Geometry of Impressionist Music

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Abstract

My project, using both geometrical and statistical methods, finds an appropriate way of determining distances between scales, calculated using appropriate metrics, in the context of impressionist music.

1 Introduction

Although the ancient Chinese, Egyptians and Mesopotamians are known to have studied the mathematical principles of sound, music remains one of the most transcendental forms of art. Throughout the history, musicians or music critics have tried to classify music. The genre of "Classical music" was defined, in descriptive language, in the 19^{th} century to differentiate the "antique" music such as Mozart's works from the new genre of Romantic music. Contemporary music theorists are no longer satisfied with such generalized classification but more interested in the details of the compositional patterns in different music eras.

The term *Impressionism* was first used by Louis Leroy in application to the famous painter Monet in a derogatory way over the vague nature of his work Sunrise. The aim of impressionists was to "suggest rather than to depict; to mirror not the object but the emotional reaction to the object; to interpret a fugitive impression rather than to seize upon and fix the permanent reality."[1] It is an art of abstraction where mystery and vagueness are desired.

Impressionist music, with the same idea, focuses on creating a sense of the theme by using varied scales and delicate shadings of sound rather than relying on standard forms and a strong, clear rhythmic beat. Impressionist composers in the twentieth century extended the nineteenth-century chordal practices to a scalar domain by using efficient voice leading to connect scales rather than merely chords. While classical music such as works by Bach and Mozart has been widely studied (more details in Section 2), there is less contemporary theoretical study on Impressionistic music. My project aims to characterize Impressionist music through the investigation of interscalar distances.

As we know, Classical music has its characteristic chord progressions. If we view the 24 major and minor triads[1] as a group, then the operations could be inversion, transposition, modulation between relative keys or parallel keys, etc. We can also geometrically represent any ordered chord in a torus, i.e. the product of circles, and any unordered chord in a quotient space of a torus (see Section 2.3). Similar to a chord, a scale is a collection of notes, too. Unlike Classical music in which all scales have seven notes, however, Impressionist music is marked by the use of exotic

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[1] A triad is a three-note chord

scales consisting of different numbers of notes. My job then is to figure out ways of reconciling the difference so that we have a most appropriate space to represent scales in Impressionist music. Once we are able to represent scales as points in a common space, the next step is to determine the distance between them using appropriate metric. Choice of metric is based on my assumption that modulation is more likely to happen between scales that are closer and each method is tested on 32 Impressionist pieces. More details can be found in Section 3.

Tymoczko investigated the relationship between modulation frequency and voice-leading distance[2] in Baroque and Classical music and obtained Pearson correlation coefficients[3] (absolute value) of at least 0.91[9]. My results (Section 5) show that the best method gives a Pearson correlation coefficient of -0.4, indicating a moderate inverse relationship between modulation frequency and interscalar distance in Impressionist music. Musically speaking, this means that modulations are more likely to happen between scales that are closer to one another using the distances I define here. I believe that direct comparison of numbers could be unfair and misleading. Section 6 is dedicated to the discussion of this difference.

2 Mathematics and Music Background

Before we look at any music at all, we need to translate music to algebra. Using equal tempered tuning system, we can divide an octave into twelve pitch classes. The interval between two consecutive pitch classes is a half-step. Therefore, we translate pitch classes to integers modulo 12 and take C to be 0. Notes that are enharmonically equivalent[4] are represented by the same number. For instance, C♯ and D♭ are both 1. As a result[2], we obtain a quotient space $\mathbb{R}/12$, which is also recognized as the pitch-class space. This step allows us to digitize music and newly define many musical terms such as transposition and inversion.

We can represent an interval using two numbers from 0 to 11. One thing to note is that we only consider intervals smaller than or equal to tritone so the order of the two numbers does not matter. For instance, 40 is the same as 04, representing a major third instead of a minor sixth. Additionally, to avoid confusion, we write t instead of 10 and e instead of 11. For example, 7t stands for the minor third between G and B. Similarly, we use 3 numbers in a row to represent a triad and use 7 numbers to represent a 7-note scale. Again the order of the three numbers does not matter.

This step allows us to represent music using numbers and newly define many musical terms such as transposition and inversion.

Now we spread the 12 numbers evenly on a circle like a clock and connect the three numbers of the triad. For example, the C major chord 047 is shown in Figure 1.

2.1 Dihedral groups

The 12 numbers in the circle are also the vertices of a regular 12-gon with all sides of the same length and all angles of the same measure. The dihedral group of order 24 is the group of

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[2] Voice-leading distance is a termed used in Classical music and is analogous to interscalar distance.

[3] Pearson correlation shows the linear relationship between two sets of data. The Pearson correlation coefficient could range from -1 to 1, with 1 indicating total positive linear correlation, 0 indicating no correlation and -1 indicating total negative linear correlation.

[4] Two enharmonically equivalent notes are essentially the same note but with different representations or names.

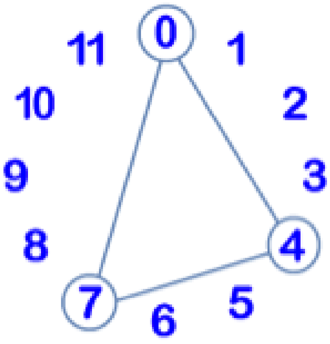

Figure 1: Graphic representation of the C major chord

symmetries of such a regular 12-gon. We are able to perform several kinds of group action on the triads. The first action of the dihedral group of order 24 on the set of major and minor triads is defined via the T/I-group, where T and I stand for transposition and inversion respectively.

2.1.1 Transposition

Musicians need to raise or lower the pitch of a song for different purposes. Such change of the overall pitch is called transposition. For example, a tired singer may want to lower a song previously composed in D major by one semitone and sing it in C♯ major instead. The tonic moves from 2 to 1 and every other note moves down by 1 unit. Geometrically, transposition is done by rotation. In this case, the triangle of every triad rotates by 1 unit counterclockwise about the center of the 12-gon.

Mathematically, we define transposition by an interger n mod 12 by the function

$$T_n : \mathbb{Z}_{12} \to \mathbb{Z}_{12} \tag{1}$$

$$T_n(x) := x + n \pmod{12} \tag{2}$$

where T_1 corresponds to clockwise rotation of the clock by 30 degrees.

2.1.2 Inversion

In music theory, inversion has a lot of meanings. For an interval, an inversion may refer to setting the lower pitch higher than the other pitch, without changing either pitch class. Like intervals, triads can be inverted by moving the lowest note up octave. The lowest note, called the bass note, determines the name of the inversion. Here, we define an inversion about n mod 12 as the function

$$I_n : \mathbb{Z}_{12} \to \mathbb{Z}_{12} \tag{3}$$

$$T_n(x) := -x + n \pmod{12} \tag{4}$$

where I_0 corresponds to a reflection of the clock about the 0-6 axis.

