On Large Values of Non-Gaussian Logarithmically Correlated Fields

by

Haotian Gu

Department of Mathematics
Duke University

Date: March 25, 2025

Approved:

Nicholas A. Cook

Alexander J. Dunlap

Gregory J. Herschlag

James H. Nolen

Dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mathematics in the Graduate School of Duke University

2025

ABSTRACT

On Large Values of Non-Gaussian Logarithmically Correlated Fields

by

Haotian Gu

Department of Mathematics
Duke University

Date: March 25, 2025

Approved:

Nicholas A. Cook

Alexander J. Dunlap

Gregory J. Herschlag

James H. Nolen

Dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mathematics in the Graduate School of Duke University

2025

Abstract

Random field models with logarithmic correlation structure appear widely across probability theory and there has been great progress in understanding the large value statistics of many Gaussian (or nearly Gaussian) log-correlated fields in the past decades. Studying the universality and non-universality for non-Gaussian log-correlated field models is a natural follow-up question, and recent studies on the characteristic polynomials of the sparse random matrices raised people's attention to Poissonian log-correlated field models. In this dissertation, we establish results for two classes of non-Gaussian log-correlated field models.

(i) Poissonian logarithmically correlated fields arise naturally in the study of the characteristic polynomials of the sparse random matrices, including random permutation matrices, their finite independent sums, and the sparse IID Bernoulli random matrices. We introduce a prototypical random function series model with log-correlation and Poissonian tail, make a connection with a branching random walk model with random time-varying branching law, and establish an asymptotic upper bound to the third order on the maximum of this random function series on the unit circle.

(ii) Gaussian holomorphic multiplicative chaos (HMC) is a random distribution arises from the study of the characteristic polynomials of circular ensembles and the partial sums of the random multiplicative functions. In particular, the fractional moments of the Fourier coefficients of Gaussian HMC and the aforementioned related models are shown to have a "better-than-square-root" cancellation. We study the fractional moments of the Fourier coefficients of the non-Gaussian version of HMC and obtain the full universality result for the "better-than-square-root" cancellation phenomenon in our setting, along with a double-layer phase transition.

Contents

List of Figures

Acknowledgements

First, I wanted to thank Prof. Cook and Prof. Durrett for using their NSF grants to support me during my graduate studies.

With respect to this thesis, I am most indebted to my advisor Prof. Nicholas Cook, for all sorts of support he has given me. Academically, he is the person who guided me, a fresh Ph.D. student with almost no research experience, to the current stage. He introduced me to lots of interesting and important ideas in various areas of mathematics, including high-dimensional probability, large deviation theory, graphon theory, connections between random matrix theory and number theory, and many other topics in random matrices. Every week in our meetings I could learn something new from him. Mentally, he is very patient and helpful and always helps me avoid panic. There have been many times when I walked into his office with bad news on my research and a pessimistic mindset, but he always stayed calm and figured out some possible new directions for me to explore. Having a role model like him, I learned to be a resilient and determined researcher. In life, he demonstrated the art of work-life balance to me through his own actions. He is certainly the kind of person I would like to become.

I would like to express my gratitude to Prof. James Nolen for being my first-year advisor especially during the difficult time of pandemic (and for showing me around on my first day at Duke), to Prof. Alexander Dunlap, Prof. Richard Durrett, Prof. Gregory Herschlag, and Prof. James Nolen for serving on my thesis committee, and to Prof. Richard Durrett, Prof. James Nolen, and Prof. Jonathan Mattingly for serving on my preliminary committee. They made valuable feedback and insights on my work.

I also want to give my warm thanks to all the organizers of Duke math events,

including the probability seminars, the high-dimensional probability seminars, the combinatorics seminars, and numerous teas for graduate students and the faculty. These events are among my most cherished memories at Duke, and I truly enjoy talking and listening to other math researchers. In particular, the probability seminar dinner every Thursday brought me closer to many researchers and also introduced me to many good restaurants in Durham.

I want to express my deep gratitude to Prof. Yuan Zhang, who mentored me in the academic year 2020-2021 at Peking University, and the Beijing International Center of Mathematical Research for hosting me during that time. During the pandemic, it was their sincere and warm support that helped me grow academically despite the difficult circumstances. I am also grateful to Prof. Jian Ding and Prof. Xinyi Li at Peking University for hosting me during the summer of 2023, who also kindly gave me the chance to participate in the national probability conference in Fuzhou.

Among my friends, I want to thank my collaborator Zhenyuan Zhang for many hours of enjoyable discussions. It feels almost miraculous for me, as a young graduate student, to have found both a collaborator and a close friend in my first in-person summer school. I would also like to thank Dr. Max Wenqiang Xu for giving me an in-depth introduction to the beautiful connection between the random multiplicative functions and the multiplicative chaos theory. He has also given me a lot of advice over the past years.

I am very grateful to my friends and fellow graduate students at Duke. Especially, I would like to say thank you to Dr. Ziang Chen and Yupei Huang for countless drives around Durham, and Victor Amaya, Dr. Chongyao Chen, Rena Chu, Dr. Haohua Deng, Victor Gardner, Kevin Zhongtian Hu, Jiayi Wang, and Bobby Zhang for their company in numerous holiday gatherings, many late-night board games, and a handful of football games. Also, I would like to thank Dr. Zhenhao Cai and Shuyang

Gong for many meaningful discussions and enjoyable company during their visits.

I want to save a whole paragraph for saying thank you to my roommates Kai Xu and Zijun Li. Living abroad is certainly a big challenge, but the energy of Kai and Zijun has made it truly enjoyable. It is a miracle that three of us, who barely knew each other despite all being from Peking University, have gotten along so well under one roof. Although we are working in different areas of mathematics, our shared passion about cooking and daily conversations during dinners made me feel less alone. Duke alumni Prof. Yuan Zhang and Prof. Ruibo Ma told me the first thing I needed to do upon arriving was not to find an advisor, but to get a car. Over the past four years, the three of us managed to live without a single drivers' license, proving that you can indeed survive in Durham without driving if you have two close friends going everywhere together with you (and of course, with the help of Instacart).

Finally, I would like to thank my family for all their support throughout my journey. There were times when I doubted whether I could really do math research, and I even thought about giving up–especially in my first year of doing research, when I struggled for months without making much progress. During those difficult times, my family seemed to believe in me more than I believed in myself. Their love and encouragement have given me the strength to keep going. In particular, I would like to dedicate my thesis to my parents, Yongjun Tian and Chengyin Gu, for their unwavering care and support in the past 27 years. My mother's resilient spirit and broad-minded approach to life, along with my father's contentment and calmness, have profoundly shaped who I am today.

Chapter 1

Introduction

For a random field $\{X_u : u \in \mathcal{M}\}$ on a metric space $(\mathcal{M}, \mathrm{d})$, we say it is a *log-correlated field* if there is a bounded continuous function $g : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ such that

$$\mathrm{Cov}(X_u, X_v) = -\log \mathrm{d}(u, v) + g(u, v), \quad \forall u, v \in \mathcal{M}.$$

The terminology of log-correlated field may also be used if the above identity holds asymptotically as $\mathrm{d}(u, v) \to 0$, or the size of level set

$$S_u(c) := \{v \in \mathcal{M} : \mathrm{Cov}(X_u, X_v) \geqslant c\}$$

decays roughly exponentially as c increases. Random field models with logarithmic correlation structure appear widely across probability theory, including branching random walk/Brownian motion (BRW/BBM), two-dimensional Gaussian free field/Ginzburg-Landau field, random walk cover times/local times, characteristic polynomials of many random matrix ensembles, and the Riemann ζ function on the critical line, etc., and there has been great progress in understanding the extreme value statistics of many Gaussian (or nearly Gaussian) log-correlated fields in the past decades (see [BK22] for a survey).

A natural question is whether these Gaussian results are universal for other log-correlated fields with different, non-Gaussian tails, and what happens outside the universality regime. Below, I outline my work on understanding the extreme value properties of two non-Gaussian log-correlated field models.

Poissonian log-correlated fields. In recent years, the extreme value statistics of log-correlated fields received intense interests, but the sharpest results are for Gaus-

sian or almost-Gaussian fields, including the recent break-throughs on the Fyodorov-Hiary-Keating conjecture [ABR20, ABR23, PZ22] and universality results on the random matrix side [BLZ23, CL24].

On the other hand, characteristic polynomials of sparse random matrices give rise to log-correlated fields with Poissonian tails. Earlier on permutation matrices, Cook and Zeitouni [CZ20] obtained the leading order of the maximum. In Chapter 2, we obtain new refined results on the maximum for a related class of random function series with Poissonian tails. We find the sub-leading order behavior is significantly different from the ubiquitous "Bramson's correction" term for Gaussian log-correlated fields, and can be modeled by a branching random walk with a randomly time-varying offspring distribution (see [BK04, HLL14, MM19, Kri22] for works on this BRW model). This is the first known non-universality result for log-correlated fields arising from the characteristic polynomial of random matrix models. Our proof is inspired by the analogue between our model and the branching random walk model mentioned above, and adapted the approach in [MM19] to our setting.

Chapter 2 is based on an upcoming joint work with Nicholas A. Cook.

Non-Gaussian holomorphic multiplicative chaos. Holomorphic multiplicative chaos is a class of random distributions closely linked to various topics, including random matrix theory, probabilistic number theory, log-correlated fields, and Gaussian multiplicative chaos. Its Fourier coefficients are analogous to the secular coefficients of Circular Beta Ensembles (CβE) [NPS23] and partial sums of Steinhaus random multiplicative functions [SZ22], both of which have attracted intensive attention recently. Among many advances, Harper's resolution of Helson's "better-than-square-root" cancellation conjecture [Har20] stands out both on its own and for its broad applications. The same phenomenon is shown for Gaussian HMC and CβE [NPS23].

In Chapter 3, we established the full universality regime for this phenomenon in

the setting of non-Gaussian holomorphic multiplicative chaos. Moreover, we unveiled for the first time a double-layer phase transition for the fractional moments of Fourier coefficients (or the so-called *secular coefficients*) of heavy-tailed holomorphic multiplicative chaos, implying a new "super-exponential-cancellation" phenomenon. Our proof is based on Soundararajan and Zaman's (which goes back to Harper [Har20]) idea on connecting the Fourier coefficients of the critical holomorphic chaos with the total mass of the critical Gaussian multiplicative chaos [SZ22], and a careful analysis of the dominating mechanism of the fractional moments of the Fourier coefficients under each situation.

Chapter 3 is based on the joint work with Zhenyuan Zhang [GZ24].

Chapter 2

Maximum of Poissonian Log-Correlated Fields

2.1 Introduction

Background. The maximum of various Gaussian (or analogous to Gaussian) logarithmically correlated fields, including the branching Brownian motion [Bra78, Bra83], branching random walks [Bac00, Aï13, BDZ16a], two-dimensional discrete Gaussian free fields [BDZ16b], characteristic polynomials of random matrices [ABB17, PZ18, CMN18, PZ22, BLZ23, CL24], ζ functions [ABB$^+$19, Naj18, ABR20, ABR23], etc., has been extensively studied in the past decades, and exhibit a universal behavior. In many models above the convergence in law of the maximum is established. For example, Fyodorov, Hiary, and Keating (FHK) [FHK12, FK14] made a surprisingly precise conjecture on the maximum of the log-characteristic polynomials $\log \chi_N(z) = \log|\det(I_N - zU_N)|$ of the circular unitary ensemble (CUE), that is

$$\max_{|z|=1} \log \chi_N(z) = \log N - \frac{3}{4}\log\log N + M_N, \tag{2.1}$$

where M_N is a sequence of random variables converging in law to some explicit distribution. This conjecture is verified up to the law of the limiting distribution in a recent work of Paquette and Zeitouni [PZ22][1]. See also [DRZ17] for the convergence in law of a general class of log-correlated Gaussian fields, and [BK22] for a recent survey.

[1] Through private communications, authors of [PZ22] claimed to bridge this gap and fully confirmed the FHK conjecture for circular ensembles.

Recently, the studies on the characteristic polynomials of *sparse* random matrices bring the attention to some non-Gaussian random fields, and in particular, random fields with Poissonian tails. For example, in [CZ20], Cook and Zeitouni linked the log-characteristic polynomial $\log \widetilde{\chi}_N(z) = \log|\det(I_N - zP_N)|$ of an $N \times N$ random permutation matrix P_N to the Poissonian log-correlated fields

$$\sum_{\ell=1}^{N} Z_\ell \log|1 - z^\ell| \tag{2.2}$$

where Z_ℓ is a sequence of independent Poisson$(1/\ell)$ variables, and established the analogue of (2.1) to the leading order

$$\lim_{N\to\infty} \frac{\max_{|z|=1} \log \widetilde{\chi}_N(z)}{\log N} = x_0 \qquad \text{in probability,} \tag{2.3}$$

where x_0 is some explicit numerical constant. We briefly comment that the effect of the sparsity of $Z_\ell - s$ is not negligible even in the study of the leading order of the maximum. Later on, [CLZ22] established the convergence of (rescaled) characteristic polynomials of finite sum of independent random permutation matrices towards an random analytic function with Poissonian randomness related to (2.2) inside the unit disk, and [Cos23] showed the convergence in law of characteristic polynomials of sparse IID Bernoulli random matrices to the Poissonian analogue of Gaussian holomorphic chaos (introduced in [NPS23] as the distributional limit of characteristic polynomials of circular ensembles).

In this chapter, we introduce a new type of log-correlated fields, called *Poissonian log-correlated fields*, as a prototypical model of characteristic polynomials of sparse random matrices mentioned above. It is particularly inspired by (2.2) (we will discuss the connection later). We establish an upper bound of its maximum to the third order, showing a significantly different behavior compared to the Gaussian case (2.1).

Set-up of Poissonian log-correlated fields. Let $0 < \xi_1 < \xi_2 < \cdots$ be a rate-1 homogeneous Poisson arrivals process and $\Xi = \{\xi_i : i \in \mathbb{N}\}$ be their collection. Let $\{U_i : i \in \mathbb{N}\}$ be a sequence of i.i.d. random variables distributed uniformly on the torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. Fix a function $f \in C^2(\mathbb{T})$ so that (w.l.o.g.) $\|f\|_\infty \leqslant 1$. For a uniformly distributed random variable U on $\mathbb{T}$, we further assume $\mathbb{E}[f(U)] = 0$ and the cumulative distribution function (CDF) of $f(U)$ being Hölder continuous. One prototypical example is $f(x) = \cos(2\pi x)$, where the CDF of $f(U)$ is $\frac{1}{2}$-Hölder continuous.

We consider a random field X_n on $\mathbb{T}$ given by the series

$$X_n(t) := \sum_{j \leqslant n} f(e^{\xi_j} t + U_j), \qquad t \in \mathbb{T}\,. \tag{2.4}$$

In the sequel we may write $\sum_{\xi \in A}$ for $\sum_{j:\xi_j \in A}$ and U_ξ for U_j when $\xi = \xi_j$.

The field is centered by our assumption $\mathbb{E}f(U) = 0$, with correlation structure

$$\mathrm{Cov}(X_n(s), X_n(t)) = \begin{cases} 2\sum_{\zeta \geqslant 1} |\widehat{f}(\zeta)|^2 \log \frac{1}{\zeta|t-s|} + O(1), \text{ if } \log \frac{1}{|t-s|} = o(n), \\ n\|f\|_2^2 + O(1), \text{ if } \log \frac{1}{|t-s|} = \omega(n), \end{cases} \tag{2.5}$$

where $\widehat{f}(\zeta) := \int_{\mathbb{T}} f(x) e^{-2\pi \mathrm{i} x\zeta} \mathrm{d}x$; see Proposition 2.2.4 for the proof. Thus, for the case $f(x) = \cos(2\pi x)$ the field is exactly log-correlated down to the scale e^{-n}, i.e. $\mathrm{Cov}(X_n(s), X_n(t)) = \log(\frac{1}{|s-t|}) + O(1)$ for $-\log|t-s| = o(n)$, while for general f we have a combination of periodic log-correlation structures depending on the Fourier decomposition of f. This is to be expected – for instance, when $f(x) = \cos(2\pi\zeta x)$ for fixed $\zeta \geqslant 2$ then the field is ζ^{-1}-periodic, with a log-correlation structure on $[0, \zeta^{-1})$, and the maximum over $[0, \zeta^{-1})$ will behave the same up to a small error as the maximum for the case $\zeta = 1$ taken over the full torus $\mathbb{T}$.

We now set the stage for our main theorem. For U uniformly distributed in $\mathbb{T}$ let λ be the log-Laplace transform of $f(U)$, and λ^* be its Legendre transform, that is

$$\lambda(\theta) = \log \mathbb{E}\left[e^{\theta f(U)}\right] \quad \text{and} \quad \lambda^*(x) = \sup_{\theta \geqslant 0}\{x\theta - \lambda(\theta)\}. \tag{2.6}$$

Since λ^* is strictly increasing and onto $\mathbb{R}_+$, there are unique $x_0, \theta_* > 0$ such that

$$\lambda^*(x_0) = x_0\theta_* - \lambda(\theta_*) = 1, \qquad \lambda'(\theta_*) = x_0. \tag{2.7}$$

It is shown in [MM18, Theorem 1.1] that there is a symmetric, convex function $\gamma : \mathbb{R} \to \mathbb{R}^+$ with $\gamma(\beta) > \gamma(0) = 1/2$ for all $\beta \neq 0$ such that for independent standard Brownian motions B_t, W_t, we have

$$\lim_{t\to\infty} \frac{1}{\log t} \log \mathbb{P}(B_s + 1 \geqslant \beta W_s,\ \forall\, s \in [0,t] \mid W) = -\gamma(\beta) \tag{2.8}$$

almost surely. Define

$$\phi_* := 2\gamma\left(\frac{1}{\theta_*\sqrt{\lambda''(\theta_*)}}\right) + \frac{1}{2}. \tag{2.9}$$

The main theorem of this chapter is then as follows.

Theorem 2.1.1 (Main Result)**.** *We have for any $\eta > 0$,*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{1}{\log n}\left(\theta_* \max_{t\in\mathbb{R}/\mathbb{Z}} X_n(t) - \xi_n - \lambda(\theta_*)n\right) \leqslant -\phi_* + \eta\right) = 1. \tag{2.10}$$

We make some comments on the main theorem, especially the quantity $m_n := (\xi_n + \lambda(\theta_*)n - \phi_* \log n)/\theta_*$. Note that ξ_n is the sum of n i.i.d. Exp(1) random variables, thus the leading order of m_n is $(\frac{1+\lambda(\theta_*)}{\theta_*})n = x_0 n$, which is reminiscent of (2.3). However, due to the fluctuation of $(\xi_n - n)$, there is a $O_{\mathbb{P}}(\sqrt{n})$ random second order term in m_n, significantly different from the Gaussian case (see eg. (2.1)). In particular, we have $(m_n - x_0 n)/\sqrt{n}$ converges in law to the centered real Gaussian distribution with certain variance. When the randomness of ξ_n is quenched, the third term has the same $(\log n)$ order as for the Gaussian log-correlated fields (see eg. (2.1)), but the occurence of γ function in the multiplicative constant makes it also significantly different.

Connection to random permutation matrices. Let P_N be a random permutation matrix drawn uniformly at random, corresponding to the random permutation $\pi_N \in S_N$. Since we are interested in the modulus of its characteristic polynomial $|\det(zI_N - P_N)|$ on the unit circle $|z| = 1$, it is more convenient to work with $\widetilde{\chi}_N(z) = |\det(I_N - zP_N)| = |z|^N |\det(\overline{z}I_N - P_N)| = |\det(\overline{z}I_N - P_N)|$.

Let $\{C_\ell(P_N) : 1 \leqslant \ell \leqslant N\}$ be the cycle structure of P_N, i.e. $C_\ell(P_N)$ is the number of ℓ-cycles in the permutation π_N. Then we have (write $z = \exp(2\pi i t) =: e(t), t \in \mathbb{T}$)

$$\log \widetilde{\chi}_N(z) = \log \prod_{\ell=1}^{N} |1 - z^\ell|^{C_\ell(P_N)} = \sum_{\ell=1}^{N} C_\ell(P_N) \log |1 - z^\ell| = \sum_{\ell=1}^{N} C_\ell(P_N) \log |1 - e(\ell t)|.$$

Note that the joint distribution $(C_\ell(P_N))_{\ell \leqslant M}$ is close to that of independent Poisson variables $(Z_\ell)_{\ell \leqslant M}$ with $\mathbb{E}[Z_\ell] = 1/\ell$, if $M = o(N)$ [AT92, Theorem 2]. With the help of this result, the study of $\log \widetilde{\chi}_N(z)$ (with appropriate cut-offs) is essentially the study of the Poissonian field (2.2).

However, the correlation structure of such a field is very complicated due to the $\log |1 - e(\ell t)|$ term. Intense number theoretic considerations are needed. While this is indeed an important feature of the model, in this chapter, we would like to emphasize the role of the Poissonian randomness Z_ℓ-s. To do so, we first consider the *modified random permutation matrices* $\widetilde{P}_N$, which are P_N with entries weighted by i.i.d. random variables uniformly distributed on the unit circle. Then the above analysis leads to the study of the Poissonian random field

$$\sum_{\ell=1}^{N} Z_\ell \log |1 - e(\ell t + U_\ell)|, \tag{2.11}$$

where $\{U_\ell\}$ is a sequence of i.i.d. variables uniformly distributed on the unit circle. This kills some (but not all) of the number theoretic properties of the random field. Moreover, the new random field (2.11) is now stationary under translation of t.

Finally, we pass to the continuous setting (2.4) from (2.11) to fully eliminate the number theory involved in the analysis. Recall $\{\xi_j\}$ are rate-1 homogeneous Poisson arrivals. If we round-up the frequencies $\{e^{\xi_j}\}$ in (2.4), then for the frequency ℓ we have

$$\sum_{\xi\in[\log\ell,\log(\ell+1))} f(\lfloor e^{\xi}\rfloor t+U_\ell)=\widetilde{Z}_\ell f(\ell t+U_\ell),$$

where $\widetilde{Z}_\ell := |\Xi\cap[\log\ell,\log(\ell+1))|\sim \mathrm{Poisson}(\log(1+1/\ell))$, close to the distribution of $Z_\ell\sim\mathrm{Poisson}(1/\ell)$.

In summary, the new continuous model is inspired by and closely related to the characteristic polynomials of random permutation matrices, emphasizing the Poissonian nature of the latter rather than the number theoretic nature. Moreover, our assumptions on the function f also rules out the log-singularity of $\log\widetilde{\chi}_N(z)$, and we intend to modify our approach to accommodate it in the future.

Difference to Gaussian log-correlated fields and proof strategy. For simplicity we take the discrete field (2.11) as an example. Even without the number theoretic properties of the random field $\log\widetilde{\chi}_N(z)$, the behavior of the random field is drastically different from its Gaussian series analogue

$$\sum_{\ell=1}^{N}\frac{N_\ell}{\sqrt{\ell}}f(\ell t+U_\ell).$$

where $\{N_\ell:\ell\geqslant 1\}$ is a sequence of standard real normal variables.

The key difference is the sparsity of the "effective frequencies", i.e. non-zero terms in the sum. Since $\mathbb{E}[Z_\ell]=1/\ell$, we can only expect a non-zero frequency ℓ in an interval with exponentially growing length, while in the above Gaussian case almost surely every frequency is effective.

The sparsity of Poissonian log-correlated fields has several consequences, and we

illustrate them with an analogue to a specific branching random walk model. First of all, at frequency ℓ, to approximate the function $f(\ell t + U_\ell)$ to the $O(1)$ order, one needs a uniform mesh of size ℓ by the Lipschitz property of f. This discretization implies that, if ℓ is effective (i.e. $Z_\ell > 0$), then the "population size" here is ℓ (for the continuous model, we argue with the correlation structure (2.5) instead). Hence, the growth of the population depends on the position of effective frequencies, and due to the randomness of the $\{Z_\ell\}$ sequence (resp. the Poisson arrivals in the continuous setting) we have a time-varying random growth of the population. So if we further model this population growth by a branching process, and treat the non-zero summand $f(\ell t + U_\ell)$ (or $f(e^\xi t + U_\xi)$) as the random walk increment, this would be a time-varying branching random walk, or a *branching process in a time-inhomogeneous random environment* as defined in [BK04, HLL14, MM19, Kri22].

Our strategy is then to take advantage of the branching random walk with random time-varying branching law analogue and adapt the first moment method, with the estimate of the probability for one random walk to stay below (or above) another quenched random walk developed in [MM18, MM19].

Organization. We introduce a parametrization of the Poissonian field and the main theorem, together with some preliminary lemmas in Section 2.2. We prove a quenched ballot-type probability estimate, reminiscent of [MM19, Theorem 3.3], in Section 2.3. Finally, we prove the parametrized version of the upper bound theorem for the maximum of the Poissonian field in Section 2.4.

Notation. We use the notation $h \lesssim g$ or $h = O(g)$ if there is a constant $C > 0$ such that $|h| \leqslant Cg$. We allow the constant C to depend implicitly on the function f. If the constant depends on some other parameter p, we will use $h \lesssim_p g$ or $h = O_p(g)$ instead. We also write $h = o(g)$ or $g = \omega(h)$ if $\lim_{n\to\infty} h/g = 0$, and we only add

subscripts when other types of limit is in consideration, eg. $h = o_{\beta\to 0}(g)$. Asymptotic constants and implicit constants may vary from line to line.

We write $\mathbb{T}$ for the one-dimensional torus $\mathbb{R}/\mathbb{Z}$. We use $\sum_{j\leqslant n}$ for $\sum_{j=1}^{n}$. We denote the Fourier coefficients of $h : \mathbb{T} \to \mathbb{C}$ as $\widehat{h}(\zeta) = \int_{\mathbb{T}} e^{-2\pi \mathrm{i}\zeta x} h(x)\mathrm{d}x$.

2.2 Preliminaries

We introduce a variant of the upper bound of the maximum of the Poissonian field conditioned on the Poisson arrivals, and we use it to prove Theorem 2.1.1. In other subsections we collect some preliminary lemmas used later in the proof, which can be skipped on a first read.

2.2.1 Proof of Theorem 2.1.1 via quenching the Poisson arrivals

In this section we state a quenched upper bound result for the maximum of the Poissonian field and prove Theorem 2.1.1. Let $\mathbb{P}^{\Xi}(\cdot) := \mathbb{P}(\cdot \mid \Xi)$, and define

$$W_j := \xi_j + \lambda(\theta_*)j,\ j \geqslant 1 \tag{2.12}$$

with $W_0 := 0$.

Proposition 2.2.1. *For any fixed $\eta > 0$,*

$$\lim_{n\to\infty} \mathbb{P}\left(\mathbb{P}^{\Xi}\Big(\theta_* \max_{t\in\mathbb{T}} X_n(t) > W_n - (\phi_* - \eta)\log n\Big) \leqslant n^{-\eta/4}\right) = 1.$$

We prove Proposition 2.2.1 in Section 2.4.

Proof of Theorem 2.1.1. Fix some $\eta > 0$. Denote $\mathcal{E}_n^{\mathrm{upper}}(\eta)$ the event $\{\theta_* \max_{t\in\mathbb{T}} X_n(t) \geqslant \xi_n + \lambda(\theta_*)n - (\phi_* - \eta)\log n\}$. Let $\mathcal{G}$ be the event

$$\mathcal{G} := \left\{\mathbb{P}^{\Xi}\Big(\mathcal{E}_n^{\mathrm{upper}}\Big) \leqslant n^{-\eta/4}\right\}.$$

By Proposition 2.2.1, we know $\lim_{n\to\infty}\mathbb{P}(\mathcal{G})=1$. Then we have

$$\begin{aligned}\mathbb{P}\left(\mathcal{E}_n^{\text{upper}}(\eta)\right) &\leqslant \mathbb{E}\left[\mathbb{P}^{\Xi}\left(\mathcal{E}_n^{\text{upper}}(\eta)\right)1(\mathcal{G})\right]+\mathbb{P}(\mathcal{G}^c)\\ &\leqslant n^{-\eta/4}+\mathbb{P}(\mathcal{G}^c),\end{aligned}$$

and the theorem follows by taking the limit in n. □

2.2.2 Discretization of the parametrized field

In this subsection we give a discretization of the Poissonian field on a specific typical event stated below.

Lemma 2.2.2. *Let $\mathcal{G}$ be the following event: for any $1\leqslant i<j\leqslant n$, if $j-i>(\log n)^2$ then $\xi_j-\xi_i\geqslant c(j-i)$. Then there exists some absolute constant $c,c'>0$ such that $\mathbb{P}(\mathcal{G})\geqslant 1-e^{-c'(\log n)^2}$.*

Proof. Let $N:=3(n-1)/(\log n)^2$, and $\ell_k:=1+k(\log n)^2/3$, for all $0\leqslant k\leqslant N$. We claim that if there is some absolute constant $c>0$ such that $\xi_{\ell_{k+1}}-\xi_{\ell_k}\geqslant c(\log n)^2$ for any $k\leqslant N$ then $\mathcal{G}$ holds as well. Indeed, if $j-i>(\log n)^2$, then there are $k<k'$ such that $k'-k\geqslant 2$ and $\ell_k\leqslant i<\ell_{k+1}<\ell_{k'}\leqslant j<\ell_{k'+1}$. So $(j-1)\leqslant\frac{(\log n)^2}{3}(k'+1-k)$. Then we have

$$\xi_j-\xi_i\geqslant\xi_{\ell_{k'}}-\xi_{\ell_{k+1}}\geqslant c(\log n)^2(k'-k-1)\geqslant c\frac{(\log n)^2}{3}(k'+1-k)\geqslant c(j-i)$$

as desired.

So we have

$$\begin{aligned}1-\mathbb{P}(\mathcal{G}) &\leqslant \mathbb{P}\left(\exists 0\leqslant k\leqslant N:\xi_{\ell_{k+1}}-\xi_{\ell_k}<c(\log n)^2\right)\\ &\leqslant 3n\mathbb{P}\left(\sum_{j\leqslant(\log n)^2/3}E_j<c(\log n)^2\right),\end{aligned}$$

where $\{E_j : j \geqslant 1\}$ is a sequence of i.i.d. Exp(1) random variables. Take $c = 1/6$, we have for any $\varepsilon > 0$

$$\begin{aligned}\mathbb{P}\left(\sum_{j\leqslant(\log n)^2/3} E_j < c(\log n)^2\right) &= \mathbb{P}\left(\sum_{j\leqslant(\log n)^2/3} (1-E_j) > (\log n)^2/6\right)\\ &\leqslant e^{-\varepsilon(\log n)^2/6}\left(\mathbb{E}[e^{\varepsilon(1-E_1)}]\right)^{(\log n)^2/3}\\ &= \exp\left(-\frac{(\log n)^2}{3}(\log(1+\varepsilon)-\frac{\varepsilon}{2})\right).\end{aligned}$$

The claim follows by choosing $\varepsilon > 0$ to be a sufficiently small. □

We then state the discretization lemma.

Lemma 2.2.3. *On the event $\mathcal{G}$ as in Lemma 2.2.2, for any $j \leqslant n$ and any uniform mesh $\mathbb{T}_j \subset \mathbb{T}$ with $|\mathbb{T}_j| = (\log n)^2 e^{\xi_j}$, we have*

$$\max_{j\leqslant n}\max_{t\in\mathbb{T}}\min_{s\in\mathbb{T}_j}|X_j(t)-X_j(s)| = O(1) \qquad a.s.$$

Proof. For any $j \leqslant n$ and any $t \in \mathbb{T}$, by definition there is some $s \in \mathbb{T}_j$ such that $|t-s| \leqslant (\log n)^{-2}e^{-\xi_j}$. Recall $X_j(t) := \sum_{i\leqslant j} f(e^{\xi_i}t + U_i)$. Since f is Lipschitz, we have uniformly in t

$$|X_j(t)-X_j(s)| \lesssim (\log n)^{-2}e^{-\xi_j}\sum_{i\leqslant j} e^{\xi} \lesssim (\log n)^{-2}\sum_{i\leqslant j} e^{-(\xi_j-\xi_i)}.$$

For $j \leqslant 2(\log n)^2$, we have a trivial upper bound

$$\sum_{i\leqslant j} e^{-(\xi_j-\xi_i)} \leqslant \sum_{i\leqslant j} 1 \leqslant 2(\log n)^2.$$

For $j > 2(\log n)^2$, by the condition in Lemma 2.2.2, we have some absolute constant $c > 0$ such that

$$\sum_{i\leqslant j} e^{-(\xi_j-\xi_i)} = \sum_{i\leqslant j-(\log n)^2} e^{-(\xi_j-\xi_i)} + \sum_{j-(\log n)^2<i\leqslant j} e^{-(\xi_j-\xi_i)}$$

$$\leqslant \sum_{k=(\log n)^2}^{\infty} e^{-ck} + (\log n)^2$$

$$\leqslant 2(\log n)^2.$$

The desired result then follows. □

2.2.3 Logarithmically correlated structure of the Poissonian field

This section is irrelevant to the proof of the main theorem, but we include it for completeness. Let $\widehat{f}(\zeta) := \int_{\mathbb{T}} f(x)e^{-2\pi \mathrm{i}\zeta x}\mathrm{d}x$. The correlation structure of the Poissonian log-correlated field can then be written as a combination of exactly log-correlation according to the Fourier expansion of f.

Proposition 2.2.4 (Logarithmic correlation structure)**.** *For any $s, t \in \mathbb{T}$, we have*

$$\mathrm{Cov}(X_n(s), X_n(t)) = 2n \sum_{\zeta \geqslant 1} |\widehat{f}(\zeta)|^2 \mathbb{E}\left[\min\left\{1, \frac{1}{\xi_n}\log\frac{1}{\zeta|t-s|}\right\}\right] + O(1). \tag{2.13}$$

Based on (2.13)*, we have the following observation:*

1. *if $\log\frac{1}{|t-s|} = o(n)$, we have*

$$\mathrm{Cov}(X_n(s), X_n(t)) = 2\sum_{\zeta \geqslant 1} |\widehat{f}(\zeta)|^2 \log\frac{1}{\zeta|t-s|} + O(1); \tag{2.14}$$

2. *if $\log\frac{1}{|t-s|} = \omega(n)$, we have*

$$\mathrm{Cov}(X_n(s), X_n(t)) = n\|f\|_2^2 + O(1). \tag{2.15}$$

Remark 1. Note that for the case $f(v) = \cos(2\pi v)$ we obtain

$$\mathrm{Cov}(X_n(s), X_n(t)) = \begin{cases} -\frac{1}{2}\log|t-s| + O(1), \text{ if } -\log|t-s| = o(n), \\ n/2 + O(1), \text{ if } -\log|t-s| = \omega(n), \end{cases}$$

so that X_n is truly log-correlated down to the microscopic scale e^{-n} (below which X_n is strongly correlated). For general f the correlation structure is a mixture of periodic log-correlation structures depending on the Fourier decomposition of f. This is to be expected: note for instance that for $f(v) = \cos(2\pi kv)$ for some integer $k > 1$, the field is periodic with period $1/k$, and it is most natural to consider it on a sub-interval of the torus of length $1/k$.

Proof. Let $r := |t - s| \in \mathbb{T}$. The translation invariance of $\{U_k : k \in \mathbb{N}\}$ gives

$$\mathrm{Cov}(X_n(t), X_n(s)) = \sum_{j=1}^{n} \mathbb{E}\left[f(e^{\xi_j} t + U_j) f(e^{\xi_j} s + U_j)\right] \tag{2.16}$$

$$= \sum_{j=1}^{n} \mathbb{E}\left[f(e^{\xi_j} r + U_j) f(U_j)\right] \tag{2.17}$$

$$= \sum_{j=1}^{n} \mathbb{E}\left[g(e^{\xi_j} r)\right] \tag{2.18}$$

where $g(v) := \mathbb{E}[f(v + U)f(U)] = f * \tilde{f}(v)$ with $\tilde{f}(v) := f(-v)$. The function g has absolutely convergent Fourier expansion with coefficients $\widehat{g}(\zeta) = |\widehat{f}(\zeta)|^2 = |\widehat{f}(-\zeta)|^2$ for $\zeta \in \mathbb{Z}$, and since $\int_{\mathbb{T}} f(v)dv = 0$ we have $\widehat{g}(0) = \widehat{f}(0) = 0$. Thus,

$$\mathbb{E}\left[g(e^{\xi_j} r)\right] = 2 \sum_{\zeta \geqslant 1} |\widehat{f}(\zeta)|^2 \mathbb{E}\left[e(e^{\xi_j} \zeta r)\right], \tag{2.19}$$

where $e(x) = \exp(2\pi \mathrm{i} x)$. Since $\{\xi_j\}$ are Poisson arrivals with rate-1, when conditioned on ξ_n, the arrivals $\{\xi_1, \cdots, \xi_{n-1}\}$ have the same distribution as the ordered statistics of $(n-1)$ i.i.d. uniform random variables on $[0, \xi_n]$. Let L has the law of such uniform variable. Then we have for any $v > 0$:

$$\mathbb{E}\left[\sum_{j=1}^{n} e(e^{\xi_j} v) \mid \xi_n\right] = (n-1)\mathbb{E}\left[e(e^{L} v) \mid \xi_n\right] + O(1)$$

$$= \frac{n-1}{\xi_n} \int_0^{\xi_n} e(e^{x} v) \mathrm{d}x + O(1).$$

For any $0 < M \leqslant \xi_n$, we trivially obtain

$$\int_M^{\xi_n} e(e^x v)\mathrm{d}x = O(\xi_n - M).$$

By change of variable and integration by parts we have

$$\begin{aligned}\int_M^{\xi_n} e(e^x v)\mathrm{d}x &= \int_{e^M}^{e^{\xi_n}} \frac{1}{y} e(vy)\mathrm{d}y \\ &= \frac{1}{2\pi v i}\left(e^{-\xi_n} e(ve^{\xi_n}) - e^{-M} e(ve^M)\right) + \frac{1}{2\pi v i}\int_{e^M}^{e^{\xi_n}} \frac{1}{y^2} e(vy)\mathrm{d}y.\end{aligned}$$

From the last equality, we have

$$\int_M^{\xi_n} e(e^x v)\mathrm{d}x = O(e^{-M}/v).$$

As a consequence:

$$\int_M^{\xi_n} e(e^x v)\mathrm{d}x = O(\min\{\xi_n - M, e^{-M}/v\}).$$

Taking $M = \min\{\xi_n, \log(10/v)\}$, we obtain

$$\int_M^{\xi_n} e(e^x v)\mathrm{d}x = O(1),$$

and

$$\int_0^{M} e(e^x v)\mathrm{d}x = M + O(1),$$

so

$$\mathbb{E}\left[\sum_{j=1}^{n} e(e^{\xi_j} v) \mid \xi_n\right] = \frac{n-1}{\xi_n} M + O(\frac{n-1}{\xi_n}).$$

Since $\xi_n \sim \Gamma(n, 1)$, we have $\mathbb{E}[1/\xi_n] = 1/(n-1)$. Substituting this estimate with $v = \zeta r$ in (2.19) completes the proof.

Now we check the two observations. Let $\tau(\zeta) = \log \frac{1}{\zeta|t-s|}$. Splitting the expectation into $\{\xi_n > \tau(\zeta)\}$ and $\{\xi_n \leqslant \tau(\zeta)\}$ we have

$$\mathbb{E}\left[\min\left\{1, \frac{1}{\xi_n}\log\frac{1}{\zeta|t-s|}\right\}\right] = \mathbb{P}(\xi_n \leqslant \tau(\zeta)) + \tau(\zeta)\mathbb{E}\left[\frac{1}{\xi_n}1(\xi_n > \tau(\zeta))\right].$$

1) If $-\log|t-s| = o(n)$, $\tau(\zeta) \leqslant -\log|t-s| = o(n)$ for all $\zeta \geqslant 1$. Recall the density function of $\Gamma(n,1)$ is $\frac{x^{n-1}}{(n-1)!}e^{-x}1(x>0)$. Then we have

$$\begin{aligned}\mathbb{P}(\xi_n \leqslant \tau(\zeta)) &= \int_0^{\tau(\zeta)} \frac{x^{n-1}}{(n-1)!}e^{-x}\mathrm{d}x \\ &\leqslant \frac{\tau(\zeta)^{n-1}}{(n-1)!} \\ &= O\left(\left(\frac{e\tau(\zeta)}{n}\right)^{n-1}\right),\end{aligned}$$

and by Cauchy-Schwarz inequality

$$\begin{aligned}\tau(\zeta)\mathbb{E}\left[\frac{1}{\xi_n}1(\xi_n > \tau(\zeta))\right] &= \tau(\zeta)\mathbb{E}[1/\xi_n] - \tau(\zeta)\mathbb{E}\left[\frac{1}{\xi_n}1(\xi_n \leqslant \tau(\zeta))\right] \\ &= \frac{\tau(\zeta)}{n} + O\left(\tau(\zeta)\sqrt{\mathbb{E}[\xi_n^{-2}]}\sqrt{\mathbb{P}(\xi_n \leqslant \tau(\zeta))}\right) \\ &= \frac{\tau(\zeta)}{n} + O\left(\left(\frac{e\tau(\zeta)}{n}\right)^{n/2}\right).\end{aligned}$$

The statement then follows.

2) If $-\log|t-s| = nA_n$ with $\lim_{n\to\infty} A_n = \infty$, we split the sum over ζ into two parts: $\zeta \leqslant e^{nA_n/2}$ or not. For the second case, since $f \in C^2(\mathbb{T})$, we have $|\widehat{f}(\zeta)| \lesssim |\zeta|^{-2}$, and therefore

$$2n\sum_{\zeta \geqslant e^{nA_n/2}} |\widehat{f}(\zeta)|^2 \mathbb{E}\left[\min\left\{1, \frac{1}{\xi_n}\log\frac{1}{\zeta|t-s|}\right\}\right]$$

$$\lesssim n \sum_{\zeta \geqslant e^{nA_n/2}} \zeta^{-4} \lesssim e^{-nA_n}.$$

For the first case, for each $\zeta \leqslant e^{nA_n/2}$ we have $nA_n = \tau(1) \geqslant \tau(\zeta) = -\log|t-s| - \log\zeta \geqslant nA_n/2$. Therefore for any $0 < t < 1$, by Markov's inequality:

$$1 - \mathbb{P}(\xi_n \leqslant \tau(\zeta)) = \mathbb{P}(\xi_n > \tau(\zeta)) \leqslant \mathbb{P}(\xi_n > nA_n/2) \leqslant e^{-\frac{nA_n}{2}t - n\log(1-t)}.$$

Take $t = 1/2$ we obtain for n large enough

$$1 - \mathbb{P}(\xi_n \leqslant \tau(\zeta)) = \mathbb{P}(\xi_n > \tau(\zeta)) \leqslant e^{-nA_n/5}.$$

Then by Cauchy-Schwarz inequality again, we have

$$\tau(\zeta)\mathbb{E}\left[\frac{1}{\xi_n}1(\xi_n > \tau(\zeta))\right] \lesssim \frac{\tau(\zeta)}{n}e^{-nA_n/10} \lesssim A_n e^{-nA_n/10} \lesssim e^{-nA_n/11}.$$

Plugging these estimates into (2.13), by Plancherel's identity we obtain

$$\mathrm{Cov}(X_n(s), X_n(t)) = 2n \sum_{\zeta \leqslant e^{nA_n/2}} |\widehat{f}(\zeta)|^2 + O(1) = n||f||_2^2 + O(1).$$

□

2.3 Quenched ballot-type probability estimate

In this section we gather estimates of some ballot-type events under both quenched and annealed probability measures. Throughout this section we fix some $\varepsilon > 0$ sufficiently small. Recall λ, x_0, θ_* from (2.6), (2.7).

Definition 2.3.1. We define a tilted measure for the random field X_n as

$$\frac{\mathrm{d}\mathbb{P}_{x_0}}{\mathrm{d}\mathbb{P}} := \exp\left(\theta_* X_n - \lambda(\theta_*)n\right).$$

Under the tilted measure we have for any $t \in \mathbb{T}$ and $j \leqslant n$, $\mathbb{E}_{x_0}[X_j(t)] = x_0 j$ and $\mathrm{Var}_{x_0}(X_j(t)) = \lambda''(\theta_*)j$.

Recall $W_j = \xi_j + \lambda(\theta_*)j$. In Section 2.4, we will need an estimate of the upper bound of a quenched ballot-type probability reminiscent of [MM19, Theorem 3.3].

Proposition 2.3.2. *Let $\phi_* = 2\gamma((\lambda''(\theta_*)\theta_*^2)^{-1/2}) + \frac{1}{2}$. For any fixed $\eta > 0$, we have*

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{x,y\in[0,(\log n)^3]} \frac{\log \mathbb{P}^{\Xi}_{x_0}(\theta_* X_n(t) - W_n \geqslant x - y, \forall j \leqslant n : \theta_* X_j(t) \leqslant x + W_j)}{\log n} \right.$$

$$\left. \leqslant -\phi_* + \eta \right) = 1. \tag{2.20}$$

For the sake of completeness, we also quote a version of [MM19, Theorem 3.3, equation (3.4)] needed here, which is a special case of the first half of their result. Recall the definition of $\gamma(\cdot)$ from (2.8).

Proposition 2.3.3 (Theorem 3.3, [MM19])**.** *Let $S^{(1)}, S^{(2)}$ be two centered independent random walks with incremental variances σ_1^2, σ_2^2 resp., and $\phi = 2\gamma(\sigma_2/\sigma_1) + 1/2$. Assume there is some $\varepsilon > 0$ such that $\mathbb{E}[e^{\varepsilon|S_1^{(i)}|}] < \infty$, $i = 1, 2$. Set $(x_n) \in \mathbb{R}_+^{\mathbb{N}}$ such that $\lim_{n\to\infty} \frac{\log x_n}{\log n} = 0$. Then for any $\eta > 0$ we have*

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{x,y\in[0,x_n]} \mathbb{P}\left(S_n^{(1)} - S_n^{(2)} \geqslant x - y, S_j^{(1)} \leqslant x + S_j^{(2)}, \forall j \leqslant n \mid S^{(2)} \right) \leqslant n^{-\phi+\eta} \right) = 1.$$

Proof of Proposition 2.3.2. Let $S_j = \theta_* X_j(t) - \theta_* x_0 j$ and $\widetilde{W}_j = W_j - \theta_* x_0 j = \xi_j - j$ (since $x_0\theta_* - \lambda(\theta_*) = 1$). We omit the dependence of t in S_j for the sake of simplicity (and we can do so by the stationarity of the random field). S_j is then a random walk with i.i.d. increments with the same law as $S_1 = \theta_* X_1(t) - x_0\theta_*$ under $\mathbb{P}_{x_0}$, and $\widetilde{W}_j$ is a random walk with i.i.d. centered Exp(1) increments. The definition of $\mathbb{P}_{x_0}$ along with the fact that $\{\xi_j - \xi_{j-1} : j \geqslant 1\}$ is a sequence of i.i.d. Exp(1) variables give $\mathbb{E}_{x_0}[S_j] = \mathbb{E}[\widetilde{W}_j] = 0$, $\mathrm{Var}_{x_0}(S_1) = \theta_*^2\lambda''(\theta_*)$, and $\mathrm{Var}(\widetilde{W}_1) = 1$.

With notations above, we can rewrite the quenched barrier event in Proposition 2.3.2 as

$$\mathcal{E}^{\text{barr}}(x,y) = \left\{ S_n - \widetilde{W}_n \geqslant x - y, \forall j \leqslant n : S_j \leqslant x + \widetilde{W}_j \right\}.$$

Since S_1 is bounded and $\widetilde{W}_1$ has a centered exponential distribution, they satisfy the moment condition in 2.3.3. Moreover, by the stationarity of the random field the two random walks S and $\widetilde{W}$ are independent and the latter is measurable with respect to Ξ. Therefore, the desired result is a direct consequence of Proposition 2.3.3. $\square$

2.4 Upper bound proof

In this section we prove Proposition 2.2.1.

