Realizing Hecke Actions on Modular Forms via Cohomology of Dessins d′Enfants

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Dissertation submitted in partial fulfillment of the
requirements for the degree of Doctor of Philosophy
in the Department of Mathematics
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ABSTRACT

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Abstract

A well known action on the space of Modular forms is done by Hecke operators, which also play an important role in modularity. This action can be further break down into steps, and form what we call a Hecke correspondence. On the other hand, through Belyi and Grothendieck there is a one to one correspondence between the equivalence classes of algebraic curves defined over **Q** equipped with Belyi functions and equivalence classes of dessins d'enfants. This applies in particular to modular curves. In my dissertation work, I will study the action of Hecke operators on certain dessins, namely, those that correspond to $X_0(N)$. This is done by defining a cohomology with coefficients in a local system on dessins, and have Hecke operators act on it. We will also construct a Hecke-equivalent isomorphism of the cohomology group with the space of cusp forms. We hope that this work can present the first step in studying the Hecke action on more general dessins.

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Chapter 1

Introduction

Modular forms have been a subject of interest for many number theorists and algebraic geometers. A well known action on modular forms are the so-called Hecke operators T_p (p prime), whose eigenvalues can be used to compute the q-expansion coefficients of modular forms which are eigenvectors for the Hecke action. On the other hand, Belyi's theorem presents us a one-to-one correspondence between equivalence classes of algebraic curves defined over $\overline{\mathbf{Q}}$ and equivalence classes of dessins d'enfants, which are bicolored graphs consisting of edges and vertices and embeded in topological surfaces. The main goal of my dissertation work is to study Hecke operator actions on (the cohomology of) dessins that correspond to modular curves. Our hope however is that these techniques can generalize to study analogues of Hecke operators on the dessins of more general curves.

The connection between modular forms, Hecke operators, and the cohomology of dessins of modular curves is a combination of several strands of mathematical research which we now briefly recall.

In [16, 15] Hecke constructed a commuting family of operators T_n on modular forms which were diagonalizable. The eigenvalues of a joint eigenvector for the Hecke operators (a Hecke eigenform) satisfies the same relations as the Hecke operators. These eigenvalues appear as coefficients in the q-expansion of the normalized Hecke eigenform. The importance of Hecke operators is illustrated by the Modularity Theorem [6] which built on the work of Wiles [29] and Taylor and Wiles [26]. Basically it says that for any elliptic curve E defined over $\mathbf{Q}$, the coefficients of the Dirichlet series associated to E (which encode the number of points of E reduced to finite fields) agree with the coefficients of a Hecke eigenform.

Now modular forms and more generally automorphic forms for reductive groups G can be viewed as certain L^2 functions on $\Gamma \setminus G$, where $\Gamma \subset G$ is an arithmetic subgroup. (For modular forms, $G = \mathrm{SL}_2(\mathbf{R})$ and Γ is a finite index of $\mathrm{SL}_2(\mathbf{Z})$.) Thus they are related to the L^2-cohomology $H^\bullet_{(2)}(\Gamma \setminus X, \mathbf{V})$, where X is the symmetric space associated to G (when $G = \mathrm{SL}_2(\mathbf{R})$, X is the complex upper half plane $\mathfrak{h}$). This is made explicit by Borel and Casselman [4] based on the structure of $L^2(\Gamma \setminus G)$ determined by Langlands [18]; they show that $H^\bullet_{(2)}(\Gamma \setminus X, \mathbf{V})$ is finite-dimensional and represented by harmonic forms when X is a hermitian symmetric space (e.g., $\mathfrak{h}$). In fact Zucker conjectured [33] that in the hermitian case $H^\bullet_{(2)}(\Gamma \setminus X, \mathbf{V})$ represents the intersection cohomology [12, 13] (a topological invariant) of the Baily-Borel compactification $\Gamma \setminus X$ [3], a projective algebraic variety defined over a number field. (When $X = \mathfrak{h}$, cusp points from $\Gamma \setminus \big(\mathbf{Q} \cup \{\infty\}\big)$ are adjoined to yield the modular curves $X(\Gamma) = \Gamma \setminus \mathfrak{h}^*$.) Zucker's conjecture was first proved independently by Looijenga [19] and Saper and Stern [23]; the modular curve case is covered by Zucker in [32].

Since in general the locally symmetric space $\Gamma \setminus X$ is non-compact, one is interested in ways to replace it by a compact space. This may be done by enlarging the space, such as the Borel-Serre compactification $\Gamma \setminus \overline{X}$ [5] which is a manifold-with-corners of the same homotopy type as $\Gamma \setminus X$ or (in the hermitian case) the Baily-Borel compactification $\Gamma \setminus X^*$ noted above. In this dissertation our focus is on a compact explicit deformation retract $\Gamma \setminus X_0 \subset \Gamma \setminus X$ of least possible dimension, a *spine*. Spines have been used in many cases to compute the cohomology of $\Gamma \setminus X$. For locally symmetric spaces arising from a general linear group of a division algebra, such retracts were constructed by Ash [1], the *well-rounded* retracts. In the case of $\Gamma \setminus \mathfrak{h}$, the well-rounded retract is $\Gamma \setminus \mathcal{T}$ where $\mathcal{T}$ is the *Serre tree.* In this dissertation we use the fact that $\Gamma \setminus \mathcal{T}$ is the underlying simplicial complex of a dessin for the modular curve $X(\Gamma)$. Explicit spines have also been constructed in a few non-linear hermitian cases [20], [30].

In this dissertation we will develop an algorithm to calculate a) a parabolic cohomology group of the dessin $\Gamma_0(N) \setminus \mathcal{T}$ that represents $\Gamma_0(N)$-cusp forms, and b) the action of Hecke operators $T_p (p \nmid N)$ on it. Although the Hecke operators come from correspondences on $\Gamma_0(N) \setminus \mathfrak{h}$ and hence act on the cohomology of $\Gamma_0(N) \setminus \mathfrak{h}$, they do not directly restrict to correspondences of $\Gamma_0(N) \setminus \mathcal{T}$. As a result our algorithm for (b) is most clear for $p < 7$; we hope to remove this restriction. A further hope is to apply these techniques to study analogues of Hecke operators on the dessins of more general algebraic curves.

The usual approach to computing Hecke operators is through Manin's theory of *modular symbols* [21]. It would be interesting to compare the algorithms in detail in the future.

We now discuss the content of the dissertation in more detail. We start by considering the cohomology $H^\bullet(\Gamma \setminus \mathfrak{h}, \mathbf{V})$ where $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ is a finite-index subgroup and $\mathbf{V}$ is a local coefficient sheaf arising from a representation $\rho \colon \mathrm{SL}_2(\mathbf{R}) \longrightarrow \mathrm{GL}(V)$. We view this as the cohomology of a complex of V-valued differential forms $A^\bullet(\Gamma, \mathfrak{h}, V)$ on the upper-half plane which satisfy $L_\gamma^* \phi = \rho(\gamma)\phi$ for $\gamma \in \Gamma$ as in [22]. The advantage of this complex is that it automatically handles elliptic points (points with nontrivial stabilizer subgroups in Γ).

To every cusp form $f \in S_k(\Gamma)$ for Γ of weight k there exists a closed differential form $\phi_f \in A^1(\Gamma, \mathfrak{h}, V_k)$ for $V_k = \mathrm{Sym}^{k-2}(\mathbf{C}^2)$ and a resulting cohomology class $[\phi_f] \in H^1(\Gamma \setminus \mathfrak{h}, \mathbf{V}_k)$. On the other hand, the Hecke correspondence induces an action of T_p on $A^1(\Gamma, \mathfrak{h}, V)$ and hence on $H^1(\Gamma \setminus \mathfrak{h}, \mathbf{V}_k)$. Our first result (Theorem 7.3.7) is that the action of T_p on the cohomology classes of cusp forms corresponds to the multiple P^{k-2} times the action of T_p on cusp forms. (For simplicity we restrict attention to $\Gamma = \Gamma_0(N)$ and $p \nmid N$.)

Eichler and Shimura constructed [24] a cohomology group based on V_k-valued group cochains of Γ and proved that it was isomorphic to $S_k(\Gamma) \oplus \overline{S_k(\Gamma)}$. For our eventual applications to dessins, we prefer to work with differential forms. We impose conditions on forms in order to define two subcomplexes of $A^\bullet(\Gamma, \mathfrak{h}, V_k)$ and their corresponding cohomologies: parabolic cohomology $H^\bullet_P(\Gamma, \mathfrak{h}, V_k)$ and L^2- cohomology $H^\bullet_{(2)}(\Gamma, \mathfrak{h}, V_k)$. The differential form version of parabolic cohomology we construct inspires the combinatorial cohomology we later construct on the dessins. On the other hand, the L^2 cohomology provides a Hecke-equivalent Eichler-Shimura isomorphism (Theorem 4.2.1, compare [32, §12]). We prove there is a Hecke-equivariant isomorphism of parabolic and L^2 cohomology (chapter 6). Putting all these together yields a Hecke-equivariant isomorphism

$$H^1_P(\Gamma, \mathfrak{h}, V) \cong S_k(\Gamma) \bigoplus \overline{S_k(\Gamma)} \ .$$

We now turn to the dessins of modular curves $\Gamma_0(N) \setminus \mathfrak{h}^*$. The (normalized) j-function is a Belyi function for this curve and the corresponding dessin can be realized on the quotient $\Gamma_0(N) \setminus \mathcal{T}$ of the Serre tree $\mathcal{T} \subset \mathfrak{h}$. The cohomology

$H^\bullet(\Gamma_0(N) \setminus \mathcal{T}, \mathbf{V}) \cong H^\bullet(\Gamma_0(N) \setminus \mathfrak{h}, \mathbf{V})$ may be calculated from a finite dimensional complex $C^\bullet(\Gamma_0(N), \mathcal{T}, V)$ of simplicial cochains.

We construct in section 7.2 a subcomplex $C_P^\bullet(\Gamma_0(N), \mathcal{T}, V) \subset C^\bullet(\Gamma_0(N), \mathcal{T}, V)$ (inspired by our differential form construction) whose cohomology is isomorphic to $H_P^\bullet(\Gamma_0(N), \mathfrak{h}, V)$. A major goal of this dissertation is to describe the Hecke action via a computable action on the finite-dimensional complex $C_P^\bullet(\Gamma_0(N), \mathcal{T}, V)$. This however is difficult since the Hecke correspondence of $\Gamma_0(N) \setminus \mathfrak{h}$ does <u>not</u> preserve $\Gamma_0(N) \setminus \mathcal{T}$.

In section 7.3 we nonetheless determine an algorithm for calculating T_p on $C_P^\bullet(\Gamma_0(N), \mathcal{T}, V)$, at least for $p = 2, 3$ and 5. We use a $\mathrm{SL}_2(\mathbf{Z})$-equivariant deformation retract $\mathfrak{h} \longrightarrow \mathcal{T}$ to overcome the difficulty noted above. The restriction to $p < 7$ was needed at one point to easily obtain a simplicial action; hopefully this restriction can be removed.

A significant portion of the latter part of this dissertation (section 7.4, 7.5, and 8) is devoted to providing the readers with ample examples on studying the Hecke action on cohomology of dessins with coefficients. Main tools used are Sage for getting necessary information on congruent subgroups in order to build their corresponding dessins, and Mathematica for computing eigenvalues for T_p for different values of weight k. Both formulas and Mathematica codes are provided.

Here is the outline of this dissertation. Chapter 2 gives the background on modular forms $M_k(\Gamma)$ and the subspace cusp forms $S_k(\Gamma)$ associated to a congruence subgroup $\Gamma \in \mathrm{SL}_2(\mathbf{Z})$. It also describes the classical action of Hecke operators T_p on the forms in $M_k(\Gamma)$ as a double coset operator. It then introduces the Hecke correspondence on modular curves $X_0(N) = \Gamma_0(N) \setminus \mathfrak{h}^*$ as described by William Stein in his lecture notes [25] at Harvard University. The Hecke correspondence is a composition

$$T_p = \pi_g \circ \alpha \circ \pi^{-1}$$

where $\pi_g \colon g\Gamma_0(pN)g^{-1}\backslash\mathfrak{h} \longrightarrow X_0(N)$ and $\pi \colon X_0(pN) \longrightarrow X_0(N)$ are projection maps and $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} : X_0(pN) \longrightarrow g\Gamma_0(pN)g^{-1} \setminus \mathfrak{h}$ (α is later replaced by $g = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ but they represent the same double coset for $\Gamma_0(N)$.) This correspondence can also be described as a double coset operator that sends a modular form f to $\sum_j f[\alpha\beta_j]_k$ where $[\cdot]_k$ is the weight k bracket operator on modular forms and $\{\beta_j\}$ forms a set of coset representatives of $\Gamma_0(pN) \setminus \Gamma_0(N)$. The same Hecke correspondence is applied on the different cohomologies that are later constructed, and their similarities are presented as theorems at the end of each chapter describing the cohomologies.

Chapters 3, 4, 5, and 6 can be considered as one group, dealing with different cohomologies defined via differential forms. Chapter 3 starts by building a complex $A^\bullet(\Gamma, \mathfrak{h}, V)$ of differential forms in $\Gamma \setminus \mathfrak{h}$ with coefficients in a vector space V (usually $\mathrm{Sym}^{k-2}(\mathbf{C}^2)$ in our case) using the approach of Matsushima and Murakami [22], and the cohomology $H^\bullet(\Gamma, \mathfrak{h}, V) \cong H^\bullet(\Gamma\backslash\mathfrak{h}, \mathbf{V}_k)$ of this complex. The forms in $A^\bullet(\Gamma, \mathfrak{h}, V)$ must satisfy a transformation rule, namely

$$L_\gamma^*(\phi) = \rho(\gamma)\phi \tag{1.0.1}$$

for $\gamma \in \Gamma$. The pullback maps π_g^*, g^* and pushforward map π_* are described and allow one to give the action of Hecke operator

$$T_p = \pi_* \circ g^* \circ \pi_g^*$$

on $\phi \in A^\bullet(\Gamma, \mathfrak{h}, V)$; it is shown that (1.0.1) is preserved under T_p. In this section, commutativity between T_p and the differential d is also presented. This led to the description of the relationship between Hecke correspondence on $H^\bullet(\Gamma, \mathfrak{h}, V)$ and the classical Hecke action on modular forms. Define

$$\phi_f(\tau) = f(\tau) \begin{pmatrix} \tau \\ 1 \end{pmatrix}^{k-2} d\tau$$

where $f \in S_k(\Gamma)$. It is shown that $[\phi_f]$ represents a class in $H^1(\Gamma, \mathfrak{h}, V)$, and moreover,

$$T_p\left([\phi_f]\right) = p^{2-k}[\phi_{T_p(f)}]$$

(Theorem 3.4.1), which relates the classical Hecke operator and the Hecke action on cohomology.

Chapter 4 and chapter 5 give two different cohomologies that both build upon the one presented in chapter 3. In each case, there is a Hecke action on each cohomology. More specifically, chapter 4 defines a pointwise norm on V and an L^2 form using the Riemannian metric. It imposes an L^2 condition on the differential forms and builds an L^2 complex

$$A^\bullet_{(2)}(\Gamma, \mathfrak{h}, V) = \left\{\phi \in A^\bullet(\Gamma, \mathfrak{h}, V) \mid \|\phi\|^2_{L^2} < \infty \;\&\; \|d\phi\|^2_{L^2} < \infty\right\} .$$

This allows us to use Hodge theory to prove an L^2 version of the Eichler-Shimura Isomorphism [24], that is

$$H^1_{(2)}\big(\Gamma, \mathfrak{h}, V_k\big) \cong S_k(\Gamma) \bigoplus \overline{S_k(\Gamma)} \;, \tag{1.0.2}$$

where $H^1_{(2)}$ is the degree 1 cohomology of the L^2 complex.

Briefly the proof of 1.0.2 uses Proposition (1.11) in [33] and Hodge theory to see that

$$H^1_{(2)}(\Gamma, \mathfrak{h}, V) \cong \ker d \cap \ker d^* \ .$$

On the other hand, since $\Gamma \setminus \mathfrak{h}$ is complete,

$$\ker d \cap \ker d^* = \{\phi \in L^2 A^1(\Gamma, \mathfrak{h}, V) \mid \Delta\phi = 0\}$$

by section 3.2 in [7] where $\Delta = dd^* + d^*d$ is the Laplacian and $L^2 A^1(\Gamma, \mathfrak{h}, V)$ denotes L^2 1-forms. The Eichler-Shimura Isomorphism then follows by showing the map $(f, \bar{g}) \mapsto \phi_f + \bar{\phi}_g \in A^\bullet(\Gamma, \mathfrak{h}, V)$ is a bijection onto L^2-harmonic forms. Chapter 4 ends by showing that step by step, ϕ_g^*, g^* and π_* preserve the L^2 condition of $A^\bullet_{(2)}(\Gamma, \mathfrak{h}, V)$ and hence the Hecke operators act on $H^\bullet_{(2)}(\Gamma, \mathfrak{h}, V)$.

Chapter 5 defines a parabolic condition which defines a complex $A^\bullet_P(\Gamma, \mathfrak{h}, V) \subset A^\bullet(\Gamma, \mathfrak{h}, V)$. The condition ensures that any degree 1 cohomology class becomes 0 in a small horoball neighborhood at each cusp. It starts by defining the horoball neighborhood of a cusp $\Gamma c \in \Gamma \setminus \mathfrak{h}$. Fix y_{suff} large enough such that the sets

$$\gamma\left(\{\tau \in \mathfrak{h} | \Im(\tau) \geq y_{suff}\} \cup \{\infty\}\right)$$

are disjoint for all cosets $\gamma\,\mathrm{Stab}_{\mathrm{SL}_2(\mathbf{Z})}(\infty) \in \mathrm{SL}_2(\mathbf{Z}) \,/\, \mathrm{Stab}_{\mathrm{SL}_2(\mathbf{Z})}(\infty)$ and pick $M > y_{suff}$. Then for $C \in \mathrm{SL}_2(\mathbf{Z})$ such that $C\infty = c$, define

$$V_\infty = \{\tau \in \mathfrak{h} | \Im(\tau) \geq M\} \cup \{\infty\} \subset W_\infty = \{\tau \in \mathfrak{h} | \Im(\tau) \geq y_{suff}\} \cup \{\infty\}$$

and

$$V_c = C \cdot V_\infty \subset W_c = C \cdot W_\infty \ .$$

Then $U_{\Gamma c} = \Gamma \backslash \Gamma V_c$ is a horoball neighborhood of the cusp Γc. The parabolic complex is then given by $A^0_P = A^0(\Gamma, \mathfrak{h}, V)$,

$$A^1_P = \left\{ \phi \in A^1(\Gamma, \mathfrak{h}, V) \mid \int_0^{w_c} \left(\iota_{\frac{\partial}{\partial x}} L^*_C \phi \right)(t, y) dt \in \mathrm{Im}(\rho(\gamma_c) - \mathrm{I}), \right.$$

$$\left. \forall \text{ cusps } c \text{ and } y \geq M \right\},$$

and $A^2_P = d(A^1_P)$ where γ_c is a generator of $\mathrm{Stab}_\Gamma(c)$, $C \in \mathrm{SL}_2(\mathbf{Z})$ sends ∞ to c and $w_c \in \mathbf{Z}^{>0}$ is the width of cusp Γc. Let $H^\bullet_P(\Gamma, \mathfrak{h}, V)$ be the corresponding cohomology. As it is shown, ϕ_f defined in chapter 3 represents a class in $H^1_P(\Gamma, \mathfrak{h}, V)$. Chapter 4 then shows the Hecke operator T_p acts on $A^1_P(\Gamma, \mathfrak{h}, V)$, and more specifically, that π^*_g, g^* and π_* preserve the parabolic condition.

Chapter 6 then establishes an isomorphism between the L^2 cohomology and the parabolic cohomology constructed in chapters 4 and 5. It does so by utilizing a homotopy formula

$$Hd + dH = \mathrm{I} - P \tag{1.0.3}$$

where H is a homopoty operator and P is a projection operator. If H and P are defined on a complex Y, and if $X \subset Y$ is a subcomplex such that

$$H(X) \subset X \quad \text{and} \quad P(Y) \subset X\ ,$$

there $H^\bullet(X) \cong H^\bullet(Y)$. The isomorphism is established in 3 steps with 3 homotopy operators. First is a homotopy H_x in the x direction with projection P_x defined by equation (1.0.3) that is related to a result of van Est [27], firstly defined in the region W_∞ and then generalized to all of $\mathfrak{h}$. Then the homotopy $\mathcal{H} = H_x + P_x H_x + P^2_x H_x + \cdots + P^{k-2}_x H_x$ shows that $A^\bullet_P(\Gamma, \mathfrak{h}, V)$ and $A^\bullet_{(2)}(\Gamma, \mathfrak{h}, V)$ have the same cohomology as their intersection with

$$A^\bullet_{\mathrm{inv}}(\Gamma, \mathfrak{h}, V) = \left\{ \phi \in A^\bullet(\Gamma, \mathfrak{h}, V) \colon \left(\rho\left(\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \right) \rho(g^{-1}) L^*_g \phi(x, y) \right)\Big|_{V_\infty} \right.$$

$$\left. \text{is constant in } x \, \forall \, g \in \mathrm{SL}_2(\mathbf{Z}) \right\}.$$

Next a homotopy operator H_{sp} with its projection P_{sp} on the coefficients acts to obtain something that is similar to the special differential forms of Goresky, Harder and MacPherson [11]. This further projects $A^\bullet_P(\Gamma, \mathfrak{h}, V)$ and $A^\bullet_{(2)}(\Gamma, \mathfrak{h}, V)$ to their intersection with

$$A^\bullet_{\mathrm{sp}}(\Gamma, \mathfrak{h}, V) = \left\{ \phi \in A^\bullet_{\mathrm{inv}}(\Gamma, \mathfrak{h}, V) \colon \left(\rho \left(\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \right) \rho(g^{-1})(L^*_g \phi)_0 \right) \Big|_{V_\infty} \in \ker N \text{ and} \right.$$
$$\left. \left(\rho \left(\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \right) \rho(g^{-1})(L^*_g \phi)_1 \right) \Big|_{V_\infty} \in (\mathrm{Im}\, N)^\perp \text{ w.r.t. } \langle \cdot, \cdot \rangle_0 \ \forall g \in \mathrm{SL}_2(\mathbf{Z}) \right\}.$$

On $A^\bullet_{\mathrm{sp}}$, a final homotopy H_y in the y direction is defined with the projection map P_y. This is done separately on $A^\bullet_{\mathrm{sp}} \cap A^\bullet_P$ and $A^\bullet_{\mathrm{sp}} \cap A^\bullet_{(2)}$. In each case, the intersection is then projected down to

$$A^\bullet_M(\Gamma, \mathfrak{h}, V) = \left\{ \phi \in A^\bullet_{\mathrm{sp}}(\Gamma, \mathfrak{h}, V) \colon \left(\iota_{\frac{\partial}{\partial x}} \rho(g^{-1})(L^*_g \phi) \right) \Big|_{V_\infty} = 0, \left(\iota_{\frac{\partial}{\partial y}} \rho(g^{-1})(L^*_g \phi) \right) \Big|_{V_\infty} \right.$$
$$\left. = 0 \text{and} \left(\rho(g^{-1})(L^*_g \phi) \right) \Big|_{V_\infty} \text{is constant in } y \, \forall g \in \mathrm{SL}_2(\mathbf{Z}) \right\}$$

showing that the cohomology of $A^\bullet_M$ is isomorphic to that of $A^\bullet_{\mathrm{sp}} \cap A^\bullet_P$ and that of $A^\bullet_{\mathrm{sp}} \cap A^\bullet_{(2)}$. Therefore the isomorphism between H_P and $H_{(2)}$ is established.

Chapter 7 switches focus to dessins d'enfants and the construction of its cohomology. It starts by introducing Belyi's theorem and dessins d'enfants. The j invariant function sends the modular curve $X(\Gamma)$ to $\mathbf{P}^1$ with at most 3 branching points, and with proper scaling, $\tilde{j}$ branches at $\{0, 1, \infty\}$ and so can be used as a Belyi function for $X(\Gamma)$. On the other hand, the Serre tree $\mathcal{T}$ in the upper half plane $\mathfrak{h}$ is a bicolored graph composed of vertices and edges as the infinite preimage of $[0, 1]$ under the map $\tilde{j}$. Dessins d'enfants corresponding to modular curves $X(\Gamma)$ can then be thought of as quotients of the Serre tree $\Gamma \setminus \mathcal{T}$. More specifically, for a dessin corresponding to $X(\Gamma)$, its white vertices are preimages of 0 under the $\tilde{j}$ map, or equivalently, the Γ-orbits of points in $\mathrm{SL}_2(\mathbf{Z}) \cdot \mu$. Its black vertices are preimages of 1 under the map $\tilde{j}$, or equivalently, the Γ-orbits of points in $\mathrm{SL}_2(\mathbf{Z})\cdot i$. The edges are then the preimages of the interval $[0,1]$ under the $\tilde{j}$ map, and the centers of faces are orbits of cusps, i.e. $\mathrm{SL}_2(\mathbf{Z})\cdot\infty$ of Γ. Chapter 7 then constructs the final cohomology of this dissertation on dessins using the Serre tree, whose definition is a combinatorial version of the one defined in chapter 5 that satisfies an analogous parabolic condition. To describe the parabolic condition for dessins, the concept of a *link* is introduced - it can be viewed as a sum of edges surrounding a cusp in a counter-clockwise fashion and with alternating signs to correct orientation. Then the parabolic condition for combinatorial 1-forms is given by

$$\phi(link_c) \in \mathrm{Im}(\rho(\gamma_c) - \mathrm{I})$$

where $link_c$ is the link surrounding cusp $\Gamma c \in \Gamma \setminus \mathfrak{h}$, and γ_c as before, is an element of $\mathrm{Stab}_\Gamma(c)$. With the construction of the combinatorial complex on dessins, chapter 7 then gives the description of Hecke correspondence on the cohomology of this complex, which involves an extra step than the previous ones because of a homotopy of the Serre tree. Because of this extra step, this correspondence is only proved for $p = 2, 3$ and 5; the condition on p is needed in Proposition 7.3.1, which shows that the map g maps the vertices of the Serre tree to points in $\mathfrak{h}$ that retract to vertices of the Serre tree under the standard deformation retract $\mathfrak{h} \longrightarrow \mathcal{T}$. With this proposition, we can study the action of Hecke operator T_p on combinatorial 1-forms and obtain that

$$H^1_P(\Gamma_0(N), \mathfrak{h}, V) \cong H^1_P(\Gamma_0(N), \mathcal{T}, V)$$

is T_p-equivariant for at least $p = 2, 3, 5$ as stated in Theorem 7.3.7. Conveniently with this setup, chapter 7 gives another way to compute the dimensions of this cohomology group in degree 1 and its corresponding cusp forms. The results agree with the statement of Eichler-Shimura Isomorphism stated in chapter 4. A great remaining part of chapter 7 is then given to examples on how to compute the eigenvalues of T_p

on dessins associated with different cusp forms $S_k(\Gamma_0(N))$, for $p = 3, 5, k = 8, 10, 12$ and $N = 2, 3$, following each step of the Hecke correspondence as discussed in section 7.3 (Even though we have only proved and shown for the cases $p = 2, 3$, and 5, we expect the algorithm can be generalized for all p with some changes). The codes presented in this chapter are from Mathematica.

Lastly in chapter 8, an observation is given and proved, that the same Hecke correspondence can be achieved via Atkin-Lehner Involution [17]. An example of T_3 acting on $X_0(2)$ is given as an illustration.

Our hope is that the techniques presented in this dissertation can be applied further. One future goal is to have analogous Hecke correspondences on algebraic curves other than $X(\Gamma)$ and to calculate Hecke action on cohomology of their corresponding dessins. Another application is to use the combinatorial tools on dessins to study the Galois actions on their corresponding curves. Last but not least, is to construct maps between inequivalent dessins, and in turn getting morphisms between different algebraic curves.

Chapter 2

Classical Hecke Actions on Modular Forms

2.1 Modular Forms and Classical Hecke Operators

Based on Diamond and Shurman's discussion in [9], Hecke operators can be represented as double coset operators on the space of modular forms, denoted as $\mathcal{M}_k(\Gamma_0(N))$, where $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}$, and k is the weight of the cusp forms.

If f is a weight k modular form in $\mathcal{M}_k(\Gamma_0(N))$, f is holomorphic on the upper half plane, satisfying the relation

$$f(\tau) = f[\gamma]_k(\tau) = (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right)$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, and $f[\alpha]_k$ is holomorphic at ∞ for all $\alpha \in \mathrm{SL}_2(\mathbf{Z})$. Any modular form f also has the following Fourier expansion

$$f(\tau) = a_0(f) + a_1(f)q + a_2(f)q^2 + \cdots = \sum_{n=0}^{\infty} a_n q^n, \quad q = e^{2\pi i \tau}.$$

If $a_0 = 0$ in the Fourier expansion of $f[\alpha]_k$ for all $\alpha \in \mathrm{SL}_2(\mathbf{Z})$, then f is a cusp form, denoted as $f \in S_k(\Gamma_0(N))$.

In [9], Diamond and Shurman defined the Hecke operator T_p on the space $M_k(\Gamma_1(N))$. Analogously, we can define the Hecke operator T_p on the space $M_k(\Gamma_0(N))$, since $\Gamma_1(N) \subseteq \Gamma_0(N)$, and thus $M_k(\Gamma_0(N)) \subseteq M_k(\Gamma_1(N))$. T_p can be realized as a weight$k$ double coset operator $[\Gamma_0 \alpha \Gamma_0]$ where

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, \quad p \text{ prime.}$$

Thus

$$T_p \colon M_k(\Gamma_0(N)) \longrightarrow M_k(\Gamma_0(N)), \quad p \text{ prime}$$

is given by

$$\begin{aligned} T_p f &= f[\Gamma_0(N) \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} \Gamma_0(N)]_k \\ &= \begin{cases} \displaystyle\sum_{j=0}^{p-1} f[\left(\begin{smallmatrix} 1 & j \\ p & 1 \end{smallmatrix}\right)]_k \ , & \text{if } p \mid N \ , \\ \displaystyle\sum_{j=0}^{p-1} f[\left(\begin{smallmatrix} 1 & j \\ p & 1 \end{smallmatrix}\right)]_k + f[\left(\begin{smallmatrix} m & n \\ N & p \end{smallmatrix}\right)\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)]_k \ , & \text{if } p \nmid N, \text{where } mp - nN = 1 \ . \end{cases} \end{aligned}$$

Recall a modular form f has Fourier expansion $f(\tau) = \sum_{n=0}^{\infty} a_n q^n$. Then $T_p f$ has Fourier expansion

$$\begin{aligned} (T_p f)(\tau) &= \sum_{n=0}^{\infty} a_{np}(f) q^n + \mathbf{1}_N(p) p^{k-1} \sum_{n=0}^{\infty} a_n(\langle p \rangle f) q^{np} \\ &= \sum_{n=0}^{\infty} \big(a_{np}(f) + \mathbf{1}_N(p) p^{k-1} a_{n/p}(\langle p \rangle f)\big) q^n \end{aligned}$$

where $\mathbf{1}_N \colon (\mathbf{Z}/N\mathbf{Z})^* \longrightarrow \mathbf{C}^*$ is the trivial character modulo N. This also gives us the relationship of the coefficients,

$$a_n(T_p f) = a_{np}(f) + \mathbf{1}_N(p) p^{k-1} a_{n/p}(\langle p \rangle f) \ . \tag{2.1.1}$$

In particular, if the modular form f is an eigenform for the Hecke operator T_p, it is called a Hecke eigenform. It is normalized when $a_1(f) = 1$. A newform is a normalized eigenform in $S_k(\Gamma_0(N))^{\text{new}}$. When $p \nmid N$, we have $T_p f = c_p f$ where $c_n \in \mathbf{C}$ is an eigenvalue. Consequently formula 2.1.1 says

$$a_1(T_p) = a_p(f) \ .$$

Since $p \nmid N$, we also have

$$a_1(T_p f) = c_p a_1(f) \ ,$$

so

$$a_p(f) = c_p a_1(f) \ .$$

Thus when $a_1 = 1$, T_p gives us information of the coefficient a_p in the Fourier expansion of f.

2.2 Hecke Correspondences

The action of Hecke operators on modular forms can also be seen through the Hecke correspondence. As described in [25], a correspondence between two curves C_1 and C_2 is a curve C together with nonconstant morphisms $\alpha \colon C \to C_1$ and $\beta \colon C \to C_2$, which can be represented by the following diagram

$$\begin{array}{ccc} & C & \\ {}^{\alpha}\swarrow & & \searrow^{\beta} \\ C_1 & & C_2 \ . \end{array}$$

The curve of interest for us are modular curves. For any congruence subgroup Γ of $\mathrm{SL}_2(\mathbf{Z})$ the corresponding modular curve can be defined as the quotient of the upper half plane $\mathfrak{h}$ by Γ, namely $Y(\Gamma) = \Gamma \backslash \mathfrak{h}$, and can be thought as the set of orbits

$$Y(\Gamma) = \{\Gamma\tau \colon \tau \in \mathfrak{h}\} \ .$$

To give modular curve a Riemann surface structure, we compactify $Y(\Gamma)$ by filling in points at the cusps. The resulting compact curve is denoted as $X(\Gamma)$. If we define

$\mathfrak{h}^* = \mathfrak{h} \cup \mathbf{Q} \cup \{\infty\}$, then

$$X(\Gamma) = \Gamma \backslash \mathfrak{h}^* = Y(\Gamma) \cup \Gamma \backslash (\mathbf{Q} \cup \{\infty\}) \ ,$$

and the modular curve associated with $\Gamma_0(N)$ is denoted as $X_0(N)$.

For a Hecke operator T_p where $p \nmid N$, we can view it as a correspondence $X_0(N) \rightsquigarrow X_0(N)$. In this case, there is a curve $C = X_0(pN)$ and maps α and β, giving the following diagram

$$\begin{array}{ccc} & X_0(pN) & \\ \alpha \swarrow & & \searrow \beta \\ X_0(N) & & X_0(N) \ . \end{array}$$

The map α is the projection map

$$\alpha \colon \Gamma_0(pN) \setminus \mathfrak{h} \to \Gamma_0(N) \setminus \mathfrak{h}$$

induced by the inclusion $\Gamma_0(pN) \hookrightarrow \Gamma_0(N)$. The map β is constructed by composing the isomorphism

$$\Gamma_0(pN) \setminus \mathfrak{h} \xrightarrow{\sim} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0(pN) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1} \setminus \mathfrak{h}$$

$$z \mapsto \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} z = pz$$

with the projection map to $\Gamma_0(N) \setminus \mathfrak{h}$ induced by the inclusion

$$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0(pN) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1} \subset \Gamma_0(N) \ .$$

Hence, the Hecke correspondence of modular curves can be viewed as

$$\begin{array}{ccc} \Gamma_0(pN) \setminus \mathfrak{h} & \xrightarrow{g} & g\Gamma_0(pN)g^{-1} \setminus \mathfrak{h} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_g} \\ \Gamma_0(N) \setminus \mathfrak{h} & & \Gamma_0(N) \setminus \mathfrak{h} \end{array}$$

where $g = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$. The composition of the three maps will recover the double coset Hecke operator on modular forms $T_p \colon M_k(\Gamma_0(N)) \longrightarrow M_k(\Gamma_0(N))$ with $p \nmid N$, as shown in Diamond and Shurman's book [9].

Note. When $p \nmid N$, matrices $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ and $g = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ generate the same double coset for $\Gamma_0(N)$, i.e. $\Gamma_0(N)\alpha\Gamma_0(N) = \Gamma_0(N)g\Gamma_0(N)$.

Recall the following definition from Diamond and Shurman:

Definition 2.2.1. For congruence subgroups Γ_1 and Γ_2 of $\mathrm{SL}_2(\mathbf{Z})$ and $\alpha \in \mathrm{GL}_2^+(\mathbf{Q})$, the weight$k$ $\Gamma_1\alpha\Gamma_2$ operator takes functions $f \in M_k(\Gamma_1)$ to $M_k(\Gamma_2)$ via

$$f[\Gamma_1\alpha\Gamma_2]_k = \sum_j f[\beta_j]_k$$

where $\{\beta_j\}$ are orbit representatives, i.e., $\Gamma_1\alpha\Gamma_2 = \cup_j \Gamma_1\beta_j$ is a disjoint union.

Moreover, if $f \in S_k(\Gamma_1)$, the transformed $f[\Gamma_1\alpha\Gamma_2]_k$ also vanishes at the cusps. Thus

$$[\Gamma_1\alpha\Gamma_2]_k \colon S_k(\Gamma_1) \longrightarrow S_k(\Gamma_2)\ .$$

The process of transferring functions forward by the double coset operator also has a geometric interpretation in terms of transferring points back between the corresponding modular curves. The configuration of modular curves is

$$\begin{array}{ccc} X_3 & \xrightarrow{\ \alpha\ } & X_3' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_g} \\ X_1 & & X_2 \end{array}$$

where α induces an isomorphism by $\gamma \mapsto \alpha\gamma\alpha^{-1}$. When $X_1 = X_2$ and $\alpha = g$, this is the same configuration as the Hecke correspondence. The double coset operator $[\Gamma_1\alpha\Gamma_2]_k$ along the maps are:

1. Since $\Gamma_1 \supset \Gamma_3$, taking $\alpha = \mathrm{I}_2$ makes the double coset operator be $f[\Gamma_1 \alpha \Gamma_3]_k = f$, the natural inclusion of the subspace $S_k(\Gamma_1)$ in $S_k(\Gamma_3)$, an injection.

2. $\Gamma_3' = \alpha^{-1}\Gamma_3\alpha$. The double coset operator is $f[\Gamma_3 \alpha \Gamma_3']_k = f[\alpha]_k$, the natural translation from $S_k(\Gamma_3)$ to $S_k(\Gamma_3')$, an isomorphism.

3. $\Gamma_3' \subset \Gamma_2$. Take $\alpha = \mathrm{I}_2$ and let β_j be a set of coset representatives for $\Gamma_3' \setminus \Gamma_2$. The double coset operator then becomes $f[\Gamma_3' \alpha \Gamma_2]_k = \sum_j f[\beta_j]_k$, the natural trace map that projects $S_k(\Gamma_3')$ onto its subspace $S_k(\Gamma_2)$ by symmetrizing over the quotient, a surjection.

The corresponding composition of double coset operators is

$$f \mapsto f \mapsto f[\alpha]_k \mapsto \sum_j f[\alpha\beta_j]_k \ .$$

Chapter 3

Hecke Actions on Cohomology from Matsushima and Murakami

3.1 Cohomology with Vector Valued Coefficients

We first adopt the method presented in Matsushima and Murakami [22] to construct a cohomology on the upper half plane with vector valued coefficients. We apply the method to the case of $\mathfrak{h}$, a subgroup $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$, and a representation of $\mathrm{SL}_2(\mathbf{Z})$ on V. We will describe what Matsushima and Murakami did in that case. We denote by L_γ the operation on $\mathfrak{h}$ of an element $\gamma \in \Gamma$. Let $\rho\colon \Gamma \longrightarrow \mathrm{Aut}(V)$ be a representation of Γ in a vector space V. As in Matsushima and Murakami, we use $A^\bullet(\Gamma, \mathfrak{h}, V)$ to denote the vector space consisting of all vector valued i - forms η on $\mathfrak{h}$ that satisfy condition

$$L_\gamma^*(\eta) = \rho(\gamma)\eta \tag{3.1.1}$$

for all $\gamma \in \Gamma$. Note that if η satisfies equation (3.1.1), so does $d\eta$ because

$$L_\gamma^*(d\eta) = dL_\gamma^*(\eta) = d\rho(\gamma)\eta = \rho(\gamma)d\eta\ .$$

Specifically, we have

$$A^0(\Gamma, \mathfrak{h}, V) = \{f\colon \mathfrak{h} \longrightarrow V \mid L_\gamma^*(f) = \rho(\gamma)f,\ \forall\gamma \in \Gamma\}\ ,$$

$$A^1(\Gamma, \mathfrak{h}, V) = \{\phi = f d\tau + g d\bar\tau \mid f, g\colon \mathfrak{h} \longrightarrow V\ \&\ L_\gamma^*(\phi) = \rho(\gamma)\phi,\ \forall\gamma \in \Gamma\}\ ,$$

$$A^2(\Gamma, \mathfrak{h}, V) = \{\omega = h d\tau \wedge d\bar\tau \mid h\colon \mathfrak{h} \longrightarrow V\ \&\ L_\gamma^*(\omega) = \rho(\gamma)\omega,\ \forall\gamma \in \Gamma\}\ ,$$

and thus we have a complex

$$0 \xrightarrow{d^{-1}} A^0(\Gamma, \mathfrak{h}, V) \xrightarrow{d^0} A^1(\Gamma, \mathfrak{h}, V) \xrightarrow{d^1} A^2(\Gamma, \mathfrak{h}, V) \xrightarrow{d^2} 0\ .$$

The cohomology groups are denoted by $H^{\bullet}(\Gamma, \mathfrak{h}, V) = \dfrac{\ker(d)}{\mathrm{Im}(d)}$.

If Γ has no fixed points, i.e., $\Gamma \setminus \mathfrak{h}$ does not have any elliptic points, we can define a local system $\mathbf{V}$ on $\Gamma \setminus \mathfrak{h}$ as

$$\Big(\mathfrak{h} \times V\Big) \Big/ \Big((\tau, v) \sim (\gamma\tau, \rho(\gamma)v)\Big)$$

where $(\tau, v) \in \mathfrak{h} \times V$. Let $A^{\bullet}(\Gamma \setminus \mathfrak{h}, \mathbf{V})$ denote the space of differential forms ϕ on $\Gamma \setminus \mathfrak{h}$ with coefficient in $\mathbf{V}$. We can again define the cohomology group as $H^{\bullet}(\Gamma \setminus \mathfrak{h}, \mathbf{V}) = \dfrac{\ker(d)}{\mathrm{Im}(d)}$ where $d^s \colon A^{s-1}(\Gamma \setminus \mathfrak{h}, \mathbf{V}) \longrightarrow A^s(\Gamma \setminus \mathfrak{h}, \mathbf{V})$ is a coboundary operator. By de Rham's theorem this is the topological cohomology of $\Gamma \setminus \mathfrak{h}$ with coefficients in $\mathbf{V}$. Matsushima and Murakami show there exists an isomorphism of the cohomology $H^{\bullet}(\Gamma, \mathfrak{h}, V) \cong H^{\bullet}(\Gamma \setminus \mathfrak{h}, \mathbf{V})$. More generally, let $\mathbf{V}$ be the sheafification of the presheaf on $\Gamma \setminus \mathfrak{h}$ given by

$$U \mapsto \{f \colon \tilde{U} \longrightarrow V \text{ locally constant s.t. } f(\gamma\tau) = \rho(\gamma)f(\tau) \text{ for all } \tau \in \tilde{U}, \gamma \in \Gamma.\}$$

Note here $\tilde{U} = \{\tau \in \mathfrak{h} \mid \Gamma\tau \in U\}$. Then even when there are elliptic points in $\Gamma \setminus \mathfrak{h}$, there is an isomorphism $H^{\bullet}(\Gamma, \mathfrak{h}, V) \cong H^{\bullet}(\Gamma \setminus \mathfrak{h}, \mathbf{V})$. In this case, $\mathbf{V}$ is sometimes called an orbifold local system.

3.2 The Action of Hecke Operators on $A^1(\Gamma, \mathfrak{h}, V)$

We could work with Hecke operators in general, but for the rest of the section, we will focus on T_p acting on cohomology with respect to $\Gamma_0(N)$ where $p \nmid N$. Recall the following diagram for the correspondence

$$\begin{array}{ccc}
\Gamma_0(pN) \setminus \mathfrak{h} & \xrightarrow{\;g\;} & g\Gamma_0(pN)g^{-1} \setminus \mathfrak{h} \\
\pi \big\downarrow & & \big\downarrow \pi_g \\
\Gamma_0(N) \setminus \mathfrak{h} & & \Gamma_0(N) \setminus \mathfrak{h}
\end{array}$$

where $g = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$. For all i, we will define the action of T_p on $A^\bullet(\Gamma_0(N), \mathfrak{h}, V)$ as $T_p = \pi_* \circ g^* \circ \pi_g^*$ where

$$\begin{aligned} \pi_g^* &\colon A^\bullet(\Gamma_0(N), \mathfrak{h}, V) \longrightarrow A^\bullet(g\Gamma_0(pN)g^{-1}, \mathfrak{h}, V) \ , \\ g^* &\colon A^\bullet(g\Gamma_0(pN)g^{-1}, \mathfrak{h}, V) \longrightarrow A^\bullet(\Gamma_0(pN), \mathfrak{h}, V) \ , \\ \pi_* &\colon A^\bullet(\Gamma_0(pN), \mathfrak{h}, V) \longrightarrow A^\bullet(\Gamma_0(N), \mathfrak{h}, V) \end{aligned} \tag{3.2.1}$$

will be defined below. The main point is to show each map preserves the condition in equation (3.1.1) on a form for all γ in the respective subgroups.

Proposition 3.2.1. If $\phi \in A^\bullet(\Gamma_0(N), \mathfrak{h}, V)$, then $\pi_g^*(\phi) = \phi \in A^\bullet(g\Gamma_0(pN)g^{-1}, \mathfrak{h}, V)$.

Proof. We claim that $g\Gamma_0(pN)g^{-1} \subseteq \Gamma_0(N)$. If $\gamma_p \in \Gamma_0(pN)$, $\gamma_p = \begin{pmatrix} a & b \\ pNc & d \end{pmatrix}$ where $ad - pNbc = 1$. We have

$$\begin{aligned} g\gamma_p g^{-1} &= \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ pNc & d \end{pmatrix} \begin{pmatrix} \frac{1}{p} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} ap & bp \\ pNc & d \end{pmatrix} \begin{pmatrix} \frac{1}{p} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a & bp \\ Nc & d \end{pmatrix} \in \Gamma_0(N) \ . \end{aligned} \tag{3.2.2}$$

This proves the claim. If $\phi \in A^\bullet(\Gamma_0(N), \mathfrak{h}, V)$, it satisfies

$$L_\gamma^*(\phi) = \rho(\gamma)\phi \tag{3.2.3}$$

for all $\gamma \in \Gamma_0(N)$, and hence for all $\gamma \in g\Gamma_0(pN)g^{-1}$. By equation 3.1.1, this shows that $\pi_g^*(\phi) = \phi \in A^1(g\Gamma_0(pN)g^{-1}, \mathfrak{h}, V)$. □

Next we define $g^* = \rho(g^{-1})L_g^*$ such that

$$\rho(g^{-1})L_g^* \colon A^\bullet(g\Gamma_0(pN)g^{-1}, \mathfrak{h}, V) \longrightarrow A^\bullet(\Gamma_0(pN), \mathfrak{h}, V) \ .$$

Proposition 3.2.2. If $\phi \in A^\bullet(g\Gamma_0(pN)g^{-1}, \mathfrak{h}, V)$, then $g^*(\phi) \in A^\bullet(\Gamma_0(pN), \mathfrak{h}, V)$.

Proof. We will show that $g^*(\phi) \in A^\bullet(\Gamma_0(pN), \mathfrak{h}, V)$ by showing it satisfies the condition in equation (3.1.1) for $\gamma \in \Gamma_0(pN)$. We have

$$\begin{aligned} L_\gamma^*(g^*(\psi)) &= L_\gamma^*\left(\rho(g^{-1})L_g^*(\phi)\right) \\ &= \rho(g^{-1})L_g^*L_{g^{-1}}^*L_\gamma^*L_g^*(\phi) \\ &= \rho(g^{-1})L_g^*L_{g\gamma g^{-1}}^*(\phi) \\ &= \rho(g^{-1}g\gamma g^{-1})L_g^*(\phi) \\ &= \rho(\gamma)\rho(g^{-1})L_g^*(\phi) \\ &= \rho(\gamma)g^*(\phi)\ , \end{aligned}$$

where for the fourth equality, we used the fact that $L_{g\gamma g^{-1}}^*(\phi) = \rho(g\gamma g^{-1})\phi$ for $\gamma \in \Gamma_0(pN)$ by equation (3.2.3). □

Lastly, given $\psi \in A^\bullet(\Gamma, \mathfrak{h}, V)$, we define

$$\pi_*(\psi) = \sum_{\gamma \in \Gamma_0(pN)\setminus\Gamma_0(N)} \rho(\gamma^{-1})L_\gamma^*(\psi)\ .$$

Lemma 3.2.3. $\pi_*(\psi)$ is independent of the choice of coset representatives for $\Gamma_0(pN)\setminus \Gamma_0(N)$.

Proof. Since $\psi \in A_i(\Gamma_0(pN), \mathfrak{h}, V)$, $L_{\gamma_{pN}}^*(\psi) = \rho(\gamma_{pN})\psi$ for $\gamma_{pN} \in \Gamma_0(pN)$. If $\gamma \in \Gamma_0(pN) \setminus \Gamma_0(N)$, let $\tilde{\gamma} = \gamma_{pN} \cdot \gamma$, and then $\tilde{\gamma} \in \Gamma_0(pN) \setminus \Gamma_0(N)$. We compute

$$\begin{aligned} \rho(\tilde{\gamma}^{-1})L_{\tilde{\gamma}}^*(\psi) &= \rho(\gamma^{-1})\rho(\gamma_{pN}^{-1})L_{\gamma_{pN}\cdot\gamma}^*(\psi) \\ &= \rho(\gamma^{-1})\rho(\gamma_{pN}^{-1})\rho(\gamma_{pN})L_\gamma^*(\psi) \\ &= \rho(\gamma^{-1})L_\gamma^*(\psi)\ . \end{aligned}$$

On the other hand, if γ runs through a set of representatives in $\Gamma_0(pN) \setminus \Gamma_0(N)$, $\tilde{\gamma}$ also run through another set of representatives in $\Gamma_0(pN) \setminus \Gamma_0(N)$. Thus

$$\sum_{\tilde{\gamma} \in \Gamma_0(pN) \setminus \Gamma_0(N)} \rho(\tilde{\gamma}^{-1}) L^*_{\tilde{\gamma}}(\psi) = \sum_{\gamma \in \Gamma_0(pN) \setminus \Gamma_0(N)} \rho(\gamma^{-1}) L^*_{\gamma}(\psi) \ . \qquad \square$$

Proposition 3.2.4. If $\psi \in A^\bullet(\Gamma_0(pN), \mathfrak{h}, V)$, then $\pi_*(\psi) \in A^\bullet(\Gamma_0(N), \mathfrak{h}, V)$.

Proof. We will show $\pi_*(\psi)$ satisfies the condition in equation (3.1.1). Let $\gamma_N \in \Gamma_0(N)$, we compute the following

$$L^*_{\gamma_N} \pi_*(\psi) = \sum_{\gamma \in \Gamma_0(pN) \setminus \Gamma_0(N)} \rho(\gamma^{-1}) L^*_{\gamma \cdot \gamma_N}(\psi) \ ,$$

write $\tilde{\gamma} = \gamma \cdot \gamma_N$, thus $\gamma = \tilde{\gamma} \cdot \gamma_N^{-1}$, the above becomes

$$\begin{aligned} &= \sum_{\tilde{\gamma} \in \Gamma_0(pN) \setminus \Gamma_0(N)\gamma_N} \rho(\gamma_N) \rho(\tilde{\gamma}^{-1}) L^*_{\tilde{\gamma}}(\psi) \\ &= \rho(\gamma_N) \sum_{\tilde{\gamma} \in \Gamma_0(pN) \setminus \Gamma_0(N)} \rho(\tilde{\gamma}^{-1}) L^*_{\tilde{\gamma}}(\psi) \\ &= \rho(\gamma_N) \pi_*(\psi) \ . \end{aligned} \qquad \square$$

3.3 Hecke Operators Commute with the Differentials d

In this section we show that the Hecke operators $T_p = \pi_* \circ g^* \circ \pi_g^*$ mapping $A^\bullet(\Gamma_0(N), \mathfrak{h}, V)$ to $A^\bullet(\Gamma_0(N), \mathfrak{h}, V)$ commute with the differentials d. Thus they induce a Hecke action on $H^\bullet(\Gamma_0(N), \mathfrak{h}, V)$.

Proposition 3.3.1. d commutes with the pullback map π_g^*.

Proof. Let $\eta \in A^\bullet(\Gamma_0(N), \mathfrak{h}, V)$ be a differential form. We want to show the following diagram

$$
\begin{array}{ccccc}
\pi_g^*\eta & \xrightarrow{d} & d\pi_g^*\eta & \overset{=}{\dashrightarrow} & \pi_g^* d\eta \\
\uparrow{\scriptstyle \pi_g^*} & & & \nearrow{\scriptstyle \pi_g^*} & \\
\eta & \xrightarrow{d} & d\eta & &
\end{array}
$$

commutes by showing the equality holds in the upper right. Since π_g acts as the identity map, in the upper half plane, thus $d\pi_g^*\eta = d\eta = \pi_g^* d\eta$. We have established the above commutative diagram. Thus d commutes with π_g^*. □

Proposition 3.3.2. d commutes with the pullback map $g^* = \rho(g^{-1})L_g^*$.

Proof. Like in section 3.2, the pullback map g^* is defined as $\rho(g^{-1})L_g^*$. Let $\eta \in A^\bullet(g\Gamma_0(pN)g^{-1}, \mathfrak{h}, V)$, thus to show the following diagram commute,

$$
\begin{array}{ccccc}
 & & \rho(g^{-1})L_g^*(\eta) & \xleftarrow{g^*} & \eta \\
 & \swarrow{\scriptstyle d} & & & \downarrow{\scriptstyle d} \\
d\left(\rho(g^{-1})L_g^*(\eta)\right) & \overset{=}{\dashleftarrow} & \rho(g^{-1})L_g^*(d\eta) & \xleftarrow{g^*} & d\eta
\end{array}
$$

we want to show the equality holds in the bottom left. This is immediate since d commutes with L_g^* and $\rho(g^{-1})$. □

Proposition 3.3.3. d commutes with π_*.

Proof. Like in section 3.2, the pushforward map π_* is defined as $\pi_*(\eta) = \sum_{\gamma \in \Gamma_0(pN)\backslash\Gamma_0(N)} \rho(\gamma^{-1})L_\gamma^*(\eta)$ for $\eta \in A^\bullet(\Gamma_0(pN), \mathfrak{h}, V)$. We will show the following diagram

$$
\begin{array}{ccccc}
\eta & \xrightarrow{d} & d\eta & & \\
\downarrow{\scriptstyle \pi} & & & \searrow{\scriptstyle \pi} & \\
\pi_*(\eta) & \xrightarrow{d} & d(\pi_*(\eta)) & \overset{=}{\dashrightarrow} & \pi_*(d\eta)
\end{array}
$$

commutes by showing the equality holds in the bottom right. This is true because

$$\begin{aligned} d(\pi_*(\eta)) &= d\left(\sum_{\gamma\in\Gamma_0(pN)\backslash\Gamma_0(N)} \rho(\gamma^{-1})L_\gamma^*(\eta)\right) \\ &= \sum_{\gamma\in\Gamma_0(pN)\backslash\Gamma_0(N)} \rho(\gamma^{-1})L_\gamma^*(d\eta) \qquad \text{since } d \text{ commutes with } L_\gamma^* \text{ and } \rho(\gamma^{-1}), \\ &= \pi_*(d\eta) \ . \end{aligned}$$

□

3.4 Relationship to Classic Hecke Operators

In this section, we calculate the relationship between Hecke operators on modular forms and Hecke operators on cohomology with vector valued coefficients. Consider a weight k cusp form $f(\tau)$ in $S_k(\Gamma)$ where Γ is a finite index subgroup of $\mathrm{SL}_2(\mathbf{Z})$. We define the associated vector-valued 1-form

$$\phi_f(\tau) = f(\tau)\begin{pmatrix}\tau\\1\end{pmatrix}^{k-2} d\tau \ .$$

We will show that ϕ_f represents a class in $H^1\big(\Gamma, \mathfrak{h}, V_k\big)$ where $V_k = \mathrm{Sym}^{k-2}(\mathbf{C}^2)$.

First we will show that $\phi_f \in A^1(\Gamma, \mathfrak{h}, V_k)$ by showing ϕ_f satisfies equation (3.1.1). Since f is a weight k modular form, it satisfies $f(\gamma\tau) = (c\tau + d)^k f(\tau)$ for any $\gamma =$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. We thus have

$$
\begin{aligned}
L_\gamma^*(\phi_f)_\tau &= f(\gamma\tau) \begin{pmatrix} \gamma\tau \\ 1 \end{pmatrix}^{k-2} d(\gamma\tau) \\
&= (c\tau+d)^k f(\tau)(c\tau+d)^{-(k-2)} \begin{pmatrix} a\tau+b \\ c\tau+d \end{pmatrix}^{k-2} (c\tau+d)^{-2} d\tau \\
&= f(\tau) \begin{pmatrix} a\tau+b \\ c\tau+d \end{pmatrix}^{k-2} d\tau \\
&= \rho(\gamma) f(\tau) \begin{pmatrix} \tau \\ 1 \end{pmatrix}^{k-2} d\tau \\
&= \rho(\gamma) \left(\phi_f\right)_\tau \ . \qquad (3.4.1)
\end{aligned}
$$

On the other hand, since f and $\begin{pmatrix} \tau \\ 1 \end{pmatrix}$ are holomorphic, $\phi_f = f \begin{pmatrix} \tau \\ 1 \end{pmatrix}^{k-2} d\tau$ lies in $\ker(d)$. Therefore ϕ_f represents a class of $H^1(\Gamma, \mathfrak{h}, V_k)$. We thus have a map from

$$S_k(\Gamma) \longrightarrow H^1(\Gamma, \mathfrak{h}, V_k)$$

that takes $f \mapsto [\phi_f]$.

We now restrict to the case $\Gamma = \Gamma_0(N)$, and compute the action of T_p where $p \nmid N$ on $S_k(\Gamma_0(N))$ and $H^1(\Gamma_0(N), \mathfrak{h}, V_k)$. We have the following maps:

$$
\begin{array}{ccc}
\psi & \xleftarrow{\ g^*\ } & \phi \\
{\scriptstyle \pi_*}\downarrow & & \uparrow{\scriptstyle \pi_g^*} \\
\pi_*(\psi) & & \phi\ .
\end{array}
$$

Consider now $\phi = \phi_f$. Then

$$\begin{aligned}
\psi &= g^*(\phi_f) \\
&= \rho(g^{-1}) L_g^*(\phi_f) \\
&= \rho(g^{-1}) f(g\tau) \begin{pmatrix} g\tau \\ 1 \end{pmatrix}^{k-2} d(g\tau) \\
&= \rho(g^{-1}) f(g\tau) \rho(g) \begin{pmatrix} \tau \\ 1 \end{pmatrix}^{k-2} p \cdot d\tau \qquad\qquad \text{since } g = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \\
&= p f(g\tau) \begin{pmatrix} \tau \\ 1 \end{pmatrix}^{k-2} d\tau \ .
\end{aligned}$$

To find $\pi_*(\psi)$, first let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and compute

$$\begin{aligned}
\rho(\gamma^{-1}) L_\gamma^*(\psi) &= \rho(\gamma^{-1}) p f(g\gamma\tau) \begin{pmatrix} \gamma\tau \\ 1 \end{pmatrix}^{k-2} d(\gamma\tau) \\
&= \rho(\gamma^{-1}) p f(g\gamma\tau)(c\tau + d)^{-(k-2)} \rho(\gamma) \begin{pmatrix} \tau \\ 1 \end{pmatrix}^{k-2} (c\tau + d)^{-2} d(\tau) \\
&= p f(g\gamma\tau)(c\tau + d)^{-k} \begin{pmatrix} \tau \\ 1 \end{pmatrix}^{k-2} d(\tau) \ .
\end{aligned}$$

Recall that for $g\gamma = \begin{pmatrix} ap & bp \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Q})$, the weight$k$ operator $[g\gamma]_k$ is defined as

$$f[g\gamma]_k(\tau) = \det(g\gamma)^{k-1}(c\tau + d)^{-k} f(g\gamma\tau) \ .$$

We can then rewrite the above equation as

$$\rho(\gamma^{-1}) L_\gamma^*(\psi) = p^{2-k} \left(f[g\gamma]_k(\tau) \right) \begin{pmatrix} \tau \\ 1 \end{pmatrix}^{k-2} d\tau \ .$$

Thus we have

$$\begin{aligned}\pi_*(\psi) &= \sum_{\gamma\in\Gamma_0(pN)\backslash\Gamma_0(N)} \rho(\gamma^{-1})L_\gamma^*(\psi)\\ &= p^{2-k}\left(\sum_{\gamma\in\Gamma_0(pN)\backslash\Gamma_0(N)} f[g\gamma]_k(\tau)\right)\begin{pmatrix}\tau\\1\end{pmatrix}^{k-2}\ d\tau\end{aligned}$$

where $\sum_{\gamma\in\Gamma_0(pN)\backslash\Gamma_0(N)} f[g\gamma]_k(\tau)$ is the expression for classical Hecke operators. We have proven

Theorem 3.4.1. If $f \in S_k(\Gamma_0(N))$ and $[\phi_f] \in H^1(\Gamma_0(N), \mathfrak{h}, V_k)$ is the corresponding cohomology class, then

$$T_p\left([\phi_f]\right) = p^{2-k}[\phi_{T_p(f)}]\ .$$

Chapter 4

L^2 Cohomology and Eichler-Shimura Isomorphism

In the previous section, we saw how cusp forms can yield classes in the cohomology $H^1(\Gamma, \mathfrak{h}, V)$, and we saw an Hecke action. However, it is not clear if these classes and their conjugates are none zero or span the cohomology. We want a variant of cohomology more tightly related to modular forms. Shimura in his book on *Introduction to the Arithmetic Theory of Automorphic Functions* [24], chapter 8, defined a parabolic cohomology based on results from Eichler by modifying group cohomology $H^\bullet(\Gamma, \mathfrak{h}, V)$. Here instead, we will define an L^2 cohomology $H^\bullet_{(2)}(\Gamma, \mathfrak{h}, V)$ and prove an analogue of Eichler-Shimura isomorphism. This result is not new, but we give a full proof in the lack of reference.

