SOLITONS FOR THE CLOSED G_2 LAPLACIAN FLOW

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A thesis submitted to the Department of Mathematics for honors
Duke University, Durham, NC
April 2021

1. Introduction

Geometric flows, such as the Ricci flow, are partial differential equations on smooth manifolds which describe the time evolution of some geometric structure on the manifold, such as a Riemannian metric. Many geometric flows are heuristically similar to the heat equation as they are often parabolic in some suitable interpretation, and hence can be thought of as deforming an initial geometric structure to one that is nicer. For instance, the curve-shortening flow is a geometric flow for planar curves that evolves a closed convex embedded curve to a round point. Unlike strictly parabolic linear equations like the heat equation, the nonlinearity of many geometric flows frequently leads to the formation of singularities in finite time. To understand these singularities, we study special, self-similar solutions called solitons, with the expectation that developing singularities of the flow are modeled on these soliton solutions.

In the setting of G_2 geometry, which is specific to seven dimensions, we consider a particular type of a differential three-form φ called a G_2 structure, which satisfies a certain pointwise algebraic nondegeneracy condition. In particular φ nonlinearly induces a metric and orientation, and hence a Laplace operator Δ_φ on k-forms. Thus we may consider the geometric flow/initial value problem

$$\begin{aligned} &\frac{\partial}{\partial t}\varphi = \Delta_\varphi \varphi \\ &\varphi(0) = \varphi_0, \quad d\varphi_0 = 0. \end{aligned}$$

This flow, introduced by Bryant in [2], is called the closed G_2 Laplacian flow. Critical points of this flow correspond to torsion-free G_2 structures, which satisfy a system of nonlinear partial differential equations. We are interested in such G_2 structures since they give rise to metrics with holonomy contained in the exceptional Lie group G_2. Thus, akin to how we expect solutions of the heat equation to converge to harmonic functions in infinite time, we hope the Laplacian flow, under appropriate conditions, will evolve closed G_2 structures to torsion-free ones. While there are results regarding short-time existence/uniqueness of the flow, and fewer regarding long-time existence and finite-time singularities, we are still far from the more advanced methods in Ricci flow such as surgery and classification of singularities.

In order to gain insight about singularities of the flow, we consider solitons of the flow, which are given by a G_2 structure φ, a vector field X, and a real constant λ, satisfying

$$\Delta_\varphi \varphi = \mathcal{L}_X \varphi + \lambda \varphi.$$

Solitons give rise to solutions of the flow that evolve only by diffeomorphisms and homotheties. We say a soliton is shrinking if $\lambda < 0$. Since the soliton equation is a nonlinear system of PDE, in order to get a feel for concrete solutions we consider a dimensional reduction of the equation by studying cohomogeneity-one solitons. We consider the cohomogeneity-one setting where the group G of isometries is either $SU(3)$ or $Sp(2)$. In these regimes, we show that the soliton equation is equivalent to a (nonlinear) first-order system of real-analytic ODEs. We then describe the space of

closed G-invariant cones, identifying it with a smooth surface/curve in $\mathbb{R}^3$. We then investigate the uniqueness of an explicit shrinking $Sp(2)$-invariant soliton, and conjecture that it is rigid to first order. We finally discuss ways to study the general closed Laplacian soliton equation, reviewing previous results by others in related directions.

2. Preliminaries

Here we introduce notation and basics from G_2 geometry. Further details can be found in [7] and [2].

Let $\{e_i\}_{i=1}^7$ denote the standard orthonormal basis of $\mathbb{R}^7$, and denote $e^{ijk} \coloneqq e^i \wedge e^j \wedge e^k$. Consider the three-form

$$\varphi_{\mathrm{Euc}} \coloneqq e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}.$$

Then G_2 is defined to be $\{A \in \mathrm{GL}(7,\mathbb{R}) \mid A^*\varphi = \varphi\}$. Moreover G_2 is a compact, connected, fourteen-dimensional Lie group. A G_2 structure on a manifold M is then a smooth three-form φ such that at each point $p \in M$ there exists a linear isomorphism $A\colon \mathbb{R}^7 \to T_pM$ such that $A^*\varphi = \varphi_{\mathrm{Euc}}$. This is equivalent to a G_2 structure on M in the sense of a reduction of the structure group of the frame bundle of the tangent bundle from $GL(7,\mathbb{R})$ to G_2. Then since $\mathrm{G}_2 \subset SO(7)$, φ determines a metric g_φ and orientation (given by the volume form vol_φ) in the following nonlinear fashion: letting u, v denote tangent vectors, we have

$$g_\varphi(u,v)\mathrm{vol}_\varphi = (u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi.$$

Since φ induces a metric and orientation, it also induces a Hodge star operator $*_\varphi$, codifferential d^*_φ, and a Laplace operator Δ_φ on k-forms. We will frequently abuse notation by dropping the subscript φ because it will be clear from context what G_2 structure we are referring to. We will also denote $\psi \coloneqq *\varphi$. Now, the action of G_2 determines a decomposition of $\wedge^k T^*M, 1 \le k \le 7$ into irreducible representations of G_2, and likewise for sections of these bundles, akin to how $\mathrm{GL}(n,\mathbb{C})$ structures induce the decomposition of complex differential forms into (p,q) forms. Explicitly, we have

$$\begin{aligned}\Omega^2(M) &= \Omega^2_7(M) \oplus \Omega^2_{14}\\ \Omega^3(M) &= \Omega^3_1(M) \oplus \Omega^3_7(M) \oplus \Omega^3_{27}(M),\end{aligned}$$

where

$$\begin{aligned}\Omega^2_7(M) &= \{X \lrcorner \varphi \mid X \in \mathfrak{X}(M)\}\\ \Omega^2_{14}(M) &= \{\omega \in \Omega^2(M) \mid \omega \wedge \varphi = - * \omega\}\\ \Omega^3_1(M) &= \{f\varphi \mid f \in C^\infty(M)\}\\ \Omega^3_7(M) &= \{X \lrcorner \psi \mid X \in \mathfrak{X}(M)\}\\ \Omega^3_{27}(M) &= \{\eta \in \wedge^3 T^*M \mid \eta \wedge \varphi = 0 = \eta \wedge \psi\}.\end{aligned}$$

Here, the lower subscript indicates the pointwise dimension of the subspace. We remark that the above decompositions are orthogonal with respect to g and preserved by $*$. We further remark that (pointwise) Ω^2_{14} corresponds the Lie algebra $\mathfrak{g}_2$, and we have G_2-invariant maps from the space of symmetric two-forms to the space of alternating three-forms given as follows. Define the linear map $i_\varphi\colon S^2T^*M \to \wedge^3 T^*M$ by setting

$$i_\varphi(\alpha \circ \beta) = \alpha \wedge *(\beta \wedge \psi) + \beta \wedge *(\alpha \wedge \psi)$$

on decomposable elements and extending linearly. Now define the map $j_\varphi\colon \wedge^3 T^*M \to S^2T^*M$ by

$$j_\varphi(\eta)(u,v) = *((u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \eta).$$

As in section 2.6 of [2], $\ker(j_\varphi) = \wedge^3_7$, and j_φ is an isomorphism between $\wedge^3_{27}$ and the space of traceless symmetric two-tensors.