Proof of Proposition 2.2.1. We prove the upper bound by combining a random quenched barrier argument with the first moment method. We decompose the event according to whether the random walk $X_j(t)$ ever crosses a quenched random barrier or not. That is:

$$\begin{aligned}
&\mathbb{P}^{\Xi}\left(\exists t \in \mathbb{T} : \theta_* X_n(t) \geqslant W_n - (\phi_* - \eta)\log n\right) \\
&\leqslant \mathbb{P}^{\Xi}\left(\exists j \leqslant n, t \in \mathbb{T} : \theta_* X_j(t) > W_j + (\log n)^2\right) \\
&+ \mathbb{P}^{\Xi}\left(\exists t \in \mathbb{T} : \theta_* X_n(t) \geqslant W_n - (\phi_* - \eta)\log n, \theta_* X_j(t) \leqslant W_j + (\log n)^2, \forall j \leqslant n\right).
\end{aligned} \tag{2.21}$$

We first show that with high probability in $\mathbb{P}$ the first quenched probability is small. Let $\mathcal{G}_1$ be the event as in Lemma 2.2.2 with $c = 1/6$. Then by Lemma 2.2.3, for any $j \leqslant n$ and any uniform mesh $\mathbb{T}_j \in \mathbb{T}$ with size $|\mathbb{T}_j| = (\log n)^2 e^{\xi_j}$, we have

$$|\max_{t\in\mathbb{T}} X_j(t) - \max_{t\in\mathbb{T}_j} X_j(t)| = O(1) \qquad a.s.$$

Therefore, by Lemma 2.2.2, the union bound, and the Girsanov change-of-measure in Definition 2.3.1, we have for any Ξ satisfying $\mathcal{G}_1$:

$$\mathbb{P}^{\Xi}\left(\exists j \leqslant n, t \in \mathbb{T} : \theta_* X_j(t) > W_j + (\log n)^2\right) 1(\mathcal{G}_1)$$

$$
\begin{aligned}
&\leqslant \sum_{j\leqslant n} \mathbb{P}^{\Xi}\left(\exists t\in\mathbb{T}_j : \theta_* X_j(t) > W_j + (\log n)^2/2\right) 1(\mathcal{G}_1)\\
&\leqslant \sum_{j\leqslant n} (\log n)^2 e^{\xi_j} \mathbb{P}^{\Xi}\left(\theta_* X_j(0) > W_j + (\log n)^2/2\right) 1(\mathcal{G}_1)\\
&= \sum_{j\leqslant n} (\log n)^2 e^{\xi_j} \mathbb{E}^{\Xi}_{x_0}\left[e^{\lambda(\theta_*)j-\theta_* X_j(0)} 1\left(\theta_* X_j(0) > W_j + (\log n)^2/2\right)\right] 1(\mathcal{G}_1)\\
&\leqslant n(\log n)^2 e^{-(\log n)^2/2}. \qquad (2.22)
\end{aligned}
$$

For the second probability, let $\mathcal{E}(t;\eta)$ be the event

$$
\left\{\theta_* X_n(t) \geqslant W_n - (\phi_* - \frac{2}{3}\eta)\log n, \theta_* X_j(t) \leqslant W_j + 2(\log n)^2, \forall j \leqslant n\right\},
$$

and define $\mathcal{G}_2$ to be the event

$$
\mathbb{P}^{\Xi}_{x_0}\left(\mathcal{E}(0;\eta)\right) \leqslant n^{-\phi_*+\eta/3}.
$$

By Proposition 2.3.2, we know $\lim_{n\to\infty}\mathbb{P}(\mathcal{G}_2) = 1$. Using again the discretization, the Girsanov change of measure and consider Ξ satisfying $\cap_{i=1,2}\mathcal{G}_i$, we have

$$
\begin{aligned}
&\mathbb{P}^{\Xi}\left(\exists t\in\mathbb{T} : \theta_* X_n(t) \geqslant W_n - (\phi_* - \eta)\log n, \theta_* Y_{\leqslant j}(t) \leqslant W_j + (\log n)^2, \forall j\leqslant n\right)\\
&\qquad\qquad \times 1(\mathcal{G}_1\cap\mathcal{G}_2)\\
&\leqslant \mathbb{P}^{\Xi}\left(\cup_{t\in\mathbb{T}_n}\mathcal{E}(t;\eta)\right) 1(\mathcal{G}_1\cap\mathcal{G}_2)\\
&\leqslant (\log n)^2 e^{\xi_n}\mathbb{P}^{\Xi}\left(\mathcal{E}(0;\eta)\right) 1(\mathcal{G}_1\cap\mathcal{G}_2)\\
&= (\log n)^2 e^{\xi_n}\mathbb{E}^{\Xi}_{x_0}\left[e^{\lambda(\theta_*)n-\theta_* X_n(0)} 1(\mathcal{E}(0;\eta))\right] 1(\mathcal{G}_1\cap\mathcal{G}_2)\\
&\leqslant (\log n)^2 n^{\phi_*-\frac{2}{3}\eta}\mathbb{P}^{\Xi}_{x_0}(\mathcal{E}(0;\eta)) 1(\mathcal{G}_1\cap\mathcal{G}_2)\\
&\leqslant (\log n)^2 n^{-\eta/3}. \qquad (2.23)
\end{aligned}
$$

In summary, by (2.21), (2.22), and (2.23), we have

$$
\begin{aligned}
&\mathbb{P}^{\Xi}\left(\exists t\in\mathbb{T} : \theta_* X_n(t) \geqslant W_n - (\phi_* - \eta)\log n\right) 1(\mathcal{G}_1\cap\mathcal{G}_2)\\
&\leqslant n(\log n)^2 e^{-(\log n)^2/2} + (\log n)^2 n^{-\eta/3}
\end{aligned}
$$

$$\leqslant n^{-\eta/4}$$

for n large enough depending on η. The original statement then follows. $\square$

Chapter 3

Universality and Phase Transitions in Low Moments of Secular Coefficients of Critical Holomorphic Multiplicative Chaos

3.1 Introduction

For polynomials ϕ on the closed unit disc $\overline{\mathbb{D}}$ and a fixed constant $\vartheta > 0$, the *holomorphic multiplicative chaos* HMC_ϑ is defined as a random distribution (or functional)

$$(\mathrm{HMC}_\vartheta, \phi) := \lim_{r\uparrow 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left(\sqrt{\vartheta} \sum_{k=1}^{\infty} \frac{X_k}{\sqrt{k}} (re^{i\theta})^k\right) \overline{\phi(re^{i\theta})}\, \mathrm{d}\theta, \tag{3.1}$$

where $\{X_k\}_{k\geqslant 1}$ is a sequence of i.i.d. rotationally invariant unit variance ("standardized") complex random variables. The existence and uniqueness of the limit is straightforward for trigonometric polynomials since it is determined by the *secular coefficients*

$$A_N := (\mathrm{HMC}_\vartheta, \theta \mapsto e^{iN\theta}) = [z^N] \exp\left(\sqrt{\vartheta} \sum_{k=1}^{\infty} \frac{X_k}{\sqrt{k}} z^k\right), \quad N \in \mathbb{N}, \tag{3.2}$$

where $[z^N]f(z)$ denotes the coefficient of z^N in the power series expansion of f around $z = 0$, i.e.,

$$\exp\left(\sqrt{\vartheta} \sum_{k=1}^{\infty} \frac{X_k}{\sqrt{k}} z^k\right) = \sum_{N=0}^{\infty} A_N z^N.$$

For instance, $A_2 = \vartheta X_1^2/2 + X_2\sqrt{\vartheta/2}$.

The case of a general ϑ where X_k is standard complex Gaussian has been recently investigated by Najnudel, Paquette, and Simm [NPS23], where the random measure

(3.1) is defined in (1.11) therein as a distributional limit of the characteristic polynomial of *Circular Beta Ensemble* (CβE) inside the unit disc ($\beta = 2/\vartheta$) and belongs to the Sobolev space H^s almost surely for any $s < s(\vartheta)$, see Theorem 1.4 therein. The critical case of $\vartheta = 1$ is of particular interest, due to its connections to the *Circular Unitary Ensemble* (CUE) and random multiplicative functions. Moreover, the behavior of secular coefficients is more subtle in the critical case. We summarize the relevant literature in Section 3.1.3.

In this work, our main focus will be the critical case $\vartheta = 1$. If each X_k is standard complex Gaussian, HMC_1 first appeared in Appendix C of Saksman and Webb [SW20] and was shown to be H^s for any $s < -1/2$ a.s. while also a.s. not H^s for any $s > -1/2$ [NPS23]. Recently, the work of Soundararajan and Zaman [SZ22] studied the *low moments* $\mathbb{E}[|A_N|^{2q}]$, $q \in (0, 1]$ of secular coefficients of HMC_1 as a model problem for multiplicative chaos in number theory. By exploiting the connections to Harper's remarkable result [Har20] on "better-than-square-root" cancellation for random multiplicative functions, they proved the following.

Theorem 3.1.1 (Theorem 2.1 of [SZ22])**.** *For standard complex Gaussian variables* $\{X_k\}_{k\geqslant 1}$ *and* $\vartheta = 1$*, uniformly in* $q \in (0, 1]$ *and* $N \geqslant 1$*,*

$$\mathbb{E}[|A_N|^{2q}] \asymp \left(\frac{1}{1 + (1-q)\sqrt{\log N}}\right)^q.$$

In other words, A_N is typically smaller than what one expects from the central limit theorem, i.e. applying Hölder's inequality to its second moment, indicating a complicated limiting behavior for secular coefficients at $\vartheta = 1$. Parallel work of [NPS23] also established the sharp tightness of secular coefficients.

Theorem 3.1.2 (Theorem 1.11 of [NPS23])**.** *If* $\vartheta = 1$*, both the families* $\{A_N/(\log(1+N))^{-1/4}\}_{N\in\mathbb{N}}$ *and* $\{(\log(1+N))^{-1/4}/A_N\}_{N\in\mathbb{N}}$ *are tight.*

It is worth noticing that all aforementioned studies focused on the Gaussian setting. A natural question would be to examine the universality of current results and what happens beyond that regime. In this article, we initiate the study of non-Gaussian holomorphic multiplicative chaos and investigate the low moments $\mathbb{E}[|A_N|^{2q}],\ q \in [0,1]$ when $|X_k|$ follows a generic standardized distribution. A rather surprising phenomenon is that, even for the simplest setting of $q = 1$, the second moments of A_N may no longer remain of constant order as in the Gaussian case. For example, our results imply that $\mathbb{E}[|A_N|^2]$ grows exponentially if each $|X_k| \sim \mathrm{Exp}(1)$, polynomially if $|X_k| \sim \mathrm{Exp}(2)$, and remains of constant order if $|X_k| \sim \mathrm{Exp}(3)$.[1]

Our main result consists of two parts. First, Theorem 3.1.3 below establishes a sharp criterion for *Gaussian universality regime* of the asymptotics of $\mathbb{E}[|A_N|^{2q}],\ q \in [0,1]$. Second, in Theorem 3.1.4 below we completely characterize a subtle *double-layer phase transition* associated with the tail of $|X_k|$; see Figure 3.1 below for the phase diagram. As a result, we fully establish the asymptotics for low moments $\mathbb{E}[|A_N|^{2q}]$ with non-Gaussian inputs $\mathbb{P}(|X_k| > t) \sim \exp(-\gamma t^p)$ for all triples $(q, \gamma, p) \in [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+$. Our main theorems generalize Theorem 3.1.1 and show a new phenomenon of *(super-)exponential cancellation* (that is, $(\mathbb{E}[|A_N|])^2/\mathbb{E}[|A_N|^2]$ decays (super-)exponentially as $N \to \infty$) emerges as the tail of $|X_k|$ becomes heavier.

We establish universality by utilizing the connection between secular coefficients and the total mass of critical multiplicative chaos, which resembles a delicate joint effect of all inputs X_k's. The double-layered phase transition, on the other hand, is a product of an intricate interplay between the dominance of X_1 (which occurs when the tail of $|X_1|$ is heavy) and a similar yet different joint effect as in the universality phase. We expand our discussions in Section 3.1.2 and a more detailed description of our approach can be found in Section 3.2.

[1]Strictly speaking, these "exponential distributions" need to be normalized to have unit variance, e.g. their tails have the form $\mathbb{P}(|X_k| > t) = \exp(-\gamma(t - c_\gamma)) \wedge 1$ for $\gamma = 1, 2, 3$.

Several corollaries are in place. First, we obtain a *universality result* for the tightness of normalized secular coefficients of non-Gaussian holomorphic chaos (Corollary 3.1.5), extending Theorem 3.1.2. Second, we derive both a universal and a non-universal regularity result for some critical non-Gaussian holomorphic chaos (Corollary 3.1.6), partially extending the regularity of Gaussian HMC_1.

Finally, based on these new observations, we propose a list of future research directions of broad interest, ranging from random matrix theory, and multiplicative chaos, to probabilistic number theory. These questions will be discussed in detail in Section 3.1.3.

3.1.1 Statement of main results

Informally speaking, our main results capture the asymptotics of all $(2q)$-moments ($q \in (0,1]$) of A_N for X_k satisfying any *stretched exponential tail*, i.e. $\mathbb{P}(|X_k| > t) \sim \exp(-t^p)$ for all $p \in (0,\infty)$. For example, $p = 2$ resembles the Gaussian case studied in Theorems 3.1.1 and 3.1.2.

We now set the stage for formalizing our theorems. Let $\{\tau_k\}$ be i.i.d. uniformly distributed on $[-\pi,\pi)$ and $\{R_k\}$ be i.i.d. real random variables with $\mathbb{E}[|R_k|^2] = 1$ and independent of $\{\tau_k\}$. Let $X_k = e^{i\tau_k}R_k$. We may assume that R_k is symmetric by a standard symmetrization procedure. We introduce a few short-hand notations for the cases of interest, covering (roughly speaking) i) $p \in (1,\infty)$ case, ii) $p = 1$ case, and iii) $p \in (0,1)$ case:

- (q-UNIV) For a given $q \in (0,1]$, R_k has a finite γ-exponential moment for some $\gamma > 2q$. That is, $\mathbb{E}[e^{\gamma|R_k|}]$ is finite for some $\gamma > 2q$.

- (EXP) R_k is a two-sided shifted *exponential* random variable with unit variance. That is, $\mathbb{P}(|R_k| \geqslant u) = \exp(-\gamma(u - c_\gamma)) \wedge 1$ for $u \geqslant 0$, where $\gamma \in (0, 2q]$ and $c_\gamma := \log(\gamma^2/2)/\gamma$;

- (SE) R_k follows a (symmetric) *stretched exponential* distribution with exponent $p \in (0,1)$, i.e. $\mathbb{P}(|R_k| \geqslant u) = \exp(-(u/c_p)^p)$ for $u \geqslant 0$, where $c_p := (2\Gamma(2/p)/p)^{-1/2}$.

Here, we have implicitly imposed the constraints on the law of R_k that R_k is symmetric and $\mathbb{E}[|R_k|^2] = 1$, which explains the shift c_γ and the scaling parameter c_p.

The cases (EXP) and (SE) are introduced as prototypes, where we do not attempt to achieve the greatest generality by considering the largest class of laws. Indeed, our proofs only rely on the absolute moment asymptotics of X_1 (along with rotational symmetry and the unit variance property). Our main results are as follows. We refer to Section 3.1.3 for asymptotic notations; in particular, we allow the asymptotic constants to depend on q and the distribution of X_k.

Theorem 3.1.3 (Primary phase transition)**.** *(i) Fix any $q \in (0,1]$. Suppose that condition (q-UNIV) holds for the i.i.d. input $\{X_k = e^{i\tau_k}R_k\}$. We have*

$$\mathbb{E}[|A_N|^{2q}] \asymp \left(\frac{1}{1+(1-q)\sqrt{\log N}}\right)^q. \tag{3.3}$$

(ii) Fix any $q > 0$. Suppose that (SE) holds for the i.i.d. input $\{X_k = e^{i\tau_k}R_k\}$. It holds that

$$\mathbb{E}[|A_N|^{2q}] = (1+o(1))(2\pi)^{1/2-q}\sqrt{\frac{2q}{p}}\left(\frac{2qc_p^p}{pe^{1-p}}\right)^{2qN/p} N^{1/2-q+2qN(1/p-1)}. \tag{3.4}$$

Note that $2qc_p^p/(pe^{1-p}) < 1$ for all $0 < p, q < 1$.

The asymptotic (3.3) shows that the better-than-square-root cancellation phenomenon can still be quantitatively analyzed for non-Gaussian inputs in (3.2) under a suitable light-tailed assumption. This considerably extends the results in the Gaussian case (cf., Theorem 3.1.1). The result of (3.4) in the heavy-tailed case indicates that the Gaussian-type behavior is limited to light-tailed distributions and does not

apply in general to heavier tails (say, heavier than the exponential distribution). This gives rise to a phase transition in the thickness of the tail of $|R_k|$, leading naturally to the question of the *primary criticality*: case (EXP). The following result completely characterizes the behavior in this critical phase, showcasing a second phase transition within this critical phase.

Theorem 3.1.4 (Secondary phase transition)**.** *Fix $q \in (0, 1]$. Suppose that (EXP) holds for the i.i.d. input $\{X_k = e^{i\tau_k} R_k\}$. It holds that if $\gamma < 2q$,*

$$\mathbb{E}[|A_N|^{2q}] \asymp N^{1/2-q} \left(\frac{2q}{\gamma}\right)^{2qN}; \tag{3.5}$$

if $\gamma = 2q$,

$$\mathbb{E}[|A_N|^{2q}] \asymp \frac{N^{1-q+q^2/2}}{(1+(1-q)\sqrt{\log N})^q}; \tag{3.6}$$

and if $\gamma > 2q$,

$$\mathbb{E}[|A_N|^{2q}] \asymp \left(\frac{1}{1+(1-q)\sqrt{\log N}}\right)^q. \tag{3.7}$$

Indeed, (3.7) follows directly from Theorem 3.1.3 (i), and is included here for the sake of completeness. Asymptotic constants in (3.5) and (3.7) may depend on γ which characterizes the distribution of X_k. The critical phase (EXP) showcases a further phase transition. There exists a critical moment exponent, which coincides with the parameter of the (shifted) exponential distribution $|R_k|$, that governs the asymptotic of the moments: Gaussian-type asymptotic of Theorem 3.1.1 below such exponent ($\gamma > 2q$) and exponential growth of moments above the exponent ($\gamma < 2q$). At the *secondary criticality*, where the moment exponent coincides with the parameter of the exponential distribution ($\gamma = 2q$), the moment exhibits a regularly varying growth in N.

To visualize the double-layer phase transition, we may restrict the class (q-UNIV) to laws satisfying

$$\mathbb{P}(|X_k| > t) \asymp \exp(-\gamma t^p) \tag{3.8}$$

as $t \to \infty$ for some $\gamma > 0$ and $p \geqslant 1$. In this setting, the phases of (EXP), (SE), and restricted (q-UNIV) have the alternative representation (3.8) for some $\gamma, p > 0$, and the pair (γ, p) uniquely determines which phase the distribution of R_k belongs to. Figure 3.1 shows the phase diagram that illustrates our main results. Note that γ is only effective in the phase diagram when $p = 1$.

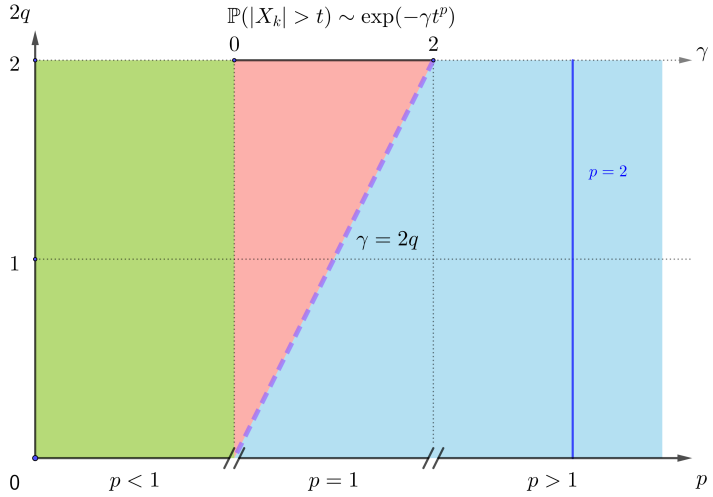

Figure 3.1: An illustration of the phase transitions of low moments for secular coefficients depending on the law of $|X_1|$ and the moment exponent $(2q)$. The tail satisfying $-\log \mathbb{P}(|X_1| > t) \asymp t^p$ is heavier on the left side and lighter on the right. The light blue regime refers to the universality phase, i.e. (q-UNIV) case, which considerably extends the dark blue line that represents the Gaussian case in previous studies. The green and red regimes correspond respectively to the (SE) case and (EXP) case with $\gamma < 2q$. The purple dashed line refers to the secondary critical phase with $\mathbb{P}(|X_1| > t) \asymp \exp(-\gamma t)$ where $\gamma = 2q$.

As a consequence of Theorem 3.1.3, we find the typical order of magnitude of A_N by considering the tightness of the family $\{A_N/w_N\}_{N\in\mathbb{N}}$ for some deterministic sequence w_N, partially generalizing Theorem 3.1.2. Our result only requires the existence of some exponential moment; see the proof in Section 3.6.1 for details.

Corollary 3.1.5. *Suppose that there exists $\varepsilon > 0$ such that $\mathbb{E}[e^{\varepsilon|X_k|}] < \infty$ for the i.i.d. input $\{X_k = e^{i\tau_k}R_k\}$. The followings hold:*

1. *for $w_N = (\log(1+N))^{-1/4}$, $\{A_N/w_N\}_{N\in\mathbb{N}}$ is tight;*

2. *for $w_N = o((\log(1+N))^{-1/4})$, $\{A_N/w_N\}_{N\in\mathbb{N}}$ is not tight.*

Unfortunately, proving the tightness of $\{(\log(1+N))^{-1/4}/A_N\}_{N\in\mathbb{N}}$ remains a difficult task based on existing techniques, mainly due to the absence of distributional convergence of the total mass of critical non-Gaussian multiplicative chaos. The convergence for critical Gaussian multiplicative chaos was shown in [JS17].

We can also obtain the regularity of some critical non-Gaussian HMC from Theorem 3.1.3. The proof and a remark on irregularity are deferred to Section 3.6.2.

Corollary 3.1.6. *If there is some $\gamma > 2$ such that $\mathbb{E}[e^{\gamma|X_k|}] < \infty$, then the corresponding* HMC_1 *is in H^s almost surely for any $s < -1/2$. Moreover, if X_k satisfies (EXP) with $\gamma \in (0,2]$, i.e. $\mathbb{P}(|X_k| \geqslant u) \asymp e^{-\gamma u}$, then the corresponding* HMC_1 *is in H^s almost surely for any $s < -1/\gamma$.*

We also mention that the super-exponential growth of the second moment of secular coefficients in the (SE) case suggests an extremely irregular behavior, in the sense that HMC_1 may not even be a tempered distribution almost surely.

3.1.2 Technical challenges

In what follows, we attempt to unveil the intricacies of low moments behind the distinct phases, without involving too much technicality. The approach of [SZ22] in the Gaussian case is to connect $\mathbb{E}[|A_N|^{2q}]$ to low moments of the total mass of a suitably constructed (finite) multiplicative chaos, of the form

$$\mathbb{E}\bigg[\Big(\int_{-\pi}^{\pi}\Big|\prod_{k=1}^{K}\exp\Big(\frac{X_k r^k e^{ik\theta}}{\sqrt{k}}\Big)\Big|^2\mathrm{d}\theta\Big)^q\bigg], \tag{3.9}$$

where $K \in \mathbb{N},\ q \in (0,1]$, and $r \in [e^{-1/K}, e^{1/K}]$. However, as the tail of $|X_k|$ becomes heavier, the contribution of the random variable X_1 in (3.9) becomes more prominent and will need to be isolated from other variables $\{X_k\}_{k\geqslant 2}$. In the cases of (SE) and (EXP) with $\gamma < 2q$, our proofs depend on a delicate but direct analysis of such domination effect of X_1, without delving into multiplicative chaos. The secondary critical case ((EXP) with $\gamma = 2q$) turns out more intricate and requires both the domination effect and the analysis of critical multiplicative chaos. In this scenario, even speculating the correct asymptotic is a non-trivial task.

Our approach is to first condition on $|X_1|$ (or equivalently, $|R_1|$) *before* connecting $\mathbb{E}[|A_N|^{2q}]$ to multiplicative chaos. This reduces our problem into studying the low moments of a *randomly weighted* total mass of a multiplicative chaos, of the form

$$\mathbb{E}\bigg[\bigg(\int_{-\pi}^{\pi}\Big|U(\theta)\times\prod_{k=2}^{K}\exp\Big(\frac{X_k r^k e^{ik\theta}}{\sqrt{k}}\Big)\Big|^2\mathrm{d}\theta\bigg)^q\bigg], \tag{3.10}$$

where $K \in \mathbb{N},\ q \in (0,1],\ r \in [e^{-1/K}, e^{1/K}]$, and the (random) weight function $U(\theta)$ resembles a discrete Fourier transform of a random function sufficiently close to a scaled Gaussian density, evaluated at the random frequency $\theta + \tau_1$, where τ_1 is uniformly distributed on $[-\pi, \pi]$. Intuitively, $U(\theta)$ arises from the remaining randomness of τ_1 after conditioning on $|R_1|$. To study the magnitude of (3.10), we need both a careful analysis of $U(\theta)$ and a *uniform* version of the estimates of the *partial mass* of the multiplicative chaos,

$$\mathbb{E}\bigg[\bigg(\int_{I}\Big|\prod_{k=1}^{K}\exp\Big(\frac{X_k r^k e^{ik\theta}}{\sqrt{k}}\Big)\Big|^2\mathrm{d}\theta\bigg)^q\bigg],$$

where I is an interval in $[-\pi, \pi]$ whose length is of a much smaller order. That is, the integration range in (3.9) is restricted to an asymptotically smaller interval, and the difficulty lies in obtaining uniformity in all those intervals of suitable length. A

technical highlight while tackling such difficulty is performing generic chaining under a proper change of measure.

A more detailed description of our approach can be found in Section 3.2.2, where we also discuss intuitively how the regularly varying formula in (3.6) arises.

3.1.3 Related works and outlook

In this section, we review a few connections of holomorphic multiplicative chaos to random matrix theory and number theory. Along the way, we also discuss a few further directions of interest that are beyond the scope of this paper.

Connections to random matrix theory and (complex) Gaussian multiplicative chaos. HMC_ϑ with Gaussian inputs $\{X_k\}_{k\geqslant 1}$ is closely related to random matrix theory, especially to the characteristic polynomial of CβE with $\vartheta = 2/\beta$. A joint distribution of N points with parameter $\beta > 0$ is said to be CβE if

$$\mathrm{C}\beta\mathrm{E}_N(\theta_1, \cdots, \theta_N) \propto 1_{\{\forall j\in\{1,\ldots,N\},\, \theta_j\in[-\pi,\pi)\}} \prod_{1\leqslant j<k\leqslant N} |e^{i\theta_j} - e^{i\theta_k}|^\beta,$$

and for $\beta = 1, 2, 4$, it is also known as circular orthogonal/unitary/symplectic ensembles (COE/CUE/CSE), respectively. For $\beta = 2$ (CUE), it is the distribution of the eigenvalues of a Haar-distributed unitary matrix, and for $\beta \neq 2$, [KN04] constructed explicit matrix models $U_N^{(\beta)}$ whose eigenvalues have law CβE. Therefore, we can define and study the characteristic polynomial

$$\chi_N^{(\beta)}(z) = \det(I - zU_N^{(\beta)}) = \sum_{n=0}^{N} a_n^{(N,\beta)} z^n$$

on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. We may drop the superscripts if $\beta = 2$.

Secular coefficients $a_n^{(N)}$ of χ_N were first explicitly studied, to our best knowledge, by Haake, Kus, and Sommers [HKS+96], where they delved into various theoretical

and numerical properties of secular coefficients for CUE, including an explicit formula of the second moment. Later, Diaconis and Gamburd [DG04] computed $(2k)$-th absolute moments of secular coefficients $a_n^{(N)}$ of CUE in terms of *magic squares*, which is the number of $N \times N$ square matrices with nonnegative integer entries summing up to n in each row and column. For a fixed n, the convergence result of $a_n^{(N,\beta)}$ is established via the formula connecting it to the first n power traces $T_k := \mathrm{Tr}((U_N^{(\beta)})^k)$:

$$a_n^{(N,\beta)} = \frac{1}{n!} \det \begin{pmatrix} T_1 & 1 & 0 & \cdots & 0 \\ T_2 & T_1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T_{n-1} & T_{n-2} & T_{n-3} & \cdots & n-1 \\ T_n & T_{n-1} & T_{n-2} & \cdots & T_1 \end{pmatrix},$$

along with the convergence result for power traces by Diaconis and Shahshahani [DS94] for $\beta = 2$, and Jiang and Matsumoto [JM15] for a general $\beta > 0$ that jointly,

$$\{T_k\}_{k=1}^n \implies \sqrt{\frac{2}{\beta}} \{\sqrt{k} \mathcal{N}_k^{\mathbb{C}}\}_{k=1}^n, \ N \to \infty,$$

where $\{N_k^{\mathbb{C}}\}_{k=1}^n$ are i.i.d. standard complex Gaussian variables.

A natural question is whether we can find connections between other ensembles of random matrices and non-Gaussian HMC_ϑ, mirroring Theorem 1.3 of [NPS23]. Some inspiring examples with non-Gaussian (but weakly dependent) coefficients are the sparse Bernoulli matrix studied by Coste [Cos23] and sums of random permutation matrices by Coste, Lambert, and Zhu [CLZ22], which can both be considered as random regular digraph models. As a simplified model in [Cos23], let B_N be an $N \times N$ matrix with i.i.d. Bernoulli$(1/n)$ entries, then $\det(I_N - zB_N)$ converges weakly in the open unit disk $\mathbb{D}$ to the so-called *Poissonian holomorphic chaos* $F(z) = \exp(-\sum_{k=1}^\infty X_k z^k/\sqrt{k})$ with $X_k = \sum_{\ell|k} \ell Y_\ell/\sqrt{k}$, where $\{Y_\ell\}_{\ell=1}^\infty$ is a family of independent Poisson variables with $\mathbb{E}[Y_\ell] = 1/\ell$. Moreover, a similar convergence in

law result (and towards a similar Poissonian chaos) was established in [CLZ22] if B_n is replaced by the average of a fixed number of i.i.d. random permutation matrices; see Theorem 2.2 therein for details.

Question 1. Find other ensembles of random matrices $V_N^{(\vartheta)}$ such that its characteristic polynomial converges (in some sense) to $\exp(\sqrt{\vartheta}\sum_{k\geqslant 1} X_k z^k/\sqrt{k})$ uniformly in $|z|\leqslant r$, for any $r\in(0,1)$.

Yet another connection exists between secular coefficients of random matrices (or HMC) and the total mass of the *Gaussian multiplicative chaos* (GMC). A GMC_ϑ of parameter $\vartheta\in(0,1]$ can be defined as the limiting random measure (see e.g. Appendix B of [NPS23])

$$\mathrm{GMC}_\vartheta(\mathrm{d}\theta) := \lim_{r\to 1}(1-r^2)^\vartheta(-\log(1-r^2))^{\frac{1}{2}1_{\vartheta=1}}|e^{\sqrt{\vartheta}\sum_{k=1}^\infty \frac{\mathcal{N}_k^{\mathbb{C}}}{\sqrt{k}}(re^{i\theta})^k}|^2\mathrm{d}\theta,$$

where the convergence holds in L^q for any $q\in(0,1)$. Its *total mass* $\mathcal{M}_\vartheta$ is then defined as $\mathcal{M}_\vartheta := \frac{1}{2\pi}\int_0^{2\pi}\mathrm{GMC}_\vartheta(\mathrm{d}\theta)$ in the in-probability, weak-$*$ convergence sense. Recently, Najnudel, Paquette, and Simm [NPS23] established the distributional convergence of $a_n^{(N,\beta)}/\sqrt{\mathbb{E}[(a_n^{(N,\beta)})^2]}$ as $n,N\to\infty$ jointly for $\beta>4$ ($0<\vartheta<1/2$), and expressed the limiting distribution as $\sqrt{\mathcal{M}_\vartheta}$ times an independent standard complex normal. See also [GV23] for the distributional convergence of Fourier coefficients of GMC_ϑ on the unit circle for $0<\vartheta<1/\sqrt{2}$. Moreover, [NPS23] also established tightness for a general $\beta>0$. As an example, for $\beta=2$ they showed that $\{(\log n)^{1/4}a_n^{(N)} : N\geqslant 2n\}$ and $\{(\log n)^{-1/4}/a_n^{(N)} : N\geqslant N_0(n)\}$ are both tight (for some $N_0(n)$ growing faster than $n\sqrt{\log n}\,(\log\log n)$). While the normalization and limiting behavior are not clear yet in the critical case ($\vartheta=1$), a question of universality and phase transition could be asked, reminiscent of our Theorem 3.1.3 and 3.1.4.

Question 2. When $\vartheta = 1$, does $A_N(\log N)^{1/4}$ converge in distribution to the same limiting random variable for any X_k as in Corollary 3.1.5? If the tail of $|X_k|$ gets heavier and reaches the (SE) phase, is there a phase transition in the limiting behaviors as well?

The convergence result of secular coefficients stated above also indicates an impressive fact, that a single A_N (when N is large) already contains some information of the total mass $\mathcal{M}_\vartheta$. Actually, [NPS23] utilized this connection and existing results on convergence of critical GMC to prove the irregularity of HMC_1 and the sharp tightness of $\{(\log(1+N))^{-1/4}/A_N\}$. To study these properties in a non-Gaussian setting, one has to understand the critical non-Gaussian multiplicative chaos.

Question 3. Prove the convergence in L^q for all $q \in (0,1)$ for

$$\lim_{r\to 1}(1-r^2)(-\log(1-r^2))^{\frac{1}{2}}|e^{\sum_{k=1}^{\infty}\frac{X_k}{\sqrt{k}}(re^{i\theta})^k}|^2\mathrm{d}\theta$$

when $\{X_k\}_{k\in\mathbb{N}}$ are i.i.d. non-Gaussian, and apply it to study the irregularity of HMC and sharp tightness of A_N in more general settings.

Another promising future direction is to explore the connection between complex Gaussian multiplicative chaos (CGMC) and HMC. Roughly speaking, the complex Gaussian field defining CGMC has independent real and imaginary parts, while HMC has uncorrelated ones, which might suggest some commonalities in between. We refer interested readers to Section 1.6 of [NPS23] for a more detailed discussion on this possible connection, and to [Lac22, LRV15] for references on CGMC.

Finally, we summarize some related results on characteristic polynomials of random matrices. The celebrated work of Hughes, Keating, and O'Connell [HKO01] introduced the log-correlated Gaussian field $G^{\mathbb{C}}(z) = \sum_{k=1}^{\infty}\mathcal{N}_k^{\mathbb{C}}z^k/\sqrt{k}$ as the limiting random distribution of the log-characteristic polynomial of CUE. For general surveys

on log-correlated Gaussian fields, see [DRSV17]. Meanwhile, convergence results towards GMC were established in [CN19, NSW20, Web15] for characteristic polynomial of CβE on the unit circle. For general surveys on GMC, see [Pow21, RV14]. We also point interested readers to the survey by Bailey and Keating [BK22] and the most recent result by Paquette and Zeitouni [PZ22] and references therein for studies towards a conjecture in [FHK12, FK14] by Fyodorov, Hiary, and Keating on the maximum of the characteristic polynomial of CUE.

Connections to number theory. A *random multiplicative function* is a completely multiplicative function $f : \mathbb{N} \to \mathbb{C}$, i.e. $f(mn) = f(m)f(n)$ for $m, n \in \mathbb{N}$, where $f(p)$ are i.i.d. random variables for primes p. An important application is modeling arithmetic functions, such as Dirichlet characters (Steinhaus case, with $f(p)$ uniformly distributed on the unit circle $\mathbb{T}$) and the Möbius function (Rademacher case, with $f(p)$ uniformly distributed on $\{\pm 1\}$ and supported on square-free numbers). The recent celebrated work of Harper [Har20] investigated the better-than-square-root cancellation phenomenon of random multiplicative functions through computing low moments of the partial sums, thus resolving Helson's conjecture [Hel10]. More precisely, [Har20] showed that

$$\mathbb{E}\Big[\Big|\sum_{n\leqslant x} f(n)\Big|\Big] \asymp \frac{\sqrt{x}}{(\log\log x)^{1/4}},$$

while $\mathbb{E}[|\sum_{n\leqslant x} f(n)|^2] \asymp x$. Harper's approach elegantly connects random multiplicative functions and critical multiplicative chaos, which we summarize in Section 3.2.2. Similar ideas are also exploited in the studies of deterministic multiplicative functions of interest in number theory, such as better-than-square-root cancellation for typical non-principal Dirichlet characters [Har23b].

The recent work of Soundararajan and Zaman [SZ22] proposed that the secular

coefficients A_N of Gaussian HMC_1 constitute a model that describes the mathematical structure of (Steinhaus) random multiplicative functions. As commented in [SZ22], the same model also serves as the function field counterpart of Harper's result [Har20], as follows. Let $\mathcal{M}_n$ denote the set of monic polynomials of degree n over the ring $\mathbb{F}_q[t]$ where q is a prime power and $\mathbb{F}_q$ is a finite field with q elements. Consider a random multiplicative function f on $\mathbb{F}_q[t]$ defined analogously, with irreducible monic polynomials as "primes". It follows that if $\bar{A}_n = q^{-n/2} \sum_{F \in \mathcal{M}_n} f(F)$, then

$$\sum_{n=0}^{\infty} \bar{A}_n z^n = \exp\left(\sum_{k=1}^{\infty} \frac{\bar{X}_k}{\sqrt{k}} z^k \right) \quad \text{where} \quad \bar{X}_k = \frac{\sqrt{k}}{q^{k/2}} \sum_{\substack{P \text{ irred.} \\ \deg(P) | k \\ r = k/\deg(P)}} \frac{f(P)^r}{r}. \tag{3.11}$$

Therefore, the partial sums $\bar{A}_n$ (in the limit case of $q \to \infty$) mirror the secular coefficients A_n defined in (3.2). We refer to the introduction of [SZ22] for related discussions around this analogy.

We speculate that secular coefficients arising from non-Gaussian chaos may connect to certain structured sparse partial sums of random multiplicative functions with heavy-tailed inputs $\{f(P)\}$. For instance, by considering partial sums of f on smooth polynomials (that factor into primes of small degrees, say $\leqslant d$), the last sum of (3.11) consists of a uniformly bounded number of summands (consisting of irreducible polynomials of degrees $\leqslant d$). In this case, heavy-tailed inputs $\{f(P)\}$ may dominate the central limit effect, and weak dependence within $\{\bar{X}_k\}$ is expected as the power r increases with k. We refer to [SX23, Xu23] for studies on sparse partial sums of random multiplicative functions.

Question 4. Find a formal analogous model of some random multiplicative function for secular coefficients arising from non-Gaussian chaos.

The secular coefficients $\{A_N\}_{N \geqslant 0}$ (with Gaussian inputs $\{X_k\}_{k \geqslant 1}$) capture other aspects of random multiplicative functions as well, such as almost sure fluctuations.

For instance, [Cai23b] established the almost sure upper bound $|\sum_{n\leqslant x} f(n)| \ll \sqrt{x}(\log\log x)^{1/4+\varepsilon}$ for random multiplicative functions f, which mirrors the result $|A_N| \ll (\log N)^{1/4+\varepsilon}$ in [Cai23a] for secular coefficients. In addition, for any function $V(x) \to \infty$, [Har23a] proved the almost sure existence of large values x satisfying $|\sum_{n\leqslant x} f(n)| \geqslant \sqrt{x}(\log\log x)^{1/4}/V(x)$. The parallel for secular coefficients was established by [Ger22]: there exist almost surely large values of N such that $|A_N| \geqslant (\log N)^{1/4}/V(N)$. We leave the investigation for the almost sure fluctuations of secular coefficients with non-Gaussian inputs to future research.

Question 5. Establish almost sure upper and lower bounds for secular coefficients arising from non-Gaussian chaos. Characterize a phase transition as the tail of $|X_1|$ becomes heavier.

Let us also mention the work of [ASV+22] that numerically computes the secular coefficients in polynomial time, supporting conjectures on finer asymptotics for their low moments. Naturally, one would also ask if the asymptotics in the (q-UNIV) and (EXP) phases can be improved to $1 + o(1)$ asymptotics.

Question 6. Find precise asymptotics of (3.3), (3.5), and (3.6).

Moreover, [ASV+22] also conjectured a similar behavior of low moments of secular coefficients for *real* standard Gaussian or Rademacher inputs $\{X_k\}_{k\geqslant 1}$. While we do not directly resolve their conjecture, we shall illustrate in Remark 5 below that the asymptotics (3.4) and (3.5) hold also in the real case, with essentially the same proof. However, this does not apply to (3.6).

Question 7. Do (3.3) and (3.6) of Theorem 3.1.4 hold if one replaces the complex inputs $\{X_k\}_{k\geqslant 1}$ by their real part $\{R_k\}_{k\geqslant 1}$?

Finally, in addition to random multiplicative functions, critical multiplicative chaos also connects to the distribution of values of the Riemann zeta function on the

critical line. According to the FHK conjecture, the local maxima of $\log|\zeta(1/2+it)|$ deviate from what one would predict from Selberg's central limit theorem, due to the log-correlated structure of the zeta values. We refer to [ABR20, ABR23, BK22, Sou22] and the references therein for the interplay between the Riemann zeta function, multiplicative chaos, and log-correlated fields.

Organization. The rest of this chapter is organized as follows. Section 3.2 is devoted to the intuition of the phase transition regime and the main ideas of the proofs. In Sections 3.3–3.5, we consider respectively the (q-UNIV), (SE), and (EXP) phases. Section 3.7 is devoted to some technical computations regarding Laplace functionals of $\{X_k\}_{k\geqslant 1}$ under distinct probability measures. Section 3.8 collects some deterministic calculations.

Notation. For quantities or functions A, B, We use Vinogradov's symbol $A \ll B$ (or $A = O(B)$) to denote $|A| \leqslant CB$ with some constant $C > 0$ that depends only on the distribution of R_k (equivalently, X_k) and the moment exponent q. Write $A \asymp B$ if $A \ll B \ll A$. If $A(N)/B(N) \to 0$ we write $A(N) = o(B(N))$. We will denote by $L > 0$ a universal constant that *may not be the same on each occurrence*, where the same applies for $C > 0$. Vectors are typically denoted by bold symbols in this paper. Denote by $\Re z$ the real part of a complex number z. When dealing with events (or expectations) involving $\{(R_k, \tau_k)\}_{k\geqslant 1}$, we will use $\mathbb{P}$ (or $\mathbb{E}$) to denote the original probability measure. We will also use $\mathbb{P}$ to denote the probability measure of any Gaussian random variable (or vector) with a specified mean and variance (or covariance matrix).

3.2 Intuition behind phase transitions and proof strategy

To understand the mechanism behind the phase transition phenomenon for low moments of secular coefficients A_N, let us first rewrite it in a more tractable form in terms of partitions. By a *partition* λ we mean a non-increasing sequence of integers (parts) $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$ with $\lambda_n = 0$ from some n onward. For a partition λ and $k \in \mathbb{N}$, we denote by $m_k(\lambda)$ the number of parts in λ that are equal to k, and $|\lambda|$ the sum of the parts in λ. For example, the all-one partition $\lambda^* := (1, \cdots, 1)$ has $|\lambda^*| = n$ and $m_k(\lambda^*) = n1_{\{k=1\}}$. Let $\mathcal{P}_N = \{\lambda : |\lambda| = N\}$, and $p_N = |\mathcal{P}_N|$. It follows from (3.2) that

$$\exp\left(\sum_{k=1}^{\infty} \frac{X_k}{\sqrt{k}} z^k\right) = \sum_{\lambda} a(\lambda) z^{|\lambda|},$$

where the sum is over the set of all partitions and

$$a(\lambda) := \prod_{k \geqslant 1} \left(\frac{X_k}{\sqrt{k}}\right)^{m_k} \frac{1}{m_k!}. \tag{3.12}$$

In particular, we arrive at the tractable form as a sum of monomials in $\{X_k\}_{k \geqslant 1}$:

$$A_N = \sum_{\lambda \in \mathcal{P}_N} a(\lambda). \tag{3.13}$$

We note the orthogonality relation that $\mathbb{E}[a(\lambda)\overline{a(\lambda')}] = 0$ for $\lambda \neq \lambda'$.

3.2.1 A phase transition of the domination regime in the monomial decomposition

In different phases, the quantity $\mathbb{E}[|A_N|^{2q}]$ is dominated by different interactions among parts in the sum over partitions of N. This can be intuitively seen by comparing the magnitudes of

$$\mathbb{E}[|a(\lambda)|^{2q}] = \prod_{k \geqslant 1} \frac{\mathbb{E}[|X_k|^{2qm_k}]}{k^{qm_k}(m_k!)^{2q}}, \quad \lambda \in \mathcal{P}_N, \tag{3.14}$$

using independence, moment asymptotics for $|X_k|$, and Stirling's approximation (see Chapter 3 of [Art64])

$$1 < (2\pi)^{-1/2} x^{1/2-x} e^x \Gamma(x) < e^{1/(12x)}, \ x \geqslant 1. \tag{3.15}$$

To be more specific, in the (SE) case,

$$\mathbb{E}[|X_k|^{2qm_k}] = \int_0^\infty e^{-(u^{1/(2qm_k)}/c_p)^p} \mathrm{d}u = \frac{2qm_k}{p} c_p^{2qm_k} \Gamma\Big(\frac{2qm_k}{p}\Big). \tag{3.16}$$

A finer computation using (3.15) reveals that $\mathbb{E}[|a(\lambda)|^{2q}]$ grows fastest in N when $\lambda = \lambda^* = (1, \cdots, 1)$ (i.e. $m_1(\lambda^*) = N$), and is significantly larger than the $(2q)$-th moments of $a(\lambda')$ at any other $\lambda' \in \mathcal{P}_N$ for N large. Therefore, the moments of A_N almost only result from $a(\lambda^*)$, which contains a single random variable X_1. The same arguments apply to even heavier tails than (SE).

On the other hand, as a prototypical example in the (q-UNIV) case, we consider the setting where X_k is a standard complex Gaussian variable. Computing the absolute moments of Gaussian variables (see e.g. [Win12]) gives

$$\mathbb{E}[|X_k|^{2qm_k}] \asymp 2^{qm_k} \Gamma\Big(qm_k + \frac{1}{2}\Big) = o\big(k^{qm_k} (m_k!)^{2q}\big),$$

indicating that every $\mathbb{E}[|a(\lambda)|^{2q}]$ is of a vanishing order and therefore the main contribution to the moments of A_N does not arise from a single partition. Instead, [SZ22] found that, as inspired by Harper's remarkable paper on low moments of partial sum of random multiplicative functions [Har20], the main contribution to the moments of A_N (in the complex Gaussian case) comes from those partitions λ with a large part, i.e. $\lambda_1 \geqslant N/(\log N)^C$ for some large constant $C > 0$, indicating an intricate interplay among all X_k.

As the exponent p of $\mathbb{P}(|X_k| > t) \asymp \exp(-ct^p)$ decreases from above 1 (e.g. Gaussian) to below 1 (stretched exponential), the contribution from X_1 becomes more prominent and experiences a phase transition at $p = 1$. At the critical phase of exponential distributions where $\mathbb{P}(|X_k| > t) \asymp \exp(-\gamma t)$, more elaborate dependence on

variables X_k emerges. Fix a moment exponent $q \in (0, 1]$. If the exponent $\gamma > 2q$, the tail of X_k decays fast enough to suppress the growth of each single $\mathbb{E}[|a(\lambda)|^{2q}]$. As will be shown in Section 3.3, the mechanism of Gaussian variables (i.e. (q-UNIV) case) remains true. If $\gamma < 2q$, on the other hand, the tail decays slowly enough to *partially* restore the dominance of $a(\lambda^*)$ among all partitions. Now the dominant parts comprise not only the all-one partition λ^* but also the partitions λ with almost all ones, i.e. $m_1(\lambda) > N - C_*$ for some large constant $C_* > 0$. This hints at a second phase transition in the behavior of A_N and the interplay structure among the i.i.d. inputs $\{X_k\}_{k \geqslant 1}$ at $\gamma = 2q$.

At the secondary criticality where $\mathbb{P}(|X_k| > t) \asymp \exp(-2qt)$, a blend of the two aforementioned scenarios influences the $2q$-th moment of A_N. The dominating terms now appear randomly and depend on the value of $|R_1| = |X_1|$. Roughly speaking, on the event that $|R_1|$ is close to a fixed $x_1 \in [0, N]$, the primary contribution to $\mathbb{E}[|A_N|^{2q}]$ stems from the $a(\lambda)$ with $m_1(\lambda)$ close to x_1. After conditioning on R_1 and fixing $m_1(\lambda)$, the rest terms can be described in terms of partitions without ones and will behave similarly as in the (q-UNIV) case.

3.2.2 Main ideas of the proofs

In the following, we delve into further details of the different cases described by our main results, and illustrate the main ideas of the proofs.

(SE) phase. The orders of the moments $\mathbb{E}[|X_k|^{2qm_k}]$ are given by (3.16). Using (3.15) and that $0 < p < 1$, one observes that the quantity (3.14) with $\lambda = \lambda^* = (1, \ldots, 1)$ (i.e., $m_1(\lambda^*) = N$) is of an order larger than that with any other $\lambda \in \mathcal{P}_N$ as $N \to \infty$. This can be directly quantified by Minkowski's inequality for $q > 1/2$ and concavity for $q \leqslant 1/2$.

(q-UNIV) phase, universality. We adapt the proof of [SZ22] to derive the proposed universality result of (3.3), utilizing also the robustness of the multiplicative chaos approach in computing low moments of random multiplicative functions in [Har20], together with several new observations and technical improvements. The first observation is that low moments of A_N concentrate around partitions such that no part is too small, similar to the complex Gaussian model. Using our assumption $\mathbb{E}[e^{\gamma|R_k|}] < \infty$ and Markov's inequality, we have $\mathbb{E}[|X_k|^{2qm_k}] \leqslant \Gamma(2qm_k+1)\gamma^{-2qm_k}$. Inserting in (3.14), it follows from (3.15) that

$$\mathbb{E}[|a(\lambda)|^{2q}] \leqslant \prod_{k\geqslant 1} \frac{C(2q/\gamma)^{2qm_k} m_k^{1/2-q}}{k^{qm_k}}. \tag{3.17}$$

Since $\gamma > 2q$, we expect that the contribution from $a(\lambda)$ is small unless m_k is in general very small (for instance, $\lambda = (N)$). Indeed, Proposition 3.3.2 below states that it suffices to consider only partitions λ with $\lambda_1 \geqslant \sqrt{N}/C(q)$ for some large constant $C(q)$.

Next, recalling that A_N can be viewed as a "Fourier coefficient" of the holomorphic multiplicative chaos, we may apply Parseval's identity to connect the sum over λ to the moments of the *total mass* of a (finite) multiplicative chaos, which is of the form

$$\mathbb{E}\left[\left(\int_{-\pi}^{\pi} |F_K(re^{i\theta})|^2 \mathrm{d}\theta\right)^q\right], \tag{3.18}$$

where $K \in \mathbb{N}$, $F_K(z) := \exp\left(\sum_{k=1}^K \frac{X_k}{\sqrt{k}} z^k\right)$, and $r \in [e^{-1/K}, e^{1/K}]$. However, a technical difficulty arises: since we only assumed $(2q+\varepsilon)$-th exponential moment exists, certain Laplace functionals of $X_k/\sqrt{k}$ may not be well-defined for a small k. Therefore, it is necessary to eliminate the dependence on X_k for small k. Fortunately, this is possible by the rotational invariance of X_k and using $\gamma > 2q$. Indeed, we show

in Section 3.3.1 that it suffices to study the *truncated* secular coefficients

$$A_{N,M_*} := \sum_{\substack{\lambda\in\mathcal{P}_N \\ m_1=\cdots=m_{M_*-1}=0}} a(\lambda) = \sum_{\substack{\lambda\in\mathcal{P}_N \\ m_1=\cdots=m_{M_*-1}=0}} \prod_{k\geqslant M_*} \left(\frac{X_k}{\sqrt{k}}\right)^{m_k} \frac{1}{m_k!} \tag{3.19}$$

of the *truncated* chaos

$$F_{K,M_*}(z) := \exp\Big(\sum_{k=M_*}^{K} \frac{X_k}{\sqrt{k}} z^k\Big). \tag{3.20}$$

Here, M_* is a large constant that depends only on q and the law of X_1. After truncation at M_*, we can apply Parseval's identity to convert the moments of A_{N,M_*} to that of the total mass of $|F_{N,M_*}|^2$ over the (scaled) unit circle; see Propositions 3.3.2 and 3.3.10 below.