Hence, T_1 and I_0 generate the dihedral group of symmetries of the 12-gon. We can easily verify the following relations:

$$T_m \circ T_n = T_{m+n} \pmod{12} \tag{5}$$
$$T_m \circ I_n = I_{m+n} \pmod{12} \tag{6}$$
$$I_m \circ T_n = I_{m-n} \pmod{12} \tag{7}$$
$$I_m \circ I_n = T_{m-n} \pmod{12} \tag{8}$$

This group is called the T/I-group.

2.2 Neo-Riemannian Theory

Now we let S denote the set of consonant triads, including both major and minor triads. It is easy to see that S has 24 elements. We have seen the action of the dihedral group of order 24 on S via transposition and inversion. Now we shall explore a second musical action defined in terms of the PLR-group. As a subgroup of the symmetric group on S, the PLR-group, or the subgroup neo-Riemannian group, is generated by operations P, L, and R.

The parallel operation P maps a major triad to its parallel minor and vice versa. The leading tone exchange operation L maps a major triad to a minor triad by lowering only the root note by a half step. The operation L raises the fifth note of a minor triad by a semitone. The relative operation R maps a major triad to its relative minor, and vice versa.

Graphically, any triangle representing a triad has three vertices. We can fix any two vertices and draw a perpendicular bisector to the side containing these two vertices. Then we reflect the third vertex about the perpendicular bisector. The reflected vertex and the two fixed vertices create a new triad. Since we could draw three different perpendicular bisectors to the original triangle, the new triad has three possibilities that are exactly the images of P, L, R actions respectively. In other words, PLR operations are contextual inversions with respect to different axes.

We can show that the PLR-group is generated by L and R by applying R and L alternately on any consonant triad. It is easy to calculate that $R(LR)^3 =$ P. If we let s = LR and t = L, then $s^{12} = 1$, $t^2 = 1$ and $tst = L(LR)L = RL = S^{-1}$. It is shown in Crans et al. [3] that the PLR-group has order 24 so it is dihedral as on page 68 of Rotman [4]. Additionally, the PLR-group acts transitively on S.

Now we come to the relationship between the PLR-group and the T/I-group. As a result, the PLR-group and the T/I-group are dual. This means that each acts transitively on S and is the centralizer of the other in the symmetric group Sym(S).

Neo-Riemannian theory could be a good way to model the triadic chord progressions in Classical and Romantic music. It has also inspired numerous subsequent investigations including the study of pop music today.

2.3 Geometric Representation

An elegant geometric depiction of the PLR-group is the Riemann Tonnetz. Tonnetz, German for “tone network”, is generally defined as a conceptual lattice diagram representing tonal space and was first described by Leonhard Euler [5]in 1739. The vertices of the graph below are pitch classes and each smallest triangle represents either a major triad or a minor triad. Each of the operations P, L and R is able to flip the triangle about one edge, arriving at the reflected triangle. For example, the triangle with sides in bold is a c minor chord and the P operation flips it downwards with respect to the C-G interval, obtaining a C major chord. The entire graph is infinite in all directions. Horizontally, there is a circle of fifths. On the diagonal axes, we have the circles of major and minor thirds [6]. Since both circles repeat themselves, the Tonnetz is doubly periodic.

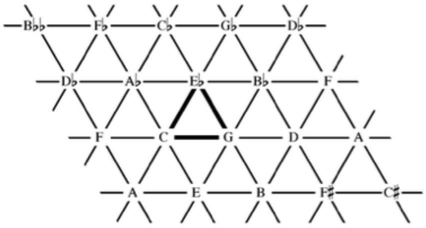

Figure 2: A part of the Tonnetz [10]

By gluing the top and bottom edges as well as the left and right edges of a fundamental domain, we obtain a torus shown in Figure 3. The blue line is the circle of fifths. The red line connects notes that are a major third apart while the green line connects notes that are a minor third apart.

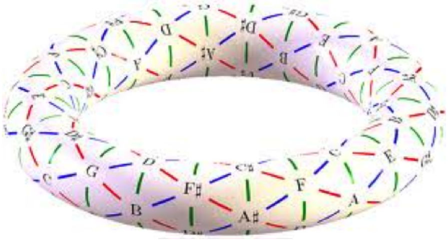

Figure 3: Musical Torus [12]

Such geometry of musical chords can be generalized in dimension n [2]. First, we represent an ordered sequence of n pitches as a point in R^n and model it by forming the quotient space $(R/12\mathbb{Z})^n$ which is also known as the n-torus. To model unordered n-note chords of pitch classes, we use the global-quotient orbifold T^n/S_n [8][11] which is the n-torus T^n modulo the symmetric group S_n. To construct the orbifold, we can take an n-dimensional prism whose base is an $(n-1)$ simplex, twist the base so as to cyclically permute its vertices and identify it with the opposite face. The boundaries of the orbifold are singular.

For example, Figure 4 is the orbifold T_2/S^2, the space of unordered pairs of pitch classes or intervals. The directed line segments in the space represent voice leadings. Any bijective voice leading between pairs of pitch classes (eg. 70 to 16) can be represented by an arrowed path. The orbifold is singular at its top and bottom edges, which act like mirrors. From the arrows on the left and right edges, we can see that the left edge is given a half twist and identified with the right. Therefore, the space is a Mobius strip.

In the case of T_2/S^2, the distance between any two intervals can be physically measured by a ruler. Based on the same idea, we may represent each scale by a point in a certain orbifold so that we can measure the distance between different scales. Details about the appropriate

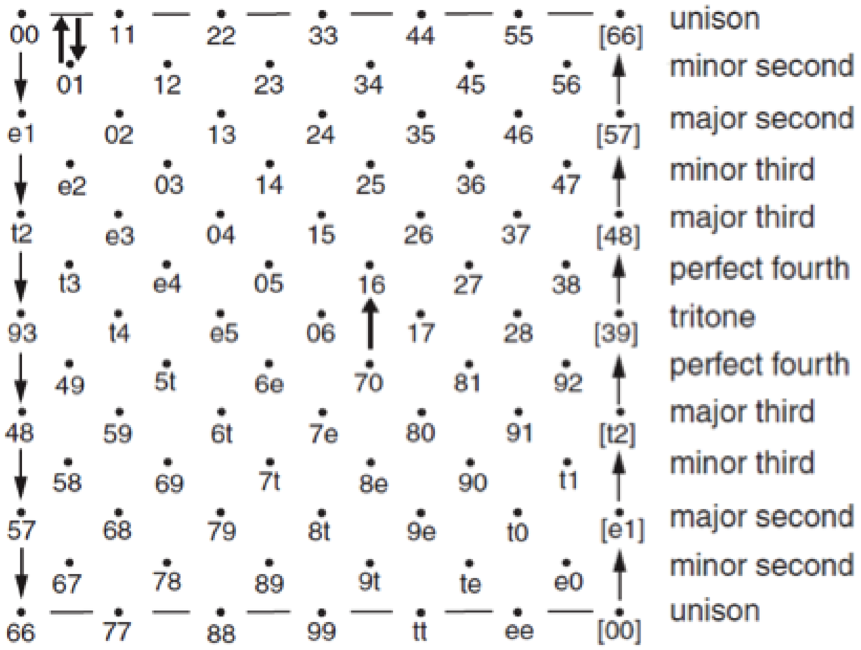

Figure 4: Orbifold T_2/S^2

orbifold for Impressionist music is covered in Section 3.

