Homogenization: in Mathematics or Physics?

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Abstract

Homogenization appeared more than 100 years ago. It is an approach to study the macro-behavior of a medium by its micro-properties. In mathematics, homogenization theory considers the limitations of the sequences of the problems and its solutions when a parameter tends to zero. This parameter is regarded as the ratio of the characteristic size in the micro scale to that in the macro scale. So what is considered is a sequence of problems in a fixed domain while the characteristic size in micro scale tends to zero. But for the real situations in physics or engineering, the micro scale of a medium is fixed and can not be changed. In the process of homogenization, it is the size in macro scale which becomes larger and larger and tends to infinity. We observe that the homogenization in physics is not equivalent to the homogenization in mathematics up to some simple rescaling. With some direct error estimates, we explain in what means we can accept the homogenized problem as the limitation of the original real physical problems. As a byproduct, we present some results on the mathematical homogenization of some problems with source term being only weakly compacted in H^{-1}, while in standard homogenization theory, the source term is assumed to be at least compacted in H^{-1}. A real example is also given to show the validation of our observation and results.

Keywords: Homogenization, re-scaling, unit transformation

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1 Introduction

Homogenization appeared more than 100 years ago. It is an approach to study the macro-behavior of a medium by its micro-properties. The origin of this word is related to the question of replacement of the heterogenous material by an "equivalent" homogenous one. The earliest papers dealing with the problem of this type are [1], [2], and the good survey of the results until 1925 is in [3]. The name of "homogenization" was first introduced by I. Babuska [4]. In physics, mechanics and engineering, homogenization is widely used to study the property of medium or material by the macro-behavior in stead of the complicated micro structure. The systematic mathematical theory of homogenization was built in [5]-[15] and so on. But, does the mathematical theory describe the physics or engineering questions exactly? It seems hard to give a positive answer. In this paper we take the flow transport problem in the periodic heterogenous porous medium as an example to demonstrate the difference between the homogenization in mathematics and physics. Any other examples such as heat or electric conductivity and mass transfer will lead to the same conclusion.

In mathematics, we consider the limitation of a sequence $\{u^\epsilon\}_{\epsilon>0}$ such that

$$\begin{cases} -\nabla\cdot(A^\varepsilon(x)\nabla u^\varepsilon(x)) = f, \quad \text{in } \Omega, \\ \\ u^\varepsilon = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \in \mathbb{R}^d (d=3)$ is occupied by the heterogenous porous medium and the permeability coefficient $A^\varepsilon(x) = A(\frac{x}{\varepsilon})$ with $A(y)$ being periodic with respect to $y \in Y = [0,1]^d$. u^ε is the flow pressure in the medium and f is the source. Just for simplicity, we take $\Omega = (0,1)^d$. By [5] and [15], under the following assumption:

$$(H0): \quad f \in L^2(\Omega) \textit{ and there exist two positive constants } \lambda,\ \Lambda \textit{ independent of } \varepsilon, \textit{ such that } \lambda|\xi|^2 \le \xi^T K(y)\xi \le \Lambda|\xi|^2, \quad \forall y, \xi \in \mathbb{R}^d, \tag{1.2}$$

we have that as $\varepsilon \to 0$, there exists a $u^0 \in H_0^1(\Omega)$ such that $u^\varepsilon \rightharpoonup u^0$ *weakly in* $H_0^1(\Omega)$, where the u^0 is the solution of the homogenized problem

$$\begin{cases} -\nabla\cdot(A^0\nabla u^0(x)) = f, \quad \text{in } \Omega, \\ \\ u^0 = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

with $A^0_{ij} = \frac{1}{|Y|}\int_Y (A_{ij} + A_{ik}\frac{\partial N^j}{\partial y_k})dy$ and N^j being the solution of the cell problem :

$$\begin{cases} -\nabla_y\cdot(A(y)\nabla_y N^j(y)) = \nabla_y\cdot(A(y)e_j), \quad \text{in } Y=[0,1]^d, \\ N^j(y) \ \text{ is Y-periodic and } < N^j > \triangleq \frac{1}{|Y|}\int_Y N^j(y)dy = 0. \end{cases} \tag{1.4}$$

Please note that in the above limit process, when the period size ε tends to zero, the domain Ω does not change, so there will be more and more ε-periods contained in the whole domain. But this is not the case in the physics. In a physical or engineering problem, the size of the periodic micro cell structure can not be changed. What we consider in the physics is the following problem (take the flow in porous media as an example)

$$\begin{cases} -\nabla \cdot \left(K\left(\frac{x}{l}\right)\nabla p(x)\right) = f, \quad \text{in } \Omega_D = (0, D)^d, \\ \\ p = 0, \quad \text{on } \partial\Omega_D, \end{cases} \tag{1.5}$$

where $l(m)$ and $D(m)$ are the characteristic lengthes of the periodic micro cell and the whole medium respectively and $l \ll D$, p $(N/m^2 = Kg/m/s^2)$ is the pressure, K (m^3s/kg) is the permeability coefficient and the source term f $\left(\frac{1}{s}\right) = \frac{f_0}{\rho}$ with f_0 $(kg/m^3/s)$ and the density ρ (kg/m^3).

We want to get the effective coefficient and the homogenized problem in physics and try to explain in what sense we can expect this result is valid. In the early work of I. Babuska [16] on the homogenization approach in engineering, he first pointed out that: 'l is a given parameter, with physical meaning which cannot be changed, e.g. cannot be made "sufficiently" small'. In his other work [17], [18], he mentioned that ε in (1.1) is the ratio of micro-scale (cell scale) to the macro-scale. If we set $\varepsilon = \frac{l}{D}$ and l is fixed, then $\varepsilon \to 0$ means that D tends to infinity. The problem is if we take the transformation $\varepsilon = \frac{l}{D}$, can we transfer the physical problem (1.5) to the mathematical problem (1.1) while the Assumption (H0) is still valid?

Let us take a variable transformation $\widehat{x} = \frac{x}{D}$ *mathematically*, then the pressure $p(\widehat{x})$ satisfies that

$$\begin{cases} -\nabla_{\widehat{x}} \cdot \left(D^{-2}K\left(\frac{\widehat{x}}{\varepsilon}\right)\nabla_{\widehat{x}} p(\widehat{x})\right) = f(\widehat{x}), \quad \text{in } \Omega = (0, 1)^d, \\ \\ p(\widehat{x}) = 0. \quad \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

It is obvious that the Assumption (H0) can not be satisfied even for a simple example as $f \equiv 1$, since $D^{-2}K(\frac{\widehat{x}}{\varepsilon}) \to 0$, as $\varepsilon \to 0$.

Let's go back to *physics*. What's the meaning of the variable transformation $\widehat{x} = \frac{x}{D}$ from (1.5) to (1.6)? The only difference is the *unit* of length: in (1.5), the length unit is one meter; in (1.6), the length unit is D meters.