The non-trivial part of (3.3) is that the low moments of A_N (and hence of the total mass of $|F_{N,M_*}|^2$ by the aforementioned connection) are asymptotically smaller than those predicted by the second moment (a.k.a. better-than-square-root cancellation). The fundamental reason is that on certain unlikely events, some values of $|F_{N,M_*}|^2$ become exceptionally large and dominate the second moment. To intuitively see this, let us consider the following random field

$$G(z) := \log F_{N,M_*}(z) = \sum_{k=M_*}^{N} \frac{X_k}{\sqrt{k}} z^k, \ z\in\mathbb{T}, \tag{3.21}$$

where $\mathbb{T}$ denotes the unit circle, together with a *branching random walk* (BRW) analogue which also guides the intuition in (EXP) $\gamma = 2q$ phase.[2] For more information on the analogy between BRW and other log-correlated fields, see Arguin's survey [Arg16] and references therein.

[2] A BRW process with d offsprings, L generations, and standard Gaussian increments is a Gaussian process on a d-ary tree of depth L, assigning i.i.d. standard Gaussian variables to each edge of the tree. The value assigned to each leaf is then the sum of independent Gaussian variables along the shortest path from the root to that leaf.

Since the variance of each summand in (3.21) is of size $\asymp 1/k$, it is natural to consider the partial sum over an exponentially growing interval, i.e. $\sum_{k\in[e^{n-1},e^n)} X_k z^k/\sqrt{k}$ and rewrite the field as

$$G(z) \approx \sum_{n=\lfloor \log M_* \rfloor}^{\lfloor \log N \rfloor} Y_n(z) := \sum_{n=\lfloor \log M_* \rfloor}^{\lfloor \log N \rfloor} \left(\sum_{k=e^{n-1}}^{e^n-1} \frac{X_k}{\sqrt{k}} z^k \right), \ z \in \mathbb{T}. \tag{3.22}$$

The first observation is that for each $z \in \mathbb{T}$, the distribution of each *increment* $Y_n(z)$ approaches a standard complex Gaussian as n increases. Next, if we consider two points $z, z' \in \mathbb{T}$ with $|z - z'|$ close to e^{-m}, then for $n \leqslant m$, one expects $Y_n(z)$ to strongly correlate with $Y_n(z')$. On the other hand, for n much larger than m, the increments $Y_n(z)$ and $Y_n(z')$ should be sufficiently decorrelated and are asymptotically independent as $n \to \infty$. From these observations, the random field $G(z)$ on $\mathbb{T}$ can be viewed as a BRW with "e offsprings" and $\lfloor \log N \rfloor$ generations, with standard complex Gaussian increments $Y_n(z)$ at each step. The latest common ancestor of $z, z' \in \mathbb{T}$ is of generation $-\log|z - z'|$. In other words, the values $G(z)$ and $G(z')$ share a common part of

$$\sum_{n=\lfloor \log M_* \rfloor}^{-\log|z-z'|} Y_n(z) \approx \sum_{n=\lfloor \log M_* \rfloor}^{-\log|z-z'|} Y_n(z') \tag{3.23}$$

and the rest are approximately independent. In particular, if $|z - z'| = o(1/N)$, $G(z)$ is almost indistinguishable from $G(z')$.

Consequently, an exceptionally large value of (3.23) will affect a spectrum of z of length $\asymp e^{-m}$, leading to an exceptionally large second moment. To circumvent such an issue, the works of [Har20] and [SZ22] borrowed a barrier argument from BRW (see e.g. [Shi15]), using a ballot event to discard the (unlikely) large values of (3.23). Their first step is to view the term $|F_{N,M_*}|^2$ as a Girsanov-type change of measure,

under which Y_n has a non-zero expectation μ_n. Next, they set up the barrier event

$$\mathcal{G}(A;N) := \left\{ \forall \log M_* < n \leqslant \log N,\ \forall z \in \mathbb{T},\ \sum_{j=\lfloor \log M_* \rfloor}^{n} (Y_j(z) - \mu_j) \leqslant A \right\}$$

and show that $\mathcal{G}(A;N)^c$ holds with probability exponentially decaying in A. On the event $\mathcal{G}(A;N)$, one writes

$$\mathbb{E}[1_{\mathcal{G}(A;N)} |F_{N,M_*}|^2] = \mathbb{E}[|F_{N,M_*}|^2]\, \mathbb{Q}(\mathcal{G}(A;N)),$$

where $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \asymp |F_{N,M_*}|^2$ and the ballot-type probability $\mathbb{Q}(\mathcal{G}(A;N))$ accounts for the $(\log N)^{-q/2}$ correction term arising in (3.4).

The technical difficulty in carrying over the above arguments to the non-Gaussian case is twofold. First, we need to replace the precise computation of $\mathbb{E}[|F_{N,M_*}|^2]$ in the Gaussian case with a slightly more involved asymptotic computation using Taylor's expansion. This will be detailed in Section 3.7. Second, it is harder to quantify the dependence structure within $G(z)$ and $G(z')$ for $z, z' \in \mathbb{T}$ as discussed near (3.23). To overcome this issue, we apply the slicing argument of [Har20] and a two-dimensional Berry-Esseen estimate to analyze quantitatively the dependence structure via normal approximation. However, Harper's original proof relies on the double-exponential growth of the number of summands defining Y_n, which leads to handy convergence results. This approach unfortunately does not suffice as our Y_n only consists of an exponential number of summands. Instead, we apply yet another change of measure to re-center the vector $(Y_n(z), Y_n(z'))$ for each pair of (z, z'), which guarantees a good enough quantitative approximation.

(EXP) phase with $\gamma < 2q$. We compute directly that

$$\mathbb{E}[|X_k|^{2qm_k}] = 2\int_{c_\gamma}^{\infty} u^{2qm_k} \gamma e^{-\gamma(u-c_\gamma)} \mathrm{d}u \asymp \gamma^{-2qm_k} \Gamma(2qm_k + 1), \tag{3.24}$$

which amounts to (using (3.15))

$$\mathbb{E}[|a(\lambda)|^{2q}] \asymp \prod_{k\geqslant 1} \frac{\Gamma(2qm_k+1)}{\gamma^{2qm_k} k^{qm_k} (m_k!)^{2q}} \asymp \prod_{k\geqslant 1} m_k^{1/2-q} \Big(\frac{2q}{\gamma\sqrt{k}}\Big)^{2qm_k}.$$

Since $\gamma < 2q$, large values of m_k are favorable. In other words, partitions $\lambda \in \mathcal{P}_N$ with $m_1(\lambda)$ close to N dominate. On the other hand, for those λ, the magnitudes of $\mathbb{E}[|a(\lambda)|^{2q}]$ are comparable (contrary to the case of (SE)), and hence a naïve Minkowski's inequality argument (and concavity when $q < 1/2$) similar to the (SE) phase suffices for the upper bound but does not conclude the lower bound.

Observe that for the all-one partition λ^*, $a(\lambda^*) = X_1^N/N!$. It follows that

$$\mathbb{E}[|a(\lambda^*)|^{2q}] = \mathbb{E}\bigg[\Big|\frac{X_1^N}{N!}\Big|^{2q}\bigg] \asymp \frac{1}{(N!)^{2q}} \int_0^\infty x^{2qN} e^{-\gamma x} \mathrm{d}x. \tag{3.25}$$

The contribution to the latter integral stems mainly from the range where x is close to $2qN/\gamma$. This motivates us to restrict the expectation $\mathbb{E}[|A_N|^{2q}]$ to the event $|R_1| = |X_1| \approx 2qN/\gamma$. By restricting to such an event, we gain better control of the dominance of $\lambda \in \mathcal{P}_N$ with $m_1(\lambda)$ close to N.

We first use Minkowski's inequality (and concavity when $q < 1/2$) to exclude the partitions $\lambda \in \mathcal{P}_N$ with $m_1(\lambda) \leqslant N - C_*$ for some large constant $C_* > 0$. For λ such that $m_1(\lambda) > N - C_*$, we may extract a common large power of $|X_1|$, that of $|X_1|^{N-C_*}$. Let us restrict to the event $|X_1| = 2qN/\gamma + O(\sqrt{N})$. For the rest power of $|X_1|^{m_1(\lambda)-(N-C_*)}$, we may substitute $|X_1|$ with $2qN/\gamma$ with a negligible error term. This leads to tractable control on

$$\mathbb{E}\bigg[\Big||X_1|^{N-C_*} \sum_{\substack{\lambda\in\mathcal{P}_N \\ m_1(\lambda)\geqslant N-C_*}} \widetilde{a}(\lambda)\Big|^{2q} 1_{\{|X_1|\approx 2qN/\gamma\}}\bigg], \tag{3.26}$$

where

$$\widetilde{a}(\lambda) := \frac{e^{i\tau_1 m_1} |2qN/\gamma|^{m_1-(N-C_*)}}{m_1!} \prod_{k\geqslant 2} \Big(\frac{X_k}{\sqrt{k}}\Big)^{m_k} \frac{1}{m_k!}.$$

The finite sum in (3.26), which is independent of R_1, can be separated from the expectation and approximated in L^2 (for C_* large) by an infinite series that one can offer a lower bound in probability.

The above arguments can also be adapted to the case of real inputs $\{X_k\}_{k\geqslant 1}$; see Remark 5.

(EXP) phase with $\gamma = 2q$. A similar computation as in the case $\gamma < 2q$ using (3.24) yields

$$\mathbb{E}[|a(\lambda)|^{2q}] \asymp \prod_{k\geqslant 1} \frac{\Gamma(2qm_k+1)}{\gamma^{2qm_k} k^{qm_k} (m_k!)^{2q}} \asymp \prod_{k\geqslant 1} m_k^{1/2-q} k^{-qm_k}.$$

Compared to the above phases, we observe a new phenomenon here: the main contribution in (3.13) stems randomly from a spectrum of λ in a different way from the other cases. The dominating term cannot be described deterministically and is governed by the random variable $|R_1| = |X_1|$. For $x_1 \in [0, N]$, conditioned on $|R_1| = x_1$, we will show that the sum in (3.13) contributes mostly when $m_1(\lambda)$ is close to x_1. The contributions from different values of $|R_1|$ can be roughly described by a slowly varying function in $|R_1|$. If both values of $|R_1|$ and m_1 are fixed, the situation is close to (q-UNIV): among partitions of $N - m_1$ without ones, those with a large component dominate.

We sketch the ideas for the lower bound of (3.6), which is the harder part. For $m \in [N/6, N/3] \cap \mathbb{Z}$, we consider the disjoint events that $|R_1| \in [m, m+1)$. On such an event, we show that the main contribution to

$$\mathbb{E}\left[\Big| \sum_{\lambda\in\mathcal{P}_N} \prod_{k\geqslant 1} \left(\frac{X_k}{\sqrt{k}}\right)^{m_k} \frac{1}{m_k!} \Big|^{2q} 1_{\{|X_1|\in[m,m+1)\}} \right] \tag{3.27}$$

stems from the sum over λ such that $m_1(\lambda) = m + O(N^{9/10})$, by arguing similarly as

(3.25). Conditioning on $|R_1| = x_1$, such quantity can be further rewritten into

$$\mathbb{E}\left[\Big| \sum_{\substack{\lambda\in\mathcal{P}_N \\ |m_1(\lambda)-m|\leqslant N^{9/10}}} u(m_1)\prod_{k\geqslant 2}\left(\frac{X_k}{\sqrt{k}}\right)^{m_k}\frac{1}{m_k!}\Big|^{2q}\right], \tag{3.28}$$

where

$$u(j) = e^{ij\tau_1}\,\frac{|x_1|^{j-m}}{j!/m!}$$

and $x_1 \in [m, m+1)$. Note that we condition only on $|R_1|$, and τ_1 is uniformly distributed on $[-\pi, \pi]$ independent of anything else. The quantity (3.28) can be viewed as a generalization of $\mathbb{E}[|A_N|^{2q}]$ (which is essentially $u(j) \equiv 1$). To derive its asymptotic, we need to apply a finer version of the multiplicative chaos approach in the (q-UNIV) case. First, applying Parseval's identity allows us to reduce our problem into studying the asymptotic of the low moments of a *randomly weighted mass* of truncated chaos,

$$\mathbb{E}\left[\left(\int_{-\pi}^{\pi} |\widetilde{F}_{K,M_*,m}(re^{i\theta})|^2 \mathrm{d}\theta\right)^q\right], \tag{3.29}$$

where

$$\widetilde{F}_{K,M_*,m}(z) := F_{K,M_*}(z) \times \Big(\sum_{j:|j-m|\leqslant N^{9/10}} u(j) z^j\Big),$$

cf. (3.18) and (3.20). For r close enough to 1, the second term on the right-hand side with $z = re^{i\theta}$ can be seen as roughly the discrete Fourier transform of the function $j \mapsto r^m m^{j-m}\Gamma(m)/\Gamma(j)$ at frequency $\tau_1 + \theta$, where one expects that roughly,

$$\Big|\sum_{j:|j-m|\leqslant N^{9/10}} u(j)(re^{i\theta})^j\Big| \asymp \min\left\{\frac{r^m}{|\tau_1+\theta|}, \sqrt{N}r^m\right\}.$$

Therefore, it is natural to decompose the integral in (3.29) depending on the magnitude of $|\theta + \tau_1|$, and reduce the problem to estimating (uniformly) the low moments

of the *partial* mass of the truncated chaos,

$$\mathbb{E}\bigg[\Big(\int_{|\theta|\leqslant\frac{1}{K_*}}|F_{K,M_*}(re^{i\theta})|^2\mathrm{d}\theta\Big)^q\bigg],$$

where $K_* \ll \sqrt{N}$. This is central to the proof and will be the goal of Section 3.5.2. As an example, we briefly illustrate the ideas behind estimating

$$\mathbb{E}\bigg[\Big(\int_{|\theta|\leqslant\frac{1}{\sqrt{N}}}|F_{K,M_*}(re^{i\theta})|^2\mathrm{d}\theta\Big)^q\bigg], \tag{3.30}$$

using the aforementioned branching random walk analogue. The integration over $\{\theta : |\theta| \leqslant 1/\sqrt{N}\}$ can be translated as considering only those points $z' \in \mathbb{T}$ having the latest common ancestor with $z = 0$ of generation no earlier than $(\log N)/2$, which form a sub-tree of the whole family. This suggests that we should 1) consider a decomposition

$$F_{K,M_*}(z) = \overline{F}_{K,M_*}(z) \times \underline{F}_{K,M_*}(z) := \exp\Big(\sum_{k=M_*}^{\sqrt{N}}\frac{X_k}{\sqrt{k}}z^k\Big) \times \exp\Big(\sum_{k=\sqrt{N}}^{K}\frac{X_k}{\sqrt{k}}z^k\Big). \tag{3.31}$$

so that if $|\theta|, |\theta'| \leqslant 1/\sqrt{N}$, uniformly $\overline{F}_{K,M_*}(re^{i\theta}) \asymp \overline{F}_{K,M_*}(re^{i\theta'})$; and 2) the integration of $\underline{F}_{K,M_*}$ over $\{|\theta| \leqslant 1/\sqrt{N}\}$ should parallel that of F_{K,M_*} over $\mathbb{T}$, since each sub-tree of a branching random walk can be considered as independently a new BRW with fewer generations. In other words, the term $\overline{F}_{K,M_*}(re^{i\theta})$ shares roughly the same value (up to multiplicative constants that are bounded in probability) for all θ with $|\theta| \leqslant 1/\sqrt{N}$. This can be quantified using the generic chaining technique, which gives tail bounds of

$$\sup_{|\theta|\leqslant 1/\sqrt{N}}\Big|\sum_{k=M_*}^{\sqrt{N}}\frac{r^k}{\sqrt{k}}\Re(X_k e^{ik\theta} - X_k)\Big|$$

under a suitable change of measure, allowing us to pull the term $|\overline{F}_{K,M_*}(re^{i\theta})|^2$ out from the integral in (3.30). With a change of variable $\theta \mapsto \pi\sqrt{N}\theta$, the remaining term

$$\mathbb{E}\left[\left(\int_{|\theta|\leqslant\frac{1}{\sqrt{N}}} |\underline{F}_{K,M_*}(re^{i\theta})|^2 \mathrm{d}\theta\right)^q\right] \asymp N^{-q/2}\mathbb{E}\left[\left(\int_{-\pi}^{\pi} |\underline{F}_{K,M_*}(re^{i\theta/(\pi\sqrt{N})})|^2 \mathrm{d}\theta\right)^q\right]$$

shares the same log-correlated structure as the (q-UNIV) case, and can be analyzed in a similar way. For details we refer to Proposition 3.5.11, which establishes that roughly,

$$\mathbb{E}\left[\left(\int_{-\pi}^{\pi} |\widetilde{F}_{K,M_*,m}(re^{i\theta})|^2 \mathrm{d}\theta\right)^q\right] \asymp N^{q^2/2}\left(\frac{r^{2m}K}{1+(1-q)\sqrt{\log K}}\right)^q.$$

Moreover, restricting to the event $|X_1| \in [m, m+1)$ in (3.27) loses a factor of m^{-q}, which can be intuitively verified by the following property of the Gamma function:

$$\int_m^{m+1} e^{-x}x^m \mathrm{d}x \asymp m^{-1/2}\int_0^{\infty} e^{-x}x^m \mathrm{d}x.$$

Altogether, we obtain

$$\mathbb{E}[|A_N|^{2q}] \gg \sum_{m=N/6}^{N/3} \frac{m^{-q}m^{q^2/2}}{(1+(1-q)\sqrt{\log(N-m)})^q} \asymp \frac{N^{1-q+q^2/2}}{(1+(1-q)\sqrt{\log N})^q},$$

leading to the lower bound of (3.6).

The upper bound follows a similar approach: condition on $|R_1|$, construct (3.28) for all $m \in \mathbb{N}$, use the multiplicative chaos upper bound for the (q-UNIV) case, and finally apply Minkowski's inequality ($q \geqslant 1/2$) or concavity ($q < 1/2$).

3.3 The universality phase

3.3.1 Reducing the proof to Proposition 3.3.1

To prove the asymptotics for $\mathbb{E}[|A_N|^{2q}]$ under the (q-UNIV) condition, we first show that, given any constant $M_* > 0$, removing summands $a(\lambda)$ in A_N with $m_i(\lambda) = 0$

for all $i < M_*$ from (3.13) costs at most a constant factor depending only on the distribution of X_k (equivalently R_k), q, and M_*. This will be applied later with M_* picked in terms of the distribution of X_k and q only. The same idea will be recycled later a few times, for instance, for the (EXP) case with $\gamma = 2q$ in Section 3.5.2.

Recall (3.19). We use the following proposition on the low moments of A_{N,M_*} with a suitable M_* to deduce the asymptotics of low moments of A_N. We will prove the upper and lower bound parts of this proposition in Sections 3.3.3 and 3.3.4 respectively.

Proposition 3.3.1. *Fix an integer M_* larger than some constant depending only on the distribution of X_k. For any large N and any $q \in (0, 1]$, under (q-UNIV) we have*

$$\mathbb{E}[|A_{N,M_*}|^{2q}] \asymp \left(\frac{1}{1 + (1-q)\sqrt{\log N}}\right)^q,$$

where the implied constants may depend on the distribution of X_k (equivalently R_k), q, and M_*, but not on N.

Remark 2. The truncation parameter M_* is chosen so that every exponential moment regarding X_k used in the proof of Proposition 3.3.1 is finite for any $k \geqslant M_*$. It is possible to track down this threshold in terms of the constant k_0 defined in Lemma 3.7.2, but we omit the work here.

Deducing equation (3.3) *of Theorem 3.1.3 from Proposition 3.3.1.* We first establish the upper bound of $\mathbb{E}[|A_N|^{2q}]$. Define

$$\mathcal{P}^{<}_{N,M_*} := \left\{(m_i)_{1\leqslant i<M_*} : \forall 1 \leqslant i < M_*,\ m_i \in \{0, \cdots, N\} \text{ and } \sum_{1\leqslant i<M_*} i m_i < N\right\}. \tag{3.32}$$

For $\mathbf{m} \in \mathcal{P}^{<}_{N,M_*}$, we define

$$A_{\mathbf{m};N,M_*} := \sum_{\substack{\lambda\in\mathcal{P}_N \\ (m_1,\ldots,m_{M_*-1})=\mathbf{m}}} a(\lambda).$$

In particular, $A_{\mathbf{0};N,M_*} = A_{N,M_*}$. For $q \leqslant 1/2$, using concavity we have

$$\begin{aligned}\mathbb{E}[|A_N|^{2q}] &\leqslant \sum_{\mathbf{m}\in\mathcal{P}^<_{N,M_*}} \mathbb{E}[|A_{\mathbf{m};N,M_*}|^{2q}] + \sum_{\lambda:\lambda_1<M_*} \mathbb{E}[|a(\lambda)|^{2q}] \\ &\ll \sum_{\mathbf{m}\in\mathcal{P}^<_{N,M_*}} \mathbb{E}\left[\left|A_{N-\sum_{j=1}^{M_*-1} jm_j,M_*}\right|^{2q}\right] \prod_{k=1}^{M_*-1} \mathbb{E}\left[\left|\left(\frac{X_k}{\sqrt{k}}\right)^{m_k} \frac{1}{m_k!}\right|^{2q}\right] \\ &\quad + \sum_{\lambda:\lambda_1<M_*} \prod_{k=1}^{M_*-1} \mathbb{E}\left[\left|\left(\frac{X_k}{\sqrt{k}}\right)^{m_k} \frac{1}{m_k!}\right|^{2q}\right].\end{aligned}$$

By Proposition 3.3.1 and (3.17) and since M_* is fixed, the first sum is bounded by

$$\begin{aligned}&\sum_{\mathbf{m}\in\mathcal{P}^<_{N,M_*}} \mathbb{E}\left[\left|A_{N-\sum_{j=1}^{M_*-1} jm_j,M_*}\right|^{2q}\right] \prod_{k=1}^{M_*-1} \mathbb{E}\left[\left|\left(\frac{X_k}{\sqrt{k}}\right)^{m_k} \frac{1}{m_k!}\right|^{2q}\right] \\ &\ll \sum_{\mathbf{m}\in\mathcal{P}^<_{N,M_*}} \frac{1}{(1+(1-q)\sqrt{\log(N-\sum_{j=1}^{M_*-1} jm_j)})^q} \prod_{k=1}^{M_*-1} Cm_k^{1/2-q} \left(\frac{2q}{\gamma\sqrt{k}}\right)^{2qm_k} \\ &\ll \sum_{\mathbf{m}\in\mathcal{P}^<_{N,M_*}} \frac{1}{(1+(1-q)\sqrt{\log(N-\sum_{j=1}^{M_*-1} jm_j)})^q} \prod_{k=1}^{M_*-1} \left(\frac{1}{e^\delta\sqrt{k}}\right)^{2qm_k},\end{aligned}$$

where $\delta > 0$ depends on γ, q. Next, since M_* is fixed,

$$\begin{aligned}&\sum_{\lambda:\lambda_1<M_*} \prod_{k=1}^{M_*-1} \mathbb{E}\left[\left|\left(\frac{X_k}{\sqrt{k}}\right)^{m_k} \frac{1}{m_k!}\right|^{2q}\right] \\ &\ll \sum_{\lambda:\lambda_1<M_*} \prod_{k=1}^{M_*-1} Cm_k^{1/2-q} \left(\frac{2q}{\gamma\sqrt{k}}\right)^{2qm_k} \\ &\ll \sum_{\lambda:\lambda_1<M_*} \prod_{k=1}^{M_*-1} e^{-\delta m_k} \\ &\ll \frac{1}{N},\end{aligned}$$

where we used in the last step that $\sum_{k<M_*} m_k \geqslant N/M_*$ for $\lambda \in \mathcal{P}_N$ with $\lambda_1 < M_*$.

We thus arrive at

$$\mathbb{E}[|A_N|^{2q}] \ll \sum_{\mathbf{m}\in\mathcal{P}^{<}_{N,M_*}} \frac{1}{(1+(1-q)\sqrt{\log(N-\sum_{j=1}^{M_*-1} jm_j)})^q} \prod_{k=1}^{M_*-1}\left(\frac{1}{e^\delta\sqrt{k}}\right)^{2qm_k} + \frac{1}{N}.$$

Let us divide the above sum over $\mathbf{m}\in\mathcal{P}^{<}_{N,M_*}$ into two parts depending on whether $\sum_{j=1}^{M_*-1} jm_j \leqslant N/2$ or not. First,

$$\begin{aligned}
&\sum_{\substack{\mathbf{m}\in\mathcal{P}^{<}_{N,M_*}\\ \sum_{j=1}^{M_*-1} jm_j\leqslant N/2}} \frac{1}{(1+(1-q)\sqrt{\log(N-\sum_{j=1}^{M_*-1} jm_j)})^q} \prod_{k=1}^{M_*-1}\left(\frac{1}{e^\delta\sqrt{k}}\right)^{2qm_k}\\
&\ll \frac{1}{(1+(1-q)\sqrt{\log N})^q} \sum_{\substack{\mathbf{m}\in\mathcal{P}^{<}_{N,M_*}\\ \sum_{j=1}^{M_*-1} jm_j\leqslant N/2}} \prod_{k=1}^{M_*-1}\left(\frac{1}{e^\delta\sqrt{k}}\right)^{2qm_k}\\
&\leqslant \frac{1}{(1+(1-q)\sqrt{\log N})^q} \prod_{k=1}^{M_*-1}\sum_{m_k=0}^{\infty}\left(\frac{1}{e^\delta\sqrt{k}}\right)^{2qm_k}\\
&\ll \frac{1}{(1+(1-q)\sqrt{\log N})^q}.
\end{aligned}$$

Second, if $\sum_{j=1}^{M_*-1} jm_j > N/2$, then $\sum_{j=1}^{M_*-1} m_j > N/(2M_*)$. This implies

$$\begin{aligned}
&\sum_{\substack{\mathbf{m}\in\mathcal{P}^{<}_{N,M_*}\\ \sum_{j=1}^{M_*-1} jm_j> N/2}} \frac{1}{(1+(1-q)\sqrt{\log(N-\sum_{j=1}^{M_*-1} jm_j)})^q} \prod_{k=1}^{M_*-1}\left(\frac{1}{e^\delta\sqrt{k}}\right)^{2qm_k}\\
&\qquad\ll \sum_{\substack{\mathbf{m}\in\mathcal{P}^{<}_{N,M_*}\\ \sum_{j=1}^{M_*-1} m_j> N/(2M_*)}} \prod_{k=1}^{M_*-1}\left(\frac{1}{e^\delta\sqrt{k}}\right)^{2qm_k} \leqslant N^{M_*}\exp\left(-\frac{2\delta qN}{2M_*}\right) \ll \frac{1}{N}.
\end{aligned} \tag{3.33}$$

Altogether, these yield $\mathbb{E}[|A_N|^{2q}] \ll (1+(1-q)\sqrt{\log N})^{-q}$ for $q \leqslant 1/2$. The case $q\in(1/2,1]$ is similar by looking at $\mathbb{E}[|A_N|^{2q}]^{1/(2q)}$ and using Minkowski's inequality instead of concavity.

Next we show a matching lower bound on $\mathbb{E}[|A_N|^{2q}]$ by using that of $\mathbb{E}[|A_{N,M_*}|^{2q}]$. The first observation is that we can, with the cost of a constant factor depending on L, remove those $a(\lambda)$ such that $0 < m_1(\lambda) < 2^L$ for some large integer L to be determined later. Denote by $\mathcal{A}_k \subset \mathcal{P}_N$ the set of partitions of N such that $2^k | m_1(\lambda)$, for $0 \leqslant k < L$. Using $X_1 \overset{\mathrm{d}}{=} -X_1$, we get

$$A_N = \sum_{\lambda \in \mathcal{A}_0} a(\lambda) + \sum_{\lambda \in \mathcal{A}_0^c} a(\lambda) \overset{\mathrm{d}}{=} -\sum_{\lambda \in \mathcal{A}_0} a(\lambda) + \sum_{\lambda \in \mathcal{A}_0^c} a(\lambda),$$

and therefore,

$$\begin{aligned} \mathbb{E}[|A_N|^{2q}] &= \frac{1}{2}\left(\mathbb{E}\left[\Big| \sum_{\lambda \in \mathcal{A}_0} a(\lambda) + \sum_{\lambda \in \mathcal{A}_0^c} a(\lambda) \Big|^{2q} + \Big| -\sum_{\lambda \in \mathcal{A}_0} a(\lambda) + \sum_{\lambda \in \mathcal{A}_0^c} a(\lambda) \Big|^{2q} \right] \right) \\ &\geqslant \frac{1}{2}\mathbb{E}\left[\Big| \sum_{\lambda \in \mathcal{A}_0^c} a(\lambda) \Big|^{2q} \right] = \frac{1}{2}\mathbb{E}\left[\Big| \sum_{\lambda \in \mathcal{A}_1} a(\lambda) \Big|^{2q} \right], \end{aligned} \tag{3.34}$$

where we used that $\max(|w-z|, |w+z|) \leqslant |z|$ for complex numbers z, w. Further, to get rid of $\lambda \in \mathcal{A}_k^c \cap \mathcal{A}_{k-1}$, we use $X_1 \overset{\mathrm{d}}{=} e^{i\pi/2^k} X_1$ and apply the same argument. By induction, finally we have as claimed above

$$\mathbb{E}[|A_N|^{2q}] \geqslant 2^{-L}\mathbb{E}\left[\Big| \sum_{\lambda \in \mathcal{A}_L} a(\lambda) \Big|^{2q} \right].$$

Continuing this procedure for $m_2, \cdots, m_{M_*-1}$, we get

$$\mathbb{E}[|A_N|^{2q}] \geqslant 2^{-M_* L}\mathbb{E}\left[\Big| \sum_{\lambda \in \mathcal{B}_{M_*,L}} a(\lambda) \Big|^{2q} \right] =: 2^{-M_* L}\mathbb{E}[|\widehat{A}_N|^{2q}],$$

where

$$\mathcal{B}_{M_*,L} = \left\{ \lambda \in \mathcal{P}_N : \forall 1 \leqslant k < M_*,\, 2^L \mid m_k(\lambda) \right\}$$

which is a subset of

$$\left\{ \lambda \in \mathcal{P}_N : \forall 1 \leqslant k < M_*,\, m_k(\lambda) \neq 0 \implies m_k(\lambda) \geqslant 2^L \right\}.$$

Since we assume the unit variance condition $\mathbb{E}[|X_1|^2] = \mathbb{E}[|R_1|^2] = 1$, we may assume there is some $\varepsilon_0 \in (0,1)$ such that $\mathbb{P}(|R_1| < \varepsilon_0) > 0$ (otherwise $|R_1| \equiv 1$ being sub-Gaussian, Theorem 3.1.3 equation (3.3) follows from Proposition 3.3.1 with $M_* = 1$). The strategy is then to restrict to the event that $|X_j| = |R_j| < \varepsilon_0$ for each $1 \leqslant j \leqslant M_* - 1$, that is, with $\widehat{\mathbb{E}}$ denoting the conditional expectation on the event $\{|R_j| < \varepsilon_0,\ 1 \leqslant j \leqslant M_* - 1\}$,

$$\mathbb{E}[|\widehat{A}_N|^{2q}] \gg \mathbb{E}[|\widehat{A}_N|^{2q} \mid |R_j| < \varepsilon_0,\ 1 \leqslant j \leqslant M_* - 1] = \widehat{\mathbb{E}}[|\widehat{A}_N|^{2q}].$$

Using either concavity or conditional Minkowski's inequality, it then suffices to prove

$$\widehat{\mathbb{E}}\Bigg[\Bigg| \sum_{\substack{\lambda \in \mathcal{B}_{M_*,L} \\ \exists j \in [1,M_*),\ m_j \neq 0}} \prod_{k \geqslant 1} \left(\frac{X_k}{\sqrt{k}}\right)^{m_k} \frac{1}{m_k!} \Bigg|^{2q}\Bigg] \leqslant \frac{1}{C}\mathbb{E}[|A_{N,M_*}|^{2q}] \tag{3.35}$$

where $C > 0$ is a fixed large constant that depends only on M_*. Let $C_0 > 0$ be the implied constant in Proposition 3.3.1. Let $\mathcal{P}^{<,L}_{N,M_*}$ be the subset of $\mathcal{P}^{<}_{N,M_*}$ with $m_k(\lambda) \geqslant 2^L$ for all $1 \leqslant k < M_*$ such that $m_k(\lambda) \neq 0$. We have for $q \leqslant 1/2$ (and similarly using Minkowski's inequality for $q > 1/2$),

$$\begin{aligned}
&\widehat{\mathbb{E}}\Bigg[\Bigg| \sum_{\substack{\lambda \in \mathcal{B}_{M_*,L} \\ \exists j \in [1,M_*),\ m_j \neq 0}} \prod_{k \geqslant 1} \left(\frac{X_k}{\sqrt{k}}\right)^{m_k} \frac{1}{m_k!} \Bigg|^{2q}\Bigg] \\
&\leqslant \sum_{\substack{\mathbf{m} \in \mathcal{P}^{<,L}_{N,M_*} \\ \mathbf{m} \neq \mathbf{0}}} \widehat{\mathbb{E}}\Bigg[\Bigg| \sum_{\substack{\lambda \in \mathcal{P}_N \\ (m_1,\dots,m_{M_*-1}) = \mathbf{m}}} \prod_{k \geqslant 1} \left(\frac{X_k}{\sqrt{k}}\right)^{m_k} \frac{1}{m_k!} \Bigg|^{2q}\Bigg] \\
&= \sum_{\substack{\mathbf{m} \in \mathcal{P}^{<,L}_{N,M_*} \\ \mathbf{m} \neq \mathbf{0}}} \prod_{k=1}^{M_*-1} \frac{\widehat{\mathbb{E}}[|X_k|^{2qm_k}]}{k^{qm_k}(m_k!)^{2q}} \mathbb{E}\Bigg[\Bigg| \sum_{\substack{\lambda \in \mathcal{P}_N \\ (m_1,\dots,m_{M_*-1}) = \mathbf{m}}} \prod_{k \geqslant M_*} \left(\frac{X_k}{\sqrt{k}}\right)^{m_k} \frac{1}{m_k!} \Bigg|^{2q}\Bigg] \\
&\leqslant \sum_{\substack{\mathbf{m} \in \mathcal{P}^{<,L}_{N,M_*} \\ \mathbf{m} \neq \mathbf{0}}} \frac{C_0}{(1 + (1-q)\sqrt{\log(N - \sum_{j=1}^{M_*-1} j m_j)})^q} \prod_{k=1}^{M_*-1} \frac{\varepsilon_0^{2qm_k}}{k^{qm_k}}.
\end{aligned}$$

The sum over $\mathbf{m}$ with $\sum_{j=1}^{M_*-1} jm_j > N/2$ can be controlled similarly as in (3.33). The rest terms are bounded by

$$\sum_{\substack{\mathbf{m}\in\mathcal{P}^{<,L}_{N,M_*}\\ \mathbf{m}\neq\mathbf{0}\\ \sum_{j=1}^{M_*-1} jm_j\leqslant N/2}} \frac{C_0}{(1+(1-q)\sqrt{\log(N-\sum_{j=1}^{M_*-1} jm_j)})^q}\prod_{k=1}^{M_*-1}\frac{\varepsilon_0^{2qm_k}}{k^{qm_k}}$$

$$\leqslant \frac{2C_0}{(1+(1-q)\sqrt{\log N})^q}\sum_{\substack{\mathbf{m}\in\mathcal{P}^{<,L}_{N,M_*}\\ \mathbf{m}\neq\mathbf{0}}}\prod_{k=1}^{M_*-1}\varepsilon_0^{2qm_k}.$$

Note that

$$\sum_{\substack{\mathbf{m}\in\mathcal{P}^{<,L}_{N,M_*}\\ \mathbf{m}\neq\mathbf{0}}}\prod_{k=1}^{M_*-1}\varepsilon_0^{2qm_k}\leqslant\prod_{k=1}^{M_*-1}\Big(1+\sum_{m_k=2^L}^{\infty}\varepsilon_0^{2qm_k}\Big)-1\ll\frac{M_*\varepsilon_0^{2^{L+1}q}}{1-\varepsilon_0^{2q}}.$$

By picking L large enough, we obtain

$$\mathbb{E}\Bigg[\Big|\sum_{\substack{\lambda\in\mathcal{B}_{M_*,L}\\ \exists j\in[1,M_*),\ m_j\neq 0}}\prod_{k\geqslant 1}\Big(\frac{X_k}{\sqrt{k}}\Big)^{m_k}\frac{1}{m_k!}\Big|^{2q}\ \Big|\ |R_j|<\varepsilon_0,\ 1\leqslant j\leqslant M_*-1\Bigg]$$

$$\leqslant\frac{1}{C_0C(1+(1-q)\sqrt{\log N})^q}$$

$$\leqslant\frac{1}{C}\mathbb{E}[|A_{N,M_*}|^{2q}],$$

thus proving (3.35) and hence the desired lower bound. □

3.3.2 Setting the stage for proving Proposition 3.3.1

We now proceed to the proof of Proposition 3.3.1. In this short section, we prepare ourselves with a few notations. First, motivated by our discussions on page 45, we define two probability measures as follows.

For any K, M, r satisfying $\log M_* \leqslant M < \log K$ and $e^{-1/K} \leqslant r \leqslant e^{1/K}$, define the measure $\mathbb{Q}^{(1)}_{r,M,K}$ by

$$\frac{\mathrm{d}\mathbb{Q}^{(1)}_{r,M,K}}{\mathrm{d}\mathbb{P}} := \frac{\exp(2\sum_{k=e^M}^{K-1} \frac{r^k}{\sqrt{k}} R_k \cos(\tau_k))}{\mathbb{E}\left[\exp(2\sum_{k=e^M}^{K-1} \frac{r^k}{\sqrt{k}} R_k \cos(\tau_k))\right]}. \tag{3.36}$$

In addition, let K_r be such that $\log K_r$ is the largest integer with $K_r \leqslant \min\{\frac{-1}{4\log r}, K\}$, and $M = M(r,\theta)$ be the smallest integer such that $e^M \geqslant \min\{10^3/|\theta|, K_r/e, M_*\}$. For $e^{-1/K} \leqslant r \leqslant e^{1/K}$ and $\theta \in [-\pi, \pi)$, define the measure $\mathbb{Q}^{(2)}_{r,M,K,\theta}$ by

$$\frac{\mathrm{d}\mathbb{Q}^{(2)}_{r,M,K,\theta}}{\mathrm{d}\mathbb{P}} := \frac{\exp(2\sum_{m=M+1}^{\log K_r}(Z_0(m) + Z_\theta(m)))}{\mathbb{E}[\exp(2\sum_{m=M+1}^{\log K_r}(Z_0(m) + Z_\theta(m)))]}, \tag{3.37}$$

where for any $M < m \leqslant \log K_r$ and $\theta \in [-\pi, \pi)$,

$$Z_\theta(m) := \Re \sum_{e^{m-1} \leqslant k < e^m} \frac{X_k r^k e^{ik\theta}}{\sqrt{k}} = \sum_{e^{m-1} \leqslant k < e^m} \frac{r^k}{\sqrt{k}} R_k \cos(\tau_k + k\theta). \tag{3.38}$$

When the values of r, M, K, θ are clear from the context, we will drop the subscripts and write instead $\mathbb{Q}^{(1)}$ and $\mathbb{Q}^{(2)}$.

To further understand the sums Y_n defined in (3.22) under the new measures, we define and compute using Lemma 3.7.4 that

$$\mu_k := \mathbb{E}^{\mathbb{Q}^{(1)}}\left[\frac{r^k}{\sqrt{k}} R_k \cos(\tau_k)\right] = \frac{r^{2k}}{k} + O(k^{-3/2}) \tag{3.39}$$

and

$$\nu_k = \nu_k(\theta) := \mathbb{E}^{\mathbb{Q}^{(2)}}\left[\frac{r^k}{\sqrt{k}} R_k \cos(\tau_k)\right] = \frac{r^{2k}}{k} + \frac{\cos(k\theta) r^{2k}}{k} + O(k^{-3/2}). \tag{3.40}$$

We also refresh ourselves with the truncated multiplicative chaos. Recall (3.19) and (3.20). It holds that

$$A_{N,M_*} = [z^N] \exp\Big(\sum_{k=M_*}^{\infty} \frac{X_k}{\sqrt{k}} z^k\Big).$$

Moreover, inserting $K = N/2$ into (3.19), we have

$$F_{N/2,M_*}(z) = \exp\bigg(\sum_{k=M_*}^{N/2} \frac{X_k}{\sqrt{k}} z^k\bigg) = \sum_{n=0}^{\infty} \bigg(\sum_{\substack{\lambda\in\mathcal{P}_n \\ \lambda_1\leqslant N/2 \\ \forall 1\leqslant k<M_*,\, m_k(\lambda)=0}} a(\lambda)\bigg) z^n. \tag{3.41}$$

In the next two subsections, we establish the upper and lower bounds for Proposition 3.3.1 respectively.

3.3.3 Upper bound of Proposition 3.3.1

By Hölder's inequality, it suffices to prove the upper bound for $1/2 \leqslant q \leqslant 1$. Proposition 3.3.1 then follows from the two propositions below.

Proposition 3.3.2. *Suppose that $\mathbb{E}[e^{\gamma|R_1|}] < \infty$ for some $\gamma > 2q$. For $1/2 \leqslant q \leqslant 1$ and N large enough, we have for some $C(q) > 0$,*

$$\mathbb{E}[|A_{N,M_*}|^{2q}]^{1/(2q)} \ll \frac{1}{\sqrt{N}} \sum_{j=1}^{J} \mathbb{E}\bigg[\bigg(\frac{1}{2\pi}\int_{-\pi}^{\pi} |F_{N/2^j,M_*}(\exp(j/N + i\theta))|^2 \mathrm{d}\theta\bigg)^q\bigg]^{1/(2q)} + \frac{1}{N},$$

where $J = \lceil \log(C(q)\sqrt{N})/\log 2 \rceil$.

Proof. The main idea resembles that of [SZ22, Proposition 3.1]: separate the sum over $\lambda \in \mathcal{P}_N$ according to the values of λ_1, control the contributions from partitions λ with a small λ_1 (and hence some $m_k(\lambda)$ is large), and apply Parseval's identity to the rest λ.

Denote by $\mathcal{P}^0_{N,M_*}$ the subset of $\mathcal{P}_N$ satisfying $m_1(\lambda) = \cdots = m_{M_*-1}(\lambda) = 0$. Recall (3.12) and (3.13). Suppose that $\mathbb{E}[e^{\gamma|R_1|}] < \infty$ for some $\gamma > 2q$. Then $\mathbb{E}[|R_1|^\ell] \ll \gamma^{-\ell}\Gamma(\ell+1)$ for $\ell \geqslant 0$. Therefore,

$$\mathbb{E}[|a(\lambda)|^{2q}] = \prod_{k:m_k>0} \frac{\mathbb{E}[|R_k|^{2qm_k}]}{(m_k!)^{2q} k^{qm_k}}$$

$$\ll \prod_{k:m_k>0} \frac{C\gamma^{-2qm_k}\Gamma(2qm_k+1)}{(m_k!)^{2q}k^{qm_k}}$$

$$\ll \prod_{k:m_k>0} \frac{C(2q/\gamma)^{2qm_k}}{k^{qm_k}}, \tag{3.42}$$

where we have used (3.15). By Minkowski's inequality,

$$\begin{aligned}
\mathbb{E}\Bigg[\Big|\sum_{\substack{\lambda\in\mathcal{P}^0_{N,M_*}\\ \lambda_1\leqslant N/2^J}} a(\lambda)\Big|^{2q}\Bigg]^{1/(2q)} &= \mathbb{E}\Bigg[\Big|\sum_{\substack{\lambda\in\mathcal{P}^0_{N,M_*}\\ \lambda_1\leqslant \sqrt{N}/C(q)}} a(\lambda)\Big|^{2q}\Bigg]^{1/(2q)} \\
&\leqslant \sum_{\substack{\lambda\in\mathcal{P}^0_{N,M_*}\\ \lambda_1\leqslant \sqrt{N}/C(q)}} \mathbb{E}[|a(\lambda)|^{2q}]^{1/(2q)} \\
&\leqslant \sum_{\substack{\lambda\in\mathcal{P}^0_{N,M_*}\\ \lambda_1\leqslant \sqrt{N}/C(q)}} \prod_{k:m_k>0} C\left(\frac{2q}{\gamma}\right)^{m_k} \\
&\leqslant p_N C^{\sqrt{N}/C(q)} \left(\frac{2q}{\gamma}\right)^{\sum_{k=M_*}^{\sqrt{N}/C(q)} m_k},
\end{aligned} \tag{3.43}$$

where p_N is the number of partitions of N. Note that for partitions $\lambda \in \mathcal{P}^0_{N,M_*}$ with the largest component at most $\sqrt{N}/C(q)$, we have

$$\sum_{k=M_*}^{\sqrt{N}/C(q)} m_k \geqslant \frac{C(q)}{\sqrt{N}} \sum_{k=M_*}^{\sqrt{N}/C(q)} km_k = C(q)\sqrt{N}. \tag{3.44}$$

Inserting back to (3.43), together with the bound $p_N \ll e^{\pi\sqrt{2/3}\sqrt{N}}$ (see [And98]), we have

$$\mathbb{E}\Bigg[\Big|\sum_{\substack{\lambda\in\mathcal{P}^0_{N,M_*}\\ \lambda_1\leqslant N/2^J}} a(\lambda)\Big|^{2q}\Bigg]^{1/(2q)} \ll \left(\frac{2q}{\gamma}\right)^{C(q)\sqrt{N}/2} \ll \frac{1}{N},$$

where we take $C(q)$ large enough in terms of the fixed constant C and q, γ. The rest of the arguments follow similarly as in Section 4 of [SZ22], which we sketch below for

completeness. By Minkowski's inequality,

$$\mathbb{E}[|A_{N,M_*}|^{2q}]^{1/(2q)} \ll \sum_{j=1}^{J} \mathbb{E}\Bigg[\Big|\sum_{\substack{\lambda\in\mathcal{P}^0_{N,M_*}\\ N/2^j<\lambda_1\leqslant N/2^{j-1}}} a(\lambda)\Big|^{2q}\Bigg]^{1/(2q)} + \frac{1}{N}. \tag{3.45}$$

For a fixed $j \in \{1,\dots,J\}$, we may decompose

$$\sum_{\substack{\lambda\in\mathcal{P}^0_{N,M_*}\\ N/2^j<\lambda_1\leqslant N/2^{j-1}}} a(\lambda) = \sum_{\substack{\rho,\sigma\\ |\rho|+|\sigma|=N\\ |\rho|>0}} a(\rho)a(\sigma),$$

where the parts of ρ lie in $(N/2^j, N/2^{j-1}]$ and of σ lie in $[M_*, N/2^j]$ (here and later, we keep such constraints in the sums over ρ or σ). Let $\mathbb{E}_j$ denote the expectation in $\{X_k\}_{N/2^j<k\leqslant N/2^{j-1}}$. Note that if $\lambda \neq \lambda'$ are distinct partitions, then $\mathbb{E}[a(\lambda)\overline{a(\lambda')}] = 0$ by independence of $\{X_k\}_{k\geqslant 1}$. Therefore, by applying Jensen's inequality and expanding the square, we obtain

$$\mathbb{E}_j\Bigg[\Big|\sum_{\substack{\lambda\in\mathcal{P}^0_{N,M_*}\\ N/2^j<\lambda_1\leqslant N/2^{j-1}}} a(\lambda)\Big|^{2q}\Bigg]^{1/q} \leqslant \sum_{N/2^j<n\leqslant N} \Big|\sum_{\sigma\in\mathcal{P}^0_{N-n,M_*}} a(\sigma)\Big|^2 \sum_{\rho\in\mathcal{P}_n} \mathbb{E}_j\Big[|a(\rho)|^2\Big]. \tag{3.46}$$

For a partition ρ whose parts lie in $(N/2^j, N/2^{j-1}]$, we have

$$\mathbb{E}_j\Big[|a(\rho)|^2\Big] \leqslant \prod_{N/2^j<k\leqslant N/2^{j-1}} \frac{C\gamma^{-2m_k}(2m_k)!}{(m_k!)^2 k^{m_k}} \leqslant \Big(\frac{C2^j}{N}\Big)^r,$$

where r is the number of parts in ρ. Using $r \leqslant 2^j \leqslant C(q)^2 N/2^j$ for $1 \leqslant j \leqslant J$, it follows that

$$\begin{aligned}\sum_{\rho\in\mathcal{P}_n} \mathbb{E}_j\Big[|a(\rho)|^2\Big] &\leqslant \sum_{2^{j-1}n/N\leqslant r<2^j n/N} \Big(\frac{C2^j}{N}\Big)^r \binom{\lfloor N/2^j\rfloor + r}{r-1}\\ &\leqslant \frac{2^j}{N} \sum_{2^{j-1}n/N\leqslant r<2^j n/N} \frac{C^{r-1}}{(r-1)!} \ll \frac{1}{N}\exp\Big(\frac{2j(N-n)}{N}\Big).\end{aligned}$$

Inserting in (3.46) leads to

$$\mathbb{E}_j\Big[\Big|\sum_{\substack{\lambda\in\mathcal{P}^0_{N,M_*}\\ N/2^j<\lambda_1\leqslant N/2^{j-1}}} a(\lambda)\Big|^{2q}\Big]^{1/q} \ll \frac{1}{N}\sum_{N/2^j<n\leqslant N}\Big|\sum_{\sigma\in\mathcal{P}^0_{N-n,M_*}} a(\sigma)\Big|^2 \exp\Big(\frac{2j(N-n)}{N}\Big)$$

$$\ll \frac{1}{2\pi N}\int_{-\pi}^{\pi}\Big|F_{N/2^j,M_*}\Big(\exp\Big(\frac{j}{N}+i\theta\Big)\Big)\Big|^2\mathrm{d}\theta, \tag{3.47}$$

where the last step follows from Parseval's identity applied to

$$F_{N/2^j,M_*}(z)=\sum_r\Big(\sum_{\sigma\in\mathcal{P}^0_{r,M_*}} a(\sigma)\Big)z^r.$$

Taking expectations on the q-th powers of both sides of (3.47) completes the proof.

