^{mn}\tau_k \widetilde{S}^{(1,k-1)}_{h\tau} V_k \widetilde{S}^{(k,mn)}_{h\tau} = \sum_{k=1}^{mn}\tau_k V_k \widetilde{S}^m_{h\tau} + \tau_k[\widetilde{S}^{(1,k-1)}h\tau, V_k]\widetilde{S}^{(k,mn)}_{h\tau} \\ &= mV\widetilde{S}^m_{h\tau} + \sum_{k=1}^{mn}(\tau_k - 1)V_k\widetilde{S}^m_{h\tau} + \sum_{k=1}^{mn}\tau_k[\widetilde{S}^{(1,k-1)}_{h\tau}, V_k]\widetilde{S}^{(k,mn)}_{h\tau},\end{aligned}$$

where the second equality is obtained by commuting $\widetilde{S}^{(1,k-1)}_{h\tau}$ and V_k, and the third by replacing τ_k with $\tau_k - 1 + 1$. So, setting $E_{h\tau} := \sum_{k=1}^{mn}\tau_k[\widetilde{S}^{(1,k-1)}_{h\tau}, V_k]\widetilde{S}^{(k,mn)}_{h\tau}$, variation of constants implies

$$\widetilde{S}^m_{h\tau} - S_{hm} = \int_0^h S_{m(h-r)}\left(\sum_{k=1}^{mn}(\tau_k - 1)V_k\right)\widetilde{S}^m_{r\tau}dr + \int_0^h S_{m(h-r)}E_{r\tau}dr.$$

And since $\|S_{m(h-r)}\|_{0\to 0} = 1$,

$$\|\widetilde{S}^m_{h\tau} - S_{hm}\|_{2\to 0} \leq \int_0^h \left\|\sum_{k=1}^{mn}(\tau_k - 1)V_k\right\|_{1\to 0}\left\|\widetilde{S}^m_{r\tau}\right\|_{2\to 1}dr + \int_0^h \|E_{r\tau}\|_{2\to 0}dr.$$

Let $I_1(h)$ and $I_2(h)$ denote the first and second integrals, respectively. Then for any $\varepsilon > 0$,

$$\mathbb{P}\left(\|\widetilde{S}^m_{h\tau} - S_{hm}\|_{2\to 0} > \frac{\varepsilon}{m}\right) \leq \mathbb{P}\left(I_1(h) > \frac{\varepsilon}{2m}\right) + \mathbb{P}\left(I_2(h) > \frac{\varepsilon}{2m}\right). \qquad \text{(A.15)}$$

We consider the two probabilities on the right, called the *first* and *second probabilities*, separately.

First probability. Note $\sum_{k=1}^{mn}(\tau_k - 1)V_k = \sum_{k=1}^{n}\sum_{j=1}^{m}(\tau^{(k)}_j - 1)V_k$ where $\tau^{(k)}_j := \tau_{(j-1)n+k}$. So, we have that

$$\left\|\sum_{k=1}^{mn}(\tau_k - 1)V_k\right\|_{1\to 0} \leq C_*\sum_{k=1}^{n}\left|\sum_{j=1}^{m}\tau^{(k)}_j - 1\right|$$

and together with Lemma A.12 and Equation (A.3),

$$I_1(h) \le C_* \sum_{k=1}^n \left| \sum_{j=1}^m \tau_j^{(k)} - 1 \right| \int_0^h \prod_{k=1}^n e^{C_* r \sum_{j=1}^m \tau_j^{(k)}} dr\,.$$

Therefore

$$\begin{aligned}\mathbb{P}\left(I_1(h) > \frac{\varepsilon}{2m}\right) &\le \mathbb{P}\left(C_* \sum_{k=1}^n \left| \sum_{j=1}^m \tau_j^{(k)} - 1 \right| \int_0^h \prod_{k=1}^n e^{C_* r \sum_{j=1}^m \tau_j^{(k)}} dr > \frac{\varepsilon}{2m} \right) \\ &\le \sum_{k=1}^n \mathbb{P}\left(\left| \sum_{j=1}^m \tau_j^{(k)} - 1 \right| \int_0^h \prod_{k=1}^n e^{C_* r \sum_{j=1}^m \tau_j^{(k)}} dr > \frac{\varepsilon}{2C_* mn} \right).\end{aligned}$$

The second inequality follows from a union bound together with the fact that for any nonnegative random variables X_k and constant c, $\{\sum_{k=1}^n X_k > c\} \subseteq \cup_{k=1}^n \{X_k > c/n\}$. Set

$$A(h) := \bigcap_{k=1}^n \left\{ h \sum_{j=1}^m \tau_j^{(k)} \le \alpha \right\} \quad \text{and} \quad B_k(h) := \left\{ \left| \sum_{j=1}^m \tau_j^{(k)} - 1 \right| \int_0^h \prod_{k=1}^n e^{C_* r \sum_{j=1}^m \tau_j^{(k)}} dr > \frac{\varepsilon}{2C_* mn} \right\}$$

for arbitrary $\alpha > 0$ and note that

$$A(h) \cap B_k(h) \subseteq \left\{ \left| \sum_{j=1}^m \tau_j^{(k)} - 1 \right| h e^{C_* n\alpha} > \frac{\varepsilon}{2C_* mn} \right\} =: B(h).$$