2.4 Modulation frequency and key distance in Classical music

Although different genres involve different modulatory norms, there are a few basic modulatory principles in Classical music. For example, music tends to start and end in the same key and the first modulatory destination of a piece in a major key is usually the dominant key. Based on the conjecture that Classical music often modulate between "closely related keys", Tymoczko studied the correlation between modulation frequency and voice-leading distance between keys in Baroque and Classical music[9].

Here, modulation frequency refers to the total number of occurrences of the equivalent modulation in his sample pieces. Tymoczko did not give a rigorous mathematical definition of voice-leading distance. Instead, he gave a brief description in words[7]: "for major keys, the distances are simply the voice-leading distances between the relevant diatonic collections; for distances between major and minor, we calculate the size of the voice leadings from the major scale to each minor scale[5], and take the average; for minor scales, we take the average of the three voice leadings in the most efficient paring of the scales in one key with those in the other." According to the examples he gave, I assume that a major or minor scale is represented as a point in T^7/S_7 and the distance between two points is measured by Manhattan metric (which he calls the "smoothness" metric).

Testing on works of Bach, Beethoven, Mozart and Haydn, Tymoczko obtained high Pearson correlation coefficients of at least 0.91 (absolute value). High correlations suggest that composers

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[5] Melodic minor scale, harmonic minor scale and natural minor scale

typically modulate between keys whose associated scales may be related to efficient voice leading or have small interscalar distances. This was certainly an inspiration of my research.

3 Methods

3.1 Five-Note Representation of Scales

Impressionist music is marked by the use of Greek modes and exotic scales. In my analysis, I identify all Greek modes with their associated major scales since they share the same notes although each mode has characteristic intervals and chords that give it its distinctive sound. For example, the G mixolydian mode (G, A, B, C, D, E, F) is identified with the C major scale (C, D, E, F, G, A, B). Other frequently used scales in impressionist music include pentatonic and whole tone scales.

Since we want to measure the distance between different scales, it is helpful to represent each scale by points in a common ambient space. Since the scale with the fewest notes, the pentatonic scale, consists of five notes, we can always extract five notes each from any other scale. Then we may represent each scale by a point in a 5-dimensional space. Since the notes are pitch classes denoted by numbers in the range of 0 to 11 modulo 12, the space is 5-way periodic and has to be a torus. In this case, the torus is a 5-torus:

$$T^5 := S^1 \times S^1 \times S^1 \times S^1 \times S^1 \tag{9}$$

where $S^1 = T^1$ is a circle $R/12Z$. Equivalently, the 5-torus is obtained from the 5-dimensional hypercube by gluing the opposite faces together with no rotation or any other transformation.

I also consider permutations of the same 5 notes to be equivalent representations, i.e. C, D, E, G, A and A, C, E, D, G are equivalent representations of a scale. Therefore, the space is T^5/S_5 where S_5 is the symmetric group of order 5!.

3.2 Two Metrics

Given two points in T^5, we use two ways to calculate the distance between them. Tymoczko uses the Manhattan metric (or taxicab metric) that adds up the steps moved by each coordinate. In other words,

$$\begin{aligned} &d[(a_1, b_1, c_1, d_1, e_1), (a_2, b_2, c_2, d_2, e_2)] \\ &\qquad = |a_1 - a_2| + |b_1 - b_2| + |c_1 - c_2| + |d_1 - d_2| + |e_1 - e_2|. \end{aligned}$$

We may also use the Euclidean metric, where

$$\begin{aligned} &d[(a_1, b_1, c_1, d_1, e_1), (a_2, b_2, c_2, d_2, e_2)] = \\ &\qquad \sqrt{|a_1 - a_2|^2 + |b_1 - b_2|^2 + |c_1 - c_2|^2 + |d_1 - d_2|^2 + |e_1 - e_2|^2}. \end{aligned}$$

Since the points representing scales live in T^5/S_5, each point has 5! pre-images in T^5 as each scale consists of 5 distinct notes. We may calculate the distance between any pair of pre-images (one from each scale) using either metric. The distance between two scales is then defined to be the shortest distance achieved by some pair of pre-images in T^5.

Now the question of determining the five-note representation arises and again we have two methods of representation.

3.3 Two Methods of Representation

3.3.1 The Stationary Method: Removing "Stationary" Notes

Considering all seven-note scales with the same tonic, we notice that the second and fourth notes of each seven-note scale are the same. Therefore, it is reasonable to represent a seven-note scale by all but the "stationary" notes of it. In other words, any seven-note scale is represented as a collection of its 1^{st}, 3^{rd}, 5^{th}, 6^{th} and 7^{th} notes. I treat the minor scales in the same ways as Tymoczko did.

For whole tone scales, which contain six notes each, we omit one note to form a five-note representation. Since all notes in the whole tone scale are equally spaced (every pair of adjacent notes are a whole tone apart), there is no natural choice of which note is stationary. Therefore, we have to consider 6 possibilities of five-note representation with one note omitted at a time. For each possible representation, we can easily calculate its distance to every other scale with the same tonic. Summing up the distances, we obtain the total distance from that particular representation to all other scales in the same key. It is natural to consider the representation that yields the shortest total distance as the most appropriate one to honor the identicalness of tonic. Calculations by either the Manhattan metric or the Euclidean metric show that the 2^{nd} note of a whole-tone scale should be omitted.

3.3.2 The Average Method: Averaging Distances of Minimum Matching

Instead of having a unique five-note representation for each scale, the second method generates 6 five-note representations for each scale. To achieve this for a seven-note scale, we fix three notes and choose two other notes each time from the leftover. It is then natural to fix the tonic chord that contains the 1^{st}, 3^{rd} and 5^{th} notes in the scale because the tonic chord defines the scale in some way. The other two varying notes are chosen from the 2^{nd}, 4^{th}, 6^{th} and 7^{th} notes. Therefore we have 6 (= 4 choose 2) combinations resulting in 6 distinct representations. For a whole tone scale, we may just omit one note from the scale each time. The pentatonic scale that contains five notes only is a trivial case here as we simply generate 6 identical representations.

For any pair of scales, we obtain up to 6^6 distance measures by calculating the distance between any one of the 6 representations of one scale and any representation of the other scale. We draw a bipartite graph whose vertices are the representations and the vertices are grouped by their corresponding scale. For every minimum matching possible, we obtain 6 edges that are equivalently 6 distance measures and we are interested in the average of the 6 distances. Our purpose is to find a perfect matching that yields the smallest average distance or simply the minimum perfect matching. This minimum perfect matching is considered the distance between that particular pair of scales.

