ON THE GEOMETRY OF ALMOST COMPLEX 6-MANIFOLDS

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This article is dedicated to the memory of Shiing-Shen Chern, whose beautiful works and gentle encouragement have had the most profound influence on my own research.

Abstract. This article discusses some basic geometry of almost complex 6-manifolds.

A 2-parameter family of intrinsic first-order functionals on almost complex structures on 6-manifolds is introduced and their Euler-Lagrange equations are computed.

A natural generalization of holomorphic bundles over complex manifolds to the almost complex case is introduced. The general almost complex manifold will not admit any nontrivial bundles of this type, but there is a class of nonintegrable almost complex manifolds for which there are such nontrivial bundles. For example, the G_2-invariant almost complex structure on the 6-sphere admits such nontrivial bundles. This class of almost complex manifolds in dimension 6 will be referred to as *quasi-integrable* and a corresponding condition for unitary structures is considered.

Some of the properties of quasi-integrable structures (both almost complex and unitary) are developed and some examples are given.

However, it turns out that quasi-integrability is not an involutive condition, so the full generality of these structures in Cartan's sense is not well-understood. The failure of this involutivity is discussed and some constructions are made to show, at least partially, how general these structures can be.

Contents

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1. Introduction

In the theory of almost complex manifolds, dimension 6 is special for a number of reasons.

First, there is a famous non-integrable almost complex structure on the 6-sphere and the tantalizing problem of determining whether or not there exists an integrable complex structure on the 6-sphere remains open as of this writing, despite much interest and effort.

Second, there exist nontrivial functorial constructions of top-degree differential forms associated to an almost complex structure J on a 6-manifold that are first-order in the almost complex structure, something that does not happen in dimension 4. These are reviewed in §2.

In particular, there is a 2-parameter family of first-order 6-forms invariantly associated to an almost complex structure in dimension 6 and hence these can be regarded as defining a 2-parameter family of natural Lagrangians for 6-dimensional almost complex structures. The Euler-Lagrange equations of these functionals are computed and some of their properties and some examples are discussed. Integrable almost complex structures are critical points for these functionals, but it turns out that there are many nonintegrable almost complex structures that are also critical points for all of these functionals.

Third, there is a natural generalization of the notion of holomorphic bundles to almost complex manifolds (this is true in any dimension), one that is based on replacing the holomorphic transition function definition with the structure of a connection whose curvature is of type $(1,1)$. This opens the way to discussing a generalization of the notion of Hermitian-Yang-Mills connections to almost complex manifolds. I explain this in §3.

For the general nonintegrable almost complex manifold, this notion turns out to be too rigid to admit any nontrivial examples, but, in dimension 6, there is a special class of almost complex manifolds, which I have called *quasi-integrable*, for which the overdetermined system that defines these generalized Hermitian-Yang-Mills connections turns out to be just as well-behaved, locally, as the familiar system on an integrable complex manifold. In particular, important examples, such as the standard nonintegrable structure on the 6-sphere, turn out to be quasi-integrable (more generally, nearly Kähler almost complex manifolds turn out to be quasi-integrable).[1]

[1]It was Gang Tian, in 1998, who pointed out to me that the (pseudo-)Hermitian-Yang-Mills connections on complex vector bundles over S^6 would be of interest in understanding the singular behavior of certain generalized Yang-Mills connections associated to calibrated geometries. I would like to thank him for his suggestion.

My original goal was to regard quasi-integrable almost complex structures as a generalization of integrable ones to which the methods of Gromov [11] might be applicable for establishing existence on compact 6-manifolds. Then one might be able to use the category of pseudo-holomorphic bundles on such structures as a replacement for the usual geometry of holomorphic mappings and submanifolds. One might even be able to use the functionals described above to look for particularly good quasi-integrable structures on 6-manifolds.

Such a program raises the natural question of how general or 'flexible' the quasi-integrable almost complex structures are. Unfortunately, quasi-integrability for almost complex structures turns out not to be a simple matter of imposing a smooth set of first-order equations on an almost complex structure. Instead, the space of 1-jets of quasi-integrable almost complex structures is a singular subset of codimension 8 in the space of all 1-jets of all almost complex structures, the singularity being the worst exactly along the locus of 1-jets of integrable almost complex structures. Moreover, unless the first Chern class $c_1(TM, J)$ is 3-torsion (a condition that depends only on the homotopy class of the almost complex structure J), the 1-jet of a quasi-integrable almost complex structure J on a compact manifold M^6 cannot avoid this singular locus. Such behavior does not bode well for application of Gromov's methods.

Instead it turns out to be better to work with a closely related notion, that of quasi-integrable $\mathrm{U}(3)$-structures on 6-manifolds (and their hyperbolic analogs, quasi-integrable $\mathrm{U}(1,2)$-structures), since this condition *is* defined by a smooth set of 16 first-order equations on the $\mathrm{U}(3)$-structure (respectively, $\mathrm{U}(1,2)$-structure).

In fact, what I have called the class of quasi-integrable $\mathrm{U}(3)$-structures was already considered by Hervella and Vidal [12, 13] and by Gray and Hervella [9], who developed a nomenclature that would describe these $\mathrm{U}(3)$-structures as being of type $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$.

The relation between the two notions of quasi-integrability is that, away from the singular locus, a quasi-integrable J has a canonical reduction to a quasi-integrable unitary structure (J, η) (where η is a nonsingular form of type $(1, 1)$ with respect to J) while, for any quasi-integrable unitary structure (J, η), the underlying almost complex structure J is quasi-integrable. (However, examples show that when J is quasi-integrable, there may not be a nonsingular η defined across the singular locus such that (J, η) is quasi-integrable.)

Unfortunately, it turns out that although quasi-integrable unitary structures are well-behaved up to second order, the PDE system that defines them is *not* involutive. It is not even formally integrable, as obstructions appear at the third order. This was something that I did not know (or even suspect) at the time of my lectures on this material in 1998 and 2000. In fact, in my lecture in 2000, I erroneously claimed that quasi-integrability for $\mathrm{U}(3)$-structures was an involutive condition. It was only later, when I was writing up a careful proof up for publication, that I discovered my mistake. This is explained fully in §4.5, where a third-order obstruction to formal integrability is derived.

This third-order obstruction shows that the quasi-integrable $\mathrm{U}(3)$-structures on a connected M^6 split into two disjoint classes: Those for which the Nijenhuis tensor vanishes identically (and hence the underlying almost complex structure is integrable) and those for which the Nijenhuis tensor vanishes only along a proper subset of real codimension 2. The former class is, of course, very well understood and forms an involutive class of its own. The latter class is much less understood. Indications are that the latter class is, in fact, *less* general than the former class, as counter-intuitive as this may seem.[2] For example, any smooth 1-parameter deformation (J_t, η_t) through quasi-integrable U(3)-structures of the generic U(3)-structure (J_0, η_0) with J_0 integrable will necessarily have J_t be integrable for t sufficiently small.

I have not proved that the third-order obstruction that I identify is the only third-order obstruction, much less have I shown that, when this third-order obstruction is taken into account, there will be no higher order obstructions to formal integrability. This would require some rather complex calculations that, so far, I have not been able to do. This remains an interesting problem for the future.

Remark 1 (Sources). The results in this article were mostly reported on in two lectures [3, 4] on aspects of the geometry of almost complex 6-manifolds. Of course, this written article contains many details and proofs that could not be given in those lectures for want of time. Moreover, the results on noninvolutivity were not known at the time of the lectures.

I discussed this work with Professor Chern from time to time and intended to write the results up for publication, but, after discovering the obstructions to formal integrabilty discussed above, I intended to resolve that issue first. Unfortunately, I was not able to do so before moving on to other projects, so the write-up languished in my private files.

Since I am no longer working in this area and interest continues to be expressed in my results, I have decided to release this article in its present form.

Those familiar with Professor Chern's unrivaled mastery of the method of equivalence and his use of differential forms will see immediately what an enormous debt I owe to him in this and many other works. He will continue to be sorely missed.

Remark 2 (Verbitsky's eprint). On 28 July 2005, Mikhail Verbitsky wrote to me and informed me of his eprint [21], which contains several results that overlap those of my own, though they were arrived at independently. The main overlaps appear to me to be these: First, he rediscovered one of the two first-order functionals that I list in Theorem 1. Second, he identified the class of almost complex structures that I have called elliptically quasi-integrable as being of interest (though for different reasons than I have). Third, he showed that, among the strictly quasi-integrable almost complex structures, the only ones that are critical for the functional that he defines are the ones that underlie nearly Kähler structures.[3]

2. Almost complex 6-manifolds

Let M^6 be a 6-manifold and let $J : TM \to TM$ be an almost complex structure on M.[4] As usual, one has the splitting

$$\mathcal{A}^1(M) = \mathcal{A}^{1,0}(M) \oplus \mathcal{A}^{0,1}(M), \tag{2.1}$$

[2] I realize that I am not being very precise about the notion of 'generality' being employed here. Roughly speaking, what I mean is that, for $k \geq 3$, the space of k-jets of structures in the former class has larger dimension than the space of k-jets of structures in the latter class.

[3] His result is loosely related to my Proposition 2, though our hypotheses and conclusions are quite different.

[4] Throughout this article, when M is a manifold, the space of complex-valued smooth alternating p-forms on M will be denoted by $\mathcal{A}^p(M)$.

where $\mathcal{A}^{1,0}(M)$ consists of the complex-valued 1-forms α on M that satisfy $\alpha(Jv) = \mathrm{i}\,\alpha(v)$ for all $v \in TM$ and $\mathcal{A}^{0,1}(M) = \overline{\mathcal{A}^{1,0}(M)}$. This induces a type decomposition

$$\mathcal{A}^m(M) = \bigoplus_{p+q=m} \mathcal{A}^{p,q}(M) \tag{2.2}$$

for $m \geq 0$ in the usual way, where

$$\mathcal{A}^{p,q}(M) = \Lambda^p\big(\mathcal{A}^{1,0}(M)\big) \otimes \Lambda^q\big(\mathcal{A}^{0,1}(M)\big). \tag{2.3}$$

(Strictly speaking, one should write $\mathcal{A}^{p,q}_J(M)$ for $\mathcal{A}^{p,q}(M)$, but this will be done only when the almost complex structure J is not clear from context.)

The exterior derivative $\mathrm{d} : \mathcal{A}^*(M) \to \mathcal{A}^*(M)$ splits into a sum of terms $\mathrm{d}^{r,s} : \mathcal{A}^{p,q} \to \mathcal{A}^{p+r,q+s}$ where, of course, $r+s = 1$ and, as is easy to see, $r, s \geq -1$. Thus, one can write

$$\mathrm{d} = \mathrm{d}^{2,-1} + \mathrm{d}^{1,0} + \mathrm{d}^{0,1} + \mathrm{d}^{-1,2}. \tag{2.4}$$

I will also use the common notation $\mathrm{d}^{1,0} = \partial$ and $\mathrm{d}^{0,1} = \bar{\partial}$.

The Leibnitz rule implies that the operator $\mathrm{d}^{-1,2} : \mathcal{A}^{1,0}(M) \to \mathcal{A}^{0,2}(M)$ is linear over the C^∞ functions on M and so represents a tensor. This tensor is known as the *Nijenhuis* tensor and, by the famous Newlander-Nirenberg Theorem [16], it vanishes identically if and only if the almost complex structure is actually complex.

Remark 3 (Existence). In dimension 6, it is easy to determine when a given manifold M carries an almost complex structure. The classical condition in terms of the characteristic classes is that $w_1(M) = \beta(w_2(M)) = 0$, where $w_1(M)$ and $w_2(M)$ are the Stiefel-Whitney classes and $\beta : H^2(M, \mathbb{Z}_2) \to H^3(M, \mathbb{Z})$ is the Bockstein homomorphism. Equivalently, M must be orientable (i.e., $w_1(M) = 0$) and $w_2(M)$ must be of the form $w_2(M) \equiv c_1(K) \mod (2)$, where K is some complex line bundle over M, (i.e., M must possess a $\mathrm{Spin}^c(6)$-structure).

The set $\mathcal{J}(M)$ of almost complex structures on M is the space of sections of a bundle $\mathsf{J}(M) \to M$ whose fiber is modeled on $\mathrm{GL}(6,\mathbb{R})/\mathrm{GL}(3,\mathbb{C})$. In fact, if $\mathsf{F}(M) \to M$ is the right principal $\mathrm{GL}(6,\mathbb{R})$-bundle of coframes $u : T_xM \to \mathbb{R}^6$, then $\mathsf{J}(M) = \mathsf{F}(M)/GL(3,\mathbb{C})$. Unless otherwise stated, the topology on $\mathcal{J}(M)$ will be the Whitney C^∞ topology.

Each section J of $\mathsf{J}(M)$ determines an orientation of M and a line bundle $K = \Lambda^3_{\mathbb{C}}(T^{1,0}_J M)^*$ such that $w_2(M) \equiv c_1(K) \mod (2)$. Moreover, when M is compact, the set $\mathcal{J}(M, o, K)$ of sections of $\mathsf{J}(M)$ that determine a given orientation o and line bundle K is easily seen to be nonempty.

Remark 4 (First-order invariants). The group $\mathrm{Diff}(M)$ acts transitively on $\mathsf{J}(M)$ in the obvious way, but it does not act transitively on $J^1\big(\mathsf{J}(M)\big)$.

In fact, the 1-jet of J at $x \in M$ determines the complex linear mapping

$$d^{-1,2}_x : (T^{1,0}_x M)^* \to \Lambda^{2,0}(T^{0,2}_x M)^*. \tag{2.5}$$

Conversely, given two almost complex structures I and J defined in a neighborhood of x that have the same 0-jet at x (i.e., $I_x = J_x$) and that determine the same linear mapping $d^{-1,2}_x$, there exists a diffeomorphism $\phi : M \to M$ that is the identity at x to first-order and has the property that $\phi^*(J)$ and I have the same 1-jet at x.

In this sense, the Nijenhuis tensor is a complete first-order invariant of almost complex structures under the action of the diffeomorphism group.

It is, perhaps, also worth remarking that the map $d_x^{-1,2}$ is essentially arbitrary. (This is true in any even dimension, not just (real) dimension 6): For any complex constants $C^i_{\bar{j}\bar{k}} = -C^i_{\bar{k}\bar{j}}$, consider the complex valued 1-forms on $\mathbb{C}^n$ defined by

$$\alpha^i = \mathrm{d}z^i + \tfrac{1}{2}C^i_{\bar{j}\bar{k}}\,\bar{z}^j\,\mathrm{d}\bar{z}^k. \tag{2.6}$$

On an open neighborhood $U \subset \mathbb{C}^n$ of $z = 0$, these n 1-forms and their complex conjugates are linearly independent. As a consequence, there is a unique almost complex structure J_C on U for which the α^i are a basis of the J_C-linear 1-forms on U. A straightforward calculation yields

$$d_0^{-1,2}\bigl(\alpha^i|_0\bigr) = \tfrac{1}{2}C^i_{\bar{j}\bar{k}}\,\overline{\alpha^j}|_0 \wedge \overline{\alpha^k}|_0\,, \tag{2.7}$$

thus verifying that $d_0^{-1,2}$ can be arbitrarily prescribed.

2.1. **Invariant forms constructed from the Nijenhuis tensor.** Let $U \subset M$ be an open set on which there exist linearly independent 1-forms $\alpha^1, \alpha^2, \alpha^3$ in $\mathcal{A}^{1,0}(U)$. Write $\alpha = (\alpha^i) : TU \to \mathbb{C}^3$ and note that α maps each fiber T_pU isomorphically (and complex linearly) onto $\mathbb{C}^3$. I will say that α is a *J-complex local coframing on M with domain U*.

There exists a unique smooth mapping $N(\alpha) : U \to M_{3\times 3}(\mathbb{C})$ such that

$$\mathrm{d}^{-1,2}\alpha = \mathrm{d}^{-1,2}\begin{pmatrix}\alpha^1\\ \alpha^2\\ \alpha^3\end{pmatrix} = N(\alpha)\begin{pmatrix}\overline{\alpha^2\wedge\alpha^3}\\ \overline{\alpha^3\wedge\alpha^1}\\ \overline{\alpha^1\wedge\alpha^2}\end{pmatrix}. \tag{2.8}$$

If $\beta = (\beta^i) : TU \to \mathbb{C}^3$ is any other J-linear coframing on U, then there exists a unique function $g : U \to \mathrm{GL}(3, \mathbb{C})$ such that $\beta = g\alpha$ and computation yields

$$N(\beta) = N(g\alpha) = \det(\bar{g})^{-1}\,g\,N(\alpha)\,{}^t\bar{g}\,. \tag{2.9}$$

This motivates investigating the representation $\rho : \mathrm{GL}(3, \mathbb{C}) \to \mathrm{End}\bigl(M_{3\times 3}(\mathbb{C})\bigr)$ defined by

$$\rho(g)(a) = \det(\bar{g})^{-1}\,g\,a\,{}^t\bar{g}. \tag{2.10}$$

In particular, it will be important to understand its orbit structure.

2.1.1. *The form Φ.* Clearly, one has $\bigl|\det(\rho(g)a)\bigr|^2 = |\det(g)|^{-2}|\det(a)|^2$, so that

$$\bigl|\det\bigl(N(g\alpha)\bigr)\bigr|^2 = |\det(g)|^{-2}\,\bigl|\det\bigl(N(\alpha)\bigr)\bigr|^2, \tag{2.11}$$

implying that the nonnegative 6-form

$$\Phi = \tfrac{\mathrm{i}}{8}\,\bigl|\det\bigl(N(\alpha)\bigr)\bigr|^2\,\alpha^1\wedge\alpha^2\wedge\alpha^3\wedge\overline{\alpha^1}\wedge\overline{\alpha^2}\wedge\overline{\alpha^3} \tag{2.12}$$

is well-defined on U independent of the choice of α. Consequently, Φ is the restriction to U of a nonnegative 6-form, also denoted Φ, that is well-defined globally on M. When it is necessary to make the dependence on the almost complex structure J explicit, I will denote this 6-form by $\Phi(J)$.

2.1.2. *The form ω.* Another invariant is the canonical $(1,1)$-form ω that is defined as follows: Let

$$Q : M_{3\times3}(\mathbb{C}) \times M_{3\times3}(\mathbb{C}) \to M_{3\times3}(\mathbb{C}) \tag{2.13}$$

be the unique bilinear map satisfying $Q(a,b) = Q(b,a)$ and $Q(a,a) = \det(a)a^{-1}$. This Q satisfies $Q({}^ta, {}^tb) = {}^tQ(a,b)$ and

$$Q\big(gah, gbh\big) = \det(g)\det(h)\, h^{-1}\, Q(a,b)\, g^{-1} \tag{2.14}$$

for all $g, h \in \mathrm{GL}(3,\mathbb{C})$ and $a, b \in M_{3\times3}(\mathbb{C})$. As a result, the $(1,1)$-form ω defined on U by

$$\omega = \tfrac{\mathrm{i}}{2}\ {}^t\alpha \wedge Q\big({}^tN(\alpha), \overline{N(\alpha)}\,\big) \wedge \overline{\alpha}, \tag{2.15}$$

does not depend on the choice of the coframing α and, since ${}^tQ({}^ta, \overline{a}) = \overline{Q({}^ta, \overline{a})}$, this form is real-valued. Consequently, it is the restriction to U of a real-valued $(1,1)$-form that is well-defined globally on M. When it is necessary to make the dependence on J explicit, this 2-form will be denoted by $\omega(J)$.