Owing to the above decompositions, we can express $d\varphi$ and $d\psi$ in terms of uniquely determined differential forms $\tau_1, \tau_7, \tau_{14}, \tau_{27}$, where the subscript k indicate which piece of the decomposition τ_k lies in. Explicitly, we have the orthogonal decompositions

$$d\varphi = \tau_1\psi + 3\tau_7 \wedge \varphi + *\tau_{27}$$
$$d\psi = 4\tau_7 \wedge \psi + \tau_{14} \wedge \varphi.$$

Strictly speaking, it is not immediate that the same one-form τ_7 should appear in both equations, but this is indeed the case (see Remark 5 in [2]). Now, it is the case that $d\varphi = 0 = d^*\varphi$ if and only if $\nabla\varphi = 0$. Here ∇ is the Levi-Civita connection determined by g_φ. The reverse direction is clear; for a proof of the forward direction, see Corollary 2.28 in [7]. The latter condition is equivalent to the manifold M having Riemannian holonomy contained in G_2, and we say such a G_2 structure is torsion-free. Observe that the condition that φ be closed is locally given by 35 equations in 35 variables, and the condition ψ be closed is locally given by 21 equations in 35 variables, with the overlap of seven equations corresponding to τ_7 appearing twice in the above. Hence the torsion-free condition is locally given by 49 equations in 35 variables.

Laplacian Solitons

Consider the following geometric flow:

Definition 2.1. We say a family of G_2 structures $\varphi(t)$ which depend differentiably on a real parameter t is a solution to the Laplacian flow with initial value φ_0 if for all t

$$\frac{\partial}{\partial t}\varphi(t) = \Delta_{\varphi(t)}\varphi(t)$$
$$\varphi(0) = \varphi_0.$$

If φ_0 is closed, then $d\varphi(t) = 0$ for all t (see Section 1.4 of [1]). Thus for closed G_2 structures, it suffices to consider the flow

$$\frac{\partial}{\partial t}\varphi = dd^*_\varphi\varphi = d\tau_{14}.$$

As with other geometric flows, we consider soliton solutions to the Laplacian flow.

Definition 2.2. A triple (φ, X, λ), where φ is a G_2 structure, X is a smooth vector field, and λ is a constant, is called a Laplacian soliton if the triple satisfies

$$\Delta_\varphi\varphi = \mathcal{L}_X\varphi + \lambda\varphi. \tag{2.3}$$

We say a soliton is a gradient soliton if $X = \operatorname{grad} f$ for some smooth function f.

Now suppose we have a soliton. Then defining

$$\rho(t) = (1 + \frac{2}{3}\lambda t)^{\frac{3}{2}}, \quad X(t) = \rho(t)^{-\frac{2}{3}}X,$$

and letting ϕ_t denote the family of diffeomorphisms generated by $X(t)$ with $\phi_0 = \mathrm{id}_M$, we set $\varphi(t) = \rho(t)\phi_t^*\varphi_0$ and observe that $\varphi(t)$ is a solution to the Laplacian flow. Note how λ affects the interval on which $\rho(t)$ may be defined. From this observation, we label a soliton as:

$$\lambda < 0 : \text{shrinker (ancient)}$$
$$\lambda = 0 : \text{steady (eternal)}$$
$$\lambda > 0 : \text{expander (immortal)}.$$

Since $\varphi(t)$ is only changing by diffeomorphisms and homotheties, it is a "self-similar" solution to the flow, which is why we call it a soliton solution. Since the exterior derivative commutes with pullbacks, we also have the soliton equation for the closed Laplacian flow

$$d(\tau_{14} - X \lrcorner \varphi) = \lambda\varphi.$$

We note the follow scaling property for solitons: If (φ, X, λ) is a soliton, then so is $(c^3\varphi, c^{-2}X, c^{-2}\lambda)$, for nonzero $c \in \mathbb{R}$. We also remark that for compact manifolds, we have the interpretation of the Laplacian flow as the upward gradient flow for Hitchin's volume functional, defined for closed G_2 structures φ on M in a fixed cohomology class by

$$\varphi \mapsto \int_M \varphi \wedge *_\varphi \varphi.$$

Because of this fact, compact shrinking Laplacian solitons cannot exist, and steady compact solitons are in fact torsion-free. This should be contrasted with the setting of Ricci flow, which does not arise as the gradient flow for some functional, where there are plenty of compact Ricci shrinkers (for instance, any complete Einstein manifold with positive Einstein constant).

3. Cohomogeneity-one Setting

Because the soliton equation is a highly nonlinear PDE, we consider a dimensional reduction of the equation in order to construct concrete examples and get a feel for the behavior of solutions. There has been prior work on homogeneous Laplacian solitons, so we consider the next simplest case, which are cohomogeneity-one solitons.

Definition 3.1. A compact Lie group G acts on a connected Riemannian manifold M with cohomogeneity-one if G acts by isometries and the largest orbits of the action have codimension one.

Note that the orbit space M/G of such an action is a one-dimensional manifold with potentially empty boundary. Interior points correspond to principal orbits, whose isotropy group we denote K, while boundary points correspond to singular orbits, whose isotropy groups H_i contain K. When $M/G \neq S^1$, we can identify the union of principal orbits M^o with $I \times (G/K)$, where I is an open interval, in almost a canonical fashion. Explicitly, given a point on a principal orbit, we consider the unique unit speed geodesic $\gamma(t)$ passing through p that is orthogonal to the principal orbits.

Remark 3.2. To see why such a geodesic exists, consider the following heuristic argument. Given p in a principal orbit, there is a unique orthogonal complement $\langle v_p \rangle$ to the image of T_eG, which is isomorphic to $T_e(G/K)$, in T_pM under the differential of the map $G \to M$ defined by $g \mapsto gp$. Thus we have a rank one distribution of TM^o given by $\{(p, \langle v_p \rangle)\}$. Since this distribution is rank one, it is involutive, so by the Frobenius theorem we have a foliation of M^o by smooth curves. We just need to show that these curves are totally geodesic submanifolds. Suppose γ is a geodesic such that $\gamma(0) = 0$ and $\gamma'(0) \in \langle v_p \rangle$, but $\gamma'(t) \notin \langle v_{\gamma(t)} \rangle$ for all t. We can then argue that γ cannot be a geodesic since it will not be a critical point for the length functional, since we may consider a variation field along γ taking value in $\langle v_{\gamma(t)} \rangle^\perp$.

Extending the interval of existence of the geodesic maximally to the open interval I by requiring that the geodesic remains in the union of principal orbits, we can map $I \times (G/K) \to M^o$ via $(t, gK) \mapsto g \cdot \gamma(t)$. From this perspective, tensors on M^o invariant under the cohomogeneity-one action of G may be identified with time-dependent G-invariant tensors on G/K.

