

Spontaneous Pair Creation revisited

Pickl, P.* and Dürr, D.†

*Mathematisches Institut der Universität München
Theresienstr. 39
80333 München
Germany*

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Recently the so called Spontaneous Pair Creation of electron positron pairs in a strong external field has been rigorously established. We give here the heuristic core of the proof, since the results differ from those given in earlier works.

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I. INTRODUCTION

The creation of an electron positron pair in an almost stationary very strong external electromagnetic field has been called—unfortunately misleading—spontaneous pair creation (SPC) ([3] - [10]). The phenomenon emerges straight forwardly from the Dirac sea interpretation of negative energy states: An adiabatically increasing field lifts a particle from the sea to the positive energy subspace where it scatters and when the potential is gently switched off one has one free electron and one unoccupied state—a hole—in the sea. The experimental verification has been sought in heavy ion collisions, but without success so far [11], [12]. A rigorous proof of the existence of SPC has been lacking until recently [13]. Surprisingly our rigorous proof—the heuristics of which we give here—yields time estimates different from what has been reported earlier and which may be relevant for experiments. The rigorous result concerns the so called external field problem, i.e. interactions between the charges are neglected. Vacuum polarization will in general perturb the external field (see [14] for a rigorous attempt) and one may think of the external field as an effective field. The description of SPC in second quantized external field Dirac theory is known to be equivalent to the existence of certain type of solutions of the one particle Dirac equation. We shall only discuss the latter.

II. FORMULATION OF THE PROBLEM

Consider the one particle Dirac equation with external electric potential A , a real valued multiple of the 4×4 unit matrix. Amc^2 gives the potential in the units eV .

On microscopic timescales $\tau = \frac{mc^2}{\hbar}t$ the equation reads

$$\begin{aligned} i\frac{\partial\psi_\tau}{\partial\tau} &= -i\frac{\hbar}{mc}\sum_{l=1}^3\alpha_l\partial_l\psi_\tau + A_{\varepsilon\tau}(\mathbf{x})\psi_\tau + \beta\psi_\tau \\ &\equiv (D^0 + A_{\varepsilon\tau}(\mathbf{x})\psi_\tau = D_{\varepsilon\tau}\psi_\tau \end{aligned} \quad (1)$$

where ε is a parameter representing the slow time variation of the external potential.

We introduce in (1) the macroscopic time scale $s = \varepsilon\tau$:

$$i\frac{\partial\psi_s}{\partial s} \equiv \frac{1}{\varepsilon}D_s\psi_s. \quad (2)$$

We will describe pair creation using the Dirac sea interpretation of the Dirac equation. The spectrum of the Dirac equation without external field is $(-\infty, -1] \cap [1, \infty)$. Wave functions which lie in the corresponding positive energy subspace of the free Dirac operator are interpreted as wave functions of particles or electrons. The Dirac sea is a many particle wave function built out of wave functions of negative energy (states) of the free Dirac operator. In the so called vacuum all states in the sea are occupied by particles. “Holes” in the Dirac sea are unoccupied states which are interpreted as anti-particles or positrons.

Suppose now that the potential is adiabatically switched on and later switched off. When the potential is non zero there may be bound states in the gap, $[-1, 1]$. The eigenvalues of these bound states change with the strength of the potential. The adiabatic theorem (see e.g. [15]) ensures that there is no tunnelling across spectral gaps. So as long as all bound states are isolated ζ from the upper continuous spectrum, transitions from the negative to the positive energy subspace are adiabatically not possible and the probability of creating a pair is zero (undercritical case). When the external field becomes overcritical (at time $s_c = 0$), i.e. when a bound state vanishes in the upper energy subspace, there exists a solution of the Dirac equation which follows the path of this bound state from the negative to the positive energy subspace. If the wave function (or a part of it) does not “follow” the bound state back into the negative energy subspace when the potential is switched off again, but stays in the positive energy subspace, there will be a hole

*Electronic address: pickl@mathematik.uni-muenchen.de

†Electronic address: duerr@mathematik.uni-muenchen.de

in the sea and a freely moving particle: SPC is achieved. The scenario is symmetric under change of sign of the potential: It then transports an unoccupied state (a hole) from the positive energy subspace to the sea and catches a particle from the sea when it is switched off. The hole (positron) then scatters.