Remark 5 (Other dimensions). Although this will have no bearing on the rest of this article, it should be remarked that a corresponding construction of a canonical $(1,1)$-form exists in other dimensions as well: If J is an almost complex structure on M^{2n}, then, for any local J-linear coframing $\alpha = (\alpha^i) : TU \to \mathbb{C}^n$, one will have

$$\mathrm{d}^{-1,2}\alpha^i = \tfrac{1}{2} N^i_{\bar{\jmath}\bar{k}}\, \overline{\alpha^j} \wedge \overline{\alpha^k} \tag{2.16}$$

where $N^i_{\bar{\jmath}\bar{k}} = -N^i_{\bar{k}\bar{\jmath}}$. Then the $(1,1)$-form

$$\omega = -\tfrac{\mathrm{i}}{4} N^i_{\bar{\ell}\bar{k}} \overline{N^{\ell}_{\bar{\imath}\bar{\jmath}}}\ \alpha^j \wedge \overline{\alpha^k} \tag{2.17}$$

is easily seen to be real-valued and well-defined, independent of the choice of α. When $n = 3$, this reduces to the definition of ω already given above.

2.1.3. *The form ψ.* A third invariant can be defined as follows: Let

$$P : M_{3\times3}(\mathbb{C}) \times M_{3\times3}(\mathbb{C}) \times M_{3\times3}(\mathbb{C}) \to \mathbb{C} \tag{2.18}$$

be the symmetric trilinear map satisfying $P(a,a,a) = \det(a)$ for $a \in M_{3\times3}(\mathbb{C})$. Note the identities $P({}^ta, {}^tb, {}^tc) = P(a,b,c)$ and

$$P(a,b,c) = \tfrac{1}{3}\,\mathrm{tr}\big(Q(a,b)c\big) = \tfrac{1}{3}\,\mathrm{tr}\big(Q(a,c)b\big), \tag{2.19}$$

which will be useful below. The map P has the equivariance

$$P(gah, gbh, gch) = \det(g)\det(h)\, P(a,b,c) \tag{2.20}$$

for all $g, h \in \mathrm{GL}(3,\mathbb{C})$. Thus, the $(3,0)$-form on U defined by

$$\psi = P\big(\,\overline{N(\alpha)}, \overline{N(\alpha)}, {}^tN(\alpha)\,\big)\, \alpha^1 \wedge \alpha^2 \wedge \alpha^3, \tag{2.21}$$

is independent of the choice of coframing α on U. Consequently, it is the restriction to U of a well-defined $(3,0)$-form on M. When it is necessary to make the dependence on J explicit, this $(3,0)$-form will be denoted by $\psi(J)$.

2.1.4. *Tautological forms.* One way to interpret the above constructions is that there exist, on $J^1\big(\mathsf{J}(M)\big)$, unique, M-semi-basic, smooth differential forms Φ (of degree 6), ψ (of degree 3), and ω (of degree 2) such that, for any almost complex structure $J : M \to \mathsf{J}(M)$, one has

$$j^1(J)^*(\Phi) = \Phi(J), \qquad j^1(J)^*(\psi) = \psi(J), \qquad j^1(J)^*(\omega) = \omega(J), \tag{2.22}$$

where $j^1(J) : M \to J^1\big(\mathsf{J}(M)\big)$ is the 1-jet lifting of $J : M \to \mathsf{J}(M)$.

2.1.5. *Bi-forms.* For later use, I will point out the existence of two further first-order invariant 'bi-form' tensors that will turn out to be important. These two forms $E_1(J)$ and $E_2(J)$ are sections of the bundle $A^{1,1}(M) \otimes A^{3,0}(M)$ and are given in terms of a local complex coframing α by the formulae

$$E_1(J) = \tfrac{\mathrm{i}}{2}\,{}^t\alpha \wedge Q\big({}^tN(\alpha), {}^tN(\alpha)\big) \wedge \overline{\alpha} \otimes P\big(\overline{N(\alpha)}, \overline{N(\alpha)}, \overline{N(\alpha)}\big)\, \alpha^1 \wedge \alpha^2 \wedge \alpha^3 \tag{2.23}$$

and

$$E_2(J) = \tfrac{\mathrm{i}}{2}\,{}^t\alpha \wedge Q\big(\overline{N(\alpha)}, \overline{N(\alpha)}\big) \wedge \overline{\alpha} \otimes P\big({}^tN(\alpha), {}^tN(\alpha), \overline{N(\alpha)}\big)\, \alpha^1 \wedge \alpha^2 \wedge \alpha^3. \tag{2.24}$$

The change-of-coframing formulae already discussed show that these bi-forms are globally defined on M. The reader will note the similarity of these definitions to

$$\begin{aligned} E_0(J) &= \omega(J) \otimes \psi(J), \\ &= \tfrac{\mathrm{i}}{2}\,{}^t\alpha \wedge Q\big({}^tN(\alpha), \overline{N(\alpha)}\big) \wedge \overline{\alpha} \otimes P\big(\overline{N(\alpha)}, \overline{N(\alpha)}, {}^tN(\alpha)\big)\, \alpha^1 \wedge \alpha^2 \wedge \alpha^3, \end{aligned} \tag{2.25}$$

which is also a section of $A^{1,1}(M) \otimes A^{3,0}(M)$.

2.2. **Relations and Inequalities.** There are some relations among these invariant forms:

Proposition 1 (Relations)**.** *The invariants ω, ψ, and Φ satisfy*

$$\tfrac{9}{8}\mathrm{i}\,\psi \wedge \overline{\psi} = \Phi + \tfrac{4}{3}\omega^3. \tag{2.26}$$

In particular,

$$-\tfrac{1}{8}\,\Phi \le \tfrac{1}{6}\,\omega^3. \tag{2.27}$$

Proof. The equation (2.26) is equivalent to

$$9\big|P(a, a, {}^t\bar{a})\big|^2 = \big|\det(a)\big|^2 + 8\,\det\big(Q(a, {}^t\bar{a})\big) \tag{2.28}$$

for $a \in M_{3\times3}(\mathbb{C})$ while the inequality (2.27) is equivalent to the homogeneous polynomial inequality

$$-\tfrac{1}{8}\,\big|\det(a)\big|^2 \le \det\big(Q(a, {}^t\bar{a})\big), \tag{2.29}$$

Obviously (2.28) implies (2.29), so it suffices to establish (2.28). This latter relation can be proved directly simply by expanding out both sides, but this is something of a mess and is somewhat unconvincing.

To establish (2.28), it suffices to prove it on an open subset of $M_{3\times3}(\mathbb{C}) \simeq \mathbb{C}^9$. Moreover, because of the equivariance of the mappings P and Q, if this relation holds for a given $a \in M_{3\times3}(\mathbb{C})$, then it holds on the entire ρ-orbit of a. Thus, I will describe a 'normal form' for the ρ-action on an open set in $M_{3\times3}(\mathbb{C})$. (This normal form will also come in handy for other reasons.)

First, consider the restriction of the ρ-action to $\mathrm{SL}(3,\mathbb{C}) \subset \mathrm{GL}(3,\mathbb{C})$. This restricted action is just $\rho(g)(a) = g a\, {}^t\bar{g}$ and hence $M_{3\times3}(\mathbb{C})$ is reducible as a real $\mathrm{SL}(3,\mathbb{C})$-representation. Namely, write $a = h + \mathrm{i}k$ where h and k are Hermitian symmetric, $h = {}^t\bar{h}$ and $k = {}^t\bar{k}$, and note that this splitting is preserved by the ρ-action of $\mathrm{SL}(3,\mathbb{C})$.

As is well-known in linear algebra, there is an open set in the space of pairs of 3-by-3 Hermitian symmetric matrices that can be simultaneously diagonalized by an element of $\mathrm{SL}(3,\mathbb{C})$.[5] In particular, a nonempty open set of elements of $M_{3\times3}(\mathbb{C})$ are ρ-equivalent to diagonal elements of the form

$$a = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \tag{2.30}$$

where the a_i are nonzero. Now, after ρ-acting by a diagonal element of $\mathrm{GL}(3,\mathbb{C})$, one can clearly reduce such an element to the form

$$a = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\lambda_1} & 0 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\lambda_2} & 0 \\ 0 & 0 & \mathrm{e}^{\mathrm{i}\lambda_3} \end{pmatrix} \tag{2.31}$$

where $\lambda_1 + \lambda_2 + \lambda_3 = 0$. Thus, the union of the ρ-orbits of such elements contains an open set in $M_{3\times3}(\mathbb{C})$, so it suffices to prove (2.28) for a of this form.

However, for a in the form (2.31), one has $\det(a) = 1$ and computation yields

$$P(a, a, {}^t\bar{a}) = \tfrac{1}{3}\big(\mathrm{e}^{-2\mathrm{i}\lambda_1} + \mathrm{e}^{-2\mathrm{i}\lambda_2} + \mathrm{e}^{-2\mathrm{i}\lambda_3}\big) \tag{2.32}$$

and

$$Q(a, {}^t\bar{a}) = \begin{pmatrix} \cos(\lambda_2{-}\lambda_3) & 0 & 0 \\ 0 & \cos(\lambda_3{-}\lambda_1) & 0 \\ 0 & 0 & \cos(\lambda_1{-}\lambda_2) \end{pmatrix}. \tag{2.33}$$

The relation (2.28) for a of the form (2.31) now follows trivially. □

Remark 6 (More identities). Although the structure equations will be explored more thoroughly in a later section, the reader may be interested to note the identity

$$\mathrm{d}^{-1,2}\omega = \frac{\mathrm{i}}{2}\,\mathrm{tr}\big({}^tN(\alpha)Q({}^tN(\alpha), \overline{N(\alpha)}\,)\big)\,\overline{\alpha^1\wedge\alpha^2\wedge\alpha^3}, \tag{2.34}$$

which follows immediately from (2.8) and (2.15).

Since the identity $\mathrm{tr}\big({}^ta\,Q({}^ta, \bar{a})\big) = 3P(a, a, {}^t\bar{a})$ holds for all $a \in M_{3\times3}(\mathbb{C})$, one has

$$\mathrm{d}^{-1,2}\omega = \tfrac{3}{2}\mathrm{i}\,\overline{\psi}, \tag{2.35}$$

so that one has the useful identity

$$\mathrm{d}\omega = 3\,\mathrm{Im}(\psi) + \partial\omega + \bar{\partial}\omega = 3\,\mathrm{Im}(\psi) + 2\,\mathrm{Re}(\bar{\partial}\omega). \tag{2.36}$$

Of course, $\bar{\partial}\omega = \mathrm{d}^{0,1}\omega = \overline{\partial\omega}$ is a second-order invariant of J.

Similarly, the $(2,2)$-form $\mathrm{d}^{-1,2}\psi$ is first-order in J while the expression $\bar{\partial}\psi$ is second-order. In fact, with respect to any complex coframing α, one has

$$\mathrm{d}^{-1,2}\psi = \big(\alpha^2{\wedge}\alpha^3 \quad \alpha^3{\wedge}\alpha^1 \quad \alpha^1{\wedge}\alpha^2\big)\,R\big(N(\alpha)\big)\begin{pmatrix} \overline{\alpha^2{\wedge}\alpha^3} \\ \overline{\alpha^3{\wedge}\alpha^1} \\ \overline{\alpha^1{\wedge}\alpha^2} \end{pmatrix} \tag{2.37}$$

[5]For example, any pair (h,k) for which the cubic polynomial $p(t) = \det(h + tk)$ has three real distinct roots can be simultaneously diagonalized by this action and the set of such pairs is open.

where $R : M_{3\times 3}(\mathbb{C}) \to M_{3\times 3}(\mathbb{C})$ is the function

$$R(a) = P(\bar{a}, \bar{a}, {}^t a)\, a. \tag{2.38}$$

Thus, the identity $\operatorname{tr}\big({}^t a\, Q({}^t a, \bar{a})\big) = 3P(a, a, {}^t \bar{a})$ implies

$$\omega \wedge \mathrm{d}^{-1,2}\psi = \tfrac{3}{2}\mathrm{i}\, \psi \wedge \overline{\psi}, \tag{2.39}$$

which also follows from (2.35) by differentiating the identity $\omega\wedge\psi = 0$.

It seems likely (though I have not written out a proof) that the ring of smooth first-order invariant forms definable for an almost complex structure J on a 6-manifold is generated by ω, ψ, $d^{-1,2}\psi$, Φ, and their conjugates.

Note, by the way, that (2.26) implies that all of these first-order forms can be generated from ω by repeatedly applying the operator $d^{-1,2}$, conjugation, and exterior multiplication. Thus, in some sense, ω is the fundamental object.[6]

Remark 7 (Nonvanishing invariants). If J is an integrable almost complex structure, then $d^{-1,2} = 0$, so, of course, all of the invariant forms constructed above from the Nijenhuis tensor vanish. In the other direction, one might be interested to know when it is possible to have one of these invariant forms be nowhere vanishing.

Now, $\Phi(J)$ is nowhere vanishing if and only if the bundle map

$$\mathrm{d}^{-1,2} : (T^{1,0}M)^* \to \Lambda^2\big(T^{0,1}M\big)^* \tag{2.40}$$

is an isomorphism. This would imply that the bundles $(T^{1,0}M)^*$ and $\Lambda^2\big(T^{0,1}M\big)^*$ have the same Chern classes, i.e., that $3c_1(J) = 0$ and $c_1(J)^2 = c_1(J)c_2(J) = 0$ (where, of course, the Chern classes are computed for the bundle TM with the complex structure J).

Thus, for example, any almost complex structure on $\mathbb{CP}^3$ that is homotopic to the standard complex structure cannot have a nonvanishing Φ, since the first Chern class of the standard almost complex structure is nontrivial.

If $\psi(J)$ were to be nonvanishing, then one clearly would have $c_1(J) = 0$, for, in this case, the 'canonical bundle' $K = \Lambda^3(T^{1,0}M)^*$ would be trivial (and not just 3-torsion) because $\psi(J)$ would be a nonvanishing section of it.

Finally, note that, if $\omega(J)$ were everywhere nondegenerate, then either $\omega(J)^3 > 0$, in which case, by (2.26) one sees that $\psi(J)$ is nowhere vanishing, or else $\omega(J)^3 < 0$, in which case, again by (2.26), one sees that $\Phi(J)$ is nowhere vanishing.

One has a stronger result in the case that $\omega(J)$ is a positive $(1,1)$-form: It is easy to prove that, if $Q({}^t a, \bar{a})$ is a positive definite Hermitian symmetric matrix, then a is ρ-equivalent to a matrix of the form (2.31) in which all of the λ_i satisfy $|\lambda_i| < \frac{1}{2}\pi$. Consequently, if $\omega(J)$ is positive, then both $\psi(J)$ and $\Phi(J)$ are nonvanishing and one has the inequality[7]

$$\tfrac{1}{6}\omega(J)^3 \le \Phi(J). \tag{2.41}$$

The case of equality in this last inequality will turn out to be important below, where it will be seen to be equivalent to *strict quasi-integrability* (when ω is positive).

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[6] Of course, in other dimensions as well, the $(1,1)$-form ω can be used to generate a collection of invariant first-order forms. See Remark 5.

[7] N.B.: The assumption that $\omega(J)$ be a positive $(1,1)$-form is essential for the inequality (2.41), since it does not hold in general.

2.3. **First-order functionals.** The above constructions yield the following result:

Theorem 1 (Invariant functionals)**.** *If M^6 is compact and oriented, then for any constants c_1 and c_2, the functional*

$$F_c(J) = \int_M c_1\,\Phi(J) + c_2\,\tfrac{\mathrm{i}}{8}\,\psi(J)\wedge\overline{\psi(J)} \tag{2.42}$$

is a diffeomorphism-invariant, first-order functional on the space $\mathcal{J}_+(M)$ of almost complex structures on M that induce the given orientation. The functional F_c belongs to the class L^6_1. □

Remark 8 (A classification). It is not difficult to show that any smooth function $L : M_{3\times 3}(\mathbb{C}) \to \mathbb{R}$ that satisfies

$$L\big(\rho(g)a\big) = |\det(g)|^{-2} L(a) \tag{2.43}$$

for all $g \in \mathrm{GL}(3,\mathbb{C})$ is of the form

$$L(a) = c_1\,\big|\det(a)\big|^2 + c_2\,\big|P(\bar a, \bar a, {}^t a)\big|^2 \tag{2.44}$$

for some real constants c_1 and c_2. From this and the discussion in Remark 4, it follows without difficulty that the functionals listed in Theorem 1 are the only smooth, diffeomorphism-invariant, first-order functionals on $\mathcal{J}_+(M)$.

Note that the map $(x,y) = \big(|\det(a)|^2, |P(\bar a, \bar a, {}^t a)|^2\big) : M_{3\times 3}(\mathbb{C}) \to \mathbb{R}^2$ has, as its image, the entire closed quadrant $x \geq 0$, $y \geq 0$, so that neither of the terms in L dominates the other in general.

The fact that these first-order functionals are in L^6_1 should be no surprise; as is usual with geometrically interesting functionals, they are on the boundary of Sobolev embedding.

2.4. **The Euler-Lagrange system.** It is natural to ask about the critical points of the functionals F_c introduced in Theorem 1. Because these functionals are first-order, their Euler-Lagrange equations can be computed using the method of Poincaré-Cartan forms [6].

To implement this method, I will first exhibit the 1-jet bundle $J^1\big(\mathsf{J}(M)\big) \to M$ as a quotient of a prolonged coframe bundle $\mathsf{F}^{(1)}(M) \to M$ and then use the canonical forms on $\mathsf{F}^{(1)}(M)$ to do the necessary computations.

2.4.1. $(0,2)$-*connections.* To motivate this, I will begin by reminding the reader about $(0,2)$-connections associated to almost complex structures. If J is an almost complex structure on M^6, then one has the $\mathrm{GL}(3,\mathbb{C})$-bundle $\mathsf{F}(J)$ of J-linear coframes $u : T_xM \to \mathbb{C}^3$ and the canonical (i.e., tautological) $\mathbb{C}^3$-valued 1-form η on $\mathsf{F}(J)$. A connection on $\mathsf{F}(J)$ is a $\mathrm{GL}(3,\mathbb{C})$-equivariant 1-form κ on $\mathsf{F}(J)$ with values in $\mathfrak{gl}(3,\mathbb{C})$ such that the 2-form $\mathrm{d}\eta + \kappa\wedge\eta$ is semi-basic for the projection $\mathsf{F}(J) \to M$. One says that such a connection is a $(0,2)$-*connection* if it satisfies the stronger condition that

$$\mathrm{d}\eta^i = -\kappa^i_j \wedge \eta^j + \tfrac{1}{2} C^i_{\bar j \bar k}\,\overline{\eta^j \wedge \eta^k}. \tag{2.45}$$

for some (necessarily unique) functions $C^i_{\bar j\bar k} = -C^i_{\bar k\bar j}$ on $\mathsf{F}(J)$. Such connections always exist and, in fact, the set of such connections is the space of sections of an affine bundle $\mathsf{F}^{(1)}(J) \to \mathsf{F}(J)$ of real rank 36.

2.4.2. *The prolonged coframe bundle.* For the next step, recall that $\mathrm{GL}(6,\mathbb{R})$ can be embedded into $\mathrm{GL}(6,\mathbb{C})$ as the subgroup of invertible matrices of the form

$$\begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} \tag{2.46}$$

where A and B lie in $M_{3\times 3}(\mathbb{C})$.

Now, let $\pi : \mathsf{F}(M) \to M$ be the bundle of $\mathbb{C}^3$-valued coframes $u : T_xM \to \mathbb{C}^3$. When there is no danger of confusion, I will simply write F for $\mathsf{F}(M)$. Let $\omega : T\mathsf{F} \to \mathbb{C}^3$ be the canonical $\mathbb{C}^3$-valued 1-form on F. Sometimes, $\omega = (\omega^i)$ is called the 'soldering form'.