4.1 L^2 Cohomology

Before defining an L^2 cohomology, we have to define an appropriate inner product on the coefficient system. For a vector space V, let $\rho\colon \mathrm{SL}_2(\mathbf{R}) \longrightarrow \mathrm{Aut}(V)$ be a finite dimensional representation, and let $\langle \cdot, \cdot \rangle_0$ denote an inner product on V that satisfies for all $v, w \in V$,

$$\langle \rho(k)v, \rho(k)w \rangle_0 = \langle v, w \rangle_0 \tag{4.1.1}$$

for any $k \in \mathrm{SO}(2) = \mathrm{Stab}_{\mathrm{SL}_2(\mathbf{R})}(i)$. Thus the inner product is invariant by SO(2). We notice that $\mathbf{C}^2$ with the standard representation and dot product satisfies this condition, and so does $V_k = \mathrm{Sym}^{k-2}(\mathbf{C}^2)$ with the standard induced representation and the induced inner product. We define

$$\langle v, w \rangle_\tau = \left\langle \rho(g^{-1}) \cdot v, \rho(g^{-1}) \cdot w \right\rangle_0 \tag{4.1.2}$$

where $g \in \mathrm{SL}_2(\mathbf{R})$ such that $\tau = g \cdot i$.

Proposition 4.1.1. $\langle \cdot, \cdot \rangle_\tau$ is well-defined, independent of the choice of g.

Proof. Suppose both g and $\tilde{g}$ takes i to τ, then we have

$$g \cdot i = \tilde{g} \cdot i \ ,$$

$$\tilde{g}^{-1} g \cdot i = i \ ,$$

and thus

$$\tilde{g}^{-1} g \in \mathrm{Stab}_{\mathrm{SL}_2(\mathbf{R})}(i) \ .$$

We compute

$$\begin{aligned}
\langle v, w \rangle_\tau &= \left\langle \rho(\tilde{g}^{-1}) v, \rho(\tilde{g}^{-1}) w \right\rangle_0 \\
&= \left\langle \rho(\tilde{g}^{-1} g g^{-1}) v, \rho(\tilde{g}^{-1} g g^{-1}) w \right\rangle_0 \\
&= \left\langle \rho(\tilde{g}^{-1} g) \rho(g^{-1}) v, \rho(\tilde{g}^{-1} g) \rho(g^{-1}) w \right\rangle_0 \ .
\end{aligned}$$

Since $\tilde{g}^{-1} g \in \mathrm{Stab}_{\mathrm{SL}_2(\mathbf{R})}(i) = \mathrm{SO}(2)$, we have by equation (4.1.1),

$$\langle v, w \rangle_\tau = \left\langle \rho(g^{-1}) v, \rho(g^{-1}) w \right\rangle_0 \ . \qquad \square$$

For $\eta, \xi \in A^\bullet(\Gamma, h, V)$, denote $\langle \eta, \xi \rangle_\tau$ the point-wise inner product induced by the Riemannian metric $y^{-2}(dx^2 + dy^2)$ on $d\tau$ and $\langle \cdot, \cdot \rangle_\tau$ on V. Then define the L^2 inner product of the differential forms as follows:

$$\langle \eta, \xi \rangle_{L^2} = \int_D \langle \eta, \xi \rangle_\tau \frac{dx\, dy}{y^2}$$

where $D \in \mathfrak{h}$ is a fundamental domain of $\Gamma \setminus \mathfrak{h}$. If $\langle \eta, \eta \rangle_{L^2}$ is finite, we say η is L^2. Let $L^2 A^\bullet(\Gamma, \mathfrak{h}, V)$ denote the space of all L^2 i-forms, then $\langle \cdot, \cdot \rangle_{L^2}$ defines an inner product on $L^2 A^\bullet(\Gamma, \mathfrak{h}, V)$.

Remark. We always use $\langle \cdot, \cdot \rangle_{L^2}$ and $\|\cdot\|_{L^2}^2$ to denote L^2 norms while $\langle \cdot, \cdot \rangle_\tau$ and $\|\cdot\|_\tau^2$ to denote point-wise norms.

Proposition 4.1.2. This L^2 inner product is unchanged if one replaces D by $\gamma(D)$ for any $\gamma \in \Gamma$.

Proof. Note that the volume form $\dfrac{dx\,dy}{y^2}$ is invariant under $\gamma \in \Gamma$. We have

$$\begin{aligned}
\int_{\gamma(D)} \langle \eta, \xi \rangle_\tau \frac{dx\,dy}{y^2} &= \int_D \left\langle L_\gamma^*(\eta), L_\gamma^*(\xi) \right\rangle_{\gamma(\tau)} \frac{dx\,dy}{y^2} \\
&= \int_D \langle \rho(\gamma)\eta, \rho(\gamma)\xi \rangle_{\gamma(\tau)} \frac{dx\,dy}{y^2} && \text{by equation (3.1.1)} \\
&= \int_D \langle \rho(\gamma^{-1})\rho(\gamma)\eta, \rho(\gamma^{-1})\rho(\gamma)\xi \rangle_\tau \frac{dx\,dy}{y^2} && \text{by equation (4.1.2)} \\
&= \int_D \langle \eta, \xi \rangle_\tau \frac{dx\,dy}{y^2} .
\end{aligned}$$

□

We define the L^2 complex by

$$A_{(2)}^\bullet(\Gamma, \mathfrak{h}, V) = \left\{ \phi \in A^\bullet(\Gamma, \mathfrak{h}, V) \mid \|\phi\|_{L^2}^2 < \infty \ \&\ \|d\phi\|_{L^2}^2 < \infty \right\} , \tag{4.1.3}$$

and we define the L^2 cohomology

$$H_{(2)}^i(\Gamma, \mathfrak{h}, V) = \frac{\ker \Big(d\colon A_{(2)}^i(\Gamma, \mathfrak{h}, V) \longrightarrow A_{(2)}^{i+1}(\Gamma, \mathfrak{h}, V) \Big)}{\operatorname{Im} \Big(d\colon A_{(2)}^{i-1}(\Gamma, \mathfrak{h}, V) \longrightarrow A_{(2)}^i(\Gamma, \mathfrak{h}, V) \Big)} .$$

4.2 An L^2 Eichler-Shimura Isomorphism

The original Eichler-Shimura Isomorphism ([24]) establishes an isomorphism between the space of cusp forms of weight k (holomorphic and anti-holomorphic) for an arithmetic subgroup $\Gamma \in \mathrm{SL}_2(\mathbf{Z})$ and a cohomology group based on a certain group cocycles. We will give a version of this isomorphism for L^2 cohomology $H_{(2)}^1(\Gamma, \mathfrak{h}, V_k)$.

Let $V_k = \mathrm{Sym}^{k-2}(\mathbf{C}^2)$ as a representation of $\mathrm{SL}_2(\mathbf{R})$. This gives rise to a local system $\mathbf{V}_k$ on $\Gamma \setminus \mathfrak{h}$ which can be represented as

$$(\mathfrak{h} \times V_k) \big/ \big((\tau, v) \sim (\gamma\tau, \rho(\gamma)v)\big)$$

where $(\tau, v) \in \mathfrak{h} \times V_k$. Let $M_k(\Gamma)$ be the space of modular forms of weight k and $S_k(\Gamma) \subset M_k(\Gamma)$ be the subspace of cuspidal modular forms. We now state the L^2 Eichler-Shimura isomorphism.

Theorem 4.2.1 (L^2 Eichler-Shimura)**.**

$$H^1_{(2)}\big(\Gamma, \mathfrak{h}, V_k\big) \cong S_k(\Gamma) \bigoplus \overline{S_k(\Gamma)} \ .$$

In this section, we will give a proof using Hodge theory and classes of differential forms in the L^2 cohomology.

Consider the complex

$$0 \xrightarrow{d} A^0_{(2)}(\Gamma, \mathfrak{h}, V_k) \xrightarrow{d} A^1_{(2)}(\Gamma, \mathfrak{h}, V_k) \xrightarrow{d} A^2_{(2)}(\Gamma, \mathfrak{h}, V_k) \xrightarrow{d} 0$$

where d is the exterior derivative. Let $\delta \colon A^i(\Gamma, \mathfrak{h}, V_k) \longrightarrow A^{i-1}(\Gamma, \mathfrak{h}, V_k)$ be the formal adjoint of d, that is, $\delta\psi$ satisfies

$$\langle d\phi, \psi \rangle_{L^1} = \langle \phi, \delta\psi \rangle_{L^2}$$

for all $\phi \in A^{i-1}(\Gamma, \mathfrak{h}, V_k)$ such that $\mathrm{Supp}(\phi)$ is compact modulo Γ and $\psi \in A^i(\Gamma, \mathfrak{h}, V_k)$. Let d^* be the restriction of δ to the domain

$$\mathrm{Dom}(d^*) = \{\psi \in A^i(\Gamma, \mathfrak{h}, V_k) \mid \|\psi\|^2_{L^2} < \infty \ \& \ \|\delta\psi\|^2_{L^2} < \infty\} \ .$$

On the other hand, we also have

$$\mathrm{Dom}(d) = A^{i-1}_{(2)}(\Gamma, \mathfrak{h}, V_k) \ .$$

Since $\Gamma \setminus \mathfrak{h}$ is complete,

$$\langle d\phi, \psi \rangle_{L^1} = \langle \phi, d^* \psi \rangle_{L^2}$$

is true for all $\phi \in \mathrm{Dom}(d)$ and $\psi \in \mathrm{Dom}(d^*)$. If $H^1_{(2)}(\Gamma, \mathfrak{h}, V_k)$ is finite dimensional, by proposition (1.11) in [33] and Hodge theory, we have

$$H^1_{(2)}(\Gamma, \mathfrak{h}, V) \cong \ker d \cap \ker d^* \ .$$

However, since $\Gamma \setminus \mathfrak{h}$ is complete,

$$\ker d \cap \ker d^* = \{\phi \in L^2 A^1(\Gamma, \mathfrak{h}, V) \mid \Delta\phi = 0\}$$

by section 3.2 in [7] where $\Delta = dd^* + d^*d$ is the Laplacian. Write

$$\begin{aligned} A^2(\Gamma, \mathfrak{h}, V) &= A^{1,1}(\Gamma, \mathfrak{h}, V) \\ A^1(\Gamma, \mathfrak{h}, V) &= A^{1,0}(\Gamma, \mathfrak{h}, V) \oplus A^{0,1}(\Gamma, \mathfrak{h}, V) \\ A^0(\Gamma, \mathfrak{h}, V) &= A^{0,0}(\Gamma, \mathfrak{h}, V) \end{aligned}$$

where forms in $A^{1,0}(\Gamma, \mathfrak{h}, V)$ (respectively $A^{0,1}(\Gamma, \mathfrak{h}, V)$) are multiples of $d\tau$ (respectively $d\bar{\tau}$). Write $d = \partial + \bar{\partial}$ where

$$\partial \colon A^{i,j}(\Gamma, \mathfrak{h}, V) \longrightarrow A^{i+1,j}(\Gamma, \mathfrak{h}, V) \ , \quad \bar{\partial} \colon A^{i,j}(\Gamma, \mathfrak{h}, V) \longrightarrow A^{i,j+1}(\Gamma, \mathfrak{h}, V) \ .$$

Then formally (ignoring domains),

$$\begin{aligned} \Delta &= dd^* + d^*d \\ &= (\partial + \bar{\partial})(\partial^* + \bar{\partial}^*) + (\partial^* + \bar{\partial}^*)(\partial + \bar{\partial}) \\ &= (\partial\partial^* + \bar{\partial}^*\bar{\partial}) + (\bar{\partial}\bar{\partial}^* + \partial^*\partial) \\ &= \Delta^{1,0} \oplus \Delta^{0,1} \end{aligned}$$

where the third step is obtained through Kähler Identities [28]. Thus

$$\begin{aligned} H^1_{(2)}(\Gamma, \mathfrak{h}, V) &= \{\phi \in A^{1,0}_{(2)}(\Gamma, \mathfrak{h}, V) \mid \Delta^{1,0}\phi = 0\} \oplus \{\phi \in A^{0,1}_{(2)}(\Gamma, \mathfrak{h}, V) \mid \Delta^{0,1}\phi = 0\} \\ &= \{\phi \in A^{1,0}_{(2)}(\Gamma, \mathfrak{h}, V) \mid d\phi = d^*\phi = 0\} \oplus \{\phi \in A^{0,1}_{(2)}(\Gamma, \mathfrak{h}, V) \mid d\phi = d^*\phi = 0\} \ . \end{aligned}$$

However, $d|_{A^{1,0}} = \bar{\partial}$ (respectively $d|_{A^{0,1}} = \partial$) and $d^*|_{A^{1,0}} = \partial^*$ (respectively $d^*|_{A^{0,1}} = \bar{\partial}^*$). So

$$H^1_{(2)}(\Gamma, \mathfrak{h}, V) = \{\phi \in A^{1,0}_{(2)}(\Gamma, \mathfrak{h}, V) \mid \bar{\partial}\phi = \partial^*\phi = 0\} \oplus \{\phi \in A^{0,1}_{(2)}(\Gamma, \mathfrak{h}, V) \mid \partial\phi = \bar{\partial}^*\phi = 0\}.$$

Then the isomorphism becomes

$$\begin{aligned} S_k(\Gamma) &\longrightarrow \{\phi \in A^{1,0}_{(2)}(\Gamma, \mathfrak{h}, V) \mid \bar{\partial}\phi = \partial^*\phi = 0\} \\ f &\mapsto \phi_f \ , \end{aligned}$$

and

$$\begin{aligned} \overline{S_k(\Gamma)} &\longrightarrow \{\phi \in A^{0,1}_{(2)}(\Gamma, \mathfrak{h}, V) \mid \partial\phi = \bar{\partial}^*\phi = 0\} \\ \bar{g} &\mapsto \bar{\phi}_g \ . \end{aligned}$$

where $\phi_f = f \begin{pmatrix} \tau \\ 1 \end{pmatrix}^{k-2} d\tau$. We know from section 3.4 that ϕ_f represents a class in $H^1(\Gamma, \mathfrak{h}, V_k)$. The cuspidal condition on f also satisfies the L^2 condition, and thus ϕ_f is also a class in $H^1_{(2)}(\Gamma, \mathfrak{h}, V_k)$. For the remaining of the section, we will show that the above maps are injective and surjective, thus giving us the isomorphism in Theorem 4.2.1.

First we will show that $\phi_f \in \ker(\partial^*)$.

Example 4.2.2. Let $v, w \in \mathbf{C}^2$, and $\tau = g \cdot i$ for some $g \in \mathrm{SL}_2(\mathbf{R})$. The inner product on the standard representation of $\mathbf{C}^2$ is given by

$$\langle v, w \rangle_0 = v^T \bar{w} \ .$$

Then for $v^{k-2}, w^{k-2} \in \mathbf{V}_k$, we have

$$\begin{aligned} \left\langle v^{k-2}, w^{k-2} \right\rangle_\tau &= \left\langle \rho(g^{-1}) \cdot v^{k-2}, \rho(g^{-1}) \cdot w^{k-2} \right\rangle_0 \\ &= \left(\left(\rho(g^{-1}) \cdot v\right)^T \cdot \rho(g^{-1}) \cdot \bar{w} \right)^{k-2} \\ &= \left(v^T \cdot (g^{-1})^T \cdot g^{-1} \cdot \bar{w} \right)^{k-2} \ . \end{aligned}$$

If we write $\tau = x + iy$, then $g = \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix}$ and thus

$$\begin{aligned}(g^{-1})^T \cdot g^{-1} &= \begin{pmatrix} y^{-1/2} & -xy^{-1/2} \\ 0 & y^{1/2} \end{pmatrix}^T \cdot \begin{pmatrix} y^{-1/2} & -xy^{-1/2} \\ 0 & y^{1/2} \end{pmatrix} \\ &= y^{-1} \begin{pmatrix} 1 & -x \\ -x & x^2 + y^2 \end{pmatrix} .\end{aligned}$$

In the case where $v = w = \begin{pmatrix} \tau \\ 1 \end{pmatrix}$, we have

$$\begin{aligned}\left\| \begin{pmatrix} \tau \\ 1 \end{pmatrix}^{k-2} \right\|_\tau^2 &= \left\langle \begin{pmatrix} \tau \\ 1 \end{pmatrix}^{k-2}, \begin{pmatrix} \tau \\ 1 \end{pmatrix}^{k-2} \right\rangle_\tau \\ &= \left(\begin{pmatrix} \tau & 1 \end{pmatrix} \cdot (g^{-1})^T \cdot g^{-1} \cdot \begin{pmatrix} \bar{\tau} \\ 1 \end{pmatrix} \right)^{k-2} \\ &= (2y)^{k-2} .\end{aligned}$$

To find $d\tau$, recall that for $\tau \in \mathbf{C}, \tau = x + iy$, we have

$$\frac{\partial}{\partial \tau} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big) , \qquad \frac{\partial}{\partial \bar{\tau}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big) .$$

Therefore

$$\begin{aligned}\left\| \frac{\partial}{\partial \tau} \right\|^2 &= \Big\langle \frac{\partial}{\partial \tau}, \frac{\partial}{\partial \bar{\tau}} \Big\rangle \\ &= \frac{1}{4}\Big\langle \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}, \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \Big\rangle \\ &= \frac{1}{4}\Big(\left\| \frac{\partial}{\partial x} \right\|^2 + \left\| \frac{\partial}{\partial y} \right\|^2 \Big) .\end{aligned}$$

According to the Riemannian metric, the first fundamental form can be written as

$$ds^2 = \frac{dx^2 + dy^2}{y^2} .$$

Thus $\left\|\frac{\partial}{\partial x}\right\|^2 = \left\|\frac{\partial}{\partial y}\right\|^2 = \frac{1}{y^2}$. Therefore, $\left\|\frac{\partial}{\partial \tau}\right\|^2 = \frac{1}{2}\cdot\frac{1}{y^2}$, and this gives us as the dual,

$$\|d\tau\|^2 = 2y^2.$$

Now to show that $\phi_f \in \ker(\partial^*)$, we apply lemma 4.2 from [8]. A 0-form in $A^0_{(2)}(\Gamma, \mathfrak{h}, V_k)$ can be expressed as

$$\sum_{i=1}^{s} g_i(\tau) v_i^{k-2}$$

where $g_i(\tau)$ are cusp forms in S_k, integer s is a finite and non-zero vectors $v_1, \cdots, v_s \in \mathbf{C}^2$. Write f and g in short for $f(\tau)$ and $g(\tau)$, and let w denote $\begin{pmatrix}\tau \\ 1\end{pmatrix}$, we have

$$\begin{aligned}\left\langle \partial\left(\sum_{i=1}^{s} g_i v_i^{k-2}\right), \phi_f \right\rangle_{L^2} &= \int_D \left\langle \frac{\partial}{\partial\tau}\left(\sum_{i=1}^{s} g_i v_i^{k-2}\right) d\tau, f w^{k-2} d\tau \right\rangle_\tau \frac{dx\,dy}{y^2} \\ &= \int_D \left(\frac{\partial}{\partial\tau}\left(\sum_{i=1}^{s} g_i v_i^{k-2}\right)\right)^T Q \cdot \bar{f}\bar{w}^{k-2} 2y^2 \frac{dxdy}{y^2}\ .\end{aligned}$$

Here Q is defined by

$$\begin{aligned}(v_1 v_2 \cdots v_{k-2})^T Q\overline{(w_1 w_2 \cdots w_{k-2})} &= \langle v_1 v_2 \cdots v_{k-2}, w_1 w_2 \cdots w_{k-2}\rangle_\tau \\ &= \prod_{i=1}^{k-2} \langle v_i, w_i\rangle_\tau \\ &= \prod_{i=1}^{k-2} v_i^T \cdot q \cdot \bar{w}\end{aligned}$$

where $q = \left(g^{-1}\right)^T \cdot g^{-1} = y^{-1}\begin{pmatrix}1 & -x \\ -x & x^2+y^2\end{pmatrix}$.

Using integration by parts and quotient rule,

$$
\begin{aligned}
\left\langle \partial \left(\sum_{i=1}^{s} g_i v_i^{k-2} \right), \phi_f \right\rangle_{L^2} &= -2 \int_D \left(\sum_{i=1}^{s} g_i v_i^{k-2} \right)^T \frac{\partial}{\partial \tau} \left(Q \cdot \bar{f} \bar{w}^{k-2} \right) dxdy \\
&= -2 \int_D \left(\sum_{i=1}^{s} g_i v_i^{k-2} \right)^T \frac{\partial Q}{\partial \tau} \cdot \bar{f} \bar{w}^{k-2} dxdy \\
&\quad - 2 \int_D \left(\sum_{i=1}^{s} g_i v_i^{k-2} \right)^T Q \cdot \frac{\partial \bar{f}}{\partial \tau} \cdot \bar{w}^{k-2} dxdy \\
&\quad - 2 \int_D \left(\sum_{i=1}^{s} g_i v_i^{k-2} \right)^T Q \cdot \bar{f} \cdot \frac{\partial \left(\bar{w}^{k-2} \right)}{\partial \tau} dxdy \ .
\end{aligned}
$$

Since f and w are holomorphic, $\frac{\partial \bar{f}}{\partial \tau} = \frac{\partial \left(\bar{w}^{k-2} \right)}{\partial \tau} = 0$. To compute $\frac{\partial Q}{\partial \tau}$, we have

$$
Q = \underbrace{qq \cdots q}_{k-2 \text{ times}} \ .
$$

Thus

$$
\frac{\partial Q}{\partial \tau} = \sum_{j=1}^{k-2} q \cdots \underbrace{\frac{\partial q}{\partial \tau}}_{j^{\text{th}} \text{ place}} \cdots q \ .
$$

The symmetric powers of q is

$$
\begin{aligned}
\mathrm{Sym}^{k-2}(q)(v_j \cdots v_j, w \cdots w) &= \sum_{\sigma \in S_{k-2}} \prod_{l=1}^{k-2} \left\langle v_j, w_{\sigma(1)} \right\rangle_\tau \\
&= (k-2)!(v_j^T q \bar{w})^{k-2} \ ,
\end{aligned}
$$

therefore we have

$$\begin{aligned}
\left\langle \partial\left(\sum_{i=1}^{s} g_i v_i^{k-2}\right), \phi_f \right\rangle_{L^2} &= -2\int_D \left(\sum_{i=1}^{s} g_i v_i^{k-2}\right)^T \frac{\partial Q}{\partial \tau}\cdot \bar{f}\bar{w}^{k-2} dxdy \\
&= -2\int_D \left(\sum_{i=1}^{s} g_i v_i^{k-2}\right)^T \left(\sum_j q\cdots \frac{\partial q}{\partial \tau}\cdots q\right)\cdot \bar{f}\bar{w}^{k-2} dxdy \\
&= -2(k-2)!\int_D \sum_{i=1}^{s} \left(g_i \bar{f}(k-2)(v_i^T q\bar{w})^{k-2}\cdot v_i^T \frac{\partial q}{\partial \tau}\bar{w}\right)\, dxdy \\
&= -2(k-2)\int_D \sum_{i=1}^{s} \left\langle g_i v_i^{k-2}, f w^{k-3}\cdot\left(\bar{q}^{-1}\frac{\partial q}{\partial \bar{\tau}} w\right)\right\rangle_\tau \, dxdy
\end{aligned}$$

Here we have $\bar{q}^{-1} = q^{-1} = y^{-1}\begin{pmatrix} x^2+y^2 & x \\ x & 1\end{pmatrix}$ and

$$\begin{aligned}
\frac{\partial q}{\partial \bar{\tau}} &= \frac{1}{2}\left(\frac{\partial q}{\partial x} + i\frac{\partial q}{\partial y}\right) \\
&= \frac{1}{2}\left[\begin{pmatrix} 0 & -y^{-1} \\ -y^{-1} & 2xy^{-1}\end{pmatrix} + i\begin{pmatrix} -y^{-2} & xy^{-2} \\ xy^{-2} & 1-x^2y^{-2}\end{pmatrix}\right] \\
&= \frac{1}{2}\begin{pmatrix} -iy^{-2} & -y^{-1}+ixy^{-2} \\ -y^{-1}+ixy^{-2} & 2xy^{-1}+i(1-x^2y^{-2})\end{pmatrix} .
\end{aligned}$$

Thus

$$\begin{aligned}
\bar{q}^{-1}\cdot\frac{\partial q}{\partial \bar{\tau}} w &= \frac{1}{2y}\begin{pmatrix} \frac{-(x+iy)}{y} & \frac{(x+iy)^2}{y} \\ -\frac{1}{y} & \frac{(x+iy)}{y}\end{pmatrix}\begin{pmatrix}\tau \\ 1\end{pmatrix} \\
&= \frac{1}{2y^2}\begin{pmatrix} -\tau & \tau^2 \\ -1 & \tau\end{pmatrix}\begin{pmatrix}\tau \\ 1\end{pmatrix} \\
&= 0 .
\end{aligned}$$

Therefore,

$$\left\langle \left(\sum_{i=1}^{s} g_i v_i^{k-2} \right), \partial^*(\phi_f) \right\rangle_{L^2} = \left\langle \partial \left(\sum_{i=1}^{s} g_i v_i^{k-2} \right), \phi_f \right\rangle_{L^2} = 0$$

for any $\sum_{i=1}^{s} g_i v_i^{k-2} \in A^0_{(2)}(\Gamma, \mathfrak{h}, V_k)$. Thus $\phi_f = f \begin{pmatrix} \tau \\ 1 \end{pmatrix}^{k-2} d\tau \in \ker(\partial^*)$.

Next we will show that $\phi_f \in \ker(\bar{\partial})$. This is a direct result from

$$\bar{\partial}\phi_f = \frac{\partial \phi_f}{\partial \bar{\tau}} d\bar{\tau} = \frac{\partial}{\partial \bar{\tau}} \left(f(\tau) \begin{pmatrix} \tau \\ 1 \end{pmatrix} \right) d\tau \wedge d\bar{\tau} \ .$$

Since both $f(\tau)$ and $\begin{pmatrix} \tau \\ 1 \end{pmatrix}$ are holomorphic, $\frac{\partial}{\partial \bar{\tau}} \left(f(\tau) \begin{pmatrix} \tau \\ 1 \end{pmatrix} \right) = 0$.

Similarly, for an antiholomorphic cusp form $\bar{g} \in \overline{S_k(\Gamma)}$, we can construct a 1-form $\bar{\phi_g} = \bar{g} \begin{pmatrix} \bar{\tau} \\ 1 \end{pmatrix} d\bar{\tau}$ that lies in $\ker(\Delta)$. The result comes from negating the form for $S_k(\Gamma)$, and we obtain that

$$\bar{\phi_g} \in \ker(\partial) \cap \ker(\bar{\partial}^*)$$

which implies

$$\bar{\phi_g} \in \ker(\Delta) \ .$$

Therefore we have constructed another injective map

$$\overline{S_k(\Gamma)} \longrightarrow H^1_{(2)}(\Gamma, \mathfrak{h}, V_k) \ ,$$

and together, we have shown that the map

$$S_k(\Gamma) \bigoplus \overline{S_k(\Gamma)} \longrightarrow H^1_{(2)}(\Gamma, \mathfrak{h}, V_k)$$

is injective.

Next, we will show the map is also surjective to establish the isomorphism between $S_k(\Gamma) \bigoplus \overline{S_k(\Gamma)}$ and $H^1_{(2)}(\Gamma, \mathfrak{h}, V_k)$. First, we note the following claim.

Remark. Let η denote a generic 1-form in $A^1_{(2)}$ with the natural decomposition into $\eta = \eta^{1,0} \oplus \eta^{0,1}$. Then $\eta^{1,0} \in A^1_{(2)}$ and $\eta^{0,1} \in A^1_{(2)}$.

We will first consider holomorphic forms in $H^1_{(2)}(\Gamma, \mathfrak{h}, V_k)$ and show that the map

$$S_k(\Gamma) \longrightarrow H^1_{(2)}(\Gamma, \mathfrak{h}, V_k)_{holom.}$$

is surjective. The same computation can be easily carried over to antiholomorphic forms in $H^1_{(2)}(\Gamma, \mathfrak{h}, V_k)$, thus establishing our desired surjectivity.

By [8], a general 1-form $\eta \in H^1_{(2)}(\Gamma, \mathfrak{h}, V_k)$ where $V_k = \mathrm{Sym}^{k-2}(\mathbf{C}^2)$ can be written as

$$\eta = \sum_{i=1}^{s} f_i(\tau)\; w_i^{k-2}\; d\tau$$

where $f_i(\tau)\colon \mathfrak{h} \longrightarrow \mathbf{C}$, $w_i = \begin{pmatrix} w_{i1}(\tau) \\ w_{i2}(\tau) \end{pmatrix} \in \mathbf{C}^2$ where $w_{i1}, w_{i2}\colon \mathfrak{h} \longrightarrow \mathbf{C}$ are holomorphic functions, and s is a finite number such that η satisfies

1. $L^*_\gamma(\eta) = \rho(\gamma)\eta$ for $\gamma \in \Gamma$.
2. the parabolic condition such that $\int_0^w \left(\iota_{\frac{\partial}{\partial x}} L^*_C \eta\right)(t, y)dt \in \mathrm{Im}\left(\rho(\gamma_c) - \mathrm{I}\right)$.
3. $\bar{\partial}\eta = 0$.
4. $\partial^*\eta = 0$.

Condition 3 implies that

$$0 = \bar{\partial}\left(\sum_{i=1}^{s} f_i \; w_i^{k-2} \; d\tau\right)$$

$$= \sum_{i=1}^{s} \bar{\partial}\left(f_i \; w_i^{k-2}\right) \; d\tau \; ,$$

since w_i are holomorphic,

$$0 = \sum_{i=1}^{s} \frac{\partial f_i}{\partial \bar{\tau}} \; w_i^{k-2} \; d\bar{\tau} \wedge d\tau \; .$$

We can choose a set of $\{w_1, w_2, \cdots, w_s\}$ such that $w_i(\tau)$ are pairwise linearly independent vectors, decreasing s if necessary, and by Corollary 4.4 of [8], $\{w_1^{k-2}, w_2^{k-2}, \cdots, w_s^{k-2}\}$ are linearly independent. Suppose $\dfrac{\partial f_i}{\partial \bar{\tau}} \neq 0$ at $x \in \mathfrak{h}$ for some i, then

$$0 = \sum_{i=1}^{t} \frac{\partial f_i}{\partial \bar{\tau}}(x) \; w_i^{k-2} \; d\bar{\tau} \wedge d\tau$$

implies

$$\frac{\partial f_i}{\partial \bar{\tau}}(x) = 0$$

which is a contradiction. Thus f_i are holomorphic for all i.

Next to analyze condition 4, let $\sum_{j=1}^{r} g_j v_j^{k-2}$. We start by finding a formula for ∂^*

on one of the term $fw^{k-2}d\tau$,

$$\begin{aligned}
\left\langle \sum_{j=1}^{r} g_j v_j^{k-2}, \partial^* \left(fw^{k-2}\right) \right\rangle_{L^2} &= \left\langle \partial \left(\sum_{j=1}^{r} g_j v_j^{k-2} \right), fw^{k-2} \right\rangle_{L^2} \\
&= \int_D \left\langle \sum_{j=1}^{r} \frac{\partial}{\partial \tau} \left(g_j v_j^{k-2}\right) d\tau, fw^{k-2}\; d\tau \right\rangle_{\tau} \frac{dx\,dy}{y^2} \\
&= 2\int_D \left\langle \sum_{j=1}^{r} \frac{\partial}{\partial \tau} \left(g_j v_j^{k-2}\right), fw^{k-2} \right\rangle_{\tau} dxdy \\
&= 2\int_D \sum_{j=1}^{r} \frac{\partial}{\partial \tau} \left(g_j v_j^{k-2}\right)^T Q \bar{f} \bar{w}^{k-2}\; dxdy\;,
\end{aligned}$$

since both f and w are holomorphic,

$$= -2\int_D \sum_{j=1}^{r} \left(g_j v_j^{k-2}\right)^T \frac{\partial Q}{\partial \tau} \bar{f} \bar{w}^{k-2}\; dxdy$$

Here again

$$\frac{\partial Q}{\partial \tau} = \sum_{j=1}^{k-2} q \cdots \underbrace{\frac{\partial q}{\partial \tau}}_{j^{\text{th}} \text{ place}} \cdots q$$

where

$$q = y^{-1} \begin{pmatrix} 1 & -x \\ -x & x^2+y^2 \end{pmatrix}\;,$$

and the symmetric powers of q is

$$\begin{aligned}
\mathrm{Sym}^{k-2}(q)(v_j \cdots v_j, w \cdots w) &= \sum_{\sigma \in S_{k-2}} \prod_{l=1}^{k-2} \left\langle v_j, w_{\sigma(1)} \right\rangle_{\tau} \\
&= (k-2)!(v_j^T q \bar{w})^{k-2}\;.
\end{aligned}$$

Thus the inner product becomes

$$= -2(k-2)! \int_D \sum_{j=1}^{r} \left(g_j \bar{f} (k-2) (v_j^T q \bar{w})^{k-2} \cdot v_i^T \frac{\partial q}{\partial \tau} \bar{w} \right) \, dxdy$$

$$= -2(k-2) \int_D \sum_{j=1}^{r} \left\langle g_j v_j^{k-2}, f w^{k-3} \cdot \left(\bar{q}^{-1} \frac{\partial q}{\partial \bar{\tau}} w \right) \right\rangle_\tau y^2 \, \frac{dx\, dy}{y^2}$$

$$= \left\langle \sum_{j=1}^{r} g_j v_j^{k-2}, -2(k-2) y^2 f w^{k-3} \cdot \left(\bar{q}^{-1} \frac{\partial q}{\partial \bar{\tau}} w \right) \right\rangle_{L^2} .$$

Since ∂^* is a linear operator, we have

$$\partial^* \left(\sum_{i=1}^{s} f_i \, w_i^{k-2} \, d\tau \right) = \sum_{i=1}^{s} \partial^* \left(f_i \, w_i^{k-2} \, d\tau \right)$$

$$= \sum_{i=1}^{s} -2(k-2) y^2 f_i w_i^{k-3} \cdot \left(\bar{q}^{-1} \frac{\partial q}{\partial \bar{\tau}} w_i \right) .$$

Now condition 4 implies

$$0 = \sum_{i=1}^{s} f_i w_i^{k-3} \cdot \left(\bar{q}^{-1} \frac{\partial q}{\partial \bar{\tau}} w_i \right) .$$

When $s = 1$, the condition above simplifies to

$$0 = f w^{k-3} \cdot \left(\bar{q}^{-1} \frac{\partial q}{\partial \bar{\tau}} w \right) . \tag{4.2.1}$$

At points τ where $f(\tau) \neq 0$ and $w(\tau) \neq 0$, (4.2.1) implies

$$w \in \ker \left(\bar{q}^{-1} \frac{\partial q}{\partial \bar{\tau}} \right)$$

$$\in \ker \left(\frac{1}{2y^2} \begin{pmatrix} -\tau & \tau^2 \\ -1 & \tau \end{pmatrix} \right)$$

$$= c(\tau) \begin{pmatrix} \tau \\ 1 \end{pmatrix}$$

where $c\colon \mathfrak{h} \to \mathbf{C}$, and since w is a holomorphic vector function, c is also holomorphic. Since $w(\tau) = 0$ only at finite points, $w(\tau) = c(\tau)\begin{pmatrix}\tau\\1\end{pmatrix}$ extends to all $\tau \in \mathfrak{h}$.

When $s \geq 2$, we have

$$0 = \sum_{i=1}^{s} f_i w_i^{k-3} \cdot \left(\bar{q}^{-1}\frac{\partial q}{\partial \bar{\tau}} w_i\right) \ . \tag{4.2.2}$$

We use the following proposition from tensor product.

Proposition 4.2.3. Let X and Y be normed spaces and let $\{x_1, x_2, \cdots, x_n\} \subset X, \{y_1, y_2, \cdots, y_n\} \subset Y$. If $\sum_{i=1}^{n} x_i \otimes y_i = 0$ and $\{x_1, x_2, \cdots, x_n\}$ is linearly independent in X, then $y_1, y_2, \cdots, y_n = 0$.

Proof. For normed linear spaces X, Y we can define the tensor product $x \otimes y\colon X^* \times Y^* \to \mathbf{F}$ for all $f \in X^*$ and all $g \in Y^*$ by:

$$(x \otimes y)(f, g) = f(x)g(y) \ .$$

Since $\sum_{i=1}^{n} x_i \otimes y_i = 0$ we have that

$$\begin{aligned}
0 \cdot (f, g) &= \left(\sum_{i=1}^{n} x_i \otimes y_i = 0\right)(f, g) \\
&= (x_1 \otimes y_1 + x_2 \otimes y_2 + \cdots + x_n \otimes y_n)(f, g) \\
&= f(x_1)g(y_1) + f(x_2)g(y_2) + \cdots + f(x_n)g(y_n) \\
&= f(g(y_1)x_1 + g(y_2)x_2 + \cdots + g(y_n)x_n) \\
&= f\left(\sum_{i=1}^{n} g(y_i)x_i\right) \ .
\end{aligned}$$

On the other hand, we have $0 \cdot (f, g) = f(0)g(0) = 0$ for all $f \in X^*$ and $g \in Y^*$. Thus for all $f \in X^*$ we have

$$f\left(\sum_{i=1}^{n} g(y_i)x_i\right) = 0$$

which implies

$$\sum_{i=1}^{n} g(y_i)x_i = 0 \ .$$

Since $\{x_1, x_2, ..., x_n\}$ is a linearly independent set, the above equation implies that $g(y_i) = 0$ for all $i = 1, \cdots, n$. Since this holds for all $g \in Y^*$, we conclude that $y_i = 0$ for all $i = 1, \cdots, n$. □

To apply the above proposition to symmetric tensor products in $\mathrm{Sym}^{k-2}(\mathbf{C}^2)$, we recall that $\mathrm{Sym}^{k-2}(\mathbf{C}^2)$ naturally embeds into $(\mathbf{C}^2)^{k-2}$ that preserves all the topological structure. Therefore the above proposition applies to $\mathrm{Sym}^{k-2}(\mathbf{C}^2)$ as well.

Since we have chosen w_i to be pairwise linearly independent, by Corollary 4.4 of [8] we have all the w_i^{k-3} in equation (4.2.2) are linearly independent in $\mathrm{Sym}^{k-3}(\mathbf{C}^2)$. By the above proposition, we then have

$$\bar{q}^{-1}\frac{\partial q}{\partial \bar{\tau}} w_i = 0 \quad \text{for all } i = 1, \cdots, s \ .$$

However, this implies

$$\begin{aligned} w_i &\in \ker\left(\bar{q}^{-1}\frac{\partial q}{\partial \bar{\tau}}\right) \\ &= c_i(\tau)\begin{pmatrix}\tau \\ 1\end{pmatrix} \end{aligned}$$

for some $c_i \colon \mathfrak{h} \to \mathbf{C}$. Therefore w_i are linearly dependent, contradicting our claim that they are pairwise linearly independent. Thus $s = 1$ and $\partial^*(\eta)$ implies $w = c(\tau)\begin{pmatrix}\tau \\ 1\end{pmatrix}$.

We can further combine $c(\tau)$ with $f(\tau)$. Thus from condition (4) we have

$$\eta = f(\tau) \begin{pmatrix} \tau \\ 1 \end{pmatrix}^{k-2} d\tau \ .$$

Now condition (1) provides us the rule in working in the upper half plane, and condition (2) implies η vanishes at cusps. Therefore we have $f \in S_k(\Gamma)$, a cusp form of weight k. We thus have shown that the map

$$S_k(\Gamma) \longrightarrow H^1_{(2)}(\Gamma, \mathfrak{h}, V_k)_{holom.}$$

is surjective. The same analysis can be carried over to anti-holomorphic cusp forms in $S_k\bar{(}\Gamma)$, and we can show that the map

$$\overline{S_k(\Gamma)} \longrightarrow H^1_{(2)}(\Gamma, \mathfrak{h}, V_k)_{anti-holom.}$$

is also surjective. Therefore we have established the Eichler-Shimura isomorphism in Theorem 4.2.1.

4.3 How Does Eichler-Shimura Isomorphism Respect Hecke Operators on $H^1_{(2)}(\Gamma_0(N), \mathfrak{h}, V)$

In this section we will show that each of the maps π_g^*, g^*, π_* defined in (3.2.1) preserve the L^2 condition of $A^\bullet_{(2)}(\Gamma_0(N), \mathfrak{h}, V)$ as outlined in equation 4.1.3. Thus the Hecke operator $T_p = \pi_* \circ g^* \circ \pi_g^*$ preserves $A^\bullet_{(2)}(\Gamma_0(N), \mathfrak{h}, V)$. It further induced an action on $H^1_{(2)}(\Gamma_0(N), \mathfrak{h}, V)$ since T_p commutes with the differentials d.

Proposition 4.3.1. If $\phi \in A^\bullet_{(2)}(\Gamma_0(N), \mathfrak{h}, V)$, then ϕ naturally pulls back to $\pi_g^*(\phi) = \phi \in A^\bullet_{(2)}(g\Gamma_0(pN)g^{-1}, \mathfrak{h}, V)$.

Proof. $\phi \in A^{\bullet}_{(2)}(\Gamma_0(N), \mathfrak{h}, V)$, thus we have

$$\|\phi\|^2_{L^2} = \int_D \|\phi\|^2_\tau \frac{dx\,dy}{y^2} < \infty \ ,$$

$$\text{and } \|d\phi\|^2_{L^2} = \int_D \|d\phi\|^2_\tau \frac{dx\,dy}{y^2} < \infty$$

where D is a fundamental domain of $\Gamma_0(N) \setminus \mathfrak{h}$. Again we have

$$\Gamma_0(N) = \coprod_i g\Gamma_0(pN)g^{-1}\gamma_i$$

where γ_i is a member of a set of coset representatives of $g\Gamma_0(pN)g^{-1} \setminus \Gamma_0(N)$. Let D_g denote a fundamental domain for $g\Gamma_0(pN)g^{-1}$, we then have

$$D_g = \coprod_i \gamma_i D \ ,$$

and the number of coset representatives is determined by $[\Gamma_0(N) \colon g\Gamma_0(pN)g^{-1}]$. We now compute

$$\begin{aligned}
\int_{D_g} \|\phi\|^2_\tau \frac{dx\,dy}{y^2} &= \int_{\coprod_i \gamma_i D} \|\phi\|^2_\tau \frac{dx\,dy}{y^2} \\
&= \sum_i \int_{\gamma_i D} \|\phi\|^2_\tau \frac{dx\,dy}{y^2} \\
&= \sum_i \int_D \left\|L^*_{\gamma_i}(\phi)\right\|^2_\tau \frac{dx\,dy}{y^2} \qquad \tfrac{dx\,dy}{y^2} \text{ is invaraint under } \Gamma_0(N), \\
&= \sum_i \|\rho(\gamma_i)\|^2_\tau \int_D \|(\phi)\|^2_\tau \frac{dx\,dy}{y^2} \quad \text{by equation 3.2.3,} \\
&< \infty \ .
\end{aligned}$$

Similarly we can show that $\int_{D_g} \|d\phi\|^2_\tau \frac{dx\,dy}{y^2} < \infty$. Thus $\pi^*_g(\phi) = \phi \in A^{\bullet}_{(2)}(g\Gamma_0(pN)g^{-1}, \mathfrak{h}, V)$. □

The next pullback map is again given by

$$g^* = \rho(g^{-1})L_g^*\colon A^\bullet(g\Gamma_0(pN)g^{-1}, \mathfrak{h}, V) \longrightarrow A^\bullet(\Gamma_0(pN), \mathfrak{h}, V) \ .$$

Proposition 4.3.2. If $\phi \in A^\bullet_{(2)}(g\Gamma_0(pN)g^{-1}, \mathfrak{h}, V)$, then $g^*(\phi) \in A^\bullet_{(2)}(\Gamma_0(pN), \mathfrak{h}, V)$.

Proof. $\phi \in A^\bullet_{(2)}(g\Gamma_0(pN)g^{-1}, \mathfrak{h}, V)$, thus we have

$$\|\phi\|^2_{L^2} = \int_D \|\phi\|^2_\tau \frac{dx\,dy}{y^2} < \infty \ ,$$

$$\text{and } \|d\phi\|^2_{L^2} = \int_D \|d\phi\|^2_\tau \frac{dx\,dy}{y^2} < \infty$$

where D is a fundamental domain of $g\Gamma_0(pN)g^{-1} \setminus \mathfrak{h}$. Let D_{pN} be a fundamental domain of $\Gamma_0(pN)$, we have

$$D_{pN} = g^{-1}D \ .$$

We now compute

$$\begin{aligned}
\int_{D_{pN}} \|g^*(\phi)\|^2_\tau \frac{dx\,dy}{y^2} &= \int_{g^{-1}D} \left\|\rho(g^{-1})L_g^*(\phi)\right\|^2_\tau \frac{dx\,dy}{y^2} \\
&= \int_D \left\|\rho(g^{-1})L_g^*L_{g^{-1}}^*(\phi)\right\|^2_\tau \frac{d(\frac{1}{p}x)d(\frac{1}{p}y)}{\left(\frac{1}{p}\right)^2 y^2} \quad \text{since } g^{-1} = \begin{pmatrix} \frac{1}{p} & 0 \\ 0 & 1 \end{pmatrix}, \\
&= \left\|\rho(g^{-1})\right\|^2_\tau \int_D \|\phi\|^2_\tau \frac{dx\,dy}{y^2} \\
&< \infty \ .
\end{aligned}$$

Similarly we can show that $\displaystyle\int_{D_{pN}} \|dg^*(\phi)\|^2_\tau \frac{dx\,dy}{y^2} < \infty$. Thus $\psi \in A^\bullet_{(2)}(\Gamma_0(pN), \mathfrak{h}, V)$.

□

The last map is given by the pushforward defined as before. For $\psi \in A^\bullet(\Gamma_0(pN), \mathfrak{h}, V)$,

$$\pi_*(\psi) = \sum_{\gamma \in \Gamma_0(pN)\setminus\Gamma_0(N)} \rho(\gamma^{-1})L_\gamma^*(\psi) \ .$$

Proposition 4.3.3. If $\psi \in A^{\bullet}_{(2)}(\Gamma_0(pN), \mathfrak{h}, V)$, then $\pi_*(\psi) \in A^{\bullet}_{(2)}(\Gamma_0(N), \mathfrak{h}, V)$.

Proof. $\psi \in A^{\bullet}_{(2)}(\Gamma_0(pN), \mathfrak{h}, V)$, thus we have

$$\|\psi\|^2_{L^2} = \int_D \|\psi\|^2_\tau \frac{dx\,dy}{y^2} < \infty \ ,$$

$$\text{and } \|d\psi\|^2_{L^2} = \int_D \|d\psi\|^2_\tau \frac{dx\,dy}{y^2} < \infty$$

where D is a fundamental domain of $\Gamma_0(pN) \setminus \mathfrak{h}$. Let D_N denote a fundamental domain of $\Gamma_0(N) \setminus \mathfrak{h}$, we have

$$D = \coprod_i \gamma_i D_N \ .$$

We can rewrite

$$\begin{aligned}
\int_D \|\psi\|^2_\tau \frac{dx\,dy}{y^2} &= \int_{\coprod_i \gamma_i D_N} \|\psi\|^2_\tau \frac{dx\,dy}{y^2} \\
&= \sum_i \int_{\gamma_i D_N} \|\psi\|^2_\tau \frac{dx\,dy}{y^2} \\
&= \sum_i \int_{D_N} \left\|L^*_{\gamma_i}(\psi)\right\|^2_\tau \frac{dx\,dy}{y^2} < \infty \ .
\end{aligned}$$

Act by $\left\|\rho(\gamma_i^{-1})\right\|^2_\tau$, we obtain

$$\begin{aligned}
\infty &> \sum_i \left\|\rho(\gamma_i^{-1})\right\|^2_\tau \int_{D_N} \left\|L^*_{\gamma_i}(\psi)\right\|^2_\tau \frac{dx\,dy}{y^2} \\
&\geq \int_{D_N} \left\|\sum_i \rho(\gamma_i^{-1}) L^*_{\gamma_i}(\psi)\right\|^2_\tau \frac{dx\,dy}{y^2} \\
&= \int_{D_N} \|\pi_*(\psi)\|^2_\tau \frac{dx\,dy}{y^2} \ .
\end{aligned}$$

Similarly we can show that $\int_{D_N} \|d\pi_*(\psi)\|^2_\tau \frac{dx\,dy}{y^2} < \infty$. Thus $\psi \in A^{\bullet}_{(2)}(\Gamma_0(N), \mathfrak{h}, V)$.

□

Recall that the 1-form $\phi_f = f(\tau)\begin{pmatrix}\tau\\1\end{pmatrix}^{k-2} d\tau$ where $f \in S_k(\Gamma)$ represents a class in $H^1_{(2)}(\Gamma, \mathfrak{h}, V_k)$. If we denote T_p as a Hecke operator, we have shown in section 3.4 that

$$T_p(\phi_f) = p^{2-k} T_p(f) \ .$$

Similar results hold for $\phi_{\bar{f}}$ as well. This shows that the Eichler-Shimura Isomorphism preserves the Hecke actions modulo p^{2-k}.

Chapter 5

Hecke Actions on Parabolic Cohomology

In this chapter, we want to build another cohomology from the one presented in section 3.1, and we will denote it as parabolic cohomology. This will give us some insight and motivation for the combinatorial cohomology that will be defined in the next chapter.

5.1 Parabolic Cohomology

By reduction theory for $\mathrm{SL}_2(\mathbf{Z})$, for all $y_0 > 0$ sufficiently large, the sets

$$\gamma\left(\{\tau \in \mathfrak{h} | \Im(\tau) \geq y_0\} \cup \{\infty\}\right)$$

are disjoint for all cosets $\gamma\,\mathrm{Stab}_{\mathrm{SL}_2(\mathbf{Z})}(\infty) \in \mathrm{SL}_2(\mathbf{Z})\,/\,\mathrm{Stab}_{\mathrm{SL}_2(\mathbf{Z})}(\infty)$. We fix such a y_0 once for all and denote it y_{suff}. We will also fix some $M > y_{suff}$ once for all.

To define a parabolic cohomology, we need the following definition.

Definition 5.1.1. Let Γc be a cusp in $\Gamma \setminus \mathfrak{h}^*$ where $c \in \mathbf{Q} \cup \{\infty\}$. Let $C \in \mathrm{SL}_2(\mathbf{Z})$ be such that $C \cdot \infty = c$. Define

$$V_\infty = \{\tau \in \mathfrak{h} | \Im(\tau) \geq M\} \cup \{\infty\} \subset W_\infty = \{\tau \in \mathfrak{h} | \Im(\tau) \geq y_{suff}\} \cup \{\infty\}$$

and

$$V_c = C \cdot V_\infty \subset W_c = C \cdot W_\infty \ .$$

Finally, let $U_{\Gamma c} = \Gamma \setminus \Gamma V_c$ be a horoball neighborhood of $\Gamma c \in \Gamma \setminus \mathfrak{h}^*$.

Now the same definition for $A^\bullet(\Gamma, \mathfrak{h}, V)$ applies when we replace $\mathfrak{h}$ by W_∞, and Γ by $\mathrm{Stab}_\Gamma(\infty)$. Before we define the parabolic cohomology, we want to use the following example for motivation.

Example 5.1.2. We will illustrate with the cohomology group of $\mathbf{Z} \setminus \mathbf{R}$ with coefficients in a vector space V, namely $H^\bullet(\mathbf{Z}, \mathbf{R}, V)$, where $\mathbf{Z}$ acts on $\mathbf{R}$ by translation by $w \in \mathbf{Z}$, and $\rho\colon \mathbf{Z} \mapsto V$ is a representation. This computation is fairly straightforward, but will provide the outline for future analysis.

Consider a 1-form $f(x)dx$ in $A^1(\mathbf{Z}, \mathbf{R}, V)$ where $f\colon \mathbf{R} \longrightarrow V$ is a holomorphic function. Define $L_k\colon \mathbf{R} \longrightarrow \mathbf{R}$ to be $L_k(x) = x + kw$. Thus it acts on V via the action of $\begin{pmatrix} 1 & kw \\ 0 & 1 \end{pmatrix}$. Hence we have

$$\begin{aligned} L_k^*(f(x)dx) &= f(x+kw)d(x+kw) \\ &= f(x+kw)dx \\ &= \rho\left(\begin{pmatrix} 1 & kw \\ 0 & 1 \end{pmatrix}\right) f(x)dx \end{aligned}$$

where the last step comes from condition (3.1.1). Thus we have $f(x+kw)dx = \rho\left(\begin{pmatrix} 1 & kw \\ 0 & 1 \end{pmatrix}\right) f(x)dx$ for all k, or simply $f(x+w) = \rho\left(\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}\right) f(x)$. We now can define

$$A^0(\mathbf{Z}, \mathbf{R}, V) = \left\{ g\colon \mathbf{R} \longrightarrow V \mid g(x+w) = \rho\left(\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}\right) g(x) \right\} ,$$

$$A^1(\mathbf{Z}, \mathbf{R}, V) = \left\{ f dx \mid f\colon \mathbf{R} \longrightarrow V \ \& \ f(x+w) = \rho\left(\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}\right) f(x) \right\} .$$

We then have the following chain complex,

$$0 \xrightarrow{d^{-1}} A^0(\mathbf{Z}, \mathbf{R}, V) \xrightarrow{d^0} A^1(\mathbf{Z}, \mathbf{R}, V) \xrightarrow{d^1} 0 .$$

Next we will compute $H^0(\mathbf{Z}, \mathbf{R}, V)$ and $H^1(\mathbf{Z}, \mathbf{R}, V)$ as the cohomology of other degrees is 0.

$$H^0(\mathbf{Z}, \mathbf{R}, V) = \frac{\ker(d^0)}{\mathrm{Im}(d^{-1})} = \text{constant functions } g \in A^0 = \ker\left(\rho\left(\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}\right) - \mathrm{I}\right) .$$

On the other hand, we have

$$H^1(\mathbf{Z}, \mathbf{R}, V) = \frac{\ker(d^1)}{\mathrm{Im}(d^0)} \ .$$

To find $H^1(\mathbf{Z}, \mathbf{R}, V)$, Suppose $fdx \in A^1(\mathbf{Z}, \mathbf{R}, V)$ and $fdx \in \mathrm{Im}(d^0)$, then $f(x)dx = d(g(x))$ where $g(x) \in A^0(\mathbf{Z}, \mathbf{R}, V)$ and satisfies

$$g(x+w) = \rho\left(\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}\right) g(x) \ . \tag{5.1.1}$$

Since $g(x)$ is an antiderivative of $f(x)$, it can be expressed as

$$\int_{x_0}^{x} f(t)dt + v_0$$

for some $x_0 \in \mathbf{R}$ and $v_0 \in V$. Thus for $f(x)dx \in \mathrm{Im}(d^0)$, equation (5.1.1) implies

$$\begin{aligned} \int_{x_0}^{x+w} f(t)dt + v_0 &= \int_{x_0}^{x} \rho\left(\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}\right) f(t)dt + \rho\left(\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}\right) v_0 \\ \Longleftrightarrow \int_{x_0}^{x+w} f(t)dt &= \int_{x_0}^{x} f(t+w)dt + \left(\rho\left(\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}\right) - \mathrm{I}\right) v_0 \\ \Longleftrightarrow \int_{x_0}^{x_0+w} f(t)dt &\in \mathrm{Im}\left(\rho\left(\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}\right) - \mathrm{I}\right) \end{aligned} \tag{5.1.2}$$

and the integral is independent of x_0. Thus to establish the following short exact sequence,

$$0 \longrightarrow \mathrm{Im}(d^0) \longrightarrow A^1(\mathbf{Z}, \mathbf{R}, V) \longrightarrow V/\,\mathrm{Im}\left(\rho\left(\left(\begin{smallmatrix} 1 & w \\ 0 & 1 \end{smallmatrix}\right)\right) - \mathrm{I}\right) \longrightarrow 0 \ ,$$

we only need to check that the map $A^1(\mathbf{Z}, \mathbf{R}, V) \longrightarrow \mathbf{V}/\,\mathrm{Im}\left(\rho\left(\left(\begin{smallmatrix} 1 & w \\ 0 & 1 \end{smallmatrix}\right)\right) - \mathrm{I}\right)$ is surjective. By setting $x_0 = 0$, and letting $f(x) = \frac{w-x}{w^2} \cdot v + \frac{x}{w^2}\rho\left(\left(\begin{smallmatrix} 1 & w \\ 0 & 1 \end{smallmatrix}\right)\right) v$ with $v \in V$, we can show that indeed

$$\begin{array}{ccc} A^1(\mathbf{Z}, \mathbf{R}, V) & \longrightarrow & V/\,\mathrm{Im}\left(\rho\left(\left(\begin{smallmatrix} 1 & w \\ 0 & 1 \end{smallmatrix}\right)\right) - \mathrm{I}\right) \\ f(x)dx & \mapsto & \displaystyle\int_0^w f(t)dt \end{array}$$

is surjective. Thus we have

$$H^1(\mathbf{Z},\mathbf{R},V) = \frac{\ker(d^1)}{\mathrm{Im}(d^0)} = \frac{A^1(\mathbf{Z},\mathbf{R},V)}{\mathrm{Im}\,(d^0)} = V/\,\mathrm{Im}\,(\rho\,((\begin{smallmatrix}1 & w\\ 0 & 1\end{smallmatrix})) - \mathrm{I})\,.$$

This provides us the structure to build the parabolic cohomology. First note that in a horoball neighborhood of each cusp $\Gamma c \in \Gamma \setminus \mathfrak{h}^*$, $A^\bullet(\mathrm{Stab}_\Gamma(c), W_c, V)$ and $A^\bullet(\mathbf{Z},\mathbf{R},V)$ give isomorphic cohomology since $U_{\Gamma c} \cong [M,\infty) \times S^1$ is homotopically equivalent to S^1. The parabolic condition implies 0 cohomology in degree 1 near the cusps. First we want to provide some notations that will be used throughout this section.

Notation 5.1.3. For every cusp $\Gamma c \in \Gamma \setminus \mathfrak{h}^*$, we have the following notations.

Notation	**Definition**
$\gamma_c \in \Gamma$	A generator of $\mathrm{Stab}_\Gamma(c)$.
$C \in \mathrm{SL}_2(\mathbf{Z})$	An element which sends ∞ to c .
$w_c \in \mathbf{Z}^{>0}$	The width of cusp Γc defined by $\begin{pmatrix}1 & \pm w_c\\ 0 & 1\end{pmatrix} = C^{-1}\gamma_c C \mod \pm\,\mathrm{I}_2$.

Remark. If it is necessary to indicate the group Γ, we write $\gamma_{c,\Gamma}$ and $w_{c,\Gamma}$.

Claim 5.1.4. w_c is independent of the choice of C.

Proof. Assume $C = C_1, C_2 \in \mathrm{SL}_2(\mathbf{Z})$ and both send ∞ to c. Then $C_2^{-1}C_1 \in \mathrm{Stab}_{\mathrm{SL}_2(\mathbf{Z})}(\infty)$. Say $C_2^{-1}C_1 = \begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix} \mod \pm\,\mathrm{I}_2$ for some integer h. We have

$$\begin{pmatrix}1 & \pm w_c\\ 0 & 1\end{pmatrix} = C_1^{-1}\gamma_c C_1 \mod \pm\,\mathrm{I}_2\ . \qquad (5.1.3)$$

Conjugate both side of equation (5.1.3) by $C_2^{-1}C_1$, we obtain

$$\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & \pm w_c\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}^{-1} = C_2^{-1}C_1C_1^{-1}\gamma_c C_1\left(C_2^{-1}C_1\right)^{-1} \mod \pm \mathrm{I}_2$$

$$\begin{pmatrix}1 & \pm w_c\\ 0 & 1\end{pmatrix} = C_2^{-1}\gamma_c C_2 \mod \pm \mathrm{I}_2 \ . \qquad \square$$

With these notations, we first define

$$A_P^0 = A^0(\Gamma, \mathfrak{h}, V).$$

Then for $\Gamma c \in \Gamma \setminus \mathfrak{h}^*$ any cusp, let $\iota_{\frac{\partial}{\partial x}}$ be the operator that isolates the coefficient of the dx term, we then define

$$A_P^1 = \Big\{\phi \in A^1(\Gamma, \mathfrak{h}, V) \mid \int_0^{w_c} \left(\iota_{\frac{\partial}{\partial x}} L_C^* \phi\right)(t, y)dt \in \mathrm{Im}(\rho\left(\gamma_c\right) - \mathrm{I}), \forall \text{ cusps } c \text{ and } y \geq M\Big\}, \tag{5.1.4}$$

and

$$A_P^2 = d(A_P^1) \ .$$

Claim 5.1.5. A_P^1 is independent of the choice of C.

