

Unthinkable: Mathematics and the Rise of the West

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Dissertation submitted in partial fulfillment of
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ABSTRACT

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Abstract

This dissertation explores the ideational underpinnings of the rise of the west through a comparison of ancient Greek geometry, medieval Arabic algebra, and early modern European calculus. Blending insights from Thomas Kuhn, Michel Foucault, and William H. Sewell, I assert that there is an underlying logic, however clouded, to the unfolding of a given civilization, governed by a cultural episteme that delineates the boundaries of rational thought and the accepted domain of human endeavor. Amid a certain conceptual configuration, the rise of the west happens; under other circumstances, it does not. Mathematics, as an explicit exhibition of logic premised on culturally determined axioms, presents an outward manifestation of the lens through which a civilization surveys the world, and as such offers a window on the fundamental assumptions from which a civilization's trajectory proceeds. To identify the epistemological conditions favorable to the rise of the west, I focus specifically on three mathematical divergences that were integral to the development of calculus, namely analytic geometry, trigonometry, and the fundamental theorem of calculus. Through a comparative/historical analysis of original source documents in mathematics, I demonstrate that the logic in the earlier cases is fundamentally different from that of calculus, and furthermore, incompatible with the key developments that constitute the rise of the west. I then examine the conceptual similarities between calculus and several features of the rise of the west to articulate a description of the early modern episteme.

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1. Introduction

One of the most enduring questions in comparative historical research revolves around the sudden prosperity experienced by western Europe in relation to the rest of the world following the Middle Ages. While the economic, political, infrastructural, and institutional components of society have all been painstakingly dissected in search of the elusive explanation for the rise of the west, the concomitant development of modern mathematics has received scant notice. Yet, the changes wrought on society by the development of calculus have been far-reaching in their consequences, and the timing is provocative. Why was calculus suddenly invented nearly simultaneously by two independent mathematicians, Newton and Leibniz, in the seventeenth century after remaining unnoticed by the medieval Arabic mathematicians, pioneers in algebra, as well as their predecessors in ancient Greece, inventors of geometry? After centuries of restricting their inquiries to space and structure, why did mathematicians abruptly embrace the study of change? Moreover, free of the material constraints that inhibited other aspects of the rise of the west from emerging elsewhere, why did calculus only develop in early modern Europe?

While previous studies have examined what set the west apart from its contemporaries, as well as differential rise within the west, none have ventured particularly far into the past. Yet, despite the temporal gulf that separates the ancient Greek, medieval Arabic, and early modern European societies, there are some tantalizing parallels. Like early modern Europe, both ancient Greece and medieval Arabia were awash in the exchange of goods and ideas from near and far, and both presided over an

intellectual golden age that ushered in unprecedented advances. In many respects, both stood on the cusp of modernity, but for some reason, retreated back into antiquity. Specifically, there are five developments, generally considered to be uniquely European, that exemplify the rise of the west: global expansion, the emergence of nation-states, the onset of capitalism, industrialization, and the scientific revolution. Each of these developments is echoed, though never fully realized, in the histories of both ancient Greek and medieval Arabic civilization.

From the fifteenth century onward, western European powers began mounting sailing expeditions to establish direct trading links with suppliers of spices and silk in Asia. Previously, trade between east and west was brokered in the Mediterranean, enriching the Italian and Ottoman regimes that controlled the region. Western Europe sought to disrupt this monopoly, first through the Portuguese exploration of the African coast and eventual rounding of the Cape of Good Hope, and later by means of westward routes, such as those pursued by Columbus and Magellan. In the process, they forged expansive overseas empires and global trading networks, and stimulated advances in navigation and cartography. However, they were not the first to engage in global exploration or to establish far-flung empires. Alexander the Great led expeditions and eventually built an empire that extended from the Balkan peninsula across Mesopotamia to the edge of India, and included much of the eastern Mediterranean, Asia Minor, and Egypt. He founded cities across his empire, many of which were settled by veterans of his campaigns, and through which Greek culture diffused (Arrian 1976). Following his death, the empire fractured into the competing Diadochi kingdoms, but under their administration, Greek colonists continued to pour into the assimilated territories (Shipley

2000). Likewise, in the seventh century, the Umayyads built an empire that encompassed the Arabian peninsula, Mesopotamia, northern Africa, and the better part of the Iberian peninsula before ceding power to the Abbasids in the eighth century (Lapidus 2002). Under the Umayyad, Abbasid, and subsequent Fatamid empires trade networks were established across three continents as well as the eastern Atlantic Ocean, Mediterranean Sea, Indian Ocean, and China Sea (Hobson 2004). While neither ancient Greek nor medieval Arabic sailors circumnavigated the globe, they were no less engaged in outward expansion than early modern Europe. Why, then, is only the early modern European case considered to be the Age of Discovery?

Another seemingly peculiar feature of early modern Europe is its geopolitical division into nation-states. Ideally, nation-states represent the geographic coincidence of an ethnic or cultural entity and political sovereignty, and typically are characterized by territorial integrity, political self-determination, and a predominantly national identity. While the full articulation of European nation-states is generally dated to the nineteenth century, the new political reality began to emerge during the early modern period. The familiar political boundaries of western Europe were largely established by the 1648 Peace of Westphalia. In addition, England, Portugal, and the Dutch Republic were among the first to coalesce into nation-states, and exhibited both a relatively centralized state structure and a sense of common identity in their early modern incarnations. But some earlier political arrangements also resemble nation-states. In the ninth and tenth centuries, the Abbasid empire grew steadily more nominal as its provinces assumed greater independence, eventually operating as de facto states with separate armies, economies, and lines of succession (Brauer 1995). Following Alexander's death, the

Greek city-states, appealing to Greek solidarity, made several, ultimately unsuccessful, attempts to unite in order to regain independence from Macedon (Errington 2008). While a nation-state is not fully realized in either case, the spectre of consolidation lingers, begging the question, why not?

Early modern Europe was also the birthplace of capitalism. As set forth by Adam Smith (Smith and Wight 2007), free market economics rests on a triad of the division of labor, self-interest, and free trade. The division of labor increases productivity which gives rise to surplus, self-interest motivates exchange, and the aggregate of self-interested exchanges, if left unfettered, will correct for market fluctuations, tending overall to benefit society. The Dutch Republic in the seventeenth century offers the first example of this system at work (de Vries and van der Woude 1997), and the rest of western Europe soon followed suit. The rise of capitalism in early modern Europe has been variously attributed to the adoption of double-entry bookkeeping (Weber 1981; Schumpeter 1950; Sombart 1953), the proliferation of financial institutions and availability of credit (de Roover 1963; Teichova, Kurgan-van Hentenryk and Ziegler 1997), and colonial expansion (Wallerstein 1974; Marx & Engels 1998). However, vestiges of the free market system are evident in the earlier cases as well. A heavy emphasis on trade and a form of mercantilism, the precursor of capitalism proper in Europe, were present in both the ancient Greek and medieval Arabic economies (Finley 1973; Labib 1969; Heck 2006). Both also employed a relatively sophisticated financial system to facilitate trade (Finley 1973; Cohen 1992; Banaji 2007; Labib 1969), and as noted above, engaged in aggressive empire building. Though neither economy developed into thoroughgoing capitalism, the potential for such a system was clearly there.

Industrialization is roughly synonymous with a shift away from an agrarian economy towards manufacturing, and the concomitant transition from subsistence-level to a higher standard of living. The onset of industrialization is historically associated with eighteenth century innovations in three areas, namely textile manufacturing, steam power, and metallurgy. The development of labor-saving devices such as the water frame and the spinning jenny enabled increased production of textiles, particularly cotton, for less time and cost. Steam power was first used to pump water out of mines, allowing more coal to be extracted, and later employed to power machines. Aside from increasing production, steam also displaced water wheels as a source of energy, freeing factories from the necessity of locating next to a water supply. Finally, the substitution of coke for charcoal in iron smelting resulted in greater quantities of inexpensive iron, and by extension, industrial and consumer goods. However, industrialization encompasses more than just a collection of technological wonders; also crucial to its success is the division of labor. As the name implies, the division of labor refers to a cooperative production system wherein the manufacturing process is divided into several component tasks. Workers perform one of those tasks exclusively, leading to greater productivity and efficiency. While industrialization is generally viewed as a phenomenon unique to Europe, there are some signs of it in earlier civilizations. In his writings, Hero of Alexandria described both a steam engine (Hero, Greenwood, and Woodcroft 1971) and windwheel (Drachmann 1961), though neither was used for industrial applications, and Xenophon mentions the division of labor with reference to shoe manufacture in ancient Greek cities (Xenophon and Ambler 2001). Industrial mills producing everything from sugar to steel, and powered by water and wind, dotted the medieval Arabic landscape

(Lucas 2005). Furthermore, the labor force was increasingly specialized, particularly in the textile industry (Shatzmiller 1994). Yet, industrialization never quite matured as it did in early modern Europe.

Science in Europe underwent several major changes in the early modern period, beginning with Copernicus' unveiling of a heliocentric model of the universe in the sixteenth century (Copernicus 1995). Other key developments included Francis Bacon's introduction of the scientific method (Bacon and Fowler 1889), Isaac Newton's enunciation of classical mechanics (Newton 1999), and William Harvey's exposition on the circulation of blood (Harvey 1962). What is interesting about all of these innovations is that they are largely reactions to ancient and medieval thought. Copernicus overturns the planetary model developed by Ptolemy, and elaborated on by medieval Arabic astronomers. Harvey offers a refutation of the dual blood system advocated by Galen, and later Avicenna. Bacon's inductive empiricism stands in contrast to Aristotle's deductive syllogisms, though it is not so different from medieval scientific methods that stress experimentation (Alhazen and Smith 2006; McGinnis 2003). While Newton alone articulated the complete formulation, fragments of classical mechanics surface in earlier works, most notably the first law of motion in Aristotle's *Physics*, and an earlier version of the first law and part of the second in ibn al-Haytham's and Avicenna's work. In short, scientists in all three cases are studying the same phenomena, but drawing different conclusions.

What is fascinating about the preceding examples is that the knowledge and technology were not completely unknown in the earlier cases, just unrealized. The same is true in the realm of mathematics. Calculus, another "uniquely European" innovation,

has two main branches. Differential calculus is primarily concerned with ascertaining a function's instantaneous rate of change, which is synonymous with the slope of the tangent to a curve at any given point. Integral calculus, on the other hand, focuses on cumulative change over time, commonly represented as the area under a curve. Early modern European mathematicians, most notably Isaac Newton and Gottfried Leibniz, honed strategies for determining the values of those tangent slopes and areas, and consequently perceived the inverse relationship between differentials and integrals, the cornerstone of calculus. Tangents and areas are, of course, prevalent in most mathematical systems, however the fundamentals of calculus are also evident in the works of ancient Greek and medieval Arabic mathematicians. They recognized that a tangent could denote the motion of a curve at the point of intersection, and in some cases, even derived differential equations. Likewise, in order to find the area or volume circumscribed by a curve, they devised methods that incorporated infinitesimal calculations similar to the early moderns Europeans' use of evanescent rectangles. Yet only Newton and Leibniz were able to coalesce these concepts into a coherent whole.

This dissertation will compare the assumptions underlying ancient Greek geometry, medieval Arabic algebra, and early modern European calculus in order to explain the variation in the approaches to mathematics taken by these three civilizations, and ultimately, variation in the civilizations themselves. This research asserts that mathematics, as a model of human understanding premised on culturally determined axioms, is an outward manifestation of the lens through which civilizations survey the world, and as such, constitutes the fundamental assumptions from which a civilization's trajectory proceeds. In other words, there is an underlying logic, however clouded, to the

unfolding of a given civilization, governed by a cultural episteme that delineates the boundaries of rational thought and the accepted domain of human endeavor, and the mathematical system that correspondingly burgeons within that civilization offers a unique insight into the contours of that episteme by its very nature as a schematic of the understanding. Mathematics presents an explicit exhibition of what is thinkable and what is not, given a set of premises believed to be self-evident, informing perceptions of the elemental possibilities and limitations of the world, and this assessment, in turn, shapes and directs the essays of human progress. Examining the paradigms that underpin the mathematical epochs most closely associated with three progressive civilizations should, in turn, illuminate the countenance of the governing episteme. In sum, then, this research constitutes a sociology of mathematics to explain the rise of the west through a comparison of ancient Greek geometry, medieval Arabic algebra, and early modern European calculus.

1.1 The Rise of the West

The “rise of the west” refers to the unprecedented development and influence accrued to western Europe relative to the rest of the world following the end of the Middle Ages. In particular, there are several phenomena unique to the west that constitute its rise, namely, global exploration and dominance, the emergence of nation-states, industrialization, the rapid expansion of capitalism, and the preeminence of science and rationalism. To explain the advent of these trends, as well as their temporal and geographic concentration, sociologists have traditionally appealed to three types of accounts. The first suggests western exceptionalism, in the form of idiosyncratic

institutions and innovations, is at the root of western success. The second proposes a version of western exceptionalism that emphasizes aggression and exploitation as the crucial factors enabling the west's rise, and in particular the unique institutions and innovations that furthered their rapacious ambitions. The third type of account contends that the west was the unwitting beneficiary of a fortuitous confluence of external forces. Variations on these three main themes abound, and while each commands a distinctive set of strengths and weaknesses, they are all nonetheless ultimately unable to fully explain the rise of the west.

Beginning with Weber's (1998) correlation of the Protestant ethic and the spirit of capitalism, scholars have sought an elusive "special something" that the west possessed and the rest lacked, deducing that a peculiar effect demands a unique cause. Theories as to the identity of this mysterious element vary in their particulars, but in the first type of account, it generally takes the form of relatively benevolent institutions or innovations that unexpectedly precipitated the rise of the west. For instance, Landes (1999), while acknowledging the impact of geography, points to the invention of property rights, free enterprise, and civil liberties as the secret to western prosperity. Essentially, he argues that this fragmentation of land, commerce, and power ensured competition, which in turn stimulated progress. Echoing Weber, he traces the genesis of this competitive environment to the dictums of another early modern European institution, namely Protestantism. Similarly, Gorski (2003) asserts that a disciplinary revolution, in concert with military and economic transformations, was the key factor in early modern state formation. Specifically, he suggests that the proliferation of communal surveillance, incarceration, and bureaucracy gave rise to an infrastructure of social control which, in

turn, strengthened state power. The widespread discipline necessary to sustain this system he attributes to the influence of Protestantism. Alternately, Jones (2003) believes that western economic success was the result of the relative stability experienced by early modern Europe. He argues that environmental good fortune combined with ameliorating political forces, and in particular decentralized authority, allowed for an unprecedented continuity of development. The cumulative effect of technological innovations building upon each other for several hundred years eventually propelled industrialization.

The primary challenge this type of account presents is in isolating a characteristic that is truly singular. In the above examples the rise of the west is ascribed to competition, social control, and stability, yet these phenomena are hardly unique in the history of the world. More to the point, these other instances did not result in global exploration, nation-states, capitalism, industrialization, or the scientific revolution, much less all of them. In other words, if relative stability spurs progress, why is there no evidence of similar advances during the Pax Romana? Why was stagnation so often the result of strict social control and centralized power in both past and future civilizations, such as ancient Sparta or the eastern bloc nations in the twentieth century? However, even when the cause appears to be unique, as in the case of Protestantism, questions remain. Why does Protestantism arise only in western Europe, or put another way, why does only Protestantism, of all the religions in the world, provoke these effects? Did Protestantism change its adherents, or was Protestantism adopted because its tenets coincided with their preexisting attitudes? If applied elsewhere, would it produce the same results? In short, is Protestantism the source or another symptom of the rise of the west?

In response to these shortcomings, another type of account evolved. In contrast to theories that tend to focus disproportionately on the auspicious facets of western society, the second approach perceives that the reasons for the rise of the west are mired in the corresponding negative aspects. This type of account presents a variation of western exceptionalism that emphasizes those innovations and institutions which fueled elite aggression and exploitation as the cause of divergent regional outcomes. Whereas the former perspective understands the rise of the west to be an unintended consequence of another process, the second type of explanation, in the Marxist tradition, posits that, while the degree of Europe's success was unexpected, the object was always domination. For example, Parker (1996) attributes western dominance to martial innovations. Better training, tactics, weaponry, defenses, and larger armies, as well as the explicit military goal of annihilating an enemy accustomed to fighting with the aim of seizing slaves, gave the west a tremendous military advantage which allowed Europe to overpower much of the world. Similarly, Abernethy (2002) suggests a trifecta of power, profit, and piety was the key to Europe's potent imperialism. He contends that strong states, expansionist economies and proselytizing religion enabled Europeans to overwhelm nations unprepared for the onslaught of these mutually reinforcing institutions. Alternately, Pomeranz (2000) credits the exploitation of colonial empires abroad and the availability of coal at home for Europe's success relative to China. Specifically, access to natural resources and slave labor overseas, as well as captive colonial markets for surplus goods, provoked rapid economic growth in Europe, while the accessibility of domestic coal deposits facilitated industrialization. Lacking both, China consequently experienced limited industrial development.

Much like their positive counterparts, the arguments forwarded by the proponents of negative western exceptionalism are not entirely satisfactory. Were the early modern Europeans really more ferocious than other conquerers, such as Genghis Khan, Attila the Hun, or Tamerlane? If global domination hinges on mutually reinforcing power, profit, and piety, then why did the conquests of Muslim empires such as the Baghdad Caliphate, the Ottomans, Safavids, and Moghuls, which blended political pursuits, mercantile interests, and the spread of Islam, fail to achieve the same result? The above examples also raise the question, why do these innovations and institutions develop only in Europe, but at the same time, all over Europe? Even if they do originate in Europe, why do they remain uniquely European? In other words, why do these innovations fail to spread beyond Europe's borders, particularly after they prove to be so effective? Why is the rest of the world seemingly so unable or unwilling to adapt?

Crafted largely in reaction to the reigning Euro-centric paradigm, the third type of account recasts the rise of the west as the result of a fortuitous confluence of external forces. This approach often underscores the historical contributions and earlier supremacy of non-western nations, as well as the highly contingent nature of the west's ascendancy. For instance, Abu-Lughod (1989) maintains that, prior to the rise of the west, China presided over an integrated global trading network. However, following a series of misfortunes, including a plague, the collapse of the Mongol Empire, and the blockage of key Asian trade routes, China withdrew from trade, creating a power vacuum. Due to a concurrent run of good fortune, western Europe was able to assume control of China's established trade routes, and was then in a position to enforce its hegemony. Similarly, Frank (1998) suggests that Europe gained ascendancy in what had

been the Asian world economy only after China suffered a natural downturn in a long economic cycle. He credits Europe with the development of many labor saving devices, asserting that China's abundant supply of cheap labor hindered technological advances, but argues that it was silver plundered from the Americas which facilitated the west's entry into the Asian economy. Conversely, Hobson (2004) asserts that the rise of the west was entirely due to the assimilation of eastern innovations brought to Europe by way of a global market dominated by Asia, and the appropriation of eastern resources through imperialism. Alternately, Goldstone (2002; 2000), while acknowledging the impact of the decline of the east, proposes that a series of unrelated chance circumstances arose, which when taken altogether, triggered the rise of the west. In brief, England was unable to implement a conventional orthodoxy, and thus to ensure stability, instead promoted a Newtonian tradition that quickly pervaded the populace. At the same time, mining coal, a vital native source of energy, was becoming increasingly difficult due to groundwater flooding in the shafts. Applying the Newton's mechanical worldview to the water problem resulted in the genesis of the steam engine, which in turn gave rise to the industrial revolution.

Criticism of this approach often surrounds its proponents' tendency to discount or ignore European innovations. Specifically, if the rise of the west is entirely due to the decline of the east, what accounts for the scientific and technological development observed in Europe? If Europe was simply assimilating and appropriating inventions from Asia, why were the outcomes different? If cheap labor inhibited technological progress in China, why did slave labor not have a similar effect in Europe? It is also at times unclear why the east suddenly ceases to be competitive. Why, for instance, did

China opt not to explore overseas after witnessing the infusion of American silver into Europe, particularly when its economy began to slow? Why did the productive capacity of industrialization fail to tempt China, regardless of how cheap labor was? Finally, explanations that rely on happenstance by definition must assume the decisive factors arise at random, but it is unclear if they, in fact, do. For example, with regard to Goldstone's theory, why were the people of England religiously and politically divided, prompting the substitution of the Newtonian framework in place of a traditional orthodoxy? Did Newton's work appear by chance at this particular juncture, or was it influenced by the context in which Newton found himself? Why should the populace embrace Newton when they were so unwilling to accept a classical doctrine? In short, under what conditions do coincidences occur?