Using the identification of $\mathfrak{gl}(6,\mathbb{R})$ with pairs of matrices in $\mathfrak{gl}(3,\mathbb{C})$ implicit in (2.46), one can see that specifying a pseudo-connection on M is equivalent to specifying a pair of 1-forms α and β with values in $M_{3\times 3}(\mathbb{C})$ such that the 2-form

$$\mathrm{d}\omega + \alpha\wedge\omega + \beta\wedge\bar{\omega} \tag{2.47}$$

is semi-basic. I will say that such a pseudo-connection is $\mathbb{C}$-*compatible* if it satisfies the stronger condition that

$$\mathrm{d}\omega^i + \alpha^i_j\wedge\omega^j + \beta^i_{\bar{\jmath}}\wedge\overline{\omega^j} = \tfrac{1}{2}T^i_{\bar{\jmath}\bar{k}}\,\overline{\omega^j\wedge\omega^k} \tag{2.48}$$

for some functions $T^i_{\bar{\jmath}\bar{k}} = -T^i_{\bar{k}\bar{\jmath}}$ on F. (Note that it is necessary to consider pseudo-connections here because the only actual connections satisfying such a condition are the ones that satisfy $T^i_{\bar{\jmath}\bar{k}} = 0$.)

As is easy to see, the pseudo-connections on F that are $\mathbb{C}$-compatible are the sections of an affine bundle $\mathsf{F}^{(1)} \to \mathsf{F}$ of real rank 144. On $\mathsf{F}^{(1)}$, there exist canonical complex-valued 1-forms ϕ^i_j and $\theta^i_{\bar{\jmath}}$ and complex-valued functions $N^i_{\bar{\jmath}\bar{k}} = -N^i_{\bar{k}\bar{\jmath}}$ such that the following structure equations hold

$$\mathrm{d}\omega^i = -\phi^i_j\wedge\omega^j - \theta^i_{\bar{\jmath}}\wedge\overline{\omega^j} + \tfrac{1}{2}N^i_{\bar{\jmath}\bar{k}}\,\overline{\omega^j\wedge\omega^k}. \tag{2.49}$$

These canonical forms and functions are defined by the property that, when a $\mathbb{C}$-compatible pseudo-connection defined by an α and β as in (2.48) is regarded as a section $\nabla : \mathsf{F} \to \mathsf{F}^{(1)}$, then ∇ pulls back ϕ to be α, θ to be β, and N to be T.

It is convenient to think of N as taking values in $M_{3\times 3}(\mathbb{C})$ as earlier, i.e., $N : \mathsf{F}^{(1)} \to M_{3\times 3}(\mathbb{C})$, since $\mathrm{GL}(3,\mathbb{C})$ clearly acts on $\mathsf{F}^{(1)}$ in such a manner that this mapping is ρ-equivariant.

Given a choice of $\mathbb{C}$-compatible pseudo-connection on F defined by 1-forms α and β as above (and, therefore, defining the function T on F), one can consider the plane field $\beta^\perp \subset T\mathsf{F}$ of corank 18 on F defined by the equations $\beta = (\beta^i_{\bar{\jmath}}) = 0$. At $u \in \mathsf{F}$, the plane $\beta^\perp_u \subset T_u\mathsf{F}$ is tangent to the $\mathrm{GL}(3,\mathbb{C})$-structure associated to an almost complex structure. In fact, $\beta^\perp_u$ determines a 1-jet of an almost complex structure at u. Thus, there is a natural submersion $\xi : \mathsf{F}^{(1)} \to J^1\bigl(\mathsf{J}(M)\bigr)$. The map ξ has been defined in such a way that it pulls back the canonical contact system on $J^1\bigl(\mathsf{J}(M)\bigr)$ to be the ideal $\mathcal{I}$ generated by the 1-forms $\theta^i_{\bar{\jmath}}$. In fact, the maximal integral manifolds of $\mathcal{I}$ on which

$$\Omega = \tfrac{\mathrm{i}}{8}\,\omega^1\wedge\omega^2\wedge\omega^3\wedge\overline{\omega^1\wedge\omega^2\wedge\omega^3} \neq 0, \tag{2.50}$$

are precisely the natural embeddings of the bundles $\mathsf{F}^{(1)}(J)$ into $\mathsf{F}^{(1)}$ for almost complex structures J.

Now computations that could be carried out abstractly on $J^1\bigl(\mathsf{J}(M)\bigr)$ can be carried out explicitly on $\mathsf{F}^{(1)}$ using the canonical forms and functions. It is this technique that will now be employed.

Recall that there are canonical forms Φ and ψ on $J^1\bigl(\mathsf{J}(M)\bigr)$ that satisfy (2.22). These forms pull back to $\mathsf{F}^{(1)}$ via ξ to satisfy

$$\xi^*\Phi = |\det(N)|^2\,\Omega \tag{2.51}$$

and

$$\xi^*\psi = P(\overline{N},\overline{N},{}^tN)\,\omega^1\wedge\omega^2\wedge\omega^3. \tag{2.52}$$

In particular, for any constants c_1 and c_2, one has

$$\xi^*(c_1\Phi + c_2\tfrac{\mathrm{i}}{8}\,\psi\wedge\overline{\psi}) = \bigl(c_1|\det(N)|^2 + c_2|P(\overline{N},\overline{N},{}^tN)|^2\bigr)\,\Omega. \tag{2.53}$$

Set $L = L(N) = c_1|\det(N)|^2 + c_2|P(\overline{N},\overline{N},{}^tN)|^2$ and note that L has the equivariance

$$L\bigl(\rho(g)a\bigr) = |\det(g)|^{-2}L(a) \tag{2.54}$$

for all $a\in M_{3\times3}(\mathbb{C})$. I am now going to compute the Euler-Lagrange system for the functional

$$\Lambda = L(N)\,\Omega. \tag{2.55}$$

To do this effectively, I will need the second structure equations on $\mathsf{F}^{(1)}$. To compute these, write

$$\mathrm{d}\omega^i = -\phi^i_j\wedge\omega^j - \bigl(\theta^i_{\bar{\jmath}} + \tfrac12 N^i_{\bar{\jmath}\bar{k}}\,\overline{\omega^k}\bigr)\wedge\overline{\omega^j}. \tag{2.56}$$

and, temporarily, set $\psi^i_{\bar{\jmath}} = \theta^i_{\bar{\jmath}} + \frac12 N^i_{\bar{\jmath}\bar{k}}\,\overline{\omega^k}$, so that the above equation takes the form

$$\mathrm{d}\omega^i = -\phi^i_j\wedge\omega^j - \psi^i_{\bar{\jmath}}\wedge\overline{\omega^j}. \tag{2.57}$$

Taking the exterior derivative of this expression gives

$$0 = -(\mathrm{d}\phi^i_j + \phi^i_k\wedge\phi^k_j + \psi^i_{\bar{k}}\wedge\overline{\psi^k_{\bar{\jmath}}})\wedge\omega^j - (\mathrm{d}\psi^i_{\bar{\jmath}} + \phi^i_k\wedge\psi^k_{\bar{\jmath}} + \psi^i_{\bar{k}}\wedge\overline{\phi^k_j})\wedge\overline{\omega^j}, \tag{2.58}$$

so that, by Cartan's Lemma, there must exist 1-forms $\pi^i_{jk} = \pi^i_{kj}$, $\pi^i_{j\bar{k}}$, and $\pi^i_{\bar{\jmath}\bar{k}} = \pi^i_{\bar{k}\bar{\jmath}}$ so that

$$\begin{aligned}\mathrm{d}\phi^i_j + \phi^i_k\wedge\phi^k_j + \psi^i_{\bar{k}}\wedge\overline{\psi^k_{\bar{\jmath}}} &= -\pi^i_{jk}\wedge\omega^k - \pi^i_{j\bar{k}}\wedge\overline{\omega^k},\\ \mathrm{d}\psi^i_{\bar{\jmath}} + \phi^i_k\wedge\psi^k_{\bar{\jmath}} + \psi^i_{\bar{k}}\wedge\overline{\phi^k_j} &= -\pi^i_{k\bar{\jmath}}\wedge\omega^k - \pi^i_{\bar{\jmath}\bar{k}}\wedge\overline{\omega^k}.\end{aligned} \tag{2.59}$$

(The forms π are *not* canonically defined on $\mathsf{F}^{(1)}$, but this ambiguity will not cause a problem.) Setting

$$\nu^i_{\bar{\jmath}\bar{k}} = \mathrm{d}N^i_{\bar{\jmath}\bar{k}} + N^\ell_{\bar{\jmath}\bar{k}}\,\phi^i_\ell - N^i_{\bar{\ell}\bar{k}}\,\overline{\phi^\ell_j} - N^i_{\bar{\jmath}\bar{\ell}}\,\overline{\phi^\ell_k} = -\nu^i_{\bar{k}\bar{\jmath}}\,, \tag{2.60}$$

one finds that the above equations imply

$$\mathrm{d}\theta^i_{\bar{\jmath}} \equiv -\phi^i_k\wedge\theta^k_{\bar{\jmath}} - \theta^i_{\bar{k}}\wedge\overline{\phi^k_j} - \bigl(\pi^i_{\bar{\jmath}\bar{k}} + \tfrac12\nu^i_{\bar{\jmath}\bar{k}}\bigr)\wedge\overline{\omega^k} \quad \bmod \omega^1,\omega^2,\omega^3. \tag{2.61}$$

2.4.3. *The Euler-Lagrange ideal.* Now, given $L = L(N) = \bar{L}$ as above, let $L_i^{\bar{\jmath}\bar{k}} = -L_i^{\bar{k}\bar{\jmath}}$ be the functions such that

$$\mathrm{d}L = \mathrm{Re}\big(L_i^{\bar{\jmath}\bar{k}}\,\mathrm{d}N^i_{\bar{\jmath}\bar{k}}\big) \tag{2.62}$$

(Since L is a polynomial of degree 6 on $M_{3\times3}(\mathbb{C})$, the functions $L_i^{\bar{\jmath}\bar{k}}$ are polynomials of degree 5.)

The equivariance of L and Ω and the definitions made so far imply that

$$\mathrm{d}\big(L\,\Omega\big) = \mathrm{Re}\big(L_i^{\bar{\jmath}\bar{k}}\,\nu^i_{\bar{\jmath}\bar{k}}\big)\wedge\Omega. \tag{2.63}$$

Let $\omega_{[\bar{\imath}]}$ be the 5-form that satisfies

$$\overline{\omega^j}\wedge\omega_{[\bar{\imath}]} = \delta^j_i\,\Omega \tag{2.64}$$

and $\omega^j\wedge\omega_{[\bar{\imath}]} = 0$ for all i and j. Consider the 6-form

$$\Lambda' = L\,\Omega + 2\,\mathrm{Re}\left(L_i^{\bar{\jmath}\bar{k}}\,\theta^i_{\bar{\jmath}}\wedge\omega_{[\bar{k}]}\right). \tag{2.65}$$

The structure equations derived so far and the definitions made imply that[8]

$$\mathrm{d}\Lambda' = 2\,\mathrm{Re}\left(\theta^i_{\bar{\jmath}}\wedge\Upsilon^{\bar{\jmath}}_i\right) \tag{2.66}$$

where

$$\Upsilon^{\bar{\jmath}}_i = \mathrm{d}\big(L_i^{\bar{\jmath}\bar{k}}\,\omega_{[\bar{k}]}\big) + L_\ell^{\bar{\jmath}\bar{k}}\,\phi^\ell_i\wedge\omega_{[\bar{k}]} - L_i^{\bar{\ell}\bar{k}}\,\overline{\phi^j_\ell}\wedge\omega_{[\bar{k}]}\,. \tag{2.67}$$

Consequently, the Euler-Lagrange ideal $\mathcal{E}_\Lambda$ on $J^1(\mathsf{J}(M))$ for the first-order functional defined by $\Lambda = L\,\Omega$ pulls back to $\mathsf{F}^{(1)}$ to be generated by the 1-forms $\theta^i_{\bar{\jmath}}$, their exterior derivatives, and the 6-forms $\Upsilon^{\bar{\jmath}}_i$ (whose real and imaginary parts constitute 18 independent forms).

2.4.4. *The Euler-Lagrange equations.* The upshot of these calculations is the following result, whose proof has just been given in the course of the above discussion.

Theorem 2 (Euler-Lagrange Equations). *Let $L(a) = c_1|\det(a)|^2 + c_2|P(\bar{a},\bar{a},{}^t a)|^2$ for $a \in M_{3\times3}(\mathbb{C})$ and real constants c_1 and c_2. Define the polynomial functions $L_i^{\bar{\jmath}\bar{k}}(a) = -L_i^{\bar{k}\bar{\jmath}}(a)$ as in* (2.62). *Then an almost complex structure J on M^6 is a critical point of the functional F_c as defined in* (2.42) *if an only if, for any local J-linear coframing $\alpha : TU \to \mathbb{C}^3$ and $(0,2)$-connection κ satisfying*

$$\mathrm{d}\alpha = -\kappa\wedge\alpha + N(\alpha)\begin{pmatrix}\overline{\alpha^2\wedge\alpha^3}\\ \overline{\alpha^3\wedge\alpha^1}\\ \overline{\alpha^1\wedge\alpha^2}\end{pmatrix}, \tag{2.68}$$

the following identities hold:

$$\mathrm{d}\big(L_i^{\bar{\jmath}\bar{k}}(N(\alpha))\,\alpha_{[\bar{k}]}\big) + L_\ell^{\bar{\jmath}\bar{k}}(N(\alpha))\,\kappa^\ell_i\wedge\alpha_{[\bar{k}]} - L_i^{\bar{\ell}\bar{k}}(N(\alpha))\,\overline{\kappa^j_\ell}\wedge\alpha_{[\bar{k}]} = 0. \tag{2.69}$$

---

[8]The reader familiar with the theory of Poincaré-Cartan forms for first-order functionals will recognize this formula as the first step in computing the so-called *Betounes form* of the Lagrangian Λ. While I could, at this point, compute the entire Betounes form, I will not need it in this article, so I leave this aside. For more information about the Betounes form, see [6].

Remark 9 (The nature of the equations). As expected, Theorem 2 shows that the Euler-Lagrange equations for F_c are second-order and quasi-linear (i.e., linear in the second derivatives of J). The reader will also note that they constitute a system of 18 equations. Of course, these equations can be expressed as the vanishing of a globally defined tensor on M and that will be done below in Corollary 1, but this is not particularly illuminating. The above formulation is good for practical calculations.

These equations cannot be elliptic since they are invariant under the diffeomorphism group. It seems reasonable to expect the functional F_c to have 'better' convexity properties when both c_1 and c_2 are positive, but I do not know whether this is the case. However, an analysis that is somewhat long does show that, when c_1 and c_2 are positive, F_c is elliptic for variations transverse to the action of the diffeomorphism group when linearized at any J for which $\omega(J)$ is positive definite. (Note that, when c_1 and c_2 are nonnegative, the functional F_c is bounded below by 0 and hence all integrable almost complex structures are absolute minima for F_c. Of course, the integrable almost complex structures are critical points of the functional F_c for any value of c.)

Finally, note that, even though κ is not uniquely determined by (2.68), this ambiguity in κ does not show up in the Euler-Lagrange equations (2.69). The reason is that (2.68) determines κ up to replacement by ${}^*\kappa$ where

$$ {}^*\kappa^i_j = \kappa^i_j + s^i_{jk}\,\alpha^k \tag{2.70} $$

for some functions $s^i_{jk} = s^i_{kj}$. Since $\alpha_{[\bar k]}$ contains $\alpha^1{\wedge}\alpha^2{\wedge}\alpha^3$ as a factor, ${}^*\kappa^\ell_i{\wedge}\alpha_{[\bar k]} = \kappa^\ell_i{\wedge}\alpha_{[\bar k]}$, so that the second term in the left hand side of (2.69) is unaffected by the ambiguity in the choice of κ. It also follows that

$$ \overline{{}^*\kappa^i_\ell} \wedge \alpha_{[\bar k]} = \overline{\kappa^i_\ell} \wedge \alpha_{[\bar k]} + \overline{s^i_{\ell k}}\,\Omega \tag{2.71} $$

and this, coupled with the skewsymmetry $L_i^{\bar j \bar k} = -L_i^{\bar k \bar j}$, shows that the ambiguity in the choice of κ does not affect the final term of the left hand side of (2.69).

Remark 10 (Degenerate critical points). Of course, any J that satisfies $\Phi(J) = 0$ is a critical point of the functional F_c when $c_2 = 0$, just as any J that satisfies $\psi(J) = 0$ is a critical point of F_c when $c_1 = 0$.

In particular, if $d^{-1,2} : \mathcal{A}^{1,0} \to \mathcal{A}^{0,2}$ has rank at most 1 at each point, then one has both $\Phi(J) = 0$ and $\psi(J) = 0$, implying that all such almost complex structures are critical for all of the F_c.

On the other hand, as will be seen below in Proposition 8, the almost complex structure underlying a nearly Kähler structure will have $\omega(J) > 0$ and yet will be critical for all of the functionals F_c. Thus, nondegenerate 'supercritical' (i.e., critical for all F_c) almost complex structures do occur.

2.4.5. *The tensorial version.* Finally, I give the 'tensorial' version of the Euler-Lagrange equations. First, though, I note that because of the canonical isomorphism

$$ A^{p,q}(M) = A^{p,0}(M) \otimes A^{0,q}(M), \tag{2.72} $$

the $\bar\partial$-operators $\bar\partial : A^{1,1}(M) \to A^{1,2}(M)$ and $\bar\partial : A^{3,0}(M) \to A^{3,1}(M)$ can be combined by the Leibnitz rule and exterior product to yield a well-defined, first-order, linear operator

$$ \bar\partial : A^{1,1}(M) \otimes A^{3,0}(M) \to A^{1,0}(M) \otimes A^{3,2}(M). \tag{2.73} $$

Then Theorem 2 can be expressed in the following terms.

Corollary 1 (The Euler-Lagrange equations)**.** *An almost complex structure J is critical for the functional F_c if and only if it satisfies*

$$\bar\partial\bigl(c_1\,E_1(J)+c_2\bigl(\tfrac{1}{3}E_2(J)+\tfrac{2}{3}E_0(J)\bigr)\bigr)=0, \tag{2.74}$$

where the $E_i(J)$ are defined as in (2.23), (2.24), *and* (2.25).

Proof. This is an exercise in unraveling the definitions and applying Theorem 2. The essential part of the calculation is noting that the elementary identity

$$\mathrm{d}P(a,b,c)=\tfrac{1}{3}\,\mathrm{tr}\bigl(Q(b,c)\,\mathrm{d}a+Q(c,a)\,\mathrm{d}b+Q(a,b)\,\mathrm{d}c\bigr) \tag{2.75}$$

can be expanded to yield a formula for $\mathrm{d}L$ when

$$L(a)=c_1\,\bigl|\det(a)\bigr|^2+c_2\,\bigl|P(\bar a,\bar a,{}^t a)\bigr|^2. \tag{2.76}$$

Substituting this formula for $\mathrm{d}L$ into the explicit equations of Theorem 2 and writing out the definition of $\bar\partial$ in a coframing then yields the desired result. □

Remark 11 (Reality and the tensors $E_i(J)$). By properties of the representation ρ, it is an invariant condition on J that, at each point, the Nijenhuis matrix $N(\alpha)$ be a complex multiple of a Hermitian symmetric matrix.