□

Proposition 3.3.3. *Fix any $q \in (0,1]$. Suppose that $\mathbb{E}[e^{\gamma|R_1|}] < \infty$ for some $\gamma > 2q$.*[3] *For any K sufficiently large (in terms of M_*) and $1 \leqslant r \leqslant e^{1/K}$, we have*

$$\mathbb{E}\left[\left(\int_{-\pi}^{\pi} |F_{K,M_*}(re^{i\theta})|^2 \mathrm{d}\theta\right)^q\right] \ll \left(\frac{K}{1+(1-q)\sqrt{\log K}}\right)^q, \tag{3.48}$$

where the implied constant depends on q, γ but is independent of K, r.

Inserting Proposition 3.3.3 into Proposition 3.3.2 then yields the upper bound of Proposition 3.3.1. To set up the proof of Proposition 3.3.3 we need the following two-sided ballot estimate for a centered Gaussian random walk, which is due to [Har20].

Lemma 3.3.4. *There is a large universal constant $L_1 > 0$ such that the following holds. Consider a sequence of independent centered Gaussian random variables $\{G_n\}_{n\in\mathbb{N}}$ with variances between $1/20$ and 20. Then uniformly for any functions $h(m)$, $g(m)$ satisfying $|h(m)| \leqslant 10\log m$ and $g(m) \leqslant -L_1 m$, and for a, n large enough,*

$$\mathbb{P}\left(\forall 1 \leqslant m \leqslant n,\ g(m) \leqslant \sum_{j=1}^{m} G_j \leqslant \min\{a, L_1 m\} + h(m)\right) \asymp \min\left\{1, \frac{a}{\sqrt{n}}\right\}.$$

The same conclusion holds if we replace $\min\{a, L_1 m\}$ above by a.

Proof. The first claim is [Har20, Probability Result 2]. For the second claim, the upper bound follows from [Har20, Probability Result 1], and the lower bound follows from the first claim. □

Definition 3.3.5. Fix a large universal constant $L_1 > 20$ as in Lemma 3.3.4. Let K be large enough (in terms of M_*) and $1 \leqslant r \leqslant e^{1/K}$, and suppose that $1 \leqslant A \leqslant \sqrt{\log K}$. Define the event $\mathcal{G}_r(A, \theta; K)$ as

$$\left\{\forall \log M_* \leqslant n \leqslant \log K,\ -A - L_1 n \leqslant \sum_{M_* \leqslant k < e^n} \left(\Re \frac{X_k r^k e^{ik\theta}}{\sqrt{k}} - \mu_k\right) \leqslant A + 10\log n\right\}.$$

[3] Strictly speaking, Proposition 3.3.3 does not rely on $\gamma > 2q$. Assuming only $\gamma > 0$, the same conclusion holds if M_* is large enough in terms of γ.

Define also the event $\mathcal{G}_r(A;K)$ that $\mathcal{G}_r(A,\theta;K)$ holds for all $\theta \in [-\pi,\pi)$.

Proposition 3.3.3 then follows from two propositions below.

Proposition 3.3.6. *For any K sufficiently large, $1 \leqslant r \leqslant e^{1/K}$, and $1 \leqslant A \leqslant \sqrt{\log K}$,*

$$\mathbb{P}(\mathcal{G}_r(A;K)^c) \ll \exp(-A).$$

Proposition 3.3.7. *For any K sufficiently large, $1 \leqslant r \leqslant e^{1/K}$, $\theta \in [-\pi,\pi)$, and $1 \leqslant A \leqslant \sqrt{\log K}$,*

$$\mathbb{E}[1_{\mathcal{G}_r(A,\theta;K)}|F_{K,M_*}(re^{i\theta})|^2] \ll \frac{AK}{\sqrt{\log K}},$$

and therefore

$$\mathbb{E}\left[1_{\mathcal{G}_r(A;K)}\int_{-\pi}^{\pi}|F_{K,M_*}(re^{i\theta})|^2\mathrm{d}\theta\right] \ll \frac{AK}{\sqrt{\log K}}.$$

Deducing Proposition 3.3.3 from Propositions 3.3.6 and 3.3.7. If $q = 1$, the upper bound of (3.48) is equivalent to K^q (up to constant) and the desired claim follows from Lemma 3.7.3. If $q \in (0,1)$, it suffices to prove

$$\mathbb{E}\left[\left(\int_{-\pi}^{\pi}|F_{K,M_*}(re^{i\theta})|^2\mathrm{d}\theta\right)^q\right] \ll \left(\frac{K}{\sqrt{\log K}}\right)^q.$$

Next, we partition the whole probability space into the events

$$\mathcal{G}_r(1;K),\ \mathcal{G}_r(2^J;K)^c,\ \text{and}\ \mathcal{G}_r(2^j;K)\backslash\mathcal{G}_r(2^{j-1};K),\ 1 \leqslant j \leqslant J := \lfloor \log\log K \rfloor.$$

Then we have by Proposition 3.3.7 that

$$\begin{aligned}&\mathbb{E}\left[1_{\mathcal{G}_r(1;K)}\left(\int_{-\pi}^{\pi}|F_{K,M_*}(re^{i\theta})|^2\mathrm{d}\theta\right)^q\right]\\ &\leqslant \left(\mathbb{E}\left[1_{\mathcal{G}_r(1;K)}\int_{-\pi}^{\pi}|F_{K,M_*}(re^{i\theta})|^2\mathrm{d}\theta\right]\right)^q\end{aligned}$$

$$\ll \left(\frac{K}{\sqrt{\log K}}\right)^q,$$

and by Hölder's inequality and Propositions 3.3.6 and 3.3.7,

$$\begin{aligned}
&\mathbb{E}\left[1_{\mathcal{G}_r(2^j;K)\setminus\mathcal{G}_r(2^{j-1};K)}\left(\int_{-\pi}^{\pi}|F_{K,M_*}(re^{i\theta})|^2\mathrm{d}\theta\right)^q\right] \\
&\leqslant (\mathbb{P}(\mathcal{G}_r(2^{j-1};K)^c))^{1-q}\left(\mathbb{E}\left[1_{\mathcal{G}_r(2^j;K)}\int_{-\pi}^{\pi}|F_{K,M_*}(re^{i\theta})|^2\mathrm{d}\theta\right]\right)^q \\
&\ll \left(\frac{K}{\sqrt{\log K}}\right)^q 2^{jq}\exp(-(1-q)2^{j-1}).
\end{aligned}$$

Note that

$$\sum_{j\geqslant 1}2^{jq}\exp(-(1-q)2^{j-1}) \ll \sum_{1\leqslant j\leqslant -\log_2(1-q)}2^{jq}\exp(-(1-q)2^{j-1}) \ll \left(\frac{1}{1-q}\right)^q.$$

Finally, by Hölder's inequality, Lemma 3.7.3, and Proposition 3.3.6,

$$\mathbb{E}\left[1_{\mathcal{G}_r(2^J;K)^c}\left(\int_{-\pi}^{\pi}|F_{K,M_*}(re^{i\theta})|^2\mathrm{d}\theta\right)^q\right] \ll \left(\frac{K}{\sqrt{\log K}}\right)^q.$$

Combining the above estimates concludes the proof. □

Proof of Proposition 3.3.6

By definition, if $\mathcal{G}_r(A;K)$ fails, then there must be some $\log M_* \leqslant n \leqslant \log K$ such that either

$$\max_{\theta\in[-\pi,\pi)}\sum_{k=M_*}^{e^n-1}\left(\Re\frac{X_k r^k e^{ik\theta}}{\sqrt{k}} - \mu_k\right) > A + 10\log n \tag{3.49}$$

or

$$\min_{\theta\in[-\pi,\pi)}\sum_{k=M_*}^{e^n-1}\left(\Re\frac{X_k r^k e^{ik\theta}}{\sqrt{k}} - \mu_k\right) < -A - L_1 n. \tag{3.50}$$

Applying the union bound, we can bound the probability of the first event (3.49) by

$$\mathbb{P}\left(\exists \log M_* \leqslant n \leqslant \log K : \max_{\theta\in[-\pi,\pi)} \sum_{k=M_*}^{e^n-1} \left(\Re \frac{X_k r^k e^{ik\theta}}{\sqrt{k}} - \mu_k\right) > A + 10\log n\right)$$

$$\leqslant \sum_{n=\log M_*}^{\log K} \mathbb{P}\left(\max_{\theta\in[-\pi,\pi)} \sum_{k=M_*}^{e^n-1} \left(\Re \frac{X_k r^k e^{ik\theta}}{\sqrt{k}} - \mu_k\right) > A + 10\log n\right) =: \sum_{n=\log M_*}^{\log K} \mathcal{P}_n.$$

The quantity $\mathcal{P}_n$ is then controlled using a chaining argument. First, we discretize the interval $[-\pi, \pi)$ into ne^n points. Define $\theta_j = 2\pi j/(ne^n)$, $0 \leqslant j < ne^n$ (and identifying θ with $\theta - 2\pi$). Then on the event in the definition of $\mathcal{P}_n$, it holds that either

$$\sum_{k=M_*}^{e^n-1} \left(\Re \frac{X_k r^k e^{ik\theta_j}}{\sqrt{k}} - \mu_k\right) \geqslant \frac{A}{2} + 5\log n \tag{3.51}$$

for some $0 \leqslant j < ne^n$ or

$$\Re \sum_{k=M_*}^{e^n-1} \frac{X_k r^k}{\sqrt{k}}(e^{ik\theta} - e^{ik\theta_j}) = \Re \int_{\theta_j}^{\theta} \sum_{k=M_*}^{e^n-1} X_k r^k (i\sqrt{k} e^{iky}) \mathrm{d}y \geqslant \frac{A}{2} + 5\log n$$

for some $0 \leqslant j < ne^n$ and some $\theta \in [\theta_j, \theta_{j+1})$. In particular, the second case implies

$$\int_{\theta_j}^{\theta_{j+1}} \Big| \sum_{k=M_*}^{e^n-1} X_k r^k \sqrt{k} e^{iky} \Big| \,\mathrm{d}y \geqslant \frac{A}{2} + 5\log n \tag{3.52}$$

for some $0 \leqslant j < ne^n$. Therefore, by the rotational invariance of X_k, it holds that

$$\mathcal{P}_n \leqslant ne^n(\mathcal{P}'_n + \mathcal{P}''_n),$$

where $\mathcal{P}'_n$ (resp. $\mathcal{P}''_n$) is the probability that (3.51) (resp. (3.52)) holds with $j = 0$.

For $\mathcal{P}'_n$, using Markov's inequality and Lemma 3.7.3 we have

$$\mathcal{P}'_n \leqslant \exp\left(-2\Big(\frac{A}{2} + 5\log n + \sum_{k=M_*}^{e^n-1} \mu_k\Big)\right) \mathbb{E}\left[\exp\left(2\sum_{k=M_*}^{e^n-1} \frac{r^k}{\sqrt{k}} R_k \cos(\tau_k)\right)\right]$$

$$\ll \exp(-A - 10\log n - 2n)e^n = \frac{e^{-n-A}}{n^{10}}.$$

Next, we estimate $\mathcal{P}_n''$. Applying in order Jensen's inequality, Markov's inequality, and the fact that the law of X_k is rotational invariant, we have for $\beta > 0$,

$$\begin{aligned}
\mathcal{P}_n'' &\leqslant \mathbb{P}\left(\frac{1}{\theta_1}\int_0^{\theta_1} \exp\Big(\beta\Big|\sum_{k=M_*}^{e^n-1} X_k r^k\sqrt{k}e^{iky}\Big|\Big)\mathrm{d}y \geqslant \exp\Big(\frac{A+10\log n}{2\theta_1}\beta\Big)\right) \\
&\leqslant \exp\left(-\frac{A+10\log n}{2\theta_1}\beta\right)\mathbb{E}\left[\frac{1}{\theta_1}\int_0^{\theta_1} \exp\Big(\beta\Big|\sum_{k=M_*}^{e^n-1} X_k r^k\sqrt{k}e^{iky}\Big|\Big)\mathrm{d}y\right] \\
&\leqslant \exp\left(-\frac{A+10\log n}{2\theta_1}\beta\right)\mathbb{E}\left[\exp\Big(\beta\Big|\sum_{k=M_*}^{e^n-1} X_k r^k\sqrt{k}\Big|\Big)\right] \\
&\ll \exp\left(-\frac{A+10\log n}{2\theta_1}\beta\right)\mathbb{E}\left[\exp\Big(2\beta\Big|\sum_{k=M_*}^{e^n-1} R_k\cos(\tau_k) r^k\sqrt{k}\Big|\Big)\right].
\end{aligned}$$

Since R_k are i.i.d. sub-exponential, so is $\sum_{k=M_*}^{e^n-1} R_k\cos(\tau_k)r^k\sqrt{k}$. Then by Bernstein's inequality, there is some constant $C > 0$ depending on the distribution of R_1 such that

$$\mathbb{P}\left(\Big|\sum_{k=M_*}^{e^n-1} R_k\cos(\tau_k)r^k\sqrt{k}\Big| \geqslant u\right) \ll \exp\left(-\frac{1}{C}\min\{\frac{u^2}{e^{2n}}, \frac{u}{e^{n/2}}\}\right),$$

which implies

$$\mathbb{E}\left[\exp\left(2\beta\Big|\sum_{k=M_*}^{e^n-1} R_k\cos(\tau_k)r^k\sqrt{k}\Big|\right)\right] \ll \exp(C\beta^2 e^{2n} + \log\beta).$$

Finally, plugging in $\beta = e^{-n}$ we get the desired bound $\mathcal{P}_n'' \ll e^{-A-n}/n^{10}$ (which is loose but sufficient). Combining this with the bound of $\mathcal{P}_n'$, we get $\mathcal{P}_n \ll e^{-A}/n^9$, and therefore

$$\mathbb{P}\left(\forall 1 \leqslant n \leqslant \log K : \max_{\theta\in[-\pi,\pi)} \sum_{k=M_*}^{e^n-1}\left(\Re\frac{X_k r^k e^{ik\theta}}{\sqrt{k}} - \mu_k\right) > A + 10\log n\right) \ll e^{-A}.$$

By replacing R_k with $-R_k$ and also μ_k with $-\mu_k$, the probability of the lower tail event (3.50) can be bounded by the same argument. This concludes the proof of Proposition 3.3.6.

Proof of Proposition 3.3.7

Let $M = M(A) = \max\{2\sqrt{A}, 20, \log M_*\}$, and define

$$A_\theta(M) := \sum_{k=M_*}^{e^M-1} \left(\Re \frac{X_k r^k e^{ik\theta}}{\sqrt{k}} - \mu_k \right).$$

Define the event

$$\mathcal{E}_r(A, \theta; M) := \{-A - L_1 M \leqslant A_\theta(M) \leqslant A + 10 \log M\},$$

and for $B > 0$, define the event $\mathcal{L}_r(B, \theta; K)$ as

$$\left\{ \forall M < n \leqslant \log K,\ -B - L_1 n \leqslant \sum_{e^M \leqslant k < e^n} \left(\Re \frac{X_k r^k e^{ik\theta}}{\sqrt{k}} - \mu_k \right) \leqslant B + 10 \log n \right\}. \tag{3.53}$$

Then we can replace the event $\mathcal{G}_r(A, \theta; K)$ by the less restricted event $\mathcal{E}_r(A, \theta; M) \cap \mathcal{L}_r(2A + L_1 M, \theta; K)$, where we used $10 \log x \leqslant 2x$ when $x \geqslant 20$. By rotational symmetry and independence,

$$\begin{aligned}
&\mathbb{E}[1_{\mathcal{G}_r(A,\theta;K)} |F_{K,M_*}(re^{i\theta})|^2] \\
=&\mathbb{E}[1_{\mathcal{G}_r(A,0;K)} |F_{K,M_*}(r)|^2] \\
\leqslant&\mathbb{E}\left[1_{\mathcal{E}_r(A,0;M)} \exp\left(2\Re \sum_{k=M_*}^{e^M-1} \frac{X_k r^k}{\sqrt{k}} \right) 1_{\mathcal{L}_r(2A+L_1M,0;K)} \exp\left(2\Re \sum_{k=e^M}^{K-1} \frac{X_k r^k}{\sqrt{k}} \right) \right] \\
=&\mathbb{E}\left[1_{\mathcal{E}_r(A,0;M)} \exp\left(2\Re \sum_{k=M_*}^{e^M-1} \frac{X_k r^k}{\sqrt{k}} \right) \right] \mathbb{E}\left[1_{\mathcal{L}_r(2A+L_1M,0;K)} \exp\left(2\Re \sum_{k=e^M}^{K-1} \frac{X_k r^k}{\sqrt{k}} \right) \right].
\end{aligned}$$

Proposition 3.3.8. *With the notations above and let $B := 2A + L_1 M$, we have*

$$\mathbb{E}\left[1_{\mathcal{L}_r(B,0;K)} \exp\left(2\Re \sum_{k=e^M}^{K-1} \frac{X_k r^k}{\sqrt{k}}\right)\right] \ll \frac{K}{e^M} \frac{B}{\sqrt{\log(K/e^M)}}. \tag{3.54}$$

Deducing Proposition 3.3.7 from Proposition 3.3.8. Note that $B = 2A + L_1 M = O(\sqrt{\log K})$. By definition of M and Proposition 3.3.8, we have

$$\begin{aligned}\mathbb{E}[1_{\mathcal{G}_r(A,\theta;K)} |F_{K,M_*}(re^{i\theta})|^2] &\ll \frac{K(A+M)}{e^M \sqrt{\log(K/e^M)}} \mathbb{E}\left[1_{\mathcal{E}_r(A,0;M)} \exp\left(2\Re \sum_{k=M_*}^{e^M-1} \frac{r^k}{\sqrt{k}} X_k\right)\right] \\ &\ll \frac{AK}{e^M \sqrt{\log K}} \times \mathbb{E}\left[\exp\left(2\Re \sum_{k=M_*}^{e^M-1} \frac{r^k}{\sqrt{k}} X_k\right)\right] \ll \frac{AK}{\sqrt{\log K}},\end{aligned}$$

where the last step is due to Lemma 3.7.3. This concludes the proof. □

Proof of Proposition 3.3.8

Recall our definition of $\mathbb{Q}^{(1)} = \mathbb{Q}^{(1)}_{r,M,K}$ from (3.36) and the event $\mathcal{L}_r(B, 0; K)$ from (3.53). The constant M_* is chosen depending on the law of R_1 so that for all $k \geqslant M_*$, every Laplace functional below (i.e. expectations of the form $\mathbb{E}[\exp(a_k R_k \cos(\tau_k))]$ for some a_k) is well defined, and the conditions in Lemmas 3.7.2 and 3.7.4 hold. Using Lemma 3.7.3, the left-hand side of (3.54) becomes

$$\mathbb{E}\left[\exp\left(\sum_{k=e^M}^{K-1} \frac{2r^k}{\sqrt{k}} R_k \cos(\tau_k)\right)\right] \mathbb{Q}^{(1)}(\mathcal{L}_r(B,0;K)) \asymp \frac{K}{e^M} \mathbb{Q}^{(1)}(\mathcal{L}_r(B,0;K)).$$

It then suffices to calculate the ballot-type probability $\mathbb{Q}^{(1)}(\mathcal{L}_r(B, 0; K))$. Define

$$Y_m := \sum_{e^{m-1} \leqslant k < e^m} \left(\frac{r^k}{\sqrt{k}} R_k \cos(\tau_k) - \mu_k\right). \tag{3.55}$$

Next, we approximate these random batches Y_m with centered Gaussian variables with uniformly bounded variance, which allows us to use Lemma 3.3.4 to compute

the probability

$$\mathbb{Q}^{(1)}\bigg(\forall M < n \leqslant \log K,\ -B - L_1 n \leqslant \sum_{m=M+1}^{n} Y_m \leqslant B + 10\log n\bigg).$$

Lemma 3.3.9. *For any r_m such that $|r_m| \ll m^2$, there is some centered Gaussian random variable N_m with variance $\sigma_m^2 = \mathbb{E}^{\mathbb{Q}^{(1)}}[Y_m^2] = \frac{1}{2} + O(e^{-m})$ such that*

$$\mathbb{Q}^{(1)}(r_m \leqslant Y_m \leqslant r_m + m^{-4}) = (1 + O(m^{-2}))\mathbb{P}(r_m \leqslant N_m \leqslant r_m + m^{-4}).$$

Deducing Proposition 3.3.8 from Lemma 3.3.9. We use a slicing argument as in [Har20]. Note that on the event $\mathcal{L}_r(B, \theta; K)$, it holds $-B - L_1(n-1) \leqslant \sum_{m=M+1}^{n-1} Y_m \leqslant B + 10\log(n-1)$ and $-B - L_1 n \leqslant \sum_{m=M+1}^{n} Y_m \leqslant B + 10\log n$, then there must be

$$|Y_n| \leqslant 2B + L_1 n + 10\log n \ll A + n,\ M < n \leqslant \log K.$$

Since $M \geqslant \max\{2\sqrt{A}, 20\}$, we know further that $|Y_n| \ll n^2$ uniformly in $M < n \leqslant \log K$. Then there must be some $r_n \in \mathcal{R}_n := \{r \in n^{-4}\mathbb{Z} : |r| \ll n^2\}$ such that $r_n \leqslant Y_n < r_n + n^{-4}$. In particular,

$$-B - L_1 n - \sum_{m=M+1}^{n} \frac{1}{m^4} \leqslant \sum_{m=M+1}^{n} r_m \leqslant B + 10\log n,\ M < n \leqslant \log K. \tag{3.56}$$

Denote by $\mathcal{D}(M, K)$ the set of all possible vectors $(r_n)_{M<n\leqslant\log K},\ r_n \in \mathcal{R}_n$ that satisfy (3.56), then Lemma 3.3.9 yields

$$\begin{aligned}
&\mathbb{Q}^{(1)}\left(\forall M < n \leqslant \log K : -B - L_1 n \leqslant \sum_{m=M+1}^{n} Y_m \leqslant B + 10\log n\right) \\
&\leqslant \sum_{(r_n)\in\mathcal{D}(M,K)} \prod_{m=M+1}^{\log K} \mathbb{Q}^{(1)}\Big(r_m \leqslant Y_m < r_m + \frac{1}{m^4}\Big) \\
&= \sum_{(r_n)\in\mathcal{D}(M,K)} \prod_{m=M+1}^{\log K} (1 + O(m^{-2}))\,\mathbb{P}\Big(r_m \leqslant N_m < r_m + \frac{1}{m^4}\Big)
\end{aligned}$$

$$\ll \sum_{(r_n)\in\mathcal{D}(M,K)} \prod_{m=M+1}^{\log K} \mathbb{P}\Big(r_m \leqslant N_m < r_m + \frac{1}{m^4}\Big)$$

$$\leqslant \mathbb{P}\left(\forall M < n \leqslant \log K : -B - L_1 n - \sum_{m=M+1}^{n} \frac{1}{m^4} \right.$$

$$\left. \leqslant \sum_{m=M+1}^{n} N_m \leqslant B + 10\log n + \sum_{m=M+1}^{n} \frac{1}{m^4}\right)$$

$$\ll \frac{B}{\sqrt{\log(K/e^M)}},$$

where the last step follows from Lemma 3.3.4. This completes the proof of Proposition 3.3.8. □

Proof of Lemma 3.3.9. That $\sigma_m^2 = \mathbb{E}^{\mathbb{Q}^{(1)}}[Y_m^2] = 1/2 + O(e^{-m})$ follows from Lemmas 3.7.1 and 3.7.4. We first compute the characteristic function of Y_m under $\mathbb{Q}$. Using Lemma 3.3.19 of [Dur19] and Lemma 3.7.4, we have

$$\mathbb{E}^{\mathbb{Q}^{(1)}}\left[\exp\left(it\Big(\frac{r^k}{\sqrt{k}}R_k\cos(\tau_k) - \mu_k\Big)\right)\right] = 1 - \frac{t^2}{2}\mathrm{Var}^{\mathbb{Q}^{(1)}}\left[\frac{r^k}{\sqrt{k}}R_k\cos(\tau_k)\right] + D_k(t)$$

$$= 1 - \left(\frac{r^{2k}}{4k} + O(k^{-3/2})\right)t^2 + D_k(t)$$

with some $D_k(t)$ satisfying

$$|D_k(t)| \ll |t|^3\mathbb{E}^{\mathbb{Q}^{(1)}}\left[\Big|\frac{r^k}{\sqrt{k}}R_k\cos(\tau_k) - \mu_k\Big|^3\right] \ll |t|^3 k^{-3/2},$$

and similarly $|D_k'(t)| \ll |t|^2 k^{-3/2}$. Rewriting this in exponential form, we have

$$\mathbb{E}^{\mathbb{Q}^{(1)}}\left[\exp\left(it(\frac{r^k}{\sqrt{k}}R_k\cos(\tau_k) - \mu_k)\right)\right] = \exp\left(-\frac{t^2}{2}\mathrm{Var}^{\mathbb{Q}^{(1)}}\left[\frac{r^k}{\sqrt{k}}R_k\cos(\tau_k)\right] + T_k(t)\right)$$

with again $|T_k(t)| \ll |t|^3 k^{-3/2}$ and $|T_k'(t)| \ll |t|^2 k^{-3/2}$. By independence, we then have

$$\mathbb{E}^{\mathbb{Q}^{(1)}}\left[\exp(itY_m)\right] = \exp\left(-\frac{t^2\sigma_m^2}{2} + S_m(t)\right)$$

$$= \exp\left(-\frac{t^2}{2} \sum_{e^{m-1} \leqslant k < e^m} \left(\frac{r^{2k}}{2k} + O(k^{-3/2}) \right) + S_m(t) \right)$$

with $|S_m(t)| \ll |t|^3 e^{-m/2}$ and $|S_m'(t)| \ll |t|^2 e^{-m/2}$. Let $\lambda_m := r_m/\sigma_m^2 \ll m^2$ by our assumption that $|r_m| \ll m^2$. Define a new measure $\widetilde{\mathbb{Q}}^{(1)}$ by

$$\frac{\mathrm{d}\widetilde{\mathbb{Q}}^{(1)}}{\mathrm{d}\mathbb{Q}^{(1)}} = \prod_{m=M+1}^{\log K} \frac{\exp(\lambda_m Y_m)}{\mathbb{E}^{\mathbb{Q}^{(1)}}[\exp(\lambda_m Y_m)]},$$

and the characteristic function of Y_m under $\widetilde{\mathbb{Q}}^{(1)}$ is

$$\mathbb{E}^{\widetilde{\mathbb{Q}}^{(1)}}\left[\exp(itY_m)\right] = \exp\left(ir_m t - \frac{\sigma_m^2}{2} t^2 + S_m(t - i\lambda_m) - S_m(-i\lambda_m) \right).$$

Note that when $Y_m \in [r_m, r_m + m^{-4}]$,

$$|\lambda_m Y_m - \lambda_m r_m| \leqslant \frac{|r_m|}{\sigma_m^2 m^4} \ll m^{-2} \qquad \text{a.s.}$$

Together with the estimates on $S_m(t)$, we have

$$\begin{aligned}
&\mathbb{Q}^{(1)}(r_m \leqslant Y_m \leqslant r_m + m^{-4}) \\
&= \mathbb{E}^{\mathbb{Q}^{(1)}}\left[e^{\lambda_m Y_m}\right] \mathbb{E}^{\widetilde{\mathbb{Q}}^{(1)}}\left[e^{-\lambda_m Y_m} 1_{\{r_m \leqslant Y_m \leqslant r_m + m^{-4}\}}\right] \\
&= \exp\left(\frac{\sigma_m^2}{2} \lambda_m^2 + S_m(-i\lambda_m) \right) \exp(-r_m \lambda_m + O(m^{-2}))\, \widetilde{\mathbb{Q}}^{(1)}(r_m \leqslant Y_m \leqslant r_m + m^{-4}) \\
&= (1 + O(m^{-2})) \exp\Big(-\frac{r_m^2}{2\sigma_m^2} \Big) \widetilde{\mathbb{Q}}^{(1)}(r_m \leqslant Y_m \leqslant r_m + m^{-4}).
\end{aligned} \tag{3.57}$$

For each m, let $\widetilde{N}_m$ be a Gaussian random variable with mean r_m and variance σ_m^2 under $\mathbb{P}$, i.e., with characteristic function

$$\mathbb{E}[\exp(it\widetilde{N}_m)] = \exp\Big(ir_m t - \frac{\sigma_m^2}{2} t^2 \Big).$$

Applying Berry-Esseen bound (see equation (3.4.1) of [Dur19]), we obtain

$$\begin{aligned}&|\widetilde{\mathbb{Q}}^{(1)}(r_m \leqslant Y_m \leqslant r_m + m^{-4}) - \mathbb{P}(r_m \leqslant \widetilde{N}_m \leqslant r_m + m^{-4})| \\ \ll &\int_{-e^{m/9}}^{e^{m/9}} \frac{|\mathbb{E}^{\widetilde{\mathbb{Q}}^{(1)}}[e^{itY_m}] - \mathbb{E}[e^{it\widetilde{N}_m}]|}{|t|} \mathrm{d}t + e^{-m/9}.\end{aligned}$$

Further, since $|S_m(t)| \ll |t|^3 e^{-m/2} \ll 1$, we have the estimate

$$\begin{aligned}\left|\mathbb{E}^{\widetilde{\mathbb{Q}}^{(1)}}[e^{itY_m}] - \mathbb{E}[e^{it\widetilde{N}_m}]\right| &\leqslant e^{-\sigma_m^2 t^2/2} |\exp(S_m(t - i\lambda_m) - S_m(-i\lambda_m)) - 1| \\ &\ll e^{-\sigma_m^2 t^2/2} |S_m(t - i\lambda_m) - S_m(-i\lambda_m)| \\ &\ll e^{-\sigma_m^2 t^2/2} \int_0^{|t|} |S_m'(s - i\lambda_m)| \, \mathrm{d}s \\ &\ll e^{-\sigma_m^2 t^2/2} e^{-m/2} (|\lambda_m|^2 |t| + |t|^3).\end{aligned}$$

Therefore,

$$\begin{aligned}&|\widetilde{\mathbb{Q}}^{(1)}(r_m \leqslant Y_m \leqslant r_m + m^{-4}) - \mathbb{P}(r_m \leqslant \widetilde{N}_m \leqslant r_m + m^{-4})| \\ \ll &e^{-m/2} \int_{-e^{m/9}}^{e^{m/9}} (|\lambda_m|^2 + |t|^2) e^{-\sigma_m^2 t^2/2} \mathrm{d}t + e^{-m/9} \\ \ll &(1 + |r_m|^2) e^{-m/2} + e^{-m/9} \ll e^{-m/9}\end{aligned}$$

for any $|r_m| \ll m^2$. Note that $\mathbb{P}(r_m \leqslant \widetilde{N}_m \leqslant r_m + m^{-4}) \gg m^{-4}$, we have

$$\widetilde{\mathbb{Q}}^{(1)}(r_m \leqslant Y_m \leqslant r_m + m^{-4}) = (1 + O(e^{-m/10}))\mathbb{P}(r_m \leqslant \widetilde{N}_m \leqslant r_m + m^{-4}),$$

Finally, a standard Gaussian computation yields

$$\mathbb{P}(r_m \leqslant \widetilde{N}_m \leqslant r_m + m^{-4}) = (1 + O(m^{-2})) \exp\left(\frac{r_m^2}{2\sigma_m^2}\right) \mathbb{P}(r_m \leqslant N_m \leqslant r_m + m^{-4}).$$

Combining with (3.57) yields the proof. □

3.3.4 Lower bound of Proposition 3.3.1

In this section, we prove the lower bound of Proposition 3.3.1.[4] We first reduce the proof to the following two propositions.

Proposition 3.3.10. *Fix $q \in (0,1]$. Suppose that $\mathbb{E}[e^{\gamma|R_1|}] < \infty$ for some $\gamma > 2q$, i.e. (q-UNIV) holds. Then for any $0 < r < 1$,*

$$\mathbb{E}[|A_{N,M_*}|^{2q}] \gg \frac{1}{N^q}\left(\mathbb{E}\left[\left(\int_{-\pi}^{\pi}|F_{N/2,M_*}(re^{i\theta})|^2\mathrm{d}\theta\right)^q\right] - \mathbb{E}\left[r^{Nq}\left(\int_{-\pi}^{\pi}|F_{N/2,M_*}(e^{i\theta})|^2\mathrm{d}\theta\right)^q\right]\right).$$

Remark 3. Proposition 3.3.10 mirrors Proposition 8.1 of [SZ22], which focuses on $q \in [1/2,1]$ instead of $q \in (0,1]$, together with a Hölder's inequality argument for $q \in (0,1/2)$, in order to obtain uniformity of the constants in q. Here, we do not attempt to have the asymptotic constants in (3.3) independent of q.

Proof. We mainly follow the arguments in Section 9 of [SZ22]. First, it follows from the same symmetrization procedure as (3.34) that $\mathbb{E}[|A_{N,M_*}|^{2q}] \geqslant \mathbb{E}[|B_{N,M_*}|^{2q}]/2$, where

$$B_{N,M_*} := \sum_{N/2<n\leqslant N}\frac{X_n}{\sqrt{n}}A_{N-n,M_*}.$$

By Khintchine's inequality, in the form of Lemma 4.1 of [LT91],[5] we have

$$\mathbb{E}\left[|B_{N,M_*}|^{2q} \mid \{A_{n,M_*}\}_{1\leqslant n\leqslant N/2}\right] \gg \left(\sum_{N/2<n\leqslant N}\frac{|A_{N-n,M_*}|^2}{n}\right)^q \gg \left(\frac{1}{N}\sum_{n<N/2}|A_{n,M_*}|^2\right)^q.$$

[4] The proof of the lower bound does not strictly rely on the (q-UNIV) condition, but the bound may not be tight for the other cases.

[5] The proof therein is stated for Rademacher random variables, but the same arguments work for sub-exponential random variables, using a standard concentration bound (e.g., Lemma 8.2.1 of [Tal21]).

Taking expectation yields

$$\mathbb{E}[|A_{N,M_*}|^{2q}] \gg \mathbb{E}\bigg[\bigg(\frac{1}{N}\sum_{n<N/2}|A_{n,M_*}|^2\bigg)^q\bigg].$$

Recall from (3.41) that with

$$\widetilde{A}_{n,N,M_*} := \sum_{\substack{\lambda\in\mathcal{P}_n \\ \lambda_1\leqslant N/2 \\ \forall 1\leqslant k<M_*,\, m_k(\lambda)=0}} a(\lambda),$$

it holds that $F_{N/2,M_*}(z) = \sum_{n\geqslant 0}\widetilde{A}_{n,N,M_*}z^n$, and hence by Parseval's identity, for $r\in(0,1]$,

$$\begin{aligned}\sum_{n<N/2}|\widetilde{A}_{n,N,M_*}|^2 &\geqslant \sum_{n=0}^{\infty}|\widetilde{A}_{n,N,M_*}|^2r^{2n} - r^N\sum_{n=0}^{\infty}|\widetilde{A}_{n,N,M_*}|^2 \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|F_{N/2,M_*}(re^{i\theta})|^2\mathrm{d}\theta - \frac{r^N}{2\pi}\int_{-\pi}^{\pi}|F_{N/2,M_*}(e^{i\theta})|^2\mathrm{d}\theta.\end{aligned} \tag{3.58}$$

Moreover, by definition $\widetilde{A}_{n,N,M_*} = A_{n,M_*}$ for $n < N/2$. The claim then follows by applying the inequality $|z+w|^q \leqslant |z|^q + |w|^q$ for $q\in[0,1]$ to (3.58), and taking expectation. □

Proposition 3.3.11. *Fix any $q\in(0,1]$. Suppose that (q-UNIV) holds. Let K be large enough. For any $e^{-1/400}\leqslant r<1$, we have*

$$\mathbb{E}\left[\left(\int_{-\pi}^{\pi}|F_{K,M_*}(re^{i\theta})|^2\mathrm{d}\theta\right)^q\right] \gg \left(\frac{K_r}{1+(1-q)\sqrt{\log K_r}}\right)^q,$$

where $\log K_r$ is the largest integer such that $K_r\leqslant\min\{-1/(4\log r), K\}$.

Remark 4. Here, K_r is the threshold below which the variances of Y_m (defined in (3.55)) are comparable (say, between 1/20 and 20). This appears necessary to make sense of the random walk analog of (3.22) proposed in the proof sketch.

Deducing the lower bound of Proposition 3.3.1 from Propositions 3.3.10 and 3.3.11. Consider $r = e^{-C/N}$ for a large constant C to be determined. Then Proposition 3.3.11 gives that for N large enough,

$$\frac{1}{N^q}\mathbb{E}\left[\left(\int_{-\pi}^{\pi}|F_{N/2,M_*}(re^{i\theta})|^2\mathrm{d}\theta\right)^q\right] \geqslant \frac{1}{C_1}\left(\frac{1}{C(1+(1-q)\sqrt{\log N})}\right)^q.$$

On the other hand, applying Proposition 3.3.3 gives

$$\frac{1}{N^q}\mathbb{E}\left[r^{Nq}\left(\int_{-\pi}^{\pi}|F_{N/2,M_*}(e^{i\theta})|^2\mathrm{d}\theta\right)^q\right] \leqslant C_2\left(\frac{e^{-C}}{1+(1-q)\sqrt{\log N}}\right)^q.$$

Therefore, by picking $C > 0$ large enough depending on the constants C_1, C_2, Proposition 3.3.10 yields

$$\mathbb{E}[|A_{N,M_*}|^{2q}] \gg \left(\frac{1}{1+(1-q)\sqrt{\log N}}\right)^q$$

for N large enough. By adjusting constants suitably, the conclusion stands for all N. □

Proof of Proposition 3.3.11

For any (random) subset $\mathcal{L}$ of $[-\pi,\pi)$, we use Hölder's inequality to obtain

$$\mathbb{E}\left[\left(\int_{-\pi}^{\pi}|F_{K,M_*}(re^{i\theta})|^2\mathrm{d}\theta\right)^q\right] \gg \frac{\left(\mathbb{E}\left[\int_{\mathcal{L}}|F_{K,M_*}(re^{i\theta})|^2\mathrm{d}\theta\right]\right)^{2-q}}{\left(\mathbb{E}\left[\left(\int_{\mathcal{L}}|F_{K,M_*}(re^{i\theta})|^2\mathrm{d}\theta\right)^2\right]\right)^{1-q}}. \tag{3.59}$$

We then carefully choose this random set $\mathcal{L}$ as inspired by [SZ22]. Recall (3.39).

Definition 3.3.12. Fix again a universal constant $L_1 > 20$ from Lemma 3.3.4. Let A be a real number with $1 \leqslant A \leqslant \sqrt{\log K_r}$. Define $\mathcal{L}(\theta) = \mathcal{L}(A,\theta;K)$ as the event that for each $\log M_* \leqslant n \leqslant \log K_r$, one has

$$-A - L_1 n \leqslant \sum_{k=M_*}^{e^n-1}\left(\Re\frac{X_k r^k e^{ik\theta}}{\sqrt{k}} - \mu_k\right) \leqslant A - 5\log n.$$

Also, let $\mathcal{L} = \mathcal{L}(A; K)$ be the random subset of $\theta \in [-\pi, \pi)$ such that $\mathcal{L}(\theta)$ holds.

First, we give a lower bound of the numerator of (3.59).

Lemma 3.3.13. *For any $1 \leqslant A \leqslant \sqrt{\log K_r}$ and $e^{-1/400} \leqslant r < 1$, we have*

$$\mathbb{E}\left[\int_{\mathcal{L}} |F_{K,M_*}(re^{i\theta})|^2 \mathrm{d}\theta\right] \gg \frac{AK_r}{\sqrt{\log K_r}}.$$

Proof. The proof is similar to the proof of Proposition 3.3.7 in Section 3.3.3, so we only sketch the key steps. First, observe that by rotational symmetry,

$$\mathbb{E}\left[\int_{\mathcal{L}} |F_{K,M_*}(re^{i\theta})|^2 \mathrm{d}\theta\right] = \mathbb{E}\left[\int_{-\pi}^{\pi} 1_{\mathcal{L}(\theta)} |F_{K,M_*}(re^{i\theta})|^2 \mathrm{d}\theta\right] = 2\pi\, \mathbb{E}[1_{\mathcal{L}(0)} |F_{K,M_*}(r)|^2].$$

It follows from Lemmas 3.7.1 and 3.7.3, and definition of K_r that

$$\mathbb{E}[1_{\mathcal{L}(0)} |F_{K,M_*}(r)|^2] = \mathbb{E}\bigg[\exp\bigg(\sum_{M_* \leqslant k \leqslant K} \frac{2r^k}{\sqrt{k}} R_k \cos(\tau_k)\bigg)\bigg] \mathbb{Q}^{(1)}(\mathcal{L}(0)) \asymp K_r \mathbb{Q}^{(1)}(\mathcal{L}(0)).$$

Next, recalling the definition of Y_m in (3.55), we replace $\mathcal{L}(0)$ by the less restrictive event

$$\widehat{\mathcal{L}}(0) := \bigg\{\forall \log M_* < n \leqslant \log K_r,\ -L_1 n \leqslant \sum_{m=\log M_*}^{n} Y_m \leqslant \min\{A, L_1 n\} - 5\log n\bigg\}.$$

On the event $\widehat{\mathcal{L}}(0)$, we have

$$\bigg|\sum_{e^m \leqslant k < e^{m+1}} \bigg(\Re \frac{X_k r^k e^{ik\theta}}{\sqrt{k}} - \mu_k\bigg)\bigg| \ll m,\ \log M_* \leqslant m \leqslant \log K_r.$$

It then follows from Lemma 3.3.9 and a slicing argument as in Section 3.3.3 that

$$\begin{aligned}
&\mathbb{Q}^{(1)}(\mathcal{L}(0)) \\
&\geqslant \mathbb{Q}^{(1)}\bigg(\forall \log M_* < n \leqslant \log K_r : -L_1 n \leqslant \sum_{m=\log M_*}^{n} Y_m \leqslant \min\{A, L_1 n\} - 5\log n\bigg)
\end{aligned}$$

$$\gg \mathbb{P}\left(\forall \log M_* < n \leqslant \log K_r : -L_1 n + \sum_{m=\log M_*}^{n} m^{-4} \leqslant \sum_{m=\log M_*}^{n} N_m \right.$$

$$\left. \leqslant \min\{A, L_1 n\} - 5\log n - \sum_{m=\log M_*}^{n} m^{-4}\right)$$

$$\gg \frac{A}{\sqrt{\log K_r}},$$

where the last step follows from Lemma 3.3.4 and by adjusting the constant L_1 suitably, while noting that M_* is a fixed constant. This completes the proof. □

For the upper bound of the denominator in (3.59), we first expand the square to get

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_{\mathcal{L}} |F_{K,M_*}(re^{i\theta})|^2 \mathrm{d}\theta\right)^2\right] \\
=&\mathbb{E}\left[\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} 1_{\mathcal{L}(\theta_1)}|F_{K,M_*}(re^{i\theta_1})|^2 1_{\mathcal{L}(\theta_2)}|F_{K,M_*}(re^{i\theta_2})|^2 \mathrm{d}\theta_1 \mathrm{d}\theta_2\right] \\
=&\int_{-\pi}^{\pi} \mathbb{E}\left[1_{\mathcal{L}(0)\cap\mathcal{L}(\theta)}|F_{K,M_*}(r)|^2 |F_{K,M_*}(re^{i\theta})|^2\right] \mathrm{d}\theta. \qquad (3.60)
\end{aligned}$$

Proposition 3.3.14. *With notations as above, for any $\theta \in [-\pi, \pi)$ and $e^{-1/400} \leqslant r < 1$, we have*

$$\mathbb{E}\left[1_{\mathcal{L}(0)\cap\mathcal{L}(\theta)}|F_{K,M_*}(r)|^2 |F_{K,M_*}(re^{i\theta})|^2\right] \ll A^2 e^{2A} \frac{K_r^2}{\log K_r} \frac{\min\{K_r, 2\pi/|\theta|\}}{(\log \min\{K_r, 2\pi/|\theta|\})^7}.$$

Deducing Proposition 3.3.11 from Proposition 3.3.14. As in [SZ22], applying equations (3.59), (3.60), Lemma 3.3.13, and Proposition 3.3.14 with $A := \sqrt{\log K_r}/(1 + (1-q)\sqrt{\log K_r})$ completes the proof. □

Proof of Proposition 3.3.14

Define $M = M(r,\theta)$ to be the smallest integer such that

$$e^M \geqslant \max\{\min\{10^3/|\theta|, K_r/e\}, M_*\}.$$

Set for any $\theta \in [-\pi, \pi)$

$$A_\theta(M) := \Re \sum_{k=M_*}^{e^M-1} \left(\frac{X_k r^k e^{ik\theta}}{\sqrt{k}} - \mu_k \right) = \sum_{k=M_*}^{e^M-1} \left(\frac{r^k}{\sqrt{k}} R_k \cos(\tau_k + k\theta) - \mu_k \right).$$

Similarly as in the proof of Proposition 3.3.7, we replace the event $\mathcal{L}(0) \cap \mathcal{L}(\theta)$ with a less restricted event $\widetilde{\mathcal{L}}$, defined by the constraints that

$$-A - L_1 M \leqslant A_0(M), A_\theta(M) \leqslant A - 5\log M, \tag{3.61}$$

and for any $M < n \leqslant \log K_r$,

$$\begin{aligned}
&- A - L_1 n - \max\{A_0(M), A_\theta(M), 0\} \\
&\leqslant \sum_{e^M \leqslant k < e^n} \left(\Re \frac{X_k r^k}{\sqrt{k}} - \mu_k \right), \sum_{e^M \leqslant k < e^n} \left(\Re \frac{X_k r^k e^{ik\theta}}{\sqrt{k}} - \mu_k \right) \\
&\leqslant A - \min\{A_0(M), A_\theta(M), 0\}.
\end{aligned}$$

Also recalling our definition (3.38), and using $\mu_k = r^{2k}/k + O(k^{-3/2})$, we get

$$\begin{aligned}
&\mathbb{E}\left[1_{\mathcal{L}(0)\cap\mathcal{L}(\theta)} |F_{K,M_*}(r)|^2 |F_{K,M_*}(re^{i\theta})|^2\right] \\
&\ll \exp\Big(4 \sum_{k=M_*}^{e^M-1} \frac{r^{2k}}{k}\Big) \mathbb{E}\Bigg[1_{\widetilde{\mathcal{L}}}\, e^{2A_0(M)+2A_\theta(M)} \prod_{m=M+1}^{\log K_r} e^{2Z_0(m)+2Z_\theta(m)} \\
&\qquad \exp\Big(2 \sum_{k=K_r}^{K} \frac{r^k}{\sqrt{k}} \Re(X_k + X_k e^{ik\theta})\Big)\Bigg] \\
&\ll e^{4M} \mathbb{E}\left[1_{\widetilde{\mathcal{L}}}\, e^{2A_0(M)+2A_\theta(M)} \prod_{m=M+1}^{\log K_r} e^{2Z_0(m)+2Z_\theta(m)}\right]
\end{aligned}$$

$$\times \mathbb{E}\bigg[\exp\Big(2\sum_{k=K_r}^{K}\frac{r^k}{\sqrt{k}}\Re(X_k+X_ke^{ik\theta})\Big)\bigg].$$

Using Lemma 3.7.2 (i) and definition of K_r, we arrive at

$$\mathbb{E}\bigg[\exp\Big(2\sum_{k=K_r}^{K}\frac{r^k}{\sqrt{k}}\Re(X_k+X_ke^{ik\theta})\Big)\bigg] \ll \mathbb{E}\bigg[\exp\Big(4\sum_{k=K_r}^{K}\frac{r^k}{\sqrt{k}}\Re X_k\Big)\bigg]$$
$$\ll \prod_{k=K_r}^{K}\Big(1+\frac{2r^{2k}}{k}+O(k^{-3/2})\Big) \ll 1.$$

We conclude that

$$\mathbb{E}\left[1_{\mathcal{L}(0)\cap\mathcal{L}(\theta)}|F_{K,M_*}(r)|^2|F_{K,M_*}(re^{i\theta})|^2\right]$$
$$\ll e^{4M}\mathbb{E}\left[1_{\widetilde{\mathcal{L}}}\,e^{2A_0(M)+2A_\theta(M)}\prod_{m=M+1}^{\log K_r}e^{2Z_0(m)+2Z_\theta(m)}\right]. \tag{3.62}$$

We now state a two-dimensional version of Proposition 3.3.8, which suffices for proving Proposition 3.3.14.