3.4 Data of Modulation Frequency

Two metrics and two methods of representation yield 4 different sets of distance measures. To evaluate the measures, we will now need the actual musical data. My sample of Impressionist

Table 1: Sample Music

	Name	Composer	Modulations
1	ChildrensCorner1	Debussy	C E C E C F B♭ C♯ C
2	ChildrensCorner2	Debussy	B♭ A♭ wA♭ B♭
3	ChildrensCorner3	Debussy	E C♯ E B♭ C B♭ C c E
4	ChildrensCorner4	Debussy	d B d c d
5	ChildrensCorner5	Debussy	A E C♯ A

music includes piano works of two typical Impressionist musicians, Debussy and Ravel. I analyzed the modulations of 32 pieces in total, among which 23 pieces were composed by Debussy and 9 were composed by Ravel. The chosen pieces vary both in length and in style.

Impressionist music is partly characterized by its unexpected modulations. Indeed, many of the modulations in my data were achieved without being facilitated by common modulation techniques such as using a common chord. While the length of the modulation is usually at least four measures in Classical music, the length of modulation in Impressionist music could be as short as one measure only. Moreover, a modulation may or may not be designated by a change of key signature.

The table below shows a small portion of the sample pieces and their modulations.[6] The complete list of sample music can be found in Appendix A.

4 Results

4.1 Interscalar Distances

The most common modulations in Impressionist music consist of the modulations between major and major scales, major and minor, major and pentatonic, major and whole-tone, minor and minor, minor and pentatonic as wells as minor and whole-tone. It is not to say that the modulation between any other scales, such as that from one pentatonic scale to one whole-tone scale, is nonexistent. However, since my sample data contains the most common modulations only, I chose to exclude the rare modulations in my analysis. Given my sample size is not large, doing so could help avoid excessive zeroes (frequency of modulation) that would substantially skew the result.

The following table shows a portion of the interscalar distances calculated by two methods with two metrics each. The complete table is in Appendix A. Note that the interscalar distance

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[6] My original plan was to analyze the modulations by using Music21, a software for computer-aided musicology. Chordify, a function of Music21, claimed to be a powerful tool for reducing a complex score with multiple parts to a succession of chords in one part that represent everything that is happening in the score. However, although it functions on the built-in music, it does not work properly for any local music. The built-in corpora does not contain any impressionist piece. The developer of Windows Music21 admitted the incompetence of Music21 in terms of importing local music and gave me three possible ways to fix it. I tried them all but still failed to read any music of my local library. Then I switched to a Mac as Music21 was initially developed for Mac users. However, the last step of importing local music was still a failure. I emailed the developer again about the issue and he offered to debug for me. The problem was that there were two "stops" on the crescendo in the xml file that I used, but only one start. He suggested that I could file a bug report with MuseScore. Due to time constraint, I manually analyzed the modulations in all my sample pieces.

is invariant under translation and each name of the modulation represents its equivalence class. For example, the modulation from C major to C minor is equivalent to that from E major to E minor.

Table 2: Interscalar Distances

Equivalent Modulation	**Manhattan, Stationary Method**	**Euclidean, Stationary Method**	**Manhattan, Average Method**	**Euclidean, Average Method**
C-C	0	0	0	0
C-C♯	5	2.236	4	0.665
C-c	2	1.414	1.5	1.202
C-c♯	3	1.732	3.5	1.86
C-pC	3	2.236	2.833	0.683
C-pC♯	6	2.828	4.833	0.775
C-wC	3	1.732	3.167	0.626
C-wC♯	2	1.414	3.167	0.626
c-c	0	0	0	0
c-c♯	5	2.236	4	2.057
c-pC	5	2.646	3.333	2.138
c-pC♯	4	2.449	4.167	2.406
c-wC	3	1.732	3.5	1.97
c-wC♯	4	2.449	3	1.912

4.2 Kernel Density Estimations

In order to evaluate the distribution of the results, I did kernel density estimation to the distances calculated by each combination of method and metric. Since we do not consider the “modulation” from C to C or that from c to c “real” modulations, these two points are omitted for kernel density estimation.

4.2.1 Stationary Method, Manhattan Metric

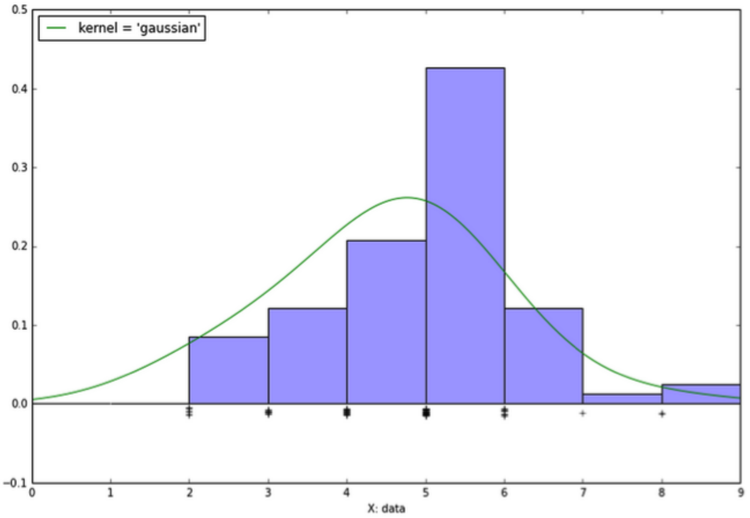

Figure 5: Stationary Method, Manhattan Metric

I obtained a nice unimodal kernel density estimation.

4.2.2 Stationary Method, Euclidean Metric

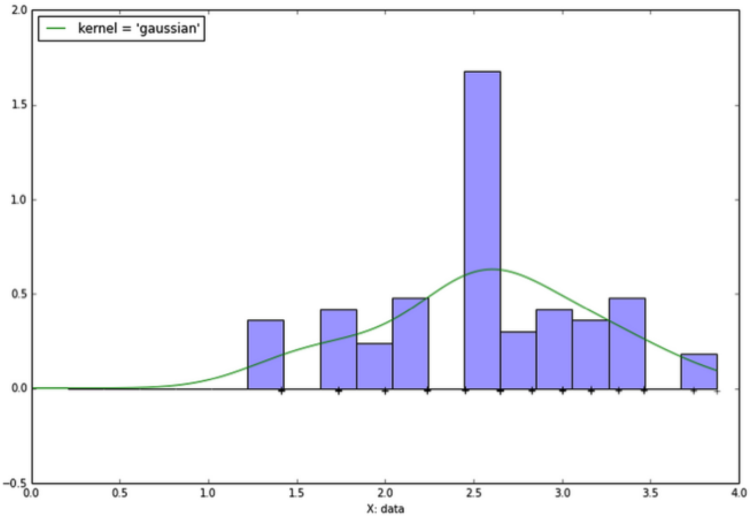

Figure 6: Stationary Method, Euclidean Metric

Here, the large bandwidth chosen might have oversmoothed the estimated density function. However, it is clear that we have a unimodal distribution.