Therefore, if we take a variable transformation $\widehat{x} = \frac{x}{D}$ *physically*, then the pressure $\widehat{p}(\widehat{x})$ should satisfy the same Conservation Law of Mass, which is independent of the unit in length,

$$\begin{cases} -\widehat{\nabla} \cdot \left(\widehat{K}\left(\frac{\widehat{x}}{\varepsilon}\right)\widehat{\nabla}\widehat{p}(\widehat{x})\right) = \widehat{f}, \quad \text{in } \Omega = (0, 1)^d, \\ \\ \widehat{p} = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

but all the physical quantities: the pressure $\widehat{p}$, the permeability coefficient $\widehat{K}$, the source term $\widehat{f}$ and the gradient operator $\widehat{\nabla}$ are all measured in the new length scale, i.e.

$$\widehat{x} = \frac{x}{D}\ (Dm),\quad \widehat{K} = \frac{K}{D^3}\ ((Dm)^3 s/kg),\quad \widehat{p} = Dp\ (kg/Dm/s^2),$$

$$\widehat{\nabla} = D\nabla\ (1/Dm),\quad \widehat{f} = f\ (1/s).$$

It should be noted that $\widehat{K} = \frac{K}{D^3}$ will tend to zero as D tending to infinity, since K is the fixed physical quantity in the original unit system. So the Assumption (H0) can not be valid either and we can not expect the validity of the homogenization theory even for a constant source term.

So far, we see that the homogenization of the physical problem (1.5) can not easily fall into the mathematical framework (1.1)-(1.4) by a direct transformation in mathematics or physics. In the following, we assume that D is sufficient large but fixed. We consider two different situations for the source term f. The first situation is that f has no micro-structure, and the other is that f has micro structure with period $(0, l)^d$, for example, the source term may have the form as $f = f_1(x, x/l) + \nabla \cdot f_2(x, x/l)$. We first give the homogenized problem with the effective coefficient K^0 for different situations. We present the error estimate between the pressure p and the first order expansion p_1, then try to understand in what sense the homogenized problem is a limitation of the original problem. It is worthwhile to point out that so far the mathematical homogenization theory (for which we consider the limitation as $l \to 0$) is still incomplete for the second situation, since the source term is only convergent weakly in H^{-1}(see [19] and [20]). The homogenization theory for this kind of problem may have independent interests.

The outline of paper is as follows: in §2 we discuss the situation that the source term has no micro-structure; in §3 we discuss the situation that the source term has micro-structure; in §4 a real example is given to show the validation of our observation and results.

2 The source term has no micro-structure

we say f has no micro-structure if f does not contain any micro-scale information at the scale comparable to or less than l. We discuss two different cases in this situation.

- Case a: $f \in L^2(R^d)$; Case b: $f \in L^\infty(R^d)$.

By unit transformation, we get (1.7), which we have known do not satisfy Assumption (H0). In order to let the Assumption (H0) be valid, we introduce a new setting of problem as

$$\begin{cases} -\widehat{\nabla} \cdot \Big(K^\varepsilon(\widehat{x})\widehat{\nabla} p^\varepsilon(\widehat{x})\Big) = \overline{f}(\widehat{x}), & \text{in } \Omega = (0,1)^d, \\ p^\varepsilon = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

with

$$\textbf{a:}\begin{cases} K^\varepsilon(\widehat{x}) = K(\frac{\widehat{x}}{\varepsilon}) = D^3\widehat{K}(\frac{\widehat{x}}{\varepsilon}),\\ p^\varepsilon(\widehat{x}) = D^{-\frac{3}{2}}\widehat{p}(\widehat{x}),\\ \overline{f}(\widehat{x}) = D^{\frac{3}{2}}f(\widehat{x}),\end{cases} \quad \textbf{b:}\begin{cases} K^\varepsilon(\widehat{x}) = K(\frac{\widehat{x}}{\varepsilon}) = D^3\widehat{K}(\frac{\widehat{x}}{\varepsilon}),\\ p^\varepsilon(\widehat{x}) = D^{-3}\widehat{p}(\widehat{x}),\\ \overline{f}(\widehat{x}) = f(\widehat{x}).\end{cases} \tag{2.2}$$

It is easy to check that Assumption (H0) is satisfied for the both cases.

Then we know from homogenization theory (see [21]) that there exists a $\widetilde{p_0}$ such that

$$\begin{cases} p^\varepsilon \rightharpoonup \widetilde{p_0} \text{ weakly in } H_0^1(\Omega),\\ \\ K^\varepsilon\widehat{\nabla}p^\varepsilon \rightharpoonup \widetilde{K^0}\widehat{\nabla}\widetilde{p_0} \text{ weakly in } (L^2(\Omega))^d,\end{cases} \tag{2.3}$$

where $\widetilde{p_0}$ is the solution of the homogenized problem of (2.1) :

$$\begin{cases} -\widehat{\nabla}\cdot\left(\widetilde{K^0}\widehat{\nabla}\widetilde{p_0}(\widehat{x})\right) = \overline{f}, \quad \text{in } \Omega = (0,1)^d,\\ \\ \widetilde{p_0} = 0, \quad \text{on } \partial\Omega,\end{cases} \tag{2.4}$$

with $\widetilde{K^0_{ij}} = \int_Y (K_{ij}(y) + K_{ik}(y)\frac{\partial\widetilde{N^j}(y)}{\partial y_k})dy$ and $\widetilde{N^j}(\frac{\widehat{x}}{\varepsilon}) = \widetilde{N^j}(y)$ solving of the cell problem:

$$\begin{cases} -\widehat{\nabla}_y\cdot(K(y)\widehat{\nabla}_y(\widetilde{N^j}(y) + y_j)) = 0, \quad \text{in } Y = (0,1)^d,\\ \\ \widetilde{N^j} \text{ is Y-periodic}, \int_Y \widetilde{N^j}(y)dy = 0.\end{cases} \tag{2.5}$$

If we denote by

$$\widetilde{p_1} = \widetilde{p_0} + \varepsilon\widetilde{N^j}\frac{\widehat{\partial}\widetilde{p_0}}{\widehat{\partial}\widehat{x}_j}, \tag{2.6}$$

then there exits a positive constant C independent of ε such that

$$\|p^\varepsilon - \widetilde{p_1}\|_{H^1(\Omega)} \le C\varepsilon^{\frac{1}{2}}. \tag{2.7}$$

After carefully checking the exact dependence of the constant C in (2.7), we have

Proposition 2.1 *If the coefficient and the source term of equation (2.1) satisfy the Assumption (H0), then there exists a positive constant C independent of* ε, p^ε, $\widetilde{p_0}$, *such that*

$$\begin{aligned} \|\widehat{\nabla}(p^\varepsilon - \widetilde{p_1})\|_{L^2(\Omega)} &\le C\varepsilon\frac{1}{\lambda^2}(\|G\|_{L^\infty(Y)} + \Lambda\|N\|_{L^\infty(Y)})\|\overline{f}(\widehat{x})\|_{L^2(\Omega)}\\ &\quad + C\varepsilon^{\frac{1}{2}}(\frac{\Lambda}{\lambda^2})(1 + \|N\|_{L^\infty(Y)})\|\overline{f}(\widehat{x})\|_{L^2(\Omega)}, \end{aligned} \tag{2.8}$$

$$\|p^\varepsilon - \widetilde{p_0}\|_{L^2(\Omega)} \le C\varepsilon\frac{1}{\lambda^2}(\|G\|_{L^\infty(Y)} + \Lambda\|N\|_{L^\infty(Y)})\|\overline{f}(\widehat{x})\|_{L^2(\Omega)}. \tag{2.9}$$

Here $G = (G^1, ..., G^d)$ with G^j being a skew-symmetrical matrix satisfying (see [15])

$$\frac{\partial}{\partial y_k} G^j_{ik} = K(y)_{ij} + K(y)_{ik} \frac{\partial N^j}{\partial y_k} - \widetilde{K}^0_{ij}, \;\; j = 1, ..., d. \tag{2.10}$$

