INFORMATION DESIGN IN CONTROLLING EPIDEMICS

by

Minjun Chang

Department of Business Administration
Duke University

Date:________________________
Approved:

____
Peng Sun, Supervisor

____
Ali Makhdoumi, Supervisor

____
Sasa Pekec

____
Can Zhang

Dissertation submitted in partial fulfillment of the
requirements for the degree of Doctor of Philosophy
in the Department of Business Administration
in the Graduate School of
Duke University

2023

ABSTRACT

INFORMATION DESIGN IN CONTROLLING EPIDEMICS

by

Minjun Chang

Department of Business Administration
Duke University

Date:_______________________
Approved:

____
Peng Sun, Supervisor

____
Ali Makhdoumi, Supervisor

____
Sasa Pekec

____
Can Zhang

An abstract of a dissertation submitted in partial fulfillment of the
requirements for the degree of Doctor of Philosophy
in the Department of Business Administration
in the Graduate School of
Duke University

2023

Abstract

This dissertation examines the use of information design as a tool to mitigate the spread of infection. Specifically, I investigate how central planners, or senders, with access to more information can utilize their informational advantage to enhance social welfare.

In the first chapter, I analyze a scenario where a single sender must determine the best information revelation policy when receivers have varying levels of social activity and must choose binary protection levels, which then determine a transmission network for the infection. I establish that, in this case, it is optimal to obfuscate information only for intermediate transmission rates and low initial infection probabilities.

In the second chapter, I further explore the use of information when there are multiple senders, each caring for their own population. I characterize the equilibrium actions of the populations given any information and the equilibrium information disclosure policies between two senders. I establish that two senders will choose to disclose no information when they are either heavily economically focused with high economic costs and a low prior belief about the disease or focused on health with low economic costs. In situations where one sender is heavily economically focused with high economic costs and a high prior belief about the disease, while the other sender is either heavily economically focused with high economic costs or focused on health with low economic costs, the optimal equilibrium strategy is to disclose partial information. Finally, the optimal strategy for the senders to disclose full information occurs when at least one sender is either concerned but not extremely concerned about the economy or focused on health with high economic costs.

Contents

List of Figures

Acknowledgements

I would like to express my deepest gratitude to my supervisors, Peng Sun and Ali Makhdoumi, for their guidance, support, and encouragement throughout my dissertation. Their knowledge, expertise, and feedback have been invaluable to the success of this research.

I am grateful to my committee members, Sasa Pekec and Can Zhang, for their valuable insights, suggestions, and feedback throughout my dissertation. Their contributions have been instrumental in improving the quality of my research.

I would also like to thank the faculty and staff of the Decision Sciences department at Duke for their support and for providing a stimulating academic environment.

I would like to thank my family and friends for their unwavering support and encouragement, especially during those moments when I needed it the most.

Lastly, upon the special request from my best sister, friend and colleague Sophie Yu, I hereby thank her for her constant support and forever friendship.

Chapter 1

Introduction

This dissertation investigates the use of information design for containing epidemics. When facing the spread of an infection, people need to make a trade-off between the costs of getting infected and adopting protective actions, which is complicated by the fact that higher protection levels from others create an incentive for individuals to free-ride. In this context, a social planner will possess more information about the severity of the infection, particularly early on during a pandemic. Therefore, a natural information design question arises: can the social planner leverage the information advantage to improve social welfare? More specifically, when is it optimal to disclose no information, full information, or partial information in the sense of generating higher collective social welfare?

In the first chapter, I study this question in a model with a single sender and a network of individuals. I develop an extension of the Reed-Frost model to capture the spread of the infection among individuals with heterogeneous social activity and protection levels. Using an information design model inspired by the Bayesian persuasion game, I show that it is optimal for the sender to fully reveal the information when the transmission rate is very high or very low and to reveal partial information when the transmission rate is intermediate. I also show that it is optimal for the sender to fully reveal the information when the initial infection probability is high and to reveal partial information otherwise.

Building on this, I extend the question to multiple social planners who possess the same information advantage and perform information design simultaneously. In the second chapter, I answer this question for a model with two senders and two populations with heterogeneous economic costs for social distancing. I analyze the equilibrium information disclosure policies between two senders who may bear different levels of priority over the health and the economy objectives. I show that the senders will reveal no information when they are heavily economically concerned with high economic costs and a low prior belief about the disease or health-concerned with low economic costs. The senders will reveal partial information when one sender is heavily economically concerned with high economic costs and a high prior belief about the disease, while the other sender is either heavily economically concerned with high economic costs or health-concerned with low economic costs. The senders will reveal full information when at least one sender is either concerned but not extremely concerned about the economy or health-concerned with high economic costs.

I am confident that these findings will be of interest to researchers and policymakers seeking to understand the optimal use of information design for controlling epidemics and will provide valuable insights into how to balance economic and health concerns during a pandemic.

Chapter 2

Controlling Epidemics: the Value of Information Design

2.1 Introduction

When facing the spread of an infection, people need to make a trade-off between the costs of getting infected and adopting some protective actions. This trade-off is more complicated than one may imagine because the spread of infection in a social network is endogenous. That is, one individual's infection probability depends on her protective action as well as the protective actions of everyone else in the society. As a result, higher protection levels from others create an incentive to free-ride. If everyone wants to free-ride, however, then the societal protection level becomes too low to curb infection.

Generally speaking, a social planner possesses more information about the severity of the infection, especially early on during a pandemic. Therefore, a natural information design question arises: can the social planner leverage the information advantage to improve social welfare? More specifically, is it possible that obfuscating information generates higher collective social welfare than being transparent?

In this paper, we develop a model to answer these questions. In particular, we study how a more informed social planner should inform a population about the severity of an infection when the individuals have heterogeneous social activity levels and decide their own protection levels. In particular, we consider a social planner who observes the infection severity, which has two possible values (high or low). We also have n individuals who have heterogeneous social activity levels. The social activity level captures the likelihood of an agent being connected to others. Additionally, the protection level of each agent captures the likelihood of that agent being present in the society and therefore susceptible to the infection. The social activity and protection levels of agents determine a social network among them that governs the spread of the infection.

To model the spread of the infection with heterogeneous social activity and protection levels, we develop an extension of the Reed-Frost model (a stochastic and discrete variation of the SIR model). Formally, each agent decides a binary protection level that captures the probability that a node representing this agent appears on the network. Furthermore, each agent has a social activity level independently and identically drawn from a uniform distribution. The product of the social activity levels between two agents determines the chance that an edge exists between them in the contact network. Before the infection starts, each agent becomes an origin of the infection independently with an *initial infection probability*. After that, each infected and unprotected agent will transmit the infection to each one of her unprotected neighbors in the contact network independently with a *transmission rate*.

Our information design model is motivated by the Bayesian persuasion game of [KG11]. The social planner (sender) commits to a messaging policy, privately observes the state of the world, and sends a one-shot public message to all agents based on the state. Each agent obtains information solely from the sender and chooses her protection level based on her belief about the state of the world and her own social activity level. A higher protection level is more costly but decreases the infection probability. In particular, the dis-utility of each agent is the protection cost plus the severity of the infection times her infection probability.

We first show some properties of the stochastic process dictating the spread of the infection. Naturally, higher social activity levels increase the infection probability, and higher protection levels decrease the infection probability. In addition, the infection probability is submodular in the protection level and the social activity level. That is, protection is more effective when social activity levels are higher. Using these properties, we establish the existence and uniqueness of Bayes Nash equilibrium among agents. We further characterize the agents' equilibrium and establish that it is a threshold strategy. That is, agents whose social activity levels are below (resp. above) a threshold choose the low (resp. high) protection level. We further explicitly characterize the equilibrium threshold as a function of the agents' posterior belief of the infection severity. Equipped with these characterizations, we then turn to our main question: can obfuscating information improve social welfare?

To gain an intuition on why obfuscating information can be helpful, let us first consider the situation in which the social planner truthfully reveals the severity of the infection, which leads to the corresponding "full disclosure equilibrium protection levels" in the society. Now imagine hypothetically that the social planner is able to dictate each agent's action in order to maximize social welfare. In this case, the corresponding "centralized protection levels" may be different from the equilibrium ones. By obfuscating information, it may be possible to distort the resulting equilibrium protection levels towards the centralized ones, hence improving social welfare. Even if one can show that such an improvement is indeed possible, how to most effectively convey the information to maximize this improvement remains a challenge. Therefore, in this paper, we attempt to address the following main questions:

> *When is it optimal to obfuscate information? How can one design the information design policy to maximize social welfare?*

In order to answer these questions, we approximate the infection probability in the limit as the number of agents goes to infinity. We first consider the situation in which information asymmetry between the social planner and the society at large only occurs at the onset of the infection. In our model, this corresponds to imposing the simplifying assumption that the infection transmits only through one step from the origins. With this one-step transmission, we show that it is optimal for the social planner to fully reveal the information to the public when the transmission rate is very high or very low and when the initial infection probability is very high or very low. We further establish that when the transmission rate and the initial infection probability are intermediate, it may be optimal to obfuscate the information.

We then extend our results to the more general model of infection, such that a node is infected as long as there is a path linking to an origin. We prove that the results of the simpler one-step transmission model in terms of the impact of the transmission rate continue to hold, while the results in terms of the impact of the initial infection probability change. In particular, it is no longer optimal for the social planner to reveal full information to the public when the initial infection probability is very small.

Our main results confirm the major benefit society can reap when the social planner reveals information about the severity of an infection strategically to the public but also lead to two new insights. First, for very high or low transmission rates, fully revealing information to the public is optimal because the social planner's information revelation cannot convince individuals to change their (equilibrium) decisions about protection levels. We also show that, for a nontrivial set of intermediate transmission rates, obfuscating information is optimal. Second, we show that when the initial infection probability of individuals is large enough, the optimal information-revealing policy is again full disclosure. When the initial infection probability is small, it becomes optimal to obfuscate information. Although our model is a simplification and abstraction from reality, we hope that our analysis sheds lights on the design of information revelation policies for pandemic control.

2.1.1 Related literature

From a methodological perspective, our paper relates to Bayesian persuasion and belongs to the rapidly growing literature on information design. Earlier works on Bayesian persuasion include [CS82], [Mye97], and [KG11]. More recently, [Ely17] extends the Bayesian persuasion framework to a dynamic setting and discusses the trade-off between current persuasion and the ability to persuade in the future. See also [BM19] and [Kam19] for a survey. Bayesian persuasion has also found several applications in operations. Representative papers include [LI19], [AIM20], [DJR21], [CD20], and [PBS18]. See [Can20] for a survey of these applications.

More specifically, our paper is related to the literature on information design in the context of epidemics. Representative papers are [AdVW20], [dVGW21], and [DR21]. The most closely related paper to ours is [dVGW21], which studies how governments may effectively inform the public about an epidemic to induce compliance under different emphasis on economy concerns or population health concerns. They model the infection probability as a linear function of the number of socially active users and prove that when the government does not distort economic versus health costs of the users in its objective, it is optimal to be fully transparent. However, when the government heavily prioritizes the economy over population health cost (or vice versa), it becomes optimal to strategically obfuscate the information. We depart from this work by considering a more explicit model of the network formation to account for the heterogeneity in the social activity level of users. We also explicitly model the spread of information by developing an extension of the Reed-Frost model. The heterogeneity of users in terms of their social activity levels and explicitly modeling the infection probability (as opposed to a linear function of the socially active users) have a profound impact on the insights we obtain and are in sharp contrast with the work of [dVGW21]. In particular, we establish that, when we account for the heterogeneity of social activity levels, fully transparent policies may no longer be optimal even if we do not distort economic versus health costs of the users.

Our paper is also related to the literature on contagion processes. Representative papers in this area include [JW96b], [BG00b], [KKT15], and [CJP09] (see [Jac08a] and [DM10] for surveys). Differently from the most common approach in this literature, which is to consider (variations of) the SIR (susceptible-infective-removed) model, we use an extension of Reed-Frost model (a discrete and stochastic version of the standard SIR model) to account for protection level and heterogeneous social activity level. As we explain in Appendix **??**, our contagion model leads to the same behavior of cumulative infections as Reed-Frost but enables a tractable and much more general analysis on how the spread of infection depends on protection levels of agents. Additionally, our contagion model is related to the independent cascade model of [KKT15] but again extends it to account for heterogeneous protection level and heterogeneous social activity level.

Lastly, our paper is related to the emerging literature on COVID-19. [FHLO20] and [JCX+20], for example, study the effect of lock-downs and confinement for slowing the spread of COVID-19. [BCF22] studies targeted closure policies in controlling epidemic spread. [HHKM+20] and [CGL+22] analyze the health benefits and the economic costs that social distancing brings about. [BDG+21] studies the design and performance of a reinforcement learning system for resource allocation during COVID-19. [BK20] studies the externality of infection and isolation during COVID-19. Closer to our work is [AMMO20], which studies the effects of testing in the spread of an infection. In particular, our network formation and the contagion model build on this paper. However, we depart from this paper in two main ways. First, we extend their model to account for users' (endogenous) protection levels. Second, we consider information design, which is not studied in [AMMO20].

The remainder of this paper is organized as follows. Section 2.2 presents the model setup. Section 2.3 describes the general information design policy and explains a reduction using the revelation principle. Section 2.4 proves key properties of the infection probability and establishes the existence and uniqueness of equilibrium among the agents. Section 2.5 studies the model with a one-step transmission setting. Section 2.6 analyzes results about the general transmission model. Section 2.7 concludes the paper. The appendix includes proofs omitted from the text.

2.2 Model Description

2.2.1 The Environment

We consider the problem of informing agents to take actions in order to minimize the spread of an infection. The problem consists of one sender (the social planner) and n receivers (the agents) represented by the set $I = \{1, \cdots, n\}$. We let $i \in I$ for a typical agent. Consider a binary set of states $\Theta = \{\theta_L, \theta_H\}$ with $0 < \theta_L < \theta_H$. The state $\theta \in \Theta$ represents the severity level of the infection, i.e., the unit loss incurred by agents from getting infected. The realized state of the world θ is private information to the sender, and agents share a common prior $\mu(\cdot) \in \Delta(\Theta)$.

We assume that agents are heterogeneous and that their types are denoted by their *social activity levels* $\mathbf{x} = (x_1, \cdots, x_n)$. Intuitively, higher social activity provides greater utility to the agents but also leads to higher probability of infection. We further assume each agent's social activity level x_i is randomly and independently drawn from a uniform distribution over $[0, 1]$. We let v denote the probability density function (pdf) of x_i and ν the joint pdf of $\mathbf{x}$. We further let ν_{-i} denote the pdf of $\mathbf{x}_{-i}$, where $\mathbf{x}_{-i}$ represents the $(n-1)$-dimensional vector that includes the social activity levels of all agents except i. The realized social activity level of each agent i (i.e., x_i) is her private information, and the sender only knows the prior ν, which is common for all agents.

Each agent $i \in I$ decides a *protection level* q_i from a binary set $Q_i = \{q_L, q_H\}$, where $0 < q_L < q_H < 1$. As we make it more precise later, a higher level of protection makes the infection probability smaller. We let $Q = Q_1 \times \cdots \times Q_n = \{q_L, q_H\}^n$ denote the action space of all agents. Intuitively, adopting a higher protection has higher cost but also reduces the spread of the infection. We normalize the cost of adopting a low protection (i.e., q_L) to 0 and the cost of adopting a high protection (i.e., q_H) to 1. We let $\mathbf{q} = (q_1, \cdots, q_n) \in Q$ denote the protection levels of all agents.

Throughout, we make use of the following notation. For any vector $\mathbf{p} \in \mathbb{R}^n$ and set $Z \subseteq I$, $\mathbf{p}_Z$ denotes the vector of entries whose indices belong to S, and $\mathbf{p}_{-Z}$ denotes the vector of entries whose indices belong to $I \setminus Z$.

We next describe how social activity levels determine the social network in the society and how the infection spreads, given the protection level of agents.

2.2.2 Network of Contacts

The social activity levels $\mathbf{x} = (x_1, \cdots, x_n)$ generate a social network in which agents i and j are connected with a probability that depends on their social activity levels x_i and x_j. Let $\mathbf{G} = (V, \mathbf{E})$ denote a random network where $V = I$ and $\mathbf{E} \in \{0, 1\}^{n \times n}$. The entries of $\mathbf{E}$ are independent binary random variables. The probability distribution of the (i, j)-th entry, denoted by $\mathbf{E}_{ij}$, is $\mathbb{P}[\mathbf{E}_{ij} = 1] = x_i x_j$. We let $G = (V, E)$

denote a realized network, where $E_{ij} = 1$ means there exists a link between agents i and j.

2.2.3 The Spread of Infection

For a given network $G = (V, E)$, the stochastic process that governs the spread of the infection is as follows. A subset of the agents, denoted by O, becomes infected uniformly at random. Then, the infection spreads according to a variation of the independent cascade model [KKT15], described in the following. At the beginning of the game, each agent chooses a protection level $q_i \in Q_i$. With probability q_i, this agent will be fully protected and no longer get the infection, i.e., node i and all edges linked to i are removed from the network. After the protection level is chosen, at time 0, each unprotected agent will become infected independently with an *initial infection probability* $\pi \in (0, 1)$. If an agent is infected, she will be active for one round and transmits infection to her unprotected neighbors with a *transmission rate* β. Given the number of agents n, the transmission probability between any pair of infected and uninfected neighbors is $\beta/n \in [0, 1]$. Given active agents, the infection simultaneously and independently transmits to each of their uninfected neighbors. If an uninfected agent is a neighbor to multiple active infected agents, then the infection is transmitted in an order-independent fashion. For example, if j is uninfected and unprotected and is connected to two active infected agents, then j becomes infected with probability $1 - (1 - \beta/n)^2$. At any round $r \geq 0$, for any uninfected agent i, let us denote by d_i^r the number of neighbors that are infected and active. Then, the probability of agent i getting infected at time $r + 1$ is

$$1 - \left(1 - \frac{\beta}{n}\right)^{d_i^r}.$$

Because an infected agent stays active for only one period, if a neighbor of an active agent does not become infected at round r, then the neighbor will never become infected via that node. A formal definition for the infection probability is as follows.

Definition 1 (Infection probability)**.** *For any agent $i \in I$, social activity profile $\mathbf{x} \in [0,1]^n$, and protection profile $\mathbf{q} \in Q$, we let $\boldsymbol{P}_i^{(n)}(\mathbf{q}, \mathbf{x})$ denote the probability of infection reaching agent i. This probability is over the randomness in the formed network, the randomness over the origin of infection, and the randomness in the stochastic process governing the spread of the infection. We formally define this probability using an indicator function on whether the node i is connected to the origin in the* realized graph, *i.e.,*

$$\boldsymbol{P}_i^{(n)}(\mathbf{q}, \mathbf{x}) \doteq \mathbb{E}_V\left[\mathbb{E}_{E,O|V^R}\left[\mathbb{1}\left\{O^R \rightsquigarrow i\right\}\right]\right], \tag{2.1}$$

where

- $V = \{V_i\}_{i \in I}$, *with* $V_i = \begin{cases} 0 & \text{with probability } q_i \\ 1 & \text{with probability } 1 - q_i \end{cases}$,
- $V^R = \{i \in I : V_i = 1\}$,
- $E \mid V^R = \{E_{ij}\}_{i,j \in V^R}$, *with* $E_{ij} = \begin{cases} 0 & \text{with probability } 1 - \frac{\beta x_i x_j}{n} \\ 1 & \text{with probability } \frac{\beta x_i x_j}{n} \end{cases}$,
- $O \mid V^R = \{O_i\}_{i \in V^R}$, *with* $O_i = \begin{cases} 0 & \text{with probability } 1 - \pi \\ 1 & \text{with probability } \pi \end{cases}$,
- $O^R = \left\{i \in V^R : O_i = 1\right\}$,
- $O^R \rightsquigarrow i$ *means there exists an element $o \in O^R$ and a sequence of nodes* $i_1 =$

$o \to i_2 \to \cdots \to i_k = i$ for some $k \in [n]$ such that $i_1, \ldots, i_k \in V^R$ and $E_{i_1 i_2} = E_{i_2 i_3} = \cdots = E_{i_{k-1} i_k} = 1$. If $i \in O^R$, we say $\mathbb{1}\left\{O^R \rightsquigarrow i\right\} = 1$.

We next illustrate the stochastic process and the infection probability by means of an example.

Example 1. *We consider a setting with two agents $\{1, 2\}$ and find the probability of infection reaching agent 1. Given a protection profile $\mathbf{q} = (q_1, q_2)$ and a social activity level profile $\mathbf{x} = (x_1, x_2)$, we now find the infection probability of agent 1, denoted by $\boldsymbol{P}_1^{(2)}(\mathbf{q}, \mathbf{x})$. Note that agent 1 gets infected if and only if the agent $1 \in V^R$ and $\mathbb{1}\left\{0^R \rightsquigarrow 1\right\} = 1$, i.e., the agent is left unprotected in the contact network and is in the connected component to the origins of the infection. Thus, using Definition 1, we can write*

$$\boldsymbol{P}_1^{(2)}(\mathbf{q}, \mathbf{x}) = \mathbf{P}(1 \in V^R, \mathbb{1}\left\{O^R \rightsquigarrow 1\right\} = 1) = \mathbf{P}(1 \in V^R, 1 \in O^R) + \mathbf{P}(1 \in V^R, O^R = \{2\}, E_{12} = 1).$$

There are two scenarios where agent 1 gets infected: either agent 1 is initially infected or the infection spreads from agent 2 to agent i. In the first scenario, agent 1 is initially infected if $1 \in O^R$. The probability of infection, in this case, is

$$\mathbf{P}(1 \in O^R) = \mathbf{P}(1 \in V^R)\mathbf{P}(1 \in O^R | 1 \in V^R) = (1 - q_1)\,\pi.$$

In the second scenario, agent 1 is left unprotected in the contact network but is not initially infected. Agent 2 must be a neighbor to agent 1 and initially infected and then successfully spread the infection to agent 1. The corresponding probability of infection is

$$\begin{aligned}\mathbf{P}(1 \in V^R, O^R = \{2\}, E_{12} = 1) &= \mathbf{P}(1, 2 \in V^R)\mathbf{P}(O^R = \{2\} \mid 1, 2 \in V^R)\mathbf{P}(E_{12} = 1 \mid 1, 2 \in V^R) \\ &= (1 - q_1)(1 - q_2)\,\pi\,(1 - \pi)\,\frac{\beta}{2} x_1 x_2.\end{aligned}$$

The total infection probability of agent 1 is the sum of the probabilities calculated *in the above two cases. Thus, we have*

$$\boldsymbol{P}_1^{(2)}(\mathbf{q},\mathbf{x}) = (1-q_1)\,\pi + (1-q_1)\,(1-q_2)\,\pi\,(1-\pi)\,\frac{\beta}{2}x_1x_2.$$

This example illustrates a basic scenario of finding the probability of infection. When the number of agents is large, the number of possible cases will be large and finding the probability of infection will be analytically challenging. Later, in Section 2.4, we will prove key properties of $\mathcal{P}$ and establish how the properties help us in obtaining useful results.

One thing we want to highlight here is that our model for the spread of infection is an extension of the Reed-Frost model. We establish this connection formally in the following.

2.2.4 Reed-Frost model and its connection to our model

The Reed-Frost model is a commonly used model for epidemics that is the discrete and stochastic equivalence of the celebrated SIR (susceptible-infective-removed) model ([DM10], Chapter 2.1). Here, we briefly introduce the Reed-Frost model and establish that our model, explained in Subsection 2.2.3, is an extension of it.

The basic version of a Reed-Frost model with parameters n and p, as described in [DM10], Chapter 2.1, is as follows. A set of n individuals is given, indexed by $i \in \{1, \ldots, n\}$. At step 0, a single individual is infected. Once infected, an individual is infectious during the subsequent time slot, after which she is removed (either dead or immunized). While infectious, she will succeed in infecting a healthy individual with probability p, and this occurs independently for all target individuals and infectious individuals.

A more formal description is as follows. Let $Z_u(t) \in \{S, I, R\}$ denote the state of node u during step t. Then, the process $Z(t) = \{Z_u(t)\}_{u \in \{1,\ldots,n\}}$ is a homogeneous discrete-time Markov process; i.e., conditional on its state $Z(t)$ at time t, the future evolution of the process after t is independent of the states visited prior to t. Given two states z, z' in $\{S, I, R\}^n$, a transition from z to z' can take place only if for all $i = 1, \ldots, n$, if $z_i \in \{I, R\}$, then we have $z'_i = R$. That is, all nodes are removed after being infected and remain removed afterwards, and if $z_i = S$, then we have $z'_i \in \{S, I\}$. Denote $I(z)$ (respectively, $S(z)$ and $R(z)$) as the number of infectious (respectively, susceptible and removed) nodes in state z. The transition probability for a node u is given by

$$\mathbf{P}\left(Z_u(t+1) \mid Z_u(t)\right) = \begin{cases} (1-p)^{I(z)} & \text{if } Z_u(t) = S, Z_u(t+1) = S \\ 1-(1-p)^{I(z)} & \text{if } Z_u(t) = S, Z_u(t+1) = I \\ 1 & \text{if } Z_u(t) = I, Z_u(t+1) = R \\ 1 & \text{if } Z_u(t) = R, Z_u(t+1) = R \end{cases}.$$

Provided the pair of states z, z' satisfies the above constraints, the transition probability is given by

$$\mathbf{P}\left(Z(t+1) = z' \mid Z(t) = z\right) = \binom{S(z)}{I(z')} (1-p)^{I(z)S(z')} \left[1-(1-p)^{I(z)}\right]^{I(z')},$$

where $I(z') = S(z) - S(z')$.

Our model of the spread of infection defined in Subsection 2.2.3 is an extension of the Reed-Frost model. Our model is similar to the Reed-Frost model in the sense that the spread of the infection can be modeled by a Markov process, i.e., the future evolution of the process is independent of the states visited prior to the current time step. However, our model differs from the basic Reed-Frost model in two main aspects. First, we consider heterogeneity in terms of the social activity levels among the agents. Therefore, the identity of agents matters, and we need to keep track of the states of all agents at each time step in our model instead of the total number of agents in each state (susceptible vs infective vs removed) in the Reed-Frost model. Second, we model possible isolation of agents according to their protection levels. That is, the initial set of removed agents can be nonempty: each node is initially removed with a probability equal to its protection level. In addition to the two major differences, we also have multiple origins instead of a single one in the basic Reed-Frost model introduced above.

A formal description of our model, using the same notation as the Reed-Frost model, is as follows. Again, we let $Z_u(t) \in \{S, I, R\}$ denote the state of node u in step t. Let q_u denote the protection level of agent u, x_u denote the social activity level of agent u, β/n denote the independent pairwise transmission rate of the infection, and π denote the initial infection probability. For all $u \in \{1, \ldots, n\}$,

$$Z_u(0) = \begin{cases} R & \text{with probability } q_u \\ I & \text{with probability } (1-q_u)\,\pi \\ S & \text{otherwise} \end{cases} \cdot$$

Similarly, we let $Z(t) = \{Z_u(t)\}_{u \in \{1,\ldots,n\}}$ denote the state of the stochastic process governing the spread of the infection. Given two states z, z' in $\{S, I, R\}^n$, a transition from z to z' can take place only if for all $i = 1, \ldots, n$, if $z_i \in \{I, R\}$, then we have $z'_i = R$. That is, all nodes are removed after being infected and remain removed afterwards, and if $z_i = S$, then we have $z'_i \in \{S, I\}$. Denote $\tilde{I}(z)$ (respectively, $\tilde{S}(z)$ and $\tilde{R}(z)$) as the set of the nodes that are infectious (respectively, susceptible and removed) in state z. The transition probability for a node u is given by

$$\mathbf{P}\left(Z_u\left(t+1\right) \mid Z_u\left(t\right)\right) = \begin{cases} \prod_{v\in\tilde{I}(Z(t))}\left(1-\frac{x_u x_v \beta}{n}\right) & \text{if } Z_u\left(t\right)=S, Z_u\left(t+1\right)=S \\ 1-\prod_{v\in\tilde{I}(Z(t))}\left(1-\frac{x_u x_v \beta}{n}\right) & \text{if } Z_u\left(t\right)=S, Z_u\left(t+1\right)=I \\ 1 & \text{if } Z_u\left(t\right)=I, Z_u\left(t+1\right)=R \\ 1 & \text{if } Z_u\left(t\right)=R, Z_u\left(t+1\right)=R \end{cases} \tag{2.2}$$

Provided the pair of states z, z' satisfies the above constraints, the transition probability is given by

$$\mathbf{P}\left(Z\left(t+1\right)=z' \mid Z\left(t\right)=z\right) = \prod_{u\in\tilde{I}(z')}\left(1-\prod_{v\in\tilde{I}(z)}\left(1-\frac{x_u x_v \beta}{n}\right)\right)\prod_{u'\in\tilde{S}(z')}\left(\prod_{v\in\tilde{I}(z)}\left(1-\frac{x_{u'} x_v \beta}{n}\right)\right)$$

where $\tilde{I}(z') = \tilde{S}(z) \setminus \tilde{S}(z')$.

Next, we establish the equivalence between this extension of the Reed-Frost model and our model for the spread of infection introduced in Definition 1. We prove that the probability of infection is equal across the two models. First, let us consider our model introduced in Definition 1. Let $L\left(O^R \rightsquigarrow i\right)$ denote the length of the shortest path between i and an element in O^R. If no such path exists, we say $L\left(O^R \rightsquigarrow i\right) = \infty$. If $i \in O^R$, we say $L\left(O^R \rightsquigarrow i\right) = 0$. With this auxiliary notation, we can write

$$\mathbf{P}(\text{node } i \text{ gets infected in model with Definition 1}) = \sum_{t=0}^{n} \mathbf{P}(L\left(O^R \rightsquigarrow i\right) = t).$$

Note that by Definition 1, a node i gets infected if and only if $\mathbb{1}\left(O^R \rightsquigarrow i\right) = 1$. If such a path exists, we have $L\left(O^R \rightsquigarrow i\right) \leq n$ because there are at most n nodes in the network.

Next, let us consider the extension of the Reed-Frost model introduced above.

Note that a node i gets infected if and only if for some $t \leq n$, $Z_i(t) = I$. By definition, for each i, there exists at most one t such that $Z_i(t) = I$ (this is because once a node is infected, it will become removed in the next round). Therefore, we have

$$\mathbf{P}(\text{node } i \text{ gets infected in the extended Reed-Frost model}) = \sum_{t=0}^{n} \mathbf{P}(Z_i(t) = I)).$$

To establish the equivalence of our extension of the Reed-Frost model and Definition 1, we prove the following result.

Claim 1. *For any $t \in \mathbb{Z}^+ \leq n$, $\mathbf{P}(L\left(O^R \rightsquigarrow i\right) = t) = \mathbf{P}(Z_i(t) = I)$.*

Proof. We prove this Claim by induction on t. First, consider $t = 0$. By definition, we have $\mathbf{P}(Z_i(0) = I) = (1 - q_i)\pi$. We also have

$$\mathbf{P}(L\left(O^R \rightsquigarrow i\right) = 0) = \mathbf{P}(i \in O^R) = \mathbf{P}(i \in O^R \mid i \in V^R)\mathbf{P}(i \in V^R) = \pi\,(1 - q_i) = \mathbf{P}(Z_i(0) = I).$$

Now assume the result holds for any $t \leq k$. We prove that the result holds for $t = k + 1$ as well. By the transitional probability defined in (2.2), we have

$$\mathbf{P}(Z_i(k+1) = I) = \mathbf{P}(Z_i(k) = S, Z_i(k+1) = I) = 1 - \prod_{v \in \tilde{I}(Z(k))} \left(1 - \frac{x_i x_v \beta}{n}\right).$$

Next, let $N_t(O^R) = \left\{i \in V^R : L\left(O^R \rightsquigarrow i\right) = t\right\}$ denote the set of nodes for which there is a shortest path from an element of O^R to them with length t. Thus, we can write

$$\begin{aligned}\mathbf{P}(L\left(O^R \rightsquigarrow i\right) = k+1) &= \mathbf{P}(i \in N_{k+1}(O^R)) = \mathbf{P}\left(\exists j \in N_k(O^R) : E_{ij} = 1\right) \\ &= 1 - \prod_{j \in N_k(O^R)} \left(1 - \frac{x_i x_j \beta}{n}\right).\end{aligned}$$

By induction hypothesis, we have $\tilde{I}(Z(k)) = N_k(O^R)$, which completes the proof of the claim. $\square$

Using Claim 1, we obtain

$$\mathbf{P}(\text{node } i \text{ gets infected in model with Definition 1}) = \sum_{t=0}^{n} \mathbf{P}(L\left(O^R \rightsquigarrow i\right) = t)$$

$$= \sum_{t=0}^{n} \mathbf{P}(Z_i(t) = I)) = \mathbf{P}(\text{node } i \text{ gets infected in the extended Reed-Frost model}),$$

which proves that the probabilities of infection across our model defined in Definition 1 and the extended Reed-Frost model described above are equal.

2.2.5 The Utility of Agents

Each agent $i \in I$ has a utility function $u_i^{(n)} : Q \times X \times \Theta \to \mathbb{R}$ that is a mapping from the action profile of all agents, the (ex ante) unknown social activity profile of all agents, and the state of the world. The utility function of agent i is given by

$$u_i^{(n)}(\mathbf{q}, \mathbf{x}, \theta) \doteq -\boldsymbol{P}_i^{(n)}(\mathbf{q}, \mathbf{x})\, \theta - \mathbb{1}(q_i = q_H), \tag{2.3}$$

where the first term, $-\boldsymbol{P}_i^{(n)}(\mathbf{x}, \mathbf{q})\, \theta$, captures the loss from getting infected, and the second term, $-\mathbb{1}(q_i = q_H)$, represents the protection cost. Here, we use the superscript (n) to highlight the case with a finite number of agents. Later in the paper, we will let n approach infinity.

2.2.6 Equilibrium

We now consider the game among the agents and discuss the equilibrium of the game with incomplete information. Each agent i knows her own social activity level x_i, the distribution $\mu(\cdot) \in \Delta(\Theta)$ over the state of the world θ, and the distribution ν_{-i} over the remaining agents' social activity levels. We denote an *action strategy* by $a : [0,1] \times \Delta(\Theta) \to \{q_L, q_H\}$, which maps an agent's social activity level and belief about the state of the world into a protection level. We use the notation $\mathbf{a}(\mathbf{x}, \mu) = (a(x_1, \mu), a(x_2, \mu), \cdots, a(x_n, \mu))$ to denote the symmetric protection profile based on the action strategy a for a social activity profile $\mathbf{x}$ and denote $\mathbf{a}(\mathbf{x}_{-i}, \mu) = (a(x_1, \mu), \cdots, a(x_{i-1}, \mu), a(x_{i+1}, \mu), \cdots, a(x_n, \mu))$ in a similar way. Note that because any message from the social planner is public, everyone in the society shares the same belief about the state of the world.

Next, we formally define the notion of equilibrium in the game. We consider the symmetric Bayesian Nash equilibrium $a^E(x, \mu)$ that maximizes the expected utility for each agent under the belief μ and given that all other agents follow the same action strategy a^E. That is, a strategy profile $a^E(x, \mu)$ is a Bayesian Nash equilibrium if and only if for every agent i, keeping the strategies of every other agent fixed, action $a^E(x_i, \mu)$ maximizes the expected utility of agent i according to her belief. Notice that the symmetry assumption means that the strategy of each agent depends on her social activity level x and her belief μ, and not her identity — the function $a^E(\cdot, \cdot)$ is not indexed by i. A formal definition is as follows.

Definition 2 (Symmetric Bayesian Nash Equilibrium). *For any belief $\mu(\cdot) \in \Delta(\Theta)$, we say $a^E : [0,1] \times \Delta(\Theta) \to \{q_L, q_H\}$ is a symmetric Bayesian Nash equilibrium if for every agent i and social activity level $x_i \in [0,1]$,*

$$\mathbb{E}_{\theta\sim\mu, \mathbf{x}_{-i}\sim\nu_{-i}}\left[u_i^{(n)}\left(\mathbf{a}^E(\mathbf{x}, \mu), \mathbf{x}, \theta\right)\right] \geq \mathbb{E}_{\theta\sim\mu, \mathbf{x}_{-i}\sim\nu_{-i}}\left[u_i^{(n)}\left(\left(q_i', \mathbf{a}^E(\mathbf{x}_{-i}, \mu)\right), \mathbf{x}, \theta\right)\right], \tag{2.4}$$

for all $q_i' \in \{q_L, q_H\}$.

In Section 2.4.2, we prove the existence and uniqueness of symmetric Bayesian

Nash equilibrium.

2.3 Information Design

In this section, We discuss the information design policies by the sender to maximize social welfare. In general, the sender can choose any signal space S and announce any *message* $s \in S$ publicly. We assume S is a finite set but general enough. We let $\mathcal{F} = \{f : \Theta \to \Delta(S)\}$ denote the space for all possible information design policies. The sender commits to a specific policy before the game among agents starts. Agents will form a posterior belief $\mu_s(\cdot) \in \Delta(\Theta)$ after seeing the *information policy* f and the generated message s. More specifically, for any $\theta \in \Theta$ and $s \in S$, the agents' belief becomes

$$\mu_s(\theta) = \frac{f(s \mid \theta)\,\mu(\theta)}{\sum_{\theta' \in \Theta} f(s \mid \theta')\,\mu(\theta')}.$$

Agents will then play an equilibrium game as discussed in Subsection 2.2.6. The extensive form game between the sender and the agents is as follows:

1. The sender picks and commits to a policy $f \in \mathcal{F}$ for providing the agents with an extra message;

2. The true state θ is realized and is private information to the sender;

3. The agents see a realized message s according to the sender's information policy $f(\cdot \mid \theta)$ and form a posterior belief μ_s;

4. The social activity levels $\mathbf{x}$ are realized, and each agent $i \in I$ observes her own x_i;

5. Each agent i then plays an equilibrium protection level $q_i = a^E(x_i, \mu_s)$ based on her activity level x_i and the posterior belief μ_s;

6. The network is realized, the infection spreads, and the payoffs are realized.

Next, we define social welfare and introduce the sender's problem.

2.3.1 Social Planner's Problem

Definition 3 (Social welfare)**.** *For any state θ, social activity profile $\mathbf{x}$, and protection profile $\mathbf{q}$, the (normalized) social welfare is the average of the utilities of all agents given by*

$$W^{(n)}\left(\mathbf{q},\mathbf{x},\theta\right) \doteq \frac{1}{n}\sum_{i=1}^{n} u_i^{(n)}\left(\mathbf{q},\mathbf{x},\theta\right). \tag{2.5}$$

With the timing discussed above, the social planner's problem becomes

$$Z_0 \doteq \max_{f\in\mathcal{F},a^E} \mathbb{E}_\theta\mathbb{E}_\mathbf{x}\mathbb{E}_f\left[W^{(n)}\left(\mathbf{a}^E\left(\mathbf{x},\mu_s\right),\mathbf{x},\theta\right)\right] \tag{2.6}$$

$$\text{s.t. } \mathbb{E}_{\theta\sim\mu_s,\mathbf{x}_{-i}\sim\nu_{-i}}\left[u_i^{(n)}\left(\mathbf{a}^E\left(\mathbf{x},\mu_s\right),\mathbf{x},\theta\right)\right] \geq \mathbb{E}_{\theta\sim\mu_s,\mathbf{x}_{-i}\sim\nu_{-i}}\left[u_i^{(n)}\left(\left(q_i',\mathbf{a}^E\left(\mathbf{x}_{-i},\mu_s\right)\right),\mathbf{x},\theta\right)\right],$$

$$\text{for all } i\in I, x_i\in[0,1], s\in S, q_i'\in\{q_L,q_H\} \tag{2.7}$$

$$\sum_{s\in S} f\left(s\mid\theta\right)=1, \text{ for all } \theta\in\Theta \tag{2.8}$$

$$f\left(s\mid\theta\right)\geq 0, \text{ for all } s\in\mathcal{S},\theta\in\Theta. \tag{2.9}$$

The social planner's problem is a constrained optimization problem. The objective function (2.6) maximizes the expected social welfare. Constraint (2.7) defines the equilibrium of the Bayesian game among the agents, following Definition 6. In the case that there are multiple equilibria in the Bayesian game, the optimization problem selects the one that yields the maximum social welfare. Later, in Section 2.4.2, we prove that there exists a unique symmetric equilibrium given any belief μ. Lastly, constraints (2.8) and (2.9) ensure f satisfies general probability rules and is an element of $\mathcal{F}$. The value Z_0 is the optimal objective value of the constrained optimization problem.