4.2.3 Average Method, Manhattan Metric

Using either metric, Method 2 gives identical distance measure for any modulation from a major scale to a whole-tone scale. Moreover, Method 2 "identifies" that there are only two equivalent classes of whole-tone scales by giving two distance measures for the modulations from minor scales to whole-tone scales. Therefore, we see many repeating values for modulations that involve a whole-tone scale.

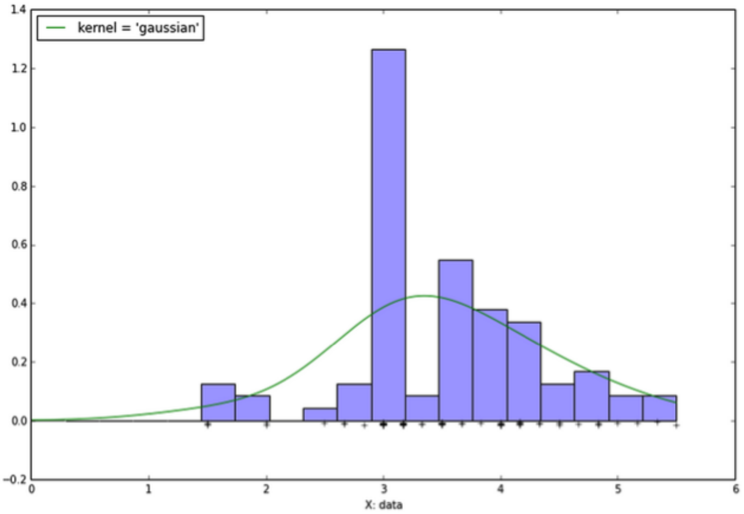

Figure 7: Average Method, Manhattan Metric, with repeating data

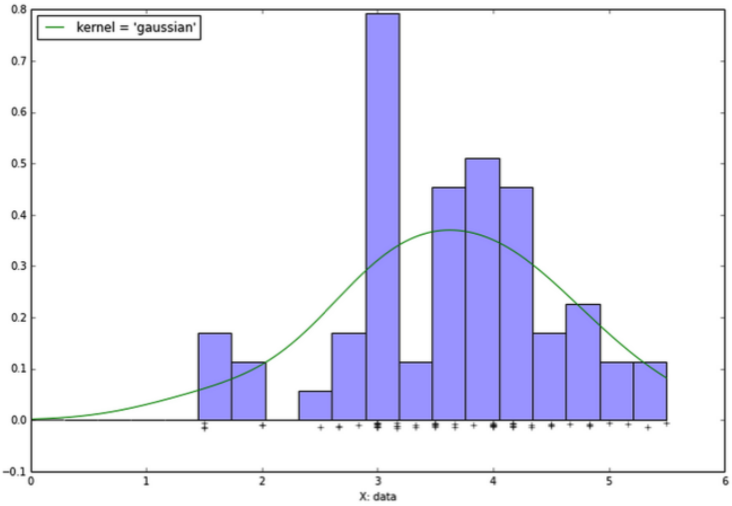

Figure 8: Average Method, Manhattan Metric, without repeating data

Both distributions are unimodal.

4.2.4 Average Method, Euclidean Metric

Similarly, we consider the cases with and without removing repeating values for whole-tone scales.

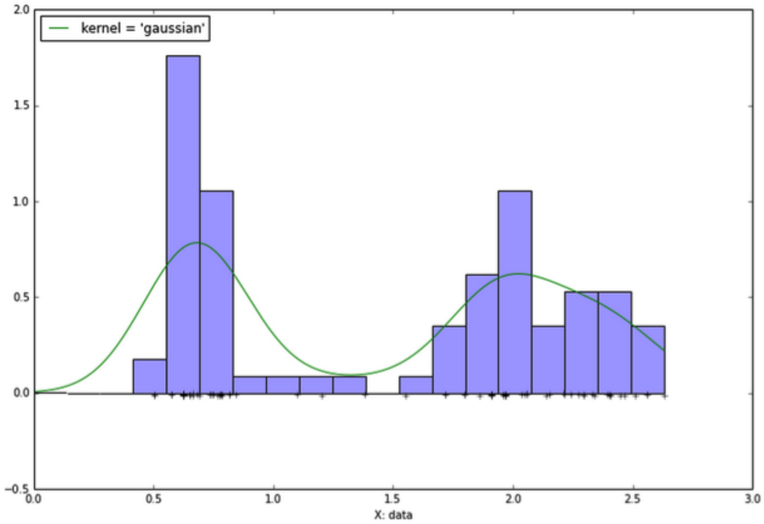

Figure 9: Average Method, Euclidean Metric, with repeating data

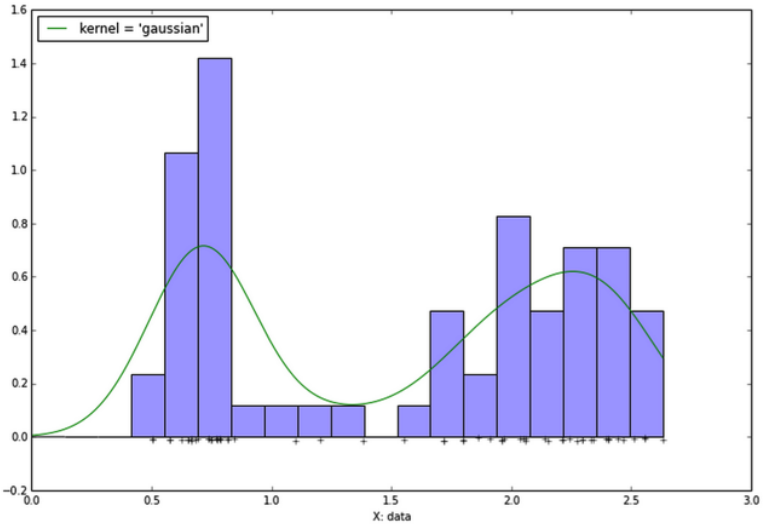

Figure 10: Average Method, Euclidean Metric, without repeating data

Both are bimodal distributions.

5 Analysis of Results

Since I have the distance measures ready now, I can study the Pearson correlations between modulation frequency and interscalar distances.

The table below shows the resulting Pearson Correlation Coefficients of my data under the operations of the following functions:

1. RemovingZeroes. This function deletes modulations “C-C” and “c-c”. This function is applied to all entries in the table.

2. MergeSymmetric. This function combines the data of symmetric modulations. For example, the numbers of occurrence of “C-D” and “C-B♭” are combined.

3. MergeWholetone. This function does not apply to the Stationary method because only the Average method produces repeated distances for modulations involving whole-tone scales.