As to energy, we have

$$\left| \int_\Omega \widehat{\nabla} p^\varepsilon \cdot K^\varepsilon \widehat{\nabla} p^\varepsilon - \int_\Omega \widehat{\nabla} \widetilde{p_0} \cdot \widetilde{K^0} \widehat{\nabla} \widetilde{p_0} \right| \to 0, \;\; \textit{as } \varepsilon \to 0. \tag{2.11}$$

By the inverse transformation of (2.2) and changing the unit in length from Dm to m, we obtain the following *homogenized* problem of (1.5) from (2.4) and (2.5)

$$\begin{cases} -\nabla \cdot (K^0 \nabla p_0(x)) = f, \;\; \text{in } \Omega_D = (0, D)^d, \\ \\ p_0 = 0, \quad \text{on } \partial\Omega_D, \end{cases} \tag{2.12}$$

with K^0 determined as follows:

$$\begin{aligned} K^0_{i,j} &= \int_Y \left(K_{ij}(y) + K_{ik}(y) \frac{\partial N^j(y)}{\partial y_k} \right) dy \\ &= \frac{1}{|l|^d} \int_{(0,l)^d} \left(K_{ij}\left(\frac{x}{l}\right) + l K_{ik}\left(\frac{x}{l}\right) \frac{\partial N^j\left(\frac{x}{l}\right)}{\partial x_k} \right) dx, \end{aligned} \tag{2.13}$$

where $N^j\left(\frac{x}{l}\right) = N^j(y)$ $(j = 1...d)$ is the solution of cell problem

$$\begin{cases} -\nabla_y \cdot (K(y) \nabla_y (N^j(y) + y_j)) = 0, \;\; \text{in } Y = (0,1)^d, \\ \\ N^j \text{ is Y-periodic and } \int_Y N^j(y) dy = 0. \end{cases} \tag{2.14}$$

It's easy to find that (2.14) is equivalent to the following problem

$$\begin{cases} -\nabla \cdot (K(\frac{x}{l}) \nabla (l N^j(\frac{x}{l}) + x_j)) = 0, \;\; \text{in } Y_l = (0,l)^d, \\ \\ N^j \text{ is } Y_l \text{ periodic and } \frac{1}{l^d} \int_{(0,l)^d} N^j(\frac{x}{l}) dx = 0. \end{cases} \tag{2.15}$$

Remark 2.2 *In fact, if we simply regard the micro size l in (1.5) as a small parameter and formally apply the mathematical homogenization theory, we would obtain the same homogenization settings for (1.5) as (2.12)-(2.15). Furthermore we would still have the following mass balance or homogenization rule as : for any $\eta \in R^d$, if p_η solves*

$$\begin{cases} -\nabla \cdot (K(\frac{x}{l}) \nabla p_\eta) = 0, \;\; \textit{in } Y_l = [0,l]^d, \\ \\ p_\eta - \eta \cdot x \;\; \textit{is periodic in } [0,l]^d, \end{cases} \tag{2.16}$$

then $\langle \nabla p_\eta \rangle = \frac{1}{l^d}\int_{Y_l} \nabla p_\eta dx = \eta$ *and by (2.13) and (2.15) ,we have*

$$\langle K(\frac{x}{l})\nabla p_\eta\rangle_{R^d} = \langle K(\frac{x}{l})\nabla p_\eta\rangle_{Y_l} = K^0\langle\nabla p_\eta\rangle_{Y_l} = K^0\langle\nabla p_\eta\rangle_{R^d}, \tag{2.17}$$

which means the mass balance between the micro and macro scales. This is the reason why the homogenized coefficient K^0 *is also called as the effective coefficient of* K*.*

The relationship between $p_0(x)$ and $\widetilde{p_0}(\widehat{x})$ are

$$\textbf{a:}\begin{cases} p_0(x) = D^{-\frac{1}{2}}\widetilde{p_0}(\widehat{x}), \\ \\ K^0 = \widetilde{K^0}, \end{cases} \qquad \textbf{b:}\begin{cases} p_0(x) = D^2\widetilde{p_0}(\widehat{x}), \\ \\ K^0 = \widetilde{K^0}. \end{cases} \tag{2.18}$$

By (2.18) and Proposition (2.1), we can obtain the next theorem

Theorem 2.3 *If* p *is the solution of* (1.5), p_1 *is defined as follows*

$$p_1 = p_0 + lN^j\frac{\partial p_0}{\partial x_j}. \tag{2.19}$$

and $f \in L^2(R^3)$*, then there exists a positive constant* C *independent of* D *such that*

$$\int_{\Omega_D} |\nabla(\frac{p}{D} - \frac{p_1}{D})|^2 dx \ \ \leq \ \ C(\frac{l}{D})^2\|f(x)\|^2_{L^2(\Omega_D)} + C\frac{l}{D}\|f(x)\|^2_{L^2(\Omega_D)}. \tag{2.20}$$

$$\int_{\Omega_D} |\frac{p}{D^2} - \frac{p_0}{D^2}|^2 dx \leq C(\frac{l}{D})^2\|f(x)\|^2_{L^2(\Omega_D)}. \tag{2.21}$$

As to energy, there exists a positive const C *independent of* D*, such that*

$$|\frac{1}{D^3}\int_{\Omega_D} \nabla p\cdot K\nabla p - \frac{1}{D^3}\int_{\Omega_D}\nabla p_0\cdot K^0\nabla p_0| \leq C\frac{1}{D}, \tag{2.22}$$

which means the convergence of the density of energy;

Theorem 2.4 *If* p *is the solution of* (1.5), p_1 *is defined in* (2.19) *and* $f \in L^\infty(R^3)$*, then there exists a positive const* C *independent of* D *such that*

$$\frac{1}{D^3}\int_{\Omega_D} |\nabla(\frac{p}{D} - \frac{p_1}{D})|^2 dx \ \ \leq \ \ C(\frac{l}{D})^2\|f(x)\|^2_{L^\infty(\Omega_D)} + C\frac{l}{D}\|f(x)\|^2_{L^\infty(\Omega_D)}. \tag{2.23}$$

$$\frac{1}{D^3}\int_{\Omega_D} |\frac{p}{D^2} - \frac{p_0}{D^2}|^2 dx \leq C(\frac{l}{D})^2\|f(x)\|^2_{L^\infty(\Omega_D)}. \tag{2.24}$$

As to energy, we have

$$|\frac{1}{D^3}\int_{\Omega_D} \nabla\frac{p}{D}\cdot K\nabla\frac{p}{D}dx - \frac{1}{D^3}\int_{\Omega_D}\nabla\frac{p_0}{D}\cdot K^0\nabla\frac{p_0}{D}dx| \to 0, \ \ as\ D\to\infty. \tag{2.25}$$

The above theorems explain in what sense we can accept the homogenized problem (2.12).