Proof. Assume $C = C_1, C_2 \in \mathrm{SL}_2(\mathbf{Z})$ and both send ∞ to c. Then $C_1^{-1}C_2 \in \mathrm{Stab}_{\mathrm{SL}_2(\mathbf{Z})}(\infty)$. Say $C_1^{-1}C_2 = \begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix} \mod \pm \mathrm{I}_2$ for some integer h. Consider the following integral

$$\int_0^{w_c} \left(\iota_{\frac{\partial}{\partial x}} L_{C_2}^* \phi\right)(t, y)dt = \int_0^{w_c} \left(\iota_{\frac{\partial}{\partial x}} L_{C_1C_1^{-1}C_2}^* \phi\right)(t, y)dt$$

$$= \int_0^{w_c} \left(\iota_{\frac{\partial}{\partial x}} L_{C_1}^* \phi\right)(t + h, y)dt \quad \text{since } C_1^{-1}C_2 = \begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix} \ .$$

By a change of variable, we obtain

$$= \int_h^{h + w_c} \left(\iota_{\frac{\partial}{\partial x}} L_{C_1}^* \phi\right)(t, y)dt \ .$$

Now calculate

$$\begin{aligned}
&\int_0^{w_c} \left(\iota_{\frac{\partial}{\partial x}} L^*_{C_1}\phi\right)(t,y)dt - \int_0^{w_c} \left(\iota_{\frac{\partial}{\partial x}} L^*_{C_2}\phi\right)(t,y)dt \\
&= \int_0^{w_c} \left(\iota_{\frac{\partial}{\partial x}} L^*_{C_1}\phi\right)(t,y)dt - \int_h^{h+w_c} \left(\iota_{\frac{\partial}{\partial x}} L^*_{C_1}\phi\right)(t,y)dt \\
&= \int_0^{h} \left(\iota_{\frac{\partial}{\partial x}} L^*_{C_1}\phi\right)(t,y)dt + \int_{h+w_c}^{w_c} \left(\iota_{\frac{\partial}{\partial x}} L^*_{C_1}\phi\right)(t,y)dt \\
&= -\int_h^{0} \left(\iota_{\frac{\partial}{\partial x}} L^*_{C_1}\phi\right)(t,y)dt + \int_h^{0} \left(\iota_{\frac{\partial}{\partial x}} L^*_{C_1}\phi\right)(t+w_c,y)dt \\
&= -\int_h^{0} \left(\iota_{\frac{\partial}{\partial x}} L^*_{C_1}\phi\right)(t,y)dt + \rho(\gamma_c)\int_h^{0} \left(\iota_{\frac{\partial}{\partial x}} L^*_{C_1}\phi\right)(t,y)dt \\
&\in \mathrm{Im}\left(\rho(\gamma_c) - \mathrm{I}\right) \ .
\end{aligned}$$

Thus either integral belonging to $\mathrm{Im}\left(\rho(\gamma_c) - \mathrm{I}\right)$ implies the other does. □

Proposition 5.1.6. $d(A_P^0) \subset A_P^1$.

Proof. Suppose $f \in A_P^0$, we would like to show that the condition in (5.1.4) holds for df. It suffices to check that it holds at cusp ∞. Let w denote w_∞ and γ denote $\gamma_\infty = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$. Note that L^*_γ preserves dx and dy. Thus equation (5.1.1) holds for both dx term and dy term. Hence the condition in (5.1.4) follows from the same reasoning as in equation (5.1.2). □

Thus we have the following chain complex

$$0 \xrightarrow{d^{-1}} A_P^0(\Gamma, \mathfrak{h}, V) \xrightarrow{d^0} A_P^1(\Gamma, \mathfrak{h}, V) \xrightarrow{d^1} A_P^2(\Gamma, \mathfrak{h}, V) \xrightarrow{d^2} 0 \ ,$$

and we define *the parabolic cohomology* in degree 1 to be

$$H_P^1\big(\Gamma, \mathfrak{h}, V\big) = \frac{\ker(d^1)}{\mathrm{Im}(d^0)} \ .$$

5.2 Cohomology Class in $H^1_P\big(\Gamma, \mathfrak{h}, V_k\big)$ associated with Cusp Forms

Consider the vector-valued 1-form $\phi_f(\tau) = f(\tau)\begin{pmatrix}\tau \\ 1\end{pmatrix}^{k-2} d\tau$ where $f(\tau)$ a weight k cusp form in $S_k(\Gamma)$. We have seen from previous chapters that ϕ_f represents a class in $H^1(\Gamma, \mathfrak{h}, V_k)$ and $H^1_{(2)}(\Gamma, \mathfrak{h}, V_k)$. Here we want to show that ϕ_f also represents a class in $H^1_P\big(\Gamma, \mathfrak{h}, V_k\big)$.

We need to show that $\phi_f \in A^1_P(\Gamma, \mathfrak{h}, V_k)$. Since $\phi_f \in A^1(\Gamma, \mathfrak{h}, V_k)$ already, we will show it by showing ϕ_f satisfies the parabolic condition defined in equation (5.1.4). Recall for any cusp $\Gamma c \in \Gamma \setminus \mathfrak{h}^*$, we have defined γ_c, L_C and w_c in notation 5.1.3. We have the following proposition.

Proposition 5.2.1. If f is a cusp form with respect to Γ, then

$$\int_0^{w_c} \left(\iota_{\frac{\partial}{\partial x}} L_C^* \phi_f\right)(t, y) dt \in \operatorname{Im}(\rho\left(\gamma_c\right) - \mathrm{I})$$

for every cusp $\Gamma c \in \Gamma \setminus \mathfrak{h}^*$.

Remark. In view of equation (5.1.4), this implies $\phi_f \in A^1_P(\Gamma, \mathfrak{h}, V_k)$.

Before beginning the proof, we give the following lemma.

Lemma 5.2.2. If $\gamma = \begin{pmatrix}1 & w \\ 0 & 1\end{pmatrix}$, then

$$\operatorname{Im}\left(\rho\left(\gamma\right) - \mathrm{I}\right) = \operatorname{Span}\{e_1^{k-2}, e_1^{k-3}e_2, \cdots, e_1 e_2^{k-3}\} \ .$$

Proof. We consider a set of basis for $\operatorname{Sym}^{k-3}(\mathbf{C}^2)$, $\{e_1^{k-2}, e_1^{k-3}e_2, \cdots, e_2^{k-2}\}$ and apply $\operatorname{Im}\left(\rho\left(\gamma\right) - \mathrm{I}\right)$ on each basis element. Write K in short for $\operatorname{Im}\left(\rho\left(\gamma\right) - \mathrm{I}\right)$, We obtain

$$K \cdot e_1^{k-3}e_2 = e_1^{k-3}(we_1 + e_2) - e_1^{k-3}e_2 = -we_1^{k-2} \ .$$

Thus $e_1^{k-2} \in \operatorname{Im}(\rho(\gamma) - \mathrm{I})$. Similarly,

$$\begin{aligned} K \cdot e_1^{k-4} e_2^2 &= e_1^{k-4}(we_1 + e_2)^2 - e_1^{k-4} e_2^2 \\ &= e_1^{k-4}(w^2 e_1^2 + 2we_1 e_2 + e_2^2) - e_1^{k-4} e_2^2 \\ &= w^2 e_1^{k-2} + 2we_1^{k-3} e_2 \\ &\in \operatorname{Im}(\rho(\gamma) - \mathrm{I}) \ . \end{aligned}$$

Since $e_1^{k-2} \in \operatorname{Im}(\rho(\gamma) - \mathrm{I})$, thus $e_1^{k-3} e_2 \in \operatorname{Im}(\rho(\gamma) - \mathrm{I})$. Using induction, assume this is true for all basis $e_1^{k-2-(s-1)} e_2^{s-1}$ where s ranges from 1 to $k-2$, we get

$$\begin{aligned} K \cdot e_1^{k-2-s} e_2^s &= e_1^{k-2-s}(we_1 + e_2)^s - e_1^{k-2-s} e_2^s \\ &= e_1^{k-2-s} \left((we_1)^s + \binom{s}{1} (we_1)^{s-1} e_2 + \cdots + \binom{s}{s-1} we_1 e_2^{s-1} + e_2^s \right) \\ &\qquad - e_1^{k-2-s} e_2^s \\ &= w^s e_1^{k-2} + \binom{s}{1} w^{s-1} e_1^{k-3} e_2 + \cdots + \binom{s}{s-1} we_1^{k-1-s} e_2^{s-1} \\ &\in \operatorname{Im}(\rho(\gamma) - \mathrm{I}) \ . \end{aligned}$$

By the induction step we know that all the terms except the last term belong to $\operatorname{Im}(\rho(\gamma) - \mathrm{I})$. Thus $e_1^{k-1-s} e_2^{s-1} \in \operatorname{Im}(\rho(\gamma) - \mathrm{I})$ for s ranges from 1 to $k-2$. When $s = 0$, we have

$$K \cdot e_1^{k-2} = e_1^{k-2} - e_1^{k-2} = 0 \ .$$

Therefore the kernel of $\operatorname{Im}(\rho(\gamma) - \mathrm{I})$ is non-trivial, i.e. $\operatorname{Im}(\rho(\gamma) - \mathrm{I})$ does not have full rank in $\mathrm{Sym}^{k-2}(\mathbf{C}^2)$. On the other hand, $\{e_1^{k-2}, e_1^{k-3} e_2, \cdots, e_1 e_2^{k-3}\}$ all belong to $\operatorname{Im}(\rho(\gamma) - \mathrm{I})$. Thus we have shown the claim. □

We now will give the proof of the proposition.

Proof of Proposition 5.2.1. Letting $\{e_1, e_2\}$ be the standard basis of $\mathbf{C}^2$, we can write

$$\begin{aligned}\begin{pmatrix}\tau\\1\end{pmatrix}^{k-2} &= (\tau e_1 + e_2)^{k-2}\\ &= \tau^{k-2}e_1^{k-2} + \binom{k-2}{1}\tau^{k-3}e_1^{k-3}e_2 + \cdots + \binom{k-2}{k-3}\tau e_1 e_2^{k-3} + e_2^{k-2}\ .\end{aligned}$$

Let us first consider the case $c = \infty$, and simply write $w = w_\infty$ in short to denote the width of ∞, and $C = \mathrm{I}$. For $\phi_f = f(\tau)\begin{pmatrix}\tau\\1\end{pmatrix}^{k-2} d\tau$ at cusp ∞,

$$\begin{aligned}\int_0^w \left(\iota_{\frac{\partial}{\partial x}}\phi_f\right)(t,y)dt &= \int_0^w f(t+iy)\begin{pmatrix}\tau\\1\end{pmatrix}^{k-2} dt\\ &= \int_0^w f(t+iy)\tau^{k-2}dt\ e_1^{k-2} + \int_0^w f(t+iy)\binom{k-2}{1}\tau^{k-3}dt\ e_1^{k-3}e_2\\ &\quad + \cdots + \int_0^w f(t+iy)\binom{k-2}{k-3}\tau dt\ e_1e_2^{k-3} + \int_0^w f(t+iy)dt\ e_2^{k-2}\ .\end{aligned}$$

By Lemma 5.2.2 applied to γ_∞, we know that the terms involving $\{e_1^{k-2}, e_1^{k-3}e_2, \cdots, e_1e_2^{k-3}\}$ belong to $\mathrm{Im}(\rho(\gamma_\infty) - \mathrm{I})$. All that is left is to show

$$\int_0^w f(t+iy)dt\ e_2^{k-2} \in \mathrm{Im}(\rho(\gamma_\infty) - \mathrm{I})\ . \tag{5.2.1}$$

Since f is a modular form, it has the following Fourier expansion

$$f(q) = \sum_{n=0}^{\infty} a_n q^n, \quad \text{where } q = e^{2\pi i\tau/w},\ \text{and } \tau = x + iy\ ,$$

and since f is cuspidal, $a_0 = 0$. By the Cauchy integral formula, we have for any y, and S^1 with radius in $(0,1)$,

$$a_0 = \frac{1}{2\pi i}\int_{S^1}\frac{f(q)}{q}dq\ ,$$

then fixing y, the integral becomes

$$= \frac{1}{2\pi i} \int_0^w \frac{f(x+iy)}{e^{2\pi ix/w} e^{-2\pi y/w}} \cdot \frac{2\pi i}{w} e^{2\pi ix/w} e^{-2\pi y/w} dx$$
$$= \frac{1}{w} \int_0^w f(x+iy) dx \ .$$

Since $a_0 = 0$, it follows that $\int_0^w f(x+iy)dx = 0$. Hence we have shown equation (5.2.1) holds and therefore $\int_0^w \left(\iota_{\frac{\partial}{\partial x}} \phi_f \right)(t,y)dt \in \mathrm{Im}(\rho(\gamma_\infty) - \mathrm{I})$.

In order to show c other than ∞, we want to show

$$\int_0^{w_c} \left(\iota_{\frac{\partial}{\partial x}} L_C^* \phi_f \right)(t,y)dt \in \mathrm{Im}(\rho\left(\gamma_c\right) - \mathrm{I})$$

holds. We consider $\phi_{f[C]_k}$ where

$$f[C]_k(\tau) = j(C,\tau)^{-k} f(C\tau), \quad \tau \in \mathfrak{h} \ ,$$

and $j(C,\tau) \in \mathbf{C}$ is the factor of automorphy. If f is a cusp form with respect to Γ, then $f[C]_k$ is a cusp form with respect to $C^{-1}\Gamma C$. Note that

$$\mathrm{Stab}_{C^{-1}\Gamma C}(\infty) = C^{-1} \, \mathrm{Stab}_\Gamma(c) C,$$

and thus $\gamma_{c,C^{-1}\Gamma C} = C^{-1}\gamma_c C$. This implies that $w_{c,C^{-1}\Gamma C} = w_c$. Since we have proved the proposition for the cusp $\Gamma\infty$, we can apply it to $f[C]_k$ for the cusp $C^{-1}\Gamma C\infty$ and obtain

$$\int_0^{w_c} \left(\iota_{\frac{\partial}{\partial x}} \phi_{f[C]_k} \right)(t,y)dt \in \mathrm{Im}(\rho(C^{-1}\gamma_c C) - \mathrm{I}) \ . \tag{5.2.2}$$

On the other hand, we have

$$\begin{aligned}
L_C^*\phi_f &= f(C\tau)\begin{pmatrix} C\tau \\ 1 \end{pmatrix}^{k-2} d(C\tau) \\
&= j(C,\tau)^k f[C]_k(\tau) j(C,\tau)^{-(k-2)}\rho(C)\begin{pmatrix} \tau \\ 1 \end{pmatrix}^{k-2} j(C,\tau)^{-2} d\tau \\
&= \rho(C)\phi_{f[C]_k} \ .
\end{aligned}$$

Thus

$$\begin{aligned}
\int_0^{w_c}\left(\iota_{\frac{\partial}{\partial x}} L_C^*\phi_f\right)(t,y)dt = \rho(C)\int_0^{w_c}\left(\iota_{\frac{\partial}{\partial x}}\phi_{f[C]_k}\right)(t,y)dt& \\
\in \rho(C)\,\mathrm{Im}(\rho(C^{-1}\gamma_c C) - \mathrm{I}) &\qquad \text{as in equation (5.2.2)} \ , \\
\in \rho(C)\rho(C^{-1})\,\mathrm{Im}(\rho(\gamma_\infty) - \mathrm{I})\rho(C)& \\
= \mathrm{Im}(\rho(\gamma_c) - \mathrm{I}) \ . &\qquad \text{since } C\colon V_k \xrightarrow{\sim} V_k \ . \qquad \square
\end{aligned}$$

Since f and $\begin{pmatrix} \tau \\ 1 \end{pmatrix}$ are holomorphic, $\phi_f = f\begin{pmatrix} \tau \\ 1 \end{pmatrix}^{k-2} d\tau$ lies in $\ker(d)$. Therefore ϕ_f represents a class of $H_P^1(\Gamma, \mathfrak{h}, V_k)$. We thus have a map from $S_k(\Gamma) \longrightarrow H_P^1(\Gamma, \mathfrak{h}, V_k)$ that takes $f \mapsto [\phi_f]$.

5.3 The Action of Hecke Operators on $A^1_P(\Gamma, \mathfrak{h}, V)$

In this section we will study the behavior of Hecke operator acting on a parabolic cohomology classes $[\phi] \in H^1_P(\Gamma, \mathfrak{h}, V)$. Recall the following diagram for the correspondence

$$\begin{array}{ccc} \Gamma_0(pN) \setminus \mathfrak{h} & \xrightarrow{g} & g\Gamma_0(pN)g^{-1} \setminus \mathfrak{h} \\ \downarrow \pi & & \downarrow \pi_g \\ \Gamma_0(N) \setminus \mathfrak{h} & & \Gamma_0(N) \setminus \mathfrak{h} \end{array}$$

where $g = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$.

First we want to show that this correspondence preserves $A^1_P(\Gamma, \mathfrak{h}, V)$. From section 3.2 we know that Hecke operators preserves $A^1(\Gamma, \mathfrak{h}, V_k)$. Thus here we will show it also preserves the parabolic condition for $A^1_P(\Gamma, \mathfrak{h}, V)$ as outlined in equation 5.1.4. Starting from the bottom right of the correspondence, we will show that in each step, the forms satisfy the parabolic condition.

Proposition 5.3.1. If $\phi \in A^1_P(\Gamma_0(N), \mathfrak{h}, V)$, then ϕ naturally pulls back to $\phi \in A^1_P(g\Gamma_0(pN)g^{-1}, \mathfrak{h}, V_k)$.

Proof. In this proof, w_∞ and γ_∞ will be with respect to $\Gamma_0(N)$; when we wish to work with respect to $g\Gamma_0(pN)g^{-1}$, we will explicitly express that. We want to show that ϕ also satisfies the parabolic condition in equation (5.1.4). We will show that ϕ satisfies this condition at the cusps of $g\Gamma_0(pN)g^{-1} \setminus \mathfrak{h}^*$ which live above the cusp $\Gamma_0(N)\infty$. The general case will follow similarly. Since $\phi \in A^1_P(\Gamma_0(N), \mathfrak{h}, V_k)$, we have

$$\int_0^{w_\infty} \left(\iota_{\frac{\partial}{\partial x}} \phi\right)(t, y)dt \in \mathrm{Im}\left(\rho(\gamma_\infty) - \mathrm{I}\right)$$

for all $y \geq y_{suff}$. Here $w_\infty = w_{\infty,\Gamma_0(N)}$ is the width of the cusp $\Gamma_0(N)\infty \in \Gamma_0(N) \setminus \mathfrak{h}^*$. A cusp above it in the space $g\Gamma_0(pN)g^{-1} \setminus \mathfrak{h}^*$ would have the form $g\Gamma_0(pN)g^{-1}(\gamma_i\infty)$, where γ_i is a member of a set of coset representatives of $g\Gamma_0(pN)g^{-1} \setminus \Gamma_0(N)$ such that

$$\Gamma_0(N)\infty = \bigcup_i g\Gamma_0(pN)g^{-1}(\gamma_i\infty) \ .$$

Write $\tilde{\Gamma} = g\Gamma_0(pN)g^{-1}$, we want to verify that

$$\int_0^{w_{\gamma_i\infty,\tilde{\Gamma}}} \left(\iota_{\frac{\partial}{\partial x}} L_C^* \phi\right)(t,y)dt \in \operatorname{Im}\left(\rho\left(\gamma_{\gamma_i\infty,\tilde{\Gamma}}\right) - \mathrm{I}\right) \tag{5.3.1}$$

where $C = \gamma_i$ is the matrix that takes ∞ to $\gamma_i\infty$, and $\gamma_{\gamma_i\infty,\tilde{\Gamma}} = \begin{pmatrix} 1 & \pm w_{\gamma_i\infty,\tilde{\Gamma}} \\ 0 & 1 \end{pmatrix}$ mod I_2 is a generator of $\operatorname{Stab}_{\tilde{\Gamma}}(\gamma_i\infty)$ where $w_{\gamma_i\infty,\tilde{\Gamma}}$ is the width of the cusp $\tilde{\Gamma}\gamma_i\infty \in \tilde{\Gamma} \setminus \mathfrak{h}^*$. Since $\operatorname{Stab}_{\tilde{\Gamma}}(\gamma_i\infty) \subset \operatorname{Stab}_{\Gamma_0(N)}(\gamma_i\infty) = \gamma_i \operatorname{Stab}_{\Gamma_0(N)}(\infty)\gamma_i^{-1}$,

$$\gamma_{\gamma_i\infty,\tilde{\Gamma}} = \gamma_i\gamma_\infty^r\gamma_i^{-1}, \quad w_{\gamma_i\infty,\tilde{\Gamma}} = rw_{\gamma_i\infty} = rw_\infty$$

for some integer $r = [\operatorname{Stab}_{\Gamma_0(N)}(\gamma_i\infty) \colon \operatorname{Stab}_{\tilde{\Gamma}}(\gamma_i\infty)] \geq 1$. Now we can compute the left hand side of equation (5.3.1),

$$\int_0^{w_{\gamma_i\infty,\tilde{\Gamma}}} \left(\iota_{\frac{\partial}{\partial x}} L_{\gamma_i}^* \phi\right)(t,y)dt = \int_0^{rw_\infty} \left(\iota_{\frac{\partial}{\partial x}} \rho(\gamma_i)\phi\right)(t,y)dt \qquad \text{since } \gamma_i \in \Gamma_0(N),$$

$$= \rho(\gamma_i)\left(\sum_{l=1}^{r} \int_{(l-1)w_\infty}^{lw_\infty} \left(\iota_{\frac{\partial}{\partial x}}\phi\right)(t,y)dt\right) \ ,$$

writing $\gamma_\infty = \gamma_{\infty,\Gamma_0(N)} = \begin{pmatrix} 1 & w_\infty \\ 0 & 1 \end{pmatrix}$, we have $\phi(t+w_\infty,y) = L_{\gamma_\infty}^*\phi(t,y)$, and thus

$$= \rho(\gamma_i)\sum_{l=1}^{r} \int_0^{w_\infty} \iota_{\frac{\partial}{\partial x}} \left(L_{\gamma_\infty}^*\right)^{l-1}\phi(t,y)dt$$

$$= \rho(\gamma_i)\sum_{l=1}^{r} \rho(\gamma_\infty^{l-1}) \int_0^{w_\infty} \iota_{\frac{\partial}{\partial x}}\phi(t,y)dt$$

$$\in \rho(\gamma_i)\sum_{l=1}^{r} \rho(\gamma_\infty^{l-1}) \operatorname{Im}\left(\rho(\gamma_\infty) - \mathrm{I}\right)$$

$$= \rho(\gamma_i)\operatorname{Im}\left(\rho(\gamma_\infty^r) - \mathrm{I}\right) \ .$$

Now the right hand side of equation (5.3.1) is

$$\begin{aligned}\operatorname{Im}\left(\rho\left(\gamma_{\gamma_i\infty,\tilde{\Gamma}}\right)-\mathrm{I}\right) &= \operatorname{Im}\left(\rho(\gamma_i)\left(\rho(\gamma_\infty^r)-\mathrm{I}\right)\rho(\gamma_i^{-1})\right)\\ &= \rho(\gamma_i)\operatorname{Im}\left(\rho(\gamma_\infty^r)-\mathrm{I}\right)\rho(\gamma_i^{-1})\end{aligned}$$

and since γ_i^{-1} is surjective unto V_k, we have

$$= \rho(\gamma_i)\operatorname{Im}\left(\rho(\gamma_\infty^r)-\mathrm{I}\right)\ .$$

Therefore equation (5.3.1) holds and thus $\phi \in A_P^1(g\Gamma_0(pN)g^{-1}, \mathfrak{h}, V_k)$. □

The next map going from $g\Gamma_0(pN)g^{-1}\setminus\mathfrak{h}$ to $\Gamma_0(pN)\setminus\mathfrak{h}$ is given by

$$\rho(g^{-1})L_g^*\colon A_P^1(g\Gamma_0(pN)g^{-1}, \mathfrak{h}, V_k) \longrightarrow A_P^1(\Gamma_0(pN), \mathfrak{h}, V_k)\ .$$

Proposition 5.3.2. If $\phi \in A_P^1(g\Gamma_0(pN)g^{-1}, \mathfrak{h}, V_k)$, then $\psi = \rho(g^{-1})L_g^*(\phi) \in A_P^1(\Gamma_0(pN), \mathfrak{h}, V_k)$.

Proof. In this proof, w_∞ and γ_∞ will be with respect to $g\Gamma_0(pN)g^{-1}$; write $\Gamma = \Gamma_0(pN)$, $w_{g^{-1}\infty,\Gamma}$ and $\gamma_{g^{-1}\infty,\Gamma}$ will be with respect to $\Gamma_0(pN)$. To show that ψ satisfies the parabolic condition in equation 5.1.4, again we will just show that ψ satisfies this condition at the cusp $\Gamma_0(pN)g^{-1}\infty \in \Gamma_0(pN)\setminus\mathfrak{h}^*$ which is sent to the cusp $g\Gamma_0(pN)g^{-1}\infty \in g\Gamma_0(pN)g^{-1}\setminus\mathfrak{h}^*$ under the g map. We want to verify that

$$\int_0^{w_{g^{-1}\infty,\Gamma}} \left(\iota_{\frac{\partial}{\partial x}} L_C^*\psi\right)(t,y)dt \in \operatorname{Im}\left(\rho\left(\gamma_{g^{-1}\infty,\Gamma}\right)-\mathrm{I}\right) \tag{5.3.2}$$

where C is the matrix that takes ∞ to $g^{-1}\infty$, and $\gamma_{g^{-1}\infty,\Gamma} = \begin{pmatrix}1 & \pm w_{g^{-1}\infty,\Gamma}\\ 0 & 1\end{pmatrix}$ mod I_2 is a generator of $\operatorname{Stab}_{\Gamma_0(pN)}(g^{-1}\infty)$ where $w_{g^{-1}\infty,\Gamma}$ is the width of the cusp $\Gamma_0(pN)g^{-1}\infty \in \Gamma_0(pN)\setminus\mathfrak{h}^*$. Recall that $g = \begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}$, thus $g^{-1}\infty = \infty$ and g induces an isomorphism of the spaces $\Gamma_0(pN)\setminus\mathfrak{h}^*$ and $g\Gamma_0(pN)g^{-1}\setminus\mathfrak{h}^*$. Thus the left hand side of equation (5.3.2) becomes

$$\int_0^{w_{g^{-1}\infty,\Gamma}} \left(\iota_{\frac{\partial}{\partial x}}\psi\right)(t,y)dt = \rho(g^{-1})\int_0^{w_{g^{-1}\infty,\Gamma}} \left(\iota_{\frac{\partial}{\partial x}}L_g^*(\phi)\right)(t,y)dt$$

$$= \rho(g^{-1})\int_0^{w_{g^{-1}\infty,\Gamma}} \left(p\cdot\iota_{\frac{\partial}{\partial x}}\phi\right)(pt,py)dt \ ,$$

by a change of variable, this becomes

$$= \rho(g^{-1})\int_0^{p\cdot w_{g^{-1}\infty,\Gamma}} \left(\iota_{\frac{\partial}{\partial x}}\phi\right)(t,py)dt$$

where $py > y_{suff}$ since $p > 1$. Since $\phi \in A_P^1(g\Gamma_0(pN)g^{-1}, \mathfrak{h}, V_k)$ we have

$$\int_0^{w_\infty} \left(\iota_{\frac{\partial}{\partial x}}\phi\right)(t,y)dt \in \mathrm{Im}\left(\rho\left(\gamma_\infty\right) - \mathrm{I}\right)$$

where $\gamma_\infty = \begin{pmatrix} 1 & \pm w_\infty \\ 0 & 1 \end{pmatrix}$ is a generator of $\mathrm{Stab}_{g\Gamma_0(pN)g^{-1}}(\infty)$ and w_∞ is the width of the cusp $g\Gamma_0(pN)g^{-1}\infty \in g\Gamma_0(pN)g^{-1}\setminus\mathfrak{h}^*$. Note that here $w_\infty = pw_{g^{-1}\infty,\Gamma}$ and $\gamma_\infty = g\gamma_{g^{-1}\infty,\Gamma}g^{-1}$. Thus the left hand side of equation (5.3.2) becomes

$$\int_0^{w_{g^{-1}\infty,\Gamma}} \left(\iota_{\frac{\partial}{\partial x}}\psi\right)(t,y)dt \in \rho(g^{-1})\,\mathrm{Im}\left(\rho\left(\gamma_\infty\right) - \mathrm{I}\right)$$

$$= \rho(g^{-1})\rho(g)\,\mathrm{Im}\left(\rho\left(\gamma_{g^{-1}\infty,\Gamma}\right) - \mathrm{I}\right)\rho(g^{-1}) \ ,$$

$\rho(g^{-1})$ sends V_k to V_k, thus does not affect the image,

$$= \mathrm{Im}\left(\rho\left(\gamma_{g^{-1}\infty,\Gamma}\right) - \mathrm{I}\right) \ .$$

□

Now to go from $\Gamma_0(pN)\setminus\mathfrak{h}$ to $\Gamma_0(N)\setminus\mathfrak{h}$ at the bottom left of the correspondence, we again take the trace operator defined as follows.

$$\pi_*(\psi) = \sum_{\gamma\in\Gamma_0(pN)\setminus\Gamma_0(N)} \rho(\gamma^{-1})L_\gamma^*(\psi) \ .$$

Proposition 5.3.3. If $\psi \in A_P^1(\Gamma_0(pN), \mathfrak{h}, V_k)$, then $\pi_*(\psi) \in A_P^1(\Gamma_0(N), \mathfrak{h}, V_k)$.

Before we give the proof to the above proposition, we first give the following lemma.

Lemma 5.3.4. Let $\{\Gamma_0(pN)\gamma_1\infty, \Gamma_0(pN)\gamma_2\infty, \cdots, \Gamma_0(pN)\gamma_m\infty\}$ denote the set of distinct cusps of $\Gamma_0(pN)\backslash\mathfrak{h}^*$ that lie above $\Gamma_0(N)\infty \in \Gamma_0(N)\backslash\mathfrak{h}^*$. For each $1 \le i \le m$, let r_i be the number of cosets $\Gamma_0(pN)\gamma \in \Gamma_0(pN) \setminus \Gamma_0(N)$ such that $\Gamma_0(pN)\gamma\infty = \Gamma_0(pN)r_i\infty$. Then if γ_∞ is a generator of $\mathrm{Stab}_{\Gamma_0(N)}(\infty)$,

$$\bigcup_{i=1}^{m}\{\gamma_i, \gamma_i\gamma_\infty, \gamma_i\gamma_\infty^2, \cdots, \gamma_i\gamma_\infty^{r_i-1}\}$$

form a complete set of coset representatives of $\Gamma_0(pN) \setminus \Gamma_0(N)$.

Proof. Note that here $r_i = [\mathrm{Stab}_{\Gamma_0(N)}(\gamma_i\infty) \colon \mathrm{Stab}_{\Gamma_0(pN)}(\gamma_i\infty)]$ and $\sum_i^m r_i = [\Gamma_0(N) \colon \Gamma_0(pN)]$. It suffices to show that γ_i and $\gamma_i\gamma_\infty^j$ are in different cosets of $\Gamma_0(pN) \setminus \Gamma_0(N)$ for $j = \{1, 2, \cdots, r_i - 1\}$ and $i = \{1, 2, \cdots, m\}$ by contradiction. If γ_i and $\gamma_i\gamma_\infty^j$ are in the same coset, then we can write

$$\mu\gamma_i = \gamma_i\gamma_\infty^j$$

for some $\mu \in \Gamma_0(pN)$. We then have

$$\mu = \gamma_i\gamma_\infty^j\gamma_i^{-1} \in \mathrm{Stab}_{\Gamma_0(pN)}(\gamma_i\infty)\ .$$

However, we have

$$\mathrm{Stab}_{\Gamma_0(pN)}(\gamma_i\infty) \subset \mathrm{Stab}_{\Gamma_0(N)}(\gamma_i\infty) = \gamma_i\,\mathrm{Stab}_{\Gamma_0(N)}(\infty)\gamma_i^{-1}$$

and thus a generator of $\mathrm{Stab}_{\Gamma_0(pN)}(\gamma_i\infty)$ is $\gamma_i\gamma_\infty^{r_i}\gamma_i^{-1}$, a contradiction. Therefore γ_i and $\gamma_i\gamma_\infty^j$ are in different cosets for $j = \{1, 2, \cdots, r_i - 1\}$. $\square$

Proof of Proposition 5.3.3. In this proof, w_∞ and γ_∞ will be with respect to $\Gamma_0(N)$; when we wish to work with respect to $\Gamma_0(pN)$, we will explicitly express that. We will again just show that $\pi_*(\psi)$ satisfies the parabolic condition as in equation (5.1.4) at the cusp $\Gamma_0(N)\infty \in \Gamma_0(N) \setminus \mathfrak{h}^*$. We want to show

$$\int_0^{w_\infty} \left(\iota_{\frac{\partial}{\partial x}} \pi_*(\psi)\right)(t,y)dt \in \mathrm{Im}(\rho\left(\gamma_\infty\right) - \mathrm{I})$$

where $\gamma_\infty = \begin{pmatrix} 1 & \pm w_\infty \\ 0 & 1 \end{pmatrix} \mod \mathrm{I}_2$ is a generator of $\mathrm{Stab}_{\Gamma_0(N)}(\infty)$ and w_∞ is the width of the cusp $\Gamma_0(N)\infty$. To consider the cusps in $\Gamma_0(pN) \setminus \mathfrak{h}^*$ that lie above $\Gamma_0(N)\infty \in \Gamma_0(N)\setminus\mathfrak{h}^*$, we use the definitions of $\{\Gamma_0(pN)\gamma_1\infty, \Gamma_0(pN)\gamma_2\infty, \cdots, \Gamma_0(pN)\gamma_m\infty\}$ and $\{r_1, r_2, \cdots, r_m\}$ as in Lemma 5.3.4. Write $\Gamma = \Gamma_0(pN)$, we let $\gamma_{\gamma_i\infty,\Gamma} = \begin{pmatrix} 1 & w_{\gamma_i\infty,\Gamma} \\ 0 & 1 \end{pmatrix}$ mod I_2 be a generator of $\mathrm{Stab}_{\Gamma_0(pN)}(\gamma_i\infty)$, and $w_{\gamma_i\infty,\Gamma}$ be the width of the cusp $\Gamma_0(pN)\gamma_i\infty \in \Gamma_0(pN) \setminus \mathfrak{h}^*$, we have

$$\gamma_{\gamma_i\infty,\Gamma} = \gamma_i\gamma_\infty^{r_i}\gamma_i^{-1}, \quad w_{\gamma_i\infty,\Gamma} = r_i w_{\gamma_i\infty} = r_i w_\infty \ .$$

Since $\psi \in A^1_P(\Gamma_0(pN), \mathfrak{h}, V_k)$,

$$\int_0^{w_{\gamma_i\infty,\Gamma}} \left(\iota_{\frac{\partial}{\partial x}} L^*_C\psi\right)(t,y)dt \in \mathrm{Im}\left(\rho(\gamma_{\gamma_i\infty,\Gamma}) - \mathrm{I}\right) \tag{5.3.3}$$

where $C = \gamma_i$ sends ∞ to $\gamma_i\infty$. Equation (5.3.3) can also be written as

$$\int_0^{r_i w_\infty} \left(\iota_{\frac{\partial}{\partial x}} L^*_{\gamma_i}\psi\right)(t,y)dt \in \mathrm{Im}\left(\rho(\gamma_i\gamma_\infty^r\gamma_i^{-1}) - \mathrm{I}\right) \ .$$

If we act by $\rho(\gamma_i^{-1})$ and sum over the unique coset representatives $\gamma_i = \{\gamma_1, \gamma_2, \cdots, \gamma_m\}$, we obtain

$$\sum_{i=1}^m \rho(\gamma_i^{-1}) \int_0^{r_i w_\infty} \left(\iota_{\frac{\partial}{\partial x}} L^*_{\gamma_i}\psi\right)(t,y)dt \in \sum_{i=1}^m \rho(\gamma_i^{-1})\,\mathrm{Im}\left(\rho(\gamma_i\gamma_\infty^r\gamma_i^{-1}) - \mathrm{I}\right) \ . \tag{5.3.4}$$

First note that the right hand side of equation (5.3.4) is simply $\operatorname{Im}(\rho(\gamma_\infty) - \mathrm{I})$ since

$$\begin{aligned}\sum_{i=1}^{m} \rho(\gamma_i^{-1}) \operatorname{Im}\left(\rho(\gamma_i \gamma_\infty^{r_i} \gamma_i^{-1}) - \mathrm{I}\right) &= \sum_{i=1}^{m} \operatorname{Im}\left(\rho(\gamma_\infty^{r_i}) - \mathrm{I}\right) \rho(\gamma_i^{-1}) \\ &\in \sum_{i=1}^{m} \operatorname{Im}\left(\rho(\gamma_\infty) - \mathrm{I}\right) \qquad \text{since } \gamma_i^{-1} \colon V_k \xrightarrow{\sim} V_k, \\ &\in \operatorname{Im}\left(\rho(\gamma_\infty) - \mathrm{I}\right) \ .\end{aligned}$$

For each i of the left hand side of equation (5.3.4), we have

$$\begin{aligned}&\rho(\gamma_i^{-1}) \int_0^{r_i w_\infty} \left(\iota_{\frac{\partial}{\partial x}} L_{\gamma_i}^* \psi\right)(t,y)dt \\ &= \sum_{j=0}^{r_i-1} \rho(\gamma_i^{-1}) \int_{j w_\infty}^{(j+1)w_\infty} \left(\iota_{\frac{\partial}{\partial x}} L_{\gamma_i}^* \psi\right)(t,y)dt \ ,\end{aligned}$$

let $\gamma_{i+j} = \gamma_i \gamma_\infty^j$, and by change of variable,

$$\begin{aligned}&= \sum_{j=0}^{r_i-1} \rho(\gamma_i^{-1}) \int_0^{w_\infty} \left(\iota_{\frac{\partial}{\partial x}} L_{\gamma_{i+j}}^* \psi\right)(t,y)dt \\ &= \sum_{j=0}^{r_i-1} \rho(\gamma_\infty^j)\rho(\gamma_{i+j}^{-1}) \int_0^{w_\infty} \left(\iota_{\frac{\partial}{\partial x}} L_{\gamma_{i+j}}^* \psi\right)(t,y)dt \\ &= \sum_{j=0}^{r_i-1} \Big(\left(\rho(\gamma_\infty^j) - \mathrm{I}\right) \rho(\gamma_{i+j}^{-1}) \int_0^{w_\infty} \left(\iota_{\frac{\partial}{\partial x}} L_{\gamma_{i+j}}^* \psi\right)(t,y)dt \\ &\qquad\qquad + \rho(\gamma_{i+j}^{-1}) \int_0^{w_\infty} \left(\iota_{\frac{\partial}{\partial x}} L_{\gamma_{i+j}}^* \psi\right)(t,y)dt \Big) \ .\end{aligned}$$

Since $\rho(\gamma_\infty^j) - \mathrm{I} \in \operatorname{Im}(\rho(\gamma_\infty) - \mathrm{I})$, we consider the remaining of the above sum and do this for all i, we get for the left hand side of equation (5.3.4),

$$\sum_{i=1}^{m} \rho(\gamma_i^{-1}) \int_0^{r_i w_\infty} \left(\iota_{\frac{\partial}{\partial x}} L_{\gamma_i}^* \psi\right)(t,y)dt = \sum_{i=1}^{m} \sum_{j=0}^{r_i-1} \rho(\gamma_{i+j}^{-1}) \int_0^{w_\infty} \left(\iota_{\frac{\partial}{\partial x}} L_{\gamma_{i+j}}^* \psi\right)(t,y)dt \ .$$

Lemma 5.3.4 tells us that γ_{i+j} for $i = 1, 2, \cdots, m$ and $j = 0, 1, \cdots, r_i - 1$ forms a complete set of coset representatives of $\Gamma_0(pN) \setminus \Gamma_0(N)$. Thus the left hand side of equation 5.3.4 becomes

$$\sum_{\gamma\in\Gamma_0(pN)\backslash\Gamma_0(N)} \rho(\gamma^{-1}) \int_0^{w_\infty} \left(\iota_{\frac{\partial}{\partial x}} L_\gamma^* \psi\right)(t,y)dt = \int_0^{w_\infty} \left(\iota_{\frac{\partial}{\partial x}} \pi_*(\psi)\right)(t,y)dt \ .$$

Therefore we have shown that

$$\int_0^{w_\infty} \left(\iota_{\frac{\partial}{\partial x}} \pi_*(\psi)\right)(t,y)dt \in \operatorname{Im}\left(\rho(\gamma_\infty) - \mathrm{I}\right) \ . \qquad \square$$

Thus we have shown the Hecke operators preserve $A_P^1(\Gamma, \mathfrak{h}, V)$. Since the Hecke operators also commute with the differential d as shown in section 3.3, we see that the Hecke operators induces an action on $H_P^1(\Gamma, \mathfrak{h}, V)$ as well.

Chapter 6

Isomorphism Between the Parabolic Cohomology and the L^2 Cohomology

In this chapter, we present a revised homotopy formula analogous to

$$Hd + dH = I - P \tag{6.0.1}$$

where H is a homopoty operator and P is a projection operator to prove $H^1_P(\Gamma, \mathfrak{h}, V) \cong H^1_{(2)}(\Gamma, \mathfrak{h}, V)$. More precisely, if H and P are defined on a complex Y, and if $X \subset Y$ is a subcomplex such that

$$H(X) \subset X \quad \text{and} \quad P(Y) \subset X\ ,$$

then the maps

$$\iota \colon X \longrightarrow Y$$

$$P \colon Y \longrightarrow X$$

induce inverse isomorphism on cohomologies $H^\bullet(X) \cong H^\bullet(Y)$. We will use this to show that $H^1_P(\Gamma, \mathfrak{h}, V) \cong H^1_{\text{sp}}(\Gamma, \mathfrak{h}, V) \cong H^1_{(2)}(\Gamma, \mathfrak{h}, V)$ where H^1_{sp} is a special cohomology that will be defined later.

6.1 x-Homotopy in the region W_∞

To start, we will define a homotopy in the x direction. Suppose $\phi \in A^\bullet(\Gamma, \mathfrak{h}, V)$. We can write ϕ as

$$\phi = \phi_0 + dx \wedge \phi_1 \tag{6.1.1}$$

where ϕ_0 and ϕ_1 do not have dx terms. Recall M and y_{suff} defined in Definition 5.1.1 and $M > y_{suff}$, we define the cutoff function $\chi(y)$ to be

$$\chi(y) = \begin{cases} 1 & y \geq M \ , \\ 0 & y \leq y_{suff} \ . \end{cases}$$

Recall $V_\infty \subset W_\infty$ as defined in Definition 5.1.1. Note $\mathrm{Supp}(\chi) \subset W_\infty$ and $\chi = 1$ in V_∞.

We will present the homotopy operator in W_∞ first and then generalize to W_c for all $c \in \mathbf{Q}$. First we define the homotopy operator in the x direction in W_∞ as

$$\left(H_x(\phi)\right)\Big|_{W_\infty} = \chi(y)\frac{1}{w}\int_0^w \left(s - \frac{w}{2}\right)\rho\left(\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\right)\iota_{\frac{\partial}{\partial x}}\phi(s+x,y)ds$$

where w is the width of the cusp $\Gamma\infty \in \Gamma \setminus \mathfrak{h}$.

On the other hand, we have

$$d\phi = dy \wedge \frac{\partial \phi_0}{\partial y} + dx \wedge \left(\frac{\partial \phi_0}{\partial x} - dy \wedge \frac{\partial \phi_1}{\partial y}\right) \ .$$

Therefore using integration by parts, we compute

$$\begin{aligned} H_x d\phi = {} & \chi(y)\frac{1}{w}\int_0^w \left(s - \frac{w}{2}\right)\rho\left(\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\right)\left(\frac{\partial \phi_0}{\partial x} - dy \wedge \frac{\partial \phi_1}{\partial y}\right)ds \\ = {} & -\chi(y)\frac{1}{w}\int_0^w \left(s - \frac{w}{2}\right)\rho\left(\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\right)dy \wedge \frac{\partial \phi_1}{\partial y}(s+x,y)ds \\ & + \chi(y)\frac{1}{w}\left(s - \frac{w}{2}\right)\rho\left(\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\right)\phi_0(s+x,y)\Big|_0^w \\ & - \chi(y)\frac{1}{w}\int_0^w \frac{\partial}{\partial s}\left[\left(s - \frac{w}{2}\right)\rho\left(\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\right)\right]\phi_0(s+x,y)ds \ , \end{aligned}$$

and

$$dH_x\phi = dy \wedge \chi'(y)\frac{1}{w}\int_0^w \left(s-\frac{w}{2}\right)\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\iota_{\frac{\partial}{\partial x}}\phi(s+x,y)ds$$

$$+\chi(y)\frac{1}{w}\int_0^w \left(s-\frac{w}{2}\right)\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)dy\wedge\frac{\partial\phi_1}{\partial y}(s+x,y)ds$$

$$+\chi(y)\frac{1}{w}dx\wedge\left(s-\frac{w}{2}\right)\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\phi_1(s+x,y)\Big|_0^w$$

$$-\chi(y)\frac{1}{w}\int_0^w \frac{\partial}{\partial s}\left[\left(s-\frac{w}{2}\right)\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\right]dx\wedge\phi_1(s+x,y)ds\ .$$

Since

$$\chi(y)\frac{1}{w}\left(s-\frac{w}{2}\right)\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\phi_0(s+x,y)\Big|_0^w = \chi(y)\phi_0(x,y)\ ,$$

$$\chi(y)\frac{1}{w}dx\wedge\left(s-\frac{w}{2}\right)\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\phi_1(s+x,y)\Big|_0^w = \chi(y)dx\wedge\phi_1(x,y)\ ,$$

and

$$\begin{aligned}
\frac{\partial}{\partial s}\Big|_s\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right) &= \frac{\partial}{\partial t}\Big|_{t=0}\rho\left(\begin{pmatrix}1 & -(s+t)\\ 0 & 1\end{pmatrix}\right)\\
&= \frac{\partial}{\partial t}\Big|_{t=0}\left[\rho\left(\begin{pmatrix}1 & -t\\ 0 & 1\end{pmatrix}\right)\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\right]\\
&= \frac{\partial}{\partial t}\Big|_{t=0}\left[\rho\left(\exp\left(\begin{pmatrix}0 & -t\\ 0 & 0\end{pmatrix}\right)\right)\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\right]\\
&= \frac{\partial}{\partial t}\Big|_{t=0}\left[\exp\left(d(\rho)_e\begin{pmatrix}0 & -t\\ 0 & 0\end{pmatrix}\right)\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\right]\\
&= \frac{\partial}{\partial t}\Big|_{t=0}\left[\exp\left(-t\cdot d(\rho)_e\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right)\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\right]\\
&= -d(\rho)_e\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\ ,
\end{aligned}$$

we have

$$H_x d\phi + dH_x\phi = \phi - \Big[\big(1-\chi(y)\big)\phi + \chi(y)\frac{1}{w}\int_0^w \rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\phi(s+x,y)ds$$

$$- \chi(y)d(\rho)_e\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\frac{1}{w}\int_0^w\left(s-\frac{w}{2}\right)\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\phi(s+x,y)ds$$

$$- dy\wedge\chi'(y)\frac{1}{w}\int_0^w\left(s-\frac{w}{2}\right)\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\iota_{\frac{\partial}{\partial x}}\phi(s+x,y)ds\Big]$$

$$= \phi - P_x\phi \tag{6.1.2}$$

where $\big(P_x\phi\big)\Big|_{W_\infty}$ is defined by this last line. Therefore in V_∞ where $\chi = 1$, from equation (6.0.1) we have

$$\begin{aligned} P_x\phi = &-d(\rho)_e\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\frac{1}{w}\int_0^w\left(s-\frac{w}{2}\right)\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\phi(s+x,y)ds \\ &+\frac{1}{w}\int_0^w\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\phi(s+x,y)ds\ . \end{aligned} \tag{6.1.3}$$

6.2 Properties of P_x in V_∞

In this section we present some properties of $P_x\phi$ in the region V_∞. To simplify notions, we will write

$$N = d(\rho)_e\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\ ,$$

$$\tilde{P}\phi = \frac{1}{w}\int_0^w\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\phi(s+x,y)ds\ ,$$

$$P_0\phi = \frac{1}{w}\int_0^w\left(s-\frac{w}{2}\right)\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\phi(s+x,y)ds\ .$$

Therefore,

$$P_x\phi = \tilde{P}\phi - NP_0\phi\ . \tag{6.2.1}$$

Remark. N is nilpotent. If $\dim(V) = k-1$ for some $k\in\mathbf{Z}$, then $N^{k-1}=0$.

Claim 6.2.1. $\operatorname{Im}\left(\rho\left(\begin{pmatrix}1 & s\\ 0 & 1\end{pmatrix}\right) - \mathrm{I}\right) = \operatorname{Im} N$ for any $s \neq 0$.

Proof. Write $T = \rho\left(\begin{pmatrix}1 & s\\ 0 & 1\end{pmatrix}\right)$, we have

$$\begin{aligned} T &= \rho\left(\exp\begin{pmatrix}0 & s\\ 0 & 0\end{pmatrix}\right) \\ &= \exp\left(d(\rho)_e\begin{pmatrix}0 & s\\ 0 & 0\end{pmatrix}\right) \\ &= \exp\left(s \cdot d(\rho)_e\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right) \ . \end{aligned}$$

Thus $\log T = s \cdot d(\rho)_e \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = s \cdot N$, which implies $\operatorname{Im} N = \operatorname{Im}(\log T)$ when $s \neq 0$.

Next we will show $\operatorname{Im}(\log T) = \operatorname{Im}(T - \mathrm{I}) = \operatorname{Im}(\mathrm{I} - T)$. We have

$$\begin{aligned} \log T &= \log(\mathrm{I} - (\mathrm{I} - T)) \\ &= -\left[(\mathrm{I} - T) + \frac{(\mathrm{I} - T)^2}{2} + \frac{(\mathrm{I} - T)^3}{3} + \cdots\right] \ . \end{aligned} \tag{6.2.2}$$

Clearly $\operatorname{Im}(\log T) \subset \operatorname{Im}(\mathrm{I} - T)$. To show the other direction, consider $(\mathrm{I} - T)v$ for some $v \in V$. By equation (6.2.2) we have

$$(\mathrm{I} - T)v \in \operatorname{Im}(\log T) \mod \operatorname{Im}(\mathrm{I} - T)^2 \ .$$

Multiplying both sides of equation (6.2.2) by $(\mathrm{I} - T)$, we have

$$(\mathrm{I} - T)^2 v \in \operatorname{Im}(\log T) \mod \operatorname{Im}(\mathrm{I} - T)^3 \ .$$

We repeat this process, and in general, we obtain

$$(\mathrm{I} - T)^i v \in \operatorname{Im}(\log T) \mod \operatorname{Im}(\mathrm{I} - T)^{i+1} \ .$$

Since T is unipotent, $(\mathrm{I} - T)$ is nilpotent. Thus

$$(\mathrm{I} - T)^{\dim V - 1} = 0 \ .$$

Therefore we have

$$(\mathrm{I}-T)^{\dim V-2}v \in \mathrm{Im}(\log T)$$

$$(\mathrm{I}-T)^{\dim V-3}v \in \mathrm{Im}(\log T)$$

$$\cdots$$

$$(\mathrm{I}-T)v \in \mathrm{Im}(\log T)\ . \qquad \square$$

Before we state the properties, we give the following definition.

Definition 6.2.2. A form $\eta \in A^\bullet(\mathrm{Stab}_\Gamma(\infty), V_\infty, V)$ is called *invariant* (in V_∞) if $\rho\left(\begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix}\right)\eta$ is independent of x in V_∞.

Proposition 6.2.3. If ϕ is invariant, then $P_x\phi = \phi$.

Proof. First we will consider $\tilde{P}\phi$. Since ϕ is invariant, we can write $\rho\left(\begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix}\right)$ $\phi(x,y) = \phi(0,y)$, which is equivalent to

$$\phi(x,y) = \rho\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\phi(0,y)\ .$$

Thus we have

$$\begin{aligned}
\tilde{P}\phi &= \frac{1}{w}\int_0^w \rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\phi(s+x,y)ds\ ,\\
&= \frac{1}{w}\int_0^w \rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\rho\left(\begin{pmatrix}1 & s+x\\ 0 & 1\end{pmatrix}\right)\phi(0,y)ds\\
&= \rho\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\phi(0,y)\\
&= \phi(x,y)\ .
\end{aligned}$$

Now for $P_0\phi$, we have

$$P_0\phi = \frac{1}{w}\int_0^w \left(s - \frac{w}{2}\right)\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\phi(s+x,y)ds$$

$$= \frac{1}{w}\int_0^w \left(s - \frac{w}{2}\right)\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\rho\left(\begin{pmatrix}1 & s+x\\ 0 & 1\end{pmatrix}\right)\phi(0,y)ds$$

$$= \phi(x,y)\frac{1}{w}\int_0^w \left(s - \frac{w}{2}\right)ds$$

$$= 0\ .$$

Therefore, $P_x\phi = \tilde{P}\phi - N \cdot P_0\phi = \phi$. □

Remark. More specifically, the above proof shows that if ϕ is invariant, then

$$\tilde{P}\phi = \phi\ ,$$

$$P_0\phi = 0\ .$$

Before we give the next proposition, we need the following lemma.

Lemma 6.2.4. $\displaystyle\int_u^{u+w} \rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\phi(s,y)ds = \int_0^w \rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\phi(s,y)ds.$

Proof. First notice that $\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\phi(s,y)$ is periodic in s with period w. This is true because

$$\rho\left(\begin{pmatrix}1 & -(s+w)\\ 0 & 1\end{pmatrix}\right)\phi(s+w,y)\ ,$$

since $\phi \in A^1(\Gamma, \mathfrak{h}, V)$ and thus satisfies equation (3.1.1),

$$=\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\rho\left(\begin{pmatrix}1 & -w\\ 0 & 1\end{pmatrix}\right)\rho\left(\begin{pmatrix}1 & w\\ 0 & 1\end{pmatrix}\right)\phi(s,y)$$

$$=\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\phi(s,y)\ .$$

Write $G(s) = \rho\left(\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\right)\phi(s,y)$ for simplification, we compute

$$\begin{aligned}
\int_u^{u+w} G(s)ds &= \int_u^{(n+1)w} G(s)ds + \int_{(n+1)w}^{u+w} G(s)ds \\
&= \int_u^{(n+1)w} G(s)ds + \int_{nw}^{u} G(s+w)ds \qquad \text{by change of variable,} \\
&= \int_{nw}^{(n+1)w} G(s)ds \qquad \text{since } G \text{ is periodic,} \\
&= \int_0^{w} G(s)ds \qquad \text{since } G \text{ is periodic.} \quad \square
\end{aligned}$$

Proposition 6.2.5. $\tilde{P}\phi$ is invariant.

Proof. Using the above lemma, we compute

$$\begin{aligned}
\rho\left(\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}\right)\tilde{P}\phi &= \rho\left(\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}\right)\frac{1}{w}\int_0^w \rho\left(\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\right) \quad \phi(s+x,y)ds \\
&= \frac{1}{w}\int_0^w \rho\left(\begin{pmatrix} 1 & -(s+x) \\ 0 & 1 \end{pmatrix}\right)\phi(s+x,y) \quad ds \\
&= \frac{1}{w}\int_x^{x+w} \rho\left(\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\right)\phi(s,y)ds \qquad \text{by change of variable} \\
&= \frac{1}{w}\int_0^{w} \rho\left(\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\right)\phi(s,y)ds \qquad \text{by Lemma (6.2.4),}
\end{aligned}$$

and thus $\tilde{P}\phi$ is independent of x. $\square$

Proposition 6.2.6. $\tilde{P}P_0\phi = 0$.

Proof. Using Fubini's Theorem and Lemma 6.2.4,

$$\begin{aligned}
\tilde{P}P_0\phi &= \frac{1}{w}\int_0^w \rho\left(\begin{pmatrix}1 & -t\\ 0 & 1\end{pmatrix}\right)\frac{1}{w}\int_0^w \left(s-\frac{w}{2}\right)\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\phi(s+x+t,y)dsdt\\
&= \frac{1}{w^2}\int_0^w \left(s-\frac{w}{2}\right)\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\int_0^w \rho\left(\begin{pmatrix}1 & -t\\ 0 & 1\end{pmatrix}\right)\phi(s+x+t,y)dtds\\
&= \frac{1}{w^2}\rho\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\int_0^w \left(s-\frac{w}{2}\right)\int_0^w \rho\left(\begin{pmatrix}1 & -(s+x+t)\\ 0 & 1\end{pmatrix}\right)\phi(s+x+t,y)dtds\\
&= \frac{1}{w^2}\rho\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\int_0^w \rho\left(\begin{pmatrix}1 & -t\\ 0 & 1\end{pmatrix}\right)\phi(t,y)dt\int_0^w \left(s-\frac{w}{2}\right)ds\\
&= 0
\end{aligned}$$

where the last step comes from $\int_0^w \left(s-\frac{w}{2}\right)ds = 0.$ □

Based on the above propositions, we then have the following theorem.

Theorem 6.2.7. $P_x^{k-1}\phi = \tilde{P}\phi \pm N^{k-1}P_0^{k-1}\phi.$

Proof. We have

$$\begin{aligned}
P_x^2\phi &= P_x(\tilde{P}\phi - NP_0\phi)\\
&= \tilde{P}\tilde{P}\phi - NP_0\tilde{P}\phi - \tilde{P}NP_0\phi + NP_0NP_0\phi\\
&= \tilde{P}\phi - 0 - 0 + N^2P_0^2\phi\ .
\end{aligned}$$

The last step is true because N commutes with $\tilde{P}$ and P_0. The theorem then follows, and $\pm$ depends on the parity of k. □

Remark. Since $N^{k-1} = 0$. We have $P_x^{k-1}\phi = \tilde{P}\phi$.

Thus we revise the homotopy formula in equation (6.0.1) as follows

$$d\mathcal{H} + \mathcal{H}d = I - \tilde{P} \tag{6.2.3}$$

where $\mathcal{H} = H_x + P_xH_x + P_x^2H_x + \cdots + P_x^{k-2}H_x$.

6.3 H_x and P_x Preserves the L^2 Condition and the Parabolic Condition

Next we show that H_x and P_x satisfy the condition in equation (3.1.1) in V_∞.

Proposition 6.3.1. If $\phi \in A^\bullet(\Gamma, \mathfrak{h}, V)$, then $H_x\phi$ and $P_x\phi$ satisfy $L_\gamma^*(H_x\phi) = \rho(\gamma)H_x\phi$ and $L_\gamma^*(P_x\phi) = \rho(\gamma)P_x\phi$ for $\gamma \in \mathrm{Stab}_\Gamma(\infty)$ in W_∞.

Proof. In V_∞, we can write $\gamma = \begin{pmatrix} 1 & nw \\ 0 & 1 \end{pmatrix}$ for $\gamma \in \mathrm{Stab}_\Gamma(\infty)$. Thus $y(L_\gamma(\tau)) = y(\tau)$ and $x(L_\gamma(\tau)) = x(\tau) + nw$ for some $n \in \mathbf{Z}$. Recall from equation (6.1.1), we can rewrite $\phi = \phi_0 + dx \wedge \phi_1$. If $\phi \in A^\bullet(\Gamma, \mathfrak{h}, V)$, it is not hard to show that $\phi_1 \in A^\bullet(\Gamma, \mathfrak{h}, V)$. Thus we have

$$
\begin{aligned}
L_\gamma^*(H_x\phi) &= \chi(y)\frac{1}{w}\int_0^w \left(s - \frac{w}{2}\right)\rho\left(\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\right)\phi_1(s + x + nw, y)ds \\
&= \chi(y)\frac{1}{w}\int_0^w \left(s - \frac{w}{2}\right)\rho\left(\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\right)\rho\left(\begin{pmatrix} 1 & nw \\ 0 & 1 \end{pmatrix}\right)\phi_1(s + x, y)ds \\
&= \rho(\gamma)\chi(y)\frac{1}{w}\int_0^w \left(s - \frac{w}{2}\right)\rho\left(\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\right)\phi_1(s + x, y)ds \\
&= \rho(\gamma)H_x\phi\ ,
\end{aligned}
$$

and

$$L_\gamma^*(P_x\phi) = \big(1-\chi(y)\big)L_\gamma^*\phi + \frac{1}{w}\int_0^w \rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right) L_\gamma^*\phi(s+x,y)ds$$

$$- \chi(y)d(\rho)_e\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\frac{1}{w}\int_0^w \left(s-\frac{w}{2}\right)\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right) L_\gamma^*\phi(s+x,y)ds$$

$$- dy\wedge\chi'(y)\frac{1}{w}\int_0^w \left(s-\frac{w}{2}\right)\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right) L_\gamma^*\phi_1(s+x,y)ds$$

$$= \big(1-\chi(y)\big)\rho(\gamma)\phi + \frac{1}{w}\int_0^w \rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right) \rho(\gamma)\phi(s+x,y)ds$$

$$- \chi(y)d(\rho)_e\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\frac{1}{w}\int_0^w \left(s-\frac{w}{2}\right)\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right) \rho(\gamma)\phi(s+x,y)ds$$

$$- dy\wedge\chi'(y)\frac{1}{w}\int_0^w \left(s-\frac{w}{2}\right)\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right) \rho(\gamma)\phi_1(s+x,y)ds$$

$$= \rho(\gamma)P_x\phi\ .$$

□

Before we give the next proposition, we need the following claim.

Claim 6.3.2. If x, s are bounded, $\exists C > 0$ such that

$$C^{-1}\left\|v\right\|_{s+iy}^2 \leq \left\|v\right\|_\tau^2 \leq C\left\|v\right\|_{s+iy}^2$$

for all $y \geq y_{suff}$ and $v \in V$.