To show to what extent the scenario in fact holds one must control the propagation of the wave function emerging from the bound state during over-criticality. The wave function will generically not be a nice scattering state, i.e. its momentum distribution will be large for small momenta, that is, the wave function lingers around the range of the potential rather than moving away. We shall describe now the heuristic core of the situation when SPC happens.

III. THE SPECTRUM OF THE DIRAC OPERATOR

We expand the wave function in generalized eigenfunctions which by themselves are time dependent. We shall need the eigenfunctions for times σ close to the critical time. Consider the eigenvalue equation

$$E\varphi = D_\sigma\varphi \quad (3)$$

for fixed $\sigma \in \mathbb{R}$. The continuous subspace determined by $\mathbf{k} \in \mathbb{R}^3 \setminus \{0\}$ or $\mathbf{k} = 0$ and $\sigma \neq 0$ is spanned by generalized eigenfunctions (not square integrable) $\varphi^j(\mathbf{k}, \sigma, \mathbf{x})$, $j = 1, 2, 3, 4$, with energy $E = \pm E_k = \pm\sqrt{k^2 + 1}$, $j = 1, 2$ being solutions with positive energy. For ease of notation we will drop the spin index j in what follows.

The generalized eigenfunctions also solve the Lippmann Schwinger equation

$$\begin{aligned} \varphi(\sigma, \mathbf{k}, \mathbf{x}) &= \varphi_0(\mathbf{k}, \mathbf{x}) \\ &+ \int G_k^+(\mathbf{x} - \mathbf{x}') A_\sigma(\mathbf{x}') \varphi(\sigma, \mathbf{k}, \mathbf{x}') d^3x', \end{aligned} \quad (4)$$

with $\varphi_0(\mathbf{k}, \mathbf{x}) = \xi(\mathbf{k}) e^{i\frac{m\epsilon}{\hbar}\mathbf{k}\cdot\mathbf{x}}$, the generalized eigenfunctions of the free Dirac operator D^0 , i.e. G_k^+ is the kernel of $(E_k - D^0)^{-1} = \lim_{\delta \rightarrow 0} (E_k - D^0 + i\delta)^{-1}$ [16].

Introducing the operator T_σ^k

$$T_\sigma^k f = \int G_k^+(\mathbf{x} - \mathbf{x}') A_\sigma(\mathbf{x}') f(\mathbf{x}') d^3x', \quad (5)$$

(4) becomes

$$(1 - T_\sigma^k)\varphi(\sigma, \mathbf{k}, \cdot) = \varphi_0(\mathbf{k}, \cdot). \quad (6)$$

There may also exist resonance-eigenfunctions which are not square integrable but which decay as $x \rightarrow \infty$.

Finally there may exist eigenfunctions, i.e. bound states $\Phi_\sigma(\kappa)$ of (3) with energies $E = \sqrt{1 - \kappa^2} \in [-1, 1]$. They satisfy instead of (6)

$$(1 - T_{\sigma(\kappa)}^\kappa)\Phi_{\sigma(\kappa)} = 0, \quad (7)$$

with imaginary $k = i\kappa$. In the following we will assume that A is such that for each time σ there exist either no or one (degenerate) bound state Φ_σ and that the bound state vanishes in the positive continuum at $\sigma = 0$. Since there generically exists a bound state Φ with energy 1 (i.e. $\mathbf{k} = 0$) at time $\sigma = 0$ and $\partial_\sigma E_\sigma|_{\sigma=0} \neq 0$ (see [17] and our argument below), we will only discuss that case.