3 The source term has micro-structure

In this situation, we will discuss the source term with the following micro-structure,

$$\begin{cases} -\nabla \cdot \left(K(\frac{x}{l})\nabla p(x)\right) = f\left(x, \frac{x}{l}\right) + \nabla \cdot F\left(x, \frac{x}{l}\right), & \text{in } \Omega_D = (0, D)^d, \\ p = 0, \quad \text{on } \partial\Omega_D. \end{cases} \tag{3.1}$$

By unit transformation, we get

$$\begin{cases} -\widehat{\nabla} \cdot \left(\widehat{K}\left(\frac{\widehat{x}}{\varepsilon}\right)\widehat{\nabla}\widehat{p}(\widehat{x})\right) = f\left(\widehat{x}, \frac{\widehat{x}}{\varepsilon}\right) + \widehat{\nabla} \cdot \widehat{F}\left(\widehat{x}, \frac{\widehat{x}}{\varepsilon}\right), & \text{in } \Omega = (0, 1)^d, \\ \widehat{p} = 0, \quad \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Setting

$$\begin{cases} K^\varepsilon(\widehat{x}) = D^3\widehat{K}(\frac{\widehat{x}}{\varepsilon}), \\ p^\varepsilon(\widehat{x}) = D^{-3}\widehat{p}(\widehat{x}), \end{cases} \tag{3.3}$$

we obtain

$$\begin{cases} -\widehat{\nabla} \cdot \left(K^\varepsilon\left(\widehat{x}\right)\widehat{\nabla}p^\varepsilon(\widehat{x})\right) = f\left(\widehat{x}, \frac{\widehat{x}}{\varepsilon}\right) + \widehat{\nabla} \cdot \widehat{F}\left(\widehat{x}, \frac{\widehat{x}}{\varepsilon}\right), & \text{in } \Omega = (0, 1)^d, \\ p^\varepsilon = 0, \quad \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

The homogenization for this kind of problem may have independent interest, since the source term here is only weakly convergent in $H^{-1}(\Omega)$ as $\varepsilon \to 0$. The standard theory only treats the case that the source term is strongly convergent in $H^{-1}(\Omega)$ see([15]).In [19] and [20] some incomplete results were present for this case. We will establish the homogenization theory for (3.4).

In the beginning, we introduce an important lemma that will be used later.

Lemma 3.1 *If $f(x, y) \in L^\infty(Y, C^{0,1}(\Omega))$ and is Y-period with respect to y, where Ω is an arbitrary bounded open subset of R^d and $Y = [0, 1]^d$, then $f(x, \frac{x}{\varepsilon}) \to Mf(x) \triangleq \frac{1}{|Y|}\int_Y f(x, y)dy$ in $H^{-1}(\Omega)$ and there exists a constant $C \geq 0$ such that*

$$\|f(x, \frac{x}{\varepsilon}) - Mf(x)\|_{H^{-1}(\Omega)} \leq C\varepsilon\|f\|_{L^\infty(Y,\ C^{0,1}(\Omega))}, \tag{3.5}$$

The proof of lemma is similar to the lemma 1.6 in [22].

Theorem 3.2 *If u^ε is the solution of the following problem*

$$\begin{cases} -\nabla \cdot (A^\varepsilon(x)\nabla u^\varepsilon) = f\left(x, \frac{x}{\varepsilon}\right) + \nabla \cdot F\left(x, \frac{x}{\varepsilon}\right), \ \mathit{in}\ \Omega, \\ u^\varepsilon = 0, \ \mathit{on}\ \partial\Omega, \end{cases} \tag{3.6}$$

where $A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right)$ *is symmetric satisfying uniformly elliptic condition, i.e. there exist two positive constants* λ, Λ *independent of* ε, *such that* $\lambda|\xi|^2 \le \xi^T K(y)\xi \le \Lambda|\xi|^2$, $\forall y, \xi \in \mathbb{R}^d$, *and* $A(y)$ *is Y-period.* $f(x, y)$ *and* $F(x, y)$ *are bounded and Y-period with respect to* y *then as* $\varepsilon \to 0$,

$$\begin{cases} u^\varepsilon \rightharpoonup u^0 \text{ weakly in } H_0^1(\Omega), \\ A^\varepsilon \nabla u^\varepsilon + F(x, \frac{x}{\varepsilon}) \rightharpoonup A^0 \nabla u^0 + F^0, \text{ weakly in } (L^2(\Omega))^d. \end{cases} \tag{3.7}$$

u^0 *is the solution of the following homogenized problem*

$$\begin{cases} -\nabla \cdot (A^0 \nabla u^0) = f^0(x) + \nabla \cdot F^0(x), \text{ in } \Omega, \\ \\ u^0 = 0, \text{ on } \partial\Omega, \end{cases} \tag{3.8}$$

with A^0, f^0, *and* F^0 *defined as follows*

$$\begin{cases} A^0_{ij} = \langle A_{ij}(y) + A_{ik} \frac{\partial N^j}{\partial y_k} \rangle_Y, \\ f^0(x) = \langle f(x, y) \rangle_Y, \\ F^0(x) = \langle F(x, y) + A(y) \nabla_y w(y) \rangle_Y, \end{cases} \tag{3.9}$$

and $N^j(y)$, $w(x, y)$ *solving the cell problems :*

$$\begin{cases} -\nabla_y \cdot (A(y) \nabla_y N^j(y)) = \nabla_y \cdot (A(y) e_j), \text{ in } Y, \\ \\ N^j(y) \text{ is periodic in } Y, \langle N^j \rangle_Y = 0, \end{cases} \tag{3.10}$$

$$\begin{cases} -\nabla_y \cdot (A(y) \nabla_y w(x, y)) = \nabla_y \cdot (F(x, y)), \text{ in } Y, \\ \\ w(x, y) \text{ is periodic in } Y, \langle w \rangle_Y = 0. \end{cases} \tag{3.11}$$

Further more, if we denote by

$$u_1 = u^0 + \varepsilon N^j \frac{\partial u^0}{\partial x_j} + \varepsilon w, \tag{3.12}$$

and assume that $A(y)$, $f(x, y)$, $F(x, y)$ *are smooth enough and* $w(x, y) \in W^{1,\infty}(\Omega \times Y)$, *then there exists a positive constant* C *independent of* ε *such that*

$$\|\nabla(u^\varepsilon - u_1)\|_{L^2(\Omega)} \le C\varepsilon^{\frac{1}{2}} \tag{3.13}$$

Proof. By the standard asymptotic expansion method, we can get the equations (3.8)-(3.12) ([19], [20]). We first use Tartar's method ([19]) to prove (3.7). We denote by $f^\varepsilon = f(x, \frac{x}{\varepsilon})$ and $F^\varepsilon = F(x, \frac{x}{\varepsilon})$ for short. By the regularity of elliptic equation, we obtain u^ε is bounded in $H_0^1(\Omega)$ and ξ^ε is bounded in $(L^2(\Omega))^d$, where $\xi^\varepsilon = A^\varepsilon \nabla u^\varepsilon + F^\varepsilon$ is a vector-function and satisfies

$$\int_\Omega \xi^\varepsilon \nabla v dx = \langle f^\varepsilon, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \forall\, v \in H_0^1(\Omega). \tag{3.14}$$

By the compact property, there exists a subsequence (still denoted by ε), such that

$$\begin{cases} u^\varepsilon \rightharpoonup u^0 \text{ weakly in } H_0^1(\Omega), \\ \xi^\varepsilon \rightharpoonup \xi^0 \text{weakly in } (L^2(\Omega))^d, \end{cases} \tag{3.15}$$

Taking $\varepsilon \to 0$ in (3.14), we have

$$\int_\Omega \xi^0 \nabla v dx = \langle f^0, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}, \quad \forall\, v \in \ H_0^1(\Omega). \tag{3.16}$$

Therefor, (3.7) is proved if we show that

$$\xi^0 = A^0 \nabla u^0 + F^0. \tag{3.17}$$

If we set

$$\chi_\lambda^\varepsilon = \lambda \cdot x + \varepsilon N_\lambda \left(\frac{x}{\varepsilon}\right), \tag{3.18}$$

with $\lambda \in R^d$ and $N_\lambda(y)$ solving the following problem

$$\begin{cases} -\nabla_y \cdot (A(y)\nabla_y N_\lambda(y)) = \nabla_y \cdot (A(y)\lambda), \text{ in } Y, \\ \\ N_\lambda \text{ is periodic in } \ Y \ and \ \langle N_\lambda\rangle_Y = 0, \end{cases} \tag{3.19}$$

the we have the following limitation:

$$\begin{cases} \chi_\lambda^\varepsilon \rightharpoonup \lambda \cdot x, \text{ weekly in} H^1(\Omega), \\ \chi_\lambda^\varepsilon \to \lambda \cdot x, \text{ strongly in} L^2(\Omega). \end{cases} \tag{3.20}$$