Proof. It is sufficient to show that $C^{-1}\left\|v\right\|_{0+iy}^2 \leq \left\|v\right\|_\tau^2 \leq C\left\|v\right\|_{0+iy}^2$. Let $g \in \mathrm{SL}_2(\mathbf{R})$ such that $g\cdot i = \tau$. Thus

$$g = \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{y} & 0\\ 0 & \frac{1}{\sqrt{y}}\end{pmatrix}$$

$$= \begin{pmatrix}\sqrt{y} & 0\\ 0 & \frac{1}{\sqrt{y}}\end{pmatrix}\begin{pmatrix}1 & \frac{x}{y}\\ 0 & 1\end{pmatrix}\ .$$

Therefore we have

$$\begin{aligned}
\|v\|_\tau^2 &= \left\|\rho(g^{-1})v\right\|_0^2 \\
&= \left\|\rho\left(\begin{pmatrix}1 & -\frac{x}{y}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{y}} & 0\\ 0 & \sqrt{y}\end{pmatrix}\right)v\right\|_0^2 \\
&= \left\|\rho\left(\begin{pmatrix}1 & -\frac{x}{y}\\ 0 & 1\end{pmatrix}\right)v\right\|_{iy}^2
\end{aligned}$$

Since x is bounded, and $\dfrac{1}{y} \leq \dfrac{1}{y_{suff}}$ is also bounded, thus $\begin{pmatrix}1 & -\frac{x}{y}\\ 0 & 1\end{pmatrix}$ lies in a compact set. Therefore

$$C^{-1}\|v\|_{iy}^2 \leq \|v\|_\tau^2 \leq C\|v\|_{iy}^2 \ .$$

□

Proposition 6.3.3. If $\phi \in A^\bullet_{(2)}(\Gamma, \mathfrak{h}, V)$, then $H_x\phi$, $P_x\phi$ and $dH_x\phi$, $dP_x\phi$ all belong to $A^\bullet_{(2)}(\mathrm{Stab}_\Gamma(\infty), W_\infty, V)$ in W_∞.

Proof. We will show both H_x and P_x are bounded operators in L^2. We compute

$$\|H_x\phi\|_{L^2}^2 = \int_{y_{suff}}^\infty \int_0^w \|H_x\phi\|_\tau^2 \frac{dx\,dy}{y^2} \ .$$

Since $x \in [0,w]$, $s \in [0,w]$, and w is a fixed value, we have

$$\begin{aligned}
\|H_x\phi\|_\tau^2 &\leq \left\|\frac{1}{w}\int_0^w \left(s - \frac{w}{2}\right)\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\phi_1(s+x,y)ds\right\|_\tau^2 \\
&\leq C\left\|\rho\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\int_0^w \rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\phi_1(s,y)ds\right\|_\tau^2 \\
&\leq C\int_0^w \|\phi_1(s,y)\|_\tau^2\, ds && \text{by Hölder's Inequality} \\
&\leq C\int_0^w \|\phi_1(s,y)\|_{s+iy}^2\, ds && \text{by Claim 6.3.2}
\end{aligned}$$

where $C > 0$ denotes a constant. Thus we have

$$\begin{aligned}
\|H_x\phi\|_{L^2}^2 &\leq C \int_{y_{suff}}^{\infty} \int_0^w \int_0^w \|\phi_1(s,y)\|_{s+iy}^2 \, ds \frac{dx\,dy}{y^2} \\
&\leq C \int_{y_{suff}}^{\infty} \int_0^w \|ds \wedge \phi_1(s,y)\|_{s+iy}^2 \frac{1}{y^2} \frac{dsdy}{y^2} \\
&\leq C \|\phi\|_{L^2}^2 \ .
\end{aligned}$$

Similarly, we can show the same for P_x. Again $x \in [0,w], s \in [0,w]$, and w is a fixed value. Write $\phi = \phi_0 + dx \wedge \phi_1$, we have

$$\|P_x\phi\|_{L^2}^2 = \int_{y_{suff}}^{\infty} \int_0^w \|P_x\phi\|_\tau^2 \frac{dx\,dy}{y^2}$$

where

$$\begin{aligned}
\|P_x\phi\|_\tau^2 \leq C \|\phi\|_\tau^2 &+ C \left\| \frac{1}{w} \int_0^w \rho\left(\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\right) \phi(s+x,y) ds \right\|_\tau^2 \\
&+ C \left\| \frac{1}{w} \int_0^w \left(s - \frac{w}{2}\right) \rho\left(\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\right) \phi(s+x,y) ds \right\|_\tau^2 \\
&+ C \cdot y^2 \left\| \frac{1}{w} \int_0^w \left(s - \frac{w}{2}\right) \rho\left(\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\right) \phi_1(s+x,y) ds \right\|_\tau^2 \\
\leq C \|\phi\|_\tau^2 &+ C \left\| \rho\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \int_0^w \rho\left(\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\right) \phi(s,y) ds \right\|_\tau^2 \\
&+ C \cdot y^2 \left\| \rho\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \int_0^w \rho\left(\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\right) \phi_1(s,y) ds \right\|_\tau^2 \\
\leq C \|\phi\|_\tau^2 &+ C \int_0^w \|\phi(s,y)\|_\tau^2 \, ds \\
&+ C \cdot y^2 \int_0^w \|dx \wedge \phi_1(s,y)\|_\tau^2 \frac{1}{y^2} ds \\
\leq C \|\phi\|_\tau^2 &+ C \int_0^w \|\phi(s,y)\|_\tau^2 \, ds \\
\leq C \|\phi\|_\tau^2 &+ C \int_0^w \|\phi(s,y)\|_{s+iy}^2 \, ds \ .
\end{aligned}$$

Again the third step is by Hölder's Inequality, and the last step is by Claim 6.3.2. Here $C > 0$ is again a constant. Thus

$$\|P_x\phi\|_{L^2}^2 \leq C \int_{y_{suff}}^{\infty} \int_0^w \|\phi\|_\tau^2 \frac{dx\,dy}{y^2} + C \int_{y_{suff}}^{\infty} \int_0^w \int_0^w \|\phi(s,y)\|_{s+iy}^2 \, ds \frac{dx\,dy}{y^2}$$

$$\leq C \, \|\phi\|_{L^2}^2 \ .$$

Thus $\left(H_x\phi\right)\Big|_{W_\infty}$ and $\left(P_x\phi\right)\Big|_{W_\infty}$ are L^2 for $\phi \in A^\bullet_{(2)}(\Gamma, \mathfrak{h}, V)$. By equation (6.1.2), this implies $\left(dH_x\phi\right)\Big|_{W_\infty}$ is L^2. Also $\left(dP_x\phi\right)\Big|_{W_\infty} = \left(P_x d\phi\right)\Big|_{W_\infty}$ and so is L^2. □

Proposition 6.3.4. If $\phi \in A^\bullet_P(\Gamma, \mathfrak{h}, V)$, then $H_x\phi$ and $P_x\phi$ satisfy the parabolic condition in equation (5.1.4) in V_∞.

Proof. Let $\gamma = \gamma_\infty = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$. Since $\iota_{\frac{\partial}{\partial x}} H_x\phi = 0$,

$$\int_0^w \iota_{\frac{\partial}{\partial x}} H_x\phi dt = 0 \in \operatorname{Im}(\rho\,(\gamma) - \mathrm{I}) \ .$$

Thus $H_x\phi$ satisfies the parabolic condition.

Recall $P_x\phi = \tilde{P}\phi - NP_0\phi$. To show $P_x\phi$ satisfies the parabolic condition, we will show both $\tilde{P}\phi$ and $NP_0\phi$ satisfy the parabolic condition. By Claim 6.2.1, we have

$$\operatorname{Im} N = \operatorname{Im}\left(\rho\left(\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\right) - \mathrm{I}\right)$$

for $s \neq 0$. Let $s = w$, thus we have $NP_0\phi \in \operatorname{Im}(\rho\,(\gamma) - \mathrm{I})$. Next consider $\tilde{P}\phi$. Write $\phi = \phi_0 + dx \wedge \phi_1$ we have

$$\int_0^w \iota_{\frac{\partial}{\partial x}} \tilde{P}\phi dt = \int_0^w \frac{1}{w} \int_0^w \rho\left(\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\right) \phi_1(s+x,y) ds\; dt$$

$$= \int_0^w \frac{1}{w} \rho\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \int_0^w \rho\left(\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\right) \phi_1(s,y) ds\; dt \qquad (6.3.1)$$

since $\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\phi_1(s,y)$ is periodic in s with period w, and independent of x. Note

$$\int_0^w \rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\phi_1(s,y)ds = \int_0^w \left(\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right) - \mathrm{I}\right)\phi_1(s,y)ds + \int_0^w \phi_1(s,y)ds$$

where $\int_0^w \phi_1(s,y)ds \in \mathrm{Im}(\rho(\gamma) - \mathrm{I})$ since ϕ_1 satisfies the parabolic condition inherited from ϕ. On the other hand,

$$\begin{aligned}\int_0^w \left(\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right) - \mathrm{I}\right)\phi_1(s,y)ds &\in \mathrm{Im}\left(\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right) - \mathrm{I}\right)\\ &= \mathrm{Im}\, N \qquad\qquad \text{by Claim 6.2.1}\\ &= \mathrm{Im}\left(\rho\left(\begin{pmatrix}1 & w\\ 0 & 1\end{pmatrix}\right) - \mathrm{I}\right)\end{aligned}$$

pointwise for all $s > 0$. Thus the double integral in equation (6.3.1) belongs to $\mathrm{Im}\left(\rho\left(\begin{pmatrix}1 & w\\ 0 & 1\end{pmatrix}\right) - \mathrm{I}\right)$. □

6.4 Extending x-Homotopy Formula to $\mathfrak{h}$

So far we have only been working in W_∞. We next extend the definition of H_x to all of $\mathfrak{h}$. Define H_x on gW_∞ as

$$H_x(\phi)\big|_{gW_\infty} = \rho(g)L^*_{g^{-1}}\left(H_x\left(\rho(g^{-1})L^*_g(\phi)\right)\Big|_{W_\infty}\right)$$

for any $g \in \mathrm{SL}_2(\mathbf{Z})$ and $\phi \in A^\bullet(\Gamma, \mathfrak{h}, V)$, and define $H_x(\phi) = 0$ on $\mathfrak{h} \setminus \cup_{g\in\mathrm{SL}_2(\mathbf{Z})} gW_\infty$. Again P_x is defined by formula in equation (6.1.2).

Proposition 6.4.1. The above definition of H_x on gW_∞ for $g \in \mathrm{SL}_2(\mathbf{Z})$ is well-defined.

Proof. Suppose $g_1 W_\infty = g_2 W_\infty$ for $g_1, g_2 \in \mathrm{SL}_2(\mathbf{Z})$. Thus $\rho(g_1^{-1}) L^*_{g_1}(\phi)$ and $\rho(g_2^{-1}) L^*_{g_2}(\phi)$ have the same width w with respect to Γ, and $g_2^{-1} g_1 \in \Gamma_\infty = \mathrm{Stab}_\Gamma(\infty)$. Write $\tilde{g} = g_1^{-1} g_2$, and $\psi = \rho(g_2^{-1}) L^*_{g_2}(\phi)$. Note that since $\phi \in A^\bullet(\Gamma, \mathfrak{h}, V), \psi \in A^\bullet(g_2^{-1}\Gamma g_2, \mathfrak{h}, V)$. We want to show

$$\begin{aligned}
& \rho(g_1) L^*_{g_1^{-1}} H_x\left(\rho(g_1^{-1}) L^*_{g_1}\phi\right) = \rho(g_2) L^*_{g_2^{-1}} H_x\left(\rho(g_2^{-1}) L^*_{g_2}\phi\right) \\
\iff \quad & H_x\left(\rho(g_1^{-1}) L^*_{g_1} L^*_{g_2^{-1}} \rho(g_2)\psi\right) = \rho(\tilde{g}) L^*_{\tilde{g}^{-1}} H_x(\psi) \\
\iff \quad & H_x\left(\rho(\tilde{g}) L^*_{\tilde{g}^{-1}}\psi\right) = \rho(\tilde{g}) L^*_{\tilde{g}^{-1}} H_x(\psi)\ .
\end{aligned}$$

The last line is true because

$$\begin{aligned}
\rho(\tilde{g}) L^*_{\tilde{g}^{-1}} H_x(\psi) &= \rho(\tilde{g}) L^*_{\tilde{g}^{-1}}\left(\chi(y)\frac{1}{w}\int_0^w \left(s - \frac{w}{2}\right)\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\psi_1(s+x, y)ds\right) \\
&= \chi(y)\frac{1}{w}\int_0^w \left(s - \frac{w}{2}\right)\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\rho(\tilde{g}) L^*_{\tilde{g}^{-1}}\psi_1(s+x, y)ds \\
&= H_x\left(\rho(\tilde{g}) L^*_{\tilde{g}^{-1}}\psi\right)
\end{aligned}$$

since $C_1\infty$ and $C_2\infty$ have the same width w with respect to Γ. □

Proposition 6.4.2. $H_x(\phi) \in A^\bullet(\Gamma, \mathfrak{h}, V)$, that is, $L^*_\gamma\big(H_x(\phi)\big) = \rho(\gamma) H_x(\phi)$ for $\gamma \in \Gamma$.

Proof. Let $g \in \mathrm{SL}_2(\mathbf{Z})$. On $\mathfrak{h} \setminus \cup_{g \in \mathrm{SL}_2(\mathbf{Z})} g W_\infty$,

$$L_\gamma(H_x(\phi)) = L_\gamma(0) = 0\ .$$

Now consider the regions gW_∞. Since ϕ satisfies the condition in equation (3.1.1) such that $\phi = \rho(\gamma)L^*_{\gamma^{-1}}\phi$, we compute

$$\begin{aligned} L^*_\gamma\big(H_x(\phi)\big|_{gW_\infty}\big) &= L^*_\gamma\left(\rho(g)L^*_{g^{-1}}\left(H_x\left(\rho(g^{-1})L^*_g\phi\right)\Big|_{W_\infty}\right)\right) \\ &= \rho(\gamma)\rho(\gamma^{-1}g)L^*_{g^{-1}\gamma}\left(H_x\left(\rho(g^{-1})L^*_g\rho(\gamma)L^*_{\gamma^{-1}}\phi\right)\Big|_{W_\infty}\right) \\ &= \rho(\gamma)\rho(\gamma^{-1}g)L^*_{g^{-1}\gamma}\left(H_x\left(\rho(g^{-1}\gamma)L^*_{\gamma^{-1}g}\phi\right)\Big|_{W_\infty}\right) \\ &= \rho(\gamma)\left(Hx(\phi)\Big|_{\gamma^{-1}gW_\infty}\right) . \end{aligned}$$

By formula (6.1.2), we get that $P_x \in A^\bullet(\Gamma, \mathfrak{h}, V)$. □

6.5 First Isomorphism of the Two Cohomology

With the above notion of H_x and P_x, we define

$$A^\bullet_{\text{inv}}(\Gamma, \mathfrak{h}, V) = \left\{\phi \in A^\bullet(\Gamma, \mathfrak{h}, V) \colon \left(\rho\left(\begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix}\right)\rho(g^{-1})L^*_g\phi(x,y)\right)\Big|_{V_\infty} \text{ is constant in } x \,\forall\, g \in \mathrm{SL}_2(\mathbf{Z})\right\} \tag{6.5.1}$$

Proposition 6.5.1. H_x and P_x preserve $A^\bullet_{\text{inv}}(\Gamma, \mathfrak{h}, V)$.

Proof. We first show this is true in W_∞. We have

$$\begin{aligned} \rho\left(\begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix}\right)H_x &= \rho\left(\begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix}\right)\chi(y)\frac{1}{w}\int_0^w\left(s-\frac{w}{2}\right)\rho\left(\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\phi_1(s+x,y)ds \\ &= \chi(y)\frac{1}{w}\int_0^w\left(s-\frac{w}{2}\right)\rho\left(\begin{pmatrix}1 & -(s+x)\\ 0 & 1\end{pmatrix}\right)\phi_1(s+x,y)ds . \end{aligned}$$

Here $\rho\left(\begin{pmatrix}1 & -(s+x)\\ 0 & 1\end{pmatrix}\right)\phi_1(s+x,y)$ is independent of x since $\phi \in A^\bullet_{\text{inv}}(\Gamma, \mathfrak{h}, V)$ and thus $\phi_1 \in A^\bullet_{\text{inv}}(\Gamma, \mathfrak{h}, V)$. Therefore the integral is independent of x, and thus $\big(H_x(\phi)\big)\big|_{W_\infty} \in A^\bullet_{\text{inv}}(\mathrm{Stab}_\Gamma(\infty), W_\infty, V)$. By the extended definition of H_x on $\mathfrak{h}$ in section 6.4, it is not hard to see that $H_x(\phi) \in A^\bullet_{\mathrm{inv}}(\Gamma, \mathfrak{h}, V)$. Similarly one can show that $P_x(\phi) \in A^\bullet_{\mathrm{inv}}(\Gamma, \mathfrak{h}, V)$. □

Therefore $\mathcal{H}$ and $\tilde{P}$ from equation (6.2.3) yield a homotopy formula

$$d\mathcal{H} + \mathcal{H}d = \mathrm{I} - \tilde{P} \ .$$

More specifically, we have $\mathcal{H}$ and $\tilde{P}$ preserve $A^\bullet_{(2)}(\Gamma, \mathfrak{h}, V)$, $A^\bullet_P(\Gamma, \mathfrak{h}, V)$ and $A^\bullet_{\mathrm{inv}}(\Gamma, \mathfrak{h}, V)$, and moreover, from Proposition (6.2.5) we have $\tilde{P}(\phi) \in A^\bullet_{\mathrm{inv}}(\Gamma, \mathfrak{h}, V)$ for $\phi \in A^\bullet(\Gamma, \mathfrak{h}, V)$. Therefore, if we denote $H^1_{\mathrm{inv}\cap 2}(\Gamma, \mathfrak{h}, V)$ as the cohomology of $A^\bullet_{\mathrm{inv}}(\Gamma, \mathfrak{h}, V) \cap A^\bullet_{(2)}(\Gamma, \mathfrak{h}, V)$, and $H^1_{\mathrm{inv}\cap P}(\Gamma, \mathfrak{h}, V)$ as the cohomology of $A^\bullet_{\mathrm{inv}}(\Gamma, \mathfrak{h}, V) \cap A^\bullet_P(\Gamma, \mathfrak{h}, V)$, we obtain

$$H^1_{\mathrm{inv}\cap(2)}(\Gamma, \mathfrak{h}, V) \cong H^1_{(2)}(\Gamma, \mathfrak{h}, V) \ ,$$

and

$$H^1_{\mathrm{inv}\cap P}(\Gamma, \mathfrak{h}, V) \cong H^1_P(\Gamma, \mathfrak{h}, V) \ .$$

6.6 A Further H_{sp} Homotopy on Space of Special Forms

To establish the isomorphism between $H^1_{\mathrm{inv}\cap(2)}(\Gamma, \mathfrak{h}, V)$ and $H^1_{\mathrm{inv}\cap P}(\Gamma, \mathfrak{h}, V)$, we next consider a subcomplex $A^\bullet_{\mathrm{sp}}$ such that

$$A^\bullet(\Gamma, \mathfrak{h}, V) \supset A^\bullet_{\mathrm{inv}}(\Gamma, \mathfrak{h}, V) \supset A^\bullet_{\mathrm{sp}}(\Gamma, \mathfrak{h}, V)$$

where $A^\bullet_{\mathrm{inv}}(\Gamma, \mathfrak{h}, V)$ is obtained through an x homotopy via a differential form approach. This is the result of van Est [27] applied to our special case. On the other hand, $A^\bullet_{\mathrm{sp}}(\Gamma, \mathfrak{h}, V)$ is similar to the special differential forms of Goresky, Harder and MacPherson [11].

For $g \in \mathrm{SL}_2(\mathbf{Z})$, write $L_g^*\phi = (L_g^*\phi)_0 + dx \wedge (L_g^*\phi)_1$, and define the space of special forms to be

$$A^\bullet_{\mathrm{sp}}(\Gamma,\mathfrak{h},V) = \Big\{\phi \in A^\bullet_{\mathrm{inv}}(\Gamma,\mathfrak{h},V)\colon \left(\rho\left(\begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix}\right)\rho(g^{-1})(L_g^*\phi)_0\right)\Big|_{V_\infty} \in \ker N \text{ and}$$
$$\left(\rho\left(\begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix}\right)\rho(g^{-1})(L_g^*\phi)_1\right)\Big|_{V_\infty} \in (\mathrm{Im}\,N)^\perp \text{ w.r.t. } \langle\cdot,\cdot\rangle_0\ \forall g \in \mathrm{SL}_2(\mathbf{Z})\Big\} \tag{6.6.1}$$

where as before,

$$N = d(\rho)_e \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} : V \longrightarrow V\ .$$

Write $V = \ker N \oplus (\ker N)^\perp = (\mathrm{Im}\,N)^\perp \oplus \mathrm{Im}\,N$, where the orthogonal complements are with respect to $\langle\cdot,\cdot\rangle_0$ on V. Then

$$N(\ker N \oplus (\ker N)^\perp) = 0 \oplus \mathrm{Im}\,N\ ,$$

and define N^+ as the pseudoinverse of N,

$$N^+\colon V \longrightarrow V$$

such that

$$N^+((\mathrm{Im}\,N)^\perp \oplus \mathrm{Im}\,N) = 0 \oplus (\ker N)^\perp\ .$$

Remark. Therefore

$$N^+N(\ker N \oplus (\ker N)^\perp) = (\ker N)^\perp\ ,$$
$$NN^+((\mathrm{Im}\,N)^\perp \oplus \mathrm{Im}\,N) = \mathrm{Im}\,N\ .$$

For $\phi \in A^\bullet_{\mathrm{inv}}(\Gamma,\mathfrak{h},V)$, we now wish to define $H_{\mathrm{sp}}\phi \in A^\bullet_{\mathrm{inv}}(\Gamma,\mathfrak{h},V)$. We define it in W_∞ by

$$(H_{\mathrm{sp}}\phi)\Big|_{W_\infty}(x,y) = \chi(y)\rho\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)N^+\rho\left(\begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix}\right)\iota_{\frac{\partial}{\partial x}}\phi(x,y) \tag{6.6.2}$$

and extend to all of $\mathfrak{h}$ as we did for H_x in section 6.4. We now calculate the resulting homotopy formula

$$dH_{\mathrm{sp}}\phi + H_{\mathrm{sp}}d\phi = \phi - P_{\mathrm{sp}}\phi$$

in V_∞. Since $\phi \in A^\bullet_{\mathrm{inv}}(\Gamma, \mathfrak{h}, V), \phi = \rho\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\phi_0(0,y) + dx \wedge \rho\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)$ $\phi_1(0,y)$ for $y \geq M$. Write ψ_0 and ψ_1 in short for $\phi_0(0,y)$ and $\phi_1(0,y)$, and γ_x for $\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$. Thus in V_∞,

$$d\phi = dy \wedge \frac{\partial}{\partial y}\left(\rho\left(\gamma_x\right)\psi_0\right) + dx \wedge \left[\frac{\partial}{\partial x}\left(\rho\left(\gamma_x\right)\psi_0\right) - dy \wedge \frac{\partial}{\partial y}\left(\rho\left(\gamma_x\right)\psi_1\right)\right]$$

$$= dy \wedge \rho\left(\gamma_x\right)\frac{\partial \psi_0}{\partial y} + dx \wedge \left[N\rho\left(\gamma_x\right)\psi_0 - dy \wedge \rho\left(\gamma_x\right)\frac{\partial \psi_1}{\partial y}\right].$$

Therefore we compute in V_∞ where $\chi(y) = 1$,

$$dH_{\mathrm{sp}}\phi = dy \wedge \frac{\partial}{\partial y}\big(\rho\left(\gamma_x\right)N^+\rho\left(\gamma_{-x}\right)\rho\left(\gamma_x\right)\psi_1\big) + dx \wedge \frac{\partial}{\partial x}\big(\rho\left(\gamma_x\right)N^+\rho\left(\gamma_{-x}\right)\rho\left(\gamma_x\right)\psi_1\big)$$

$$= dy \wedge \rho(\gamma_x)N^+\frac{\partial \psi_1}{\partial y} + dx \wedge N\rho\left(\gamma_x\right)N^+\psi_1$$

$$= dy \wedge \rho\left(\gamma_x\right)N^+\frac{\partial \psi_1}{\partial y} + dx \wedge \rho\left(\gamma_x\right)\psi_{1\mathrm{Im}}$$

since N and ρ commute., and H_{sp} does not have a dx term. Here $\psi_{1\mathrm{Im}}$ is the projection of ψ_1 onto $\operatorname{Im} N$ with respect to $\langle\cdot,\cdot\rangle_0$. On the other hand, we compute in V_∞,

$$H_{\mathrm{sp}}d\phi = \rho\left(\gamma_x\right)N^+\rho\left(\gamma_{-x}\right)\left(N\rho\left(\gamma_x\right)\psi_0 - dy \wedge \rho\left(\gamma_x\right)\frac{\partial \psi_1}{\partial y}\right)$$

$$= \rho\left(\gamma_x\right)\psi_{0\ker^\perp} - dy \wedge \rho(\gamma_x)N^+\frac{\partial \psi_1}{\partial y}.$$

Here $\psi_{0\ker^\perp}$ is the projection of ψ_0 onto $(\ker N)^\perp$ with respect to $\langle\cdot,\cdot\rangle_0$. Therefore the homotopy formula restricts in V_∞ to

$$dH_{\mathrm{sp}}\phi + H_{\mathrm{sp}}d\phi = dx \wedge \rho\left(\gamma_x\right)\psi_{1\mathrm{Im}} + \rho\left(\gamma_x\right)\psi_{0\ker^\perp}$$

$$= \phi - P_{\mathrm{sp}}\phi$$

where

$$\begin{aligned}(P_{\mathrm{sp}}\phi)\big|_{V_\infty} &= \rho\left(\gamma_x\right)\psi_{0\mathrm{ker}} + dx\wedge\rho\left(\gamma_x\right)\psi_{1\mathrm{Im}^\perp} \\ &= \rho\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\phi_0(0,y)_{\mathrm{ker}} + dx\wedge\rho\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\phi_1(0,y)_{\mathrm{Im}^\perp} \\ &\in A^\bullet_{\mathrm{sp}}(\mathrm{Stab}_\Gamma(\infty), V_\infty, V)\ .\end{aligned}$$

Although we have only calculated in V_∞, it follows by an analogue of the extended definition of H_x in section 6.4 that $P_{\mathrm{sp}}\in A^\bullet_{\mathrm{sp}}(\Gamma,\mathfrak{h},V)$. Moreover, it is not hard to see that both H_{sp} and P_{sp} preserve the L^2 condition and the parabolic condition. Hence in computing either the parabolic cohomology or the L^2 cohomology, we can restrict to forms in $A^\bullet_{\mathrm{sp}}(\Gamma,\mathfrak{h},V)$.

6.7 y-Homotopy to Establish the Isomorphism Part I

We now consider a homotopy operator in y direction on forms in $A^\bullet_{\mathrm{sp}}(\Gamma,\mathfrak{h},V)$. We will assume $V=V_k=\mathrm{Sym}^{k-2}(\mathbf{C}^2)$ and thus in terms of the standard basis $\{e_1,e_2\}$ of $\mathbf{C}^2$, we have

$$\ker N = \mathrm{Span}\{e_1^{k-2}\}\ ,$$
$$(\mathrm{Im}\,N)^\perp = \mathrm{Span}\{e_2^{k-2}\}\ .$$

We define a sub-complex of the space of special forms to be

$$A^\bullet_M(\Gamma,\mathfrak{h},V) = \Big\{\phi\in A^\bullet_{\mathrm{sp}}(\Gamma,\mathfrak{h},V)\colon \left(\iota_{\frac{\partial}{\partial x}}\rho(g^{-1})(L_g^*\phi)\right)\Big|_{V_\infty} = 0, \left(\iota_{\frac{\partial}{\partial y}}\rho(g^{-1})(L_g^*\phi)\right)\Big|_{V_\infty} = 0 \text{ and } \left(\rho(g^{-1})(L_g^*\phi)\right)\Big|_{V_\infty} \text{ is constant in } y\,\forall\, g\in\mathrm{SL}_2(\mathbf{Z})\Big\}$$

Proposition 6.7.1. $A^\bullet_M(\Gamma,\mathfrak{h},V)\subset A^\bullet_P(\Gamma,\mathfrak{h},V)$ and $A^\bullet_M(\Gamma,\mathfrak{h},V)\subset A^\bullet_{(2)}(\Gamma,\mathfrak{h},V)$.

Proof. Suppose $\phi \in A^{\bullet}_{M}(\Gamma, \mathfrak{h}, V)$, on one hand, we have

$$0 = \int_0^w \iota_{\frac{\partial}{\partial x}} \rho(g^{-1})(L_g^* \phi)(s, y) ds \in \operatorname{Im}(\rho(g\gamma_\infty) - \mathrm{I})$$

for all $g \in \mathrm{SL}_2(\mathbf{Z})$. Thus $A^{\bullet}_{M}(\Gamma, \mathfrak{h}, V) \in A^{\bullet}_{P}(\Gamma, \mathfrak{h}, V)$.

On the other hand, since ϕ does not have dx nor dy terms, in V_∞, we can write

$$\phi(x, y)\Big|_{V_\infty} = \rho\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \phi_0(0, M)$$

where ϕ_0 has no dy term and $\phi_0 \in \ker N$. Let $g \in \mathrm{SL}_2(\mathbf{Z})$ such that $gi = \tau$. Thus again

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \dfrac{1}{\sqrt{y}} \end{pmatrix} .$$

We compute in V_∞

$$\begin{aligned} \|\phi\|_\tau^2 &= \left\|\rho(g^{-1})\phi(x, y)\right\|_0^2 \\ &= \left\| \rho\left(\begin{pmatrix} \dfrac{1}{\sqrt{y}} & 0 \\ 0 & \sqrt{y} \end{pmatrix}\right) \rho\left(\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}\right) \rho\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \phi_0(0, M) \right\|_0^2 \\ &= \left\| \rho\left(\begin{pmatrix} \dfrac{1}{\sqrt{y}} & 0 \\ 0 & \sqrt{y} \end{pmatrix}\right) \phi_0(0, M) \right\|_0^2 . \end{aligned}$$

Since $\phi_0 \in \ker N = \mathrm{Span}\{e_1^{k-2}\}$, we have

$$\left\| \rho\left(\begin{pmatrix} \dfrac{1}{\sqrt{y}} & 0 \\ 0 & \sqrt{y} \end{pmatrix}\right) \phi_0(0, M) \right\|_0^2 = C \cdot \left(\frac{1}{y}\right)^{k-2}$$

where C is some constant. Thus in V_∞

$$\begin{aligned}\|\phi\|_{L^2}^2 &= \int_M^\infty \int_0^w \|\phi\|_\tau^2 \frac{dx\,dy}{y^2} \\ &= Cw \int_M^\infty \left(\frac{1}{y}\right)^k dy \\ &= \frac{Cw}{k-1} M^{-k+1} < \infty \ .\end{aligned}$$

The last step is obtained because we can set weight $k > 1$ so that $-k+1 < 0$. This result can be easily generalized to all gV_∞ for $g \in \mathrm{SL}_2(\mathbf{Z})$, and ϕ is bounded in the region $\mathfrak{h} \backslash \cup_{g \in \mathrm{SL}_2(\mathbf{Z})} gV_\infty$, thus ϕ is L^2 in all of $\mathfrak{h}$. The same is true for $d\phi$, that is, if $d\phi \in A_M^\bullet(\Gamma, \mathfrak{h}, V)$, then $d\phi \in\in A_{(2)}^\bullet(\Gamma, \mathfrak{h}, V)$. Therefore $A_M^\bullet(\Gamma, \mathfrak{h}, V) \subset A_{(2)}^\bullet(\Gamma, \mathfrak{h}, V)$. $\square$

Thus we have $A_M^\bullet(\Gamma, \mathfrak{h}, V) \in A_{\mathrm{sp}\cap P}^\bullet(\Gamma, \mathfrak{h}, V)$ and $A_M^\bullet(\Gamma, \mathfrak{h}, V) \in A_{\mathrm{sp}\cap(2)}^\bullet(\Gamma, \mathfrak{h}, V)$. Now we define the homotopy operators in y direction that establishes an equivalence between $A_P^\bullet(\Gamma, \mathfrak{h}, V)$ and $A_{(2)}^\bullet(\Gamma, \mathfrak{h}, V)$ respectively to $A_M^\bullet(\Gamma, \mathfrak{h}, V)$. First in W_∞ we define

$$\begin{aligned}(H_M\phi)\Big|_{W_\infty} &= \chi(y) \int_M^y \left(\iota_{\frac{\partial}{\partial y}}\phi\right)(x,s)ds \ , \\ \text{and} \quad (H_\infty\phi)\Big|_{W_\infty} &= -\chi(y) \int_y^\infty \left(\iota_{\frac{\partial}{\partial y}}\phi\right)(x,s)ds \ ,\end{aligned}$$

and extend to all of $\mathfrak{h}$ as we did for H_x in section 6.4. Then we define the homotopy operator on $A_{\mathrm{sp}\cap P}^\bullet(\Gamma, \mathfrak{h}, V)$ to be

$$H_{y,P}\phi = H_M\phi \ ,$$

and define the homotopy operator on $A_{\mathrm{sp}\cap(2)}^\bullet(\Gamma, \mathfrak{h}, V)$ to be

$$H_{y,(2)}\phi = \begin{cases} H_M\phi & \text{if } \deg(\phi) = 1 \\ H_\infty\phi & \text{if } \deg(\phi) = 2 \ . \end{cases}$$

We now calculate the corresponding homotopy formulas. In this case, we can rewrite ϕ as

$$\phi = \iota_{\frac{\partial}{\partial y}} dy \wedge \phi + dy \wedge \iota_{\frac{\partial}{\partial y}} \phi \ .$$

We start with

$$dH_M\phi + H_M d\phi = \phi - P_M\phi$$

in W_∞. We have

$$d\phi = dx \wedge \frac{\partial}{\partial x}\left(\iota_{\frac{\partial}{\partial y}} dy \wedge \phi\right) + dy \wedge \left(\frac{\partial}{\partial y}\left(\iota_{\frac{\partial}{\partial y}} dy \wedge \phi\right) - dx \wedge \frac{\partial}{\partial x}\left(\iota_{\frac{\partial}{\partial y}} \phi\right)\right) \ .$$

We then compute the following in W_∞. Using integration by parts, we have

$$\begin{aligned} H_M d\phi &= \chi(y) \int_M^y \left(\frac{\partial}{\partial y}\left(\iota_{\frac{\partial}{\partial y}} dy \wedge \phi\right) - dx \wedge \frac{\partial}{\partial x}\left(\iota_{\frac{\partial}{\partial y}} \phi\right)\right)(x,s)ds \\ &= \chi(y)\left[\iota_{\frac{\partial}{\partial y}} dy \wedge \phi - \iota_{\frac{\partial}{\partial y}} dy \wedge \phi(x,M)\right] - \chi(y)\int_M^y dx \wedge \frac{\partial}{\partial x}\left(\iota_{\frac{\partial}{\partial y}} \phi\right)(x,s)ds \end{aligned}$$

Define

$$\begin{aligned} \iota_M^* \colon \mathfrak{h} &\longrightarrow \mathfrak{h} \\ (x,y) &\longrightarrow (x,M) \ . \end{aligned}$$

Thus $\iota_{\frac{\partial}{\partial y}} dy \wedge \phi(x,M) = \iota_M^*\phi$ and it is independent of y. On the other hand, we compute

$$\begin{aligned} dH_M\phi = \chi(y) dy \wedge \iota_{\frac{\partial}{\partial y}} \phi + \chi(y) dx \wedge \int_M^y \frac{\partial}{\partial x}\left(\iota_{\frac{\partial}{\partial y}} \phi\right)(x,s)ds \\ + dy \wedge \chi'(y) \int_M^y \left(\iota_{\frac{\partial}{\partial y}} \phi\right)(x,s)ds \ . \end{aligned} \tag{6.7.1}$$

Thus we have in W_∞,

$$\begin{aligned} H_M d\phi + dH_M\phi &= \phi - \left((1-\chi(y))\phi + \chi(y)\iota_M^*\phi - dy \wedge \chi'(y)\int_M^y \left(\iota_{\frac{\partial}{\partial y}} \phi\right)(x,s)ds\right) \\ &= \phi - P_M\phi \ . \end{aligned}$$

where $P_M\phi\Big|_{W_\infty}$ is defined by the last line. From the definition of H_M and P_M, it is clear that both preserve $A^\bullet_{\mathrm{sp}\cap P}$ and $A^\bullet_M$ in V_∞. Moreover, for $\phi \in A^\bullet_{\mathrm{sp}\cap P}$, ϕ does not have a dx term. Thus in V_∞, $P_M\phi = \iota^*_M\phi \in A^\bullet_M$. This result can be extended to all of $\mathfrak{h}$ by a similar extension of H_M and P_M as we did for H_x in section 6.4. Therefore H_M establishes a homotopy equivalence between $A^\bullet_{\mathrm{sp}\cap P}(\Gamma, \mathfrak{h}, V)$ and $A^\bullet_M(\Gamma, \mathfrak{h}, V)$. Thus

$$H^1_{\mathrm{sp}\cap P}(\Gamma, \mathfrak{h}, V) \cong H^1_M(\Gamma, \mathfrak{h}, V)\ . \tag{6.7.2}$$

6.8 y-Homotopy to Establish the Isomorphism Part II

Next we want to calculate the homotopy formula for $H_{y,(2)}$. We have the following chain complex

$$0 \longrightarrow A^0_{\mathrm{sp}\cap(2)} \underset{H_M}{\overset{d}{\rightleftarrows}} A^1_{\mathrm{sp}\cap(2)} \underset{H_\infty}{\overset{d}{\rightleftarrows}} A^2_{\mathrm{sp}\cap(2)} \longrightarrow 0\ .$$

Before we start, we give the following observation.

Observation 1. $dx \wedge \dfrac{\partial}{\partial x}\left(\iota_{\frac{\partial}{\partial y}}\phi\right) = 0$ for $\phi \in A^\bullet_{\mathrm{sp}}(\Gamma, \mathfrak{h}, V)$.

Proof. Write $\phi = \phi_0 + dx \wedge \phi_1$. Since $\phi \in A^\bullet_{\mathrm{sp}}(\Gamma, \mathfrak{h}, V)$, we have $\phi_0 \in \ker N$ and $\phi_1 \in (\mathrm{Im}\, N)^\perp$. Thus we compute

$$\begin{aligned} dx \wedge \frac{\partial}{\partial x}\left(\iota_{\frac{\partial}{\partial y}}\phi\right) &= dx \wedge \frac{\partial}{\partial x}\left(\iota_{\frac{\partial}{\partial y}}\phi_0\right) + dx \wedge \frac{\partial}{\partial x}\left(\iota_{\frac{\partial}{\partial y}} dx \wedge \phi_1\right) \\ &= dx \wedge \frac{\partial}{\partial x}\left(\iota_{\frac{\partial}{\partial y}}\phi_0\right)\ . \end{aligned}$$

Since $\phi \in A^\bullet_{\mathrm{inv}}(\Gamma, \mathfrak{h}, V)$, ϕ_0 can also be written as

$$\phi_0 = \rho\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\phi(0, y)\ .$$

Thus $\dfrac{\partial}{\partial x}\phi_0 = N\phi_0 = 0$ since $\phi_0 \in \ker N$. Therefore

$$\begin{aligned} dx \wedge \frac{\partial}{\partial x}\left(\iota_{\frac{\partial}{\partial y}}\phi\right) &= dx \wedge \frac{\partial}{\partial x}\left(\iota_{\frac{\partial}{\partial y}}\phi_0\right) \\ &= dx \wedge \iota_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x}(\phi_0) \\ &= 0 \ . \end{aligned} \qquad \square$$

Again we first focus in the region in W_∞, and the definition and computation can be easily carried over to all of $\mathfrak{h}$ via a similar approach as we did for H_x in section 6.4. We start with $\phi \in A^0_{\mathrm{sp}\cap(2)}$, and the homotopy formula is

$$H_M d\phi = \phi - P_0\phi \ .$$

Note here P_0 is not the same as the one in section 6.2 In this section we use the subscript to denote the degree of the forms that P acts on. We have as before

$$\begin{aligned} H_M d\phi &= \chi(y)\int_M^y \left(\frac{\partial}{\partial y}\left(\iota_{\frac{\partial}{\partial y}} dy \wedge \phi\right) - dx \wedge \frac{\partial}{\partial x}\left(\iota_{\frac{\partial}{\partial y}}\phi\right)\right)(x,s)ds \\ &= \chi(y)\left[\iota_{\frac{\partial}{\partial y}} dy \wedge \phi - \iota_{\frac{\partial}{\partial y}} dy \wedge \phi(x,M)\right] - \chi(y)\int_M^y dx \wedge \frac{\partial}{\partial x}\left(\iota_{\frac{\partial}{\partial y}}\phi\right)(x,s)ds \ . \end{aligned}$$

Since ϕ is degree 0, $\iota_{\frac{\partial}{\partial y}} dy \wedge \phi = \phi$. Again write $\iota_{\frac{\partial}{\partial y}} dy \wedge \phi(x,M) = \iota_M^*\phi$. Thus

$$\begin{aligned} H_M d\phi &= \phi - \Big((1-\chi(y))\phi + \iota_M^*\phi\Big) \\ &= \phi - P_0\phi \ . \end{aligned}$$

Next move to $\phi \in A^1_{\mathrm{sp}\cap(2)}$. The homotopy formula now is

$$dH_M\phi + H_\infty d\phi = \phi - P_1\phi \ .$$

As in equation (6.7.1) in W_∞,

$$dH_M\phi = \chi(y)dy \wedge \iota_{\frac{\partial}{\partial y}}\phi + \chi(y)dx \wedge \int_M^y \frac{\partial}{\partial x}\left(\iota_{\frac{\partial}{\partial y}}\phi\right)(x,s)ds$$

$$+ dy \wedge \chi'(y)\int_M^y \left(\iota_{\frac{\partial}{\partial y}}\phi\right)(x,s)ds$$

$$= \chi(y)dy \wedge \iota_{\frac{\partial}{\partial y}}\phi + dy \wedge \chi'(y)\int_M^y \left(\iota_{\frac{\partial}{\partial y}}\phi\right)(x,s)ds \qquad \text{by Observation 1.}$$

On the other hand, we compute in W_∞,

$$H_\infty d\phi = -\chi(y)\int_y^\infty \left(\frac{\partial}{\partial y}\left(\iota_{\frac{\partial}{\partial y}}dy \wedge \phi\right) - dx \wedge \frac{\partial}{\partial x}\left(\iota_{\frac{\partial}{\partial y}}\phi\right)\right) \quad (x,s)ds$$

$$= -\chi(y)\left[\lim_{M\to\infty}\iota_{\frac{\partial}{\partial y}}dy \wedge \phi(0,M) - \iota_{\frac{\partial}{\partial y}}dy \wedge \phi\right] \qquad \text{by Observation 1,}$$

$$= \chi(y)\iota_{\frac{\partial}{\partial y}}dy \wedge \phi \qquad \text{since } \phi = 0 \text{ near } \infty.$$

Therefore

$$dH_M\phi + H_\infty d\phi = \phi - \left[(1-\chi(y))\phi - dy \wedge \chi'(y)\int_M^y \left(\iota_{\frac{\partial}{\partial y}}\phi\right)(x,s)ds\right]$$

$$= \phi - P_2\phi\ .$$

Lastly, when $\phi \in A^2_{\mathrm{sp}\cap(2)}$. The homotopy formula becomes

$$dH_\infty\phi = \phi - P_2\phi\ .$$

We compute in W_∞,

$$dH_\infty\phi = \chi(y)dy \wedge \iota_{\frac{\partial}{\partial y}}\phi - \chi(y)dx \wedge \int_y^\infty \frac{\partial}{\partial x}\Big(\iota_{\frac{\partial}{\partial y}}\phi\Big)(x,s)ds$$

$$- dy \wedge \chi'(y)\int_y^\infty Big(\iota_{\frac{\partial}{\partial y}}\phi\Big)(x,s)ds$$

$$= \chi(y)dy \wedge \iota_{\frac{\partial}{\partial y}}\phi - dy \wedge \chi'(y)\int_y^\infty Big(\iota_{\frac{\partial}{\partial y}}\phi\Big)(x,s)ds \quad \text{by Observation 1.}$$

Since ϕ is degree 2, it has both dx and dy terms. Thus $\chi(y)dy \wedge \iota_{\frac{\partial}{\partial y}}\phi = \chi(y)$ ϕ. Therefore

$$dH_\infty\phi = \phi - \left[(1-\chi(y))\phi + dy \wedge \chi'(y)\int_y^\infty Big(\iota_{\frac{\partial}{\partial y}}\phi\Big)(x,s)ds\right]$$

$$= \phi - P_2\phi \ .$$

Combining the above, we obtain

$$H_{y,(2)}\phi = \begin{cases} H_M\phi & \text{if } \deg(\phi) = 1 \\ H_\infty\phi & \text{if } \deg(\phi) = 2 \ . \end{cases}$$

The homotopy formula is

$$dH_{y,(2)}\phi + H_{y,(2)}d\phi = \phi - P_{y,(2)}\phi$$

where

$$P_{y,(2)}\phi = \begin{cases} \iota_M^*\phi & \text{if } \deg(\phi) = 0 \\ 0 & \text{if } \deg(\phi) = 2 \ . \end{cases}$$

After we extend H_M, H_∞ and $P_{y,(2)}$ by a similar approach as we did for H_x in section 6.4, it is not hard to see that they all preserve $A^\bullet_{\text{sp}}(\Gamma, \mathfrak{h}, V)$. Moreover, if $\phi \in A^\bullet_M(\Gamma, \mathfrak{h}, V)$, then $H_M\phi = H_\infty\phi = 0$, and $P_{y,(2)}\phi$ all belong to $A^\bullet_M(\Gamma, \mathfrak{h}, V)$.

Proposition 6.8.1. H_M, H_∞ and $P_{y,(2)}$ are bounded operators in L^2.

Proof. First we consider H_M defined on $\phi \in A^1_{\text{sp}}(\Gamma, \mathfrak{h}, V)$. We can write

$$\phi = dy \wedge \phi_y + dx \wedge \phi_x$$

where ϕ_y and ϕ_x are vector valued functions and by definition, $\rho\left(\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}\right)\phi_y \in$ $\ker N = \text{Span}\{e_1^{k-2}\}$. Thus we can write $\rho\left(\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}\right)\phi_y = f(y)e_1^{k-2}$ where $f(y)$ is a function that depends on y. Again write

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \dfrac{1}{\sqrt{y}} \end{pmatrix}$$

such that $g \cdot i = \tau$, we compute

$$\begin{aligned}
\left\| \iota_{\frac{\partial}{\partial y}} \phi(x,y) \right\|_\tau^2 &= \|\phi_y\|_\tau^2 \\
&= \left\| \rho(g^{-1}) \phi_y \right\|_0^2 \\
&= \left\| \rho\left(\begin{pmatrix} \frac{1}{\sqrt{y}} & 0 \\ 0 & \sqrt{y} \end{pmatrix} \right) \rho\left(\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \right) \phi_y \right\|_0^2 \\
&= \left\| \rho\left(\begin{pmatrix} \frac{1}{\sqrt{y}} & 0 \\ 0 & \sqrt{y} \end{pmatrix} \right) f(y) e_1^{k-2} \right\|_0^2 \\
&= \left(\frac{1}{y} \right)^{k-2} \|f(y)\|_0^2 \ .
\end{aligned}$$

We then have

$$\begin{aligned}
\|H_M \phi\|_{L^2}^2 &= \int_0^w \int_M^\infty \left\| \int_M^y \left(\iota_{\frac{\partial}{\partial y}} \phi \right)(x,s) ds \right\|_\tau^2 \frac{dy\,dx}{y^2} \\
&\leq \int_0^w \int_M^\infty \Big(\int_M^y \left\| \iota_{\frac{\partial}{\partial y}} \phi(x,s) \right\|_\tau ds \Big)^2 \frac{dy\,dx}{y^2} \\
&= C \int_0^w \int_M^\infty \Big(\int_M^y \left(\frac{1}{y} \right)^{\frac{k-2}{2}} \|f(s)\|_0 \, ds \Big)^2 \frac{dy\,dx}{y^2}
\end{aligned}$$

where C denotes an arbitrary constant. Since we can safely assume the weight $k \geq 2$, we can pick $\alpha \in \mathbf{R}$ such that $-1 < \alpha < k-2$. Let $\psi(s) = s^\alpha$, we use Hölder's

Inequality on $\|H_M\phi\|_{L^2}^2$ with $\psi(s)$,

$$\begin{aligned}
\|H_M\phi\|_{L^2}^2 &\leq C\int_0^w\int_M^\infty \Big(\int_M^y \left(\frac{1}{y}\right)^{k-2} \|f(s)\|_0^2\,\psi^{-1}(s)ds\Big)\Big(\int_M^y \psi(s)ds\Big)\frac{dy\,dx}{y^2}\\
&= C\int_0^w\int_M^\infty\int_s^\infty \left(\frac{1}{y}\right)^{k-2} \|f(s)\|_0^2\,\psi^{-1}(s)\Big(\int_M^y \psi(s)ds\Big)\frac{1}{y^2}dy\,ds\,dx\\
&= C\int_0^w\int_M^\infty \|f(s)\|_0^2\,\psi^{-1}(s)s^2\Big(\int_s^\infty \left(\frac{1}{y}\right)^{k}\left(\int_M^y \psi(s)ds\right)dy\Big)\frac{ds\,dx}{s^2}\ ,
\end{aligned}$$

where the second step is by Fubini's Theorem. Here

$$\begin{aligned}
\int_s^\infty \left(\frac{1}{y}\right)^{k}\left(\int_M^y \psi(s)ds\right)dy &= \int_s^\infty \left(\frac{1}{y}\right)^{k}\left(\frac{1}{\alpha+1}s^{\alpha+1}\right)\Big|_M^y dy\\
&\leq C\int_s^\infty \left(\frac{1}{y}\right)^{k} y^{\alpha+1}dy \qquad \text{since } M \text{ is a fixed number}\\
&= C\Big(\frac{1}{|-k+\alpha+2|}y^{-k+\alpha+2}\ \Big|_s^\infty\Big)\\
&\leq C\cdot s^{-k+\alpha+2} \qquad \text{since } \alpha < k-2.
\end{aligned}$$

and thus

$$\begin{aligned}
\|H_M\phi\|_{L^2}^2 &\leq C\int_0^w\int_M^\infty \|f(s)\|_0^2\,s^{-\alpha}s^2s^{-k+\alpha+2}\frac{ds\,dx}{s^2}\\
&= C\int_0^w\int_M^\infty \|f(s)\|_0^2\,s^{4-k}\frac{ds\,dx}{s^2}\ .
\end{aligned}$$

Note that

$$\begin{aligned}
\|\phi(x,s)\|_{x+is}^2 &\geq \|dy\wedge\phi_y(x,s)\|_{x+is}^2\\
&= s^2\left(\frac{1}{s}\right)^{k-2}\|f(s)\|_0^2\\
&= \|f(s)\|_0^2\,s^{4-k}\ .
\end{aligned}$$

Therefore

$$\begin{aligned}\|H_M\phi\|_{L^2}^2 &\le C\int_0^w\int_M^\infty \|\phi(x,s)\|_{x+is}^2\,\frac{ds\,dx}{s^2}\\ &= C\,\|\phi\|_{L^2}^2\ .\end{aligned}$$

Next similarly we consider $H_\infty\phi$ defined on $\phi \in A^2_{\mathrm{sp}}(\Gamma, \mathfrak{h}, V)$. We can write

$$\phi = dx \wedge dy \wedge \phi_{xy}$$

where ϕ_{xy} is a vector valued function and by definition, $\rho\left(\begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix}\right)\phi_{xy} \in (\mathrm{Im}\,N)^\perp =$ $\mathrm{Span}\{e_2^{k-2}\}$. We then write $\rho\left(\begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix}\right)\phi_{xy} = g(y)e_2^{k-2}$ where $g(y)$ is a function that depends on y. Again we compute

$$\begin{aligned}\left\|\iota_{\frac{\partial}{\partial y}}\phi(x,y)\right\|_\tau^2 &= \|dx\wedge\phi_{xy}\|_\tau^2\\ &= y^2\left\|\rho(g^{-1})\phi_{xy}\right\|_0^2\\ &= y^2\left\|\rho\left(\begin{pmatrix}\frac{1}{\sqrt{y}} & 0\\ 0 & \sqrt{y}\end{pmatrix}\right)g(y)e_2^{k-2}\right\|_0^2\\ &= y^k\,\|g(y)\|_0^2\ .\end{aligned}$$

Thus we have

$$\begin{aligned}\|H_\infty\phi\|_{L^2}^2 &= \int_0^w\int_M^\infty\left\|\int_y^\infty\left(\iota_{\frac{\partial}{\partial y}}\phi\right)(x,s)ds\right\|_\tau^2\frac{dy\,dx}{y^2}\\ &= C\int_0^w\int_M^\infty\left(\int_y^\infty y^{\frac{k}{2}}\,\|g(s)\|_0\,ds\right)^2\frac{dy\,dx}{y^2}\end{aligned}$$

where C is an arbitrary constant. Since we can safely assume the weight $k \ge 2$, we can pick $\alpha \in \mathbf{R}$ such that $-k < \alpha < -1$. Let $\psi(s) = s^\alpha$, we use Hölder's Inequality on $\|H_\infty \phi\|^2_{L^2}$ with $\psi(s)$,

$$\|H_\infty \phi\|^2_{L^2} \leq C \int_0^w \int_M^\infty \Big(\int_y^\infty y^k \|g(s)\|_0^2 \, \psi^{-1}(s) ds \Big) \Big(\int_y^\infty \psi(s) ds \Big) \frac{dy\, dx}{y^2}$$

$$= C \int_0^w \int_M^\infty \|g(s)\|_0^2 \, \psi^{-1}(s) s^2 \Big(\int_M^s y^k \frac{1}{y^2} \left(\int_y^\infty \psi(s) ds \right) dy \Big) \frac{ds\, dx}{s^2}$$

where the last step is by Fubini's Theorem. Here

$$\int_M^s y^k \frac{1}{y^2} \left(\int_y^\infty \psi(s) ds \right) dy = \int_M^s y^{k-2} \left| \frac{1}{\alpha+1} s^{\alpha+1} \Big|_M^y \right| dy$$

$$\leq C \int_M^s y^{k-2} y^{\alpha+1} dy \qquad \text{since } \alpha < -1,$$

$$= C \left(\frac{1}{|k+\alpha|} y^{k+\alpha} \Big|_M^s \right)$$

$$\leq C \cdot s^{k+\alpha} \qquad \text{since } M \text{ is a fixed number,}$$

and thus

$$\|H_\infty \phi\|^2_{L^2} \leq C \int_0^w \int_M^\infty \|g(s)\|_0^2 \, s^{-\alpha} s^2 s^{k+\alpha} \frac{ds\, dx}{s^2}$$

$$= C \int_0^w \int_M^\infty \|g(s)\|_0^2 \, s^{k+2} \frac{ds\, dx}{s^2} \ .$$

Note that

$$\|\phi(x,s)\|^2_{x+is} = \|dx \wedge dy \wedge \phi_y(x,s)\|^2_{x+is}$$

$$= s^2 s^k \|g(s)\|_0^2$$

$$= \|g(s)\|_0^2 \, s^{k+2} \ .$$

Therefore

$$\|H_M \phi\|^2_{L^2} \leq C \int_0^w \int_M^\infty \|\phi(x,s)\|^2_{x+is} \frac{ds\, dx}{s^2}$$

$$= C \|\phi\|^2_{L^2} \ .$$

Lastly to show $P_{y,(2)}$ is L^2 bounded, we only need to focus on when $\phi \in A^0_{\mathrm{sp}}(\Gamma, \mathfrak{h}, V)$ and $P_{y,(2)}\phi = \iota^*_M \phi$. We have

$$\begin{aligned} \left\| P_{y,(2)}\phi(x,y) \right\|^2_\tau &= \left\| \iota^*_M \phi \right\|^2_\tau \\ &= \left\| \phi(x,M) \right\|^2_\tau \\ &\leq C \left\| \phi(x,y) \right\|^2_\tau \qquad \text{since } M \text{ is a fixed number} . \end{aligned}$$

Therefore

$$\begin{aligned} \left\| P_{y,(2)}\phi(x,y) \right\|^2_{L^2} &= \int_0^w \int_M^\infty \left\| P_{y,(2)}\phi(x,y) \right\|^2_\tau \frac{dy\,dx}{y^2} \\ &\leq C \int_0^w \int_M^\infty \left\| \phi(x,y) \right\|^2_\tau \frac{dy\,dx}{y^2} \\ &= C \left\| \phi(x,y) \right\|^2_{L^2} . \end{aligned}$$

□

Hence H_M and H_∞ establish a homotopy equivalence between $A^\bullet_{\mathrm{sp}\cap(2)}(\Gamma, \mathfrak{h}, V)$ and

$A^\bullet_M(\Gamma, \mathfrak{h}, V)$. Thus

$$H^1_{\mathrm{sp}\cap(2)}(\Gamma, \mathfrak{h}, V) \cong H^1_M(\Gamma, \mathfrak{h}, V) .$$

Together with equation (6.7.2), we have obtained

$$H^1_{\mathrm{sp}\cap P}(\Gamma, \mathfrak{h}, V) \cong H^1_{\mathrm{sp}\cap(2)}(\Gamma, \mathfrak{h}, V) ,$$

which then tells us

$$H^1_P(\Gamma, \mathfrak{h}, V) \cong H^1_{(2)}(\Gamma, \mathfrak{h}, V) .$$

Chapter 7

Hecke Correspondence In Terms of Dessins d'Enfants

In this chapter, we will first define dessins d'enfants and their relationship to modular curves, in which case we will call the corresponding graphs modular dessins. We will then define the cohomology group of the modular dessins, where we can proceed to study the Hecke correspondence.

7.1 Belyi's Theorem and Dessins d'Enfants of Modular Curves

Theorem 7.1.1 (Belyi's Theorem)**.** Let S be a compact Riemann surface. The following statements are equivalent:

(a) S is defined over $\overline{\mathbf{Q}}$.

(b) S admits a morphism $f\colon S \longrightarrow \mathbf{P}^1$ with at most three branching values.

Under such setting, any meromorphic function with at most three critical values is called a *Belyi map.* By composing with appropriate Möbius transformations, we can always assume the branching values are subset of $\{0, 1, \infty\}$. We also use (S, f) to denote a *Belyi pair* as described in [10], where S is a compact Riemann surface and f is a Belyi function. When compactified, the modular curve $X(\Gamma) = \Gamma \setminus \mathfrak{h}^*$ is a compact Riemann surface. Then consider the Klein j function, which is a modular function of weight 0 for $\mathrm{SL}_2(\mathbf{Z})$ defined on $\mathfrak{h}$. Let $\mu = e^{\pi i/3}$, $j(\mu) = 0, j(i) = 1728$, and $j(\infty) = \infty$. We consider the normalized j function, $\tilde{j} = \dfrac{1}{1728} \cdot j$. For any congruent subgroup Γ, $\tilde{j}$ sends the modular curve $X(\Gamma)$ to $\mathbf{P}^1$ that ramifies only at points $\Gamma\mu, \Gamma i$ and $\Gamma\infty$, with branching points $0, 1$ and ∞. Thus together, $\big(X(\Gamma), \tilde{j}\big)$ form a Belyi pair. The degree of the Belyi map $\tilde{j}$ is $[\mathrm{PSL}_2(\mathbf{Z})\colon \overline{\Gamma}]$ where $\overline{\Gamma}$ is the image of Γ in $\mathrm{PSL}_2(\mathbf{Z})$. Figure 7.1 shows a fundamental domain of the action of $\mathrm{SL}_2(\mathbf{Z})$ on the upper half plane $\mathfrak{h}$; it is the union of a white (ideal) triangle and a gray (ideal) triangle.

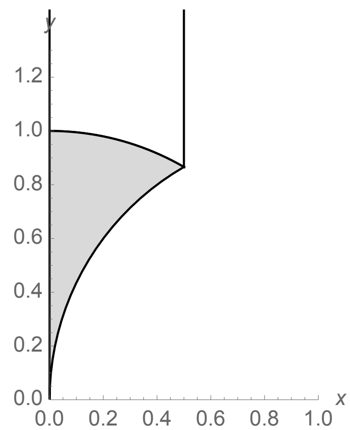


Figure 7.1: Fundamental Domain of $\mathrm{SL}_2(\mathbf{Z})\backslash\mathfrak{h}$.

Claim 7.1.2. $\tilde{j}$ sends the arc joining i and μ to the interval $[0, 1]$.

Proof. First we show that j is real on all $\mathrm{SL}_2(\mathbf{Z})$ translates of the boundaries of the standard fundamental domain. j has a power series representation

$$j(\tau) = \frac{1}{q} + \sum_{n=0}^{\infty} c_n q^n$$

where $q = e^{2\pi i\tau}$ and $c_n \in \mathbf{R}$ are the coefficients. Writing τ as $x + iy$, we can further express q as $e^{2\pi i(x+iy)} = e^{2\pi ix}e^{-2\pi y}$. Therefore

$$j \in \mathbf{R} \Longleftarrow e^{2\pi ix} \in \mathbf{R} \iff x \in \mathbf{Z} \cup \frac{1}{2}\mathbf{Z} \ .$$

Now for the arcs, we have

$$\gamma = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$$

such that $\gamma \cdot \frac{1}{2} = 1$ and $\gamma \cdot \infty = -1$. Thus γ sends the vertical line $x = \frac{1}{2}$ to the upper half of the unit circle. Since j is a modular function, $j(\gamma \cdot \tau) = j(\tau)$ for all $\gamma \in \mathrm{SL}_2(\mathbf{Z})$. Therefore j is also real on all the $\mathrm{SL}_2(\mathbf{Z})$ translates of the unit circle in the upper half plane.