If G also preserves a G_2 structure, i.e. $g^*\varphi = \varphi$ for all g in the identity component of G, then K must be a subgroup of G_2. Furthermore, since G_2, with its usual action on $\mathbb{R}^7$, acts transitively on the unit sphere with the stabilizer of a point on the sphere being isomorphic to $SU(3)$, the cohomogeneity-one condition implies that K must be a subgroup of $SU(3)$. Cleyton-Swann have previously studied cohomogeneity-one G_2 structures, and classified them in the case where G is a simple Lie group [3]. To do this, they relied on representation-theoretic computations to determine what the possible $\mathrm{Ad}_G(K)$-invariant complements to the isotropy representation at a point in a principal orbit are. In other words, at a point in a principal orbit, the Lie algebra of G decomposes into $\mathrm{Ad}_G(K)$-invariant summands $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p} \cong \mathbb{R}^6$. Cleyton-Swann also compute the G-invariant tensors on G/K, which they use to characterize the cohomogeneity-one G_2 structure on the union of principal orbits. We will consider the cases when $G = SU(3)$ and $G = Sp(2)$ (here

$Sp(2)$ denotes the compact symplectic group, i.e. 2×2 quaternion-valued invertible matrices that preserve the standard Hermitian form on $\mathbb{H}^2$), with the respective associated principal isotropy groups being isomorphic to $U(1)^2$ and $Sp(1) \times U(1)$.

From work of Cleyton-Swann, we have in the case of $G = SU(3)$, and $K = U(1)^2$, that a cohomogeneity-one G_2 structure is given by

$$\varphi = (f_1^2\omega_1 + f_2^2\omega_2 + f_3^2\omega_3) \wedge dt + f_1 f_2 f_3(\cos\theta\alpha + \sin\theta\beta),$$

where f_1, f_2, f_3, θ are functions defined on I (with f_i non-vanishing), and ω_i are invariant two-forms and α, β are invariant three-forms on the flag manifold $SU(3)/U(1)^2$ satisfying

$$\begin{aligned} d\omega_i &= \frac{1}{2}\alpha \\ d\beta &= -2(\omega_1 \wedge \omega_2 + \omega_2 \wedge \omega_3 + \omega_3 \wedge \omega_1) \\ d\alpha &= d(\omega_i \wedge \omega_j) = 0. \end{aligned} \tag{3.3}$$

Thus we have the corresponding metric and volume form

$$\begin{aligned} g &= dt^2 + f_1^2 g_1 + f_2^2 g_2 + f_3^3 g_3, \\ \mathrm{vol} &= f_1^2 f_2^2 f_3^2 \omega_1 \wedge \omega_2 \wedge \omega_3 \wedge dt, \end{aligned} \tag{3.4}$$

where g_i are invariant symmetric two-tensors on $SU(3)/U(1)^2$. Now we proceed to derive the $SU(3)$-invariant soliton equation. First we compute $\Delta_\varphi\varphi$:

$$\begin{aligned} d\varphi = &(\frac{1}{2}(f_1^2 + f_2^2 + f_3^2) - (f_1 f_2 f_3 \cos\theta)')\alpha \wedge dt - (f_1 f_2 f_3 \sin\theta)'\beta \wedge dt \\ &- 2f_1 f_2 f_3 \sin\theta(\omega_1\omega_2 + \omega_2\omega_3 + \omega_3\omega_1) \end{aligned} \tag{3.5}$$

$$\begin{aligned} d * \varphi = &((f_2^2 f_3^2)' - 2f_1 f_2 f_3 \cos\theta)w_2 w_3 \wedge dt \\ &+ ((f_3^2 f_1^2)' - 2f_1 f_2 f_3 \cos\theta)w_3 w_1 \wedge dt \\ &+ ((f_1^2 f_2^2)' - 2f_1 f_2 f_3 \cos\theta)w_1 w_2 \wedge dt. \end{aligned} \tag{3.6}$$

We now introduce notation that will make subsequent computations clearer. From (3.6) and the assumption that φ is closed, the components of τ_{14} with respect to the frame $\{\omega_i \mid i = 1, 2, 3\}$ are given by

$$\tau_i \coloneqq -\frac{f_i^2}{f_j^2 f_k^2}((f_j^2 f_k^2)' - 2\epsilon_\theta f_1 f_2 f_3), \tag{3.7}$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$. We further denote

$$\begin{aligned} g^3 &\coloneqq f_1 f_2 f_3, \\ \overline{f} &\coloneqq f_1^2 + f_2^2 + f_3^2, \\ \overline{\tau} &\coloneqq \tau_1 + \tau_2 + \tau_3 \\ |\tau|^2 &= \sum_{i=1}^{3} \frac{\tau_i^2}{f_i^4}. \end{aligned}$$

(Note that the last expression is indeed the norm of the torsion two-form with respect to the $SU(3)$-invariant metric.)

Returning to our computation of $\Delta_\varphi\varphi$, (3.5) and (3.6) implies

$$\Delta_\varphi\varphi = dd^*\varphi = -d * d * \varphi = \left(\sum_{i=1}^{3} \tau_i'\omega_i\right) \wedge dt + \frac{1}{2}\overline{\tau}\alpha.$$

We now consider the vector field in the soliton equation. Since the vector field X is also invariant under the action of G, we have $X = u(t)\partial_t$ for some function u, where ∂_t corresponds to the tangent vector of our unit-speed geodesic. Then

$$X \lrcorner \varphi = u(f_1^2\omega_1 + f_2^2\omega_2 + f_3^2\omega_3),$$

and hence since $d\varphi = 0$, we have

$$\mathcal{L}_X\varphi = d(X\lrcorner\varphi) = \left(\sum_{i=1}^{3}(uf_i^2)'\omega_i \wedge dt\right) + \frac{1}{2}u\overline{f}\alpha,$$

where we have used the structure equations (3.3). Using linear independence of the invariant forms and the fact that the f_i are nonvanishing, the soliton equation and the requirement that $d\varphi = 0$ is given by the set of equations

$$\sin\theta = 0 \tag{3.8}$$

$$\frac{1}{2}\overline{f} - (g^3\cos\theta)' = 0 \tag{3.9}$$

$$\lambda f_i^2 + (f_i^2 u)' - \tau_i' = 0 \tag{3.10}$$

$$-\frac{1}{2}\overline{\tau} + \lambda g^3\cos\theta + \frac{1}{2}u\overline{f} = 0, \tag{3.11}$$

where $i = 1, 2, 3$. We summarize the above discussion as the following

Proposition 3.12. *An $SU(3)$-invariant Laplacian soliton defined on the union of principal orbits is given by a solution to the equations* (3.8)-(3.11).

We remark that when the orbit space of a cohomogeneity-one manifold is an interval, then any cohomogeneity-one soliton is a gradient soliton since we may simply find an antiderivative for the function $u(t)$.

On the face of it, (3.8)-(3.11) suggest we *a priori* have a second-order system of six equations in five variables $(\theta, f_1, f_2, f_3, u)$, but there are clearly redundancies. Note that (3.8) implies $\theta = 0, \pi$, or in other words $\epsilon_\theta \coloneqq \cos\theta = \pm 1$. Furthermore, the assumption that f_i are nonvanishing on the principal orbits implies that we may use (3.11) to solve for u in terms of the f_i's. However, (3.10) does not give us separate differential equations for u and f_i, so it is unclear exactly how many parameters govern a $SU(3)$-invariant soliton.