To review, sociologists have traditionally proposed three types of explanations for the rise of the west: western exceptionalism, negative western exceptionalism, and the decline of the east. However, despite many excellent theories as to the particulars, they are nonetheless unable to fully account for the rise of the west. There is some crucial dynamic underlying the process that is only hinted at in the foregoing arguments, but what has been overlooked? A clue lies in Sewell's (1992) reformulation of the duality of structure, which asserts that structure derives from the interplay of schemas and resources. Schemas he defines as the tacit rules and norms that guide thoughts and behavior, while resources encompass anything that can be used to maintain or increase power, including material assets, capital, knowledge, institutions, and authority. The above accounts all attempt to forward structural explanations for the rise of the west that disproportionately focus on resources and fail to, or at best inadequately, analyze the

corresponding schemas. It is not that the innovations, institutions, and material resources outlined above are not relevant to the rise of the west; they are simply not the whole story. Just as the mere presence of ingredients will not result in a cake without the intervention of a recipe, so too are resources absent a normative understanding of how to make use of them insufficient to explain the rise of the west. What is needed is a coherent ideational theory of the rise of the west to complement the existing resource-heavy body of literature.

1.2 Theoretical Framework

The principal theory driving this research derives from traditional interpretations of epistemological obstacles and ruptures (Foucault 1994; Kuhn 1996) combined with Sewell's (1992) reformulation of the duality of structure. In short, it posits that patterns in the collective consciousness subtly frame the accepted realm of ideas by distinguishing between what is and what is not thinkable, and furthermore, that this configuration of thought varies according to space and time, resulting in differential development across epochs and civilizations. Conversely, all the artifacts generated at a given intersection of space and time bear the indelible fingerprints of the corresponding ideational framework.

Kuhn (1996) elaborates a similar understanding of the evolution of scientific theories, designating as a "paradigm" the underlying collection of beliefs and assumptions that organize scientific thought, and by extension, scientific practice. He contends that the prevailing paradigm determines not just the subject matter of science, but also the types of questions asked, their form, the kinds of answers sought, and interpretations of results. Once a paradigm has taken hold, a period of normal science

ensues, during which scientists attempt to subjugate all phenomena in accordance with that paradigm, until such time as enough significant anomalies accrue to prompt a paradigm shift (cf. Koyré 1958; Lakatos 1980). Foucault (1994) expands on this concept with respect to discourse, extending the influence of these overarching conditions of possibility, which he christens the “episteme,” beyond science to all spheres of thought. He then utilizes this framework to explicate the advent of the human sciences by comparing the epistemes of the Renaissance, Classical, and Modern epochs. Whereas Kuhn's paradigm supposes a deliberate consensus (or in the case of paradigm shifts, a conscious decision to disregard it), Foucault's model represents a sort of epistemological unconscious that exerts an ever-present, but unobtrusive, influence. Central to both theses is the notion of epistemological obstacles (cf. Bachelard 1985; 2006). These subtle wrinkles in the intellectual landscape suppress, obscure, and divert ideas in order to block the pursuit of knowledge considered to be off-limits or not worth knowing. In other words, these conceptual biases render some thoughts unthinkable, and consequently refocus attention on matters more amenable to the overall conceptual structure. The specific set of epistemological obstacles a society embraces constitutes its episteme, which in turn gives shape to its endeavors, and ultimately its historical trajectory.

For a simple illustration of epistemic influence, consider Nisbett's (2003) account of an experiment comparing the thought processes of his American and Japanese students. The participants were all shown the same series of animated underwater vignettes featuring at least one large, bright and fast fish, several slower animals, and elements such as plants and rocks. After viewing each approximately twenty second scene twice, the participants were asked to describe what they had seen. The American

students were inclined to concentrate on the individual fish, particularly the large, bright and fast focal fish, while the Japanese students were more likely to perceive the scene as a whole, noting both the moving animals and the stationary background elements, as well as the relations between them. Echoing Kuhn and Foucault, Nisbett attributes the pattern of responses to the socially transmitted “habits of thought” arising from divergent philosophical traditions that embrace different fundamental beliefs about the nature of the world, and therefore differentially prioritize individual agency and collective harmony. In other words, the participants' perceptions are a function of the external stimuli filtered through an epistemic lens.

The question remains, where does one episteme end and another begin? How does an episteme respond to innovations that fall just short of revolutionary? For the answer, it is necessary to appeal to Sewell's (1992) notion of the duality of structure. Sewell posits that structure arises from the interaction of schemas and resources. Schemas are the virtual norms that orient thoughts and actions, or more specifically, “the various conventions, recipes, scenarios, principles of action, and habits of speech and gesture” generally learned through induction or recurrent exposure. Resources, on the other hand, are actual, and comprise material objects as well as capital, knowledge, institutions, rituals, authority, and anything else that contributes to the enhancement or perpetuation of power. When novel situations develop, anomalies which confound the existing structure arise, prompting a renegotiation of the interaction between schemas and resources. An attempt is made to match the anomaly to more familiar scenarios or past experiences, and evaluate it with respect to competing norms while at the same time factoring in any material constraints on the outcome. Eventually, the resolution that best

satisfies all of these considerations is determined, and the anomaly is accordingly either assimilated into the existing structure, thereby reinforcing it, or the structure is transformed to accommodate the anomaly.

The episteme is, in fact, a specific instance of schema (though, as schemas ran the gamut from superficial and idiosyncratic to profound and universal, the term “episteme” will be retained to better signify the level of interest). Embedding epistemes in the duality of structure provides both a mechanism for fluctuation through the constant renegotiation of resources and schemas, and an explanation of continuity in light of this potential for change, allowing for a more dynamic episteme that is still susceptible to overthrow. The difference between epistemic refinement and a full-blown epistemological rupture also becomes clear; an epistemological rupture is accompanied by structural transformation. In other words, in the same way that a jeep manufactured fifty years ago and a jeep built yesterday are both obviously jeeps, even though the model has undergone numerous design changes and technological upgrades, so too can an episteme absorb innovations without altering its essential character. If jeep manufacturers were to unveil a bicycle as the next generation of the jeep, it would represent an epistemological rupture.

Additionally, merging epistemes with the duality of structure affirms that the impetus for an epistemological rupture derives from the preceding episteme. The process Sewell (1992) describes specifies that the new episteme is a product of the interaction between the old structure and an anomaly, precluding the idea of a spontaneous realignment. Even if the result is an utter rejection of the prevailing norms, the new episteme is still rooted in the old, since it was necessarily conceived under the auspices of

the old way of thinking. Otherwise, all ideas devised under previous epistemes would be completely incomprehensible, and all history prior to the institution of the current episteme would be meaningless.

In summary, the framework adopted by this research blends the epistemic model espoused by Kuhn (1996) and Foucault (1994) with Sewell's (1992) reformulation of the duality of structure. The resulting theory suggests that, at a given intersection of space and time, there exists an overarching societal perception of the fundamental possibilities and limitations of the world, which in turn, shapes and directs the endeavors of its members. This conceptual scaffold and the idiosyncratic epistemological biases that determine its contours form the basis of an ideational explanation of the rise of the west. This project adopts Foucault's term, "episteme," in view of its broader connotation, and also borrows the phrase "epistemological obstacles" from Bachelard, though it understands them to be socially rather psychologically constructed. Embedding the episteme within the duality of structure both positions this research with respect to previous studies and lends the episteme a degree of plasticity while retaining an overall continuity, which is necessary in light of the spatial and temporal breadth of each case.

That said, why look to mathematics? The difficulty inherent in juxtaposing ancient Greek and medieval Arabic societies with early modern Europe is twofold. First, how can the rise of the west, manifested as global exploration, the emergence of nation-states, the rapid expansion of capitalism, industrialization, and the scientific revolution, be compared with its non-occurrence? This issue is particularly salient given the project's focus on ideational frameworks. How can the conceptual constraints that demarcate the ideas associated with these phenomena be accessed when the concept itself is absent?

Secondly, there is the question of data. The availability of historical records is subject to the vagaries of time, and artifacts from the more distant past are frequently in shorter supply than those from more recent eras. With a fragmentary (at best) image of ancient Greek and medieval Arabic communities, the task of tracing the overarching conceptual structure organizing those civilizations is that much harder.

Mathematics satisfies both concerns. In accordance with the theory outlined above, this dissertation seeks to explicate the logic governing the unfolding of three civilizations, and mathematics is primarily an exhibition of logic. Mathematicians of all three eras attempt to adhere as closely as possible to a logical model, from elucidating at the outset the assumptions that form the foundation of their enterprises to laying bare the intricate latticework of their reasoning in proofs. By studying a manifestation of the episteme that is so close to logic in its pure form, this dissertation is able to more directly access the conditions of possibility for the rise of the west, circumventing the unquenchable need for thoroughgoing observations about social, political, and economic life in the earlier cases. Furthermore, the development of mathematics is not contingent on material resources or technological advances, suggesting that mathematical concepts are ostensibly accessible at all times in history. Focusing on mathematics also transforms the impossible dichotomy of an occurrence and its absence into an exploration of variations on a common theme. Additionally, though not every mathematical work from antiquity has survived, many have, and knowledge of some lost manuscripts persists in commentaries and compendiums. There is no shortage of source material with regard to ancient Greek and medieval Arabic mathematics.

This dissertation will consider three mathematical divergences that were integral to the development of calculus, namely analytic geometry, trigonometry, and the fundamental theorem of calculus. Specifically, the early modern incarnation of each will be compared to the corresponding examples from the earlier cases, paying particular attention to any variations in their underlying assumptions. Once the key conceptual differences in the three approaches have been identified, the implications of those differences will be analyzed with regard to the rise of the west. The final chapter will consist of a discussion of the findings and the broader implications of the epistemic theory.

2. Analytic Geometry

In the seventeenth century, within the span of a few months, two manuscripts began to circulate that would permanently alter the course of mathematics. One was René Descartes' *Geometry* (1637), a mathematical appendix to his *Discourse on the Method*, and the other was Pierre de Fermat's *Introduction to Plane and Solid Loci* (Mahoney 1994). Both authors, upon reading Book VII of Pappus' *Synagoge*, were inspired to develop an analytic method to solve the ancient locus problems that had stymied Euclid and Apollonius. Stranger still, both independently invented the same method, namely analytic geometry.

The principle at the heart of analytic geometry is simply that geometric curves can be expressed as algebraic equations, and conversely that algebraic equations can be rendered as geometric curves, by means of a coordinate system. This concept may seem obvious to anyone who has ever taken a high school algebra class and recalls using an x- and y-axis to find intercepts and calculate slopes, but in 1637, it was quite novel. The combination of geometry and algebra proved powerful, enabling Descartes and Fermat to solve the locus problems that piqued their interest and many others besides. Furthermore, several of the insights of analytic geometry directly contributed to the development of calculus. Descartes' and Fermat's successors, including Newton and Leibniz, made use of the coordinate framework to evaluate curves, and scripted their equations in terms of two variables. They adopted the familiar notation of letters for variables, superscripts for exponents, and symbols for common operations, such as addition, that was first laid out in Descartes' work. Perhaps most importantly, both Descartes and Fermat, utilizing the

principles of analytic geometry, developed simple methods for finding a tangent to any curve, which proved useful in the study of derivatives. In short, the development of analytic geometry marked the beginning of a new era in mathematics.

The primary innovation put forth by both Descartes and Fermat was the use of a coordinate system as a mediator between geometry and algebra, but the use of coordinates was not unknown to their predecessors. Most notably among the Greeks, Menaechmus first employed intersecting conics to solve the Delian problem of doubling a cube (in modern parlance, the coordinates of the intersection point indicate the solution) (Bulmer-Thomas 1939). Archimedes also alluded to intersecting conics in *The Sphere and Cylinder*, when he makes use of a double mean proportional in two propositions (Heath 2002). In *The Conics* (Apollonius and Heath 1896), Apollonius designated reference lines on the conic sections, similar to a coordinate axis, from which he derived descriptive relations. In the medieval era, Ibn al-Haytham used intersecting conics to solve his namesake problem, which required lines to be drawn from two points in the plane of a circle to a point on the circumference of the circle making equal angles with the normal at that point (Alhazen and Smith 2006). Similarly, Omar Khayyam applied intersecting conics to determine geometric solutions to cubic equations (Khayyam and Kasir 1931). Even in Europe prior to Descartes and Fermat, Nicole Oresme graphed variables in terms of “latitudes” and “longitudes,” and derived algebraic relations from the resulting figures (Oresme and Clagett 1968). Galileo also made use of rectangular coordinates in his demonstrations regarding uniformly accelerated motion (Galilei 1989). Why then was it not until Descartes and Fermat that analytic geometry was fully articulated?

To illuminate the conceptual stumbling blocks that hampered earlier discovery, the two examples thought to have come the closest to analytic geometry from the preceding epochs (Boyer 1991), namely Apollonius's use of reference lines in *The Conics* and Omar Khayyam's geometric solution to cubic equations in *Treatise on Demonstration of Problems of Algebra*, will be considered in comparison to Descartes' pioneering work in *Geometry*. The chapter will proceed with a brief description of the three cases, then identify key differences in their approaches to coordinate geometry, and finally discuss the implications of those differences with regard to the rise of the west.

2.1 Exemplars

The principal subject matter of Apollonius's *The Conics* is, as the name suggests, an exploration of the mathematical properties of conic sections. Conic sections arise from the intersection of a plane and a cone, and depending on the angle of intersection, take the form of a circle, ellipse, parabola, or hyperbola. Apollonius begins his treatment of the conics by establishing reference lines that are applicable to all the possible types of sections, namely the diameter and the tangent at its extremity. The diameter is the line that bisects the conic section, and the tangent at its extremity is a line that touches the curve at only one point, in this case the endpoint of the diameter. These two lines act as a coordinate frame that can facilitate description and comparison. Apollonius designates as “ordinates” the lines drawn parallel to the tangent, from the diameter to the curve, and names the distances measured on the diameter “abscissas” (See Figure 2.1). Having introduced this framework, he then proceeds to derive the unique relations of abscissas and ordinates that define each section. For instance, Apollonius discovers that in the case

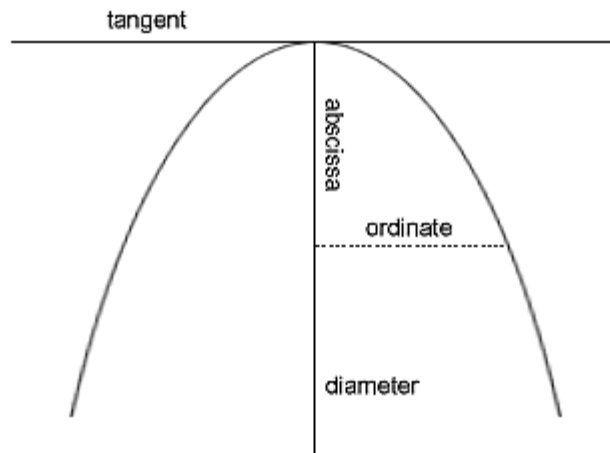


Figure 2.1: Apollonius's ordinates and abscissas

of a parabola, the square on the ordinate is equal to the rectangle formed by the abscissa and the latus rectum (a fixed length drawn perpendicular to the diameter, which has a particular relation to the axial triangle of the cone¹). By contrast, in the case of a hyperbola, the square on the ordinate is equal to the rectangle formed by the abscissa and a line greater than the latus rectum (determined by construction), and for an ellipse, the square on the ordinate is equal to the rectangle formed by the abscissa and a line shorter than the latus rectum (determined by construction). These relations, in turn, constitute the foundation on which Apollonius builds his investigation of the conics.

In his *Treatise on Demonstration of Problems of Algebra*, Omar Khayyam also verges on analytic geometry with his geometric solution to cubic equations. Khayyam apportions cubic equations into a typology based on their constituent parts, and solves each type in turn by means of intersecting conics. For instance, the first species he considers is “a cube and sides are equal to a number” (in Cartesian notation, $x^3 + bx = a$),

¹ The latus rectum is a fixed length drawn perpendicular to the diameter, that bears a particular relation to the axial triangle ABC of the cone, namely, LR (latus rectum) is to RA as the square on BC is to the rectangle with sides BA and AC.

where the problem is to find the length of a side of the cube (i.e., solve for x). Khayyam sets up the construction such that the given number of sides equals the square on the line AB ($b = AB^2$) and the given number equals a solid with a base equal to the square on AB and height equal to BC ($a = AB^2 * BC$) (See Figure 2.2). The solution (x) lies in finding

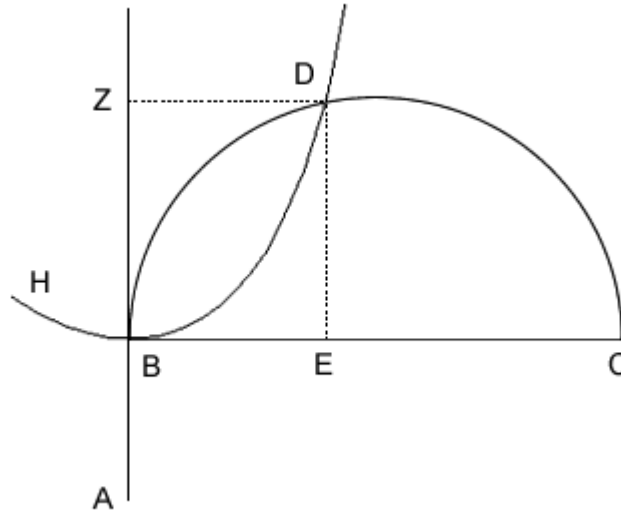


Figure 2.2: Khayyam's geometric solution to cubic equations

the double mean proportional on the given lines. The mean proportional is a magnitude whose square is equal to the rectangle constructed from two given magnitudes, or in modern notation, $l/m = m/p$ where $m^2 = l*p$. A double mean proportional involves determining two intermediary magnitudes, such that $l/m = m/n = n/p$ where $m^2 = l*n$ and $n^2 = m*p$. Khayyam makes use of intersecting conics, a parabola and circle, by exploiting their inherent mathematical qualities to find a double mean proportional.² Specifically, in

2 Why does it work? Because finding the double mean proportional is the equivalent of determining the cube root:

$$a/x = x/y = y/b$$

by cross multiplying, $x^2 = ay$ and $y^2 = xb$;

rearranging the terms of the first equation, $y = x^2/a$, and substituting for y , $y^2 = (x^2/a)^2 = x^4/a^2$;

but y^2 is also equal to xb , therefore $x^4/a^2 = xb$; and simplifying, $x^3 = b*a^2$;

therefore, solving for x is the same as finding the cube root.

a semi-circle, the square on any line drawn perpendicular to the diameter is equal to the rectangle constructed from the resulting two segments of the diameter, and likewise in a parabola, the square on an ordinate is equal to the rectangle constructed from the corresponding abscissa and the latus rectum. The intersection point of the two conics, when projected onto the given lines, indicates two new lengths (BE, BZ), which in turn satisfy the conditions of the double mean proportional. Taking the given lines (AB, BC) as axes, the coordinates of the intersection point therefore determine the solution to the equation.

Finally, Descartes uses a coordinate system to negotiate between geometry and algebra. Key to this interaction is the statement Descartes begins *Geometry* with, that any problem in geometry can be reduced to lines. He likens geometry to arithmetic and its five types of operations, namely addition, subtraction, multiplication, division, and the extraction of roots, and proceeds to demonstrate how to apply those operations to geometric lengths. Adding lines to lines or subtracting lines from lines is fairly straightforward, and the results are clearly still lines. However, in cases of multiplication, rather than generating an area, Descartes puts the two lines in question into a proportion with unity and finds a fourth line that satisfies the proportion.³ He treats division in the same way,⁴ and to attain roots, which he classifies as a type of division, Descartes simply finds the appropriate mean proportional.⁵ The significant innovation is that the resulting products and quotients are all lines. Reducing all geometric problems to lines, and

3 Essentially, $a/1 = x/b$ where $a*x = 1*b$

4 For example, $x/1 = c/b$ where $x*b = 1*c$, and therefore $x = c/b$; or $a/1 = c/x$ where $a*x = c*1$, and therefore $x = c/a$

5 Essentially, $1/x = x/n$ where $x*x = n*1$, thus x is the square root of n . Finding the double mean proportional results in a cube root, while finding the triple mean proportional returns the quartic root, and so on.

ensuring that the results of the five main operations employed to solve the problems are also lines, allows Descartes to manipulate the problems in the plane of the axes.