Table 3: Pearson Correlation Coefficients

	Removing Zeroes	**Merge: Whole-tone only**	**Merge: Symmetric only**	**Merge: Whole-tone and Symmetric**
Stationary, Manhattan	0.1192	NA	0.0954	NA
Stationary, Euclidean	0.1772	NA	0.1416	NA
Average, Manhattan	−0.2499	−0.424	−0.2386	−0.374
Average, Euclidean	−0.302	−0.423	−0.2409	−0.392

The Stationary method gives positive correlation between modulation frequency and interscalar distance. A positive correlation here implies that modulations happen more frequently between scales further apart and contradicts our assumption. However, the positive correlations are so weak that they might be ignored. Therefore, let us focus on the results given by the Average method. Using the Average method, the two metrics give comparable results. A Pearson coefficient of nearly −0.4 suggests a moderate correlation between modulation frequency and the interscalar distances calculated by the Average method.

The data plots of the better results are as follows.

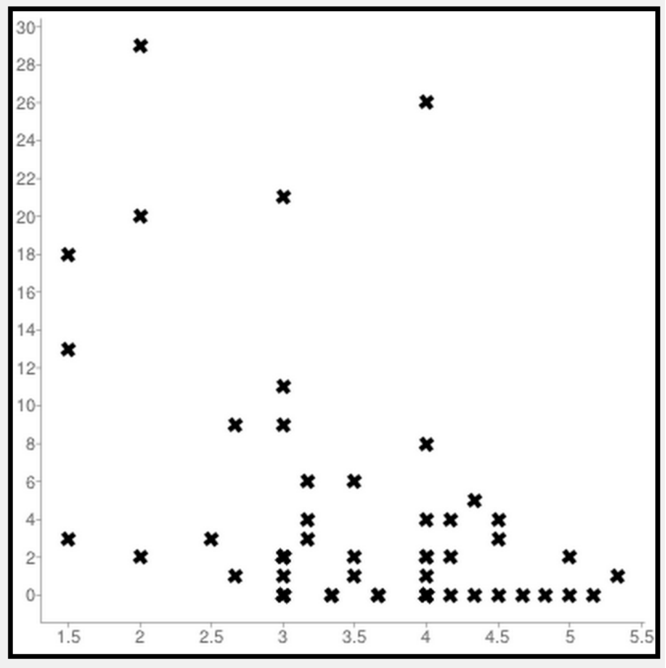

Figure 11: Average Method, Manhattan Metric, MergeWholetone

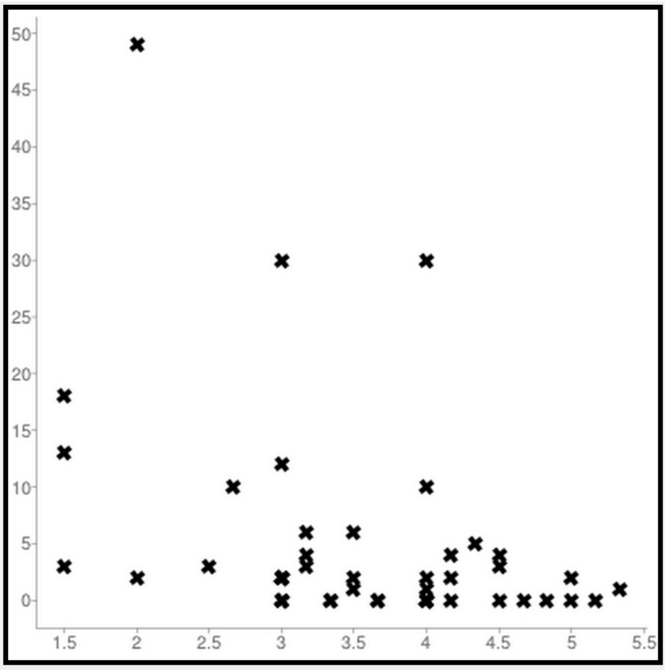

Figure 12: Average Method, Manhattan Metric, MergeWholetone, MergeSymmetric

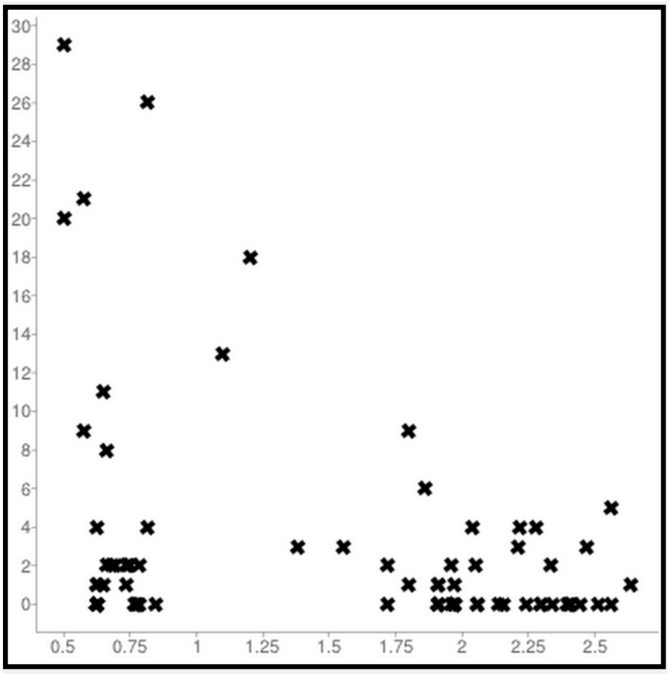

Figure 13: Average Method, Euclidean Metric, MergeWholetone

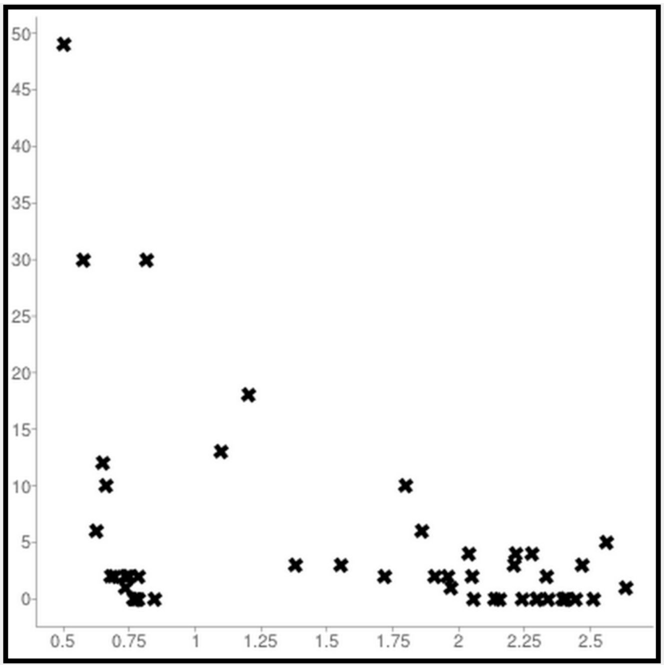

Figure 14: Average Method, Euclidean Metric, MergeWholetone, MergeSymmetric

In any of these plots, the horizontal axis represents the interscaler distances whereas the vertical axis represents the modulation frequency in the 32 sample pieces. Although the overall shapes of the scatter plots agree with my initial assumption, the distributions are unsatisfying due to the excessive "zero occurrences". One reason may be that the sample size is limited in terms of either quantity or variety. If I were able to use a software to automatically find modulations, the sample size would be much larger. Then fewer modulations would have "zero occurrence" and the data points would hopefully be more spread out.