Introduce the vector function

$$\eta_\lambda^\varepsilon = A^T \nabla_y \chi_\lambda^\varepsilon. \tag{3.21}$$

By the definition of χ_λ^ε and (3.19), we can easily obtain

$$\begin{cases} \eta_\lambda^\varepsilon \rightharpoonup (A^0)^T \lambda \text{ weakly in } \ L^2((\Omega))^d, \\ \int_\Omega \eta_\lambda^\varepsilon \cdot \nabla v dx = 0, \ \forall\, v \in \ H_0^1(\Omega). \end{cases} \tag{3.22}$$

For any $\varphi \in \mathcal{D}(\Omega)$, choose $\varphi \chi_\lambda^\varepsilon$ as the test function in (3.14) and φu^ε as the test function in (3.22). We have

$$\int_\Omega \xi^\varepsilon \cdot \nabla \chi_\lambda^\varepsilon \varphi dx + \int_\Omega \xi^\varepsilon \cdot \nabla \varphi \chi_\lambda^\varepsilon dx = \langle f^\varepsilon, \varphi \chi_\lambda^\varepsilon\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}, \ \forall\, \varphi \in \mathcal{D}(\Omega), \tag{3.23}$$

$$\int_\Omega \eta_\lambda^\varepsilon \cdot \nabla u^\varepsilon \varphi dx + \int_\Omega \eta_\lambda^\varepsilon \cdot \nabla \varphi u^\varepsilon dx = 0, \ \forall\, \varphi \in \mathcal{D}(\Omega). \tag{3.24}$$

By the definition of η_λ^ε, we have

$$A^\varepsilon \nabla u^\varepsilon \cdot \nabla \chi_\lambda^\varepsilon = (A^\varepsilon)^T \nabla \chi_\lambda^\varepsilon \cdot \nabla u^\varepsilon = \eta_\lambda^\varepsilon \cdot \nabla u^\varepsilon.$$

From (3.23)-(3.24), we have

$$\int_\Omega \xi^\varepsilon \cdot \nabla \varphi \chi_\lambda^\varepsilon dx + \int_\Omega F^\varepsilon \cdot \nabla \chi_\lambda^\varepsilon \varphi dx - \int_\Omega \eta_\lambda^\varepsilon \cdot \nabla \varphi u^\varepsilon dx = \langle f^\varepsilon, \varphi \chi_\lambda^\varepsilon \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \quad (3.25)$$

Taking $\varepsilon \to 0$, by (3.20) and (3.22), we obtain

$$\begin{aligned} &\int_\Omega \xi^0 \cdot \nabla \varphi (\lambda \cdot x) dx + \int_\Omega \varphi dx \int_Y F(x,y) \cdot (\lambda + \nabla_y N_\lambda) dy - \int_\Omega \lambda (A^0)^T \cdot \nabla \varphi u^0 dx \\ &\quad = \langle f^0, (\lambda \cdot x) \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \end{aligned} \quad (3.26)$$

which can be rewritten in the form

$$\begin{aligned} &\int_\Omega \xi^0 \cdot \nabla [\varphi (\lambda \cdot x)] dx - \int_\Omega \xi^0 \cdot \varphi \lambda dx + \int_\Omega \varphi dx \int_Y F(x,y) \cdot (\lambda + \nabla_y N_\lambda) dy \\ &\quad - \int_\Omega \lambda (A^0)^T \cdot \nabla \varphi u^0 dx = \langle f^0, (\lambda \cdot x) \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \end{aligned} \quad (3.27)$$

By (3.16), we get

$$\int_\Omega \xi^0 \cdot \varphi \lambda dx = \int_\Omega \varphi dx \int_Y F(x,y) \cdot (\lambda + \nabla_y N_\lambda) dy + \int_\Omega \lambda (A^0)^T \cdot \nabla u^0 \varphi dx. \quad (3.28)$$

If let $\lambda = e_i$, we can obtain

$$\xi_i^0 = (A^0 \nabla u^0)_i + F_i^0. \quad (3.29)$$

Here we use the following relationship ([20])

$$F_i^0 = \int_Y (F_i(x,y) + (A(y) \nabla_y w)_i) dy = \int_Y (F(x,y)(e_i + \nabla_y N^i)) dy. \quad (3.30)$$

Then we have proved (3.7).

Next we will give the error estimate

$$\|\nabla (u^\varepsilon - u_1)\|_{L^2(\Omega)} \leq C \varepsilon^{\frac{1}{2}}. \quad (3.31)$$

Following the argument in [15], we first compute

$$\begin{aligned} A^\varepsilon \nabla u_1 - A^0 \nabla u^0 &= \left(A^\varepsilon \left(e_j + \nabla_y N^j \right) - A^0 e_j \right) \frac{\partial u^0}{\partial x_j} + \varepsilon A^\varepsilon N^j \nabla \left(\frac{\partial u^0}{\partial x_j} \right) \\ &= g^j(y) \frac{\partial u^0}{\partial x_j} + \varepsilon A^\varepsilon N^j \nabla \left(\frac{\partial u^0}{\partial x_j} \right), \end{aligned} \quad (3.32)$$

with $g^j(y) = A^\varepsilon(e_j + \nabla_y N^j) - A^0 e_j$. By the cell problem (3.55) and the definition of A^0 in (3.9), we find

$$\langle g^j(y)\rangle_Y = 0 \ \ and \ \ \nabla_y \cdot g^j(y) = 0. \tag{3.33}$$

Then there exists a skew-symmetric matrix G^j ([15]), such that

$$g^j = \nabla_y \cdot G^j, \quad G^j_{ik} = -G^j_{ki}, \quad and \quad G^j_{ik} \in H^1_{per}(Y). \tag{3.34}$$

By this property, we can get:

$$g^j \frac{\partial u^0}{\partial x_j} = \varepsilon \nabla \cdot \left(\frac{\partial u^0}{\partial x_j} G^j \right) - \varepsilon G^j \nabla \left(\frac{\partial u^0}{\partial x_j} \right), \tag{3.35}$$

and

$$-\nabla \cdot (A^\varepsilon \nabla (u^\varepsilon - u_1)) \ = \ R^\varepsilon + \nabla \cdot r_\varepsilon, \tag{3.36}$$

with $R^\varepsilon = \nabla \cdot (F(x,y) + A(y)\nabla_y w(x,y) - F^0(x)) + (f(x,y) - f^0(x))$, and $r^\varepsilon = \varepsilon G^j_{ik} \frac{\partial^2 u^0}{\partial x_j \partial x_k} + \varepsilon A^\varepsilon_{ij} N^j \frac{\partial^2 u^0}{\partial x_j \partial x_k}$.