Descartes then introduces a new notation that reflects this simplified geometry. He represents variables with letters, initiating the convention of using letters from the end of the alphabet for unknowns, employs superscript numerals to indicate powers, and substitutes symbols for operations. By reducing geometry to lines and rendering equations wholly symbolic, Descartes lays the necessary groundwork for communication between the two, but still requires a means of translation.

In the second book of *Geometry*, Descartes illustrates how to extract an algebraic equation from a geometric curve by means of coordinate axes. Essentially, he captures the unknown relation between an ordinate and abscissa by manipulating known quantities and relations on the axes. Specifically, he begins with the curve CE, and chooses a reference line (AB) that will serve as the axis of abscissas (See Figure 2.3). CB and GA

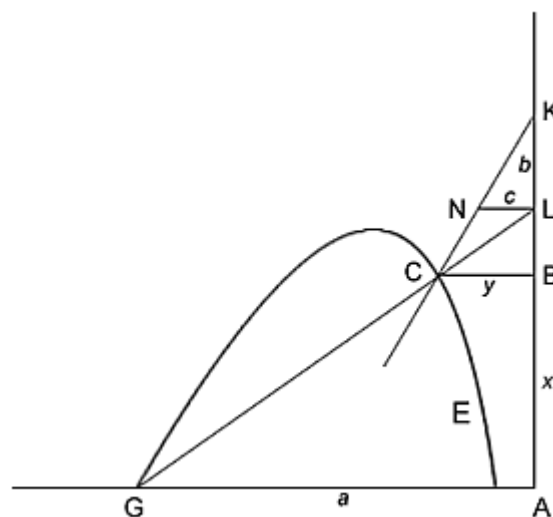


Figure 2.3: Descartes' coordinate axes

are ordinates. Descartes designates the unknown quantities AB and CB as x and y , respectively, and labels the known quantities a , b , and c . Then, appealing to similar triangles, he is able to define BL in terms of b , c and y ($BL = \frac{b}{c}y - b$), and by extension, AL (by adding x). Since CB is to LB as GA is to LA, again because of similar triangles, Descartes is able to manipulate the proportion into a more recognizable formula, namely that of a hyperbola ($y^2 = cy - \frac{cx}{b}y + ay - ac$). The equation, in turn, represents all possible points on the curve. Having established equivalence between geometric figures and algebraic equations, Descartes solves the ancient locus problem that originally provoked this analytic endeavor. He starts with the geometric given and selects principle lines, one known (x) and one unknown (y), that will serve as axes. Then, he rewrites the problem as an equation in terms of x and y , using the method described above. After solving for y by manipulating the equation until y is isolated on one side of the equal sign, he is able to discern the identity of the locus from the expression y is equal to, and furthermore derive the measures necessary to render the figure geometrically.

In summary, Apollonius designates as reference lines the diameter of a conic section and its tangent, and describes the measurements garnered from them as abscissas and ordinates, respectively. He then derives a typology of curves based on the relations he observes between the abscissas and ordinates. Khayyam appeals to a geometric construction that includes intersecting conics to find solutions to cubic equations. The intersection point projected onto two lines specified by the given yields the solution, or alternately, the coordinates of the intersection point solve the equation. Descartes reduces geometric problems to lines, and streamlines algebraic notation accordingly. He then demonstrates the interchangeability of geometric figures and algebraic equations by

means of coordinate axes, opening the door to solving geometric problems through the manipulation of equations.

2.2 Epistemological Obstacles

What, then, differentiates Descartes' use of coordinate axes from his predecessors? One difference that is readily apparent lies in the generation of the axes. In Apollonius's and Khayyam's systems, the coordinate axes derive from the geometric figures they are intended to gauge, whereas in Descartes' scheme, the axes exist independent of the curve. Apollonius selects as his axes the diameter of the section and the tangent drawn to the curve at the diameter's endpoint, both of which are clearly consequents of the conic section itself. Likewise, Khayyam's axes are composed of the sides of the solids he is studying. Additionally, in both cases the axes are necessarily at right angles to each other. By contrast, Descartes arbitrarily chooses lines to serve as his axes, and furthermore, specifies nothing as to their angle of intersection. For Descartes, any straight line will work as an axis as well as any other.

Descartes' analytic geometry also begins to blur the sharp distinction between magnitudes and multitudes evident in the earlier cases. Magnitude and multitude are the two most basic forms of quantity. Magnitudes are continuous and infinitely divisible, as for example lines, which when divided result in yet more lines. On the other hand, multitudes are collections of discrete entities that can be divided only to a certain extent, as for instance a dozen eggs, divisible into at most twelve units without fundamentally altering the character of the egg. Geometric objects such as lines, planes, and solids are magnitudes, whereas number is a type of multitude. In *The Conics*, Apollonius deals

solely with magnitudes, and does not try to introduce numbers to the conic sections. The Greeks in general were disinclined to combine numbers and magnitudes, perhaps most obvious in Euclid's separate treatments of the theory of proportions for magnitudes in book V and for numbers in book VII of *The Elements*, due to the impossibility of formulating a ratio in terms of rational numbers that describes the relation between the side of a square and its diagonal. Likewise Khayyam, though he provides both geometrical and numerical solutions for quadratic equations (derived from separate geometric and arithmetic demonstrations), finds only geometrical solutions for cubic equations, believing numerical answers impossible. In other words, he is able to ascertain the length that fulfills the conditions of the proposition, but has no means of measuring it. By contrast, Descartes' framework opens new avenues of communication between magnitudes and multitudes. In Descartes' reconsideration of arithmetical operations, he explicitly places lines in a ratio with unity, or in other words, he defines the magnitudes in terms of units. Also, geometric problems pared down to lines correspond more readily to numbers, which are likewise free from dimensionality.

Another divergence is manifest in their understandings of the terms in an equation. When Apollonius or Khayyam reference a squared term (x^2), they mean a literal square, and likewise, a cubed term (x^3) actually denotes a cube. A line multiplied by a line necessarily gives rise to an area, and an area multiplied by a line produces a volume. Descartes, on the other hand, reduces mathematical operations to ratios, and so a line multiplied by a line simply results in another line. This distinction is evident in the ways Apollonius defines conic identity relations exclusively by means of squares and rectangles, Khayyam redefines all the terms in his cubic equations as solids, and

Descartes includes a squared term in the equation of a line. These varying interpretations have consequences for the development of analytic geometry on several fronts.

First, a literal understanding of the equations complicates the notation of ancient Greek and medieval Arabic algebra. Descartes ushered in an era of symbolic algebra. He introduced superscript numerals to indicate exponents, substituted letters for variables, and employed symbols for common operations. His reduction of all problems to lines is most apparent in his rendering of a line, square, and cube as x , x^2 , and x^3 , respectively. In contrast, the ancient Greek mathematicians adopted a form of syncopated, or partially symbolic, algebra. They used symbols to represent unknowns, and abbreviations for some operations, but lacked special symbols for operations and exponentiation. Addition was implicit, and each power of x was represented by a different symbol, making the variables much harder to manipulate than in Descartes' system. Alternately, rhetorical algebra was the preferred mode of medieval Arabic mathematicians. Equations were expressed as sentences, and their constituent parts, including unknowns and operations, were written out as words. Apart from the difficulty inherent in solving equations composed entirely of words, medieval Arabic mathematicians were also hampered by their strict adherence to a geometric interpretation of the terms. They used words such as side, root, rectangle, square, solid figure, and cube to suggest variation in powers of x , which in turn limited their ability to simplify equations. These notational hurdles made it difficult for earlier mathematicians to obtain algebraic solutions, increasing their reliance on geometry to solve problems.

The geometric approach also presents some stumbling blocks. Mathematicians are first and foremost limited to solutions that can be arrived at by geometric

construction. Some problems simply will not yield to geometry, as for instance, the locus problems recorded by Pappus, which went unsolved for centuries. Other problems require multiple constructions in order to completely solve them, as opposed to a universal formula. In *The Conics*, Apollonius has to demonstrate every theorem three times, once for each conic section. Khayyam's *Treatise on Demonstration of Problems of Algebra* in fact tackles nineteen separate species of cubic equations, each involving some permutation of the unknown, its square, its cube, and a constant. Descartes' analytic method, on the other hand, applied to a four line locus problem succeeds in solving all four line locus problems, and furthermore, the method is the same for each successive type of locus problem. Additionally, insisting on the dimensionality of products necessarily restricts the scope of mathematics to dimensions that are conceptually meaningful. As Descartes points out, in the ancient Greek formulation of the locus problem, figures beyond three dimensions are disregarded because they are not comprehensible. Khayyam similarly limits his inquiry to an investigation of “measurable quantities,” or in other words lines, surfaces, and solids, and deems figures greater than three dimensions unreal. In contrast, Descartes carries the logic of his solution for solid loci forward indefinitely, navigating higher dimensions with the same ease. Finally, a literal interpretation heightens concerns about commensurability, which in turn complicates the demonstrations. This is evident in the way Apollonius speaks of areas in relation only to other areas, and also in Khayyam's rendering of all the terms of a cubic equation as solids. Descartes is freed from this constraint by treating all problems exclusively in terms of lines.

2.3 Rise of the West

Having identified several conceptual differences that stimulate and impede the development of analytic geometry, it remains to consider them with regard to the rise of the west.

There is a common theme of efficiency and standardization that echoes in both analytic geometry and industrialization. In the former, this motif is apparent in the simplification of geometry to lines, the streamlined notation, and the transformation of problems into equations in terms of two variables. In industrialization it takes the form of the division of labor and mass production. The key to understanding both lies in the conceptualization of parts and wholes.

The division of labor denotes a cooperative production scheme wherein the process for manufacturing goods is divided into a set of tasks and each worker is assigned to one of those tasks exclusively. The simplicity of the task, relative to the entire production process, combined with its repetition enable workers to perform their tasks with greater skill and productivity than would otherwise be achieved by each worker undertaking the whole process. In addition to greater productivity, there is also a tendency toward greater uniformity in the finished product due to increases in skill on the part of the workers and the cooperative nature of the production. The efficiency and standardization that result from the division of labor therefore derive from focusing on intermediary steps rather than the total process, or in other words, an emphasis on parts over wholes.

Compare that to innovations in analytic geometry. Descartes reduces geometry to lines, achieving a uniformity of components from which to build his mathematical

system. He, in fact, manages this transformation by interpreting products and quotients as constituents of a proportional expression. Descartes then converts those lines and their relations into equations in terms of two variables by means of coordinate axes.

Essentially, he again achieves a transformation by considering the mathematical elements as parts, in this case relative to the two variables. Further, by using superscripts to indicate exponents (x^2), he grants prominence to the base unit (x), in effect representing figures in terms of their parts. Finally, he solves the equations by manipulating their components with a simple set of arithmetic operations until one variable is isolated. The end result is a standardized format and method for solving. As with the division of labor, efficiency and standardization in analytic geometry arise from stressing parts over wholes.

This conceptual similarity is interesting, but more striking is the difference observed in the earlier cases. The cumbersome notation, the clutter that results from insisting on dimensional homogeneity, and the twin sets of proofs required for magnitudes and numbers, for example, all drag down the efficiency of earlier mathematics. Moreover, there exists no standard method for solving problems as in analytic geometry. Some solutions, as for instance Khayyam's treatment of cubic equations, further subdivide the problem into multiple species, which then each require a unique proof. Perhaps not surprisingly, the earlier cases favor wholes over parts. It is evident in their notation, where the representations of variables raised to a power take on the character of the resulting figure, ignoring the base component. Likewise, fragmented solutions arise as a consequence of perceiving the species equations as wholes, rather than instances of a more general case. Also, their circumscription of the mathematical

domain at three dimensions suggests an estimation of the whole of what is knowable, where for Descartes it was just a part.

There is a recognizable consonance in the logic of the division of labor and analytic geometry that stands in contrast to the conceptual underpinnings evident in earlier mathematical writings. Specifically, in the former two, parts were emphasized over wholes, whereas in the latter, wholes were given precedence, hinting at a heretofore unknown epistemological obstacle to industrialization.

Another parallel exists between analytic geometry and capitalism. In the same way that analytic geometry freed mathematics from geometric figures, so capitalism liberated economic theory from a rigid zero-sum accounting of bullion reserves. Both introduce more abstract entities, which can nonetheless be quantified, into what had been fairly intransitive formulations.

Mercantilism, the precursor to capitalism in Europe, held that economics is a zero-sum game, wherein a profit by one party necessitates a loss by another. Wealth was synonymous with bullion, and as such, economic policy was directed at amassing precious metals by maximizing exports and limiting imports. The emergence of capitalism ushered in a definition of wealth that went beyond bullion to include less tangible sources of capital. Furthermore, this new assessment of value meant that both parties can gain from exchange. Mutual benefit is achieved because it is cheaper for both to specialize in areas where they enjoy a comparative advantage and then trade to satisfy deficiencies than for both to independently produce all types of goods. Capitalism, therefore, shifts the conception of wealth away from bullion and towards a more abstract measure of value, which in turn has consequences for economic policy.

Similarly, in the development of analytic geometry, one of the key departures from existing mathematics was disassociating problem solving from geometric construction. In Descartes' work, this shift began with the construal of products and quotients as proportions instead of figures. Once the problems could be conceived of entirely in terms of lines, he then redefined them, relative to two reference lines, as equations. The equations, in turn, could be reconfigured by means of arithmetic operations, wholly without reference to the original figures, in order to isolate one of the variables and arrive at a solution. Moving letters around on a page thus supplants geometric construction as a means of solving problems. A more abstract interpretation also reorients the aims of mathematics, broadening its scope and subject matter. Analytic geometry, therefore, transforms the interpretation of mathematical problems as geometric figures into a more abstract understanding of equations, and in turn, reshapes the discipline.

In contrast, a more literal grasp of mathematics prevails in the earlier cases. A line multiplied by a line generates an area, and a quantity squared produces an actual square. This dimensionality is reflected in the notation, making the arithmetic manipulation of equations seen in analytic geometry nearly impossible. Concrete geometric construction is thus the principal method of problem solving, which in turn reinforces the literal understanding. Furthermore, the earlier cases shy away from the abstract in the sense that they restrict their inquiries to three dimensions. Everything beyond is considered to be incomprehensible, and necessarily exempt from investigation, because it cannot be represented geometrically. Just as wealth was tied to bullion under mercantilism, mathematics in the earlier cases was meaningless absent geometric figures.

In both capitalism and analytic geometry, an existing concept is redefined in more abstract terms, giving rise to new priorities and policies. Conversely, in the earlier cases a more tangible interpretation was preferred, which led to a more limited agenda, both economic and mathematical. Once again, a conceptual agreement is observed in the two early modern artifacts, in direct contrast to the perspective evident in the earlier cases.

A common thread also underlies the logic of classical mechanics and analytic geometry. In both, there is an effort to integrate previously disparate elements under a single system of rules by focusing on their extrinsic, rather than intrinsic, properties.

One of the central tenets of classical mechanics is the notion that the same physical laws apply to all bodies. Through his experiments with balls on an inclined plane, Galileo (1989) demonstrated that gravitation accelerates all objects at the same rate, contradicting the Aristotelian tradition which held that heavier objects have a greater downward tendency, and thus fall faster. Kepler (1992) explained all planetary motion by positing elliptical orbits and deducing that the areas swept by the planets relative to the focal sun in equal periods of time are equal, supplanting an increasingly complex system of circular motion and nested epicycles. Building on these efforts, Newton (1999) supplies the core of classical mechanics with his three laws of motion and inverse-square law of universal gravitation, and then proves that these laws govern both terrestrial and celestial objects. Classical mechanics represents a change in focus from the nature of objects to the forces acting on them, or in other words, from the intrinsic properties to the extrinsic.

A similar theme is evident in analytic geometry. Descartes' blurring of magnitude and multitude, as well as his reduction of geometry to lines, shift emphasis from what an

entity is to how it can be manipulated. Furthermore, he establishes a method for solving problems that draws on a few simple operations and is applicable to all types of equations. However, the clearest example is Descartes' use of axis lines. The lines are chosen arbitrarily, as one is as good as any other, and then the curves are defined in relation to them. Not only are equations derived from an external source, but inherent in the selection of any line is the power to overlay multiple curves on the same framework. In all of these instances, the meaning of the thing itself is superseded by its place in a grander scheme, or put another way, its extrinsic qualities.

Conversely, intrinsic properties are stressed in the earlier cases. The distinction between magnitudes and multitudes reflects an emphasis on their natures rather than their relations, and necessitates separate proofs for the same propositions. Multiple proofs are also required for the conic sections and all the species of equations because each is treated according to its identity, and not in light of a shared relation to an external entity. Similarly, the reference lines employed in the earlier cases are extracted from the figures under consideration, and therefore the relations obtained from them are essentially intrinsic. In other words, the character of a mathematical object takes priority over its context in the earlier cases.

Both analytic geometry and classical mechanics seek to unify their fields into a comprehensive system of rules based on the extrinsic properties of their objects. By focusing their inquiries on the relations between, rather than the fundamental nature of, entities they are able to formulate universal statements. The earlier cases, on the other hand, concentrate on the intrinsic qualities, which impedes their ability to generalize and contributes to a fragmented understanding.

To review, several parallels are revealed when the rise of the west is considered in comparison to analytic geometry. The efficiency and standardization that characterize the division of labor is also evident in analytic geometry, and in both instances can be interpreted as emphasis on parts over wholes. Capitalism brings about a redefinition of existing concepts in more abstract terms in much the same way as analytic geometry. Classical mechanics and analytic geometry both espouse universal rules, which derive from an attention to extrinsic rather than intrinsic properties. Furthermore, the conceptual similarities indicated—perceiving things as particular instances of a more general case, thinking in more abstract terms, and emphasizing relational instead of innate qualities—admit of a certain compatibility.

A pattern is beginning to emerge, wherein the logical contours of the rise of the west match those of analytic geometry, but not the earlier mathematics. Just as there are glimmers of the rise of the west in the earlier civilizations, there are sparks of something like analytic geometry in Greek and Arabic mathematics, as for example the use of coordinate axes. However, the conceptual frame necessary to support a fully articulated analytic geometry is simply not there, and moreover, the logic that is in place is at points not merely out of alignment, but in fact precludes such developments. Here, then, is evidence of an epistemic influence, and in the parallels observed between the rise of the west and analytic geometry, the first peek at its countenance. The next chapter will continue to elaborate on these findings through an examination of trigonometry.

3. Trigonometry

With the publication of *On the Revolutions of the Heavenly Spheres* in 1543, Copernicus upset the geocentric model of the solar system that had presided for centuries, and sparked a scientific revolution. While it is best remembered for its astronomical content, the book is also of significance to mathematics because of Copernicus' sophisticated use of trigonometry to bolster his claims. However, this pivotal manuscript would likely have remained unpublished if not for the persistent prodding of Copernicus' only student, Georg Rheticus. In addition to convincing a reluctant Copernicus to publish, Rheticus also revolutionized trigonometry, reconfiguring the science of triangles into the familiar system still in use today. His name is perhaps not as well known as some of the other mathematical innovators, due in large part to a papal ban on his writings, but his ideas nevertheless reverberated throughout early modern Europe, and proved integral to the development of calculus.