2.3.2 Revelation Principle and the Reduction of the Signal Space

In general, the sender can choose any signal space S, as discussed in the earlier paragraph. However, in this section, we establish that there is no loss of generality by focusing on the signal space of mappings from the social activity levels to the action set $\{q_L, q_H\}$. We consider the signal space $\mathcal{M} = \{m : [0,1] \to \{q_L, q_H\}\}$ and the space for *communication rules* $\mathcal{H} = \{h : \Theta \to \Delta(\mathcal{M})\}$. In particular, we focus on the mappings $\mathcal{M}$ because given that the social activity levels of the agents are private, the sender will need to announce the communication rule without such knowledge and as a function of the agents' social activity levels. The structure of this section will be as follows. First, we discuss the extensive form game with the reduced signal space $\mathcal{M}$. Then, we define a notion of obedience constraints imposed on the communication rule $h \in \mathcal{H}$. Lastly, we prove a revelation principle that draws an equivalence between using a general policy f followed by a Bayesian equilibrium and a communication rule h over the reduced signal space $\mathcal{M}$ that satisfies the obedience constraint.

We let $\mathbf{m}(\mathbf{x}) = (m(x_1), \cdots, m(x_n))$ denote the mapped protection profile of all agents for a social activity profile $\mathbf{x}$ and denote $\mathbf{m}(\mathbf{x}_{-i}) = (m(x_1), \cdots, m(x_{i-1}), m(x_{i+1}), \cdots, m(x_n))$ in a similar way. Given these notations, the extensive form game between the sender and the agents is as follows:

1. The sender picks and commits to a communication rule $h \in \mathcal{H}$ to send a message to the agents;

2. The true state θ is realized and is private information to the sender;

3. The agents see a realized message m according to the sender's communication rule;

4. The social activity levels $\mathbf{x}$ are realized, and each agent $i \in I$ observes her own

x_i;

5. Each agent then picks her protection level q_i based on the prior information, the message m provided by the sender, and her activity level x_i;

6. The network is realized, the infection spreads, and the payoffs are realized.

This setting is different from a typical information design problem described by [BM19] in that the communication rule cannot be conditioned on the realization of the social activity profile. We use this assumption because in the context of infection protection, it is hard for the sender to have agents reveal their social activity levels with certain precision and accuracy. Instead, the message m above can be viewed as a public announcement that specifies instructions for different types of agents. The communication rule encodes the information that the agents receive about the realized state of the world. The conditional dependence of the announced message m on the state of the world θ represents the information conveyed to the agents.

Next, we define the obedience constraint on the communication rule h. Obedience is the requirement that the message $m : [0,1] \to \{q_L, q_H\}$ announced publicly to all agents according to the pre-committed rule $h : \Theta \to \Delta(\mathcal{M})$ is such that each agent i would prefer to follow her recommended protection level $m(x_i)$ rather than choosing any other protection level q_i'. A formal definition is as follows.

Definition 4 (Obedience Constraint). *A communication rule $h : \Theta \to \Delta(\mathcal{M})$ with $\mathcal{M} = \{m : [0,1] \to \{q_L, q_H\}\}$ is obedient if, for each $i \in I$, $x_i \in [0,1]$, and $m \in M$, we have*

$$\mathbb{E}_\theta \mathbb{E}_{\mathbf{x}_{-i}} \left[h(m \mid \theta) \left(u_i^{(n)} (\mathbf{m}(\mathbf{x}), \mathbf{x}, \theta) \right] \geq \mathbb{E}_\theta \mathbb{E}_{\mathbf{x}_{-i}} \left[h(m \mid \theta) u_i^{(n)} ((q_i', \mathbf{m}(\mathbf{x}_{-i})), \mathbf{x}, \theta) \right) \right], \tag{2.10}$$

for all $q_i' \in \{q_L, q_H\}$, *i.e.,*

$$\sum_{\theta\in\Theta}\int_{\mathbf{x}_{-i}\in[0,1]^{n-1}} \mu(\theta)\,\nu_{-i}(\mathbf{x}_{-i})\,h(m\mid\theta)\,u_i^{(n)}(\mathbf{m}(\mathbf{x}),\mathbf{x},\theta)\,d\mathbf{x}_{-i}$$
$$\geq \sum_{\theta\in\Theta}\int_{\mathbf{x}_{-i}\in[0,1]^{n-1}} \mu(\theta)\,\nu_{-i}(\mathbf{x}_{-i})\,h(m\mid\theta)\,u_i^{(n)}((q_i',\mathbf{m}(\mathbf{x}_{-i})),\mathbf{x},\theta)\,d\mathbf{x}_{-i}.$$

The obedience constraint requires that after observing the public announcement m, each agent i with a social activity level x_i finds that no other action q_i' could yield her a strictly higher utility than following the recommendation $m(x_i)$. Next, using the obedience constraint and the reduced set of signals, we write the social planner's problem and establish the revelation principle: a communication rule h satisfies obedience if and only if there exists an information policy f that gives rise to this communication rule in the Bayesian Nash equilibrium.

Using the social welfare introduced in Definition 3, the social planner's problem becomes

$$Z_1 \doteq \max_{h\in\mathcal{H}}\ \mathbb{E}_\theta\mathbb{E}_{\mathbf{x}}\mathbb{E}_h\left[W^{(n)}(\mathbf{m}(\mathbf{x}),\mathbf{x},\theta)\right] \tag{2.11}$$

$$\text{s.t. } \mathbb{E}_\theta\mathbb{E}_{\mathbf{x}_{-i}}\left[h(m\mid\theta)\,u_i^{(n)}(\mathbf{m}(\mathbf{x}),\mathbf{x},\theta)\right] \geq \mathbb{E}_\theta\mathbb{E}_{\mathbf{x}_{-i}}\left[h(m\mid\theta)\,u_i^{(n)}((q_i',\mathbf{m}(\mathbf{x}_{-i})),\mathbf{x},\theta)\right],$$
$$\text{for all } i\in I, x_i\in[0,1], m\in M, q_i'\in\{q_L,q_H\} \tag{2.12}$$

$$\sum_{m\in\mathcal{M}} h(m\mid\theta) = 1,\ \text{for all } \theta\in\Theta \tag{2.13}$$

$$h(m\mid\theta)\geq 0,\ \text{for all } m\in\mathcal{M},\theta\in\Theta, \tag{2.14}$$

where

$$\mathbb{E}_\theta\mathbb{E}_{\mathbf{x}}\mathbb{E}_h\left[W^{(n)}(\mathbf{m}(\mathbf{x}),\mathbf{x},\theta)\right] = \sum_{\theta\in\Theta}\int_{\mathbf{x}}\sum_{m\in\mathcal{M}}\mu(\theta)\,\nu(\mathbf{x})\,h(m\mid\theta)\,W^{(n)}(\mathbf{m}(\mathbf{x}),\mathbf{x},\theta)\,d\mathbf{x}.$$

The social planner's problem is a constrained optimization problem. The objective function (2.11) maximizes the expected (normalized) social welfare. Constraint

(2.12) ensures that h is obedient (Definition 4). Constraints (2.13) and (2.14) ensure h satisfies general probability rules and is an element of $\mathcal{H}$. The value Z_1 is the optimal objective value of the constrained optimization problem. Note that this is an extremely large-scale optimization problem and cannot be solved for practical purposes. In the next section, we show that it is sufficient to focus on communication rules that recommend protections based on social activity thresholds.

In the next result, we establish the equivalence between the optimization problem (2.6) - (2.9) and the optimization problem (2.11) - (2.14) by proving that Z_1 is identical to Z_0 under all circumstances. The result is as follows.

Proposition 1 (Revelation Principle)**.** *$Z_0 = Z_1$ for any prior $\mu(\cdot)$.*

Proof. First, we show that $Z_1 \geq Z_0$. Let f', a^E be the optimal solution to the optimization problem (2.6) - (2.9). That is, $Z_0 = \mathbb{E}_\theta \mathbb{E}_{\mathbf{x}} \mathbb{E}_{f'} \left[W^{(n)} \left(\mathbf{a}^E (\mathbf{x}, \mu_s), \mathbf{x}, \theta \right) \right]$. Let the communication rule h' be defined as

$$h'(m \mid \theta) = \sum_{s \in S} f'(s \mid \theta) \mathbb{1} \left\{ m = a^E (\cdot, \mu_s) \right\}, \tag{2.15}$$

that is, the communication rule h' recommends the message m only if the information policy f' sends the signal s and the message m is the same as the symmetric Bayesian Nash equilibrium $a^E(\cdot, \mu_s)$ in the followed game with realized signal s and posterior μ_s. We now show that the communication rule h' is a feasible rule that satisfies the constraints (2.12)-(2.14) and also attains the same social welfare in (2.11) as Z_0. This suffices to show that $Z_1 \geq Z_0$. For now, we assume a^E is the unique symmetric equilibrium and later prove this claim. This implies that there is a unique m where $\mathbb{1}\left\{ m = a^E(\cdot, \mu_s) \right\} = 1$. Since f' is feasible, by (2.7), we have for all $i \in I$, $x_i \in [0, 1]$, and $s \in S$,

$$\mathbb{E}_{\theta \sim \mu_s, \mathbf{x}_{-i} \sim \nu_{-i}} \left[u_i^{(n)} \left(\mathbf{a}^E \left(\mathbf{x}, \mu_s \right), \mathbf{x}, \theta \right) - u_i^{(n)} \left(\left(q_i', \mathbf{a}^E \left(\mathbf{x}_{-i}, \mu_s \right) \right), \mathbf{x}, \theta \right) \right] \geq 0,$$

which implies

$$\int_{\mathbf{x}_{-i}} \nu_{-i} \left(\mathbf{x}_{-i} \right) \sum_{\theta} \frac{f' \left(s \mid \theta \right) \mu \left(\theta \right)}{\sum_{\theta'} f' \left(s \mid \theta' \right) \mu \left(\theta' \right)} \times \\ \left[u_i^{(n)} \left(\mathbf{a}^E \left(\mathbf{x}, \mu_s \right), \mathbf{x}, \theta \right) - u_i^{(n)} \left(\left(q_i', \mathbf{a}^E \left(\mathbf{x}_{-i}, \mu_s \right) \right), \mathbf{x}, \theta \right) \right] d\mathbf{x}_{-i} \geq 0,$$

for all q_i'. Multiplying $\sum_{\theta'} f' \left(s \mid \theta' \right) \mu \left(\theta' \right)$ on both sides and summing over s, we have for all $i \in I$ and $x_i \in [0, 1]$,

$$\int_{\mathbf{x}_{-i}} \nu_{-i} \left(\mathbf{x}_{-i} \right) \sum_{\theta} \mu \left(\theta \right) \sum_{s \in S} f' \left(s \mid \theta \right) \times \\ \left[u_i^{(n)} \left(\mathbf{a}^E \left(\mathbf{x}, \mu_s \right), \mathbf{x}, \theta \right) - u_i^{(n)} \left(\left(q_i', \mathbf{a}^E \left(\mathbf{x}_{-i}, \mu_s \right) \right), \mathbf{x}, \theta \right) \right] d\mathbf{x}_{-i} \geq 0,$$

which, by invoking (2.15), implies

$$\mathbb{E}_{\theta} \mathbb{E}_{\mathbf{x}_{-i}} h' \left(m \mid \theta \right) \left[u_i^{(n)} \left(\mathbf{m} \left(\mathbf{x} \right), \mathbf{x}, \theta \right) - u_i^{(n)} \left(\left(q_i', \mathbf{m} \left(\mathbf{x}_{-i} \right) \right), \mathbf{x}, \theta \right) \right] \geq 0.$$

Therefore, constraints (2.12) are satisfied.

Moreover, for all θ, we have

$$\sum_{m \in \mathcal{M}} h' \left(m \mid \theta \right) = \sum_{m \in \mathcal{M}} \sum_{s \in S} f' \left(s \mid \theta \right) \mathbb{1} \left\{ m = a^E \left(\cdot, \mu_s \right) \right\} \overset{(a)}{=} \sum_{s \in S} f' \left(s \mid \theta \right) \overset{(b)}{=} 1,$$

where (a) follows from the fact that there is a unique m where $\mathbb{1} \left\{ m = a^E \left(\cdot, \mu_s \right) \right\} = 1$, and (b) follows from equation (2.8). Therefore, constraints (2.13) are satisfied.

Constraints (2.14) are satisfied because for all $m \in \mathcal{M}$ and $\theta \in \Theta$, we can write

$$h'(m \mid \theta) = \sum_{s \in S} f'(s \mid \theta) \mathbb{1}\left\{m = a^E(\cdot, \mu_s)\right\} \overset{(a)}{\geq} 0,$$

where (a) follows from (2.9). This completes the feasibility proof for the communication rule h'. Lastly, we show that h' attains the same social welfare as f'. We can write

$$\begin{aligned}
&\mathbb{E}_\theta \mathbb{E}_{\mathbf{x}} \mathbb{E}_{h'}\left[W^{(n)}(\mathbf{m}(\mathbf{x}), \mathbf{x}, \theta)\right] \\
&= \sum_\theta \int_{\mathbf{x}} \sum_{m \in \mathcal{M}} \mu(\theta)\, \nu(\mathbf{x})\, h'(m \mid \theta)\, W^{(n)}(\mathbf{m}(\mathbf{x}), \mathbf{x}, \theta)\, dx \\
&\overset{(a)}{=} \sum_\theta \int_{\mathbf{x}} \mu(\theta)\, \nu(\mathbf{x}) \sum_{s \in S} f'(s \mid \theta)\, W^{(n)}\left(\mathbf{a}^E(\mathbf{x}, \mu_s), \mathbf{x}, \theta\right) dx \\
&= \mathbb{E}_\theta \mathbb{E}_{\mathbf{x}} \mathbb{E}_{f'}\left[W^{(n)}\left(\mathbf{a}^E(\mathbf{x}, \mu_s), \mathbf{x}, \theta\right)\right] = Z_0,
\end{aligned}$$

where (a) follows from (2.15) and the uniqueness of the equilibrium. Since (2.6) is a maximization problem, we have $Z_1 \geq Z_0$.

Next, we prove the reverse direction, i.e., $Z_0 \geq Z_1$. Let h' be the optimal communication rule for the optimization problem (2.11) - (2.14). That is, $Z_1 = \mathbb{E}_\theta \mathbb{E}_{\mathbf{x}} \mathbb{E}_{h'}\left[W^{(n)}(\mathbf{m}(\mathbf{x}), \mathbf{x}, \theta)\right]$. Note that the signal space $\mathcal{S}$ is general enough and that any message m can be considered as a signal itself. We construct the information disclosure policy f' as

$$f'(s \mid \theta) = \begin{cases} h'(m \mid \theta) & \text{if } s = m \\ 0 & \text{otherwise.} \end{cases} \tag{2.16}$$

Next, we prove that the information disclosure policy f' is a feasible policy that satisfies the constraints (2.7) - (2.9) and also attains the same social welfare in (2.6) as Z_1. This suffices to prove that $Z_0 \geq Z_1$. The non-negativity constraints (2.9) are trivially satisfied by construction. We have

$$\sum_s f'(s \mid \theta) = \sum_m h'(m \mid \theta) = 1$$

for all θ, which satisfies constraints (2.8). To prove (2.7), we construct the following action $a : [0,1] \times \Delta(\Theta) \to \{q_L, q_H\}$ as

$$a(x, \mu_m) = m(x)$$

and prove that a satisfies (2.4) and is thus a Bayesian Nash equilibrium. We have

$$\begin{aligned}
&\mathbb{E}_{\theta \sim \mu_m} \mathbb{E}_{\mathbf{x}_{-i}} \left[u_i^{(n)} \left(\mathbf{a}(\mathbf{x}, \mu_m), \mathbf{x}, \theta \right) - u_i^{(n)} \left(\left(q_i', \mathbf{a}(\mathbf{x}_{-i}, \mu_m) \right), \mathbf{x}, \theta \right) \right] \\
&= \mathbb{E}_{\mathbf{x}_{-i}} \sum_\theta \frac{f'(m \mid \theta)\mu(\theta)}{\sum_{\theta'} f'(m \mid \theta')\mu(\theta')} \left[u_i^{(n)} \left(\mathbf{m}(\mathbf{x}), \mathbf{x}, \theta \right) - u_i^{(n)} \left(\left(q_i', \mathbf{m}(\mathbf{x}_{-i}) \right), \mathbf{x}, \theta \right) \right] \\
&= \frac{1}{\sum_{\theta'} f'(m \mid \theta')\mu(\theta')} \mathbb{E}_\theta \mathbb{E}_{\mathbf{x}_{-i}} \left[h'(m \mid \theta) \left[u_i^{(n)} \left(\mathbf{m}(\mathbf{x}), \mathbf{x}, \theta \right) - u_i^{(n)} \left(\left(q_i', \mathbf{m}(\mathbf{x}_{-i}) \right), \mathbf{x}, \theta \right) \right] \right] \geq 0
\end{aligned}$$

for all $q_i' \in \{q_L, q_H\}$ and $x_i \in [0,1]$ following (2.12). Therefore, a satisfies (2.4) and is indeed a Bayesian Nash equilibrium. Constraints (2.7) are satisfied.

Lastly, we show that the constructed f' and a attain the same social welfare in (2.6) as h' attains in (2.11). We have

$$\begin{aligned}
\mathbb{E}_\theta \mathbb{E}_\mathbf{x} \mathbb{E}_f \left[W^{(n)} \left(\mathbf{a}^E(\mathbf{x}, \mu_s), \mathbf{x}, \theta \right) \right] &= \mathbb{E}_\theta \mathbb{E}_\mathbf{x} \mathbb{E}_f \left[W^{(n)} \left(a(x, \mu_m), \mathbf{x}, \theta \right) \right] \\
&= \mathbb{E}_\theta \mathbb{E}_\mathbf{x} \mathbb{E}_h \left[W^{(n)} \left(\mathbf{m}(\mathbf{x}), \mathbf{x}, \theta \right) \right] = Z_1
\end{aligned}$$

by construction. Since (2.6) is a maximization problem, we have $Z_0 \geq Z_1$. Therefore, we have $Z_1 = Z_0$. This completes the proof. $\square$

2.4 Existence and Uniqueness of Equilibrium

In this section, we prove the existence and uniqueness of the symmetric Bayesian Nash equilibrium in the incomplete information game among the agents. We start with proving some properties of the infection probability $\boldsymbol{P}$ from Definition 1.

2.4.1 Properties of the Infection Probability

In this section, we first analyze key properties of the infection probability $\boldsymbol{P}$. As illustrated in Example 1, the infection probability depends on the graph structure as well as the protection profile of the agents. We first prove that the infection probability satisfies natural monotonicity and linearity. In what follows, for two vectors $\mathbf{p}, \tilde{\mathbf{p}} \in \mathbb{R}^n$, we write $\mathbf{p} \geq \tilde{\mathbf{p}}$ to denote $\mathbf{p}_i \geq \tilde{\mathbf{p}}_i$ for all $i = 1, \cdots, n$.

Lemma 1 (Monotonicity and Linearity)**.**

(a) For any $i \in I$ and $\mathbf{q} \in Q$, we have

$$\boldsymbol{P}_i^{(n)}(\mathbf{q}, \hat{\mathbf{x}}) \geq \boldsymbol{P}_i^{(n)}(\mathbf{q}, \mathbf{x}), \quad \text{for all } \hat{\mathbf{x}} \geq \mathbf{x}.$$

(b) For any $i \in I$ and $\mathbf{x} \in X$, we have

$$\boldsymbol{P}_i^{(n)}(\hat{\mathbf{q}}, \mathbf{x}) \leq \boldsymbol{P}_i^{(n)}(\mathbf{q}, \mathbf{x}), \quad \text{for all } \hat{\mathbf{q}} \geq \mathbf{q}.$$

(c) For any $i, j \in I$, $\mathbf{x} \in X$, and any protection profile $\mathbf{q}_{-j} \in Q_{-j}$, $\boldsymbol{P}_i^{(n)}((q_j, \mathbf{q}_{-j}), \mathbf{x})$ is linear in q_j.

Proof. We prove that for any set of nodes $O \in I$ that are infected at time 0 where $i \notin O$, the probability of infection reaching node i is increasing in $\mathbf{x}$ and decreasing in $\mathbf{q}$. We denote this probability by

$$\boldsymbol{P}_i^{(n)}\left(\mathbf{q},\mathbf{x} \mid \text{origin}=O\right).$$

We have

$$\boldsymbol{P}_i^{(n)}\left(\mathbf{q},\mathbf{x}\right) = \left(1-q_i\right)\Big(\pi + \left(1-\pi\right)\boldsymbol{P}_i^{(n)}\left(\mathbf{q},\mathbf{x} \mid \text{origin}=O\right)\Big).$$

We also use an auxiliary infection probability in this lemma. In particular, for any set of nodes O, we let

$$\tilde{\boldsymbol{P}}_i^{(n)}\left(\mathbf{q},\mathbf{x} \mid \text{origin}=O\right)$$

denote the probability of infection reaching node i, $i \notin O$, in one round (i.e., only through the nodes in O).

Claim 2. *For any set O and $i \in I$, the probability $\tilde{\boldsymbol{P}}_i^{(n)}\left(\mathbf{q},\mathbf{x} \mid \text{origin}=O\right)$ is increasing in $\mathbf{x}$ and decreasing in $\mathbf{q}$.*

Proof of Claim 2: We prove this claim by induction on the size of O. If $|O|=1$ and contains only node j (i.e., $O=\{j\}$), then the probability is $x_i x_j \left(1-q_i\right)\left(1-q_j\right)\frac{\beta}{n}$, which is increasing in $\mathbf{x}$ and decreasing in $\mathbf{q}$. Now suppose $|O|>1$ and let $j \in O$. We can write

$$\begin{aligned}\tilde{\boldsymbol{P}}_i^{(n)}\left(\mathbf{q},\mathbf{x} \mid \text{origin}=O\right) &= x_i x_j \left(1-q_i\right)\left(1-q_j\right)\frac{\beta}{n} \\ &\quad + \left(1 - x_i x_j \left(1-q_i\right)\left(1-q_j\right)\frac{\beta}{n}\right)\tilde{\boldsymbol{P}}_i^{(n)}\left(\mathbf{q},\mathbf{x} \mid \text{origin}=O\setminus\{j\}\right).\end{aligned}$$

Taking derivative of this expression with respect to x_i gives

$$\frac{\partial}{\partial x_i}\tilde{\boldsymbol{P}}_i^{(n)}\left(\mathbf{q},\mathbf{x} \mid \text{origin}=O\right)$$

$$= x_j\beta\left(1-q_i\right)\left(1-q_j\right) - x_j\frac{\beta}{n}\left(1-q_i\right)\left(1-q_j\right)\tilde{\boldsymbol{P}}_i^{(n)}\left(\mathbf{q},\mathbf{x}\mid \text{origin}=O\setminus\{j\}\right)$$
$$+\left(1-x_ix_j\left(1-q_i\right)\left(1-q_j\right)\frac{\beta}{n}\right)\frac{\partial}{\partial x_i}\tilde{\boldsymbol{P}}_i^{(n)}\left(\mathbf{q},\mathbf{x}\mid \text{origin}=O\setminus\{j\}\right)\overset{(a)}{\geq} 0,$$

where (a) follows from the induction hypothesis. This establishes the monotonicity in x_i. Furthermore, the monotonicity in $\mathbf{x}_{-i}$ follows by induction hypothesis, which says $\tilde{\boldsymbol{P}}_i^{(n)}\left(\mathbf{q},\mathbf{x}\mid \text{origin}=O\setminus\{j\}\right)$ is increasing in $\mathbf{x}_{-i}$.

Taking the derivative of this expression with respect to q_i gives

$$\frac{\partial}{\partial q_i}\tilde{\boldsymbol{P}}_i^{(n)}\left(\mathbf{q},\mathbf{x}\mid \text{origin}=O\right) = -x_ix_j\frac{\beta}{n}\left(1-q_j\right)+x_ix_j\frac{\beta}{n}\left(1-q_j\right)\tilde{\boldsymbol{P}}_i^{(n)}\left(\mathbf{q},\mathbf{x}\mid \text{origin}=O\setminus\{j\}\right)$$
$$+\left(1-x_ix_j\frac{\beta}{n}\left(1-q_i\right)\left(1-q_j\right)\right)\frac{\partial}{\partial q_i}\tilde{\boldsymbol{P}}_i^{(n)}\left(\mathbf{q},\mathbf{x}\mid \text{origin}=O\setminus\{j\}\right)\overset{(a)}{\leq} 0,$$

where (a) follows from the induction hypothesis. This establishes the monotonicity in q_i. Furthermore, the monotonicity in $\mathbf{q}_{-i}$ follows by induction hypothesis, which says $\tilde{\boldsymbol{P}}_i^{(n)}\left(\mathbf{q},\mathbf{x}\mid \text{origin}=O\setminus\{j\}\right)$ is decreasing in $\mathbf{q}_{-i}$. □

We now proceed with the proof of Lemma 1 by induction on the number of nodes in the network. The lemma evidently holds for $n=2$. We can write

$$\boldsymbol{P}_i^{(n)}\left(\mathbf{q},\mathbf{x}\mid \text{origin}=O\right) = \tilde{\boldsymbol{P}}_i^{(n)}\left(\mathbf{q},\mathbf{x}\mid \text{origin}=O\right)+\left(1-\tilde{\boldsymbol{P}}_i^{(n)}\left(\mathbf{q},\mathbf{x}\mid \text{origin}=O\right)\right)$$
$$\times\sum_{O^{\text{new}}\in I\setminus O}\mathbb{P}\left(O^{\text{new}}\mid O\right)\boldsymbol{P}_i^{(n)}\left(\mathbf{q},\mathbf{x}\mid \text{origin}=O^{\text{new}}\right).$$

Taking the derivative of this expression with respect to x_i leads to

$$\frac{\partial}{\partial x_i}\boldsymbol{P}_i^{(n)}\left(\mathbf{q},\mathbf{x}\mid \text{origin}=O\right) = \frac{\partial}{\partial x_i}\tilde{\boldsymbol{P}}_i^{(n)}\left(\mathbf{q},\mathbf{x}\mid \text{origin}=O\right)$$
$$-\frac{\partial}{\partial x_i}\tilde{\boldsymbol{P}}_i^{(n)}\left(\mathbf{q},\mathbf{x}\mid \text{origin}=O\right)\sum_{O^{\text{new}}\in I\setminus S}\mathbb{P}\left(O^{\text{new}}\mid O\right)\boldsymbol{P}_i^{(n)}\left(\mathbf{q},\mathbf{x}\mid \text{origin}=O^{\text{new}}\right)$$
$$+\left(1-\tilde{\boldsymbol{P}}_i^{(n)}\left(\mathbf{q},\mathbf{x}\mid \text{origin}=O\right)\right)\sum_{O^{\text{new}}}\mathbb{P}\left(O^{\text{new}}\mid O\right)\frac{\partial}{\partial x_i}\boldsymbol{P}_i^{(n)}\left(\mathbf{q},\mathbf{x}\mid \text{origin}=O^{\text{new}}\right)$$
$$=\frac{\partial}{\partial x_i}\tilde{\boldsymbol{P}}_i^{(n)}\left(\mathbf{q},\mathbf{x}\mid \text{origin}=O\right)\left(1-\sum_{O^{\text{new}}}\mathbb{P}\left(O^{\text{new}}\mid O\right)\boldsymbol{P}_i^{(n)}\left(\mathbf{q},\mathbf{x}\mid \text{origin}=O^{\text{new}}\right)\right)$$

$$
\begin{aligned}
&+ \left(1 - \tilde{\boldsymbol{P}}_i^{(n)}\left(\mathbf{q}, \mathbf{x} \mid \text{origin} = O\right)\right) \sum_{O^{\text{new}}} \mathbb{P}\left(O^{\text{new}} \mid O\right) \frac{\partial}{\partial x_i} \boldsymbol{P}_i^{(n)}\left(\mathbf{q}, \mathbf{x} \mid \text{origin} = O^{\text{new}}\right) \\
&\overset{(a)}{\geq} 0,
\end{aligned}
$$

where (a) follows from Claim 1 and the induction hypothesis. Since

$$
\frac{\partial}{\partial x_i} \boldsymbol{P}_i^{(n)}\left(\mathbf{q}, \mathbf{x}\right) = \frac{\partial}{\partial x_i} \boldsymbol{P}_i^{(n)}\left(\mathbf{q}, \mathbf{x} \mid \text{origin} = O\right),
$$

this establishes monotonicity in x_i. Furthermore, the monotonicity in $\mathbf{x}_{-i}$ follows by induction hypothesis.

Taking the derivative of this expression with respect to q_i leads to

$$
\begin{aligned}
\frac{\partial}{\partial q_i} \boldsymbol{P}_i^{(n)}\left(\mathbf{q}, \mathbf{x} \mid \text{origin} = O\right) &= \frac{\partial}{\partial q_i} \tilde{\boldsymbol{P}}_i^{(n)}\left(\mathbf{q}, \mathbf{x} \mid \text{origin} = O\right) \\
&- \frac{\partial}{\partial q_i} \tilde{\boldsymbol{P}}_i^{(n)}\left(\mathbf{q}, \mathbf{x} \mid \text{origin} = O\right) \sum_{O^{\text{new}} \in I \setminus O} \mathbb{P}\left(O^{\text{new}} \mid O\right) \boldsymbol{P}_i^{(n)}\left(\mathbf{q}, \mathbf{x} \mid \text{origin} = O^{\text{new}}\right) \\
&+ \left(1 - \tilde{\boldsymbol{P}}_i^{(n)}\left(\mathbf{q}, \mathbf{x} \mid \text{origin} = O\right)\right) \sum_{O^{\text{new}}} \mathbb{P}\left(O^{\text{new}} \mid O\right) \frac{\partial}{\partial q_i} \boldsymbol{P}_i^{(n)}\left(\mathbf{q}, \mathbf{x} \mid \text{origin} = O^{\text{new}}\right) \\
=& \frac{\partial}{\partial q_i} \tilde{\boldsymbol{P}}_i^{(n)}\left(\mathbf{q}, \mathbf{x} \mid \text{origin} = O\right) \left(1 - \sum_{O^{\text{new}}} \mathbb{P}\left(O^{\text{new}} \mid O\right) \boldsymbol{P}_i^{(n)}\left(\mathbf{q}, \mathbf{x} \mid \text{origin} = O^{\text{new}}\right)\right) \\
&+ \left(1 - \tilde{\boldsymbol{P}}_i^{(n)}\left(\mathbf{q}, \mathbf{x} \mid \text{origin} = O\right)\right) \sum_{O^{\text{new}}} \mathbb{P}\left(O^{\text{new}} \mid O\right) \frac{\partial}{\partial q_i} \boldsymbol{P}_i^{(n)}\left(\mathbf{q}, \mathbf{x} \mid \text{origin} = O^{\text{new}}\right) \\
&\overset{(a)}{\leq} 0,
\end{aligned}
$$

where (a) follows from Claim 1 and the induction hypothesis. Therefore,

$$
\frac{\partial}{\partial q_i} \boldsymbol{P}_i^{(n)}\left(\mathbf{q}, \mathbf{x}\right) = -\pi + (1 - \pi) \frac{\partial}{\partial q_i} \boldsymbol{P}_i^{(n)}\left(\mathbf{q}, \mathbf{x} \mid \text{origin} = O\right) \leq 0.
$$

This establishes monotonicity in q_i. Furthermore, the monotonicity in $\mathbf{q}_{-i}$ follows by induction hypothesis.

To show part (c) of the lemma, we construct the following probabilities. For any $j \in [n]$, we let $\overline{\mathbb{P}}_i^{(n)}\left(\mathbf{q}_{-j}, \mathbf{x}\right)$ denote the probability of agent i getting infected when j

is not protected, i.e., $q_j = 0$. We also let $\underline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-j}, \mathbf{x})$ denote the probability of agent i getting infected when j is protected, i.e., $q_j = 1$. By construction, we have

$$\boldsymbol{P}_i^{(n)}(\mathbf{q}, \mathbf{x}) = (1 - q_i)\, \overline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-i}, \mathbf{x}) + q_i \times 0 = (1 - q_j)\, \overline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-j}, \mathbf{x}) + q_j \underline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-j}, \mathbf{x}). \tag{2.17}$$

Since $\overline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-i}, \mathbf{x})$ is irrelevant to q_i and $\overline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-j}, \mathbf{x})$, $\underline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-j}, \mathbf{x})$ is irrelevant to q_j, we have

$$\frac{\partial^2}{\partial q_i^2} \boldsymbol{P}_i^{(n)}(\mathbf{q}, \mathbf{x}) = \frac{\partial^2}{\partial q_j^2} \boldsymbol{P}_i^{(n)}(\mathbf{q}, \mathbf{x}) = 0.$$

This completes the proof. $\square$

Part (a) of Lemma 1 proves that the infection probability for an agent is increasing in the social activity levels of all agents because higher social activity leads to a denser social network over which the infection can spread. Similarly, part (b) establishes that the infection probability for an agent is decreasing in the protection levels of all agents because higher protection results in a smaller social network over which the infection can spread. Furthermore, part (c) of Lemma 1 proves that this probability is linear in any agent's protection level.

Next, we establish that the stochastic process of the spread of the infection not only satisfies natural monotonicity and linearity but also satisfies submodularity.

Lemma 2 (Submodularity)**.** *For any $i \in I$, we have*

$$\boldsymbol{P}_i^{(n)}(\mathbf{q}, \hat{\mathbf{x}}) - \boldsymbol{P}_i^{(n)}(\hat{\mathbf{q}}, \hat{\mathbf{x}}) \geq \boldsymbol{P}_i^{(n)}(\mathbf{q}, \mathbf{x}) - \boldsymbol{P}_i^{(n)}(\hat{\mathbf{q}}, \mathbf{x}), \tag{2.18}$$

for all $\hat{\mathbf{q}} \geq \mathbf{q}$ and $\hat{\mathbf{x}} \geq \mathbf{x}$.

Proof. We first show the submodularity in q_i and $\mathbf{x}$ using equation (2.17), which we proved in the proof of Lemma 1. We have

$$\boldsymbol{P}_i^{(n)}(\mathbf{q},\mathbf{x}) = (1-q_i)\,\overline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-i},\mathbf{x}).$$

Therefore,

$$\begin{aligned}
&\Big(\boldsymbol{P}_i^{(n)}((q_i=q_H,\mathbf{q}_{-i}),\hat{\mathbf{x}}) - \boldsymbol{P}_i^{(n)}((q_i=q_H,\mathbf{q}_{-i}),\mathbf{x})\Big)\\
&-\Big(\boldsymbol{P}_i^{(n)}((q_i=q_L,\mathbf{q}_{-i}),\hat{\mathbf{x}}) - \boldsymbol{P}_i^{(n)}((q_i=q_L,\mathbf{q}_{-i}),\mathbf{x})\Big)\\
&=(1-q_H)\,\overline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-i},\hat{\mathbf{x}}) - (1-q_H)\,\overline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-i},\mathbf{x})\\
&-\Big((1-q_L)\,\overline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-i},\hat{\mathbf{x}}) - (1-q_L)\,\overline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-i},\mathbf{x})\Big)\\
&=(q_L-q_H)\Big(\overline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-i},\hat{\mathbf{x}}) - \overline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-i},\mathbf{x})\Big) \le 0,
\end{aligned}$$

since $\overline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-i},\mathbf{x})$ is increasing in $\mathbf{x}$ by Lemma 1 and $q_L \le q_H$.

For any $j \neq i$, we have

$$\boldsymbol{P}_i^{(n)}(\mathbf{q},\mathbf{x}) = (1-q_j)\,\overline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-j},\mathbf{x}) + q_j\underline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-j},\mathbf{x}).$$

Therefore,

$$\begin{aligned}
&\Big(\boldsymbol{P}_i^{(n)}((q_j=q_H,\mathbf{q}_{-j}),\hat{\mathbf{x}}) - \boldsymbol{P}_i^{(n)}((q_j=q_H,\mathbf{q}_{-j}),\mathbf{x})\Big)\\
&-\Big(\boldsymbol{P}_i^{(n)}((q_j=q_L,\mathbf{q}_{-j}),\hat{\mathbf{x}}) - \boldsymbol{P}_i^{(n)}((q_j=q_L,\mathbf{q}_{-j}),\mathbf{x})\Big)\\
&=(1-q_H)\,\overline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-j},\hat{\mathbf{x}}) + q_H\underline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-j},\hat{\mathbf{x}}) - (1-q_H)\,\overline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-i},\mathbf{x}) - q_H\underline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-i},\mathbf{x})\\
&-(1-q_L)\,\overline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-i},\hat{\mathbf{x}}) - q_L\underline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-i},\hat{\mathbf{x}}) + (1-q_L)\,\overline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-i},\mathbf{x}) + q_L\underline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-i},\mathbf{x})\\
&=(q_L-q_H)\Big[\Big(\overline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-i},\hat{\mathbf{x}}) - \underline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-i},\hat{\mathbf{x}})\Big) - \Big(\overline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-i},\mathbf{x}) - \underline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-i},\mathbf{x})\Big)\Big].
\end{aligned}$$

The lemma follows by showing that $\overline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-i},\mathbf{x}) - \underline{\mathbb{P}}_i^{(n)}(\mathbf{q}_{-i},\mathbf{x})$ is increasing in x_j. Note that the infection probability is a linear expectation of the indicator functions.

Therefore, for $\bar{x}_j \geq \underline{x}_j$, we have

$$\begin{aligned}
&\overline{\mathbb{P}}_i^{(n)}\left(\mathbf{q}_{-i},\left(\bar{x}_j,\mathbf{x}_{-j}\right)\right)-\underline{\mathbb{P}}_i^{(n)}\left(\mathbf{q}_{-i},\left(\bar{x}_j,\mathbf{x}_{-j}\right)\right)-\left(\overline{\mathbb{P}}_i^{(n)}\left(\mathbf{q}_{-i},\left(\underline{x}_j,\mathbf{x}_{-j}\right)\right)-\underline{\mathbb{P}}_i^{(n)}\left(\mathbf{q}_{-i},\left(\underline{x}_j,\mathbf{x}_{-j}\right)\right)\right)\\
=&\mathbb{E}_{v\sim V',E\sim\bar{E}}\left[\mathbb{E}_O\left[\mathbb{1}\left\{O\rightsquigarrow i\right\}-\mathbb{1}\left\{O\rightsquigarrow i\text{ not through }j\right\}\right]\right]\\
&-\mathbb{E}_{v\sim V',E\sim\underline{E}}\left[\mathbb{E}_O\left[\mathbb{1}\left\{O\rightsquigarrow i\right\}-\mathbb{1}\left\{O\rightsquigarrow i\text{ not through }j\right\}\right]\right],
\end{aligned}$$

where O is the same as in definition 1; $V'=(V_1,\cdots,V_n)$ is such that for $i\neq j$, V_i is a Bernoulli random variable that is 1 with probability $1-q_i$ and V_j is 1 with probability 1; and $\overline{E}$ is defined in the same way as in Definition 1 with $x_j=\overline{x}_j$ while $\underline{E}$ is defined with $x_j=\underline{x}_j$. Note that $\mathbb{E}_{v\sim V',E\sim\overline{E}}\left[\mathbb{E}_O\left[\mathbb{1}\left\{O\rightsquigarrow i\text{ not through }j\right\}\right]\right]$ is unchanged in x_j. Therefore, we obtain

$$\begin{aligned}
&\overline{\mathbb{P}}_i^{(n)}\left(\mathbf{q}_{-i},\left(\overline{x}_j,\mathbf{x}_{-j}\right)\right)-\underline{\mathbb{P}}_i^{(n)}\left(\mathbf{q}_{-i},\left(\overline{x}_j,\mathbf{x}_{-j}\right)\right)-\left(\overline{\mathbb{P}}_i^{(n)}\left(\mathbf{q}_{-i},\left(\underline{x}_j,\mathbf{x}_{-j}\right)\right)-\underline{\mathbb{P}}_i^{(n)}\left(\mathbf{q}_{-i},\left(\underline{x}_j,\mathbf{x}_{-j}\right)\right)\right)\\
&=\overline{\mathbb{P}}_i^{(n)}\left(\mathbf{q}_{-i},\left(\overline{x}_j,\mathbf{x}_{-j}\right)\right)-\overline{\mathbb{P}}_i^{(n)}\left(\mathbf{q}_{-i},\left(\underline{x}_j,\mathbf{x}_{-j}\right)\right)\geq 0.
\end{aligned}$$

Since $\overline{\mathbb{P}}_i^{(n)}\left(\mathbf{q}_{-i},\mathbf{x}\right)$ is increasing in x_j by Lemma 1, together with the assumption $q_L<q_H$, we get

$$\begin{aligned}
&\left(\mathbb{P}_i^{(n)}\left(\left(q_j=q_H,\mathbf{q}_{-j}\right),\hat{\mathbf{x}}\right)-\mathbb{P}_i^{(n)}\left(\left(q_j=q_H,\mathbf{q}_{-j}\right),\mathbf{x}\right)\right)\\
&-\left(\mathbb{P}_i^{(n)}\left(\left(q_j=q_L,\mathbf{q}_{-j}\right),\hat{\mathbf{x}}\right)-\mathbb{P}_i^{(n)}\left(\left(q_j=q_L,\mathbf{q}_{-j}\right),\mathbf{x}\right)\right)\\
=&\left(q_L-q_H\right)\left[\left(\overline{\mathbb{P}}_i^{(n)}\left(\mathbf{q}_{-i},\hat{\mathbf{x}}\right)-\underline{\mathbb{P}}_i^{(n)}\left(\mathbf{q}_{-i},\hat{\mathbf{x}}\right)\right)-\left(\overline{\mathbb{P}}_i^{(n)}\left(\mathbf{q}_{-i},\mathbf{x}\right)-\underline{\mathbb{P}}_i^{(n)}\left(\mathbf{q}_{-i},\mathbf{x}\right)\right)\right]\leq 0.
\end{aligned}$$

This completes the proof of submodularity. $\square$

Lemma 2 proves that the infection probability for an agent is submodular in the social activity and protection levels of all agents. That is, both sides of the inequality (2.18) represent how much the infection probability is reduced due to higher protection levels. This inequality indicates that high social activity leads to a higher reduction in infection probability from additional protection. Intuitively, this is because higher social activity increases transmission paths, which makes the effectiveness of protection (removing nodes) more prominent in reducing infection.