6 Conclusion

As mentioned earlier, Tymoczko investigated correlations between modulation frequency and interscalar distances in Baroque and Classical music and found them to be rather strong. Compared with Professor Tymoczko's Pearson correlation coefficients of at least 0.91 (absolute value), 0.4 seems less exciting. However, since I was unable to find out either the quantity of sample music Tymoczko used or his way of counting modulations[7], such comparison could be misleading – we may not draw the conclusion that T_5/S_5 is an inappropriate space for scales in Impressionist music. Musically speaking, the number -0.4 tells us that modulations are somewhat more likely to happen between scales that are closer in the space of T_5/S_5, although it is not always the case. Given that Impressionist music is much less homogeneous than Classical music, we may develop a more complex model that involves other parameters.

Given the "zero occurrences" of several modulations, we may also question the possibility of an unknown rule in Impressionist music that forbids certain modulation from happening regardless of interscalar distances. For example, an Impressionist piece initially in a pentatonic scale may be more likely to modulate to a major scale rather than to a whole-tone scale. This demands special attention to the modulations that have low occurrences.

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[7] I contacted him via email and he told me to read his book[7]. However, the book does not have all the details that I need.

A

Modulations in Sample Music

Table 1: Sample Music

	Name	Composer	Modulations
1	ChildrensCorner1	Debussy	C E C E C F B♭ C♯ C
2	ChildrensCorner2	Debussy	B♭ A♭ wA♭ B♭
3	ChildrensCorner3	Debussy	E C♯ E B♭ C B♭ C c E
4	ChildrensCorner4	Debussy	d B d c d
5	ChildrensCorner5	Debussy	A E C♯ A
6	PreludeB1 1	Debussy	B♭ F B♭ F pB♭ F pE♭ F B♭
7	PreludeB1 2	Debussy	wC a♭ A
8	PreludeB1 4	Debussy	A wA A
9	PreludeB1 6	Debussy	d D d D d D C♯ d
10	PreludeB2 1	Debussy	C G C G
11	PreludeB2 3	Debussy	C♯ F G F C♯
12	PreludeB2 4	Debussy	C♯ c♯ C♯ a A C♯
13	PreludeB2 5	Debussy	A♭ B♭ C c B♭ A♭
14	ClairDeLune a	Debussy	C♯ E♭ A♭ F♯ F A♭ F♯ F e♭ C♯ f E C♯ f E C♯ f C♯ f E C♯ f C♯ f E C♯
15	Passepied a	Debussy	f♯ c♯ f♯ c♯ C♯ f♯ b f♯ c♯ c♯ f♯ B f B E B A G F E A E A♭ A♭ E B E B
16	Arabesque1	Debussy	E a♭ D E A E A E C E a♭ E A E
17	Arabesque2	Debussy	G D G B b♭ G D G D C G C G C D G B G C B C G B G
18	PourLePianoPrelude	Debussy	a C D C wC a
19	PourLePianoSarabande	Debussy	c♯ B E F♯ E F♯ E♭ c♯ a c♯ E A E
20	PourLePianoToccata	Debussy	c♯ C C♯ c♯ C♯
21	SunkenCathedral	Debussy	pG E pG B E♭ F C a♭ wG C
22	Nuages	Debussy	E A♭ C♯ E
23	Fetes	Debussy	f D A C♯ F A B E D E C a♭ C♯ D E♭ A C♯ A
24	Mirrors4	Ravel	A♭ B♭ F♯ C♯ E D G D G
25	Rapsodie3	Ravel	A D F♯ A D C A♭ C A♭
26	MotherGoose1	Ravel	a e d e a
27	MotherGoose3	Ravel	C♯ pF C♯ c♯ F♯ e♭ c♯ F♯ pF♯ F♯
28	MotherGoose5	Ravel	C e C
29	LEnfantTasse	Ravel	b♭ F b♭
30	LenfantPatres	Ravel	a A a
31	DaphnisNymphs	Ravel	c♯ g c a c
32	Pavane	Ravel	G g G

B Interscalar Distances

Table 2: Interscalar Distances

Equivalent Modulation	**Manhattan, Stationary Method**	**Euclidean, Stationary Method**	**Manhattan, Average Method**	**Euclidean, Average Method**
C-C	0	0	0	0
C-C♯	5	2.236	4	0.665
C-D	4	2.449	4	0.817
C-E♭	5	2.646	3	0.65
C-E	6	3.464	3	0.575
C-F	5	3	2	0.505
C-F♯	8	3.873	5	0.785
C-G	5	3	2	0.505
C-A♭	6	3.464	3	0.575
C-A	5	2.646	3	0.65
C-B♭	4	2.449	4	0.817
C-B	5	2.236	4	0.665
C-c	2	1.414	1.5	1.202
C-c♯	3	1.732	3.5	1.86
C-d	5	3	3.167	2.212
C-e♭	5	2.449	4.5	2.275
C-e	6	3.464	1.5	1.099
C-f	5	2.828	2.5	1.551
C-f♯	6	3.317	4.5	2.467
C-g	7	3.317	3.167	2.216
C-a♭	4	3.162	4.167	2.036
C-a	5	3	1.5	1.383
C-b♭	4	2	4.167	2.332
C-b	5	2.828	3.5	2.049
C-pC	3	2.236	2.833	0.683
C-pC♯	6	2.828	4.833	0.775
C-pD	3	2.236	3.833	0.788
C-pE♭	6	3.162	3.5	0.692
C-pE	3	1.732	4.167	0.737
C-pF	4	2.449	3.167	0.751
C-pF♯	5	2.236	5.5	0.844
C-pG	2	2	3.5	0.769
C-pA♭	5	2.646	4.167	0.737
C-pA	2	1.414	4.167	0.78
C-pB♭	5	2.646	3.5	0.745
C-pB	4	2	4.833	0.775
C-wC	3	1.732	3.167	0.626
C-wC♯	2	1.414	3.167	0.626
C-wD	5	2.646	3.167	0.626
C-wE♭	2	1.414	3.167	0.626