Therefore

$$\mathbb{P}\left(I_1(h) > \frac{\varepsilon}{m}\right) \le \sum_{k=1}^n \mathbb{P}\left(B_k(h) \cap A(h)\right) + \mathbb{P}\left(B_k(h) \cap A(h)^c\right) \le n\big[\mathbb{P}\left(B(h)\right) + \mathbb{P}\left(A(h)^c\right)\big].$$

Set $h = t/m^2$. By Lemma A.15 for all $\varepsilon > 0$ such that $K := \varepsilon(2C_* tn)^{-1} e^{-C_* n\alpha} < 1$,

$$\mathbb{P}\left(B(h)\right) = \mathbb{P}\left(\left| \sum_{j=1}^m \tau_j^{(k)} - 1 \right| > \frac{\varepsilon m}{2C_* tn e^{C_* n\alpha}} \right) \le 2e^{-\frac{1}{2}K^2 m}.$$

And by Lemma A.14,

$$\mathbb{P}\left(A(h)^c\right) = \mathbb{P}\left(\bigcup_{k=1}^n \left\{ \sum_{j=1}^m \tau_j^{(k)} > \frac{\alpha}{h} \right\} \right) \le n\mathbb{P}\left(\sum_{j=1}^m \tau_j > \frac{\alpha m^2}{t} \right) \le n2^m e^{-\frac{1}{2}K' m^2}$$

where $K' := \alpha/t$. Therefore

$$\mathbb{P}\left(I_1(h) > \frac{\varepsilon}{2m}\right) \le 2e^{-\frac{1}{2}K^2 m} + 2^m n e^{-\frac{1}{2}K' m^2} \le 2^m C e^{-\frac{1}{2}Cm^2} \tag{A.16}$$

for some positive constant C independent of m.

Second probability. Recall $E_{r\tau} := \sum_{k=1}^{mn} \tau_k [\widetilde{S}_{r\tau}^{(1,k-1)}, V_k] \widetilde{S}_{r\tau}^{(k,mn)}$. Also, from Equation (A.6),

$$\begin{aligned}[\widetilde{S}_{r\tau}^{(1,k-1)}, V_k] \widetilde{S}_{r\tau}^{(k,mn)} &= \sum_{j=1}^{k-1} \int_0^r \tau_j e^{(r-s)\sum_{j=1}^{k-1} \tau_j V_j} [V_j, V_k] \widetilde{S}_{s\tau}^{(1,k-1)} \widetilde{S}_{r\tau}^{(k,mn)} ds \\ &\quad + \sum_{j=1}^{k-1} \int_0^r \tau_j e^{(r-s)\sum_{j=1}^{k-1} \tau_j V_j} \big[[\widetilde{S}_{s\tau}^{(1,j-1)}, V_j] \widetilde{S}_{s\tau}^{(j,k-1)}, V_k \big] \widetilde{S}_{r\tau}^{(k,mn)} ds.\end{aligned}$$

Lemma A.12 together with $\|[V_j, V_k]\|_{2\to 0} \le \|V_j\|_{1\to 0}\|V_k\|_{2\to 1} + \|V_k\|_{1\to 0}\|V_j\|_{2\to 1} \le 2C_*^2$ give

$$\left\|[V_j, V_k]\widetilde{S}_{s\tau}^{(1,k-1)}\widetilde{S}_{r\tau}^{(k,mn)}\right\|_{2\to 0} \le 2C_*^2\left(1 + C_* r\sum_{j=1}^{mn}\tau_j\right)e^{3C_* r\sum_1^{mn}\tau_j}.$$

Also,

$$\begin{aligned}\left[[\widetilde{S}_{s\tau}^{(1,j-1)}, V_j]\widetilde{S}_{s\tau}^{(j,k-1)}, V_k\right] = \widetilde{S}_{s\tau}^{(1,j-1)}V_j\widetilde{S}_{s\tau}^{(j,k-1)}V_k - V_k\widetilde{S}_{s\tau}^{(1,j-1)}V_j\widetilde{S}_{s\tau}^{(j,k-1)}\\ - V_j\widetilde{S}_{s\tau}^{(1,k-1)}V_k + V_kV_j\widetilde{S}_{s\tau}^{(1,k-1)}\end{aligned}$$

together with Lemma A.12 gives

$$\left\|\left[[\widetilde{S}_{s\tau}^{(1,j-1)}, V_j]\widetilde{S}_{s\tau}^{(j,k-1)}, V_k\right]\widetilde{S}_{r\tau}^{(k,mn)}\right\|_{2\to 0} \le 4C_*^2\left(1 + C_* r\sum_{j=1}^{mn}\tau_j\right)e^{3C_* r\sum_1^{mn}\tau_j}.$$