2.4.2 Existence and Uniqueness of Equilibrium

Before analyzing the optimal messaging policy in the information design problem, we take a deeper look into the Bayesian Nash equilibrium among the agents. We start our analysis by considering the case in which the information is fully revealed and then extend it. In this case, the true state θ is public information to all agents. Following Definition 6, we define a full information equilibrium as follows.

Definition 5 (Full Information Equilibrium)**.** *For any $\theta \in \Theta$, we say a protection strategy $a^{FE} : [0,1] \times \Theta \to \{q_L, q_H\}$ is a symmetric full information equilibrium if for every agent i and social activity level $x_i \in [0,1]$,*

$$\mathbb{E}_{\mathbf{x}_{-i}} \left[u_i^{(n)} \left(\mathbf{a}^{FE} \left(\mathbf{x}, \theta \right), \mathbf{x}, \theta \right) - u_i^{(n)} \left(\left(q_i', \mathbf{a}^{FE} \left(\mathbf{x}_i, \mu \right) \right), \mathbf{x}, \theta \right) \right] \geq 0, \tag{2.19}$$

for all $q_i' \in \{q_L, q_H\}$.

We assume that whenever an agent is indifferent between exerting high and low protection levels, she will choose low protection. With this assumption, we prove that there exists a unique full information equilibrium that is a threshold strategy. That is, agent i exerts low (resp. high) protection if her social activity level is no higher (resp. lower) than the threshold. We make use of the following notation. For a *protection threshold* denoted by $\sigma \in [0,1]$, we define a *threshold protection strategy*

$\mathbf{q}(\sigma, \mathbf{x})$ as $\mathbf{q}(\sigma, \mathbf{x}) = (q_1, \cdots, q_n)$ such that

$$q_i = \begin{cases} q_L & x_i \leq \sigma \\ q_H & x_i > \sigma. \end{cases}$$

To characterize the unique full information equilibrium, we introduce the following variation of utility. For any $\tilde{\mathbf{x}} \in [0,1]^{n-1}$, $q \in \{q_L, q_H\}$ and $x \in [0,1]$, let

$$\tilde{u}^{(n)}(q, x, \sigma, \tilde{\mathbf{x}}, \theta) \doteq u_i^{(n)}\left((q_i = q, \mathbf{q}_{-i} = \mathbf{q}(\sigma, \tilde{\mathbf{x}})), (x_i = x, \mathbf{x}_{-i} = \tilde{\mathbf{x}}), \theta\right) \tag{2.20}$$

denote the utility for the agent with social activity level x and protection q, where the rest of the agents have social activity profile $\tilde{\mathbf{x}}$ and follow a threshold strategy with threshold $\sigma \in [0,1]$.

Proposition 2. *Given any $\theta \in \Theta$, there exists a unique symmetric full information equilibrium $\sigma^{FE}(\theta)$ such that for all $i \in I$,*

$$a^{FE}(x_i, \theta) = \begin{cases} q_L & x_i \leq \sigma^{FE}(\theta) \\ q_H & \textit{otherwise.} \end{cases}$$

Moreover, there are three possibilities for the threshold:

- *If $\mathbb{E}_{\tilde{\mathbf{x}}}\left[\tilde{u}^{(n)}(q_L, 1, 1, \tilde{\mathbf{x}}, \theta) - \tilde{u}^{(n)}(q_H, 1, 1, \tilde{\mathbf{x}}, \theta)\right] \geq 0$, then $\sigma^{FE}(\theta) = 1$.*
- *If $\mathbb{E}_{\tilde{\mathbf{x}}}\left[\tilde{u}^{(n)}(q_L, 0, 0, \tilde{\mathbf{x}}, \theta) - \tilde{u}^{(n)}(q_H, 0, 0, \tilde{\mathbf{x}}, \theta)\right] \leq 0$, then $\sigma^{FE}(\theta) = 0$.*
- *Otherwise, the equilibrium threshold is the solution of*

$$\mathbb{E}_{\tilde{\mathbf{x}}}\left[\tilde{u}^{(n)}\left(q_L, \sigma^{FE}(\theta), \sigma^{FE}(\theta), \tilde{\mathbf{x}}, \theta\right) - \tilde{u}^{(n)}\left(q_H, \sigma^{FE}(\theta), \sigma^{FE}(\theta), \tilde{\mathbf{x}}, \theta\right)\right] = 0. \tag{2.21}$$

Proof. First, we generalize the notation of $\tilde{u}$ to the infection probability $\tilde{\boldsymbol{P}}$ in a similar way, i.e.,

$$\tilde{u}^{(n)}\left(a, x, \sigma, \tilde{\mathbf{x}}, \theta\right) = -\tilde{\boldsymbol{P}}^{(n)}\left(a, x, \sigma, \tilde{\mathbf{x}}\right)\theta - \mathbb{1}\left\{a = q_H\right\}.$$

Note that by definition, we have

$$\tilde{\boldsymbol{P}}^{(n)}\left(a, x, \sigma, \tilde{\mathbf{x}}\right) = \boldsymbol{P}_i^{(n)}\left(\left(a, \mathbf{q}\left(\sigma, \tilde{\mathbf{x}}\right)\right), \left(x, \tilde{\mathbf{x}}\right)\right).$$

Therefore, the result follows directly from Lemma 2 that establishes the submodularity of the infection probability $\boldsymbol{P}_i$. First, let us show that a threshold t as defined in the proposition is indeed an equilibrium. Therefore, for $t \in (0, 1)$, we have

$$\mathbb{E}_{\tilde{\mathbf{x}}}\left[\tilde{u}^{(n)}\left(q_L, t, t, \tilde{\mathbf{x}}, \theta\right) - \tilde{u}^{(n)}\left(q_H, t, t, \tilde{\mathbf{x}}, \theta\right)\right] = 0.$$

Denote the protection profile induced by this threshold t as $\mathbf{q}_t = \mathbf{q}\left(t, \mathbf{x}\right)$. The above equation implies that for agent i with social activity level $x_i = t$, we have

$$\mathbb{E}_{\mathbf{x}_{-i}}\left[u_i^{(n)}\left(\mathbf{q}_t, \left(t, \mathbf{x}_{-i}\right), \theta\right) - u_i^{(n)}\left(\left(q_i', \mathbf{q}_{-i}\right), \left(t, \mathbf{x}_{-i}\right), \theta\right)\right] = 0,$$

for all $q_i' \in [q_L, q_H]$. Therefore the equilibrium condition for agents with an activity level right at the boundary is satisfied. Next, we show that the condition is also satisfied for agents on the two sides of the threshold. Substituting the equation for utility and evaluating at $x_i = t$ yields

$$\mathbb{E}_{\tilde{\mathbf{x}}}\left[\boldsymbol{P}_i^{(n)}\left(\left(q_H, \mathbf{q}\left(t, \tilde{\mathbf{x}}\right)\right), \left(t, \tilde{\mathbf{x}}\right)\right) - \boldsymbol{P}_i^{(n)}\left(\left(q_L, \mathbf{q}\left(t, \tilde{\mathbf{x}}\right)\right), \left(t, \tilde{\mathbf{x}}\right)\right)\right]\theta + 1 = 0.$$

By using Lemma 2, we have that $\boldsymbol{P}$ is submodular in $\mathbf{q}$ and $\mathbf{x}$. Therefore, for $x_j < t$, we have

$$\mathbb{E}_{\tilde{\mathbf{x}}}\left[\boldsymbol{P}_j^{(n)}\left(\left(q_H,\mathbf{q}\left(t,\tilde{\mathbf{x}}\right)\right),\left(x_j,\tilde{\mathbf{x}}\right)\right)-\boldsymbol{P}_j^{(n)}\left(\left(q_L,\mathbf{q}\left(t,\tilde{\mathbf{x}}\right)\right),\left(x_j,\tilde{\mathbf{x}}\right)\right)\right]\theta+1\geq 0.$$

Denote $\mathbf{q}_{-j}=\mathbf{q}\left(t,\tilde{\mathbf{x}}\right)$. This is equivalent to

$$\mathbb{E}_{\mathbf{x}_{-j}}\left[u_j^{(n)}\left(\left(q_L,\mathbf{q}_{-j}\right),\left(x_j,\mathbf{x}_{-j}\right),\theta\right)-u_j^{(n)}\left(\left(q_H,\mathbf{q}_{-j}\right),\left(x_j,\mathbf{x}_{-j}\right),\theta\right)\right]\geq 0,$$

since in $\mathbf{q}_t$, we have $q_j=q_L$ as $x_j<t$, and the equilibrium condition is satisfied for all agents with activity levels below the threshold t. Similarly, using the submodularity (shown in Lemma 2), for $x_j>t$, we have

$$\mathbb{E}_{\tilde{\mathbf{x}}}\left[\boldsymbol{P}_j^{(n)}\left(\left(\left(\left(q_H,\mathbf{q}\left(t,\tilde{\mathbf{x}}\right)\right),\left(x_j,\tilde{\mathbf{x}}\right)\right)-\boldsymbol{P}_j^{(n)}\left(\left(\left(\left(q_L,\mathbf{q}\left(t,\tilde{\mathbf{x}}\right)\right),\left(x_j,\tilde{\mathbf{x}}\right)\right)\right]\theta+1\leq 0,$$

which is equivalent to

$$\mathbb{E}_{\mathbf{x}_{-j}}\left[u_j^{(n)}\left(\left(q_H,\mathbf{q}_{-j}\right),\left(x_j,\mathbf{x}_{-j}\right),\theta\right)-u_j^{(n)}\left(\left(q_L,\mathbf{q}_{-j}\right),\left(x_j,\mathbf{x}_{-j}\right),\theta\right)\right]\geq 0.$$

Now in $\mathbf{q}_t$, we have $q_j=q_H$ as $x_j>t$. Therefore, the equilibrium condition is satisfied for all agents with activity levels above the threshold t. Therefore, $\mathbf{q}_t$ is an equilibrium protection profile.

Next, we show that $\mathbf{q}_t$ is the unique symmetric equilibrium. Suppose that there is a threshold equilibrium t' where the condition in the proposition is not satisfied. Then, there can be two cases. In the first case, we have

$$\mathbb{E}_{\tilde{\mathbf{x}}}\left[\tilde{u}^{(n)}\left(q_L,t',t',\tilde{\mathbf{x}},\theta\right)-\tilde{u}^{(n)}\left(q_H,t',t',\tilde{\mathbf{x}},\theta\right)\right]<0.$$

By definition, the action for an agent with activity level t' in the equilibrium is q_L.

The above equation shows that the agent will be strictly better off by exerting high protection and will then have an incentive to deviate. This contradicts t' being an equilibrium. In the second case, we have

$$\mathbb{E}_{\tilde{\mathbf{x}}}\left[\tilde{u}^{(n)}\left(q_L, t', t', \tilde{\mathbf{x}}, \theta\right) - \tilde{u}^{(n)}\left(q_H, t', t', \tilde{\mathbf{x}}, \theta\right)\right] > 0.$$

Since σ is defined continuously in $[0,1]$ and the utility function u is continuous over σ, there exists an $\epsilon > 0$ small enough where

$$\mathbb{E}_{\tilde{\mathbf{x}}}\left[\tilde{u}^{(n)}\left(q_L, t'+\epsilon, t', \tilde{\mathbf{x}}, \theta\right) - \tilde{u}^{(n)}\left(q_H, t'+\epsilon, t', \tilde{\mathbf{x}}, \theta\right)\right] > 0.$$

By definition, the action for agent with activity level $t'+\epsilon$ is q_H in the equilibrium. However, the above equation shows that the agent will be strictly better off by exerting low protection and will then have an incentive to deviate. This again contradicts t' being an equilibrium. We next show that equilibrium must satisfy the equation in the statement of the proposition. By invoking Lemma 1, we know that the infection probability is monotone decreasing in q. Accordingly, the utility function as a linear function of the infection probability is monotone increasing in q and thus decreasing in σ. Combining with the assumptions

$$\mathbb{E}_{\tilde{\mathbf{x}}}\left[\tilde{u}^{(n)}\left(q_L, 1, 1, \tilde{\mathbf{x}}, \theta\right) - \tilde{u}^{(n)}\left(q_H, 1, 1, \tilde{\mathbf{x}}, \theta\right)\right] < 0$$

and

$$\mathbb{E}_{\tilde{\mathbf{x}}}\left[\tilde{u}^{(n)}\left(q_L, 0, 0, \tilde{\mathbf{x}}, \theta\right) - \tilde{u}^{(n)}\left(q_H, 0, 0, \tilde{\mathbf{x}}, \theta\right)\right] > 0,$$

we know that there is a unique root to (2.21). $\square$

Intuitively, if the agent is very active and no one else is protecting, then the agent will prefer to exert high protection. If the agent is very inactive and everyone else is protecting, then the agent will prefer to exert low protection. We next establish the relationship between the equilibrium threshold and the state of the world θ.

Lemma 3. *$\sigma^{FE}(\theta)$ is weakly decreasing in θ, i.e., for all $\theta' \geq \theta$, we have*

$$\sigma^{FE}(\theta') \leq \sigma^{FE}(\theta).$$

Proof. From the proof of Proposition 2, we know that $\sigma^{FE}(\theta)$ is the unique solution to

$$\mathbb{E}_{\tilde{\mathbf{x}}}\left[\boldsymbol{P}_i^{(n)}\left(\left(q_H, \mathbf{q}\left(\sigma^{FE}(\theta), \tilde{\mathbf{x}}\right)\right), \left(\sigma^{FE}(\theta), \tilde{\mathbf{x}}\right)\right) - \boldsymbol{P}_i^{(n)}\left(\left(q_L, \mathbf{q}\left(\sigma^{FE}(\theta), \tilde{\mathbf{x}}\right)\right), \left(\sigma^{FE}(\theta), \tilde{\mathbf{x}}\right)\right)\right] = 0.$$

By invoking Lemma 1 and Lemma 2, we know that

$$\mathbb{E}_{\tilde{\mathbf{x}}}\left[\boldsymbol{P}_i^{(n)}\left(\left(q_H, \mathbf{q}\left(\sigma^{FE}(\theta), \tilde{\mathbf{x}}\right)\right), \left(\sigma^{FE}(\theta), \tilde{\mathbf{x}}\right)\right) - \boldsymbol{P}_i^{(n)}\left(\left(q_L, \mathbf{q}\left(\sigma^{FE}(\theta), \tilde{\mathbf{x}}\right)\right), \left(\sigma^{FE}(\theta), \tilde{\mathbf{x}}\right)\right)\right]$$

is decreasing in the value of $\sigma^{E}(\theta)$. Thus, for $\theta' > \theta$, we have $\sigma^{E}(\theta') \leq \sigma^{E}(\theta)$. □

Lemma 3 establishes that the equilibrium attained by the agents in the full information benchmark is weakly decreasing in the severity level of the infection. Intuitively, when the impact from the infection becomes larger, more agents will choose to protect themselves.

Furthermore, we prove that the Bayesian Nash equilibrium following any belief $\mu(\cdot) \in \Delta(\Theta)$ is also unique and admits a threshold strategy, as proved next.

Proposition 3. *Given any belief $\mu \in \Delta(\theta)$, there exists a unique symmetric Bayesian*

Nash equilibrium $\sigma^E(\mu)$ *such that for all* $i \in I$,

$$a^E(x_i, \mu) = \begin{cases} q_L & x_i \le \sigma^E(\mu) \\ q_H & \textit{otherwise.} \end{cases}$$

Moreover, the Bayesian Nash equilibrium threshold satisfies

$$\sigma^E(\mu) = \sigma^{FE}\left(\tilde{\theta}\right), \tag{2.22}$$

where $\tilde{\theta} = \sum_\theta \mu(\theta)\theta$.

Proof. The proof follows by noting that

$$\tilde{u}^{(n)}(a, x, \sigma, \tilde{\mathbf{x}}, \theta) = -\tilde{\boldsymbol{P}}^{(n)}(a, x, \sigma, \tilde{\mathbf{x}})\theta - \mathbb{1}\{a = q_H\}$$

is linear in θ. Therefore, using the definition yields

$$\mathbb{E}_\theta \mathbb{E}_{\tilde{\mathbf{x}}}\left[\tilde{u}^{(n)}\left(q_L, x = \sigma^{FE}(\mu), \sigma^{FE}(\mu), \tilde{\mathbf{x}}, \theta\right) - \tilde{u}^{(n)}\left(q_H, \sigma^{FE}(\mu), \sigma^{FE}(\mu), \tilde{\mathbf{x}}, \theta\right)\right] = 0,$$

which in turn results in

$$\begin{aligned} \mathbb{E}_{\tilde{\mathbf{x}}}\Bigg[&\tilde{u}^{(n)}\left(q_L, x = \sigma^E\left(\sum_\theta \mu(\theta)\theta\right), \sigma^E\left(\sum_\theta \mu(\theta)\theta\right), \tilde{\mathbf{x}}, \theta\right) \\ &- \tilde{u}^{(n)}\left(q_H, \sigma^E\left(\sum_\theta \mu(\theta)\theta\right), \sigma^E\left(\sum_\theta \mu(\theta)\theta\right), \tilde{\mathbf{x}}, \theta\right)\Bigg] = 0. \end{aligned}$$

This completes the proof. □

We prove in Proposition 1 that a communication rule h satisfies the obedience constraint if and only if there is a Bayesian Nash equilibrium following an information policy f that induces the decision rule. In Proposition 3, we establish that the unique symmetric Bayesian Nash equilibrium follows a threshold rule. Therefore, it is without loss of generality to assume that the sender recommends only thresholds that satisfy the obedience constraint.

2.4.3 Social Planner's Problem with Thresholds

In this subsection, we write the social planner's problem using information policies that recommend only thresholds. We define the infection probability for agent i with a protection threshold σ as

$$\mathbb{P}_i^{(n)}(\sigma, \mathbf{x}) \doteq \boldsymbol{P}_i^{(n)}(\mathbf{q}(\sigma, \mathbf{x}), \mathbf{x}), \tag{2.23}$$

the utility of agent i with a protection threshold σ as

$$u_i^{(n)}(\sigma, \mathbf{x}, \theta) \doteq -\mathbb{P}_i^{(n)}(\sigma, \mathbf{x})\,\theta - \mathbb{1}(x_i > \sigma), \tag{2.24}$$

and the social welfare with a protection threshold σ as

$$W^{(n)}(\sigma, \mathbf{x}, \theta) \doteq \frac{1}{n}\sum_{i=1}^{n} u_i^{(n)}(\sigma, \mathbf{x}, \theta). \tag{2.25}$$

Recall that x_i's follow a uniform distribution over $[0,1]$. The expected social welfare with a protection threshold σ becomes

$$\mathbb{E}_{\mathbf{x}}\left[W^{(n)}(\sigma, \mathbf{x}, \theta)\right] \doteq \mathbb{E}_{\mathbf{x}}\left[\frac{1}{n}\sum_{i=1}^{n} u_i^{(n)}(\sigma, \mathbf{x}, \theta)\right] = -\frac{\theta}{n}\mathbb{E}_{\mathbf{x}}\left[\sum_{i=1}^{n} \mathbb{P}_i^{(n)}(\sigma, \mathbf{x})\right] - (1-\sigma).$$

Furthermore, we denote the expected probability of infection for any agent in the network as

$$\mathcal{P}^{(n)}(\sigma) \doteq \mathbb{E}_{\mathbf{x}}\left[\mathbb{P}_1^{(n)}(\sigma, \mathbf{x})\right], \tag{2.26}$$

where the subscript 1 in the above definition is arbitrary and the definition of $\mathcal{P}^{(n)}(\sigma, n)$ would be the same if we take expectation of any other agent's infection probability. For any $\theta \in \Theta$, we define the welfare function $A^{(n)}(\sigma, \theta)$ with a protection threshold σ and a state of the world θ as

$$A^{(n)}(\sigma, \theta) \doteq \mathbb{E}_{\mathbf{x}}\left[W^{(n)}(\sigma, \mathbf{x}, \theta)\right] = -\theta \mathcal{P}^{(n)}(\sigma) - (1-\sigma) \tag{2.27}$$

and define the incentive function $B(\sigma, \theta, n)$ with a protection threshold σ and a state of the world θ as

$$B^{(n)}(\sigma, \theta) \doteq \mathbb{E}_{\tilde{\mathbf{x}}}\left[\tilde{u}^{(n)}(q_H, \sigma, \sigma, \tilde{\mathbf{x}}, \theta) - \tilde{u}^{(n)}(q_L, \sigma, \sigma, \tilde{\mathbf{x}}, \theta)\right]. \tag{2.28}$$

To formulate the social planner's problem, using the information policies that recommend only thresholds, we need one more discussion about the dimension of the threshold space. Recall that in the social planner's problem (2.6)-(2.9) defined in Subsection 2.3.1, we assume that the signal space S is a general finite space. Here, we introduce a similar assumption for the threshold space $\sigma \in \Sigma$, where Σ is a large but finite set with its elements in $[0, 1]$. We include $\left\{0, 1, \sigma^{FE}(\theta_L), \sigma^{FE}(\theta_H)\right\}$ in Σ, where the inclusion of $\left\{\sigma^{FE}(\theta_L), \sigma^{FE}(\theta_H)\right\}$ ensures that the set of feasible solutions to the social planner's problem is nonempty. (We will later define a full disclosure policy in Subsection 2.5.3 and prove that it is always a feasible solution to the following optimization problem.) With a slight abuse of notation, we let $h : \Theta \to \Delta(\Sigma)$ denote the communication rules on thresholds and have the following result.

Proposition 4. *Consider the following optimization formulation.*

$$Z_2 \doteq \max_{h} \sum_{\theta} \sum_{\sigma} \mu(\theta) h(\sigma \mid \theta) A^{(n)}(\sigma, \theta) \tag{2.29}$$

s.t.

$$\sum_{\theta} \mu(\theta)\, h(\sigma \mid \theta)\, B^{(n)}(\sigma, \theta) = 0, \ \ \forall\, \sigma \in \Sigma \setminus \{0, 1\}, \tag{2.30}$$

$$\sum_{\theta} \mu(\theta)\, h(1 \mid \theta)\, B^{(n)}(1, \theta) \leq 0, \tag{2.31}$$

$$\sum_{\theta} \mu(\theta)\, h(0 \mid \theta)\, B^{(n)}(0, \theta) \geq 0, \tag{2.32}$$

$$\sum_{\sigma \in \Sigma} h(\sigma \mid \theta) = 1, \ \ \forall \theta \in \Theta, \tag{2.33}$$

$$h(\sigma \mid \theta) \geq 0, \ \ \forall \sigma \in \Sigma, \theta \in \Theta. \tag{2.34}$$

For any prior $\mu(\cdot)$, we have

$$Z_1 = Z_2.$$

Proof. We first prove that $Z_1 \geq Z_2$. Let h_1 be the optimal solution to the optimization problem (2.29) - (2.34). We have $Z_2 = \sum_\theta \sum_\sigma \mu(\theta)\, h_1(\sigma \mid \theta)\, A^{(n)}(\sigma, \theta)$. We then construct h_2 as

$$h_2(m \mid \theta) = \begin{cases} h_1(\sigma \mid \theta) & \text{if } m(x) = q_L \,\forall\, x \in [0, \sigma] \text{ and } m(x) = q_H \,\forall\, x \in (\sigma, 1] \\ 0 & \text{otherwise} \end{cases}$$

and prove that h_2 is a feasible solution to the problem (2.11) - (2.14) and attains the same objective function value as Z_2. This suffices to show that $Z_1 \geq Z_2$. The probability rules (2.13) and the non-negativity constraints (2.14) are trivially satisfied, where

$$\sum_m h_2(m \mid \theta) = \sum_\sigma h_1(\sigma \mid \theta) = 1$$

for all $\theta \in \Theta$. Now we look at the obedience constraints (2.12). Note that for m not following a threshold structure, i.e., there does not exist a σ where $m(x) = q_L$

for all $x \leq \sigma$ and $m(x) = q_H$ for all $x > \sigma$, $h_2(m \mid \theta) = 0$ for all θ by construction. Therefore, both sides of (2.12) are zero, and the constraints are satisfied. Consider m, which follows a threshold structure with a threshold σ. Note that by definition, we have

$$\mathbf{m}(\mathbf{x}) = \mathbf{q}(\sigma, \mathbf{x}).$$

Since h_1 is feasible and satisfies (2.30), we have

$$\begin{aligned}
&\mathbb{E}_\theta \mathbb{E}_{\mathbf{x}_{-i}} \left[h_1(\sigma \mid \theta) \left(\tilde{u}^{(n)}(q_H, \sigma, \sigma, \mathbf{x}_{-i}, \theta) - \tilde{u}^{(n)}(q_L, \sigma, \sigma, \mathbf{x}_{-i}, \theta) \right) \right] = 0 \\
&\overset{(*)}{=} \mathbb{E}_\theta \mathbb{E}_{\mathbf{x}_{-i}} \left[h_1(\sigma \mid \theta) \left(u_i^{(n)}((q_H, \mathbf{q}(\sigma, \mathbf{x}_{-i})), (x_i = \sigma, \mathbf{x}_{-i}), \theta) - u_i^{(n)}((q_L, \mathbf{q}(\sigma, \mathbf{x}_{-i})), (x_i = \sigma, \mathbf{x}_{-i}), \theta) \right) \right]
\end{aligned}$$

where $(*)$ follows from the definition of $\tilde{u}$ in (2.20). Moreover, using the submodularity in the infection probability $\mathcal{P}_i$ and following the proof process of Proposition 2, we have

$$\mathbb{E}_\theta \mathbb{E}_{\mathbf{x}_{-i}} \left[h_1(\sigma \mid \theta) \left(u_i^{(n)}((q_H, \mathbf{q}(\sigma, \mathbf{x}_{-i})), (x_i, \mathbf{x}_{-i}), \theta) - u_i^{(n)}((q_L, \mathbf{q}(\sigma, \mathbf{x}_{-i})), (x_i, \mathbf{x}_{-i}), \theta) \right) \right] \leq 0,$$

for all $x_i \leq \sigma$, and

$$\mathbb{E}_\theta \mathbb{E}_{\mathbf{x}_{-i}} \left[h_1(\sigma \mid \theta) \left(u_i^{(n)}((q_H, \mathbf{q}(\sigma, \mathbf{x}_{-i})), (x_i, \mathbf{x}_{-i}), \theta) - u_i^{(n)}((q_L, \mathbf{q}(\sigma, \mathbf{x}_{-i})), (x_i, \mathbf{x}_{-i}), \theta) \right) \right] \geq 0,$$

for all $x_i > \sigma$.

Now let us continue proving the feasibility of h_2. For $x_i \leq \sigma$,

$$\begin{aligned}
&\mathbb{E}_\theta \mathbb{E}_{\mathbf{x}_{-i}} \left[h_2(m \mid \theta) \left(u_i^{(n)}(\mathbf{m}(\mathbf{x}), \mathbf{x}, \theta) - u_i^{(n)}((q_i', \mathbf{m}(\mathbf{x}_{-i})), \mathbf{x}, \theta) \right) \right] \\
&= \mathbb{E}_\theta \mathbb{E}_{\mathbf{x}_{-i}} \left[h_2(m \mid \theta) \left(u_i^{(n)}((q_L, \mathbf{q}(\sigma, \mathbf{x})), \mathbf{x}, \theta) - u_i^{(n)}((q_H, \mathbf{q}(\sigma, \mathbf{x})), \mathbf{x}, \theta) \right) \right] \\
&= \mathbb{E}_\theta \mathbb{E}_{\mathbf{x}_{-i}} \left[h_1(\sigma \mid \theta) \left(u_i^{(n)}((q_L, \mathbf{q}(\sigma, \mathbf{x})), \mathbf{x}, \theta) - u_i^{(n)}((q_H, \mathbf{q}(\sigma, \mathbf{x})), \mathbf{x}, \theta) \right) \right] \geq 0.
\end{aligned}$$

Similarly, for $x_i \geq \sigma$,

$$\begin{aligned}
&\mathbb{E}_\theta \mathbb{E}_{\mathbf{x}_{-i}} \left[h_2 \left(m \mid \theta \right) \left(u_i^{(n)} \left(\mathbf{m} \left(\mathbf{x} \right), \mathbf{x}, \theta \right) - u_i^{(n)} \left(\left(q_i', \mathbf{m} \left(\mathbf{x}_{-i} \right) \right), \mathbf{x}, \theta \right) \right) \right] \\
&= \mathbb{E}_\theta \mathbb{E}_{\mathbf{x}_{-i}} \left[h_2 \left(m \mid \theta \right) \left(u_i^{(n)} \left(\left(q_H, \mathbf{q} \left(\sigma, \mathbf{x} \right) \right), \mathbf{x}, \theta \right) - u_i^{(n)} \left(\left(q_L, \mathbf{q} \left(\sigma, \mathbf{x} \right) \right), \mathbf{x}, \theta \right) \right) \right] \\
&= \mathbb{E}_\theta \mathbb{E}_{\mathbf{x}_{-i}} \left[h_1 \left(\sigma \mid \theta \right) \left(u_i^{(n)} \left(\left(q_H, \mathbf{q} \left(\sigma, \mathbf{x} \right) \right), \mathbf{x}, \theta \right) - u_i^{(n)} \left(\left(q_L, \mathbf{q} \left(\sigma, \mathbf{x} \right) \right), \mathbf{x}, \theta \right) \right) \right] \geq 0.
\end{aligned}$$

Therefore, h_2 satisfies all obedience constraints (2.12). Lastly, we have

$$\begin{aligned}
\mathbb{E}_\theta \mathbb{E}_\mathbf{x} \mathbb{E}_{h_2} \left[W^{(n)} \left(\mathbf{m} \left(\mathbf{x} \right), \mathbf{x}, \theta \right) \right] &= \mathbb{E}_\theta \mathbb{E}_\mathbf{x} \mathbb{E}_{h_1} \left[W^{(n)} \left(\sigma, \mathbf{x}, \theta \right) \right] = \mathbb{E}_\theta \mathbb{E}_{h_1} \left[A^{(n)} \left(\sigma, \theta \right) \right] \\
&= \sum_\theta \sum_\sigma \mu \left(\theta \right) h_1 \left(\sigma \mid \theta \right) A^{(n)} \left(\sigma, \theta \right) = Z_2
\end{aligned}$$

by the definition of the welfare function $A^{(n)} \left(\sigma, \theta \right)$ in (2.27). Since (2.11) is a maximization problem, we have $Z_1 \geq Z_2$.

Next, we prove the reverse direction, i.e., $Z_2 \geq Z_1$. Let h_2' be the optimal solution to the optimization problem (2.11) - (2.14). We have $Z_1 =$ $\mathbb{E}_\theta \mathbb{E}_\mathbf{x} \mathbb{E}_{h_2'} \left[W^{(n)} \left(\mathbf{m} \left(\mathbf{x} \right), \mathbf{x}, \theta \right) \right]$. By Proposition 1, there exists an information disclosure policy $f' : \Theta \to \Delta \left(\mathcal{S} \right)$ to the optimization problem (2.6) - (2.9), given by

$$f' \left(s \mid \theta \right) = \sum_{m : a^E \left(x_i, \mu_s \right) = m \left(x_i \right) \forall i} h_2' \left(m \mid \theta \right),$$

where

$$Z_0 = \mathbb{E}_\theta \mathbb{E}_\mathbf{x} \mathbb{E}_{f'} \left[W^{(n)} \left(\mathbf{a}^E \left(\mathbf{x}, \mu_s \right), \mathbf{x}, \theta \right) \right] = Z_1.$$

By Proposition 3, the equilibrium action a^E following any posterior belief μ_s is unique and follows a threshold strategy with threshold $\sigma^E(\mu_s)$. We then construct h_1' as

$$h_1'(\sigma \mid \theta) = \sum_{s:\sigma^E(\mu_s)=\sigma} f'(s \mid \theta)$$

and prove that h_1' is a feasible solution to the optimization problem (2.29) - (2.34) and attains the same objective function value as Z_1. This suffices to show that $Z_2 \geq Z_1$. Again, the non-negativity constraints (2.34) are trivially satisfied. Moreover, the probability rules (2.33) are satisfied as

$$\sum_{\sigma} h_1'(\sigma \mid \theta) = \sum_{s} f'(s \mid \theta) = 1,$$

for all θ given f' satisfies (2.8). For the obedience constraints (2.30) - (2.32), we have

$$\begin{aligned}
\mathbb{E}_\theta\left[h_1'(\sigma \mid \theta) B^{(n)}(\sigma,\theta)\right] &= \mathbb{E}_\theta\left[\sum_{s:\sigma^E(\mu_s)=\sigma} f'(s \mid \theta) B^{(n)}\left(\sigma^E(\mu_s),\theta\right)\right] \\
&\begin{cases} = 0, & \text{if } \sigma \in (0,1), \\ \geq 0, & \text{if } \sigma = 0, \\ \leq 0, & \text{if } \sigma = 1, \end{cases}
\end{aligned}$$

by the definition of $B^{(n)}$ in (2.28) and the characterizations of equilibrium σ^E, σ^{FE} in Proposition 2 and Proposition 3. Therefore h_1' satisfies all obedience constraints (2.30) - (2.32). Lastly, we have

$$\begin{aligned}
\sum_{\theta}\sum_{\sigma} \mu(\theta) h_1'(\sigma \mid \theta) A^{(n)}(\sigma,\theta) &= \sum_{\theta}\sum_{\sigma}\sum_{s:\sigma^E(\mu_s)=\sigma} \mu(\theta) f'(s \mid \theta) A^{(n)}(\sigma^E(\mu_s),\theta) \\
&= \sum_{\theta}\sum_{\mathbf{x}}\sum_{s} \mu(\theta) f'(s \mid \theta) A^{(n)}(\mathbf{a}^E(\mathbf{x},\mu_s),\mathbf{x},\theta)
\end{aligned}$$

$$= Z_0 = Z_1.$$

Since (2.29) is a maximization problem, we have $Z_2 \geq Z_1$. Therefore, we have $Z_2 = Z_1$. This completes the proof. $\square$

The objective function (2.29) maximizes the expected social welfare. Constraints (2.30)-(2.32) ensure that h is obedient (Definition 4 and Proposition 3). Constraints (2.33) and (2.34) ensure h satisfies general probability rules. The value Z_2 is the optimal objective value of the optimization problem (2.29) - (2.34). Recall Z_1 is the optimal objective value of the optimization problem (2.11) - (2.14). Proposition 4 establishes the equivalence between the two optimization problems and the reduction over the scale of the optimization.

2.5 One-Step Transmission and Mean-Field Approximation

In this section, we consider a one-step infection transmission process and then consider a mean-field approximation of the infection probability. On the one hand, one can consider this as a "warm up" for the general model introduced previously. On the other hand, the model can be considered as capturing the situation in which information design is only relevant at the outset of a pandemic.

Specifically, we assume the infection can only transmit one time, i.e., an agent that is not the origin can only be infected directly by the origin. To distinguish the general model from the one-step model, we define $\mathbb{P}^{(1,n)}(\sigma, x, t)$, $u_i^{(1,n)}(\sigma, \mathbf{x}, \theta)$, $W^{(1,n)}(\sigma, \mathbf{x}, \theta)$, $A^{(1,n)}(\sigma, \theta)$, and $B^{(1,n)}(\sigma, \theta)$ in the same formats as equations (2.23) - (2.28) and change the superscripts from (n) to $(1, n)$.

2.5.1 Mean-Field Approximation

To derive the mean-field approximation, we first consider a discretization of the social activity levels. That is, suppose the agents' social activity levels are uniformly drawn from t types $\{a^1, \cdots, a^t\}$. With t types, there are n/t agents of each possible type, and then we let both n and t go to infinity. For $x \in \{a^1, \cdots, a^t\}$, we have

$$\begin{aligned}
\mathbb{P}^{(1,n)}(\sigma, x, t) \doteq{} & (1 - q_H \mathbb{1}\{x > \sigma\} - q_L \mathbb{1}\{x \leq \sigma\}) \\
& \times \Bigg[\pi + (1-\pi)\Bigg(1 - \prod_{y \in \{a^1,\ldots,a^t\}\setminus\{x\}} \left(1 - \frac{\beta}{n} xy\pi (1 - q_H \mathbb{1}\{y > \sigma\} - q_L \mathbb{1}\{y \leq \sigma\})\right)^{\frac{n}{t}} \\
& \qquad \left(1 - \frac{\beta}{n} x^2 \pi (1 - q_H \mathbb{1}\{x > \sigma\} - q_L \mathbb{1}\{x \leq \sigma\})\right)^{\frac{n}{t}-1}\Bigg)\Bigg]. \qquad (2.35)
\end{aligned}$$

In (2.35), the term $(1 - q_H \mathbb{1}\{x > \sigma\} - q_L \mathbb{1}\{x \leq \sigma\})$ is the probability that the type x agent remains in the contact network. Conditional on that, this agent can be infected as the origin with probability π. If the agent is not initially infected with probability $1 - \pi$, the infection reaches her through any other agent who is an origin. In particular, the infection reaches this agent through a type y origin with probability

$$\frac{\beta}{n} xy\pi (1 - q_H \mathbb{1}\{y > \sigma\} - q_L \mathbb{1}\{y \leq \sigma\}).$$

Note that because of the limited transmission assumption, the type y agents can transmit infection only if they are present in the contact network and are initially infected, with a probability $\pi (1 - q_H \mathbb{1}\{y > \sigma\} - q_L \mathbb{1}\{y \leq \sigma\})$. Equation (2.35) then follows because there are n/t agents of type $y \neq x$ and $n/t - 1$ remaining type x agents (besides the focal one).

Notice that this infection probability only depends on the social activity level of an agent and not her identity. We further denote $\mathcal{P}^{[1]}(\sigma, x)$ as the mean-field approximation of the infection probability $\mathbb{P}^{(1,n)}(\sigma, x, t)$, i.e.,

$$\mathcal{P}^{[1]}(\sigma, x) \doteq \lim_{t\to\infty}\lim_{n\to\infty} \mathbb{P}^{(1,n)}(\sigma, x, t). \tag{2.36}$$

We can interpret the quantity $\mathcal{P}^{[1]}(\sigma, x)$ as the fraction of type x agents infected. Our next proposition characterizes this quantity.

Proposition 5. *For any social activity level x and protection threshold σ, we have*

$$\mathcal{P}^{[1]}(\sigma, x) = (1 - q_H \mathbb{1}\{x > \sigma\} - q_L \mathbb{1}\{x \le \sigma\}) \left[\pi + (1-\pi)\left(1 - e^{-\frac{x\beta\pi}{2}\left[1 - q_H + (q_H - q_L)\sigma^2\right]}\right)\right]. \tag{2.37}$$

Proof. We show the result starting with (2.35). Take n to infinity and define

$$\begin{aligned}
&\mathcal{P}^{[1]}(\sigma, x, t) \doteq \lim_{n\to\infty} \mathbb{P}^{(1,n)}(\sigma, x, t) \\
&= (1 - q_H \mathbb{1}\{x > \sigma\} - q_L \mathbb{1}\{x \le \sigma\}) \left[\pi + (1-\pi)\left(1 - \prod_{y \in \in \{a^1, \cdots, a^t\}} e^{-\frac{\beta x y \pi\left(1 - q_H \mathbb{1}\{y > \sigma\} - q_L \mathbb{1}\{y \le \sigma\}\right)}{t}}\right)\right] \\
&= (1 - q_H \mathbb{1}\{x > \sigma\} - q_L \mathbb{1}\{x \le \sigma\}) \left[\pi + (1-\pi)\left(1 - e^{-\frac{\sum_{y \in \{a^1, \cdots, a^t\}} \beta x y \pi\left(1 - q_H \mathbb{1}\{y > \sigma\} - q_L \mathbb{1}\{y \le \sigma\}\right)}{t}}\right)\right].
\end{aligned} \tag{2.38}$$

Let us introduce a notation for the exponent that appears in the above equation as

$$\mathsf{e}^{(1)}(\sigma, t) \doteq -\frac{\sum_{y \in \{a^1, \cdots, a^t\}} y\left(1 - q_H \mathbb{1}\{y > \sigma\} - q_L \mathbb{1}\{y \le \sigma\}\right)}{t}. \tag{2.39}$$

Now we take t to infinity and return to the original $[0, 1]$ space of x. With a slight abuse of notation, we define

$$\begin{aligned}
\mathsf{e}^{(1)}(\sigma) &\doteq \lim_{t\to\infty} \mathsf{e}^{(1)}(\sigma, t) = -\int_0^1 y\left(1 - q_H \mathbb{1}\{y > \sigma\} - q_L \mathbb{1}\{y \le \sigma\}\right) dy \\
&= \int_0^\sigma y(1 - q_L)\, dy + \int_\sigma^1 y(1 - q_H)\, dy = \frac{1}{2}\left[1 - q_H + (q_H - q_L)\sigma^2\right].
\end{aligned} \tag{2.40}$$

Using (2.38) and (2.40), we have

$$\begin{aligned}\mathcal{P}^{[1]}(\sigma, x) &= \lim_{t\to\infty}\lim_{n\to\infty}\mathbb{P}^{(1,n)}(\sigma, x, t)\\ &= \lim_{t\to\infty}\left(1 - q_H\mathbb{1}\{x > \sigma\} - q_L\mathbb{1}\{x \leq \sigma\}\right)\left[\pi + (1-\pi)\left(1 - e^{-x\beta\pi\mathsf{e}(\sigma,t)}\right)\right]\\ &= \left(1 - q_H\mathbb{1}\{x > \sigma\} - q_L\mathbb{1}\{x \leq \sigma\}\right)\left[\pi + (1-\pi)\left(1 - e^{-x\beta\pi\lim_{t\to\infty}\mathsf{e}(\sigma,t)}\right)\right]\\ &= \left(1 - q_H\mathbb{1}\{x > \sigma\} - q_L\mathbb{1}\{x \leq \sigma\}\right)\left[\pi + (1-\pi)\left(1 - e^{-x\beta\pi\mathsf{e}(\sigma)}\right)\right]\\ &= \left(1 - q_H\mathbb{1}\{x > \sigma\} - q_L\mathbb{1}\{x \leq \sigma\}\right)\left[\pi + (1-\pi)\left(1 - e^{-\frac{x\beta\pi}{2}\left[1 - q_H + (q_H - q_L)\sigma^2\right]}\right)\right].\end{aligned}$$

This completes the proof. □

We next characterize the fraction of infected agents (of any type) following any protection threshold.