In this case, which will be called the case of a *Nijenhuis tensor of real type*, one clearly has

$$\Phi(J)=\tfrac{\mathrm{i}}{8}\,\psi(J)\wedge\overline{\psi(J)}. \tag{2.77}$$

Moreover, by the formulae (2.23), (2.24), and (2.25), one also has

$$E_1(J)=E_2(J)=E_0(J)=\omega(J)\otimes\psi(J). \tag{2.78}$$

In particular, by Corollary 1, an almost complex structure J with Nihenhuis tensor of real type is a critical point of all of the functionals F_c if and only if it is critical for some functional F_c where $c=(c_1,c_2)$ satisfies $c_1+c_2\neq 0$. In turn, this happens if and only if J satisfies the second-order condition

$$\bar\partial\bigl(\omega(J)\,\otimes\,\psi(J)\bigr)=0, \tag{2.79}$$

i.e.,

$$\bar\partial\bigl(\omega(J)\bigr)\otimes\psi(J)+\omega(J)\,\widehat{\otimes}\,\bar\partial\bigl(\psi(J)\bigr)=0, \tag{2.80}$$

where the terms on the left hand side are regarded as sections of $A^{1,0}(M)\otimes A^{3,2}(M)$ via the canonical isomorphism

$$A^{1,2}(M)\otimes A^{3,0}(M)=A^{1,0}(M)\otimes A^{3,2}(M) \tag{2.81}$$

and the canonical homomorphism induced by wedge product

$$A^{1,1}(M)\otimes A^{3,1}(M)\longrightarrow A^{1,0}(M)\otimes A^{3,2}(M). \tag{2.82}$$

2.4.6. *The nearly pseudo-Kähler case.* An interesting special case is the following one, in which a second-order (determined) system of PDE on J coupled with a first-order open condition on J implies the Euler-Lagrange equations for all of the functionals F_c and, moreover, a first-order closed condition on J.

Proposition 2 (Nearly pseudo-Kähler structures)**.** *Suppose that the almost complex structure J on M^6 satisfies $\bar\partial\omega = 0$ while ψ is nonvanishing. Then J is critical for all of the functionals F_c and its Nijenhuis tensor is of real type.*

Proof. Since $\bar\partial\omega = 0 = \overline{\partial\omega}$, the relation (2.36) simplifies to

$$\mathrm{d}\omega = 3\,\mathrm{Im}(\psi) = 3\tfrac{\mathrm{i}}{2}(\overline{\psi} - \psi). \tag{2.83}$$

This implies $\mathrm{d}\overline{\psi} = \mathrm{d}\psi$, which, by type considerations, implies both that

$$\bar\partial\psi = 0 \tag{2.84}$$

and that $d^{-1,2}\psi$ is a real-valued $(2,2)$-form. Of course, this already implies that

$$\bar\partial\big(\omega\otimes\psi\big) = 0. \tag{2.85}$$

Now, by (2.37) and the formula for $R(a)$, it follows from the reality of $d^{-1,2}\psi$ that, for any local complex coframing $\alpha : TU \to \mathbb{C}^3$, the matrix

$$R\big(N(\alpha)\big) = P\big(\overline{N(\alpha)}, \overline{N(\alpha)}, {}^tN(\alpha)\big)\,N(\alpha), \tag{2.86}$$

is Hermitian symmetric. Since

$$\psi = P\big(\overline{N(\alpha)}, \overline{N(\alpha)}, {}^tN(\alpha)\big)\,\alpha^1\wedge\alpha^2\wedge\alpha^3 \tag{2.87}$$

is assumed to be nonvanishing, it follows that $N(\alpha)$ is a (nonzero) complex multiple of a Hermitian symmetric matrix, i.e., that J has a Nijenhuis tensor of real type, as claimed.

In particular, by Corollary 1, Remark 11, and the formula $\bar\partial\big(\omega\otimes\psi\big) = 0$, it now follows that J is critical for all F_c, as claimed. □

Remark 12 (Nomenclature). In the case that $\bar\partial\omega = 0$ while ψ is nowhere vanishing, one now sees by the formula for $R(a)$ and the reality of the Nijenhuis tensor that

$$\mathrm{d}\psi = \mathrm{d}^{-1,2}\psi = 2\,\omega^2. \tag{2.88}$$

As will be seen in §4, the equations $\mathrm{d}\omega = 3\,\mathrm{Im}(\psi)$ and $\mathrm{d}\psi = 2\,\omega^2$ characterize the so-called 'strictly nearly Kähler' case when ω is a positive $(1,1)$-form and ψ is nonvanishing. However, as will be seen, there are examples of solutions of the above equations for which ω is not a positive $(1,1)$-form, and this can be thought of as the 'strictly nearly pseudo-Kähler' case.

3. Pseudo-holomorphic bundles

When (M,J) is a complex n-manifold, there is a close relationship between holomorphic structures on complex bundles over M and connections on those bundles whose curvature satisfies certain restrictions. This relationship is made somewhat clearer in the context of Hermitian bundles over M, so that is how the discussion will begin.

Suppose, first, that $E \to M$ is a holomorphic bundle over M. Let $\mathcal{H}(E)$ denote the space of Hermitian metrics on E. As is well-known [10, p. 73], for any $h \in \mathcal{H}(E)$, there exists a unique connection ∇^h on E that is compatible with h and satisfies $(\nabla^h)^{0,1} = \bar\partial_E$. The curvature $(\nabla^h)^2$ of ∇^h is an $\mathrm{End}(E)$-valued $(1,1)$-form on M.

Conversely, suppose that E is a complex vector bundle over M, that $h \in \mathcal{H}(E)$ is an Hermitian metric on E, and that ∇ is an h-compatible connection on E whose curvature ∇^2 is of type $(1,1)$. Then there is a unique holomorphic structure on E such that $\nabla^{0,1} = \bar\partial_E$.

Of course, this correspondence between holomorphic structures and connections with curvature of type $(1,1)$ is not bijective, precisely because there are many Hermitian structures on E. One way of tightening this correspondence is to impose some extra conditions on the curvature form.

For example, suppose that M carries a Kähler structure, represented by a Kähler form η, say.[9] Then, for a given holomorphic bundle E, an Hermitian metric $h \in \mathcal{H}(E)$ is said to be *Hermitian-Yang-Mills* if it satisfies

$$\mathrm{tr}_\eta\big((\nabla^h)^2\big) = \mathrm{i}\lambda\ \mathrm{id}_E \tag{3.1}$$

for some constant $\lambda \in \mathbb{R}$. This is a determined equation for h that is elliptic if η is definite and hyperbolic if η is indefinite.

Alternatively, one can consider a complex bundle $E \to M$ endowed with a fixed Hermitian metric $h \in \mathcal{H}(E)$ and look for connections ∇ that are h-compatible and satisfy

$$\big((\nabla)^2\big)^{0,2} = 0, \qquad \mathrm{tr}_\eta\big((\nabla)^2\big)^{1,1} = \mathrm{i}\lambda\ \mathrm{id}_E \tag{3.2}$$

for some constant $\lambda \in \mathbb{R}$.

For the importance of these concepts in algebraic and complex differential geometry, see [7] and [20]. This introductory discussion is meant to motivate the following definitions.

Definition 1 (Pseudo-holomorphic bundles). When (M, J) is an almost complex manifold and $E \to M$ is a complex vector bundle over M endowed with a Hermitian metric h, an h-compatible connection ∇ on E is said to define a *pseudo-holomorphic* structure on E if ∇^2 has curvature of type $(1,1)$.

If, in addition, η is a (possibly indefinite) almost Hermitian metric on (M, J), and $\mathrm{tr}_\eta(\nabla^2) = \mathrm{i}\lambda\ \mathrm{id}_E$, the connection ∇ is said to be *pseudo-Hermitian-Yang-Mills.*

Remark 13 (Philosophy). The main reason that this generalization of holomorphic bundle and Hermitian-Yang-Mills connections might be of interest is that these can exist and be nontrivial even when (M, J) is not an integrable complex manifold. In fact, the main point of this section is to point out that there is a rather large class of almost complex structures in (real) dimension 6 that, at least locally, support 'as many' pseudo-Hermitian-Yang-Mills connections as the integrable complex structures do.

These pseudo-Hermitian-Yang-Mills connections on almost complex manifolds, particularly the G_2-invariant 6-sphere, also show up in the study of singular behaviour of Yang-Mills connections. See §5.3 of [19] for a further discussion of this.

Also, while it is not, strictly speaking, necessary to restrict attention to connections ∇ compatible with a Hermitian structure on E, it is convenient and it simplifies the theory in some respects. I will leave it to the reader to determine

[9] In what follows, one can allow the metric η to be indefinite (but always nondegenerate). Thus, the $(1,1)$-form η need not be positive. However, for the sake of brevity, I will usually only treat the definite case explicitly, only pointing out the places where the sign makes a significant difference.

when this restriction can be dropped. In any case, since E is only being regarded as a complex bundle (rather than a holomorphic one), any two Hermitian structures on E are equivalent under the action of the smooth gauge group $\mathrm{Aut}(E)$, so the specification of a Hermitian norm on E is not a delicate issue.

Proposition 3 (Curvature restrictions)**.** *If ∇ is a pseudo-holomorphic structure on $E \to M$, then $\nabla^2 = (\nabla^2)^{1,1}$ takes values in $\mathrm{Hom}(E) \otimes \mathcal{K}$ where*

$$\mathcal{K} = \ker\left(\mathrm{d}^{-1,2} + \mathrm{d}^{2,-1} : \mathcal{A}^{1,1} \to \mathcal{A}^{0,3} \oplus \mathcal{A}^{3,0}\right). \tag{3.3}$$

Proof. Let r be the rank of E and let $\mathsf{u} = (\mathsf{u}_1, \ldots, \mathsf{u}_r)$ be a unitary basis of $\Gamma(U, E)$ for some open set $U \subset M$. Then $\nabla\mathsf{u} = \mathsf{u}A$, where A is a 1-form on U with values in $\mathfrak{u}(r)$. By hypothesis, the curvature form $F = \mathrm{d}A + A{\wedge}A$ is a $(1,1)$-form with values in $\mathfrak{u}(r)$. By the Bianchi identity

$$\mathrm{d}F = F \wedge A - A \wedge F. \tag{3.4}$$

Since F is a $(1,1)$-form, the $\mathcal{A}^{0,3} \oplus \mathcal{A}^{3,0}$-component of the right hand side of this equation vanishes. Thus,

$$\left(\mathrm{d}^{-1,2} + \mathrm{d}^{2,-1}\right)(F) = 0, \tag{3.5}$$

as desired. □

Remark 14 (Generic triviality). When n is sufficiently large and J (or more, specifically, the Nijenhuis tensor of J) is sufficiently generic, the map $d^{-1,2} + d^{2,-1}$ is injective on $\mathcal{A}^{1,1}$. In such a case, any pseudo-holomorphic structure ∇ on E is necessarily a flat connection, so that E is a flat bundle. This is analogous to the fact that the sheaf of holomorphic functions on a general nonintegrable almost complex structure is just the constant sheaf.

The interesting case is going to be when there are plenty of non-flat pseudo-holomorphic structures. Note that the set of complex bundles over M that admit pseudo-holomorphic structures is closed under sums, products, and taking duals. It would appear that a good way to find almost complex structures that support many pseudo-holomorphic bundles would be to look for ones that support many pseudo-holomorphic line bundles. It is to this case that I will now turn.

3.1. **The line bundle case.**

From now on, I will restrict attention to the 6-dimensional case.

Consider the case of a line bundle $E \to M$. Then any pseudo-holomorphic structure ∇ on E has curvature of the form $\nabla^2 = \mathrm{i}\phi \otimes \mathrm{id}_E$ where ϕ is a closed, real-valued $(1,1)$-form.

Using a local complex coframing $\alpha = (\alpha^i)$ on the open set $U \subset M$, write

$$\phi = \tfrac{\mathrm{i}}{2} F_{i\bar{\jmath}}\, \alpha^i \wedge \overline{\alpha^j} \tag{3.6}$$

where $F_{i\bar{\jmath}} = \overline{F_{j\bar{\imath}}}$ can be regarded as a Hermitian symmetric 3-by-3 matrix F defined on U. One computes

$$\mathrm{d}^{-1,2}\phi = \tfrac{\mathrm{i}}{2}\,\mathrm{tr}\big({}^tN(\alpha)F\big)\,\overline{\alpha^1 \wedge \alpha^2 \wedge \alpha^3}. \tag{3.7}$$

Thus, closure implies two linear zeroth-order equations on ϕ, namely, the real and imaginary parts of the equation $\mathrm{tr}\big({}^tN(\alpha)F\big) = 0$.

This algebraic system combined with the already overdetermined system $\mathrm{d}\phi = 0$ for $\phi \in \mathcal{A}^{1,1}$ is generally not involutive.[10] However, there is a class of almost complex structures characterized by a first-order condition that generalizes the integrable case and for which the system $\mathrm{d}\phi = 0$ for $\phi \in \mathcal{A}^{1,1}$ is involutive. It is to this condition that I now turn.

Note that, because F is Hermitian-symmetric, the equation $\mathrm{tr}\big({}^tN(\alpha)F\big) = 0$ will reduce to a single linear equation on F if and only if $N(\alpha)$ is a complex multiple of a Hermitian-symmetric matrix, i.e., if and only if J has a Nijenhuis tensor of real type (see Remark 11).

It turns out that simply having its Nijenhuis tensor be of real type is a rather awkward condition on J because the union of the ρ-orbits of the Hermitian symmetric matrices in $M_{3\times3}(\mathbb{C})$ has complicated singularities. It is better to work with a somewhat stronger condition that 'tames' this singularity, which motivates the following definition:

Definition 2 (Quasi-integrability). An almost complex structure J on M^6 is said to be *elliptically quasi-integrable* if, for any local J-linear coframing α, the matrix $N(\alpha)$ takes values in the ρ-orbit of $I_3 \in M_{3\times3}(\mathbb{C})$ union with the zero element. It is said to be *strictly elliptically quasi-integrable* if, for any local J-linear coframing α, the matrix $N(\alpha)$ takes values in the ρ-orbit of $I_3 \in M_{3\times3}(\mathbb{C})$.

Similarly, an almost complex structure J on M^6 is said to be *hyperbolically quasi-integrable* if, for any local J-linear coframing α, the matrix $N(\alpha)$ takes values in the ρ-orbit of $\mathrm{diag}(1,-1,-1) \in M_{3\times3}(\mathbb{C})$ union with the zero element. It is said to be *strictly hyperbolically quasi-integrable* if, for any local J-linear coframing α, the matrix $N(\alpha)$ takes values in the ρ-orbit of $\mathrm{diag}(1,-1,-1) \in M_{3\times3}(\mathbb{C})$.

Remark 15 (Nonstrictness). The main reason for including the possibility of the Nijenhuis tensor vanishing in the definition of quasi-integrability is so that integrable almost complex structures will be quasi-integrable. The main focus of the rest of this section will be the strictly quasi-integrable case.

The justification for the term 'quasi-integrable' will appear below. Basically, the reason is that the overdetermined system describing pseudo-Hermitian-Yang-Mills connections, which would not be involutive for the general almost complex structure in dimension 6, turns out to be very well-behaved in the quasi-integrable case.

Proposition 4 (Structure reduction). *If J is an elliptic strictly quasi-integrable almost complex structure on the 6-manifold M, then there is a canonical* $\mathrm{SU}(3)$*-structure on M whose local sections consist of the local J-linear coframings $\alpha : TU \to \mathbb{C}^3$ for which $N(\alpha) = I_3$. The invariant $(1,1)$-form $\omega = \omega(J)$ is a positive $(1,1)$-form on M while ψ is a nonvanishing $(3,0)$-form on M.*

Similarly, if J is a hyperbolic strictly quasi-integrable almost complex structure on the 6-manifold M, then there is a canonical $\mathrm{SU}(1,2)$*-structure on M whose local sections consist of the local J-linear coframings $\alpha : TU \to \mathbb{C}^3$ for which $N(\alpha) = \mathrm{diag}(1,-1,-1)$. The invariant $(1,1)$-form $\omega = \omega(J)$ is a $(1,1)$-form of signature $(1,2)$ on M while ψ is a nonvanishing $(3,0)$-form on M.*

---

[10]In fact, by a somewhat long argument that involves higher order invariants of the almost complex structure, it can be shown that, for a sufficiently generic almost complex structure J on a 6-manifold M, the sheaf of closed $(1,1)$-forms on M is trivial. Since the methods of this argument will not be used further in this note, I will not give details here.

In either case (hyperbolic or elliptic), these forms satisfy

$$\tfrac{1}{6}\,\omega^3 = \tfrac{\mathrm{i}}{8}\,\psi\wedge\overline{\psi} = \Phi > 0. \tag{3.8}$$

Proof. The elliptic case follows from the evident fact that the ρ-stabilizer of $I_3 \in M_{3\times3}(\mathbb{C})$ is $\mathrm{SU}(3)$ while $P(I_3, I_3, I_3) = 1$ and $Q(I_3, I_3) = I_3$. The hyperbolic case is similar since, if $D = \mathrm{diag}(1, -1, -1)$, then $P(D, D, D) = 1$ and $Q(D, D) = D$. □

Recall that a $\mathrm{U}(3)$-structure (also called an *almost Hermitian structure*) on M^6 is a pair (J, η) consisting of an almost complex structure J on M and a 2-form η that is a positive $(1,1)$-form with respect to J. A J-linear coframing $\alpha : TU \to \mathbb{C}^3$ defined on $U \subset M$ is said to be (J, η)-*unitary* if α defines a unitary isomorphism between T_xU and $\mathbb{C}^3$ (with its standard Hermitian structure) for all $x \in U$. One has a similar description of $\mathrm{U}(1,2)$-structures on M. The transition rule (2.9) then implies that the following definitions are meaningful.

Definition 3 (Quasi-integrable unitary structures). A $\mathrm{U}(3)$-structure (J, η) on M^6 is said to be *quasi-integrable* if any local (J, η)-unitary coframing $\alpha : TU \to \mathbb{C}^3$ satisfies $N(\alpha) = \lambda I_3$ for some function $\lambda : U \to \mathbb{C}$.

Similarly, a $\mathrm{U}(1,2)$-structure (J, η) on M^6 is said to be *quasi-integrable* if any local (J, η)-unitary coframing $\alpha : TU \to \mathbb{C}^3$ satisfies $N(\alpha) = \mathrm{diag}(\lambda, -\lambda, -\lambda)$ for some function $\lambda : U \to \mathbb{C}$.

Remark 16 (Relations). If (J, η) is quasi-integrable on M, then J is quasi-integrable on M. However, there do exist quasi-integrable J for which there does not exist a corresponding η such that (J, η) is quasi-integrable.

The difficulty is caused by the places where the Nijenhuis tensor of J vanishes. If J is quasi-integrable, then on the open set $M^* \subseteq M$ where its Nijenhuis tensor is nonzero (i.e., where J is strictly quasi-integrable), the pair $\bigl(J, \omega(J)\bigr)$ is quasi-integrable. In fact, any η on M^* such that (J, η) is quasi-integrable is necessarily a multiple of $\omega(J)$. However, it can happen that one cannot smoothly extend any such positive η as a positive $(1,1)$-form on all of M. See Remark 26.

Remark 17 (The quasi-integrable condition). Because the orbit $\rho\bigl(\mathrm{GL}(3,\mathbb{C})\bigr)\cdot I_3 \subset M_{3\times3}(\mathbb{C})$ is diffeomorphic to $\mathrm{GL}(3,\mathbb{C})/\,\mathrm{SU}(3)$, this orbit has dimension 10 and codimension 8 in $M_{3\times3}(\mathbb{C})$. Thus, the condition of strict elliptic quasi-integrability can be thought of as a system of 8 first-order equations (plus some inequalities due to the fact that the orbit is not closed) for the almost complex structure J. The case of strict hyperbolic quasi-integrability is similar. The nature of these equations will be discussed below.