If we introduce the boundary corrector θ^ε as

$$\begin{cases} -\nabla \cdot (A^\varepsilon(x) \nabla \theta^\varepsilon) = 0, \text{ in } \Omega, \\ \\ \theta^\varepsilon = -\varepsilon N^j \frac{\partial u_0}{\partial x_j} - \varepsilon w, \text{ on } \partial\Omega, \end{cases} \tag{3.37}$$

then $e_\varepsilon = u^\varepsilon - u_1 - \theta^\varepsilon \in H^1_0(\Omega)$ satisfies

$$-\nabla \cdot (A^\varepsilon \nabla (u^\varepsilon - u_1 - \theta^\varepsilon)) \ = \ R^\varepsilon + \nabla \cdot r_\varepsilon, \tag{3.38}$$

and the weak form is

$$\int_\Omega A^\varepsilon \nabla e_\varepsilon \nabla \phi dx = \int_\Omega R^\varepsilon \phi dx + \int_\Omega r_\varepsilon \nabla \phi dx, \quad \forall \phi \in H^1_0(\Omega) \tag{3.39}$$

Since $e_\varepsilon \in H^1_0(\Omega)$, taking $\phi = e_\varepsilon$, we get

$$\int_\Omega A^\varepsilon \nabla e_\varepsilon \nabla e_\varepsilon dx = \int_\Omega R^\varepsilon e_\varepsilon dx + \int_\Omega r_\varepsilon \nabla e_\varepsilon dx, \quad \forall \phi \in H^1_0(\Omega) \tag{3.40}$$

By the elliptic condition and assumption that $G \in L^\infty(Y)$, the second term in the right hand can be estimated as follows

$$\|r^\varepsilon\|^2_{L^2(\Omega)} \le \varepsilon (\|G\|_{L^\infty(Y)} + \Lambda \|N\|_{L^\infty(Y)}) \| \frac{\partial^2 u^0}{\partial x_j \partial x_k} \|_{L^2(\Omega)}. \tag{3.41}$$

By the definition of $F^0(x)$ in (3.9), we obtain

$$\begin{aligned} \nabla_x \cdot (F(x,y) + A(y)\nabla_y w(x,y) - F^0(x)) &= \left(\nabla_x \cdot F - \int_Y \nabla_x \cdot F(x,y) dy \right) \\ &+ \nabla_x \cdot (A(y)\nabla_y w(x,y)) - \left(\int_Y \nabla_x \cdot (A(y) \nabla_y w(x,y)) dy \right) \end{aligned} \tag{3.42}$$

It can be easily checked that

$$\int_Y R^\varepsilon(x,y)dy = 0. \tag{3.43}$$

By lemma 3.1 and lemma 1.6 in [22], the first term in the right hand of (3.40) can be estimated as follows:

$$\int_\Omega R^\varepsilon(x,y)e_\varepsilon dx \leq C\varepsilon\|e_\varepsilon\|_{H_0^1(\Omega)} \tag{3.44}$$

So we obtain

$$\begin{aligned} \|\nabla(u^\varepsilon - u_1 - \theta^\varepsilon)\|_{L^2(\Omega)} &\leq C\varepsilon\frac{1}{\lambda}(\|G\|_{L^\infty} + \Lambda\|N\|_{L^\infty})\|\frac{\partial^2 u^0}{\partial x_j \partial x_k}\|_{L^2(\Omega)} \\ &\quad +C\varepsilon \end{aligned} \tag{3.45}$$

Next we estimate $\|\nabla\theta^\varepsilon\|_{L^2(\Omega)}$. Multiplying $\theta^\varepsilon + \varepsilon\phi^\varepsilon N^j\frac{\partial u^0}{\partial x_j} + \varepsilon\phi^\varepsilon w$ on both sides of (3.37), we obtain

$$\begin{aligned} \int_\Omega |\nabla\theta^\varepsilon|^2 dx &\leq (\frac{\Lambda}{\lambda})^2\int_\Omega |\varepsilon\nabla(N^j\frac{\partial u^0}{\partial x_j}\phi^\varepsilon)|^2 dx + (\frac{\Lambda}{\lambda})^2\int_\Omega |\varepsilon\nabla(w\phi^\varepsilon)|^2 dx \\ &= C(\frac{\Lambda}{\lambda})^2 I_1 + C(\frac{\Lambda}{\lambda})^2 I_2 \end{aligned} \tag{3.46}$$

where ϕ^ε is a cut-off function, satisfying

$$\begin{cases} \phi^\varepsilon = 1, \quad \text{on } \partial\Omega \\ 0 \leq \phi^\varepsilon \leq 1, \quad \text{on } \Omega_\varepsilon \\ |\varepsilon\nabla\phi^\varepsilon| \leq C \\ \phi^\varepsilon \in C^\infty(\overline{\Omega}), \end{cases} \tag{3.47}$$

with $\Omega_\varepsilon = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) \leq \varepsilon\}$.

For I_1, we directly use the result in [21]

$$I_1 \leq C\varepsilon(1 + \|N\|_{L^\infty(Y)})^2\|\frac{\partial u^0}{\partial x_j}\|^2_{H^1(\Omega)}. \tag{3.48}$$

Please note that the above estimate only depends on the $\|N(y)\|_{L^\infty(Y)}$ rather than $\|N(y)\|_{W^{1,\infty}(Y)}$. This is the contribution of Suslina [23].

For I_2, we have

$$\begin{aligned} I_2 &\leq \varepsilon^2\int_{\Omega_\varepsilon} |\nabla_x w(x,y)|^2 dx + \int_{\Omega_\varepsilon} |\nabla_y w(x,y)|^2 dx + \int_{\Omega_\varepsilon} |w(x,y)|^2 dx \\ &\leq C\varepsilon\|w\|^2_{W^{1,\infty}(\Omega\times Y)}. \end{aligned} \tag{3.49}$$

By (3.45), (3.46), (3.48), and (3.49), we complete the proof.

□.

Remark 3.3 *Please note that the estimate in (3.45) is of order* ε. *The final estimate* (3.13) *decays to the order* $\varepsilon^{\frac{1}{2}}$, *due to the oscillation of the corrector term* θ^ε (3.37) *at the boundary. For the one-dimension problems, the boundary decays to isolated points and this kind of oscillation at the boundary does not appear any more. The estimate* (3.13) *can be improved to*

$$\|\partial(u^\varepsilon - u_1)\|_{L^2(0,1)} \le C\varepsilon. \tag{3.50}$$

Remark 3.4 *As to the convergence of potential energy, we have as* $\varepsilon \to 0$

$$|\int_\Omega (\nabla u^\varepsilon \cdot A^\varepsilon \nabla u^\varepsilon + F^\varepsilon \cdot \nabla u^\varepsilon)\, dx - \int_\Omega \left(\nabla u^0 \cdot A^0 \nabla u^0 + F^0 \cdot \nabla u^0\right) dx| \to 0. \tag{3.51}$$

So far we have establish the mathematical homogenization theory for (3.4), we now come back to the physics problem (3.1).