Corollary 1. *For any protection threshold $\sigma \in [0, 1]$, the fraction of infected agents is*

$$\begin{aligned}\mathcal{P}^{[1]}(\sigma) &\doteq \mathbb{E}_x\left[\mathcal{P}^{[1]}(\sigma, x)\right]\\ &= 1 - q_H + (q_H - q_L)\sigma + \frac{2(1-\pi)}{\beta\pi\left(1 - q_H + (q_H - q_L)\sigma^2\right)}\\ &\quad\times\left[-(1-q_L) + (1-q_H)e^{-\frac{\beta\pi}{2}\left(1-q_H+(q_H-q_L)\sigma^2\right)} + (q_H - q_L)e^{-\frac{\sigma\beta\pi}{2}\left(1-q_H+(q_H-q_L)\sigma^2\right)}\right].\end{aligned} \tag{2.41}$$

Proof. By definition, we have

$$\begin{aligned}\mathcal{P}^{[1]}(\sigma) &= \int_0^1 \mathcal{P}^{[1]}(\sigma, x)\,dx\\ &= \int_0^1 \left(1 - q_H\mathbb{1}\{x > \sigma\} - q_L\mathbb{1}\{x \leq \sigma\}\right)\left[\pi + (1-\pi)\left(1 - e^{-\frac{x\beta\pi}{2}\left[1-q_H+(q_H-q_L)\sigma^2\right]}\right)\right]dx\\ &= 1 - q_H + (q_H - q_L)\sigma + \frac{2(1-\pi)}{\beta\pi\left(1 - q_H + (q_H - q_L)\sigma^2\right)}\\ &\quad\times\left[-(1-q_L) + (1-q_H)e^{-\frac{\beta\pi}{2}\left(1-q_H+(q_H-q_L)\sigma^2\right)} + (q_H - q_L)e^{-\frac{\sigma\beta\pi}{2}\left(1-q_H+(q_H-q_L)\sigma^2\right)}\right].\end{aligned}$$

This completes the proof. □

As we introduced in the beginning of Section 2.5, we let $A^{(1,n)}(\sigma, \theta)$ and $B^{(1,n)}(\sigma, \theta)$

represent the welfare function and the incentive function in this one-step model corresponding to the $A^{(n)}(\sigma,\theta)$ and $B^{(n)}(\sigma,\theta)$ defined in the general model.

Next, we define $A^{[1]}(\sigma,\theta)$ as the limit of the welfare function, i.e.,

$$A^{[1]}(\sigma,\theta) \doteq \lim_{n\to\infty} A^{(1,n)}(\sigma,\theta),$$

and $B^{[1]}(\sigma,\theta)$ as the limit of the incentive function, i.e.,

$$B^{[1]}(\sigma,\theta) \doteq \lim_{n\to\infty} B^{(1,n)}(\sigma,\theta).$$

The following result characterizes the limits of the welfare function and the incentive function.

Corollary 2. *For any protection threshold $\sigma \in [0,1]$ and state of the world $\theta \in \Theta$, the limit of the welfare function is*

$$A^{[1]}(\sigma,\theta) = -\,\theta\mathcal{P}^{[1]}(\sigma) - (1-\sigma). \tag{2.42}$$

The limit of the incentive function is

$$B^{[1]}(\sigma,\theta) = (q_H - q_L)\left[\pi + (1-\pi)\left(1 - e^{-\frac{\sigma\beta\pi}{2}\left(1-q_H+(q_H-q_L)\sigma^2\right)}\right)\right]\theta - 1. \tag{2.43}$$

Proof. By definition, we have

$$A^{[1]}(\sigma,\theta) = \lim_{n\to\infty} A^{(1,n)}(\sigma,\theta) = \lim_{n\to\infty} -\theta\mathcal{P}^{(1,n)}(\sigma) - (1-\sigma) = -\theta\mathcal{P}^{[1]}(\sigma) - (1-\sigma).$$

$$\begin{aligned}
B^{(1)}(\sigma,\theta) &= \lim_{n\to\infty} B^{[1,n]}(\sigma,\theta) = \lim_{n\to\infty} \mathbb{E}_{\tilde{\mathbf{x}}}\left[\tilde{u}^{(n)}(q_H,\sigma,\sigma,\tilde{\mathbf{x}},\theta) - \tilde{u}^{(n)}(q_L,\sigma,\sigma,\tilde{\mathbf{x}},\theta)\right] \\
&= -1 - (1-q_H)\left[\pi + (1-\pi)\left(1 - e^{-\frac{\beta\pi\sigma\mathrm{e}^{(1)}(\sigma)}{2}}\right)\right] - (1-q_L)\left[\pi + (1-\pi)\left(1 - e^{-\frac{\beta\pi\sigma\mathrm{e}^{(1)}(\sigma)}{2}}\right)\right] \\
&= (q_H - q_L)\left[\pi + (1-\pi)\left(1 - e^{-\frac{\beta\pi\sigma\mathrm{e}^{(1)}(\sigma)}{2}}\right)\right]\theta - 1
\end{aligned}$$

$$= (q_H - q_L) \left[\pi + (1-\pi) \left(1 - e^{-\frac{\sigma\beta\pi}{2}\left(1 - q_H + (q_H - q_L)\sigma^2\right)} \right) \right] \theta - 1.$$

This completes the proof. □

2.5.2 Information Design with Mean-Field Approximation

With the mean-field approximation, the information design problem presented in Proposition 4 for the general model becomes

$$\max_h \sum_\theta \sum_\sigma \mu(\theta)\, h(\sigma \mid \theta)\, A^{[1]}(\sigma, \theta) \tag{2.44}$$

$s.t.$

$$\sum_\theta \mu(\theta)\, h(\sigma \mid \theta)\, B^{[1]}(\sigma, \theta) = 0, \;\; \forall\, \sigma \in \Sigma \setminus \{0, 1\}, \tag{2.45}$$

$$\sum_\theta \mu(\theta)\, h(1 \mid \theta)\, B^{[1]}(1, \theta) \leq 0, \tag{2.46}$$

$$\sum_\theta \mu(\theta)\, h(0 \mid \theta)\, B^{[1]}(0, \theta) \geq 0, \tag{2.47}$$

$$\sum_{\sigma \in \Sigma} h(\sigma \mid \theta) = 1, \;\; \forall \theta \in \Theta, \tag{2.48}$$

$$h(\sigma \mid \theta) \geq 0, \;\; \forall \sigma \in \Sigma, \theta \in \Theta, \tag{2.49}$$

where $A^{[1]}, B^{[1]}$ are defined in (2.42) and (2.43). The interpretations of (2.44)-(2.49) correspond to those of (2.29)-(2.34) defined in Subsection 2.4.3 but are now with the one-step transmission and the mean-field approximation.

Next, we establish that any basic feasible solution to the optimization problem (2.44)-(2.49) contains only a few nonzero elements. Before stating the result, notice that following Propositions 2 and 3, we have

$$\sigma^{FE}(\theta) = \begin{cases} 1, & B^{[1]}(1,\theta) \leq 0, \\ 0, & B^{[1]}(0,\theta) \geq 0, \\ \sigma^* \text{ such that } B^{[1]}(\sigma^*,\theta) = 0, & \text{otherwise.} \end{cases}$$

Theorem 1. *For any basic feasible solution to the linear program* (2.44)- (2.49)*, the number of* σ *values such that* $h(\sigma \mid \theta) > 0$ *for some* $\theta \in \{\theta_L, \theta_H\}$ *is at most* $2|\Theta|$*.*

Proof. First, notice that we have $|\Theta| = 2$, but we prove that this result holds for arbitrary $|\Theta| = d$. We assume $|\Sigma| = m$. Therefore, the linear program (2.44) - (2.49) has dm variables. Any basic feasible solution to the linear program will have at least dm tight and linear independent constraints. Suppose there are $c \leq m$ σ's that satisfy $h(\sigma \mid \theta) > 0$ for some $\theta \in \Theta$. We want to prove that $c \leq 2d$ in any basic feasible solution.

We denote $\Sigma_{\text{non-trivial}} = \{\sigma \in \Sigma : h(\sigma \mid \theta) > 0 \text{ for some } \theta\}$ as the set of σ's with nontrivial recommendation probabilities, $|\Sigma_{\text{non-trivial}}| = c$. Then, we have $|\Sigma \setminus \Sigma_{\text{non-trivial}}| = m - c$. That is, we have $m - c$ σ's that satisfy $h(\sigma \mid \theta) = 0$ for all $\theta \in \Theta$, which contribute $d(m-c)$ binding constraints from non-negativity constraints (2.49).

One thing to note is that $m - c$ constraints from the obedience constraints (2.45) - (2.47) are linearly dependent because $\sum_\theta \mu(\theta) h(\sigma \mid \theta) B^{[1]}(\sigma,\theta) = \sum_\theta \mu(\theta) \cdot 0 \cdot B^{[1]}(\sigma,\theta) = 0$ for $\sigma \in \Sigma \setminus \Sigma_{\text{non-trivial}}$. This implies that constraints (2.45) - (2.47) can contribute at most c independent binding constraints.

Next, we analyze the $h(\sigma \mid \theta)$ for $\sigma \in \Sigma_{\text{non-trivial}}$. With the d θ's, we have d associated equilibrium thresholds $\{\sigma^{FE}(\theta)\}_{\theta\in\Theta}$. Denote by $\Sigma_{FE} = \{\sigma^{FE}(\theta)\}_{\theta\in\Theta}$, $|\Sigma_{FE}| = d$. Let $|\Sigma_{\text{non-trivial}} \cap \Sigma_{FE}| = c' \leq \min(c,d)$, i.e., there are at most $\min(c,d)$

σ's that both have nontrivial recommendation probabilities and are equal to a equilibrium threshold following a state of the world.

For any $\sigma \in \Sigma_{\text{non-trivial}} \cap \Sigma_{FE}$, the obedience constraints (2.45) - (2.47) can be satisfied with at most $d-1$ θ's that have $h(\sigma \mid \theta) = 0$. Denote by $\theta^{-1}(\sigma) = \left\{\theta \mid \sigma^{FE}(\theta) = \sigma\right\}$. Note that this notation is not an inverse function. It is valid by the existence and uniqueness of the full equilibrium threshold as established in Proposition 2. The obedience constraints (2.45) - (2.47) can be satisfied by having $h(\sigma \mid \theta^{-1}(\sigma)) > 0$ and all other recommendation probabilities as 0, i.e.,

$$
\begin{aligned}
&\sum_{\theta} \mu(\theta)\, h(\sigma \mid \theta)\, B^{[1]}(\sigma, \theta) \\
&= \sum_{\theta \neq \theta^{-1}(\sigma)} \mu(\theta)\, h(\sigma \mid \theta)\, B^{[1]}(\sigma, \theta) + \mu\left(\theta^{-1}(\sigma)\right) h\left(\sigma \mid \theta^{-1}(\sigma)\right) B^{[1]}\left(\sigma, \theta^{-1}(\sigma)\right) \\
&= \sum_{\theta \neq \theta^{-1}(\sigma)} \mu(\theta) \cdot 0 \cdot B^{[1]}(\sigma, \theta) + \mu\left(\theta^{-1}(\sigma)\right) h\left(\sigma \mid \theta^{-1}(\sigma)\right) \cdot 0 = 0.
\end{aligned}
$$

Therefore, each $\sigma \in \Sigma_{\text{non-trivial}} \cap \Sigma_{FE}$ can contribute at most $d-1$ binding constraints from non-negativity constraints (2.49). There are c' elements in $\Sigma_{\text{non-trivial}} \cap \Sigma_{FE}$, which amounts to at most $(d-1)\,c'$ binding constraints.

For any $\sigma \in \Sigma_{\text{non-trivial}} \setminus \Sigma_{FE}$, the obedience constraints (2.45) - (2.47) can be satisfied with at most $d-2$ θ's that have $h(\sigma \mid \theta) = 0$. $B^{[1]}(\sigma \mid \theta) \neq 0$ for all $\theta \in \Theta$. Therefore, each $\sigma \in \Sigma_{\text{non-trivial}} \setminus \Sigma_{FE}$ can contribute at most $d-2$ binding constraints from non-negativity constraints (2.49). There are $c-c'$ elements in $\Sigma_{\text{non-trivial}} \setminus \Sigma_{FE}$, which amounts to at most $(d-2)(c-c')$ binding constraints.

To summarize, the non-negativity constraints (2.49) can contribute at most

$$
d(m-c) + (d-1)\,c' + (d-2)(c-c') = dm - 2c + c'
$$

binding constraints. The obedience constraints (2.45) - (2.47) contribute at most c independent binding constraints. The probability constraints (2.48) contribute d

independent binding constraints. According to theorem 2.2 ([BT97]), for any basic feasible solutions, the number of independent active constraints must satisfy

$$dm - 2c + c' + c + d \geq dm \Rightarrow c \leq c' + d \leq 2d.$$

This completes the proof. $\square$

Theorem 1 enables us to greatly reduce the space of decision variables that we need to consider in solving problem (2.44).

In closing this subsection, we should highlight that the information design problem with one-step transmission, formulated in (2.44), is a linear program. In Section 2.6, we see that once we consider full transmission, the corresponding information design problem is no longer a linear program, which complicates its analysis.

2.5.3 The Effect of the Infection Transmission Rate

In this section, we study the benefit of information design with different transmission rates β. We formally prove that for small and large β, the optimal information policy is to reveal full information and have agents act according to their corresponding equilibria. We then illustrate that, for intermediate values of β, it may not be optimal to fully disclose information. We formally define a full disclosure policy as

$$h^E(\sigma \mid \theta) = \begin{cases} 1 & \sigma = \sigma^{FE}(\theta) \\ 0 & \text{Otherwise.} \end{cases} \tag{2.50}$$

Note that (2.50) is always a feasible policy for the optimization problem (2.44)-(2.49). This is because, by definition,

$$\sum_{\theta} \mu(\theta) h^E(\sigma \mid \theta) B^{[1]}(\sigma, \theta) = \begin{cases} \mu(\theta_L) B^{[1]}\left(\sigma^{FE}(\theta_L), \theta_L\right) & \sigma = \sigma^{FE}(\theta_L) \\ \mu(\theta_H) B^{[1]}\left(\sigma^{FE}(\theta_H), \theta_H\right) & \sigma = \sigma^{FE}(\theta_H) \\ 0 & \text{Otherwise.} \end{cases}$$

Again, by definition, if $\sigma^{FE}(\theta_H), \sigma^{FE}(\theta_L) \in (0,1)$, then $B^{[1]}\left(\sigma^{FE}(\theta_L), \theta_L\right) = B^{[1]}\left(\sigma^{FE}(\theta_H), \theta_H\right) = 0$, and thus, (2.45) is satisfied. If $\sigma^{FE}(\theta) = 0$, then $B^{[1]}(0, \theta) > 0$, so (2.47) is satisfied. If $\sigma^{FE}(\theta) = 1$, then $B^{[1]}(1, \theta) < 0$, so (2.46) is satisfied. It can be easily checked that the probability rules (2.48) and (2.49) are satisfied. This proves the feasibility of full disclosure policy (2.50).

Now the effect of the transmission rate β can be summarized as follows.

Theorem 2. *Consider the one-step transmission model with mean-field approximation.*

- *For any π, (q_L, q_H), and (θ_L, θ_H), there exist $\beta_1, \beta_2 \in (0, \infty)$ such that for $\beta < \beta_1$ and $\beta > \beta_2$, the full disclosure policy defined in* (2.50) *is an optimal solution to the linear program* (2.44)-(2.49)*, i.e., the optimal strategy is to reveal full information to the public.*

- *There exist π, (q_L, q_H), and (θ_L, θ_H) for which the full disclosure policy is not optimal for $\beta \in [\beta_3, \beta_4]$ for some β_3 and β_4.*

Proof. We prove the theorem by showing that with the two limits $\beta \to 0$ and $\beta \to \infty$, (2.50) is the optimal solution to the linear program. To clear the equations, we denote by $Q \doteq (1 - q_H + (q_H - q_L)\sigma^2)$. First, look at $\beta \to 0$. Using definition (2.41), we have

$$\lim_{\beta \to 0} \mathcal{P}^{[1]}(\sigma) = 1 - q_H + (q_H - q_L)\sigma$$

$$
\begin{aligned}
&+\lim_{\beta\to 0}\frac{2(1-\pi)\left[-(1-q_L)+(1-q_H)e^{-\frac{\beta\pi Q}{2}}+(q_H-q_L)e^{-\frac{-\sigma\beta\pi Q}{2}}\right]}{\beta\pi Q}\\
=&1-q_H+(q_H-q_L)\sigma\\
&+\lim_{\beta\to 0}\frac{2(1-\pi)\left[-\frac{(1-q_H)\pi Q}{2}e^{-\frac{\beta\pi Q}{2}}-\frac{\sigma\pi(q_H-q_L)Q}{2}e^{-\frac{\sigma\beta\pi Q}{2}}\right]}{\pi Q}\\
=&1-q_H+(q_H-q_L)\sigma+\frac{2(1-\pi)\left[-\frac{(1-q_H)\pi Q}{2}-\frac{\sigma\pi(q_H-q_L)Q}{2}\right]}{\pi Q}\\
=&\left(1-q_H+(q_H-q_L)\sigma\right)(1-(1-\pi))\\
=&\pi\left(1-q_H+(q_H-q_L)\sigma\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{\beta\to 0}A^{[1]}(\sigma,\theta)=&-\theta\pi\left(1-q_H+(q_H-q_L)\sigma\right)-(1-\sigma)\\
=&-\theta\pi(1-q_H)-1+\left(1-\theta\pi(q_H-q_L)\right)\sigma,
\end{aligned}
$$

which is linear in σ. Moreover, for fixed π and $0<\theta_L<\theta_H<\infty$, we have

$$
\lim_{\beta\to 0}B^{[1]}(\sigma,\theta)=(q_H-q_L)\pi\theta-1.
$$

There are three cases: $\theta_L<\theta_H<\frac{1}{\pi(q_H-q_L)}$, $\theta_L<\frac{1}{\pi(q_H-q_L)}<\theta_H$, and $\frac{1}{\pi(q_H-q_L)}<\theta_L<\theta_H$. For the first case $\theta_L<\theta_H<\frac{1}{\pi(q_H-q_L)}$, $B^{[1]}(\sigma,\theta)<0$ for all σ and θ. This implies $\sigma^{FE}(\theta_L)=\sigma^{FE}(\theta_H)=1$. We have shown that (2.50) is always a feasible solution to (2.44) - (2.49). Now we show its optimality by showing that (2.50) is the only feasible solution to (2.44) - (2.49). Consider any information policy $h'\neq h^E$. This implies that there exists $\sigma'\neq 1$ and $\theta'\in\Theta$ such that $h'(\sigma'\mid\theta')\neq 0$. If $\sigma'=0$, we have $\sum_\theta\mu(\theta)h(0\mid\theta)B^{[1]}(0,\theta)=-\mu(\theta')h(0\mid\theta')<0$, which violates (2.47). If $\sigma'\in(0,1)$, we have $\sum_\theta\mu(\theta)h(\sigma'\mid\theta)B^{[1]}(\sigma',\theta)=-\mu(\theta')h(0\mid\theta')<0$, which violates (2.45). Therefore, (2.50) is the only feasible solution to the linear program and is thus optimal.

For the second case $\theta_L < \frac{1}{\pi(q_H - q_L)} < \theta_H$, we have $\sigma^{FE}(\theta_L) = 1$ and $\sigma^{FE}(\theta_H) = 0$. Moreover, notice that in this case, $\arg\max_\sigma A^{[1]}(\sigma, \theta_L) = 1$ since $1 - \theta_L \pi (q_H - q_L) > 0$. $\arg\max_\sigma A^{[1]}(\sigma, \theta_H) = 0$ since $1 - \theta_H \pi (q_H - q_L) < 0$. This implies that h^E defined in (2.50) is also the first best solution in this case. Since the first best solution gives an upper bound on the optimization objective, we have shown the optimality of (2.50).

Finally, for the third case $\frac{1}{\pi(q_H - q_L)} < \theta_L < \theta_H$, we have $\sigma^{FE}(\theta_L) = \sigma^{FE}(\theta_H) = 0$. This is a similar case, where (2.50) is the only feasible solution to (2.44) - (2.49). Consider any information policy $h' \neq h^E$. This implies that there exists $\sigma' \neq 0$ and $\theta' \in \Theta$ such that $h'(\sigma' \mid \theta') \neq 0$. If $\sigma' = 1$, we have for θ', $\sum_\theta \mu(\theta) h(1 \mid \theta) B^{[1]}(1, \theta) = -\mu(\theta') h(1 \mid \theta') < 0$, which violates (2.46). If $\sigma' \in (0, 1)$, we have $\sum_\theta \mu(\theta) h(\sigma' \mid \theta) B^{[1]}(\sigma', \theta) = -\mu(\theta') h(\sigma' \mid \theta') < 0$, which violates (2.45). Therefore, (2.50) is the only feasible solution to the linear program and is thus optimal.

Combining the three cases, we show that the full information policy defined in (2.50) is optimal when $\beta \to 0$.

Next, we consider the other limit, $\beta \to \infty$. By definition, this implies

$$\lim_{\beta \to \infty} \mathcal{P}^{[1]}(\sigma) = 1 - q_H + (q_H - q_L)\sigma$$

and

$$\lim_{\beta \to \infty} A^{[1]}(\sigma, \theta) = -\theta(1 - q_H + (q_H - q_L)\sigma) - (1 - \sigma) = -\theta(1 - q_H) - 1 + (1 - \theta(q_H - q_L))\sigma,$$

which is linear in σ. Moreover, for fixed π and $0 < \theta_L < \theta_H < \infty$, we have

$$\lim_{\beta \to \infty} B^{[1]}(\sigma, \theta) = \theta(q_H - q_L) - 1$$

for all σ.

There are three cases: $\theta_L < \theta_H < \frac{1}{q_H - q_L}$, $\theta_L < \frac{1}{q_H - q_L} < \theta_H$, and $\frac{1}{q_H - q_L} < \theta_L < \theta_H$. For the first case $\theta_L < \theta_H < \frac{1}{q_H - q_L}$, $B^{[1]}(\sigma, \theta) < 0$ for all σ and θ. This implies $\sigma^{FE}(\theta_L) = \sigma^{FE}(\theta_H) = 1$, and this is the same case as for $\beta \to 0$, $\theta_L < \theta_H < \frac{1}{\pi(q_H - q_L)}$. With the same proof, we can see how (2.50) is the only feasible solution and is thus optimal.

For the second case $\theta_L < \frac{1}{q_H - q_L} < \theta_H$, we have $\sigma^{FE}(\theta_L) = 1$ and $\sigma^{FE}(\theta_H) = 0$. Moreover, notice that for $\beta \to \infty$, $\arg\max_\sigma A^{[1]}(\sigma, \theta_L) = 1$ since $1 - \theta_L(q_H - q_L) > 0$. Similarly, $\arg\max_\sigma A^{[1]}(\sigma, \theta_H) = 0$ since $1 - \theta_H(q_H - q_L) < 0$. This implies that h^E defined in (2.50) is also the first best solution in this case. Since the first best solution gives an upper bound on the optimization objective, we have shown the optimality of (2.50).

Finally, for the last case $\frac{1}{q_H - q_L} < \theta_L < \theta_H$, we have $\sigma^{FE}(\theta_L) = \sigma^{FE}(\theta_H) = 0$. This is the same case as for $\beta \to 0$, $\frac{1}{\pi(q_H - q_L)} < \theta_L < \theta_H$. With the same proof, we can show (2.50) is the only feasible solution to the linear program and is thus optimal.

Combining the three cases, we show that the full information policy defined in (2.50) is optimal when $\beta \to \infty$. By definition of limit, this completes the proof for the first part of the theorem.

To show the existence in the second part of the theorem, we use an example. Take $\beta = 100$, $q_H = 0.86$, $q_L = 0.24$, $\pi = 0.2$, $\theta_L = 1.7$, and $\theta_H = 7.5$. The full information equilibrium under severe harm of infection is $\sigma^{FE}(\theta_H) = 0$, i.e., all agents will take high protection. The full information equilibrium under mild harm of infection is $\sigma^{FE}(\theta_L) = 0.502$, i.e., agents with social activity levels above 0.502 will take high protection, and the rest will take low protection. The social welfare under full information disclosure is -4.68.

Consider the following information policy:

$$h\left(0.001 \mid \theta\right)=\begin{cases}0.2075 & \theta=\theta_L\\ 0.9406 & \text{Otherwise,}\end{cases}$$

and

$$h\left(0.497 \mid \theta\right)=\begin{cases}0.7925 & \theta=\theta_L\\ 0.0594 & \text{Otherwise.}\end{cases}$$

That is, if the sender sees a mild harm of infection, with probability 0.2075, she recommends that agents with activity levels above 0.001 take high protection and, with the remaining probability, recommends that agents with activity levels above 0.497 take high protection. If the sender sees a severe harm of infection, with probability 0.9406, she recommends that agents with activity levels above 0.001 take high protection and, with the remaining probability, recommends that agents with activity levels above 0.497 take high protection. This information policy is feasible and will attain a social welfare of -0.977, which is higher than -4.68. □

The first part of Theorem 2 states that when the infection transmission rate is very large or small, the sender finds it optimal to fully disclose the information. Based on different values of other parameters, there are two intuitions to this result. On the one hand, when the infection is barely (resp. highly) transmissible, the potential threat of loss from infection is too low (resp. high) even when the state is bad (resp. good). Therefore, in both states, no agent (resp. all agents) will choose to protect, and the feasible set is a singleton, i.e., it is impossible to induce any protection behaviors other than the full information equilibrium. On the other hand, when the potential threat of loss from infection is too low or too high compared with the protection cost, the influence of free-riding diminishes, and there is no incentive misalignment between the central planner and the agents.

The second part provides a complementary observation by establishing that for intermediate values for the infection transmission rate, the full disclosure policy may no longer be optimal.

The next example illustrates the result of Theorem 2.

Example 2. *Let us define the "price of full disclosure" as the social welfare difference between the full disclosure and the optimal (i.e., the solution to the linear program* (2.44)*) policies divided by the social welfare from the full disclosure policy. The higher the price of full disclosure, the more valuable it is for the social planner to design an information policy that is not fully revealing. A zero price for full disclosure means that the sender can do nothing with the information advantage and the best strategy is to reveal the full information to the public.*

With this definition in hand, we next illustrate the impact of intermediate β in the context of an example. We set the high protection level as $q_H = 0.86$, the low protection level as $q_L = 0.24$, the initial infection probability as $\pi = 0.2$, the mild harm of infection as $\theta_L = 1.7$, and the severe harm of infection as $\theta_H = 7.5$. The price of full disclosure for this setting as a function of different transmission rates β is depicted in Figure 2.1. As we observe, when the transmission rate is low, in particular, lower than 38*, the price of full disclosure is* 0*, which means that the full information disclosure policy is the optimal policy. Moreover, the price of full disclosure decreases to* 0 *as β increases over* 490*, confirming the results of part (a) of Theorem 2. For an intermediate transmission rate, however, as established in part (b) of Theorem 2, we see that it is socially beneficial to disclose partial information. In particular, for certain transmission rates, mixing information can result in more than* 30 *percent of improvement in social welfare.*

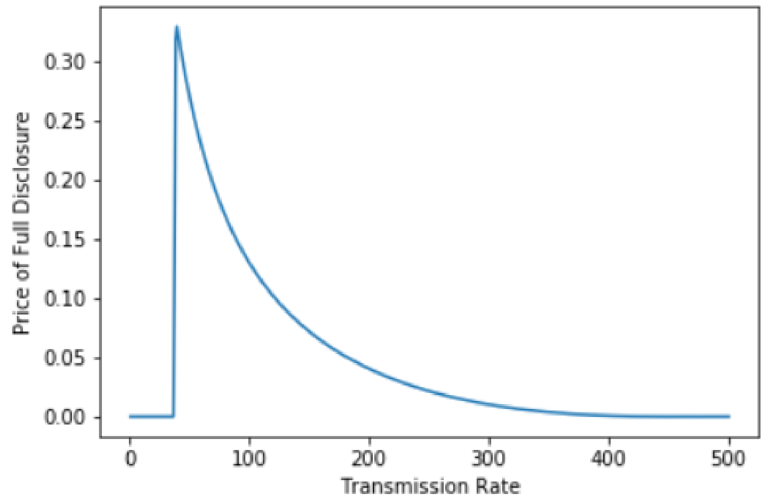


Figure 2.1: Price of full disclosure under different β with $(q_H, q_L, \pi, \theta_H, \theta_L) = (0.86, 0.24, 0.2, 7.5, 1.7)$

2.5.4 The Effect of the Initial Infection Probability

Next, we study the benefit of information design under different values for the initial infection probability π. We formally prove that for small and large π, the optimal information policy is to reveal full information, and consequently, the agents play according to the full disclosure equilibrium. We then illustrate that, for intermediate values of π, the full disclosure policy may not be optimal.

Theorem 3. *Consider the one-step transmission model with mean-field approximation.*

- *For any β, (q_L, q_H), and (θ_L, θ_H), there exist $\pi_1, \pi_2 \in (0, 1)$ such that if $\pi \in [0, \pi_1)$ or $\pi \in (\pi_2, 1]$, the full disclosure policy defined in* (2.50) *is an optimal solution to the linear program* (2.44) - (2.49)*, i.e., the optimal strategy is to reveal full information to the public.*

- *There exist β, (q_L, q_H), and (θ_L, θ_H) for which the full disclosure policy is not optimal for $\pi \in [\pi_3, \pi_4]$ for some π_3 and π_4.*

Proof. We prove the first part of the theorem by showing that in the two limits, $\pi \to 0$ and $\pi \to 1$, (2.50) is the optimal solution to the linear program. First, let us consider $\pi \to 0$. For fixed β and $0 < \theta_L < \theta_H < \infty$, we have

$$\lim_{\pi \to 0} B^{[1]}(\sigma, \theta) = -1 < 0$$

for all θ. This implies that $\sigma^{FE}(\theta_L) = \sigma^{FE}(\theta_H) = 1$. Moreover, $\lim_{\pi \to 0} \mathcal{P}^{(1)}(\sigma) = 0$ and

$$\lim_{\pi \to 0} A^{[1]}(\sigma, \theta) = -1 + \sigma,$$

which implies that $\arg\max_{\sigma \in [0,1]} A^{[1]}(\sigma, \theta) = 1$ for all θ. Therefore, (2.50) is the optimal solution to the linear program.

Next, let us consider $\pi \to 1$. By definition, this implies

$$\lim_{\pi \to 1} \mathcal{P}^{[1]}(\sigma) = 1 - q_H + (q_H - q_L)\sigma$$

and

$$\lim_{\pi \to 1} A^{[1]}(\sigma, \theta) = -\theta(1 - q_H + (q_H - q_L)\sigma) - (1 - \sigma) = -\theta(1 - q_H) - 1 + (1 - \theta(q_H - q_L))\sigma,$$

which is linear in σ. Moreover, for fixed β and $0 < \theta_L < \theta_H < \infty$, we have

$$\lim_{\pi \to 1} B^{[1]}(\sigma, \theta) = (q_H - q_L)\theta - 1.$$

There are three cases: $\theta_L < \theta_H < \frac{1}{(q_H - q_L)}$, $\theta_L < \frac{1}{(q_H - q_L)} < \theta_H$, and $\frac{1}{(q_H - q_L)} < \theta_L < \theta_H$. For the first case $\theta_L < \theta_H < \frac{1}{(q_H - q_L)}$, $B^{[1]}(\sigma, \theta) < 0$ for all σ and θ. This implies $\sigma^{FE}(\theta_L) = \sigma^{FE}(\theta_H) = 1$. We have shown that (2.50) is always a feasible solution to (2.44) - (2.49). Now we show its optimality by showing that (2.50) is the only feasible solution to (2.44) - (2.49). Consider any information policy $h' \neq h^E$. This implies that there exists $\sigma' \neq 1$ and $\theta' \in \Theta$ such that $h'(\sigma' \mid \theta') \neq 0$. If $\sigma' = 0$, we have $\sum_\theta \mu(\theta) h(0 \mid \theta) B^{[1]}(0, \theta) = -\mu(\theta') h(0 \mid \theta') < 0$, which violates (2.47). If $\sigma' \in (0,1)$, we have $\sum_\theta \mu(\theta) h(\sigma' \mid \theta) B^{[1]}(\sigma', \theta) = -\mu(\theta') h(0 \mid \theta') < 0$, which violates (2.45). Therefore, (2.50) is the only feasible solution to the linear program and is thus optimal.

For the second case $\theta_L < \frac{1}{(q_H - q_L)} < \theta_H$, we have $\sigma^{FE}(\theta_L) = 1$ and $\sigma^{FE}(\theta_H) = 0$. Moreover, notice that in this case, $\arg\max_\sigma A^{[1]}(\sigma, \theta_L) = 1$ since $1 - \theta_L(q_H - q_L) > 0$. $\arg\max_\sigma A^{[1]}(\sigma, \theta_H) = 0$ since $1 - \theta_H(q_H - q_L) < 0$. This implies that h^E defined in (2.50) is also the first best solution in this case. Since the first best solution gives an upper bound on the optimization objective, we have shown the optimality of (2.50).

Finally, for the last case $\frac{1}{(q_H - q_L)} < \theta_L < \theta_H$, we have $\sigma^{FE}(\theta_L) = \sigma^{FE}(\theta_H) = 0$. This is a similar case where (2.50) is the only feasible solution to (2.44) - (2.49). Consider any information policy $h' \neq h^E$. This implies that there exists $\sigma' \neq 0$ and $\theta' \in \Theta$ such that $h'(\sigma' \mid \theta') \neq 0$. If $\sigma' = 1$, we have for θ', $\sum_\theta \mu(\theta) h(1 \mid \theta) B^{[1]}(1, \theta) = -\mu(\theta') h(1 \mid \theta') < 0$, which violates (2.46). If $\sigma' \in (0,1)$, we have $\sum_\theta \mu(\theta) h(\sigma' \mid \theta) B^{[1]}(\sigma', \theta) = -\mu(\theta') h(\sigma' \mid \theta') < 0$, which violates (2.45). Therefore, (2.50) is the only feasible solution to the linear program and is thus optimal.

Combining the three cases, we show that the full information policy defined in (2.50) is optimal when $\pi \to 1$. By definition of limit, this completes the proof for the first part of the theorem.

The same example that we used in the proof of Theorem 2 establishes the second part of the theorem. $\square$

The first part of Theorem 3 states that when the initial infection probability π

is too small or too large, there is no need for obfuscation. In these two cases, the best policy is to completely reveal the information to the public. When π is small, the economic concern of the protection costs dominates the health concern of getting infected. There is no incentive misalignment between the sender and the agents, and therefore, the optimal policy is full disclosure. When π is large, two phenomena can happen. The first is that the potential threat of loss from the infection is too high even when $\theta = \theta_L$. Therefore, in both states of the world, all agents choose to protect. That is, it is impossible to induce any other protection behaviors than everyone protecting from the full information equilibrium. The second is that the potential threat of loss from infection is too high compared to the protection cost. In this situation, the health concern dominates the economic one. Again, there is no incentive misalignment between the sender and the agents, and therefore, full disclosure becomes the optimal policy.

The second part provides a complementary observation by establishing that for intermediate values of the initial infection probability π, the full disclosure policy may no longer be optimal.

The next example illustrates the result of Theorem 3.

Example 3. *Recall the price of full disclosure that we introduced in Example 2. Similar to Example 2, we set the high protection level as* $q_H = 0.86$*, the low protection level as* $q_L = 0.24$*, the mild harm of infection as* $\theta_L = 1.7$*, and the severe harm of infection as* $\theta_H = 7.5$*. We fix the transmission rate* $\beta = 39$ *and vary the initial infection probability* π*. The price of full disclosure in this model as a function of* π *is depicted in Figure 2.2. We observe that when the initial infection probability is smaller than* 0.19*, the price of full disclosure is* 0*. In this case, a full information disclosure policy is optimal. When the initial infection probability is larger than* 0.43*, the price of full disclosure is again* 0*. The sender cannot do anything with her infor-*

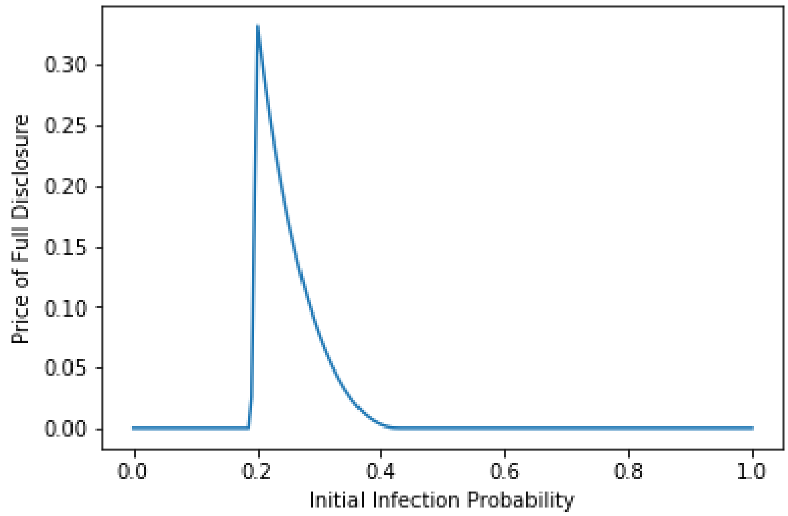


Figure 2.2: Price of full disclosure under different π with $(q_H, q_L, \beta, \theta_H, \theta_L) = (0.86, 0.24, 39, 7.5, 1.7)$

mation advantage, and the best strategy is full disclosure. These observations confirm the results of part (a) of Theorem 3. For an intermediate initial infection probability, as established in part (b) of Theorem 3, we observe that it is socially beneficial to disclose partial information. In particular, for intermediate values of π*, mixing information can result in more than* 30% *improvement in social welfare.*

2.6 Full Transmission

In this section, we return to the full transmission model and analyze the mean-field approximation of the infection probability. We first note that, unlike with the one-step transmission, the social planner's information problem is no longer a linear program. Nevertheless, we show that the impact of the transmission rate β that we obtained in the simpler one-step transmission model continues to hold in this more general setting. In contrast, we establish that the impact of the initial infection probability π changes.