Somewhat better behaved is the quasi-integrability condition for a pair (J, η). In this case, quasi-integrability is a closed condition and is easily seen to be a system of 16 first-order equations for the $\mathrm{U}(3)$- or $\mathrm{U}(1,2)$-structure (J, η). Since quasi-integrability is unaffected by replacing (J, η) by $(J, \mu\,\eta)$ for any smooth positive μ, it would be more natural to formulate this concept for $\mathbb{R}^+\cdot\mathrm{U}(3{-}q, q)$-structures, rather than $\mathrm{U}(3{-}q, q)$-structures, but I will wait until the discussion in §4.5 to do this.

Remark 18 (Other nomenclatures). It should also be noted that quasi-integrability for a $\mathrm{U}(3)$-structure is what Gray and Hervella [9] refer to as type $\mathcal{W}_1\oplus\mathcal{W}_3\oplus\mathcal{W}_4$ (or, alternatively $\mathcal{G}_1$, see [12, 13]), since, quasi-integrability for a $\mathrm{U}(3)$-structure is the vanishing of the $\mathrm{U}(3)$-torsion component that Gray and Hervella denote by W_2.

Proposition 5. *Let (J,η) be a strictly quasi-integrable $\mathrm{U}(3{-}q,q)$-structure on M^6. Then any pseudo-holomorphic stucture ∇ on an Hermitian bundle $E \to M$ is pseudo-Hermitian-Yang-Mills. In fact,*

$$\mathrm{tr}_\eta\big(\nabla^2\big) = 0. \tag{3.9}$$

Proof. Under these conditions, the formula (3.7) implies that the kernel space $\mathcal{K}$ defined in Proposition 3 is equal to the set of $\phi \in \mathcal{A}^{1,1}$ such that $\mathrm{tr}_\eta(\phi) = 0$. □

The main interest in quasi-integrability comes from the following elliptic result and its hyperbolic analog, which ensure that, at least locally, there are many pseudo-holomorphic line bundles on quasi-integrable manifolds.

Proposition 6 (The fundamental exact sequence)**.** *Let (J,η) be a quasi-integrable $\mathrm{U}(3)$-structure on M^6 and let*

$$\mathcal{A}^{1,1}_0(M) = \big\{\phi \in \mathcal{A}^{1,1}(M) \mid \mathrm{tr}_\eta(\phi) = 0\big\}\,, \tag{3.10}$$

be the η-primitive part of the $(1,1)$-forms on M. Let $\mathcal{Z}^{1,1}_0(M) \subset \mathcal{A}^{1,1}_0(M)$ denote the closed forms in $\mathcal{A}^{1,1}_0(M)$.

Then $d^{-1,2} + d^{2,-1}$ vanishes on $\mathcal{A}^{1,1}_0$, the subcomplex

$$0 \longrightarrow \mathcal{A}^{1,1}_0 \longrightarrow \mathcal{A}^{2,1}{\oplus}\mathcal{A}^{1,2} \longrightarrow \mathcal{A}^4 \longrightarrow \mathcal{A}^5 \longrightarrow \mathcal{A}^6 \longrightarrow 0 \tag{3.11}$$

of the deRham complex on M is a locally exact elliptic complex, and, consequently, the sheaf complex

$$0 \longrightarrow \mathcal{Z}^{1,1}_0 \longrightarrow \mathcal{A}^{1,1}_0 \longrightarrow \mathcal{A}^{2,1}{\oplus}\mathcal{A}^{1,2} \longrightarrow \mathcal{A}^4 \longrightarrow \mathcal{A}^5 \longrightarrow \mathcal{A}^6 \longrightarrow 0 \tag{3.12}$$

is a fine resolution of $\mathcal{Z}^{1,1}_0$.

Proof. To begin, note that $\mathcal{A}^{1,1}(M)$ is the space of sections of a (complex) vector bundle of rank 9 over M and that $\mathcal{A}^{1,1}_0(M)$ is the space of sections of a (complex) subbundle of rank 8. If $U \subseteq M$ is an open subset on which there exists a (J,η)-unitary coframing $\alpha : TU \to \mathbb{C}^3$, then one has $N(\alpha) = \lambda I_3$ for some $\lambda \in C^\infty(U)$ and, moreover, for any $\phi \in \mathcal{A}^{1,1}_0(U)$, one has

$$\phi = \tfrac{\mathrm{i}}{2}\, F_{i\bar{j}}\, \alpha^i \wedge \overline{\alpha^j} \tag{3.13}$$

for some 3-by-3 matrix of functions $F = \big(F_{i\bar{j}}\big)$ satisfying $\mathrm{tr}(F) = 0$. Of course, by the formula (3.7), it follows that $(\mathrm{d}^{-1,2} + \mathrm{d}^{2,-1})(\phi) = 0$.

Thus, $\mathrm{d}\big(\mathcal{A}^{1,1}_0(M)\big)$ lies in $\mathcal{A}^{2,1}(M) \oplus \mathcal{A}^{1,2}(M)$ and it follows that (3.11) is indeed a subcomplex of the (complexified) deRham complex. (Note, that, moreover, each of the vector spaces involved is the complexification of the real subspace consisting of the real-valued forms in that subspace.)

Since it is a subcomplex of the deRham complex, for any nonzero (real) cotangent vector ξ, the symbol σ_ξ of the operators in the sequence (3.11) is simply left exterior multiplication by ξ. This symbol sequence is obviously exact at $\mathcal{A}^5$ and $\mathcal{A}^6$. Moreover, elementary linear algebra shows that $\mathcal{A}^{1,1}_0$ contains no nonzero decomposable 2-forms, so the symbol σ_ξ is injective at $\mathcal{A}^{1,1}_0$. By rank count (the bundles have ranks 8, 18, 15, 6 and 1 in order), then, it follows that the symbol sequence will be exact at $\mathcal{A}^{2,1}(M) \oplus \mathcal{A}^{1,2}(M)$ if and only if it is exact at $\mathcal{A}^4(M)$. To prove that it is exact at $\mathcal{A}^4$, it suffices to show that any 4-form of the form $\xi{\wedge}\Upsilon$ can be written in this form where Υ lies in $\mathcal{A}^{2,1}(M) \oplus \mathcal{A}^{1,2}(M)$. However, this is immediate from exterior algebra considerations. Thus, the sequence (3.11) is an elliptic complex.

To prove local exactness is somewhat more subtle and I will not give full details here. For the results needed in the following argument and the definitions of the terms involved see [5], especially Chapter 10, §3. (Note, however, that local exactness is obvious at the places $\mathcal{A}^5$ and $\mathcal{A}^6$ since this is just the Poincaré Lemma. The issue is local exactness at the other places.)

First of all, a calculation shows that the Cartan characters of the overdetermined system $\mathrm{d}\phi = 0$ for $\phi \in \mathcal{A}^{1,1}_0$ are

$$(s_0, s_1, s_2, s_3, s_4, s_5, s_6) = (0, 0, 2, 4, 6, 4, 0) \tag{3.14}$$

and that this system is indeed involutive. It then follows that the Spencer resolution of the overdetermined operator $\mathrm{d} : \mathcal{A}^{1,1}_0 \to \mathcal{A}^{2,1} \oplus \mathcal{A}^{1,2}$ is formally exact and a computation verifies that (3.11) is, in fact, this Spencer resolution.

Finally, the operator $\mathrm{d} : \mathcal{A}^{1,1}_0 \to \mathcal{A}^{2,1} \oplus \mathcal{A}^{1,2}$ is seen to satisfy Singer's δ-estimate, which, together with the formal exactness already derived, implies that (3.11) is indeed locally exact (see [5, Theorem 3.10]). □

Remark 19 (The hyperbolic case). In the case of a hyperbolic quasi-integrable (J, η), the complex (3.11) is still formally exact locally. After all, the overdetermined system $\mathrm{d}\phi = 0$ for $\phi \in \mathcal{A}^{1,1}_0$ is still involutive and has Cartan characters given by (3.14). However, the local exactness is not obvious since one does not have an analog of the δ-estimate.

Remark 20 (Relation with Hodge theory). Note that, when $M^* \subseteq M$, the open set on which J is strictly quasi-integrable, is dense in M, the space $\mathcal{A}^{1,1}_0(M)$ is equal to the kernel of $\mathrm{d}^{-1,2} + \mathrm{d}^{2,-1}$ on $\mathcal{A}^{1,1}(M)$. In particular, in this case, the vector space $\mathcal{Z}^{1,1}_0(M)$ is equal to the space of closed $(1,1)$-forms on M.

By Proposition 6, when M is compact and (J, η) is elliptically quasi-integrable, the space $\mathcal{Z}^{1,1}_0(M)$ is finite dimensional and consists of the closed 'primitive' $(1,1)$-forms on M. In general, one does not know that these forms are harmonic in the usual sense. The reason is that, when $\phi \in \left(\mathcal{A}^{1,1}_0\right)_{\mathbb{R}}$, one has

$$*\phi = -\phi \wedge \eta, \tag{3.15}$$

where $*$ is the Hodge star operator for the underlying metric and orientation associated to the U(3)-structure (J, η). When ϕ is closed, this only gives

$$\mathrm{d}(*\phi) = -\phi \wedge \mathrm{d}\eta = \phi \wedge (\partial\eta + \bar\partial\eta), \tag{3.16}$$

so that, unless (J, η) satisfies $\bar\partial\eta = 0$ (which is not automatic), one will not generally have ϕ be coclosed.

In any case, if $\phi = \bar\phi \in \mathcal{Z}^{1,1}_0(M)$ satisfies the condition that $[\phi] \in H^2(M, \mathbb{R})$ is an integral cohomology class, then there will exist a complex line bundle E endowed with an Hermitian structure h and an h-compatible connection ∇ such that $\nabla^2 = 2\pi \mathrm{i} \phi \otimes \mathrm{id}_E$. In particular, ∇ will define a pseudo-holomorphic structure on E, one that is, moreover, pseudo-Hermitian-Yang-Mills.

Remark 21 (Potentials). As the reader will recall, in the integrable case, a closed, smooth $(1,1)$-form ϕ can be locally written in the form $\phi = \mathrm{i}\,\partial\bar\partial f$ for some smooth 'potential' function f. Moreover, when (J, η) is Hermitian (i.e., J is integrable), an η-primitive $(1,1)$ form is locally expressible as $\mathrm{i}\,\partial\bar\partial f$ where f is harmonic with respect to the underlying metric.

Unfortunately, in the non-integrable case, there does not appear to be such a local potential formula for the closed $(1,1)$-forms.

3.2. **General rank.** In fact, in the quasi-integrable case, the above results for line bundles generalize in a natural way to all ranks:

Proposition 7. *Let (J,η) be a quasi-integrable structure on M^6 and let $E\to M$ be a rank r complex bundle endowed with an Hermitian structure h. The overdetermined system*

$$(\nabla^2)^{0,2} = \mathrm{tr}_\eta(\nabla^2) = 0 \tag{3.17}$$

for h-compatible connections ∇ on E is involutive, with Cartan characters

$$(s_0,s_1,s_2,s_3,s_4,s_5,s_6) = (0,0,0,0,2r^2,3r^2,r^2). \tag{3.18}$$

Proof. I will give the proof in the case that (J,η) is a U(3)-structure. The U(1, 2)-structure case is entirely analogous.

The claimed result is local, so let $U\subset M$ be an open set over which both U and $T^{1,0}(M)$ are trivial as complex bundles. Let $\mathsf{u}=(\mathsf{u}_1,\dots,\mathsf{u}_r)$ be an h-unitary basis of the section of E over U and let $\alpha: TU\to\mathbb{C}^3$ be a (J,η)-unitary coframing on U. Then there exist 1-forms ϕ^i_j and a smooth function λ on U such that α satisfies

$$\mathrm{d}\alpha^i = -\phi^i_l\wedge\alpha^l + \lambda\,\overline{\alpha^j\wedge\alpha^k} \tag{3.19}$$

where (i,j,k) is any even permutation of $(1,2,3)$. In particular, note that

$$\mathrm{d}(\alpha^1\wedge\alpha^2\wedge\alpha^3) = -\,\mathrm{tr}(\phi)\wedge\alpha^1\wedge\alpha^2\wedge\alpha^3 + 2\lambda\,\eta^2, \tag{3.20}$$

since $\eta = \frac{\mathrm{i}}{2}\big(\alpha^1\wedge\overline{\alpha^1}+\alpha^2\wedge\overline{\alpha^2}+\alpha^3\wedge\overline{\alpha^3}\big)$.

Now, any h-compatible connection ∇ on E over U will be of the form

$$\nabla\mathsf{u} = \mathsf{u}\otimes\big(\pi - {}^t\bar\pi\big) \tag{3.21}$$

where

$$\pi = p_1\,\alpha^1 + p_2\,\alpha^2 + p_3\,\alpha^3 \tag{3.22}$$

for some $p_i: U\to M_{r\times r}(\mathbb{C})$. Letting

$$F = \mathrm{d}(\pi - {}^t\bar\pi) + (\pi - {}^t\bar\pi)\wedge(\pi - {}^t\bar\pi) = -{}^t\overline{F} \tag{3.23}$$

denote the curvature of ∇ as usual, the condition that $F=F^{1,1}$ can be written as

$$F\wedge\alpha^1\wedge\alpha^2\wedge\alpha^3 = 0 \tag{3.24}$$

and the condition that $\mathrm{tr}_\eta(F)=0$ can be written in the form

$$F\wedge\eta^2 = 0. \tag{3.25}$$

This motivates the following construction: Let $X = U\times\big(M_{r\times r}(\mathbb{C})\big)^3$, and regard the projections $p_i: X\to M_{r\times r}(\mathbb{C})$ as vector-valued functions on X. Define $\pi = p_1\,\alpha^1+p_2\,\alpha^2+p_3\,\alpha^3$ and regard (3.23) as defining a $\mathfrak{u}(r)$-valued 2-form F on X. Let $\mathcal{I}$ be the ideal on X algebraically generated by the $2r^2$ components of the 5-form $\Upsilon = F\wedge\alpha^1\wedge\alpha^2\wedge\alpha^3$ and the r^2 components of the 6-form $\Psi = F\wedge\eta^2$. Note that, by the Bianchi identity and (3.20), one has the crucial identity

$$\begin{aligned}\mathrm{d}\Upsilon &= \mathrm{d}F\wedge\alpha^1\wedge\alpha^2\wedge\alpha^3 + F\wedge d\big(\alpha^1\wedge\alpha^2\wedge\alpha^3\big)\\ &= \Big(F\wedge(\pi - {}^t\bar\pi) - (\pi - {}^t\bar\pi)\wedge F\Big)\wedge\alpha^1\wedge\alpha^2\wedge\alpha^3\\ &\qquad + F\wedge\big(-\,\mathrm{tr}(\phi)\wedge\alpha^1\wedge\alpha^2\wedge\alpha^3 + 2\lambda\,\eta^2\big)\\ &= -\Upsilon\wedge(\pi - {}^t\bar\pi) - (\pi - {}^t\bar\pi)\wedge\Upsilon - \mathrm{tr}(\phi)\wedge\Upsilon + 2\lambda\,\Psi,\end{aligned} \tag{3.26}$$

which shows that $\mathcal{I}$ is differentially closed. (It is for this that the hypothesis of quasi-integrability is needed.)

The integral manifolds of $(\mathcal{I},\, \mathrm{i}\,\alpha^1\wedge\alpha^2\wedge\alpha^3\wedge\overline{\alpha^1\wedge\alpha^2\wedge\alpha^3})$ in X are, by construction, locally the graphs of the h-compatible connections on E over U that satisfy (3.17). The verification that this ideal with independence condition is involutive and has characters as described in (3.18) is now routine. □

Remark 22 (Gauge fixing). The reader should not be surprised that the last nonzero character in (3.18) is $s_6 = r^2$ rather than 0. This is to be expected since the equations (3.17) are gauge invariant. In terms of the local representation defined above, this implies that if $\pi = p_i\,\alpha^i$ does satisfy these equations, then, for any smooth map $g : U \to \mathrm{U}(r)$, the $\mathfrak{gl}(r,\mathbb{C})$-valued $(1,0)$-form

$$\pi^g = g^{-1}\,\partial g + g^{-1}\pi g \tag{3.27}$$

will also satisfy these equations. As usual, one can break this gauge-invariance by imposing an additional equation, such as the 'Coulomb gauge' condition

$$\mathrm{d}(\pi + {}^t\bar\pi)\wedge\eta^2 = 0, \tag{3.28}$$

and the reader can check that this yields a system whose characters are

$$(s_0, s_1, s_2, s_3, s_4, s_5, s_6) = (0, 0, 0, 0, 2r^2, 4r^2, 0). \tag{3.29}$$

Moreover, when η is positive definite, this is an elliptic system, which implies, in particular, that one has elliptic regularity for Coulomb-gauged pseudo-Hermitian-Yang-Mills connections over a quasi-integrable U(3)-structure.

In particular, note that this shows that pseudo-Hermitian-Yang-Mills connections have the same degree of local generality for quasi-integrable U(3)-structures as for integrable U(3)-structures. In the case of real-analytic quasi-integrable U(3)-structures, the Coulomb-gauged pseudo-Hermitian-Yang-Mills connections are also real-analytic.

It seems very likely, though I have not tried to verify this, that there would be analogs of the Uhlenbeck removable singularities theorem generalizing to the quasi-integrable case the ones known in the integrable case [7, 19, 20].

Naturally, in the hyperbolic case, i.e., for quasi-integrable $\mathrm{U}(1,2)$-structures, one does not have elliptic regularity.

4. Examples and Generality

In this section, after some examples have been given of (strictly) quasi-integrable almost complex structures, the question of how general such structures are will be addressed.

4.1. **The 6-sphere.** The most familiar example of of a non-integrable almost complex structure is the G_2-invariant almost complex structure on the 6-sphere.

The group G_2 is the subgroup of $\mathrm{GL}(7,\mathbb{R})$ that preserves the 3-form

$$\phi = \mathrm{d}x^{123} + \mathrm{d}x^{145} + \mathrm{d}x^{167} + \mathrm{d}x^{246} - \mathrm{d}x^{257} - \mathrm{d}x^{356} - \mathrm{d}x^{347}, \tag{4.1}$$

where $x = (x^i)$ are the standard coordinates on $\mathbb{R}^7$ and $\mathrm{d}x^{ijk} = \mathrm{d}x^i\wedge\mathrm{d}x^j\wedge\mathrm{d}x^k$. It is not difficult to show that G_2 is a subgroup of SO(7), i.e., that it preserves the standard inner product and orientation on $\mathbb{R}^7$. It is connected and has dimension 14. It acts transitively on $S^6 \subset \mathbb{R}^7$ and the stabilizer of a point in S^6 is isomorphic to SU(3), i.e., $\mathrm{G}_2 \cap \mathrm{SO}(6) = \mathrm{SU}(3)$.