We now wish to prolong the $SU(3)$-invariant soliton equations to obtain a first-order order system of ordinary differential equations so as to apply standard existence/uniqueness/regularity results. To do this, we rewrite the soliton equations considering τ_i now as an independent variable. (3.5) implies that for (i, j, k) a cyclic permutation of $(1, 2, 3)$, we have

$$f_i' = \epsilon_\theta\frac{\overline{f}}{2f_jf_k} + \frac{\tau_i}{2f_i} - \epsilon_\theta\frac{f_i^2}{f_jf_k}.$$

Thus we may rewrite the soliton equations as

$$f_i' = \frac{\tau_i}{2f_i} + \epsilon_\theta\frac{\overline{f} - 2f_i^2}{2f_jf_k} \tag{3.13}$$

$$\tau_i' = \lambda f_i^2 + f_i^2u' + u(\tau_i + \frac{f_i}{f_jf_j}(\overline{f} - 2f_i^2)) \tag{3.14}$$

$$0 = \overline{\tau} - 2\lambda g^3 - u\overline{f} \tag{3.15}$$

$$0 = \sum_{i=1}^{3}\frac{\tau_i}{f_i^2}. \tag{3.16}$$

Differentiating (3.16), plugging in (3.13) and (3.14), and then simplifying, we obtain

$$u' = \frac{1}{3}\left(-3\lambda - u\frac{\overline{f}}{g^3} + |\tau|^2 + \frac{\epsilon_\theta}{g}\sum_{i=1}^{3}(\overline{f} - 2f_i^2)\tau_i f_i^2\right)$$

Now note that we may solve for say τ_3 in terms of the f_i's and τ_1, τ_2 using (3.16), and thus we may solve for u in terms of the f_i's and τ_1, τ_3 using (3.15). Therefore, we have obtained the first order system of five equations in variables $(f_1, f_2, f_3, \tau_1, \tau_2)$

$$\begin{aligned} f_i' &= \frac{\tau_i}{2f_i} + \epsilon_\theta \frac{\overline{f} - 2f_i^2}{2f_j f_k} \\ \tau_i' &= \lambda f_i^2 + f_i^2 u' + u(\tau_i + \frac{f_i}{f_j f_j}(\overline{f} - 2f_i^2)) \end{aligned}$$

where we understand u, u', τ_3 to be functions of the five variables. We summarize the above discussion as the following

Proposition 3.17. *An $SU(3)$-invariant Laplacian soliton defined on the union of principal orbits is equivalently described as a first-order autonomous system of ODEs in five variables.*

Since an $SU(3)$-invariant soliton corresponds to a one-parameter family in the state space $(f_1, f_2, f_3, \tau_1, \tau_2)$, we expect a general $SU(3)$-invariant soliton to depend on four real parameters. However, one may check that the following scaling preserves solutions to the cohomogeneity-one solition equations (with different λ's):

$$(t, f_i, \tau_i, u, \lambda) \mapsto (\mu^{-1}t, \mu f_i, \mu\tau_i, \mu^{-1}u, \mu^{-2}\lambda).$$

Since this scaling maps steady solitons to steady solitons, we see that when $\lambda = 0$ we expect a steady $SU(3)$-invariant soliton to depend on three real parameters. Furthermore, the expressions defining the first-order system are real-analytic in the five variables, so by standard ODE theory we have that a solution to the system is also real-analytic (in time).

When $G = Sp(2)$, Cleyton-Swann observe that the expressions for (closed) G-invariant G_2 structures can be viewed as special cases of the $SU(3)$-invariant equations where we take $f_2 \equiv f_3$. With this remark in hand, we relabel $x \coloneqq f_1, w \coloneqq f_2^2, z \coloneqq \tau_2$ and obtain the first-order system for $Sp(2)$-invariant solitons

$$\begin{aligned} x' &= 1 - \frac{x}{2w}(x + 2z) \\ w' &= x + z \\ z' &= \frac{4(\lambda x w - 3z)(w - x^2 - \frac{3}{2}xz)}{3x(x^2 + 2w)}. \end{aligned}$$

In this case, the other component τ_1 of the torsion two-form and the function u defining the vector field are given by

$$\begin{aligned} \tau_1 &= -\frac{2x^2}{w}z \\ u &= \frac{2(w - x^2)z - 2\lambda x w^2}{w(x^2 + 2w)}. \end{aligned}$$

A solution to the G-invariant soliton equations is *a priori* only defined on a subinterval $J \subset I$. Thus to determine when such a local solution extends to a complete soliton on all of M, we must consider two separate geometric problems:

(i) Does the soliton extend over the singular orbit? This is similar to the question of determining when a radial function defined on $\mathbb{R}^n \setminus \mathbf{0}$ extends to a smooth function over all of $\mathbb{R}^n$. Cleyton-Swann give such conditions for when a general G-invariant G_2 structure extends smoothly

across the singular orbit. In the case of G-invariant solitons, Haskins-Nordstrom solve this by considering the equivalent problem of finding initial conditions for which a singular initial value problem of the form

$$X' = \frac{1}{t}A(X) + B(t, X), \quad X(0) = X_0, \quad A, B \text{ smooth in neighborhood of } (t, Y_0)$$

has a solution [6].

(ii) Does the solution, defined in some neighborhood of $t_0 > 0$, extend to a solution defined for all $t > t_0$ (that is, does it extend to a forward-complete solution)? Haskins-Nordstrom solve this by studying formal series solutions in powers of $r = \frac{1}{t}$ and by relating scale-invariant quantities constructed from the f_i, τ_i to the asymptotics of soliton solutions [6].

4. Geometry of Closed G-Invariant Cones

In order to better geometrically interpret solutions of the the soliton system, it is useful to consider the space of closed cohomogeneity-one G_2 cones. Before we do this, we state some general definitions for G_2 conifolds that will be useful in the sequel.

Definition 4.1. Let (N, g_N) be a connected, compact, smooth Riemmanian six-manifold. We say $C = (0, \infty) \times N$ with G_2 structure φ_C is a G_2 cone if the induced metric g_C is the conical metric

$$g_C = dr^2 + r^2 g_N$$

We call N the link of the cone.

Definition 4.2. An asymptotically conical (AC) G_2 manifold (with rate μ) is the following data: a manifold M with G_2 structure φ, G_2 cone C, compact set $K \subset M$ (possibly empty), $k_0 \in \{0, 1, 2, \dots\} \cup \{\infty\}$, real number $\mu < 0$, and diffeomorphism $h \colon C \to M \setminus K$ such that for all nonnegative integers $k \leq k_0$

$$|\nabla^k (h^* \varphi - \varphi_C)|_{g_C} = O(r^{\mu - k})$$

Proposition 4.3. *The set of closed $SU(3)$-invariant G_2 cones is naturally parametrized by a smooth, noncompact surface A in $\mathbb{R}^3$. The set of $Sp(2)$-invariant cones is then naturally parametrized by a smooth, noncompact curve C lying inside A. The torsion-free $SU(3)$-invariant/$Sp(2)$-invariant G_2 cone corresponds to the point in A (and in C) lying closest to the origin.*

Proof. If $M = (0, \infty) \times SU(3)/T^2$ is a G_2 cone (with vertex at $t = 0$), (3.4) implies $f_i(t) = ta_i$, where a_i is a nonzero constant. Then (3.5) is equivalent to $\cos\theta \equiv \pm 1$ and $(\cos\theta 3) a_1 a_2 a_3 = \frac{1}{2}(a_1^2 + a_2^2 + a_3^2)$. We focus on the $\theta \equiv 0$ case; the $\theta \equiv \pi$ case is equivalent by inversion about the origin in $\mathbb{R}^3$. Define

$$F(a) = F(a_1, a_2, a_3) \coloneqq 3a_1 a_2 a_3 - \frac{1}{2}(a_1^2 + a_2^2 + a_3^2).$$