2.6.1 Mean-Field Approximation for Full Transmission

We consider the same discretization of the social activity levels as developed in Subsection 2.5.1. To derive the mean-field approximation, we discretize the agents' social activity levels so that they are uniformly drawn from t types $\{a^1, \cdots, a^t\}$. With t types, there are n/t agents of each possible type, and then we let both n and t go to infinity. We use the same notation $\mathbb{P}^{(n)}(\sigma, x, t)$ for the infection probability of each agent with social activity level x when there are t types and the protection threshold is σ. For $x \in \{a^1, \cdots, a^t\}$, we have

$$\mathbb{P}^{(n)}(\sigma, x, t) \doteq (1 - q_H \mathbb{1}\{x > \sigma\} - q_L \mathbb{1}\{x \leq \sigma\})$$
$$\left[\pi + (1-\pi)\left(1 - \prod_{y \in \{a^1, \ldots, a^t\} \setminus \{x\}} \left(1 - \frac{\beta}{n} xy \mathbb{P}^{(n)}(\sigma, y, t)\right)^{\frac{n}{t}} \left(1 - \frac{\beta}{n} x^2 \mathbb{P}^{(n)}(\sigma, x, t)\right)^{\frac{n}{t}-1}\right)\right]. \quad (2.51)$$

In (2.51), the term $(1 - q_H \mathbb{1}\{x > \sigma\} - q_L \mathbb{1}\{x \leq \sigma\})$ is the probability that the type x agent remains in the contact network. Conditional on that, this agent can be infected as the origin with probability π. If the agent is not initially infected with probability $1 - \pi$, the infection reaches her through any other agent in an order-independent way. In particular, the infection reaches this agent through a type y agent with probability

$$\frac{\beta}{n} xy \mathbb{P}^{(n)}(\sigma, y, t).$$

Note that the type y agents can transmit infection only if they are present in the contact network and are infected with probability $\mathbb{P}^{(n)}(\sigma, y, t)$. Equation (2.51) then follows by noting that there are n/t agents of type $y \neq x$ and $n/t - 1$ remaining type x agents.

Next, we denote $\mathcal{P}(\sigma, x)$ as the mean-field approximation of the infection probability $\mathbb{P}^{(n)}(\sigma, x, t)$ corresponding to what we defined in (2.36) in the one-step model. Similarly, we interpret it as the fraction of type x agents infected. Our next proposition characterizes this quantity.

Proposition 6. *For any protection threshold σ and social activity level x, we have*

$$\mathcal{P}(\sigma,x) \doteq \lim_{t\to\infty}\lim_{n\to\infty}\mathbb{P}^{(n)}(\sigma,x,t) = (1-q_H\mathbb{1}\{x>\sigma\}-q_L\mathbb{1}\{x\leq\sigma\})\left(\pi+(1-\pi)\left(1-e^{-\beta x\mathsf{e}(\sigma)}\right)\right), \tag{2.52}$$

where $\mathsf{e}(\sigma)$ is the unique positive root of the equation

$$\begin{aligned}\mathsf{e}(\sigma) =& \frac{1-q_H+(q_H-q_L)\sigma^2}{2} + \frac{1-\pi}{\beta^2\mathsf{e}^2(\sigma)}\Bigg(-(1-q_L)+(q_H-q_L)\times\\ &(1+\beta\sigma\mathsf{e}(\sigma))e^{-\beta\sigma\mathsf{e}(\sigma)}+(1-q_H)(1+\beta\mathsf{e}(\sigma))e^{-\beta\mathsf{e}(\sigma)}\Bigg).\end{aligned} \tag{2.53}$$

Proof. We start by proving the following claim.

Claim 3. *For $x\in\{a^1,\cdots,a^t\}$, the following is well defined:*

$$\mathcal{P}(\sigma,x,t) \doteq \lim_{n\to\infty}\mathbb{P}^{(n)}(\sigma,x,t).$$

Moreover, we have

$$\begin{aligned}\mathcal{P}(\sigma,x,t) =& (1-q_H\mathbb{1}\{x>\sigma\}-q_L\mathbb{1}\{x\leq\sigma\})\times\\ &\left(\pi+(1-\pi)\left(1-\prod_{y\in\{a^1,\cdots,a^t\}}e^{-\frac{\beta}{t}xy\mathcal{P}(\sigma,y,t)}\right)\right).\end{aligned} \tag{2.54}$$

Proof of Claim 3: For all $j\in\{1,\cdots,t\}$, let $y_n^j \doteq \mathbb{P}^{(n)}(\sigma,a^j,t)$ and

$$f_n^j(y_n^1,\cdots,y_n^t) \doteq (1-q_H\mathbb{1}\{a^j>\sigma\}-q_L\mathbb{1}\{a^j\leq\sigma\})\left(\pi+(1-\pi)\left(1-\prod_{k=1}^{t}(1-\frac{\beta}{n}a^ja^ky_n^k)^{\frac{n}{t}}\right)\right).$$

We also let $f_n:[0,1]^t\to[0,1]^t$ be $f_n=(f_n^1,\cdots,f_n^t)$. Note that the fixed points for f_n are the solutions we want for (2.51). It can be seen that f_n^j is continuous. Next, we show that for all j and $(z^1, \cdots, z^t)$, we have

$$f_n^j\left(z^1, \cdots, z^t\right) \to f^j\left(z^1, \cdots, z^t\right) \quad \text{uniformly},$$

where

$$\begin{aligned} f^j\left(z^1, \cdots, z^t\right) = &\left(1 - q_H \mathbb{1}\left\{a^j > \sigma\right\} - q_L \mathbb{1}\left\{a^j \leq \sigma\right\}\right) \times \\ &\left(\pi + (1-\pi)\left(1 - \prod_{k=1}^{t} e^{-\frac{\beta}{t} a^j a^k z^k}\right)\right). \end{aligned} \tag{2.55}$$

The statement follows from

$$\left(1 - \frac{x}{n}\right)^n \to e^{-x} \text{ uniformly.}$$

Using Taylor expansion, we have

$$e^x - \left(1 + \frac{x}{n}\right)^n = \sum_{k>n} \frac{1}{k!} x^k + \sum_{k=0}^{n} \left(\frac{1}{k!} - \binom{n}{k} \frac{1}{n^k}\right) x^k = \sum_{k=0}^{\infty} a_{n,k} \frac{1}{k!} x^k,$$

where

$$a_{n,k} = \begin{cases} 1 - \left(\left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right)\right) & k \leq n \\ 1 & \text{otherwise} \end{cases}.$$

Note that $0 \leq a_{n,k} \leq 1$ for all k, n. And for any fixed k, we have $\lim_{n\to\infty} a_{n,k} = 0$. Let $\epsilon > 0$ and choose N such that for $n \geq N$, $\sum_{k>n} \frac{1}{k!} b^k < \frac{1}{2}\epsilon$. Then,

$$\left|e^x - \left(1 + \frac{x}{n}\right)^n\right| \leq \sum_{k=0}^{N} a_{n,k} \frac{1}{k!} |x|^k + \frac{1}{2}\epsilon \leq \sum_{k=0}^{N} a_{n,k} \max\left(1, b^N\right) + \frac{1}{2}\epsilon.$$

Now choose $N' \geq N$ such that $\sum_{k=0}^{N} a_{n,k} \leq \frac{1}{2\max(1,b^N)}\epsilon$ for $n \geq N'$. We have

$$\left| e^x - \left(1 + \frac{x}{n}\right)^n \right| \leq \epsilon$$

for all x. So $\left(1 - \frac{x}{n}\right)^n \to e^{-x}$ uniformly.

Now let $y_n = (y_n^1 \cdots y_n^t)$. Since f_n is continuous, there exists δ such that for $|y_n - y| \leq \delta$, $|f_n(y_n) - f_n(y)| \leq \frac{\epsilon}{2}$. Since $f_n \to f$ uniformly, there exists N such that for all $n \geq N$, $|f_n(x) - f(x)| \leq \frac{\epsilon}{2}$ for all x. Taking the two together, we have

$$\begin{aligned} |f_n(y_n) - f(y)| &= |f_n(y_n) - f_n(y) + f_n(y) - f(y)| \\ &\leq |f_n(y_n) - f_n(y)| + |f_n(y) - f(y)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Finally, we show that there exist fixed points for functions f_n and f. We have

$$\begin{aligned} \frac{\partial f_n^j\left(y_n^1, \cdots, y_n^t\right)}{\partial y_n^i} &= \left(1 - q_H \mathbb{1}\left\{a^j > \sigma\right\} - q_L \mathbb{1}\left\{a^j \leq \sigma\right\}\right)(1-\pi) \\ &\quad \frac{\beta a^j a^i}{t}\left(1 - \frac{\beta}{n} a^j a^i y_n^i\right)^{\frac{n}{t}-1} \prod_{k \neq i}\left(1 - \frac{\beta}{n} a^j a^k y_n^k\right)^{\frac{n}{t}} \geq 0, \\ \frac{\partial f^j\left(y^1, \cdots, y^t\right)}{\partial y^i} &= \left(1 - q_H \mathbb{1}\left\{a^j > \sigma\right\} - q_L \mathbb{1}\left\{a^j \leq \sigma\right\}\right)(1-\pi) \\ &\quad \frac{\beta a^j a^i}{t} e^{-\frac{\beta}{t} a^j a^i y_n^i - 1} \prod_{k \neq i} e^{-\frac{\beta}{t} a^j a^k y_n^k} \geq 0 \end{aligned}$$

for all i and j. By the Knaster–Tarski theorem ([Tar55]), f_n and f being order preserving and the function space $[0,1]^t$ being a complete lattice together imply that there exist fixed points $y_n = f_n(y_n)$ and $y = f(y)$. Therefore, there exists N such that for all $n \to N$, $|y_n - y| \leq \epsilon$. By definition, this means that $y_n \to y$ uniformly, where y is the fixed point for $f = (f^1 \cdots f^t)$ as described in (2.55). This completes the proof of Claim 3. □

We now proceed with the proof of Proposition 6. Again, we introduce a notation for the exponent that appears in the above equation as

$$\mathsf{e}\left(\sigma,t\right) \doteq \frac{\sum_{y\in\{a^1,\cdots,a^t\}} y\mathcal{P}\left(\sigma,y,t\right)}{t}. \tag{2.56}$$

Now we take t to infinity and return to the original $[0,1]$ space of x. With a slight abuse of notation, we define

$$\begin{aligned}
\mathsf{e}\left(\sigma\right) \doteq \lim_{t\to\infty} \mathsf{e}\left(\sigma,t\right) = &\int_0^1 x\mathcal{P}\left(\sigma,x,t\right)dx \\
= &\int_0^1 x\left(1-q_H\mathbb{1}\left\{x>\sigma\right\}-q_L\mathbb{1}\left\{x\le\sigma\right\}\right)\left(\pi+\left(1-\pi\right)\left(1-e^{-\beta x\mathsf{e}(\sigma)}\right)\right)dx \\
= &\frac{1-q_H+\left(q_H-q_L\right)\sigma^2}{2}+\frac{1-\pi}{\beta^2\mathsf{e}^2\left(\sigma\right)}\Bigg(-\left(1-q_L\right)+\left(q_H-q_L\right)\left(1+\beta\sigma\mathsf{e}\left(\sigma\right)\right)e^{-\beta\sigma\mathsf{e}(\sigma)} \\
&+\left(1-q_H\right)\left(1+\beta\mathsf{e}\left(\sigma\right)\right)e^{-\beta\mathsf{e}(\sigma)}\Bigg),
\end{aligned}$$

as illustrated in (2.53). Now we show that (2.53) has a unique root between 0 and 1. We denote

$$f(x) = \frac{1-q_H+\left(q_H-q_L\right)\sigma^2}{2}+\frac{(1-\pi)(-\left(1-q_L\right)+\left(q_H-q_L\right)\left(1+\beta\sigma x\right)e^{-\beta\sigma x}+\left(1-q_H\right)\left(1+\beta x\right)e^{-\beta x})}{\beta^2x^2}$$

as the right-hand side of equation (2.53) and show that $f(x)=x$ has a unique root in $[0,1]$. We have

$$\begin{aligned}
f(0) = &\frac{1-q_H+\left(q_H-q_L\right)\sigma^2}{2}+\lim_{x\to0}\frac{(1-\pi)(-(q_H-q_L)\beta^2\sigma^2xe^{-\beta\sigma x}-(1-q_H)\beta^2xe^{-betax})}{2\beta^2x} \\
= &\frac{1-q_H+\left(q_H-q_L\right)\sigma^2}{2}+\lim_{x\to0}\frac{-(1-\pi)((q_H-q_L)\sigma^2e^{-\beta\sigma x}+(1-q_H)e^{-\beta x})}{2} \\
= &\frac{\pi(1-q_H+\left(q_H-q_L\right)\sigma^2)}{2}\ge0, \\
f(1) = &\frac{1-q_H+\left(q_H-q_L\right)\sigma^2}{2} \\
&+\frac{(1-\pi)(-(1-q_L)+(q_H-q_L)(1+\beta\sigma)e^{-\beta\sigma}+(1-q_H)(1+\beta)e^{-\beta})}{\beta^2} \\
< &\frac{1-q_H+\left(q_H-q_L\right)\sigma^2}{2}+\frac{(1-\pi)(-(1-q_L)+(q_H-q_L)+(1-q_H))}{\beta^2} \\
= &\frac{1-q_H+\left(q_H-q_L\right)\sigma^2}{2}<1,
\end{aligned}$$

and

$$
\begin{aligned}
f'(x) =& \frac{\beta^2 x^2 (1-\pi)(-\beta^2\sigma^2 x (q_H - q_L) e^{-\beta\sigma x} - \beta^2 x (1-q_H) e^{-\beta x})}{\beta^4 x^4} \\
&- \frac{2\beta^2 x (1-\pi)(-(1-q_L) + (q_H - q_L)(1+\beta\sigma x) e^{-\beta\sigma x} + (1-q_H)(1+\beta x) e^{-\beta x})}{\beta^4 x^4} \\
=& \frac{(1-\pi)(2(1-q_L) - (q_H - q_L) e^{-\beta\sigma x}(\beta^2\sigma^2 x^2 + 2\beta\sigma x + 2) - (1-q_H) e^{-\beta x}(\beta^2 x^2 + 2\beta x + 2))}{\beta^2 x^3} \\
\geq& \frac{(1-\pi)(2(1-q_L) - 2(q_H - q_L) - 2(1-q_H))}{\beta^2 x^3} = 0
\end{aligned}
$$

for all x, which implies that $f(x)$ is monotone increasing between 0 and 1. This implies that there exists a unique positive root for equation (2.53) between 0 and 1.

Using (2.54) and (2.53), we have

$$
\begin{aligned}
\mathcal{P}(\sigma, x) &= \lim_{t\to\infty} \left(1 - q_H \mathbb{1}\{x > \sigma\} - q_L \mathbb{1}\{x \leq \sigma\}\right) \left[\pi + (1-\pi)\left(1 - e^{-x\beta\pi \mathsf{e}(\sigma,t)}\right)\right] \\
&= \left(1 - q_H \mathbb{1}\{x > \sigma\} - q_L \mathbb{1}\{x \leq \sigma\}\right) \left[\pi + (1-\pi)\left(1 - e^{-\beta x \mathsf{e}(\sigma)}\right)\right].
\end{aligned}
$$

This completes the proof. □

We next characterize the fraction of infected agents (of any type) following any protection threshold.

Corollary 3. *For any protection threshold $\sigma \in [0,1]$, the fraction of infected agents is*

$$
\begin{aligned}
\mathcal{P}(\sigma) \doteq& \mathbb{E}_x\left[\mathcal{P}(\sigma, x)\right] \\
=& 1 - q_H + (q_H - q_L)\sigma + \frac{(1-\pi)}{\beta \mathsf{e}(\sigma)} \left(-(1-q_L) + (q_H - q_L) e^{-\beta\sigma \mathsf{e}(\sigma)} + (1-q_H) e^{-\beta \mathsf{e}(\sigma)} \right),
\end{aligned}
\tag{2.57}
$$

where $\mathsf{e}(\sigma)$ is defined in (2.53).

Proof. By definition, we have

$$
\begin{aligned}
\mathcal{P}(\sigma) =& \int_0^1 \lim_{t\to\infty} \mathcal{P}(\sigma, x, t)\, dx \\
=& \int_0^1 \left(1 - q_H \mathbb{1}\{x > \sigma\} - q_L \mathbb{1}\{x \le \sigma\}\right) \left(\pi + (1-\pi)\left(1 - e^{-\beta x \mathsf{e}(\sigma)}\right)\right) dx \\
=& 1 - q_H + (q_H - q_L)\,\sigma + \frac{(1-\pi)}{\beta \mathsf{e}(\sigma)} \left(-(1-q_L) + (q_H - q_L)\, e^{-\beta\sigma\mathsf{e}(\sigma)} + (1-q_H)\, e^{-\beta\mathsf{e}(\sigma)} \right).
\end{aligned}
$$

This completes the proof. □

Next, similar to what we defined in the one-step model in (2.42) and (2.43), we let $A(\sigma,\theta)$ and $A(\sigma,\theta)$ be the limit of the welfare and inventive functions, respectively. That is,

$$
A(\sigma,\theta) \doteq \lim_{n\to\infty} A^{(n)}(\sigma,\theta)\,, \text{ and } B(\sigma,\theta) \doteq \lim_{n\to\infty} B^{(n)}(\sigma,\theta)\,,
$$

in which $A^{(n)}(\sigma,\theta)$ and $B^{(n)}(\sigma,\theta)$ are defined in (2.27) and (2.28), respectively.

The next result characterizes the limits of the welfare function and the incentive function.

Corollary 4. *For any protection threshold $\sigma \in [0,1]$ and state of the world $\theta \in \Theta$, the welfare function is*

$$
A(\sigma,\theta) = -\,\theta\mathcal{P}(\sigma) - (1-\sigma)\,. \tag{2.58}
$$

The incentive function is

$$
B(\sigma,\theta) = (q_H - q_L)\left[\pi + (1-\pi)\left(1 - e^{-\beta\sigma\mathsf{e}(\sigma)}\right)\right]\theta - 1, \tag{2.59}
$$

where $\mathsf{e}(\sigma)$ is defined in (2.53).

Proof. By definition, we have

$$A(\sigma,\theta) = \lim_{n\to\infty} A^n(\sigma,\theta) = \lim_{n\to\infty} -\theta\mathcal{P}^n(\sigma) - (1-\sigma) = -\theta\mathcal{P}(\sigma) - (1-\sigma).$$

$$\begin{aligned}
B(\sigma,\theta) &= \lim_{n\to\infty} B^n(\sigma,\theta) = \lim_{n\to\infty} \mathbb{E}_{\tilde{\mathbf{x}}}\left[\tilde{u}^n(q_H,\sigma,\sigma,\tilde{\mathbf{x}},\theta) - \tilde{u}^n(q_L,\sigma,\sigma,\tilde{\mathbf{x}},\theta)\right] \\
&= -1-(1-q_H)\left[\pi+(1-\pi)\left(1-e^{-\frac{\beta\pi\sigma\mathsf{e}(\sigma)}{2}}\right)\right] - (1-q_L)\left[\pi+(1-\pi)\left(1-e^{-\frac{\beta\pi\sigma\mathsf{e}(\sigma)}{2}}\right)\right] \\
&= (q_H-q_L)\left[\pi+(1-\pi)\left(1-e^{-\frac{\beta\pi\sigma\mathsf{e}(\sigma)}{2}}\right)\right]\theta - 1.
\end{aligned}$$

This completes the proof. □

Following A and B as defined in (2.58) and (2.59), the full transmission information design problem with the mean-field approximation is

$$\max_h \sum_\theta \sum_\sigma \mu(\theta)\, h(\sigma \mid \theta)\, A(\sigma,\theta) \tag{2.60}$$

s.t.

$$\sum_\theta \mu(\theta)\, h(\sigma \mid \theta)\, B(\sigma,\theta) = 0, \;\; \forall\, \sigma \in \Sigma \setminus \{0,1\}, \tag{2.61}$$

$$\sum_\theta \mu(\theta)\, h(1 \mid \theta)\, B(1,\theta) \le 0, \tag{2.62}$$

$$\sum_\theta \mu(\theta)\, h(0 \mid \theta)\, B(0,\theta) \ge 0, \tag{2.63}$$

$$\sum_{\sigma\in\Sigma} h(\sigma \mid \theta) = 1, \;\; \forall\theta \in \Theta, \tag{2.64}$$

$$h(\sigma \mid \theta) \ge 0, \;\; \forall\sigma \in \Sigma, \theta \in \Theta. \tag{2.65}$$

Concluding this subsection, we would like to highlight that the social planner's problem with full transmission, formulated in (2.60), unlike the one-step transmission, is not a linear program. This can be seen by noting that maximizing (2.60) involves $h(\sigma \mid \theta)\,\mathcal{P}(\sigma)$, where $\mathcal{P}(\sigma)$ is a nonlinear function of $\mathsf{e}(\sigma)$ as established in (2.53).

Moreover, substituting (2.59) into (2.61), we have

$$\mathsf{e}(\sigma) = -\frac{\ln\left(\frac{(q_H - q_L)\sum_\theta[\mu(\theta)h(\sigma|\theta)\theta] - 1}{(1-\pi)(q_H - q_L)\sum_\theta[\mu(\theta)h(\sigma|\theta)\theta]}\right)}{\beta\sigma},$$

which establishes that $\mathsf{e}(\sigma)$ is a nonlinear function of $h(\sigma \mid \theta)$. The fact that the social planner's problem with the full transmission is no longer a linear program complicates the analysis. Nevertheless, we prove in the next subsection that certain insights from the simpler one-step transmission model, in terms of the impact of the transmission rate β, continue to hold in this more general setting.

2.6.2 The Effect of the Transmission Rate

In this section, we study the benefit of information design with different transmission rates β. Extending the result we found with the one-step transmission in Section 2.5.3, we formally prove that for small and large β, the optimal information policy is to reveal full information and have agents act according to their equilibrium. We then illustrate that, for intermediate values of β, the optimal information design policy is not a full disclosure policy. The effect of the transmission rate β is stated in the following theorem.

Theorem 4. *Consider full transmission and mean-field approximation.*

- *For any π, (q_L, q_H), and (θ_L, θ_H), there exist $0 < \beta_1 < \beta_2 < \infty$ such that for $\beta < \beta_1$ and $\beta > \beta_2$, the full disclosure policy defined in* (2.50) *is an optimal solution to the social planner's problem* (2.60) - (2.65)*, i.e., the optimal strategy is to reveal full information to the public.*

- *There exist π, (q_L, q_H), and (θ_L, θ_H) for which the full disclosure policy is not* *optimal for $\beta \in [\beta_3, \beta_4]$ for some β_3 and β_4.*

Proof. We prove the theorem by showing that in the two extremes of $\beta \to 0$ and $\beta \to \infty$, (2.50) is the optimal solution to the social planner's problem. First, let us consider $\beta \to 0$. To study the limit, we define the sequence f_n with $n = \frac{1}{\beta}$ as

$$f_n(x) = \frac{1 - q_H + (q_H - q_L)\sigma^2}{2} + \frac{(1-\pi)n^2\left(-(1-q_L) + (q_H - q_L)\left(1 + \frac{\sigma x}{n}\right)e^{-\frac{\sigma x}{n}} + (1-q_H)\left(1+\frac{x}{n}\right)e^{-\frac{x}{n}}\right)}{x^2}.$$

Fixing x and considering point-wise convergence, we have

$$\begin{aligned}
\lim_{n\to\infty} f_n(x) =& \frac{1 - q_H + (q_H - q_L)\sigma^2}{2} \\
&+ \lim_{n\to\infty} \frac{(1-\pi)\left((q_H - q_L)\left(-\frac{\sigma x}{n^2} + \frac{\sigma x}{n^2} + \frac{\sigma^2 x^2}{n^3}\right)e^{-\frac{\sigma x}{n}} + (1-q_H)\left(-\frac{x}{n^2} + \frac{x}{n^2} + \frac{x^2}{n^3}\right)e^{-\frac{x}{n^2}}\right)}{-\frac{2x^2}{n^3}} \\
=& \frac{1 - q_H + (q_H - q_L)\sigma^2}{2} - \lim_{n\to\infty} \frac{(1-\pi)\left((q_H - q_L)\sigma^2 e^{-\frac{\sigma x}{n}} + (1-q_H)e^{-\frac{x}{n}}\right)}{2} \\
=& \frac{1 - q_H + (q_H - q_L)\sigma^2}{2} - \frac{(1-\pi)\left(1 - q_H + (q_H - q_L)\sigma^2\right)}{2} \\
=& \frac{\pi\left(1 - q_H + (q_H - q_L)\sigma^2\right)}{2}.
\end{aligned}$$

Therefore, the limit function is a constant function and is continuous. The function space is $[0, 1]$ and is compact. Dini's theorem states that given a sequence of functions, if the limit function is continuous, the function space is compact, and the sequence of functions is order preserving, then the sequence of functions converges uniformly to the limit function. Lastly, we show that the sequence of functions is monotone, i.e., $f_n(x)$ is monotone in n. We have

$$\begin{aligned}
\frac{\partial f_n(x)}{\partial n} =& \frac{x^2(1-\pi)}{x^4}\Bigg(2n\left(-(1-q_L) + (q_H - q_L)\left(1 + \frac{\sigma x}{n}\right)e^{-\frac{\sigma x}{n}} + (1-q_H)\left(1 + \frac{x}{n}\right)e^{-\frac{x}{n}}\right) \\
&+ n^2\left((q_H - q_L)\frac{\sigma^2 x^2}{n^3}e^{-\frac{\sigma x}{n}} + (1-q_H)\frac{x^2}{n^3}e^{-\frac{x}{n}}\right)\Bigg) \\
=& \frac{2(1-\pi)n}{x^2}\Bigg(-(1-q_L) + (q_H - q_L)\left(1 + \frac{\sigma x}{n} + \frac{1}{2}\frac{\sigma^2 x^2}{n^2}\right)e^{-\frac{\sigma x}{n}} \\
&+ (1-q_H)\left(1 + \frac{x}{n} + \frac{1}{2}\frac{x^2}{n^2}\right)e^{-\frac{x}{n}}\Bigg) < 0.
\end{aligned}$$

Therefore, by Dini's theorem, f_n converges uniformly to $\frac{\pi\left(1-q_H+(q_H-q_L)\sigma^2\right)}{2}$. By definition, this implies that $\mathsf{e}(\sigma) = \frac{\pi\left(1-q_H+(q_H-q_L)\sigma^2\right)}{2}$ at $\beta \to 0$. Plugging into (2.57), (2.58), and (2.59), we have

$$\begin{aligned}
\lim_{\beta\to 0} \mathcal{P}(\sigma) =& 1 - q_H + (q_H - q_L)\,\sigma \\
&+ \lim_{\beta\to 0} \frac{(1-\pi)}{\beta \mathsf{e}(\sigma)} \left(-(1-q_L) + (q_H - q_L)\, e^{-\beta\sigma\mathsf{e}(\sigma)} + (1-q_H)\, e^{-\beta\mathsf{e}(\sigma)} \right) \\
=& \pi\left(1 - q_H + (q_H - q_L)\,\sigma\right), \\
\lim_{\beta\to 0} A(\sigma,\theta) =& -\theta\pi\left(1 - q_H + (q_H - q_L)\,\sigma\right) - (1-\sigma) \\
=& -\theta\pi\,(1-q_H) - 1 + (1 - \theta\pi\,(q_H - q_L))\,\sigma, \\
\lim_{\beta\to 0} B(\sigma,\theta) =& (q_H - q_L)\,\pi\theta - 1.
\end{aligned}$$

Note that the limits of functions A and B are the same regardless of one-step or full transmission. This means that the result for $\beta \to 0$ remains the same when we extend the model from one-step to full transmission. This completes the proof of the first part of the theorem.

Next, look at $\beta \to \infty$. We now define another sequence of functions f_β as

$$\begin{aligned}
f_\beta(x) =& \frac{1 - q_H + (q_H - q_L)\,\sigma^2}{2} \\
&+ \frac{(1-\pi)\left(-(1-q_L) + (q_H - q_L)(1+\beta\sigma x)\, e^{-\beta\sigma x} + (1-q_H)(1+\beta x)\, e^{-\beta x} \right)}{\beta^2 x^2}.
\end{aligned}$$

Fixing x and considering point-wise convergence, we have

$$\begin{aligned}
&\lim_{\beta\to\infty} f_\beta(x) = \frac{1 - q_H + (q_H - q_L)\,\sigma^2}{2} \\
&+ \lim_{\beta\to\infty} \frac{(1-\pi)\left(-(1-q_L) + (q_H - q_L)(1+\beta\sigma x)\, e^{-\beta\sigma x} + (1-q_H)(1+\beta x)\, e^{-\beta x} \right)}{\beta^2 x^2} \\
&= \frac{1 - q_H + (q_H - q_L)\,\sigma^2}{2}.
\end{aligned}$$

Again, the limit function is a constant function and is continuous. The function space is $[0,1]$ and is compact. Lastly, we show that the sequence of functions is monotone, i.e., $f_\beta(x)$ is monotone in β:

$$
\begin{aligned}
\frac{\partial f_\beta(x)}{\partial \beta} =& \frac{\beta^2 x^2 (1-\pi)\left(-(q_H-q_L)\beta\sigma^2 x^2 e^{-\beta\sigma x} - (1-q_H)\beta x^2 e^{-\beta x}\right)}{\beta^4 x^4} \\
&- \frac{2\beta x^2 (1-\pi)\left(-(1-q_L) + (q_H-q_L)(1+\beta\sigma x)e^{-\beta\sigma x} + (1-q_H)(1+\beta x)e^{-\beta x}\right)}{\beta^4 x^4} \\
=& \frac{2(1-\pi)\left(1-q_L-(q_H-q_L)\left(1+\beta\sigma x+\frac{1}{2}\beta^2\sigma^2x^2\right)e^{-\beta\sigma x} - (1-q_H)\left(1+\beta x+\frac{1}{2}\beta^2 x^2\right)e^{-\beta x}\right)}{\beta^3 x^2} \\
>& 0.
\end{aligned}
$$

Therefore, by Dini's theorem, f_β converges uniformly to $\frac{1-q_H+(q_H-q_L)\sigma^2}{2}$. By definition, this implies that $\mathsf{e}(\sigma) = \frac{1-q_H+(q_H-q_L)\sigma^2}{2}$ at $\beta \to \infty$. Plugging into (2.57), (2.58), and (2.59), we have

$$
\begin{aligned}
\mathcal{P}(\sigma) =& 1-q_H+(q_H-q_L)\sigma \\
&+ \lim_{\beta\to\infty} \frac{(1-\pi)}{\beta \mathsf{e}(\sigma)}\left(-(1-q_L)+(q_H-q_L)e^{-\beta\sigma\mathsf{e}(\sigma)} + (1-q_H)e^{-\beta\mathsf{e}(\sigma)}\right) \\
=& 1-q_H+(q_H-q_L)\sigma,
\end{aligned}
$$

$$
\begin{aligned}
\lim_{\beta\to\infty} A(\sigma,\theta) =& -\theta(1-q_H+(q_H-q_L)\sigma) - (1-\sigma) = -\theta(1-q_H) - 1 + (1-\theta(q_H-q_L))\sigma, \\
\lim_{\beta\to\infty} B(\sigma,\theta) =& \theta(q_H-q_L) - 1.
\end{aligned}
$$

Note that again the limits of functions A and B are the same regardless of one-step or full transmission. Therefore, the result for $\beta \to \infty$ is preserved in the extension. This completes the proof for the theorem. □

The first part of Theorem 4 states that when the infection transmission rate is very large or small, the sender finds it optimal to fully disclose the information. Based on different values of other parameters, there are two intuitions to this result. On the one hand, when the infection is barely (resp. highly) transmissible, the potential threat of loss from infection is too low (resp. high) even when the state is bad (resp. good). Therefore, in both states, no agent (resp. all agents) will choose to protect, i.e., it is impossible to induce any protection behaviors other than the full information equilibrium. On the other hand, when the potential threat of loss from infection is too low or too high compared with the protection cost, the impact of the free-riding diminishes, and there is no incentive misalignment between the sender and the agents.

The second part provides a complementary observation by establishing that for intermediate values for the infection transmission rate, the full disclosure policy may no longer be optimal.

Next, we look at how the effect of the initial infection probability extends to the full transmission model.

2.6.3 The Effect of the Initial Infection Probability

In this section, we study the benefit of information design under different initial infection probabilities π. We formally prove that for large π, the optimal information policy is to reveal full information and have agents play the full disclosure equilibrium. More importantly, we prove that no matter how small π is, there exist cases such that the optimal information policy generates strictly higher social welfare than the equilibrium. That is, it is socially beneficial to obfuscate the information. The formal result is as follows.

Theorem 5. *Consider full transmission and mean-field approximation.*

1. *For any β, (q_L, q_H), and (θ_L, θ_H), there exists π_1 such that for $\pi > \pi_1$, the full disclosure policy defined in* (2.50) *is an optimal solution to the social planner's problem* (2.60) - (2.65)*, i.e., the optimal strategy is to reveal full information to the public.*

2. *There exist $(\beta, \theta_L, \theta_H, q_L, q_H)$ and π_2 such that for all $\pi \leq \pi_2$, the optimal solution to the social planner's problem* (2.60) - (2.65) *is strictly better than the full disclosure policy defined in* (2.50)*, i.e., it is beneficial to obfuscate the information.*

Proof. We prove part 1 of the theorem by showing that in the limit $\pi \to 1$, (2.50) is the optimal solution to the social planner's problem. For part 2, we use an example to prove the existence. Consider $\pi \to 1$. We have

$$\lim_{\pi \to 1} f_\pi (x) = \frac{1 - q_H + (q_H - q_L)\,\sigma^2}{2}.$$

The limit function is a constant function and is continuous. The function space is $[0, 1]$ and is compact. Lastly, we show that the sequence of functions is monotone, i.e., $f_\pi (x)$ is monotone in π.

$$\frac{\partial f_\pi (0)}{\partial x} \frac{(1 - q_L) - (q_H - q_L)(1 + \beta\sigma x)\, e^{-\beta\sigma x} - (1 - q_H)(1 + \beta x)\, e^{-\beta x}}{\beta^2 x^2} > 0.$$

By Dini's theorem, f_π converges uniformly to $\frac{1-q_H+(q_H-q_L)\sigma^2}{2}$. By definition, this implies that $\mathsf{e}(\sigma) = \frac{1-q_H+(q_H-q_L)\sigma^2}{2}$ at $\pi \to 1$. Plugging into (2.57), (2.58), and (2.59), we have

$$\lim_{\pi \to 1} \mathcal{P}(\sigma) = 1 - q_H + (q_H - q_L)\,\sigma$$

$$
\begin{aligned}
&+\lim_{\pi\to 1}\frac{(1-\pi)}{\beta \mathsf{e}(\sigma)}\left(-(1-q_L)+(q_H-q_L)\,e^{-\beta\sigma \mathsf{e}(\sigma)}+(1-q_H)\,e^{-\beta \mathsf{e}(\sigma)}\right)\\
=&1-q_H+(q_H-q_L)\,\sigma,
\end{aligned}
$$

$$
\begin{aligned}
\lim_{\pi\to 1}A(\sigma,\theta)=&\lim_{\pi\to 1}A(\sigma,\theta)=-\theta\,(1-q_H+(q_H-q_L)\,\sigma)-(1-\sigma)\\
=&-\theta\,(1-q_H)-1+(1-\theta\,(q_H-q_L))\,\sigma,\\
\lim_{\pi\to 1}B(\sigma,\theta)=&(q_H-q_L)\,\theta-1.
\end{aligned}
$$

Note that the limits of functions A and B are the same regardless of one-step or full transmission. Therefore, the result for $\pi\to 1$ is preserved in the extension. This completes the proof for the first part of the result.

Part 2 of the theorem is shown by the example in Section 5.2. Next, we show a result that studies the transformation of the obedience constraints.

Claim 4. *A message rule defined in* (2.66) *is feasible in the social planner's problem* (2.60) - (2.65) *if and only if*

$$
\begin{aligned}
\sigma_1&=\sigma^{FE}\left(\frac{m_1\theta_L+m_2\theta_H}{m_1+m_2}\right),\\
\sigma_2&=\sigma^{FE}\left(\frac{(1-m_1)\,\theta_L+(1-m_2)\,\theta_H}{2-m_1-m_2}\right).
\end{aligned}
$$

Proof of Claim 4: Consider a message rule defined as (2.66). By definition, the probability constraints (2.64) and (2.65) are satisfied. Therefore, the message rule is feasible if and only if it satisfies

$$
\begin{aligned}
\frac{1}{2}h(\sigma_1\mid\theta_L)\,B(\sigma_1,\theta_L)+\frac{1}{2}h(\sigma_1\mid\theta_H)\,B(\sigma_1,\theta_H)=0,\\
\frac{1}{2}h(\sigma_2\mid\theta_L)\,B(\sigma_2,\theta_L)+\frac{1}{2}h(\sigma_2\mid\theta_H)\,B(\sigma_2,\theta_H)=0.
\end{aligned}
$$

Taking the first equation, we have

$$m_1 B(\sigma_1, \theta_L) + m_2 B(\sigma_1, \theta_H) = 0,$$

which results in

$$\begin{aligned} &m_1\left((q_H - q_L)\left[\pi + (1-\pi)\left(1 - e^{-\beta\sigma\mathsf{e}(\sigma)}\right)\right]\theta_L - 1\right) \\ +&m_2\left((q_H - q_L)\left[\pi + (1-\pi)\left(1 - e^{-\beta\sigma\mathsf{e}(\sigma)}\right)\right]\theta_H - 1\right) = 0. \end{aligned}$$

This in turn implies

$$(q_H - q_L)\left[\pi + (1-\pi)\left(1 - e^{-\beta\sigma\mathsf{e}(\sigma)}\right)\right]\frac{m_1\theta_L + m_2\theta_H}{m_1 + m_2} - 1 = 0.$$

Therefore, we have

$$B\left(\sigma_1, \frac{m_1\theta_L + m_2\theta_H}{m_1 + m_2}\right) = 0$$

or equivalently

$$\sigma_1 = \sigma^{FE}\left(\frac{m_1\theta_L + m_2\theta_H}{m_1 + m_2}\right).$$

Using a similar process, we will get $\sigma_2 = \sigma^{FE}\left(\frac{(1-m_1)\theta_L + (1-m_2)\theta_H}{2 - m_1 - m_2}\right)$. This completes the proof for the claim and thus the proof for the theorem. $\square$

The first part of Theorem 5 states that when the initial infection probability π is too large, there is no need for the sender to obfuscate. The intuition behind this part is similar to that of Theorem 3 for the one-step case. When π is large, two phenomena can happen. The first is that the potential threat of loss from infection is too high even when the state is $\theta = \theta_L$. Therefore, in both states of the world, all agents will choose to protect, and it is impossible to induce any protection behaviors other than the full information equilibrium. The second is that the potential threat of loss from infection is too high compared to the protection cost. In this situation, the health concern outweighs the economic one. There is no incentive misalignment between the sender and the agents, and therefore, full disclosure becomes the optimal policy.

The second part of Theorem 5 provides an interesting contrast between the one-step transmission model and the full transmission one. Theorem 3 for the one-step model shows that when π is small, the optimal policy is always to reveal full information to the public. In contrast, the second part of Theorem 5 states that there exist scenarios for which no matter how small π is, the full disclosure policy is no longer optimal. The next example illustrates Theorem 5.

Example 4. *We let $\mu = 1/2$, the high protection level $q_H = 0.8$, the low protection level $q_L = 0.6$, the mild harm of infection $\theta_L = 8$, and the severe harm of the infection $\theta_H = 14$. We fix the transmission rate $\beta = 24$ and let π approach 0. We consider all message rules that mix two signals. Any rule can be defined by two probabilities (m_1, m_2) as follows.*

$$\begin{aligned} h(\sigma_1 \mid \theta_L) &= m_1, \quad h(\sigma_2 \mid \theta_L) = 1 - m_1, \\ h(\sigma_1 \mid \theta_H) &= m_2, \quad h(\sigma_2 \mid \theta_H) = 1 - m_2. \end{aligned} \tag{2.66}$$

Note that the message rules are symmetric and $(m_1, m_2) = (1, 0)$ represents the full disclosure policy. Further discussions about the transformation of obedience constraints are included in the appendix. Figure 2.3 depicts achievable social welfare

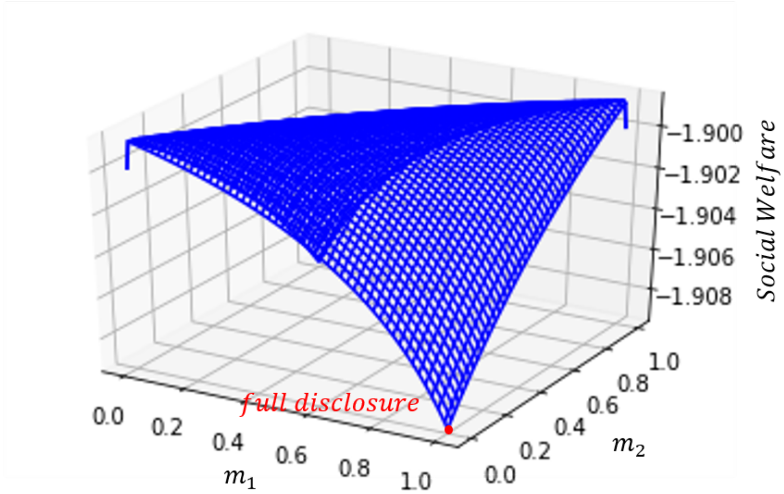


Figure 2.3: The objective of feasible two-signal rules when $(q_H, q_L, \beta, \pi, \theta_H, \theta_L) = (0.8, 0.6, 24, 0, 14, 8)$

following all feasible two-signal rules. The red dot points to the full disclosure policy. From the plot, we observe that there exist mixing policies that are strictly better than the full disclosure one in terms of social welfare. This implies that the optimal information policy is to obfuscate and reveal partial information. Note that this observation is different from what we see in the one-step transmission model in Example 3. This observation implies that when an infection spreads in a network, there always exist circumstances such that, no matter how small the initial infection probability is, it is optimal for the social planner to intervene and strategically reveal information.