Using (7) for $\sigma = \kappa = 0$ observing the explicit form of G_0^+ (see e.g. [18]) one can easily estimate Φ for large x . Assuming Φ bounded, a term falling off like x^{-1} appears. Using that this term has to be equal to zero (so that Φ is square integrable) one gets the crucial identity

$$\int (1 + \beta) A_0(\mathbf{x}) \Phi(\mathbf{x}) d^3x = 0. \quad (8)$$

A. Propagation estimate

We estimate the propagation of a wave function generated by the *static* Dirac Operator $D_\sigma = D^0 + A_\sigma(\mathbf{x})$, where $\sigma > 0$ should be thought of as near the critical value (the relevant regime turns out to be of order $\sigma = \mathcal{O}(\epsilon^{1/3})$).

Since the generalized eigenfunctions for $(\sigma, \mathbf{k}) \approx (0, 0)$ are close to the bound state Φ it is reasonable to write in leading order:

$$\varphi(\sigma, \mathbf{k}, \mathbf{x}) \approx \eta_\sigma(\mathbf{k}) \Phi(\mathbf{x}). \quad (9)$$

Since they solve (4), the first summand of (4) must become negligible with respect to $\eta_\sigma(\mathbf{k}) \Phi$, which is part of the second summand. Hence $\eta_\sigma(\mathbf{k})$ must diverge for $(\sigma, \mathbf{k}) \rightarrow (0, 0)$. For the outgoing asymptote of the state Φ (generalized Fourier transform) evolved with D_σ near criticality we have with (9) that

$$\begin{aligned} \widehat{\Phi}_{out}(\sigma, \mathbf{k}) &:= \int (2\pi)^{-\frac{3}{2}} \Phi(\mathbf{x}) \overline{\varphi}(\sigma, \mathbf{k}, \mathbf{x}) d^3x \\ &\approx (2\pi)^{-\frac{3}{2}} \overline{\eta}_\sigma(\mathbf{k}). \end{aligned} \quad (10)$$

Now, for (σ, k) close to but different from $(0, 0)$, $\eta_\sigma(\mathbf{k}) \sim \widehat{\Phi}_{out}(\sigma, \mathbf{k})$ will be peaked around a value $k(\sigma)$ with width $\Delta(\sigma)$ (determined below) defined by

$$\eta_\sigma(k(\sigma) \pm \Delta(\sigma)) \approx \eta_\sigma(k(\sigma)) / \sqrt{2}. \quad (11)$$

We may use the width for the rough estimate

$$|\partial_k \widehat{\Phi}_{out}(\sigma, \mathbf{k})| < \Delta(\sigma)^{-1} \widehat{\Phi}_{out}(\sigma, k_\sigma), \quad (12)$$

where the right hand side should be multiplied by some appropriate constant which we—since it is not substantial—take to be one. Using (9), (10), $d^3k = k^2 d\Omega dk$ and partial integration (observing $\frac{\epsilon}{-iks} \partial_k e^{-i(1+\frac{k^2}{2})\frac{\epsilon}{s}} = e^{-i(1+\frac{k^2}{2})\frac{\epsilon}{s}}$) we get

$$\begin{aligned} U_\sigma(s, 0)\Phi &= e^{-isD_\sigma}\Phi \approx \frac{1}{(2\pi)^{\frac{3}{2}}} \int e^{-i(1+\frac{k^2}{2})\frac{\epsilon}{s}} \widehat{\Phi}_{out}\varphi d^3k \\ &= \frac{-i\epsilon}{s} \int e^{-i(1+\frac{k^2}{2})\frac{\epsilon}{s}} \partial_k \left(|\widehat{\Phi}_{out}|^2 \Phi(x) k d\Omega \right) dk \end{aligned}$$

By (12), (10) and (11), assuming that $\Delta(\sigma) \ll k(\sigma)$

$$\begin{aligned} |\partial_k \left(|\widehat{\Phi}_{out}|^2 k \right) | d\Omega dk &\approx |\widehat{\Phi}_{out}|^2 \left(\frac{2}{\Delta(\sigma)k} + \frac{1}{k^2} \right) d^3k \\ &\approx |\widehat{\Phi}_{out}|^2 \frac{2}{\Delta(\sigma)k(\sigma)} d^3k. \end{aligned}$$