If $F(a) = 0$ and $a_i = 0$ for some i, then we must have $a = \mathbf{0}$. Thus the space of closed $SU(3)$-invariant G_2 cones is parametrized by the set $A = F^{-1}(0) \setminus \mathbf{0} \in \mathbb{R}^3$. The nonvanishing condition on a_i implies A must be contained in the union of the eight open octants of $\mathbb{R}^3$. In particular $a = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$ are in A, so A is disconnected. However, observe that F is invariant under the action of $a_i \mapsto -a_i, a_j \mapsto -a_j$ for $i \neq j$. Furthermore, $F(a) = 0$ implies only either zero or two of a_1, a_2, a_3 can be negative. Therefore we restrict out attention to the intersection of A with the positive octant. Now, observe that $DF(a) = (3a_2 a_3 - a_1, 3a_3 a_1 - a_2, 3a_1 a_2 - a_3)$. Then the only point satisfying $DF(a) = \mathbf{0}$ is $a = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, which does not satisfy $F(a) = 0$. Hence A is a smooth surface in $\mathbb{R}^3$. Let

$$U = \{(x, y) \in \mathbb{R}^2 \mid x, y > 0, 9x^2 y^2 - x^2 - y^2 \geq 0\}$$
$$U_0 = \{(x, y) \in U \mid 9x^2 y^2 - x^2 - y^2 = 0\}$$

Then A is homeomorphic to the topological pushout $U \cup_\iota U$, where $\iota\colon U_0 \to U$ is inclusion, via the maps $G_-, G_+ \colon U \to \mathbb{R}^3$ given by

$$G_-(x,y) = (x, y, 3xy - \sqrt{9x^2y^2 - x^2 - y^2}),$$
$$G_+(x,y) = (x, y, 3xy + \sqrt{9x^2y^2 - x^2 - y^2}).$$

In particular, A is noncompact and connected.

We now find the point in A closest to the origin using Lagrange multipliers. Our function to minimize is $f(x,y,z) = x^2 + y^2 + z^2$ and our constraint is $g(x,y,z) = 3xyz - \frac{1}{2}(x^2 + y^2 + z^2) = 0$. Thus our equations are

$$\begin{aligned} x &= \lambda(3yz - x) \\ y &= \lambda(3zx - y) \\ z &= \lambda(3xy - z) \\ 0 &= 3xyz - \frac{1}{2}(x^2 + y^2 + z^2). \end{aligned}$$

When $\lambda = 0, -1$, we get $x = y = z = 0$ which we disallow. Hence we may assume that $\lambda \neq -1, 0$, in addition to the usual $x, y, z \neq 0$. The first three equations yield

$$x, y, z = \pm\frac{1+\lambda}{3\lambda},$$

which upon plugging into the last equation shows that the point closest to the origin is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Noticeably, this corresponds to the torsion-free $SU(3)$-invariant G_2 cone. From this we see that the spheres around the origin that intersect A have radius greater than or equal to $\frac{\sqrt{3}}{2}$.

As in the case of the $Sp(2)$-invariant soliton equation, the $Sp(2)$-invariant cones may be identified with the $SU(3)$-invariant cones satisfying $a_2 = a_3$, i.e. the intersection of A with the hyperplane $y - z = 0$, which we will call C. Note that $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \in C$, and under our identification this point corresponds to the torsion-free $Sp(2)$-invariant G_2 cone. □

5. Explicit Solutions of the Soliton Equations

Returning to the first-order system for the $Sp(2)$-invariant equations, one can check that, for $b > 0$ and $\lambda = -\frac{9}{4b^2}$ we have the explicit solution

$$\begin{aligned} x &= t \\ w &= b^2 + \frac{1}{4}t^2 \\ z &= -\frac{1}{2}t \\ \lambda &= -\frac{9}{4b^2}, b > 0. \end{aligned}$$

which is C^0-asymptotic to the closed $Sp(2)$-invariant cone $(1, \frac{1}{2})$ with rate $\mu = -2$. This solution is furthermore complete, i.e. forward-complete and smoothly extends across the singular orbit. Recently Haskins-Nordstrom showed that given any G-invariant cone C with closed G_2 structure, there exists a unique forward-complete shrinking G-invariant soliton that is AC to the cone C in a C^0 sense. [6]. To better understand the uniqueness of the explicit shrinker, we consider the

linearization of the $Sp(2)$-invariant system about this solution. Computing, we obtain

$$(x', w', z')^T = \begin{pmatrix} -\frac{x+z}{w} & \frac{x(x+2z)}{2w^2} & -\frac{x}{w} \\ 1 & 0 & 1 \\ \frac{2\lambda w x^2\left(x^2 z - 2w(2x+z)\right) - 4z\left(-2w^2 - 5wx^2 + x^3(x+3z)\right)}{(2wx+x^3)^2} & \frac{\lambda\left(8w^2+8wx^2-2x^3(2x+3)\right)-36z(x+z)}{3(2w+x^2)^2} & \frac{4x(x+3z)-2w\left(\lambda x^2+2\right)}{2wx+x^3} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Plugging in the original solution, our defining matrix is

$$\begin{pmatrix} -\frac{2t}{4b^2+t^2} & 0 & -\frac{4t}{4b^2+t^2} \\ 1 & 0 & 1 \\ \frac{5}{8}t\left(\frac{8}{4b^2+3t^2}+\frac{1}{b^2}\right)-\frac{1}{t} & -\frac{4}{4b^2+3t^2}-\frac{1}{2b^2} & \frac{3}{4}t\left(\frac{8}{4b^2+3^2}+\frac{1}{b^2}\right)-\frac{2}{t} \end{pmatrix}$$

Since the soliton equations have the translation symmetry $t \mapsto t+s$, we have

$$\gamma'(t) = F(\gamma(t)) \implies \gamma'(t+s) = F(\gamma(t+s))$$
$$\implies \frac{d}{ds}\gamma'(t+s) = \frac{d}{ds}(F(\gamma(t+s))) = (\nabla F)(\gamma(t+s)) \cdot \frac{d}{ds}\gamma(t+s).$$

Therefore

$$\gamma_1(t) \coloneqq \frac{d}{ds}\Big|_{s=0} \gamma'(t+s) = \left(1, \frac{1}{2}t, -\frac{1}{2}\right)$$

is one solution to the linearized system. Let $X(t)$ denote a fundamental solution of the linearized problem, i.e. a 3×3 matrix whose columns are solutions of the linearized ODE, and let $W(t) \coloneqq \det(X(t))$ be the Wronskian of X. Recall Wronski's identity

$$W(t) = W(t_0)\exp\left(\int_{t_0}^{t} \operatorname{tr}(X(s))\,ds\right).$$

Computing, we obtain

$$W(t) = \left(\frac{3e^{-\frac{3t_0^2}{8b^2}} t_0^3(4b^2+t_0^2)}{b^2(4b^2+3t_0^2)^2}\right)\frac{3t^2+4b^2}{t^2(t^2+4b^2)}e^{\frac{3t^2}{8b^2}}$$

Recall that for any square matrix A with columns $v_1, \ldots, v_n$, we have the inequality

$$|\det A| \le \prod_{i=1}^{n} \|v_i\|_2.$$

Suppose $X(t)$ is a fundamental matrix with one of the columns equal to $\gamma_1(t)$. Since $\|\gamma_1(t)\| = O(t)$ as $t\to\infty$ and $O(1)$ as $t\to 0$, we see that at least one of the other columns of X must be exponentially increasing as $t\to\infty$, and at least one of the columns must diverge like $\frac{C}{t^2}$ as $t\to 0$. This serves as a suggestion that the explicit $Sp(2)$ shrinker is rigid in the sense that deformations other than the trivial one given by time translation are large. We summarize this suggestion as

Conjecture 5.1. The explicit $Sp(2)$-invariant shrinker is rigid to first-order about complete $Sp(2)$-invariant AC shrinkers.