Proposition 3.3.15. *Notations as above, and let $M'=\max\{M,A\}$. Given any B,B' satisfying $B'\leqslant 0\leqslant B$ and $B,-B'\leqslant LM'$ with some absolute constant $L>0$, define the event $\mathcal{E}$ to be*

$$\bigcap_{M<n\leqslant\log K_r}\left\{B'-L_1n\leqslant\sum_{e^M\leqslant k<e^n}\left(\Re\frac{X_kr^k}{\sqrt{k}}-\nu_k\right),\sum_{e^M\leqslant k<e^n}\left(\Re\frac{X_kr^ke^{ik\theta}}{\sqrt{k}}-\nu_k\right)\leqslant B\right\}. \tag{3.63}$$

Then

$$\mathbb{E}\bigg[1_{\mathcal{E}}\prod_{m=M+1}^{\log K_r}\exp(2Z_0(m)+2Z_\theta(m))\bigg]\ll\frac{K_r^2}{e^{2M}}\left(\frac{M'}{\sqrt{1+\log(K_r/e^{M'})}}\right)^2.$$

Deducing Proposition 3.3.14 from Proposition 3.3.15. Our plan is to condition on $\{X_k\}_{1\leqslant k<e^M}$ and insert Proposition 3.3.15 into (3.62). First we claim that, with

a constant $L_0 > 0$ large enough, setting $B = A - \min\{A_0(M), A_\theta(M), 0\} + L_0$ and $B' = -A - \max\{A_0(M), A_\theta(M), 0\} - L_0$ in the definition of event $\mathcal{E}$ gives that $\widetilde{\mathcal{L}} \subseteq \mathcal{E}$. Indeed, this is a consequence of Lemma 3.7.5, with

$$L_0 = \sum_{m \geqslant M} \left| \sum_{e^{m-1} \leqslant k < e^m} (\mu_k - \nu_k) \right| \ll 1.$$

Moreover, we have by (3.61) that on the event $\widetilde{\mathcal{L}}$, $B - B' \leqslant LM'$ and $M' \ll A + M$ with some large enough absolute constant $L > 0$. Together with (3.62) yield that

$$\begin{aligned}
&\mathbb{E}\left[1_{\mathcal{L}(0)\cap\mathcal{L}(\theta)} |F_{K,M_*}(r)|^2 |F_{K,M_*}(re^{i\theta})|^2\right] \\
\ll &\frac{K_r^2 e^{2M}}{1 + \log(K_r/e^{M'})} \mathbb{E}\left[1_{\widetilde{\mathcal{L}}}\, e^{2(A_0(M)+A_\theta(M))} (A+M)^2\right] \\
\ll &\frac{K_r^2 e^{2M}}{1 + \log(K_r/e^{M'})} \mathbb{E}\left[1_{\widetilde{\mathcal{L}}}\, e^{2(A_0(M)+A_\theta(M))} A^2 M^2\right] \\
\ll &\frac{K_r^2 e^{2M}}{1 + \log(K_r/e^{M'})} \frac{e^{2A} A^2}{M^8} \mathbb{E}\left[1_{\{A_0(M) \leqslant A - 5\log M\}}\, e^{2A_0(M)}\right],
\end{aligned}$$

where in the last step we used rotational symmetry while bounding $A_\theta(M) \leqslant A - 5\log M$. On the other hand, Lemma 3.7.2 yields

$$\mathbb{E}\left[1_{\{A_0(M) \leqslant A - 5\log M\}}\, e^{2A_0(M)}\right] \leqslant \mathbb{E}[e^{2A_0(M)}] \ll e^{-M}. \tag{3.64}$$

Next, using $M' = \max\{M, A\} \leqslant \max\{M, \sqrt{\log K_r}\}$ we obtain $(1 + \log(K_r/e^{M'}))M \gg \log K_r$. Combining the above and using the definition of M completes the proof. $\square$

Proof of Proposition 3.3.15

Suppose first that $\log(K_r/e^M) \leqslant 10$. Using rotational symmetry and Lemma 3.7.2 (i),

$$\mathbb{E}\Big[1_{\mathcal{E}} \prod_{m=M+1}^{\log K_r} \exp(2Z_0(m) + 2Z_\theta(m))\Big] \ll \mathbb{E}\Big[\prod_{m=M+1}^{\log K_r} \exp(4Z_0(m))\Big]$$

$$= \mathbb{E}\Big[\exp\Big(4\Re\sum_{m=M+1}^{\log K_r}\frac{X_k r^k}{\sqrt{k}}\Big)\Big]$$

$$\ll 1.$$

Now if $K_r > e^{M+10}$, we can assume θ satisfies $10^3/|\theta| \leqslant K_r/e$ and $e^M|\theta| \geqslant 10^3$. Recall (3.37). In view of Lemma 3.7.6, it suffices to bound $\mathbb{Q}^{(2)}(\mathcal{E})$. Recall from (3.40) that

$$\nu_k = \mathbb{E}^{\mathbb{Q}^{(2)}}\left[\Re\frac{X_k r^k}{\sqrt{k}}\right] = \frac{r^{2k}}{k} + \frac{\cos(k\theta)r^{2k}}{k} + O(k^{-3/2}),$$

and define

$$X_{k,0} := \Re\frac{X_k r^k}{\sqrt{k}} - \nu_k \qquad \text{and} \qquad X_{k,\theta} := \Re\frac{X_k r^k e^{ik\theta}}{\sqrt{k}} - \nu_k, \qquad k \in \mathbb{N}.$$

We apply the same strategy as in the proof of Proposition 3.3.8: approximate the batched sums of $X_{k,\theta}$ using Gaussians, and apply Lemma 3.3.4. We first use Lemma 3.7.4 to compute the joint characteristic function of $(X_{k,0}, X_{k,\theta})$ as

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^{(2)}}[\exp(iuX_{k,0} + ivX_{k,\theta})] \\
&= 1 + \frac{-u^2\mathbb{E}^{\mathbb{Q}^{(2)}}[X_{k,0}^2] - v^2\mathbb{E}^{\mathbb{Q}^{(2)}}[X_{k,\theta}^2] - 2uv\mathbb{E}^{\mathbb{Q}^{(2)}}[X_{k,0}X_{k,\theta}]}{2} \\
&+ \sum_{j=3}^{\infty}\frac{1}{j!}\mathbb{E}^{\mathbb{Q}^{(2)}}[(iuX_{k,0} + ivX_{k,\theta})^j] \\
&= 1 - \frac{(u^2+v^2)r^{2k}}{4k} - \frac{uvr^{2k}\cos(k\theta)}{2k} + (u^2+v^2)O\Big(\frac{r^{3k}}{k^{3/2}}\Big) \\
&+ \sum_{j=3}^{\infty}\frac{1}{j!}\mathbb{E}^{\mathbb{Q}^{(2)}}[(iuX_{k,0} + ivX_{k,\theta})^j] \\
&=: 1 - \frac{(u^2+v^2)r^{2k}}{4k} - \frac{uvr^{2k}\cos(k\theta)}{2k} + D_k(u,v),
\end{aligned}$$

and using Lemma 3.3.19 of [Dur19] and Lemma 3.7.4, we have

(a) $|D_k(u,v)| \ll (u^2+v^2+|u|^3+|v|^3)r^{3k}k^{-3/2}$;

(b) $|\frac{\partial D_k(u,v)}{\partial u}|, |\frac{\partial D_k(u,v)}{\partial v}| \ll (|u|+|v|+|u|^2+|v|^2)r^{3k}k^{-3/2}$;

(c) $|\frac{\partial^2 D_k(u,v)}{\partial u \partial v}| \ll (1+|u|+|v|)r^{3k}k^{-3/2}$.

Before using independence to form a product of the characteristic functions over k, we need to transform our expression into an exponential form. We have

$$\mathbb{E}^{\mathbb{Q}^{(2)}}[\exp(iuX_{k,0}+ivX_{k,\theta})] = \exp\left(-\frac{(u^2+v^2)r^{2k}}{4k} - \frac{uvr^{2k}\cos(k\theta)}{2k} + T_k(u,v)\right),$$

where the above (a)–(c) hold with $D_k(u,v)$ replaced by $T_k(u,v)$.

Next, we group the random variables $X_{k,0}, X_{k,\theta}$ and approximate their sums using Gaussian distributions. Define

$$Y_{m,0} = \sum_{e^{m-1}\leqslant k<e^m} X_{k,0} \quad \text{and} \quad Y_{m,\theta} = \sum_{e^{m-1}\leqslant k<e^m} X_{k,\theta}, \qquad m \in \mathbb{N}.$$

After summing over $e^{m-1} \leqslant k < e^m$,

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}^{(2)}}\left[e^{iuY_{m,0}+ivY_{m,\theta}}\right]\\ =&\exp\left(\sum_{e^{m-1}\leqslant k<e^m}\Big(-\frac{(u^2+v^2)r^{2k}}{4k} - \frac{uvr^{2k}\cos(k\theta)}{2k}\Big) + S_m(u,v)\right), \qquad (3.65)\end{aligned}$$

where

(a) $|S_m(u,v)| \ll (u^2+v^2+|u|^3+|v|^3)e^{-m/2} \ll (1+|u|^3+|v|^3)e^{-m/2}$;

(b) $|\frac{\partial S_m(u,v)}{\partial u}|, |\frac{\partial S_m(u,v)}{\partial v}| \ll (|u|+|v|+|u|^2+|v|^2)e^{-m/2}$;

(c) $|\frac{\partial^2 S_m(u,v)}{\partial u \partial v}| \ll (1+|u|+|v|)e^{-m/2}$.

Our Gaussian approximation will have the same covariance structure of $(Y_{m,0}, Y_{m,\theta})$ under $\mathbb{Q}^{(2)}$, which we compute first. Define the covariance matrix

$$\Sigma_m := \begin{pmatrix} \sigma_m^2 & \rho_m \sigma_m^2 \\ \rho_m \sigma_m^2 & \sigma_m^2 \end{pmatrix},$$

where σ_m, ρ_m are defined such that Σ_m is the covariance matrix of $(Y_{m,0}, Y_{m,\theta})$ under $\mathbb{Q}^{(2)}$. It follows from Lemmas 3.7.4 and 3.7.5 that

$$\sigma_m^2 = \sum_{e^{m-1} \leqslant k < e^m} \left(\frac{r^{2k}}{2k} + O(k^{-3/2}) \right) = \frac{1}{2} + O(e^{-m/2})$$

and

$$\rho_m \sigma_m^2 = \sum_{e^{m-1} \leqslant k < e^m} \left(\frac{r^{2k} \cos(k\theta)}{2k} + O(k^{-3/2}) \right) = O(e^{M-m} + e^{-m/2}).$$

In particular,

$$\rho_m \ll \frac{e^{M-m} + e^{-m/2}}{\frac{1}{2} + O(e^{-m/2})} \ll e^{M-m} + e^{-m/2}. \tag{3.66}$$

Let $\mathbf{N}_m := (N_{m,1}, N_{m,2})$ be a two-dimensional centered Gaussian vector with covariance matrix Σ_m. That is,

$$\mathbb{E}[e^{i\mathbf{x}\cdot\mathbf{N}_m}] = \exp\left(-\frac{1}{2} \mathbf{x}^\top \Sigma_m \mathbf{x} \right), \ \mathbf{x} \in \mathbb{R}^2. \tag{3.67}$$

Lemma 3.3.16. *For any $|u_m|, |v_m| \ll m$, it holds that*

$$\begin{aligned} &\mathbb{Q}^{(2)}(u_m \leqslant Y_{m,0} \leqslant u_m + m^{-3}, \ v_m \leqslant Y_{m,\theta} \leqslant v_m + m^{-3}) \\ &= (1 + O(m^{-2}))\mathbb{P}(u_m \leqslant N_{m,1} \leqslant u_m + m^{-3}, \ v_m \leqslant N_{m,2} \leqslant v_m + m^{-3}). \end{aligned}$$

Proof of Proposition 3.3.15. We use a slicing argument to bound $\mathbb{Q}^{(2)}(\mathcal{E})$. A direct argument as in the proof of Proposition 3.3.8 would not work properly since our bound on $|Y_{m,0}|, |Y_{m,\theta}|$ is not yet uniform in $m \in (M, \log K_r]$. For this reason, we

condition on the values $\{Y_{m,0}, Y_{m,\theta}\}_{M<m\leqslant M'}$ and consider the ballot event with partial sums of $\{Y_{m,0}, Y_{m,\theta}\}_{M'<m\leqslant \log K_r}$ where $M' = \max\{M, A\}$. Observe that by definition (3.63), on the event $\mathcal{E}$,

$$B' - L_1 M' \leqslant \sum_{M<j\leqslant M'} Y_{j,0}, \sum_{M<j\leqslant M'} Y_{j,\theta} \leqslant B,$$

and $|Y_{m,0}|, |Y_{m,\theta}| \ll m$ for $M' < m \leqslant \log K_r$, using $M' \gg B - B'$. Therefore,

$$\mathbb{Q}^{(2)}(\mathcal{E}) = \mathbb{E}\Big[\mathbb{Q}^{(2)}(\mathcal{E} \mid \{Y_{m,0}, Y_{m,\theta}\}_{M<m\leqslant M'})\Big]$$

$$\leqslant \mathbb{Q}^{(2)}\Bigg(\forall M' < m \leqslant \log K_r,\ B' - L_1 m - B \leqslant \sum_{M'<j\leqslant m} Y_{j,0}, \sum_{M'<j\leqslant m} Y_{j,\theta} \leqslant B - B' + L_1 M'$$

$$\text{and} \quad \forall M' < m \leqslant \log K_r,\ |Y_{m,0}|, |Y_{m,\theta}| \ll m\Bigg). \tag{3.68}$$

Given a set of values $\{Y_{m,0}, Y_{m,\theta}\}_{M<m\leqslant \log K_r}$ satisfying the event in (3.68), there exist numbers $u_m, v_m \in \mathcal{S}_m := \{t \in (1/m^3)\mathbb{Z} : |t| \ll m\}$, $M' < m \leqslant \log K_r$ such that for all $M' < m \leqslant \log K_r$,

$$u_m \leqslant Y_{m,0} < u_m + m^{-3};\ v_m \leqslant Y_{m,\theta} < v_m + m^{-3},$$

and in particular,

$$B' - L_1 m - B - \sum_{M'<j\leqslant m} m^{-3} \leqslant \sum_{M'<j\leqslant m} u_j, \sum_{M'<j\leqslant m} v_j \leqslant B - B' + L_1 M'. \tag{3.69}$$

Denote by $\mathcal{C}(M', K_r)$ the set of all possible vectors $(u_m, v_m)_{M'\leqslant m\leqslant \log K_r},\ u_m, v_m \in \mathcal{S}_m$ satisfying (3.69). Using Lemma 3.3.16 and (3.68), we have

$$\mathbb{Q}^{(2)}(\mathcal{E})$$

$$\leqslant \mathbb{Q}^{(2)}\Bigg(\forall M' < m \leqslant \log K_r,\ B' - L_1 m - B \leqslant \sum_{M'<j\leqslant m} Y_{j,0}, \sum_{M'<j\leqslant m} Y_{j,\theta} \leqslant B - B' + L_1 M'$$

$$\text{and } \forall M' < m \leqslant \log K_r,\ |Y_{m,0}|, |Y_{m,\theta}| \ll m\Bigg)$$

$$\leqslant \sum_{(u_m,v_m)\in\mathcal{C}(M',K_r)} \prod_{M'<m\leqslant \log K_r} \mathbb{Q}^{(2)}\left(Y_{m,0}\in[u_m,u_m+\frac{1}{m^3}],\ Y_{m,\theta}\in[v_m,v_m+\frac{1}{m^3}]\right)$$

$$\leqslant \sum_{(u_m,v_m)\in\mathcal{C}(M',K_r)} \prod_{M'<m\leqslant \log K_r} (1+O(m^{-2}))$$

$$\mathbb{P}\left(N_{m,1}\in[u_m,u_m+\frac{1}{m^3}],\ N_{m,2}\in[v_m,v_m+\frac{1}{m^3}] \text{ for all } M'<m\leqslant \log K_r\right)$$

$$\ll\mathbb{P}\Bigg(B'-L_1m-2\sum_{j=1}^{m}\frac{1}{j^3}-B\leqslant \sum_{M'<\ell\leqslant m} N_{\ell,1},\ \sum_{M'<\ell\leqslant m} N_{\ell,2}\leqslant B+2\sum_{j=1}^{m}\frac{1}{j^3}-B'+L_1M'$$

$$\text{for all } M'<m\leqslant \log K_r\Bigg)$$

$$\leqslant \mathbb{P}\left(-Lm\leqslant \sum_{M'<\ell\leqslant m} N_{\ell,1},\ \sum_{M'<\ell\leqslant m} N_{\ell,2}\leqslant LM' \text{ for all } M'<m\leqslant \log K_r\right),$$

where in the last step we used that $B, B'\ll M'<m$ and L is some other absolute constant.

Before applying the Gaussian ballot theorem, we shall decorrelate each pair of random variables $(N_{m,1},N_{m,2})$. Recall from [SZ22, Section 12] that for any Borel set $B\subseteq\mathbb{R}^2$,

$$\mathbb{P}((N_{m,1},N_{m,2})\in B)\leqslant\sqrt{\frac{1+|\rho_m|}{1-|\rho_m|}}\,\mathbb{P}((\widetilde{N}_{m,1},\widetilde{N}_{m,2})\in B),$$

where $\widetilde{N}_{m,1},\widetilde{N}_{m,2}$ are i.i.d. $N(0,\sigma_m^2(1+|\rho_m|))$ distributed. Using (3.66), it is straightforward to see that

$$\prod_{M'<m\leqslant \log K_r}\sqrt{\frac{1+|\rho_m|}{1-|\rho_m|}}\ll 1. \tag{3.70}$$

By Lemma 3.3.4, we conclude that

$$\mathbb{Q}^{(2)}(\mathcal{E})\ll\mathbb{P}\left(-Lm\leqslant \sum_{M'<\ell\leqslant m} N_{\ell,1},\ \sum_{M'<\ell\leqslant m} N_{\ell,2}\leqslant LM' \text{ for } M'<m\leqslant \log K_r\right)$$

$$\ll \mathbb{P}\left(-Lm \leqslant \sum_{M'<\ell\leqslant m} \widetilde{N}_{\ell,1},\ \sum_{M'<\ell\leqslant m} \widetilde{N}_{\ell,2} \leqslant LM' \text{ for } M' < m \leqslant \log K_r\right)$$

$$= \left(\mathbb{P}\left(-Lm \leqslant \sum_{M'<\ell\leqslant m} \widetilde{N}_{\ell,1} \leqslant LM' \text{ for } M' < m \leqslant \log K_r\right)\right)^2$$

$$\ll \left(\frac{M'}{\sqrt{1+\log(K_r/e^{M'})}}\right)^2.$$

This yields the desired result. □

Proof of Lemma 3.3.16. Let $\mathbf{Y}_m := (Y_{m,0}, Y_{m,\theta})^\top, \mathbf{u} := (u_m, v_m)^\top$, and $\boldsymbol{\lambda} := \Sigma_m^{-1}\mathbf{u}$, where we recall that Σ_m is the covariance matrix of $(Y_{m,0}, Y_{m,\theta})$ under $\mathbb{Q}^{(2)}$. First, we apply an exponential tilt of the measure so that $(Y_{m,0}, Y_{m,\theta})$ is centered at (u_m, v_m). Define the tilted measure $\widetilde{\mathbb{Q}}^{(2)}$ by

$$\frac{\mathrm{d}\widetilde{\mathbb{Q}}^{(2)}}{\mathrm{d}\mathbb{Q}^{(2)}} := \frac{\exp(\boldsymbol{\lambda}\cdot\mathbf{Y}_m)}{\mathbb{Q}^{(2)}(\exp(\boldsymbol{\lambda}\cdot\mathbf{Y}_m))}.$$

Let $\widehat{\mathbf{N}}_m := (\widehat{N}_{m,1}, \widehat{N}_{m,2})$ be a two-dimensional Gaussian vector with the same mean and covariance matrix as $\mathbf{Y}_m$ under $\widetilde{\mathbb{Q}}^{(2)}$. By (3.67) and since exponential tilts preserves the covariance for Gaussians, we have

$$\mathbb{E}[e^{i\mathbf{x}\cdot\widehat{\mathbf{N}}_m}] = \exp\left(-\frac{1}{2}\mathbf{x}^\top\Sigma_m\mathbf{x} + i\mathbf{u}\cdot\mathbf{x}\right),\ \mathbf{x}\in\mathbb{R}^2.$$

The corresponding characteristic function of $\mathbf{Y}_m$ under the tilted measure $\widetilde{\mathbb{Q}}^{(2)}$ is given by

$$\mathbb{E}^{\widetilde{\mathbb{Q}}^{(2)}}[e^{i\mathbf{x}\cdot\mathbf{Y}_m}] = \exp\left(-\frac{1}{2}\mathbf{x}^\top\Sigma_m\mathbf{x} + i\mathbf{u}\cdot\mathbf{x} + S_m(\mathbf{x}-i\boldsymbol{\lambda}) - S_m(-i\boldsymbol{\lambda})\right),\ \mathbf{x}\in\mathbb{R}^2, \tag{3.71}$$

where we recall S_m from (3.65). Following [Sad66], we have

$$\begin{aligned}
&\Big|\widetilde{\mathbb{Q}}^{(2)}(u_m \leqslant Y_{m,0} \leqslant u_m + m^{-3},\ v_m \leqslant Y_{m,\theta} \leqslant v_m + m^{-3}) \\
&\qquad\qquad - \mathbb{P}(u_m \leqslant \widehat{N}_{m,1} \leqslant u_m + m^{-3},\ v_m \leqslant \widehat{N}_{m,2} \leqslant v_m + m^{-3})\Big| \\
&\ll \int_{-e^{m/9}}^{e^{m/9}} \int_{-e^{m/9}}^{e^{m/9}} \left|\frac{\Delta(s,t)}{st}\right| \mathrm{d}s\,\mathrm{d}t + \int_{-e^{m/9}}^{e^{m/9}} \left|\frac{\mathbb{E}^{\widetilde{\mathbb{Q}}^{(2)}}[\exp(isY_{m,0})] - \mathbb{E}[\exp(is\widehat{N}_{m,1})]}{s}\right| \mathrm{d}s \\
&\qquad + \int_{-e^{m/9}}^{e^{m/9}} \left|\frac{\mathbb{E}^{\widetilde{\mathbb{Q}}^{(2)}}[\exp(itY_{m,\theta})] - \mathbb{E}[\exp(it\widehat{N}_{m,2})]}{t}\right| \mathrm{d}t + e^{-m/9},
\end{aligned} \tag{3.72}$$

where

$$\begin{aligned}
\Delta(s,t) &:= \mathbb{E}^{\widetilde{\mathbb{Q}}^{(2)}}\left[\exp\left(isY_{m,0} + itY_{m,\theta}\right)\right] - \mathbb{E}[\exp(is\widehat{N}_{m,1} + it\widehat{N}_{m,2})] \\
&\quad - \mathbb{E}^{\widetilde{\mathbb{Q}}^{(2)}}[\exp(isY_{m,0})]\,\mathbb{E}^{\widetilde{\mathbb{Q}}^{(2)}}[\exp(itY_{m,\theta})] + \mathbb{E}[\exp(is\widehat{N}_{m,1})]\,\mathbb{E}[\exp(it\widehat{N}_{m,2})].
\end{aligned}$$

For $\mathbf{x}_1 = (s,0)$, we estimate

$$\begin{aligned}
|S_m(\mathbf{x}_1 - i\boldsymbol{\lambda}) - S_m(-i\boldsymbol{\lambda})| &= \left|\int_0^s \frac{\mathrm{d}S_m(t - i\lambda_1, -i\lambda_2)}{\mathrm{d}t}\mathrm{d}t\right| \\
\ll& e^{-m/2}\int_0^{|s|}(1 + |t|^2 + \|\boldsymbol{\lambda}\|^2)\mathrm{d}t \ll e^{-m/2}(1 + \|\mathbf{u}\|^2)(|s| + |s|^3),
\end{aligned} \tag{3.73}$$

where we used $\|\boldsymbol{\lambda}\| \leqslant \|\Sigma_m^{-1}\|_{\mathrm{op}}\|\mathbf{u}\| = (\sigma_m^2 - |\rho_m\sigma_m^2|)^{-1}\|\mathbf{u}\| \ll \|\mathbf{u}\|$, since $\sigma_m^2 \gg 1$ and $\rho_m\sigma_m^2 = o(1)$ by Lemma 3.7.5. Inserting back into (3.71), the second integral of (3.72) can be bounded by

$$\begin{aligned}
&\int_{-e^{m/9}}^{e^{m/9}} \left|\frac{\mathbb{E}^{\widetilde{\mathbb{Q}}^{(2)}}[\exp(isY_{m,0})] - \mathbb{E}[\exp(is\widehat{N}_{m,1})]}{s}\right| \mathrm{d}s \\
\leqslant& \int_{-e^{m/9}}^{e^{m/9}} \left|\frac{e^{S_m(\mathbf{x}_1 - i\boldsymbol{\lambda}) - S_m(-i\boldsymbol{\lambda})} - 1}{s}\right| e^{-\sigma_m^2 s^2/2}\mathrm{d}s \\
\ll& \int_{-e^{m/9}}^{e^{m/9}} \left|\frac{S_m(\mathbf{x}_1 - i\boldsymbol{\lambda}) - S_m(-i\boldsymbol{\lambda})}{s}\right| e^{-\sigma_m^2 s^2/2}\mathrm{d}s
\end{aligned}$$

$$\leqslant e^{-m/2}(1+u_m^2+v_m^2)\int_{-e^{m/9}}^{e^{m/9}}(1+s^2)e^{-\sigma_m^2 s^2/2}\mathrm{d}s$$

$$\ll e^{-m/2}m^2 \ll e^{-m/9}.$$

A similar estimate holds for the third term. We now bound the first integral of (3.72), where we consider $\mathbf{x}_2=(0,t)$ and $\mathbf{x}_3=(s,t)$ in (3.71):

$$\begin{aligned}
|\Delta(s,t)| &= \Big|\exp\Big(-\frac{\sigma_m^2}{2}\|\mathbf{x}_3\|^2+i\mathbf{u}\cdot\mathbf{x}_3\Big)\Big[\exp(-\rho_m\sigma_m^2 st+S_m(\mathbf{x}_3-i\boldsymbol{\lambda})-S_m(-i\boldsymbol{\lambda}))\\
&\quad +1-\exp\Big(-\rho_m\sigma_m^2 st\Big)-\exp(S_m(\mathbf{x}_1-i\boldsymbol{\lambda})+S_m(\mathbf{x}_2-i\boldsymbol{\lambda})-2S_m(-i\boldsymbol{\lambda}))\Big]\Big|\\
&\leqslant \exp\Big(-\frac{\sigma_m^2}{2}\|\mathbf{x}_3\|^2\Big)\Big[\Big|e^{-\rho_m\sigma_m^2 st}-1\Big|\Big|e^{S_m(\mathbf{x}_3-i\boldsymbol{\lambda})-S_m(-i\boldsymbol{\lambda})}-1\Big|\\
&\quad +\Big|e^{S_m(\mathbf{x}_3-i\boldsymbol{\lambda})-S_m(-i\boldsymbol{\lambda})}\Big|\Big|e^{S_m(\mathbf{x}_1-i\boldsymbol{\lambda})+S_m(\mathbf{x}_2-i\boldsymbol{\lambda})-S_m(\mathbf{x}_3-i\boldsymbol{\lambda})-S_m(-i\boldsymbol{\lambda})}-1\Big|\Big]\\
&\ll \exp\Big(-\frac{\sigma_m^2}{2}\|\mathbf{x}_3\|^2\Big)\Big[|st||S_m(\mathbf{x}_3-i\boldsymbol{\lambda})-S_m(-i\boldsymbol{\lambda})|\\
&\quad +|S_m(\mathbf{x}_1-i\boldsymbol{\lambda})+S_m(\mathbf{x}_2-i\boldsymbol{\lambda})-S_m(\mathbf{x}_3-i\boldsymbol{\lambda})-S_m(-i\boldsymbol{\lambda})|\Big]
\end{aligned}$$

where we used Lemma 3.7.5 in the last step. Similarly to (3.73), we have

$$\begin{aligned}
&|S_m(\mathbf{x}_3-i\boldsymbol{\lambda})-S_m(-i\boldsymbol{\lambda})|\\
\leqslant&|S_m(\mathbf{x}_3-i\boldsymbol{\lambda})-S_m(\mathbf{x}_1-i\boldsymbol{\lambda})|+|S_m(\mathbf{x}_1-i\boldsymbol{\lambda})-S_m(-i\boldsymbol{\lambda})|\\
\ll&e^{-m/2}(1+s^2+\|\mathbf{u}\|^2)(|t|+|t|^3)+e^{-m/2}(1+\|\mathbf{u}\|^2)(|s|+|s|^3)
\end{aligned}$$

and also

$$\begin{aligned}
&|S_m(\mathbf{x}_1-i\boldsymbol{\lambda})+S_m(\mathbf{x}_2-i\boldsymbol{\lambda})-S_m(\mathbf{x}_3-i\boldsymbol{\lambda})-S_m(-i\boldsymbol{\lambda})|\\
=&\Big|\int_0^s\int_0^t\frac{\partial^2 S_m(u-i\lambda_1,v-i\lambda_2)}{\partial u\partial v}\mathrm{d}u\mathrm{d}v\Big|\\
\ll&e^{-m/2}(1+\|\boldsymbol{\lambda}\|_1)(1+|s|+|t|)|st|\\
\ll&me^{-m/2}(1+|s|+|t|)|st|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{-e^{m/9}}^{e^{m/9}}\int_{-e^{m/9}}^{e^{m/9}}\left|\frac{\Delta(s,t)}{st}\right|\mathrm{d}s\mathrm{d}t\\
\ll& e^{-m/2}\int_{-e^{m/9}}^{e^{m/9}}\int_{-e^{m/9}}^{e^{m/9}}\exp\Big(-\frac{\sigma_m^2}{2}\|\mathbf{x}_3\|^2\Big)\Big[(1+s^2+\|\mathbf{u}\|^2)(|t|+|t|^3)\\
&+(1+\|\mathbf{u}\|^2)(|s|+|s|^3)+m(1+|s|+|t|)|st|\Big]\mathrm{d}s\mathrm{d}t\\
\ll& m^2e^{-m/2}\ll e^{-m/9}.
\end{aligned}$$

Plugging back to (3.72), we obtain

$$\begin{aligned}
\Big|&\widetilde{\mathbb{Q}}^{(2)}(u_m\leqslant Y_{m,0}\leqslant u_m+m^{-3},\ v_m\leqslant Y_{m,\theta}\leqslant v_m+m^{-3})\\
&-\mathbb{P}(u_m\leqslant \widehat{N}_{m,1}\leqslant u_m+m^{-3},\ v_m\leqslant \widehat{N}_{m,2}\leqslant v_m+m^{-3})\Big|\ll e^{-m/9}
\end{aligned}$$

uniformly for all $|u_m|,|v_m|\ll m$, $M\leqslant m\leqslant \log K_r$. We may replace the absolute error by a multiplier of $(1+O(e^{-m/10}))$, since $(\widehat{N}_{m,1},\widehat{N}_{m,2})$ is centered at (u_m,v_m) and has constant order variances with vanishing correlation. This gives that for $|u_m|,|v_m|\ll m$ and $M\leqslant m\leqslant \log K_r$,

$$\begin{aligned}
&\widetilde{\mathbb{Q}}^{(2)}(u_m\leqslant Y_{m,0}\leqslant u_m+m^{-3},\ v_m\leqslant Y_{m,\theta}\leqslant v_m+m^{-3})\\
=&(1+O(e^{-m/10}))\mathbb{P}(u_m\leqslant \widehat{N}_{m,1}\leqslant u_m+m^{-3},\ v_m\leqslant \widehat{N}_{m,2}\leqslant v_m+m^{-3}).
\end{aligned}$$

Now note that if $Y_{m,0}\in[u_m,u_m+m^{-3}]$ and $Y_{m,\theta}\in[v_m,v_m+m^{-3}]$, it holds that $|\boldsymbol{\lambda}\cdot\boldsymbol{Y}_m-\boldsymbol{\lambda}\cdot\boldsymbol{u}|\ll\|\boldsymbol{u}\|/m^3\ll m^{-2}$. Therefore, we have

$$\begin{aligned}
&\mathbb{Q}^{(2)}(u_m\leqslant Y_{m,0}\leqslant u_m+m^{-3},\ v_m\leqslant Y_{m,\theta}\leqslant v_m+m^{-3})\\
=&\mathbb{E}^{\mathbb{Q}^{(2)}}[e^{\boldsymbol{\lambda}\cdot\mathbf{Y}_m}]\,\widetilde{\mathbb{Q}}^{(2)}\Big(e^{-\boldsymbol{\lambda}\cdot\mathbf{Y}_m}1_{\{u_m\leqslant Y_{m,0}\leqslant u_m+m^{-3},\ v_m\leqslant Y_{m,\theta}\leqslant v_m+m^{-3}\}}\Big)\\
=&\exp\Big(-\frac{1}{2}\mathbf{u}^\top\Sigma_m^{-1}\mathbf{u}+S_m(-i\boldsymbol{\lambda})\Big)(1+O(m^{-2}))\\
&\times\widetilde{\mathbb{Q}}^{(2)}(u_m\leqslant Y_{m,0}\leqslant u_m+m^{-3},\ v_m\leqslant Y_{m,\theta}\leqslant v_m+m^{-3})
\end{aligned}$$

$$=(1+O(m^{-2}))\exp\Big(-\frac{1}{2}\mathbf{u}^\top\Sigma_m^{-1}\mathbf{u}\Big)(1+O(e^{-m/10}))$$
$$\times\,\mathbb{P}(u_m\leqslant\widehat{N}_{m,1}\leqslant u_m+m^{-3},\ v_m\leqslant\widehat{N}_{m,2}\leqslant v_m+m^{-3}).$$

Thus,

$$\mathbb{Q}^{(2)}(u_m\leqslant Y_{m,0}\leqslant u_m+m^{-3},\ v_m\leqslant Y_{m,\theta}\leqslant v_m+m^{-3})$$
$$=(1+O(m^{-2}))e^{-\frac{1}{2}\mathbf{u}^\top\Sigma_m^{-1}\mathbf{u}}\,\mathbb{P}(u_m\leqslant\widehat{N}_{m,1}\leqslant u_m+m^{-3},\ v_m\leqslant\widehat{N}_{m,2}\leqslant v_m+m^{-3}).$$

By a standard Gaussian computation,

$$\mathbb{P}(u_m\leqslant N_{m,1}\leqslant u_m+m^{-3},\ v_m\leqslant N_{m,2}\leqslant v_m+m^{-3})$$
$$=(1+O(m^{-2}))e^{-\frac{1}{2}\mathbf{u}^\top\Sigma_m^{-1}\mathbf{u}}\,\mathbb{P}(u_m\leqslant\widehat{N}_{m,1}\leqslant u_m+m^{-3},\ v_m\leqslant\widehat{N}_{m,2}\leqslant v_m+m^{-3}).$$

The proof is then complete. □

3.4 The stretched exponential phase

Recall in the stretched exponential case (SE), $\mathbb{P}(|R_k|\geqslant u)=\exp(-(u/c_p)^p)$ with $0<p<1$ and $c_p=(2\Gamma(2/p)/p)^{-1/2}$. Denoting by $\lambda^*=(1,\cdots,1)$ the all-one partition of N, Theorem 3.1.3 in the (SE) case follows from the following proposition, which shows that the contribution from λ^* alone is the main term in low moments of A_N.

Proposition 3.4.1. *For all N large enough and any $q>0$,*

$$\mathbb{E}\left[|A_N|^{2q}\right]=(1+o(1))\mathbb{E}\left[|a(\lambda^*)|^{2q}\right]=(1+o(1))(2\pi)^{1/2-q}\sqrt{\frac{2q}{p}}C_{p,q}^{2qN/p}N^{2q(1/p-1)N+1/2-q},\tag{3.74}$$

where $C_{p,q}=2qc_p^p/(pe^{1-p})$.

Proof. Recall that $a(\lambda^*)=X_1^N/N!$. The asymptotic formula of $\mathbb{E}\left[|a(\lambda^*)|^{2q}\right]$ then follows from (3.16). For the first asymptotic relation, by concavity, we have for

$0 < q < 1/2$,

$$\mathbb{E}\left[|a(\lambda^*)|^{2q}\right] - \sum_{\substack{|\lambda|=N \\ m_1<N}} \mathbb{E}\left[|a(\lambda)|^{2q}\right] \leqslant \mathbb{E}\left[|A_N|^{2q}\right] \leqslant \mathbb{E}\left[|a(\lambda^*)|^{2q}\right] + \sum_{\substack{|\lambda|=N \\ m_1<N}} \mathbb{E}\left[|a(\lambda)|^{2q}\right],$$

and by Minkowski's inequality, for $q \geqslant 1/2$,

$$\begin{aligned}
&\mathbb{E}\left[|a(\lambda^*)|^{2q}\right]^{1/(2q)} - \sum_{\substack{|\lambda|=N \\ m_1<N}} \mathbb{E}\left[|a(\lambda)|^{2q}\right]^{1/(2q)} \\
\leqslant &\mathbb{E}\left[|A_N|^{2q}\right]^{1/(2q)} \\
\leqslant &\mathbb{E}\left[|a(\lambda^*)|^{2q}\right]^{1/(2q)} + \sum_{\substack{|\lambda|=N \\ m_1<N}} \mathbb{E}\left[|a(\lambda)|^{2q}\right]^{1/(2q)}.
\end{aligned}$$

Therefore it suffices to show for $0 < q < 1/2$,

$$\sum_{\substack{|\lambda|=N \\ m_1<N}} \mathbb{E}\left[|a(\lambda)|^{2q}\right] = o\Big(\mathbb{E}\left[|a(\lambda^*)|^{2q}\right]\Big) \tag{3.75}$$

and for $q \geqslant 1/2$,

$$\sum_{\substack{|\lambda|=N \\ m_1<N}} \mathbb{E}\left[|a(\lambda)|^{2q}\right]^{1/(2q)} = o\Big(\mathbb{E}\left[|a(\lambda^*)|^{2q}\right]^{1/(2q)}\Big) \tag{3.76}$$

as $N \to \infty$. Fix $q \in (0, 1/2)$, by (3.15) we have

$$\begin{aligned}
\sum_{\substack{|\lambda|=N \\ m_1<N}} \mathbb{E}\left[|a(\lambda)|^{2q}\right] &\leqslant \sum_{k=1}^{N} \sum_{\substack{|\lambda'|=k \\ m_1(\lambda')=0}} \mathbb{E}\left[\left|\frac{X_1^{N-k}}{(N-k)!} a(\lambda')\right|^{2q}\right] \\
&\ll \sum_{k=1}^{N} C_{p,q}^{2q(N-k)/p} (N-k)^{2q(1/p-1)(N-k)+1/2-q} \sum_{\substack{|\lambda'|=k \\ m_1(\lambda')=0}} \mathbb{E}\left[|a(\lambda')|^{2q}\right].
\end{aligned} \tag{3.77}$$

For each λ' as a partition of $k \in \{1, \dots, N\}$ with $m_1(\lambda') = 0$, we have with some constant $c = c_{p,q} > 0$ (which may vary from line to line) that

$$\mathbb{E}\left[|a(\lambda')|^{2q}\right] \ll \prod_{j=2}^{k} c C_{p,q}^{2qm_j/p} m_j^{2q(1/p-1)m_j+1} \ll c^k \Big(\frac{k}{2}\Big)^{q(1/p-1)k+\sqrt{2k}},$$

where in the last inequality we use the fact that $|\lambda'| = k = \sum_{j=2}^{k} j m_j \geqslant 2\sum_{j=2}^{k} m_j$, and that $\sum_{j=2}^{k} 1_{\{m_j>0\}} \leqslant \sqrt{2k}$. Recall the partition number $p_k \ll e^{\pi\sqrt{2/3}\sqrt{k}}$, we can bound (3.77) from above by

$$\begin{aligned}
&\sum_{\substack{|\lambda|=N \\ m_1<N}} \mathbb{E}\left[|a(\lambda)|^{2q}\right] \\
\ll &\sum_{k=1}^{N} C_{p,q}^{2q(N-k)/p} (N-k)^{2q(1/p-1)(N-k)+1/2-q} e^{\pi\sqrt{2/3}\sqrt{k}} c^k \Big(\frac{k}{2}\Big)^{q(1/p-1)k+\sqrt{2k}} \\
\ll &\mathbb{E}\left[|a(\lambda^*)|^{2q}\right] \sum_{k=1}^{\infty} C_{p,q}^{-2qk/p} e^{ck} \frac{(N-k)^{2q(1/p-1)(N-k)+1/2-q} (k/2)^{q(1/p-1)k+\sqrt{2k}}}{N^{2q(1/p-1)N+1/2-q}} \\
\ll &\mathbb{E}\left[|a(\lambda^*)|^{2q}\right] \sum_{k=1}^{\infty} \exp\left(ck - q\Big(\frac{1}{p}-1\Big)(2N\log N - 2(N-k)\log(N-k) - k\log k)\right) \\
\ll &\mathbb{E}\left[|a(\lambda^*)|^{2q}\right] \sum_{k=1}^{\infty} \exp\left(ck - q\Big(\frac{1}{p}-1\Big) k\log N\right) \\
\ll &\mathbb{E}\left[|a(\lambda^*)|^{2q}\right] \sum_{k=1}^{\infty} N^{-ck} = o\left(\mathbb{E}\left[|a(\lambda^*)|^{2q}\right]\right),
\end{aligned}$$

establishing (3.75) and hence (3.74). A similar computation proves (3.76) and hence the result follows. □

3.5 The exponential phase

Let us recall that in the exponential phase (EXP), $\mathbb{P}(|R_k| \geqslant u) = \exp(-\gamma(u - c_\gamma)) \wedge 1$ for $u \geqslant 0$, where $\gamma \in (0, 2q]$ and $c_\gamma = \log(\gamma^2/2)/\gamma$. Our goal is to provide asymptotics

for $\mathbb{E}[|A_N|^{2q}]$ as $N \to \infty$ where $q \in (0,1]$. To this end, we split into two cases: $\gamma < 2q$ and $\gamma = 2q$.

3.5.1 The case $\gamma < 2q$

In this section, we prove Theorem 3.1.4 for $\gamma < 2q$, that

$$\mathbb{E}[|A_N|^{2q}] \asymp \phi(N) = \phi_{q,\gamma}(N) := N^{1/2-q}\left(\frac{2q}{\gamma}\right)^{2qN}, \quad q \in (0,1],\ \gamma < 2q. \tag{3.78}$$

Proof of the upper bound

A direct application of Minkowski's inequality together with (3.24) yields for $q \geqslant 1/2$,

$$\begin{aligned}
&\mathbb{E}[|A_N|^{2q}]^{1/(2q)} \\
&\leqslant \sum_{\lambda\in\mathcal{P}_N} \mathbb{E}\left[\left|\prod_{k\geqslant 1}\left(\frac{X_k}{\sqrt{k}}\right)^{m_k}\frac{1}{m_k!}\right|^{2q}\right]^{1/(2q)} \\
&\ll \sum_{\lambda\in\mathcal{P}_N}\prod_{\substack{k\geqslant 1\\ m_k\neq 0}}\frac{C\gamma^{-m_k}\Gamma(2m_kq+1)^{1/(2q)}}{k^{m_k/2}m_k!} \\
&\ll \frac{\Gamma(2Nq+1)^{1/(2q)}}{\gamma^N N!} + \sum_{j=2}^{N}\sum_{\substack{\lambda\in\mathcal{P}_N\\ m_1(\lambda)=N-j}}\frac{\Gamma(2(N-j)q+1)^{1/(2q)}}{\gamma^{N-j}(N-j)!}\prod_{\substack{k\geqslant 2\\ m_k\neq 0}}\frac{C\Gamma(2m_kq+1)^{1/(2q)}}{\gamma^{m_k}k^{m_k/2}m_k!}.
\end{aligned}$$

By (3.15),

$$\frac{\Gamma(2m_kq+1)^{1/(2q)}}{m_k!} \leqslant C(2q)^{m_k}m_k^{1/(4q)-1/2}.$$

On the other hand,

$$\prod_{\substack{k\geqslant 2\\ m_k\neq 0}}\frac{Cm_k^{1/(4q)-1/2}}{k^{m_k/2}} \ll 1,$$

since the product on $k \geqslant C^2$ can be bounded by one and the product on $2 \leqslant k \leqslant C^2$ is bounded by a constant, where C may depend on q. Therefore, for $\lambda \in \mathcal{P}_N$ with

$m_1(\lambda) = N - j$, we have

$$\prod_{\substack{k\geqslant 2\\ m_k\neq 0}} \frac{C\gamma^{-m_k}\Gamma(2m_kq+1)^{1/(2q)}}{k^{m_k/2}m_k!} \ll \prod_{\substack{k\geqslant 2\\ m_k\neq 0}} \left(\frac{2q}{\gamma}\right)^{m_k} \leqslant \left(\frac{2q}{\gamma}\right)^{j/2}.$$

Combining the above leads to

$$\begin{aligned}
&\mathbb{E}[|A_N|^{2q}]^{1/(2q)} \\
&\ll \frac{\gamma^{-N}\Gamma(2Nq+1)^{1/(2q)}}{N!} + \sum_{j=2}^{N} \sum_{\substack{\lambda\in\mathcal{P}_N\\ m_1(\lambda)=N-j}} \frac{\gamma^{-(N-j)}\Gamma(2(N-j)q+1)^{1/(2q)}(2q/\gamma)^{j/2}}{(N-j)!} \\
&\ll \gamma^{-N}(2q)^N N^{1/(4q)-1/2} + \sum_{j=2}^{N} p_j \gamma^{-(N-j/2)}(2q)^{N-j/2}(N-j)^{1/(4q)-1/2} \\
&\ll \gamma^{-N}(2q)^N N^{1/(4q)-1/2}\left(1 + \sum_{j=2}^{N} e^{C\sqrt{j}}\Big(\frac{\gamma}{2q}\Big)^{j/2}\Big(\frac{N-j}{N}\Big)^{1/(4q)-1/2}\right) \\
&\ll \left(\frac{2q}{\gamma}\right)^N N^{1/(4q)-1/2} = \phi(N)^{1/(2q)},
\end{aligned}$$

where we have used (3.15) again. This leads to the upper bound in (3.78). The case $q \in (0, 1/2)$ is similar by using concavity instead of Minkowski's inequality.

Proof of the lower bound

Recall from (3.78) that our goal is to show $\mathbb{E}[|A_N|^{2q}] \gg \phi(N)$. Observe the decomposition

$$A_N = \sum_{\substack{\lambda\in\mathcal{P}_N\\ m_1(\lambda)\geqslant N-C}} \prod_{k\geqslant 1}\left(\frac{X_k}{\sqrt{k}}\right)^{m_k}\frac{1}{m_k!} + \sum_{\substack{\lambda\in\mathcal{P}_N\\ m_1(\lambda)< N-C}} \prod_{k\geqslant 1}\left(\frac{X_k}{\sqrt{k}}\right)^{m_k}\frac{1}{m_k!} \tag{3.79}$$

for $C \geqslant 0$. A careful examination of the proof of the upper bound yields the following lemma, giving an upper bound for the second term in (3.79).

Lemma 3.5.1. *For any $\varepsilon > 0$, there exists $C = C(\varepsilon) > 0$ such that*

$$\mathbb{E}\Big[\Big| \sum_{\substack{\lambda\in\mathcal{P}_N \\ m_1(\lambda)<N-C}} \prod_{k\geqslant 1} \Big(\frac{X_k}{\sqrt{k}}\Big)^{m_k} \frac{1}{m_k!}\Big|^{2q}\Big] \leqslant \varepsilon\phi(N).$$

We will focus on the sum over partitions that have a large number of ones, and use Lemma 3.5.1 to show that the remaining terms are negligible. To this end, let us fix a large constant $C_* > 0$ to be determined. For a given $\lambda \in \mathcal{P}_N$ with $m_1(\lambda) \geqslant N - C_*$, we let λ_* denote the partition after removing $N - C_*$ ones from λ. In particular, $\lambda_* \in \mathcal{P}_{C_*}$ and there is a bijective correspondence between such λ and λ_*.

Consider $M = M(\gamma, q) \in \mathbb{N}$ such that

$$\Big(\frac{2q}{\gamma}\Big)^{2M} > \Big(\frac{2}{\gamma}\Big)^{2}. \tag{3.80}$$

We will restrict to the event that $|X_1| \in [(2qN-\sqrt{N})/\gamma, 2qN/\gamma]$ and $|X_2|, \dots, |X_M| \leqslant 1$. More precisely, we have

$$\begin{aligned}\mathbb{E}[|A_N|^{2q}] &\geqslant \mathbb{E}\Big[|A_N|^{2q} 1_{\{|R_1|\in[(2qN-\sqrt{N})/\gamma, 2qN/\gamma],\, |R_j|\leqslant 1,\, 2\leqslant j\leqslant M\}}\Big] \\ &\geqslant e^{-LM}\mathbb{E}\left[|A_N|^{2q} 1_{\{|R_1|\in[(2qN-\sqrt{N})/\gamma, 2qN/\gamma]\}} \mid \{|R_j| \leqslant 1,\, 2 \leqslant j \leqslant M\}\right].\end{aligned} \tag{3.81}$$

Therefore, in the following, we may without loss of generality condition on the event $\{|R_j| \leqslant 1,\, 2 \leqslant j \leqslant M\}$. The resulting probability measure and expectation operator will be denoted by $\widetilde{\mathbb{P}}$ and $\widetilde{\mathbb{E}}$ respectively.