Trigonometry is principally concerned with solving triangles, which entails determining the values of all three sides and all three angles. As such, the relationships between the sides and angles of a triangle, expressed as trigonometric functions (e.g., sine and cosine), provide a powerful tool for computing the unknown elements from incomplete data. Previous incarnations of trigonometry rendered the triangle as a sector of a circle, inscribing the third side as a chord, and focused on the relations between the chords and arcs. Rheticus's innovation was to define trigonometric functions as ratios of the sides of right triangles, such that the sine of an angle equals the length of the side opposite it divided by the hypotenuse, the cosine equals the length of the adjacent side

divided by the hypotenuse, the tangent equals the opposite over the adjacent, and so on. Reconceptualizing trigonometric functions, and in particular the tangent, as ratios eases their translation onto the coordinate plane, which in turn, simplifies their manipulation. Furthermore, the idea of the tangent as a ratio figured heavily into both Newton and Leibniz's development of calculus, and specifically their discovery of the inverse relation between differentials and integrals.

Determining the measures of a triangle's sides and angles has long occupied mathematicians. Among the Greeks, Hipparchus, who bears the honorific “father of trigonometry,” generated the first trigonometric table of chord values and their corresponding arcs. Menelaus of Alexandria conducted the earliest known investigations into spherical trigonometry, and devised a theorem that relates the arcs of great circles by means of the chord function (Heath 1981). However, the most significant work on trigonometry in antiquity is contained in Ptolemy's *Almagest* (Boyer 1991). Not only did Ptolemy elucidate trigonometric formulas for computing chord values in half degree increments and furnish tables of his results, but the *Almagest* also preserves the otherwise lost work of his predecessors. In the medieval period, al-Battani produced an astronomical handbook based on Ptolemy's theory which substituted trigonometric methods for Ptolemy's geometric procedures, and also was the first to adopt the use of sines (half-chords) rather than chords (Bond 1921). Abu al-Wafa worked with all six trigonometric functions, refined angle addition identities, and derived the law of sines for spherical trigonometry (Boyer 1991). In his *Treatise on the Quadrilateral*, Nasir al-Din al-Tusi made the case for trigonometry as a discipline in its own right, rather than as an adjunct of geometry and astronomy. Though he explored both plane and spherical

trigonometry, he is most famous for his derivation of the law of sines from plane triangles (Berggren 2003; Bond 1921). Of the early modern mathematicians, Regiomontanus published a systematic account of methods for solving triangles, *De triangulis omnimodis*, drawing heavily on Greek and Arabic sources (Regiomontanus 1967). Copernicus devoted several chapters of *On the Revolutions of the Heavenly Spheres* to explaining the trigonometry necessary for understanding his argument for the heliocentric universe (Copernicus 1995). Rheticus, Copernicus's only student, was the first to define trigonometric functions as ratios of the sides of right triangles. His major work, *Opus Palatinum*, was completed in 1596 by his student Otho, and later revised by Pitiscus, who in fact coined the term “trigonometry” in his own work on the subject (Van Brummelen 2009).

To illuminate the conceptual stumbling blocks that hampered earlier discovery, two classic examples of trigonometry from the earlier periods, namely Ptolemy's derivation of the difference formula for sines in *Almagest* and Nasir al-Din al-Tusi's proof of the Law of Sines from his *Treatise on the Quadrilateral*, will be considered in comparison to a demonstration from Rheticus's *Opus Palatinum*. This chapter will briefly describe the three cases, then indicate the key differences in their approaches to trigonometry, and conclude with a discussion of the implications of those differences with respect to rise of the west.

3.1 Exemplars

Ptolemy's *Almagest* (Ptolemy 1998; Heath 1981) contains a method, thought to have originated with Hipparchus, for solving triangles by likening their sides to the

chords of a circle. Key to this method is a demonstration of the relation between the sides and diagonals of a cyclic quadrilateral, commonly known as Ptolemy's Theorem. He begins with quadrilateral ABCD inscribed in a circle, and wishes to show that the sum of the rectangles formed by its two pairs of opposite sides is equal to the rectangle formed by the diagonals, or in other words, that $AC \cdot BD = AB \cdot CD + BC \cdot AD$ (See Figure 3.1).

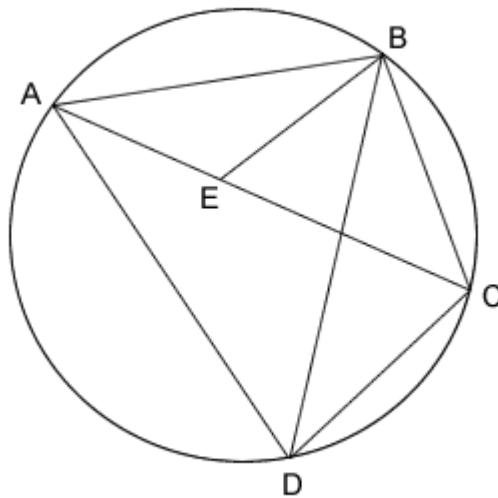


Figure 3.1: Ptolemy's Theorem

Ptolemy constructs BE such that angle ABE is equal to angle DBC, and then adds angle EBD to both, establishing that angle ABD equals angle EBC. However, angle BDA is equal to angle BCE because they subtend the same arc, and the equiangular triangles ABD and EBC are therefore similar. Thus, BC is to CE as BD is to AD, and the rectangle formed by the extremes is equal to the rectangle formed by the means, or in other words, $BC \cdot AD = CE \cdot BD$. Likewise, since triangles BAE and BDC are similar, $BA \cdot DC = AE \cdot BD$. Therefore, $AE \cdot BD + CE \cdot BD = BA \cdot DC + BC \cdot AD$. But $CE \cdot BD + AE \cdot BD$ is equal to the rectangle formed by AC and BD, the diagonals, and $BA \cdot DC$ and $BC \cdot AD$ are

the rectangles formed by the opposite sides of the quadrilateral. Having proved that the rectangle formed by the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the rectangles formed by its opposite sides, Ptolemy employs the theorem to demonstrate what is essentially the difference formula for sines. Assuming AD is the diameter of the circle with center O, the equation $AC \cdot BD = BA \cdot DC + BC \cdot AD$ can be rewritten in terms of the chord functions of angles¹ about the center: $\text{crd } AOC \cdot \text{crd } BOD = \text{crd } BOA \cdot \text{crd } DOC + \text{crd } BOC \cdot \text{diameter } AD$ (See Figure 3.2). Substituting a for angle

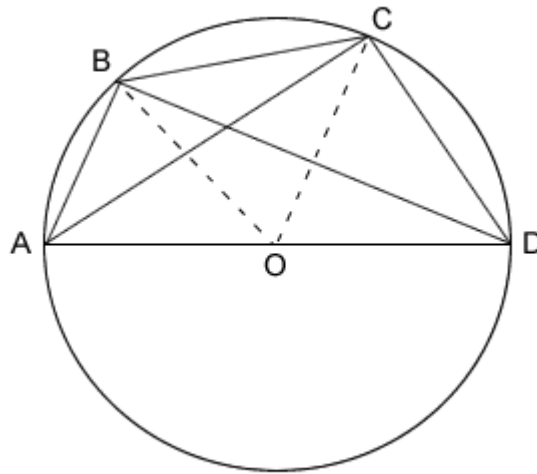


Figure 3.2: Ptolemy's difference formula for sines

AOB and b for angle AOC, the equation can be rendered as $\text{crd } b \cdot \text{crd } (180 - a) = \text{crd } a \cdot \text{crd } (180 - b) + \text{diameter} \cdot \text{crd } (b - a)$, which rearranged as $\text{crd } (b - a) = \text{crd } b \cdot \text{crd } (180 - a) - \text{crd } a \cdot \text{crd } (180 - b)$, where the diameter is equal to one, bears a remarkable similarity to the difference formula for sines, $\sin (b - a) = \sin a \cdot \cos b - \cos a \cdot \sin b$. Similarly, with

¹ It is important to note that neither the ancient Greek nor medieval Arabic mathematicians conceived of the trigonometric functions as functions of angles about the center of a circle, but rather as functions of arc lengths measured on a circle with a circumference of 360 degrees. Since both angles and arcs are measured on the same sexagesimal scale, their values are synchronized (i.e., a 30 degree angle corresponds to a 30 degree arc). For the sake of clarity, the modern convention of specifying in terms of angles is employed in these examples.

a different interpretation of a and b the sum formula for sines results. These formulas, in turn, enable Ptolemy to compute the table of chords that comprises foundation of his subsequent astronomical calculations.

In his *Treatise on the Quadrilateral* (al-Tusi 1891; Berggren 2003; Bond 1921), Nasir al-Din al-Tusi also devises methods for solving triangles, including his derivation of the law of sines, which states that for any triangle ABC , the sines of any two angles are in the same ratio with each other as the sides opposite them.² Al-Tusi considers two cases, depending on whether triangle ABC is acute or obtuse; the procedure is identical, but the figures are arrayed slightly differently. Assuming ABC is acute, al-Tusi begins by prolonging CB to E and BC to H , so that CE and BH are each sixty units long. Then, taking CE as a radius with center C , he describes an arc that meets CA produced to D , and likewise taking BH as a radius with center B , he describes an arcs that meets BA produced to T . From point D he drops DF perpendicular to CE , from T , TK perpendicular to BH , and finally from point A , AL perpendicular to BC (See Figure 3.3). Since

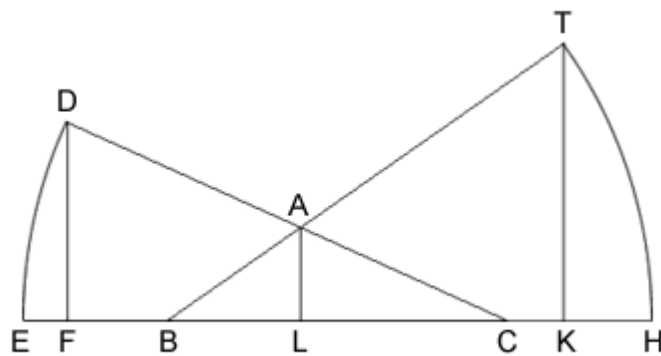


Figure 3.3: Nasir al-Din al-Tusi's derivation of the law of sines

² In modern form, for any triangle ABC with sides a , b , and c and the angles opposite them A , B , and C :

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

triangles ACL and DCF are similar, AL is to AC as DF is to DC, and likewise because triangles ABL and TBK are similar, AB is to AL as TB is to TK. However, DC and TB are radii, and therefore equal to sixty units. Thus, multiplying both sides of the two proportions, AB is to AC as DF is to TK. But AB and AC are the sides opposite angles C and B, respectively, and DF and TK are equal to the sine of C and sine of B, respectively. In other words, the sines of the two angles are in the same ratio with each other as the sides opposite them. Using this theorem, al-Tusi can solve all possible triangles where two angles and one side are known, where one angle and two sides are known, or where all three sides are known.

In contrast, though circles appear in Rheticus's initial descriptions of trigonometric functions, they are absent from his methods of solving triangles. In his *Opus Palatinum* (Van Brummelen 2009), Rheticus identifies three species of right triangles, and defines six trigonometric functions in terms of the sides of those triangles depending on the location of the side designated as the unit measure relative to the angle being considered. Specifically, if the unit side is the hypotenuse, then the side opposite the angle is the sine and the adjacent side is the cosine. If the unit side comprises the side adjacent to the angle, then the opposite side is the tangent and the hypotenuse is the secant. Finally, if the unit side is opposite the angle, then the adjacent side is the cotangent and the hypotenuse is the cosecant.³ Rheticus then utilizes these definitions to manipulate incomplete data in order to solve triangles. Consider, for example, Rheticus's

³ In other words, sine = opposite/hypotenuse, cosine = adjacent/hypotenuse, tangent = opposite/adjacent, secant = hypotenuse/adjacent, cotangent = adjacent/opposite, and cosecant = hypotenuse/opposite, where the denominator in each case is the radius, which is set equal to one.

treatment of obtuse triangle ABC where sides AB, BC, and angle B are known (See Figure 3.4). From the obtuse angle A, he drops line AD perpendicular to BC, creating

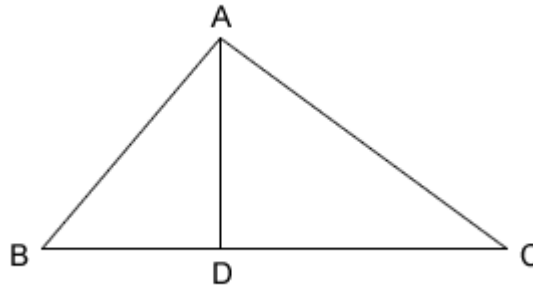


Figure 3.4: Rheticus' method for solving triangles

two right triangles, ABD and ADC. Since angle B is given and angle BDA is right, the remaining angle DAB is also known. Rheticus next ascertains the ratio of the sides in triangle ABD by setting one side as the unit measure and employing the appropriate trigonometric functions. For instance, if hypotenuse AB is designated as the unit, then calculating the sine and cosine of angle B will reveal the ratio of the three sides. Further, since the length of AB was given, Rheticus can then determine the lengths of BD and AD based on that ratio. However, BC is equal to the sum of BD and DC, and since BC was given and BD has been determined, the length of DC is also known, meaning two sides of triangle ADC are known (DC, AD). Taking AD as the unit measure, Rheticus can deduce the cotangent of angle C from the ratio of lengths DC and AD, and by extension the value of angle C and the ratio of side AC to AD and DC. Like before, since the length of AD is known, he can compute the value of AC based on the ratio, and because two angles (B and C) of the original triangle ABC are known, the third angle BAC is also known.

Rheticus has thus solved the triangle without reference to circles by appealing to the internal dynamics of right triangles.

In summary, Ptolemy solves triangles by equating their sides with the chords of a circle and exploiting the geometric properties inherent in such a configuration. Once he extricates the relationship between the sides and diagonals of a cyclic quadrilateral, he deduces from it a set of formulas that enable him to calculate the values of unknown chords and arcs, which by design correspond to the sides and angles of triangles. Al-Tusi also uses circles to establish a relation between the sides and angles of triangles for the purpose of solving them. He devises a general rule that states the sides of a triangle are in the same proportion as the sines of their corresponding angles, meaning that if any three elements of the triangle are known he can determine the remaining unknown values. Rheticus reconceptualizes trigonometric functions as the ratios of sides of right triangles, and eschews the use of circles and chords. He solves triangles by dividing them into right triangles, and then leveraging his definitions of the trigonometric functions to calculate the unknowns.

3.2 Epistemological Obstacles

What, then, distinguishes Rheticus from his predecessors? The most immediate difference is the absence of circles in Rheticus's trigonometry. In the earlier cases, circles fulfill a number of interrelated functions. The chord and half-chord lengths, the major trigonometric identities employed by Ptolemy and al-Tusi respectively, are geometrically determined by circles. Similarly, trigonometric functions in the earlier cases are defined in terms of arc lengths. Furthermore, both use circles to facilitate comparisons between

triangles; Ptolemy confines all of the triangles of interest to a single circle while al-Tusi traces two sectors with equal radii in order to ensure commensurability. Finally, their proofs rely heavily on the geometric properties of circles, and in particular the manipulation of their chords, to establish the relations between a triangle's angles and sides. In Rheticus's work, on the other hand, circles are superseded by right triangles. Specifically, he defines trigonometric functions as the ratio of a right triangle's sides, and in terms of the angular measures of their inclination; because the triangles are right, certain information about the angles and sides (e.g., the Pythagorean theorem) is known, rendering them comparable; and finally, his proofs are almost entirely dependent on the geometric properties of triangles.

Interestingly, right triangles also appear in the earlier cases, though the circles take precedence. In Ptolemy's demonstration involving the quadrilateral inscribed in a semicircle, triangles ABD and ACD are right, because any angle inscribed in a semicircle is right. Ptolemy utilizes this fact to prove the Pythagorean trigonometric identity, $\sin^2 x + \cos^2 x = 1$. Likewise, to construct the sine of a given angle, al-Tusi traces an arc and drops a perpendicular, resulting in right triangles CFD and BKT. Although the sines could be determined by simply constructing two right triangles with equal hypotenuses, al-Tusi insists on applying circles, in accordance with the definition of sines as half-chords. In both cases, a relation between right triangles and trigonometric functions is evident, yet the earlier mathematicians instead choose to employ circles.

The other major difference exhibited by Rheticus is his conception of trigonometric functions as ratios. Rheticus defines the six major trigonometric functions as ratios of the sides of any right triangle, such that the sine is equal to the opposite side

divided by the hypotenuse, the cosine is equal to the adjacent side divided by the hypotenuse, and so on. To simplify his calculations, he would sometimes specify as a unit measure the side of the triangle that constituted the second quantity in the ratio (equivalent to setting the denominator of a fraction equal to one), seemingly rendering the trigonometric function equal to a single side of the triangle, however the ratio was always central to his understanding. In contrast, while the earlier mathematicians place trigonometric functions into ratios with other quantities, they do not apprehend the functions as ratios. Instead, they conceive of trigonometric functions as fixed lengths, which take the form of chords or half-chords. As the size of a circle varies, so too do the chord lengths associated with any given angle, and therefore the chord functions are only comparable when derived from the same circle or a circle of an identical size. While this would appear to hint at a potential ratio between the chord and diameter, the ancient and medieval mathematicians understood a diameter of 120 parts to be the standard, and concentrated solely on the lengths of chords.

These conceptual differences, in turn, have consequences for the development of trigonometry. For example, treating trigonometric functions as ratios greatly simplifies the computation of tangents. To determine the length of a tangent using the earlier methods, first a line must be drawn tangent to the circle at the point where the sine meets it. The length in question is the portion of the tangent that falls between that point and the extended cosine line. Since the radius of the circle is perpendicular to the tangent, the triangle formed by the radius, the tangent, and the extended cosine is right. Next, a semicircle must be traced around the new right triangle using the extended cosine line as its base (scaled to 120 units), rendering the tangent a chord (See Figure 3.5). From there,

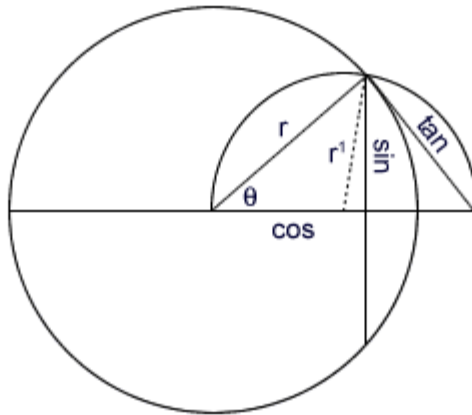


Figure 3.5: Earlier mathematicians' method of calculating tangents

the length can be found by doubling the angle opposite the tangent, because angles at the circumference of the circle are half the size of angles at the center when they subtend the same arc, and referring to a table of chords. The length of the other chord, which was a radius of the circle in the original figure, can be ascertained in the same manner. Then, the ratio of the tangent to the original radius can be calculated and, finally, the length of the tangent relative to the original figure determined.

In contrast to this rather lengthy process, Rheticus defines the tangent of an angle as the ratio of the side opposite to the side adjacent, and obtains the measure of a tangent by means of simple division. It is interesting to note that this interpretation of the tangent, as opposite over adjacent, is equivalent to the ratio of the sine to the cosine, which in turn is reminiscent of a concept central to both analytic geometry and calculus, namely the slope of a function. The understanding of the tangent as equal to sine divided by cosine is in many ways analogous to the definition, upheld by analytic geometry, of a slope as rise over run. Similarly, the idea of the tangent as a ratio of two variables resonates with the perception, evident in calculus, of the tangent as representative of the

rate of change of a function. With its redefinition as a ratio, the tangent is beginning to assume a more dynamic character, unique to early modern mathematics.

3.3 Rise of the West

In many respects, trigonometry is intertwined with the rise of the west. For instance, it underlies Copernicus's astronomical calculations, early modern navigational techniques, and the mechanical engineering driving industrial technology. However, in addition to these instrumental ties, the conceptual differences that promote the development of trigonometry are also mirrored in the rise of the west.

In both trigonometry and early modern exploration unexpected advantages result from an inversion of the traditional order of operations. Specifically, early modern shipbuilders constructed their ships from the inside out, letting the skeleton rather than the shell determine the shape of the ship, and likewise, trigonometry employs right triangles within rather than circles without to determine the measures of sides and angles. In both cases, the reversal takes the form of framing from within as opposed to circumscribing.