$\tilde{j} = \frac{1}{1728} \cdot j$, thus $\tilde{j}$ is real on all $\mathrm{SL}_2(\mathbf{Z})$ translates of the boundaries of the standard fundamental domain. We also have $\tilde{j}(\mu) = 0$, $\tilde{j}(i) = 1$ and $\tilde{j}$ is a continuous map, thus $\tilde{j}$ maps the arc joining μ and i to the interval $[0, 1]$. □

On the other hand, we also have the notion of a dessin $\mathcal{D}$ associated with a Belyi pair. Girondo and González-Diez in their book [10] has established a detailed description of the one to one correspondence between equivalent dessins (X, D) and equivalent Belyi pairs (S_D, f_D). We start with the following definition:

Definition 7.1.3. A *dessin d'enfant*, or simply a *dessin*, is a pair (X, D) where X is an oriented compact topological surface, and $D \subset X$ is a finite graph such that:

(a) D is connected.

(b) D is bicoloured, i.e. the vertices have been given either white or black colour and vertices connected by an edge have different colours.

(c) $X \setminus D$ is the union of finitely many topological discs, which we call *faces* of D.

As stated in Chapter 4 of [10], let $D_f = f^{-1}([0, 1])$, where (S, f) is a Belyi pair. We can consider D_f as a bicoloured graph embedded in S where its white vertices are $f^{-1}(0)$ and its black vertices are $f^{-1}(1)$. It is shown in the same chapter that D_f is a dessin d'enfant. Moreover we have map

$$\{\text{Belyi Pairs}\} \longrightarrow \{\text{Dessins}\}$$
$$(S, f) \mapsto (S, D_f)$$

that sends equivalent Belyi pairs to equivalent dessins. On the other hand, we also have map

$$\{\text{Dessins}\} \longrightarrow \{\text{Belyi Pairs}\}$$
$$(X, D) \mapsto (S_D, f_D)$$

that sends equivalent dessins to equivalent Belyi pairs, where (S_D, f_D) is obtained from dessin D through triangulation of D as described in section 4.2.1 and 4.2.2 in [10]. Therefore we also have the following theorem.

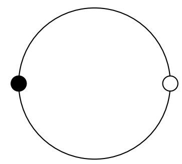
Theorem 7.1.4. The above maps define a one-to-one correspondences

$$\{\text{ Equiv. classes of dessins}\} \to \{\text{Equiv. classes of Belyi pairs}\}$$
$$(X, D) \mapsto (S_D, f_D)$$
$$(S, D_f) \leftarrow\!\shortmid (S, f)\ .$$

Proof. See proof to Theorem 4.25 in [10]. □

Thus in our case, to obtain a dessin for a corresponding modular curve $X(\Gamma)$ with the Belyi function $\tilde{j}$, we let white vertices to be the preimages of 0 under the $\tilde{j}$ map, that is, the Γ-orbits of points in $\mathrm{SL}_2(\mathbf{Z})\cdot\mu$, and black vertices to be preimages of 1, which are the Γ-orbits of points in $\mathrm{SL}_2(\mathbf{Z})\cdot i$. The edges are then the preimages of the interval $[0, 1]$ under the $\tilde{j}$ map, and the centers of faces are orbits of cusps, i.e. $\mathrm{SL}_2(\mathbf{Z})\cdot\infty$ of Γ. In general, white vertices have degree 3, i.e. there are three edges coming out of each white vertex, and black vertices have degree 2. However, this will change with elliptic points. The number of edges coming out from each elliptic vertex is $3-(\nu-1)$ for white vertices, and $2-(\nu-1)$ for black vertices, where ν is the order of the elliptic point.

Remark. With the above construction, the dessin for $\mathrm{SL}_2(\mathbf{Z})\backslash\mathfrak{h}^*$ is simply the following graph.

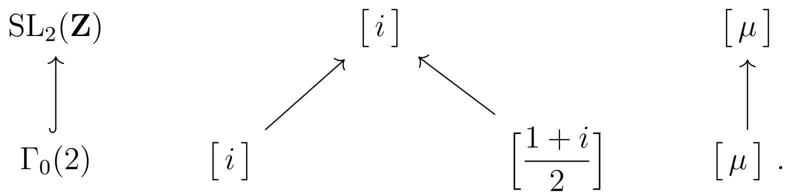


.

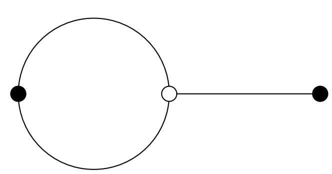
Example 7.1.5. The simplest dessin of modular curves aside from the one associated to $\mathrm{SL}_2(\mathbf{Z})$ is the one associated with $\Gamma_0(2)$, where $[\mathrm{SL}_2(\mathbf{Z})\colon \Gamma_0(2)] = 3$. Under the normalized $\tilde{j}$ map, the white and black vertices are the preimages of 0 and 1, which are the $\Gamma_0(2)$ orbits of μ and i. In order to find the orbits, we use Sage to compute the coset representatives of $\Gamma_0(2)$ in $\mathrm{SL}_2(\mathbf{Z})$. We obtain the following list:

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \quad \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}.$$

Thus the $\Gamma_0(2)$-orbits of i and μ are



Here $\dfrac{1+i}{2}$ is an order 2 elliptic point. Hence for $\Gamma_0(2)$, the dessin has 3 edges, one white vertex of degree 3, two black vertices, one of degree 2 and one of degree 1 as follows:



Associated with $\Gamma_0(2)$ are two cusps 0 and ∞, which agrees with the two faces of the dessin above. We can also see it from the fundamental domain of $\Gamma_0(2)$ obtained through Sage as in figure 7.2.

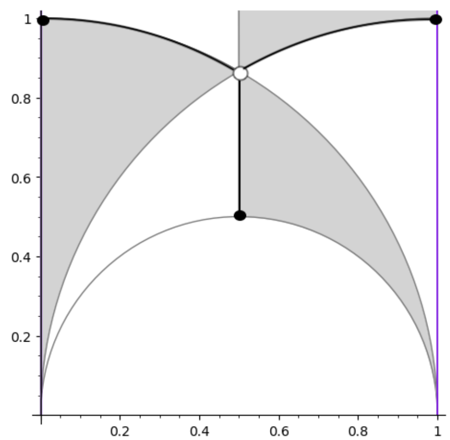


Figure 7.2: Fundamental Domain of $\Gamma_0(2)$.

Here i and $i+1$ are identified under $\Gamma_0(2)$, thus forming a circle when drawing abstractly. We see that dessins for modular curves with respect to $\tilde{j}$ drawn in the upper half plane are a quotient of the Serre tree to be described in the next section.

7.2 Cohomology of Combinatorial Forms on Dessins

We want to build a complex from the Serre tree $\mathcal{T}$ (figure below) whose cohomology is isomorphic to $H^\bullet_P(\Gamma\backslash\mathfrak{h}, \mathbf{V}_k)$. The Serre Tree in the upper half plane $\mathfrak{h}$ is a bicolored graph composed of vertices $\{p_i\}$ and edges $\{e_j\}$ obtained as the infinite preimage of $[0,1]$ under the $\tilde{j}$ map. We denote the vertices as $\mathrm{V}(\mathcal{T})$ and the edges as $\mathrm{E}(\mathcal{T})$. Based on the $\tilde{j}$ map, the black vertices of $\mathcal{T}$ are the preimages of 1, and the white vertices of $\mathcal{T}$ are the preimages of 0. In another word, the black vertices of $\mathcal{T}$ are the $\mathrm{SL}_2(\mathbf{Z})$ translates of i and the white vertices are the $\mathrm{SL}_2(\mathbf{Z})$ translates of μ in the upper half plane.

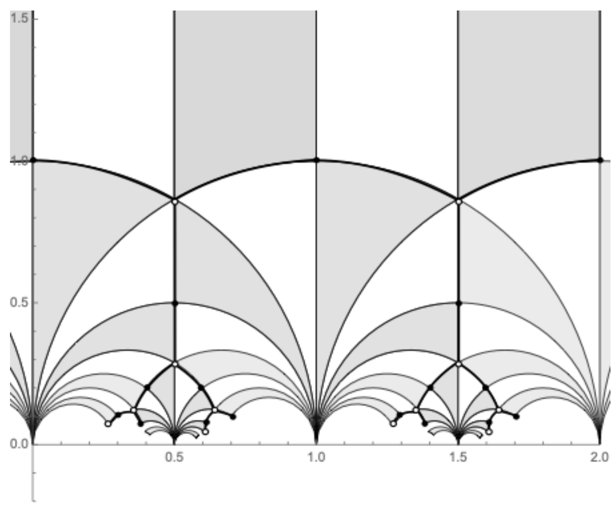


Figure 7.3: The Serre tree $\mathcal{T}$.

Thus the dessins of $\Gamma\setminus\mathfrak{h}^*$ associated to $\tilde{j}$ is $\Gamma\setminus\mathcal{T}\subset\Gamma\setminus\mathfrak{h}$. It is useful to choose representations of the Γ-orbits of edges and vertices, sometimes within a fundamental domain for Γ. See figure 7.2 for the case $\Gamma=\Gamma_0(2)$.

To build a cohomology on dessins, we consider the Serre tree $\mathcal{T}$ in the upper half plane. If we set the orientation of each edge to be from white vertex to black vertex, then we can define the complex by

$$C^0(\Gamma,\mathcal{T})=\{f\colon\ \mathrm{V}(\mathcal{T})\to\mathbf{C}\}\ ,\qquad C^1(\Gamma,\mathcal{T})=\{\phi\colon\ \mathrm{E}(\mathcal{T})\to\mathbf{C}\}\ .$$

For any $e \in \mathrm{E}(\mathcal{T})$ with white vertex p_0 and black vertex p_1, we then set the differential map $d\colon C^0 \to C^1$ as

$$df(e) = f(p_1) - f(p_0)\ .$$

Now we generalized to coefficients from a vector space $V = \mathrm{Sym}^{k-2}(\mathbf{C}^2)$ with the representation induced from the standard representation $\rho\colon\ \mathrm{SL}_2(\mathbf{Z}) \longrightarrow \mathrm{Aut}(V)$. We sometimes write $V = V_k$ in short when we do not need to specify the weight k. Analogously to $A^\bullet(\Gamma, \mathfrak{h}, V)$, we define the complex with coefficients as

$$\begin{aligned} C^0(\Gamma, \mathcal{T}, V) &= \{f\colon \mathrm{V}(\mathcal{T}) \to V \mid f(\gamma p) = \rho(\gamma) f(p)\ \forall\, p \in \mathrm{V}(\mathcal{T})\ \&\ \gamma \in \Gamma\}\ , \\ C^1(\Gamma, \mathcal{T}, V) &= \{\phi\colon\ \mathrm{E}(\mathcal{T}) \to V \mid \phi(\gamma e) = \rho(\gamma)\phi(e)\ \forall\, e \in \mathrm{E}(\mathcal{T})\ \&\ \gamma \in \Gamma\}\ . \end{aligned} \tag{7.2.1}$$

Again, the differential map $d\colon C^0 \to C^1$ is given by $df(e) = f(p_1) - f(p_0)$ for any edge e with white vertex p_0 and black vertex p_1.

Note that $C^0(\Gamma, \mathcal{T}, V)$ and $C^1(\Gamma, \mathcal{T}, V)$ are actually finite dimensional since although there are infinitely many edges and vertices in $\mathcal{T}$, there are only a finite number modulo the action of Γ. Thus knowing the value of f on a finite set of representatives determines it elsewhere. It is convenient to let these representatives lie in a strict fundamental domain D for Γ. In the case where any endpoint p lies outside of the strict fundamental domain D of $\Gamma \setminus \mathfrak{h}$, or on an edge that is not included in D, there is a $\gamma \in \Gamma$ such that $\gamma^{-1} \cdot p$ is a vertex within the fundamental domain. In this case, for every $e \in \mathrm{E}(\mathcal{T})$ with white vertex p_0 and black vertex p_1, the differential map is given by

$$df(e) = \rho(\gamma_1) f(\gamma_1^{-1} \cdot p_1) - \rho(\gamma_2) f(\gamma_2^{-1} \cdot p_0)$$

where $\gamma_1, \gamma_2 \in \Gamma$ and γ_1^{-1} sends p_1 into D and γ_2^{-1} sends p_0 into D. If either endpoint is already in D, then γ_1 or γ_2 is just $\mathbb{1}_2$.

Let $H^\bullet(\Gamma, \mathcal{T}, V)$ denote the cohomology of $C^\bullet(\Gamma, \mathcal{T}, V)$. It is isomorphic to $H^\bullet(\Gamma \setminus \mathcal{T}, \mathbf{V}_0)$ where $\mathbf{V}_0$ is defined analogously to $\mathbf{V}$ in section 3.1.

In fact there is a $\mathrm{SL}_2(\mathbf{Z})$-equivalent deformation retract $r\colon \mathfrak{h} \longrightarrow \mathcal{T}$ defined as follows. Let D be the strict fundamental domain for $\mathrm{SL}_2(\mathbf{Z})$ whose closure is $\{x+iy \in \mathfrak{h} \mid -\frac{1}{2} \le x \le \frac{1}{2}, x^2+y^2 \ge 1\}$. Then define for $g \in \mathrm{SL}_2(\mathbf{Z})$,

$$r(\tau) = \begin{cases} x + i\sqrt{1-x^2} & \text{for } \tau \in D \ , \\ gr(g^{-1}\tau) & \text{for } \tau \in gD \ . \end{cases}$$

The homotopy r_t between the identity and r is given on D as

$$\begin{aligned} &r_t\colon \mathfrak{h} \longrightarrow \mathcal{T} \\ &x+iy \mapsto x + i\big((1-t)y + t\sqrt{1-x^2}\big) \ , \end{aligned} \tag{7.2.2}$$

and extended to all gD to be $\mathrm{SL}_2(\mathbf{Z})$-equivariant. It follows that $H^\bullet(\Gamma \setminus \mathfrak{h}, \mathbf{V}) \cong H^\bullet(\Gamma \setminus \mathcal{T}, \mathbf{V})$. (See Theorem 10.5 in [31].)

We now want to define a subcomplex $C_P^\bullet\big(\Gamma, \mathcal{T}, V\big) \subset C^\bullet\big(\Gamma, \mathcal{T}, V\big)$ so that the map on cohomology corresponds to $H_P^\bullet\big(\Gamma, \mathfrak{h}, V\big) \longrightarrow H^\bullet\big(\Gamma, \mathfrak{h}, V\big)$. To do this, we want $C_P^\bullet(\Gamma, \mathcal{T}, V)$ to consist of forms that are exact when restricted to the circle of edges bound a face, that is, the boundary of a neighborhood of a cusp point.

Pick a cusp (face), and consider the circle of edges surround it counter-clockwise. This is called the *link* of the cusp. We view this circle as a sum of the edges but with alternating signs to correct orientation. We cut this circle link at a vertex which is on the boundary of the fundamental domain D to get a line and let $\gamma^{-1} \in \Gamma$ be the element in the Γ-stabilizer of the cusp that sends the final vertex which is not in D to the starting vertex which is in D. For example, if the face has 4 edges and we cut at a black vertex, we get

$$\begin{array}{ccccccccc} & e_1 & & e_2 & & e_3 & & e_4 & \\ \mathrm{B} & \longleftarrow & \mathrm{W} & \longrightarrow & \mathrm{B} & \longleftarrow & \mathrm{W} & \longrightarrow & \mathrm{B} \\ p_1 & & p_2 & & p_3 & & p_4 & & \gamma p_1 \end{array}$$

where B and W denote the black and white vertices. Thus the cut circle link is $(-e_1 + e_2 - e_3 + e_4)$. Suppose $f(p_i) = v_i \in C^0_P\big(\Gamma, \mathcal{T}, V_k\big)$ for $p_i \in \mathrm{V}(\mathcal{T})$ and $v_i \in \mathbf{V}_k$. We have

$$\begin{aligned} df(-e_1 + e_2 - e_3 + e_4) &= -df(e_1) + df(e_2) - df(e_3) + df(e_4) \\ &= -(f(p_1) - f(p_2)) + (f(p_3) - f(p_2)) \\ &\qquad - (f(p_3) - f(p_4)) + (f(\gamma \cdot p_1) - f(p_4)) \\ &= \rho(\gamma) v_1 - v_1 \ . \end{aligned}$$

Thus we want $C^1_P(\Gamma, \mathcal{T}, V)$ to consist of ϕ of this form when restricted to the circle bounding every face. For each cusp c, let $\gamma_c \in \mathrm{Stab}_\Gamma(c)$ be the element that closes up the circle around c; in practice, those are the generators of the stabilizers of c. Let $link_c$ denote the cut circle link of c. Thus for every cusp c, define

$$\begin{aligned} &C^0_P(\Gamma, \mathcal{T}, V) = C^0_P(\Gamma, \mathcal{T}, V) \ , \\ &C^1_P(\Gamma, \mathcal{T}, V) = \{\phi\colon\ \mathrm{E}(\mathcal{T}) \to V \mid \phi(\gamma e) = \rho(\gamma)\phi(e)\ \forall\ e \in \mathrm{E}(\mathcal{T})\ \&\ \gamma \in \Gamma \\ &\qquad\qquad\qquad\qquad \&\forall c, \phi(link_c) \in \mathrm{Im}(\rho(\gamma_c) - \mathrm{I})\} \ . \end{aligned} \tag{7.2.3}$$

Lemma 7.2.1. For $f \in C^0_P(\Gamma, \mathcal{T}, V)$, df necessarily lies in $C^1_P(\Gamma, \mathcal{T}, V)$. Thus $C^\bullet_P(\Gamma, \mathcal{T}, V)$ is a complex.

Proof. Consider any cut circle link around a cusp. If it starts at a white vertex, then it ends at a white vertex. Similarly if it starts at a black vertex, then it will necessarily end at a black vertex. This forces there to be even number of edges in each cut circle link. Without loss of generality, assume we cut the link at a black vertex p_1, and number the vertices as $\{p_1, \cdots, p_{2n}\}$ and the edges as $\{e_1, \cdots, e_{2n}\}$ where the odd numbered vertices are black and even numbered vertices are white , then we have from the above example

$$\begin{aligned} df(-e_1 + \cdots + e_{2n}) &= -df(e_1) + \cdots + df(e_{2n}) \\ &= -(f(p_1) - f(p_2)) + \cdots + (f(\gamma_c \cdot p_1) - f(p_{2n})) \\ &= \rho(\gamma_c)v_1 - v_1 \end{aligned}$$

for some $\gamma_c \in \mathrm{Stab}_\Gamma(c)$ and $v_1 \in V$. Thus $df \in C^1_P\big(\Gamma, \mathcal{T}, V_k\big)$. $\square$

The cohomology in degree 1 is then given by $H^1_P(\Gamma, \mathcal{T}, V) = \dfrac{C^1_P(\Gamma, \mathcal{T}, V)}{\mathrm{Im}(d)}$. It can be shown to be isomorphic to $H^1_P(\Gamma, \mathfrak{h}, V)$.

7.3 Hecke Correspondence of Combinatorial 1-Forms

In this section, we start a study of the behavior of the combinatorial 1-forms in $C^1_P(\Gamma, \mathcal{T}, V)$ from section 7.2 under the Hecke correspondence T_p.

The difficulty with doing this is that the Hecke operator does not preserve the dessin $\Gamma_0(N)\backslash\mathcal{T}$ as a correspondence on points of $\Gamma_0(N)\backslash\mathfrak{h}$. The problems is common to other attempts to calculate Hecke operators using a lower dimensional "spine"; a common approach to resolve the problem is to use the "sharbly complex" [14] [2].

Our approach is more geometric. At the moment the algorithm applies to T_p for $p = 2, 3, 5$ and relies on the following

Proposition 7.3.1. Let $g = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ where $p = 2, 3,$ or 5. For any vertex $v \in \mathrm{V}(\mathcal{T}), r(gv) \in \mathrm{V}(\mathcal{T})$.

We expect the algorithm (though not the proposition) can be generalized for all p.

Proof. We prove by using the map $\tilde{j}$. Recall from Claim 7.1.2, $\tilde{j}$ sends the arc joining i and μ to the interval $[0, 1]$. Moreover, $\tilde{j}(iy) > 1$ for $y > 1$ and $\tilde{j}(\frac{1}{2}+iy) < 0$ for $y > \frac{\sqrt{3}}{2}$ and r maps τ to $\mathrm{V}(\mathcal{T})$ where $\tilde{j}(\tau) \geq 1$ or $\tilde{j}(\tau) \leq 0$. Any vertex of $\mathcal{T}$ can be written in the form $A \cdot i$ or $A \cdot \mu$ for some matrix $A \in SL$. Write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$. We have

$$\begin{aligned} gA &= \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} a & bp \\ \frac{c}{p} & d \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \\ &= A'g \ . \end{aligned}$$

Thus if $A \in \Gamma_0(p)$ and $v = Ai$, we have

$$\tilde{j}(gv) = \tilde{j}(gAi) = \tilde{j}(A'gi) = \tilde{j}(gi)$$

since $\tilde{j}$ is an $\mathrm{SL}_2(\mathbf{Z})$ invariant function. Since $gi = pi$, $\tilde{j}(gi) > 1$ since $p > 1$. Thus $r(gv) \in \mathrm{V}(\mathcal{T})$. Similarly, if $A \in \Gamma_0(p)$ and $v = A\mu$, we have

$$\tilde{j}(gv) = \tilde{j}(gA\mu) = \tilde{j}(A'g\mu) = \tilde{j}(g\mu) \ .$$

When $p = 2$, $\tilde{j}(g\mu) = \tilde{j}(1 + \sqrt{3}i) = \tilde{j}(\sqrt{3}i) > 1$. Thus $r(gv) \in \mathrm{V}(\mathcal{T})$.

So far we have dealt with all the points of the form $\Gamma_0(p)i$ and $\Gamma_0(p)\mu$. Next we consider points in $\mathrm{SL}_2(\mathbf{Z})\, i$ and $\mathrm{SL}_2(\mathbf{Z})\, \mu$. First by Proposition 1.43 in [24], a set of coset representatives for $\Gamma_0(p) \setminus \mathrm{SL}_2(\mathbf{Z})$ is

$$\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & p \end{pmatrix}$$

where $r \in \{1, 2, \cdots, p-1\}$. By the calculation with $\Gamma_0(p)$, suffices to just consider these coset representatives. First we work with $\begin{pmatrix} 0 & -1 \\ 1 & p \end{pmatrix} i$ and $\begin{pmatrix} 0 & -1 \\ 1 & p \end{pmatrix} \mu$. We have

$$\begin{aligned} g \begin{pmatrix} 0 & -1 \\ 1 & p \end{pmatrix} &= \begin{pmatrix} 0 & -p \\ 1 & p \end{pmatrix} \\ &= S \begin{pmatrix} 1 & p \\ 0 & p \end{pmatrix} \end{aligned}$$

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Note that $\tilde{j}(\tau \pm 1) = \tilde{j}(\tau)$Thus we have

$$\tilde{j}\left(g \begin{pmatrix} 0 & -1 \\ 1 & p \end{pmatrix} i\right) = \tilde{j}\left(S \begin{pmatrix} 1 & p \\ 0 & p \end{pmatrix} i\right) = \tilde{j}\left(S \cdot \frac{i+p}{p}\right) = \tilde{j}\left(S \cdot \frac{i}{p}\right) = \tilde{j}(pi) > 1 \ ,$$

and

$$\tilde{j}\left(g \begin{pmatrix} 0 & -1 \\ 1 & p \end{pmatrix} \mu\right) = \tilde{j}\left(S \begin{pmatrix} 1 & p \\ 0 & p \end{pmatrix} \mu\right) = \tilde{j}\left(S \cdot \frac{\mu}{p}\right) = \tilde{j}\left(\frac{p(-1 + \sqrt{3}i)}{2}\right) \ .$$

When $p = 2$,this becomes

$$\tilde{j}\left(-1+\sqrt{3}i\right) = \tilde{j}\left(\sqrt{3}i\right) > 1.$$

When $p > 2$, write $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, it becomes

$$\tilde{j}\left(T^{\frac{p+1}{2}}\frac{p(-1+\sqrt{3}i)}{2}\right) = \tilde{j}\left(\frac{1+p\sqrt{3}i}{2}\right) < 0$$

since p is prime and thus $\frac{p+1}{2}$ is an integer. Thus $r\left(g\begin{pmatrix} 0 & -1 \\ 1 & p \end{pmatrix} i\right) \in \mathrm{V}(\mathcal{T})$ and $r\left(g\begin{pmatrix} 0 & -1 \\ 1 & p \end{pmatrix} \mu\right) \in \mathrm{V}(\mathcal{T})$.

Next we will show that $r\left(g\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} i\right) \in \mathrm{V}(\mathcal{T})$ and $r\left(g\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} \mu\right) \in \mathrm{V}(\mathcal{T})$ for $r = 1, \cdots, p-1$. We have

$$g\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} = S\begin{pmatrix} r & 1 \\ -p & 0 \end{pmatrix}.$$

Thus

$$\tilde{j}\left(g\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} i\right) = \tilde{j}\left(S\begin{pmatrix} r & 1 \\ -p & 0 \end{pmatrix} i\right) = \tilde{j}\left(\frac{-r+i}{p}\right)$$

and

$$\tilde{j}\left(g\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} \mu\right) = \tilde{j}\left(S\begin{pmatrix} r & 1 \\ -p & 0 \end{pmatrix} \mu\right) = \tilde{j}\left(\frac{-r+\mu}{p}\right).$$

Using S and T it is not hard to show that $\frac{-r+x}{p}$ can be moved to iy where $y \geq 1$ or $\frac{1}{2} + iy$ where $y \geq \frac{\sqrt{3}}{2}$ for $x = i$ or μ, when $p = 2, 3, 5$. This fact can also be verified using Sage or Mathematica to compute their $\tilde{j}$ value. Note that the result does not work in general for $p \geq 7$. $\square$

Recall the following diagram for the correspondence:

$$\begin{array}{ccc} \Gamma_0(pN)\setminus\mathfrak{h} & \xrightarrow{\ g\ } & g\Gamma_0(pN)g^{-1}\setminus\mathfrak{h} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi_g} \\ \Gamma_0(N)\setminus\mathfrak{h} & & \Gamma_0(N)\setminus\mathfrak{h} \end{array}$$

where $g = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$. Starting from $\phi \in C^1_P(\Gamma_0(N), \mathcal{T}, V)$ in the bottom right, we will see what happens to it as we travel back to $C^1_P(\Gamma_0(N), \mathcal{T}, V)$ in the bottom left.

Notation 7.3.2. To emphasize the analogy with differential forms, for $\phi\colon \mathrm{E}(\mathcal{T}) \longrightarrow V$ and $g \in \mathrm{GL}_2(\mathbf{Q})$, define L^*_g by

$$L^*_g\phi(e) = \phi(ge)$$

for $e \in g^{-1}(\mathrm{E}(\mathcal{T}))$. Similarly define $L^*_g f$ for $f\colon \mathrm{V}(\mathcal{T}) \longrightarrow V$.

Proposition 7.3.3. The 1-form ϕ associated with $\Gamma_0(N)\setminus\mathfrak{h}$ naturally pulls back to ϕ associated with $g\Gamma_0(pN)g^{-1}\setminus\mathfrak{h}$, that is, $\phi \in C^1_P(g\Gamma_0(pN)g^{-1}, \mathcal{T}, V)$.

Proof. From the proof of Proposition 5.3.1, we know that $g\Gamma_0(pN)g^{-1} \subset \Gamma_0(N)$ and thus

$$L^*_\gamma(\phi) = \rho(\gamma)\phi\ .$$

This shows that $\phi \in C^1(g\Gamma_0(pN)g^{-1}, \mathcal{T}, V)$ by equation (7.2.1).

For the rest of the proof, $link_\infty$, D_N w_∞ and γ_∞ will be with respect to $\Gamma_0(N)$; when we wish to work with respect to $g\Gamma_0(pN)g^{-1}$, we will explicitly express that. We want to show that ϕ also satisfies the parabolic condition in equation (7.2.3), and we will consider the cusps of $g\Gamma_0(pN)g^{-1}\setminus\mathfrak{h}^*$ which live above the cusp $\Gamma_0(N)\infty$. The general case will follow similarly. Since $\phi \in C^1_P(\Gamma_0(N), \mathcal{T}, V)$, we have

$$\phi(link_\infty) \in \mathrm{Im}(\rho(\gamma_\infty) - \mathrm{I})$$

where $link_\infty$ is the cut circle link around ∞ of the Serre tree $\mathcal{T}$ in a fundamental domain D_N of $\Gamma_0(N)\backslash\mathfrak{h}$, and $\gamma_\infty = \begin{pmatrix} 1 & \pm w_\infty \\ 0 & 1 \end{pmatrix} \mod \mathrm{I}_2$ is a generator of $\mathrm{Stab}_{\Gamma_0(N)}(\infty)$ where w_∞ is the width of ∞ with respect to $\Gamma_0(N)$. As in the proof of Proposition 5.3.1, a cusp above $\Gamma_0(N)\infty$ in the space $g\Gamma_0(pN)g^{-1}\setminus\mathfrak{h}^*$ would have the form $g\Gamma_0(pN)g^{-1}(\gamma_i\infty)$, where γ_i is a member of a set of coset representatives of $g\Gamma_0(pN)g^{-1}\setminus\Gamma_0(N)$ such that

$$\Gamma_0(N)\infty = \bigcup_i g\Gamma_0(pN)g^{-1}(\gamma_i\infty)\ .$$

Write $\tilde{\Gamma} = g\Gamma_0(pN)g^{-1}$, we want to verify that

$$\phi(link_{\gamma_i\infty,\tilde{\Gamma}}) \in \mathrm{Im}(\rho(\gamma_{\gamma_i\infty,\tilde{\Gamma}}) - \mathrm{I}) \tag{7.3.1}$$

where $link_{\infty,\tilde{\Gamma}}$ is the cut circle link around ∞ of the Serre tree $\mathcal{T}$ in a fundamental domain $D_{\tilde{\Gamma}}$ of $\tilde{\Gamma}\setminus\mathfrak{h}$, and $\gamma_{\gamma_i\infty,\tilde{\Gamma}} = \begin{pmatrix} 1 & \pm w_{\gamma_i\infty,\tilde{\Gamma}} \\ 0 & 1 \end{pmatrix} \mod \mathrm{I}_2$ is a generator of $\mathrm{Stab}_{\tilde{\Gamma}}(\gamma_i\infty)$ where $w_{\gamma_i\infty,\tilde{\Gamma}}$ is the width of the cusp $\tilde{\Gamma}\gamma_i\infty \in \tilde{\Gamma}\setminus\mathfrak{h}^*$. Since $\mathrm{Stab}_{\tilde{\Gamma}}(\gamma_i\infty) \subset \mathrm{Stab}_{\Gamma_0(N)}(\gamma_i\infty) = \gamma_i\,\mathrm{Stab}_{\Gamma_0(N)}(\infty)\gamma_i^{-1}$,

$$\gamma_{\gamma_i\infty,\tilde{\Gamma}} = \gamma_i\gamma_\infty^r\gamma_i^{-1}, \quad w_{\gamma_i\infty,\tilde{\Gamma}} = rw_{\gamma_i\infty} = rw_\infty$$

for some integer $r = [\mathrm{Stab}_{\Gamma_0(N)}(\gamma_i\infty)\colon \mathrm{Stab}_{\tilde{\Gamma}}(\gamma_i\infty)] \geq 1$.

If $link_\infty$ has the following form

$$\begin{array}{ccccccc} & e_1 & & e_2 & & e_w & \\ \mathrm{B} & \longleftarrow & \mathrm{W} & \longrightarrow & \cdots & \longrightarrow & \mathrm{B} \\ p_1 & & p_2 & & & & \gamma_\infty\cdot p_1\ , \end{array}$$

then $link_{\gamma_i\infty,\tilde{\Gamma}}$ has the form

$$\begin{array}{ccccccc} & e_1' & & e_2' & & e_{rw}' & \\ \mathrm{B} & \longleftarrow & \mathrm{W} & \longrightarrow & \cdots & \longrightarrow & \mathrm{B} \\ p_1' & & p_2' & & & & \gamma_{\gamma_i\infty,\tilde{\Gamma}}\cdot p_1'\ . \end{array}$$

around the cusp $\tilde{\Gamma}\gamma_i\infty$. Therefore we have

$$\begin{aligned}
\phi(link_{\gamma_i\infty,\tilde{\Gamma}}) &= \phi(-e'_1 + e'_2 - \cdots + e'_{rw}) \\
&= L^*_{\gamma_i}(\phi)(-e_1 + e_2 - \cdots + e_{rw}) \\
&= \rho(\gamma_i) \sum_{l=1}^{r} \phi(-e_{(l-1)w+1} + e_{(l-1)w+2} - \cdots + e_{lw}) \\
&= \rho(\gamma_i) \sum_{l=1}^{r} \left(L^*_{\gamma_\infty}\right)^{l-1} (\phi)(-e_1 + e_2 - \cdots + e_w) \\
&= \rho(\gamma_i) \sum_{l=1}^{r} \rho(\gamma_\infty^{l-1})\phi(-e_1 + e_2 - \cdots + e_w) \\
&\in \rho(\gamma_i) \sum_{l=1}^{r} \rho(\gamma_\infty^{l-1}) \operatorname{Im}(\rho(\gamma_\infty) - \mathrm{I}) \\
&= \rho(\gamma_i) \operatorname{Im}(\rho(\gamma_\infty^r) - \mathrm{I})
\end{aligned}$$

On the other hand, the right hand side of equation (7.3.1) is

$$\begin{aligned}
&\operatorname{Im}(\rho(\gamma_i\gamma_\infty^r\gamma_i^{-1}) - \mathrm{I}) \\
=&\rho(\gamma_i) \operatorname{Im}(\rho(\gamma_\infty^r) - \mathrm{I})\rho(\gamma_i^{-1}) \\
=&\rho(\gamma_i) \operatorname{Im}(\rho(\gamma_\infty^r) - \mathrm{I}) \qquad \text{since } \gamma_i^{-1} \colon V_k \xrightarrow{\sim} V_k \ . \qquad \square
\end{aligned}$$

In order to now pullback a form in $C^1_P(g\Gamma_0(pN)g^{-1}, T, V)$ to one in $C^\bullet_P(\Gamma_0(pN), T, V)$, we cannot simply use $\rho(g^{-1})L^*_g$ as we did for differential forms. The problem is that acting by g does not send $\mathcal{T}$ to $\mathcal{T}$ but to a different tree $g\mathcal{T}$.

To resolve this we consider the two dessins for $g\Gamma_0(pN)g^{-1} \setminus \mathfrak{h}^*$ associated to the Belyi functions $\tau \mapsto \tilde{j}(\tau)$ and $\tau \mapsto \tilde{j}(g^{-1}\tau)$, namely $g\Gamma_0(pN)g^{-1}\setminus\mathcal{T}$ and $g\Gamma_0(pN)g^{-1}\setminus g\mathcal{T}$. The cohomology of these two dessins (which are both isomorphic to $H^\bullet(g\Gamma_0(pN)g^{-1}, \mathfrak{h}, V)$) are calculated via the complexes

$$C^\bullet(g\Gamma_0(pN)g^{-1}, \mathcal{T}, V) \text{ and } C^\bullet(g\Gamma_0(pN)g^{-1}, g\mathcal{T}, V)$$

respectively.

The part of the Hecke operator corresponding to pullback by g is given by the composition

$$C^1(g\Gamma_0(pN)g^{-1}, \mathcal{T}, V) \xrightarrow{r^*} C^1(g\Gamma_0(pN)g^{-1}, g\mathcal{T}, V) \xrightarrow{\rho(g^{-1})L_g^*} C^1(\Gamma_0(pN), \mathcal{T}, V) .$$

We first describe r^*. Recall the deformation retract $r\colon \mathfrak{h} \longrightarrow \mathcal{T}$ defined in section 7.2. Under the assumption that $p < 7$, Proposition 7.3.1 implies that if ge is an edge of $g\mathcal{T}$, then $r(ge)$ is image of an oriented path in $\mathcal{T}$ between two vertices. Define

$$r_*(ge) = e_1 - e_2 + e_3 - \cdots - e_{k-1} + e_k$$

to be the simplicial chain in $\mathcal{T}$ representing this oriented path. Then for $\phi \in C^1_P(g\Gamma_0(pN)g^{-1}, T, V)$, define

$$\begin{aligned}(r^*\phi)(ge) &= \phi(r_*(ge)) \\ &= \phi(e_1) - \phi(e_2) + \cdots + \phi(e_k) .\end{aligned}$$

Proposition 7.3.4. For $\phi \in C^1_P(g\Gamma_0(pN)g^{-1}, T, V)$, $r^*(\phi) \in C^1_P(g\Gamma_0(pN)g^{-1}, gT, V)$.

Proof. To start,

$$L^*_{g\gamma g^{-1}}(r^*\phi)(ge) = (r^*\phi)(g\gamma g^{-1}ge) = r^*\phi(g\gamma e)$$

for $g\gamma g^{-1} \in g\Gamma_0(pN)g^{-1}$ and $ge \in \mathrm{E}(g\mathcal{T})$. Thus

$$\begin{aligned}L^*_{g\gamma g^{-1}}(r^*\phi)(ge) &= \phi(r_*(g\gamma e)) \\ &= \phi(r_*(g\gamma g^{-1}ge)) \\ &= \phi(g\gamma g^{-1}r_*(ge)) \\ &= \rho(g\gamma g^{-1})\phi(r_*(ge)) \\ &= \rho(g\gamma g^{-1})r^*\phi(ge) .\end{aligned}$$

This shows that $r^*\phi \in C^1(g\Gamma_0(pN)g^{-1}, gT, V)$. The parabolic condition on $r^*\phi$ follows from the for ϕ since the link of a cusp with respect to $g\mathcal{T}$ is homotopic to the link with respect to $\mathcal{T}$ via r. □

We then apply the pullback map induced by g defined as

$$\rho(g^{-1})L_g^*\colon C_P^1(g\Gamma_0(pN)g^{-1}, \mathcal{T}, V) \longrightarrow C_P^1(\Gamma_0(pN), \mathcal{T}, V)\ .$$

Proposition 7.3.5. If $\eta \in C_P^1(g\Gamma_0(pN)g^{-1}, g\mathcal{T}, V)$, then $\psi = \rho(g^{-1})L_g^*(\eta) \in C_P^1(\Gamma_0(pN), \mathcal{T}, V)$.

Proof. Use the same reasoning as in the proof of Proposition 3.2.2, we can show that $\psi \in C^1(\Gamma_0(pN), \mathcal{T}, V)$ because it satisfies $L_\gamma^*(\psi) = \rho(\gamma)\psi$ for all $\gamma \in \Gamma_0(pN)$.

For the rest of the proof, write $link_\infty$, D_∞, w_∞ and γ_∞ will be with respect to $g\Gamma_0(pN)g^{-1}$ and $g\mathcal{T}$; write $\Gamma = \Gamma_0(pN)$, $link_{g^{-1}\infty,\Gamma}$, D_Γ, $w_{g^{-1}\infty,\Gamma}$ and $\gamma_{g^{-1}\infty,\Gamma}$ will be with respect to $\Gamma_0(pN)$ and $\mathcal{T}$. To show that ψ satisfies the parabolic condition in equation 7.2.3, again we will just show that ψ satisfies this condition at the cusp $\Gamma_0(pN)g^{-1}\infty \in \Gamma_0(pN)\backslash\mathfrak{h}^*$ which is sent to the cusp $g\Gamma_0(pN)g^{-1}\infty \in g\Gamma_0(pN)g^{-1}\backslash\mathfrak{h}^*$ under the g map. We want to verify that

$$\psi(link_{g^{-1}\infty,\Gamma}) \in \mathrm{Im}(\rho(\gamma_{g^{-1}\infty,\Gamma}) - \mathrm{I}) \tag{7.3.2}$$

where $link_{g^{-1}\infty,\Gamma}$ is the cut circle link around $g^{-1}\infty$ of the Serre tree $\mathcal{T}$ in a fundamental domain D_Γ of $\Gamma_0(pN)\backslash\mathfrak{h}$, and $\gamma_{g^{-1}\infty,\Gamma} = \begin{pmatrix} 1 & \pm w_{g^{-1}\infty,\Gamma} \\ 0 & 1 \end{pmatrix} \mod \mathrm{I}_2$ is a generator of $\mathrm{Stab}_{\Gamma_0(pN)}(g^{-1}\infty)$ where $w_{g^{-1}\infty,\Gamma}$ is the width of the cusp $\Gamma_0(pN)g^{-1}\infty \in \Gamma_0(pN)\backslash\mathfrak{h}^*$. Since $g = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$, we have $g^{-1}\infty = \infty$ and g induces an isomorphism of the spaces $\Gamma_0(pN)\backslash\mathfrak{h}^*$ and $g\Gamma_0(pN)g^{-1}\backslash\mathfrak{h}^*$ as well as the dessins $\Gamma_0(pN)\backslash\mathcal{T}$ and $g\Gamma_0(pN)g^{-1}\backslash g\mathcal{T}$.

From the hypothesis, we have $\eta \in C^1_P(g\Gamma_0(pN)g^{-1}, g\mathcal{T}, V)$, thus

$$\eta(link_\infty) \in \mathrm{Im}(\rho(\gamma_\infty) - \mathrm{I})$$

where $link_\infty$ is the cut circle link around ∞ of the tree $g\mathcal{T}$ in a fundamental domain D_∞ of $g\Gamma_0(pN)g^{-1} \setminus \mathfrak{h}$, and $\gamma_\infty = \begin{pmatrix} 1 & \pm w_\infty \\ 0 & 1 \end{pmatrix} \bmod \mathrm{I}_2$ is a generator of $\mathrm{Stab}_{g\Gamma_0(pN)g^{-1}}(\infty)$ where w_∞ is the width of the cusp $g\Gamma_0(pN)g^{-1}\infty \in g\Gamma_0(pN)g^{-1} \setminus \mathfrak{h}^*$. Note $w_\infty = pw_{g^{-1}\infty,\Gamma}$ and $\gamma_\infty = g\gamma_{g^{-1}\infty,\Gamma}g^{-1}$.

If $link_{g^{-1}\infty,\Gamma}$ has the following form

$$\begin{array}{ccccccc} & e_1 & & e_2 & & e_w & \\ \mathrm{B} & \longleftarrow & \mathrm{W} & \longrightarrow & \cdots & \longrightarrow & \mathrm{B} \\ p_1 & & p_2 & & & & \gamma_{g^{-1}\infty,\Gamma} \cdot p_1 \ , \end{array}$$

then $link_\infty$ has the form

$$\begin{array}{ccccccc} & e'_1 & & e'_2 & & e'_w & \\ \mathrm{B} & \longleftarrow & \mathrm{W} & \longrightarrow & \cdots & \longrightarrow & \mathrm{B} \\ p'_1 & & p'_2 & & & & \gamma_\infty \cdot p'_1 \ . \end{array}$$

around the cusp $g\Gamma_0(pN)g^{-1}\infty$ where $e'_i = ge_i, p'_i = gp_i$. Therefore we have

$$\begin{aligned} \psi(link_{g^{-1}\infty,\Gamma}) &= \rho(g^{-1})L^*_g(\phi)(-e_1 + e_2 - \cdots + e_w) \\ &= \rho(g^{-1})\phi(-e'_1 + e'_2 - \cdots + e'_w) \\ &= \rho(g^{-1})\phi(link_\infty) \\ &\in \rho(g^{-1})\,\mathrm{Im}(\rho(\gamma_\infty) - \mathrm{I}) \\ &= \rho(g^{-1})\rho(g)\,\mathrm{Im}(\rho(\gamma_{g^{-1}\infty,\Gamma}) - \mathrm{I})\rho(g^{-1}) \\ &= \mathrm{Im}(\rho(\gamma_{g^{-1}\infty,\Gamma}) - \mathrm{I}) \qquad \text{since } g^{-1}\colon V_k \xrightarrow{\sim} V_k \ . \quad \square \end{aligned}$$

To go from 1-forms associated with $\Gamma_0(pN)\backslash\mathfrak{h}$ to 1-forms associated with $\Gamma_0(N)\backslash\mathfrak{h}$ at the bottom left of the correspondence, we again take the trace operator defined as before.

$$\pi_*(\psi) = \sum_{\gamma \in \Gamma_0(pN) \backslash \Gamma_0(N)} \rho(\gamma^{-1}) L^*_\gamma(\psi) \ .$$

As shown in Proposition 5.3.3, the trace operator is well-defined and satisfies the condition that $L^*_\gamma(\pi_*(\psi)) = \rho(\gamma)\pi_*(\psi)$ for all $\psi \in C^1_P(\Gamma_0(pN), \mathcal{T}, V)$ and $\gamma \in \Gamma_0(N)$.

Proposition 7.3.6. If $\psi \in C^1_P(\Gamma_0(pN), \mathcal{T}, V)$, then $\pi_*(\psi) \in C^1_P(\Gamma_0(N), \mathcal{T}, V)$.

Proof. For this proof, $link_\infty$, D_∞, w_∞, γ_∞ will be with respect to $\Gamma_0(N)$; when we wish to work with respect to $\Gamma_0(pN)$, we will explicitly express that. To show that $\pi_*(\psi)$ satisfies the parabolic condition as in equation (7.2.3), we will again just how that the condition holds at the cusp $\Gamma_0(N)\infty \in \Gamma_0(N) \setminus \mathfrak{h}^*$, that is

$$\pi_*(\psi)(link_\infty) \in \mathrm{Im}(\rho(\gamma_\infty) - \mathrm{I})$$

where $link_\infty$ is the cut circle link around ∞ of the Serre tree $\mathcal{T}$ in a fundamental domain D_N of $\Gamma_0(N) \setminus \mathfrak{h}$, and $\gamma_\infty = \begin{pmatrix} 1 & \pm w_\infty \\ 0 & 1 \end{pmatrix}$ mod I_2 is a generator of $\mathrm{Stab}_{\Gamma_0(N)}(\infty)$ where w_∞ is the width of the cusp $\Gamma_0(N)\infty$. To consider the cusps in $\Gamma_0(pN) \setminus \mathfrak{h}^*$ that lie above $\Gamma_0(N)\infty \in \Gamma_0(N) \setminus \mathfrak{h}^*$, again we use the definitions of $\{\Gamma_0(pN)\gamma_1\infty, \Gamma_0(pN)\gamma_2\infty, \cdots, \Gamma_0(pN)\gamma_m\infty\}$ and $\{r_1, r_2, \cdots, r_m\}$ as in Lemma 5.3.4. Write $\Gamma = \Gamma_0(pN)$, we let $\gamma_{\gamma_i\infty,\Gamma} = \begin{pmatrix} 1 & w_{\gamma_i\infty,\Gamma} \\ 0 & 1 \end{pmatrix}$ mod I_2 be a generator of $\mathrm{Stab}_{\Gamma_0(pN)}(\gamma_i\infty)$, and $w_{\gamma_i\infty,\Gamma}$ be the width of the cusp $\Gamma_0(pN)\gamma_i\infty \in \Gamma_0(pN) \setminus \mathfrak{h}^*$, we have

$$\gamma_{\gamma_i\infty,\Gamma} = \gamma_i \gamma_\infty^{r_i} \gamma_i^{-1}, \quad w_{\gamma_i\infty,\Gamma} = r_i w_{\gamma_i\infty} = r_i w_\infty \ .$$

Since $\psi \in C^1_P(\Gamma_0(pN), \mathbf{V}_k)$, we have

$$\psi(link_{\gamma_i\infty,\Gamma}) \in \mathrm{Im}\left(\rho(\gamma_{\gamma_i\infty,\Gamma}) - \mathrm{I}\right) \tag{7.3.3}$$

where $link_{\gamma_i\infty,\Gamma}$ is the cut circle link around $\gamma_i\infty$ of the Serre tree $\mathcal{T}$ in a fundamental domain D_Γ of $\Gamma_0(pN) \setminus \mathfrak{h}$, and $\gamma_{\gamma_i\infty,\Gamma} = \begin{pmatrix} 1 & \pm w_{\gamma_i\infty,\Gamma} \\ 0 & 1 \end{pmatrix} \mod \mathrm{I}_2$ is a generator of $\mathrm{Stab}_{\Gamma_0(pN)}(\gamma_i\infty)$ where $w_{\gamma_i\infty,\Gamma}$ is the width of the cusp $\Gamma_0(pN)\gamma_i\infty$.

If $link_\infty$ has the following form

$$\begin{array}{ccccccc} & e_1 & & e_2 & & e_w & \\ \mathrm{B} & \longleftarrow & \mathrm{W} & \longrightarrow & \cdots & \longrightarrow & \mathrm{B} \\ p_1 & & p_2 & & & & \gamma_\infty \cdot p_1 \end{array} ,$$

then $link_{\gamma_i\infty,\Gamma}$ has the form

$$\begin{array}{ccccccc} & e'_1 & & e'_2 & & e'_{r_iw} & \\ \mathrm{B} & \longleftarrow & \mathrm{W} & \longrightarrow & \cdots & \longrightarrow & \mathrm{B} \\ p'_1 & & p'_2 & & & & \gamma_{\gamma_i\infty,\Gamma} \cdot p'_1 \end{array} .$$

around the cusp $\Gamma_0(pN)\gamma_i\infty$. Now equation (7.3.3) becomes

$$\begin{aligned} \psi(-e'_1 + e'_2 - \cdots + e'_{r_iw}) &= (L^*_{\gamma_i}\psi)(-e_1 + e_2 - \cdots + e_{r_iw}) \\ &\in \mathrm{Im}\left(\rho(\gamma_i\gamma_\infty^{r_i}\gamma_i^{-1}) - \mathrm{I}\right) . \end{aligned}$$

If we act by $\rho(\gamma_i^{-1})$ and sum over the unique coset representatives $\gamma_i = \{\gamma_1, \gamma_2, \cdots, \gamma_m\}$, we obtain

$$\sum_{i=1}^{m} \rho(\gamma_i^{-1})(L^*_{\gamma_i}\psi)(-e_1 + e_2 - \cdots + e_{r_iw}) \in \sum_{i=1}^{m} \rho(\gamma_i^{-1})\,\mathrm{Im}\left(\rho(\gamma_i\gamma_\infty^{r}\gamma_i^{-1}) - \mathrm{I}\right) . \quad (7.3.4)$$

First note that the right hand side of equation (7.3.4) is simply $\mathrm{Im}\left(\rho(\gamma_\infty) - \mathrm{I}\right)$ since

$$\begin{aligned} \sum_{i=1}^{m} \rho(\gamma_i^{-1})\,\mathrm{Im}\left(\rho(\gamma_i\gamma_\infty^{r_i}\gamma_i^{-1}) - \mathrm{I}\right) &= \sum_{i=1}^{m} \mathrm{Im}\left(\rho(\gamma_\infty^{r_i}) - \mathrm{I}\right)\rho(\gamma_i^{-1}) \\ &\in \sum_{i=1}^{m} \mathrm{Im}\left(\rho(\gamma_\infty) - \mathrm{I}\right) \qquad \text{since } \gamma_i^{-1} \colon V_k \xrightarrow{\sim} V_k, \\ &\in \mathrm{Im}\left(\rho(\gamma_\infty) - \mathrm{I}\right) . \end{aligned}$$

For each i of the left hand side of equation (7.3.4), we have

$$\rho(\gamma_i^{-1})(L^*_{\gamma_i}\psi)(-e_1+e_2-\cdots+e_{r_iw})$$

$$=\sum_{j=0}^{r_i-1}\rho(\gamma_i^{-1})(L^*_{\gamma_i}\psi)(-e_{jw+1}+e_{jw+2}-\cdots+e_{jw})\ ,$$

let $\gamma_{i+j}=\gamma_i\gamma_\infty^j$,

$$=\sum_{j=0}^{r_i-1}\rho(\gamma_\infty^j)\rho(\gamma_{i+j}^{-1})(L^*_{\gamma_{i+j}}\psi)(-e_1+e_2-\cdots+e_w)$$

$$=\sum_{j=0}^{r_i-1}\Big(\left(\rho(\gamma_\infty^j)-\mathrm{I}\right)\rho(\gamma_{i+j}^{-1})(L^*_{\gamma_{i+j}}\psi)(-e_1+e_2-\cdots+e_w)$$

$$+\rho(\gamma_{i+j}^{-1})(L^*_{\gamma_{i+j}}\psi)(-e_1+e_2-\cdots+e_w)\Big)\ .$$

Since $\rho(\gamma_\infty^j)-\mathrm{I}\in\mathrm{Im}\,(\rho(\gamma_\infty)-\mathrm{I})$, we consider the remaining of the above sum and do this for all i, we get for the left hand side of equation (7.3.4),

$$\sum_{i=1}^{m}\rho(\gamma_i^{-1})(L^*_{\gamma_i}\psi)(-e_1+e_2\cdots+e_{r_iw})=\sum_{i=1}^{m}\sum_{j=0}^{r_i-1}\rho(\gamma_{i+j}^{-1})(L^*_{\gamma_{i+j}}\psi)(-e_1+e_2\cdots+e_w)\ .$$

Lemma 5.3.4 tells us that γ_{i+j} for $i=1,2,\cdots,m$ and $j=0,1,\cdots,r_i-1$ forms a complete set of coset representatives of $\Gamma_0(pN)\setminus\Gamma_0(N)$. Thus the left hand side of equation 7.3.4 becomes

$$\sum_{\gamma\in\Gamma_0(pN)\setminus\Gamma_0(N)}\rho(\gamma^{-1})(L^*_\gamma\psi)(-e_1+e_2-\cdots+e_w)=\pi_*(\psi)(link_\infty)\ .$$

Therefore we have shown that

$$\pi_*(\psi)(link_\infty)\in\mathrm{Im}(\rho(\gamma_\infty)-\mathrm{I})\ .$$ □

Theorem 7.3.7. The isomorphism

$$H^1_P(\Gamma_0(N),\mathfrak{h},V)\cong H^1_P(\Gamma_0(N),\mathcal{T},V)$$

is equivariant for the action of T_p (at least for $p=2,3,$ and 5).

Furthermore, the discussion in this section gives an algorithm for calculating T_p on $C^1_P(\Gamma_0(N), \mathcal{T}, V)$, at least for $p \nmid N, p < 7$. We work through some examples in section 7.5.

7.4 Dimension Formula

Since

$$H^1_P(\Gamma, \mathcal{T}, V_k) \cong H^1_P(\Gamma, \mathfrak{h}, V_k) \cong S_k(\Gamma) \bigoplus \overline{S_k(\Gamma)} \ ,$$

we should have

$$\dim(H^1_P(\Gamma, \mathcal{T}, V_k)) = 2\dim(S_k(\Gamma)) \ .$$

Here we will verify this where $\Gamma = \Gamma_0(2)$ and $\Gamma = \Gamma_0(3)$ for $k \leq 20$, but the general ideas are the same.

7.4.1 Case 1: $\Gamma = \Gamma_0(2)$.

The space $\Gamma_0(2) \setminus \mathfrak{h}$ has two cusps, namely ∞ and 0. Let $\gamma_\infty \in \mathrm{Stab}_{\Gamma_0(2)}(\infty)$ denote the element in the stabilizers of ∞ in $\Gamma_0(2) \setminus \mathfrak{h}$, and $\gamma_0 \in \mathrm{Stab}_{\Gamma_0(2)}(0)$ be the element in the stabilizers of 0 in $\Gamma_0(2) \setminus \mathfrak{h}$. Thus

$$\gamma_\infty = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

$\Gamma_0(2) \setminus \mathfrak{h}$ also has one elliptic point, $\dfrac{1+i}{2}$. Let $\gamma_e \in \mathrm{Stab}_{\Gamma_0(2)}\left(\dfrac{1+i}{2}\right)$ be the element in $\Gamma_0(2)$ that fixes the elliptic point. Thus

$$\gamma_e = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}.$$

Figure 7.4 shows part of the Serre Tree for $\Gamma_0(2)$. $\{e_1, e_2, e_3\}$ together with their vertices, form a fundamental set of edges and vertices to form the dessin for $\Gamma_0(2)$ by identifying p_1 with p_1' under the action of γ_∞. Here we have $\gamma_\infty\colon p_1 \to p_1'$, $\gamma_0\colon \tilde{p}_3 \to p_3$ and $\gamma_e\colon p_3 \to p_3$.

We will compute $H^\bullet_P(\Gamma, \mathcal{T}, V_k)$ using the formula $H^\bullet_P(\Gamma, \mathcal{T}, V_k) = \dfrac{C^1_P(\Gamma, \mathcal{T}, V_k)}{\mathrm{Im}(d)}$. Let $V_k = \mathrm{Sym}^{k-2}(\mathbf{C}^2)$ be defined as before, where k is the weight of cusp forms in

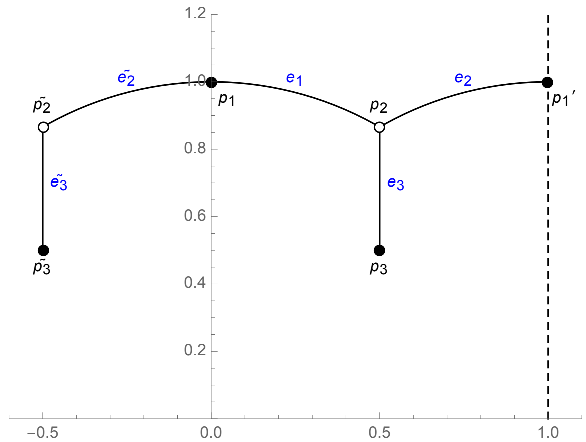


Figure 7.4: Part of the Serre Tree for $\Gamma_0(2)$.

$S_k(\Gamma)$. Hence $\dim(V_k) = k-1$. Use the same orientation as in section 7.2, where the direction of edges is from white vertex to black vertex. If $\phi \in C^1_P\big(\Gamma, \mathcal{T}, V_k\big)$, then we have

$$\begin{aligned}\phi(link_\infty) &= \phi(-e_1 + e_2)\\ &= -\phi(e_1) + \phi(e_2)\\ &\in \operatorname{Im}\big(\rho(\gamma_\infty) - \mathrm{I}_{k-1}\big)\ ,\end{aligned}$$

and

$$\begin{aligned}\phi(link_0) &= \phi(-e_3 + e_1 - \tilde{e_2} + \tilde{e_3})\\ &= -\phi(e_3) + \phi(e_1) - \phi(\tilde{e_2}) + \phi(\tilde{e_3})\\ &\in \operatorname{Im}\big(\rho(\gamma_0^{-1}) - \mathrm{I}_{k-1}\big)\ .\end{aligned}$$

Here $\rho(\gamma_\infty)$ is an upper triangular matrix, and $\rho(\gamma_0^{-1})$ is a lower triangular matrix both with ones on the diagonal and nonzero elements on the superdiagonal and subdiagonal respectively. Thus both $\operatorname{Im}\big(\rho(\gamma_\infty) - \mathrm{I}_{k-1}\big)$ and $\operatorname{Im}(\rho(\gamma_0^{-1}) - \mathrm{I}_{k-1})$ have co-dimension 1 in V_k. Since the dessin for $\Gamma_0(2)$ has 3 edges, namely $\{e_1, e_2, e_3\}$,

$$\dim\big(C^1_P\big(\Gamma, \mathcal{T}, V_k\big)\big) = \dim\big(V_k\big)\cdot 3 - 2\ .$$

To compute $\dim\big(\operatorname{Im}(d)\big)$, we have the following short exact sequence

$$0 \xrightarrow{d} \ker(d) \xrightarrow{d} C^0_P\big(\Gamma, \mathcal{T}, V_k\big) \xrightarrow{d} C^1_P\big(\Gamma, \mathcal{T}, V_k\big) \xrightarrow{d} 0$$

giving

$$C^0_P\big(\Gamma, \mathcal{T}, V_k\big) = \ker(d) \bigoplus \operatorname{Im}(d) \ .$$

Thus

$$\dim\big(\operatorname{Im}(d)\big) = \dim\big(C^0_P\big(\Gamma, \mathcal{T}, V_k\big)\big) - \dim\big(\ker(d)\big) \ .$$

If $f \in C^0_P\big(\Gamma, \mathcal{T}, V_k\big)$, then

$$f(p_3) = f(\gamma_e \cdot p_3) = \rho(\gamma_e) \cdot f(p_3)$$

since p_3 is an elliptic point of $\Gamma_0(2) \setminus \mathfrak{h}$. Let $w_3 = f(p_3)$, thus $w_3 \in \ker\big(\rho(\gamma_e) - \mathrm{I}_{k-1}\big)$. The dessin for $\Gamma_0(2)$ has 3 vertices, namely $\{p_1, p_2, p_3\}$. This gives

$$\dim\big(C^0_P\big(\Gamma, \mathcal{T}, V_k\big)\big) = \dim\big(V_k\big) \cdot 2 + \dim\big(\ker\big(\rho(\gamma_e) - \mathrm{I}_{k-1}\big)\big) \ .$$

Now for f to be in $\ker(d)$, we need

$$\begin{cases} df(e_1) & = f(p_1) - f(p_2) = 0 \ , \\ df(e_2) & = f(p_1') - f(p_2) = \rho(\gamma_\infty) f(p_1) - f(p_2) = 0 \ , \\ df(e_3) & = f(p_3) - f(p_2) = 0 \ . \end{cases}$$

Hence we have $f(p_1) = f(p_2) = f(p_3) \in \ker\big(\rho(\gamma_e) - \mathrm{I}_{k-1}\big) \bigcap \ker\big(\rho(\gamma_\infty) - \mathrm{I}_{k-1}\big)$, which can be verified to be 0, since $\rho(\gamma_\infty) - \mathrm{I}_{k-1}$ is a strictly upper triangular matrix and $\rho(\gamma_e) - \mathrm{I}_{k-1}$ is not. Therefore

$$\dim(\operatorname{Im}(d)) = \dim(V_k) \cdot 2 + \dim\big(\ker\big(\rho(\gamma_e) - \mathrm{I}_{k-1}\big)\big) \ .$$

Combining the above results, we have

$$\begin{aligned} \dim\big(H^\bullet_P\big(\Gamma, \mathcal{T}, V_k\big)\big) &= \dim\big(C^1_P\big(\Gamma, \mathcal{T}, V_k\big)\big) - \dim\big(\operatorname{Im}(d)\big) \\ &= \dim\big(V_k\big) \cdot 3 - 2 - \big(\dim(V_k) \cdot 2 + \dim\big(\ker\big(\rho(\gamma_e) - \mathrm{I}_{k-1}\big)\big)\big) \\ &= \dim(V_k) - \dim\big(\ker\big(\rho(\gamma_e) - \mathrm{I}_{k-1}\big)\big) - 2 \ . \end{aligned}$$

Observation 2. $\ker\left(\rho(\gamma_e) - \mathrm{I}_{k-1}\right)$ has dimension 0 when weight k is odd.