2.7 Conclusion

We considered the use of information design to control the spread of an infection. In our model, each user has a social activity level that determines the social networks among agents. Additionally, each agent takes a binary protection action after observing the information provided by the social planner, which implies how susceptible this agent is. Our main objective is to identify when obfuscating information about the severity of the infection helps to increase the social welfare of agents compared to full information revelation. In this regard, we quantify the price of full disclosure between disclosing full information and obfuscating partial information in different cases. In particular, we found that when the infection transmission is limited (one step), revealing full information is optimal when the infection is barely or highly transmissible and when the initial transmission probability is high or low. In comparison, we established that when the infection transmission is unlimited over a social network, it is sometimes beneficial for the sender to intervene and recommend instructions even when the initial transmission probability is small. In computational examples, we found that mixing information can improve social welfare by more than 30%.

There are a few directions motivated by our analysis that are worth exploring. First, obfuscating information will have some negative reputation effect for the social planner. Therefore, a natural question is whether the gain in social welfare (obtained by obfuscating information) is worthwhile. Studying this question would require a dynamic model of interaction between the social planner and the agents. Second, collecting additional information improves social welfare but is also costly. This motivates the study of quantifying the value of information, maybe empirically.

Chapter 3

Controlling Epidemics in a Connected World: Information Design with Multiple Senders

3.1 Introduction

When facing the spread of an infection, people need to make a trade-off between the costs of getting infected and adopting some protective actions. In particular, the social network is endogenous because more social distancing from others incentivizes an individual to free-ride. The free-ride incentive and the level of social welfare a central government can improve using its information advantage about the severity of the infection have been well studied by our earlier work and by [dVGW21].

Built on this basis, we develop a further question. What if there are multiple governments that possess the same information advantage and perform information design simultaneously? What will be the level of information disclosure in equilibrium among the senders, and how will that differ from the optimal information policy in a single sender case?

In this paper, we develop a model to answer these questions. In particular, we study the Nash equilibrium between two more-informed governments to inform two populations about the severity of an infection when the individuals have heterogeneous economic costs for social distancing and decide whether they want to self-isolate. Moreover, we study the scenario where information can travel across the population while infection cannot. That is, all individuals in the two populations will see messages from both governments, while an individual's infection probability depends only on her social distancing action and the social distancing actions of the other individuals in her own population. We use this assumption to model the conditions where lockdown policies block physical contact and transmission of the disease while information can still circulate over the Internet.

In particular, we consider two senders who observe the infection severity, which has two possible values (high or low). We also have two continuums of receivers who have heterogeneous economic costs. The economic costs capture the receivers' economic loss from choosing self-isolation. An individual's infection probability depends on her social distancing action and the social distancing actions of all the other individuals in her own population. The more individuals isolate themselves, the less exposed the population is, which decreases the expected infection probability for everyone within the same population.

Our multiple senders' information design model is motivated by the Bayesian persuasion game of [GK17]. The governments (senders) each commit to a messaging policy, privately observes the same state of the world, and send a one-shot public message to all receivers based on the state. Each receiver obtains information from all the senders and chooses whether to self-isolate based on her belief about the state of the world and her own economic cost. Self-isolation is costly but will completely protect the individual from the risk of getting infected. In particular, the dis-utility of each individual is either the economic cost if the individual chooses to self-isolate or the healthcare cost times her infection probability if the individual chooses not to self-isolate, where her infection probability is in turn, her perceived infectiousness of the disease times the size of the socially active population.

We first characterize the individuals' equilibrium within each population. We show the existence and uniqueness of a Bayes Nash equilibrium among individuals in a population and establish that it is a threshold strategy. That is, individuals whose economic costs are below (resp. above) a threshold choose to (resp. not to) self-isolate. We further explicitly characterize the equilibrium threshold as a function of the individuals' perceived infectiousness of the disease based on their posterior belief. Equipped with these characterizations, we then turn to our main question: what are the equilibrium information disclosure policies among the two senders? Moreover, what is the optimal one if there are multiple?

In particular, we say each government needs to balance its population's health with the economy, and different governments assign different levels of priority over the two conflicting objectives to reflect different long-term and short-term political goals. We aim to find the equilibrium that minimizes the collective social costs across two populations.

To find the optimal equilibrium, we study the set of distributions of posterior belief and perceived infectiousness that can be induced by two senders and convert the model to a game that designs the distributions of belief ([GK17]). This establishment of equivalence allows us to solve the informational content of equilibria by just inspecting each sender's social cost function in turn without worrying about strategic considerations.

Our main result characterizes the optimal information disclosure under different scenarios. We prove that it is optimal for the senders to obfuscate information when both senders are heavily economically focused and have relatively high economic costs in each population, and the prior infectiousness of the disease is high. It is also optimal to obfuscate information when the prior infectiousness of the disease is high, one sender is heavily economically focused and has relatively high economic costs in his population, while the other sender is more biased toward the population's health and has relatively small economic costs in his population. When both senders are either heavily economically focused, have relatively high economic costs, and the prior infectiousness of the disease is relatively low or more biased toward the population's health and have relatively small economic costs, the optimal policy is to disclose no information. When at least one sender is weakly more biased toward his population's economy but not extremely economically focused or is more biased toward the population's health but has relatively high economic costs, the optimal policy is to reveal full information to the public.

The remainder of this paper is organized as follows. Section 3.2 presents the model setup and establishes the existence and uniqueness of the continuation equilibrium among the individuals. Section 3.3 characterizes the equilibrium outcomes among the senders' game. Section 3.4 analyzes the impact of parameters over the optimal information disclosure strategy. Section 3.5 concludes the paper. The appendix includes proofs omitted from the text.

3.2 Model Description

3.2.1 The Environment

We consider an environment where two governments (which we refer to as senders) seek to induce certain levels of social distancing in their own population (which we refer to as receivers) that are experiencing the spread of an infectious disease. Both senders possess the same level of information advantage over the receivers. First, the senders inform the receivers about the epidemic's severity in a way that may or may not reflect its proprietary information. The message of the two senders reaches the receivers in each population, but the spread of the infection in each population does not reach the other population.

Observing both messages from the two senders, each receiver decides about the level of social interactions. To make this choice, receivers need to balance the economic costs of social isolation with the risk of getting the infection. Importantly, each receiver's choice affects the probability of infection in their own population, but not the other population. This gives rise to two different incentives depending on the population each receiver belongs to.

3.2.2 Spread of infection and the receiver's utility

We refer to the *infectiousness* of the disease as ω, which is a binary random variable $\omega \in \Omega = \{\omega_l, \omega_h\}$ taking a high value ω_h (representing a highly severe epidemic) with prior belief $p_0 \in (0, 1)$ and a low value ω_l (representing a less severe epidemic) with the complement probability. Thus, the perceived *prior infectiousness* of the disease is given by $\mu_0 = p_0\omega_h + (1 - p_0)\,\omega_l$, which captures the expected severity of the disease according to the prior belief. The realized state of the world ω is private information to the two senders, and all receivers share the common prior p_0.

We have two continua of receivers, indexed by $i \in \{1, 2\}$. Specifically, we normalize both population sizes to 1. We consider receivers that are heterogeneous in their economic costs. The costs account for the economic loss when individuals take actions to prevent the spread of an infectious disease. We allow for the distribution of economic cost to be heterogeneous across the two populations. In particular, for each receiver in population $i \in \{1, 2\}$, we let the economic cost be drawn from a uniform distribution with support $\mathcal{C}^i = [0, b^i]$. Notice that, the larger b^i is, the more unequal the economic impact of the epidemic on the population is.

Each receiver decides $\alpha \in \{0, 1\}$, such that the individual remains isolated and avoids the disease when $\alpha = 1$ or engages in social interactions otherwise. If $\alpha = 1$, the receiver is completely isolated from the disease and will only pay her economic cost. Thus to make this choice, a receiver balances the trade off between her economic cost and her potential loss from getting the infection, which we introduce in detail in the next paragraph.

We assume that an infected receiver incurs a (per unit) *healthcare cost* $k \in \mathbb{R}_+$. This cost captures the symptoms' seriousness and may include additional economic costs due to hospitalization. The health risk then corresponds to the probability of being infected times the cost k. The probability of being infected for a receiver engaging in social interactions is the product of the *perceived infectiousness* of the disease, denoted by μ, which captures the expected severity of the disease according to a belief and the size of the *socially active population*, denoted by P^i for population $i \in \{1, 2\}$. This yields $\mu P^i k$ as the expected healthcare loss for a receiver in population i who has decided not to isolate.

Note that the perceived infectiousness of the disease is a linear function of the belief about the severity of the disease, given by $\mu = p\omega_h + (1-p)\omega_l$. Thus, given any belief p, the socially active population P^i, and the economic cost c, the cost function for a receiver in population i taking action $\alpha \in \{0, 1\}$ is

$$u(p, P^i, c, \alpha) = \alpha c + (1 - \alpha)(p\omega_h + (1 - p)\omega_l)P^i k. \tag{3.1}$$

Again, for a receiver in population i, the first term of the above utility is the economic cost when she isolates, and the second term is the expected healthcare loss if she does not. The healthcare loss depends on the receiver's belief about the severity of the disease, the size of the active receivers in population i and the (per unit) healthcare cost k.

3.2.3 Continuation Equilibrium Among Receivers

We now consider the game among the receivers and characterize the equilibrium of the game with incomplete information. To distinguish the equilibrium of the game between the two senders, which we discuss later, we name the equilibrium of the game among receivers as the *continuation equilibrium*.

Recall that messages from the two senders are public. Thus, every receiver shares the same belief which we denote by p. Each receiver knows her own economic cost c, the belief p, and the distribution $\mathcal{C}^i$ over her population's remaining economic costs. We denote an action strategy in population i by $a^i : \mathcal{C}^i \times [0,1] \to \{0,1\}$ that maps a receiver's economic cost and her belief about the severity of disease into a choice to social distancing. We consider the *symmetric Bayesian Nash* equilibrium $a^{i*}(c,p)$, which is the strategy that minimizes the expected cost for each receiver in population i when the common belief is p and all other receivers follow the same strategy $a^{i*}(c,p)$. That is, a strategy profile $a^{i*}(c,p)$ is a Bayesian Nash equilibrium if and only if for every receiver with arbitrary economic cost c in the support $[0,b^i]$, keeping the strategies of every other receivers fixed, action $a^{i*}(c,p)$ minimizes the expected cost of this receiver. Notice that the symmetry assumption means that the strategy of each receiver depends on her economic cost c and her belief p, and not her identity. A formal definition is as follows.

Definition 6 (Continuation Equilibrium)**.** *For any belief $p \in [0,1]$, we say $a^{i*} : \mathcal{C}^i \times [0,1] \to \{0,1\}$ is a symmetric Bayesian Nash equilibrium if for all $c \in \mathcal{C}^i$*

$$a^{i*}(c,p) \in_{\alpha \in \{0,1\}} \{\alpha c + (1-\alpha)(p\omega_h + (1-p)\omega_l)P^i_{a^{i*},p}k\}, \tag{3.2}$$

where

$$P^i_{a^{i*},p} \doteq \mathbb{E}_{c \sim \mathcal{C}^i}[1 - a^{i*}(c,p)]. \tag{3.3}$$

We say $a^{i*}(c,p)$ defined in (3.2) is the population i's individual optimal choice, and $P^i_{a^{i*},p}$ defined in (3.3) is the size of the socially active population in the continuation equilibrium of population i. The probability of a receiver becoming infected, $(1-\alpha)(p\omega_h + (1-p)\omega_l)P^i_{a^{i*},p}$, depends on $P^i_{a^{i*},p}$, which captures the externalities that engaging in social interactions create. As fewer people self-isolate (i.e., as $P^i_{a^{i*},p}$ increases), a socially active receiver is more likely to become infected, raising her incentives to self-isolate, giving rise to the game among receivers in the same population. We denote the set of continuation equilibrium induced by belief p as $\mathcal{B}(p)$.

The next result establishes the existence and uniqueness of the continuation equilibrium given any belief p. This equilibrium is characterized by a threshold in the population's individual economic cost. The result is as follows.

Proposition 7 (Uniqueness of the Continuation Equilibrium)**.** *Suppose the posterior belief is $p \in [0,1]$ and the population's economic cost is drawn from $\mathcal{C}^i = \text{Uniform}\,[0,b^i]$. For any $i \in \{1,2\}$, a unique continuation equilibrium $a^{i*} : \mathcal{C}^i \times [0,1] \to \{0,1\}$ exists and is given by threshold $c^{i*}(p)$ as*

$$a^{i*}(c,p) = \begin{cases} 1 & \text{if } c \leq c^{i*}(p) \\ 0 & \text{if } c \geq c^{i*}(p), \end{cases} \tag{3.4}$$

where

$$c^{i*}(p) = \frac{(p\omega_h + (1-p)\omega_l)kb^i}{(p\omega_h + (1-p)\omega_l)k + b^i}.$$

The size of the socially active population with the continuation equilibrium is

$$P_{a^{i*},p} = \frac{b^i}{(p\omega_h + (1-p)\omega_l)k + b^i}. \tag{3.5}$$

Proof. Let us fix p. Notice that the two populations are isolated, which means two separate continuation equilibria a^{1*}, a^{2*} are played in each population as described in (3.2) and (3.3). We now prove that they all follow the form described in (3.4). The identity of the population group does not affect it. Therefore, we write the proof without the index i. It is straightforward to see from (3.2) and (3.3), the continuation equilibrium $a^*(c,p)$ in (3.4) of an individual with cost c is the unique solution to

$$\min_{\alpha\in\{0,1\}} \alpha c + (1-\alpha)\left(p\omega_h + (1-p)\omega_l\right)P_{a^*,p}k$$

for $c^*(p) = (p\omega_h + (1-p)\omega_l)P_{a^*,p}k$. We next show that $P_{a^*,p}$ is uniquely determined by $a^*(c,p)$ with $c^*(p) = (p\omega_h + (1-p)\omega_l)P_{a^*,p}k$ and (3.3). Let $F(\cdot)$ denote the cumulative distribution function of the uniform distribution with support $[0,b]$. We can write

$$P_{a^*,p} = 1 - F\left((p\omega_h + (1-p)\omega_l)P_{a^*,p}k\right). \tag{3.6}$$

Let $h(x) \triangleq x - 1 + F\left((p\omega_h + (1-p)\omega_l)xk\right)$ for all $x \in [0,1]$. The function h is continuous and strictly increasing. Moreover, $h(0) = -1 < 0$ and $h(1) = F\left((p\omega_h + (1-p)\omega_l)k\right) > 0$. Thus, the mean-value theorem asserts that $P_{a^*,p}$ is the unique fixed point for $h(x) = 0$. Solving this fixed point equation, we obtain $P_{a^*,p} = \frac{b}{(p\omega_h+(1-p)\omega_l)k+b}$. The fact that $c^*(p) = (p\omega_h + (1-p)\omega_l)P_{a^*,p}k$ yields $c^*(p) = \frac{(p\omega_h+(1-p)\omega_l)kb}{\mu k+b}$. This completes the proof. □

The result indicates that in the unique equilibrium, only individuals who suffer an economic cost less than the infection loss should self-isolate. The rest of the population with higher economic costs will still engage in social interactions.

Proposition 7 also sheds light on the role of the externalities that individual choices create in the population. Indeed, an individual in population i whose economic cost is exactly at the threshold c^{i*} is indifferent between avoiding and engaging in social interactions, i.e., $c^{i*}(p) = (p\omega_h + (1-p)\omega_l)kP_{a^{i*},p} = (p(\omega_h - \omega_l) + \omega_l)kP_{a^{i*},p}$ given (3.5). Thus, two countervailing forces shape the behavior of c^{i*} as a function of the belief p. On the one hand, the higher the belief p is, the higher the value of c^{i*} for a fixed size $P_{a^{i*},p}$, i.e., more receivers have a high incentive to isolate themselves. On the other hand, a higher belief p lowers the size of the socially active population $P_{a^{i*},p}$ in the continuation equilibrium because fewer receivers engage in social interactions. The net effect is that c^{i*} increases in p at a diminishing rate. In other words, due to the externalities that individual choices create (via $P_{a^{i*},p}$), the additional number of self-isolated receivers decreases when the epidemic appears to be more severe.

3.2.4 Information Structure and Public Information Policies

Now, we consider the information design game among the two senders to minimize the collective social costs. As we mentioned in Subsection 3.2.1, the two senders observe the actual realization of $\omega \in \{0, 1\}$, and thus can potentially influence a receiver's behavior by strategically disclosing this information. Formally, we let $\mathcal{M} = (\mathcal{M}^1, \mathcal{M}^2)$ denote the pair of *message space* for the two senders and assume $\mathcal{M}^i$ is finite but general enough. Before ω is realized, each sender $i \in \{1, 2\}$ commits to a public information policy π^i which specifies the probability $\pi^i(m \mid \omega)$ of sending message $m \in \mathcal{M}^i$ given each realization of $\omega \in \{0, 1\}$. Let $\Pi = (\pi^1, \pi^2)$ denote the *information policy* employed by the two senders.

Receivers will form a *posterior belief* $p_{\mathbf{m}} \in (0, 1)$ after seeing the information policy Π and the generated message $\mathbf{m} = (m^1, m^2)$. More specifically, by using Bayes' rule, for any p_0, Π and $\mathbf{m} \in \mathcal{M}$, the individual's posterior belief becomes

$$p_{\mathbf{m}} = \frac{p_0\pi^1(m^1 \mid \omega_h)\pi^2(m^2 \mid \omega_h)}{p_0\pi^1(m^1 \mid \omega_h)\pi^2(m^2 \mid \omega_h) + (1-p_0)\pi^1(m^1 \mid \omega_l)\pi^2(m^2 \mid \omega_l)}.$$

Receivers will then play a continuation equilibrium as discussed in Subsection 3.2.3. The extensive form game between the two senders and the two populations of receivers is as follows:

1. The senders pick and commit to an information policy Π for providing the receivers with message $\mathbf{m} = (m^1, m^2)$.
2. The true state ω is realized and is private information to the two senders.
3. The individuals see the realized messages $\mathbf{m} = (m^1, m^2)$ according to the pre-commited information policy Π and form a posterior belief $p_{\mathbf{m}}$.
4. Receivers in each population i play a continuation equilibrium a^{i*} according to the common belief $p_{\mathbf{m}}$ and the economic costs they have.

Next, we define the population's social cost and introduce the game between the two senders.

3.2.5 Information design and the senders' objective

When managing an epidemic, each government faces a tradeoff between health costs and economic costs. Different governments assign different weights to these conflicting objectives, which may reflect different long-term and short-term goals. Our model allows for a choice of such parameter, and in particular, we refer to C_h^i and C_e^i as the population i's total expected health costs and economic costs, respectively. Given the belief p and the two population's corresponding continuation equilibria $a^* = (a^{1*}, a^{2*})$, as defined in Proposition 7, the sum of the expected health cost of individuals in population i is

$$C_h^i(p) = \mathbb{E}_{c \sim \mathcal{C}_i}[\left(1 - a^{i*}(c, p)\right) (p\omega_h + (1-p)\omega_l) P_{a^{i*},p} k], \tag{3.7}$$

and the sum of the expected economic cost of individuals in population i is

$$C_e^i(p) = \mathbb{E}_{c \sim \mathcal{C}_i}[a^{i*}(c,p)c]. \tag{3.8}$$

We consider a weighted sum of the health cost and the economic cost as the senders' objectives which we refer to as *social cost*. In particular, sender i's objective, following belief p, denoted by $C^i(p)$, is equal to

$$C^i(p) = \lambda_i C_h^i(p) + (1-\lambda_i)C_e^i(p), \tag{3.9}$$

where $\lambda_i \in [0,1]$ is a *weight factor*, capturing the priority of the sender in minimizing the health cost versus the economic cost. A sender is more biased toward the population's health when $\lambda_i > \frac{1}{2}$ and toward the economy when $\lambda_i < \frac{1}{2}$. At the extremes, the sender's priority lies solely in reducing either health costs C_h^i (when $\lambda_i = 1$) or economic costs C_e^i (when $\lambda_i = 0$).

The two senders are playing a game among themselves. We consider the Nash equilibrium Π^* that minimizes the expected social cost for each sender, given that the other sender follows the same information policy Π^*. A formal definition is as follows.

Definition 7 (Nash Equilibrium)**.** *For any prior belief p_0, we say $\Pi^* = (\pi^{1*}, \pi^{2*})$ is a Nash equilibrium if*

$$\mathbb{E}_\omega \mathbb{E}_{m^j}[C^i(p_{(m^i,m^j)}) - C^i(p_{(m',m^j)})] \leq 0, \tag{3.10}$$

where

$$p_{(m^1,m^2)} = \frac{p_0 \pi^{1*}(m^1 \mid \omega_h)\pi^{2*}(m^2 \mid \omega_h)}{p_0 \pi^{1*}(m^1 \mid \omega_h)\pi^{2*}(m^2 \mid \omega_h) + (1-p_0)\pi^{1*}(m^1 \mid \omega_l)\pi^{2*}(m^2 \mid \omega_l)},$$

for all $i \neq j \in \{1,2\}$*,* $m' \in \mathcal{M}^i$*, and* m^i *for which there exists* ω *such that* $\pi^{i*}(m^i \mid \omega) > 0$*.*

The inequality (3.10) ensures that for any message that has a chance to be delivered to the receivers, no sender has an incentive to deviate and deliver another message.

Applying Proposition 7, we can explicitly write down the population's health cost, economic cost, and the sender's social cost using equations (3.7) - (3.9). This is formalized in the next corollary.

Corollary 5. *For any posterior* p*, population* i*'s health cost, economic cost, and the social cost with the continuation equilibrium can be written as*

$$\mathrm{C}_h^i(p) = \frac{(p\omega_h + (1-p)\omega_l)kb_i^2}{((p\omega_h + (1-p)\omega_l)k + b_i)^2}, \tag{3.11}$$

$$\mathrm{C}_e^i(p) = \frac{(p\omega_h + (1-p)\omega_l)^2k^2b_i}{2((p\omega_h + (1-p)\omega_l)k + b_i)^2}, \tag{3.12}$$

$$\mathrm{C}^i(p) = \frac{(1-\lambda_i)(p\omega_h + (1-p)\omega_l)^2k^2b_i + 2\lambda_i(p\omega_h + (1-p)\omega_l)kb_i^2}{2((p\omega_h + (1-p)\omega_l)k + b_i)^2}. \tag{3.13}$$

Proof. The proof follows from invoking Proposition 7. We have

$$\begin{aligned}
\mathrm{C}_h^i(p) &= \int_{\frac{(p\omega_h+(1-p)\omega_l)kb_i}{(p\omega_h+(1-p)\omega_l)k+b_i}}^{b_i} \frac{(p\omega_h + (1-p)\omega_l)k(1 - \frac{(p\omega_h+(1-p)\omega_l)k}{(p\omega_h+(1-p)\omega_l)k+b_i})}{b_i} dc \\
&= \frac{(p\omega_h + (1-p)\omega_l)kb_i^2}{((p\omega_h + (1-p)\omega_l)k + b_i)^2}, \\
\mathrm{C}_e^i(p) &= \int_0^{\frac{(p\omega_h+(1-p)\omega_l)kb_i}{(p\omega_h+(1-p)\omega_l)k+b_i}} \frac{c}{b_i} dc = \frac{(p\omega_h + (1-p)\omega_l)^2k^2b_i}{2((p\omega_h + (1-p)\omega_l)k + b_i)^2}, \\
\mathrm{C}^i(p) &= \frac{(1-\lambda_i)(p\omega_h + (1-p)\omega_l)^2k^2b_i + 2\lambda_i(p\omega_h + (1-p)\omega_l)kb_i^2}{2((p\omega_h + (1-p)\omega_l)k + b_i)^2}.
\end{aligned}$$

This completes the proof. □

There may be multiple equilibria for the game between the senders. Following the literature on information design that considers "sender-preferred" equilibrium, here we focus on an equilibrium that minimizes the overall social cost of the two populations. In what follows, we focus on characterizing such equilibrium. Therefore, the equilibrium of the game between the senders becomes

$$
\begin{aligned}
Z_0 \doteq \min_{\Pi}\ & \mathbb{E}_{\mathbf{m}\sim\Pi}[\mathrm{C}^1(p_{\mathbf{m}}) + \mathrm{C}^2(p_{\mathbf{m}})] && (3.14)\\
s.t.\ & \\
& \Pi \text{ satisfies (3.10)}, && (3.15)\\
& \sum_{m^i} \pi^i(m^i \mid \omega) = 1,\ \forall i \in \{1,2\}, \omega \in \{0,1\}, && (3.16)\\
& \pi^i(m^i \mid \omega) \geq 0,\ \forall i \in \{1,2\}, m^i \in \mathcal{M}_i, \omega \in \{0,1\}. && (3.17)
\end{aligned}
$$

The objective function of the above optimization (3.14) minimizes the expected sum of social costs. Constraint (3.15) ensures that the information policy Π is a Nash equilibrium for the game between the two senders. Constraints (3.16) and (3.17) ensure Π is a valid information policy, i.e., π^1 and π^2 satisfy natural probability rules.

Note that since the signal space $\mathcal{M}$ is large, this is an extremely large-scale optimization problem and cannot be solved for practical purposes. In the next section, we apply findings in [GK17] to convert this optimization problem to a problem that involves designing posteriors instead of messaging policies, which will largely reduce the scale and make the problem solvable. We then use this reformulation to characterize the solution of problem (3.14).

3.3 Senders' Game: Equilibrium Characterization

3.3.1 Going from message spaces to designing posteriors

Recall that $\Omega = \{\omega_l, \omega_h\}$ is the set of possible states. We let $\Delta(\Delta(\Omega))$ denote the set of distributions over beliefs. Using the results in [GK17], we consider an equivalent game that directly designs posteriors instead of messages. The formal result is as follows.

Proposition 8. *Consider the following optimization problem.*

$$Z_1 \doteq \min_{\tau \in \Delta(\Delta(\Omega))} \mathbb{E}_\tau[\mathrm{C}^1(p) + \mathrm{C}^2(p)] \tag{3.18}$$

$$s.t.\ \mathbb{E}_\tau[p] = p_0, \tag{3.19}$$

$$\mathbb{E}_{\tau'}\left[\mathrm{C}^i(p')\right] \geq \mathrm{C}^i(p) \quad \textit{for all } i, p \textit{ for which } \tau(p) > 0,$$

$$\textit{and } \tau' \textit{ for which } \mathbb{E}_{\tau'}[p'] = p. \tag{3.20}$$

We have

$$Z_1 = Z_0.$$

Proof. This result credits to directly applying Lemma 6 and Proposition 2 in [GK17].

We first prove that $Z_1 \leq Z_0$. We let $\pi^* = (\pi^{1*}, \pi^{2*})$ denote one optimal solution to the optimization problem (3.14) - (3.17). Now, we construct a distribution τ as

$$\tau(p) = \sum_{(m^1,m^2):p_{(m^1,m^2)}=p} [p_0\pi^{1*}(m^1 \mid \omega_h)\pi^{2*}(m^2 \mid \omega_h) + (1-p_0)\pi^{1*}(m^1 \mid \omega_l)\pi^{2*}(m^2 \mid \omega_l)]. \tag{3.21}$$

We next prove that the τ constructed above is a feasible solution to the minimization problem (3.18) - (3.20). We have

$$\begin{aligned}\mathbb{E}_\tau[p] &= \sum_p p \sum_{(m^1,m^2):p_{(m^1,m^2)}=p} [p_0\pi^{1*}(m^1 \mid \omega_h)\pi^{2*}(m^2 \mid \omega_h) + (1-p_0)\pi^{1*}(m^1 \mid \omega_l)\pi^{2*}(m^2 \mid \omega_l)] \\ &= \sum_{(m^1,m^2)} [p_0\pi^{1*}(m^1 \mid \omega_h)\pi^{2*}(m^2 \mid \omega_h) + (1-p_0)\pi^{1*}(m^1 \mid \omega_l)\pi^{2*}(m^2 \mid \omega_l)]p_{(m^1,m^2)} \\ &= \sum_{(m^1,m^2)} p_0\pi^{1*}(m^1 \mid \omega_h)\pi^{2*}(m^2 \mid \omega_h) \\ &= p_0 \sum_{m_2}[\sum_{m_1} \pi^{1*}(m^1 \mid \omega_h)]\pi^{2*}(m^1 \mid \omega_h) = p_0 \sum_{m_2} \pi^{2*}(m^2 \mid \omega_h) = p_0,\end{aligned}$$

where the last two equations follow from (3.16).

Next, we prove that τ satisfies (3.15) by contradiction. Suppose not. That is, there exists p and τ' such that for some i $\sum_x \tau'(x)x = p$ and $\tau(p)[\mathbb{E}_{\tau'}[C^i(x)] - C^i(p)] < 0$. Since the identity i doesn't matter here, WLOG, we take $i = 1$ for the remaining proof. Let $\mathcal{M}_p = \{\mathbf{m} : p_{\mathbf{m}} = p\}$ denote the set of messages that result in the posterior p under Bayesian update. We now construct a deviated information plan and show that the sender 1 will strictly benefit from deviation. WLOG, we can assume that $\mathcal{M}_p = (m^1, m^2)$ contains only one message. $\tau(p)[\mathbb{E}_{\tau'}[C^i(x)] - C^i(p)] < 0$ implies that $\tau(p) = p_0\pi^1(m^1 \mid \omega_h)\pi^2(m^2 \mid \omega_h) + (1-p_0)\pi^1(m^1 \mid \omega_l)\pi^2(m^2 \mid \omega_l) > 0$. We assume τ' takes a finite support and denote it as $supp(\tau') = \{p_1, ..., p_n\}$ and say $\tau'(p_i) = x_i$. Now we construct $\tilde{\tau}^1$ as follows. Take n messages $\{m^1_1, ..., m^1_n\}$ from sender 1's message space $\mathcal{M}^1$. Let $\tilde{\tau}^1(m^1_i \mid \omega_h) = \frac{\tau(p)x_i p_i}{p_0\pi^2(m^2|\omega_h)}$ and $\tilde{\tau}^1(m^1_i \mid \omega_l) = \frac{\tau(p)x_i(1-p_i)}{(1-p_0)\pi^2(m^2|\omega_l)}$ for all i. By this construction, individuals will form a posterior p_i when they see message (m^1_i, m^2) since

$$\frac{p_0\tilde{\pi}^1(m^1_i \mid \omega_h)\pi^2(m^2 \mid \omega_h)}{p_0\tilde{\pi}^1(m^1_i \mid \omega_h)\pi^2(m^2 \mid \omega_h) + (1-p_0)\tilde{\pi}^1(m^1_i \mid \omega_l)\pi^2(m^2 \mid \omega_l)} = p_i.$$

We now have

$$\sum_{m' \in \{m^1_1, \ldots, m^1_n\}} [p_0\pi^2(m^2 \mid \omega_h)\tilde{\pi}^1(m' \mid \omega_h) + (1-p_0)\pi^2(m^2 \mid \omega_l)\tilde{\pi}^1(m' \mid \omega_l)](C^1(p_{(m^1,m^2)}) - C^1(p_{(m',m^2)}))$$

$$
\begin{aligned}
&= \sum_i [p_0\pi^2(m^2 \mid \omega_h)\frac{x_i p_i \tau(p)}{p_0\pi^2(m^2 \mid \omega_h)} + (1-p_0)\pi^2(m^2 \mid \omega_l)\frac{x_i(1-p_i)\tau(p)}{(1-p_0)\pi^2(m^2 \mid \omega_l)}](C^1(p_{(m^1,m^2)}) - C^1(p_{(m',m^2)}))\\
&= \tau(p)\sum_i [\tau'(p_i)p_i + \tau'(p_i)(1-p_i)](C^1(p) - C^1(p_i))\\
&= \tau(p)[C^i(p) - \mathbb{E}_{\tau'}[C^1(x)]] > 0
\end{aligned}
$$

by contradiction assumption. Since the summation is less than 0, this implies that there exist at least one $m' \in \{m_1^1, ..., m_n^1\}$ where $\mathbb{E}_\omega \mathbb{E}_{m^2}[C^i(p_{(m^1,m^2)}) - C^i(p_{(m',m^2)})] > 0$, contradicting to τ being a feasible solution to (3.15). Therefore, τ constructed in (3.21) satisfies (3.19) and (3.20) and is thus a feasible solution to the minimization problem (3.18). Since π^* is an optimal solution to (3.14), we have

$$
Z_0 = \mathbb{E}_{\pi^*}[C^1(p_{\mathbf{m}}) + C^2(p_{\mathbf{m}})] = \mathbb{E}_\tau[C^1(p) + C^2(p)] \geq Z_1.
$$

This completes the first half of the proof.

Next, we prove $Z_0 \leq Z_1$. Similarly, we start with one optimal solution τ^* to the optimization problem (3.18) - (3.20). Again, we assume τ^* takes a finite support and denote it as $supp(\tau^*) = \{p_1, ..., p_n\}$ and say $\tau^*(p_i) = x_i$. Now, we construct a message policy π as follows. Let $\pi^2(m^* \mid \omega_h) = \pi^2(m^* \mid \omega_l) = 1$ for a fixed message m^*. That is, the sender 2 will always announce the same message, which provides no additional information to the individuals. Next, take n messages $\{m_1, ..., m_n\}$ from sender 1's message space $\mathcal{M}^1$ and let

$$
\pi^1(m_i \mid \omega_h) = \frac{x_i p_i}{p_0}, \;\; \pi^1(m_i \mid \omega_l) = \frac{x_i(1-p_i)}{(1-p_0)},
$$

for all i. That is, seeing a realized message (m_i, m^*), individuals will form a posterior

p_i since

$$\frac{p_0\pi^1(m_i \mid \omega_h)\pi^2(m^* \mid \omega_h)}{p_0\pi^1(m_i \mid \omega_h)\pi^2(m^* \mid \omega_h) + (1-p_0)\pi^1(m_i \mid \omega_l)\pi^2(m^* \mid \omega_l)} = p_i.$$

Moreover, the message (m_i, m^*) will be realized with the same probability as p_i in τ^* since

$$p_0\pi^1(m_i \mid \omega_h)\pi^2(m^* \mid \omega_h) + (1-p_0)\pi^1(m_i \mid \omega_l)\pi^2(m^* \mid \omega_l) = x_1.$$

We now remain to show that the constructed (π^1, π^2) is a feasible solution to the minimization problem (3.14) - (3.17). It is straightforward to see that π^2 satisfies (3.16) and (3.17). For π^1, we have $\pi^1(m_i \mid \omega) \geq 0$ for all i and

$$\sum_i \pi^1(m_i \mid \omega_h) = \sum_i \frac{\tau^*(p_i)p_i}{p_0} = \frac{p_0}{p_0} = 1,$$

$$\sum_i \pi^1(m_i \mid \omega_l) = \sum_i \frac{\tau^*(p_i)(1-p_i)}{1-p_0} = \frac{1-p_0}{1-p_0} = 1,$$

where the equalities follow from (3.19). Next, we have

$$\mathbb{E}_\omega \mathbb{E}_{m^{-i}}[C^i(p_{(m^i,m^{-i})}) - C^i(p_{(m',m^{-i})})] = C^i(p = p_{(m^i,m^{-i})}) - \mathbb{E}[C^i(p')] \leq 0$$

by (3.20) for all i. Lastly, we have

$$Z_1 = \mathbb{E}_{\tau^*}[C^1(p) + C^2(p)] = \mathbb{E}_\pi[C^1(p_m) + C^2(p_m)] \geq Z_0.$$

Since $Z_0 \geq Z_1$ and $Z_1 \geq Z_0$, we have $Z_0 = Z_1$. This completes the proof.

$\square$

Proposition 8 simplifies the space of decision variables and enables us to focus on finding a distribution over beliefs instead of searching over arbitrary message spaces.

In Section 3.3.3, we discuss how to map the solution of the above optimization back to information disclosure policies.

Again, the senders' optimization problem in terms of choosing posteriors, characterized in Proposition 8, is a constrained optimization. The objective function (3.18) minimizes the sum of the expected social costs of the two populations. Constraint (3.19) is called *Bayes plausibility* and ensures that the expected posterior belief is the same as the prior belief. Constraint (3.20) ensures that no sender has an incentive to deviate and obtain a smaller expected social cost.

In view of the constraint (3.20), we consider the convexification of each sender's social cost function. For each $i \in \{1, 2\}$, we denote by V^i the convexification of C^i as

$$V^i(p) \doteq \inf \left\{ z \mid (p, z) \in \mathrm{co}(\mathrm{C}^i) \right\},$$

where $\mathrm{co}(\mathrm{C}^i)$ denotes the convex hull of the epigraph of C^i. Note that for each $i \in \{1, 2\}$, the function $V^i(\cdot)$ is convex by construction. In fact, it is the largest convex function that is everywhere weakly lower than the function $\mathrm{C}^i(\cdot)$. We refer to the posterior belief p such that $\mathrm{C}^i(p) = V^i(p)$ as *coincident*. We also let N^i denote the set of coincident beliefs for sender i and $N = N^1 \cap N^2$ denote the *coincident belief* for the two senders' game. The following Corollary further simplifies the senders' optimization problem in terms of this definition.

Corollary 6. *For a given $\tau \in \Delta(\Delta(\Omega))$, Constraint* (3.20) *holds if and only if the support of τ lies in the coincident belief N.*

Proof. Suppose the support of τ lies in the coincident belief N. This implies that for any p such that $\tau(p) > 0$, $C^i(p) = \inf\{z \mid (p, z) \in \mathrm{co}(C^i)\}$ for all $i \in \{1, 2\}$. By definition, any τ' such that $\mathbb{E}_{\tau'}[p'] = p$ is in the convex hull $\mathrm{co}(C^i)$ of the epigraph of C^i. Therefore, we have $C^i(p) \le \mathbb{E}_{\tau'}[C^i(p')]$ for all such τ'.

Now it remains to show that if constraint (3.20) holds, the support of τ lies in the coincident belief N. We prove this statement by contradiction. Suppose not. That is, there exists p for which $\tau(p) > 0$ and $V^i(p) < C^i(p)$ for some i. By definition, $V^i(p)$ is in the convex hull of the epigraph of C^i, i.e., there exists a convex combination of points in the support of C^i that gives the value $V^i(p)$. Let τ' be this convex combination. We then have $\mathbb{E}_{\tau'}[C^i(p')] = V^i(p) < C^i(p)$, which violates constraint (3.20). This completes the proof.

$\square$

Corollary 6 and Proposition 8 enable us to further simplify the sender's optimization problem because characterizing the set of coincident beliefs is typically straightforward. We next use this observation to characterize the equilibrium outcome. Finally, we use our characterization to find the equilibrium information design policies of the two senders in the original problem.

3.3.2 Equilibrium characterization

Our main theorem, presented next, characterizes the optimal solution of problem (3.18) for any set of parameters. We make use of the following notations in the statement of this result:

$$p_i^* = \frac{(1-5\lambda_i)b_i^2 - 2\lambda_i b_i k\omega_h - k\omega_l[(1-\lambda_i)b_i + 2(1-2\lambda_i)k\omega_h]}{k(\omega_h - \omega_l)[(1-\lambda_i)b_i + 2(1-2\lambda_i)k\omega_h]} \text{ for } i \in \{1,2\}$$

and $p^* = \min\{p_1^*, p_2^*\}$.