6. General Case

We now move towards describing the soliton equation on a general manifold with closed G_2 structure. We first make some elementary observations.

Proposition 6.1. *Suppose $(\varphi, \operatorname{grad} f, \lambda)$ is a shrinking gradient Laplacian soliton. Then f is subharmonic.*

Proof. We compute

$$\Delta_\varphi \varphi = \lambda\varphi + \mathcal{L}_X\varphi \text{ (shrinking if } \lambda < 0),$$

$$\implies \mathrm{Ric}(g_\varphi) + \nabla^2 f = -\frac{\lambda}{3}g_\varphi + \frac{4}{21}|\tau_\varphi|^2 g_\varphi + \frac{1}{8}j_\varphi(*_\varphi(\tau_\varphi \wedge \tau_\varphi)) \tag{6.2}$$

Taking the metric trace of the last equation yields

$$\Delta f = -R - \frac{7\lambda}{3} + \frac{4}{3}|\tau_\varphi|^2 + \frac{1}{8}\mathrm{tr}_g\left(j_\varphi(*_\varphi(\tau_\varphi \wedge \tau_\varphi))\right)$$

Recall that for closed G_2 structures, $R = -\frac{1}{2}|\tau|^2$ which is nonpositive. Then since the soliton being shrinking implies $\lambda < 0$, we should analyze the $\tau \wedge \tau$ term to see if $\Delta f \geq 0$. Indeed, using the decomposition of $\Lambda^3(T^*M)$ into G_2 irreps we can write $*(\tau \wedge \tau) = a\varphi + *(\alpha \wedge \varphi) + \eta$, with $\eta \in \Omega^3_{27}(M)$. Recall that $\ker(j_\varphi) = \Lambda^3_7(T^*M)$ and that the image of $\Lambda^3_{27}(T^*M)$ under j_φ is the traceless symmetric tensors. Thus we just need to compute a. Observe that $a = \frac{1}{7} * ((*(\tau \wedge \tau)) \wedge \psi)$ and recall that

$$\begin{aligned} -|\tau_\varphi|^2 \mathrm{vol}_\varphi &= \tau_\varphi \wedge \tau_\varphi \wedge \varphi \\ &= \tau_\varphi \wedge \tau_\varphi \wedge *\psi \\ &= \langle \tau_\varphi \wedge \tau_\varphi, \psi\rangle_\varphi \mathrm{vol}_\varphi \\ &= \langle \psi, \tau_\varphi \wedge \tau_\varphi\rangle_\varphi \mathrm{vol}_\varphi \\ &= \psi \wedge *(\tau_\varphi \wedge \tau_\varphi) \end{aligned} \tag{6.3}$$

so $a = -\frac{1}{7}|\tau|^2$ and

$$\begin{aligned} \frac{1}{8}\mathrm{tr}_g\left(j_\varphi(*_\varphi(\tau_\varphi \wedge \tau_\varphi))\right) &= \frac{1}{8}\mathrm{tr}_g(j_\varphi(a\varphi)) \\ &= \frac{1}{8}\mathrm{tr}_g(6ag_\varphi) \\ &= -\frac{3}{4}|\tau_\varphi|^2. \end{aligned}$$

Thus despite this contribution being negative, it still holds that $\Delta f \geq 0$. □

We remark that since subharmonic functions on compact manifolds are constant, the above proposition provides another reason why compact gradient shrinkers do not exist. Now we consider the linearization of the soliton equation. The highest order terms have appeared in the literature (see for instance section 2.4 of [1]); here we give the full linearization for the sake of completeness.

Proposition 6.4. *The linearization of the Laplacian soliton equation at a solution (φ, X, λ) in the direction of closed forms η and arbitrary vector fields Y is given by*

$$d(L\eta) = L_Y\varphi + L_X\eta + \lambda\eta,$$

where

$$L\eta = L(3f^0\varphi + *(f^1 \wedge \varphi) + f^3) = *d*\eta + *d*(f^0\varphi - 2f^3) - 3f^0\tau_2 - \frac{5}{2} * \mathrm{Alt}(tr((j(f^3))^\sharp \otimes (d*\varphi))).$$

Proof. Suppose

$$\frac{\partial}{\partial s}\varphi = \eta, \frac{\partial}{\partial s}X = Y, \frac{\partial}{\partial s}f = b.$$

are deformations of the G_2 structure φ, vector field X, and potential function f (for the case of gradient solitons). Using the conventions in section 2 of [1], we have

$$\begin{aligned}\eta &:= 3f^0\varphi + *(f^1 \wedge \varphi) + f^3 = i(h) + *(f^1 \wedge \varphi)\\ \frac{\partial}{\partial s} g &= 2f^0 g + \frac{1}{2} j(f^3) = 4h\\ \frac{\partial}{\partial s}(*\varphi) &= 4f^0 * \varphi + f^1 \wedge \varphi - *f^3.\end{aligned}$$

The linearization of the Lie derivative term in the soliton equation at (φ, X) in the general case is

$$d(Y \lrcorner \varphi + X \lrcorner \eta).$$

In the gradient case

$$d\left((-4h^{ij}(\partial_j f)\partial_i) \lrcorner \varphi + \operatorname{grad} b \lrcorner \varphi + \operatorname{grad} f \lrcorner \eta\right).$$

Now we consider the full linearization of $\Delta_\varphi \varphi$ (when $d\varphi = d\eta = 0$). Recall that $*$ can be thought of as

$$\alpha \mapsto \alpha^\sharp \lrcorner \mathrm{vol},$$

where

$$\left(\frac{1}{k!}\alpha_{i_1 \dots i_k} dx^{i_1} \wedge \dots dx^{i_k}\right)^\sharp = \frac{1}{k!}\alpha_{i_1 \dots i_k} g^{i_1 j_1} \dots g^{i_k j_k} \delta^{r_1 \dots r_k}_{j_1 \dots j_k} \partial_{r_1} \wedge \dots \wedge \partial_{r_k}.$$