We first make a few simplifications to the first sum of (3.79). It holds that

$$\begin{aligned}&\Big| \sum_{\substack{\lambda\in\mathcal{P}_N \\ m_1(\lambda)\geqslant N-C_*}} \prod_{k\geqslant 1} \Big(\frac{X_k}{\sqrt{k}}\Big)^{m_k(\lambda)} \frac{1}{m_k(\lambda)!}\Big| \\ =&|X_1|^{N-C_*}\Big| \sum_{\lambda_*\in\mathcal{P}_{C_*}} \frac{m_1(\lambda_*)!}{(m_1(\lambda_*)+N-C_*)!} \prod_{k\geqslant 1} \Big(\frac{X_k}{\sqrt{k}}\Big)^{m_k(\lambda_*)} \frac{1}{m_k(\lambda_*)!}\Big|.\end{aligned} \tag{3.82}$$

Here, with the bijective correspondence between λ and λ_* described above, we have $m_k(\lambda) = m_k(\lambda_*)$ except when $k = 1$. When the situation is clear, we omit writing the dependence on λ or λ_*. With $|X_1| \approx 2qN/\gamma$, we expect that

$$\sum_{\lambda_* \in \mathcal{P}_{C_*}} \frac{m_1!}{(m_1 + N - C_*)!} \prod_{k \geqslant 1} \left(\frac{X_k}{\sqrt{k}}\right)^{m_k} \frac{1}{m_k!}$$

$$\approx \sum_{\lambda_* \in \mathcal{P}_{C_*}} \left(\frac{2qNe^{i\tau_1}}{\gamma}\right)^{m_1} \frac{1}{(m_1 + N - C_*)!} \prod_{k \geqslant 2} \left(\frac{X_k}{\sqrt{k}}\right)^{m_k} \frac{1}{m_k!}.$$

The goal of the next lemma is to make this precise.

Lemma 3.5.2. *On the event $|R_1| = |X_1| \in [(2qN - \sqrt{N})/\gamma, 2qN/\gamma]$, it holds that*

$$\Big| \sum_{\lambda_* \in \mathcal{P}_{C_*}} \frac{m_1!}{(m_1 + N - C_*)!} \prod_{k \geqslant 1} \left(\frac{X_k}{\sqrt{k}}\right)^{m_k} \frac{1}{m_k!}$$

$$- \sum_{\lambda_* \in \mathcal{P}_{C_*}} \left(\frac{2qNe^{i\tau_1}}{\gamma}\right)^{m_1} \frac{1}{(N - C_*)! N^{m_1}} \prod_{k \geqslant 2} \left(\frac{X_k}{\sqrt{k}}\right)^{m_k} \frac{1}{m_k!} \Big|$$

$$\ll \sum_{\lambda_* \in \mathcal{P}_{C_*}} \frac{(2q/\gamma)^{m_1} m_1 C_*}{\sqrt{N}(N - C_*)!} \prod_{k \geqslant 2} \left(\frac{X_k}{\sqrt{k}}\right)^{m_k} \frac{1}{m_k!}.$$

Proof. Consider $\lambda_* \in \mathcal{P}_{C_*}$. By triangle inequality,

$$\left| \frac{(N - C_*)!}{(m_1 + N - C_*)!} X_1^{m_1} - N^{-m_1} e^{i\tau_1 m_1} \left(\frac{2qN}{\gamma}\right)^{m_1} \right|$$

$$\leqslant \left| \frac{(N - C_*)!}{(m_1 + N - C_*)!} - N^{-m_1} \right| \cdot |X_1|^{m_1} + N^{-m_1} \Big| |R_1|^{m_1} - \left(\frac{2qN}{\gamma}\right)^{m_1} \Big|.$$

Using Taylor's expansion, we have for N large that

$$\prod_{j=1}^{m_1} \frac{N}{N - C_* + j} \leqslant \exp\left(\sum_{j=1}^{m_1} \frac{C_* - j}{N - C_* + j}\right) \ll \frac{m_1 C_*}{N - C_*}.$$

In addition, by the mean-value theorem

$$\left| |R_1|^{m_1} - \left(\frac{2qN}{\gamma}\right)^{m_1} \right| \ll m_1 \sqrt{N} \left(\frac{2qN}{\gamma}\right)^{m_1 - 1}.$$

Altogether, we have

$$\left|\frac{(N-C_*)!}{(m_1+N-C_*)!}\,X_1^{m_1}-N^{-m_1}e^{i\tau_1 m_1}\left(\frac{2qN}{\gamma}\right)^{m_1}\right|\ll\frac{(2q/\gamma)^{m_1}m_1C_*}{\sqrt{N}}. \tag{3.83}$$

Next, we use (3.83) and the triangle inequality to obtain

$$\begin{aligned}
&\Big|\sum_{\lambda_*\in\mathcal{P}_{C_*}}\frac{m_1!}{(m_1+N-C_*)!}\prod_{k\geqslant 1}\left(\frac{X_k}{\sqrt{k}}\right)^{m_k}\frac{1}{m_k!}\\
&-\sum_{\lambda_*\in\mathcal{P}_{C_*}}\left(\frac{2qNe^{i\tau_1}}{\gamma}\right)^{m_1}\frac{1}{(N-C_*)!N^{m_1}}\prod_{k\geqslant 2}\left(\frac{X_k}{\sqrt{k}}\right)^{m_k}\frac{1}{m_k!}\Big|\\
&\leqslant\sum_{\lambda_*\in\mathcal{P}_{C_*}}\Big|\frac{X_1^{m_1}}{(m_1+N-C_*)!}-\left(\frac{2qNe^{i\tau_1}}{\gamma}\right)^{m_1}\frac{1}{(N-C_*)!N^{m_1}}\Big|\prod_{k\geqslant 2}\left(\frac{X_k}{\sqrt{k}}\right)^{m_k}\frac{1}{m_k!}\\
&\ll\sum_{\lambda_*\in\mathcal{P}_{C_*}}\frac{(2q/\gamma)^{m_1}m_1C_*}{\sqrt{N}(N-C_*)!}\prod_{k\geqslant 2}\left(\frac{X_k}{\sqrt{k}}\right)^{m_k}\frac{1}{m_k!},
\end{aligned}$$

as desired. □

Lemma 3.5.3. *There exists $\delta>0$ such that for C large enough,*

$$\widetilde{\mathbb{P}}\left(\left|\sum_{\lambda_*\in\mathcal{P}_C}\prod_{k\geqslant 2}\left(\frac{X_k}{\sqrt{k}(2qe^{i\tau_1}/\gamma)^k}\right)^{m_k}\frac{1}{m_k!}\right|>\delta\right)>\delta. \tag{3.84}$$

Proof. Denote by

$$\xi_C:=\sum_{\lambda_*\in\mathcal{P}_C}\prod_{k\geqslant 2}\left(\frac{X_k}{\sqrt{k}(2qe^{i\tau_1}/\gamma)^k}\right)^{m_k}\frac{1}{m_k!}.$$

We show that $\xi_C\xrightarrow{L^2}\xi$ for some random variable $\xi\neq 0$ as $C\to\infty$. Since a partition $\lambda_*\in\mathcal{P}_C$ is uniquely determined by the vector $\mathbf{m}=\mathbf{m}(\lambda_*):=(m_2(\lambda_*),\dots,m_C(\lambda_*))$, we can rewrite ξ_C as

$$\xi_C=\sum_{\substack{\mathbf{m}\in\mathbb{N}_0^{\{2,\dots,C\}}\\ \sum_{j=2}^C jm_j\leqslant C}}\prod_{k\geqslant 2}\left(\frac{X_k}{\sqrt{k}(2qe^{i\tau_1}/\gamma)^k}\right)^{m_k}\frac{1}{m_k!},$$

where $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. This motivates us to compute the tail L^2 norm using (3.24), that

$$\widetilde{\mathbb{E}}\left[\Big| \sum_{\substack{\mathbf{m}\in\mathbb{N}_0^{\{2,3,\dots\}} \\ \sum_{j=2}^{\infty} jm_j > C}} \prod_{k\geqslant 2} \left(\frac{X_k}{\sqrt{k}(2qe^{i\tau_1}/\gamma)^k}\right)^{m_k} \frac{1}{m_k!}\Big|^2\right] \tag{3.85}$$

$$= \sum_{\substack{\mathbf{m}\in\mathbb{N}_0^{\{2,3,\dots\}} \\ \sum_{j=2}^{\infty} jm_j > C}} \prod_{k\geqslant 2} \left(\frac{1}{\sqrt{k}(2q/\gamma)^k}\right)^{2m_k} \frac{1}{(m_k!)^2} \widetilde{\mathbb{E}}[|X_k|^{2m_k}] \tag{3.86}$$

$$\leqslant \sum_{\substack{\mathbf{m}\in\mathbb{N}_0^{\{2,3,\dots\}} \\ \sum_{j=2}^{\infty} jm_j > C}} \prod_{\substack{k\geqslant M \\ m_k\neq 0}} \frac{L'\gamma^{-2m_k}}{k^{m_k}(2q/\gamma)^{2km_k}} \binom{2m_k}{m_k} \prod_{2\leqslant k<M} \frac{1}{(m_k!)^2},$$

where we recall that we conditioned on the event $\{|R_j| \leqslant 1,\ 2 \leqslant j \leqslant M\}$ and that $\mathbb{E}[e^{im\tau_k}\overline{e^{in\tau_k}}] = 0$ for $m \neq n$. Suppose we have proved that

$$\sum_{\substack{\mathbf{m}\in\mathbb{N}_0^{\{2,3,\dots\}} \\ \sum_{j=2}^{\infty} jm_j > C}} \prod_{\substack{k\geqslant M \\ m_k\neq 0}} \frac{L'\gamma^{-2m_k}}{k^{m_k}(2q/\gamma)^{2km_k}} \binom{2m_k}{m_k} \prod_{2\leqslant k<M} \frac{1}{(m_k!)^2} \to 0. \tag{3.87}$$

Then it follows that $\xi_C \xrightarrow{L^2} \xi$ where

$$\widetilde{\mathbb{E}}[|\xi|^2] = \widetilde{\mathbb{E}}\left[\Big| \sum_{\mathbf{m}\in\mathbb{N}_0^{\{2,3,\dots\}}} \prod_{\substack{k\geqslant M \\ m_k\neq 0}} \left(\frac{X_k}{\sqrt{k}(2qe^{i\tau_1}/\gamma)^k}\right)^{m_k} \frac{1}{m_k!}\Big|^2\right] \geqslant \frac{1}{C(q)} > 0.$$

Since L^2 convergence implies convergence in probability, (3.84) follows. Therefore, it suffices to establish (3.87). Using (3.80) and (3.15), we have for some $\varepsilon = \varepsilon(\gamma, q) > 0$,

$$\sum_{\substack{\mathbf{m}\in\mathbb{N}_0^{\{2,3,\dots\}} \\ \sum_{j=2}^{\infty} jm_j > C}} \prod_{\substack{k\geqslant M \\ m_k\neq 0}} \frac{L'\gamma^{-2m_k}}{k^{m_k}(2q/\gamma)^{2km_k}} \binom{2m_k}{m_k} \prod_{2\leqslant k<M} \frac{1}{(m_k!)^2}$$

$$\ll \sum_{\substack{\mathbf{m}\in\mathbb{N}_0^{\{2,3,\dots\}} \\ \sum_{j=2}^{\infty} jm_j > C}} \prod_{\substack{k\geqslant M \\ m_k\neq 0}} \frac{1}{e^{\varepsilon km_k}} \prod_{2\leqslant k<M} \frac{1}{e^{\varepsilon m_k}}, \tag{3.88}$$

where we may without loss of generality replace M by $\max(M, L')$. Note that for each $\mathbf{m} \in \mathbb{N}_0^{\{2,3,\dots\}}$ with $\sum_{j=2}^{\infty} jm_j > C$, either $m_k \geqslant C^{1/3}$ for some $k \in [2, C^{1/3})$, or $m_k \geqslant 1$ for some $k \geqslant C^{1/3}$. Therefore, for $C > M^3$,

$$\begin{aligned}
&\sum_{\substack{\mathbf{m}\in\mathbb{N}_0^{\{2,3,\dots\}} \\ \sum_{j=2}^{\infty} jm_j > C}} \prod_{\substack{k\geqslant M \\ m_k\neq 0}} \frac{1}{e^{\varepsilon k m_k}} \prod_{2\leqslant k<M} \frac{1}{e^{\varepsilon m_k}} \\
&\leqslant e^{-\varepsilon C^{1/3}} \sum_{\mathbf{m}\in\mathbb{N}_0^{\{2,3,\dots\}}} \prod_{k\geqslant M} \frac{1}{e^{\varepsilon k m_k}} \prod_{2\leqslant k<M} \frac{1}{e^{\varepsilon m_k}} \\
&\leqslant e^{-\varepsilon C^{1/3}} \prod_{2\leqslant k<M} \left(\sum_{m_k\geqslant 0} \frac{1}{e^{\varepsilon m_k}} \right) \prod_{k\geqslant M} \left(\sum_{m_k\geqslant 0} \frac{1}{e^{\varepsilon k m_k}} \right) \ll e^{-\varepsilon C^{1/3}}.
\end{aligned}$$

This proves (3.87) and hence (3.84). □

In the rest of this subsection, we prove the lower bound of Theorem 3.1.4 in the case $\gamma < 2q$. First, combining (3.82) and Lemma 3.5.2 yields

$$\begin{aligned}
&\widetilde{\mathbb{E}}\left[\left| \sum_{\substack{\lambda\in\mathcal{P}_N \\ m_1(\lambda)\geqslant N - C_*}} \prod_{k\geqslant 1} \left(\frac{X_k}{\sqrt{k}} \right)^{m_k(\lambda)} \frac{1}{m_k(\lambda)!} \right|^{2q} 1_{\{|R_1|\in[(2qN-\sqrt{N})/\gamma, 2qN/\gamma]\}} \right] \\
&= \widetilde{\mathbb{E}}\left[|X_1|^{2q(N-C_*)} \left| \sum_{\lambda_*\in\mathcal{P}_{C_*}} \frac{m_1(\lambda_*)!}{(m_1(\lambda_*)+N-C_*)!} \prod_{k\geqslant 1} \left(\frac{X_k}{\sqrt{k}} \right)^{m_k(\lambda_*)} \frac{1}{m_k(\lambda_*)!} \right|^{2q} \right. \\
&\qquad \left. 1_{\{|R_1|\in[(2qN-\sqrt{N})/\gamma, 2qN/\gamma]\}} \right] \\
&\geqslant \widetilde{\mathbb{E}}\left[|X_1|^{2q(N-C_*)} \left| \sum_{\lambda_*\in\mathcal{P}_{C_*}} \left(\frac{2qNe^{i\tau_1}}{\gamma} \right)^{m_1} \frac{1}{(N-C_*)!N^{m_1}} \prod_{k\geqslant 2} \left(\frac{X_k}{\sqrt{k}} \right)^{m_k} \frac{1}{m_k!} \right|^{2q} \right. \\
&\qquad \left. 1_{\{|R_1|\in[(2qN-\sqrt{N})/\gamma, 2qN/\gamma]\}} \right] \\
&\quad - C\widetilde{\mathbb{E}}\left[|X_1|^{2q(N-C_*)} \left| \sum_{\lambda_*\in\mathcal{P}_{C_*}} \frac{(2q/\gamma)^{m_1} m_1 C_*}{\sqrt{N}(N-C_*)!} \prod_{k\geqslant 2} \left(\frac{X_k}{\sqrt{k}} \right)^{m_k} \frac{1}{m_k!} \right|^{2q} \right.
\end{aligned}$$

$$1_{\{|R_1|\in[(2qN-\sqrt{N})/\gamma,2qN/\gamma]\}}\Bigg], \tag{3.89}$$

where C is the implied constant in Lemma 3.5.2 and in the last line we assumed $q \leqslant 1/2$, and the case $q \in (1/2, 1]$ follows similarly by Minkowski's inequality. In the first expectation of (3.89), the absolute value is independent of the rest. This yields

$$\begin{aligned}
&\widetilde{\mathbb{E}}\Bigg[\Bigg|\sum_{\lambda_*\in\mathcal{P}_{C_*}}\left(\frac{2qNe^{i\tau_1}}{\gamma}\right)^{m_1}\frac{1}{(N-C_*)!N^{m_1}}\prod_{k\geqslant 2}\left(\frac{X_k}{\sqrt{k}}\right)^{m_k}\frac{1}{m_k!}\Bigg|^{2q}|X_1|^{2q(N-C_*)} \\
&\qquad 1_{\{|R_1|\in[(2qN-\sqrt{N})/\gamma,2qN/\gamma]\}}\Bigg] \\
&= \widetilde{\mathbb{E}}\Bigg[\Bigg|\sum_{\lambda_*\in\mathcal{P}_{C_*}}\left(\frac{2qNe^{i\tau_1}}{\gamma}\right)^{m_1}\frac{1}{(N-C_*)!N^{m_1}}\prod_{k\geqslant 2}\left(\frac{X_k}{\sqrt{k}}\right)^{m_k}\frac{1}{m_k!}\Bigg|^{2q}\Bigg]\mathbb{E}\Bigg[|X_1|^{2q(N-C_*)} \\
&\qquad 1_{\{|R_1|\in[(2qN-\sqrt{N})/\gamma,2qN/\gamma]\}}\Bigg] \\
&= \frac{1}{(N-C_*)!^{2q}}\widetilde{\mathbb{E}}\Bigg[\Bigg|\sum_{\lambda_*\in\mathcal{P}_{C_*}}\left(\frac{2qe^{i\tau_1}}{\gamma}\right)^{m_1}\prod_{k\geqslant 2}\left(\frac{X_k}{\sqrt{k}}\right)^{m_k}\frac{1}{m_k!}\Bigg|^{2q}\Bigg]\mathbb{E}\Bigg[|X_1|^{2q(N-C_*)} \\
&\qquad 1_{\{|R_1|\in[(2qN-\sqrt{N})/\gamma,2qN/\gamma]\}}\Bigg].
\end{aligned} \tag{3.90}$$

Here, we supply a simple lower bound for the latter expectation in (3.90):

$$\begin{aligned}
\mathbb{E}\Bigg[|X_1|^{2q(N-C_*)}1_{\{|R_1|\in[(2qN-\sqrt{N})/\gamma,2qN/\gamma]\}}\Bigg] &\gg \int_{(2qN-\sqrt{N})/\gamma}^{2qN/\gamma} x^{2q(N-C_*)}e^{-\gamma x}\mathrm{d}x \\
&= \gamma^{-2q(N-C_*)}\int_{2qN-\sqrt{N}}^{2qN} y^{2q(N-C_*)}e^{-y}\mathrm{d}y \\
&\gg \sqrt{N}\left(\frac{2qN}{\gamma}\right)^{2q(N-C_*)}e^{-2qN},
\end{aligned}$$

where in the last step we use the inequality $y^{2q(N-C_*)}e^{-y} \gg (2qN)^{2q(N-C_*)}e^{-2qN}$ which is a consequence of Taylor's expansion when N is large enough compared to C_*.

We next bound the first expectation in (3.90). Since for $\lambda_* \in \mathcal{P}_{C_*}$, $m_1 = C_* - \sum km_k$, we have

$$\widetilde{\mathbb{E}}\left[\Big|\sum_{\lambda_*\in\mathcal{P}_{C_*}}\left(\frac{2qe^{i\tau_1}}{\gamma}\right)^{m_1}\prod_{k\geqslant 2}\left(\frac{X_k}{\sqrt{k}}\right)^{m_k}\frac{1}{m_k!}\Big|^{2q}\right]$$

$$=\left(\frac{2q}{\gamma}\right)^{2qC_*}\widetilde{\mathbb{E}}\left[\Big|\sum_{\lambda_*\in\mathcal{P}_{C_*}}\prod_{k\geqslant 2}\left(\frac{X_k}{\sqrt{k}(2qe^{i\tau_1}/\gamma)^{km_k}}\right)^{m_k}\frac{1}{m_k!}\Big|^{2q}\right].$$

In view of Lemma 3.5.3, there exists $\delta_0 > 0$ (independent of C_*) such that

$$\widetilde{\mathbb{E}}\left[\Big|\sum_{\lambda_*\in\mathcal{P}_{C_*}}\prod_{k\geqslant 2}\left(\frac{X_k}{\sqrt{k}(2qe^{i\tau_1}/\gamma)^{km_k}}\right)^{m_k}\frac{1}{m_k!}\Big|^{2q}\right] > \delta_0$$

for any C_* large enough. In this case,

$$\widetilde{\mathbb{E}}\left[\Big|\sum_{\lambda_*\in\mathcal{P}_{C_*}}\left(\frac{2qNe^{i\tau_1}}{\gamma}\right)^{m_1}\frac{1}{(N-C_*)!N^{m_1}}\prod_{k\geqslant 2}\left(\frac{X_k}{\sqrt{k}}\right)^{m_k}\frac{1}{m_k!}\Big|^{2q}|X_1|^{2q(N-C_*)}\right.$$

$$\left.1_{\{|R_1|\in[(2qN-\sqrt{N})/\gamma,2qN/\gamma]\}}\right]$$

$$\gg \frac{1}{(N-C_*)!^{2q}}\left(\frac{2q}{\gamma}\right)^{2qC_*}\mathbb{E}\left[|X_1|^{2q(N-C_*)}1_{\{|R_1|\in[(2qN-\sqrt{N})/\gamma,2qN/\gamma]\}}\right]$$

$$\gg \frac{1}{(N-C_*)!^{2q}}\left(\frac{2q}{\gamma}\right)^{2qC_*}\sqrt{N}\left(\frac{2qN}{\gamma}\right)^{2q(N-C_*)}e^{-2qN}$$

$$\gg \phi(N).$$

On the other hand, for a fixed C_*, the second term of (3.89) is bounded by

$$\widetilde{\mathbb{E}}\left[|X_1|^{2q(N-C_*)}\Big|\sum_{\lambda_*\in\mathcal{P}_{C_*}}\frac{(2q/\gamma)^{m_1}m_1C_*}{\sqrt{N}(N-C_*)!}\prod_{k\geqslant 2}\left(\frac{X_k}{\sqrt{k}}\right)^{m_k}\frac{1}{m_k!}\Big|^{2q}\right.$$

$$\left.1_{\{|R_1|\in[(2qN-\sqrt{N})/\gamma,2qN/\gamma]\}}\right]$$

$$\ll \left(\frac{2qN}{\gamma}\right)^{2q(N-C_*)} \widetilde{\mathbb{E}}\left[\Big|\sum_{\lambda_*\in\mathcal{P}_{C_*}} \frac{(2q/\gamma)^{m_1}m_1C_*}{\sqrt{N}(N-C_*)!}\prod_{k\geqslant 2}\left(\frac{X_k}{\sqrt{k}}\right)^{m_k}\frac{1}{m_k!}\Big|^{2q}\right.$$

$$\left. 1_{\{|R_1|\in[(2qN-\sqrt{N})/\gamma,2qN/\gamma]\}}\right]$$

$$\ll_{C_*} \left(\frac{2qN}{\gamma}\right)^{2q(N-C_*)}\left(\frac{(2q/\gamma)^{C_*}C_*^2}{\sqrt{N}(N-C_*)!}\right)^{2q}$$

$$= o_{C_*}(\phi(N))$$

for N large enough.

Altogether, we conclude from (3.89) that for some constant $M' = M'(\gamma, q)$ independent of C_* and N,

$$\widetilde{\mathbb{E}}\left[\Big|\sum_{\substack{\lambda\in\mathcal{P}_N\\ m_1(\lambda)\geqslant N-C_*}}\prod_{k\geqslant 1}\left(\frac{X_k}{\sqrt{k}}\right)^{m_k}\frac{1}{m_k!}\Big|^{2q} 1_{\{|R_1|\in[(2qN-\sqrt{N})/\gamma,2qN/\gamma]\}}\right] \geqslant \frac{\phi(N)}{M'} - o_{C_*}(\phi(N)).$$

Pick $\varepsilon = 1/(2M'e^{LM})$ in Lemma 3.5.1 and let C_* be the implied constant C therein. Recalling (3.81), we have

$$\mathbb{E}[|A_N|^{2q}] \geqslant e^{-LM}\left(\frac{\phi(N)}{M'} - o_{C_*}(\phi(N))\right) - \varepsilon\phi(N) \gg \phi(N)$$

as $N \to \infty$. This completes the proof of the lower bound.

Remark 5. The same arguments in this section carry through if we replace the rotationally invariant complex inputs $\{X_k\}_{k\geqslant 1}$ with real ones. Suppose that $\{X_k\}_{k\geqslant 1}$ forms a sequence of i.i.d. two-sided symmetric shifted exponential random variables with unit variance. That is, $\mathbb{P}(|X_k| \geqslant u) = \exp(-\gamma(u - c_\gamma)) \wedge 1$ for $u \geqslant 0$, where $\gamma \in (0, 2q)$ and $c_\gamma = \log(\gamma^2/2)/\gamma$. Note that the same moment asymptotic (3.24) still applies. The only argument in the current section that depends on the rotation invariance is (3.86), whereas in the real case, cross-terms may exist. Nevertheless, the cross-terms can be estimated similarly: the left-hand side of (3.86) can be bounded

by

$$\widetilde{\mathbb{E}}\Bigg[\Big|\sum_{\substack{\mathbf{m}\in\mathbb{N}_0^{\{2,3,\dots\}}\\ \sum_{j=2}^{\infty} jm_j>C}}\prod_{k\geqslant 2}\Big(\frac{X_k}{\sqrt{k}(2q/\gamma)^k}\Big)^{m_k}\frac{1}{m_k!}\Big|^2\Bigg]$$

$$\ll \sum_{\substack{\mathbf{m},\mathbf{n}\in\mathbb{N}_0^{\{2,3,\dots\}}\\ \min\{\sum_{j=2}^{\infty} jm_j,\sum_{j=2}^{\infty} jn_j\}>C\\ \forall j,\, m_j+n_j\in 2\mathbb{N}_0}}\prod_{\substack{k\geqslant M\\ m_k\neq 0}}\frac{1}{e^{\varepsilon k(m_k+n_k)}}\prod_{2\leqslant k<M}\frac{1}{e^{\varepsilon(m_k+n_k)}}$$

$$\ll e^{-2\varepsilon C^{1/3}},$$

for $C > M^3$ (possibly with a smaller ε and a larger M than (3.88)), which replaces (3.88) and the arguments that follow.

3.5.2 The case $\gamma = 2q$

Fix $q \in (0, 1]$. Our goal in this section is to prove (3.6) that under the case (EXP) with $\gamma = 2q$,

$$\mathbb{E}[|A_N|^{2q}] \asymp \frac{N^{1-q+q^2/2}}{(1+(1-q)\sqrt{\log N})^q}.$$

Let us recall the philosophy for the case $\gamma = 2q$ that the main contribution to A_N from the sum over partitions (3.13) is random and depends on the value of $|X_1|$. More precisely, on the event that $|X_1|$ is close to m, we expect that the partitions $\lambda \in \mathcal{P}_N$ with $m_1(\lambda) = m + O(\sqrt{N})$ dominate A_N. After applying the decomposition

$$\mathbb{E}[|A_N|^{2q}] = \sum_{m=0}^{\infty}\mathbb{E}[|A_N|^{2q}1_{|X_1|\in[m,m+1)}],$$

we will show that for $m \leqslant N-2$,

$$\mathbb{E}\big[|A_N|^{2q}1_{|X_1|\in[m,m+1)}\big] \asymp \frac{m^{-q+q^2/2}}{(1+(1-q)\sqrt{\log(N-m)})^q}. \tag{3.91}$$

The desired asymptotic then follows from Karamata's theorem (Theorem 1.5.11 of [BGT89]):

$$\sum_{m=0}^{N-2} \frac{m^{-q+q^2/2}}{(1+(1-q)\sqrt{\log(N-m)})^q} \asymp \frac{N^{1-q+q^2/2}}{(1+(1-q)\sqrt{\log N})^q}. \tag{3.92}$$

As illustrated in Section 3.2.2, a key step towards (3.91) is reproducing the same multiplicative chaos approach from (q-UNIV) case to estimate (3.28) after conditioning on R_1. For this reason, we first prepare ourselves with a stronger version of Proposition 3.3.1, which is the focus of Section 3.5.2. When performing the multiplicative chaos analysis, an essential ingredient is establishing uniform estimates of a weighted mass of truncated multiplicative chaos (cf. (3.29)). This will be the goal of Sections 3.5.2 and 3.5.2.

In this section, we will use $C_1, C_2, \dots$ to denote large positive constants (greater than one) that only depend on q and each other, *which may be different from constants defined in other sections*. Throughout, we fix $q \in (0,1]$, and the notation $\ll$ will always denote asymptotic constants that depend only on q.

Strengthening Proposition 3.3.1

In this section, we show that Proposition 3.3.1 also extends to the (EXP) case with $\gamma = 2q$, if we restrict to partitions without ones (i.e., $m_1(\lambda) = 0$). Recall (3.19).

Proposition 3.5.4. *Fix an integer $M_* \geqslant 2$ larger than some constant depending only on the distribution of X_k. For any large N and any $q \in (0,1]$, under (EXP) with $\gamma = 2q$ we have*

$$\mathbb{E}[|A_{N,M_*}|^{2q}] \asymp \left(\frac{1}{1+(1-q)\sqrt{\log N}}\right)^q.$$

Proof. The proof is very similar to the proof of Proposition 3.3.1, and the only modification needed lies in the upper bound of Proposition 3.3.2, where we denote by

$J = \lceil \log(C_1\sqrt{N})/\log 2 \rceil$. Indeed, (3.42) becomes

$$\mathbb{E}[|a(\lambda)|^{2q}] \ll \prod_{k:m_k>0} \frac{C_2}{k^{qm_k}}.$$

Since $k \geqslant M_* \geqslant 2$, (3.43) becomes

$$\mathbb{E}\Big[\Big|\sum_{\substack{|\lambda|=N \\ \lambda_1 \leqslant N/2^J}} a(\lambda)\Big|^{2q}\Big]^{1/(2q)} \leqslant \sum_{\substack{|\lambda|=N \\ \lambda_1 \leqslant \sqrt{N}/C_1}} \prod_{k:m_k>0} \frac{C_2}{k^{qm_k}} \leqslant p_N C_2^{\sqrt{N}/C_1} 2^{-q\sum_{k=M_*}^{\sqrt{N}/C_1} m_k}.$$

Using (3.44), we obtain

$$\mathbb{E}\Big[\Big|\sum_{\substack{|\lambda|=N \\ \lambda_1 \leqslant N/2^J}} a(\lambda)\Big|^{2q}\Big]^{1/(2q)} \ll \frac{1}{N}$$

for C_1 picked large enough. The rest follows line by line as in the proof of Proposition 3.3.1. □

Proposition 3.5.5. *Suppose that (EXP) holds with $\gamma = 2q$. It holds that*

$$\mathbb{E}\Big[\Big|\sum_{\substack{\lambda\in\mathcal{P}_N \\ m_1(\lambda)=0}} \prod_{k\geqslant 2} \Big(\frac{X_k}{\sqrt{k}}\Big)^{m_k} \frac{1}{m_k!}\Big|^{2q}\Big] \asymp \frac{1}{(1+(1-q)\sqrt{\log N})^q}.$$

Proof. Recall (3.32). We follow a similar argument that deduced Theorem 3.1.3 equation (3.3) from Proposition 3.3.1 in Section 3.3.1, using now Proposition 3.5.4 instead. For the upper bound when $q \leqslant 1/2$ (and similarly for $q > 1/2$ using instead Minkowski's inequality), by Proposition 3.5.4,

$$\mathbb{E}\Big[\Big|\sum_{\substack{\lambda\in\mathcal{P}_N \\ m_1(\lambda)=0}} \prod_{k\geqslant 2} \Big(\frac{X_k}{\sqrt{k}}\Big)^{m_k} \frac{1}{m_k!}\Big|^{2q}\Big]$$

$$\ll \sum_{\mathbf{m}\in\mathcal{P}^{<}_{N,M_*}} \mathbb{E}\left[\left|A_{N-\sum_{j=1}^{M_*-1} jm_j,M_*}\right|^{2q}\right] \prod_{k=1}^{M_*-1} \mathbb{E}\left[\left|\left(\frac{X_k}{\sqrt{k}}\right)^{m_k}\frac{1}{m_k!}\right|^{2q}\right]$$

$$+ \sum_{\lambda:\lambda_1<M_*} \prod_{k=1}^{M_*-1} \mathbb{E}\left[\left|\left(\frac{X_k}{\sqrt{k}}\right)^{m_k}\frac{1}{m_k!}\right|^{2q}\right].$$

$$\ll \sum_{\mathbf{m}\in\mathcal{P}^{<}_{N,M_*}} \frac{1}{(1+(1-q)\sqrt{\log(N-\sum_{j=1}^{M_*-1} jm_j)})^q} \prod_{k=2}^{M_*-1} C_3 m_k^{1/2-q} k^{-qm_k}$$

$$+ \sum_{\lambda:\lambda_1<M_*} \prod_{k=2}^{M_*-1} C_3 m_k^{1/2-q} k^{-qm_k}.$$

Since the product ranges in $k \geqslant 2$, the second sum and the first sum restricted to λ such that $\sum_{j=1}^{M_*-1} jm_j > N/2$ can be controlled in the same way as before by $1/N$. The rest is then

$$\sum_{\substack{\mathbf{m}\in\mathcal{P}^{<}_{N,M_*}\\ \sum_{j=1}^{M_*-1} jm_j\leqslant N/2}} \frac{1}{(1+(1-q)\sqrt{\log(N-\sum_{j=1}^{M_*-1} jm_j)})^q} \prod_{k=2}^{M_*-1} C_3 m_k^{1/2-q} k^{-qm_k}$$

$$\ll \frac{1}{(1+(1-q)\sqrt{\log N})^q} \sum_{\substack{\mathbf{m}\in\mathcal{P}^{<}_{N,M_*}\\ \sum_{j=1}^{M_*-1} jm_j\leqslant N/2}} \prod_{k=2}^{M_*-1} k^{-qm_k/2}$$

$$\leqslant \frac{1}{(1+(1-q)\sqrt{\log N})^q} \prod_{k=2}^{M_*-1} \sum_{m_k=0}^{\infty} k^{-qm_k/2} \ll \frac{1}{(1+(1-q)\sqrt{\log N})^q}.$$

The lower bound follows from precisely the same proof in Section 3.3.1. □

Low moments of partial mass of truncated chaos

The goal of this section is to establish the following two propositions. Fix large constants C_4, C_5 to be determined. The reason why we introduce these constants will become apparent later in Section 3.5.2.

Proposition 3.5.6. *It holds that uniformly for* $r \in [e^{-C_5/K}, e^{C_5/K}]$,

$$\mathbb{E}\Big[\Big(\int_{|\theta|\leqslant \frac{1}{C_4\sqrt{N}}} |F_{K,M_*}(re^{i\theta})|^2 \mathrm{d}\theta\Big)^q\Big] \gg \frac{K_r^q N^{-q+q^2/2}}{(1+(1-q)\sqrt{\log N})^q}.$$

Proposition 3.5.7. *Uniformly in* $1 \leqslant j \leqslant \log(\pi C_4\sqrt{N})$, $K \gg \sqrt{N}$, *and* $r \in [e^{-C_5/K}, e^{C_5/K}]$,

$$\mathbb{E}\Big[\Big(\int_{\frac{e^{j-1}}{C_4\sqrt{N}}\leqslant|\theta|\leqslant\frac{e^j}{C_4\sqrt{N}}} |F_{K,M_*}(re^{i\theta})|^2 \mathrm{d}\theta\Big)^q\Big] \ll \Big(\frac{e^{j-1}}{\sqrt{N}}\Big)^{2q-q^2} \frac{C_4^{q^2}K^q}{(1+(1-q)\sqrt{\log N})^q}. \tag{3.93}$$

Moreover,

$$\mathbb{E}\Big[\Big(\int_{|\theta|\leqslant \frac{1}{C_4\sqrt{N}}} |F_{K,M_*}(re^{i\theta})|^2 \mathrm{d}\theta\Big)^q\Big] \ll \frac{C_4^{q^2}N^{-q+q^2/2}K^q}{(1+(1-q)\sqrt{\log N})^q}. \tag{3.94}$$

A key strategy of proving Propositions 3.5.6 and 3.5.7 is to decompose F_{K,M_*} into two terms depending on the region of integration of θ; cf. (3.31). Consider a large constant C_6 to be determined and our goal is to show that with high probability,

$$\Big|\exp\Big(\sum_{k=M_*}^{K_*/C_6} \frac{X_k}{\sqrt{k}}(re^{i\theta})^k\Big)\Big| \approx \Big|\exp\Big(\sum_{k=M_*}^{K_*/C_6} \frac{X_k}{\sqrt{k}}r^k\Big)\Big|$$

uniformly for $|\theta| \leqslant 1/K_*$ and $K_* \leqslant \sqrt{N}$. To this end, we take the logarithm and compute the difference

$$\Re\Big(\sum_{k=M_*}^{K_*/C_6} \frac{X_k}{\sqrt{k}}(re^{i\theta})^k\Big) - \Re\Big(\sum_{k=M_*}^{K_*/C_6} \frac{X_k}{\sqrt{k}}r^k\Big) = \sum_{k=M_*}^{K_*/C_6} \frac{r^k}{\sqrt{k}}\Re(X_k e^{ik\theta} - X_k).$$

It then suffices to understand the magnitudes of

$$\sup_{|\theta|\leqslant 1/K_*} \sum_{k=M_*}^{K_*/C_6} \frac{r^k}{\sqrt{k}}\Re(X_k e^{ik\theta} - X_k) \quad \text{and} \quad \inf_{|\theta|\leqslant 1/K_*} \sum_{k=M_*}^{K_*/C_6} \frac{r^k}{\sqrt{k}}\Re(X_k e^{ik\theta} - X_k).$$

However, bounds under the original probability $\mathbb{P}$ turn out not sufficient for our purpose. To this end, we define a new probability measure $\widehat{\mathbb{Q}}_{K_*/C_6,M_*}$, where

$$\frac{\mathrm{d}\widehat{\mathbb{Q}}_{K_*/C_6,M_*}}{\mathrm{d}\mathbb{P}} = \frac{\exp(2q\sum_{k=M_*}^{K_*/C_6}\frac{r^k}{\sqrt{k}}\Re(X_k))}{\mathbb{E}[\exp(2q\sum_{k=M_*}^{K_*/C_6}\frac{r^k}{\sqrt{k}}\Re(X_k))]}. \tag{3.95}$$

If the situation is clear we write instead $\widehat{\mathbb{Q}}$. The denominator in (3.95) can be estimated using Lemma 3.7.2 (in a similar way that derives Lemma 3.7.3), which gives

$$\mathbb{E}\bigg[\exp\Big(2q\sum_{k=M_*}^{K_*/C_6}\frac{r^k}{\sqrt{k}}\Re(X_k)\Big)\bigg] \asymp \Big(\frac{K_*}{C_6M_*}\Big)^{q^2} \tag{3.96}$$

uniformly in K_* large enough.

Proposition 3.5.8. *There exists a constant $C_7 > 0$ depending only on q, independent of $K_* \leqslant C_4\sqrt{N}$, such that uniformly in $r \in [e^{-C_5/N}, e^{C_5/N}]$ and C_6, N large enough,*

$$\widehat{\mathbb{Q}}\bigg(\sup_{|\theta|\leqslant 1/K_*}\Big|\sum_{k=M_*}^{K_*/C_6}\frac{r^k}{\sqrt{k}}\Re(X_ke^{ik\theta}-X_k)\Big| \geqslant u\bigg) \leqslant C_7e^{-10u}, \quad u \geqslant 0. \tag{3.97}$$

In particular, uniformly in $r \in [e^{-C_5/N}, e^{C_5/N}]$ and C_6, N large enough,

$$\mathbb{E}^{\widehat{\mathbb{Q}}}\bigg[\exp\Big(2q\sup_{|\theta|\leqslant 1/K_*}\sum_{k=M_*}^{K_*/C_6}\frac{r^k}{\sqrt{k}}\Re(X_ke^{ik\theta}-X_k)\Big)\bigg] \ll 1 \tag{3.98}$$

and

$$\mathbb{E}^{\widehat{\mathbb{Q}}}\bigg[\exp\Big(2q\inf_{|\theta|\leqslant 1/K_*}\sum_{k=M_*}^{K_*/C_6}\frac{r^k}{\sqrt{k}}\Re(X_ke^{ik\theta}-X_k)\Big)\bigg] \gg 1. \tag{3.99}$$

To prove Proposition 3.5.8, we need the following preliminary result on generic chaining that offers tail bounds arising from (3.97). For a thorough treatment on chaining, we refer to [Tal21].

Lemma 3.5.9 (Theorem 3.2 of [Dir15]). *Consider a continuous stochastic process* $\{Z_t\}_{t\in T}$ *indexed by a bounded metric space* (T,d) *satisfying*

$$\forall s,t\in T,\ \forall u\geqslant 0,\ \mathbb{P}(|Z_s-Z_t|\geqslant u\,d(s,t))\leqslant 2e^{-u}. \tag{3.100}$$

Then for any $t_0\in T$,

$$\mathbb{P}\bigg(\sup_{t\in T}|Z_t-Z_{t_0}|\geqslant L\Big(\inf_{\mathbb{T}}\sup_{t\in T}\sum_{n\geqslant 1}2^n d(t,T_n)+u\sup_{s,t\in T}d(s,t)\Big)\bigg)\leqslant e^{-u}, \tag{3.101}$$

where $\mathbb{T}$ *is the set of all admissible sequence of subsets* $\{T_n\}_{n\geqslant 0}$ *of* T *satisfying* $|T_0|=1$ *and* $|T_n|\leqslant 2^{2^n}$ *for* $n\geqslant 1$.

Proof of Proposition 3.5.8. Consider large constants C_8, C_9 to be determined. To apply Lemma 3.5.9, we let $T=\{\theta:|\theta|\leqslant 1/K_*\}$ with $d(\theta,\theta')=K_*|\theta-\theta'|/C_8$, $t_0=0$, and

$$Z_\theta=\sum_{k=M_*}^{K_*/C_6}\frac{r^k}{\sqrt{k}}\Re(X_k e^{ik\theta}).$$

Note that the chaining is performed under the tilted probability measure $\widehat{\mathbb{Q}}$. We first verify (3.100), with $\mathbb{P}=\widehat{\mathbb{Q}}$ therein. Consider $\theta,\theta'\in T$ and define $\beta=10C_9(K_*|\theta-\theta'|)^{-1}$. We have

$$\begin{aligned}
&\widehat{\mathbb{Q}}\left(\Big|\sum_{k=M_*}^{K_*/C_6}\frac{r^k}{\sqrt{k}}\Re(X_k(e^{ik\theta}-e^{ik\theta'}))\Big|\geqslant\frac{uK_*|\theta-\theta'|}{C_8}\right)\\
&=\widehat{\mathbb{Q}}\left(\sum_{k=M_*}^{K_*/C_6}\frac{r^k}{\sqrt{k}}\Re(X_k(e^{ik\theta}-e^{ik\theta'}))\geqslant\frac{uK_*|\theta-\theta'|}{C_8}\right)\\
&+\widehat{\mathbb{Q}}\left(\sum_{k=M_*}^{K_*/C_6}\frac{r^k}{\sqrt{k}}\Re(X_k(e^{ik\theta'}-e^{ik\theta}))\geqslant\frac{uK_*|\theta-\theta'|}{C_8}\right)\\
&\leqslant\frac{\mathbb{E}^{\widehat{\mathbb{Q}}}[\exp(2\beta\sum_{k=M_*}^{K_*/C_6}\frac{r^k}{\sqrt{k}}\Re(X_k(e^{ik\theta}-e^{ik\theta'})))]}{\exp(2\beta uK_*|\theta-\theta'|/C_8)}
\end{aligned}$$

$$+\frac{\mathbb{E}^{\widehat{\mathbb{Q}}}[\exp(2\beta\sum_{k=M_*}^{K_*/C_6}\frac{r^k}{\sqrt{k}}\Re(X_k(e^{ik\theta'}-e^{ik\theta})))]}{\exp(2\beta u K_*|\theta-\theta'|/C_8)}. \tag{3.102}$$

To offer an upper bound for the numerators, we apply the same asymptotic computation in Lemma 3.7.4, which gives

$$\begin{aligned}&\mathbb{E}^{\widehat{\mathbb{Q}}}\bigg[\exp\Big(2\beta\frac{r^k}{\sqrt{k}}\Re(X_k(e^{ik\theta}-e^{ik\theta'}))\Big)\bigg]\\&=\frac{\mathbb{E}\bigg[\exp\Big(2\beta\frac{r^k}{\sqrt{k}}\Re(X_k(e^{ik\theta}-e^{ik\theta'}))+2q\frac{r^k}{\sqrt{k}}\Re X_k\Big)\bigg]}{\mathbb{E}[\exp(2q\frac{r^k}{\sqrt{k}}\Re X_k)]}.\end{aligned}$$

Recalling $X_k=e^{i\tau_k}R_k$, we have by Taylor's expansion that

$$\begin{aligned}&\mathbb{E}\bigg[\exp\Big(2\beta\frac{r^k}{\sqrt{k}}\Re(X_k(e^{ik\theta}-e^{ik\theta'}))+2q\frac{r^k}{\sqrt{k}}\Re X_k\Big)\bigg]\\&=1+2\frac{r^{2k}}{k}\mathbb{E}[|R_k(q\cos(\tau_k)+\beta(\cos(\tau_k+k\theta)-\cos(\tau_k+k\theta')))|^2]\\&\quad+\sum_{j=3}^{\infty}\frac{(2r^k/\sqrt{k})^j}{j!}\mathbb{E}[|R_k(q\cos(\tau_k)+\beta(\cos(\tau_k+k\theta)-\cos(\tau_k+k\theta')))|^j].\end{aligned} \tag{3.103}$$

For the second moment in (3.103), write $\Delta_k:=k(\theta-\theta')/2, \alpha_k:=k(\theta+\theta')/2$. Using independence of R_k and τ_k, a direct computation gives

$$\begin{aligned}&2\frac{r^{2k}}{k}\mathbb{E}[|R_k(q\cos(\tau_k)+\beta(\cos(\tau_k+k\theta)-\cos(\tau_k+k\theta')))|^2]\\&=2\frac{r^{2k}}{k}\mathbb{E}[|q\cos(\tau_k)+\beta(\cos(\tau_k+k\theta)-\cos(\tau_k+k\theta'))|^2]\\&=(q^2+4\beta^2\sin^2(\Delta_k)-4q\beta\sin(\Delta_k)\sin(\alpha_k))\frac{r^{2k}}{k}.\end{aligned}$$

For the remainder of (3.103), the inequality $|a+b|^j\leqslant 2^j(|a|^j+|b|^j)$ implies

$$|q\cos(\tau_k)+\beta(\cos(\tau_k+k\theta)-\cos(\tau_k+k\theta'))|^j\leqslant(2q)^j+(4\beta\Delta_k)^j.$$

Similarly as in the proof of Lemma 3.7.2 (i), we have

$$\Big|\sum_{j=3}^{\infty}\frac{(2r^k/\sqrt{k})^j}{j!}\mathbb{E}[|R_k(q\cos(\tau_k)+\beta(\cos(\tau_k+k\theta)-\cos(\tau_k+k\theta')))|^j]\Big|$$

$$\leqslant \sum_{j=3}^{\infty} \frac{(2r^k/\sqrt{k})^j}{j!} \mathbb{E}[|R_k|^j]\big((2q)^j + (4\beta\Delta_k)^j\big)$$

$$\ll k^{-3/2}\left((2q)^3 + (4\beta\Delta_k)^3 + \mathbb{E}[e^{\frac{6q}{k^{1/8}}|R_1|}] + \mathbb{E}[e^{\frac{12\beta\Delta_k}{k^{1/8}}|R_1|}]\right),$$

where in the last step the asymptotic constant is absolute given that N is large enough (in terms of constants C_4, C_5 that depend only on q), where we recall that $k \leqslant K_* \leqslant C_4\sqrt{N}$ while $|\log r| \leqslant C_5/N$. Since $\beta\Delta_k = 10C_9k/K_* \leqslant 10C_9$, $k \geqslant M_* \geqslant (120C_9/q)^8$ will suffice to ensure $k^{-1/8}\max\{6q, 120C_9\} < \gamma = 2q$. Therefore,

$$\Big|\sum_{j=3}^{\infty} \frac{(2r^k/\sqrt{k})^j}{j!} \mathbb{E}[|R_k(q\cos(\tau_k) + \beta(\cos(\tau_k + k\theta) - \cos(\tau_k + k\theta')))|^j]\Big| \ll C_9^3 k^{-3/2}.$$

Inserting these estimates back into (3.103), we have

$$\mathbb{E}\bigg[\exp\Big(2\beta\frac{r^k}{\sqrt{k}}\Re(X_k(e^{ik\theta} - e^{ik\theta'})) + 2q\frac{r^k}{\sqrt{k}}\Re X_k\Big)\bigg]$$

$$= 1 + (q^2 + 4\beta^2\sin^2(\Delta_k) - 4q\beta\sin(\Delta_k)\sin(\alpha_k))\frac{r^{2k}}{k} + C_9^3 O(k^{-3/2}).$$

Taking products over k, and using (3.95) and (3.96), we have

$$\mathbb{E}^{\widehat{\mathbb{Q}}}\bigg[\exp\Big(2\beta\sum_{k=M_*}^{K_*/C_6}\frac{r^k}{\sqrt{k}}\Re(X_k(e^{ik\theta} - e^{ik\theta'}))\Big)\bigg]$$

$$\ll \exp\left(\sum_{k=M_*}^{K_*/C_6}\frac{4\beta^2\sin^2(\Delta_k)r^{2k}}{k} + \sum_{k=M_*}^{K_*/C_6}\frac{4q\beta|\sin(\Delta_k)|r^{2k}}{k}\right)$$

$$\ll \exp\left(\beta^2|\theta - \theta'|^2\sum_{k=M_*}^{K_*/C_6} kr^{2k} + 2q\beta|\theta - \theta'|\sum_{k=M_*}^{K_*/C_6} r^{2k}\right)$$

$$\ll \exp\left(\frac{50C_9^2}{C_6^2} + \frac{20C_9}{C_6}\right).$$

Note that the asymptotic constants do not depend on C_9 since we assumed $k \geqslant M_* \geqslant (120C_9/q)^8$. Therefore, there exists a constant C_{10} depending only on q, we may pick

C_6 large enough depending on C_9 so that

$$\mathbb{E}^{\widehat{\mathbb{Q}}}\Big[\exp\Big(2\beta\sum_{k=M_*}^{K_*/C_6}\frac{r^k}{\sqrt{k}}\Re(X_k(e^{ik\theta}-e^{ik\theta'}))\Big)\Big]\leqslant C_{10}.$$

Inserting into (3.102) while interchanging θ,θ' yields

$$\widehat{\mathbb{Q}}\left(\Big|\sum_{k=M_*}^{K_*/C_6}\frac{r^k}{\sqrt{k}}\Re(X_k(e^{ik\theta}-e^{ik\theta'}))\Big|\geqslant\frac{uK_*|\theta-\theta'|}{C_8}\right)\leqslant 2C_{10}e^{-5C_9u/C_8}.$$

Picking C_9 large depending on C_8, C_{10} gives that $\min\{1, 2C_{10}e^{-5C_9u/C_8}\}\leqslant 2e^{-u}$, as desired.