By Theorem 3.2 and unit transformation, we get the *homogenized* problem of (3.1).

$$\begin{cases} -\nabla \cdot (K^0 \nabla p_0(x)) = f^0(x) + \nabla \cdot F^0(x), \quad \text{in } \Omega_D = (0, D)^d, \\ \\ p_0 = 0, \quad \text{on } \partial\Omega_D, \end{cases} \tag{3.52}$$

where

$$\begin{cases} p_0(x) = D^2 \widetilde{p_0}(\widehat{x}) \\ \\ K^0 = \widetilde{K^0} \end{cases} \tag{3.53}$$

with K^0, f^0, and F^0 defined as follows

$$\begin{cases} K^0_{ij} = \langle K_{ij}(y) + K_{ik}\frac{\partial N^j}{\partial y_k}\rangle_Y, \\ f^0(x) = \langle f(x,y)\rangle_Y, \\ F^0(x) = \langle F(x,y) + K(y)\nabla_y w(y)\rangle_Y, \end{cases} \tag{3.54}$$

and $N^j(\frac{x}{l}) = N^j(y)$, $w(x, \frac{x}{l}) = w(x,y)$ solving the cell problems

$$\begin{cases} -\nabla_y \cdot (K(y)\nabla_y N^j(y)) = \nabla_y \cdot (K(y)e^j), \text{ in } Y, \\ \\ N^j(y) \text{ is periodic in Y}, \langle N^j\rangle_Y = 0 \end{cases} \tag{3.55}$$

$$\begin{cases} -\nabla_y \cdot (K(y)\nabla_y w(x,y)) = \nabla_y \cdot (F(x,y)), \text{ in } Y, \\ \\ w(x,y) \text{ is periodic in Y}, \langle w\rangle_Y = 0. \end{cases} \tag{3.56}$$

By unit transformation, (3.53), and (3.13), we can obtain the next theorem

Theorem 3.5 *If p is the solution of* (3.1), p_1 *is defined as follows*

$$p_1 = p_0 + lN^j \frac{\partial p_0}{\partial x_j} + lw, \tag{3.57}$$

with p_0 solving (3.52) *and $f(x,y)$, $F(x,y)$ are bounded, smooth enough and Y-period with respect to y, then there exists a positive number C independent of D, such that*

$$\frac{1}{D^3} \int_{\Omega_D} |\nabla(\frac{p}{D} - \frac{p_1}{D})|^2 dx \ \leq \ C\frac{l}{D}, \tag{3.58}$$

$$\frac{1}{D^3} \int_{\Omega_D} |\frac{p}{D^2} - \frac{p_0}{D^2}|^2 dx \ \leq \ C(\frac{l}{D})^2. \tag{3.59}$$

As to the convergence of potential energy, we have

$$|E(p) - E_0(p_0)| \to 0 \quad asD \to \infty, \tag{3.60}$$

with

$$E(p) = \frac{1}{D^3} \int_{\Omega_D} \left(\nabla \left(\frac{p}{D} \right) K \nabla \left(\frac{p}{D} \right) + \frac{F}{D} \nabla \left(\frac{p}{D} \right) \right) dx,$$

$$E_0(p_0) = \frac{1}{D^3} \int_{\Omega_D} \left(\nabla \left(\frac{p_0}{D} \right) K^0 \nabla \left(\frac{p_0}{D} \right) + \frac{F^0}{D} \nabla \left(\frac{p_0}{D} \right) \right) dx.$$

Above theorem explains in what sense we can accept the homogenized problem (3.52).

Remark 3.6 *In one-dimension case, by Remark 3.3, our results are changed to be:*

$$\frac{1}{D} \int_0^D |\partial_x(\frac{p}{D} - \frac{p_0}{D})|^2 dx \leq C \left(\frac{l}{D} \right)^2, \tag{3.61}$$

$$\frac{1}{D} \int_0^D |\frac{p}{D^2} - \frac{p_0}{D^2})|^2 dx \leq C \left(\frac{l}{D} \right)^2, \tag{3.62}$$

$$|E(p) - E_0(p_0)| \to 0 \quad as\ D \to \infty, \tag{3.63}$$

with

$$E(p) = \frac{1}{D} \int_0^D \left(\partial_x \left(\frac{p}{D} \right) K \partial_x \left(\frac{p}{D} \right) + \frac{F}{D} \partial_x \left(\frac{p}{D} \right) \right) dx,$$

$$E_0(p_0) = \frac{1}{D} \int_0^D \left(\partial_x \left(\frac{p_0}{D} \right) K^0 \partial_x \left(\frac{p_0}{D} \right) + \frac{F^0}{D} \partial_x \left(\frac{p_0}{D} \right) \right) dx.$$

4 Example

In this part, we show a one-dimensional example to verify our results . We consider the following problem:

$$\begin{cases} -\partial_x(K\partial_x p) = -1 + \partial_x\left(\left(x + \frac{D}{2} + C\right)K\right) & \text{in } (0, D), \\ \\ p(0) = p(D) = 0, \end{cases} \tag{4.1}$$

where $K = \dfrac{1}{2 + cos(\frac{2\pi x}{l})}$ and $C = \dfrac{l}{2\pi}\sin\left(\dfrac{2\pi D}{l}\right) + \left(\dfrac{l}{2\pi}\right)^2 \dfrac{1}{D}\cos\left(\dfrac{2\pi D}{l}\right) - \left(\dfrac{l}{2\pi}\right)^2 \dfrac{1}{D}$. It is clear that the source term has micro-structure, which we have discussed in §3.

The solution is

$$\begin{aligned} p \;=\; & \frac{x^2}{2} - \frac{D}{2}x + x\frac{l}{2\pi}\sin\left(\frac{2\pi x}{l}\right) + \left(\frac{l}{2\pi}\right)^2 \cos\left(\frac{2\pi x}{l}\right) \\ & -Cx - \left(\frac{l}{2\pi}\right)^2. \end{aligned} \tag{4.2}$$

The homogenized problem has the form as

$$\begin{cases} -\partial_x(K^0\partial_x p_0) = -1 + \partial_x\left(\frac{x}{2}\right), & \text{in } (0, D), \\ \\ p_0(0) = p_0(D) = 0, \end{cases} \tag{4.3}$$

with $K^0 = \frac{1}{2}$ and the solution $p_0 = \dfrac{x^2}{2} - \dfrac{D}{2}x$. $N(\dfrac{x}{l}) = \dfrac{\sin(\frac{2\pi x}{l})}{4\pi}$ is the solution of cell problem

$$\begin{cases} -\partial_y(K\partial_y(N + y)) = 0, & \text{in } Y = (0, 1), \\ \\ N(y) \text{ is Y-periodic with respect to y and } \langle N(y)\rangle_Y = 0, \end{cases} \tag{4.4}$$

and $w(x, \dfrac{x}{l}) = (x + \dfrac{D}{2} + C)\dfrac{\sin(\frac{2\pi x}{l})}{4\pi}$ is the solution of cell problem

$$\begin{cases} -\partial_y(K(y)\partial_y w(x, y)) = \partial_y(F(x, y)), \text{ in } Y = (0, 1), \\ w(x, y) \text{ is Y-periodic with respect to y and } \langle w(x, y)\rangle_Y = 0. \end{cases} \tag{4.5}$$

By direct computation, we obtain

$$\begin{aligned} & \int_0^D (p - p_0)^2 dx \\ = & \int_0^D \left(x\frac{l}{2\pi}\sin\left(\frac{2\pi x}{l}\right) + \left(\frac{l}{2\pi}\right)^2 \cos\left(\frac{2\pi x}{l}\right)\right)^2 dx + \int_0^D (Cx)^2\, dx + \int_0^D \left(\frac{l}{2\pi}\right)^4 dx \\ = & D^3\left(\frac{l^2}{24\pi^2} + \frac{C^2}{3}\right) - D^2\left(\frac{l^3}{16\pi^3}\sin\left(\frac{4\pi D}{l}\right)\right) + D\left(\frac{l^4}{16\pi^4} - \frac{l^4}{32\pi^4}\cos\left(\frac{4\pi D}{l}\right) - \frac{l^2}{8\pi^2}\right) \\ & + \left(\frac{l^5}{128\pi^5}\sin\left(\frac{4\pi D}{l}\right) - \frac{l^3}{32\pi^3}\sin\left(\frac{4\pi D}{l}\right)\right). \end{aligned} \tag{4.6}$$

So we have

$$\frac{1}{D}\int_0^D (\frac{p-p_0}{D^2})^2 dx = \left(\frac{l}{D}\right)^2\left(\left(\frac{l^2}{24\pi^2}+\frac{C^2}{3}\right)+o(1)\right), \tag{4.7}$$

which is consistent with the theoretical result (3.62).