C-wE	5	2.646	3.167	0.626
C-wF	4	2.449	3.167	0.626
C-wF♯	5	2.646	3.167	0.626
C-wG	6	3.162	3.167	0.626
C-wA♭	5	2.646	3.167	0.626
C-wA	6	3.742	3.167	0.626
C-wB♭	5	2.646	3.167	0.626
C-wB	4	3.162	3.167	0.626
c-c	0	0	0	0
c-c♯	5	2.236	4	2.057
c-d	4	2.449	4.333	2.558
c-e♭	5	2.646	3	1.961
c-e	5	3.464	3	1.718
c-f	5	3	2.667	1.799
c-f♯	8	3.742	5.333	2.632
c-g	5	3	2.667	1.799
c-a♭	5	3.464	3	1.718
c-a	5	2.646	3	1.961
c-b♭	4	2.449	4.333	2.558
c-b	5	2.236	4	2.057
c-pC	5	2.646	3.333	2.138
c-pC♯	4	2.449	4.167	2.406
c-pD	5	2.646	4.667	2.403
c-pE♭	4	2.449	3	2.153
c-pE	3	1.732	4.833	2.393
c-pF	5	2.828	3.667	2.404
c-pF♯	3	1.732	4.5	2.446
c-pG	4	2.449	4	2.297
c-pA♭	3	2.236	3.667	2.34
c-pA	4	2	5.167	2.513
c-pB♭	5	2.646	3.333	2.296
c-pB	2	1.414	4	2.242
c-wC	3	1.732	3.5	1.97
c-wC♯	4	2.449	3	1.912
c-wD	3	1.732	3.5	1.97
c-wE♭	2	1.414	3	1.912
c-wE	5	3	3.5	1.97
c-wF	4	2.449	3	1.912
c-wF♯	6	3.317	3.5	1.97
c-wG	4	2.646	3	1.912
c-wA♭	5	2.828	3.5	1.97
c-wA	6	3.162	3	1.912
c-wB♭	5	2.646	3.5	1.97
c-wB	5	3.162	3	1.912

C Data for Kernel Density Estimations

Since we do not consider the “modulation” from C to C or that from c to c “real” modulations, these two points are omitted for kernel density estimation.

C.1 Stationary Method, Manhattan Metric

distances = [5, 4, 5, 6, 5, 8, 5, 6, 5, 4, 5, 2, 3, 5, 5, 6, 5, 6, 7, 4, 5, 4, 5, 3, 6, 3, 6, 3, 4, 5, 2, 5, 2, 5, 4, 3, 2, 5, 2, 5, 4, 5, 6, 5, 6, 5, 4, 5, 4, 5, 5, 5, 8, 5, 5, 5, 4, 5, 5, 4, 5, 4, 3, 5, 3, 4, 3, 4, 5, 2, 3, 4, 3, 2, 5, 4, 6, 4, 5, 6, 5, 5]

C.2 Stationary Method, Euclidean Metric

distances = [2.236, 2.449, 2.646, 3.464, 3.0, 3.873, 3.0, 3.464, 2.646, 2.449, 2.236, 1.414, 1.732, 3.0, 2.449, 3.464, 2.828, 3.317, 3.317, 3.162, 3.0, 2.0, 2.828, 2.236, 2.828, 2.236, 3.162, 1.732, 2.449, 2.236, 2.0, 2.646, 1.414, 2.646, 2.0, 1.732, 1.414, 2.646, 1.414, 2.646, 2.449, 2.646, 3.162, 2.646, 3.742, 2.646, 3.162, 2.236, 2.449, 2.646, 3.464, 3.0, 3.742, 3.0, 3.464, 2.646, 2.449, 2.236, 2.646, 2.449, 2.646, 2.449, 1.732, 2.828, 1.732, 2.449, 2.236, 2.0, 2.646, 1.414, 1.732, 2.449, 1.732, 1.414, 3.0, 2.449, 3.317, 2.646, 2.828, 3.162, 2.646, 3.162]

C.3 Average Method, Manhattan Metric

Without removing the repeating values of distances, we have distances = [4.0, 4.0, 3.0, 3.0, 2.0, 5.0, 2.0, 3.0, 3.0, 4.0, 4.0, 1.5, 3.5, 3.167, 4.5, 1.5, 2.5, 4.5, 3.167, 4.167, 1.5, 4.167, 3.5, 2.833, 4.833, 3.833, 3.5, 4.167, 3.167, 5.5, 3.5, 4.167, 4.167, 3.5, 4.833, 3.167, 3.167, 3.167, 3.167, 3.167, 3.167, 3.167, 3.167, 3.167, 3.167, 3.167, 3.167, 4.0, 4.333, 3.0, 3.0, 2.667, 5.333, 2.667, 3.0, 3.0, 4.333, 4.0, 3.333, 4.167, 4.667, 3.0, 4.833, 3.667, 4.5, 4.0, 3.667, 5.167, 3.333, 4.0, 3.5, 3.0, 3.5, 3.0, 3.5, 3.0, 3.5, 3.0, 3.5, 3.0, 3.5, 3.0]

Removing repeating values, we have distance = [4.0, 4.0, 3.0, 3.0, 2.0, 5.0, 2.0, 3.0, 3.0, 4.0, 4.0, 1.5, 3.5, 3.167, 4.5, 1.5, 2.5, 4.5, 3.167, 4.167, 1.5, 4.167, 3.5, 2.833, 4.833, 3.833, 3.5, 4.167, 3.167, 5.5, 3.5, 4.167, 4.167, 3.5, 4.833, 3.167, 4.0, 4.333, 3.0, 3.0, 2.667, 5.333, 2.667, 3.0, 3.0, 4.333, 4.0, 3.333, 4.167, 4.667, 3.0, 4.833, 3.667, 4.5, 4.0, 3.667, 5.167, 3.333, 4.0, 3.5, 3.0]

C.4 Average Method, Euclidean Metric

Without removing the repeating values of distances, we have distances = [0.665, 0.817, 0.65, 0.575, 0.505, 0.785, 0.505, 0.575, 0.65, 0.817, 0.665, 1.202, 1.86, 2.212, 2.275, 1.099, 1.551, 2.467, 2.216, 2.036, 1.383, 2.332, 2.049, 0.683, 0.775, 0.788, 0.692, 0.737, 0.751, 0.844, 0.769, 0.737, 0.78, 0.745, 0.775, 0.626, 0.626, 0.626, 0.626, 0.626, 0.626, 0.626, 0.626, 0.626, 0.626, 0.626, 0.626, 2.057, 2.558, 1.961, 1.718, 1.799, 2.632, 1.799, 1.718, 1.961, 2.558, 2.057, 2.138, 2.406, 2.403, 2.153, 2.393, 2.404, 2.446, 2.297, 2.34, 2.513, 2.296, 2.242, 1.97, 1.912, 1.97, 1.912, 1.97, 1.912, 1.97, 1.912, 1.97, 1.912, 1.97, 1.912]

Removing repeating values, we have

distances = [0.665, 0.817, 0.65, 0.575, 0.505, 0.785, 0.505, 0.575, 0.65, 0.817, 0.665, 1.202, 1.86, 2.212, 2.275, 1.099, 1.551, 2.467, 2.216, 2.036, 1.383, 2.332, 2.049, 0.683, 0.775, 0.788, 0.692, 0.737, 0.751, 0.844, 0.769, 0.737, 0.78, 0.745, 0.775, 0.626, 2.057, 2.558, 1.961, 1.718, 1.799, 2.632, 1.799, 1.718, 1.961, 2.558, 2.057, 2.138, 2.406, 2.403, 2.153, 2.393, 2.404, 2.446, 2.297, 2.34, 2.513, 2.296, 2.242, 1.97, 1.912]

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