Applying (3.101) with $t_0=0$ leads to

$$\mathbb{P}\left(\sup_{\theta\in T}\Big|\sum_{k=M_*}^{K_*/C_6}\frac{r^k}{\sqrt{k}}\Re(X_ke^{ik\theta}-X_k)\Big|\geqslant L\Big(\inf_{\mathbb{T}}\sup_{t\in T}\sum_{n\geqslant 1}2^nd(t,T_n)+u\sup_{s,t\in T}d(s,t)\Big)\right)\leqslant e^{-u}.$$

Recalling that $d(\theta,\theta')=K_*|\theta-\theta'|/C_8$ and $K_*|\theta-\theta'|\leqslant 2$, we have $\sup_{s,t\in T}d(s,t)\leqslant 2/C_8$. Next, we claim that

$$\inf_{\mathbb{T}}\sup_{t\in T}\sum_{n\geqslant 1}2^nd(t,T_n)\ll\frac{1}{C_8}.$$

Indeed, this follows by taking uniform nets of $\{T_n\}$, so that $\sup_{t\in T}d(t,T_n)\ll 2^{-2^n}/C_8$ for $n\geqslant 1$. Therefore, by picking C_8 large enough (depending only on the universal constant L), we have

$$\mathbb{P}\left(\sup_{\theta\in T}\Big|\sum_{k=M_*}^{K_*/C_6}\frac{r^k}{\sqrt{k}}\Re(X_ke^{ik\theta}-X_k)\Big|\geqslant 1+u\right)\leqslant e^{-20u}.$$

This establishes (3.97) by picking e.g. $C_7=e^{20}$. Note that by our construction, M_* depends only on C_8, C_9, and hence only on q. Directly integrating (3.97) yields (3.98),

since $q \leqslant 1$. Moreover, by taking $u = \log C_7$ in (3.97), we have

$$\widehat{\mathbb{Q}}\Big(\inf_{|\theta| \leqslant 1/K_*} \sum_{k=M_*}^{K_*/C_6} \frac{r^k}{\sqrt{k}} \Re(X_k e^{ik\theta} - X_k) \geqslant -\log C_7 \Big)$$

$$\geqslant \widehat{\mathbb{Q}}\Big(\sup_{|\theta| \leqslant 1/K_*} \Big| \sum_{k=M_*}^{K_*/C_6} \frac{r^k}{\sqrt{k}} \Re(X_k e^{ik\theta} - X_k) \Big| \leqslant \log C_7 \Big) \geqslant \frac{1}{2}.$$

Since C_7 is a constant, (3.99) follows. □

Proof of Proposition 3.5.6. We introduce the decomposition

$$F_{K,M_*}(z) = \overline{F}_{K,M_*}(z) \times \underline{F}_{K,M_*}(z) := \exp\Big(\sum_{k=M_*}^{C_4\sqrt{N}/C_6} \frac{X_k}{\sqrt{k}} z^k \Big) \times \exp\Big(\sum_{k=C_4\sqrt{N}/C_6}^{K} \frac{X_k}{\sqrt{k}} z^k \Big).$$

By independence,

$$\mathbb{E}\left[\left(\int_{|\theta| \leqslant \frac{1}{C_4\sqrt{N}}} |F_{K,M_*}(re^{i\theta})|^2 \mathrm{d}\theta \right)^q \right] \geqslant \mathbb{E}\left[\left(\int_{|\theta| \leqslant \frac{1}{C_4\sqrt{N}}} |\underline{F}_{K,M_*}(re^{i\theta})|^2 \mathrm{d}\theta \right)^q \right]$$

$$\times \mathbb{E}\left[\inf_{|\theta| \leqslant \frac{1}{C_4\sqrt{N}}} |\overline{F}_{K,M_*}(re^{i\theta})|^{2q} \right].$$

We first analyze the second term. Using (3.95) and (3.96), we write

$$\mathbb{E}\left[\inf_{|\theta| \leqslant \frac{1}{C_4\sqrt{N}}} |\overline{F}_{K,M_*}(re^{i\theta})|^{2q} \right]$$

$$= \mathbb{E}\left[|\overline{F}_{K,M_*}(r)|^{2q} \inf_{|\theta| \leqslant \frac{1}{C_4\sqrt{N}}} \Big| \frac{\overline{F}_{K,M_*}(re^{i\theta})}{\overline{F}_{K,M_*}(r)} \Big|^{2q} \right]$$

$$= \Big(\frac{C_4\sqrt{N}}{C_6 M_*} \Big)^{q^2} \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\inf_{|\theta| \leqslant \frac{1}{C_4\sqrt{N}}} \Big| \frac{\overline{F}_{K,M_*}(re^{i\theta})}{\overline{F}_{K,M_*}(r)} \Big|^{2q} \right]$$

$$= \Big(\frac{C_4\sqrt{N}}{C_6 M_*} \Big)^{q^2} \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\exp\Big(2q \inf_{|\theta| \leqslant \frac{1}{C_4\sqrt{N}}} \sum_{k=M_*}^{C_4\sqrt{N}/C_6} \frac{r^k}{\sqrt{k}} \Re(X_k e^{ik\theta} - X_k) \Big) \right].$$

As a consequence of (3.99) and since C_6 and M_* depend only on q, we have

$$\mathbb{E}\Big[\inf_{|\theta|\leqslant\frac{1}{C_4\sqrt{N}}}|\overline{F}_{K,M_*}(re^{i\theta})|^{2q}\Big]\gg\Big(\frac{C_4\sqrt{N}}{C_6M_*}\Big)^{q^2}\gg N^{q^2/2}.$$

It remains to show

$$\mathbb{E}\Big[\Big(\int_{|\theta|\leqslant\frac{1}{C_4\sqrt{N}}}|\underline{F}_{K,M_*}(re^{i\theta})|^2\mathrm{d}\theta\Big)^q\Big]\gg\frac{K_r^qN^{-q}}{(1+(1-q)\sqrt{\log N})^q}. \tag{3.104}$$

The case of $q=1$ follows by a direction computation in the form of Lemma 3.7.3. Let us assume $q<1$. The approach is to perform a change of variable and then adapt the proof of Proposition 3.3.11. Using the change of variable $\theta\mapsto\pi C\theta\sqrt{N}$, we have

$$\mathbb{E}\left[\left(\int_{|\theta|\leqslant\frac{1}{C_4\sqrt{N}}}|\underline{F}_{K,M_*}(re^{i\theta})|^2\mathrm{d}\theta\right)^q\right]\gg N^{-q/2}\mathbb{E}\left[\left(\int_{-\pi}^{\pi}|\underline{F}_{K,M_*}(re^{i\theta/(\pi C_4\sqrt{N})})|^2\mathrm{d}\theta\right)^q\right]. \tag{3.105}$$

Define the tilted measures

$$\frac{\mathrm{d}\mathbb{Q}^{(1)}}{\mathrm{d}\mathbb{P}}:=\frac{\exp(2\sum_{k=C_4\sqrt{N}/C_6}^{K-1}\frac{r^k}{\sqrt{k}}R_k\cos(\tau_k))}{\mathbb{E}\left[\exp(2\sum_{k=C_4\sqrt{N}/C_6}^{K-1}\frac{r^k}{\sqrt{k}}R_k\cos(\tau_k))\right]}$$

and

$$\frac{\mathrm{d}\mathbb{Q}^{(2)}_{r,M,K,\theta}}{\mathrm{d}\mathbb{P}}:=\frac{\exp(2\sum_{m=\log(C_4\sqrt{N}/C_6)+1}^{\log K_r}(Z_0(m)+Z_\theta(m)))}{\mathbb{E}[\exp(2\sum_{m=\log(C_4\sqrt{N}/C_6)+1}^{\log K_r}(Z_0(m)+Z_\theta(m)))]},$$

where

$$Z_\theta(m):=\Re\sum_{e^{m-1}\leqslant k<e^m}\frac{X_kr^ke^{ik\theta/(\pi C_4\sqrt{N})}}{\sqrt{k}}=\sum_{e^{m-1}\leqslant k<e^m}\frac{r^k}{\sqrt{k}}R_k\cos\Big(\tau_k+\frac{k\theta}{\pi C_4\sqrt{N}}\Big).$$

The definition (3.39) still stands, and (3.40) is replaced by

$$\nu_k=\nu_k(\theta):=\mathbb{E}^{\mathbb{Q}^{(2)}}\left[\frac{r^k}{\sqrt{k}}R_k\cos(\tau_k)\right]=\frac{r^{2k}}{k}+\frac{\cos(k\theta/(\pi C_4\sqrt{N}))r^{2k}}{k}+O(k^{-3/2}).$$

Next, we need to adapt Definition 3.3.12 by the following.

Definition 3.5.10. Fix $L_1 > 20$. Let A be a real number with $1 \leqslant A \leqslant \sqrt{\log K_r}$. Define $\mathcal{L}(\theta) = \mathcal{L}(A, \theta; K)$ as the event that for each $\log(C_4\sqrt{N}/C_6) \leqslant n \leqslant \log K_r$, one has

$$-A - L_1 n \leqslant \sum_{k=C_4\sqrt{N}/C_6}^{e^n-1} \left(\Re \frac{X_k r^k e^{ik\theta/(\pi C_4\sqrt{N})}}{\sqrt{k}} - \mu_k \right) \leqslant A - 5\log\Big(n - \log\Big(\frac{\sqrt{N}}{C_6}\Big)\Big).$$

Also, let $\mathcal{L} = \mathcal{L}(A; K)$ be the random subset of $\theta \in [-\pi, \pi)$ such that $\mathcal{L}(\theta)$ holds.

We apply Hölder's inequality to obtain

$$\mathbb{E}\left[\left(\int_{-\pi}^{\pi} |\underline{F}_{K,M_*}(re^{i\theta/(\pi C_4\sqrt{N})})|^2 \mathrm{d}\theta\right)^q\right] \gg \frac{\left(\mathbb{E}\left[\int_{\mathcal{L}} |\underline{F}_{K,M_*}(re^{i\theta/(\pi C_4\sqrt{N})})|^2 \mathrm{d}\theta\right]\right)^{2-q}}{\left(\mathbb{E}\left[\left(\int_{\mathcal{L}} |\underline{F}_{K,M_*}(re^{i\theta/(\pi C_4\sqrt{N})})|^2 \mathrm{d}\theta\right)^2\right]\right)^{1-q}}. \tag{3.106}$$

The lower bound of the numerator follows in exactly the same way as Lemma 3.3.13 (the normal approximation argument remains the same, since the angle θ has not come into play yet). The factor arising from the change of measure is now $K_r/(C_4\sqrt{N}/C_6) \asymp K_r/\sqrt{N}$ (instead of K_r), and thus

$$\mathbb{E}\left[\int_{\mathcal{L}} |\underline{F}_{K,M_*}(re^{i\theta/(\pi C_4\sqrt{N})})|^2 \mathrm{d}\theta\right] \gg \frac{AK_r}{\sqrt{N\log(K_r/\sqrt{N})}}. \tag{3.107}$$

To give an upper bound of the denominator, we follow the proof of Proposition 3.3.14 and indicate the necessary changes. First, we redefine the threshold M as the smallest integer such that $e^M \geqslant \max\{\min\{10^3 C_4\sqrt{N}/(C_6|\theta|), K_r/e\}, M_*\}$. In the definition of the event $\widetilde{\mathcal{L}}$, we replace (3.61) by

$$-A - L_1 M \leqslant A_0(M), A_\theta(M) \leqslant A - 5\log\Big(M - \log\Big(\frac{C_4\sqrt{N}}{C_6}\Big)\Big).$$

As the constant arising from the change of measure alters, (3.62) is replaced by

$$\mathbb{E}\left[1_{\mathcal{L}(0)\cap\mathcal{L}(\theta)} |\underline{F}_{K,M_*}(r)|^2 |\underline{F}_{K,M_*}(re^{i\theta/(\pi C_4\sqrt{N})})|^2\right]$$

$$\ll \frac{e^{4M}}{N^2}\mathbb{E}\left[1_{\tilde{\mathcal{L}}}\, e^{2A_0(M)+2A_\theta(M)} \prod_{m=M+1}^{\log K_r} e^{2Z_0(m)+2Z_\theta(m)}\right].$$

Next, the form of Proposition 3.3.15 remains the same, whose proof can be adapted with θ replaced by $\theta/(\pi C_4\sqrt{N})$, as long as we show the equivalent of decorrelation step (3.70). For this, let us first compute using Lemmas 3.7.4 and 3.7.5 that

$$\sigma_m^2 = \sum_{e^{m-1}\leqslant k<e^m}\left(\frac{r^{2k}}{2k}+O(k^{-3/2})\right) = \frac{1}{2}+O(e^{-m/2})$$

and

$$\rho_m\sigma_m^2 = \sum_{e^{m-1}\leqslant k<e^m}\left(\frac{r^{2k}\cos(k\theta/(\pi C_4\sqrt{N}))}{2k}+O(k^{-3/2})\right) = O(\sqrt{N}e^{-m}+e^{-m/2}).$$

In particular,

$$\rho_m \ll \sqrt{N}e^{-m}+e^{-m/2},$$

and hence

$$\prod_{\log(C_4\sqrt{N}/C_6)<m\leqslant\log K_r}\sqrt{\frac{1+|\rho_m|}{1-|\rho_m|}} \ll 1,$$

which serves as the equivalent of (3.70) and suffices for our purpose.

In the proof that deduces Proposition 3.3.14 from Proposition 3.3.15, the only change is in (3.64), which is now replaced by

$$\mathbb{E}\left[1_{\{A_0(M)\leqslant A-5\log M\}}\, e^{2A_0(M)}\right] \leqslant \mathbb{E}[e^{2A_0(M)}] \ll \sqrt{N}e^{-M}.$$

These altogether result in the following equivalent of Proposition 3.3.14:

$$\mathbb{E}\left[1_{\mathcal{L}(\theta_1)}|\underline{F}_{K,M_*}(re^{i\theta_1/(\pi C_4\sqrt{N})})|^2 1_{\mathcal{L}(\theta_2)}|\underline{F}_{K,M_*}(re^{i\theta_2/(\pi C_4\sqrt{N})})|^2\right]$$

$$\ll A^2e^{2A}\frac{K_r^2/N^{3/2}}{\log K_r}\frac{\min\{K_r,2\pi\sqrt{N}/|\theta_1-\theta_2|\}}{(\log(\min\{K_r,2\pi\sqrt{N}/|\theta_1-\theta_2|\})-\log(C_4\sqrt{N}/C_6))^7}$$

$$\ll A^2 e^{2A} \frac{K_r^2/N}{\log N} \frac{\min\{\sqrt{N}, 2\pi/|\theta_1-\theta_2|\}}{(\log(\min\{K_r, 2\pi\sqrt{N}/|\theta_1-\theta_2|\}) - \log(C_4\sqrt{N}/C_6))^7}.$$

Integrating with respect to θ_1, θ_2 yields

$$\mathbb{E}\left[\left(\int_{\mathcal{L}} |\underline{F}_{K,M_*}(re^{i\theta/(\pi C_4\sqrt{N})})|^2 \mathrm{d}\theta\right)^2\right] \ll A^2 e^{2A} \frac{K_r^2/N}{\log N}. \tag{3.108}$$

Finally, inserting (3.107) and (3.108) into (3.106) leads to (note that we apply with $A \asymp 1$ for $q < 1$)

$$\mathbb{E}\left[\left(\int_{-\pi}^{\pi} |\underline{F}_{K,M_*}(re^{i\theta/(\pi C_4\sqrt{N})})|^2 \mathrm{d}\theta\right)^q\right] \gg \frac{(K_r/\sqrt{N\log K_r})^{2-q}}{(K_r^2/(N\log N)^{1-q}} \gg \frac{K_r^q N^{-q/2}}{(1+(1-q)\sqrt{\log N})^q}.$$

Combined with (3.105) proves (3.104), and hence completing the proof of Proposition 3.5.6. □

Proof of Proposition 3.5.7. For $1 \leqslant j \leqslant \log(\pi C_4\sqrt{N})$, we introduce the decomposition

$$\begin{aligned} F_{K,M_*}(z) &= \overline{F}_{K,M_*;j}(z) \times \underline{F}_{K,M_*;j}(z) \\ &:= \exp\Big(\sum_{k=M_*}^{C_4\sqrt{N}/(C_6e^j)} \frac{X_k}{\sqrt{k}} z^k\Big) \times \exp\Big(\sum_{k=C_4\sqrt{N}/(C_6e^j)}^{K} \frac{X_k}{\sqrt{k}} z^k\Big). \end{aligned}$$

Applying independence and rotational symmetry, we have

$$\begin{aligned} &\mathbb{E}\left[\left(\int_{\frac{e^j-1}{C_4\sqrt{N}}\leqslant\theta\leqslant\frac{e^j}{C_4\sqrt{N}}} |F_{K,M_*}(re^{i\theta})|^2 \mathrm{d}\theta\right)^q\right] \\ \leqslant\ &\mathbb{E}\left[\left(\int_{\frac{e^j-1}{C_4\sqrt{N}}\leqslant\theta\leqslant\frac{e^j}{C_4\sqrt{N}}} |\underline{F}_{K,M_*;j}(re^{i\theta})|^2 \mathrm{d}\theta\right)^q\right] \times \mathbb{E}\Big[\sup_{\frac{e^j-1}{C_4\sqrt{N}}\leqslant\theta\leqslant\frac{e^j}{C_4\sqrt{N}}} |\overline{F}_{K,M_*;j}(re^{i\theta})|^{2q}\Big] \\ =\ &\mathbb{E}\left[\left(\int_{\frac{e^j-1}{C_4\sqrt{N}}\leqslant\theta\leqslant\frac{e^j}{C_4\sqrt{N}}} |\underline{F}_{K,M_*;j}(re^{i\theta})|^2 \mathrm{d}\theta\right)^q\right] \times \mathbb{E}\Big[\sup_{|\theta|\leqslant\frac{e^j+e^{j-1}}{2C_4\sqrt{N}}} |\overline{F}_{K,M_*;j}(re^{i\theta})|^{2q}\Big]. \end{aligned}$$

We decompose

$$\sup_{|\theta|\leqslant \frac{e^j+e^{j-1}}{2C_4\sqrt{N}}} \frac{|\overline{F}_{K,M_*;j}(re^{i\theta})|^{2q}}{|\overline{F}_{K,M_*;j}(r)|^{2q}} = \exp\left(2q \sup_{|\theta|\leqslant \frac{e^j+e^{j-1}}{2C_4\sqrt{N}}} \sum_{k=M_*}^{C_4\sqrt{N}/(C_6e^j)} \frac{r^k}{\sqrt{k}} \Re(X_k e^{ik\theta} - X_k)\right).$$

The change of measure formula (3.95) together with (3.96) yield

$$\mathbb{E}\Big[\sup_{|\theta|\leqslant \frac{e^j+e^{j-1}}{2C_4\sqrt{N}}} |\overline{F}_{K,M_*;j}(re^{i\theta})|^{2q}\Big]$$

$$= \Big(\frac{C_4\sqrt{N}}{C_6M_*e^j}\Big)^{q^2} \mathbb{E}^{\widehat{\mathbb{Q}}}\Big[\exp\left(2q \sup_{|\theta|\leqslant \frac{e^j+e^{j-1}}{2C_4\sqrt{N}}} \sum_{k=M_*}^{C_4\sqrt{N}/(C_6e^j)} \frac{r^k}{\sqrt{k}} \Re(X_k e^{ik\theta} - X_k)\right)\Big].$$

Moreover, by (3.98),

$$\mathbb{E}\Big[\sup_{|\theta|\leqslant \frac{e^j+e^{j-1}}{2C_4\sqrt{N}}} |\overline{F}_{K,M_*;j}(re^{i\theta})|^{2q}\Big] \ll \Big(\frac{C_4\sqrt{N}}{C_6M_*e^j}\Big)^{q^2} \ll \Big(\frac{C_4\sqrt{N}}{e^j}\Big)^{q^2}.$$

Therefore,

$$\mathbb{E}\left[\left(\int_{\frac{e^{j-1}}{C_4\sqrt{N}}\leqslant\theta\leqslant\frac{e^j}{C_4\sqrt{N}}} |F_{K,M_*}(re^{i\theta})|^2 \mathrm{d}\theta\right)^q\right]$$

$$\leqslant \mathbb{E}\left[\left(\int_{\frac{e^{j-1}}{C_4\sqrt{N}}\leqslant\theta\leqslant\frac{e^j}{C_4\sqrt{N}}} |\underline{F}_{K,M_*;j}(re^{i\theta})|^2 \mathrm{d}\theta\right)^q\right] \times \Big(\frac{C_4\sqrt{N}}{e^j}\Big)^{q^2}.$$

To show (3.93), it remains to prove

$$\mathbb{E}\left[\left(\int_{\frac{e^{j-1}}{C_4\sqrt{N}}\leqslant\theta\leqslant\frac{e^j}{C_4\sqrt{N}}} |\underline{F}_{K,M_*;j}(re^{i\theta})|^2 \mathrm{d}\theta\right)^q\right] \ll \frac{e^{2qj}(K/N)^q}{(1+(1-q)\sqrt{\log N})^q}, \tag{3.109}$$

where the asymptotic constant does not depend on j, K, N. The case of $q = 1$ being a consequence of Lemma 3.7.3, we assume that $q < 1$. To this end, we adapt the proof of Proposition 3.3.3. Recall Definition 3.3.5. We first claim that

$$\mathbb{E}\left[1_{\mathcal{G}_r(A;K)} \int_{\frac{e^{j-1}}{C_4\sqrt{N}}\leqslant\theta\leqslant\frac{e^j}{C_4\sqrt{N}}} |\underline{F}_{K,M_*;j}(re^{i\theta})|^2 \mathrm{d}\theta\right] \ll \frac{AKe^{2j}}{N\sqrt{\log N}}. \tag{3.110}$$

The only change compared to Proposition 3.3.7 is that we replace the definition of M therein by $\max\{M, \log(C_4\sqrt{N}/(C_6 e^j))\}$. This leads to

$$\mathbb{E}\left[1_{\mathcal{G}_r(A,\theta;K)}|\underline{F}_{K,M_*;j}(re^{i\theta})|^2\right] \ll \frac{AKe^j}{\sqrt{N\log K}}.$$

Integration with respect to θ then gives (3.110). Applying the same argument that deduces Proposition 3.3.3 from Propositions 3.3.6 and 3.3.7 then leads to (3.109) and hence (3.93). A similar argument establishes (3.94) and can be omitted. □

Low moments of weighted mass of truncated chaos

Define

$$\widetilde{F}_{K,M_*,m}(z) := \exp\Big(\sum_{M_*\leqslant k\leqslant K}\frac{X_k}{\sqrt{k}}z^k\Big)\times\Big(\sum_{j:|j-m|\leqslant N^{9/10}}u(j)z^j\Big),$$

our goal is to show the following.

Proposition 3.5.11. *Let K_r be such that $\log K_r$ is the largest integer with $K_r \leqslant \min\{\frac{-1}{4\log r}, K\}$. Uniformly for $r \in [e^{-C_5/K}, e^{C_5/K}]$ and $K \gg \sqrt{N}$,*

$$\begin{aligned}
&N^{q^2/2}\left(\frac{r^{2m}K_r}{1+(1-q)\sqrt{\log K_r}}\right)^q\\
\ll&\mathbb{E}\left[\left(\int_{-\pi}^{\pi}|\widetilde{F}_{K,M_*,m}(re^{i\theta})|^2\mathrm{d}\theta\right)^q\right]\\
\ll&N^{q^2/2}\left(\frac{r^{2m}K}{1+(1-q)\sqrt{\log K}}\right)^q,
\end{aligned}\tag{3.111}$$

where the asymptotic constants depend on q only. In particular, with the choice $K = N/2$,

$$\begin{aligned}
&N^{q^2/2}\left(\frac{r^{2m}K_r}{1+(1-q)\sqrt{\log K_r}}\right)^q\\
\ll&\mathbb{E}\left[\left(\int_{-\pi}^{\pi}|\widetilde{F}_{K,M_*,m}(re^{i\theta})|^2\mathrm{d}\theta\right)^q\right]
\end{aligned}$$

$$\ll N^{q^2/2}\left(\frac{r^{2m}N}{1+(1-q)\sqrt{\log N}}\right)^q.$$

Recall the truncated chaos F_{K,M_*} from (3.20). We have

$$\frac{|\widetilde{F}_{K,M_*,m}(re^{i\theta})|}{|F_{K,M_*}(re^{i\theta})|}=\Big|\sum_{j:|j-m|\leqslant N^{9/10}} e^{ij(\tau_1+\theta)}\,\frac{|x_1|^{j-m}}{j!/m!}r^j\Big|. \tag{3.112}$$

The following two lemmas deal with the right-hand side of (3.112), whose proofs are elementary and deferred to Section 3.8. Recall the constants C_4, C_5.

Lemma 3.5.12. *Uniformly in N large enough, $m\in[N/6,N/3]$, $x_1\in[m,m+1)$, $\tau_1\in[-\pi,\pi)$, and $r\in[e^{-C_5/N},e^{C_5/N}]$, we have*

$$\Big|\sum_{j:|j-m|\leqslant N^{9/10}} e^{ij\tau_1}\,\frac{|x_1|^{j-m}}{j!/m!}r^j\Big|\ll\min\left\{\frac{r^m}{|\tau_1|},\sqrt{N}r^m\right\}.$$

Lemma 3.5.13. *There exists a large universal constant $C_4>0$ such that the following holds. Uniformly in N large enough, $m\in[N/6,N/3]$, $x_1\in[m,m+1)$, $|\tau|\leqslant 1/(C_4\sqrt{N})$, and $r\in[e^{-C_5/N},e^{C_5/N}]$, we have*

$$\Big|\sum_{j:|j-m|\leqslant N^{9/10}} e^{ij\tau}\,\frac{|x_1|^{j-m}}{j!/m!}r^j\Big|\gg\sqrt{N}r^m.$$

We define C_4 so that the conditions in Lemma 3.5.13 are satisfied. In particular, we may remove the C_4 that appears in (3.93) and (3.94).

Proof of Proposition 3.5.11. For the first part of (3.111), we apply Lemma 3.5.13 and Proposition 3.5.6 to obtain

$$\begin{aligned}\mathbb{E}\left[\left(\int_{-\pi}^{\pi}|\widetilde{F}_{K,M_*,m}(re^{i\theta})|^2\mathrm{d}\theta\right)^q\right]&\geqslant\mathbb{E}\left[\left(\int_{|\theta+\tau_1|\leqslant\frac{1}{C_4\sqrt{N}}}|\widetilde{F}_{K,M_*,m}(re^{i\theta})|^2\mathrm{d}\theta\right)^q\right]\\&\gg N^q r^{2qm}\mathbb{E}\left[\left(\int_{|\theta+\tau_1|\leqslant\frac{1}{C_4\sqrt{N}}}|F_{K,M_*}(re^{i\theta})|^2\mathrm{d}\theta\right)^q\right]\end{aligned}$$

$$\gg N^{q^2/2}\left(\frac{r^{2m}K_r}{1+(1-q)\sqrt{\log N}}\right)^q,$$

where in the last step we use rotational symmetry conditionally on τ_1. For the second part of (3.111), we apply Lemma 3.5.12, Proposition 3.5.7 (conditionally on τ_1 and using rotational symmetry), and $|a+b|^q \leqslant |a|^q + |b|^q$ for $q \in (0,1]$ to get

$$\begin{aligned}
&\mathbb{E}\bigg[\bigg(\int_{-\pi}^{\pi}|\widetilde{F}_{K,M_*,m}(re^{i\theta})|^2\mathrm{d}\theta\bigg)^q\bigg] \\
&\leqslant \sum_{j=1}^{\log(\pi C_4\sqrt{N})}\mathbb{E}\bigg[\bigg(\int_{\frac{e^{j-1}}{C_4\sqrt{N}}\leqslant|\theta+\tau_1|\leqslant\frac{e^j}{C_4\sqrt{N}}}|\widetilde{F}_{K,M_*,m}(re^{i\theta})|^2\mathrm{d}\theta\bigg)^q\bigg] \\
&\qquad\qquad + \mathbb{E}\bigg[\bigg(\int_{|\theta+\tau_1|\leqslant\frac{1}{C_4\sqrt{N}}}|\widetilde{F}_{K,M_*,m}(re^{i\theta})|^2\mathrm{d}\theta\bigg)^q\bigg] \\
&\ll \sum_{j=1}^{\log(\pi C_4\sqrt{N})} r^{2qm}\Big(\frac{e^j}{C_4\sqrt{N}}\Big)^{-2q}\mathbb{E}\bigg[\bigg(\int_{\frac{e^{j-1}}{C_4\sqrt{N}}\leqslant|\theta+\tau_1|\leqslant\frac{e^j}{C_4\sqrt{N}}}|F_{K,M_*}(re^{i\theta})|^2\mathrm{d}\theta\bigg)^q\bigg] \\
&\qquad\qquad + N^q r^{2qm}\mathbb{E}\bigg[\bigg(\int_{|\theta+\tau_1|\leqslant\frac{1}{C_4\sqrt{N}}}|F_{K,M_*}(re^{i\theta})|^2\mathrm{d}\theta\bigg)^q\bigg] \\
&\ll \sum_{j=1}^{\log(\pi C_4\sqrt{N})} r^{2qm}e^{-2qj}N^q\times\Big(\frac{e^{j-1}}{\sqrt{N}}\Big)^{2q-q^2}\frac{K^q}{(1+(1-q)\sqrt{\log N})^q} \\
&\qquad\qquad + N^q r^{2qm}\times\frac{N^{-q+q^2/2}K^q}{(1+(1-q)\sqrt{\log N})^q} \\
&\ll N^{q^2/2}\left(\frac{r^{2m}K}{1+(1-q)\sqrt{\log N}}\right)^q.
\end{aligned}$$

This completes the proof. □

Proof of the lower bound

Observe from (3.92) that

$$\begin{aligned}
&\sum_{m=N/6}^{N/3} \frac{m^{-q+q^2/2}}{(1+(1-q)\sqrt{\log(N-m)})^q} \\
\gg &\frac{N^{1-q+q^2/2}}{(1+(1-q)\sqrt{\log N})^q} \\
\asymp &\sum_{m=0}^{N-2} \frac{m^{-q+q^2/2}}{(1+(1-q)\sqrt{\log(N-m)})^q}.
\end{aligned}$$

This hints at the strategy of restricting to the event $m \approx |X_1| \in [N/6, N/3]$ in order to achieve an asymptotic lower bound for $\mathbb{E}[|A_N|^{2q}]$, which has considerably reduced technicality. To this end, let us write

$$\mathbb{E}[|A_N|^{2q}] \geqslant \sum_{m=N/6}^{N/3} \mathbb{E}\left[\left| \sum_{\lambda\in\mathcal{P}_N} \prod_{k\geqslant 1} \left(\frac{X_k}{\sqrt{k}}\right)^{m_k} \frac{1}{m_k!}\right|^{2q} 1_{\{|X_1|\in[m,m+1)\}}\right]. \tag{3.113}$$

Fix $m \in [N/6, N/3]$. In view of the discussions in Section 3.2.2, we expect that

$$\begin{aligned}
&\mathbb{E}\left[\left| \sum_{\lambda\in\mathcal{P}_N} \prod_{k\geqslant 1} \left(\frac{X_k}{\sqrt{k}}\right)^{m_k} \frac{1}{m_k!}\right|^{2q} 1_{\{|X_1|\in[m,m+1)\}}\right] \\
\asymp &\mathbb{E}\left[\left| \sum_{\substack{\lambda\in\mathcal{P}_N \\ |m_1(\lambda)-m|\leqslant N^{9/10}}} \prod_{k\geqslant 1} \left(\frac{X_k}{\sqrt{k}}\right)^{m_k} \frac{1}{m_k!}\right|^{2q} 1_{\{|X_1|\in[m,m+1)\}}\right].
\end{aligned} \tag{3.114}$$

This is confirmed by the following lemma.

Lemma 3.5.14. *For $m \in [N/6, N/3]$,*

$$\mathbb{E}\left[\left| \sum_{\substack{\lambda\in\mathcal{P}_N \\ |m_1(\lambda)-m|> N^{9/10}}} \prod_{k\geqslant 1} \left(\frac{X_k}{\sqrt{k}}\right)^{m_k} \frac{1}{m_k!}\right|^{2q} 1_{\{|X_1|\in[m,m+1)\}}\right] = o(N^{-2}).$$

Proof. We focus first on the sum over $m_1(\lambda) \geqslant m$. Using in turn concavity (and similarly Minkowski's inequality for $q > 1/2$), independence, (3.15), and Proposition 3.5.5, we have

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\sum_{\substack{\lambda\in\mathcal{P}_N\\ m_1(\lambda)-m>N^{9/10}}}\prod_{k\geqslant 1}\Big(\frac{X_k}{\sqrt{k}}\Big)^{m_k}\frac{1}{m_k!}\Big|^{2q}1_{\{|X_1|\in[m,m+1)\}}\Big]\\
&\leqslant \sum_{j=m+N^{9/10}}^{N}\mathbb{E}\Big[\Big|\sum_{\substack{\lambda\in\mathcal{P}_N\\ m_1(\lambda)=j}}\prod_{k\geqslant 2}\Big(\frac{X_k}{\sqrt{k}}\Big)^{m_k}\frac{1}{m_k!}\Big|^{2q}\frac{|X_1|^{2qj}}{(j!)^{2q}}1_{\{|X_1|\in[m,m+1)\}}\Big]\\
&\ll \sum_{j=m+N^{9/10}}^{N}\frac{m^{2qj}e^{-\gamma m}}{(j!)^{2q}}\mathbb{E}\Big[\Big|\sum_{\substack{\lambda\in\mathcal{P}_{N-j}\\ m_1(\lambda)=0}}\prod_{k\geqslant 2}\Big(\frac{X_k}{\sqrt{k}}\Big)^{m_k}\frac{1}{m_k!}\Big|^{2q}\Big]\\
&\ll \sum_{j=m+N^{9/10}}^{N}\Big(\frac{m}{j}\Big)^{\gamma j}e^{\gamma(j-m)}j^{-q}.
\end{aligned}$$

Using Taylor's expansion (that $-\log(1-x) \leqslant -x - x^2/2$ for $0 < x < 1$), we have the bound that for $j' \geqslant 0$,

$$e^{j'}\Big(\frac{m}{m+j'}\Big)^{m+j'} \leqslant \exp\Big(-\frac{(j')^2}{2(m+j')}\Big). \tag{3.115}$$

Inserting $j' = j - m$ and using $m \geqslant N/6$ lead to

$$\sum_{j=m+N^{9/10}}^{N}\Big(\frac{m}{j}\Big)^{\gamma j}e^{\gamma(j-m)} \leqslant \sum_{j'=N^{9/10}}^{N-m}\exp\Big(-\frac{\gamma(j')^2}{14m}\Big).$$

Since $m \in [N/6, N/3]$, we conclude that the above is $O(\exp(-\gamma N^{1/3})) = o(N^{-2})$ for N large. The sum over $m_1(\lambda) < m$ is similar and omitted. □

We will apply the same second moment approach from Section 3.3.4 to estimate the right-hand side of (3.114). We need a few preliminary computations, the proofs of which are elementary and deferred to the .

Lemma 3.5.15. *For $m \in [N/6, N/3]$ and a (random) function u of the form*

$$u(j) = e^{ij\tau_1} \frac{|x_1|^{j-m}}{j!/m!},$$

where $|x_1| \in [m, m+1)$ and we recall that τ_1 is uniformly distributed on $[-\pi, \pi]$ independent of anything else, we have

$$\mathbb{E}\Bigg[\Big| \sum_{\substack{\lambda \in \mathcal{P}_N \\ |m_1(\lambda)-m| \leqslant N^{9/10}}} u(m_1) \prod_{k \geqslant 2} \Big(\frac{X_k}{\sqrt{k}}\Big)^{m_k} \frac{1}{m_k!}\Big|^{2q}\Bigg] \gg \frac{N^{q^2/2}}{(1+(1-q)\sqrt{\log N})^q}.$$

Proof. First, taking advantage of $|u(j)| \leqslant 1$, we may apply a similar symmetrization argument that deduced Theorem 3.1.3 equation (3.3) from Proposition 3.3.1 in Section 3.3.1, so that it suffices to prove

$$\mathbb{E}\Bigg[\Big| \sum_{\substack{\lambda \in \mathcal{P}_N \\ |m_1(\lambda)-m| \leqslant N^{9/10} \\ \forall 2 \leqslant k < M_*, m_k(\lambda)=0}} u(m_1) \prod_{k \geqslant M_*} \Big(\frac{X_k}{\sqrt{k}}\Big)^{m_k} \frac{1}{m_k!}\Big|^{2q}\Bigg] \gg \frac{N^{q^2/2}}{(1+(1-q)\sqrt{\log N})^q}. \quad (3.116)$$

The proof of (3.116) follows from a straightforward adaptation of the lower bound part for the universality phase. First, in the proof of Proposition 3.3.10, we have used the relation (3.41). If we instead define

$$\widetilde{F}_{K,M_*,m}(z) := \exp\Big(\sum_{M_* \leqslant k \leqslant K} \frac{X_k}{\sqrt{k}} z^k\Big) \times \Big(\sum_{j:|j-m| \leqslant N^{9/10}} u(j) z^j\Big), \quad (3.117)$$

then (3.41) has the analogue

$$\widetilde{F}_{N/2,M_*,m}(z) = \sum_{n=0}^{\infty} \Bigg(\sum_{\substack{\lambda \in \mathcal{P}_n \\ \lambda_1 \leqslant N/2 \\ |m_1(\lambda)-m| \leqslant N^{9/10} \\ \forall 2 \leqslant k < M_*, m_k(\lambda)=0}} u(m_1) \prod_{k=M_*}^{N/2} \Big(\frac{X_k}{\sqrt{k}}\Big)^{m_k} \frac{1}{m_k!} \Bigg) z^n.$$

Therefore, the following equivalent of Proposition 3.3.10 holds:

$$\mathbb{E}\Bigg[\Big|\sum_{\substack{\lambda\in\mathcal{P}_N\\ |m_1(\lambda)-m|\leqslant N^{9/10}\\ \forall 2\leqslant k<M_*,\, m_k(\lambda)=0}} u(m_1)\prod_{k\geqslant 2}\Big(\frac{X_k}{\sqrt{k}}\Big)^{m_k}\frac{1}{m_k!}\Big|^{2q}\Bigg]$$

$$\gg \frac{1}{N^q}\left(\mathbb{E}\left[\left(\int_{-\pi}^{\pi}|\widetilde{F}_{N/2,M_*,m}(re^{i\theta})|^2\mathrm{d}\theta\right)^q\right]-\mathbb{E}\left[r^{Nq}\left(\int_{-\pi}^{\pi}|\widetilde{F}_{N/2,M_*,m}(e^{i\theta})|^2\mathrm{d}\theta\right)^q\right]\right), \tag{3.118}$$

and hence it suffices to prove the counterparts of Proposition 3.3.11 and Proposition 3.3.3 with F_{K,M_*} replaced by $\widetilde{F}_{K,M_*,m}$. Indeed, by Proposition 3.5.11,

$$\mathbb{E}\left[\left(\int_{-\pi}^{\pi}|\widetilde{F}_{K,M_*,m}(re^{i\theta})|^2\mathrm{d}\theta\right)^q\right]\gg N^{q^2/2}\left(\frac{r^{2m}K_r}{1+(1-q)\sqrt{\log K_r}}\right)^q, \tag{3.119}$$

where $r=e^{-C_5/N}$ for some fixed large constant C_5, and

$$\mathbb{E}\left[\left(\int_{-\pi}^{\pi}|\widetilde{F}_{K,M_*,m}(e^{i\theta})|^2\mathrm{d}\theta\right)^q\right]\ll N^{q^2/2}\left(\frac{K}{1+(1-q)\sqrt{\log K}}\right)^q. \tag{3.120}$$

Note that the constants in (3.119) and (3.120) do not depend on C_5. Inserting (3.119) and (3.120) into (3.118) yields that for some constant $C_{11}>0$,

$$\mathbb{E}\Bigg[\Big|\sum_{\substack{\lambda\in\mathcal{P}_N\\ |m_1(\lambda)-m|\leqslant N^{9/10}\\ \forall 2\leqslant k<M_*,\, m_k(\lambda)=0}} u(m_1)\prod_{k\geqslant 2}\Big(\frac{X_k}{\sqrt{k}}\Big)^{m_k}\frac{1}{m_k!}\Big|^{2q}\Bigg]$$

$$\geqslant \frac{1}{N^q}\left(\frac{N^{q^2/2}}{C_{11}}\Big(\frac{e^{-2C_5m/N}K_r}{1+(1-q)\sqrt{\log K_r}}\Big)^q-C_{11}e^{-C_5q}N^{q^2/2}\Big(\frac{K}{1+(1-q)\sqrt{\log K}}\Big)^q\right)$$

Since $m/N\in[1/6,1/3]$, the right-hand side is $\gg N^{q^2/2}/(1+(1-q)\sqrt{\log N})^q$ for C_5 picked large enough. This completes the proof. □

Proof of lower bound of (3.6). By (3.113) and Lemmas 3.5.14 and 3.5.15, we have

$$\mathbb{E}[|A_N|^{2q}]\geqslant\sum_{m=N/6}^{N/3}\mathbb{E}\Bigg[\Big|\sum_{\substack{\lambda\in\mathcal{P}_N\\ |m_1(\lambda)-m|\leqslant N^{9/10}}}\prod_{k\geqslant 1}\Big(\frac{X_k}{\sqrt{k}}\Big)^{m_k}\frac{1}{m_k!}\Big|^{2q}1_{\{|X_1|\in[m,m+1)\}}\Bigg]-o(N^{-1})$$

$$\geqslant \sum_{m=N/6}^{N/3} \mathbb{E}\Bigg[\Bigg| \sum_{\substack{\lambda\in\mathcal{P}_N \\ |m_1(\lambda)-m|\leqslant N^{9/10}}} u(m_1) \prod_{k\geqslant 2} \left(\frac{X_k}{\sqrt{k}}\right)^{m_k} \frac{1}{m_k!}\Bigg|^{2q}\Bigg]$$

$$\times \mathbb{E}\Big[\Big(\frac{|X_1|^m}{m!}\Big)^{2q} 1_{\{|X_1|\in[m,m+1)\}}\Big] - o(N^{-1})$$

$$\gg \sum_{m=N/6}^{N/3} \frac{N^{q^2/2}}{(1+(1-q)\sqrt{\log N})^q} N^{-q} - o(N^{-1})$$

$$\gg \frac{N^{1-q+q^2/2}}{(1+(1-q)\sqrt{\log N})^q},$$

as desired. □

Proof of the upper bound

Fix a large constant $C_{12} > 0$ to be determined. We expect that for each $m \in [C_{12}, N - C_{12}]$,

$$\mathbb{E}\Bigg[\Bigg| \sum_{\lambda\in\mathcal{P}_N} \prod_{k\geqslant 1} \left(\frac{X_k}{\sqrt{k}}\right)^{m_k} \frac{1}{m_k!}\Bigg|^{2q} 1_{\{|X_1|\in[m,m+1)\}}\Bigg]$$

$$\approx \mathbb{E}\Bigg[\Bigg| \sum_{\substack{\lambda\in\mathcal{P}_N \\ |m_1-m|\leqslant N^{9/10}}} \prod_{k\geqslant 1} \left(\frac{X_k}{\sqrt{k}}\right)^{m_k} \frac{1}{m_k!}\Bigg|^{2q} 1_{\{|X_1|\in[m,m+1)\}}\Bigg], \tag{3.121}$$

and that the contribution to $\mathbb{E}[|A_N|^{2q}]$ from the event $\{|X_1| \notin [C_{12}, N - C_{12}]\}$ does not exceed the order of the lower bound established in Section 3.5.2. This is the purpose of the following result.

Lemma 3.5.16. *It holds that*

$$\mathbb{E}[|A_N|^{2q}] \ll \sum_{m=C_{12}}^{N-C_{12}} \mathbb{E}\Bigg[\Bigg| \sum_{\substack{\lambda\in\mathcal{P}_N \\ |m_1-m|\leqslant N^{9/10}}} \prod_{k\geqslant 1} \left(\frac{X_k}{\sqrt{k}}\right)^{m_k} \frac{1}{m_k!}\Bigg|^{2q} 1_{\{|X_1|\in[m,m+1)\}}\Bigg]$$

$$+ \frac{N^{1-q+q^2/2}}{(1+(1-q)\sqrt{\log N})^q}.$$

Proof. The first step is to control the sum

$$\sum_{m\notin[C_{12},N-C_{12}]}\mathbb{E}\bigg[\Big|\sum_{\lambda\in\mathcal{P}_N}\prod_{k\geqslant 1}\Big(\frac{X_k}{\sqrt{k}}\Big)^{m_k}\frac{1}{m_k!}\Big|^{2q}1_{\{|X_1|\in[m,m+1)\}}\bigg].$$

We use the same approach as in Lemma 3.5.14. We illustrate the case $0<q<1/2$ first using concavity. Using Proposition 3.5.5 and (3.115), the contribution from the sum over $m\geqslant N-C_{12}$ can be bounded by

$$\begin{aligned}
&\sum_{m\geqslant N-C_{12}}\mathbb{E}\bigg[\Big|\sum_{\lambda\in\mathcal{P}_N}\prod_{k\geqslant 1}\Big(\frac{X_k}{\sqrt{k}}\Big)^{m_k}\frac{1}{m_k!}\Big|^{2q}1_{\{|X_1|\in[m,m+1)\}}\bigg]\\
&\leqslant\sum_{m\geqslant N-C_{12}}\sum_{j=0}^{N}\mathbb{E}\bigg[\Big|\sum_{\substack{\lambda\in\mathcal{P}_N\\ m_1(\lambda)=j}}\prod_{k\geqslant 2}\Big(\frac{X_k}{\sqrt{k}}\Big)^{m_k}\frac{1}{m_k!}\Big|^{2q}\bigg]\times\mathbb{E}\bigg[\frac{|X_1|^{2qj}}{(j!)^{2q}}1_{\{|X_1|\in[m,m+1)\}}\bigg]\\
&\leqslant\sum_{m\geqslant N-C_{12}}\sum_{j=0}^{N}\Big(\frac{m}{j}\Big)^{\gamma j}e^{\gamma(j-m)}j^{-q}(1+(1-q)\sqrt{\log(1+N-j)})^{-q}\\
&\ll\sum_{m\geqslant N-C_{12}}\sum_{j=0}^{N}j^{-q}e^{-(j-m)^2/(2j)}(1+(1-q)\sqrt{\log(1+N-j)})^{-q}\\
&\ll N^{1-q}(1+(1-q)\sqrt{\log N})^{-q}.
\end{aligned}$$

Likewise,

$$\sum_{m\leqslant C_{12}}\mathbb{E}\bigg[\Big|\sum_{\lambda\in\mathcal{P}_N}\prod_{k\geqslant 1}\Big(\frac{X_k}{\sqrt{k}}\Big)^{m_k}\frac{1}{m_k!}\Big|^{2q}1_{\{|X_1|\in[m,m+1)\}}\bigg]\ll\sum_{m\leqslant C_{12}}\sum_{j=0}^{N}j^{-q}e^{-(j-m)^2/(2j)}\ll 1.$$

Suppose now that $q\geqslant 1/2$. By Minkowski's inequality, Proposition 3.5.5, and (3.115), there are constants $C_{13},C_{14}>0$ such that for each $m\geqslant N-C_{12}$,

$$\begin{aligned}
&\mathbb{E}\bigg[\Big|\sum_{\lambda\in\mathcal{P}_N}\prod_{k\geqslant 1}\Big(\frac{X_k}{\sqrt{k}}\Big)^{m_k}\frac{1}{m_k!}\Big|^{2q}1_{\{|X_1|\in[m,m+1)\}}\bigg]^{1/(2q)}\\
&\leqslant\sum_{j=0}^{N}\bigg(\mathbb{E}\bigg[\Big|\sum_{\substack{\lambda\in\mathcal{P}_N\\ m_1(\lambda)=j}}\prod_{k\geqslant 2}\Big(\frac{X_k}{\sqrt{k}}\Big)^{m_k}\frac{1}{m_k!}\Big|^{2q}\bigg]\times\mathbb{E}\bigg[\frac{|X_1|^{2qj}}{(j!)^{2q}}1_{\{|X_1|\in[m,m+1)\}}\bigg]\bigg)^{1/(2q)}
\end{aligned}$$

$$\leqslant \sum_{j=0}^{N} \left(\left(\frac{m}{j}\right)^{\gamma j} e^{\gamma(j-m)} j^{-q} (1+(1-q)\sqrt{\log(1+N-j)})^{-q} \right)^{1/(2q)}$$

$$\ll \sum_{j=0}^{N} j^{-1/2} e^{-(j-m)^2/(4qj)} (1+(1-q)\sqrt{\log(1+N-j)})^{-1/2}.$$

Note that

$$\frac{j^{-1/2} e^{-(j-m)^2/(4qj)}}{(1+(1-q)\sqrt{\log(1+N-j)})^{1/2}}$$

$$\ll \begin{cases} j^{-1/2}(1+(1-q)\sqrt{\log(1+N-j)})^{-1/2} & \text{if } j \geqslant m - C_{14}\sqrt{N}; \\ O(e^{-(m-j-C_{14}\sqrt{N})/C_{13}}) & \text{if } j < m - C_{14}\sqrt{N}. \end{cases}$$

Therefore,

$$\mathbb{E}\left[\sum_{m \geqslant N - C_{12}} \left| \sum_{\lambda \in \mathcal{P}_N} \prod_{k \geqslant 1} \left(\frac{X_k}{\sqrt{k}}\right)^{m_k} \frac{1}{m_k!} \right|^{2q} 1_{\{|X_1| \in [m, m+1)\}} \right]$$

$$\ll 1 + \sum_{N - C_{12} \leqslant m \leqslant N + C_{14}\sqrt{N}} \left((1+(1-q)\sqrt{\log N})^{-q} + 1 \right)$$

$$\ll \sqrt{N} \ll \frac{N^{1-q+q^2/2}}{(1+(1-q)\sqrt{\log N})^q}.$$

The sum over $m \leqslant C_{12}$ can be similarly bounded by $\ll 1$.