Thus the linearization of $\varphi \mapsto \tau_2$ is given by

$$\begin{aligned}\frac{\partial}{\partial s}((d*\varphi)^\sharp \lrcorner \mathrm{vol}) &= \left(\frac{\partial}{\partial s}((d*\varphi)^\sharp)\right) \lrcorner \mathrm{vol} + (d*\varphi)^\sharp \lrcorner (\frac{\partial}{\partial s}\mathrm{vol})\\ &= \left(\frac{\partial}{\partial s}((d*\varphi)^\sharp)\right) \lrcorner \mathrm{vol} + (d*\varphi)^\sharp \lrcorner 2\mathrm{tr}_g(h)\mathrm{vol}\\ &= \left(\frac{\partial}{\partial s}((d*\varphi)^\sharp)\right) \lrcorner \mathrm{vol} + 2\mathrm{tr}_g(h)\tau_2\\ &= 2\mathrm{tr}_g(h)\tau_2 + *d(4f^0 * \varphi + f^1 \wedge \varphi - *f^3)\\ &+ \left(\frac{1}{5!}\alpha_{i_1 \dots i_5} \frac{\partial}{\partial s}(g^{i_1 j_1}) \dots g^{i_5 j_5} \delta^{r_1 \dots r_5}_{j_1 \dots j_5} \partial_{r_1} \wedge \dots \wedge \partial_{r_5}\right) \lrcorner \mathrm{vol} + \dots\\ &+ \left(\frac{1}{5!}\alpha_{i_1 \dots i_5} g^{i_1 j_1} \dots \frac{\partial}{\partial s}(g^{i_5 j_5}) \delta^{r_1 \dots r_5}_{j_1 \dots j_5} \partial_{r_1} \wedge \dots \wedge \partial_{r_5}\right) \lrcorner \mathrm{vol}.\end{aligned}$$

In general we have

$$\frac{\partial}{\partial s} g_{ij} = \omega_{ij} \implies \frac{\partial}{\partial s} g^{ij} = -g^{ik}\omega_{kl}g^{lj}.$$

In our case

$$\begin{aligned}\omega_{ij} &= 2f^0 g_{ij} + \frac{1}{2}(j(f^3))_{ij}\\ \implies \frac{\partial}{\partial s} g^{ij} &= -2f^0 g^{ij} - \frac{1}{2}(j(f^3))^{ij}.\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{\partial}{\partial s}((d*\varphi)^\sharp \lrcorner \mathrm{vol}) \\
&= 2\mathrm{tr}_g(h)\tau_2 + *d(4f^0 * \varphi + f^1 \wedge \varphi - *f^3) - 10f^0\tau_2 \\
&- \left(\frac{1}{5!}\alpha_{i_1\dots i_5}\frac{1}{2}(j(f^3))^{i_1j_1}\dots g^{i_5j_5}\delta^{r_1\dots r_5}_{j_1\dots j_5}\partial_{r_1}\wedge\dots\wedge\partial_{r_5}\right)\lrcorner\mathrm{vol} - \dots \\
&- \left(\frac{1}{5!}\alpha_{i_1\dots i_5}g^{i_1j_1}\dots\frac{1}{2}(j(f^3))^{i_5j_5}\delta^{r_1\dots r_5}_{j_1\dots j_5}\partial_{r_1}\wedge\dots\wedge\partial_{r_5}\right)\lrcorner\mathrm{vol}
\end{aligned}$$

$$\begin{aligned}
&= 2\mathrm{tr}_g(h)\tau_2 + *d(4f^0 * \varphi + f^1 \wedge \varphi - *f^3) - 10f^0\tau_2 \\
&- *\left(\left(\frac{1}{5!}\alpha_{i_1\dots i_5}\frac{1}{2}(j(f^3))^{i_1j_1}\dots g^{i_5j_5}\delta^{r_1\dots r_5}_{j_1\dots j_5}\partial_{r_1}\wedge\dots\wedge\partial_{r_5}\right)^\flat\right) - \dots \\
&- *\left(\left(\frac{1}{5!}\alpha_{i_1\dots i_5}g^{i_1j_1}\dots\frac{1}{2}(j(f^3))^{i_5j_5}\delta^{r_1\dots r_5}_{j_1\dots j_5}\partial_{r_1}\wedge\dots\wedge\partial_{r_5}\right)^\flat\right) \\
&= 2\mathrm{tr}_g(h)\tau_2 + *d(4f^0 * \varphi + f^1 \wedge \varphi - *f^3) - 10f^0\tau_2 - \frac{5}{2} * \mathrm{Alt}(\mathrm{tr}((j(f^3))^\sharp \otimes (d*\varphi))) \\
&= *d(4f^0 * \varphi + f^1 \wedge \varphi - *f^3) - 3f^0\tau_2 - \frac{5}{2} * \mathrm{Alt}(\mathrm{tr}((j(f^3))^\sharp \otimes (d*\varphi))) \\
&= *d*\eta + *d*(f^0\varphi - 2f^3) - 3f^0\tau_2 - \frac{5}{2} * \mathrm{Alt}(\mathrm{tr}((j(f^3))^\sharp \otimes (d*\varphi))).
\end{aligned}$$

□

We now discuss directions of study for understanding the general case of Laplacian shrinkers. While no results towards these points were obtained during the duration of the project, understanding the following topics would be crucial to understanding shrinking solitons. Thus we record here results in the literature that model the approaches we attempted in understanding Laplacian shrinkers, and furthermore explain why we could not immediately apply these results to our problem at hand.

In order to understand the moduli space of Laplacian solitons, it would a useful first step to have an idea of what the tangent space to a point in the moduli space would look like. One way to do this is to consider infinitesimal deformations of the defining equation. This approach was used by Moroianu-Nagy-Semmelman [10] in the setting of nearly Kahler six-manifolds, and in by Dwivedi-Singal [4] in the setting of nearly G_2 manifolds. We adapt the basic set up from [10]. Let $S = (\varphi, X, \lambda)$ be a Laplacian soliton.

Definition 6.5. The virtual tangent space $T_S\mathfrak{M}$ at S is the vector space $(\eta, Y) \in \omega^3(M) \oplus \mathfrak{X}(M)$ of solutions to the linearized soliton equation (with the same choice of λ).

Now by diffeomorphism-invariance of the soliton equation, we have that for any vector field Z, $(\mathcal{L}_Z\varphi, \mathcal{L}_ZX)$ is a solution to the linearized equation. In particular, we have a map from the space of vector fields $\mathfrak{X}(M)$ to the virtual tangent space given by $Z \mapsto (\mathcal{L}_Z\varphi, \mathcal{L}_ZX)$ whose image, which we denote $\chi^S(M)$, is isomorphic to $\mathfrak{X}(M)/\mathfrak{k}_X(M)$, where $\mathfrak{k}_X(M)$ is the set of vector fields which are Killing fields for the soliton, i.e. Z such that $\mathcal{L}_Z\varphi = 0 = \mathcal{L}_ZX$. Since deformations in $\chi^S(M)$ correspond to diffeomorphisms (isotopic to the identity), we do not consider them to be interesting deformations. Therefore we make the following definition:

Definition 6.6. We define the space of infinitesimal soliton deformations to be the quotient $T_S\mathfrak{M}/\chi^S(M)$.

Because this description of the space of infinitesimal soliton deformations is fairly abstract, it would be useful to identify this space with a more concrete space. This is in fact what Moroianu-Nagy-Semmelman and Dwivedi-Singal do. In [10], the space of infinitesimal deformations of a nearly

Kahler structure is identified with an eigenspace for the Laplacian restricted to a certain subspace of coclosed $(1,1)$-forms, and in [4], the space of infinitesimal deformations of a nearly G_2 structure is identified with a certain space of Killing fields for the metric plus a certain subspace of coclosed Ω^3_{27} forms.