If we denote by

$$\begin{aligned} p_1 &= p_0 + lN\partial_x p_0 + lw \\ &= p_0 + x\frac{l}{2\pi}\sin\left(\frac{2\pi x}{l}\right)+\frac{lC}{4\pi}\sin\left(\frac{2\pi x}{l}\right), \end{aligned} \tag{4.8}$$

then we have

$$p - p_1 = \left(\frac{l}{2\pi}\right)^2\cos\left(\frac{2\pi x}{l}\right)-\left(\frac{l}{2\pi}\right)^2 - Cx + \frac{lC}{4\pi}\sin\left(\frac{2\pi x}{l}\right). \tag{4.9}$$

$$\begin{aligned} \int_0^D (\partial_x(p-p_1))^2 dx &= \int_0^D \left(-\frac{l}{2\pi}\sin\left(\frac{2\pi x}{l}\right)-C+\frac{C}{2}\cos\left(\frac{2\pi x}{l}\right)\right)^2 dx \\ &= D\left(\frac{l^2}{8\pi^2}+\frac{9C^2}{8}\right)+\left(\frac{C^2 l}{32\pi}-\frac{l^3}{32\pi^2}\right)\sin\left(\frac{4\pi D}{l}\right). \end{aligned} \tag{4.10}$$

We obtain

$$\frac{1}{D}\int_0^D (\partial_x(\frac{p-p_1}{D}))^2 dx = \left(\frac{l}{D}\right)^2\left(\frac{l^2}{8\pi^2}+\frac{9C^2}{8}+o(1)\right). \tag{4.11}$$

This is also consistent with the theoretical result (3.61).

As to the convergence of the energy, we have

$$\begin{aligned} &\int_0^D \partial_x p \cdot K\partial_x p dx + \int_0^D F\partial_x p dx \\ =&\int_0^D fp dx \\ =&\frac{D^3}{12}+\frac{CD^2}{2}+D\left(\frac{l^2}{4\pi^2}\cos\left(\frac{2\pi D}{l}\right)+\frac{l^2}{4\pi^2}\right)-\frac{l}{2\pi}\sin\left(\frac{2\pi D}{l}\right), \end{aligned} \tag{4.12}$$

$$\int_0^D \partial_x p_0 \cdot K^0 \partial_x p_0 dx + \int_0^D F^0 \partial_x p_0 dx = \frac{D^3}{12}, \tag{4.13}$$

so we have

$$|E(p)-E_0(p_0)| = \frac{C}{D2}+\frac{1}{D^2}\left(\frac{l^2}{4\pi^2}\cos\left(\frac{2\pi D}{l}\right)+\frac{l^2}{4\pi^2}\right)-\frac{l}{2\pi D^3}\sin\left(\frac{2\pi D}{l}\right)\to 0$$

as $D\to\infty$, with

$$E(p) = \frac{1}{D}\int_0^D \left(\partial_x\left(\frac{p}{D}\right)K\partial_x\left(\frac{p}{D}\right)+\frac{F}{D}\partial_x\left(\frac{p}{D}\right)\right)dx,$$

$$E_0(p_0) = \frac{1}{D}\int_0^D \left(\partial_x\left(\frac{p_0}{D}\right)K^0\partial_x\left(\frac{p_0}{D}\right) + \frac{F^0}{D}\partial_x\left(\frac{p_0}{D}\right)\right)dx.$$

This is also consistent with the theoretical result (3.63).

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References

[1] J. C. Maxwell, *A treatise on electricity and magnetism*, 3rd Ed. , Clarendon Press, Oxford, 1881.

[2] S. Poission, *Second mémoire sur la théorie du magnétisme*, Mem. Acad. France **5**, 1822.

[3] K. Lichtenecker, *Die dielektrizitätskonstante natürlicher und künstlicher mischkörper*, Phys. Zeitschr. XXVII (1926), 115-158.

[4] I. Babuška, *Solution of problem with interfaces and singularities*, in Mathematical aspects of finite elements in partial differential equations, C. de Boor ed., Academic Press, New York (1974), 213-277.

[5] A. Bensousan, J. L. Lions and G. Papanicolaou, *Asymptotic analysis for periodic structures*, North-Holland, Amsterdam, 1978.

[6] G. Allaire, *Homogenization et convergence a deux echelles*, application a un probleme de convection diffusion. C.R.Acad. Sci. Paris **6** (1991), 312-581.

[7] G. Allaire, *Homogenization and two-scale convergence*, SIAM J. Math. Anal. **23:6** (1992), 1482-1518.

[8] F. Murat and L. Tartar, *H-convergence*, Séminaire d'Analyse Fonctionnelle et Numérique de l'Université d'Alger, mimeographed notes, 1978.

[9] L. Tartar, *Compensated compactness and partial differential equations*, in Nolinear Analysis and Mechanics: Heriot-Watt Symposium, Pitman **IV** (1979), 136-212.

[10] L. Tartar, *H-measure, a new approach for studying homogenization, oscillations and concentration effects in partial differential equations*, Proc. Roy. Soc. Edinburgh **115 A** (1990), 193-230.

[11] S. Spagnolo, *Sul limite delle soluzioni di problemi di Cauchy relativi all' equatione del calore*, Ann. Scuola Norm. Sup. Pisa **3** (1967), 657-699.

[12] G. Nguestseng, *A general convergence result for a functional related to the theory of homogenization*, SIAM J. Math. Anal. **20:3** (1989), 608-623.

[13] S. M. Kozlov, *The averaging of random operators*, Mat.Sb.(N.S) **109(151):2(6)** (1979), 188-202.

[14] V. V. Zhikov and O. A. Oleinik, *Homogenization and G-convergence of differential operators*, Russ. Math. Surv. **34** (1979), 65-147.

[15] V. V. Zhikov, S. M. Kozlov and O. A. Oleinik, *Homogenization of differential operators and integral functionals*, Springer Berlin, 1994.

[16] I. Babuška, *Homogenization approach in engineering*, Lecture notes in economics and mathematical systems, M. Beckman and H. P. Kunzi(eds.), Springer-Verlag (1975), 137-153.

[17] I. Babuška, *Homogenization and its application. Mathematical and computational problems*, Numerical solution of partial differential equations **III**, Academic Press (1976), 89-116.

[18] I. Babuška, *The computational aspects of the homogenization problem*, Computing methods in applied sciences and engineering **I**, Lecture notes in mathematics, Springer-Verlag,Berlin Heidelberg New York (1977), 309-316.

[19] D. Cioranescu and P. Donato, *An introduction to homogenization*, Oxford Lecture Series in Mathematics and Its Applications **17**, Oxford university press, 1999.

[20] T. Yu and X. Yue, *Residual-free bubble methods for numerical homogenization of elliptic problems*, Commun. Math. Sci., **9** (2011), 1163-1176.

[21] V. V. Zhikov, *Some estimates from homogenization theory*, Doklady Mathematics, **73:1** (2006), 96-99.

[22] O. A. Olenik and A.S. Shamaev, G.A. Yosifian, *Mathematical problems in elasticity and homogenization*, Studies in mathematics and its applications, J.L. Lions, G.Papanicolaou, H. Fujita, H.B. Keller **26**, North-Holland, 1992.

[23] T. A. Suslina, *Homogenization of a stationary periodic maxwell system*, St. Petersburg Math. J. **16** (2005), 863-922.