Finally, for $m \in [C_{12}, N - C_{12}]$, we apply the same technique in Lemma 3.5.14 that leads to

$$\mathbb{E}\left[\left| \sum_{\substack{\lambda \in \mathcal{P}_N \\ |m_1 - m| > N^{9/10}}} \prod_{k \geqslant 1} \left(\frac{X_k}{\sqrt{k}}\right)^{m_k} \frac{1}{m_k!} \right|^{2q} 1_{\{|X_1| \in [m, m+1)\}} \right] \ll N^{-2}.$$

This completes the proof by concavity if $q < 1/2$ and Minkowski's inequality if $q \geqslant 1/2$. □

Proof of upper bound of (3.6). Conditioning on R_1, the right-hand side of (3.121) can be rewritten using independence as

$$\mathbb{E}\left[\frac{|X_1|^{2qm}}{(m!)^{2q}}1_{\{|X_1|\in[m,m+1)\}}\right]\mathbb{E}\left[\Big|\sum_{\substack{\lambda\in\mathcal{P}_N\\|m_1-m|\leqslant N^{9/10}}}\frac{e^{i\tau_1(m_1-m)}|x_1|^{m_1-m}}{m_1!/m!}\prod_{k\geqslant 2}\left(\frac{X_k}{\sqrt{k}}\right)^{m_k}\frac{1}{m_k!}\Big|^{2q}\right], \tag{3.122}$$

where $x_1 \in [m, m+1)$. The first expectation contributes $\asymp m^{-q}$, which directly follows from (3.15). For the second expectation, we use the same multiplicative chaos approach as in the universality phase. Similarly as done in Section 3.5.2, we first apply the same argument that deduced Theorem 3.1.3 equation (3.3) from Proposition 3.3.1 in Section 3.3.1, to start the product in (3.122) from $k = M_*$ instead of from $k = 2$. Define

$$A_{N,M_*,m} := \sum_{\substack{\lambda\in\mathcal{P}_N\\|m_1-m|\leqslant N^{9/10}\\m_1=\cdots=m_{M_*-1}=0}} u(m_1)\prod_{k\geqslant M_*}\left(\frac{X_k}{\sqrt{k}}\right)^{m_k}\frac{1}{m_k!}.$$

It remains to bound

$$\mathbb{E}\left[\Big|\sum_{\substack{\lambda\in\mathcal{P}_N\\|m_1-m|\leqslant N^{9/10}\\m_1=\cdots=m_{M_*-1}=0}}\frac{e^{i\tau_1(m_1-m)}|x_1|^{m_1-m}}{m_1!/m!}\prod_{k\geqslant M_*}\left(\frac{X_k}{\sqrt{k}}\right)^{m_k}\frac{1}{m_k!}\Big|^{2q}\right] = \mathbb{E}\left[|A_{N,M_*,m}|^{2q}\right]$$

from above. First, the same argument of Proposition 3.3.2 applies (possibly with a different constant $C(q)$) with F_{K,M_*} replaced by $\widetilde{F}_{K,M_*,m}$ defined in (3.117): for $1/2 \leqslant q \leqslant 1$,

$$\mathbb{E}\left[|A_{N,M_*,m}|^{2q}\right]^{1/(2q)} \ll \frac{1}{\sqrt{N}}\sum_{j=1}^{J}\mathbb{E}\left[\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|\widetilde{F}_{N/2^j,M_*,m}(\exp(j/N+i\theta))|^2\mathrm{d}\theta\right)^q\right]^{1/(2q)} + \frac{1}{N}, \tag{3.123}$$

and for $0 < q < 1/2$,

$$\mathbb{E}\left[|A_{N,M_*,m}|^{2q}\right] \ll \frac{1}{N^q}\sum_{j=1}^{J}\mathbb{E}\left[\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|\widetilde{F}_{N/2^j,M_*,m}(\exp(j/N+i\theta))|^2\mathrm{d}\theta\right)^q\right] + \frac{1}{N^{2q}}. \tag{3.124}$$

These follow by arguing similarly as in the proof of Proposition 3.5.4, and adapting Proposition 3.3.2 to the case $0 < q < 1/2$ by applying concavity instead of Minkowski's inequality in (3.45). Inserting the upper bound of Proposition 3.5.11 (second inequality of (3.111)) into (3.123) and (3.124) yields that for $q \in (0,1]$ and $m \in [C_{12}, N - C_{12}]$,

$$\mathbb{E}\left[\Big|\sum_{\substack{\lambda\in\mathcal{P}_N\\ |m_1-m|\leqslant N^{9/10}\\ m_1=\cdots=m_{M_*-1}=0}} u(m_1)\prod_{k\geqslant 2}\left(\frac{X_k}{\sqrt{k}}\right)^{m_k}\frac{1}{m_k!}\Big|^{2q}\right] \ll \frac{m^{q^2/2}}{(1+(1-q)\sqrt{\log(N-m)})^q}. \tag{3.125}$$

Combining Lemma 3.5.16 with (3.125), we have

$$\begin{aligned}
\mathbb{E}[|A_N|^{2q}] &\ll \sum_{m=C_{12}}^{N-C_{12}}\mathbb{E}\left[\Big|\sum_{\substack{\lambda\in\mathcal{P}_N\\ |m_1-m|\leqslant N^{9/10}}}\prod_{k\geqslant 1}\left(\frac{X_k}{\sqrt{k}}\right)^{m_k}\frac{1}{m_k!}\Big|^{2q}1_{\{|X_1|\in[m,m+1)\}}\right]\\
&\qquad + O\Big(\frac{N^{1-q+q^2/2}}{(1+(1-q)\sqrt{\log N})^q}\Big)\\
&\ll \sum_{m=C_{12}}^{N-C_{12}} m^{-q}\,\frac{m^{q^2/2}}{(1+(1-q)\sqrt{\log(N-m)})^q} + O\Big(\frac{N^{1-q+q^2/2}}{(1+(1-q)\sqrt{\log N})^q}\Big)\\
&\ll \frac{N^{1-q+q^2/2}}{(1+(1-q)\sqrt{\log N})^q},
\end{aligned}$$

as desired. □

3.6 Remaining proofs

3.6.1 Sharp tightness of scaled secular coefficients

In this subsection, we prove Corollary 3.1.5 using the universality result of Theorem 3.1.3 (i).

Proof of Corollary 3.1.5. Suppose that $\mathbb{E}[e^{\varepsilon|X_k|}] < \infty$, where we may without loss of generality assume that $\varepsilon < 1$. For (i), we apply Markov's inequality to $|A_N|^{\varepsilon/3}$, which yields

$$\mathbb{P}\big(|A_N|(\log N)^{1/4} > C\big) \leqslant \frac{\mathbb{E}[|A_N|^{\varepsilon/3}]}{C^{\varepsilon/3}(\log N)^{-\varepsilon/12}} \leqslant \frac{C(\varepsilon)}{C^{\varepsilon/3}},$$

where $C(\varepsilon)$ is the asymptotic constant in (3.3). Picking C large enough gives the tightness of $|A_N|(\log N)^{1/4}$.

For (ii), we apply the Paley-Zygmund inequality to $|A_N|^{\varepsilon/3}$, which gives that for some large constant $C(\varepsilon) > 0$, uniformly in N,

$$\mathbb{P}\bigg(|A_N| > \frac{(\log N)^{-1/4}}{C(\varepsilon)}\bigg) \geqslant \mathbb{P}\bigg(|A_N|^{\varepsilon/3} > \frac{\mathbb{E}[|A_N|^{\varepsilon/3}]}{2}\bigg) \geqslant \frac{(\mathbb{E}[|A_N|^{\varepsilon/3}])^2}{4\mathbb{E}[|A_N|^{2\varepsilon/3}]} \geqslant \frac{1}{C(\varepsilon)},$$

where the last step follows from Theorem 3.1.3 (i). □

3.6.2 Regularity of critical non-Gaussian holomorphic chaos

In this subsection, we prove Corollary 3.1.6 using the universality result Theorem 3.1.3 (i).

Proof of Corollary 3.1.6. By definition of the Sobolev space H^s, it suffices to show that the series

$$C_s := \sum_{n=0}^{\infty}(1+n^2)^s|A_n|^2$$

converges almost surely for desired s. If there is some $\gamma > 2$ such that $\mathbb{E}[e^{\gamma|X_k|}] < \infty$, then by Theorem 3.1.3 (i) we have

$$\mathbb{E}[C_s] \ll \sum_{n=0}^{\infty}(1+n)^{2s} < \infty$$

for $s < -1/2$. Therefore, the corresponding HMC_1 is in H^s almost surely for $s < -1/2$. Similarly, if X_k satisfies (EXP) with $\gamma \in (0, 2]$, then for any $0 < q < \gamma/2$, Theorem 3.1.3 (i) implies

$$\mathbb{E}[|C_s|^q] \leqslant \sum_{n=0}^{\infty}(1+n^2)^{qs}\mathbb{E}[|A_n|^{2q}] \ll \sum_{n=0}^{\infty}\frac{(1+n)^{2qs}}{(1+(1-q)\sqrt{\log(1+n)})^q} < \infty$$

for $s < -1/(2q)$. Since $q < \gamma/2$ is arbitrary the result follows. □

Remark 6. To show the other direction of irregularity, i.e. non-Gaussian HMC_1 is almost surely not H^s for $s > -1/2$ in the first case of Corollary 3.1.6 (and respectively $s > -1/\gamma$ in the second case), one may need a convergence in probability result of the total mass of critical non-Gaussian multiplicative chaos

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left(2\Re\sum_{k=1}^{\infty}\frac{X_k}{\sqrt{k}}(re^{i\theta})^k\right)\mathrm{d}\theta$$

as $r \to 1$; see the proof of irregularity of Gaussian HMC in Section 6 of [NPS23]. This problem, to the best of our knowledge, remains open. We refer to [Jun18] and references therein for studies on sub-critical non-Gaussian multiplicative chaos.

3.7 Some technical computations regarding exponential moments

We collect some technical computations of (exponential) moments under distinct probability measures in this section. Recall the definitions in Section 3.3.2.

Lemma 3.7.1. *Fix $M_* \in \mathbb{N}$. For any K large enough and $e^{-1/K} \leqslant r \leqslant e^{1/K}$,*

$$\exp\Big(\sum_{k=M_*}^{K} \frac{r^{2k}}{k}\Big) \asymp_{M_*} K.$$

Remark 7. In practice, M_* is a constant depending only on q and the distribution of X_k, while K grows to infinity with $N \to \infty$.

Proof. Suppose that $1 \leqslant r \leqslant e^{1/K}$. Then the lower bound $\exp\Big(\sum_{k=M_*}^{K} \frac{r^{2k}}{k}\Big) \gg K$ is trivial and there exists some constant $L > 0$ such that

$$\sum_{k=M_*}^{K} \frac{r^{2k}}{k} \leqslant \sum_{n=\lfloor \log M_* \rfloor}^{\lceil \log K \rceil} \sum_{e^{n-1} \leqslant k < e^n} \frac{r^{2k}}{k} \leqslant \sum_{n=\lfloor \log M_* \rfloor}^{\lceil \log K \rceil} e^{2e^n/K}$$

$$\leqslant \log K + 1 + L \sum_{n=\lfloor \log M_* \rfloor}^{\lceil \log K \rceil} \frac{e^n}{K} \leqslant \log K + L.$$

The other case $e^{-1/K} \leqslant r \leqslant 1$ is similarly established. □

Lemma 3.7.2. *Suppose that $\mathbb{E}[e^{c_0|R_1|}] < \infty$ for some $c_0 > 0$. For any $\beta \in (0, 2]$, $K \geqslant 1$, and $e^{-1/K} \leqslant r \leqslant e^{1/K}$, we have the following asymptotics for $k \geqslant k_0 = k(c_0)$, some constant depending only on c_0.*

(i) $\mathbb{E}\Big[\exp\Big(2\beta \Re \frac{X_k r^k}{\sqrt{k}}\Big)\Big] = \mathbb{E}\Big[\exp\Big(2\beta \frac{r^k}{\sqrt{k}} R_k \cos(\tau_k)\Big)\Big] = 1 + \beta^2 \frac{r^{2k}}{k} + O(k^{-3/2})$;

(ii) $\mathbb{E}\Big[\exp\Big(2\beta \frac{r^k}{\sqrt{k}} R_k \cos(\tau_k)\Big) R_k \cos(\tau_k)\Big] = \beta \frac{r^k}{\sqrt{k}} + O(k^{-1})$;

(iii) $\mathbb{E}\Big[\exp\Big(2\beta \frac{r^k}{\sqrt{k}} R_k \cos(\tau_k)\Big) R_k^2\Big] = 1 + O(k^{-1/2})$;

(iv) For any $\alpha \in \mathbb{R}$, $\mathbb{E}\Big[\exp\Big(2\beta \frac{r^k}{\sqrt{k}} R_k \cos(\tau_k)\Big) R_k^2 \cos^2(\tau_k + \alpha)\Big] = \frac{1}{2} + O(k^{-1/2})$;

(v) $\mathbb{E}\Big[\exp\Big(2\beta \frac{r^k}{\sqrt{k}} R_k \cos(\tau_k)\Big) |R_k \cos(\tau_k)|^3\Big] \ll 1$.

Proof. Recall that $\mathbb{E}[\cos(\tau_k)^2] = \frac{1}{2}$. For (i), by Taylor's expansion and Fubini's theorem,

$$\mathbb{E}\left[\exp\left(2\beta\Re\frac{X_k r^k}{\sqrt{k}}\right)\right] = \mathbb{E}\left[\exp\left(2\beta\frac{r^k}{\sqrt{k}}R_k\cos(\tau_k)\right)\right] =: 1+\beta^2\frac{r^{2k}}{k}+E_r(k,\beta),$$

where, after noticing $k^{(m-3)/(2m)} \geqslant k^{1/8}$ for $m \geqslant 4$, we have

$$\begin{aligned}
|E_r(k,\beta)| &= \left|\sum_{m=3}^{\infty}\frac{2^m\beta^m r^{mk}}{m!k^{m/2}}\mathbb{E}\left[(R_k\cos(\tau_k))^m\right]\right| \\
&\ll k^{-3/2}\left(\mathbb{E}[|R_k|^3]+\sum_{m=4}^{\infty}\frac{(2e\beta/k^{1/8})^m}{m!}\mathbb{E}[|R_k|^m]\right) \\
&\leqslant k^{-3/2}\left(\mathbb{E}[|R_k|^3]+\mathbb{E}\left[e^{c_0|R_k|}\right]\right) = O(k^{-3/2})
\end{aligned}$$

for k larger than some universal constant. Likewise, we have for (ii),

$$\begin{aligned}
\mathbb{E}\left[\exp\left(2\beta\Re\frac{X_k r^k}{\sqrt{k}}\right)\Re X_k\right] &= \mathbb{E}\left[\exp\left(2\beta\frac{r^k}{\sqrt{k}}R_k\cos(\tau_k)\right)R_k\cos(\tau_k)\right] \\
&=: \beta\frac{r^k}{\sqrt{k}}+\widetilde{E}_r(k,\beta),
\end{aligned}$$

where for k larger than some universal constant (and additionally the fact $m \leqslant (\frac{3}{2})^m$ for $m \geqslant 1$)

$$\begin{aligned}
|\widetilde{E}_r(k,\beta)| &= \left|\sum_{m=2}^{\infty}\frac{2^m\beta^m r^{mk}}{m!k^{m/2}}\mathbb{E}\left[(R_k\cos(\tau_k))^{m+1}\right]\right| \\
&\leqslant k^{-1}\left(\mathbb{E}[|R_k|^3]+\mathbb{E}\left[e^{c_0|R_k|}\right]\right) \\
&= O(k^{-1}).
\end{aligned}$$

For (iv), we use only the first term in the expansion and get

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(2\beta\frac{r^k}{\sqrt{k}}R_k\cos(\tau_k)\right)R_k^2\cos^2(\tau_k+\alpha)\right] \\
&= \mathbb{E}[R_k^2\cos^2(\tau_k+\alpha)]+\mathbb{E}\left[\sum_{m=1}^{\infty}\frac{(2\beta r^k)^m}{m!k^{m/2}}R_k^{m+2}\cos^m(\tau_k)\cos^2(\tau_k+\alpha)\right] = \frac{1}{2}+O(k^{-\frac{1}{2}}).
\end{aligned}$$

For (iii), similarly as in (iv):

$$\mathbb{E}\left[\exp\left(2\beta\frac{r^k}{\sqrt{k}}R_k\cos(\tau_k)\right)R_k^2\right] = \mathbb{E}[R_k^2] + \mathbb{E}\left[\sum_{m=1}^{\infty}\frac{(2\beta r^k)^m}{m!k^{m/2}}R_k^{m+2}\cos^m(\tau_k)\right]$$
$$= 1 + O(k^{-1/2}).$$

For (v), we have by the simple bound $|x| \leqslant e^{|x|}$, for k larger than a certain constant,

$$\mathbb{E}\left[\exp\left(2\beta\Re\frac{X_k r^k}{\sqrt{k}}\right)|\Re X_k|^3\right] \ll \mathbb{E}\left[\exp\left(2\beta\frac{|\Re X_k|r^k}{\sqrt{k}} + 3|\Re X_k|\right)\right]$$
$$\ll \mathbb{E}[e^{2e\beta+3|R_k|/\sqrt{k}}] \ll 1.$$

This completes the proof. □

Lemma 3.7.3. *Assume the same settings of Lemma 3.7.2. For $\theta \in [-\pi, \pi)$, $e^{-1/K} \leqslant r \leqslant e^{1/K}$, and any M_* large enough depending on c_0,*

$$\mathbb{E}\big[|F_{K,M_*}(re^{i\theta})|^2\big] \asymp_{M_*} K.$$

Proof. Using independence of $\{(R_k, \tau_k)\}_{k\geqslant 1}$ and rotational invariance of X_k, as well as Lemma 3.7.2 (i), we have

$$\mathbb{E}\big[|F_{K,M_*}(re^{i\theta})|^2\big] = \mathbb{E}\big[|F_{K,M_*}(r)|^2\big]$$
$$= \mathbb{E}\Big[\exp\Big(2\Re\sum_{k=M_*}^{K}\frac{X_k r^k}{\sqrt{k}}\Big)\Big]$$
$$= \prod_{k=M_*}^{K}\mathbb{E}\Big[\exp\Big(2\frac{r^k}{\sqrt{k}}R_k\cos(\tau_k)\Big)\Big]$$
$$= \exp\Big(\sum_{k=M_*}^{K}\frac{r^{2k}}{k} + O(1)\Big) \asymp_{M_*} \exp\Big(\sum_{k=M_*}^{K}\frac{r^{2k}}{k}\Big) \asymp_{M_*} K,$$

where the last step uses Lemma 3.7.1. □

Let us recall definitions of tilted probability measures $\mathbb{Q}^{(1)}_{r,M,K}$ and $\mathbb{Q}^{(2)}_{r,M,K,\theta}$ in (3.36) and (3.37).

Lemma 3.7.4. *Assume the same settings of Lemma 3.7.2. We have for* $\mathbb{Q}^{(1)} = \mathbb{Q}^{(1)}_{r,M,K}$, *and any* $e^M \vee k_0 \leqslant k < K$ *and* $e^{-1/K} \leqslant r \leqslant e^{1/K}$,

(i) $\mathbb{E}^{\mathbb{Q}^{(1)}}[R_k \cos(\tau_k)] = \frac{r^k}{\sqrt{k}} + O(k^{-1})$;

(ii) $\mathbb{E}^{\mathbb{Q}^{(1)}}[R_k^2 \cos^2(\tau_k)] = \frac{1}{2} + O(k^{-1/2})$;

(iii) $\mathbb{E}^{\mathbb{Q}^{(1)}}[|R_k \cos(\tau_k)|^3] \ll 1$.

We also have for $\mathbb{Q}^{(2)} = \mathbb{Q}^{(2)}_{r,M,K,\theta}$, *and any* $e^M \vee k_0 \leqslant k < K_r$, $\theta \in [-\pi, \pi)$, *and* $e^{-1/K} \leqslant r \leqslant e^{1/K}$,

(i) $\mathbb{E}^{\mathbb{Q}^{(2)}}[R_k \cos(\tau_k)] = (1 + \cos(k\theta)) \frac{r^k}{\sqrt{k}} + O(k^{-1})$;

(ii) $\mathbb{E}^{\mathbb{Q}^{(2)}}[R_k^2 \cos^2(\tau_k)] = \frac{1}{2} + O(k^{-1/2})$ *and* $\mathbb{E}^{\mathbb{Q}^{(2)}}[R_k^2 \cos(\tau_k)\cos(\tau_k + k\theta)] = \frac{\cos(k\theta)}{2} + O(k^{-1/2})$;

(iii) $\mathbb{E}^{\mathbb{Q}^{(2)}}[|R_k \cos(\tau_k)|^3] \ll 1$.

Proof. By definition, Taylor's expansion, Fubini's theorem, and Lemma 3.7.2, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^{(1)}}[R_k \cos(\tau_k)] &= \frac{\mathbb{E}\left[\exp(\frac{2r^k}{\sqrt{k}} R_k \cos(\tau_k)) R_k \cos(\tau_k)\right]}{\mathbb{E}\left[\exp(\frac{2r^k}{\sqrt{k}} R_k \cos(\tau_k))\right]} \\
&= \frac{\frac{r^k}{\sqrt{k}} + O(k^{-1})}{1 + \frac{r^{2k}}{k} + O(k^{-3/2})} = \frac{r^k}{\sqrt{k}} + O(k^{-1}), \\
\mathbb{E}^{\mathbb{Q}^{(1)}}\left[R_k^2 \cos^2(\tau_k)\right] &= \frac{\mathbb{E}\left[\exp(\frac{2r^k}{\sqrt{k}} R_k \cos(\tau_k)) R_k^2 \cos^2(\tau_k)\right]}{\mathbb{E}\left[\exp(\frac{2r^k}{\sqrt{k}} R_k \cos(\tau_k))\right]} \\
&= \frac{\frac{1}{2} + O(k^{-1/2})}{1 + \frac{r^{2k}}{k} + O(k^{-3/2})} = \frac{1}{2} + O(k^{-1/2}),
\end{aligned}$$

and

$$\mathbb{E}^{\mathbb{Q}^{(1)}}\left[|R_k\cos(\tau_k)|^3\right] = \frac{\mathbb{E}\left[\exp(\frac{2r^k}{\sqrt{k}}R_k\cos(\tau_k))|R_k\cos(\tau_k)|^3\right]}{\mathbb{E}\left[\exp(\frac{2r^k}{\sqrt{k}}R_k\cos(\tau_k))\right]}$$

$$\ll \frac{1}{1+\frac{r^{2k}}{k}+O(k^{-3/2})} \ll 1.$$

Computation of moments under $\mathbb{Q}^{(2)}$ is similar. Using the fact $\mathbb{E}[e^{\beta\cos(\tau_k)}\sin(\tau_k)] = \mathbb{E}[e^{\beta\sin(\tau_k)}\cos(\tau_k)] = 0$ for any fixed β in the second equality, and Lemma 3.7.2 in the third, we obtain

$$\mathbb{E}^{\mathbb{Q}^{(2)}}\left[R_k\cos(\tau_k)\right] = \frac{\mathbb{E}\left[\exp(\frac{2r^kR_k}{\sqrt{k}}(\cos(\tau_k)+\cos(\tau_k+k\theta)))R_k\cos(\tau_k)\right]}{\mathbb{E}\left[\exp(\frac{2r^kR_k}{\sqrt{k}}(\cos(\tau_k)+\cos(\tau_k+k\theta)))\right]}$$

$$= \frac{\mathbb{E}\left[\exp(\frac{4r^k\cos(\frac{k\theta}{2})}{\sqrt{k}}R_k\cos(\tau_k))R_k\cos(\tau_k)\right]}{\mathbb{E}\left[\exp(\frac{4r^k\cos(\frac{k\theta}{2})}{\sqrt{k}}R_k\cos(\tau_k))\right]}\cos(\frac{k\theta}{2})$$

$$= \frac{2\cos^2(\frac{k\theta}{2})\frac{r^k}{\sqrt{k}}+O(k^{-1})}{1+4\cos^2(\frac{k\theta}{2})\frac{r^{2k}}{k}+O(k^{-3/2})} = (1+\cos(k\theta))\frac{r^k}{\sqrt{k}}+O(k^{-1}),$$

and analogously, taking $\beta = 2\cos(\frac{k\theta}{2}),\ \alpha = \frac{k\theta}{2}$ in Lemma 3.7.2 (iv),

$$\mathbb{E}^{\mathbb{Q}^{(2)}}\left[R_k^2\cos^2(\tau_k)\right] = \frac{\mathbb{E}\left[\exp(\frac{4r^k\cos(\frac{k\theta}{2})}{\sqrt{k}}R_k\cos(\tau_k))R_k^2\cos^2(\tau_k+\frac{k\theta}{2})\right]}{\mathbb{E}\left[\exp(\frac{4r^k\cos(\frac{k\theta}{2})}{\sqrt{k}}R_k\cos(\tau_k))\right]}$$

$$= \frac{\frac{1}{2}+O(k^{-1/2})}{1+4\cos^2(\frac{k\theta}{2})\frac{r^{2k}}{k}+O(k^{-3/2})} = \frac{1}{2}+O(k^{-1/2}),$$

and using Lemma 3.7.2 (iii), (iv),

$$\mathbb{E}^{\mathbb{Q}^{(2)}}\left[R_k^2\cos(\tau_k)\cos(\tau_k+k\theta)\right] = \frac{\cos(k\theta)-1}{2}\mathbb{E}^{\mathbb{Q}^{(2)}}\left[R_k^2\right] + \mathbb{E}^{\mathbb{Q}^{(2)}}\left[R_k^2\cos^2(\tau+\frac{k\theta}{2})\right]$$

$$= \frac{\cos(k\theta)}{2}+O(k^{-1/2}).$$

Finally,

$$\mathbb{E}^{\mathbb{Q}^{(2)}}\left[|R_k\cos(\tau_k)|^3\right] \ll \frac{\mathbb{E}[|R_k|^3]+O(\frac{1}{\sqrt{k}})}{1+4\cos^2(\frac{k\theta}{2})\frac{r^{2k}}{k}+O(k^{-3/2})} \ll 1.$$

This completes the proof. □

Recall (3.39) and (3.40). The discrepancy of the sums over μ_k and ν_k is small due to the fluctuation of the cosine function, as pointed out in the next lemma. We refer to [SZ22, equation (12.7)] for a simple proof of this elementary estimate.

Lemma 3.7.5. *We have for any $m \in \mathbb{N}$ and $\theta \in [-\pi,\pi)\setminus\{0\}$ that*

$$\left|\sum_{e^{m-1}\leqslant k<e^m}\frac{r^{2k}\cos(k\theta)}{k}\right| \ll \frac{1}{|\theta|e^m}.$$

In particular, for any $\theta \in [-\pi,\pi)\setminus\{0\}$,

$$\sum_{m\geqslant \log\frac{1}{|\theta|}}\left|\sum_{e^{m-1}\leqslant k<e^m}(\mu_k-\nu_k)\right| \ll 1.$$

Finally, we record the following estimate for the denominator of (3.37).

Lemma 3.7.6. *For any $\theta \in [-\pi,\pi)\setminus\{0\}$, $M \geqslant \log(10^3/|\theta|)$, and $e^{-1/K}\leqslant r\leqslant e^{1/K}$,*

$$\mathbb{E}\left[\exp\left(2\sum_{m=M+1}^{\log K_r}(Z_0(m)+Z_\theta(m))\right)\right] \asymp \frac{K_r^2}{e^{2M}}.$$

Proof. Recall in the proof of Lemma 3.7.4 we computed

$$\begin{aligned}\mathbb{E}\left[\exp\Big(\frac{2r^kR_k}{\sqrt{k}}(\cos(\tau_k)+\cos(\tau_k+k\theta))\Big)\right] &= 1+4\cos^2(\frac{k\theta}{2})\frac{r^{2k}}{k}+O(k^{-3/2})\\ &= 1+\frac{2r^k}{k}+\frac{2r^k}{k}\cos(k\theta)+O(k^{-3/2}).\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big(2\sum_{m=M+1}^{\log K_r}(Z_0(m)+Z_\theta(m))\Big)\Big]\\
&=\prod_{e^M\leqslant k<K_r}\Big(1+\frac{2r^k}{k}+\frac{2r^k}{k}\cos(k\theta)+O(k^{-3/2})\Big)\\
&\asymp\exp\Big(\sum_{e^M\leqslant k<K_r}\Big(\frac{2r^k}{k}+\frac{2r^k}{k}\cos(k\theta)\Big)\Big).
\end{aligned}$$

By Lemmas 3.7.1 and 3.7.5, the desired statement follows. □

3.8 Deferred proofs from Section 3.5.2

Proof of Lemma 3.5.12. Suppose first that $|\tau_1|\leqslant N^{-1/2}$. Then

$$\Big|\sum_{j:|j-m|\leqslant N^{9/10}}e^{ij\tau_1}\,\frac{|x_1|^{j-m}}{j!/m!}r^j\Big|\leqslant\sum_{j:|j-m|\leqslant N^{9/10}}\frac{|x_1|^{j-m}}{j!/m!}r^j\ll\sum_{j:|j-m|\leqslant N^{9/10}}\frac{m^{j-m}}{j!/m!}r^m. \tag{3.126}$$

Using (3.115), the sum over $j\geqslant m$ can be bounded by

$$\sum_{j:0\leqslant j-m\leqslant N^{9/10}}\frac{m^{j-m}}{j!/m!}\ll\sum_{j'=0}^{N^{9/10}}e^{j'}\Big(\frac{m}{m+j'}\Big)^{m+j'}\leqslant\sum_{j'=0}^{N^{9/10}}\exp\Big(-\frac{(j')^2}{m}\Big)\ll\sqrt{N}. \tag{3.127}$$

The other sum over $j<m$ can be bounded similarly, leading to

$$\Big|\sum_{j:|j-m|\leqslant N^{9/10}}e^{ij\tau_1}\,\frac{|x_1|^{j-m}}{j!/m!}r^j\Big|\ll\sqrt{N}r^m\ll\frac{r^m}{|\tau_1|}$$

in the case $|\tau_1|\leqslant N^{-1/2}$.

Now suppose that $|\tau_1|\geqslant N^{-1/2}$. A consequence of (3.126) and (3.127) is that

$$\Big|\sum_{j:|j-m|\leqslant N^{9/10}}e^{ij\tau_1}\,\frac{|x_1|^{j-m}}{j!/m!}\,r^j\Big|\ll\Big|\sum_{j:|j-m|\leqslant\sqrt{N}}e^{ij\tau_1}\,\frac{|x_1|^{j-m}}{j!/m!}\Big|\,r^m+r^m.$$

On the other hand, summation by parts yields (without loss of generality, assume $\sqrt{N} \in \mathbb{Z}$)

$$\sum_{j:0\leqslant j-m\leqslant\sqrt{N}} e^{ij\tau_1}\,\frac{|x_1|^{j-m}}{j!/m!}$$

$$=\frac{|x_1|^{\sqrt{N}}}{(\sqrt{N}+m)!/m!}\sum_{j=0}^{\sqrt{N}} e^{ij\tau_1} - \sum_{k=1}^{\sqrt{N}}\Big(\sum_{j=0}^{k-1} e^{ij\tau_1}\Big)\Big(\frac{|x_1|^k}{(k+m)!/m!} - \frac{|x_1|^{k-1}}{(k-1+m)!/m!}\Big).$$

Applying triangle inequality then leads to

$$\Big|\sum_{j:0\leqslant j-m\leqslant\sqrt{N}} e^{ij\tau_1}\,\frac{|x_1|^{j-m}}{j!/m!}\Big|$$

$$\leqslant\frac{|x_1|^{\sqrt{N}}}{|\tau_1|(\sqrt{N}+m)!/m!} + \frac{1}{|\tau_1|}\sum_{k=1}^{\sqrt{N}}\Big|\frac{|x_1|^k}{(k+m)!/m!} - \frac{|x_1|^{k-1}}{(k-1+m)!/m!}\Big|$$

$$\ll\frac{1}{|\tau_1|}\ll\sqrt{N}.$$

The other sum over $j : -\sqrt{N} \leqslant j-m < 0$ is similar. This finishes the proof. □

Proof of Lemma 3.5.13. The main idea is that, under our assumptions, $e^{ij\tau} \approx e^{im\tau}$, $|x_1| \approx m$, and $r^j \approx r^m$ for $|j-m| \leqslant \sqrt{C_4 N}$, and the sum over j with $|j-m| \geqslant \sqrt{C_4 N}$ is negligible. Let us make these approximations precise. First, for N large enough,

$$\Big|\sum_{\sqrt{C_4N}\leqslant|j-m|\leqslant N^{9/10}} e^{ij\tau_1}\,\frac{|x_1|^{j-m}}{j!/m!}r^j\Big| \leqslant \sum_{\sqrt{C_4N}\leqslant|j-m|\leqslant N^{9/10}}\frac{|x_1|^{j-m}}{j!/m!}r^j$$

$$\leqslant Lr^m\sum_{\sqrt{C_4N}\leqslant|j-m|\leqslant N^{9/10}}\frac{m^{j-m}}{j!/m!},$$

since $|\log r| = O(1/N)$. Applying the same argument in (3.127) leads to

$$\sum_{\sqrt{C_4N}\leqslant|j-m|\leqslant N^{9/10}}\frac{m^{j-m}}{j!/m!} \ll \sum_{j=\sqrt{C_4N}}^{N^{9/10}}\exp\Big(-\frac{j^2}{m}\Big) \ll e^{-3C_4}.$$

It follows that

$$\Big| \sum_{\sqrt{C_4 N} \leqslant |j-m| \leqslant N^{9/10}} e^{ij\tau_1} \, \frac{|x_1|^{j-m}}{j!/m!} r^j \Big| \leqslant L r^m e^{-3C_4}. \tag{3.128}$$

Second, for N large enough,

$$\sum_{m-\sqrt{C_4 N} < j < m+\sqrt{C_4 N}} \frac{|x_1|^{j-m}}{j!/m!} r^j |e^{ij\tau} - e^{im\tau}|$$

$$\ll r^m \sum_{j=1}^{\sqrt{C_4 N}} \exp\Big(- \frac{j^2}{m} \Big) C_4^{-1/2} \ll r^m \sqrt{N} C_4^{-1/2}.$$

By the triangle inequality,

$$\Big| \sum_{m-\sqrt{C_4 N} < j < m+\sqrt{C_4 N}} e^{ij\tau} \frac{|x_1|^{j-m}}{j!/m!} r^j - \sum_{m-\sqrt{C_4 N} < j < m+\sqrt{C_4 N}} e^{im\tau} \frac{|x_1|^{j-m}}{j!/m!} r^j \Big|$$

$$\leqslant L r^m \sqrt{N} C_4^{-1/2}. \tag{3.129}$$

In addition,

$$\Big| \sum_{m-\sqrt{C_4 N} < j < m+\sqrt{C_4 N}} e^{im\tau} \frac{|x_1|^{j-m}}{j!/m!} r^j \Big| = \sum_{m-\sqrt{C_4 N} < j < m+\sqrt{C_4 N}} \frac{|x_1|^{j-m}}{j!/m!} r^j \geqslant \frac{1}{L} r^m \sqrt{N}. \tag{3.130}$$

Combining (3.128)–(3.130) and applying the triangle inequality, we conclude that

$$\Big| \sum_{j : |j-m| \leqslant N^{9/10}} e^{ij\tau} \, \frac{|x_1|^{j-m}}{j!/m!} r^j \Big| \geqslant \frac{1}{L} r^m \sqrt{N} - L r^m e^{-3C_4} - L r^m \sqrt{N} C_4^{-1/2}.$$

The claim then follows by picking C_4 large enough. □

Chapter 4

Conclusion

In this dissertation, we established several results related to the large values of non-Gaussian logarithmically correlated fields, including Theorem 2.1.1, Theorem 3.1.3, and Theorem 3.1.4.

Bibliography

[ABB17] Louis-Pierre Arguin, David Belius, and Paul Bourgade. Maximum of the characteristic polynomial of random unitary matrices. *Communications in Mathematical Physics*, 349(2):703–751, 2017.

[ABB+19] Louis-Pierre Arguin, David Belius, Paul Bourgade, Maksym Radziwiłł, and Kannan Soundararajan. Maximum of the Riemann zeta function on a short interval of the critical line. *Communications on Pure and Applied Mathematics*, 72(3):500–535, 2019.

[ABR20] Louis-Pierre Arguin, Paul Bourgade, and Maksym Radziwiłł. The Fyodorov-Hiary-Keating conjecture. i. *ArXiv preprint arXiv:2007.00988*, 2020.

[ABR23] Louis-Pierre Arguin, Paul Bourgade, and Maksym Radziwiłł. The Fyodorov-Hiary-Keating conjecture. ii. *ArXiv preprint arXiv:2307.00982*, 2023.

[And98] George E Andrews. *The Theory of Partitions*. 2. Cambridge University Press, 1998.

[Arg16] Louis-Pierre Arguin. Extrema of log-correlated random variables. *Advances in disordered systems, random processes and some applications*, page 166, 2016.

[Art64] Emil Artin. *The Gamma Function*. Holt, Rinehart & Winston, New York, 1964.

[ASV+22] Daksh Aggarwal, Unique Subedi, William Verreault, Asif Zaman, and Chenghui Zheng. A conjectural asymptotic formula for multiplicative chaos in number theory. *Research in Number Theory*, 8(2):1–19, 2022.

[AT92] Richard Arratia and Simon Tavare. The Cycle Structure of Random Permutations. *The Annals of Probability*, 20(3):1567–1591, July 1992. Publisher: Institute of Mathematical Statistics.

[Aï13] Elie Aïdékon. Convergence in law of the minimum of a branching random walk. *The Annals of Probability*, 41(3A):1362–1426, May 2013. Publisher: Institute of Mathematical Statistics.

[Bac00] Markus Bachmann. Limit theorems for the minimal position in a branching random walk with independent logconcave displacements. *Advances in Applied Probability*, 32(1):159–176, 2000.

[BDZ16a] Maury Bramson, Jian Ding, and Ofer Zeitouni. Convergence in law of the maximum of nonlattice branching random walk. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 52(4):1897–1924, November 2016. Publisher: Institut Henri Poincaré.

[BDZ16b] Maury Bramson, Jian Ding, and Ofer Zeitouni. Convergence in law of the maximum of the two-dimensional discrete gaussian free field. *Communications on Pure and Applied Mathematics*, 69(1):62–123, 2016.

[BGT89] Nicholas H Bingham, Charles M Goldie, and Jef L Teugels. *Regular Variation*. 27. Cambridge University Press, 1989.

[BK04] John D Biggins and Andreas E Kyprianou. Measure change in multitype branching. *Advances in Applied Probability*, 36(2):544–581, 2004.

[BK22] Emma C Bailey and Jonathan P Keating. Maxima of log-correlated fields: some recent developments. *Journal of Physics A: Mathematical and Theoretical*, 55(5):053001, 2022.

[BLZ23] Paul Bourgade, Patrick Lopatto, and Ofer Zeitouni. Optimal rigidity and maximum of the characteristic polynomial of Wigner matrices, December 2023. arXiv:2312.13335 [math-ph].

[Bra78] Maury D Bramson. Maximal displacement of branching brownian motion. *Communications on Pure and Applied Mathematics*, 31(5):531–581, 1978.

[Bra83] Maury Bramson. Convergence of solutions of the Kolmogorov equation to travelling waves, 1983. ISBN: 9780821822852 9781470406950 ISSN: 0065-9266, 1947-6221 Issue: 285 Publisher: American Mathematical Society Series: Memoirs of the American Mathematical Society Volume: 44.

[Cai23a] Rachid Caich. Almost sure upper bound for a model problem for multiplicative chaos in number theory. *ArXiv preprint arXiv:2304.01632*, 2023.

[Cai23b] Rachid Caich. Almost sure upper bound for random multiplicative functions. *ArXiv preprint arXiv:2304.00943*, 2023.

[CL24] Giorgio Cipolloni and Benjamin Landon. Maximum of the Characteristic Polynomial of I.I.D. Matrices, May 2024. arXiv:2405.05045 [math-ph].

[CLZ22] Simon Coste, Gaultier Lambert, and Yizhe Zhu. The characteristic polynomial of sums of random permutations and regular digraphs. *arXiv preprint arXiv:2204.00524*, 2022.

[CMN18] Reda Chhaibi, Thomas Madaule, and Joseph Najnudel. On the maximum of the CβE field. *Duke Mathematical Journal*, 167(12):2243–2345, 2018.

[CN19] Reda Chhaibi and Joseph Najnudel. On the circle, $GMC^\gamma = \varprojlim C\beta E_n$ for $\gamma = \sqrt{\frac{2}{\beta}}$ ($\gamma \leqslant 1$). *ArXiv preprint arXiv:1904.00578*, 2019.

[Cos23] Simon Coste. Sparse matrices: convergence of the characteristic polynomial seen from infinity. *Electronic Journal of Probability*, 28:1–40, 2023.

[CZ20] Nicholas Cook and Ofer Zeitouni. Maximum of the characteristic polynomial for a random permutation matrix. *Communications on Pure and Applied Mathematics*, 73(8):1660–1731, 2020.

[DG04] Persi Diaconis and Alex Gamburd. Random matrices, magic squares and matching polynomials. *The Electronic Journal of Combinatorics*, page R2, 2004.

[Dir15] Sjoerd Dirksen. Tail bounds via generic chaining. *Electronic Journal of Probability*, 20, 2015.

[DRSV17] Bertrand Duplantier, Rémi Rhodes, Scott Sheffield, and Vincent Vargas. Log-correlated Gaussian fields: an overview. *Geometry, Analysis and Probability: In Honor of Jean-Michel Bismut*, pages 191–216, 2017.

[DRZ17] Jian Ding, Rishideep Roy, and Ofer Zeitouni. Convergence of the centered maximum of log-correlated Gaussian fields. *The Annals of Probability*, 45(6A), November 2017.

[DS94] Persi Diaconis and Mehrdad Shahshahani. On the eigenvalues of random matrices. *Journal of Applied Probability*, 31(A):49–62, 1994.

[Dur19] Rick Durrett. *Probability: Theory and Examples*, volume 49. Cambridge University Press, 2019.

[FHK12] Yan V Fyodorov, Ghaith A Hiary, and Jonathan P Keating. Freezing transition, characteristic polynomials of random matrices, and the Riemann zeta function. *Physical Review Letters*, 108(17):170601, 2012.

[FK14] Yan V Fyodorov and Jonathan P Keating. Freezing transitions and extreme values: random matrix theory, and disordered landscapes. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 372(2007):20120503, 2014.

[Ger22] Maxim Gerspach. Almost sure lower bounds for a model problem for multiplicative chaos in number theory. *Mathematika*, 68(4):1331–1363, 2022.

[GV23] Christophe Garban and Vincent Vargas. Harmonic analysis of gaussian multiplicative chaos on the circle. *arXiv preprint arXiv:2311.04027*, 2023.

[GZ24] Haotian Gu and Zhenyuan Zhang. Universality and Phase Transitions in Low Moments of Secular Coefficients of Critical Holomorphic Multiplicative Chaos, March 2024. arXiv:2401.05681 [math].

[Har20] Adam J Harper. Moments of random multiplicative functions, I: Low moments, better than squareroot cancellation, and critical multiplicative chaos. In *Forum of Mathematics, Pi*, volume 8. Cambridge University Press, 2020.

[Har23a] Adam J Harper. Almost sure large fluctuations of random multiplicative functions. *International Mathematics Research Notices*, 2023(3):2095–2138, 2023.

[Har23b] Adam J Harper. The typical size of character and zeta sums is $o(\sqrt{x})$. *arXiv preprint arXiv:2301.04390*, 2023.

[Hel10] Henry Helson. Hankel forms. *Studia Mathematica*, 1(198):79–84, 2010.

[HKO01] Christopher P Hughes, Jonathan P Keating, and Neil O'Connell. On the characteristic polynomial of a random unitary matrix. *Communications in Mathematical Physics*, 220(2):429–451, 2001.

[HKS+96] Fritz Haake, Marek Kus, Hans-Jürgen Sommers, Henning Schomerus, and Karol Zyczkowski. Secular determinants of random unitary matrices. *Journal of Physics A: Mathematical and General*, 29(13):3641, 1996.

[HLL14] Chunmao Huang, Xingang Liang, and Quansheng Liu. Branching random walks with random environments in time. *Frontiers of Mathematics in China*, 9:835–842, 2014.

[JM15] Tiefeng Jiang and Sho Matsumoto. Moments of traces of circular beta-ensembles. *The Annals of Probability*, 43(6):3279–3336, 2015.

[JS17] Janne Junnila and Eero Saksman. Uniqueness of critical Gaussian chaos. *Electronic Journal of Probability*, 22(11):1–31, 2017.

[Jun18] Janne Junnila. On the Multiplicative Chaos of Non-Gaussian Log-Correlated Fields. *International Mathematics Research Notices*, 2020(19):6169–6196, 08 2018.

[KN04] Rowan Killip and Irina Nenciu. Matrix models for circular ensembles. *International Mathematics Research Notices*, 2004(50):2665–2701, 2004.

[Kri22] Xaver Kriechbaum. Tightness for branching random walk in time-inhomogeneous random environment. *arXiv preprint arXiv:2205.11643*, 2022.

[Lac22] Hubert Lacoin. A universality result for subcritical complex gaussian multiplicative chaos. *The Annals of Applied Probability*, 32(1):269–293, 2022.

[LRV15] Hubert Lacoin, Rémi Rhodes, and Vincent Vargas. Complex gaussian multiplicative chaos. *Communications in Mathematical Physics*, 337:569–632, 2015.

[LT91] Michel Ledoux and Michel Talagrand. *Probability in Banach Spaces: Isoperimetry and Processes*, volume 23. Springer Science & Business Media, 1991.

[MM18] Bastien Mallein and Piotr Miłoś. Brownian motion and random walk above quenched random wall. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 54(4):1877–1916, November 2018. Publisher: Institut Henri Poincaré.

[MM19] Bastien Mallein and Piotr Miłoś. Maximal displacement of a supercritical branching random walk in a time-inhomogeneous random environment. *Stochastic Processes and their Applications*, 129(9):3239–3260, 2019.

[Naj18] Joseph Najnudel. On the extreme values of the Riemann zeta function on random intervals of the critical line. *Probability Theory and Related Fields*, 172:387–452, 2018.

[NPS23] Joseph Najnudel, Elliot Paquette, and Nick Simm. Secular coefficients and the holomorphic multiplicative chaos. *The Annals of Probability*, 51(4):1193–1248, 2023.

[NSW20] Miika Nikula, Eero Saksman, and Christian Webb. Multiplicative chaos and the characteristic polynomial of the CUE: The l^1-phase. *Transactions of the American Mathematical Society*, 373(6):3905–3965, 2020.

[Pow21] Ellen Powell. Critical Gaussian multiplicative chaos: a review. *Markov Processes and Related Fields*, 27(4):557–606, 2021.

[PZ18] Elliot Paquette and Ofer Zeitouni. The maximum of the CUE field. *International Mathematics Research Notices*, 2018(16):5028–5119, 2018.

[PZ22] Elliot Paquette and Ofer Zeitouni. The extremal landscape for the CβE ensemble. *ArXiv preprint arXiv:2209.06743*, 2022.

[RV14] Rémi Rhodes and Vincent Vargas. Gaussian multiplicative chaos and applications: a review. *Probability Surveys*, 11:315–392, 2014.

[Sad66] SM Sadikova. Two-dimensional analogues of an inequality of Esseen with applications to the central limit theorem. *Theory of Probability & Its Applications*, 11(3):325–335, 1966.

[Shi15] Zhan Shi. *Branching Random Walks*, volume 2151 of *Lecture Notes in Mathematics*. Springer International Publishing, Cham, 2015.

[Sou22] Kannan Soundararajan. The distribution of values of zeta and l-functions. *Plenary LectureS*, 2:1260–1310, 2022.

[SW20] Eero Saksman and Christian Webb. The Riemann zeta function and Gaussian multiplicative chaos: Statistics on the critical line. *The Annals of Probability*, 48(6):2680–2754, 2020.

[SX23] Kannan Soundararajan and Max Wenqiang Xu. Central limit theorems for random multiplicative functions. *Journal d'Analyse Mathématique*, 151:343–374, 2023.

[SZ22] Kannan Soundararajan and Asif Zaman. A model problem for multiplicative chaos in number theory. *L'Enseignement Mathématique*, 68(3):307–340, 2022.

[Tal21] Michel Talagrand. *Upper and Lower Bounds for Stochastic Processes: Decomposition Theorems*, volume 60. Springer Science & Business Media, 2021.

[Web15] Christian Webb. The characteristic polynomial of a random unitary matrix and Gaussian multiplicative chaos-the l^2-phase. *Electronic Journal of Probability*, 20:1–21, 2015.

[Win12] Andreas Winkelbauer. Moments and absolute moments of the normal distribution. *ArXiv preprint arXiv:1209.4340*, 2012.

[Xu23] Max Wenqiang Xu. Better than square-root cancellation for random multiplicative functions. *ArXiv preprint arXiv:2303.06774*, 2023.