However, we know that S_k has dimension 0 for odd k. Thus we only focus on cases where k is even. Since $\dim(V_k) = k - 1$, it has little value when V_k has dimension 1. We start with $k = 4$. In the following table, we computed the dimensions for small values of k for the congruence subgroup $\Gamma_0(2)$.

Table 7.1: Dimension Formula for $\Gamma_0(2)$.

k	$\dim(V_k)$	$\dim\left(\ker\left(\rho(\gamma_e) - \mathrm{I}_{k-1}\right)\right)$	$\dim(H_P^\bullet(\Gamma, \mathcal{T}, V_k))$	$\dim(S_k(\Gamma))^*$
4	3	1	0	0
6	5	3	0	0
8	7	3	2	1
10	9	5	2	1
12	11	5	4	2
14	13	7	4	2
16	15	7	6	3
18	17	9	6	3
20	19	9	8	4

*Dimension of S_k computed by Sage.

7.4.2 Case 2: $\Gamma = \Gamma_0(3)$.

The space $\Gamma_0(3) \setminus \mathfrak{h}$ has two cusps, namely ∞ and 0. The stabilizer of each cusp is thus represented as

$$\gamma_\infty = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}.$$

$\Gamma_0(2) \setminus \mathfrak{h}$ also has one elliptic point, $\dfrac{3+\sqrt{3}i}{6}$. The stabilizer of the elliptic point is represented as

$$\gamma_e = \begin{pmatrix} -1 & 1 \\ -3 & 2 \end{pmatrix}.$$

Figure 7.5 shows part of the Serre Tree for $\Gamma_0(3)$. $\{e_1, e_2, e_3, e_4\}$ together with their vertices, form a fundamental set of edges and vertices to form the dessin for $\Gamma_0(3)$ by identifying p_1 with p_1' under the action of γ_∞. Here we have $\gamma_\infty\colon p_1 \to p_1'$, $\gamma_0\colon \tilde{p_4} \to p_4$ and $\gamma_e\colon p_4 \to p_4$.

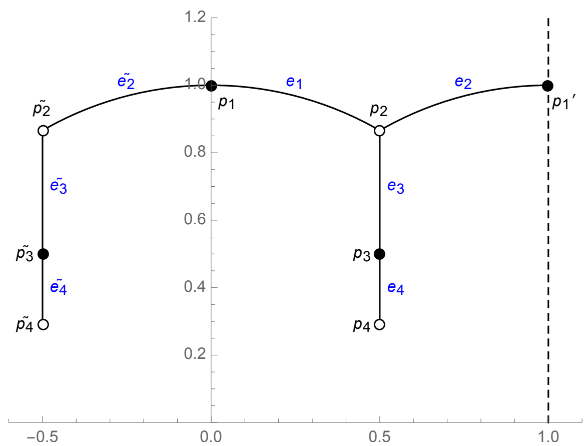


Figure 7.5: Part of the Serre Tree for $\Gamma_0(3)$

As before, we use the formula $H^\bullet_P\big(\Gamma, \mathcal{T}, V_k\big) = \dfrac{C^1_P\big(\Gamma, \mathcal{T}, V_k\big)}{\mathrm{Im}(d)}$, and V_k where k is the weight of cusp forms in $S_k(\Gamma)$. In this case, we have for $\phi \in C^1_P\big(\Gamma, \mathcal{T}, V_k\big)$,

$$\begin{aligned}\phi(link_\infty) &= \phi(-e_1 + e_2)\\ &= -\phi(e_1) + \phi(e_2)\\ &\in \mathrm{Im}\left(\rho(\gamma_\infty) - \mathrm{I}_{k-1}\right)\ ,\end{aligned}$$

and

$$\begin{aligned}\phi(link_0) &= \phi(e_4 - e_3 + e_1 - \tilde{e_2} + \tilde{e_3} - \tilde{e_4})\\ &= \phi(e_4) - \phi(e_3) + \phi(e_1) - \phi(\tilde{e_2}) + \phi(\tilde{e_3}) - \phi(\tilde{e_4})\\ &\in \mathrm{Im}\left(\rho(\gamma_0^{-1}) - \mathrm{I}_{k-1}\right)\ .\end{aligned}$$

Again, $\rho(\gamma_\infty)$ is an upper triangular matrix, and $\rho(\gamma_0^{-1})$ is a lower triangular matrix both with ones on the diagonal and nonzero elements on the superdiagonal and subdiagonal respectively, and both $\mathrm{Im}\left(\rho(\gamma_\infty) - \mathrm{I}_{k-1}\right)$ and $\mathrm{Im}\left(\rho(\gamma_0^{-1}) - \mathrm{I}_{k-1}\right)$ have co-dimension 1 in V_k. The dessin for $\Gamma_0(3)$ has 4 edges, namely $\{e_1, e_2, e_3, e_4\}$,

$$\dim(C_P^1(\Gamma, \mathcal{T}, V_k)) = \dim(V_k) \cdot 4 - 2 \ .$$

To compute $\dim\left(\mathrm{Im}(d)\right)$, again, we use the relation

$$C_P^0(\Gamma, \mathcal{T}, V_k) = \ker(d) \bigoplus \mathrm{Im}(d)$$

from the short exact sequence

$$0 \xrightarrow{d} \ker(d) \xrightarrow{d} C_P^0(\Gamma, \mathcal{T}, V_k) \xrightarrow{d} C_P^1(\Gamma, \mathcal{T}, V_k) \xrightarrow{d} 0 \ .$$

For $f \in C_P^0(\Gamma, \mathcal{T}, V_k)$,

$$f(p_4) = f(\gamma_e \cdot p_4) = \rho(\gamma_e) f(p_4)$$

since p_4 is an elliptic point of $\Gamma_0(3) \setminus \mathfrak{h}$. Let $w_4 = f(p_4)$, thus $w_4 \in \ker\left(\rho(\gamma_e) - \mathrm{I}_{k-1}\right)$. The dessin for $\Gamma_0(3)$ has 4 vertices, namely $\{p_1, p_2, p_3, p_4\}$, and thus

$$\begin{aligned} \dim\left(C_P^0(\Gamma, \mathcal{T}, V_k)\right) &= \dim\left(V_k\right) \cdot 3 + \dim\left(\ker\left(\rho(\gamma_e) - \mathrm{I}_{k-1}\right)\right) \\ &= \dim\left(V_k\right) \cdot 3 + \dim(w_4) \ . \end{aligned}$$

Now for f to be in $\ker(d)$, we again need

$$\begin{cases} df(e_1) &= f(p_1) - f(p_2) = 0 \ , \\ df(e_2) &= f(p_1') - f(p_2) = \rho(\gamma_\infty) \cdot f(p_1) - f(p_2) = 0 \ , \\ df(e_3) &= f(p_3) - f(p_2) = 0 \ , \\ df(e_4) &= f(p_3) - f(p_4) = 0 \ . \end{cases}$$

Again, $\ker(d) = \ker\left(\rho(\gamma_e) - \mathrm{I}_{k-1}\right) \bigcap \ker\left(\rho(\gamma_\infty) - \mathrm{I}_{k-1}\right)$, which can be verified to be 0, since $\rho(\gamma_\infty) - \mathrm{I}_{k-1}$ is a strictly upper triangular matrix and $\rho(\gamma_e) - \mathrm{I}_{k-1}$ is not. Therefore

$$\dim(\mathrm{Im}(d)) = \dim(V_k) \cdot 3 + \dim(w_4) \ .$$

Combining the above results, we get

$$\begin{aligned}\dim\Big(H_P^\bullet\big(\Gamma,\mathcal{T},V_k\big)\Big) &= \dim\Big(C_P^1\big(\Gamma,\mathcal{T},V_k\big)\Big)-\dim\Big(\operatorname{Im}(d)\Big)\\ &= \dim\big(V_k\big)\cdot 4-2-\Big(\dim(V_k)\cdot 3+\dim(w_4)\Big)\\ &= \dim(V_k)-\dim(w_4)-2\ .\end{aligned}$$

Observation 3. w_4 has dimension 0 when weight k is odd.

Since S_k has dimension 0 for odd k, we only focus on cases where k is even. Again, $\dim(\mathbb{V}_2)=1$ has little value to us. Thus we start with $k=4$. In the following table, we computed the dimensions for small values of k for the congruence subgroup $\Gamma_0(3)$.

Table 7.2: Dimension Formula for $\Gamma_0(3)$.

k	$\dim(V_k)$	$\dim(w_3)$	$\dim(H_P^\bullet(\Gamma,\mathcal{T},V_k))$	$\dim(S_k(\Gamma))^*$
4	3	1	0	0
6	5	1	2	1
8	7	3	2	1
10	9	3	4	2
12	11	3	6	3
14	13	5	6	3
16	15	5	8	4
18	17	5	10	5
20	19	7	10	5

*Dimension of S_k computed by Sage.

7.5 Examples of Hecke Correspondence of 1-Forms

From section 7.3 we can calculate the Hecke correspondence $T_p (p < 7, p \nmid N)$ on 1-forms in $C^1_P(\Gamma_0(N), \mathcal{T}, V_k)$ associated with modular curves in the following diagram

$$\begin{array}{ccc} \Gamma_0(pN) \setminus \mathfrak{h} & \xrightarrow{g} & g\Gamma_0(pN)g^{-1} \setminus \mathfrak{h} \\ \pi\downarrow & & \downarrow\pi_g \\ \Gamma_0(N) \setminus \mathfrak{h} & & \Gamma_0(N) \setminus \mathfrak{h} \end{array}$$

where $g = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$. We will illustrate the steps through examples of T_3 and T_5 with different base curves, which can be generalized to other cases as well.

7.5.1 T_3 Acting on $X_0(2)$

We start with the curve $X_0(2) = \Gamma_0(2) \setminus \mathfrak{h}$. We have the following diagram for T_3:

$$\begin{array}{ccc} \Gamma_0(6) \setminus \mathfrak{h} & \xrightarrow{g} & g\Gamma_0(6)g^{-1} \setminus \mathfrak{h} \\ \pi\downarrow & & \downarrow\pi_g \\ \Gamma_0(2) \setminus \mathfrak{h} & & \Gamma_0(2) \setminus \mathfrak{h} \end{array}$$

where $g = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$.

We start from the bottom right and work our way back to bottom left. As shown in example 7.1.5, the dessin for $\Gamma_0(2)$ embedded in $\mathfrak{h}$ is

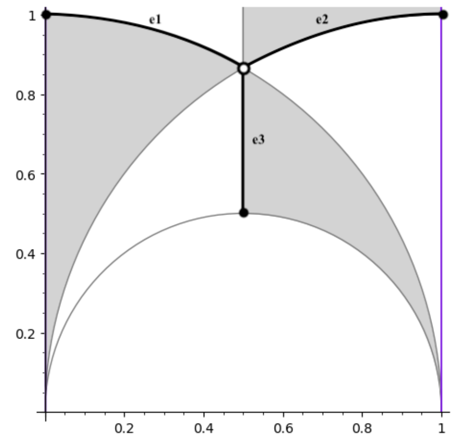
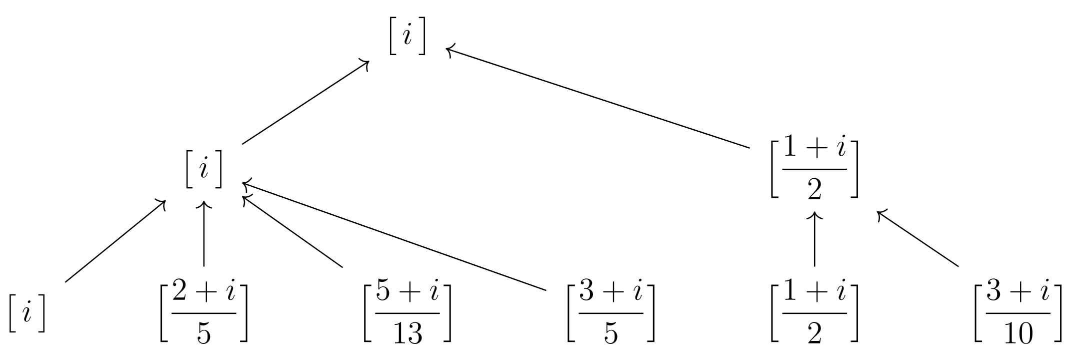


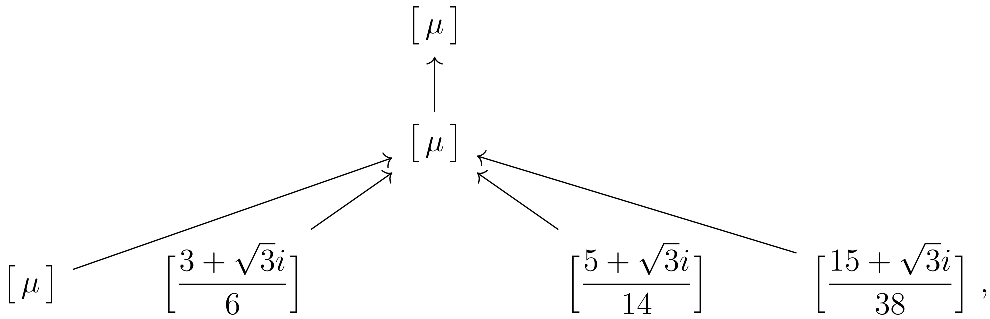
Figure 7.6: Dessin of $\Gamma_0(2)$ embedded in $\mathfrak{h}$.

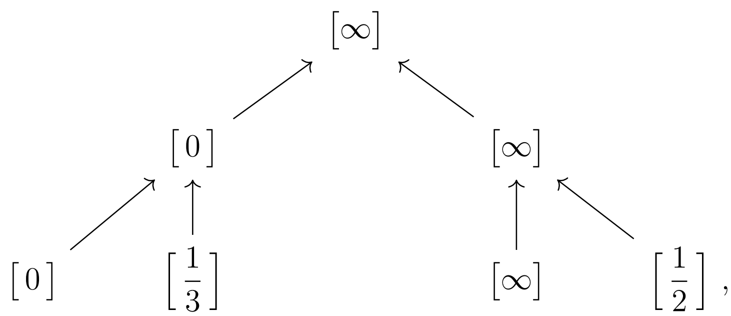
Here we also labeled the edges $e_1, \cdots, e_3$. To get the dessin for $g\Gamma_0(6)g^{-1}$, we start with the embedded dessin for $\Gamma_0(6)$ and map it under g. The Belyi map for $\Gamma_0(6)$ has degree $[\mathrm{PSL}_2(\mathbf{Z})\colon \Gamma_0(6)] = 12$. Thus this dessin has 12 edges. The coset representatives of $\Gamma_0(6)$ in $\Gamma_0(2)$ are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}.$$

Thus the $\Gamma_0(6)$-orbits of i, μ and ∞ are







where the top rows are $\mathrm{SL}_2(\mathbf{Z})$-orbits, the middle rows are $\Gamma_0(2)$-orbits, and the bottom rows are $\Gamma_0(6)$-orbits. These give us 6 black vertices, 4 white vertices, and 4 faces together with the 12 edges we computed from before. First we have the embedded dessin in $\mathfrak{h}$ -

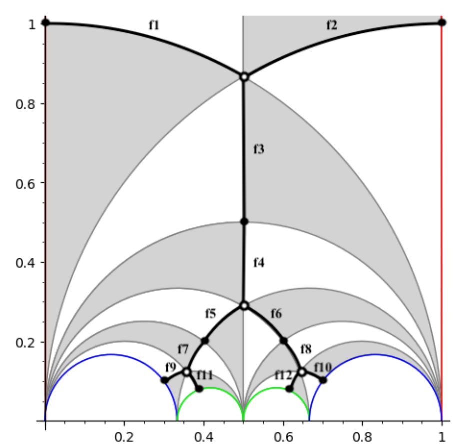
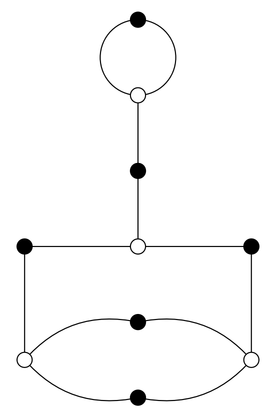


Figure 7.7: Dessin of $\Gamma_0(6)$ embedded in $\mathfrak{h}$.

As before, we labeled the edges $f_1, \cdots, f_{12}$. The colored boundaries are identified to the same color. Hence we also have the abstract dessin for $\Gamma_0(6)$:



To get the embedded dessin for $g\Gamma_0(6)g^{-1}$, we apply g to the embedded dessin for $\Gamma_0(6)$, which acts on the graph by multiplication by 3. We obtain the following embedded dessin for $g\Gamma_0(6)g^{-1}$:

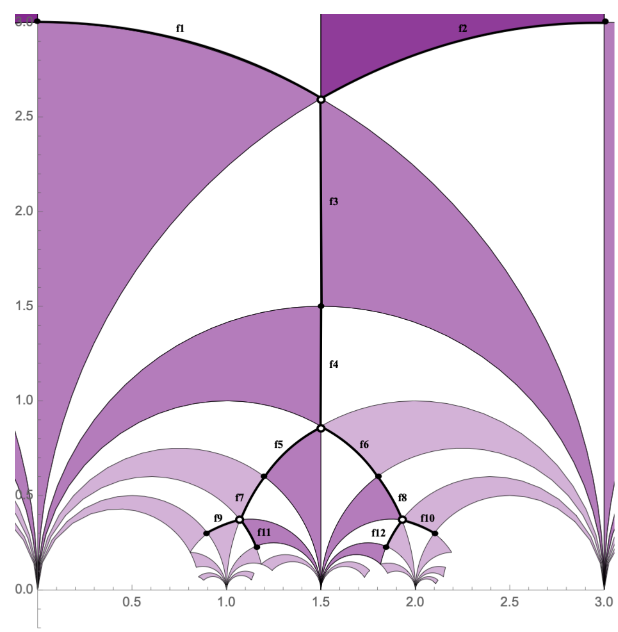


Figure 7.8: Dessin of $g\Gamma_0(6)g^{-1}$ Embedded in $\mathfrak{h}$.

We then overlay the embedded dessin for $\Gamma_0(2)$ with the embeded dessin for $g\Gamma_0(6)g^{-1}$. We get

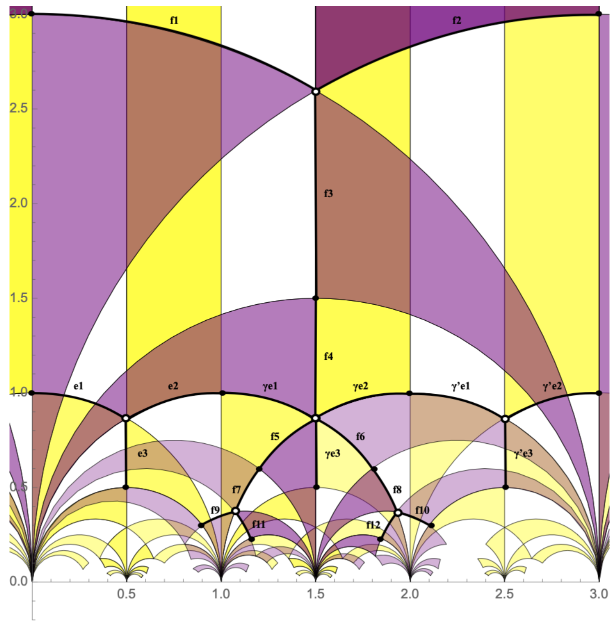


Figure 7.9: Overlay of Dessins of $\Gamma_0(2)$ and $g\Gamma_0(6)g^{-1}$ Embedded in $\mathfrak{h}$.

where the yellow part comes from expanding the embedded dessin of $\Gamma_0(2)$ along the Serre tree:

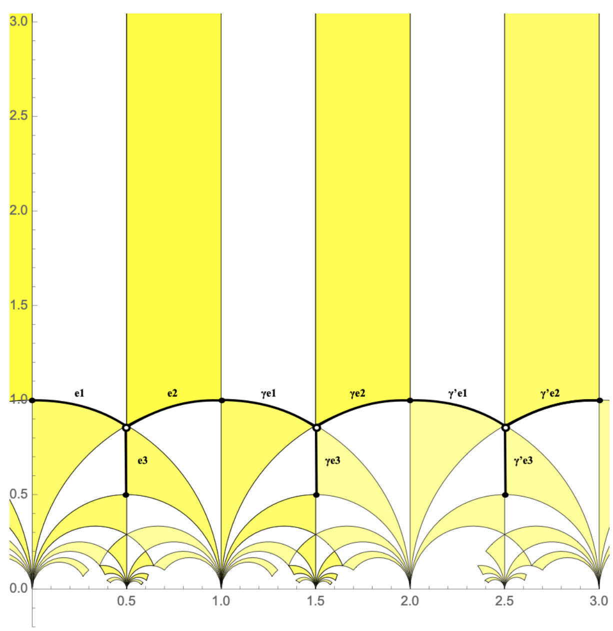


Figure 7.10: Expanded Dessin of $\Gamma_0(2)$ Embedded in $\mathfrak{h}$.

Here e_1, e_2, e_3 are of the original dessin. $\gamma = \gamma_\infty = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ shifts the edges to the right by 1 and $\gamma' = \gamma_\infty^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ shifts the edges to the right by 2.

If we have a 1-form in $C_P^1(\Gamma_0(2), \mathcal{T}, V_k)$, it naturally pulls back to a 1-form in $C_P^1(g\Gamma_0(6)g^{-1}, \mathcal{T}, V_k)$, since if $\gamma \in \Gamma_0(6)$, $\gamma = \begin{pmatrix} a & b \\ 6c & d \end{pmatrix}$ where $a, b, c, d \in \mathbf{Z}$ such that $ad - 6bc = 1$. We have

$$\begin{aligned} g\gamma g^{-1} &= \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 6c & d \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3a & 3b \\ 6c & d \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a & 3b \\ 2c & d \end{pmatrix} \in \Gamma_0(2) \ . \end{aligned}$$

Thus $g\Gamma_0(6)g^{-1} \subseteq \Gamma_0(2)$. If ϕ is a 1-form in $C_P^1(\Gamma_0(2), \mathcal{T}, V_k)$, $\phi(\gamma e) = \rho(\gamma)\phi(e)$ for $\gamma \in \Gamma_0(2)$ and $e \in \mathrm{E}(\mathcal{T})$. Since $g\Gamma_0(6)g^{-1} \in \Gamma_0(2)$, for any $e \in \mathrm{E}(\mathcal{T})$, $\phi(\gamma e) = \rho(\gamma)\phi(e)$ for $\gamma \in g\Gamma_0(6)g^{-1}$. Thus ϕ is necessarily a 1-form in $C_P^1(g\Gamma_0(6)g^{-1}, \mathcal{T}, V_k)$.

We then have 1-forms at each edge. To see how the 1-forms $\phi \in C_P^1(\Gamma_0(2), \mathcal{T}, V_k)$ pull back to 1-forms in $C_P^1(g\Gamma_0(6)g^{-1}, \mathcal{T}, V_k)$, we apply the deformation retract r as defined in equation (7.2.2) with respect to the orientation of the edges. More specifically, we deform the oriented edges gf_i of the dessin for $g\Gamma_0(6)g^{-1}$ via the deformation retract r to linear combinations of the oriented edges e_j of the dessin for $\Gamma_0(2)$. For example, edge gf_1 can be deformed to edges e_1, e_2, and $\gamma_\infty e_1$ with the following orientation

$$\begin{aligned} \phi(gf_1) &= \phi(\gamma_\infty e_1) - \phi(e_2) + \phi(e_1) \\ &= \rho(\gamma_\infty)\phi(e_1) - \phi(e_2) + \phi(e_1) \end{aligned}$$

as shown in Figure 7.9. There f_1 passes through a white, a yellow and a white ideal triangle, and the deformation r pushes f_1 down to the bottom edges of those triangles.

Similarly, we can homotope all 1-forms associated with $f_2, \cdots f_{12}$ to 1-forms associated with e_1, e_2, and e_3 as follows:

$$\begin{aligned}
\phi(gf_2) &= \rho(\gamma_\infty)\phi(e_2) - \rho(\gamma_\infty^2)\phi(e_1) + \rho(\gamma_\infty^2)\phi(e_2)\ , \\
\phi(gf_3) &= \phi(gf_4) = \phi(gf_5) = \phi(gf_6) = \phi(gf_7) = \phi(gf_8) = 0\ , \\
\phi(gf_9) &= \rho(\gamma_\infty)\phi(e_1) - \phi(e_2) + \phi(e_3)\ , \\
\phi(gf_{10}) &= \rho(\gamma_\infty)\phi(e_2) - \rho(\gamma_\infty^2)\phi(e_1) + \rho(\gamma_\infty^2)\phi(e_3)\ , \\
\phi(gf_{11}) &= \rho(\gamma_\infty)\phi(e_3) - \rho(\gamma_\infty\gamma_e^{-1})\phi(e_3) + \rho(\gamma_\infty\gamma_e^{-1})\phi(e_2)\ , \\
\phi(gf_{12}) &= \rho(\gamma_\infty)\phi(e_3) - \rho(\gamma_\infty\gamma_e^{-1})\phi(e_3) + \rho(\gamma_\infty\gamma_e^{-1})\phi(e_1)
\end{aligned}$$

where $\gamma_e = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \in \mathrm{Stab}_{\Gamma_0(2)}\left(\frac{1+i}{2}\right)$. γ_e maps e_1, e_2, e_3 to the 3 edges right below e_3, connecting at the black vertex $\frac{1+i}{2}$, with a counterclockwise order of $\gamma_e e_3, \gamma_e e_2, \gamma_e e_1$. Thus $\gamma_\infty\gamma_e e_1, \gamma_\infty\gamma_e e_2, \gamma_\infty\gamma_e e_3$ refer to the edges that will homotope to part of f_{11} or f_{12}.

Now to go from $C^1_P(g\Gamma_0(6)g^{-1}, \mathcal{T}, V_k)$ to $C^1_P(\Gamma_0(6), \mathcal{T}, V_k)$, we apply g^{-1}. Assume the edges of the dessin for $\Gamma_0(6)$ are labeled as in Figure 7.7, and we denote ψ as

1-forms in $C^1_P(\Gamma_0(6), \mathcal{T}, V_k)$. We get

$$\psi(f_1) = \rho(g^{-1})\phi(gf_1) = \rho(g^{-1}\gamma_\infty)\phi(e_1) - \rho(g^{-1})\phi(e_2) + \rho(g^{-1})\phi(e_1) \ ,$$

$$\psi(f_2) = \rho(g^{-1})\phi(gf_2) = \rho(g^{-1}\gamma_\infty)\phi(e_2) - \rho(g^{-1}\gamma_\infty^2)\phi(e_1) + \rho(g^{-1}\gamma_\infty^2)\phi(e_2) \ ,$$

$$\psi(f_3) = \psi(f_4) = \psi(f_5) = \psi(f_6) = \psi(f_7) = \psi(f_8) = 0 \ ,$$

$$\psi(f_9) = \rho(g^{-1})\phi(gf_9) = \rho(g^{-1}\gamma_\infty)\phi(e_1) - \rho(g^{-1})\phi(e_2) + \rho(g^{-1})\phi(e_3) \ ,$$

$$\psi(f_{10}) = \rho(g^{-1})\phi(gf_{10}) = \rho(g^{-1}\gamma_\infty)\phi(e_2) - \rho(g^{-1}\gamma_\infty^2)\phi(e_1) + \rho(g^{-1}\gamma_\infty^2)\phi(e_3) \ ,$$

$$\psi(f_{11}) = \rho(g^{-1})\phi(gf_{11}) = \rho(g^{-1}\gamma_\infty)\phi(e_3) - \rho(g^{-1}\gamma_\infty\gamma_e^{-1})\phi(e_3) + \rho(g^{-1}\gamma_\infty\gamma_e^{-1})\phi(e_2) \ ,$$

$$\psi(f_{12}) = \rho(g^{-1})\phi(gf_{12}) = \rho(g^{-1}\gamma_\infty)\phi(e_3) - \rho(g^{-1}\gamma_\infty\gamma_e^{-1})\phi(e_3) + \rho(g^{-1}\gamma_\infty\gamma_e^{-1})\phi(e_1) \ .$$

Lastly, we want to go from $C^1_P(\Gamma_0(6), \mathcal{T}, V_k)$ down to the bottom left $C^1_P(\Gamma_0(2), \mathcal{T}, V_k)$. For 1-forms $\psi \in C^1_P(\Gamma_0(6), \mathcal{T}, V_k)$, define 1-forms in $C^1_P(\Gamma_0(2), \mathcal{T}, V_k)$ to be $\pi_*(\psi)(e) = \displaystyle\sum_{\gamma \in \Gamma_0(6)\backslash\Gamma_0(2)} \rho(\gamma^{-1})\psi(\gamma e)$ where $e \in \mathrm{E}(\mathcal{T})$.

Now we can compute 1-forms in $C^1_P(\Gamma_0(2), \mathcal{T}, V_k)$. Recall the coset representatives of $\Gamma_0(6) \setminus \Gamma_0(2)$ are

$$\overset{\mathrm{I}_2}{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}}, \quad \overset{\gamma_1}{\begin{pmatrix}1 & 0\\2 & 1\end{pmatrix}}, \quad \overset{\gamma_2}{\begin{pmatrix}1 & 1\\2 & 3\end{pmatrix}}, \quad \overset{\gamma_3}{\begin{pmatrix}1 & 2\\2 & 5\end{pmatrix}},$$

and the generators of $\Gamma_0(6)$ are

$$\overset{\gamma_x}{\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}}, \quad \overset{\gamma_y}{\begin{pmatrix}5 & -1\\6 & -1\end{pmatrix}}, \quad \overset{\gamma_z}{\begin{pmatrix}7 & -3\\12 & -5\end{pmatrix}}.$$

Let the edges of the dessin for $\Gamma_0(2)$ be labeled as in figure 7.6 and the edges of the dessin for $\Gamma_0(6)$ be labeled as in figure 7.7. We have

$$\begin{array}{lll} I_2 \cdot e_1 = f_1 \ , & I_2 \cdot e_2 = f_2 \ , & I_2 \cdot e_3 = f_3 \ , \\ \gamma_1 \cdot e_1 = f_7 \ , & \gamma_1 \cdot e_2 = f_{11} \ , & \gamma_1 \cdot e_3 = f_9 \ , \\ \gamma_2 \cdot e_1 = \gamma_z^{-1} f_{12} \ , & \gamma_2 \cdot e_2 = \gamma_z^{-1} f_8 \ , & \gamma_2 \cdot e_3 = \gamma_z^{-1} f_{10} \ , \\ \gamma_3 \cdot e_1 = \gamma_z^{-1} f_6 \ , & \gamma_3 \cdot e_2 = \gamma_z^{-1} f_5 \ , & \gamma_3 \cdot e_3 = \gamma_z^{-1} f_4 \ . \end{array}$$

Combining all above, we get

$$\begin{aligned}
\pi_*(\psi)(e_1) &= \rho(\mathrm{I}_2^{-1})\psi(\mathrm{I}_2\, e_1) + \rho(\gamma_1^{-1})\psi(\gamma_1 e_1) + \rho(\gamma_2^{-1})\psi(\gamma_2 e_1) + \rho(\gamma_3^{-1})\psi(\gamma_3 e_1) \\
&= \psi(f_1) + \rho(\gamma_1^{-1})\psi(f_7) + \rho(\gamma_2^{-1}\gamma_z^{-1})\psi(f_{12}) + \rho(\gamma_3^{-1}\gamma_z^{-1})\psi(f_6) \\
&= \left(\rho(g^{-1}\gamma_\infty) + \rho(g^{-1}) + \rho(\gamma_2^{-1}\gamma_z^{-1}g^{-1}\gamma_\infty\gamma_e^{-1})\right)\phi(e_1) \\
&\quad - \rho(g^{-1})\phi(e_2) \\
&\quad + \left(\rho(\gamma_2^{-1}\gamma_z^{-1}g^{-1}\gamma_\infty) - \rho(\gamma_2^{-1}\gamma_z^{-1}g^{-1}\gamma_\infty\gamma_e^{-1})\right)\phi(e_3)\ ,
\end{aligned}$$

$$\begin{aligned}
\pi_*(\psi)(e_2) &= \rho(\mathrm{I}_2^{-1})\psi(\mathrm{I}_2\, e_2) + \rho(\gamma_1^{-1})\psi(\gamma_1 e_2) + \rho(\gamma_2^{-1})\psi(\gamma_2 e_2) + \rho(\gamma_3^{-1})\psi(\gamma_3 e_2) \\
&= \psi(f_2) + \rho(\gamma_1^{-1})\psi(f_{11}) + \rho(\gamma_2^{-1}\gamma_z^{-1})\psi(f_8) + \rho(\gamma_3^{-1}\gamma_z^{-1})\psi(f_5) \\
&= -\rho(g^{-1}\gamma_\infty^2)\phi(e_1) \\
&\quad + \left(\rho(g^{-1}\gamma_\infty) + \rho(g^{-1}\gamma_\infty^2) + \rho(\gamma_1^{-1}g^{-1}\gamma_\infty\gamma_e^{-1})\right)\phi(e_2) \\
&\quad + \left(\rho(\gamma_1^{-1}g^{-1}\gamma_\infty) - \rho(\gamma_1^{-1}g^{-1}\gamma_\infty\gamma_e^{-1})\right)\phi(e_3)\ ,
\end{aligned}$$

$$\begin{aligned}
\pi_*(\psi)(e_3) &= \rho(\mathrm{I}_2^{-1})\psi(\mathrm{I}_2\, e_3) + \rho(\gamma_1^{-1})\psi(\gamma_1 e_3) + \rho(\gamma_2^{-1})\psi(\gamma_2 e_3) + \rho(\gamma_3^{-1})\psi(\gamma_3 e_3) \\
&= \psi(f_3) + \rho(\gamma_1^{-1})\psi(f_9) + \rho(\gamma_2^{-1}\gamma_z^{-1})\psi(f_{10}) + \rho(\gamma_3^{-1}\gamma_z^{-1})\psi(f_4) \\
&= \left(\rho(\gamma_1^{-1}g^{-1}\gamma_\infty) - \rho(\gamma_2^{-1}\gamma_z^{-1}g^{-1}\gamma_\infty^2)\right)\phi(e_1) \\
&\quad + \left(\rho(\gamma_2^{-1}\gamma_z^{-1}g^{-1}\gamma_\infty) - \rho(\gamma_1^{-1}g^{-1})\right)\phi(e_2) \\
&\quad + \left(\rho(\gamma_1^{-1}g^{-1}) + \rho(\gamma_2^{-1}\gamma_z^{-1}g^{-1}\gamma_\infty^2)\right)\phi(e_3)\ .
\end{aligned}$$

We now can express the Hecke operator on classes of cohomology in a matrix form with bases $\phi(e_1), \phi(e_2)$ and $\phi(e_3)$ and their coefficients as entries. We use Mathematica to carry out the computation. From table 7.1 we know that $H_P^1(\Gamma_0(2), \mathcal{T}, V_k)$ has nonzero dimension for weight $k \geq 8$. We will illustrate the computation for $k = 8, 10$, and 12, and this computation can be easily generalized.

Case 1 : k = 8.

In this case, $V_8 = \mathrm{Sym}^6(\mathbf{C}^2)$ with basis $\{x^6y^0, x^5y^1, \cdots, x^0y^6\}$ where $\{x, y\}$ are the standard basis of $\mathbf{C}^2$. Using Mathematica, we first find the representations of the involved matrices in V_8.

```
gamma1inv = Transpose[Table[Coefficient[(x - 2 y) ^ (6 - i) (y) ^ i, x ^ (6 - j) * y ^ j], {i, 0, 6}, {j, 0, 6}]]
gamma2inv =
 Transpose[Table[Coefficient[(3 x - 2 y) ^ (6 - i) (-x + y) ^ (i), x ^ (6 - j) * y ^ (j)], {i, 0, 6}, {j, 0, 6}]]
gamma3inv =
 Transpose[Table[Coefficient[(5 x - 2 y) ^ (6 - i) (-2 x + y) ^ (i), x ^ (6 - j) * y ^ (j)], {i, 0, 6}, {j, 0, 6}]]
gammainfinity =
 Transpose[Table[Coefficient[(x) ^ (6 - i) (x + y) ^ (i), x ^ (6 - j) * y ^ (j)], {i, 0, 6}, {j, 0, 6}]]
gammazinv =
 Transpose[Table[Coefficient[(-5 x - 12 y) ^ (6 - i) (3 x + 7 y) ^ (i), x ^ (6 - j) * y ^ (j)], {i, 0, 6}, {j, 0, 6}]]
gammaeinv =
 Transpose[Table[Coefficient[(-x - 2 y) ^ (6 - i) (x + y) ^ (i), x ^ (6 - j) * y ^ (j)], {i, 0, 6}, {j, 0, 6}]]
ginv = Transpose[Table[Coefficient[(x/3) ^ (6 - i) (y) ^ (i), x ^ (6 - j) * y ^ (j)], {i, 0, 6}, {j, 0, 6}]]
```

We then compute the entries of the matrix representing the Hecke operator on cohomology

```
t11 = ginv.gammainfinity + ginv + gamma2inv.gammazinv.ginv.gammainfinity.gammaeinv
t12 = -ginv
t13 = gamma2inv.gammazinv.ginv.(gammainfinity - gammainfinity.gammaeinv)
t21 = -ginv.gammainfinity.gammainfinity
t22 = ginv.(gammainfinity + gammainfinity.gammainfinity) + gamma1inv.ginv.gammainfinity.gammaeinv
t23 = gamma1inv.ginv.gammainfinity.(IdentityMatrix[7] - gammaeinv)
t31 = gamma1inv.ginv.gammainfinity - gamma2inv.gammazinv.ginv.gammainfinity.gammainfinity
t32 = gamma2inv.gammazinv.ginv.gammainfinity - gamma1inv.ginv
t33 = gamma1inv.ginv + gamma2inv.gammazinv.ginv.gammainfinity.gammainfinity
```

and obtain the final matrix

```
T3 = ArrayFlatten[ {{t11, t12, t13}, {t21, t22, t23}, {t31, t32, t33}}] // MatrixForm
```

rixForm=

$\frac{731}{729}$	$\frac{1}{729}$	$\frac{1}{729}$	$\frac{1}{729}$	$\frac{1}{729}$	$\frac{1}{729}$	$\frac{1}{729}$	$-\frac{1}{729}$	0	0	0	0	0	0	0	-1	1	-1	1	-1	1
0	$\frac{83}{243}$	$\frac{2}{243}$	$\frac{1}{81}$	$\frac{4}{243}$	$\frac{5}{243}$	$\frac{2}{81}$	0	$-\frac{1}{243}$	0	0	0	0	0	4	-4	$\frac{10}{3}$	-3	$\frac{8}{3}$	$-\frac{7}{3}$	2
0	0	$\frac{11}{81}$	$\frac{1}{27}$	$\frac{2}{27}$	$\frac{10}{81}$	$\frac{5}{27}$	0	0	$-\frac{1}{81}$	0	0	0	0	$\frac{20}{3}$	$-\frac{50}{9}$	$\frac{40}{9}$	$-\frac{11}{3}$	$\frac{26}{9}$	$-\frac{20}{9}$	$\frac{5}{3}$
0	0	0	$\frac{1}{9}$	$\frac{4}{27}$	$\frac{10}{27}$	$\frac{20}{27}$	0	0	0	$-\frac{1}{27}$	0	0	0	$\frac{160}{27}$	$-\frac{40}{9}$	$\frac{88}{27}$	$-\frac{64}{27}$	$\frac{44}{27}$	$-\frac{10}{9}$	$\frac{20}{27}$
0	0	0	0	$\frac{19}{81}$	$\frac{5}{9}$	$\frac{5}{3}$	0	0	0	0	$-\frac{1}{9}$	0	0	$\frac{80}{27}$	$-\frac{160}{81}$	$\frac{104}{81}$	$-\frac{22}{27}$	$\frac{40}{81}$	$-\frac{25}{81}$	$\frac{5}{27}$
0	0	0	0	0	$\frac{163}{243}$	2	0	0	0	0	0	$-\frac{1}{3}$	0	$\frac{64}{81}$	$-\frac{112}{243}$	$\frac{64}{243}$	$-\frac{4}{27}$	$\frac{20}{243}$	$-\frac{4}{81}$	$\frac{2}{81}$
0	0	0	0	0	0	$\frac{1459}{729}$	0	0	0	0	0	0	-1	$\frac{64}{729}$	$-\frac{32}{729}$	$\frac{16}{729}$	$-\frac{8}{729}$	$\frac{4}{729}$	$-\frac{2}{729}$	0
$-\frac{1}{729}$	$-\frac{2}{729}$	$-\frac{4}{729}$	$-\frac{8}{729}$	$-\frac{16}{729}$	$-\frac{32}{729}$	$-\frac{64}{729}$	$\frac{731}{729}$	$-\frac{161}{243}$	$\frac{329}{729}$	$-\frac{23}{81}$	$\frac{161}{729}$	$-\frac{7}{81}$	$\frac{43}{243}$	$-\frac{728}{729}$	$\frac{487}{729}$	$-\frac{323}{729}$	$\frac{217}{729}$	$-\frac{143}{729}$	$\frac{97}{729}$	$-\frac{7}{81}$
0	$-\frac{1}{243}$	$-\frac{4}{243}$	$-\frac{4}{81}$	$-\frac{32}{243}$	$-\frac{80}{243}$	$-\frac{64}{81}$	0	$\frac{83}{243}$	$-\frac{34}{81}$	$\frac{41}{81}$	$-\frac{20}{81}$	$\frac{55}{81}$	$\frac{134}{243}$	$-\frac{4}{243}$	$-\frac{28}{81}$	$\frac{106}{243}$	$-\frac{109}{243}$	$\frac{32}{81}$	$-\frac{79}{243}$	$\frac{22}{81}$
0	0	$-\frac{1}{81}$	$-\frac{2}{27}$	$-\frac{8}{27}$	$-\frac{80}{81}$	$-\frac{80}{27}$	0	0	$\frac{11}{81}$	$-\frac{1}{9}$	$\frac{2}{3}$	$\frac{190}{243}$	$\frac{845}{243}$	$\frac{20}{243}$	$\frac{10}{243}$	$-\frac{8}{81}$	$\frac{53}{243}$	$-\frac{74}{243}$	$\frac{80}{243}$	$-\frac{25}{81}$
0	0	0	$-\frac{1}{27}$	$-\frac{8}{27}$	$-\frac{40}{27}$	$-\frac{160}{27}$	0	0	0	$\frac{1}{9}$	$\frac{28}{81}$	$\frac{490}{243}$	$\frac{4700}{729}$	$-\frac{160}{729}$	$-\frac{40}{729}$	$\frac{8}{729}$	$-\frac{16}{729}$	$\frac{68}{729}$	$-\frac{130}{729}$	$\frac{20}{81}$
0	0	0	0	$-\frac{1}{9}$	$-\frac{10}{9}$	$-\frac{20}{3}$	0	0	0	0	$\frac{19}{81}$	$\frac{395}{243}$	$\frac{2045}{243}$	$\frac{80}{243}$	0	$-\frac{8}{243}$	$\frac{2}{243}$	0	$\frac{5}{243}$	$-\frac{5}{81}$
0	0	0	0	0	$-\frac{1}{3}$	-4	0	0	0	0	0	$\frac{163}{243}$	$\frac{1454}{243}$	$-\frac{64}{243}$	$\frac{16}{243}$	0	$-\frac{4}{243}$	$\frac{4}{243}$	$-\frac{4}{243}$	$\frac{2}{81}$
0	0	0	0	0	0	-1	0	0	0	0	0	0	$\frac{1459}{729}$	$\frac{64}{729}$	$-\frac{32}{729}$	$\frac{16}{729}$	$-\frac{8}{729}$	$\frac{4}{729}$	$-\frac{2}{729}$	0
$-\frac{728}{729}$	$\frac{1}{729}$	$\frac{1}{729}$	$\frac{1}{729}$	$\frac{1}{729}$	$\frac{1}{729}$	$\frac{1}{729}$	$\frac{728}{729}$	-1	1	-1	1	-1	1	$\frac{730}{729}$	0	0	0	0	0	0
$-\frac{976}{243}$	$-\frac{28}{81}$	$-\frac{2}{243}$	$-\frac{1}{243}$	0	$\frac{1}{243}$	$\frac{2}{243}$	$\frac{976}{243}$	$-\frac{892}{243}$	$\frac{10}{3}$	-3	$\frac{8}{3}$	$-\frac{7}{3}$	2	$\frac{968}{243}$	$\frac{82}{243}$	0	0	0	0	0
$-\frac{1600}{243}$	$-\frac{260}{243}$	$-\frac{8}{81}$	$-\frac{1}{243}$	$-\frac{2}{243}$	0	$\frac{5}{243}$	$\frac{1600}{243}$	$-\frac{1340}{243}$	$\frac{368}{81}$	$-\frac{11}{3}$	$\frac{26}{9}$	$-\frac{20}{9}$	$\frac{5}{3}$	$\frac{1640}{243}$	$\frac{260}{243}$	$\frac{10}{81}$	0	0	0	0
$-\frac{4480}{729}$	$-\frac{1120}{729}$	$-\frac{208}{729}$	$-\frac{16}{729}$	$-\frac{4}{729}$	$-\frac{10}{729}$	$\frac{20}{729}$	$\frac{4480}{729}$	$-\frac{1120}{243}$	$\frac{272}{81}$	$-\frac{64}{27}$	$\frac{44}{27}$	$-\frac{10}{9}$	$\frac{20}{27}$	$\frac{4160}{729}$	$\frac{400}{243}$	$\frac{16}{81}$	$\frac{2}{27}$	0	0	0
$-\frac{640}{243}$	$-\frac{80}{81}$	$-\frac{80}{243}$	$-\frac{16}{243}$	0	$-\frac{5}{243}$	$\frac{5}{243}$	$\frac{640}{243}$	$-\frac{400}{243}$	$\frac{80}{81}$	$-\frac{16}{27}$	$\frac{32}{81}$	$-\frac{25}{81}$	$\frac{5}{27}$	$\frac{800}{243}$	$\frac{160}{243}$	$\frac{16}{27}$	$-\frac{4}{27}$	$\frac{10}{81}$	0	0
$-\frac{256}{243}$	$-\frac{64}{243}$	$-\frac{32}{243}$	$-\frac{16}{243}$	0	$-\frac{4}{243}$	$\frac{2}{243}$	$\frac{256}{243}$	$-\frac{64}{81}$	$\frac{160}{243}$	$-\frac{16}{27}$	$\frac{128}{243}$	$-\frac{92}{243}$	$\frac{2}{81}$	$\frac{128}{243}$	$\frac{160}{243}$	$-\frac{64}{243}$	$\frac{40}{81}$	$-\frac{104}{243}$	$\frac{82}{243}$	0
0	$-\frac{64}{729}$	0	$-\frac{16}{729}$	0	$-\frac{4}{729}$	0	0	$\frac{64}{729}$	$-\frac{128}{729}$	$\frac{208}{729}$	$-\frac{320}{729}$	$\frac{484}{729}$	$-\frac{728}{729}$	$\frac{128}{729}$	$-\frac{64}{729}$	$\frac{160}{729}$	$-\frac{208}{729}$	$\frac{328}{729}$	$-\frac{484}{729}$	$\frac{730}{729}$

Since each entry is a 7 by 7 matrix, the final matrix is 21 by 21. We then compute the eigenvalues of this matrix.

```
Eigenvalues[T3]
```

$$\left\{\frac{2188}{729}, \frac{2188}{729}, \frac{1}{243}\left(155+\sqrt{12361}\right), \frac{730}{729}, \frac{730}{729}, \frac{730}{729}, \frac{1}{81}\left(-25-\sqrt{3001}\right), \frac{1}{81}\left(25+\sqrt{2785}\right), \frac{1}{81}\left(-25+\sqrt{3001}\right), \frac{1}{81}\left(25-\sqrt{2785}\right), \frac{82}{243}, \frac{82}{243}, \frac{82}{243}, \frac{2}{9}, \frac{1}{243}\left(155-\sqrt{12361}\right), \frac{10}{81}, \frac{10}{81}, \frac{10}{81}, \frac{2}{27}, \frac{4}{243}, \frac{4}{243}\right\}$$

To find out which eigenvalues correspond to the 1-forms in $H^1_P(\Gamma_0(2), \mathcal{T}, V_8)$, we need to find eigenforms. To do this, we will first compute a set of bases for $H^1_P(\Gamma_0(2), \mathcal{T}, V_8)$. Recall the part of the Serre tree that is the dessin for $\Gamma_0(2)$ embedded in $\mathfrak{h}$ in Figure 7.11.

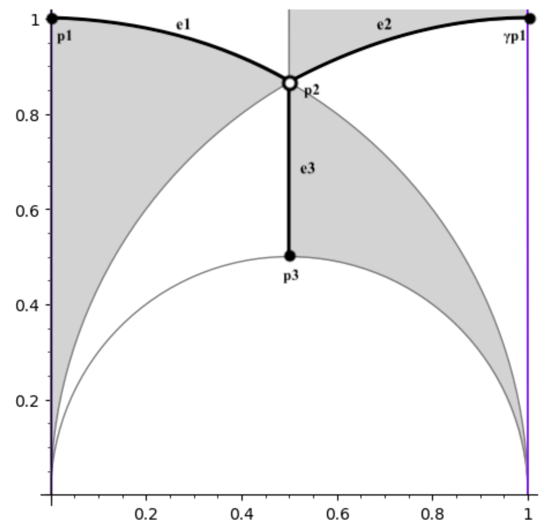


Figure 7.11: Dessin of $\Gamma_0(2)$ embedded in $\mathfrak{h}$.

We denote the vectors associated with e_1, e_2, e_3 as v_1, v_2, v_3, and the vectors associated with p_1, p_2, p_3 as w_1, w_2, w_3. In this way, the vector associated with $\gamma_\infty \cdot p_1$ is $\rho(\gamma_\infty)w_1$. From section 7.4 we know that for a 1-form ϕ to be in $C^1_P(\Gamma_0(2), \mathcal{T}, V_8)$, ϕ has to satisfy

$$\begin{aligned}\phi(link_\infty) &= \phi(-e_1 + e_2)\\ &= -\phi(e_1) + \phi(e_2)\\ &= -v_1 + v_2\\ &\in \mathrm{Im}\left(\rho(\gamma_\infty) - \mathrm{I}_{k-1}\right)\ ,\end{aligned}$$

and from figure 7.4 we also have

$$\begin{aligned}\phi(link_0) &= \phi(-e_3 + e_1 - \tilde{e_2} + \tilde{e_3})\\ &= -\phi(e_3) + \phi(e_1) - \phi(\tilde{e_2}) + \phi(\tilde{e_3})\\ &= -v_3 + v_1 - \rho(\gamma_\infty^{-1})v_2 + \rho(\gamma_\infty^{-1})v_3\\ &\in \mathrm{Im}\left(\rho(\gamma_0^{-1}) - \mathrm{I}_{k-1}\right)\end{aligned}$$

where

$$\gamma_\infty = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Thus the last entry of $\mathrm{Im}\left(\rho(\gamma_\infty) - \mathrm{I}_{k-1}\right)$ is 0, and similarly the first entry of $\mathrm{Im}\left(\rho(\gamma_0^{-1}) - \mathrm{I}_{k-1}\right)$ is also 0.

For $\phi \in H^1_P(\Gamma_0(2), \mathcal{T}, V_8)$, we also need $\phi \in (\mathrm{Im}(d))^\perp$, which means $\langle \phi, df \rangle = 0 = \langle d^*\phi, f \rangle$, $\forall f \in C^0_P\big(\Gamma_0(2), \mathcal{T}, V_8\big)$. This implies $d^*\phi = 0$. To find $d^*\phi$, we define

$$\langle \phi, \psi \rangle = \frac{1}{n} \sum_i^n \langle \phi(e_i), \psi(e_i) \rangle$$

where $\phi, \psi \in C^1_P(\Gamma_0(2), \mathcal{T}, V_8)$ and n is the number of edges of the dessin for $\Gamma_0(2)$, in which case $n = 3$. Thus we have

$$\begin{aligned}
\langle \phi, df \rangle &= \frac{1}{n} \sum_i^n \langle \phi(e_i), df(e_i) \rangle \\
&= \frac{1}{n}\big(\langle \phi(e_1), df(e_1) \rangle + \langle \phi(e_2), df(e_2) \rangle + \langle \phi(e_3), df(e_3) \rangle \big) \\
&= \frac{1}{n}\big(\langle v_1, w_1 - w_2 \rangle + \langle v_2, \rho(\gamma_\infty) w_1 - w_2 \rangle + \langle v_3, w_3 - w_2 \rangle \big) \\
&= \frac{1}{n}\big(\langle v_1 + \rho(\gamma_\infty^*) v_2, w_1 \rangle + \langle -v_1 - v_2 - v_3, w_2 \rangle + \langle v_3, w_3 \rangle \big) .
\end{aligned}$$

Since γ_∞ is a real matrix, γ_∞^* is the transpose of γ_∞. Thus ϕ also satisfy

$$\begin{cases} d^*\phi(p_1) & = v_1 + \rho(\gamma_\infty^*) v_2 \ , \\ d^*\phi(p_2) & = -v_1 - v_2 - v_3 \ , \\ d^*\phi(p_3) & = v_3 \in \ker\left(\rho(\gamma_e) - \mathrm{I}_{k-1}\right) \ . \end{cases}$$

The last relation also comes from section 7.4, the condition on elliptic points. The above relations can then be translated as

$$\begin{cases} v_1 + \rho(\gamma_\infty^*) v_2 = 0 \ , \\ v_1 + v_2 \in \ker\left(\rho(\gamma_e) - \mathrm{I}_{k-1}\right)^\perp \ . \end{cases}$$

Combining the conditions for $\phi \in C^1_P(\Gamma_0(2), \mathcal{T}, V_8)$, we have for $\phi \in H^1_P(\Gamma_0(2), \mathcal{T}, V_8)$

$$\begin{cases} -v_1 + v_2 \in \operatorname{Im}\left(\rho(\gamma_\infty) - \mathrm{I}_{k-1}\right) , \\ v_1 - \rho(\gamma_\infty^{-1})v_2 + \rho(\gamma_\infty^{-1})v_3 - v_3 \in \operatorname{Im}\left(\rho(\gamma_0^{-1}) - \mathrm{I}_{k-1}\right) , \\ v_1 + \rho(\gamma_\infty^*)v_2 = 0 , \\ v_1 + v_2 \in \ker\left(\rho(\gamma_e) - \mathrm{I}_{k-1}\right)^\perp \end{cases} \tag{7.5.1}$$

where $\operatorname{Im}\left(\rho(\gamma_\infty) - \mathrm{I}_{k-1}\right)$ are vectors with 0 on the last row, and $\operatorname{Im}\left(\rho(\gamma_0^{-1}) - \mathrm{I}_{k-1}\right)$ are vectors with 0 on the first row. We then implement the computation in Mathematica. First we find the representations for the involved matrices.

```
gammae = Transpose[Table[Coefficient[(x + 2 y) ^ (6 - i) (-x - y) ^ i, x ^ (6 - j) * y ^ j], {i, 0, 6}, {j, 0, 6}]]
gammainfinv = Transpose[Table[Coefficient[(x) ^ (6 - i) (-x + y) ^ i, x ^ (6 - j) * y ^ j], {i, 0, 6}, {j, 0, 6}]]
gammainfstar = Transpose[gammainfinity]
```

We then find $\ker\left(\rho(\gamma_e) - \mathrm{I}_{k-1}\right)$.

```
NullSpace[gammae - IdentityMatrix[7]]
{{2, 9, 15, 10, 0, 0, 4}, {-5, -26, -50, -40, 0, 16, 0}, {1, 6, 14, 16, 8, 0, 0}}

p1 = NullSpace[gammae - IdentityMatrix[7]][[1]]
p2 = NullSpace[gammae - IdentityMatrix[7]][[2]]
p3 = NullSpace[gammae - IdentityMatrix[7]][[3]]
```

Now we can set up the conditions for $\phi \in H^1_P(\Gamma_0(2), \mathcal{T}, V_8)$.

```
v1 = {a1, a2, a3, a4, a5, a6, a7}
v2 = {b1, b2, b3, b4, b5, b6, b7}
s = Solve[{v1 + gammainfstar.v2 == {0, 0, 0, 0, 0, 0, 0}, (v1 + v2).p1 == {0, 0, 0, 0, 0, 0, 0},
   (v1 + v2).p2 == {0, 0, 0, 0, 0, 0, 0}, (v1 + v2).p3 == {0, 0, 0, 0, 0, 0, 0}, -a7 + b7 == 0,
   a1 - gammainfinv[[1]].v2 + (gammainfinv - IdentityMatrix[7])[[1]].(-v1 - v2) == 0},
  {a1, a2, a3, a4, a5, a6, a7, b1, b2, b3, b4, b5, b6, b7}]
```

{a1, a2, a3, a4, a5, a6, a7}

{b1, b2, b3, b4, b5, b6, b7}

Solve: Equations may not give solutions for all "solve" variables.

$$\left\{\left\{a3 \to -\frac{54\,a1}{179}+\frac{147\,a2}{358},\ a4 \to -\frac{3\,a2}{8},\ a5 \to \frac{27\,a1}{179}-\frac{147\,a2}{716},\ a6 \to \frac{a2}{4},\ a7 \to \frac{113\,a1}{179}-\frac{2205\,a2}{1432},\ b1 \to -a1,\ b2 \to a1 - a2,\ b3 \to -\frac{125\,a1}{179}+\frac{569\,a2}{358},\ b4 \to \frac{17\,a1}{179}-\frac{1995\,a2}{1432},\ b5 \to \frac{118\,a1}{179}+\frac{173\,a2}{716},\ b6 \to -\frac{226\,a1}{179}+\frac{1131\,a2}{716},\ b7 \to \frac{113\,a1}{179}-\frac{2205\,a2}{1432}\right\}\right\}$$

We see that a_1, a_2 are independent, which allows us to pick two bases for ϕ. This also agrees with the dimension formula where $\dim(H^1_P(\Gamma_0(2), \mathcal{T}, V_8)) = 2$. We continue to construct the bases.

```
colvecu = Join[u1, u2, u3]
colvecw = Join[w1, w2, w3]
phi1 = colvecu
phi2 = colvecw - (colvecu.colvecw / colvecu.colvecu) * colvecu
```

$$\left\{1, 0, -\frac{54}{179}, 0, \frac{27}{179}, 0, \frac{113}{179}, -1, 1, -\frac{125}{179}, \frac{17}{179}, \frac{118}{179}, -\frac{226}{179}, \frac{113}{179}, 0, -1, 1, -\frac{17}{179}, -\frac{145}{179}, \frac{226}{179}, -\frac{226}{179}\right\}$$

$$\left\{0, 1, \frac{147}{358}, -\frac{3}{8}, -\frac{147}{716}, \frac{1}{4}, -\frac{2205}{1432}, 0, -1, \frac{569}{358}, -\frac{1995}{1432}, \frac{173}{716}, \frac{1131}{716}, -\frac{2205}{1432}, 0, 0, -2, \frac{633}{358}, -\frac{13}{358}, -\frac{655}{358}, \frac{2205}{716}\right\}$$

$$\left\{1, 0, -\frac{54}{179}, 0, \frac{27}{179}, 0, \frac{113}{179}, -1, 1, -\frac{125}{179}, \frac{17}{179}, \frac{118}{179}, -\frac{226}{179}, \frac{113}{179}, 0, -1, 1, -\frac{17}{179}, -\frac{145}{179}, \frac{226}{179}, -\frac{226}{179}\right\}$$

$$\left\{\frac{137781}{116672}, 1, \frac{3171}{58336}, -\frac{3}{8}, -\frac{3171}{116672}, \frac{1}{4}, -\frac{92673}{116672}, -\frac{137781}{116672}, \frac{21109}{116672}, \frac{89221}{116672}, -\frac{149457}{116672}, \frac{59509}{58336}, \frac{5169}{58336}, -\frac{92673}{116672}, 0, -\frac{137781}{116672}, -\frac{95563}{116672}, \frac{193209}{116672}, -\frac{115847}{116672}, -\frac{19753}{58336}, \frac{92673}{58336}\right\}$$

Here we obtain a set of orthogonal bases, and we can check indeed ϕ_1 and ϕ_2 are in $\ker(d^*)$.

```
phi1[[1 ;; 7]] + gammainfstar.phi1[[8 ;; 14]] == ConstantArray[0, 7]
phi2[[1 ;; 7]] + gammainfstar.phi2[[8 ;; 14]] == ConstantArray[0, 7]
```

```
True

True
```

We then apply the Hecke operator and check whether the resulting forms lie in $H^1_P(\Gamma_0(2), \mathcal{T}, V_8)$.

```
tphi1 = T3.phi1
tphi2 = T3.phi2
e11 = tphi1[[1 ;; 7]]
e12 = tphi1[[8 ;; 14]]
e13 = tphi1[[15 ;; 21]]
e21 = tphi2[[1 ;; 7]]
e22 = tphi2[[8 ;; 14]]
e23 = tphi2[[15 ;; 21]]

-e11 + e12
e11 - gammainfinv.e12 + (gammainfinv - IdentityMatrix[7]).e13
-e21 + e22
e21 - gammainfinv.e22 + (gammainfinv - IdentityMatrix[7]).e23
```

$$\left\{-\frac{282\,824}{130\,491}, -\frac{32\,633}{43\,497}, -\frac{116\,213}{43\,497}, -\frac{318\,604}{43\,497}, -\frac{176\,926}{43\,497}, -\frac{86\,380}{43\,497}, 0\right\}$$

$$\left\{0, \frac{382\,684}{14\,499}, -\frac{1\,031\,810}{43\,497}, \frac{11\,612\,068}{130\,491}, -\frac{668\,428}{14\,499}, \frac{1\,220\,168}{43\,497}, 0\right\}$$

$$\left\{-\frac{12\,821\,737}{10\,631\,736}, -\frac{5\,222\,479}{28\,351\,296}, \frac{10\,154\,197}{3\,150\,144}, \frac{40\,644\,833}{21\,263\,472}, \frac{1\,694\,183}{14\,175\,648}, -\frac{222\,877}{787\,536}, 0\right\}$$

$$\left\{0, \frac{28\,745\,467}{7\,087\,824}, \frac{113\,620\,841}{14\,175\,648}, \frac{95\,299\,871}{21\,263\,472}, \frac{113\,620\,841}{7\,087\,824}, -\frac{29\,699\,857}{3\,543\,912}, 0\right\}$$

```
e11 + gammainfstar.e12 == ConstantArray[0, 7]
e21 + gammainfstar.e22 == ConstantArray[0, 7]
```

```
False

False
```

We see the resulting forms $T_3\phi_1$ and $T_3\phi_2$ lie in $C^1_P(\Gamma_0(2), \mathcal{T}, V_8)$ but not in $\ker(d^*)$. We use projection unto $\ker(d^*)$,

$$\mathrm{proj}_{\ker(d^*)} T_3\phi_i = \underbrace{\frac{\langle T_3\phi_i, \phi_1\rangle}{\|\phi_1\|^2}}_{a_i} \cdot \phi_1 + \underbrace{\frac{\langle T_3\phi_i, \phi_2\rangle}{\|\phi_2\|^2}}_{b_i} \cdot \phi_2$$

with the resulting matrix

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}.$$

Using Mathematica, we get

```
a = tphi1.phi1 / (Norm[phi1]) ^ 2
b = tphi1.phi2 / (phi2.phi2)
atilde = tphi2.phi1 / (phi1.phi1)
btilde = tphi2.phi2 / (phi2.phi2)
mat22 = {{a, atilde}, {b, btilde}} // MatrixForm
```

atrixForm=

$$\begin{pmatrix} \frac{4}{243} & 0 \\ 0 & \frac{4}{243} \end{pmatrix}$$

Therefore the eigenvalue we obtained for the Hecke operator T_3 on cohomology with weight 8 is $\frac{4}{243}$, or $\frac{1}{3^6} \cdot 12$.