Hence

$$|U_\sigma(s, 0)\Phi(\mathbf{x})| \leq \frac{2\varepsilon|\Phi(x)|}{s\Delta(\sigma)k(\sigma)} \int |\widehat{\Phi}_{out}|^2 d^3k.$$

Since Φ is normalized we get for the decay time s_d , defined by $|\langle U(s_d, 0)\Phi, \Phi \rangle| \approx 1/2$

$$s_d \approx 4\varepsilon(k(\sigma)\Delta(\sigma))^{-1}. \quad (13)$$

B. Control of the Generalized Eigenfunctions

Let us now estimate $\eta_\sigma(\mathbf{k})$ for $\sigma \approx 0$. In view of (6) and (9) we have that

$$(1 - T_\sigma^k)\eta_\sigma(\mathbf{k})\Phi \approx \varphi_0(\mathbf{k}, \cdot). \quad (14)$$

We can estimate $\eta_\sigma(\mathbf{k})$ by considering the scalar product of (14) with $A_0\Phi$:

$$\eta_\sigma(\mathbf{k})\langle (1 - T_\sigma^k)\Phi, A_0\Phi \rangle \approx \langle \varphi_0(\mathbf{k}, \cdot), A_0\Phi \rangle.$$

Since $\varphi_0(0, \mathbf{x}) = \xi(0)$ for which

$$\xi(0) = D^0\xi(0) = \beta\xi(0) \quad (15)$$

we have that $\varphi_0(0, \mathbf{x}) = \frac{1+\beta}{2}\xi(0)$ and in view of (8)

$$\langle \varphi_0, A_0\Phi \rangle |_{k=0} = \frac{\xi(0)^\dagger}{2} \int (1 + \beta)A_0(\mathbf{x})\Phi(\mathbf{x})d^3x = 0. \quad (16)$$

Hence $\langle \varphi_0(\mathbf{k}, \cdot), A_0\Phi \rangle = \mathcal{O}(k)$, and it turns out that $\mathcal{O}(k) = Ck + \mathcal{O}(k^2)$ with an appropriate $C \neq 0$. Thus

$$\eta_\sigma(\mathbf{k}) \approx Ck \langle (1 - T_\sigma^k)\Phi, A_0\Phi \rangle^{-1}$$

Expanding T_σ^k in orders of k around $k = 0$ until fourth order yields

$$\eta_\sigma(\mathbf{k}) \approx Ck \left(S_0^\sigma + kS_1^\sigma + k^2S_2^\sigma + k^3S_3^\sigma + k^4S_4^\sigma \right)^{-1}. \quad (17)$$

By virtue of (5) one easily shows

$$\langle T_\sigma^k\Phi, A_\sigma\Phi \rangle = \langle A_\sigma\Phi, T_\sigma^k\Phi \rangle \quad (18)$$

and we get with (7), using $A_\sigma > A_0$, that

$$\begin{aligned} S_0^\sigma &= \langle (1 - T_\sigma 0)\Phi, A_0\Phi \rangle = \langle (T_0 0 - T_\sigma 0)\Phi, A_0\Phi \rangle \\ &= \langle (A_0 - A_\sigma)\Phi, T_0 0\Phi \rangle = \langle (A_0 - A_\sigma)\Phi, \Phi \rangle \\ &= -\| \sqrt{A_\sigma - A_0}\Phi \|^2 = -C_0\sigma. \end{aligned}$$

with $C_0 > 0$. Computing that $\partial_k G_k^+ |_{k=0} = 1 + \beta$, and observing (5) and (8), $\partial_k T_k\Phi |_{k=0} = 0$, hence with (18)

$$S_1^\sigma = \partial_k \langle T_\sigma^k\Phi, A_0\Phi \rangle |_{k=0} = \partial_k \langle T_0^k\Phi, A_\sigma\Phi \rangle |_{k=0} = 0.$$