In particular, it follows that G_2 preserves a SU(3)-structure on S^6. One can define this structure by defining its associated invariant differential forms. The invariant $(1,1)$-form ω is defined to be the pullback to S^6 of the G_2-invariant 2-form $R \lrcorner \phi$, where R is the radial vector field on $\mathbb{R}^7$. The invariant $(3,0)$-form ψ is defined to be the pullback to S^6 of the complex-valued 3-form $R \lrcorner (*\phi) + \mathrm{i}\phi$, where $*\phi$ is the 4-form that is Hodge dual to ϕ in $\mathbb{R}^7$. The Euler identities

$$\mathrm{d}(R \lrcorner \phi) = 3\,\phi \qquad \text{and} \qquad \mathrm{d}\big(R \lrcorner (*\phi)\big) = 4\,{*\phi} \tag{4.2}$$

together with a little algebra then imply the relations

$$\mathrm{d}\omega = 3\,\mathrm{Im}(\psi) \qquad \text{and} \qquad \mathrm{d}\psi = 2\,\omega^2. \tag{4.3}$$

(The almost complex structure J is defined to be the unique one for which ψ is of type $(3,0)$.)

Let $u : \mathrm{G}_2 \to S^6$ be defined by $u(g) = g{\cdot}u_0$ for some fixed unit vector $u_0 \in S^6$. It is then easy to verify that on G_2 there exist complex-valued left-invariant forms α_i and $\kappa_{i\bar{j}} = -\overline{\kappa_{j\bar{i}}}$ satisfying $\kappa_{1\bar{1}} + \kappa_{2\bar{2}} + \kappa_{3\bar{3}} = 0$ and

$$\begin{aligned} \mathrm{d}\alpha_i &= -\kappa_{i\bar{l}} \wedge \alpha_l + \overline{\alpha_j \wedge \alpha_k} \\ \mathrm{d}\kappa_{i\bar{j}} &= -\kappa_{i\bar{k}} \wedge \kappa_{k\bar{j}} + \tfrac{3}{4}\alpha_i \wedge \overline{\alpha_j} - \tfrac{1}{4}\delta_{i\bar{j}}\,\alpha_l \wedge \overline{\alpha_l} \end{aligned} \tag{4.4}$$

where, in the first equation, (i,j,k) is an even permutation of $(1,2,3)$ and where

$$u^*\omega = \tfrac{\mathrm{i}}{2}\left(\alpha_1 \wedge \overline{\alpha_1} + \alpha_2 \wedge \overline{\alpha_2} + \alpha_3 \wedge \overline{\alpha_3}\right) \qquad \text{and} \qquad u^*\psi = \alpha_1 \wedge \alpha_2 \wedge \alpha_3\,. \tag{4.5}$$

For details, see[11] [2, Proposition 2.3]. In particular, it follows that the almost complex structure J is strictly quasi-integrable and that $\omega = \omega(J)$ and $\psi = \psi(J)$.

Note that the $\mathfrak{su}(3)$-valued matrix $\kappa = (\kappa_{i\bar{j}})$ is a connection matrix for the tangent bundle $E = T^{1,0}S^6$, and that this is compatible with the given Hermitian structure on E. The second formula in (4.4) then shows that this connection defines a pseudo-holomorphic structure on E and, in agreement with Proposition 5, it is indeed pseudo-Hermitian-Yang-Mills.

Note also that because $\bar{\partial}\omega = 0$, it follows (see Remark 20) that the closed $(1,1)$-forms on S^6 are also coclosed. In particular, the group $Z^{1,1}_0(S^6)$ injects into the trivial deRham group $H^2(S^6,\mathbb{C})$ and hence is trivial. Thus, the only global pseudo-holomorphic line bundles over S^6 are trivial and, indeed, any pseudo-holomorphic structure on a unitary bundle E of rank r must have the trace of its curvature vanishing, so that the holonomy of such a connection necessarily lies in $\mathrm{SU}(r)$.

4.2. **The flag manifold SU(3)/$\mathbb{T}^2$.** As an example of a different sort, consider the flag manifold $M = \mathrm{SU}(3)/\mathbb{T}^2$, where $\mathbb{T}^2 \subset \mathrm{SU}(3)$ is the maximal torus consisting of diagonal matrices. The canonical left invariant form on SU(3) can be written in the form

$$\gamma = g^{-1}\,\mathrm{d}g = \begin{pmatrix} \mathrm{i}\,\beta_1 & \alpha_3 & -\overline{\alpha_2} \\ -\overline{\alpha_3} & \mathrm{i}\,\beta_2 & \alpha_1 \\ \alpha_2 & -\overline{\alpha_1} & \mathrm{i}\,\beta_3 \end{pmatrix} \tag{4.6}$$

[11] Note, however, that, in order to make that notation match with the notation in this article, one must take $\alpha_i = -2\theta_i$.

where the β_i are real and satisfy $\beta_1+\beta_2+\beta_3=0$. The structure equation $\mathrm{d}\gamma=-\gamma\wedge\gamma$ is then equivalent to

$$\begin{aligned} \mathrm{d}\alpha_i &= -\mathrm{i}(\beta_j-\beta_k)\wedge\alpha_i+\overline{\alpha_j\wedge\alpha_k} \\ \mathrm{d}(\mathrm{i}\beta_i) &= \alpha_k\wedge\overline{\alpha_k}-\alpha_j\wedge\overline{\alpha_j} \end{aligned} \tag{4.7}$$

where (i,j,k) is an even permutation of $(1,2,3)$. It follows that, if $\pi:\mathrm{SU}(3)\to\mathrm{SU}(3)/\mathbb{T}^2$ is the coset projection, then there is a unique almost complex structure J on $\mathrm{SU}(3)/\mathbb{T}^2$ such that

$$\begin{aligned} \pi^*\bigl(\omega(J)\bigr) &= \tfrac{\mathrm{i}}{2}\left(\alpha_1\wedge\overline{\alpha_1}+\alpha_2\wedge\overline{\alpha_2}+\alpha_3\wedge\overline{\alpha_3}\right) \\ \pi^*\bigl(\psi(J)\bigr) &= \alpha_1\wedge\alpha_2\wedge\alpha_3\,. \end{aligned} \tag{4.8}$$

As the structure equations indicate, $\bigl(J,\omega(J)\bigr)$ is strictly quasi-integrable and satisfies

$$\mathrm{d}\bigl(\omega(J)\bigr)=3\,\mathrm{Im}\bigl(\psi(J)\bigr) \qquad \text{and} \qquad \mathrm{d}\bigl(\psi(J)\bigr)=2\left(\omega(J)\right)^2. \tag{4.9}$$

In particular, $\bar\partial\bigl(\omega(J)\bigr)=0$, so that, just as in the case of S^6 (again via Remark 20), closed $(1,1)$-forms are both primitive and coclosed. Thus, $Z^{1,1}_0(M)$ injects into $H^2(M,\mathbb{C})\simeq\mathbb{C}^2$ via the map that sends closed forms to their cohomology classes. By (4.7), there are closed $(1,1)$-forms B_i on M that satisfy $\pi^*(B_i)=\mathrm{d}\beta_i$. These $(1,1)$-forms satisfy $B_1+B_2+B_3=0$, but are otherwise indepdendent as forms and (hence) as cohomology classes. In particular, it follows that every element of $H^2(M,\mathbb{Z})$ can be represented by a closed $(1,1)$-form. Thus, every complex line bundle over M carries a pseudo-holomorphic structure and this is unique up to gauge equivalence. In particular, it follows that the complex vector bundles generated by these line bundles (and their duals) all carry pseudo-holomorphic structures.

Note that these pseudo-holomorphic line bundles exist even though there are clearly no almost complex hypersurfaces (even locally) in M.

4.3. **Nearly Kähler structures.** Both of the above examples fall into a larger class of elliptic quasi-integrable structures that has been studied extensively, that of *nearly Kähler structures*. For references on what is known about these structures, the reader might consult [8] or [18].

For the purposes of this paper, it will be convenient to take a definition of 'nearly Kähler' that is well-adapted for use in dimension 6. (This is not the standard definition, but see [18].) An SU(3)-structure on a 6-manifold M can be specified by giving its fundamental $(1,1)$-form ω (which defines the Hermitian structure) and a its fundamental $(3,0)$-form ψ. These forms are required to satisfy the equations

$$\mathrm{d}\omega=3c\,\mathrm{Im}(\psi) \qquad \text{and} \qquad \mathrm{d}\psi=2c\,\omega^2 \tag{4.10}$$

for some real constant c.

When $c=0$, one sees immediately that the underlying almost complex structure is integrable, that ω defines a Kähler metric for this complex structure, and that ψ is a parallel holomorphic volume form on this Kähler manifold. In other words, such an SU(3)-structure is what is usually called 'Calabi-Yau'. When $c\neq0$ (in which case, one says that the structure is *strictly nearly Kähler*), one can consider instead the scaled pair $\bigl(c^2\omega,c^3\psi\bigr)$ (which will have the same underlying almost complex structure, but a metric scaled by c^2) and see that one is reduced to the case $c=1$.

A calculation using the structure equations [18] shows that, if $\pi : \mathsf{F} \to M$ is the SU(3)-bundle over M whose local sections are the special unitary coframings $\alpha : TU \to \mathbb{C}^3$, then the tautological forms and connections on F satisfy

$$\pi^*\omega = \tfrac{\mathrm{i}}{2}\left(\alpha_1\wedge\overline{\alpha_1}+\alpha_2\wedge\overline{\alpha_2}+\alpha_3\wedge\overline{\alpha_3}\right) \qquad \text{and} \qquad \pi^*\psi = \alpha_1\wedge\alpha_2\wedge\alpha_3 \tag{4.11}$$

and

$$\begin{aligned} \mathrm{d}\alpha_i &= -\kappa_{i\bar l}\wedge\alpha_l + c\,\overline{\alpha_j\wedge\alpha_k} \\ \mathrm{d}\kappa_{i\bar j} &= -\kappa_{i\bar k}\wedge\kappa_{k\bar j} + c^2\left(\tfrac{3}{4}\alpha_i\wedge\overline{\alpha_j} - \tfrac{1}{4}\delta_{i\bar j}\,\alpha_l\wedge\overline{\alpha_l}\right) + K_{i\bar j p\bar q}\,\alpha_q\wedge\overline{\alpha_p}\,, \end{aligned} \tag{4.12}$$

where, in the first line of (4.12), (i,j,k) is any even permuation of $(1,2,3)$ and where $K_{i\bar j p\bar q} = K_{p\bar j i\bar q} = K_{i\bar q p\bar j} = \overline{K_{j\bar i q\bar p}}$ and $K_{i\bar i p\bar q} = 0$.

In particular, note that the $\mathfrak{su}(3)$-valued connection form κ has its curvature of type $(1,1)$. Thus, κ defines a pseudo-holomorphic structure on the tangent bundle of M, one that is, in fact, pseudo-Hermitian-Yang-Mills.

Note also that the curvature tensor K that shows up in these structure equations takes values in an SU(3)-irreducible bundle of (real) rank 27 and that it vanishes exactly when the structure is either flat (if $c = 0$) or, up to a constant scalar factor, equivalent to the G_2-invariant structure on S^6 (if $c \neq 0$).

In particular, the well-known result that the underlying metric of a strictly nearly Kähler structure on a 6-manifold is Einstein (with positive Einstein constant) follows immediately from this. It also follows that all such structures are real analytic in, say, coordinates harmonic for the metric.

A straightforward calculation using exterior differential systems shows that, modulo diffeomorphism, the local nearly Kähler structures in dimension 6 depend on two arbitrary functions of 5 variables, i.e., they have the same local generality as the well-known $c = 0$ case.

Meanwhile, no compact example with $c \neq 0$ that is not homogeneous appears to be known.

Proposition 8 (Supercriticality)**.** *Any nearly Kähler structure is critical for all of the functionals F_c of Theorem 1.*

Proof. This follows immediately from the definitions and Proposition 2. □

4.4. **Projectivized tangent bundles.** Let S be a complex surface endowed with a Kähler form η and let $\pi : F \to S$ be the associated U(2)-coframe bundle, with structure equations

$$\mathrm{d}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} = -\begin{pmatrix}\psi_{1\bar 1} & \psi_{1\bar 2}\\ \psi_{2\bar 1} & \psi_{2\bar 2}\end{pmatrix}\wedge\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} \tag{4.13}$$

where $\pi^*\eta = \tfrac{\mathrm{i}}{2}\big(\eta_1\wedge\overline{\eta_1} + \eta_1\wedge\overline{\eta_1}\big)$ and $\overline{\psi_{i\bar j}} = -\psi_{j\bar i}$ and

$$\mathrm{d}\psi_{i\bar j} = -\psi_{i\bar k}\wedge\psi_{k\bar j} + \tfrac{1}{2}K_{i\bar j\ell\bar k}\,\eta_k\wedge\overline{\eta_\ell} \tag{4.14}$$

where, as usual $K_{i\bar j\ell\bar k} = K_{\ell\bar j i\bar k} = K_{i\bar k\ell\bar j} = \overline{K_{j\bar i k\bar\ell}}\,$.

Setting $\alpha_1 = \eta_1$, $\alpha_2 = \overline{\eta_2}$, and $\alpha_3 = -\psi_{2\bar 1}$, one computes

$$\left.\begin{aligned} \mathrm{d}\alpha_1 &\equiv \overline{\alpha_2}\wedge\overline{\alpha_3} \\ \mathrm{d}\alpha_2 &\equiv \overline{\alpha_3}\wedge\overline{\alpha_1} \\ \mathrm{d}\alpha_3 &\equiv \tfrac{1}{2}K_{1\bar 1 2\bar 2}\,\overline{\alpha_1}\wedge\overline{\alpha_2} \end{aligned}\right\} \mod \alpha_1,\ \alpha_2,\ \alpha_3\,. \tag{4.15}$$

Letting $\mathbb{T}^2 \subset \mathrm{U}(2)$ be the subgroup of diagonal unitary matrices, it follows that there is a well-defined almost complex structure J on $M = F/\mathbb{T}^2$ whose $(1,0)$-forms pull back to F to be linear combinations of α_1, α_2, and α_3.

Since, after pullback to F, one has the formula,

$$\omega(J) = \tfrac{\mathrm{i}}{2}\left(\tfrac{1}{2}K_{1\bar{1}2\bar{2}}\,\alpha_1\wedge\overline{\alpha_1} + \tfrac{1}{2}K_{1\bar{1}2\bar{2}}\,\alpha_2\wedge\overline{\alpha_2} + \alpha_3\wedge\overline{\alpha_3}\right), \tag{4.16}$$

it follows that $\omega(J)$ will be nondegenerate on M (which is the projectivized tangent bundle of S) as long as the Kähler metric η on S has nonvanishing holomorphic bisectional curvature. In particular, J is elliptically quasi-integrable if η has positive holomorphic bisectional curvature and hyperbolically quasi-integrable if η has negative holomorphic bisectional curvature. In any case, the Nijenhuis tensor of J has real type.

Using the structure equations on F, one also sees that, if η has constant holomorphic bisectional curvature c, then the almost complex structure J will be supercritical. When $c > 0$, this is just the nearly Kähler case of $\mathrm{SU}(3)/\mathbb{T}^2$ already discussed and is quasi-integrable. When $c < 0$, i.e., when (S,η) is a ball quotient (i.e., locally isometric to the Hermitian symmetric space $\mathrm{SU}(1,2)/\mathrm{U}(2) = \mathbb{B}^2$), this gives examples (some of which are compact) of supercritical almost complex structures that are hyperbolically quasi-integrable.

Remark 23 (Hyperbolic nearly Kähler structures). This last example points out that there is a hyperbolic analog of nearly Kähler structures that is worth defining: An $\mathrm{SU}(1,2)$-structure on M^6, with defining $(1,1)$-form ω of Hermitian signature $(1,2)$ and $(3,0)$-form ψ will be said to be *hyperbolically nearly Kähler* if it satisfies the structure equations

$$\mathrm{d}\omega = 3c\,\mathrm{Im}(\psi) \qquad \text{and} \qquad \mathrm{d}\psi = 2c\,\omega^2 \tag{4.17}$$

for some (real) constant c. Again, the usual methods of exterior differential systems shows that the local (i.e., germs of) hyperbolic nearly Kähler structures modulo diffeomorphism depend on 2 arbitrary functions of 5 variables, just as in the elliptic case.

4.5. **Generality.** It is natural to ask how 'general' (local) quasi-integrable almost complex structures are. This is something of a vague question, but the notion of generality can be made precise in a number of ways, usually (when considering the generality of a space of solutions of a system of PDE) by an appeal to Cartan's theory of systems in involution [5].

For the sake of simplicity, I will only consider the elliptic quasi-integrable case and only occasionally remark on the analogous results in the hyperbolic case.

Since quasi-integrability is a first-order condition on J, the set of 1-jets of quasi-integrable structures is a well-defined subset $\mathsf{QJ}(M) \subset J^1\big(\mathsf{J}(M)\big)$. Unfortunately, $\mathsf{QJ}(M)$ is not smooth, but, at its smooth points, it has codimension 8 in $J^1\big(\mathsf{J}(M)\big)$.

By contrast, quasi-integrability for $\mathrm{U}(3)$-structures is better behaved as a first-order PDE system: Let $\mathsf{U}(M) \to M$ be the bundle of unitary structures over M, whose general fiber is modeled on $\mathrm{GL}(6,\mathbb{R})/\mathrm{U}(3)$. Then the set of 1-jets of quasi-integrable $\mathrm{U}(3)$-structures on M is a smooth submanifold $\mathsf{QU}(M) \subset J^1\big(\mathsf{U}(M)\big)$ of codimension 16.

Unfortunately, the condition of quasi-integrability for $\mathrm{U}(3)$-structures turns out not to be involutive, and a full analysis has yet to be done. The point of this subsection is to explain at least where the first obstruction appears. Along the way, some useful information about quasi-integrable U(3)-structures will be uncovered.

4.5.1. *The structure equations.* Since quasi-integrability for a U(3)-structure (J,η) is well-defined as a condition on the underlying $\mathbb{R}^+\cdot\mathrm{U}(3)$-structure, it is natural to study the structure equations on the associated $\mathbb{R}^+\cdot\mathrm{U}(3)$-bundle $\pi:\mathsf{F}\to M$ whose local sections are the 'conformally unitary' coframings $\alpha : TU\to\mathbb{C}^3$, i.e., α is J-linear and satisfies $\eta_U = \mu\,\frac{\mathrm{i}}{2}\,(\alpha^1{\wedge}\overline{\alpha^1}+\alpha^2{\wedge}\overline{\alpha^2}+\alpha^3{\wedge}\overline{\alpha^3})$ for some positive function μ.

A straightforward analysis of the intrinsic torsion of the $\mathbb{R}^+\cdot\mathrm{U}(3)$-structure F shows that $\zeta=(\zeta_i)$, the canonical $\mathbb{C}^3$-valued 1-form on F, satisfies unique first structure equations of the form

$$
\mathrm{d}\zeta_i = -\rho\wedge\zeta_i - \kappa_{i\bar\jmath}\wedge\zeta_j - B_{ij\bar k}\,\overline{\zeta_j}\wedge\zeta_k + \tfrac12\varepsilon_{ijk}\,\lambda\,\overline{\zeta_j\wedge\zeta_k} \tag{4.18}
$$

where $\rho=\overline{\rho}$ is real, $\kappa_{i\bar\jmath}=-\overline{\kappa_{j\bar\imath}}$, and $B_{i\ell\bar\jmath}=-B_{\ell i\bar\jmath}$ satisfying $B_{ij\bar\jmath}=0$ and λ are complex functions defined on F. (N.B.: The unitary summation convention is being employed and, as usual, the symbol ε_{ijk} is completely antisymmetric and satisfies $\varepsilon_{123}=1$.) The 1-forms ρ and $\kappa=(\kappa_{i\bar\jmath})=-{}^t\bar\kappa$ are the $\mathbb{R}$- and $\mathfrak{u}(3)$-components of the canonical connection form on the $\mathbb{R}^+\cdot\mathrm{U}(3)$-bundle $\pi:\mathsf{F}\to M$. The functions λ and $B_{ij\bar k}$ represent the components of the first-order tensorial (torsion) invariants of the $\mathbb{R}^+\cdot\mathrm{U}(3)$-structure.