Case 2 : $\mathbf{k = 10}$.

In this case, $V_{10} = \text{Sym}^8(\mathbf{C}^2)$ with basis $\{x^8y^0, x^7y^1, \cdots, x^0y^8\}$ where $\{x, y\}$ are the standard basis of $\mathbf{C}^2$. The analysis are similar to the case of $k = 8$. From table 7.1, dimension of $H^1_P(\Gamma_0(2), \mathcal{T}, V_{10})$ is again 2. So we can follow the exact procedure as in the case of $k = 8$. We obtain the eigenvalue for the Hecke operator T_3 on cohomology with weight 10 is $-\frac{52}{2187}$, or $-\frac{1}{3^8} \cdot 156$.

Case 3 : $\mathbf{k = 12}$.

In this case, $V_{12} = \text{Sym}^{10}(\mathbf{C}^2)$ with basis $\{x^{10}y^0, x^9y^1, \cdots, x^0y^{10}\}$ where $\{x, y\}$ are the standard basis of $\mathbf{C}^2$. The analysis are similar to the case of $k = 8$. But here dimension of $H^1_P(\Gamma_0(2), \mathcal{T}, V_{12})$ is 4, so some additional analysis would be required. First, we have a similar set up in Mathematica.

```
gamma1inv =
 Transpose[Table[Coefficient[(x - 2 y) ^ (10 - i) (y) ^ i, x ^ (10 - j) * y ^ j], {i, 0, 10}, {j, 0, 10}]]
gamma2inv =
 Transpose[Table[Coefficient[(3 x - 2 y) ^ (10 - i) (-x + y) ^ (i), x ^ (10 - j) * y ^ (j)], {i, 0, 10},
   {j, 0, 10}]]
gamma3inv =
 Transpose[Table[Coefficient[(5 x - 2 y) ^ (10 - i) (-2 x + y) ^ (i), x ^ (10 - j) * y ^ (j)],
   {i, 0, 10}, {j, 0, 10}]]
gammainfinity =
 Transpose[Table[Coefficient[(x) ^ (10 - i) (x + y) ^ (i), x ^ (10 - j) * y ^ (j)], {i, 0, 10}, {j, 0, 10}]]
gammazinv =
 Transpose[Table[Coefficient[(-5 x - 12 y) ^ (10 - i) (3 x + 7 y) ^ (i), x ^ (10 - j) * y ^ (j)],
   {i, 0, 10}, {j, 0, 10}]]
gammaeinv =
 Transpose[Table[Coefficient[(-x - 2 y) ^ (10 - i) (x + y) ^ (i), x ^ (10 - j) * y ^ (j)], {i, 0, 10},
   {j, 0, 10}]]
ginv = Transpose[Table[Coefficient[(x/3) ^ (10 - i) (y) ^ (i), x ^ (10 - j) * y ^ (j)], {i, 0, 10}, {j, 0, 10}]]
```

```
t11 = ginv.gammainfinity + ginv + gamma2inv.gammazinv.ginv.gammainfinity.gammaeinv
t12 = -ginv
t13 = gamma2inv.gammazinv.ginv.(gammainfinity - gammainfinity.gammaeinv)
t21 = -ginv.gammainfinity.gammainfinity
t22 = ginv.(gammainfinity + gammainfinity.gammainfinity) + gamma1inv.ginv.gammainfinity.gammaeinv
t23 = gamma1inv.ginv.gammainfinity.(IdentityMatrix[11] - gammaeinv)
t31 = gamma1inv.ginv.gammainfinity - gamma2inv.gammazinv.ginv.gammainfinity.gammainfinity
t32 = gamma2inv.gammazinv.ginv.gammainfinity - gamma1inv.ginv
t33 = gamma1inv.ginv + gamma2inv.gammazinv.ginv.gammainfinity.gammainfinity
T3 = ArrayFlatten[{{t11, t12, t13}, {t21, t22, t23}, {t31, t32, t33}}]
Eigenvalues[T3]
```

This gives us the list of eigenvalues.

$$\Big\{\frac{177148}{59049}, \frac{177148}{59049}, \frac{1}{729}\left(425+\sqrt{95089}\right), \frac{59050}{59049}, \frac{59050}{59049}, \frac{59050}{59049},$$
$$-0.999\ldots, 0.997\ldots, 0.339\ldots, -0.335\ldots, \frac{6562}{19683}, \frac{6562}{19683}, \frac{6562}{19683}, \frac{26}{81},$$
$$\frac{1}{729}\left(425-\sqrt{95089}\right), \frac{730}{6561}, \frac{730}{6561}, \frac{730}{6561}, 0.0805\ldots, -0.0795\ldots, -0.0378\ldots,$$
$$\frac{82}{2187}, \frac{82}{2187}, \frac{82}{2187}, 0.0348\ldots, \frac{10}{729}, \frac{10}{729}, \frac{10}{729}, \frac{2}{243}, \frac{28}{6561}, \frac{28}{6561}, \frac{28}{6561}, \frac{28}{6561}\Big\}$$

To find the eigenvalue for the associated Hecke operator on cohomology, next we find a set of bases for $H^1_P(\Gamma_0(2), \mathcal{T}, V_{12})$. As before, the bases can be determined by conditions in (7.5.1).

```
gammae = Transpose[Table[Coefficient[(x + 2 y) ^ (10 - i) (-x - y) ^ i, x ^ (10 - j) * y ^ j],
   {i, 0, 10}, {j, 0, 10}]]
gammainfinv =
 Transpose[Table[Coefficient[(x) ^ (10 - i) (-x + y) ^ i, x ^ (10 - j) * y ^ j], {i, 0, 10}, {j, 0, 10}]]
gammainfstar = Transpose[gammainfinity]
NullSpace[gammae - IdentityMatrix[11]]
```

Here the nullspace has dimension 5.

```
p1 = NullSpace[gammae - IdentityMatrix[11]][[1]]
p2 = NullSpace[gammae - IdentityMatrix[11]][[2]]
p3 = NullSpace[gammae - IdentityMatrix[11]][[3]]
p4 = NullSpace[gammae - IdentityMatrix[11]][[4]]
p5 = NullSpace[gammae - IdentityMatrix[11]][[5]]
v1 = {a1, a2, a3, a4, a5, a6, a7, a8, a9, a10, a11}
v2 = {b1, b2, b3, b4, b5, b6, b7, b8, b9, b10, b11}
s = Solve[{v1 + gammainfstar.v2 == ConstantArray[0, 11], (v1 + v2).p1 == ConstantArray[0, 11],
   (v1 + v2).p2 == ConstantArray[0, 11], (v1 + v2).p3 == ConstantArray[0, 11],
   (v1 + v2).p4 == ConstantArray[0, 11], (v1 + v2).p5 == ConstantArray[0, 11], -a11 + b11 == 0,
   a1 - gammainfinv[[1]].v2 + (gammainfinv - IdentityMatrix[11])[[1]].(-v1 - v2) == 0},
  {a1, a2, a3, a4, a5, a6, a7, a8, a9, a10, a11, b1, b2, b3, b4, b5, b6, b7, b8, b9, b10, b11}]
```

$$\Big\{\Big\{a5 \to -\frac{1032\,a1}{14\,501} + \frac{45\,705\,a2}{232\,016} - \frac{164\,301\,a3}{203\,014} + \frac{27\,423\,a4}{72\,505},\ a6 \to -\frac{5\,a2}{48} - a4,$$
$$a7 \to \frac{1290\,a1}{14\,501} - \frac{228\,525\,a2}{928\,064} + \frac{172\,749\,a3}{203\,014} - \frac{27\,423\,a4}{58\,004},\ a8 \to \frac{125\,a2}{768} + \frac{21\,a4}{16},$$
$$a9 \to -\frac{2107\,a1}{14\,501} + \frac{746\,515\,a2}{1\,856\,128} - \frac{19\,429\,a3}{14\,501} + \frac{447\,909\,a4}{580\,040},\ a10 \to -\frac{21\,a2}{64} - \frac{51\,a4}{20},$$
$$a11 \to \frac{13\,067\,a1}{14\,501} - \frac{14\,397\,075\,a2}{3\,712\,256} + \frac{47\,520\,a3}{14\,501} - \frac{1\,727\,649\,a4}{232\,016},\ b1 \to -a1,\ b2 \to a1 - a2,\ b3 \to -a1 + 2\,a2 - a3,$$
$$b4 \to a1 - 3\,a2 + 3\,a3 - a4,\ b5 \to -\frac{13\,469\,a1}{14\,501} + \frac{882\,359\,a2}{232\,016} - \frac{1\,053\,783\,a3}{203\,014} + \frac{262\,597\,a4}{72\,505},$$
$$b6 \to \frac{9341\,a1}{14\,501} - \frac{170\,135\,a2}{43\,503} + \frac{1\,208\,635\,a3}{203\,014} - \frac{103\,086\,a4}{14\,501},\ b7 \to -\frac{311\,a1}{14\,501} + \frac{2\,474\,569\,a2}{928\,064} - \frac{376\,722\,a3}{101\,507} + \frac{510\,403\,a4}{58\,004},$$
$$b8 \to -\frac{12\,589\,a1}{14\,501} + \frac{2\,179\,979\,a2}{11\,136\,768} - \frac{19\,857\,a3}{14\,501} - \frac{1\,249\,213\,a4}{232\,016},$$
$$b9 \to \frac{23\,726\,a1}{14\,501} - \frac{7\,105\,471\,a2}{1\,856\,128} + \frac{89\,408\,a3}{14\,501} - \frac{2\,035\,929\,a4}{580\,040},$$
$$b10 \to -\frac{26\,134\,a1}{14\,501} + \frac{11\,786\,895\,a2}{1\,856\,128} - \frac{95\,040\,a3}{14\,501} + \frac{7\,072\,137\,a4}{580\,040},$$
$$b11 \to \frac{13\,067\,a1}{14\,501} - \frac{14\,397\,075\,a2}{3\,712\,256} + \frac{47\,520\,a3}{14\,501} - \frac{1\,727\,649\,a4}{232\,016}\Big\}\Big\}$$

The solution has 4 degree of freedom, enabling us to pick 4 independent bases $\{u, w, x, y\}$.

```
s1 = s /. {a1 :> 1, a2 :> 0, a3 :> 0, a4 :> 0}
u1 = v1 /. s1 /. {a1 :> 1, a2 :> 0, a3 :> 0, a4 :> 0} //. {x_List} :> x
u2 = v2 /. s1 //. {x_List} :> x
u3 = - (u1 + u2)
s2 = s /. {a1 :> 0, a2 :> 1, a3 :> 0, a4 :> 0}
w1 = v1 /. s2 /. {a1 :> 0, a2 :> 1, a3 :> 0, a4 :> 0} //. {x_List} :> x
w2 = v2 /. s2 //. {x_List} :> x
w3 = - (w1 + w2)
s3 = s /. {a1 :> 0, a2 :> 0, a3 :> 1, a4 :> 0}
x1 = v1 /. s3 /. {a1 :> 0, a2 :> 0, a3 :> 1, a4 :> 0} //. {x_List} :> x
x2 = v2 /. s3 //. {x_List} :> x
x3 = - (x1 + x2)
s4 = s /. {a1 :> 0, a2 :> 0, a3 :> 0, a4 :> 1}
y1 = v1 /. s4 /. {a1 :> 0, a2 :> 0, a3 :> 0, a4 :> 1} //. {x_List} :> x
y2 = v2 /. s4 //. {x_List} :> x
y3 = - (y1 + y2)
```

Now we can apply the Hecke operator to the orthogonalized basis elements.

```
colvecu = Join[u1, u2, u3]
colvecw = Join[w1, w2, w3]
colvecx = Join[x1, x2, x3]
colvecy = Join[y1, y2, y3]
phi1 = colvecu
phi2 = colvecw - (colvecw.phi1 / phi1.phi1) * phi1
phi3 = colvecx - (colvecx.phi1 / phi1.phi1) * phi1 - (colvecx.phi2 / phi2.phi2) * phi2
phi4 = colvecy - (colvecy.phi1 / phi1.phi1) * phi1 - (colvecy.phi2 / phi2.phi2) * phi2 - (colvecy.phi3 / phi3.phi3) * phi3
tphi1 = T3.phi1
tphi2 = T3.phi2
tphi3 = T3.phi3
tphi4 = T3.phi4
```

Again, we can use the relations in (7.5.1) to check that $T_3\phi_1, \cdots, T_3\phi_4$ lie in $C^1_P(\Gamma_0(2), \mathcal{T}, V_12)$ but not in $\ker(d^*)$. We use projection unto $\ker(d^*)$,

```
c11 = tphi1.phi1 / (phi1.phi1)
c12 = tphi2.phi1 / (phi1.phi1)
c13 = tphi3.phi1 / (phi1.phi1)
c14 = tphi4.phi1 / (phi1.phi1)
c21 = tphi1.phi2 / (phi2.phi2)
c22 = tphi2.phi2 / (phi2.phi2)
c23 = tphi3.phi2 / (phi2.phi2)
c24 = tphi4.phi2 / (phi2.phi2)
c31 = tphi1.phi3 / (phi3.phi3)
c32 = tphi2.phi3 / (phi3.phi3)
c33 = tphi3.phi3 / (phi3.phi3)
c34 = tphi4.phi3 / (phi3.phi3)
c41 = tphi1.phi4 / (phi4.phi4)
c42 = tphi2.phi4 / (phi4.phi4)
c43 = tphi3.phi4 / (phi4.phi4)
c44 = tphi4.phi4 / (phi4.phi4)
mat44 = {{c11, c12, c13, c14}, {c21, c22, c23, c24}, {c31, c32, c33, c34}, {c41, c42, c43, c44}}
Eigenvalues[mat44]
```

$$\left\{\frac{28}{6561}, \frac{28}{6561}, \frac{28}{6561}, \frac{28}{6561}\right\}$$

We obtain the eigenvalue for the Hecke operator T_3 on cohomology with weight 12 is $\dfrac{28}{6561}$, or $\dfrac{1}{3^{10}} \cdot 252$.

7.5.2 T_5 Acting on $X_0(2)$

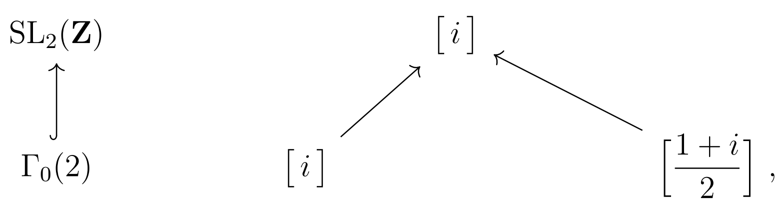
To see the action of T_5 on $X_0(2)$, we have the following diagram.

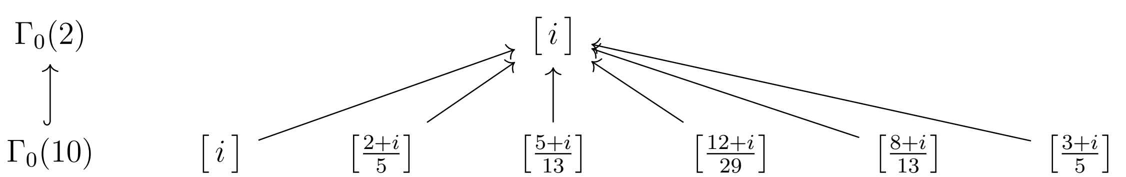
$$\begin{array}{ccc} \Gamma_0(10)\setminus\mathfrak{h} & \xrightarrow{g} & g\Gamma_0(10)g^{-1}\setminus\mathfrak{h} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_g} \\ \Gamma_0(2)\setminus\mathfrak{h} & & \Gamma_0(2)\setminus\mathfrak{h} \end{array}$$

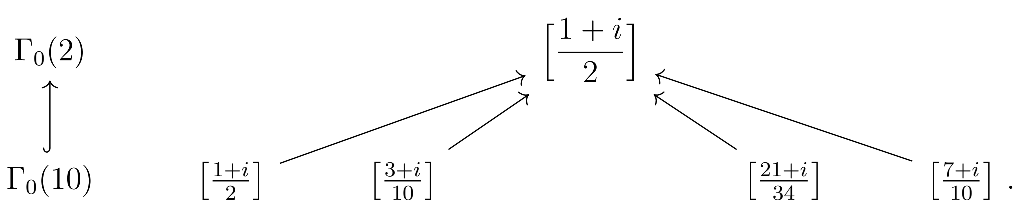
where $g = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$. The dessin for $\Gamma_0(2)$ is as shown in Figure 7.6. To find the dessin for $\Gamma_0(10)$, first we have$[\mathrm{PSL}_2(\mathbf{Z})\colon \Gamma_0(10)] = 18$. Thus the dessin for $\Gamma_0(10)$ has 18 edges. The coset representatives of $\Gamma_0(10)$ in $\Gamma_0(2)$ are The coset representatives of $\Gamma_0(6)$ in $\Gamma_0(2)$ are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}, \quad \begin{pmatrix} 1 & 4 \\ 2 & 9 \end{pmatrix}.$$

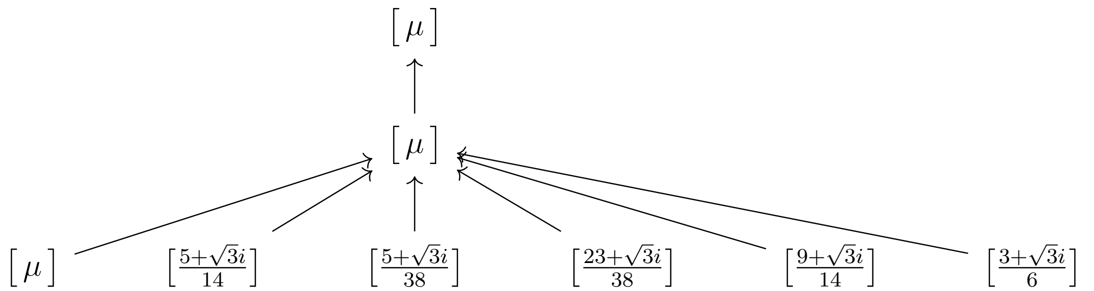
Thus the $\Gamma_0(10)$-orbits of i are



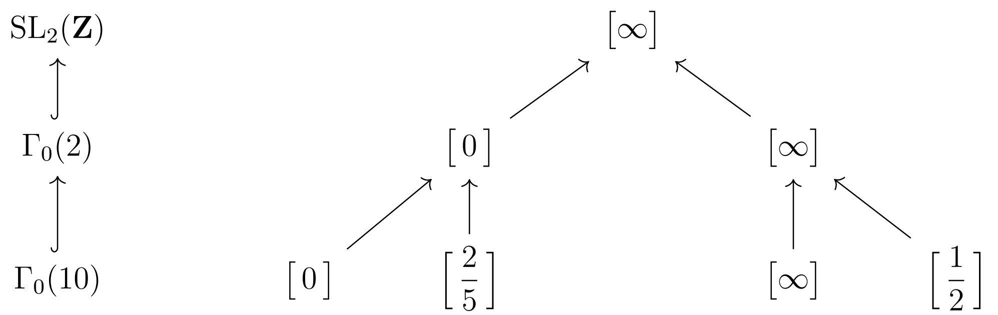




The $\Gamma_0(2)$-orbits of μ is μ, and the $\Gamma_0(10)$-orbits of μ are



The $\Gamma_0(10)$-orbits of ∞ are



These give us 10 black vertices, 6 white vertices, and 4 faces together with the 18 edges we computed from before. The embedded dessin in $\mathfrak{h}$ is shown below.

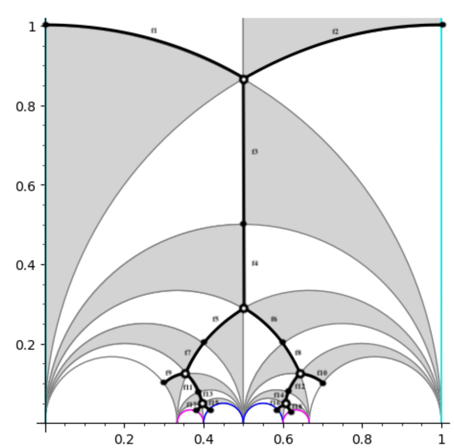
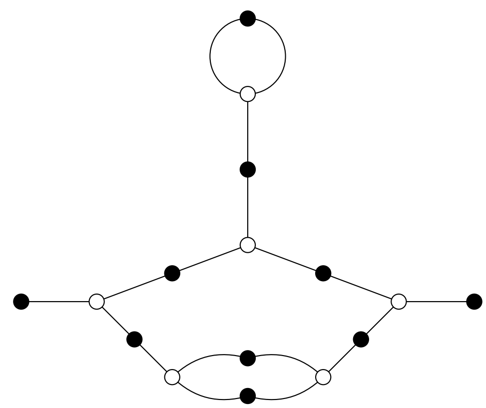


Figure 7.12: Dessin of $\Gamma_0(10)$ embedded in $\mathfrak{h}$.

Here the points on the same colored boundaries are identified to one another. As before, we labeled the edges $f_1, \cdots, f_{18}$. We also have the abstract dessin for $\Gamma_0(10)$:



The embedded dessin for $g\Gamma_0(10)g^{-1}$ is then obtained by applying g to the embedded dessin for $\Gamma_0(10)$, which acts on the graph by multiplication by 5. We obtain the following embedded dessin for $g\Gamma_0(10)g^{-1}$:

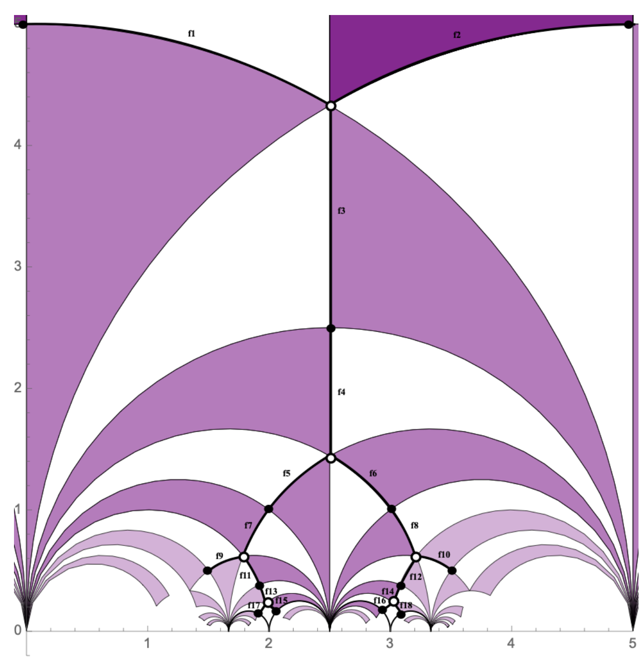


Figure 7.13: Dessin of $g\Gamma_0(10)g^{-1}$ Embedded in $\mathfrak{h}$

We then overlay the embedded dessin for $\Gamma_0(2)$ with the embeded dessin for

$g\Gamma_0(10)g^{-1}$. We get

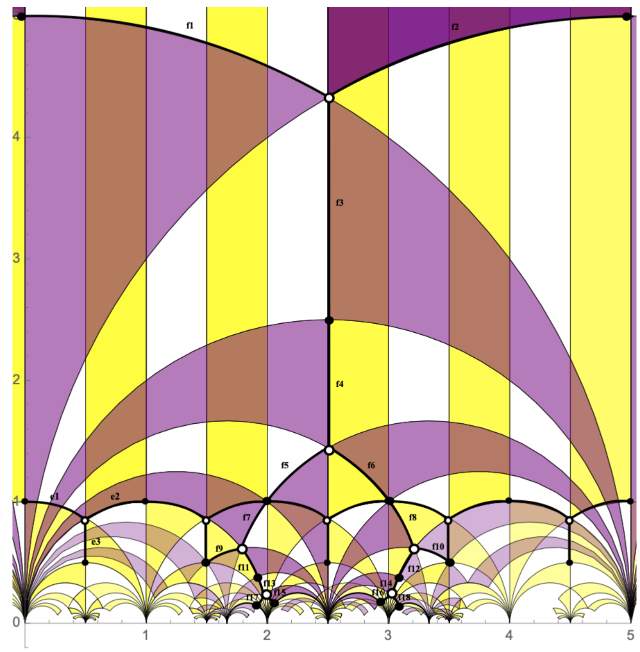


Figure 7.14: Overlay of Dessins of $\Gamma_0(2)$ and $g\Gamma_0(10)g^{-1}$ Embedded in $\mathfrak{h}$.

where the yellow part comes from expanding the embedded dessin of $\Gamma_0(2)$ along the Serre tree:

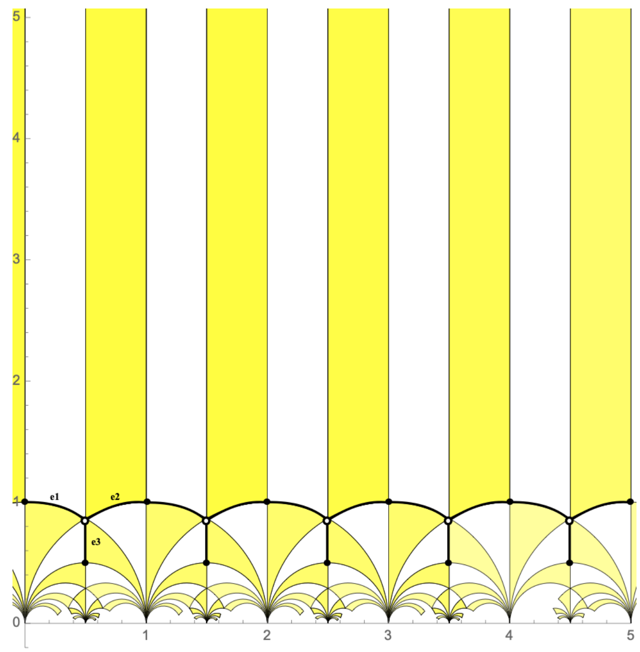


Figure 7.15: Expanded Dessin of $\Gamma_0(2)$ Embedded in $\mathfrak{h}$.

Here e_1, e_2, e_3 are of the original dessin for $\Gamma_0(10)$. We use $\gamma_\infty = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ to shift them to the right by 1 unit, and γ_∞^2 to shift them to the right by 2 units and so on to obtain the other unlabeled edges.

Let ϕ denote 1-forms in $C_P^1(\Gamma_0(2), \mathcal{T}, V_k)$. We use deformation retract r again to pull back to 1-forms ϕ in $C^1_P((g\Gamma_0(10)g^{-1}, \mathcal{T}, V_k)$. We get

$$\phi(gf_1) = \rho(\gamma_\infty^2)\phi(e_1) - \rho(\gamma_\infty)\phi(e_2) + \rho(\gamma_\infty)\phi(e_1) - \phi(e_2) + \phi(e_1)\ ,$$

$$\phi(gf_2) = \rho(\gamma_\infty^2)\phi(e_2) - \rho(\gamma_\infty^3)\phi(e_1) + \rho(\gamma_\infty^3)\phi(e_2) - \rho(\gamma_\infty^4)\phi(e_1) + \rho(\gamma_\infty^4)\phi(e_2)\ ,$$

$$\phi(gf_3) = \phi(gf_4) = \phi(gf_{11}) = \phi(gf_{12}) = \phi(gf_{13}) = \phi(gf_{14}) = 0\ ,$$

$$\phi(gf_5) = \rho(\gamma_\infty^2)\phi(e_1)\ ,$$

$$\phi(gf_6) = \rho(\gamma_\infty^2)\phi(e_2)\ ,$$

$$\phi(gf_7) = \rho(\gamma_\infty)\phi(e_2)\ ,$$

$$\phi(gf_8) = \rho(\gamma_\infty^3)\phi(e_1)\ ,$$

$$\phi(gf_9) = \rho(\gamma_\infty)\phi(e_3)\ ,$$

$$\phi(gf_{10}) = \rho(\gamma_\infty^3)\phi(e_3)\ ,$$

$$\phi(gf_{15}) = \rho(\gamma_\infty)\phi(e_2) - \rho(\gamma_\infty^2)\phi(e_1) + \rho(\gamma_\infty^2)\phi(e_3) - \rho(\gamma_\infty^2\gamma_e)\phi(e_3) + \rho(\gamma_\infty^2\gamma_e)\phi(e_2)\ ,$$

$$\phi(gf_{16}) = \rho(\gamma_\infty^3)\phi(e_1) - \rho(\gamma_\infty^2)\phi(e_2) + \rho(\gamma_\infty^2)\phi(e_3) - \rho(\gamma_\infty^2\gamma_e)\phi(e_3) + \rho(\gamma_\infty^2\gamma_e)\phi(e_1)\ ,$$

$$\begin{aligned}\phi(gf_{17}) = \rho(\gamma_\infty)\phi(e_3) - \rho(\gamma_\infty\gamma_e)\phi(e_3) + \rho(\gamma_\infty\gamma_e)\phi(e_1) - \rho(\gamma_\infty\gamma_e\gamma_\infty^{-1})\phi(e_2) \\ + \rho(\gamma_\infty\gamma_e\gamma_\infty^{-1})\phi(e_3)\ ,\end{aligned}$$

$$\begin{aligned}\phi(gf_{18}) = \rho(\gamma_\infty^3)\phi(e_3) - \rho(\gamma_\infty^3\gamma_e)\phi(e_3) + \rho(\gamma_\infty^3\gamma_e)\phi(e_2) - \rho(\gamma_\infty^3\gamma_e\gamma_\infty)\phi(e_1) \\ + \rho(\gamma_\infty^3\gamma_e\gamma_\infty)\phi(e_3)\end{aligned}$$

where $\gamma_e = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \in \mathrm{Stab}_{\Gamma_0(2)}\left(\frac{1+i}{2}\right)$. γ_e maps e_1, e_2, e_3 to the 3 edges right below e_3, connecting at the black vertex $\frac{1+i}{2}$, with a counterclockwise order of $\gamma_e e_3, \gamma_e e_2, \gamma_e e_1$.

To go from $C^1_P((g\Gamma_0(10)g^{-1}, \mathcal{T}, V_k)$ to $C^1_P((\Gamma_0(10), \mathcal{T}, V_k)$, we apply g^{-1} to the forms. Let ψ denote 1-forms in $C^1_P(\Gamma_0(10), \mathcal{T}, V_k)$. Since the edges are labeled the same, we have

$$\psi(f_1) = \rho(g^{-1})\left(\rho(\gamma_\infty^2)\phi(e_1) - \rho(\gamma_\infty)\psi(e_2) + \rho(\gamma_\infty)\phi(e_1) - \phi(e_2) + \phi(e_1)\right) \ ,$$

$$\psi(f_2) = \rho(g^{-1})\left(\rho(\gamma_\infty^2)\phi(e_2) - \rho(\gamma_\infty^3)\phi(e_1) + \rho(\gamma_\infty^3)\phi(e_2) - \rho(\gamma_\infty^4)\phi(e_1) + \rho(\gamma_\infty^4)\phi(e_2)\right) \ ,$$

$$\psi(f_3) = \psi(f_4) = \psi(f_{11}) = \psi(f_{12}) = \psi(f_{13}) = \psi(f_{14}) = 0 \ ,$$

$$\psi(f_5) = \rho(g^{-1})\left(\rho(\gamma_\infty^2)\phi(e_1)\right) \ ,$$

$$\psi(f_6) = \rho(g^{-1})\left(\rho(\gamma_\infty^2)\phi(e_2)\right) \ ,$$

$$\psi(f_7) = \rho(g^{-1})\left(\rho(\gamma_\infty)\phi(e_2)\right) \ ,$$

$$\psi(f_8) = \rho(g^{-1})\left(\rho(\gamma_\infty^3)\phi(e_1)\right) \ ,$$

$$\psi(f_9) = \rho(g^{-1})\left(\rho(\gamma_\infty)\phi(e_3)\right) \ ,$$

$$\psi(f_{10}) = \rho(g^{-1})\left(\rho(\gamma_\infty^3)\phi(e_3)\right) \ ,$$

$$\psi(f_{15}) = \rho(g^{-1})\big(\rho(\gamma_\infty)\phi(e_2) - \rho(\gamma_\infty^2)\phi(e_1) + \rho(\gamma_\infty^2)\phi(e_3) - \rho(\gamma_\infty^2\gamma_e)\phi(e_3) + \rho(\gamma_\infty^2\gamma_e)\phi(e_2)\big) \ ,$$

$$\psi(f_{16}) = \rho(g^{-1})\big(\rho(\gamma_\infty^3)\phi(e_1) - \rho(\gamma_\infty^2)\phi(e_2) + \rho(\gamma_\infty^2)\phi(e_3) - \rho(\gamma_\infty^2\gamma_e)\phi(e_3) + \rho(\gamma_\infty^2\gamma_e)\phi(e_1)\big) \ ,$$

$$\psi(f_{17}) = \rho(g^{-1})\big(\rho(\gamma_\infty)\phi(e_3) - \rho(\gamma_\infty\gamma_e)\phi(e_3) + \rho(\gamma_\infty\gamma_e)\phi(e_1) - \rho(\gamma_\infty\gamma_e\gamma_\infty^{-1})\phi(e_2) + \rho(\gamma_\infty\gamma_e\gamma_\infty^{-1})\phi(e_3)\big) \ ,$$

$$\psi(f_{18}) = \rho(g^{-1})\big(\rho(\gamma_\infty^3)\phi(e_3) - \rho(\gamma_\infty^3\gamma_e)\phi(e_3) + \rho(\gamma_\infty^3\gamma_e)\phi(e_2) - \rho(\gamma_\infty^3\gamma_e\gamma_\infty)\phi(e_1) + \rho(\gamma_\infty^3\gamma_e\gamma_\infty)\phi(e_3)\big) \ .$$

We then take the trace operator to go from $C_P^1(\Gamma_0(10), \mathcal{T}, V_k)$ back to $C_P^1(\Gamma_0(2), \mathcal{T}, V_k)$. Recall the coset representatives of $\Gamma_0(10) \setminus \Gamma_0(2)$ are

$$\overset{\mathrm{I}_2}{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}},\quad \overset{\gamma_1}{\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}},\quad \overset{\gamma_2}{\begin{pmatrix}1 & 1\\ 2 & 3\end{pmatrix}},\quad \overset{\gamma_3}{\begin{pmatrix}1 & 2\\ 2 & 5\end{pmatrix}},\quad \overset{\gamma_4}{\begin{pmatrix}1 & 3\\ 2 & 7\end{pmatrix}},\quad \overset{\gamma_5}{\begin{pmatrix}1 & 4\\ 2 & 9\end{pmatrix}},$$

and the generators of $\Gamma_0(10)$ are

$$\overset{\gamma_x}{\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}}, \quad \overset{\gamma_y}{\begin{pmatrix}3 & -1\\ 10 & -3\end{pmatrix}}, \quad \overset{\gamma_z}{\begin{pmatrix}19 & -7\\ 30 & -11\end{pmatrix}}, \quad \overset{\gamma_s}{\begin{pmatrix}11 & -5\\ 20 & -9\end{pmatrix}}, \quad \overset{\gamma_t}{\begin{pmatrix}7 & -5\\ 10 & -7\end{pmatrix}}.$$

Let the edges of the dessin for $\Gamma_0(2)$ be labeled as in figure 7.6 and the edges of the dessin for $\Gamma_0(10)$ be labeled as in figure 7.12. We have

$$\begin{array}{lll}
I_2 \cdot e_1 = f_1 \ , & I_2 \cdot e_2 = f_2 \ , & I_2 \cdot e_3 = f_3 \ , \\
\gamma_1 \cdot e_1 = f_7 \ , & \gamma_1 \cdot e_2 = f_{11} \ , & \gamma_1 \cdot e_3 = f_9 \ , \\
\gamma_2 \cdot e_1 = f_{13} \ , & \gamma_2 \cdot e_2 = f_{15} \ , & \gamma_2 \cdot e_3 = f_{17} \ , \\
\gamma_3 \cdot e_1 = \gamma_s^{-1} f_{16} \ , & \gamma_3 \cdot e_2 = \gamma_s^{-1} f_{14} \ , & \gamma_3 \cdot e_3 = \gamma_s^{-1} f_{18} \ , \\
\gamma_4 \cdot e_1 = \gamma_s^{-1} f_{12} \ , & \gamma_4 \cdot e_2 = \gamma_s^{-1} f_8 \ , & \gamma_4 \cdot e_3 = \gamma_s^{-1} f_{10} \ , \\
\gamma_5 \cdot e_1 = \gamma_s^{-1} f_6 \ , & \gamma_5 \cdot e_2 = \gamma_s^{-1} f_5 \ , & \gamma_5 \cdot e_3 = \gamma_s^{-1} f_4 \ .
\end{array}$$

Combining all above, we get

$$\begin{aligned}
\pi_*(\psi)(e_1) &= \psi(e_1) + \rho(\gamma_1^{-1})\psi(\gamma_1 e_1) + \rho(\gamma_2^{-1})\psi(\gamma_2 e_1) + \rho(\gamma_3^{-1})\psi(\gamma_3 e_1) \\
&\qquad + \rho(\gamma_4^{-1})\psi(\gamma_4 e_1) + \rho(\gamma_5^{-1})\psi(\gamma_5 e_1) \\
&= \psi(f_1) + \rho(\gamma_1^{-1})\psi(f_7) + \rho(\gamma_2^{-1})\psi(f_{13}) + \rho(\gamma_3^{-1}\gamma_s^{-1})\psi(f_{16}) \\
&\quad + \rho(\gamma_4^{-1}\gamma_s^{-1})\psi(f_{12}) + \rho(\gamma_5^{-1}\gamma_s^{-1})\psi(f_6) \\
&= \rho\left(g^{-1}(\gamma_\infty^2 + \gamma_\infty + \mathrm{I}_{k-1}) + \gamma_3^{-1}\gamma_s^{-1}g^{-1}(\gamma_\infty^3 + \gamma_\infty^2\gamma_e)\right)\phi(e_1) \\
&\quad + \rho\left(g^{-1}(-\gamma_\infty - \mathrm{I}_{k-1}) + \gamma_1^{-1}g^{-1}\gamma_\infty - \gamma_3^{-1}\gamma_s^{-1}g^{-1}\gamma_\infty^2 + \gamma_5^{-1}\gamma_s^{-1}g^{-1}\gamma_\infty^2\right)\phi(e_2) \\
&\quad + \rho\left(\gamma_3^{-1}\gamma_s^{-1}g^{-1}(\gamma_\infty^2 - \gamma_\infty^2\gamma_e)\right)\phi(e_3)\ ,
\end{aligned}$$

$$\begin{aligned}
\pi_*(\psi)(e_2) &= \psi(e_2) + \rho(\gamma_1^{-1})\psi(\gamma_1 e_2) + \rho(\gamma_2^{-1})\psi(\gamma_2 e_2) + \rho(\gamma_3^{-1})\psi(\gamma_3 e_2) \\
&\qquad + \rho(\gamma_4^{-1})\psi(\gamma_4 e_2) + \rho(\gamma_5^{-1})\psi(\gamma_5 e_2) \\
&= \psi(f_2) + \rho(\gamma_1^{-1})\psi(f_{11}) + \rho(\gamma_2^{-1})\psi(f_{15}) + \rho(\gamma_3^{-1}\gamma_s^{-1})\psi(f_{14}) \\
&\quad + \rho(\gamma_4^{-1}\gamma_s^{-1})\psi(f_8) + \rho(\gamma_5^{-1}\gamma_s^{-1})\psi(f_5) \\
&= +\rho\left(g^{-1}(-\gamma_\infty^3 - \gamma_\infty^4) - \gamma_2^{-1}g^{-1}\gamma_\infty^2 + \gamma_4^{-1}\gamma_s^{-1}g^{-1}\gamma_\infty^3 + \gamma_5^{-1}\gamma_s^{-1}g^{-1}\gamma_\infty^2\right)\phi(e_1) \\
&\quad + \rho\left(g^{-1}(\gamma_\infty^2 + \gamma_\infty^3 + \gamma_\infty^4) + \gamma_2^{-1}g^{-1}(\gamma_\infty + \gamma_\infty^2\gamma_e)\right)\phi(e_2) \\
&\quad + \rho\left(\gamma_2^{-1}g^{-1}(\gamma_\infty^2 - \gamma_\infty^2\gamma_e)\right)\phi(e_3)\ ,
\end{aligned}$$

$$\begin{aligned}
\pi_*(\psi)(e_3) &= \psi(e_3) + \rho(\gamma_1^{-1})\psi(\gamma_1 e_3) + \rho(\gamma_2^{-1})\psi(\gamma_2 e_3) + \rho(\gamma_3^{-1})\psi(\gamma_4 e_3) \\
&\qquad + \rho(\gamma_4^{-1})\psi(\gamma_3 e_3) + \rho(\gamma_5^{-1})\psi(\gamma_5 e_3) \\
&= \psi(f_3) + \rho(\gamma_1^{-1})\psi(f_9) + \rho(\gamma_2^{-1})\psi(f_{17}) + \rho(\gamma_3^{-1}\gamma_s^{-1})\psi(f_{18}) \\
&\quad + \rho(\gamma_4^{-1}\gamma_s^{-1})\psi(f_{10}) + \rho(\gamma_5^{-1}\gamma_s^{-1})\psi(f_4) \\
&= \rho\left(\gamma_2^{-1}g^{-1}\gamma_\infty\gamma_e - \gamma_3^{-1}\gamma_s^{-1}g^{-1}\gamma_\infty^3\gamma_e\gamma_\infty\right)\phi(e_1) \\
&\quad + \rho\left(-\gamma_2^{-1}g^{-1}\gamma_\infty\gamma_e\gamma_\infty^{-1} - \gamma_3^{-1}\gamma_s^{-1}g^{-1}\gamma_\infty^3\gamma_e\right)\phi(e_2) \\
&\quad + \rho\big(\gamma_1^{-1}g^{-1}\gamma_\infty + \gamma_2^{-1}g^{-1}(\gamma_\infty - \gamma_\infty\gamma_e + \gamma_\infty\gamma_e\gamma_\infty^{-1}) + \gamma_3^{-1}\gamma_s^{-1}g^{-1}(\gamma_\infty^3 \\
&\qquad - \gamma_\infty^3\gamma_e + \gamma_\infty^3\gamma_e\gamma_\infty) + \gamma_4^{-1}\gamma_s^{-1}g^{-1}\gamma_\infty^3\big)\phi(e_3)\ .
\end{aligned}$$

Here the operator of the representation ρ is $\rho(A \pm B) = \rho(A) \pm \rho(B)$. Now we can use Mathematica to compute the eigenvalues for the Hecke operator T_5 with different weights.

Case 1 : k = 8.

We first set up the representations for the matrices where $V_8 = \mathrm{Sym}^6(\mathbf{C}^2)$.

```
gamma1inv =
 Transpose[Table[Coefficient[(x - 2 y) ^ (6 - i) (y) ^ i, x ^ (6 - j) * y ^ j], {i, 0, 6}, {j, 0, 6}]]
gamma2inv =
 Transpose[Table[Coefficient[(3 x - 2 y) ^ (6 - i) (-x + y) ^ i, x ^ (6 - j) * y ^ j], {i, 0, 6}, {j, 0, 6}]]
gamma3inv =
 Transpose[Table[Coefficient[(5 x - 2 y) ^ (6 - i) (-2 x + y) ^ i, x ^ (6 - j) * y ^ j], {i, 0, 6}, {j, 0, 6}]]
gamma4inv =
 Transpose[Table[Coefficient[(7 x - 2 y) ^ (6 - i) (-3 x + y) ^ i, x ^ (6 - j) * y ^ j], {i, 0, 6}, {j, 0, 6}]]
gamma5inv =
 Transpose[Table[Coefficient[(9 x - 2 y) ^ (6 - i) (-4 x + y) ^ i, x ^ (6 - j) * y ^ j], {i, 0, 6}, {j, 0, 6}]]
gammainf =
 Transpose[Table[Coefficient[(x) ^ (6 - i) (x + y) ^ (i), x ^ (6 - j) * y ^ (j)], {i, 0, 6}, {j, 0, 6}]]
gammainfinv =
 Transpose[Table[Coefficient[(x) ^ (6 - i) (-x + y) ^ (i), x ^ (6 - j) * y ^ (j)], {i, 0, 6}, {j, 0, 6}]]
gammae = Transpose[Table[Coefficient[(x + 2 y) ^ (6 - i) (-x - y) ^ (i), x ^ (6 - j) * y ^ (j)],
   {i, 0, 6}, {j, 0, 6}]]
gammasinv =
 Transpose[Table[Coefficient[(-9 x - 20 y) ^ (6 - i) (5 x + 11 y) ^ (i), x ^ (6 - j) * y ^ (j)],
   {i, 0, 6}, {j, 0, 6}]]
ginv = Transpose[Table[Coefficient[(x/5) ^ (6 - i) (y) ^ (i), x ^ (6 - j) * y ^ (j)], {i, 0, 6}, {j, 0, 6}]]
```

```
t11 = ginv.(gammainf.gammainf + gammainf + IdentityMatrix[7]) +
  gamma3inv.gammasinv.ginv.(gammainf.gammainf.gammainf + gammainf.gammainf.gammae)
t12 = ginv.(-gammainf - IdentityMatrix[7]) + gamma1inv.ginv.gammainf -
  gamma3inv.gammasinv.ginv.gammainf.gammainf + gamma5inv.gammasinv.ginv.gammainf.gammainf
t13 = gamma3inv.gammasinv.ginv.(gammainf.gammainf - gammainf.gammainf.gammae)
t21 = ginv.(-gammainf.gammainf.gammainf - gammainf.gammainf.gammainf.gammainf) -
  gamma2inv.ginv.gammainf.gammainf + gamma4inv.gammasinv.ginv.gammainf.gammainf.gammainf +
  gamma5inv.gammasinv.ginv.gammainf.gammainf
t22 = ginv.(gammainf.gammainf + gammainf.gammainf.gammainf + gammainf.gammainf.gammainf.gammainf) +
  gamma2inv.ginv.(gammainf + gammainf.gammainf.gammae)
t23 = gamma2inv.ginv.(gammainf.gammainf - gammainf.gammainf.gammae)
t31 = gamma2inv.ginv.gammainf.gammae -
  gamma3inv.gammasinv.ginv.gammainf.gammainf.gammainf.gammae.gammainf
t32 = -gamma2inv.ginv.gammainf.gammae.gammainfinv +
  gamma3inv.gammasinv.ginv.gammainf.gammainf.gammainf.gammae
t33 = gamma1inv.ginv.gammainf +
  gamma2inv.ginv.(gammainf - gammainf.gammae + gammainf.gammae.gammainfinv) +
  gamma3inv.gammasinv.ginv.
   (gammainf.gammainf.gammainf - gammainf.gammainf.gammainf.gammae +
     gammainf.gammainf.gammainf.gammae.gammainf) +
  gamma4inv.gammasinv.ginv.gammainf.gammainf.gammainf
T5 = ArrayFlatten[ {{t11, t12, t13}, {t21, t22, t23}, {t31, t32, t33}}]
Eigenvalues[T5]
```

We then can compute the matrix for T_5 and find its eigenvalues as follows.

$$\left\{\frac{78126}{15625}, \frac{78126}{15625}, \frac{8833+\sqrt{64531489}}{15625}, \frac{4\left(2057+8\sqrt{55366}\right)}{15625}, \frac{4\left(2057+8\sqrt{55366}\right)}{15625}, \frac{9889+3\sqrt{3672769}}{15625}, \frac{4\left(2045+24\sqrt{5815}\right)}{15625}, \frac{9713+3\sqrt{3554041}}{15625}, \frac{9889-3\sqrt{3672769}}{15625}, \frac{9713-3\sqrt{3554041}}{15625}, \frac{4\left(433+24\sqrt{214}\right)}{15625}, \frac{612}{3125}, \frac{612}{3125}, \frac{308}{3125}, \frac{4\left(2045-24\sqrt{5815}\right)}{15625}, \frac{8833-\sqrt{64531489}}{15625}, \frac{4\left(2057-8\sqrt{55366}\right)}{15625}, \frac{4\left(2057-8\sqrt{55366}\right)}{15625}, \frac{4\left(433-24\sqrt{214}\right)}{15625}, -\frac{42}{3125}, -\frac{42}{3125}\right\}$$

Next, to find out which eigenvalues correspond to 1-forms in $H^1_P(\Gamma_0(2), \mathcal{T}, V_8)$, we find a set of basis for $H^1_P(\Gamma_0(2), \mathcal{T}, V_8)$. Again we use conditions in 7.5.1 for the computation.

```
gammainfstar = Transpose[gammainf]
NullSpace[gammae - IdentityMatrix[7]]
```

```
{{1, 0, 0, 0, 0, 0, 0}, {1, 1, 0, 0, 0, 0, 0}, {1, 2, 1, 0, 0, 0, 0}, {1, 3, 3, 1, 0, 0, 0},
 {1, 4, 6, 4, 1, 0, 0}, {1, 5, 10, 10, 5, 1, 0}, {1, 6, 15, 20, 15, 6, 1}}

{{2, 9, 15, 10, 0, 0, 4}, {-5, -26, -50, -40, 0, 16, 0}, {1, 6, 14, 16, 8, 0, 0}}
```

Thus the kernel has dimension 3.

```
p1 = NullSpace[gammae - IdentityMatrix[7]][[1]]
p2 = NullSpace[gammae - IdentityMatrix[7]][[2]]
p3 = NullSpace[gammae - IdentityMatrix[7]][[3]]
v1 = {a1, a2, a3, a4, a5, a6, a7}
v2 = {b1, b2, b3, b4, b5, b6, b7}
s = Solve[{v1 + gammainfstar.v2 == {0, 0, 0, 0, 0, 0, 0}, (v1 + v2).p1 == {0, 0, 0, 0, 0, 0, 0},
   (v1 + v2).p2 == {0, 0, 0, 0, 0, 0, 0}, (v1 + v2).p3 == {0, 0, 0, 0, 0, 0, 0}, -a7 + b7 == 0,
   a1 - gammainfinv[[1]].v2 + (gammainfinv - IdentityMatrix[7])[[1]].(-v1 - v2) == 0},
  {a1, a2, a3, a4, a5, a6, a7, b1, b2, b3, b4, b5, b6, b7}]
```

$$\left\{\left\{a3 \to -\frac{54\,a1}{179} + \frac{147\,a2}{358},\ a4 \to -\frac{3\,a2}{8},\ a5 \to \frac{27\,a1}{179} - \frac{147\,a2}{716},\ a6 \to \frac{a2}{4},\ a7 \to \frac{113\,a1}{179} - \frac{2205\,a2}{1432},\ b1 \to -a1,\ b2 \to a1 - a2,\ b3 \to -\frac{125\,a1}{179} + \frac{569\,a2}{358},\ b4 \to \frac{17\,a1}{179} - \frac{1995\,a2}{1432},\ b5 \to \frac{118\,a1}{179} + \frac{173\,a2}{716},\ b6 \to -\frac{226\,a1}{179} + \frac{1131\,a2}{716},\ b7 \to \frac{113\,a1}{179} - \frac{2205\,a2}{1432}\right\}\right\}$$

Here the degree of freedom for the vectors v_i is 2, which agrees with the dimension of $H^1_P(\Gamma_0(2), \mathcal{T}, V_8)$ which is 2 as well.

```
s1 = s /. {a1 :> 1, a2 :> 0}
u1 = v1 /. s1 /. {a1 :> 1, a2 :> 0} //. {x_List} :> x
u2 = v2 /. s1 //. {x_List} :> x
u3 = -(u1 + u2)
s2 = s /. {a1 :> 0, a2 :> 1}
w1 = v1 /. s2 /. {a1 :> 0, a2 :> 1} //. {x_List} :> x
w2 = v2 /. s2 //. {x_List} :> x
w3 = -(w1 + w2)
```

We assign a set of basis, and then orthogonalize them.

```
colvecu = Join[u1, u2, u3]
colvecw = Join[w1, w2, w3]
phi1 = colvecu
phi2 = colvecw - (colvecu.colvecw / colvecu.colvecu) * colvecu
tphi1 = T5.phi1
tphi2 = T5.phi2
```

After applying the T_5 operator on the basis elements, we can check that $T\phi_1$ and

$T\phi_2$ are in $C^1_P(\Gamma_0(2), \mathcal{T}, V_8)$ but not in $\ker(d^*)$. We again use projection onto $\ker(d^*)$.

```
a = tphi1.phi1 / (Norm[phi1]) ^ 2
atilde = tphi2.phi1 / (phi1.phi1)
b = tphi1.phi2 / (phi2.phi2)
btilde = tphi2.phi2 / (phi2.phi2)
mat22 = {{a, atilde}, {b, btilde}}
Eigenvalues[mat22]
```

$$\left\{\left\{-\frac{42}{3125}, 0\right\}, \left\{0, -\frac{42}{3125}\right\}\right\}$$

$$\left\{-\frac{42}{3125}, -\frac{42}{3125}\right\}$$

Thus the eigenvalue for T_5 with weight 8 is $-\dfrac{42}{3125}$ or $-\dfrac{1}{5^6} \cdot 210$..

Case 2 : $\mathbf{k = 10}$.

Similarly as the T_3 with base curve $X_0(2)$ case. Here we set $V_{10} = \mathrm{Sym}^8(\mathbf{C}^2)$, and the dimension of $H^1_P(\Gamma_0(2), \mathcal{T}, V_{10})$ is again 2. Follow the same procedure as above, we can find the eigenvalue to be $\dfrac{174}{78125}$ or $\dfrac{1}{5^8} \cdot 870$..

Case 3 : $\mathbf{k = 12}$.

Similarly as the T_3 with base curve $X_0(2)$ case. Here we set $V_{12} = \mathrm{Sym}^{10}(\mathbf{C}^2)$. Here the dimension of $H^1_P(\Gamma_0(2), \mathcal{T}, V_{12})$ is 4. Follow the same procedure as above, we can find the eigenvalue to be $\dfrac{966}{1953125}$ or $\dfrac{1}{5^{10}} \cdot 4830$.

7.5.3 T_3 Acting on $X_0(3)$

To see the action of T_3 on $X_0(3)$, we have the following Hecke correspondence.

$$\begin{array}{ccc} \Gamma_0(9)\backslash\mathfrak{h} & \xrightarrow{g} & g\Gamma_0(9)g^{-1}\backslash\mathfrak{h} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_g} \\ \Gamma_0(3)\backslash\mathfrak{h} & & \Gamma_0(3)\backslash\mathfrak{h} \end{array}$$

where $g = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$.

To find the dessin for $\Gamma_0(3)$, first we have $[\mathrm{PSL}_2(\mathbf{Z}) \colon \Gamma_0(3)] = 4$, so the dessin has 4 edges. The coset representatives for $\Gamma_0(3)$ in $\mathrm{PSL}_2(\mathbf{Z})$ are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}.$$

Thus the $\Gamma_0(3)$-orbits of i and μ are

$$\begin{array}{ccc} & [\,i\,] & \\ \nearrow & & \nwarrow \\ [\,i\,] & & \left[\frac{1+i}{2}\right] \end{array}, \qquad \begin{array}{ccc} & [\,\mu\,] & \\ \nearrow & & \nwarrow \\ [\,\mu\,] & & \left[\frac{3+\sqrt{3}i}{6}\right] \end{array}.$$

with $\frac{3+\sqrt{3}i}{6}$ being an elliptic point. Now the dessin for $\Gamma_0(3)$ embedded in $\mathfrak{h}$ is as follows. We will label the edges $e_1, \cdots, e_4$.

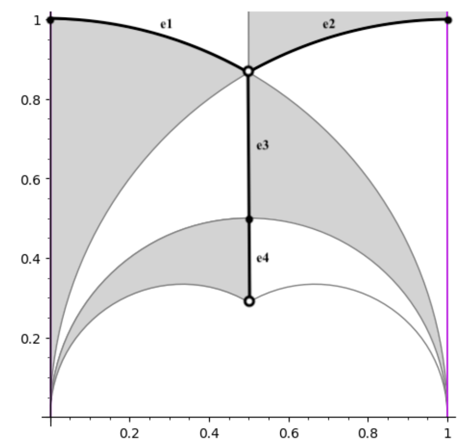
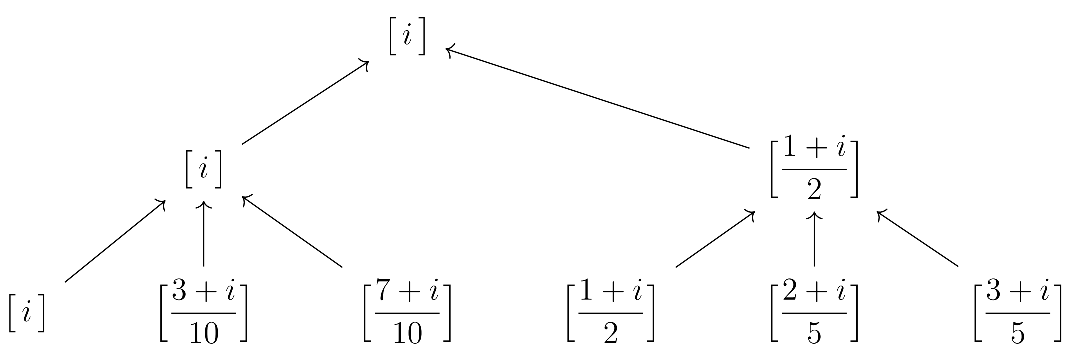


Figure 7.16: Dessin of $\Gamma_0(3)$ embedded in $\mathfrak{h}$.