Theorem 6 (Equilibrium characterization)**.** *Given the healthcare cost k, the prior belief p_0, the two population's economic cost support $\mathcal{C}^i = [0, b^i]$, and the weight factors*

λ_i, *the optimal* τ^* *that solves* (3.18) *is as follows:*

1. *If there exists* $i \neq j \in \{1, 2\}$ *such that* $\lambda_i \leq \frac{1}{5}$, $\frac{b_i(1-5\lambda_i)}{(1-2\lambda_i)} > 2k\omega_l$, $0 < p_i^* < p_0$, *and either* $\lambda_j \leq \frac{1}{5}$ *and* $\frac{b_j(1-5\lambda_j)}{(1-2\lambda_j)} > 2k\omega_l$, *or* $\lambda_j > \frac{1}{2}$ *and* $\frac{b_j(1-5\lambda_j)}{(1-2\lambda_j)} < 2k\omega_l$ *hold, then we have*

$$\tau^*(p) = \begin{cases} \frac{1-p_0}{1-p^*} & p = p^* \\ \frac{p_0-p^*}{1-p^*} & p = 1 \\ 0 & \textit{otherwise} . \end{cases}$$

2. *If for* $i = 1$ *and* 2, *either* $\lambda_i \leq \frac{1}{5}$, $\frac{b_i(1-5\lambda_i)}{(1-2\lambda_i)} > 2k\omega_l$, *and* $p_i^* \geq p_0$, *or* $\lambda_i > \frac{1}{2}$, $\frac{b_i(1-5\lambda_i)}{(1-2\lambda_i)} < 2k\omega_l$ *hold, then we have*

$$\tau^*(p) = \begin{cases} 1 & p = p_0 \\ 0 & \textit{otherwise} . \end{cases}$$

3. *Otherwise, we have*

$$\tau^*(p) = \begin{cases} 1 - p_0 & p = 0 \\ p_0 & p = 1 \\ 0 & \textit{otherwise} . \end{cases}$$

Proof. We prove the result by studying the social cost functions of the two senders and the coincident belief given the social cost functions. By (3.13), we have

$$\mathrm{C}^i(p) = \frac{(1-\lambda_i)(p\omega_h + (1-p)\omega_l)^2 k^2 b_i + 2\lambda_i (p\omega_h + (1-p)\omega_l) k b_i^2}{2((p\omega_h + (1-p)\omega_l)k + b_i)^2}. \tag{3.22}$$

Taking derivatives with respect to p, we get

$$\frac{\partial}{\partial p}\mathrm{C}^i(p) = \frac{(\omega_h - \omega_l) b_i^2 k(\lambda_i b_i + (1-2\lambda_i)k\omega_l + (1-2\lambda_i)k(\omega_h - \omega_l)p)}{(b_i + k\omega_l + k(\omega_h-\omega_l)p)^3}, \tag{3.23}$$

$$\frac{\partial^2}{\partial p^2}\mathrm{C}^i(p) = \frac{(\omega_h - \omega_l) b_i^2 k^2(b_i(1-5\lambda_i) - 2k(1-2\lambda_i)\omega_l - 2k(1-2\lambda_i)(\omega_h - \omega_l)p)}{(b_i + k\omega_l + k(\omega_h-\omega_l)p)^4}. \tag{3.24}$$

When $\lambda_i \leq \frac{1}{5}$, we have that $\frac{\partial}{\partial p}\mathrm{C}^i(p) \geq 0$ for all p. That is, the social cost function will be monotone increasing for both i. Moreover,

$$\frac{\partial^2}{\partial p^2} C^i(p) \begin{cases} > 0 & \text{for } p < \frac{b_i(1-5\lambda_i) - 2k(1-2\lambda_i)\omega_l}{2k(1-2\lambda_i)(\omega_h-\omega_l)} \\ < 0 & \text{for } p > \frac{b_i(1-5\lambda_i) - 2k(1-2\lambda_i)\omega_l}{2k(1-2\lambda_i)(\omega_h-\omega_l)}. \end{cases}$$

Since $\frac{b_i(1-5\lambda_i)}{(1-2\lambda_i)} > 2k\omega_l$, the social cost functions C^i will be first convex increasing up to the point $\frac{b_i(1-5\lambda_i) - 2k(1-2\lambda_i)\omega_l}{2k(1-2\lambda_i)(\omega_h-\omega_l)} > 0$ and then concave increasing. Thus, the coincident belief $N^i = [0, \max\{p_i^*, 0\}] \cup \{1\}$, where p_i^* can be found by

$$\frac{\partial}{\partial p}\mathrm{C}^i(p) \mid_{p=p_i^*} = \frac{\mathrm{C}^i(1) - \mathrm{C}^i(p_i^*)}{1-p_i^*}. \tag{3.25}$$

Substituting (3.22) and (3.23) into (3.25), we solve

$$\frac{(1-p_i^*)(\omega_h - \omega_l) b_i^2 k(\lambda_i b_i + (1-2\lambda_i)k\omega_l + (1-2\lambda_i)k(\omega_h-\omega_l)p_i^*)}{(b_i + k\omega_l + k(\omega_h - \omega_l)p_i^*)^3} -$$
$$\frac{(1-\lambda_i)\omega_h^2 k^2 b_i + 2\lambda_i \omega_h k b_i^2}{2(\omega_h k + b_i)^2} + \frac{(1-\lambda_i)(p_i^*(\omega_h-\omega_l)+\omega_l)^2 k^2 b_i + 2\lambda_i(p_i^*(\omega_h - \omega_l) + \omega_l)k b_i^2}{2(p_i^*(\omega_h-\omega_l)k + \omega_l k + b_i)^2} = 0$$

and get

$$p_i^* = \frac{(1-5\lambda_i)b_i^2 - 2\lambda_i b_i k\omega_h - k\omega_l[(1-\lambda_i)b_i + 2(1-2\lambda_i)k\omega_h]}{k(\omega_h-\omega_l)[(1-\lambda_i)b_i + 2(1-2\lambda_i)k\omega_h]} \tag{3.26}$$

Note that $p^* = \min_i \{p_i^*\}$. When $p^* > 0$, we have $N^i = [0, p_i^*] \cup \{1\}$, and $N = [0, p^*] \cup \{1\}$. Denote by $\mathrm{C} = \mathrm{C}^1 + \mathrm{C}^2$ and $V(p) \doteq \inf \{z \mid (p,z) \in \mathrm{co}(\mathrm{C})\}$. We have

$$V(p) = \begin{cases} \mathrm{C}(p) & \text{for } p \in N, \\ \mathrm{C}(p^*) + \frac{(\mathrm{C}(1)-\mathrm{C}(p^*))(p-p^*)}{1-p^*} & \text{otherwise.} \end{cases}$$

That is, the convexification $V(p)$ of the objective function $\mathrm{C}(p)$ is the curve $\mathrm{C}(p)$ between 0 and p^* and a straight line connecting $\mathrm{C}(p^*)$ and $\mathrm{C}(1)$ between p^* and 1. Now when $p_0 > p^*$, $V(p_0) = \mathrm{C}(p^*) + \frac{(\mathrm{C}(1)-\mathrm{C}(p^*))(p_0-p^*)}{1-p^*}$. Therefore,

$$\tau^*(p) = \begin{cases} \frac{1-p_0}{1-p^*} & p = p^* \\ \frac{p_0-p^*}{1-p^*} & p = 1 \\ 0 & \text{otherwise .} \end{cases}$$

This proves the optimal characterization for the first part of the theorem.

Now if for all i, $\lambda_i \leq \frac{1}{5}$, and $\frac{b_i(1-5\lambda_i)}{(1-2\lambda_i)} > 2k\omega_l$, again we have p_i^* as described in (3.26), $p^* = \min_i \{p_i^*\}$, and $N = [0, p^*] \cup \{1\}$. When $p_0 \leq p^*$, $V(p_0) = C(p_0)$. Therefore in this case, $\tau^*(p) = \begin{cases} 1 & p = p_0 \\ 0 & \text{otherwise .} \end{cases}$

When $\lambda_i > \frac{1}{2}$, we have

$$\frac{\partial^2}{\partial p^2} C^i(p) \begin{cases} > 0 & \text{for } p > \frac{b_i(1-5\lambda_i)-2k(1-2\lambda_i)\omega_l}{2k(1-2\lambda_i)(\omega_h-\omega_l)} \\ < 0 & \text{for } p < \frac{b_i(1-5\lambda_i)-2k(1-2\lambda_i)\omega_l}{2k(1-2\lambda_i)(\omega_h-\omega_l)} \end{cases}.$$

When $\frac{b_i(1-5\lambda_i)}{(1-2\lambda_i)} < 2k\omega_l$, we have $\frac{\partial^2}{\partial p^2} C^i(p) > 0$ for all p. That is, the cost function will always be convex. In this case, we have $N^i = [0, 1]$. If sender i satisfies $\lambda_i \leq \frac{1}{5}$, and $\frac{b_i(1-5\lambda_i)}{(1-2\lambda_i)} > 2k\omega_l$ and sender j satisfies $\lambda_j > \frac{1}{2}$, $\frac{b_j(1-5\lambda_j)}{(1-2\lambda_j)} < 2k\omega_l$, we have $N^i = [0, p_i^*] \cup \{1\}$ and $N^j = [0, 1]$. Therefore, $N = [0, p_i^*] \cup \{1\}$. Since $p_0 \leq p_i^*$, we again have $V(p_0) = C(p_0)$ and $\tau^*(p) = \begin{cases} 1 & p = p_0 \\ 0 & \text{otherwise} \end{cases}.$

If for all i, $\lambda_i > \frac{1}{2}$, and $\frac{b_i(1-5\lambda_i)}{(1-2\lambda_i)} < 2k\omega_l$, we have $N = [0, 1]$, i.e., $V(p) = C(p)$ for all p. Therefore, $\tau^*(p) = \begin{cases} 1 & p = p_0 \\ 0 & \text{otherwise} \end{cases}.$ The three cases together prove the optimal characterization for the second part of the theorem.

Lastly, we look at all rest possibilities. Note that if for any i, $N^i = \{0\} \cup \{1\}$, we will have $N = \{0\} \cup \{1\}$. For all the following cases, we will have $N^i = \{0\} \cup \{1\}$. When $\lambda_i \leq \frac{1}{5}$, $\frac{b_i(1-5\lambda_i)}{(1-2\lambda_i)} > 2k\omega_l$, and $p_i^* < 0$, $N^i = \{0\} \cup \{1\}$. When $\lambda_i \leq \frac{1}{5}$, $\frac{b_i(1-5\lambda_i)}{(1-2\lambda_i)} \leq 2k\omega_l$, $\frac{\partial}{\partial p}\mathrm{C}^i(p) > 0$ for all p and $\frac{\partial^2}{\partial p^2}\mathrm{C}^i(p) \leq 0$ for all p. That is, the cost function will always be concave increasing. Thus we have $N^i = \{0\} \cup \{1\}$. When $\frac{1}{5} < \lambda_i \leq \frac{1}{2}$, $\frac{\partial}{\partial p}\mathrm{C}^i(p) > 0$ for all p and $\frac{\partial^2}{\partial p^2}\mathrm{C}^i(p) \leq 0$ for all p. That is, the cost function will always be concave increasing. Again we have $N^i = \{0\} \cup \{1\}$.

When $\lambda_i > \frac{1}{2}$ and $\frac{\lambda_i b_i}{(2\lambda_i - 1)} \geq \omega_l k$, $\frac{\lambda_i b_i}{(2\lambda_i-1)k} < \frac{b_i(1-5\lambda_i)}{2k(1-2\lambda_i)}$ and $\frac{b_i(1-5\lambda_i)-2k(1-2\lambda_i)\omega_l}{2k(1-2\lambda_i)(\omega_h-\omega_l)} > \frac{\lambda_i b_i - k\omega_l(2\lambda_i-1)}{(2\lambda_i-1)k(\omega_h-\omega_l)}$. Therefore, the cost function will be concave increasing for $p < \frac{\lambda_i b_i - k\omega_l(2\lambda_i-1)}{(2\lambda_i-1)k(\omega_h-\omega_l)}$, then concave decreasing for $\frac{\lambda_i b_i - k\omega_l(2\lambda_i-1)}{(2\lambda_i-1)k(\omega_h-\omega_l)} \leq p < \frac{b_i(1-5\lambda_i)-2k(1-2\lambda_i)\omega_l}{2k(1-2\lambda_i)(\omega_h-\omega_l)}$, and finally convex decreasing for $p > \frac{b_i(1-5\lambda_i)-2k(1-2\lambda_i)\omega_l}{2k(1-2\lambda_i)(\omega_h-\omega_l)}$.There will be three possible cases and we will illustrate all in the following. First if $\frac{b_i(1-5\lambda_i)}{(1-2\lambda_i)} \geq 2k\omega_h$, $\frac{b_i(1-5\lambda_i)-2k(1-2\lambda_i)\omega_l}{2k(1-2\lambda_i)(\omega_h-\omega_l)} > 1$ and $\frac{\partial^2}{\partial p^2}\mathrm{C}^i(p) < 0$ for all p. Thus, the cost function is always concave and $N^i = \{0\} \cup \{1\}$.Now look at $\frac{b_i(1-5\lambda_i)}{(1-2\lambda_i)} < 2k\omega_h$. The second case is if $C^i(0) \leq C^i(1)$, i.e., $\frac{(1-\lambda_i)\omega_l^2k^2b_i+2\lambda_i\omega_lkb_i^2}{2(\omega_lk+b_i)^2} \leq \frac{(1-\lambda_i)\omega_h^2k^2b_i+2\lambda_i\omega_hkb_i^2}{2(\omega_hk+b_i)^2}$, we have $C^i(0) \leq C^i(p)$ for all p since the function is first increasing then decreasing and $C^i(0) \leq C^i(1)$. Moreover, the convex part of the function is decreasing. This implies for any point $p \in (0,1)$ and on the convex part, there exists two points $p_1 < p$ and $p_2 > p$ such that $C^i(p_1) < C^i(p)$ and $C^i(p_2) < C^i(p)$. Therefore, the coincident belief is $N^i = \{0\} \cup \{1\}$. Lastly if $C^i(0) > C^i(1)$, in this case, the coincident belief is $N^i = \{0\} \cup [p_i^{**}, 1]$ if $0 < p_i^{**} < 1$ and $N^i = \{0\} \cup \{1\}$ otherwise, where p_i^{**} can be found by the following equation,

$$\frac{\partial}{\partial p}C^i(p) \mid_{p=p_i^{**}} = \frac{C^i(p_i^{**}) - C^i(0)}{p_i^{**}}. \tag{3.27}$$

Substitute (3.22) and (3.23) into (3.27), we have

$$\begin{aligned}&\frac{p_i^{**}(\omega_h-\omega_l)b_i^2k(\lambda_ib_i+(1-2\lambda_i)k\omega_l+(1-2\lambda_i)k(\omega_h-\omega_l)p_i^{**})}{(b_i+k\omega_l+k(\omega_h-\omega_l)p_i^{**})^3} - \\ &\frac{(1-\lambda_i)(p_i^{**}(\omega_h-\omega_l)+\omega_l)^2k^2b_i+2\lambda_i(p_i^{**}(\omega_h-\omega_l)+\omega_l)kb_i^2}{2(p_i^{**}(\omega_h-\omega_l)k+\omega_lk+b_i)^2} \\ &+ \frac{(1-\lambda_i)\omega_l^2k^2b_i+2\lambda_i\omega_lkb_i^2}{2(\omega_lk+b_i)^2} = 0\end{aligned}$$

Solving the above equation, we get

$$p_i^{**} = \frac{(b+k\omega_l)[(1-5\lambda_i)b_i-2(1-2\lambda_i)k\omega_l]}{k(\omega_h-\omega_l)[(1-\lambda_i)b_i-2(1-2\lambda_i)k\omega_l]}. \tag{3.28}$$

Now given $\lambda_i > \frac{1}{2}$, $(1-\lambda_i)b_i - 2(1-2\lambda_i)k\omega_l > 0$. Moreover, given $\omega_lk \leq \frac{\lambda_ib_i}{(2\lambda_i-1)}$, we have

$$\begin{aligned}(1-5\lambda_i)b_i - 2(1-2\lambda_i)k\omega_l &= (1-5\lambda_i)b_i + 2(2\lambda_i - 1)k\omega_l \\ &\leq (1-5\lambda_i)b_i + 2(2\lambda_i-1)k\frac{\lambda_i b_i}{(2\lambda_i-1)k} \\ &= (1-3\lambda_i)b_i < 0,\end{aligned}$$

as $\lambda_i > \frac{1}{2}$. This implies that $p_i^{**} < 0$ for all $\lambda_i > \frac{1}{2}$. Therefore, the coincident belief will be $N^i = \{0\} \cup \{1\}$. To conclude, we prove that when $\lambda_i > \frac{1}{2}$ and $\frac{\lambda_i b_i}{(2\lambda_i - 1)} \geq \omega_l k$, the coincident belief in all the three cases is $N^i = \{0\} \cup \{1\}$.

When $\lambda_i > \frac{1}{2}$ and $\frac{\lambda_i b_i}{(2\lambda_i-1)k} < \omega_l \leq \frac{b_i(1-5\lambda_i)}{2k(1-2\lambda_i)}$, the cost function is concave decreasing up to $\frac{b_i(1-5\lambda_i)-2k(1-2\lambda_i)\omega_l}{2k(1-2\lambda_i)(\omega_h-\omega_l)}$ and then convex decreasing. In this case, the coincident belief is $N^i = \{0\} \cup [p_i^{**}, 1]$ if $0 < p_i^{**} < 1$ and $N^i = \{0\} \cup \{1\}$ otherwise, where p_i^{**} can be found by (3.28). Given $\lambda_i > \frac{1}{2}$, $(1-\lambda_i)b_i - 2(1-2\lambda_i)k\omega_l > 0$. Moreover, given $\omega_l \leq \frac{b_i(1-5\lambda_i)}{2k(1-2\lambda_i)}$, we have

$$\begin{aligned}(1-5\lambda_i)b_i - 2(1-2\lambda_i)k\omega_l &= (1-5\lambda_i)b_i + 2(2\lambda_i - 1)k\omega_l \\ &\leq (1-5\lambda_i)b_i + 2(2\lambda_i-1)k\frac{b_i(1-5\lambda_i)}{2k(1-2\lambda_i)} \\ &= 0,\end{aligned}$$

as $\lambda_i > \frac{1}{2}$. This implies that the root $p_i^{**} \leq 0$ for all $\lambda_i > \frac{1}{2}$. Therefore, the coincident belief will be $N^i = \{0\} \cup \{1\}$.

When $N = \{0\} \cup \{1\}$, we have

$$V(p) = (1-p)\mathrm{C}(0) + p\mathrm{C}(1).$$

That is, the convexification $V(p)$ of the objective function $\mathrm{C}(p)$ is the straight line connecting C(0) and C(1). Therefore,

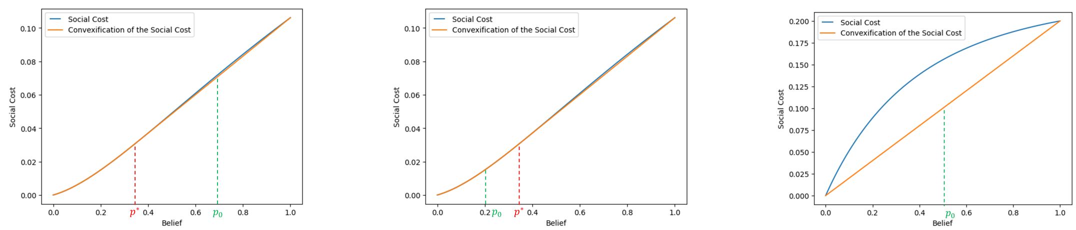
$$\tau^*(p) = \begin{cases} 1 - p_0 & p = 0 \\ p_0 & p = 1 \\ 0 & \text{otherwise .} \end{cases}$$

This completes all the proof for the theorem.

$\square$

Theorem 6 states three possible cases in the two senders' information design game, each of which gives rise to a unique form of equilibrium.

In the first case of the result, one sender i is heavily focused on economic concerns (i.e., $\lambda_i \leq \frac{1}{5}$), and the product of the infectiousness of the disease in the mild state and the healthcare cost is smaller than the ratio of the highest possible economic cost in population i (i.e., $\frac{b_i(1-5\lambda_i)}{(1-2\lambda_i)} > 2k\omega_l$). The latter condition states that the economic cost of the population and the healthcare cost are relatively balanced. In this case, sender i's social cost will be convex and increasing with small posterior belief, and concave and increasing with large posterior beliefs. The other sender j is one of two possibilities. In the first possibility, the sender j is also heavily focused on economic concerns (i.e., $\lambda_j \leq \frac{1}{5}$), and the ratio of the highest possible economic cost in the population is also larger than the product of the infectiousness of the disease in the mild state and the healthcare cost (i.e., $\frac{b_j(1-5\lambda_j)}{(1-2\lambda_j)} > 2k\omega_l$). Here, the interpretation is similar to what we described for sender i. In the second possibility, sender j is more biased toward the population's health ($\lambda_j > \frac{1}{2}$), and the ratio of the highest possible economic cost in the population is instead smaller than the product of the infectiousness of the disease in the mild state and the healthcare cost ($\frac{b_j(1-5\lambda_j)}{(1-2\lambda_j)} < 2k\omega_l$). In this case, sender j's social cost will always be convex and



(a) Mixing (b) No Disclosure (c) Full Disclosure

Figure 3.1: Optimal Equilibria Under Different Cost Functions

decreasing with the posterior belief. If both senders satisfy the above conditions, and the prior infectiousness of the disease is relatively high ($0 < p_i^* < p_0$), the optimal equilibrium is to mix between two posterior beliefs. Specifically, with probability $\frac{1-p_0}{1-p^*}$, the optimal equilibrium will induce a posterior belief p^*, where p^* is the minimum fraction $\frac{(1-5\lambda_i)b_i^2-2\lambda_i b_i k\omega_h - k\omega_l[(1-\lambda_i)b_i+2(1-2\lambda_i)k\omega_h]}{k(\omega_h-\omega_l)[(1-\lambda_i)b_i+2(1-2\lambda_i)k\omega_h]}$ among the two senders. With the rest probability $\frac{p_0-p^*}{1-p^*}$, the optimal equilibrium will induce a posterior belief 1. Note that a posterior belief of 1 means that individuals believe the disease is severe for sure. This optimal equilibrium is a case where the two senders will obfuscate information.

In Figure 3.1, we plot the social cost of a sender as a function of beliefs in blue and the convexification of the social cost in orange. In 3.1a, we set $\omega_l = 0$, $\omega_h = 1$, $b = 1$, $k = 0.8$, $\lambda = 0.05$, and $p_0 = 0.7$. In this case, we have $p^* = 0.35 < p_0$ and the coincident belief is $N = [0, p^*] \cup \{1\}$. As shown in the plot, the convexification between p^* and 1 is the straight line connecting the two points. Therefore, the optimal equilibrium, in this case, is a mixing, i.e., $\tau^*(p^*) = \frac{1-p_0}{1-p^*}$ and $\tau^*(1) = \frac{p_0-p^*}{1-p^*}$. In the next subsection, We will discuss the corresponding optimal information disclosure policy and show that this optimal mixing equilibrium is equivalent to an optimal partial information disclosure policy.

In the second case of Theorem 6, both senders fall into one of two possibilities. In the first possibility, the sender is heavily focused on economic concerns ($\lambda_i \leq \frac{1}{5}$), and the ratio of the highest possible economic cost in the population is larger than the product of the infectiousness of the disease in the mild state and the healthcare cost ($\frac{b_i(1-5\lambda_i)}{(1-2\lambda_i)} > 2k\omega_l$), and the prior infectiousness of the disease is relatively low ($p_i^* \geq p_0$). In this case, the sender's social cost will be convex and increasing with a small posterior belief and concave and increasing with a large posterior belief. In the second possibility, the sender is more biased toward the population's health ($\lambda_i > \frac{1}{2}$), and the ratio of the highest possible economic cost in the population is smaller than the product of the infectiousness of the disease in the mild state and the healthcare cost ($\frac{b_i(1-5\lambda_i)}{(1-2\lambda_i)} < 2k\omega_l$). In this case, the sender's social cost will always be convex and decreasing in the posterior belief. If both senders fall into one of these two possibilities, the optimal equilibrium is to induce a posterior belief equal to the prior belief p_0 always, i.e., $\tau^*(p_0) = 1$. Translating back to the information design game, the optimal equilibrium in this case refers to a no information disclosure policy as the optimal information policy. That is, each sender will provide the same message to the individuals no matter what the severity level of the disease is. Thus, the only information among the individuals will be the prior belief, and the posterior belief will simply be the prior. We call this case an optimal no disclosure case.

We plot an example of the optimal no disclosure case in Figure 3.1b. We set $\omega_l = 0$, $\omega_h = 1$, $b = 1$, $k = 0.8$, $\lambda = 0.05$ the same as in 3.1a, but $p_0 = 0.2 < p^*$. In this case, the coincident belief is again $N = [0, p^*] \cup \{1\}$. As shown in the plot, the convexification between 0 and p^* is the social cost function itself. Therefore, the optimal equilibrium, in this case, is a no disclosure, i.e., $\tau^*(p_0) = 1$.

The last case of Theorem 6 encompasses multiple possibilities that give rise to the same format of optimal equilibrium. In this case, there exists at least one sender that satisfy one of the following conditions: (1) the sender is heavily focused on economic concerns ($\lambda_i \leq \frac{1}{5}$) but the healthcare cost and the economic cost are unbalanced such that the fraction p^* is less than zero; (2) the sender is weakly more biased toward the population's economy but not extremely focused on economic concerns ($\frac{1}{5} < \lambda_i \leq \frac{1}{2}$); (3) the sender is more biased toward the population's health ($\lambda_i > \frac{1}{2}$) but the ratio of the highest possible economic cost in the population is weakly larger than the product of the infectiousness of the disease in the mild state and the healthcare cost ($\frac{b_i(1-5\lambda_i)}{(1-2\lambda_i)} \geq 2k\omega_l$). In these cases, the optimal equilibrium is to induce a perceived infectiousness equal to the true severity level of the disease. That is, with probability p_0, the optimal equilibrium will induce a posterior belief 1. With probability $1-p_0$, the optimal equilibrium will induce a posterior belief 0. Note that by construction, the true severity level of the disease is 1 with probability p_0 and is 0 with probability $1-p_0$. This implies that an information policy where both senders fully disclose the true state of the world as their messages will be an optimal policy. We call this case an optimal full disclosure case.

We plot an example of the optimal full disclosure case in Figure 3.1c. We set $\omega_l = 0$, $\omega_h = 1$, $b = 1$, $k = 1$, $\lambda = 0.6$, and $p_0 = 0.5$. In this case, the coincident belief is $N = \{0\} \cup \{1\}$. As shown in the plot, the convexification of the social cost is the straight line connecting 0 and 1. Therefore, the optimal equilibrium in this case is full disclosure, i.e., $\tau^*(0) = 1 - p_0$ and $\tau^*(1) = p_0$.

We have discussed the corresponding optimal information disclosure policy to the optimal equilibrium in the second no-disclosure case and the last full-disclosure case. In the next subsection, we will explicitly write out the equivalency and also discuss the corresponding optimal information disclosure policy for the first mixing case in Theorem 6.

3.3.3 Optimal Information Disclosure Policy

We start with formally stating the result.

Corollary 7 (Optimal Information Disclosure Policy)**.** *Given the optimal equilibrium τ^*, one optimal Γ^* in (3.14) that attains τ^* is as follows:*

1. *For* $\tau^*(p) = \begin{cases} \frac{1-p_0}{1-p^*} & p = p^* \\ \frac{p_0-p^*}{1-p^*} & p = 1 \\ 0 & \text{otherwise} \end{cases}$ *, we have* $\mathcal{M}^{1*} = \{l, h\}$*,* $\mathcal{M}^{2*} = \{L, H\}$*,*

$$\pi^{1*}(l \mid \omega_l) = 1, \quad \pi^{1*}(h \mid \omega_h) = \frac{p_0 - p^*}{1 - p^*},$$
$$\pi^{2*}(L \mid \omega_l) + \pi^{2*}(H \mid \omega_h) = 1.$$

2. *For* $\tau^*(p) = \begin{cases} 1 & p = p_0 \\ 0 & \text{otherwise} \end{cases}$ *, we have*

$$\pi^{1*}(m \mid \omega_l) = \pi^{1*}(m \mid \omega_h), \text{ for all } m \in \mathcal{M}^1,$$
$$\pi^{2*}(m \mid \omega_l) = \pi^{2*}(m \mid \omega_h), \text{ for all } m \in \mathcal{M}^2.$$

3. *For* $\tau^*(p) = \begin{cases} 1 - p_0 & p = 0 \\ p_0 & p = 1 \\ 0 & \text{otherwise .} \end{cases}$ *, we have* $\mathcal{M}^{1*} = \{l, h\}$*,* $\mathcal{M}^{2*} = \{L, H\}$*,*

$$\pi^{1*}(l \mid \omega_l) = \pi^{1*}(h \mid \omega_h) = 1$$
$$\pi^{2*}(L \mid \omega_l) = \pi^{2*}(H \mid \omega_h) = 1.$$

Proof. We first prove the result for $\tau^*(p) = \begin{cases} \frac{1-p_0}{1-p^*} & p = p^* \\ \frac{p_0-p^*}{1-p^*} & p = 1 \\ 0 & \text{otherwise} \end{cases}$ Since τ^* mixes two posterior beliefs, we can derive a Γ^* that attains τ^* by letting $\mathcal{M}^{1*} = \{l, h\}$, $\mathcal{M}^{2*} = \{L, H\}$,

$$\begin{cases} \pi^{1*}(l \mid \omega_l) = x_1, & \pi^{1*}(h \mid \omega_l) = 1 - x_1, \\ \pi^{1*}(l \mid \omega_h) = 1 - y_1, & \pi^{1*}(h \mid \omega_h) = y_1, \\ \pi^{2*}(L \mid \omega_l) = x_2, & \pi^{2*}(H \mid \omega_l) = 1 - x_2, \\ \pi^{2*}(L \mid \omega_h) = 1 - y_2, & \pi^{2*}(H \mid \omega_h) = y_2. \end{cases}$$

With this message space, there are four possible messages $(l, L), (l, H), (h, L), (h, H)$. There are multiple ways to associate the posteriors with the four messages. Here, we assume that the individuals form a posterior of p^* after seeing $(l, L), (l, H)$ and a posterior of 1 after seeing $(h, L), (h, H)$. Then, we solve the following system of equations.

$$\frac{p_0(1-y_1)(1-y_2)}{p_0(1-y_1)(1-y_2) + (1-p_0)x_1x_2} = \frac{p_0(1-y_1)y_2}{p_0(1-y_1)y_2 + (1-p_0)x_1(1-x_2)} = p^*, \quad (3.29)$$

$$\frac{p_0y_1(1-y_2)}{p_0y_1(1-y_2) + (1-p_0)(1-x_1)x_2} = \frac{p_0y_1y_2}{p_0y_1y_2 + (1-p_0)(1-x_1)(1-x_2)} = 1, \quad (3.30)$$

$$p_0y_1(1-y_2) + (1-p_0)(1-x_1)x_2 + p_0y_1y_2 + (1-p_0)(1-x_1)(1-x_2) = \frac{p_0 - p^*}{1-p^*}. \quad (3.31)$$

Here, equation (3.29) is the Bayesian update for the posterior belief after messages $(l, L), (l, H)$. Equation (3.30) is the Bayesian update for the posterior belief after messages $(h, L), (h, H)$. Equation (3.31) is the probability rule specified in τ^*. Solving the equations, we get

$$\begin{cases} x_1 = 1, \\ y_1 = \frac{\mu_0 - \mu^*}{\mu_0(1-\mu^*)}, \\ x_2 + y_2 = 1. \end{cases}$$

This completes the proof for the first part of Corollary 7.

The second and third parts of the result are straightforward and simple Bayesian updates for beliefs.

□

Note that in Corollary 7, we only characterize one possible Γ^* that can attain the corresponding optimal τ^* because there exist multiple forms of Γ^* that can give rise to the same optimal equilibrium τ^*. The result describes three possible scenarios for information disclosure in the two-sender game. The first case corresponds to a partial information disclosure policy. In this case, one sender always says the state is low when the disease is not severe and mixes with probability $\frac{p_0 - p^*}{1-p^*}$ when the disease is severe. This mixing can be seen as downplaying the health risk in the severe disease case to induce more protective actions in the mild one. The other sender can choose any announcement strategies that satisfy $\pi^{2*}(L \mid \omega_l) + \pi^{2*}(H \mid \omega_h) = 1$. This announcement strategy essentially discloses no additional information because the sender provides messages with the same probability in the two different states. Therefore, the information that individuals receive is reduced to a single sender case.

The second case corresponds to a no information disclosure policy. Each sender will provide the same message to the individuals no matter what the severity level of the disease is. Thus, the only information among the individuals is the prior belief, and the posterior belief will be the same as the prior.

The last case corresponds to a full information disclosure policy, where both

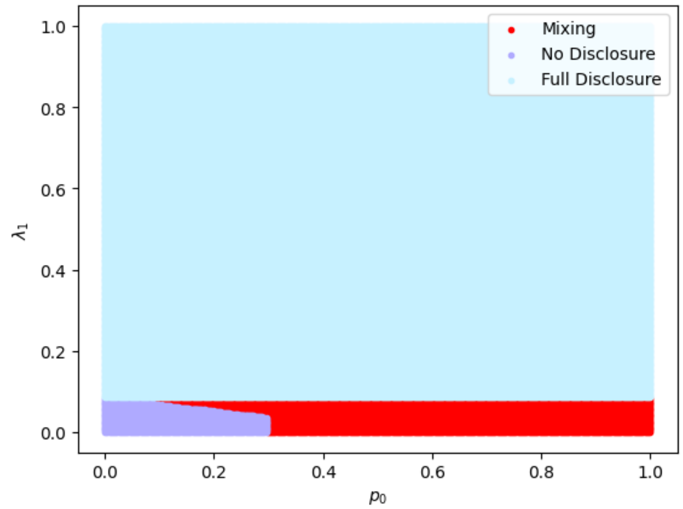


Figure 3.2: Optimal information disclosure policy with varying p_0 and λ_1, $\omega_l = 0.3, \omega_h = 0.8, k = 0.8, \lambda_2 = 0.15, b_1 = 1, b_2 = 2.5$

senders truthfully announce the severity of the disease. As a result, the posterior belief will be the same as the true state of the disease.

Now that we have analyzed the optimal information disclosure policy in the different scenarios, the next section will discuss the impact of different parameters on the optimal information disclosure policies.

3.4 Comparative Analysis

3.4.1 Prior p_0 vs one sender's weight factor λ_i

In this subsection, we discuss how the prior p_0 and one sender's weight factor λ_i affect the optimal information disclosure policies. In Figure 3.2, we set $\omega_l = 0.3, \omega_h = 0.8, k = 0.8, \lambda_2 = 0.15, b_1 = 1, b_2 = 2.5$ and plot the optimal information policies under different $p_0 \in (0, 1)$ and $\lambda_1 \in (0, 1)$. Given the other sender is biased toward the economy ($\lambda_2 = 0.15$), if the sender is not extremely focused on economic concerns, the optimal policy is always to reveal full information. If the sender is extremely focused

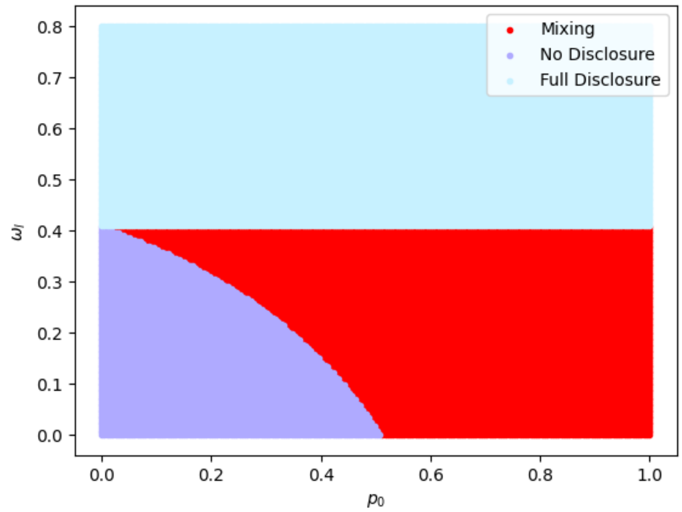


Figure 3.3: Optimal information disclosure policy with varying p_0 and ω_l, $\omega_h = 0.8, k = 0.8, \lambda_1 = 0.05, \lambda_2 = 0.15, b_1 = 1, b_2 = 2.5$

on economic concerns (λ_1 is close to 0), then the optimal policy is to reveal no information when the prior belief is small and to reveal partial information when the prior belief is large. There exists a threshold value of prior p_0 where the optimal policy switches between a no disclosure one and a partial disclosure policy. Moreover, we observe that the threshold value decreases as the sender's weight factor λ_1 increases.

If we fix any belief p_0, we see that as the sender's weight factor increases, the optimal policy tends to reveal more information. That is, as the sender becomes more concerned about the population's health, the two senders will reveal more information in the optimal equilibrium.

On the other hand, if we fix any sender's weight factor λ_i, we observe that as the prior belief increases, the optimal policy reveals more information. In other words, as the infection becomes more severe, the two senders will reveal more information in the optimal equilibrium.

3.4.2 Prior p_0 vs one state ω_l

In this subsection, we discuss how the prior p_0 and the infectiousness of the disease in the mild state ω_l change the optimal information disclosure policies. In Figure 3.3, we use $\omega_h = 0.8, k = 0.8, \lambda_1 = 0.05, \lambda_2 = 0.15, b_1 = 1, b_2 = 2.5$ and plot the optimal information policies under different $p_0 \in (0, 1)$ and $\omega_l \in (0, 0.8)$. Note that in this case, both senders are biased toward the economy. When the infectiousness of the disease in the mild state is above a half of the infectiousness in the severe state, the optimal information policy is to reveal full information. When the infectiousness of the disease in the mild state is relatively small compared to the infectiousness in the severe state, the senders will not reveal full information in the optimal strategy. Instead, the optimal policy is to reveal no information when the prior belief is small and to reveal partial information when the prior belief is large. Again, there exists a threshold value of prior p_0 where the optimal policy switches between a no disclosure policy and a partial disclosure policy. Moreover, the threshold value decreases as the infectiousness in the mild state ω_l increases.

The effect of p_0 on the optimal information policy fixing any state ω_l is unchanged compared to what we discussed in the earlier subsection. If we fix any p_0, we see that as the state ω_l increases, the optimal policy will tend to reveal more information. That is, as the infectiousness of the disease becomes higher, the two senders will reveal more information in the optimal equilibrium.

3.4.3 Prior p_0 vs one population's economic cost bound b_i

In this subsection, we discuss how the prior p_0 and the highest economic cost in a population b_i change the optimal information disclosure policies. In Figure 3.4, we use $\omega_l = 0.3, \omega_h = 0.8, k = 0.8, \lambda_1 = 0.05, \lambda_2 = 0.15, b_2 = 2.5$ and plot the optimal

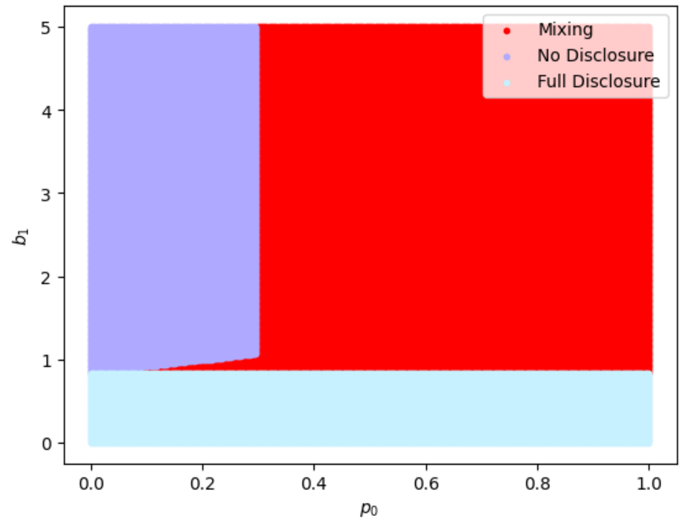
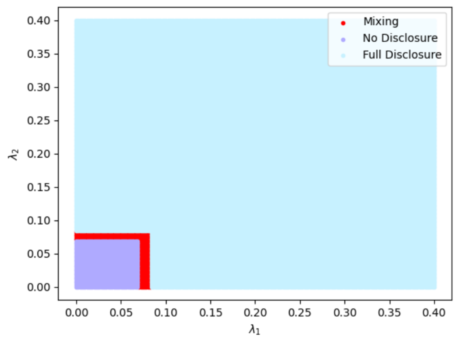


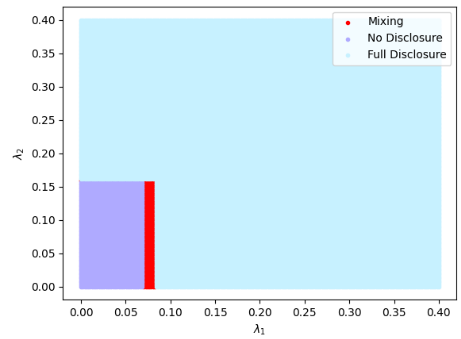
Figure 3.4: Optimal information disclosure policy with varying p_0 and b_1, $\omega_l = 0.3, \omega_h = 0.8, k = 0.8, \lambda_1 = 0.05, \lambda_2 = 0.15, b_2 = 2.5$

information policies under different $p_0 \in (0, 1)$ and $b_1 \in (0, 5)$. Again in this case, both senders are biased toward the economy. When the highest economic cost in a population is low, i.e., the economic impact of the disease on the population is relatively similar and small, the optimal information policy is to reveal full information. When the economic cost in the population gets high, the senders will not reveal full information in the optimal strategy. Instead, the optimal policy is to reveal no information when the prior belief is small and to reveal partial information when the prior belief is large. There exists a threshold value of prior p_0 where the optimal policy switches between a no disclosure policy and a partial disclosure policy. This threshold value increases as the highest economic cost b_1 in the population increases. Moreover, the increasing rate is large with medium values of b_1 and gets smaller with larger values of b_1.