The clearest example of early modern explorers inverting the customary process is, of course, their decision to sail west to go east. However, the pattern extends even to the construction of their sailing vessels. Advances in shipbuilding were critical to the success experienced by early modern European explorers. In particular, the speed and agility of the caravel, a ship developed by the Portuguese in the fifteenth century, enabled explorers to sail against the wind (and thereby sail beyond Cape Bojador), navigate the rocky coast of Africa, and weather transatlantic voyages. Of Columbus's three ships, the

Nina and the Pinta were both caravels, while the Santa Maria was a carrack, another type of Portuguese ship developed in the fifteenth century. One of the major differences that sets both caravels and carracks apart from earlier ship designs lies in their method of construction. On earlier ships, the shell of the hull was built first, and the skeleton added later. The Portuguese reversed the order of construction and built the frame first, which in turn allowed them to build larger ships with sleeker designs. In other words, because the method of fabrication was inverted, these new ships were determined by their internal, rather than their outer, structures.

Similarly, Rheticus's conceptualization of trigonometry shifts focus from circles ringing the triangle to its interior. His method for solving a triangle consists of dividing it into right triangles and then manipulating the known relations of a right triangle, expressed as trigonometric functions, to determine the unknown measures of the original triangle. Unlike earlier mathematicians who sought the measures of the sides and angles in the external scaffolding of lines and circles they built up around the triangle, Rheticus inverts the framework and traces the structure inside the triangle. He not only reinterprets the trigonometric functions as ratios of the sides of right triangles, but also measures the inclination of one side to another with angles in the triangle, thus eliminating the need for circles. In Rheticus's scheme, all the measures of a given triangle derive from its chassis of right triangles and internal angle measures, or put another way, it is these inner structures that give meaning to the outer cortex.

In contrast, the ancient Greek and medieval Arabic mathematicians solve triangles using a framework built around the outside of the figure. Because they define trigonometric functions in relation to circles, their principal means of solving a triangle

involves situating it inside a circle, and extrapolating values based on the geometric properties of circles. For instance, the proof of Ptolemy's theorem rests on the fact that angles subtending the same arc in a circle are equal to each other. Similarly, Ptolemy's sum and difference formulas for sines recast the relationship between the sides and diagonals of a cyclic quadrilateral in terms of arc lengths measured on a circle. Likewise, in his proof of the law of sines, al-Tusi determines the lengths of the sines by tracing circles with equal radii around the angles of the triangle, and later relies on those equal radii to manipulate the proportions resulting from similar triangles. In both cases, the properties of triangles emerge only in conjunction with an outer structure of circles.

Both trigonometry and early modern shipbuilding focus on internal structure to determine the dimensions of their respective objects, which represents an inversion of the customary approach. The earlier cases, on the other hand, stress an understanding based on an outer scaffold in accordance with a more traditional understanding.

There is also a common theme of competing identities apparent in both nation-states and trigonometry. Citizens of nation-states can evoke either a regional or national identity. Likewise, trigonometric functions can be defined as lengths, in terms of other trigonometric functions, or as ratios of the sides of a right triangle. Furthermore, in both cases the more particular identity is superseded by the more general.

One of the key features of nation-states that distinguishes them from empires and city-states is a sense of national unity. Empires consisted of sundry populations that retained their various cultural identities but submitted to an extraneous political administration, while city-states were smaller and relied on common interest rather than common identity for citizen solidarity. Although some nation-states arose from a cultural

or ethnic nation seeking political sovereignty, many came about as the result of deliberate policies instituted by the state to cultivate a national identity among regional factions. In other words, in order to reinforce their overarching political authority, states established a corresponding national culture to compete with their citizens' preexistent regional identities, often through the promotion of a national language and compulsory education. Serving as the sovereign of a cultural nation confers legitimacy on a state, and thus for the nation-state to thrive, regional identity is necessarily subordinate to national identity.

Trigonometric functions admit of multiple interpretations as well. They represent the line segments associated with a particular angle measure, determined by geometrical construction. They express the relations between other trigonometric functions, as for instance the tangent being equal to the ratio of the sine to the cosine. Trigonometric functions also denote ratios between the hypotenuse, opposite, and adjacent sides of a right triangle. Of these three definitions, the first is the most particular, in that it specifies a single numerical measure for each value of the angle, whereas the third is the most general since the ratios can be derived from any right triangle. Furthermore, the third definition unifies the six functions into a simple scheme by describing all of them in terms of the same three variables. Early modern mathematicians acknowledge all three definitions, but give precedence to the characterization of trigonometric functions as ratios of the sides of a right triangle, which is to say, the most general identity.

Alternately, the earlier cases recognize only one interpretation of trigonometric functions. Both the ancient Greek and medieval Arabic mathematicians define trigonometric functions solely as lengths that correspond to specific angles. While this understanding is comparable to the least general of the three early modern identities,

since there is only one option it is impossible to make claims regarding their preference for general or particular definitions. However, a certain computational disunity is evident. Though the trigonometric functions can all be articulated in a single geometrical diagram, to ascertain the values of all six requires multiple constructions and then additional calculations to render the results commensurable. In other words, the earlier mathematicians cannot express all of the trigonometric functions in terms of the same variables, and must rely instead on a piecemeal method of computation.

Just as nation-states must contend with competing identities, so too does early modern trigonometry, and in both cases the more general definition prevails, which in turn promotes internal cohesion. The earlier mathematicians, by contrast, endorse only one definition of trigonometric functions that corresponds with the least general of the three early modern possibilities, and their trigonometric calculations coincidentally exhibit a degree of incongruity.

Another parallel is evident between trigonometry and industrialization, in that both apprehend key concepts in terms of ratios. In early modern trigonometry, sines, cosines, and other functions are defined as ratios of the sides of a right triangle, while the onset of industrialization entails a greater focus on productivity and efficiency ratios. The emphasis in both cases is on relative value.

The processes commonly associated with industrialization, namely specialization, standardization, and rationalization, are all fundamentally directed at achieving greater productivity and efficiency, or in other words, increasing output and eliminating waste. Both productivity and efficiency are generally conceptualized as ratios; productivity is gauged by analyzing output produced per unit of input, while efficiency is appraised by

measuring the costs of production relative to the output. Calculating the ratios of inputs and outputs gained significance during industrialization due to the concomitant transition from subsistence living to an economic model that prioritized profit, which derives from the difference between costs and revenues. Manipulating the productivity and efficiency ratios, then, has the potential to increase profits. Put another way, in order to effectively accumulate wealth through profit, industrialists were forced to consider the relative value of production.

As indicated above, the sine from a geometrical standpoint is half of the chord determined by a given angle (twice as large as the angle associated with the sine) and a circle with radius r . The cosine, originally the line that bisected the chord and the angle it subtended, extends from the origin and meets the sine at a right angle. The line tangent to the circle at the point where the sine meets it is divided into two components: the tangent runs from that point until it meets a line extended from the cosine, while the cotangent stretches between the point of intersection with the sine and a line drawn from the origin at a right angle to the cosine. The length between the origin and the tangent, that overlaps the cosine, is known as the secant, and the line between the origin and the cotangent, perpendicular to the secant, is the cosecant (See Figure 3.6). The right triangles formed by the cosine, sine and radius, the radius, tangent and secant, and the cotangent, radius, and cosecant are similar, meaning the same three angles appear in all three triangles, and their sides are proportional. Thus, the tangent bears the same relation to radius r that the sine has to the cosine, the secant is to the radius as the radius is to the cosine, the cosecant is to the radius as the radius is the sine, and the cotangent is the radius as the cosine is to the sine, and in this way all six trigonometric functions can be expressed in terms of sine

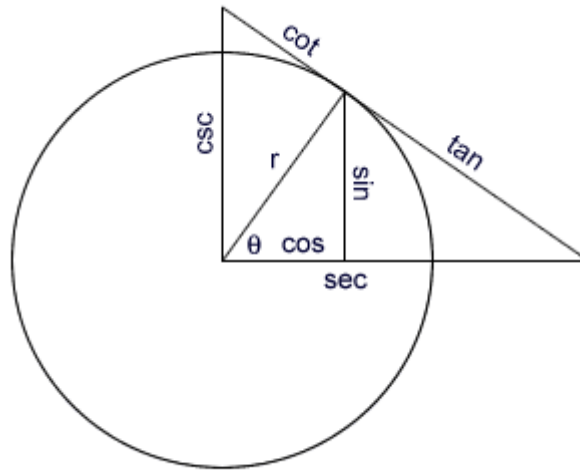


Figure 3.6: The unit circle

(opposite), cosine (adjacent), and radius (hypotenuse). The early modern mathematicians chose to interpret trigonometric functions as expressions of their value relative to this fundamental triangle formed by the sine, cosine, and unit measure.

Conversely, the ancient Greek and medieval Arabic mathematicians comprehend trigonometric functions as fixed lengths that correspond to a particular arc measure. Although the right triangles, and the relations they imply, are evident from the figure, which represents a purely geometrical rendering of the six major trigonometric functions that earlier mathematicians would have been familiar with, they nonetheless interpret those functions in absolute terms. Furthermore, even when algebraic manipulation makes the equivalence of a trigonometric function and the ratio of two others obvious, as for example in both Abu al-Wafa's and al-Biruni's work (Van Brummelen 2009), the reluctance to define one magnitude as the relation of two others persists. In short, the earlier mathematicians refused to portray trigonometric functions in relative terms.

Early modern trigonometry and industrialization both stress the relative value by perceiving trigonometric functions and productivity, respectively, as ratios. In contrast, the ancient Greek and medieval Arabic mathematicians grasp trigonometric functions as lengths, devoid of relative attributes.

To review, several parallels become apparent when trigonometry and the rise of the west are evaluated together. Early modern shipbuilders and trigonometers both deduce the external features from the internal structure of their ships and triangles, respectively. Nation-states and early modern mathematicians both adopt the more general of several competing identities. Both trigonometry and industrialization emphasize relative value by appealing to ratios. Moreover, these conceptual similarities, namely reversing perspective, unifying through the adoption of a more comprehensive definition, and assessing relative worth, seem somewhat congruous, lending further credence to the theory of epistemic influence. The next chapter will continue to build on these findings through an examination of the fundamental theorem of calculus.

4. The Fundamental Theorem of Calculus

The invention of calculus represents one of the greatest advances in mathematics, but it also sparked one of its bitterest controversies. Who conceived of calculus first, Isaac Newton or Gottfried Leibniz? Newton's supporters asserted that he began to formulate his version of calculus no later than 1666, though he did not publish any explanation of it until 1693. Leibniz's defenders countered that Leibniz published first in 1684, and insisted that his research, which can be dated to no earlier than 1675, was not privy to Newton's unpublished ideas (Boyer 1959). Ultimately, it was determined that each had discovered calculus independently, coincidentally at roughly the same time near the end of the seventeenth century.

There are two main branches of calculus, differential and integral. Differential calculus is principally concerned with instantaneous rate of change, which is generally conceptualized as the slope of the tangent to a curve at a given point. Integral calculus focuses on cumulative change over time, which is often represented as the sum of the area under a curve. The breakthrough attributed to Newton and Leibniz lies in the enunciation of the relation between differentials and integrals, otherwise known as the fundamental theorem of calculus. Newton and Leibniz perceived that differentiation and integration are inverse operations, much like addition and subtraction or multiplication and division. In other words, differentiating an integral restores the original equation, and likewise integrating a differential. Similarly, if both operations occur simultaneously, they cancel out each other. This insight was of practical significance because it enabled mathematicians to compute integrals by reversing the much simpler process of

differentiation. The fundamental theorem of calculus also established a comprehensive framework for incorporating the study of change and motion into mathematics, which in turn led to numerous advances, most notably in physics.

While Newton and Leibniz were the first to bridge differentiation and integration, the rudiments of calculus are evident in both Greek and Arabic mathematics. Tangents and areas are, of course, ubiquitous in both corpora, but it is their evaluation in conjunction with infinitesimals that is most of interest to the history of calculus. Among the Greeks, Archimedes accomplished the most in this vein. He used infinitesimals extensively in *The Method of Mechanical Theorems* to find areas and volumes, though he generally demanded more rigorous methods in his mathematical works. Additionally, he calculated the quadrature of the parabola using an infinite series of diminishing triangles, and devised a procedure to locate the tangent at any point on a spiral, which indicates the direction of motion at any given instant (Heath 2002). Eudoxus's method of exhaustion, which involves determining areas by means of converging polygons, also resembles later developments in integral calculus, and was frequently employed in ancient Greek mathematical proofs (Boyer 1959). Similarly, in Book XII of *The Elements*, Euclid used inscribed polygons to determine the area of a circle, laying the foundation for his exploration of the volumes of regular solids (Euclid 1956).

Medieval Arabic mathematics also presaged the development of calculus. Thabit ibn Qurra, the Banu Musa, al-Quhi, and Ibrahim ibn Sinan, among others, all incorporated infinitesimals in their investigations of areas and volumes (Rashed 1993). Sharaf al-Din al-Tusi found the derivatives of cubic polynomials in his solutions utilizing maxima (al-Tusi 1985). Additionally, ibn al-Haytham contrived a formula for

determining the integrals of polynomials up to the fourth degree (Alhazen and Smith 2006). In the early modern era as well a number mathematicians anticipated the development of calculus. Most notably, Galileo investigated the instantaneous velocities of falling bodies (Galilei 1989). Fermat invented a means of determining maxima, minima, and tangents for various curves, and discovered a trick for evaluating the integral of any power function directly (Mahoney 1994). Isaac Barrow in fact rendered a geometrical proof of the fundamental theorem of calculus, but did not grasp its significance (Boyer 1959). In all three historical periods, traces of both differential and integral calculus are evident, and in several cases, the mathematicians in question explicitly dabbled in both branches. Why, then, did no one see the connection prior to Newton and Leibniz?

To illuminate the conceptual stumbling blocks that hampered earlier discovery, two examples from the earlier periods that best approximate differential calculus, namely Archimedes' method for finding tangents to spiral curve in *On Spirals* and Sharaf al-Din al-Tusi's use of maxima to solve cubic polynomials in his *Treatise on Equations*, will be considered in comparison to Newton's process for determining the instantaneous rate of change outlined in *The Principia*. Additionally, two earlier methods of calculating the quadrature of the parabola which resemble integral calculus will be compared with Newton's formulation of the same problem. Following a brief description of the cases, this chapter will highlight the differences in their approaches to both differentials and integrals that inhibited an earlier realization of the fundamental theorem of calculus, and conclude with a discussion of the implications of those differences for the rise of the west.

4.1 Differentials

In his work *On Spirals* (Heath 2002), Archimedes develops a means of finding the tangent to any point on a spiral curve by geometric construction. Because the angle (OPT) formed by the tangent (PR) and the line connecting the origin (O) and the point of tangency (P) is acute in the backward (counterclockwise) direction, the tangent PR must intersect at some point with a straight line (OT) drawn perpendicular to OP at the origin (See Figure 4.1). Archimedes proves that the length of line OT is equal to the arc swept

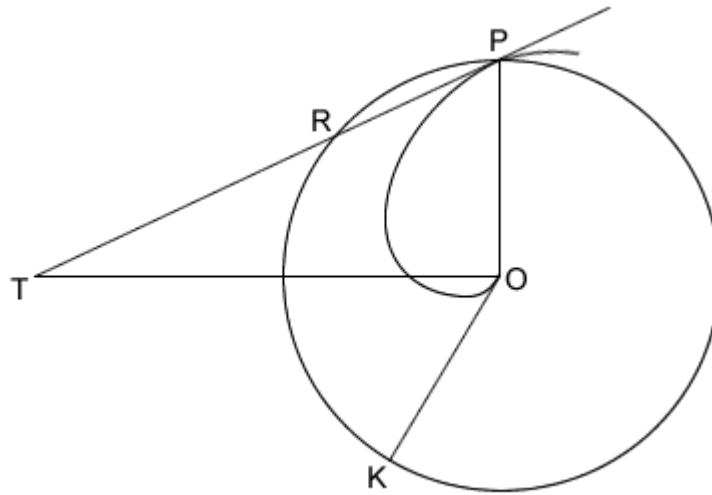


Figure 4.1: Archimedes' tangent to a spiral

by a circle with radius OP from the initial line (OK) to point P in the forward (clockwise) direction (arc KRP). Thus, given any point P on the spiral, finding the tangent is just a matter of measuring the circular arc swept by radius OP from the origin line to the point of tangency, constructing a straight line of equal length perpendicular to the radius, and connecting points P and T. The tangent indicates the direction of motion at any given point on the spiral, and since each point on the curve possesses a unique tangent, the direction of motion is constantly changing. Basically, Archimedes decomposes the

motion that describes the spiral into two components, circular revolution around the origin and a uniform radial movement away from the origin, with the tangent representing their combined effects at any given point. In other words, the tangent's spatial orientation corresponds to the spiral's instantaneous rate of change. Furthermore, although the concept of a slope, which derives from the coordinate plane, was foreign to Archimedes, his depiction of the tangent as the hypotenuse of a right triangle whose other two legs measure its progress in two directions mirrors the definition of a slope in terms of rise and run. Archimedes has essentially found the derivative of a spiral.

Al-Tusi takes a different approach, and in an effort to determine whether cubic polynomials have (positive) solutions, derives the equivalent of differential equations. In his *Treatise on Equations* (al-Tusi 1985; Hogendijk 1989), he identifies five types of cubic equations that only have a solution when certain conditions are met, namely $x^3 + c = ax^2$, $x^3 + c = bx$, $x^3 + ax^2 + c = bx$, $x^3 + bx + c = ax^2$, and $x^3 + c = ax^2 + bx$. Note that all five equations are written in terms of a single variable (x), and each contains a constant (c). Essentially, the question al-Tusi seeks to answer is whether an x exists such that the sum of the variable terms of the equation will ever equal c .¹ To determine this, al-Tusi finds the maximum value of x , then substitutes it into the equation and compares the result with c ; if c is greater, then the function has no solution, whereas if c is less than or equal to the function of the maximum, the equation has at least one solution. The modern method for finding the maximum of an equation is to take the derivative and set it equal

1 In other words, is there a value for x such that:
 $-x^3 + ax^2 = c$
 $-x^3 + bx = c$
 $-x^3 - ax^2 + bx = c$
 $-x^3 - bx + ax^2 = c$
 $-x^3 + ax^2 + bx = c$

to zero. Al-Tusi arrives at the equivalent differential equation for all five cases by geometrical and algebraic means. Consider, for example, the fifth type of cubic equation, $x^3 + c = ax^2 + bx$. Al-Tusi sets length BC equal to a^2 and length BA equal to the square root of b (See Figure 4.2), rendering the equation $c = BCx^2 + BA^2x - x^3$. He next selects

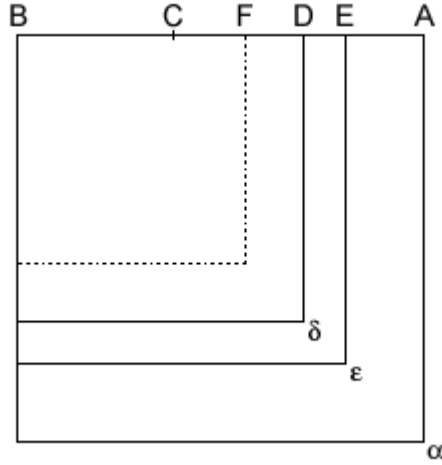


Figure 4.2: Sharaf al-Din al-Tusi's function of the maximum

three lengths as potential x values, BE, BD and BF, and plugs them into the equation, such that $f(BE) = BC \cdot BE^2 + BA^2 \cdot BE - BE^3$, $f(BD) = BC \cdot BD^2 + BA^2 \cdot BD - BD^3$, and so on. Al-Tusi then evaluates $f(BD) - f(BE)$, and after a series of algebraic manipulations,³ determines that $f(BD) - f(BE) = DE[CD(BD + BE) - (BA^2 - BE^2)]$. Similarly, he finds $f(BF) - f(BD) = FD[CF(BF + BD) - (BA^2 - BD^2)]$. To maximize $f(BD)$, $CD \cdot (BD + BE)$

2 This demonstration will only consider the case in which $a < \sqrt{b}$, but al-Tusi, in a similar fashion, evaluates all three cases, $a < \sqrt{b}$, $a = \sqrt{b}$, and $a > \sqrt{b}$.