Remark 24 (Relation with Gray-Hervella torsion). The reader who is familiar with the canonical connection of a U(3)-structure and its four irreducible torsion components as described by Gray and Hervella in [9] may be wondering how these structure equations correspond. The function λ represents their tensor W_1, the quasi-integrable assumption is the vanishing of their tensor W_2, and the functions $B_{ij\bar k}$ represent the components of their tensor W_3. Their tensor W_4 is not conformally invariant and, as a result, does not appear in the structure equations (4.18). Instead, one has the uniqueness of the conformal connection form ρ, corresponding to the fact that the first prolongation of the Lie algebra of $\mathbb{R}^+\cdot\mathrm{U}(3)\subset\mathrm{GL}(6,\mathbb{R})$ is trivial.

Note that, for the underlying almost complex structure J, the canonical forms ω and ψ satisfy

$$
\pi^*\omega = \tfrac{\mathrm{i}}{2}|\lambda|^2\,\zeta_i\wedge\overline{\zeta_i} \qquad\text{and}\qquad \pi^*\psi = |\lambda|^2\overline{\lambda}\,\zeta_1\wedge\zeta_2\wedge\zeta_3\,. \tag{4.19}
$$

The second structure equations are computed by taking the exterior derivative of the first structure equations. In order to do this most efficiently, first set

$$
\begin{aligned}
D\lambda &= \mathrm{d}\lambda - \lambda\big(\rho - \operatorname{tr}\kappa\big),\\
DB_{ij\bar k} &= \mathrm{d}B_{ij\bar k} + B_{pj\bar k}\kappa_{i\bar p} + B_{ip\bar k}\kappa_{j\bar p} - B_{ij\bar p}\kappa_{p\bar k} - B_{ij\bar k}\,\rho,\\
K_{i\bar\jmath} &= \mathrm{d}\kappa_{i\bar\jmath} + \kappa_{i\bar k}\wedge\kappa_{k\bar\jmath} = -\overline{K_{j\bar\imath}}.
\end{aligned} \tag{4.20}
$$

Then the exterior derivative of (4.18) becomes, after some rearrangment and term collection,[12]

$$\begin{aligned} 0 = &-\mathrm{d}\rho\wedge\zeta_i - K_{i\bar{\jmath}}\wedge\zeta_j - DB_{ij\bar{k}}\wedge\overline{\zeta_j}\wedge\zeta_k + \tfrac{1}{2}\varepsilon_{ijk}\,D\lambda\wedge\overline{\zeta_j}\wedge\overline{\zeta_k} \\ &+ \big(B_{i\ell\bar{p}}\overline{B_{\ell q\bar{\jmath}}} - \tfrac{1}{2}\varepsilon_{ijk}\overline{\varepsilon_{pqk}}\,|\lambda|^2\big)\,\zeta_p\wedge\zeta_q\wedge\overline{\zeta_j} \\ &+ \big(B_{iq\bar{\jmath}}B_{jp\bar{k}} + \varepsilon_{iq\ell}\lambda\,\overline{B_{\ell k\bar{p}}}\,\big)\,\zeta_k\wedge\overline{\zeta_p}\wedge\overline{\zeta_q}\,. \end{aligned} \tag{4.21}$$

By exterior algebra, there exist unique functions λ_ℓ, $\lambda_{\bar\ell}$, $B_{ij\bar{k}\bar{\ell}}$, $B_{ij\bar{k}\ell}$, $S_{ij} = -S_{ji}$, $R_{i\bar{\jmath}} = \overline{R_{j\bar{\imath}}}$, $L_{i\bar{\jmath}k\ell} = -L_{i\bar{\jmath}\ell k}$, and $M_{i\bar{\jmath}k\bar{\ell}} = \overline{M_{j\bar{\imath}\ell\bar{k}}}$ on F such that

$$\begin{aligned} D\lambda &= \lambda_{\bar\ell}\,\zeta_\ell + \lambda_\ell\,\overline{\zeta_\ell}\,, \\ DB_{ij\bar{k}} &= B_{ij\bar{k}\bar{\ell}}\,\zeta_\ell + B_{ij\bar{k}\ell}\,\overline{\zeta_\ell}\,, \\ \mathrm{d}\rho &= \tfrac{1}{2}S_{k\ell}\,\overline{\zeta_k}\wedge\overline{\zeta_\ell} + \tfrac{\mathrm{i}}{2}\,R_{k\bar{\jmath}}\,\zeta_j\wedge\overline{\zeta_k} + \tfrac{1}{2}\overline{S_{k\ell}}\,\zeta_k\wedge\zeta_\ell\,, \\ K_{i\bar{\jmath}} &= \tfrac{1}{2}L_{i\bar{\jmath}k\ell}\,\overline{\zeta_k}\wedge\overline{\zeta_\ell} + \tfrac{1}{2}\,M_{i\bar{\jmath}k\bar{\ell}}\,\zeta_\ell\wedge\overline{\zeta_k} - \tfrac{1}{2}\overline{L_{j\bar{\imath}k\ell}}\,\zeta_k\wedge\zeta_\ell\,. \end{aligned} \tag{4.22}$$

The coefficients appearing in the terms on the right hand side of (4.22) are the second-order invariants of the $\mathbb{R}^+\!\cdot\mathrm{U}(3)$-structure F.

Substituting these equations into (4.21) yields the *first Bianchi identities*, which can be sorted into (p,q)-types in the obvious way. I now want to consider some of the consequences of these identities.

First, let $\sigma:\mathbb{R}^+\!\cdot\mathrm{U}(3)\to\mathbb{C}^*$ be the representation that satisfies $\sigma(tg) = t\det(\bar{g})$ for $t>0$ and $g\in\mathrm{U}(3)$ and define the complex line bundle $\Lambda = \mathsf{F}\times_\sigma\mathbb{C}$ over M. Then the 1-form $\rho - \operatorname{tr}\kappa$ can be viewed as defining a connection ∇ on the line bundle Λ and the equation

$$\mathrm{d}\lambda = \lambda(\rho - \operatorname{tr}\kappa) + \lambda_{\bar\ell}\,\zeta_\ell + \lambda_\ell\,\overline{\zeta_\ell} \tag{4.23}$$

shows that λ represents a smooth section L of Λ. Consequently, the 1-forms $\lambda_{\bar\imath}\,\zeta_i$ and $\lambda_i\,\overline{\zeta_i}$ represent, respectively $\nabla^{1,0}L$ and $\nabla^{0,1}L = \bar\partial L$.

Proposition 9 (Holomorphicity of L)**.** *Let M^6 be endowed with an $\mathbb{R}^+\!\cdot\mathrm{U}(3)$-structure that is quasi-integrable. Then the section L is pseudo-holomorphic, i.e., $\bar\partial L = 0$. In particular, if M is connected, then either L vanishes identically* (*in which case the almost complex structure J is integrable*) *or else the zero locus of L* (*and hence, of the Nijenhuis tensor*) *is of* (*real*) *codimension at least* 2 *in M.*

Proof. One immediately sees that the $(0,3)$-component of the first Bianchi identities is the three equations

$$\lambda_\ell = 0. \tag{4.24}$$

Thus $\bar\partial L = 0$, as claimed.

The remainder of the proposition now follows by a standard argument about pseudo-holomorphic sections of line bundles endowed with connections over almost complex manifolds, so I will only sketch it.

First, note that, for any connected pseudo-holomorphic curve $C\subset M$, the restriction of (Λ,∇) to C becomes a complex line bundle Λ_C that carries a unique holomorphic structure for which $\nabla^{0,1} = \bar\partial_C$ and that the restriction of the section L

[12]Although $\overline{\varepsilon_{ijk}} = \varepsilon_{ijk}$, I use the former when needed to maintain the unitary summation convention that summation is implied when an index appears both barred and unbarred in a given term. N.B.: In an expression like $\overline{a_{\bar\imath}}$, the index i is to be regarded as unbarred.

to C becomes a holomorphic section of Λ_C with respect to this holomorphic structure. In particular, L either vanishes identically on C or else has only isolated zeros of finite order.

Consequently, if L vanishes to infinite order at any point $p \in M$, it must vanish identically on any connected pseudo-holomorphic curve C in M that passes through p. Since the union of the pseudo-holomorphic discs in M passing through p contains an open neighborhood of p, it follows that the set of points $Z_\infty(L)$ at which L vanishes to infinite order is an open set. Since $Z_\infty(L)$ is clearly closed and M is connected, it follows that either $Z_\infty(L)$ is all of M, so that L vanishes identically, or else that $Z_\infty(L)$ is empty.

Now suppose that $Z_\infty(L)$ is empty. If $Z(L)$, the set of points in M at which L vanishes is empty, then there is nothing to prove, so suppose that $L(p) = 0$. Since L vanishes to finite order at p, there is an immersed pseudo-holomorphic disk $D \subset M$ passing through p such that L does not vanish identically on D. Since L restricted to D is a holomorphic section of a holomorphic line bundle, it follows that, by shrinking D if necessary, it can be assumed that p is the only zero of L on D and that $\nu_p(L_D) = k$ for some integer $k > 0$.

One can now embed D into a (real) 4-parameter family of pseudo-holomorphic disks $D_t \subset M$ with $t \in \mathbb{R}^4$ that foliate a neighborhood of p and satisfy $D_0 = D$. One can further assume that L does not vanish on any of the boundaries ∂D_t (since it does not vanish on ∂D_0). Then for each t, the section L restricted to D_t has a zero locus consisting of k points (counted with multiplicity). Thus, the intersection $Z(L) \cap D_t$ is a finite set of points whose total multiplicity is k. That the real codimension of $Z(L)$ is at least 2 now follows from this. □

Remark 25 (Speculation about $Z(L)$). It is tempting to conjecture that if $Z(L)$ is neither empty nor all of M, then it has a stratification

$$Z(L) = Z^1(L) \cup Z^2(L) \cup Z^3(L) \tag{4.25}$$

where $Z^i(L) \subset M$ is a smooth almost complex submanifold of (M, J) of complex codimension i, with $Z(L) = \overline{Z^1(L)}$. One might even expect that, when M is compact, $Z^1(L)$, counted with the appropriate multiplicity, would define a class in $H_4(M, \mathbb{Z})$ that would be dual to $c_1(\Lambda)$.

In fact, let $\nu : Z(L) \to \mathbb{Z}^+$ be the upper semicontinuous function giving the order of vanishing of L. Then the set $Z^*(L) \subset Z(L)$ on which ν is locally constant is open and dense in $Z(L)$ and it is not difficult to show that $Z^*(L)$ is a smooth 4-dimensional submanifold of M whose tangent spaces are everywhere complex. The difficulty apprears to be in understanding the 'singular' part of L, i.e., the set of points at which ν is not locally constant.

Note that, in the open dense set $M^* = M \setminus Z(L)$, the almost complex structure J is elliptically strictly quasi-integrable and hence there cannot be any complex codimension 1 almost complex submanifolds in M^*, even locally.

Remark 26 (Singularity issues). Let S be a complex surface with a Kähler form η whose bisectional curvature is everywhere nonnegative. Then, as explained in §4.4, the projectivized tangent bundle $M = F/\mathbb{T}^2$ carries an almost complex structure J whose Nijenhuis tensor is everywhere of real type. On the open set $M^* \subset M$ that represents the points in the projectivized tangent bundle where the holomorphic bisectional curvature is strictly positive, J is strictly elliptically quasi-integrable.

However, the set of points in M where the holomorphic bisectional curvature (represented by $K_{1\bar{1}2\bar{2}} \geq 0$) vanishes can be quite wild. In particular, it does not have to have real codimension 2 in M. For example, it can easily happen that this zero locus is a single point in M. By Proposition 9, such elliptically quasi-integrable almost complex structures do not underlie an elliptically quasi-integrable unitary structure on M. This example gives an indication of what sort of singularities are being avoided by working with quasi-integrable unitary structures rather quasi-integrable almost complex structures.

Remark 27 (Second-order forms). Note that a consequence of the holomorphicity of L is the following formulae: First, $\mathrm{d}\psi = \bar{\partial}\psi + 2\,\omega^2$, where

$$\pi^*(\bar{\partial}\psi) = 2|\lambda|^2\,\overline{(\lambda_{\bar{k}}\,\zeta_k)} \wedge (\zeta_1 \wedge \zeta_2 \wedge \zeta_3). \tag{4.26}$$

Second $\mathrm{d}\omega = 3\,\mathrm{Im}(\psi) + \partial\omega + \bar{\partial}\omega$, where

$$\pi^*(\bar{\partial}\omega) = \tfrac{\mathrm{i}}{2}\lambda\,\overline{(\lambda_{\bar{k}}\,\zeta_k)} \wedge (\zeta_\ell \wedge \overline{\zeta_\ell}) - \tfrac{\mathrm{i}}{2}|\lambda|^2 B_{ij\bar{k}}\,\zeta_k \wedge \overline{\zeta_i \wedge \zeta_j}. \tag{4.27}$$

The two terms on the right hand side represent the decomposition of $\bar{\partial}\omega$ into its primitive types, as defined by the underlying $\mathbb{R}^+ \cdot \mathrm{U}(3)$-structure.

I will now resume the investigation of the first Bianchi identities. The $(3,0)$-part of the first Bianchi identity becomes

$$\overline{\varepsilon_{ijk}}\,S_{jk} - \overline{\varepsilon_{jk\ell}}\,L_{j\bar{\imath}k\ell} = 0, \tag{4.28}$$

which shows that S_{jk} is determined in terms of $L_{j\bar{\imath}k\ell}$. The $(2,1)$-part of the first Bianchi identity is the more complicated

$$\begin{aligned} &\tfrac{\mathrm{i}}{2}\big(\delta_{i\bar{\jmath}}R_{k\bar{\ell}} - \delta_{i\bar{\ell}}R_{k\bar{\jmath}}\big) + \tfrac{1}{2}\big(M_{i\bar{\jmath}k\bar{\ell}} - M_{i\bar{\ell}k\bar{\jmath}}\big) + B_{ik\bar{\jmath}\bar{\ell}} - B_{ik\bar{\ell}\bar{\jmath}} \\ &\qquad + B_{ip\bar{\ell}}\overline{B_{pj\bar{k}}} - B_{ip\bar{\jmath}}\overline{B_{p\ell\bar{k}}} + \varepsilon_{ikp}\overline{\varepsilon_{j\ell p}}\,|\lambda|^2 = 0. \end{aligned} \tag{4.29}$$

The $(1,2)$-part of the first Bianchi identity is

$$\begin{aligned} &\delta_{i\bar{\ell}}S_{jk} + L_{i\bar{\ell}jk} + \big(B_{ik\bar{\ell}j} - B_{ij\bar{\ell}k}\big) - \varepsilon_{ijk}\lambda_{\bar{\ell}} \\ &\qquad + B_{ij\bar{p}}B_{pk\bar{\ell}} - B_{ik\bar{p}}B_{pj\bar{\ell}} + \big(\varepsilon_{ijp}\overline{B_{p\ell\bar{k}}} - \varepsilon_{ikp}\overline{B_{p\ell\bar{\jmath}}}\,\big)\,\lambda = 0. \end{aligned} \tag{4.30}$$

Finally, remember that the $(0,3)$-part of the Bianchi identity has already been seen to be

$$\lambda_\ell = 0. \tag{4.31}$$

For what I have in mind, it will be necessary to solve these equations more-or-less explicitly in order to understand how many free derivatives there are at second-order for quasi-integrable structures. After some exterior algebra and representation theoretic arguments, one finds that the following method works:

It is useful to introduce notation for the traces of the covariant derivatives of the B-tensor. Thus, set

$$B_{ip\bar{\jmath}\bar{p}} = u_{i\bar{\jmath}} + \mathrm{i}\,v_{i\bar{\jmath}} \tag{4.32}$$

where $u = (u_{i\bar{\jmath}}) = {}^t\bar{u}$ and $v = (v_{i\bar{\jmath}}) = {}^t\bar{v}$ take values in Hermitian 3-by-3 matrices. Note that, because $B_{ji\bar{\jmath}} = 0$ by definition, it follows that u and v are themselves traceless. Next, write

$$B_{ij\bar{k}k} = -B_{ji\bar{k}k} = 4\varepsilon_{ijp}\,a_{\bar{p}} \tag{4.33}$$

for functions $a = (a_{\bar p})$. Then one finds that

$$B_{ij\bar k\ell} = B^*_{ij\bar k\ell} + \varepsilon_{ijp}\big(\delta_{\ell\bar p}\, a_{\bar k} + \delta_{\ell\bar k}\, a_{\bar p}\big) \tag{4.34}$$

where $B^*_{ij\bar k\ell} = -B^*_{ji\bar k\ell}$ satisfies $B^*_{ij\bar k k} = B^*_{ik\bar k\ell} = 0$.

With these quantities in hand, one finds that the Bianchi identities are equivalent to the following relations:

$$S_{ij} = \varepsilon_{ijk}\big(4\, a_{\bar k} + \tfrac{3}{2}\lambda_{\bar k}\big), \tag{4.35}$$

$$L_{i\bar j k\ell} = B^*_{ik\bar j\ell} - B^*_{i\ell\bar j k} + \delta_{i\bar j}\, Q_{k\ell} + \delta_{k\bar j}\, P_{i\ell} - \delta_{\ell\bar j}\, P_{ik} + \delta_{k\bar j}\, F_{i\ell} - \delta_{\ell\bar j}\, F_{ik}\,, \tag{4.36}$$

where

$$\begin{aligned} Q_{ij} &= -\varepsilon_{ijk}\big(2\, a_{\bar k} + \tfrac{1}{2}\lambda_{\bar k}\big) = -Q_{ji}\\ P_{ij} &= -\varepsilon_{ijk}\big(3\, a_{\bar k} + \quad \lambda_{\bar k}\big) = -P_{ji}\\ F_{ij} &= \tfrac{1}{2} B_{ip\bar q} B_{jq\bar p} + \tfrac{1}{2}\varepsilon_{pqi}\, \lambda\, \overline{B_{pq\bar j}} = F_{ji}\,, \end{aligned} \tag{4.37}$$

$$R_{i\bar j} = -2\, v_{i\bar j} \tag{4.38}$$

(in particular, note that R is traceless), and, finally,

$$\begin{aligned} M_{i\bar j k\bar\ell} = K_{i\bar j k\bar\ell} &- \delta_{i\bar j}\, u_{k\bar\ell} - \delta_{k\bar\ell}\, u_{i\bar j} - \mathrm{i}\big(\delta_{i\bar\ell}\, v_{k\bar j} - \delta_{k\bar j}\, v_{i\bar\ell}\big)\\ &+ 2\delta_{i\bar j}\, B_{pq\bar\ell}\overline{B_{pqk}} + 2\delta_{k\bar\ell}\, B_{pq\bar j}\overline{B_{pq\bar\imath}} - \delta_{i\bar j}\delta_{k\bar\ell}\big(B_{pq\bar r}\overline{B_{pq\bar r}} + 2|\lambda|^2\big)\,, \end{aligned} \tag{4.39}$$

where $K_{i\bar j k\bar\ell} = K_{k\bar j i\bar\ell} = K_{i\bar\ell k\bar j} = \overline{K_{j\bar\imath\ell\bar k}}$ has the symmetries of a Kähler curvature tensor.