Expanding S_2^σ and S_3^σ around $\sigma = 0$ we thus obtain

$$\begin{aligned} \eta_\sigma(\mathbf{k}) & \quad (19) \\ & \approx -Ck \left(C_0\sigma + (C_2 + \mathcal{O}(\sigma))k^2 + (C_3 + \mathcal{O}(\sigma))k^3 \right)^{-1}. \end{aligned}$$

We shall now determine the constants C_2 and C_3 . For that we do a similar expansion for (7), i.e. for $\sigma < 0$ and for $k = i\kappa$. Dividing (14) by $\eta_\sigma(\mathbf{k})$ and replacing Φ by $\Phi_{\sigma(\kappa)} (\approx \Phi)$ yields the left hand side of (7) and thus $\eta_{\sigma(\kappa)}(i\kappa) = \infty$ (otherwise the right hand side of (7) would not be zero). Therefore for all κ

$$-C_0|\sigma(\kappa)| - (C_2 + \mathcal{O}(\sigma(\kappa)))\kappa^2 - i(C_3 + \mathcal{O}(\sigma(\kappa)))\kappa^3 + \dots = 0$$

Since $C_0 > 0$, assuming $C_2, C_3 \neq 0$ (see [13] for a proof) we conclude that $C_2 < 0$ and C_3 must be imaginary. Note that near $k = 0$, $\sigma \sim k^2 = E - m$, while if $C_1 \neq 0$, $E - m \sim -\sigma^2$, i.e. $dE/d\sigma = 0$ at $\sigma = 0$, suggesting that a transversal crossing of the eigenvalue into the upper continuum goes together with a bound state at $E = 1$.

Hence for (19) we get

$$\eta_\sigma(\mathbf{k}) \approx \frac{-Ck}{C_0\sigma - (|C_2| + \mathcal{O}(\sigma))k^2 - i(|C_3| + \mathcal{O}(\sigma))k^3}.$$

For $C_0\sigma \approx C_2k^2$ the denominator behaves like C_3k^3 , otherwise it behaves like $C_0\sigma - C_2k^2$. Hence by (10)

$$|\widehat{\Phi}_{out}(\sigma, \mathbf{k})|^2 \approx Ck^2 \left((C_0\sigma - |C_2|k^2)^2 + |C_3|^2k^6 \right)^{-1} \quad (20)$$

This result [19] differs from the results given in the literature (see e.g. formula (7) in [3]). The right hand side of (20) obviously diverges for $(\sigma, k) \rightarrow (0, 0)$. For fixed $0 \neq \sigma \approx 0$ the divergent behavior is strongest close to the resonance at $(C_0\sigma - |C_2|k(\sigma)^2 = 0)$

$$k(\sigma) = \sqrt{\sigma C_0|C_2|^{-1}} = \mathcal{O}(\sqrt{\sigma}). \quad (21)$$

In view of (11) $\Delta(\sigma)$ can be roughly estimated by setting the right hand side of (20) equal to 1/2 of its maximal size, i.e

$$C_0\sigma - |C_2|(k(\sigma) + \Delta(\sigma))^2 \approx |C_3|k^3(\sigma)$$

hence

$$\Delta(\sigma) \approx k(\sigma)^2 |C_3| (2|C_2|)^{-1} = \mathcal{O}(\sigma). \quad (22)$$

C. Estimating the decay time s_d

Let us first approximate U by U_{s_d} , i.e. the time propagator for the time independent Dirac operator present at s_d .

Using (13), (21) and (22) we have $s_d \propto \varepsilon s_d^{-\frac{3}{2}}$, hence $s_d \propto \varepsilon^{\frac{2}{5}}$. The rigorous estimate, taking into account the time dependence of the external field, yields $s_d \propto \varepsilon^{\frac{1}{3}}$ ([13]). Hence if the field stays overcritical for times $S \gg \varepsilon^{\frac{1}{3}}$ the probability of pair creation is one. Note that in the adiabatic case $S = \mathcal{O}(1)$, and thus this is well satisfied.

The distribution of the outgoing momenta will be discussed below.