3 $f(BE) = BC \cdot BE^2 + BA^2 \cdot BE - BE^3 = BC \cdot BE^2 + BE(BA^2 - BE^2)$;
 $f(BD) = BC \cdot BD^2 + BA^2 \cdot BD - BD^3 = BC \cdot BD^2 + BD(BA^2 - BD^2)$;
therefore, $f(BD) - f(BE) = BC \cdot BD^2 + BD(BA^2 - BD^2) - [BC \cdot BE^2 + BE(BA^2 - BE^2)]$;
but, $BD(BA^2 - BD^2) = BD[DE(BD + BE)] + BD(BA^2 - BE^2)$
and, $BC \cdot BD^2 - BC \cdot BE^2 = -BC[DE(BD + BE)]$
therefore, $f(BD) - f(BE) = BD[DE(BD + BE)] + BD(BA^2 - BE^2) - BC[DE(BD + BE)] - BE(BA^2 - BE^2)$;
but, $BD[DE(BD + BE)] - BC[DE(BD + BE)] = CD[DE(BD + BE)]$
and, $BD(BA^2 - BE^2) - BE(BA^2 - BE^2) = -DE(BA^2 - BE^2)$
therefore, $f(BD) - f(BE) = CD[DE(BD + BE)] - DE(BA^2 - BE^2) = DE[CD(BD + BE) - (BA^2 - BE^2)]$

must be greater than $BA^2 - BE^2$ for all E between D and A, and $CF*(BF + BD)$ must be less than $BA^2 - BD^2$ for all F between D and C. However, it is also known that $CD*(BD + BE) > 2CD*BD$, $CF*(BF + BD) < 2CD*BD$, and $BA^2 - BD^2 > BA^2 - BE^2$, by definition, thus when $2CD*BD$ is greater than or equal to $BA^2 - BD^2$, $CD*(BD + BE)$ must be greater than $BA^2 - BE^2$ and likewise when $2CD*BD$ is less than or equal to $BA^2 - BD^2$, $CF*(BF + BD)$ must be less than $BA^2 - BD^2$. Therefore, $f(BD)$ reaches its maximum value when $2CD*BD$ equals $BA^2 - BD^2$. Rearranging the terms, and substituting m for BD , the equation for the maximum of $x^3 + c = BCx^2 + xBA^2$ is $3m^2 = 2mBC + BA^2$, which is the same equation arrived at by taking the derivative.

In *The Principia* (Newton 1999), Newton formulates a rule for determining the instantaneous rate of change for any function, namely for any x^n the differential is equal to $n*x^{(n-1)}$. His reasoning is built upon the idea that quantities are increased or decreased by moments, by which he means instantaneous increments or decrements that have no magnitude. To demonstrate the relation between moments and functions, Newton first supposes a rectangle with sides A and B, increased by continual motion (see Figure 4.3).

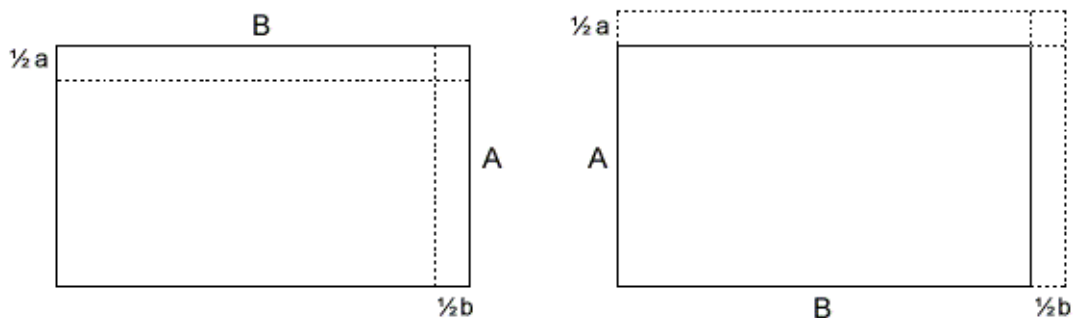


Figure 4.3: Newton's instantaneous rate of change

When sides A and B are decreased by half a moment ($\frac{1}{2} a$ and $\frac{1}{2} b$, respectively), the

rectangle is equal to $A - \frac{1}{2}a$ multiplied by $B - \frac{1}{2}b$, or $AB - \frac{1}{2}aB - \frac{1}{2}bA + \frac{1}{4}ab$.

Similarly, when A and B are increased by half a moment, the rectangle is equal to $A + \frac{1}{2}a$ multiplied by $B + \frac{1}{2}b$, or $AB + \frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab$. Subtract the smaller rectangle from the larger and the remainder is $aB + bA$, which is the increase to the rectangle generated by the total increments a and b . By the same token, solid ABC has a moment equal to $aBC + bAC + cAB$, and so on. Next, Newton sets sides A , B , and C equal to each other. Substituting A for B and C , AB becomes A^2 and its moment, $aB + bA$, is transformed into $aA + aA$ or $2aA$. Likewise, ABC reduces to A^3 , and its moment $aBC + bAC + cAB$ becomes $aAA + aAA + aAA$ or $3aA^2$. Since a is a moment, and therefore bereft of magnitude, the instantaneous rate of change for any x^n simplifies to $n \cdot x^{(n-1)}$, as predicted.

To review, Archimedes finds the tangent to any point on a spiral by means of a geometrical construction, and the inclination of the tangent indicates the direction of motion at any given instant. Further, his decomposition of the spiral's motion into two components with the tangent forming the diagonal is reminiscent of Newton's use of parallelograms to represent the rate of change as a combination of simpler forces. Al-Tusi derives differential equations for cubic polynomials in the process of determining whether they have solutions. He leverages his understanding of solid geometry and algebraic manipulation to ascertain the maximum value of the independent variable, which results in an equation that reflects the differential when the instantaneous rate of change is zero. Essentially, he examines the change in y for the two sets of potential x 's on either side of the maximum, and determines where those two sets meet by rewriting the expressions in common terms. Newton extracts the instantaneous rate of change from

a function by examining the effects of infinitesimal increments and decrements. The relation he uncovers between functions and their moments leads him to a simple rule for determining the differential equation of any function.

What, then, sets Newton apart from his predecessors? For one, Archimedes' and al-Tusi's solutions are more or less restricted to the specific problems they address. Archimedes' method for finding a tangent is dependent on the particular relation a spiral has to a circle, and will not easily translate to other types of curves. Al-Tusi only calculates the maximums for five cases of cubic polynomials, and while his method may extend to the simpler forms of equations covered in *Treatise on Equations*, because it is based on a geometric construction, his analysis cannot extend beyond three dimensions. In contrast, Newton's approach can be applied to nearly any function, and though the proof has a geometrical basis (rectangle AB), it serves merely as a heuristic illustration of the simplest case. Secondly, the fact that Archimedes and al-Tusi discover an expression of the instantaneous rate of change seems almost incidental to their goals, whereas Newton explicitly proposes to study momentary action. Accordingly, the principal finding for Archimedes lies in the construction of the tangent and less in what it represents. Likewise, the equation of the maximum for al-Tusi is just a means of achieving his main purpose of determining whether or not the cubic polynomial has a solution. Newton, on the other hand, seeks and finds a formula for the instantaneous rate of change.

Additionally, Archimedes' tangent communicates the instantaneous direction of motion relative to the figure, not to any objective scale. Since the tangent is built on the line connecting the origin to the point of tangency, which changes in both position and

length depending on the point of interest, and a line perpendicular to it whose length is determined by a circle that also varies in size, there is no fixed standard for gauging measures. Comparison across figures and even on the same spiral is therefore difficult. Newton's moments, rendered in symbolic algebra, bear clear relations to one another. Al-Tusi's equations, though algebraic in character, were written as words, which may have obscured the relation between the original cubic polynomial and its maximum. Al-Tusi, at any rate, does not recognize that his equation for the maximum is also an equation for the instantaneous rate of change in general. This may also be a consequence of the way he arranges the equations, common among Arabic mathematicians, with terms on both sides of the equal sign. This has the effect of narrowing the scope of the problem to a single instance of the relation expressed by the equation. For instance, in the example above, he writes $x^3 + c = ax^2 + bx$ rather than consolidating the terms on either side, and then tries to find the specific value of x where the equation equals the constant c . Newton, following Descartes, considers the whole equation, and as such, finds the instantaneous rate of change for the entire function.

4.2 Integrals

The quadrature of the parabola refers to the area bound by a parabolic curve and an intersecting chord. Archimedes takes up the question of its value in his eponymous treatise (Heath 2002), and demonstrates the area under the curve is equal to four thirds the triangle inscribed within the parabolic segment whose base is the chord and whose height is determined by a line drawn from the midpoint of the chord to the vertex of the parabola (See Figure 4.4). He begins by inscribing triangle QPq, which results in two

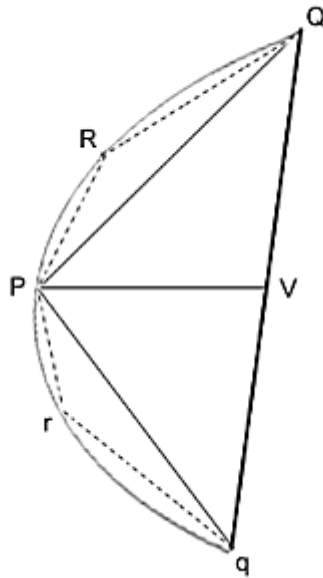


Figure 4.4: Archimedes' quadrature of the parabola

new parabolic segments, namely PRQ with base PQ and Prq with base Pq. Archimedes then inscribes triangles within the two new segments, each with a height equal to the length between the midpoint of the base and the vertex of the parabola, which in turn cuts four new parabolic segments. He proceeds in this manner indefinitely, dissecting ever smaller parabolic segments into steadily tinier triangles. Based on earlier proofs, he knows that the area encompassed by each pair of smaller triangles is the equivalent of one quarter the area of the next larger triangle, or in other words that the area of triangles PRQ and Prq together total one quarter the area of triangle PQQ. The quadrature of the parabola is therefore equal to the area of the original inscribed triangle multiplied by the geometric series that expresses the established proportion between the triangles (i.e., $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots$). Archimedes determines the value of this series by examining a series of squares in the same proportion (See Figure 4.5). As the diagram makes clear, the series of squares, each of whose area is one quarter that of its predecessor, is equal to one

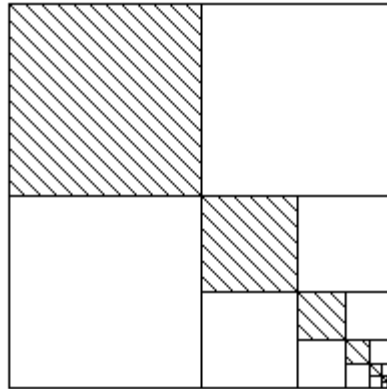


Figure 4.5: Archimedes' sum of a series

third of the original area. One third in addition to the original area amounts to four thirds, and thus the quadrature of the parabola is equal to four thirds the area of the inscribed triangle.

Ibrahim ibn Sinan adopts a somewhat similar method of determining the quadrature of the parabola (Rashed 1993). He also begins with a section of a parabola bounded by a chord, and constructs a triangle with base equal to the chord and height determined by a line drawn from the midpoint of the chord to the vertex of the parabola. He likewise shows that two smaller triangles inscribed within the parabolic segments cut off by the legs of the original triangle are together equal to one quarter of that original triangle. However, rather than rely on an infinite series of diminishing triangles, Ibrahim instead determines that the parabolic segments bear the same relation as their inscribed triangles. In other words, if the smaller triangles add up to one quarter of the larger triangle, then the smaller parabolic segments likewise total one quarter of the larger parabolic segment. To demonstrate this property, Ibrahim draws two lines, AB and CD, divided into any number of parts (in this example, by points E, G, H and I) such that the

relations between the parts is the same for both figures (i.e., BG is to GE as DI is to IH, and so on). Then parallel lines are extended from the points to any length, so long as the relations of the parallels to each other are the same in both figures, and lines are drawn connecting point A and N and points C and S (see Figure 4.6). He then proceeds to show

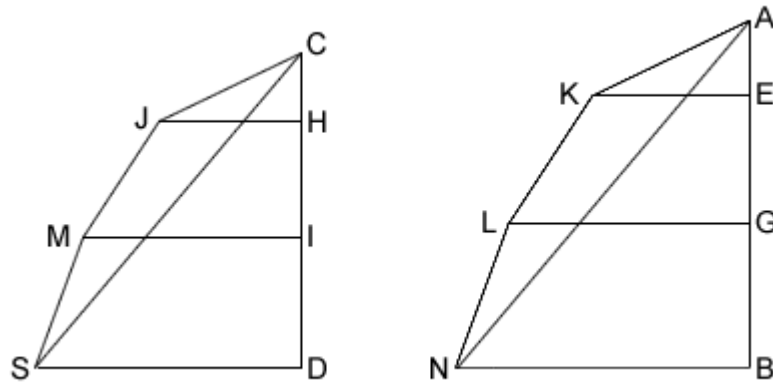


Figure 4.6: Ibrahim ibn Sinan's quadrature of the parabola

that all of the corresponding line segments bear the same relation to each other that triangle ABN has to triangle CDS, and furthermore that the areas contained by those segments are likewise to each other as triangle ABN is to triangle CDS. Thus, the total area contained within polygons AKLNB and CJMSD are related to each other as triangle ABN is to triangle CDS. The key is that, so long as the proportions are maintained, the initial line can be divided by any number of points, and therefore the polygon can have any number of sides. A parabola can be approximated by an infinitely-sided polygon, and because the ratio between a parabola's ordinates and abscissas is constant, any two segments of a parabola would necessarily be proportionate to each other. Therefore the parabolic segments are to each other as their inscribed triangles, and the original triangle

is equal to $\frac{3}{4}$ of the parabolic segment, or in other words, the area of the parabolic segment is equal to four thirds the area of the inscribed triangle.

Newton approaches the quadrature of curves from a different perspective (Newton 1999). He begins with the curvilinear figure $AacE$ composed of the curve acE and straight lines Aa and AE (see Figure 4.7), and inscribes any number of parallelograms (in

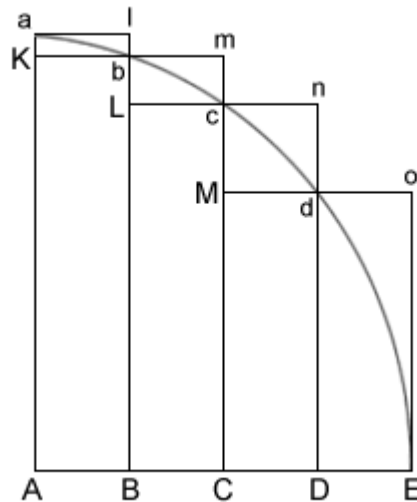


Figure 4.7: Newton's quadrature of curves

this example, Ab , Bc , Cd) on equal bases (AB , BC , CD , and DE). Then he completes parallelograms $aKbl$, $bLcm$, $cMdn$, and $dDEo$, which when added to the inscribed parallelograms circumscribe figure $AacE$. Newton contends that when the width of the parallelograms is diminished and their number increased indefinitely, the difference between the inscribed figure $AkbLcMdD$ and the circumscribed figure $AalbmcndoE$ decreases to the point where the figures are equal, both to each other and to the intermediate curvilinear figure $AacE$. He proves this by pointing out that the difference between the inscribed and circumscribed figures is equal to the sum of the parallelograms Kl , Lm , Mn , and Do , which in turn is equal to the rectangle $AalB$. As its width AB is

diminished indefinitely, rectangle AaLB becomes less than any given rectangle, and by extension the difference between the inscribed and circumscribed figures disappears as well. Therefore, the inscribed, circumscribed, and intermediate figures must ultimately be equal. To compute the value of the area under the curve (Newton 1736), Newton likens the curve itself (acE) to the instantaneous rate of change of the figure AacE. He demonstrates this relationship by taking the equation of a known curvilinear area and finding the differential, which, as expected, is the same as the equation of the curve. Thus, calculating the area under a curve is a simple matter of applying his method for determining differentials in reverse.

In summary, Archimedes finds the quadrature of the parabola by inscribing an infinite number of steadily diminishing triangles. Because he knows that each set of smaller triangles is one quarter the size of the next larger triangle, he is able to compute the area relative to the original inscribed triangle by means of an infinite series. Ibrahim ibn Sinan calculates the quadrature of the parabola by first proving that the parabolic segments are in the same relation to each other as their inscribed triangles, and then deducing the overall relation of the parabolic segment to the original inscribed triangle based on the proportions of the triangles. To prove the relation between the parabolic segments and their inscribed triangles, he first shows the relation holds for triangles and polygons, and then posits that a polygon with an infinite number of sides inscribed in a parabola is indistinguishable from the parabola itself. Newton constructs a series of parallelograms that both inscribe and circumscribe the the curvilinear figure he seeks the area of, and demonstrates that if the width of the parallelograms diminishes and their number increases indefinitely, the inscribed and circumscribed figures will ultimately

coincide with the intermediate curvilinear figure. Furthermore, he indicates that the curve which determines the curvilinear figure represents the instantaneous rate of change, suggesting the inverse relationship of differentials and integrals.

Much like the differential examples, the earlier cases have a limited application when compared to Newton's approach. Archimedes and Ibrahim determine the quadrature of the parabola, whereas Newton computes the quadrature of any curve. Also like the differential exemplars, the earlier mathematicians did not view the problem of finding the area under a curve as one of change. Whereas Newton saw the area under a curve as representative of the function's aggregate fluctuation, Archimedes and Ibrahim saw only geometric area. Another difference lies in their techniques for calculating the value of the area. Archimedes' and Ibrahim's efforts are directed at establishing a proportion between the first inscribed triangle and the parabolic segment, so that they can build a formula for the area around a known quantity (the triangle). Though Archimedes solves the problem by summing an infinite series, the summation is only possible because of what he knows about the proportion that exists between the triangles. Likewise, Ibrahim fills the parabola with an infinitely-sided polygon, but only to affirm the proportion between the inscribed triangle and parabolic segment. He finds the area of the parabolic segment by applying the proportion to the known area of the triangle. Newton, in contrast, estimates the area by summing the heights of an infinite number of rectangles, and does not attempt to describe the area under the curve by referencing another, known figure. Instead, he defines the area under the curve in terms of the curve itself.

4.3 Epistemological Obstacles

To recap, the two earlier cases of differential calculus differed from Newton's formulation in a couple of ways. Their applicability was far more limited, and the fact that they derived a measure of the instantaneous rate of change was a secondary consideration. Moreover, Archimedes lacks the means of comparing the various tangents his method produces, and al-Tusi does not recognize his equation for the maximum as that of the instantaneous rate of change in general. Likewise, the earlier examples of integral calculus diverge from Newton's model at several points. They also have a less general application, and treat the area under the curve as merely an area, as opposed to cumulative change. Additionally, both relate the area under the curve to a known polygon as opposed to the curve itself. Nevertheless, while these differences hint at factors that inhibited the realization of the inverse relationship, none precludes the discovery altogether. The question remains, what conceptual idiosyncrasies facilitated the emergence of the fundamental theorem of calculus only in the later period, given that earlier mathematicians were familiar with both components?

One of the conceits that sets Newton (as well as Leibniz) apart from his predecessors is the way he frames his inquiries in terms of change. Approaching both aspects of calculus from the perspective of change places them in a common context from the outset, and suggests some relation must exist between them, mediated by the idea of change, even before it is formally articulated. In other words, while the relation between the slope of the tangent and the area under the curve is not immediately apparent, the relation between instantaneous increments of change and the cumulative change over a given interval is less obscure. At the very least, Newton can deduce that a differential is

to an integral as the rate of generated change is to the quantity generated. Furthermore, subsuming the two branches of calculus under the concept of change means that either can serve as a starting point in the derivation of the fundamental theorem of calculus. While Newton reasons from differentials to integrals, Leibniz, in fact, does the reverse and arrives at the same conclusion.