Therefore for any $0 \le r \le h$,

$$\begin{aligned}\|E_{r\tau}\|_{2\to 0} &\le \sum_{k=1}^{mn}\sum_{j=1}^{k-1}\tau_k\tau_j\int_0^r\left\|[V_j, V_k]\widetilde{S}_{s\tau}^{(1,k-1)}\widetilde{S}_{r\tau}^{(k,mn)}\right\|_{2\to 0}\\ &\qquad + \left\|\left[[\widetilde{S}_{s\tau}^{(1,j-1)}, V_j]\widetilde{S}_{s\tau}^{(j,k-1)}, V_k\right]\widetilde{S}_{r\tau}^{(k,mn)}\right\|_{2\to 0} ds\\ &\le 6C_*^2 r\left(1 + C_* r\sum_{\ell=1}^{mn}\tau_\ell\right)e^{3C_* r\sum_1^{mn}\tau_\ell}\sum_{k=1}^{mn}\sum_{j=1}^{k-1}\tau_k\tau_j\\ &\le Ch\left(1 + Ch\sum_{\ell=1}^{mn}\tau_\ell\right)e^{Ch\sum_1^{mn}\tau_\ell}\left(\sum_{k=1}^{mn}\tau_k\right)^2\end{aligned}$$

for some $C > 0$. So, we have that

$$I_2(h) = \int_0^h\|E_{r\tau}\|_{2\to 0}dr \le Ch^2\left(1 + Ch\sum_{\ell=1}^{mn}\tau_\ell\right)e^{Ch\sum_1^{mn}\tau_\ell}\left(\sum_{k=1}^{mn}\tau_k\right)^2.$$

For arbitrary $\alpha > 0$, set

$$A(h) := \left\{h\sum_{k=1}^{mn}\tau_k \le \alpha\right\} \quad \text{and} \quad B(h) := \left\{Ch^2\left(1 + Ch\sum_{\ell=1}^{mn}\tau_\ell\right)e^{Ch\sum_1^{mn}\tau_\ell}\left(\sum_{k=1}^{mn}\tau_k\right)^2 > \frac{\varepsilon}{2m}\right\}.$$

Then taking $h = t/m^2$ as before,

$$\begin{aligned}\mathbb{P}\left(I_2(h) > \frac{\varepsilon}{2m}\right) &= \mathbb{P}\left(A(h)\cap B(h)\right) + \mathbb{P}\left(A(h)^c\cap B(h)\right)\\ &\le \mathbb{P}\left(Ch^2\left(1 + C\alpha\right)e^{C\alpha}\left(\sum_{k=1}^{mn}\tau_k\right)^2 > \frac{\varepsilon}{2m}\right) + \mathbb{P}\left(h\sum_{k=1}^{mn}\tau_k > \alpha\right)\\ &= \mathbb{P}\left(\sum_{k=1}^{mn}\tau_k > Km^{\frac{3}{2}}\right) + \mathbb{P}\left(\sum_{k=1}^{mn}\tau_k > \frac{\alpha m^2}{t}\right) \qquad \text{(A.17)}\\ &\le n\left[\mathbb{P}\left(\sum_{k=1}^{m}\tau_k > K'm^{\frac{3}{2}}\right) + \mathbb{P}\left(\sum_{k=1}^{m}\tau_k > \frac{\alpha m^2}{nt}\right)\right]\\ &\le n\left(2^m e^{-\frac{1}{2}K'm^{3/2}} + 2^m e^{-\frac{1}{2}K''m^2}\right) \le 2^m C' e^{-\frac{1}{2}C'm^{3/2}}\end{aligned}$$

for some $C' > 0$ where $K = (\varepsilon(2t^2C(1+C\alpha)e^{C\alpha})^{-1})^{1/2}$, $K' = Kn^{-1}$, $K'' = \alpha(nt)^{-1}$, and the second-to-last last inequality follows from Lemma A.14. Combining (A.15), (A.16), and (A.17) and taking $h = t/m^2$ we therefore have that for all ε sufficiently small,

$$\mathbb{P}\left(\|\widetilde{S}^m_{t\tau/m^2} - S_{t/m}\|_{2\to 0} > \tfrac{\varepsilon}{m}\right) \leq 2^m C'' e^{-\frac{1}{2}C'' m^{3/2}}$$

for some constant $C'' > 0$ independent of m. So, we have that

$$\sum_{m=1}^{\infty} \mathbb{P}\left(\|\widetilde{S}^m_{t\tau/m^2} - S_{t/m}\|_{2\to 0} > \tfrac{\varepsilon}{m}\right) \leq \sum_{m=1}^{\infty} 2^m C'' e^{-\frac{1}{2}C'' m^{3/2}} < \infty. \qquad \square$$