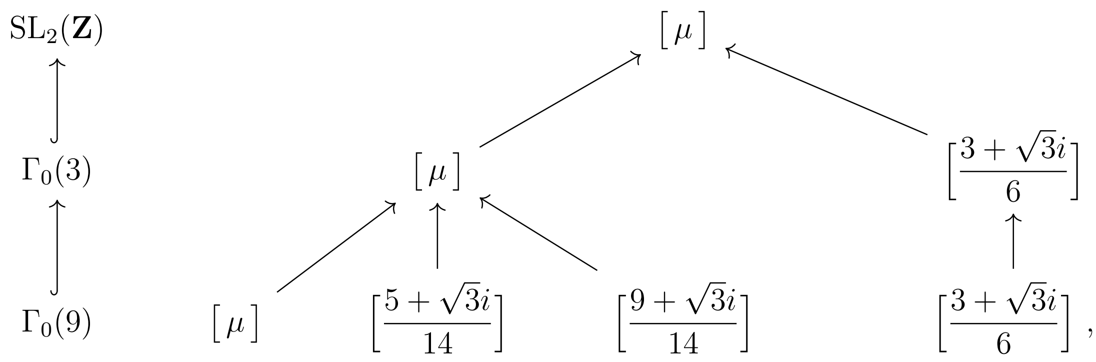
Next, we want to find the embedded dessin for $\Gamma_0(9)$. The Belyi map for $\Gamma_0(9)$ has degree $[\mathrm{PSL}_2(\mathbf{Z})\colon \Gamma_0(9)] = 12$. Thus this dessin has 12 edges. The coset representatives of $\Gamma_0(9)$ in $\Gamma_0(3)$ are

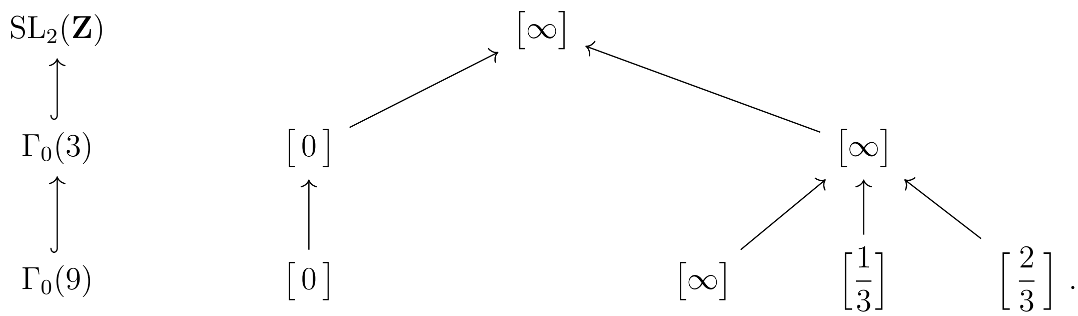
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 \\ 3 & 2 \end{pmatrix}.$$

Thus the $\Gamma_0(9)$-orbits of i are



Thus the $\Gamma_0(9)$-orbits of μ and ∞ are





These give us 6 black vertices, 4 white vertices, and 4 faces together with the 12 edges. The embedded dessin for $\Gamma_0(9)$ in $\mathfrak{h}$ is then the following:

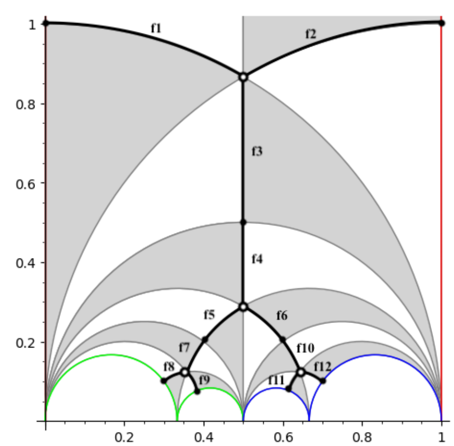
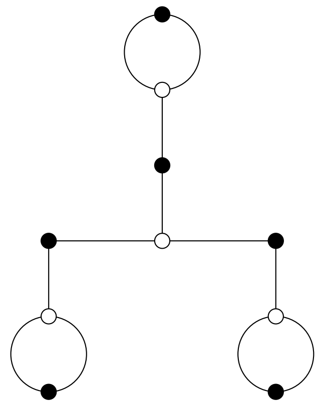


Figure 7.17: Dessin of $\Gamma_0(9)$ embedded in $\mathfrak{h}$.

As before, we labeled the edges $f_1, \cdots, f_{12}$. The colored boundaries are identified to the same color. Hence we also have the abstract dessin for $\Gamma_0(9)$:



To get the embedded dessin for $g\Gamma_0(9)g^{-1}$, we apply g to the embedded dessin for $\Gamma_0(9)$, which acts on the graph by multiplication by 3. We obtain the following embedded dessin for $g\Gamma_0(9)g^{-1}$:

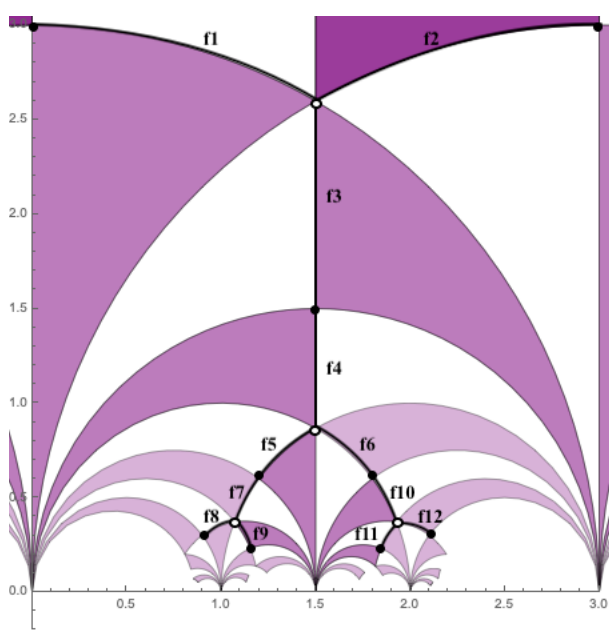


Figure 7.18: Dessin of $g\Gamma_0(9)g^{-1}$ Embedded in $\mathfrak{h}$.

We then overlay the embedded dessin for $\Gamma_0(3)$ with the embeded dessin for $g\Gamma_0(9)g^{-1}$. We get

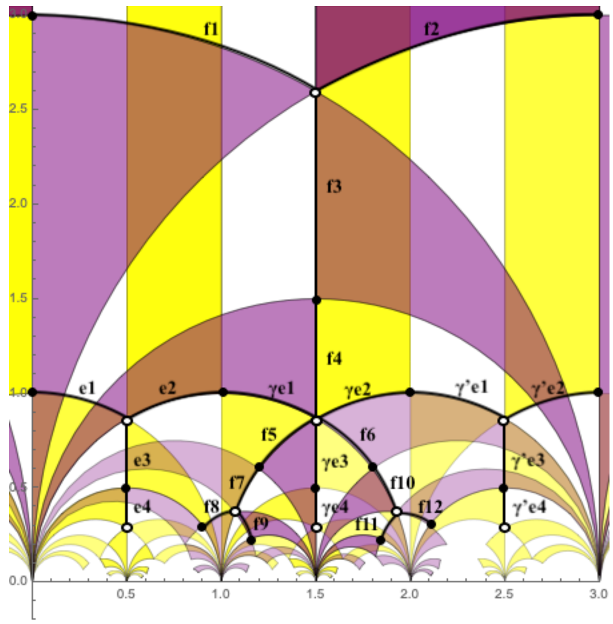


Figure 7.19: Overlay of Dessins of $\Gamma_0(3)$ and $g\Gamma_0(9)g^{-1}$ Embedded in $\mathfrak{h}$.

where the yellow part comes from expanding the embedded dessin of $\Gamma_0(3)$ along the Serre tree:

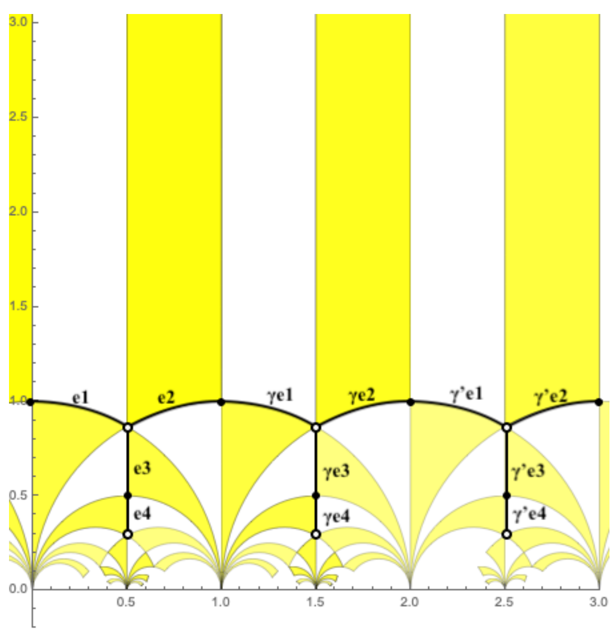


Figure 7.20: Expanded Dessin of $\Gamma_0(3)$ Embedded in $\mathfrak{h}$.

Here e_1, e_2, e_3, e_4 are of the original dessin. $\gamma = \gamma_\infty = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ shifts the edges to the right by 1 and $\gamma' = \gamma_\infty^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ shifts the edges to the right by 2.

Similarly as the case with $\Gamma_0(2)$,if we have a 1-form in $C^1_P(\Gamma_0(3), \mathcal{T}, V_k)$, it naturally pulls back to a 1-form in $C^1_P(g\Gamma_0(9)g^{-1}, \mathcal{T}, V_k)$. This is true in general as well, as it has been shown in (3.2.2).

Let ϕ be a 1-form in $C^1_P(\Gamma_0(3), \mathcal{T}, V_k)$. Again, use deformation retract r, we can get the pulled back 1-forms in $C^1_P(g\Gamma_0(9)g^{-1}, \mathcal{T}, V_k)$ as depicted in Figure 7.19 with the orientation going from white vertices to black vertices as follows.

$$\phi(f_1) = \rho(\gamma_\infty)\phi(e_1) - \phi(e_2) + \phi(e_1) \ ,$$

$$\phi(f_2) = \rho(\gamma_\infty)\phi(e_2) - \rho(\gamma_\infty^2)\phi(e_1) + \rho(\gamma_\infty^2)\phi(e_2) \ ,$$

$$\phi(f_3) = \phi(f_4) = \phi(f_5) = \phi(f_6) = \phi(f_7) = \phi(f_{10}) = 0 \ ,$$

$$\phi(f_8) = \rho(\gamma_\infty)\phi(e_1) - \phi(e_2) + \phi(e_3) \ ,$$

$$\phi(f_9) = \rho(\gamma_\infty)\phi(e_3) - \rho(\gamma_\infty)\phi(e_4) + \rho(\gamma_\infty\gamma_e)\phi(e_4) \ ,$$

$$\phi(f_{11}) = \rho(\gamma_\infty)\phi(e_3) - \rho(\gamma_\infty)\phi(e_4) + \rho(\gamma_\infty\gamma_e^2)\phi(e_4) \ ,$$

$$\phi(f_{12}) = \rho(\gamma_\infty)\phi(e_2) - \rho(\gamma_\infty^2)\phi(e_1) + \rho(\gamma_\infty^2)\phi(e_3)$$

where $\gamma_e = \begin{pmatrix} -1 & 1 \\ -3 & 2 \end{pmatrix} \in \mathrm{Stab}_{\Gamma_0(3)}\left(\dfrac{3+\sqrt{3}i}{6}\right)$. Hence $\gamma_e e_4$ and $\gamma_e^2 e_4$ are the two edges attached to the white vertex of e_4 in counterclockwise fashion.

Again we apply g^{-1} to go from $C^1_P(g\Gamma_0(9)g^{-1}, \mathcal{T}, V_k)$ to $C^1_P(\Gamma_0(9), \mathcal{T}, V_k)$. With the labeling of edges for $\Gamma_0(9)$ as in figure 7.17, and let ψ denote 1-forms in $C^1_P(\Gamma_0(9), \mathcal{T},$

V_k), we obtain

$$\psi(f_1) = \rho(g^{-1})\phi(f_1) = \rho(g^{-1}\gamma_\infty)\phi(e_1) - \rho(g^{-1})\phi(e_2) + \rho(g^{-1})\phi(e_1)\ ,$$

$$\psi(f_2) = \rho(g^{-1})\phi(f_2) = \rho(g^{-1}\gamma_\infty)\phi(e_2) - \rho(g^{-1}\gamma_\infty^2)\phi(e_1) + \rho(g^{-1}\gamma_\infty^2)\phi(e_2)\ ,$$

$$\psi(f_3) = \psi(f_4) = \psi(f_5) = \psi(f_6) = \psi(f_7) = \psi(f_{10}) = 0\ ,$$

$$\psi(f_8) = \rho(g^{-1})\phi(f_8) = \rho(g^{-1}\gamma_\infty)\phi(e_1) - \rho(g^{-1})\phi(e_2) + \rho(g^{-1})\phi(e_3)\ ,$$

$$\psi(f_9) = \rho(g^{-1})\phi(f_9) = \rho(g^{-1}\gamma_\infty)\phi(e_3) - \rho(g^{-1}\gamma_\infty)\phi(e_4) + \rho(g^{-1}\gamma_\infty\gamma_e)\phi(e_4)\ ,$$

$$\psi(f_{11}) = \rho(g^{-1})\phi(f_{11}) = \rho(g^{-1}\gamma_\infty)\phi(e_3) - \rho(g^{-1}\gamma_\infty)\phi(e_4) + \rho(g^{-1}\gamma_\infty\gamma_e^2)\phi(e_4)\ ,$$

$$\psi(f_{12}) = \rho(g^{-1})\phi(f_{12}) = \rho(g^{-1}\gamma_\infty)\phi(e_2) - \rho(g^{-1}\gamma_\infty^2)\phi(e_1) + \rho(g^{-1}\gamma_\infty^2)\phi(e_3)\ .$$

Lastly, we take the trace to go from 1-forms in $C_P^1(\Gamma_0(9), \mathcal{T}, V_k)$ back to 1-forms in $C_P^1(\Gamma_0(3), \mathcal{T}, V_k)$. Recall the coset representatives of $\Gamma_0(9) \setminus \Gamma_0(3)$ are

$$\overset{\mathrm{I}_2}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}},\quad \overset{\gamma_1}{\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}},\quad \overset{\gamma_2}{\begin{pmatrix} -1 & -1 \\ 3 & 2 \end{pmatrix}},$$

and the generators of $\Gamma_0(9)$ are

$$\overset{\gamma_x}{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}},\quad \overset{\gamma_y}{\begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}},\quad \overset{\gamma_z}{\begin{pmatrix} 7 & -4 \\ 9 & -5 \end{pmatrix}}.$$

Let the edges of the dessin for $\Gamma_0(3)$ be labeled as in Figure 7.16 and the edges of the dessin for $\Gamma_0(9)$ be labeled as in Figure 7.17. We have

$$\begin{array}{llll}
I_2 \cdot e_1 = f_1\ , & I_2 \cdot e_2 = f_2\ , & I_2 \cdot e_3 = f_3\ , & I_2 \cdot e_4 = f_4\ , \\
\gamma_1 \cdot e_1 = \gamma_y^{-1} f_9\ , & \gamma_1 \cdot e_2 = \gamma_y^{-1} f_8\ , & \gamma_1 \cdot e_3 = \gamma_y^{-1} f_7\ , & \gamma_1 \cdot e_4 = \gamma_y^{-1} f_5\ , \\
\gamma_2 \cdot e_1 = \gamma_x^{-1}\gamma_z^{-1} f_{12}\ , & \gamma_2 \cdot e_2 = \gamma_x^{-1}\gamma_z^{-1} f_{11}\ , & \gamma_2 \cdot e_3 = \gamma_x^{-1}\gamma_z^{-1} f_{10}\ , & \\
\gamma_2 \cdot e_4 = \gamma_x^{-1}\gamma_z^{-1} f_6\ . & & &
\end{array}$$

Combining all above, we get

$$
\begin{aligned}
\pi_*(\psi)(e_1) &= \rho(\mathrm{I}_2^{-1})\psi(\mathrm{I}_2\, e_1) + \rho(\gamma_1^{-1})\psi(\gamma_1 e_1) + \rho(\gamma_2^{-1})\psi(\gamma_2 e_1) \\
&= \psi(f_1) + \rho(\gamma_1^{-1}\gamma_y^{-1})\psi(f_9) + \rho(\gamma_2^{-1}\gamma_x^{-1}\gamma_z^{-1})\psi(f_{12}) \\
&= \left(\rho(g^{-1}\gamma_\infty) + \rho(g^{-1}) - \rho(\gamma_2^{-1}\gamma_x^{-1}\gamma_z^{-1}g^{-1}\gamma_\infty^2)\right)\phi(e_1) \\
&\quad + \left(-\rho(g^{-1}) + \rho(\gamma_2^{-1}\gamma_x^{-1}\gamma_z^{-1}g^{-1}\gamma_\infty)\right)\phi(e_2) \\
&\quad + \left(\rho(\gamma_1^{-1}\gamma_y^{-1}g^{-1}\gamma_\infty) + \rho(\gamma_2^{-1}\gamma_x^{-1}\gamma_z^{-1}g^{-1}\gamma_\infty^2)\right)\phi(e_3) \\
&\quad + \left(-\rho(\gamma_1^{-1}\gamma_y^{-1}g^{-1}\gamma_\infty) + \rho(\gamma_1^{-1}\gamma_y^{-1}g^{-1}\gamma_\infty\gamma_e)\right)\phi(e_4)\ ,
\end{aligned}
$$

$$
\begin{aligned}
\pi_*(\psi)(e_2) &= \rho(\mathrm{I}_2^{-1})\psi(\mathrm{I}_2\, e_2) + \rho(\gamma_1^{-1})\psi(\gamma_1 e_2) + \rho(\gamma_2^{-1})\psi(\gamma_2 e_2) \\
&= \psi(f_2) + \rho(\gamma_1^{-1}\gamma_y^{-1})\psi(f_8) + \rho(\gamma_2^{-1}\gamma_x^{-1}\gamma_z^{-1})\psi(f_{11}) \\
&= \left(-\rho(g^{-1}\gamma_\infty^2) + \rho(\gamma_1^{-1}\gamma_y^{-1}g^{-1}\gamma_\infty)\right)\phi(e_1) \\
&\quad + \left(\rho(g^{-1}\gamma_\infty) + \rho(g^{-1}\gamma_\infty^2) - \rho(\gamma_1^{-1}\gamma_y^{-1}g^{-1})\right)\phi(e_2) \\
&\quad + \left(\rho(\gamma_1^{-1}\gamma_y^{-1}g^{-1}) + \rho(\gamma_2^{-1}\gamma_x^{-1}\gamma_z^{-1}g^{-1}\gamma_\infty)\right)\phi(e_3) \\
&\quad + \left(-\rho(\gamma_2^{-1}\gamma_x^{-1}\gamma_z^{-1}g^{-1}\gamma_\infty) + \rho(\gamma_2^{-1}\gamma_x^{-1}\gamma_z^{-1}g^{-1}\gamma_\infty\gamma_e^2)\right)\phi(e_4)\ ,
\end{aligned}
$$

$$
\begin{aligned}
\pi_*(\psi)(e_3) &= \rho(\mathrm{I}_2^{-1})\psi(\mathrm{I}_2\, e_3) + \rho(\gamma_1^{-1})\psi(\gamma_1 e_3) + \rho(\gamma_2^{-1})\psi(\gamma_2 e_3) \\
&= \psi(f_3) + \rho(\gamma_1^{-1}\gamma_y^{-1})\psi(f_7) + \rho(\gamma_2^{-1}\gamma_x^{-1}\gamma_z^{-1})\psi(f_{10}) \\
&= 0\ ,
\end{aligned}
$$

$$
\begin{aligned}
\pi_*(\psi)(e_4) &= \rho(\mathrm{I}_2^{-1})\psi(\mathrm{I}_2\, e_4) + \rho(\gamma_1^{-1})\psi(\gamma_1 e_4) + \rho(\gamma_2^{-1})\psi(\gamma_2 e_4) \\
&= \psi(f_4) + \rho(\gamma_1^{-1}\gamma_y^{-1})\psi(f_5) + \rho(\gamma_2^{-1}\gamma_x^{-1}\gamma_z^{-1})\psi(f_6) \\
&= 0\ .
\end{aligned}
$$

Now we can compute the eigenvalues for the Hecke operator T_3 on cohomology using Mathematica. We will illustrate the computation with weight $k = 10$ where

$\dim\left(H_P^1\left(\Gamma_0(3), \mathcal{T}, V_{10}\right)\right) = 4$. The techniques used here are similar to the ones in the case with $\Gamma_0(2)$ and can be generalized to other weights as well. First we have the set up of the representations of the matrices.

```
gamma1inv = Transpose[Table[Coefficient[(x - 3 y) ^ (8 - i) (y) ^ i, x ^ (8 - j) * y ^ j], {i, 0, 8}, {j, 0, 8}]]
gamma2inv =
 Transpose[Table[Coefficient[(2 x - 3 y) ^ (8 - i) (x - y) ^ i, x ^ (8 - j) * y ^ j], {i, 0, 8}, {j, 0, 8}]]
gammaxinv =
 Transpose[Table[Coefficient[(x) ^ (8 - i) (-x + y) ^ (i), x ^ (8 - j) * y ^ (j)], {i, 0, 8}, {j, 0, 8}]]
gammayinv =
 Transpose[Table[Coefficient[(-2 x - 9 y) ^ (8 - i) (x + 4 y) ^ (i), x ^ (8 - j) * y ^ (j)], {i, 0, 8}, {j, 0, 8}]]
gammazinv =
 Transpose[Table[Coefficient[(-5 x - 9 y) ^ (8 - i) (4 x + 7 y) ^ (i), x ^ (8 - j) * y ^ (j)], {i, 0, 8},
   {j, 0, 8}]]
gammainfinity =
 Transpose[Table[Coefficient[(x) ^ (8 - i) (x + y) ^ (i), x ^ (8 - j) * y ^ (j)], {i, 0, 8}, {j, 0, 8}]]
gammae =
 Transpose[Table[Coefficient[(-x - 3 y) ^ (8 - i) (x + 2 y) ^ (i), x ^ (8 - j) * y ^ (j)], {i, 0, 8}, {j, 0, 8}]]
ginv = Transpose[Table[Coefficient[(x/3) ^ (8 - i) (y) ^ (i), x ^ (8 - j) * y ^ (j)], {i, 0, 8}, {j, 0, 8}]]
```

We then can find the matrix representation for T_3 and find its eigenvalues.

```
t11 = ginv.(gammainfinity + IdentityMatrix[9]) -
  gamma2inv.gammaxinv.gammazinv.ginv.gammainfinity.gammainfinity
t12 = -ginv + gamma2inv.gammaxinv.gammazinv.ginv.gammainfinity
t13 = gamma1inv.gammayinv.ginv.gammainfinity +
  gamma2inv.gammaxinv.gammazinv.ginv.gammainfinity.gammainfinity
t14 = gamma1inv.gammayinv.ginv.(-gammainfinity + gammainfinity.gammae)
t21 = -ginv.gammainfinity.gammainfinity + gamma1inv.gammayinv.ginv.gammainfinity
t22 = ginv.(gammainfinity + gammainfinity.gammainfinity) - gamma1inv.gammayinv.ginv
t23 = gamma1inv.gammayinv.ginv + gamma2inv.gammaxinv.gammazinv.ginv.gammainfinity
t24 = gamma2inv.gammaxinv.gammazinv.ginv.(-gammainfinity + gammainfinity.gammae.gammae)
t31 = ConstantArray[0, {9, 9}]
t32 = ConstantArray[0, {9, 9}]
t33 = ConstantArray[0, {9, 9}]
t34 = ConstantArray[0, {9, 9}]
t41 = ConstantArray[0, {9, 9}]
t42 = ConstantArray[0, {9, 9}]
t43 = ConstantArray[0, {9, 9}]
t44 = ConstantArray[0, {9, 9}]
T3 = ArrayFlatten[{{t11, t12, t13, t14}, {t21, t22, t23, t24}, {t31, t32, t33, t34}, {t41, t42, t43, t44}}]
Eigenvalues[T3]
```

$$\left\{3, \frac{1327+2\sqrt{200209}}{2187}, 1, \frac{1}{3}, \frac{25}{81}, \frac{1327-2\sqrt{200209}}{2187}, \frac{1}{9}, \frac{1}{27}, -\frac{1}{81}, -\frac{1}{81}, \frac{1}{81}, \frac{1}{81}, \frac{1}{81}, \frac{1}{243}, \frac{1}{729}, \frac{1}{2187}, \frac{1}{6561}, \frac{1}{6561}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right\}$$

To determine which eigenvalues correspond to classes of the cohomology, we need to find a set of basis elements. As in section 7.4.2, we denote the vectors associated with e_1, e_2, e_3, e_4 as v_1, v_2, v_3, v_4, and the vectors associated with p_1, p_2, p_3, p_4 as w_1, w_2, w_3, w_4 as follows. In this way, the vector associated with $\gamma_\infty \cdot p_1$ is $\rho(\gamma_\infty)w_1$.

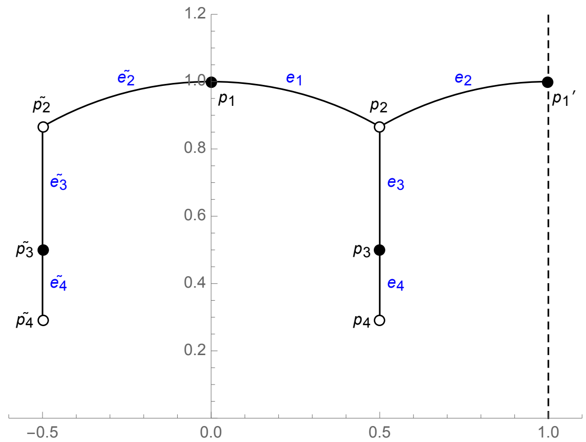


Figure 7.21: Dessin for $\Gamma_0(3)$ as Part of the Serre Tree

We have for a 1-form ϕ to be in $C^1_P(\Gamma_0(3), \mathcal{T}, V_k)$,

$$\begin{aligned}\phi(link_\infty) &= \phi(-e_1 + e_2) \\ &= -\phi(e_1) + \phi(e_2) \\ &= -f(p_1) + f(p_2) + f(p_1') - f(p_2) \\ &= f(\gamma_\infty \cdot p_1) - f(p_1) \\ &= \big(\rho(\gamma_\infty) - \mathrm{I}_{k-1}\big) \cdot f(p_1) \\ &\in \mathrm{Im}\,\big(\rho(\gamma_\infty) - \mathrm{I}_{k-1}\big)\ ,\end{aligned}$$

and

$$\begin{aligned}\phi(link_0) &= \phi(e_4 - e_3 + e_1 - \tilde{e_2} + \tilde{e_3} - \tilde{e_4}) \\ &= \phi(e_4) - \phi(e_3) + \phi(e_1) - \phi(\tilde{e_2}) + \phi(\tilde{e_3}) - \phi(\tilde{e_4}) \\ &= f(\tilde{p_4}) - f(p_4) \\ &= f(\gamma_0^{-1} \cdot p_4) - f(p_4) \\ &\in \mathrm{Im}\,\big(\rho(\gamma_0^{-1}) - \mathrm{I}_{k-1}\big)\end{aligned}$$

where

$$\gamma_\infty = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}.$$

Thus the last entry of $\operatorname{Im}\big(\rho(\gamma_\infty) - \mathrm{I}_{k-1}\big)$ is 0, and similarly the first entry of $\operatorname{Im}\big(\rho(\gamma_0^{-1}) - \mathrm{I}_{k-1}\big)$ is also 0. These combined give us

Similarly, for $\phi \in H^1_P(\Gamma_0(3), \mathcal{T}, V_k)$, we also need $\phi \in (\operatorname{Im}(d))^\perp$, which means $\langle \phi, df \rangle = 0 = \langle d^*\phi, f \rangle$, $\forall f \in C^0_P\big(\Gamma_0(3), \mathcal{T}, V_k\big)$. We need to find ϕ such that $d^*\phi = 0$. Recall the inner product on $C^0_P\big(\Gamma_0(3), \mathcal{T}, V_k\big)$ is

$$\langle \phi, \psi \rangle = \frac{1}{4} \sum_i^4 \langle \phi(e_i), \psi(e_i) \rangle$$

for $\phi, \psi \in C^1_P(\Gamma_0(3), \mathcal{T}, V_k)$. Thus we have

$$\begin{aligned}
\langle \phi, df \rangle &= \frac{1}{4} \sum_i^4 \langle \phi(e_i), df(e_i) \rangle \\
&= \frac{1}{4}\big(\langle \phi(e_1), df(e_1) \rangle + \langle \phi(e_2), df(e_2) \rangle + \langle \phi(e_3), df(e_3) \rangle + \langle \phi(e_4), df(e_4) \rangle \big) \\
&= \frac{1}{4}\big(\langle v_1, w_1 - w_2 \rangle + \langle v_2, \rho(\gamma_\infty) w_1 - w_2 \rangle + \langle v_3, w_3 - w_2 \rangle + \langle v_3, w_3 - w_2 \rangle \\
&\qquad + \langle v_4, w_3 - w_4 \rangle \big) \\
&= \frac{1}{4}\big(\langle v_1 + \rho(\gamma_\infty^*) v_2, w_1 \rangle + \langle -v_1 - v_2 - v_3, w_2 \rangle + \langle v_3 + v_4, w_3 \rangle - \langle v_4, w_4 \rangle \big) .
\end{aligned}$$

Since γ_∞ is a real matrix, γ_∞^* is the transpose of γ_∞. Thus ϕ satisfy

$$\begin{cases}
d^*\phi(p_1) & = v_1 + \rho(\gamma_\infty^*) v_2 \ , \\
d^*\phi(p_2) & = -v_1 - v_2 - v_3 \ , \\
d^*\phi(p_3) & = v_3 + v_4 \ , \\
d^*\phi(p_4) & = -v_4 \in \ker\big(\rho(\gamma_e) - \mathrm{I}_{k-1}\big) \ .
\end{cases}$$

The last relation also comes from the condition on elliptic points. The above relations can then be translated as

$$\begin{cases} v_1 + \rho(\gamma_\infty^*) v_2 = 0 \ , \\ v_1 + v_2 + v_3 = 0 \ , \\ v_3 + v_4 = 0 \ , \\ -v_4 \in \ker\left(\rho(\gamma_e) - \mathrm{I}_{k-1}\right)^\perp \ . \end{cases}$$

Combining the conditions for $\phi \in C_P^1(\Gamma_0(2), \mathcal{T}, V_8)$, we have for $\phi \in H_P^1(\Gamma_0(2), \mathcal{T}, V_8)$

$$\begin{cases} -v_1 + v_2 \in \mathrm{Im}\left(\rho(\gamma_\infty) - \mathrm{I}_{k-1}\right) \ , \\ v_1 - \rho(\gamma_\infty^{-1}) v_2 + \rho(\gamma_\infty^{-1}) v_3 - v_3 \in \mathrm{Im}\left(\rho(\gamma_0^{-1}) - \mathrm{I}_{k-1}\right) \ , \\ v_1 + \rho(\gamma_\infty^*) v_2 = 0 \ , \\ v_1 + v_2 \in \ker\left(\rho(\gamma_e) - \mathrm{I}_{k-1}\right)^\perp \end{cases} \tag{7.5.2}$$

where $\mathrm{Im}\left(\rho(\gamma_\infty) - \mathrm{I}_{k-1}\right)$ are vectors with 0 on the last row, and $\mathrm{Im}\left(\rho(\gamma_0^{-1}) - \mathrm{I}_{k-1}\right)$ are vectors with 0 on the first row. Implementing this in Mathematica, first we find the dimension of the kernel.

```
gammainfinv = Transpose[Table[Coefficient[(x)^(8 - i) (-x + y)^i, x^(8 - j) * y^j], {i, 0, 8}, {j, 0, 8}]]
gammainfstar = Transpose[gammainfinity]
NullSpace[gammae - IdentityMatrix[9]]

{{89, 960, 4200, 9450, 11151, 5670, 0, 0, 729},
 {-14, -159, -735, -1764, -2268, -1323, 0, 243, 0}, {4, 48, 237, 621, 918, 729, 243, 0, 0}}
```

Then we use conditions in (7.5.2) to compute a set of basis.

```
p1 = NullSpace[gammae - IdentityMatrix[9]][[1]]
p2 = NullSpace[gammae - IdentityMatrix[9]][[2]]
p3 = NullSpace[gammae - IdentityMatrix[9]][[3]]
v1 = {a1, a2, a3, a4, a5, a6, a7, a8, a9}
v2 = {b1, b2, b3, b4, b5, b6, b7, b8, b9}
s = Solve[{v1 + gammainfstar.v2 == ConstantArray[0, 9], (v1 + v2).p1 == ConstantArray[0, 9],
    (v1 + v2).p2 == ConstantArray[0, 9], (v1 + v2).p3 == ConstantArray[0, 9], -a9 + b9 == 0,
    a1 - gammainfinv[[1]].v2 + (gammainfinv - IdentityMatrix[9])[[1]].(-v1 - v2) +
      (IdentityMatrix[9] - gammainfinv)[[1]].(v1 + v2) == 0},
   {a1, a2, a3, a4, a5, a6, a7, a8, a9, b1, b2, b3, b4, b5, b6, b7, b8, b9}]
```

$$\Big\{\Big\{a5 \to -\frac{459\,a1}{4298}+\frac{2413\,a2}{6447}-\frac{2032\,a3}{2149}+\frac{1626\,a4}{2149},\ a6 \to -\frac{16\,a2}{243}-\frac{23\,a4}{27},\ a7 \to \frac{255\,a1}{2149}-\frac{24\,130\,a2}{58\,023}+\frac{2019\,a3}{2149}-\frac{5420\,a4}{6447},$$
$$a8 \to \frac{23\,a2}{243}+\frac{28\,a4}{27},\ a9 \to \frac{484\,a1}{307}-\frac{353\,813\,a2}{74\,601}+\frac{1824\,a3}{307}-\frac{53\,182\,a4}{8289},\ b1 \to -a1,\ b2 \to a1-a2,$$
$$b3 \to -a1+2\,a2-a3,\ b4 \to a1-3\,a2+3\,a3-a4,\ b5 \to -\frac{3839\,a1}{4298}+\frac{23\,375\,a2}{6447}-\frac{10\,862\,a3}{2149}+\frac{6970\,a4}{2149},$$
$$b6 \to \frac{2003\,a1}{4298}-\frac{1\,599\,386\,a2}{522\,207}+\frac{11\,330\,a3}{2149}-\frac{311\,293\,a4}{58\,023},\ b7 \to \frac{2077\,a1}{4298}+\frac{70\,771\,a2}{174\,069}-\frac{3774\,a3}{2149}+\frac{84\,716\,a4}{19\,341},$$
$$b8 \to -\frac{1171\,a1}{614}+\frac{333\,979\,a2}{74\,601}-\frac{1694\,a3}{307}+\frac{20\,300\,a4}{8289},\ b9 \to \frac{484\,a1}{307}-\frac{353\,813\,a2}{74\,601}+\frac{1824\,a3}{307}-\frac{53\,182\,a4}{8289}\Big\}\Big\}$$

In the solution there are 4 degrees of freedom, which matches the fact that $\dim(H^1_P(\Gamma_0(3), \mathcal{T}, V_{10})) = 4$. We now can find a set of basis elements.

```
s1 = s /. {a1 :> 1, a2 :> 0, a3 :> 0, a4 :> 0}
u1 = (v1 /. s1) /. {a1 :> 1, a2 :> 0, a3 :> 0, a4 :> 0} //. {x_List} :> x
u2 = ( v2 /. s1) //. {x_List} :> x
u3 = - (u1 + u2)
u4 = -u3
s2 = s /. {a1 :> 0, a2 :> 1, a3 :> 0, a4 :> 0}
w1 = ( v1 /. s2) /. {a1 :> 0, a2 :> 1, a3 :> 0, a4 :> 0} //. {x_List} :> x
w2 = (v2 /. s2) //. {x_List} :> x
w3 = - (w1 + w2)
w4 = -w3
s3 = s /. {a1 :> 0, a2 :> 0, a3 :> 1, a4 :> 0}
x1 = ( v1 /. s3) /. {a1 :> 0, a2 :> 0, a3 :> 1, a4 :> 0} //. {x_List} :> x
x2 = (v2 /. s3) //. {x_List} :> x
x3 = - (x1 + x2)
x4 = -x3
s4 = s /. {a1 :> 0, a2 :> 0, a3 :> 0, a4 :> 1}
y1 = v1 /. s4 /. {a1 :> 0, a2 :> 0, a3 :> 0, a4 :> 1} //. {x_List} :> x
y2 = v2 /. s4 //. {x_List} :> x
y3 = - (y1 + y2)
y4 = -y3
```

We then orthogonalize the bases and apply the T_3 operator.

```
colvecu = Join[u1, u2, u3, u4]
colvecw = Join[w1, w2, w3, w4]
colvecx = Join[x1, x2, x3, x4]
colvecy = Join[y1, y2, y3, y4]
phi1 = colvecu
phi2 = colvecw - (colvecw.phi1 / phi1.phi1) * phi1
phi3 = colvecx - (colvecx.phi1 / phi1.phi1) * phi1 - (colvecx.phi2 / phi2.phi2) * phi2
phi4 = colvecy - (colvecy.phi1 / phi1.phi1) * phi1 - (colvecy.phi2 / phi2.phi2) * phi2 - (colvecy.phi3 / phi3.phi3) * phi3
tphi1 = T3.phi1
tphi2 = T3.phi2
tphi3 = T3.phi3
tphi4 = T3.phi4
```

Again, we can use the conditions in 7.5.2 to see that $T_3\phi_1, \cdots, T_3\phi_4$ lie in $C^1_P(\Gamma_0(3), \mathcal{T}, V_{10})$ but not in $\ker(d^*)$. We use projection unto $\ker(d^*)$, and compute the eigenvalues of T_3 on cohomology.

```
c11 = tphi1.phi1 / (phi1.phi1)
c12 = tphi2.phi1 / (phi1.phi1)
c13 = tphi3.phi1 / (phi1.phi1)
c14 = tphi4.phi1 / (phi1.phi1)
c21 = tphi1.phi2 / (phi2.phi2)
c22 = tphi2.phi2 / (phi2.phi2)
c23 = tphi3.phi2 / (phi2.phi2)
c24 = tphi4.phi2 / (phi2.phi2)
c31 = tphi1.phi3 / (phi3.phi3)
c32 = tphi2.phi3 / (phi3.phi3)
c33 = tphi3.phi3 / (phi3.phi3)
c34 = tphi4.phi3 / (phi3.phi3)
c41 = tphi1.phi4 / (phi4.phi4)
c42 = tphi2.phi4 / (phi4.phi4)
c43 = tphi3.phi4 / (phi4.phi4)
c44 = tphi4.phi4 / (phi4.phi4)
mat44 = {{c11, c12, c13, c14}, {c21, c22, c23, c24}, {c31, c32, c33, c34}, {c41, c42, c43, c44}}
Eigenvalues[mat44]
```

$$\left\{-\frac{1}{81}, -\frac{1}{81}, \frac{1}{81}, \frac{1}{81}\right\}$$

Thus the eigenvalues for T_3 on the cohomology $H^1_P(\Gamma_0(3), \mathcal{T}, V_{10})$ is $\pm\frac{1}{81}$, or $\pm\frac{1}{3^8}$.

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Chapter 8

Hecke Correspondence via Atkin-Lehner Involution

To realize the Hecke operator T_p via the Hecke Correspondence, through our investigation, there is an alternate approach by Atkin-Lehner Involution [17]. We start with the following definition.

Definition 8.0.1. Given $N \in \mathbf{N}$, let $Q \| N$ be a prime power dividing N such that $(Q, N/Q) = 1$. Let $x, y, z, t \in \mathbf{Z}$ be such that the determinant of

$$w_Q = \begin{pmatrix} Qx & y \\ Nz & Qt \end{pmatrix}$$

is Q. Then w_Q defines an Atkin-Lehner involution of the modular curve $X_0(N)$ which does not depend on the choice of x, y, z and t.

The Hecke correspondence can then be realized as

$$\begin{array}{ccc} \Gamma_0(pN) \setminus \mathfrak{h} & \xrightarrow{w_Q} & \Gamma_0(pN) \setminus \mathfrak{h} \\ \pi \downarrow & & \downarrow \pi_g \\ \Gamma_0(N) \setminus \mathfrak{h} & & \Gamma_0(N) \setminus \mathfrak{h} \end{array}$$

where w_Q acts by conjugation on $\Gamma_0(N)$.

8.1 Proof that Atkin-Lehner Involution Yields the Same Hecke Correspondence

In this section, we will show that when $(p, N) = 1$, the two maps

$$\textcircled{1} \qquad\qquad\qquad\qquad\qquad\qquad \textcircled{2}$$

$$\begin{array}{c} \Gamma_0(pN)\backslash\mathfrak{h} \; \circlearrowleft^{w_p} \\ \big\uparrow\big\downarrow \\ \Gamma_0(N)\backslash\mathfrak{h} \end{array} \qquad \& \qquad \begin{array}{ccc} \Gamma_0(pN)\backslash\mathfrak{h} & \xrightarrow{g} & g\Gamma_0(pN)g^{-1}\backslash\mathfrak{h} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi_g} \\ \Gamma_0(N)\backslash\mathfrak{h} & & \Gamma_0(N)\backslash\mathfrak{h} \end{array}$$

are isomorphic.

In the upper half plane, the action of each map is

$$\begin{array}{cccc} & \mathfrak{h} & \longrightarrow & \mathfrak{h} \\ \textcircled{1} & \tau & \mapsto & w_p\tau \\ \textcircled{2} & \tau & \mapsto & g\tau \end{array}$$

where $w_p = \begin{pmatrix} px & y \\ pNz & pt \end{pmatrix}$, $g = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$, and $\det(w_p) = \det(g) = p$. We have

$$\begin{aligned} \det(w_p) &= p^2xw - pNyz \\ &= p(pxw - Nyz) \ . \end{aligned}$$

Thus we have $(pxw - Nyz) = 1$. We want to show that

$$\begin{aligned} & w_p\tau = \Gamma_0(N)g\tau \\ \Rightarrow \quad & w_p \in \Gamma_0(N)g \\ \Rightarrow \quad & w_p \cdot g^{-1} \in \Gamma_0(N) \ . \end{aligned}$$

We can compute

$$\begin{aligned} w_p \cdot g^{-1} &= \begin{pmatrix} px & y \\ pNz & pt \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} px & y \\ pNz & pt \end{pmatrix} \begin{pmatrix} \frac{1}{p} & 0 \\ 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} x & y \\ Nz & pw \end{pmatrix}^{-1} \ . \end{aligned}$$

We also have

$$\det({}_p \cdot g^{-1}) = pwx - Nyz = 1$$

and $Nz \equiv 0 \mod N$. Therefore

$$w_p \cdot g^{-1} \in \Gamma_0(N) \ .$$

8.2 Example of T_3 with Base Curve $X_0(2)$

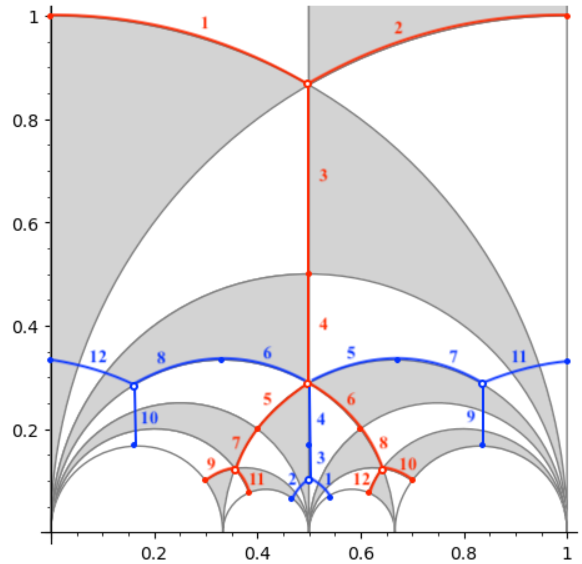
We can also think about the correspondence as follows:

$$\begin{array}{c} \Gamma_0(6) \setminus \mathfrak{h} \ \circlearrowleft \ w_Q \\ \uparrow\downarrow \\ \Gamma_0(2) \setminus \mathfrak{h} \end{array}$$

where w_Q is an Atkin-Lehner involution of the modular curve $X_0(6)$. $N = 6$, and we take $Q = 3$ and pick the matrix for w_3 to be

$$w_3 = \begin{pmatrix} 3 & 1 \\ 6 & 3 \end{pmatrix}$$

with determinant 3. The embedded dessin for $\Gamma_0(6)$ before and after the involution is depicted in the following figure:



where the blue edges form the dessin for $\Gamma_0(6)$ after the involution and the red edges form the regular dessin for $\Gamma_0(6)$. We will call the edges for the red dessin $r_1, r_2, \cdots, r_{12}$ and the edges for the blue dessin $b_1, b_2, \cdots, b_{12}$. Note that r_1, r_2, r_3 also form the set of edges for the dessin for $\Gamma_0(2)$, and we will label them e_1, e_2, and e_3 respectively when referring to $\Gamma_0(2)$.

As before, we start from $\Gamma_0(2)$, move to the blue dessin, then the red dessin, then back to $\Gamma_0(2)$. We have the coset representative for $\Gamma_0(6) \setminus \Gamma_0(2)$ are

$$\overset{\mathrm{I}_2}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}, \quad \overset{\gamma_1}{\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}}, \quad \overset{\gamma_2}{\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}}, \quad \overset{\gamma_3}{\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}},$$

and the generators of $\Gamma_0(6)$ are

$$\overset{\gamma_x}{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}, \quad \overset{\gamma_y}{\begin{pmatrix} 5 & -1 \\ 6 & -1 \end{pmatrix}}, \quad \overset{\gamma_z}{\begin{pmatrix} 7 & -3 \\ 12 & -5 \end{pmatrix}}.$$

Let $\phi \in C^1_P(\Gamma_0(2), \mathcal{T}, V_k)$ and $\psi \in C^1_P(\Gamma_0(6), \mathcal{T}, V_k)$, through deformation retract

r, we have

$$\begin{aligned}
\psi(b_1) &= \psi(r_6) - \psi(r_8) + \psi(r_{12}) \\
&= \rho(\gamma_z\gamma_3)\phi(e_1) - \rho(\gamma_z\gamma_2)\phi(e_2) + \rho(\gamma_z\gamma_2)\phi(e_1)\ , \\
\psi(b_2) &= \psi(r_5) - \psi(r_7) + \psi(r_{11}) \\
&= \rho(\gamma_z\gamma_3)\phi(e_2) - \rho(\gamma_1)\phi(e_1) + \rho(\gamma_1)\phi(e_2)\ , \\
\psi(b_3) &= \psi(b_4) = \psi(b_5) = \psi(b_6) = \psi(b_7) = \psi(b_8) = 0\ , \\
\psi(b_9) &= \psi(r_6) - \psi(r_8) + \psi(r_{10}) \\
&= \rho(\gamma_z\gamma_3)\phi(e_1) - \rho(\gamma_z\gamma_2)\phi(e_2) + \rho(\gamma_z\gamma_2)\phi(e_3)\ , \\
\psi(b_{10}) &= \psi(r_5) - \psi(r_7) + \psi(r_9) \\
&= \rho(\gamma_z\gamma_3)\phi(e_2) - \rho(\gamma_1)\phi(e_1) + \rho(\gamma_1)\phi(e_3)\ , \\
\psi(b_{11}) &= \psi(r_4) - \psi(r_3) + \psi(r_2) \\
&= \rho(\gamma_z\gamma_3)\phi(e_3) - \phi(e_3) + \phi(e_2)\ , \\
\psi(b_{12}) &= \psi(r_4) - \psi(r_3) + \psi(r_1) \\
&= \rho(\gamma_z\gamma_3)\phi(e_3) - \phi(e_3) + \phi(e_1)\ .
\end{aligned}$$

To go from 1-forms associated with the blue edges to the 1-forms associated with the red edges, we have

$$\psi(r_1) = \rho(w_3^{-1}\gamma_z^{-1})\psi(b_1) \ ,$$

$$\psi(r_2) = \rho(w_3^{-1}\gamma_z^{-1})\psi(b_2) \ ,$$

$$\psi(r_3) = \rho(w_3^{-1}\gamma_z^{-1})\psi(b_3) \ ,$$

$$\psi(r_4) = \rho(w_3^{-1}\gamma_z^{-1})\psi(b_4) \ ,$$

$$\psi(r_5) = \rho(w_3^{-1}\gamma_z^{-1})\psi(b_5) \ ,$$

$$\psi(r_6) = \rho(w_3^{-1}\gamma_z^{-1})\psi(b_6) \ ,$$

$$\psi(r_7) = \rho(w_3^{-1}\gamma_z^{-1})\psi(b_7) \ ,$$

$$\psi(r_8) = \rho(w_3^{-1}\gamma_z^{-1})\psi(b_8) \ ,$$

$$\psi(r_9) = \rho(w_3^{-1}\gamma_z^{-1})\psi(b_9) \ ,$$

$$\psi(r_{10}) = \rho(w_3^{-1}\gamma_z^{-1})\psi(b_{10}) \ ,$$

$$\psi(r_{11}) = \rho(w_3^{-1}\gamma_z^{-1})\psi(b_{11}) \ ,$$

$$\psi(r_{12}) = \rho(w_3^{-1}\gamma_z^{-1})\psi(b_{12}) \ .$$

Lastly, we use the trace operator to pull back 1-forms $\psi \in C^1_P(\Gamma_0(6), \mathcal{T}, V_k)$ to 1-forms $\phi \in C^1_P(\Gamma_0(2), \mathcal{T}, V_k)$. We have

$$\begin{aligned}
\pi_*(\psi)(e_1) = &= \rho(\mathrm{I}_2^{-1})\psi(\mathrm{I}_2\, e_1) + \rho(\gamma_1^{-1})\psi(\gamma_1 e_1) + \rho(\gamma_2^{-1})\psi(\gamma_2 e_1) + \rho(\gamma_3^{-1})\psi(\gamma_3 e_1) \\
&= \psi(r_1) + \rho(\gamma_1^{-1})\psi(r_7) + \rho(\gamma_2^{-1}\gamma_z^{-1})\psi(r_{12}) + \rho(\gamma_3^{-1}\gamma_z^{-1})\psi(r_6) \\
&= \left(\rho(w_3^{-1}\gamma_3) + \rho(w_3^{-1}\gamma_2) + \rho(\gamma_2^{-1}\gamma_z^{-1}w_3^{-1}\gamma_z^{-1})\right)\phi(e_1) \\
&\quad - \rho(w_3^{-1}\gamma_2)\phi(e_2) \\
&\quad + \left(\rho(\gamma_2^{-1}\gamma_z^{-1}w_3^{-1}\gamma_3) - \rho(\gamma_2^{-1}\gamma_z^{-1}w_3^{-1}\gamma_z^{-1})\right)\phi(e_3) \ ,
\end{aligned}$$

$$\pi_*(\psi)(e_2) = \rho(\mathrm{I}_2^{-1})\psi(\mathrm{I}_2\, e_2) + \rho(\gamma_1^{-1})\psi(\gamma_1 e_2) + \rho(\gamma_2^{-1})\psi(\gamma_2 e_2) + \rho(\gamma_3^{-1})\psi(\gamma_3 e_2)$$
$$= \psi(r_2) + \rho(\gamma_1^{-1})\psi(r_{11}) + \rho(\gamma_2^{-1}\gamma_z^{-1})\psi(r_8) + \rho(\gamma_3^{-1}\gamma_z^{-1})\psi(r_5)$$
$$= -\rho(w_3^{-1}\gamma_z^{-1}\gamma_1)\phi(e_1)$$
$$+ \left(\rho(w_3^{-1}\gamma_3) + \rho(w_3^{-1}\gamma_z^{-1}\gamma_1) + \rho(\gamma_1^{-1}w_3^{-1}\gamma_z^{-1})\right)\phi(e_2)$$
$$+ \left(\rho(\gamma_1^{-1}w_3^{-1}\gamma_3) - \rho(\gamma_1^{-1}w_3^{-1}\gamma_z^{-1})\right)\phi(e_3)\ ,$$

$$\pi_*(\psi)(e_3) = \rho(\mathrm{I}_2^{-1})\psi(\mathrm{I}_2\, e_3) + \rho(\gamma_1^{-1})\psi(\gamma_1 e_3) + \rho(\gamma_2^{-1})\psi(\gamma_2 e_3) + \rho(\gamma_3^{-1})\psi(\gamma_3 e_3)$$
$$= \psi(r_3) + \rho(\gamma_1^{-1})\psi(r_9) + \rho(\gamma_2^{-1}\gamma_z^{-1})\psi(r_{10}) + \rho(\gamma_3^{-1}\gamma_z^{-1})\psi(r_4)$$
$$\left(\rho(\gamma_1^{-1}w_3^{-1}\gamma_3) - \rho(\gamma_2^{-1}\gamma_z^{-1}w_3^{-1}\gamma_z^{-1}\gamma_1)\right)\phi(e_1)$$
$$+ \left(\rho(-\gamma_1^{-1}w_3^{-1}\gamma_2) + \rho(\gamma_2^{-1}\gamma_z^{-1}w_3^{-1}\gamma_3)\right)\phi(e_2)$$
$$+ \left(\rho(\gamma_1^{-1}w_3^{-1}\gamma_2) + \rho(\gamma_2^{-1}\gamma_z^{-1}w_3^{-1}\gamma_z^{-1}\gamma_1)\right)\phi(e_3)\ .$$

We implement the computation in Mathematica for weight $k = 8$. First, we find the representation of the involved matrices.

```
gamma1 = Transpose[Table[Coefficient[(x + 2 y) ^ (6 - i) (y) ^ i, x ^ (6 - j) * y ^ j], {i, 0, 6}, {j, 0, 6}]]
gamma2 = Transpose[Table[Coefficient[(x + 2 y) ^ (6 - i) (x + 3 y) ^ i, x ^ (6 - j) * y ^ j], {i, 0, 6},
   {j, 0, 6}]]
gamma3 = Transpose[Table[Coefficient[(x + 2 y) ^ (6 - i) (2 x + 5 y) ^ i, x ^ (6 - j) * y ^ j],
   {i, 0, 6}, {j, 0, 6}]]
gamma1inv = Transpose[Table[Coefficient[(x - 2 y) ^ (6 - i) (y) ^ i, x ^ (6 - j) * y ^ j], {i, 0, 6}, {j, 0, 6}]]
gamma2inv =
 Transpose[Table[Coefficient[(3 x - 2 y) ^ (6 - i) (-x + y) ^ i, x ^ (6 - j) * y ^ j], {i, 0, 6}, {j, 0, 6}]]
gamma3inv =
 Transpose[Table[Coefficient[(5 x - 2 y) ^ (6 - i) (-2 x + y) ^ i, x ^ (6 - j) * y ^ j], {i, 0, 6}, {j, 0, 6}]]
gammazinv =
 Transpose[Table[Coefficient[(-5 x - 12 y) ^ (6 - i) (3 x + 7 y) ^ i, x ^ (6 - j) * y ^ j], {i, 0, 6}, {j, 0, 6}]]
w3inv = Transpose[Table[Coefficient[(x - 2 y) ^ (6 - i) (-1/3 x + y) ^ i, x ^ (6 - j) * y ^ j], {i, 0, 6},
   {j, 0, 6}]]
```

With these, we can write out the coefficients of the trace operator with $\phi(e_1), \phi(e_2)$ and $\phi(e_3)$ as basis.

```
t11 = w3inv.(gamma3 + gamma2) + gamma2inv.gammazinv.w3inv.gammazinv
t12 = -w3inv.gamma2
t13 = gamma2inv.gammazinv.w3inv.(gamma3 - gammazinv)
t21 = -w3inv.gammazinv.gamma1
t22 = w3inv.(gamma3 + gammazinv.gamma1) + gamma1inv.w3inv.gammazinv
t23 = gamma1inv.w3inv.(gamma3 - gammazinv)
t31 = gamma1inv.w3inv.gamma3 - gamma2inv.gammazinv.w3inv.gammazinv.gamma1
t32 = -gamma1inv.w3inv.gamma2 + gamma2inv.gammazinv.w3inv.gamma3
t33 = gamma1inv.w3inv.gamma2 + gamma2inv.gammazinv.w3inv.gammazinv.gamma1
```

These coefficients form the entries for the matrix representation of the cohomology associated Hecke operator T_3.

```
tr = ArrayFlatten[{{t11, t12, t13}, {t21, t22, t23}, {t31, t32, t33}}]
Eigenvalues[tr]
```

$$\left\{\frac{2188}{729}, \frac{2188}{729}, \frac{1}{243}\left(155+\sqrt{12361}\right), \frac{730}{729}, \frac{730}{729}, \frac{730}{729}, \frac{1}{81}\left(-25-\sqrt{3001}\right), \frac{1}{81}\left(25+\sqrt{2785}\right), \frac{1}{81}\left(-25+\sqrt{3001}\right), \frac{1}{81}\left(25-\sqrt{2785}\right), \frac{82}{243}, \frac{82}{243}, \frac{82}{243}, \frac{2}{9}, \frac{1}{243}\left(155-\sqrt{12361}\right), \frac{10}{81}, \frac{10}{81}, \frac{10}{81}, \frac{2}{27}, \frac{4}{243}, \frac{4}{243}\right\}$$

Here we obtained the same matrix with the same eigenvalues as in section 7.5.1. Following the same techniques as in that section, we again obtain the eigenvalue for the Hecke operator T_3 on cohomology with weight 8 is $\dfrac{4}{243}$.

Chapter 9

Conclusions

In this dissertation we developed an algorithm to calculate a) a parabolic cohomology group of the dessin $\Gamma_0(N) \setminus \mathcal{T}$ that represents $\Gamma_0(N)$-cusp forms, and b) the action of Hecke operators $T_p (p \nmid N)$ on it. Although the Hecke operators come from correspondences on $\Gamma_0(N) \setminus \mathfrak{h}$ and hence act on the cohomology of $\Gamma_0(N) \setminus \mathfrak{h}$, they do not directly restrict to correspondences of $\Gamma_0(N) \setminus \mathcal{T}$. As a result our algorithm for (b) is most clear for $p < 7$; we hope to remove this restriction. A further hope is to apply these techniques to study analogues of Hecke operators on the dessins of more general algebraic curves.

We first considered the cohomology $H^\bullet(\Gamma \backslash \mathfrak{h}, \mathbf{V})$ where $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ is a finite-index subgroup and $\mathbf{V}$ is a local coefficient sheaf arising from a representation $\rho \colon \mathrm{SL}_2(\mathbf{R}) \longrightarrow \mathrm{GL}(V)$. We view this as the cohomology of a complex of V-valued differential forms $A^\bullet(\Gamma, \mathfrak{h}, V)$ on the upper-half plane which satisfy $L_\gamma^* \phi = \rho(\gamma)\phi$ for $\gamma \in \Gamma$ as in [22]. The Hecke correspondence induces an action of T_p on $A^1(\Gamma, \mathfrak{h}, V)$ and hence on $H^1(\Gamma \setminus \mathfrak{h}, \mathbf{V}_k)$. Our first result (Theorem 7.3.7) is that the action of T_p on the cohomology classes of cusp forms corresponds to the multiple P^{k-2} times the action of T_p on cusp forms. (For simplicity we restrict attention to $\Gamma = \Gamma_0(N)$ and $p \nmid N$.)

Eichler and Shimura constructed [24] a cohomology group based on V_k-valued group cochains of Γ and proved that it was isomorphic to $S_k(\Gamma) \oplus \overline{S_k(\Gamma)}$. For our eventual applications to dessins, we worked with differential forms. We imposed conditions on forms in order to define two subcomplexes of $A^\bullet(\Gamma, \mathfrak{h}, V_k)$ and their corresponding cohomologies: parabolic cohomology $H^\bullet_P(\Gamma, \mathfrak{h}, V_k)$ and L^2- cohomology $H^\bullet_{(2)}(\Gamma, \mathfrak{h}, V_k)$. The differential form version of parabolic cohomology we constructed inspired the construction of combinatorial cohomology on the dessins. On the other hand, the L^2 cohomology provided a Hecke-equivalent Eichler-Shimura isomorphism (Theorem 4.2.1, compare [32, §12]), and we proved there is a Hecke-equivariant isomorphism of parabolic and L^2 cohomology (chapter 6).

We then applied the (normalized) j-function as a Belyi function for $\Gamma_0(N) \setminus \mathfrak{h}^*$ and the corresponding dessin can be realized on the quotient $\Gamma_0(N) \setminus \mathcal{T}$ of the Serre tree $\mathcal{T} \subset \mathfrak{h}$. We calculated the cohomology $H^\bullet(\Gamma_0(N) \setminus \mathcal{T}, \mathbf{V}) \cong H^\bullet(\Gamma_0(N) \setminus \mathfrak{h}, \mathbf{V})$ from a finite dimensional complex $C^\bullet(\Gamma_0(N), \mathcal{T}, V)$ of simplicial cochains.

We constructed in section 7.2 a subcomplex $C^\bullet_P(\Gamma_0(N), \mathcal{T}, V) \subset C^\bullet(\Gamma_0(N), \mathcal{T}, V)$ (inspired by our differential form construction) whose cohomology is isomorphic to $H^\bullet_P(\Gamma_0(N), \mathfrak{h}, V)$ and described the Hecke action via a computable action on the finite-dimensional complex $C^\bullet_P(\Gamma_0(N), \mathcal{T}, V)$. This however is difficult since the Hecke correspondence of $\Gamma_0(N) \setminus \mathfrak{h}$ does <u>not</u> preserve $\Gamma_0(N) \setminus \mathcal{T}$.

In section 7.3 we nonetheless determined an algorithm for calculating T_p on $C^\bullet_P(\Gamma_0(N), \mathcal{T}, V)$, at least for $p = 2, 3$ and 5. We used a $\mathrm{SL}_2(\mathbf{Z})$-equivariant deformation retract $\mathfrak{h} \longrightarrow \mathcal{T}$ to overcome the difficulty noted above. The restriction to $p < 7$ was needed at one point to easily obtain a simplicial action; hopefully this restriction can be removed.

A significant portion of the latter part of this dissertation (section 7.4, 7.5, and 8) is devoted to providing the readers with ample examples on studying the Hecke action on cohomology of dessins with coefficients. Main tools used are Sage for getting necessary information on congruent subgroups in order to build their corresponding dessins, and Mathematica for computing eigenvalues for T_p for different values of weight k. Both formulas and Mathematica codes are provided.

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