This emphasis on change, in turn, is reflected in the various methodological approaches to both sets of problems. The earlier cases are characterized by a certain static quality, whereas Newton's solutions are imbued with motion. Archimedes and al-Tusi, for instance, seek a result that satisfies the question for a single value (Archimedes at the point of tangency and al-Tusi at the maximum). Newton, on the other hand, concentrates on what is happening between two values, and finds a formula for determining its momentary rate. Similarly, when determining the quadrature of the parabola, Archimedes and Ibrahim look for the proportion that relates the area to a fixed quantity, namely the inscribed triangle. In contrast, Newton evaluates the area under the curve relative to the curve itself, which he interprets as a measure of the area's fluctuation. Because of their narrow parameters and static solutions, the earlier iterations of differentials and integrals do not lend themselves to a broader interpretation, and any commonalities between the two branches are obfuscated by the specificity of the cases. The dynamic aspect of Newton's approach not only accentuates the potential convergence of differentials and integrals as functions of change, it also demands a level of generality which facilitates the exploration of that relationship.

Finally, the influence of developments discussed in the preceding chapters is also evident. Both Newton and Leibniz make extensive use of the coordinate system. When

discussing area under the curve, they have in mind the area contained by the curve and the axis lines. Likewise, the diminishing parallelograms they use to estimate that area have as their bases a portion of the x-axis (also called the abscissa) and are erected parallel to the y-axis in the manner of ordinates. Also, both view the differential (akin to the slope of the tangent) as a ratio. Newton, appealing to motion rather than infinitesimal quantities, defines the differential in terms of the ratio of change as the increments diminish into nothingness (Dy/Dx , as $Dx \rightarrow 0$), and describes the integral as the sum of those ratios for some interval. Leibniz, more comfortable with infinitesimals, views the differential as the ratio of ordinates to abscissas at an instant, and characterizes the integral as the sum of the ordinates for infinitesimal intervals in the abscissa. Additionally, both employ symbolic notation, which brings the inverse relation into sharper focus. Consider how much easier it is to recognize the pattern among these equations written in symbolic form, than to perceive the same relation from the equivalent described in prose or among figures rendered geometrically:

$$\begin{aligned}
 f'(x) &= 2x + 3 \\
 f(x) &= x^2 + 3x + 6 \\
 F(x) &= \frac{1}{3}x^3 + \frac{3}{2}x^2 + 6x + 2
 \end{aligned}$$

4.4 Rise of the West

How are the conceptual differences that facilitate the realization of the fundamental theorem of calculus reflected in the rise of the west? One parallel can be discerned between nation-states, which represent a coincident people and territory, and the dual nature of infinitesimals inherent in calculations based on vanishing quantities. A

common thread underlies their logic as well, in that both invoke a particular negotiation of magnitude and multitude.

As the moniker implies, in a nation-state political sovereignty and cultural identity coincide, and in fact, reinforce each other. The cultural (or ethnic) homogeneity legitimizes the state's authority, while the state promotes and consolidates a national identity. Thus, the state, a magnitude, arises from the multitude of people who comprise a nation, and likewise, the nation, a multitude, derives from the state. In other words, the nation-state has a dual character as both one and many, wherein many can come together to form one and one can also comprise many. Prior to the advent of nation-states, geopolitical entities generally took the form of multi-ethnic empires, which collected multiple cultural groups under a common administration but did not integrate them into a cohesive body, either politically or culturally. One faction rose to dominate the others, which to greater and lesser extents retained their cultural identities. Thus, empires are essentially multitudes, and incapable of being incorporated into a state-like magnitude given their cultural breadth.

Similarly, the early modern formulation of calculus muddies the traditional distinction between magnitude and multitude. Magnitudes are continuous and infinitely divisible, as for example lines, whereas multitudes are collections of discrete entities that can be divided into units and no further, such as a flock of geese. Newton transgresses this boundary in his explorations of both differentials and integrals. The momentary increments and decrements he employs to calculate the differential act as units of change despite the fact that they are fragments of a continuous magnitude. Likewise, the parallelograms he sums to determine the area under the curve are in effect lines, and add

up to an area despite having no width. This, in essence, is the paradox of infinitesimals; they represent the smallest possible division, or in other words the units, of a continuous quantity, which by definition has no unit. Newton exploits this ambiguity between one and many to derive functions for both the instantaneous rate of change and the cumulative change over time, and furthermore, it is central to his understanding of their inverse relation.

In contrast, the earlier schools observe a stricter division between magnitude and multitude, wherein a multitude can never add up to a magnitude. In other words, no matter how many points are strung together, they will never congeal into a line, and similarly no collection of lines will ever constitute a plane. Thus, Archimedes and Ibrahim estimate the quadrature of the parabola using other areas, such as inscribed triangles or polygons, rather than summing lines measuring the height of the curve at every instant. Likewise, though their analyses suggest an instantaneous rate of change, Archimedes and al-Tusi never fully articulate the idea because it implies that continuous quantities are comprised of discrete instants. The qualitative difference between magnitude and multitude cannot be bridged logically, and the earlier mathematicians were unwilling to relax their standards even at the infinite extreme. Consequently, magnitude and multitude remained distinct.

Both nation-states and early modern European calculus exhibit a particular understanding of magnitude and multitude wherein an infinite number of lines add up to an area or a political jurisdiction emerges from a cultural aggregate, and vice versa. Moreover, this overlap is absent in the earlier iterations of calculus, where a sharper distinction between magnitude and multitude is maintained.

Another analogue is evident in the early modern approaches to physics and calculus, in that both attempt to model change. While many advances in physics proceeded directly from the development of calculus, there are conceptual similarities even in the work that predates Newton and Leibniz's revelation of the fundamental theorem. Specifically, both the science and mathematics of this period attempt to incorporate variability into their analyses by evaluating multiple perspectives.

As motion (which is, essentially, a manifestation of change) leapt to the forefront of scientific investigation, frames of reference assumed greater significance in treatises on physics, including those by Galileo and Descartes. Perhaps the clearest use of perspective in early modern science is Christiaan Huygens' treatment of hard body collisions. In *On the Motion of Bodies from Impact* (Huygens 1929), first published in 1703 but composed in 1656, Huygens examines the transference of force resulting from various types of kinetic exchanges. His principal means of extrapolating the effects consists of comparing the motion that would be observed from a stationary point to the apparent motion from the perspective of someone on a boat moving at a uniform rate parallel to the shore. Because, to the observer on the bank, a collision involving a symmetric exchange of forces can be made to appear lopsided by, for instance, placing it on a boat or altering the boat's velocity, Huygens is able to predict the results of all types of collisions by simply adjusting the frame of reference.

Similarly, perspective plays a key role in Newton's derivation of the fundamental theorem of calculus. Newton's approach to the fundamental theorem begins with the differential, and he establishes a simple rule for extracting the instantaneous rate of change from any equation. He then turns to evaluating the area under a curve, but rather

than focusing on the area, he instead considers the relationship that exists between the area and the curve. Newton recognizes that, taking the curve as a starting point, the area under it represents the cumulative change over time, but from another point of view, beginning with the area suggests that the curve represents the instantaneous rate of change relative to that area. By approaching the integral from the perspective of the differential, he not only uncovers a means of quickly evaluating integrals (by reversing his method for deriving differentials), but also discerns the inverse relation that constitutes the fundamental theorem.

Conversely, change and motion are largely absent from the demonstrations of Greek and Arabic mathematicians. The figures on which they ply their logic are treated as static, even those originally described by motion such as Archimedes' spiral, in that they are assumed as given at the outset of the proof. They approach the problems from a single, omniscient point of view, as evidenced by their heavy reliance on synthetic proofs. They further minimize the potential influence of perspective by treating variations of a problem as separate cases, as for example al-Tusi's classification of cubic equations into a total of eighteen types and his individual examination of each type. For some equations, he further amplifies the number of cases by seeking separate solutions based on the value of one part of the equation, such as his three treatments of $x^3 + c = ax^2 + bx$ depending on whether a is less than, equal to, or greater than the square root of b . In short, the earlier mathematicians emphasized the unchanging, in both subject matter and methods.

By leveraging perspective, both early modern science and mathematics advanced the study of change in their respective fields. The earlier cases, by contrast, were

uninterested in change, and instead sought solutions that were unaffected by different frames of reference.

There is also a common motif of breaking from the familiar and launching into uncharted regions interwoven in both global exploration and the development of calculus. Just as European adventurers abdicate the more tractable Mediterranean routes to the east in favor of traversing the Atlantic, so too do the coeval mathematicians eschew the study of straight lines and simple shapes in favor of complex curves and evanescent proportions.

The European Age of Discovery was sparked in the 15th century when the western edge of the known world decided to search for a route to the east that did not involve going through the middle. In particular, they wanted an access point to the spice and silk trade, based in the east, that circumvented Mediterranean and Ottoman brokers. They could have maintained the status quo, and continued paying the middlemen, or they could have fought for control of the most direct route, but instead they chose to take a chance on the unknown. First, the Portuguese skirted the coast of Africa, sketching the map as they went, and ultimately gained entry to the Indian Ocean. Once they had established themselves along the southern passage, their competitors headed west across the Atlantic. They sailed into the abyss, leaving the known world behind, on the chance of finding a new solution to the problem of trade routes to the east.

The early modern mathematicians strike out into the unknown as well. Descartes' innovations in analytic geometry enabled mathematical inquiry to expand beyond the conceivable dimensions, and manipulate equations that have no meaning in previous eras. Similarly, Newton extends his analyses to incorporate not only straight lines or familiar

curves such as parabolas, but any curve whatsoever that can be expressed as an equation. Additionally, Newton's forays into nascent and evanescent quantities also signify a departure from the known, as evidenced by his development of a method for magnifying the relations as they vanished (Newton's microscope) in an effort to allay the uncertainty about what happened when they disappeared from view. Moreover, Newton (and Leibniz) pioneered calculations based on infinitesimals, whose paradoxical nature facilitated groundbreaking discoveries as well as presented new difficulties. They probed the nebulous horizon dividing finite from infinite, continuous from discrete, and harnessed that ambiguity to solve previously insoluble problems. Interestingly, the early modern mathematicians also leave behind visual landmarks, both in terms of trading geometric for symbolic representations and in their use of infinitesimals, which are far too small to see.

The ancient and medieval mathematicians pursue a more conventional tack. Regular geometric figures, such as circles, squares or triangles, form the backbone of their repertoire, from the content of their proofs to the meaning of their equations. They rely so heavily on regular geometric figures because they are known and knowable; certain qualities are inherent in any figure designated “triangle” or “square,” which can be made to serve the ambition of the proof. Additionally, the earlier periods favored compass and straight edge constructions, limiting the catalog of shapes available to mathematicians and largely excluding figures generated through motion, even regular ones. Specifically, this preference for known figures is evident in Archimedes' and al-Tusi's introduction of a circle and triangle and nested squares, respectively, in their differential demonstrations, as well as Archimedes' and Ibrahim's reference to inscribed

triangles to ascertain the quadrature of the parabola. They continually circle back to familiar shapes to prove their propositions, retreading the same ground in search of new solutions.

At roughly the same time the early modern Europeans were sailing off the edge of the map, they were also exploring new frontiers in infinitesimal mathematics. In both situations, they were motivated to investigate the unknown, even though it increased the degree of difficulty considerably, in contrast to their predecessors, who elected to adhere to more established methods.

Finally, calculus and capitalism share an underlying theme of commensurability. The onset of capitalism was accompanied by commodification, which produced an economic metric against which the value of anything could be measured. Likewise, calculus conceived of all mathematical objects in terms of infinitesimals, which provided a common means of comparison.

The cornerstone of the capitalist system is the free market where, in theory, anything can be exchanged at the value set by the law of supply and demand. Commodification refers to the process by which goods and services are redefined in terms of their economic value, for the purpose of trade. As capitalism gains momentum and the free market expands, more and more assets come to be regarded as potentially saleable, and thus are reborn as commodities. While simply viewing the world through the lens of exchange has the consequence of rendering everything fungible, commodification further reinforces commensurability by diminishing the importance of qualitative factors apart from price. In other words, goods that are otherwise

incomparable are incorporated into a common market and measured according to a single standard, resulting in some definite quantitative relation where none previously existed.

Similarly, reducing mathematical calculations to infinitesimals essentially carries the principles at the heart of analytic geometry through to their logical conclusion. Descartes' system provides a means of comparing mathematical relations by transforming them into a series of coordinates governed by a common set of axes. This translation is possible because Descartes places magnitudes in a ratio with unity, resulting in an axis divided into regular intervals, usually denoted by integers, but also comprised of all the values in-between. Since it is the smallest possible division of a magnitude, the infinitesimal represents the true unit of the coordinate system, and as such, ultimately enables any comparative analysis. Newton makes use of this infinitesimal unit in the guise of instantaneous increments in his evaluation of the differential, and as the width of parallelograms in his determination of the integral. Furthermore, transposing quantities into this system has the effect of obscuring some qualitative distinctions. For instance, higher order equations lose their dimensionality and become lines, and whether an equation represents a differential or integral becomes a matter of perspective, which in turn facilitates greater commensurability.

Alternately, incommensurability remained a stumbling block in the earlier cases. Both Greek and Arabic mathematicians are careful to respect qualitative differences in the quantities they work with, because they have no metric for comparing across types. For example, they relate the quadrature of the parabola only to other areas, and ensure that the terms in their equations are all of the same degree even when it requires acute parsing, such as al-Tusi's inference that all the terms in a cubic equation must be volumes.

Similarly, without analytic geometry as a mediator, they are forced to distinguish between numeric and geometric solutions, necessitating multiple demonstrations for what the early modern mathematicians would perceive as a single result. Moreover, though they are able to approximate both differentials and integrals, the relations that the early modern mathematicians mined so successfully for the development of calculus, such as the instant to the whole or the curve to the area, remain hidden in plain sight since they cut across different classes of magnitudes.

Commensurability is achieved in both capitalism and calculus through the reclassification of their respective objects according to a common measure. Under capitalism, this took the form of commodification, whereas in calculus, it was predicated on the understanding of infinitesimals as units. In contrast, the ancient and medieval mathematicians continued to struggle with incommensurability for lack of that common measure.

To review, several parallels are evident between calculus and the rise of the west. The coincident cultural and political aspects of nation-states recall the dual nature of infinitesimals, and both can be understood as particular negotiation of magnitude and multitude. Early modern physics and mathematics both attempt to evaluate change by drawing on multiple perspectives. Global exploration and the use of evanescent proportions both entail a willingness to launch into the unknown, leaving visual landmarks behind. Commensurability arises from both the commodification inherent in capitalism and the use of infinitesimals as units. More generally, these conceptual similarities consist of blurring distinctions, approaching problems from different angles, and assimilating disparate elements into a common understanding. The final chapter will

consider the findings observed with respect to analytic geometry, trigonometry, and the fundamental theorem of calculus in order to flesh out the contours of the episteme governing the rise of the west.

5. Discussion

Mathematics is inextricably entwined with the rise of the west. The cartography and navigation that informed early modern exploration were facilitated by advances in trigonometry. Logarithms, invented by John Napier in the early seventeenth century, made many difficult scientific and astronomical calculations possible at long last. The new mathematics of capitalism, as exemplified by the practice of double entry bookkeeping or the metric of comparative advantage, ushered in sweeping economic changes. The practical application of Newtonian mechanics, built on a foundation of calculus, led to many technological innovations, such as the steam engine. Similarly, trigonometry formed the backbone of Copernicus' argument for a heliocentric model of the universe, which in turn provoked a scientific revolution. Without mathematics, the rise of the west simply could not have happened.

Despite these strong ties, mathematics has received little attention in conjunction with the rise of the west. Furthermore, like other developments commonly associated with the rise of the west, mathematics also underwent a significant transformation during the early modern period, culminating with the invention of calculus. This dissertation posits that the changes reverberating through early modern Europe, including global expansion, the beginnings of nation-states, the spread of capitalism, industrialization, the scientific revolution, and the development of calculus, emerged in response to a common cultural episteme. Specifically, the conceptual structure associated with a given place and time influences the course of that civilization's development by rendering some ideas unthinkable while encouraging others. Thus, amid a particular configuration of

epistemological obstacles, the rise of the west happens; under other circumstances, it does not. Similarly, because they are premised on the fundamental assumptions about the possibilities and limitations of the world which comprise the episteme, all of the artifacts generated by a given society bear the indelible imprint of the corresponding ideational framework. As such, a close examination of a civilization's output, particularly that which is unique, should yield clues as to the nature of its governing episteme.

The preceding chapters have elucidated several conceptual parallels between early modern mathematics and the rise of the west, as well as conceptual dissonance between the rise of the west and the earlier cases, namely ancient Greek and medieval Arabic mathematics. Briefly, both early modern exploration and calculus exhibit a pattern of breaking from the familiar in favor of investigating uncharted regions, whereas earlier mathematicians preferred to restrict their inquiries to more conventional topics. Similarly, early modern trigonometry and exploration both reap unexpected advantages from inverting traditional approaches and framing from within, while the earlier cases adhere to the customary practice of erecting an external framework. Nation-states and trigonometry share a theme of competing identities, and in both cases, the more particular identity is superseded by the more general. The ancient and medieval mathematicians, in contrast, endorse only one definition of trigonometric functions that corresponds with the least general of the early modern identities. Both nation-states and calculus also evince a particular amalgam of magnitude and multitude that is absent in the earlier cases, where a sharper distinction prevails.

Likewise, capitalism and analytic geometry reformulate existing concepts in more abstract terms, giving rise to new priorities and policies, while the earlier cases favor a

more tangible interpretation, which in turn limits the scope of their ambitions. Also, calculus and capitalism both achieve commensurability by redefining their respective objects in terms of a common metric. The ancient and medieval mathematicians, however, continued to struggle with incommensurability for lack of a common measure. Similarly, both industrialization and analytic geometry emphasize parts rather than wholes, leading to increased efficiency and standardization, whereas the earlier cases take the reverse approach, giving precedence to wholes. Industrialization and trigonometry both apprehend key concepts in terms of ratios, stressing their relative value, while the earlier mathematicians conceive of trigonometric functions as fixed lengths, devoid of relative attributes. Classical mechanics and analytic geometry both integrate previously disparate elements into a single comprehensive system by focusing on their extrinsic properties, whereas the earlier cases concentrate on intrinsic qualities, which impedes their ability to generalize and promotes a fragmented understanding. Additionally, early modern science and calculus both attempt to model change by incorporating perspective into their analyses. In contrast, the ancient and medieval cases sought solutions that were unaffected by different frames of reference.

That these parallels can be discerned in early modern mathematics but not in the earlier cases, where in fact the opposite is observed, lends credence to the idea that there is an overarching conceptual framework, unique to early modern Europe, guiding both the development of mathematics and the transformations associated with the rise of the west. In addition, analytic geometry was concurrently developed by two mathematicians, unbeknownst to each other, at the beginning of the seventeenth century. Just a few short decades later, two other contemporaries independently discovered the fundamental

theorem of calculus. The fact that these breakthroughs occurred almost simultaneously to multiple mathematicians, but only in the early modern period, also suggests an epistemic influence. Likewise, since the precursors of analytic geometry, trigonometry, and calculus are evident in the earlier cases, and mathematics is unclouded by material considerations, the only impediment to their elaboration must be conceptual.

Having identified several points of ideational similitude between early modern mathematics and the rise of the west, now it remains to clarify what these conceptual parallels reveal about the overarching episteme. In other words, while the preceding chapters served to excavate the bare bones of the episteme, this chapter investigates how the pieces fit together and what shape they take when assembled. Several broad themes are manifest in the parallels outlined above, which give some hints as to the character of the episteme. Specifically, a tendency to assimilate elements into a common framework by redefining them as particular instances of a more general case surfaces repeatedly. Likewise, the emphasis of relative, rather than innate, characteristics and the manipulation of perspective recur with some frequency. The ability to obtain outcomes consistently takes precedence over rigorous foundations. There are also multiple instances of blurred distinctions and increased abstraction. These motifs admit of a certain harmony, and frequently reinforce each other. The next section will discuss the patterns that distinguish the early modern episteme, and their relation to the rise of the west, in greater detail.