Thus far we have been unsuccessful in trying to identify deformations of solitons with more concrete spaces, and there are several features of nearly Kahler and nearly G_2 geometry which helped the respective authors carry out their computations but which do not apply to the case of shrinking solitons. First off, complete nearly Kahler and nearly G_2 manifolds are Einstein with positive Einstein constant, and hence compact by Myer's theorem. Furthermore, nearly Kahler/G_2 structures are so called "weak" holonomy, in that they are very close to having special/exceptional holonomy; for instance, a nearly G_2 structure has $\tau_7, \tau_{14}, \tau_{27} = 0$ and τ_1 a (nonzero) constant. These two facts enable the authors to not only speak of the spectrum of the Laplacian, which is already a delicate matter on noncompact manifolds due to the presence of continuous spectrum, but also yields finer Hodge-theoretic decompositions of e.g. exact forms over M which is crucial to their analysis. Furthermore, in the nearly Kahler case, the authors utilize the fact that such structures are uniquely determined by the metric in order to use Ebin's slice theorem to give a unique representative for an infinitesimal deformation. Though Ebin's slice theorem generally applies to compact manifolds, there are results extending it to cases of complete noncompact manifolds with bounded geometry. In spite of this, we *a priori* do not know if Laplacian solitons are uniquely determined by the metric. For example, for torsion-free G_2 structures on a simply-connected manifold such that the induced metrics are not locally irreducible and are non-locally-symmetric, such structures are uniquely determined by the metric they induce since otherwise we would have more than one independent parallel three-form which would contradict Berger's list. Conversely, for general G_2 structures there are many isometric G_2 structures which are not equal to the original one, since such isometric structures are in correspondence with sections of a certain $\mathbb{RP}^7$-bundle over M which always admits sections (see Remark 4 in [2]). However, it is unclear whether the soliton equation is strong enough to force this uniqueness.

Given our difficulty in trying to describe the moduli space for general (closed, shrinking) solitons, the next step we took was to specify a geometry we wish for the solitons to have. Namely, we considered the moduli space of AC closed shrinking solitons. Karigiannis-Lotay [8] studied the deformation theory of G_2 conifolds and characterized the moduli space of *AC* torsion-free G_2 structures. Here the authors show that for certain rates μ, the moduli space of torsion-free G_2 structure that are all AC to the same torsion-free cone (in a C^∞ sense, with rate μ) is a smooth finite-dimensional manifold. To accomplish this, they carry out roughly the following (see Section 5 in [8]):

- Prove a slice theorem identifying an open set in the space of torsion-free gauge-fixed AC G_2 structures with a neighborhood of a point in the moduli space.
- Associate torsion-free gauge-fixed AC G_2 structures with solutions of a nonlinear elliptic PDE.
- Use the Banach space implicit function theorem to deduce smoothness of the map in the first bullet.

In carrying out the above analysis, Karigiannis-Lotay utilize Fredholmness of a certain Hodge-Dirac-type operator defined on weighted Sobolev spaces. This operator is in fact not always Fredholm; there is a discrete set of critical rates, as described by the theory of Lockhart-McOwen, for which the operator fails to be Fredholm. These critical rates are directly related to the presence of homogeneous harmonic differential forms defined on the torsion-free G_2 cone M is AC to. In turn, Karigiannis-Lotay are able to say a great deal about such forms largely due to special geometry torsion-free G_2 cones have. Namely, the link of such cones is strictly nearly Kahler six-fold, and so the authors are able to use Obata's theorem and other results from geometry to understand analytic properties of the AC manifolds. In our setting, the closed condition is too weak to be able to say

much about the cone our soliton is asymptotic to, and furthermore the cone itself is not a soliton, which differs in the Karigiannis-Lotay's case where both the cone and the AC manifold solve the geometric problem being studied, i.e. both are torsion-free.

Finally, we discuss the issue of unique continuation. In the setting of Ricci flow, shrinking solitons have appeared as models for finite-time singularities of the flow, and the known complete noncompact such solitons that are not locally products/reducible have an asympototically conical geometry. In this setting there are results [9] showing that if two gradient shrinking Ricci solitons are asymptotic to the same cone, then both solitons must be isometric on some end. As we look to Ricci flow as a model for understanding the Laplacian flow, we would like to see if we can adapt such unique continuation results to the G_2 setting. In fact, we are even more interested in such a result because as was mentioned earlier, there are no compact shrinking Laplacian solitons, i.e. any Laplacian shrinker is necessarily noncompact. However, even though the equation (6.2) for the Ricci tensor derived from the soliton equation for the Laplacian flow differs from the soliton equation for Ricci flow only by lower-order terms, there are heuristic reasons why there has been difficulty in adapting the result. For one, complete noncompact Ricci gradient shrinkers have been shown to have at most Euclidean volume growth. However, Fowdar recently found a gradient Laplacian shrinker with exponential volume growth (see Section 5.2 in [5]), and moreover the $SU(3)$-invariant shrinkers we consider all have at least Euclidean volume growth. Furthermore, a manifold with closed G_2 structure has nonpositive scalar curvature, which vanishes identically if and only if the G_2 structure is torsion-free. Therefore, closed shrinkers must have Ricci curvature negative somewhere. Conversely, closed Ricci shrinkers are known to have everywhere nonnegative scalar curvature. By imposing a strong enough asymptotically conical hypothesis, we could force the solitons we are considering to behave like AC Ricci shrinkers. For instance, by requiring our solitons be AC to a G_2 cone in a C^2 sense as in Definition 1.1 of [9], we can likewise obtain quadratically decaying curvature by using Kotschwar-Wang's proof combined with (6.2) and the fact that as quadratic forms

$$-2|\tau|^2 g \leq j(*(\tau \wedge \tau)) \leq \frac{2}{3}|\tau|^2 g.$$

However, in doing so we arguably lose the geometric picture. For instance, the AC solitons found by Haskins-Nordstrom are a priori AC only in a C^0 or C^1 sense, and a goal of studying the general case was to explain the rigidity of AC shrinkers in the cohomogeneity-one setting of [6].

References

[1] R. Bryant and F. Xu, *Laplacian flow for closed g_2-structures: Short time behavior*, 2011.

[2] R. L. Bryant, *Some remarks on G_2-structures*, Proceedings of Gökova Geometry-Topology Conference 2005, 2006, pp. 75–109. MR2282011

[3] R. Cleyton and A. Swann, *Cohomogeneity-one g2-structures*, Journal of Geometry and Physics **44** (2002Dec), no. 2-3, 202-220.

[4] S. Dwivedi and R. Singhal, *Deformation theory of nearly g2 manifolds*, 2020.

[5] U. Fowdar, *S1-invariant laplacian flow*, 2021.

[6] M. Haskins and J. Nordstrom, *Cohomogeneity-one g2 laplacian solitons*, In preparation.

[7] S. Karigiannis, *Flows of g2-structures, i*, The Quarterly Journal of Mathematics **60** (2008Jul), no. 4, 487-522.

[8] S. Karigiannis and J. D. Lotay, *Deformation theory of g2 conifolds*, Communications in Analysis and Geometry **28** (2020), no. 5, 1057–1210.

[9] B. Kotschwar and L. Wang, *Rigidity of asymptotically conical shrinking gradient ricci solitons*, 2013.

[10] A. Moroianu, P.-A. Nagy, and U. Semmelmann, *Deformations of nearly Kähler structures*, Pacific J. Math. **235** (2008), no. 1, 57–72. MR2379770