IV. ε IN HEAVY ION COLLISIONS

One way to create overcritical fields experimentally are heavy ion collisions. (There are other experiments, which might become more relevant for SPC [20] [21]). Since the fields of the nuclei are repulsive for positrons, it is the positron which scatters. For heavy ion collisions the adiabatic time scale on which the field increases is directly determined by the relative speed with which the heavy ions approach each other and one computes that ε is of order 10^{-1} , however the time where the field remains overcritical is very small (see [22]). The time variation of the field (even for weak fields) produces also pairs (see e.g. [23]), which may become relevant when the field strength is close to criticality and where the time duration of over-criticality is small (see below). In principle an estimate of the induced pairs is needed or an experimental measurement of this “background”: measuring once a system which is slightly under-critical and compare the rate of created pairs to a system which is slightly over-critical.

Theoretical models indicate that the collision time can be enlarged if the nuclei form a composite nuclear system [24]. This effect may be useful to increase the amount of spontaneously created pairs in U-U scattering experiments, though it seems not possible to enlarge the collision time beyond the decay time of the previous chapter.

A. SPC probability for Collision times $\ll s_d$

In the literature the SPC-probability has been computed for an overcritical field of very short life times like $S \ll \varepsilon^{\frac{1}{2}} \ll \varepsilon^{\frac{1}{3}}$ [22].

We note that this probability can be easily estimated without any reference to the resonance, contrary to [4]. We also note, that for such short life times the adiabatically changing field can only vary by $\mathcal{O}(S)$. However in the literature one considers also the case where the field changes—when it reaches over critical values—nonadiabatically to a very small overcritical stationary value of the size $A_0 + a$ with $a \ll S$. In this case only a very small part of the critical wave function Φ will scatter in the positive continuous subspace. Let P^\perp be the projection onto the subspace orthogonal to Φ . The probability for scattering may be estimated by $\|P^\perp U(s, 0)\Phi\|^2$. Now let $U_0(s, 0) = \exp -\frac{i}{\varepsilon}s$ then

$$\begin{aligned} & (U(S, 0) - U_0(S, 0))\Phi \\ &= \frac{i}{\varepsilon} \int_0^S U_0(S, s)(1 - D_s)U(s, 0)\Phi ds . \end{aligned}$$

We apply now P^\perp and replace in the integral U by U_0 , which yields by iteration an error of smaller order. Observing further that $P^\perp\Phi = 0$ and $\Phi = D_0\Phi$

$$\begin{aligned} & \|P^\perp U(S, 0)\Phi\|^2 \\ &= \|P^\perp \frac{i}{\varepsilon} \int_0^S U_0(S, s)(1 - D_s)U_0(s, 0)\Phi ds\|^2 \\ &= \|\frac{i}{\varepsilon} e^{-\frac{iS}{\varepsilon}} \int_0^S P^\perp(1 - D_s)\Phi ds\|^2 \\ &= \|\frac{i}{\varepsilon} e^{-\frac{iS}{\varepsilon}} \int_0^S P^\perp(A_0 - A_s)\Phi ds\|^2 \\ &= \mathcal{O}\left(a\frac{S}{\varepsilon}\right)^2 \ll 1 \end{aligned}$$

This differs from estimates in the literature (see for example formula (20) in [24]), where the formula different from but corresponding to (20)(see e.g. formula 2.24 in [22]) is interpreted as a Breit Wigner form, leading to false estimates. We note that for the “short time analysis” (which is $S \ll s_d$) the use of Breit-Wigner form is not reasonable anyhow.

V. ENERGY SPECTRUM OF THE OUTGOING POSITRONS

An interesting prediction is the shape of the momentum distribution of the created positron. It would be nice if the shape would be simply the resonance (20) as suggested e.g. in [22]. But that requires a somewhat different situation than what has been discussed here. It would require an overcritical static field (adiabatic is not enough) of a life time S larger than the decay time s_d . Then in fact the resonance (20) would stay more or less intact (see also [24]). In an adiabatically changing field the resonance (20) changes however with the field and it is highly unclear how. The resonance (20) will be surely washed out and one can only expect a shape which is somewhat peaked around small momenta.

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