5.1 Characteristics of the Episteme

One of the most persistent themes to emerge from the foregoing analysis is the early modern emphasis on relativity. Both classical mechanics and analytic geometry achieve results by focusing on the relations between entities rather than their innate qualities. Trigonometry and industrialization stress relative value by conceptualizing their key concepts in terms of ratios. Capitalism and calculus both attain commensurability by determining a common standard against which their objects can be measured. Additionally, the capitalist notion of comparative advantage is based on a consideration of relative opportunity costs. Competition and efficiency, as well, are concepts that are rooted in relative determinations. Relativity is particularly prominent in early modern mathematics. Analytic geometry defines curves with respect to a coordinate axis, and a coordinate axis thus serves as a means of relating curves to each other. Trigonometric functions are redefined as ratios of sides, and furthermore relate all triangles to right triangles. Calculus is fundamentally the study of change, and change implies a new condition relative to a previous state. On its own, this attention to relativity furnishes a more complicated, but better integrated perception of the world. In addition, relativity encourages the application of ideas and strategies that prove effective in one case to a broader array of related problems, as for example the transposition of the steam engine from pumping water out of mines to driving factory machinery to powering transportation. However, relativity also interacts with the other epistemic tendencies in intriguing ways.

Another trend that characterizes the early modern episteme is the inclusion of perspective in both representations and analyses. This is most apparent in the attempts of

both calculus and early modern science to model change, wherein they find solutions by alternating points of view. The inversion of traditional approaches evident in both trigonometry and exploration similarly arises from a shift in perspective. Likewise, industrialization and analytic geometry both adjust from perceiving wholes to perceiving parts. Perspective plays a role in the idea of nation-states as well, in that a state may emerge from a nation or a national culture may emanate from the state. As with relativity, perspective also figures significantly in early modern mathematics. Analytic geometry incorporates perspective both in the sense that it reconceptualizes geometric figures as lines, and in the way that the coordinate axes can be used to either derive an equation from a curve or extract a curve from an equation. Trigonometry implicitly invokes perspective by defining its functions in terms of the hypotenuse, opposite and adjacent sides, which vary depending on which angle is being considered. The fundamental theorem of calculus demonstrates the inverse relationship between differentials and integrals, such that an equation can be viewed as either depending on one's perspective.

There are a number of consequences that stem from drawing on perspective. Considering something from multiple points of view contributes to a more thorough understanding, which in turn expands the possibilities of its manipulation. For instance, Newton's derivation of the fundamental theorem of calculus, achieved through his adjustment of the frame of reference, had the practical consequence of greatly simplifying the calculation of integrals, which in turn facilitated numerous advances in engineering. Similarly, perspective is critical to the early modern adoption of a heliocentric model of the universe. By all accounts, the stars and planets appear to

revolve around the Earth, supporting a geocentric hypothesis, however with the realization that those observations are the consequence of a given perspective, the heliocentric model, premised on an alternate point of view, takes on new validity despite the apparent empirical contradiction. Additionally, the incorporation of perspective necessarily introduces more variability into descriptions, if they are to encompass shifting frames of reference. This leads to more dynamic models, and has the effect of making people more comfortable with change, as a function of familiarity. By extension, a society that exhibits less resistance to change should prove more voluble. In other words, an emphasis on perspective may presage a greater openness to revolution.

Furthermore, the overlap of relativity and perspective also elucidates the methodological back and forth that informs several features of the rise of the west. In early modern mathematics, it is perhaps best exemplified by the use of both analysis and synthesis to solve problems. With the fifth rule in *Rules for the Direction of the Mind*, Descartes (1964) explains that the method essentially consists of systematically breaking down a complex problem into simpler pieces and reassembling them until the problem is solved. The alternating use of analysis and synthesis is also evident in the scientific work of Galileo (Naylor 1990), Newton (1730), and Leibniz (1956). The division of labor similarly reflects this pattern of pulling things apart and putting them back together, as does the disintegration of empires and their reconstitution as nation-states. Perspective enables the early moderns to switch between analysis and synthesis, while relativity illuminates the inverse nature of the two techniques, and enunciates the relations between the decomposed parts and synthesized wholes.

Another strong motif that arises from the foregoing analysis is the early modern predilection for assimilating seemingly disparate elements into a comprehensive framework. Specifically, classical mechanics and analytic geometry both seek to unite their various subsets under a single system of universally applicable rules. Similarly, analytic geometry and industrialization exhibit a tendency to view things as parts of a larger whole rather than wholes unto themselves. Trigonometry and nation-states evince a pattern of particular identities being subsumed by a more general one, while nation-states and calculus both build on a particular negotiation of one and many that merges continuous magnitude with discrete multitude. Capitalism and calculus likewise consolidate their respective objects by means of a common metric that renders them commensurable. This theme of assimilation also pervades early modern mathematics. Analytic geometry unites previously incompatible parts of mathematics by reducing geometric figures to lines, reinterpreting curves as equations, and using coordinate axes to mediate between geometric figures and algebraic expressions. The six major trigonometric functions are rewritten to reflect ratios of the sides of a single right triangle. The fundamental theorem of calculus, which explicates the inverse relation between differentials and integrals, provides the basis for unifying the two branches of calculus.

This desire to integrate specific cases into a broader system has the effect of rendering all things as parts. In other words, the search for a common framework carries with it an assumption that such a framework exists, however hidden it may be, and that the set of particulars in evidence fit into it. Thus, everything comes to be regarded as part of some larger community, even when that community is more virtual than actual. In the case of nation-states, for instance, people feel solidarity with their fellow citizens, despite

never encountering most of them, by virtue of their shared national identity. Furthermore, the propensity to assimilate into a common framework increases isomorphism. When rival cases are regarded not merely as others but rather as variants on the same theme, they become more amenable to the idea of borrowing from one another. Conversely, the idea of a more general type exerts a certain degree of pressure to conform to its precepts, suggesting the particular instances may come to increasingly resemble each other through their adherence to the more general principles. Moreover, cases that aspire to be counted as particular instances adopt the characteristics of either the other specific examples or the more general form. This may explain why so many innovations spread like wildfire throughout early modern Europe but failed to catch on beyond its borders.

Relativity is inherent in the pursuit of a common framework, in the sense that the general principle being sought serves to relate the particulars to one another. Conversely, assimilation bolsters relativity by giving priority to those factors that are shared. Echoing the suggestion above that strategies can be transposed between related issues, organizing different problems as species of the same case implies that they can all be resolved with a single method. Reducing the number of tools needed to analyze or explain any given topic, in turn, increases efficiency. Furthermore, the interaction of perspective, relativity, and a comprehensive system produces some interesting effects. The combination of examining subjects from multiple angles and the impulse to relate things to one another points to a more adaptive understanding of the world. That in conjunction with a propensity for consolidation suggests that new elements will be more easily integrated into the existing system, even those that initially appear contradictory. Similarly, a more

pliable worldview and a greater capacity for absorbing new things logically implies a greater degree of comfort when exploring the unknown.

Increased abstraction also emerges as a common refrain in early modern thought. Capitalism and analytic geometry engender more flexibility by introducing greater abstraction into their calculations. Industrialization and analytic geometry likewise appeal to more abstract concepts to increase efficiency. Nation-states and trigonometry both emphasize the most abstract of several competing identities. Calculus and capitalism achieve commensurability by redefining their objects in more abstract terms. Increased abstraction also plays a significant role in early modern mathematics. Analytic geometry adopts a representational depiction of both geometric figures and algebraic equations as lines plotted on a two-dimensional graph. Trigonometry relies on a more abstract definition of trigonometric functions as ratios of the sides of a right triangle, as opposed to more readily appreciable lengths. Calculus, given its dependence on infinitesimal calculations, is inherently more abstract. The representation of the instantaneous rate of change as the slope of a tangent and the cumulative change over time as the area under a curve also bespeaks greater abstraction.

One of the key effects of this tendency is that it facilitates problem solving. For instance, it prevents an analysis from getting bogged down in contradictory details. Along the same lines, greater abstraction also streamlines ideas by eliminating extraneous information, thereby increasing efficiency and making it easier to discern a solution. Peeling away the particulars can also reduce potential sources of bias, which might otherwise render the solution invisible. Similarly, just as gaining an aerial view can divulge the path out of a maze, increased abstraction provides a way out of some logical

quandaries by adjusting the level of analysis upward. For instance, greater abstraction enabled Descartes and Fermat to solve the centuries-old riddle of the locus problem. Additionally, abstraction resonates with the other epistemic patterns. A broader perspective lends itself to a more abstract interpretation, while greater abstraction offers an alternative perspective. Increased abstraction also facilitates assimilation into a common framework by identifying the more general themes that unite seemingly dissimilar elements.

The early modern episteme is also characterized by an inclination to blur distinctions. Both calculus and nation-states muddy the distinction between one and many. Likewise, capitalism and calculus obscure qualitative differences for the sake of a common metric. Similarly, trigonometry and industrialization emphasize relative value, diminishing the distinctions that arise from innate properties. In order to institute a single system of rules, classical mechanics and analytic geometry blur the boundaries between terrestrial and celestial, and lines and more complex geometric figures, respectively. Blurred distinctions are integral to early modern mathematics as well. Analytic geometry obscures distinctions based on dimensionality by reducing geometric figures to lines. Trigonometry blurs distinctions between different types of triangles by rendering all of them in terms of the relations that exist in right triangles. Calculus, with its invocation of the infinitesimal, blurs the border between the infinite and the finite. The primary consequence of this tendency is to free the understanding from a narrower perception of boundaries, such that, for instance, the ocean ceases to be a barrier to expansion or wealth begins to encompass more than monetary assets. In addition, blurring distinctions also reinforces the patterns described above. Obscuring differences has the effect of

highlighting similarities, which makes it easier to relate things to one another. Similarly, blurring distinctions aids in generalization, which in turn facilitates incorporation into a common framework. Blurred distinctions imply a greater level of abstraction, while increased abstraction diminishes the salience of sharp divisions.

Finally, there is a pattern of building on uncertain foundations that surfaces repeatedly in the rise of the west. Specifically, this trend stems from the calculation that the potential benefit to be derived from the results outweighs the risk inherent in proceeding from dubious suppositions. It is most apparent in the early modern drive toward exploration, where a single successful voyage could yield a fortune but the probability of failure was high. Accordingly, in the above examples this pattern is best exemplified by the practice of breaking with the familiar and launching into the unknown evident in both early modern exploration and calculus. Investment, a key component of capitalism, represents a similar calculation of risk and reward, and is likewise premised on uncertainty. Along the same lines, capitalism and calculus both construct systems around unstable fundamentals, namely price and infinitesimals. Additionally, both analytic geometry and capitalism exchange a more tangible basis, bullion and geometric figures, respectively, for more abstract concepts. Industrialization and trigonometry likewise trade concrete meanings, in the form of output quantity and lengths, for more indefinite ratios. Early modern mathematics is also constructed on less rigorous foundations, relative to the earlier cases. Analytic geometry is premised on the notion that a unit can be placed in a ratio with magnitudes, which is a contradiction since units are indivisible and magnitudes are infinitely divisible. Trigonometry jettisons the circles that give its functions definite meanings in favor of relational descriptions. Calculus is

based on the idea of vanishing quantities, which have no magnitude yet paradoxically can sum to a magnitude.

A willingness to build on uncertain foundations opens up a multitude of new opportunities, but also increases the chance of collapse, either from risks not paying off or through the derivation of false conclusions due to a faulty premise. Similarly, the conceptual structures built on such foundations admit of greater flexibility, by virtue of their uncertain base and the emphasis placed on results. This theme also interacts with the others. For instance, the calculation that the rewards will outweigh the risks demands a forward-looking perspective. In addition, the risk-reward perspective represents a relative conception of value. The protection that greater abstraction affords against becoming distracted by the details also has the effect of glossing over points that might otherwise preclude the espousal of uncertain foundations. Similarly, blurring distinctions makes it easier to build on uncertainties, while imprecise foundations necessitate a certain amount of blurred distinctions.

To review, the preceding analysis reveals six facets of the early modern episteme. Specifically, there is a greater emphasis on relative characteristics. Perspective figures more prominently in both representations and reasoning. There is a tendency to assimilate seemingly disparate elements into a common framework, often by redefining them as particular instances of a more general case. There is a trend toward greater abstraction. Sharp distinctions tend to lose their hard edge. Finally, there is a pattern of building on uncertain foundations, on the expectation that the potential rewards will outweigh the risks. Furthermore, these facets interact with and reinforce each other. For instance, perspective makes it easier to see relations while relativity helps to make sense

of perspective. Blurring distinctions that would otherwise divide elements aids in uniting them under a single framework, while assimilating elements into a common system diminishes some distinctions. Greater abstraction helps obviate the need for a thoroughgoing foundation, while uncertainty demands a greater degree of abstraction, and so on. In other words, these patterns do not merely recur with some frequency in various aspects of the rise of the west, but in fact complement and build upon each other, suggesting an overall epistemic coherence.

It is not that the earlier societies never employed these ideas; just as there are traces in the ancient and medieval cases of the developments that define the rise of the west, there are conceptual glimmers as well. However, one or two instances do not constitute a pattern, and in fact, the patterns that are observed in ancient and medieval thought, as evidenced in their mathematical systems, more often than not run contrary to the principles outlined above. By the same token, it is not as though the early modern era adhered to these trends absolutely. While with the benefit of hindsight, the ideas and institutions that characterize the rise of the west are unmistakable, adoption was not perfect or universal. Resistance to the key developments that constitute the rise of the west is well documented. Colonial expansion met with resistance from both native populations abroad (Schroeder 1998; Nester 2000) and opponents at home (cf. Casas 2003; Smith 2007; Montaigne 2003). Nation-states were challenged by heterogeneous factions (cf. Goldstone 1991; Bell 2003; Tutino 2007) and regional claims to power (cf. Koenigsberger 1987; Elliott 1992). Anti-capitalist sentiments soon emerged in response to the new economic climate (Fourier 1996; Owen 1991). Campaigns against industrialization were undertaken by laborers displaced by machines (Bailey 1998) and

artists who venerated nature (Löwy and Sayre 2001). Early modern science faced numerous religious objections, from the Catholic Church's denunciation of Galileo (Blackwell 2008) to Berkeley's rejection of Newtonian mechanics (Berkeley 1991), as well as entrenched traditions within the scientific community (Kuhn 1996).

Calculus also aroused the ire of some early modern critics. Most complaints centered on the logical validity of the infinitesimal heuristic, and whether the results obtained by such methods could be trusted. The Paris Academy of Sciences, for instance, bitterly debated the rigor of infinitesimal considerations for almost six years at the beginning of the eighteenth century (Mancosu 1989). Michel Rolle, arguably the most strident opponent, assailed infinitesimal calculus for its inexactitude, which he claimed concealed mistakes, and decried calculus as “a collection of ingenious fallacies” (Blay 1986). George Berkeley delivered the most significant challenge to infinitesimal calculus with *The Analyst* (Berkeley 1991), his blistering critique of the logical foundations of Newton's method. After famously deriding infinitesimals as “the ghosts of departed quantities,” Berkeley's criticisms would continue to haunt calculus's defenders for over two hundred years. However, history indicates that calculus, like global dominance, nation-states, capitalism, industrialization, and science, ultimately prevailed. The same is true of the ideas that undergird those developments. Despite a few examples to the contrary, the overall pattern is one that emphasizes relativity, incorporates perspective, assimilates into a common framework, increases abstraction, blurs distinctions, and builds on uncertain foundations.

5.2 Conclusions

It bears repeating that the principle theory guiding this research posits that structures arise from the interaction of ideas and resources. Recall that schemas, of which the episteme is a particular instance, are the virtual norms that orient thoughts and actions, while resources consist of anything that can be used to increase or maintain power. Each entails a different set of opportunities and limitations which must be reconciled, and structure is the result of that negotiation between the ideal mode and the available means. Accordingly, the conceptual framework outlined above represents only part of the story, and is meant to complement the existing resource-heavy body of research. In other words, it is the episteme in interaction with resources that produces the rise of the west. Just as the presence of ingredients will not result in a cake without the intervention of a recipe, neither will a recipe produce a cake absent the necessary ingredients. Without resources such as ships capable of sailing the ocean, the printing press, private property, the steam engine, and telescopes, it would have been difficult for the episteme to effect the rise of the west. However, just as critical to the success of the rise of the west is how those resources were understood and how they were then put to use. It is this ideational component that has been neglected, and so that is what this dissertation focuses on, but the equally important role that resources play in the rise of the west should not be disregarded.

What, then, has this dissertation accomplished? First and foremost, it articulates a description of the early modern episteme. By analyzing the conceptual commonalities present in both the logic of early modern mathematics and the phenomena associated with the rise of the west, several contours of the episteme can be discerned. Specifically,

six overarching trends in early modern thought were identified: an emphasis on relativity, incorporation of perspective, assimilation into a common framework, increased abstraction, blurring of distinctions, and a willingness to build on uncertain foundations. However, this research does not merely identify conceptual patterns in early modern European examples; it also compares the rise of the west with two earlier cases that in many ways teetered on the cusp of modernity, but ultimately retreated. In both the ancient Greek and medieval Arabic cases, not only are the conceptual patterns that characterize the rise of the west absent, in many places their logic is the reverse of that observed in the early modern case. In other words, this research demonstrates that the episteme described above is unique to early modern Europe. Since sociology as a discipline arose largely in response to the phenomena associated with the rise of the west, sociologists of all stripes have a stake in uncovering the conceptual structure that shaped it. Although the field has changed and expanded over time, some epistemological obstacles may have lingered, such as a bias towards resources, which continue to restrict the sociological imagination.

This dissertation also introduces early modern mathematics as an understudied aspect of the rise of the west. The development of analytic geometry, trigonometry, and calculus fundamentally transformed the nature of mathematics, and this new mathematics, in turn, underlies advances in other domains, such as physics, engineering, and navigation. Despite its significance, and the timing of these transitions, mathematics has received little notice in conjunction with the rise of the west. Perhaps this lack of attention is due to the emphasis on resources that pervades the rise of the west literature and the relative unimportance of resources to the development of mathematics. However,

mathematics has also largely evaded the scrutiny of sociologists in general. For a field that makes frequent use of quantitative methods to study a society increasingly dependent on mathematics, whether in the form of more precise engineering, statistical analyses, or computer algorithms, the relative paucity of sociological investigations into mathematics is surprising. What little scholarship there is in this area is mostly an offshoot of the sociology of scientific knowledge, and as such tends to focus on the social characteristics of mathematicians (Fisher 1973; Fang 1975) or social constructivist accounts of mathematical knowledge (Bloor 1976; Restivo 1992; Ernest 1998). The potential sociology holds for mathematics, and vice versa, remains for the most part untapped.

To illustrate the cost of this oversight, consider the advantages mathematics brings to this dissertation. First, it enables comparisons across a wider temporal gulf than other data sources generally permit by virtue of the fact that so much of the mathematical record still exists. The assertion that the ancient Greek and medieval Arabic epistemes were substantively different from the early modern European episteme, and in fact conflict with the conceptual configuration underpinning the rise of the west, is based on patterns observed in their respective mathematical systems. Along the same lines, mathematics also provides an evidentiary basis for the episteme. While traces of the episteme can be detected in any artifact generated under its auspices, the influence of other factors must be controlled for to greater and lesser degrees. Since mathematics is unclouded by material constraints, the effects of the episteme are that much easier to isolate. Furthermore, given mathematics' reputation for being both objective and universal, to demonstrate that mathematics is susceptible to epistemic influence suggests that all other aspects of society are likewise affected. Mathematics makes this research

possible, and this is just one example of what the amalgam of sociology and mathematics could do.

Finally, this dissertation highlights the often overlooked role ideas play in structuring society. The epistemological obstacles that shepherd the unfolding of a society are deeply ingrained and permeate every aspect, and so are frequently taken for granted. Once people become accustomed to thinking about a resource in a particular way, it is easy to forget that that understanding is not an incontrovertible attribute of the resource itself. Yet, as the ubiquity of instruction manuals attests, resources do not automatically convey how they are to be used. There is a conceptual element that must be accounted for above and beyond the mere presence of resources, particularly when those resources are present throughout history but utilized only during certain periods. Moreover, since resources admit of multiple interpretations, when consensus is observed within but not between societies, as in the case of epistemes, it suggests there is a social component to the genesis, dissemination, and maintenance of the conceptual framework. With these considerations in mind, this research explores how ideas could influence and explain the rise of the west, and hopefully opens the door to further inquiries into the unthinkable.

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