Again, we do a similar discussion. Fix any prior belief p_0, we see that as the population's economic cost bound increases, the optimal policy will reveal less information. That is, as the population's cost for action against the disease becomes



(a) $b_1 = b_2 = 1$



(b) $b_1 = 1, b_2 = 2.5$

Figure 3.5: Optimal information disclosure policy with varying λ_1 and λ_2, λ_2, $\omega_l = 0.3, \omega_h = 0.8, k = 0.8, p_0 = 0.1$

higher, the two senders will reveal less information in the optimal equilibrium.

3.4.4 Weight factors λ_i vs λ_j

In this subsection, we discuss how the two senders' weight factors λ_1, λ_2 impact the optimal information disclosure policy. In Figure 3.5, we use $\omega_l = 0.3, \omega_h = 0.8, k = 0.8$, and $p_0 = 0.1$. Here, we also consider the impact of asymmetry among the two population. In Figure 3.5a, we use $b_1 = b_2 = 1$, i.e., the distributions of economic costs in the two populations are the same. We plot the optimal information policies under different $\lambda_1, \lambda_2 \in (0, 0.4)$. The plot is not drawn fully to $\lambda_1, \lambda_2 \in (0.4, 1)$ due to presentation clearness. But everything interesting is already captured. When $b_1 = b_2$, the optimal information disclosure policy plot is symmetric. When the two senders are both extremely economic focused ($\lambda_1, \lambda_2 \to 0$), the optimal information policy is to reveal no information. When the two senders are both biased but not extremely biased toward the economy, the optimal information policy is to mix and reveal partial information. Otherwise, the optimal information policy is to reveal full information. As the senders care more and more about their population's health, the optimal disclosure policy will tend to reveal more information.

In Figure 3.5b, we use $b_1 = 1$ and $b_2 = 2.5$, i.e., the distributions of economic costs in the two population are different and the later one bears higher economic costs. Again, we plot the optimal information policies under different $\lambda_1, \lambda_2 \in (0, 0.4)$. In this case, the plot is not symmetric anymore. When the two senders are both extremely economic focused, the optimal information policy is to reveal no information. When sender 2 is biased but not extremely biased toward the economy, the optimal policy is to reveal no information when sender 1 is extremely economic focused and to reveal partial information when sender 1 is also biased but not extremely biased toward the economy. Otherwise, the optimal information policy is to reveal full information. While the details change slightly, a similar general pattern persists. As the senders care more and more about their population's health, the optimal disclosure policy will tend to reveal more information.

3.5 Conclusion

We analyzed a game involving two senders aimed at controlling the spread of infection across separate populations using information design. In our model, messages from the two senders are visible to individuals in both populations. However, the disease only spreads within each population, creating a game with externalities. The more social distancing people within one population are taking, the less likely one individual will be infected by the disease, and thus the less willing the individual is to take self-isolation. Our main objective is to characterize the information disclosure policies of the two senders in the optimal equilibrium. Our findings suggest that when both senders are either heavily focused on economic factors and have high economic costs associated with a mild disease belief or are biased towards the population's health and have small economic costs, it is optimal for them to disclose no information.

When at least one sender is biased but not extremely focused on economic factors or is biased towards the population's health but has high economic costs, it is optimal for the senders to disclose full information. Lastly, when one sender is heavily focused on economic factors with high economic costs associated with a severe disease belief and the other sender is either heavily focused on economic factors with high economic costs or more biased towards the population's health with small economic costs, the optimal equilibrium is a mix of a nonzero perceived infectiousness and 1, with the optimal disclosure policy being to reveal partial information.

Bibliography

[AAL20] Fernando E Alvarez, David Argente, and Francesco Lippi. A simple planning problem for covid-19 lockdown. *Available at SSRN: https://ssrn.com/abstract=3574445*, 2020.

[AB02] Réka Albert and Albert-László Barabási. Statistical mechanics of complex networks. Reviews of modern physics, 74(1):47–97, 2002.

[ACWW21] Daron Acemoglu, Victor Chernozhukov, Iván Werning, and Michael D Whinston. A multi-risk sir model with optimally targeted lockdown. *American Economic Review: Insights, forthcoming*, 2021.

[AdVW20] Saed Alizamir, Francis de Véricourt, and Shouqiang Wang. Warning against recurring risks: An information design approach. *Management Science*, 66(10):4612–4629, 2020.

[AHKH20] Roy M Anderson, Hans Heesterbeek, Don Klinkenberg, and T Déirdre Hollingsworth. How will country-based mitigation measures influence the course of the covid-19 epidemic? *The Lancet*, 395(10228):931–934, 2020.

[AIM20] Jerry Anunrojwong, Krishnamurthy Iyer, and Vahideh Manshadi. Information design for congested social services: Optimal need-based persuasion. *arXiv preprint arXiv:2005.07253*, 2020.

[AM06] R. ANDERSON and T. MOORE. The economics of information security. *Science*, 314(5799):610–613, 2006.

[AMMO17] Daron Acemoglu, Ali Makhdoumi, Azarakhsh Malekian, and Asuman Ozdaglar. Privacy-constrained network formation. *Games and Economic Behavior*, 105:255–275, 2017.

[AMMO20] Daron Acemoglu, Ali Makhdoumi, Azarakhsh Malekian, and Asuman Ozdaglar. Testing, voluntary social distancing and the spread of an infection. Technical report, national bureau of economic Research, 2020.

[AMO16a] Daron Acemoglu, Azarakhsh Malekian, and Asuman Ozdaglar. Network security and contagion. *Journal of Economic Theory*, 166:536–585, 2016.

[AMO16b] Daron Acemoglu, Azarakhsh Malekian, and Asuman Ozdaglar. Network security and contagion. *Journal of Economic Theory*, 166:536–585, 2016.

[AMS09] Sinan Aral, Lev Muchnik, and Arun Sundararajan. Distinguishing influence-based contagion from homophily-driven diffusion in dynamic networks. Proceedings of the National Academy of Sciences, 106(51):21544–21549, 2009.

[AOTS13] D. ACEMOGLU, A. OZDAGLAR, and A. TAHBAZ-SALEHI. Systemic risk and stability in financial networks. NBER working paper, National Bureau of Economic Research, 2013.

[AOTS15] Daron Acemoglu, Asuman Ozdaglar, and Alireza Tahbaz-Salehi. Systemic risk and stability in financial networks. The American Economic Review, 105(2):564–608, 2015.

[AOTS17] Daron Acemoglu, Asuman Ozdaglar, and Alireza Tahbaz-Salehi. Microeconomic origins of macroeconomic tail risks. *American Economic Review*, 107(1):54–108, 2017.

[ASBS00] Luıs A Nunes Amaral, Antonio Scala, Marc Barthelemy, and H Eugene Stanley. Classes of small-world networks. Proceedings of the National Academy of Sciences, 97(21), 2000.

[Atk20] Andrew Atkeson. What will be the economic impact of covid-19 in the us? rough estimates of disease scenarios. *Available at SSRN: https://ssrn.com/abstract=3559156*, 2020.

[Aum59] Robert J Aumann. Acceptable points in general cooperative n-person games. Contributions to the Theory of Games, 4:287–324, (1959).

[B$^+$75] Norman TJ Bailey et al. *The mathematical theory of infectious diseases and its applications*. Charles Griffin & Company Ltd, 5a Crendon Street, High Wycombe, Bucks HP13 6LE., 1975.

[BA99] Albert-László Barabási and Réka Albert. Emergence of scaling in random networks. Science, 286(5439):509–512, 1999.

[BB01] Béla Bollobás and Bollobás Béla. *Random graphs*. Number 73. Cambridge university press, 2001.

[BBI08] M. BACCARA and H. BAR-ISAAC. How to organize crime. *Review of Economic Studies*, 75(4):1039–1067, 2008.

[BCAZ06] Coralio Ballester, Antoni Calvó-Armengol, and Yves Zenou. Who's who in networks. wanted: the key player. Econometrica, 74(5):1403–1417, 2006.

[BCF20] John R Birge, Ozan Candogan, and Yiding Feng. Controlling epidemic spread: Reducing economic losses with targeted closures. *Available at SSRN: https://ssrn.com/abstract=3590621*, 2020.

[BCF22] John R Birge, Ozan Candogan, and Yiding Feng. Controlling epidemic spread: Reducing economic losses with targeted closures. *Management Science*, 2022.

[BCJ+12] Yann Bramoullé, Sergio Currarini, Matthew O Jackson, Paolo Pin, and Brian W Rogers. Homophily and long-run integration in social networks. Journal of Economic Theory, 147(5):1754–1786, (2012).

[BCS19] Benjamin Bernard, Agostino Capponi, and Joseph E Stiglitz. Bail-ins and bail-outs: Incentives, connectivity, and systemic stability. *Available at SSRN: https://ssrn.com/abstract=2951448*, 2019.

[BDG12] Y. BACHRACH, M. DRAIEF, and S. GOYAL. Competig for security. working paper, 2012.

[BDG+21] Hamsa Bastani, Kimon Drakopoulos, Vishal Gupta, Ioannis Vlachogiannis, Christos Hadjicristodoulou, Pagona Lagiou, Gkikas Magiorkinis, Dimitrios Paraskevis, and Sotirios Tsiodras. Efficient and targeted covid-19 border testing via reinforcement learning. *Nature*, 599(7883):108–113, 2021.

[BDKVD20] Adam Brzezinski, Guido Deiana, Valentin Kecht, and David Van Dijcke. The covid-19 pandemic: government vs. community action across the united states. *COVID Economics: Vetted and Real-Time Papers*, 7:115–156, 2020.

[BEK+11a] L. Blume, D. Easley, J. Kleinberg, R. Kleinberg, and E. Tardos. Which networks are least susceptible to cascading failures? In *52nd Annual Symposium on Foundations of Computer Science*, pages 393–402, 2011.

[BEK+11b] Larry Blume, David Easley, Jon Kleinberg, Robert Kleinberg, and Eva Tardos. Network formation in the presence of contagious risk. In Proceedings of the 12th ACM conference on Electronic commerce, pages 1–10. ACM, (2011).

[BG97] V Bala and Sanjeev Goyal. Self-organization in communication networks. *Econometric Institute Research Papers*, (EI 9713-/A), 1997.

[BG00a] V. BALA and S. GOYAL. A noncooperative model of network formation. *Econometrica*, 68(5):1181–1230, 2000.

[BG00b] Venkatesh Bala and Sanjeev Goyal. A noncooperative model of network formation. Econometrica, 68(5):1181–1229, 2000.

[BG13] Chris T Bauch and Alison P Galvani. Social factors in epidemiology. *Science*, 342(6154):47–49, 2013.

[BHM20] David W Berger, Kyle F Herkenhoff, and Simon Mongey. An seir infectious disease model with testing and conditional quarantine. *Available at SSRN: https://ssrn.com/abstract=3561142*, 2020.

[BK07] Yann Bramoullé and Rachel Kranton. Public goods in networks. Journal of Economic Theory, 135(1):478–494, 2007.

[BK20] Zachary Bethune and Anton Korinek. Covid-19 infection externalities: Pursuing herd immunity or containment? *Covid Econ. Vetted Real Time Pap*, 11:1, 2020.

[BKST20] Luiz Brotherhood, Philipp Kircher, Cezar Santos, and Michèle Tertilt. An economic model of the covid-19 epidemic: The importance of testing and age-specific policies. *Available at SSRN: https://ssrn.com/abstract=3618840*, 2020.

[Bla77] Peter Michael Blau. *Inequality and heterogeneity: A primitive theory of social structure*, volume 7. Free Press New York, 1977.

[Bla82] Hubert M Blalock. *Race and ethnic relations*. Prentice-Hall Englewood Cliffs, NJ, 1982.

[BLH20] Shan BruSoe-Lin and Reobert Hecht. To suppress covid-19, we need to test those with no symptoms. *The New York Times*, 2020.

[BM04] Peter S Bearman and James Moody. Suicide and friendships among american adolescents. American Journal of Public Health, 94(1):89–95, 2004.

[BM19] Dirk Bergemann and Stephen Morris. Information design: A unified perspective. *Journal of Economic Literature*, 57(1):44–95, 2019.

[Bol98] Béla Bollobás. *Random graphs. Springer*, 1998.

[Bru20] Geoff Brumfiel. To end the coronavirus crisis we need widespread testing, experts say. *NPR*, 2020.

[BSI91] D. L. BRITO, E. SHESHINSKI, and M. D. INTRILIGATOR. Externalities and compulsary vaccinations. *Journal of Public Economics*, 45(1):69 – 90, 1991.

[BT97] Dimitris Bertsimas and John N Tsitsiklis. *Introduction to linear optimization*, volume 6. Athena Scientific Belmont, MA, 1997.

[BVDVVH04] Chris Baerveldt, Marijtje AJ Van Duijn, Lotte Vermeij, and Dianne A Van Hemert. Ethnic boundaries and personal choice. assessing the influence of individual inclinations to choose intra-ethnic relationships on pupils' networks. Social Networks, 26(1):55–74, 2004.

[Can20] Ozan Candogan. Information design in operations. In *Pushing the Boundaries: Frontiers in Impactful OR/OM Research*, pages 176–201. INFORMS, 2020.

[Cap16] Agostino Capponi. Systemic risk, policies, and data needs. In *Optimization Challenges in Complex, Networked and Risky Systems*, pages 185–206. INFORMS, 2016.

[CCAZ11] Antonio Cabrales, Antoni Calvó-Armengol, and Yves Zenou. Social interactions and spillovers. *Games and Economic Behavior*, 72(2):339–360, 2011.

[CCH+08] David Crandall, Dan Cosley, Daniel Huttenlocher, Jon Kleinberg, and Siddharth Suri. Feedback effects between similarity and social influence in online communities. In Proceedings of the 14th ACM SIGKDD international conference on Knowledge discovery and data mining, pages 160–168, 2008.

[CD20] Ozan Candogan and Kimon Drakopoulos. Optimal signaling of content accuracy: Engagement vs. misinformation. *Operations Research*, 68(2):497–515, 2020.

[CG84] Robert W Crandall and John D Graham. Automobile safety regulation and offsetting behavior: Some new empirical estimates. *The American Economic Review*, 74(2):328–331, 1984.

[CG94] Robert T Carter and A Lin Goodwin. Racial identity and education. Review of Research in Education, 20(1):291–336, 1994.

[CGL+22] Aldo Carranza, Marcel Goic, Eduardo Lara, Marcelo Olivares, Gabriel Y Weintraub, Julio Covarrubia, Cristian Escobedo, Natalia Jara, and Leonardo J Basso. The social divide of social distancing: Shelter-in-place behavior in santiago during the covid-19 pandemic. *Management Science*, 2022.

[CH17] Yeon-Koo Che and Johannes Hörner. Recommender systems as incentives for social learning. *The Quarterly Journal of Economics, forthcoming*, 2017.

[CJP09] Sergio Currarini, Matthew O Jackson, and Paolo Pin. An economic model of friendship: Homophily, minorities, and segregation. *Econometrica*, 77(4):1003–1045, 2009.

[CJP10] Sergio Currarini, Matthew O Jackson, and Paolo Pin. Identifying the roles of race-based choice and chance in high school friendship network formation. Proceedings of the National Academy of Sciences, 107(11):4857–4861, 2010.

[CL02a] F. CHUNG and L. LU. Connected components in random graphs with given expected degree sequences. *ANNALS OF COMBINATORICS*, 6(2):125–145, 2002.

[CL02b] Fan Chung and Linyuan Lu. The average distances in random graphs with given expected degrees. Proceedings of the National Academy of Sciences, 99(25):15879–15882, 2002.

[CL02c] Fan Chung and Linyuan Lu. Connected components in random graphs with given expected degree sequences. Annals of combinatorics, 6(2):125–145, 2002.

[CM15] L Elisa Celis and Aida Sadat Mousavifar. A model for social network formation: Efficiency, stability and dynamics. *arXiv preprint arXiv:1510.09025, (2015)*, 2015.

[CP05] Jacomo Corbo and David Parkes. The price of selfish behavior in bilateral network formation. In Proceedings of the twenty-fourth annual ACM symposium on Principles of distributed computing, pages 99–107, 2005.

[CP06] Giacomo Calzolari and Alessandro Pavan. On the optimality of privacy in sequential contracting. Journal of Economic theory, 130(1):168–204, 2006.

[CS82] Vincent P Crawford and Joel Sobel. Strategic information transmission. *Econometrica: Journal of the Econometric Society*, pages 1431–1451, 1982.

[Das20] Krishna Dasaratha. Virus dynamics with behavioral responses. *arXiv preprint arXiv:2004.14533*, 2020.

[DEB02] Jörn Davidsen, Holger Ebel, and Stefan Bornholdt. Emergence of a small world from local interactions: Modeling acquaintance networks. Physical Review Letters, 88(12), 2002.

[DJR21] Kimon Drakopoulos, Shobhit Jain, and Ramandeep Randhawa. Persuading customers to buy early: The value of personalized information provisioning. *Management Science*, 67(2):828–853, 2021.

[DKM17] Sanmay Das, Emir Kamenica, and Renee Mirka. Reducing congestion through information design. In *2017 55th annual allerton conference on communication, control, and computing (allerton)*, pages 1279–1284. IEEE, 2017.

[DM97] Bhaskar Dutta and Suresh Mutuswami. Stable networks. Journal of economic theory, 76(2):322–344, 1997.

[DM10] Moez Draief and Laurent Massouli. *Epidemics and rumours in complex networks*. Cambridge University Press, 2010.

[DM13] Sergei N Dorogovtsev and José FF Mendes. *Evolution of networks: From biological nets to the Internet and WWW*. Oxford University Press, 2013.

[DMG92] D. DE MEZA and J R GOULD. The social efficiency of private decisions to enforce property rights. *Journal of Political Economy*, 100(3):561–80, 1992.

[DOT14] Kimon Drakopoulos, Asuman Ozdaglar, and John N Tsitsiklis. An efficient curing policy for epidemics on graphs. *IEEE Transactions on Network Science and Engineering*, 1(2):67–75, 2014.

[DOT17] Kimon Drakopoulos, Asuman Ozdaglar, and John N Tsitsiklis. When is a network epidemic hard to eliminate? *Mathematics of Operations Research*, 42(1):1–14, 2017.

[DPVV20] Rahul Deb, Mallesh Pai, Akhil Vohra, and Rakesh Vohra. Testing alone is insufficient. *Available at SSRN: https://ssrn.com/abstract=3593974*, 2020.

[DR13] Cynthia Dwork and Aaron Roth. The algorithmic foundations of differential privacy. Theoretical Computer Science, 9(3-4):211–407, 2013.

[DR20] Kimon Drakopoulos and Ramandeep S Randhawa. Demand control of information products: Why perfect tests may not be worth waiting for. *Available at SSRN: https://ssrn.com/abstract=3565245*, 2020.

[DR21] Kimon Drakopoulos and Ramandeep S Randhawa. Why perfect tests may not be worth waiting for: Information as a commodity. *Management Science*, 67(11):6678–6693, 2021.

[DUR07] R. DURRETT. *Random Graph Dynamics*. Cambridge University Press, 2007.

[dVGW21] Francis de Véricourt, Huseyin Gurkan, and Shouqiang Wang. Informing the public about a pandemic. *Management Science*, 67(10):6350–6357, 2021.

[DZ98] Amir Dembo and Ofer Zeitouni. *Large deviations techniques and applications*, volume 2. *Springer*, 1998.

[EAHS20] Ali Eshragh, Saed Alizamir, Peter Howley, and Elizabeth Stojanovski. Modeling the dynamics of the covid-19 population in australia: A probabilistic analysis. *Plos one*, 15(10):e0240153, 2020.

[EGJ14] Matthew Elliott, Benjamin Golub, and Matthew O Jackson. Financial networks and contagion. The American economic review, 104(10):3115–3153, 2014.

[EGJS21] Jeffrey Ely, Andrea Galeotti, Ole Jann, and Jakub Steiner. Optimal test allocation. *Journal of Economic Theory*, 193:105236, 2021.

[EK10] David Easley and Jon Kleinberg. *Networks, crowds, and markets: Reasoning about a highly connected world. Cambridge University Press*, 2010.

[Ely17] Jeffrey C Ely. Beeps. *American Economic Review*, 107(1):31–53, 2017.

[ERT20] Martin S Eichenbaum, Sergio Rebelo, and Mathias Trabandt. The macroeconomics of epidemics. *Available at SSRN: https://ssrn.com/abstract=3560328*, 2020.

[ESZ20] Samuel Engle, John Stromme, and Anson Zhou. Staying at home: mobility effects of covid-19. *Available at SSRN*, 2020.

[Eva85] Leonard Evans. Human behavior feedback and traffic safety. *Human factors*, 27(5):555–576, 1985.

[Fag07] Giorgio Fagiolo. Clustering in complex directed networks. Physical Review E, 76(2):026107, 2007.

[FDC09] James H Fowler, Christopher T Dawes, and Nicholas A Christakis. Model of genetic variation in human social networks. Proceedings of the National Academy of Sciences, 106(6):1720–1724, 2009.

[FHLO20] James H Fowler, Seth Hill, Remy Levin, and Nick Obradovich. The effect of stay-at-home orders on covid-19 infections in the united states. *Available at SSRN 3576826*, 2020.

[FI00] Eric Fong and Wsevolod W Isajiw. Determinants of friendship choices in multiethnic society. In Sociological Forum, volume 15, pages 249–271. Springer, 2000.

[FJS21] Maryam Farboodi, Gregor Jarosch, and Robert Shimer. Internal and external effects of social distancing in a pandemic. *Journal of Economic Theory*, 196:105293, 2021.

[FKM04] A. M. FRIEZE, M. KRIVELEVICH, and R. MARTIN. The emergence of a giant component in random subgraphs of pseudo-random graphs. *Random Structures and Algorithms*, 24(1):42–50, 2004.

[FL98] Drew Fudenberg and David K Levine. *The theory of learning in games*, volume 2. *MIT press*, 1998.

[FLM+03] Alex Fabrikant, Ankur Luthra, Elitza Maneva, Christos H Papadimitriou, and Scott Shenker. On a network creation game. In Proceedings of the twenty-second annual symposium on Principles of distributed computing, pages 347–351. ACM, (2003).

[G02] Michelle Girvan and Mark EJ . Community structure in social and biological networks. Proceedings of the National Academy of Sciences, 99(12), 2002.

[Gan20a] Joshua Gans. *Economics in the Age of COVID-19*. MIT Press, 2020.

[Gan20b] Joshua S Gans. The economic consequences of r= 1: Towards a workable behavioural epidemiological model of pandemics. *Available at SSRN: https://ssrn.com/abstract=3662452*, 2020.

[Ger31] Semyon Aranovich Gershgorin. Uber die abgrenzung der eigenwerte einer matrix. (6):749–754, (1931).

[GG10] Andrea Galeotti and Sanjeev Goyal. The law of the few. The American Economic Review, 100(4):1468–1492, 2010.

[GGJ+10] Andrea Galeotti, Sanjeev Goyal, Matthew O Jackson, Fernando Vega-Redondo, and Leeat Yariv. Network games. The review of economic studies, 77(1):218–244, 2010.

[GGK06] Andrea Galeotti, Sanjeev Goyal, and Jurjen Kamphorst. Network formation with heterogeneous players. Games and Economic Behavior, 54(2):353–372, 2006.

[GHKRR20] Andrew Glover, Jonathan Heathcote, Dirk Krueger, and José-Víctor Ríos-Rull. Health versus wealth: On the distributional effects of controlling a pandemic. *Available at SSRN: https://ssrn.com/abstract=3583489*, 2020.

[GHL09] F. CHUNG GRAHAM, P. HORN, and L. LU. The giant component in a random subgraph of a given graph. In *Workshop on Algorithms and Models for the Web-Graph*, pages 38–49, 2009.

[GJ12] B. GOLUB and M. JACKSON. How homophily affects the speed of learning and best-response dynamics. *The Quarterly Journal of Economics*, 127(3), 2012.

[GK17] Matthew Gentzkow and Emir Kamenica. Bayesian persuasion with multiple senders and rich signal spaces. *Games and Economic Behavior*, 104:411–429, 2017.

[GL95] S. M. GOLDMAN and J. LIGHTWOOD. Cost optimization in the SIS model of infectious disease with treatment. Working paper, University of California at Berkeley, 1995.

[GMP20] Pietro Garibaldi, Espen R Moen, and Christopher A Pissarides. Modelling contacts and transitions in the sir epidemics model. *COVID Economics Vetted and Real-Time Papers, CEPR*, 2020.

[Goy93] Sanjeev Goyal. *Sustainable communication networks.* Econometric Institute, Erasmus University Rotterdam, 1993.

[GOY07] S. GOYAL. *Connections: An Introduction to the Economics of Networks.* Princeton University Press, 2007.

[GP97] P. GEOFFARD and T. PHILLIPSON. Disease eradication: Private versus public vaccination. *American Economic Review*, 87(1):222–30, 1997.

[Gra95] Mark Granovetter. *Getting a job: A study of contacts and careers.* University of Chicago Press, 1995.

[GV11] S. GOYAL and A. VIGIER. Robust networks. Mimeo, Cambridge University, 2011.

[GV13] S. Goyal and A. Vigier. Social interaction, vaccination, and epidemics. Working paper, 2013.

[GV14] Sanjeev Goyal and Adrien Vigier. Attack, defence, and contagion in networks. *The Review of Economic Studies*, 81(4):1518–1542, 2014.

[Hei13] Fritz Heider. *The psychology of interpersonal relations. Psychology Press*, 2013.

[HH91] Robert W Hahn and John A Hird. The costs and benefits of regulation: Review and synthesis. *Yale J. on Reg.*, 8:233, 1991.

[HHKM+20] Shaun P Hargreaves Heap, Christel Koop, Konstantinos Matakos, Asli Unan, and Nina Weber. Covid-19 and people's health-wealth preferences: Information effects and policy implications. *Available at SSRN 3605003*, 2020.

[Hir80] Jack Hirshleifer. Privacy: Its origin, function, and future. The Journal of Legal Studies, pages 649–664, 1980.

[HJ10] B. HOYER and K. DE JAEGHER. Strategic network disruption and defense. Working paper, Utrecht School of Economics, 2010.

[HK02] Petter Holme and Beom Jun Kim. Growing scale-free networks with tunable clustering. Physical review E, 65(2), 2002.

[HOF08] R. VAN DER HOFSTAD. *Random Graphs and Complex Networks.* 2008.

[HP06] Kai-Lung Hui and Ivan PL Png. The economics of privacy. Handbooks in Information System and Economics, 1:471–493, 2006.

[HT73] J. Hopcroft and R. Tarjan. Dividing a graph into triconnected components. *SIAM Journal on Computing*, 2(3):135–158, 1973.

[Jac03] Matthew O Jackson. The stability and efficiency of economic and social networks. In Advances in Economic Design, pages 319–361. 2003.

[Jac05] Matthew O Jackson. A survey of network formation models: stability and efficiency. Group Formation in Economics: Networks, Clubs, and Coalitions, pages 11–49, 2005.

[Jac08a] M Jackson. Social and economic networks. *Princeton University Press*, 2008.

[JAC08b] M. O. JACKSON. *Social and Economic Networks.* Princeton University Press, 2008.

[Jac08c] Matthew O Jackson. Average distance, diameter, and clustering in social networks with homophily. In Internet and Network Economics, pages 4–11. Springer, (2008).

[Jac10] Matthew O Jackson. *Social and economic networks. Princeton University Press*, 2010.

[JCX+20] Tuo Ji, Hai-Lian Chen, Jing Xu, Ling-Ning Wu, Jie-Jia Li, Kai Chen, and Gang Qin. Lockdown contained the spread of 2019 novel coronavirus disease in huangshi city, china: Early epidemiological findings. *Clinical Infectious Diseases*, 71(6):1454–1460, 2020.

[JLP13] Matthew O Jackson and Dunia Lopez-Pintado. Diffusion and contagion in networks with heterogeneous agents and homophily. Network Science, 1(01):49–67, 2013.

[Joh00] Scott John. Social network analysis: A handbook, 2000.

[JPV20] Callum J Jones, Thomas Philippon, and Venky Venkateswaran. Optimal mitigation policies in a pandemic: Social distancing and working from home. *Available at SSRN: https://ssrn.com/abstract=3574448*, 2020.

[JR07] Matthew O Jackson and Brian W Rogers. Meeting strangers and friends of friends: How random are social networks? *American Economic Review*, 97(3):890–915, 2007.

[JRBT12] Matthew O Jackson, Tomas Rodriguez-Barraquer, and Xu Tan. Epsilon-equilibria of perturbed games. Games and Economic Behavior, 75(1):198–216, 2012.

[JW96a] M. JACKSON and A. WOLINSKY. A strategic model of social and economic networks. *Journal of Economic Theory*, 71(1):44 – 74, 1996.

[JW96b] Matthew O Jackson and Asher Wolinsky. A strategic model of social and economic networks. Journal of economic theory, 71(1):44–74, 1996.

[JW02] Matthew O Jackson and Alison Watts. The evolution of social and economic networks. Journal of Economic Theory, 106(2):265–295, 2002.

[K$^+$41] Shizuo Kakutani et al. A generalization of brouwer's fixed point theorem. Duke mathematical journal, 8(3):457–459, 1941.

[Kam19] Emir Kamenica. Bayesian persuasion and information design. *Annual Review of Economics*, 11:249–272, 2019.

[Kap20a] Edward H Kaplan. Containing 2019-ncov (wuhan) coronavirus. *Health care management science*, pages 1–4, 2020.

[Kap20b] Edward H Kaplan. Covid-19 scratch models to support local decisions. *Available at SSRN 3577867*, 2020.

[Kee94] Theodore E Keeler. Highway safety, economic behavior, and driving environment. *The American Economic Review*, 84(3):684–693, 1994.

[KG11] Emir Kamenica and Matthew Gentzkow. Bayesian persuasion. *American Economic Review*, 101(6):2590–2615, 2011.

[KH03] H. KUNREUTHER and G. HEAL. Interdependent Security. *Journal of Risk and Uncertainty*, 26(2):231–249, 2003.

[KKT03] David Kempe, Jon Kleinberg, and Éva Tardos. Maximizing the spread of influence through a social network. In Proceedings of the ninth ACM SIGKDD international conference on Knowledge discovery and data mining, pages 137–146, 2003.

[KKT15] David Kempe, Jon Kleinberg, and Éva Tardos. Maximizing the spread of influence through a social network. *Theory of Computing*, 11(4):105–147, 2015.

[KL13] Jon Kleinberg and Katrina Ligett. Information-sharing in social networks. Games and Economic Behavior, 82:702–716, 2013.

[KPR01] Jon Kleinberg, Christos H Papadimitriou, and Prabhakar Raghavan. On the value of private information. In Proceedings of the 8th conference on Theoretical aspects of rationality and knowledge, pages 249–257. Morgan Kaufmann Publishers Inc., (2001).

[KR10] D. KOVENOCK and B. ROBERSON. The optimal defense of networks of targets. Working paper, 2010.

[KS20] Thomas Kruse and Philipp Strack. Optimal control of an epidemic through social distancing. *Available at SSRN: https://ssrn.com/abstract=3583186*, 2020.

[KT20] Maximilian Kasy and Alexander Teytelboym. Adaptive targeted infectious disease testing. *Oxford Review of Economic Policy*, 36:S77–S93, 2020.

[KW06] Gueorgi Kossinets and Duncan J Watts. Empirical analysis of an evolving social network. *Science*, 311(5757):88–90, 2006.

[LAR11] N. LARSON. Network security. MPRA paper, University Library of Munich, Germany, 2011.

[Lau96] Kenneth C Laudon. Markets and privacy. Communications of the ACM, 39(9):92–104, 1996.

[LBSB18] Ka Yin Leung, Frank Ball, David Sirl, and Tom Britton. Individual preventive social distancing during an epidemic may have negative population-level outcomes. *Journal of The Royal Society Interface*, 15(145):20180296, 2018.

[LI19] David Lingenbrink and Krishnamurthy Iyer. Optimal signaling mechanisms in unobservable queues. *Operations research*, 67(5):1397–1416, 2019.

[LMB$^+$54] P.F. Lazarsfeld, R.K. Merton, M. Berger, T. Abel, and C.H. Page. Friendship as a social process: A substantive and methodological analysis. Freedom and control in modern society, 18(1):18–66, 1954.

[LPS09] Yingbin Liang, H Vincent Poor, and S Shamai. Information theoretic security. Foundations and Trends in Communications and Information Theory, 5(4–5):355–580, 2009.

[LRSC01] Charles E Leiserson, Ronald L Rivest, Clifford Stein, and Thomas H Cormen. Introduction to algorithms, 2001.

[LS80] Björn Lindgren and Charles Stuart. The effects of traffic safety regulation in sweden. *Journal of Political Economy*, 88(2):412–427, 1980.

[LSG06] Darius Lakdawalla, Neeraj Sood, and Dana Goldman. Hiv breakthroughs and risky sexual behavior. *The Quarterly Journal of Economics*, 121(3):1063–1102, 2006.

[MAD05] A Medus, G Acuna, and CO Dorso. Detection of community structures in networks via global optimization. Physica A: Statistical Mechanics and its Applications, 358(2), 2005.

[Mag11] Trina J Magi. Fourteen reasons privacy matters: A multidisciplinary review of scholarly literature 1. *The Library Quarterly*, 81(2):187–209, 2011.

[Mar87] Peter V Marsden. Core discussion networks of americans. American sociological review, pages 122–131, 1987.

[Mar88] Peter V Marsden. Homogeneity in confiding relations. Social networks, 10(1):57–76, 1988.

[McA20] David McAdams. Economic epidemiology in the wake of covid-19. *economics*, 82120:122900, 2020.

[MD13] AD Medus and Claudio Oscar Dorso. Triadic closure mechanism in face-to-face and online social relationship networks. *arXiv preprint arXiv:1312.3496, (2013)*, 2013.

[Mid20] Catarina Midoes. Social distancing: did individuals act before governments? 2020.

[MM20] Charles F Manski and Francesca Molinari. Estimating the covid-19 infection rate: Anatomy of an inference problem. *Journal of Econometrics*, 2020.

[MMR20] Vahideh Manshadi, Sidhant Misra, and Scott Rodilitz. Diffusion in random networks: Impact of degree distribution. *Operations Research*, 68(6):1722–1741, 2020.

[Moo01] James Moody. Race, school integration, and friendship segregation in america1. American Journal of Sociology, 107(3):679–716, 2001.

[Mor00] Stephen Morris. Contagion. The Review of Economic Studies, 67(1):57–78, 2000.

[MR98] M. MOLLOY and B. REED. The size of the giant component of a random graph with a given degree sequence. *Combinatorics, Probability and Computing*, 7:295–305, 1998.

[MR00] M. MOLLOY and B. REED. A critical point for random graphs with a given degree sequence. *Random Structures and Algorithms*, (6):161–180, 2000.

[MSL87] J Miller McPherson and Lynn Smith-Lovin. Homophily in voluntary organizations: Status distance and the composition of face-to-face groups. American sociological review, pages 370–379, 1987.

[MSLC01] Miller McPherson, Lynn Smith-Lovin, and James M Cook. Birds of a feather: Homophily in social networks. Annual review of sociology, pages 415–444, 2001.

[MT20] William Maloney and Temel Taskin. Determinants of social distancing and economic activity during covid-19: A global view. *Available at SSRN: https://ssrn.com/abstract=3599572*, 2020.

[Mye97] Roger B Myerson. *Game theory: analysis of conflict*. Harvard university press, 1997.

[NEW02] M. E. J. NEWMAN. Spread of epidemic disease on networks. *Physical Review E*, 66:016128, 2002.

[New03] Mark EJ Newman. The structure and function of complex networks. SIAM review, 45(2):167–256, 2003.

[New04] Mark EJ Newman. Coauthorship networks and patterns of scientific collaboration. Proceedings of the National Academy of Sciences, 101:5200–5205, 2004.

[NP03] Mark EJ Newman and Juyong Park. Why social networks are different from other types of networks. Physical Review E, 68(3), 2003.

[NSW01a] M. E. J. NEWMAN, S. H. STROGATZ, and D. J. WATTS. Random graphs with arbitrary degree distributions and their applications. *Physical Review E*, 64:026118, 2001.

[NSW01b] Mark EJ Newman, Steven H Strogatz, and Duncan J Watts. Random graphs with arbitrary degree distributions and their applications. *Physical review E*, 64(2):026118, 2001.

[PBS18] Yiangos Papanastasiou, Kostas Bimpikis, and Nicos Savva. Crowdsourcing exploration. *Management Science*, 64(4):1727–1746, 2018.

[Pel75] Sam Peltzman. The effects of automobile safety regulation. *Journal of political Economy*, 83(4):677–725, 1975.

[Phi00] Tomas Philipson. Economic epidemiology and infectious diseases. *Handbook of health economics*, 1:1761–1799, 2000.

[PP+93] Tomas J Philipson, Richard A Posner, et al. *Private choices and public health: The AIDS epidemic in an economic perspective.* Harvard University Press, 1993.

[PQ20] Marcus Painter and Tian Qiu. Political beliefs affect compliance with covid-19 social distancing orders. *Available at SSRN 3569098*, 2020.

[PS20] Facundo Piguillem and Liyan Shi. Optimal covid-19 quarantine and testing policies. *Available at SSRN: https://ssrn.com/abstract=3594243*, 2020.

[PSV01] Romualdo Pastor-Satorras and Alessandro Vespignani. Epidemic spreading in scale-free networks. Physical review letters, 86(14):3200, 2001.

[PZ16] Eleonora Patacchini and Yves Zenou. Racial identity and education in social networks. *Social Networks*, 44:85–94, 2016.

[Rap53] Anatol Rapoport. Spread of information through a population with socio-structural bias: I. assumption of transitivity. The bulletin of mathematical biophysics, 15(4):523–533, 1953.

[Ros65] J. B. Rosen. Existence and Uniqueness of Equilibrium Points for Concave N-Person Games. *Econometrica*, 33(3):520–534, 1965.

[Sam00] Pamela Samuelson. Privacy as intellectual property? Stanford Law Review, pages 1125–1173, 2000.

[SAN71] J. L. SANDERS. Quantitative guidelines for communicable disease control programs. *Biometrics*, 27(4):883–893, 1971.

[SET74] S. P. SETHI. Quantitative guidelines for communicable disease control program: A complete synthesis. *Biometrics*, 30(4):681–691, 1974.

[SLD20] Tim Searchinger, Anthony LaMantia, and Gordon Douglas. Mass testing is the only sustainable solution to the coronavirus. *Washington Post*, 2020.

[SP09] Brian Skyrms and Robin Pemantle. A dynamic model of social network formation. In Adaptive Networks, pages 231–251. 2009.

[Sti80] George J Stigler. An introduction to privacy in economics and politics. The Journal of Legal Studies, pages 623–644, 1980.

[SYDV09] Peng Sun, Liu Yang, and Francis De Véricourt. Selfish drug allocation for containing an international influenza pandemic at the onset. *Operations Research*, 57(6):1320–1332, 2009.

[Tar55] Alfred Tarski. A lattice-theoretical fixpoint theorem and its applications. *Pacific journal of Mathematics*, 5(2):285–309, 1955.

[Tay04] Curtis R Taylor. Consumer privacy and the market for customer information. *RAND Journal of Economics*, 35(4):631–650, 2004.

[TOS+06] Riitta Toivonen, Jukka-Pekka Onnela, Jari Saramäki, Jörkki Hyvönen, and Kimmo Kaski. A model for social networks. Physica A: Statistical Mechanics and its Applications, 371(2):851–860, 2006.

[TOX09] F. TOXVAERD. Foundations of strategic epidemiology: Recurrent infection and treatment. Working paper, 2009.

[Tox20] Flavio Toxvaerd. Equilibrium social distancing. *Mimeo, Cambridge University*, 2020.

[TW07] Eva Tardos and Tom Wexler. Network formation games and the potential function method. Algorithmic Game Theory, pages 487–516, 2007.

[Var09] Hal R Varian. Economic aspects of personal privacy. In *Internet policy and economics*, pages 101–109. Springer, 2009.

[Váz03] Alexei Vázquez. Growing network with local rules: Preferential attachment, clustering hierarchy, and degree correlations. Physical Review E, 67(5), 2003.

[VR07a] VEGA-REDONDO. *Complex Social Networks*. Cambridge University Press, 2007.

[VR07b] Fernando Vega-Redondo. *Complex social networks*. Number 44. Cambridge University Press, 2007.

[Wat99] Duncan J Watts. *Small worlds: the dynamics of networks between order and randomness. Princeton university press*, 1999.

[WdVS09] Shouqiang Wang, Francis de Véricourt, and Peng Sun. Decentralized resource allocation to control an epidemic: A game theoretic approach. *Mathematical biosciences*, 222(1):1–12, 2009.

[Wes68] Alan F Westin. Privacy and freedom. Washington and Lee Law Review, 25(1):166, 1968.

[WF94] Stanley Wasserman and Katherine Faust. *Social network analysis: Methods and applications*, volume 8. *Cambridge university press*, 1994.

[WS98] Duncan J Watts and Steven H Strogatz. Collective dynamics of 'small-world'networks. Nature, 393(6684):440–442, 1998.

Biography

Minjun Chang holds a B.A. in Applied Mathematics from University of California, Berkeley and a Ph.D. in Decision Sciences from the Fuqua School of Business at Duke University.