The main use of these formule will be in the following observation: *All of the second-order invariants of a $\mathbb{R}^+\!\cdot \mathrm{U}(3)$-structure are expressed in terms of the covariant derivatives of its λ- and B-tensors plus the 'Kähler-component' of the curvature of its κ-connection. Moreover, the only relations among these latter three invariants, other than the usual symmetries of the Kähler curvature, are the relations $\lambda_i = 0$.*

Proposition 10 (Noninvolutivity of the quasi-integrable condition)**.** *The first order system on $\mathbb{R}^+\!\cdot \mathrm{U}(3)$-structures that defines quasi-integrability is not formally integrable. In particular, it is not involutive.*

Proof. The structure equations and Bianchi identities imply the formulae

$$\mathrm{d}\rho = \varepsilon_{k\ell p}\big(2\, a_{\bar p} + \tfrac{3}{4}\lambda_{\bar p}\big)\, \overline{\zeta_k}\wedge\overline{\zeta_\ell} - \mathrm{i}\, v_{i\bar j}\, \zeta_j\wedge\overline{\zeta_i} + \overline{\varepsilon_{k\ell p}\big(2\, a_{\bar p} + \tfrac{3}{4}\lambda_{\bar p}\big)}\, \zeta_k\wedge\zeta_\ell \tag{4.40}$$

and, by (4.36) and (4.37),

$$\begin{aligned} \mathrm{d}(\mathrm{tr}\,\kappa) &= \tfrac{1}{2} L_{i\bar\imath k\ell}\, \overline{\zeta_k}\wedge\overline{\zeta_\ell} + \tfrac{1}{2}\, M_{i\bar\imath k\bar\ell}\, \zeta_\ell\wedge\overline{\zeta_k} - \tfrac{1}{2}\overline{L_{i\bar\imath k\ell}}\, \zeta_k\wedge\zeta_\ell\\ &= -\varepsilon_{k\ell p}\big(6\, a_{\bar p} + \tfrac{7}{4}\lambda_{\bar p}\big)\, \overline{\zeta_k}\wedge\overline{\zeta_\ell} + \tfrac{1}{2} M_{i\bar\imath k\bar\ell}\, \zeta_\ell\wedge\overline{\zeta_k}\\ &\qquad + \overline{\varepsilon_{k\ell p}\big(6\, a_{\bar p} + \tfrac{7}{4}\lambda_{\bar p}\big)}\, \zeta_k\wedge\zeta_\ell \end{aligned} \tag{4.41}$$

Meanwhile, the exterior derivative of the equation

$$\mathrm{d}\lambda = \lambda\,\big(\rho - \mathrm{tr}\,\kappa\big) + \lambda_{\bar\imath}\, \zeta_i \tag{4.42}$$

yields

$$0 = (\lambda_{\bar\imath}\, \zeta_i)\wedge\big(\rho - \mathrm{tr}\,\kappa\big) + \lambda\,\big(\mathrm{d}\rho - \mathrm{d}(\mathrm{tr}\,\kappa)\big) + \mathrm{d}\lambda_{\bar\imath}\wedge\zeta_i + \lambda_{\bar\imath}\, \mathrm{d}\zeta_i \tag{4.43}$$

which, by the first structure equations, can be written in the form

$$0 = \lambda\bigl(\mathrm{d}\rho - \mathrm{d}(\mathrm{tr}\,\kappa)\bigr) + D\lambda_{\bar\imath}\wedge\zeta_i + \tfrac12\varepsilon_{ijk}\,\lambda\lambda_{\bar\imath}\,\overline{\zeta_j\wedge\zeta_k}\,, \tag{4.44}$$

where

$$D\lambda_{\bar\imath} = \mathrm{d}\lambda_{\bar\imath} - (2\rho - \mathrm{tr}\,\kappa)\lambda_{\bar\imath} - \kappa_{j\bar\imath}\,\lambda_{\bar\jmath} - B_{j\ell\bar\imath}\lambda_{\bar\jmath}\,\overline{\zeta_\ell} \tag{4.45}$$

is π-semi-basic. Thus, taking the $(0,2)$-part of (4.44) yields

$$\lambda\bigl(\mathrm{d}\rho - \mathrm{d}(\mathrm{tr}\,\kappa)\bigr)^{0,2} = -\tfrac12\varepsilon_{ijk}\,\lambda\lambda_{\bar\imath}\,\overline{\zeta_j\wedge\zeta_k}\,. \tag{4.46}$$

On the other hand, (4.40) and (4.41) yield

$$\lambda\bigl(\mathrm{d}\rho - \mathrm{d}(\mathrm{tr}\,\kappa)\bigr)^{0,2} = \varepsilon_{ijk}\,\lambda\bigl(8\,a_{\bar\imath} + \tfrac52\lambda_{\bar\imath}\bigr)\,\overline{\zeta_j\wedge\zeta_k}\,. \tag{4.47}$$

In other words, the equation

$$\lambda\bigl(8\,a_{\bar\imath} + 3\lambda_{\bar\imath}\bigr) = 0 \tag{4.48}$$

is an identity for quasi-integrable $\mathbb{R}^+\cdot\mathrm{U}(3)$-structures.

It is important to understand how this identity was derived. While it is a relation on the second-order invariants, it is *not* an algebraic consequence of the first Bianchi identity. Instead, it was derived by combining the first Bianchi identity with the result of differentiating the relation (4.42), which is, itself, a part of the first Bianchi identity. It is the presence of this second-order identity that *cannot* be found by differentiating only one time the first-order defining equations for quasi-integrable $\mathbb{R}^+\cdot\mathrm{U}(3)$-structures that shows that these defining equations are not formally integrable and, hence, not involutive. (See Remark 28 for a further discussion of this point.) □

Corollary 2 (Quasi-integrable dichotomy)**.** *Let M^6 be connected and let $\bigl(J,[\eta]\bigr)$ be a quasi-integrable $\mathbb{R}^+\cdot\mathrm{U}(3)$-structure on M. Then either J is integrable (i.e., λ vanishes identically) or else the structure satisfies the second-order equations*

$$\lambda_{\bar\ell} = -\tfrac13\overline{\varepsilon_{ij\ell}}\,B_{ij\bar k k}\,. \tag{4.49}$$

Proof. By Proposition 9, either λ vanishes identically or else its zero locus has no interior. In this latter case, (4.48) implies that $8\,a_{\bar\imath} + 3\lambda_{\bar\imath}$ must vanish identically. Tracing through the definitions, this is (4.49). □

Remark 28 (Formal integrability). An adequate discussion of formal integrability would be too long to include here; see [5, Ch. IX] for details. Very roughly speaking, a system Σ of first-order PDE is formally integrable if any differential equation of order q that is satisfied by all of the solutions of Σ is derivable from Σ by differentiating the equations in Σ at most $q{-}1$ times.

However, it is worth pointing out explicitly what is going on in this particular case. Let $\pi : \mathsf{CU}(M) \to M$ denote the bundle whose local sections are the local $\mathbb{R}^+\cdot\mathrm{U}(3)$-structures on M. (The notation $\mathsf{CU}(M)$ is meant to denote 'conformal unitary'.) This bundle has fibers isomorphic to the 26-dimensional homogeneous space $\mathrm{GL}(6,\mathbb{R})/\bigl(\mathbb{R}^+\cdot\mathrm{U}(3)\bigr)$. (Thus, such local structures depend on 26 functions of 6 variables.)

The 1-jets of local quasi-integrable $\mathbb{R}^+\cdot\mathrm{U}(3)$-structures form a smooth subbundle $Q \subset J^1\bigl(\mathsf{CU}(M)\bigr)$, one that has (real) codimension 16 in $J^1\bigl(\mathsf{CU}(M)\bigr)$. Its first prolongation $Q^{(1)} \subset J^2\bigl(\mathsf{CU}(M)\bigr)$ is also a smooth submanifold and the projection $Q^{(1)} \to Q$ is a smooth submersion. However, its second prolongation $Q^{(2)} = \bigl(Q^{(1)}\bigr)^{(1)} \subset J^3\bigl(\mathsf{CU}(M)\bigr)$ is *not* a smooth manifold and its projection $Q^{(2)} \to Q^{(1)}$ is not surjective and not a submersion.

In fact, the relations (4.48) show that the image of the projection of $Q^{(2)} \to Q^{(1)}$ has real codimension at least[13] 2 in $Q^{(1)}$. It is the failure of this surjectivity that shows that the system defined by Q is not formally integrable, which, of course, implies that it is not involutive.

Remark 29 (Generality revisited). In light of Corollary 2, the study of (elliptic) quasi-integrable $\mathbb{R}^+\cdot \mathrm{U}(3)$-structures $\bigl(J,[\eta]\bigr)$ can now be broken into two (overlapping) classes:

The first class is defined by $\lambda = 0$ and is easy to understand. In this case, J is integrable and it is easy now to argue that these depend on 14 functions of 6 variables, locally. To see this, note that, because J is integrable, one can imagine constructing these in two stages: First, choose an integrable almost complex structure J on M. Since these are all locally equivalent up to diffeomorphism, the general integrable almost complex structure on M is easily seen to depend on 6 arbitrary functions of 6 variables (i.e., the generality of the diffeomorphism group in dimension 6). Second, once J is fixed, choose a section $[\eta]$ up to positive multiples of the bundle of positive $(1,1)$-forms for J. Since this projectivized bundle has real rank 8 over M, it follows that this choice depends on 8 functions of 6 variables. Consequently, the general integrable conformal unitary structure on M^6 depends on $14 = 6+8$ arbitrary functions of 6 variables. (Of course, when one reduces modulo diffeomorphism, this comes back down to 8 function of 6 variables.)

The second class, defined by the first-order quasi-integrable conditions plus the 6 second-order conditions (4.49), is more problematic. If one lets $Q_1'' \subset Q^{(1)}$ denote the codimension 6 submanifold defined by these equations, one does not know that Q_1'' is formally integrable, let alone involutive. My attempts to check involutivity for Q_1'' have, so far, been unsuccessful. Thus, it is hard to say how many functions of how many variables the general solution depends on.

What one does know is that the tableau of Q_1'' is isomorphic to a proper subtableau of the system in $J^2(\mathsf{CU}(M))$ that defines the conformally unitary structures with integrable underlying almost complex structure and its last nonzero character is strictly less than 14. Consequently, generality of quasi-integrable conformally unitary structures of the second type is strictly less than 14 functions of 6 variables.

On the other hand, there is a lower bound on this generality that can be derived by a construction in the next subsubsection, where a family depending on 10 arbitrary functions of 6 variables is constructed by making use of some linear algebra constructions involving 3-forms on 6-manifolds.

4.5.2. *A construction.* I do not know an explicit construction of the general quasi-integrable confomally unitary structure. However, there is a natural construction of a large class of such structures (depending, as will be seen, on 10 functions of 6 variables) which I will now outline. This construction, which I found in 2000, has

[13]It has not been shown that (4.48) defines the image of $Q^{(2)}$ in $Q^{(1)}$, only that $Q^{(2)}$ lies in union of the two submanifolds $Q_1' \subset Q^{(1)}$ defined by $\lambda = 0$ (and hence of codimension 2) and $Q_1'' \subset Q^{(1)}$ defined by $8\,a_{\bar\imath} + 3\lambda_{\bar\imath} = 0$ (and hence of codimension 6). While I believe that the union $Q_1' \cup Q_1''$ is the image of $Q^{(2)}$ in $Q^{(1)}$, I have not written out a proof.

much in common with a construction of Hitchin's [14, §5], and the reader may want to compare his treatment.

First, recall some linear algebra:[14] For any 6-dimensional vector space V over $\mathbb{R}$, the space $A^3(V) = \Lambda^6(V^*)$ of alternating 3-forms has dimension 20 and contains two open $\mathrm{GL}(V)$-orbits $A^3_\pm(V) \subset A^3(V)$. If v^i is a basis of V^*, the open set $A^3_+(V)$ is the $\mathrm{GL}(V)$-orbit of

$$\begin{aligned}\phi_+ &= \mathrm{Re}\big((v^1 + \mathrm{i}v^2) \wedge (v^3 + \mathrm{i}v^4) \wedge (v^5 + \mathrm{i}v^6)\big)\\ &= v^1 \wedge v^3 \wedge v^5 - v^1 \wedge v^4 \wedge v^6 - v^2 \wedge v^3 \wedge v^6 - v^2 \wedge v^4 \wedge v^5\end{aligned} \tag{4.50}$$

while the open set $A^3_-(V)$ is the $\mathrm{GL}(V)$-orbit of

$$\phi_- = v^1 \wedge v^2 \wedge v^3 + v^4 \wedge v^5 \wedge v^6. \tag{4.51}$$

My interest will be in ϕ_+.

Let $J_+ : V \to V$ be the complex structure on V for which the complex-valued 1-forms $(v^1+\mathrm{i}v^2)$, $(v^3+\mathrm{i}v^4)$, and $(v^5+\mathrm{i}v^6)$ are J-linear and let $\mathrm{GL}(V, J_+) \subset \mathrm{GL}(V)$ be the commuting subgroup of J_+. Of course, $\mathrm{GL}(V, J_+)$ is isomorphic to $\mathrm{GL}(3, \mathbb{C})$ and hence contains a simple, normal subgroup $\mathrm{SL}(V, J_+) \subset \mathrm{GL}(V, J_+)$ isomorphic to $\mathrm{SL}(3, \mathbb{C})$ that consists of the elements of $\mathrm{GL}(V, J_+)$ that are complex unimodular.

If $C \in \mathrm{GL}(V)$ anticommutes with J_+ and satisfies $\det(C) = -1$, then the $\mathrm{GL}(V)$-stabilizer of ϕ_+ is easily seen to be the 2-component group $\mathrm{SL}(V, J_+) \cup C\cdot \mathrm{SL}(V, J_+)$. In fact, $\mathrm{SL}(V, J_+)$ is the intersection of the stabilizer of ϕ_+ with $\mathrm{GL}^+(V)$, the group of orientation-preserving linear transformations of V.

It follows that if V is endowed with an orientation and ϕ is any element of $A^3_+(V)$, then there is a unique element $*\phi \in A^3_+(V)$ and complex structure $J_\phi : V \to V$ such that $\phi\wedge{*\phi} > 0$ and such that $\phi + \mathrm{i}\,{*\phi}$ is a $(3, 0)$-form for J_ϕ. Note the identity $J^*_\phi(\phi) = *\phi$. Note also that reversing the chosen orientation of V will replace $(*\phi, J_\phi)$ by $(-{*\phi}, -J_\phi)$.

It is important to bear in mind that the map $* : A^3_+(V) \to A^3_+(V)$, though it is obviously smooth algebraic, is homogeneous of degree 1, and satisfies $**\phi = -\phi$, is *not* the restriction to $A^3_+(V)$ of a linear endomorphism of $A^3(V)$.

Of course, J_ϕ induces a decomposition of the complex-valued alternating forms on V into types. I will denote the associated subspaces by $A^{p,q}_\phi(V)$, as usual. For my purposes, the most important thing to note is that the space $A^{p,p}_\phi(V)$ is the complexification of the real subspace $H^p_\phi(V)$ consisting of the real-valued forms in $A^{p,p}_\phi(V)$. Moreover, $H^p_\phi(V)$ contains an open convex cone $H^p_\phi(V)^+$ consisting of the forms that are positive on all J_ϕ-complex subspaces of dimension p.

In particular, note that the normalized squaring map $s : H^2_\phi(V)^+ \to H^4_\phi(V)^+$ defined by $s(\omega) = \frac{1}{2}\omega^2$ is a diffeomorphism and hence has a smooth inverse

$$\sigma : H^4_\phi(V)^+ \to H^2_\phi(V)^+. \tag{4.52}$$

There is a hyperbolic analog of this square root: Let $H^2_\phi(V)^{r,s}$ denote the open set of nondegenerate J_ϕ-Hermitian forms of type (r, s) (so that $H^2_\phi(V)^+ = H^2_\phi(V)^{3,0}$). Then the normalized squaring map $s : H^2_\phi(V)^{1,2} \to H^4_\phi(V)$ is a diffeomorphism onto its (open) image, denoted $H^4_\phi(V)^{1,2}$. Accordingly, I extend the domain of σ to include $H^4_\phi(V)^{1,2}$ and so that it inverts s on $H^4_\phi(V)^{1,2}$.

[14] For a proof of these statements, see the Appendix.

Now let M^6 be an oriented 6-manifold and let $\mathcal{A}^3_+(M)$ denote the set of 3-forms on M that, at each point are linearly equivalent to ϕ_+. These forms are the sections of an open subset of $A^3(TM)$ and hence are *stable* in Hitchin's sense [14].

For each $\varphi \in \mathcal{A}^3_+(M)$, there is associated a smooth 3-form $*\varphi \in \mathcal{A}^3_+(M)$ and an associated almost complex structure $J_\varphi : TM \to TM$.

Using the type decomposition with respect to J_φ (and the fact that $\mathrm{d}^{-2,3} = 0$), there exist a $(1,0)$-form β and a $(2,2)$-form π on M such that

$$\mathrm{d}(\varphi + \mathrm{i}\,{*\varphi}) = \overline{\beta} \wedge (\varphi + \mathrm{i}\,{*\varphi}) + \pi. \tag{4.53}$$

Of course $\bar\partial(\varphi + \mathrm{i}\,{*\varphi}) = \overline{\beta}\wedge(\varphi + \mathrm{i}\,{*\varphi})$ and $\mathrm{d}^{-1,2}(\varphi + \mathrm{i}\,{*\varphi}) = \pi$.

Now let $\varphi \in \mathcal{A}^3_+(M)$ have the property that $*\varphi$ is closed. Then the left and right hand sides of (4.53) are both real and, by the type decomposition, it follows that $\beta = 0$ and $\pi = \bar\pi$ lies in $\mathcal{H}^4(M)$, the space of sections of the bundle $H^4_\varphi(TM)$.

I will say that φ is *positive definite* if π is a section of $H^4_\varphi(TM)^+$ and that φ is *positive indefinite* if π is a section of $H^4_\varphi(TM)^{1,2}$.

Obviously, positivity (of either type) is an open condition on the 1-jet of φ.

If φ satisfies the condition that $\mathrm{d}(*\varphi) = 0$ and is positive (definite or indefinite), then there exists a unique $(1,1)$-form η_φ satisfying $\mathrm{d}\varphi = \pi = 2\,{\eta_\varphi}^2$ that lies in $\mathcal{H}^2(M)^+$ if φ is positive definite and in $\mathcal{H}^2(M)^{1,2}$ if φ is positive indefinite.

Proposition 11 (Quasi-integrability)**.** *Let $\varphi \in \mathcal{A}^3_+(M)$ satisfy $\mathrm{d}(*\varphi) = 0$. If φ is positive definite, then the* U(3)*-structure $(J_\varphi, \eta_\varphi)$ is strictly quasi-integrable. If φ is positive indefinite, then the* U(1,2)*-structure $(J_\varphi, \eta_\varphi)$ is strictly quasi-integrable.*

Proof. First, assume that φ is positive definite. The claimed result is local, so let $\alpha = (\alpha^i) : TU \to \mathbb{C}^3$ be a local $(J_\varphi, \eta_\varphi)$-unitary coframing. Then

$$\eta_\varphi = \tfrac{\mathrm{i}}{2}\,{}^t\alpha \wedge \bar\alpha, \tag{4.54}$$

so that

$$\pi = 2\,{\eta_\varphi}^2 = \begin{pmatrix}\alpha^2\wedge\alpha^3 & \alpha^3\wedge\alpha^1 & \alpha^1\wedge\alpha^2\end{pmatrix} \wedge \begin{pmatrix}\overline{\alpha^2\wedge\alpha^3}\\ \overline{\alpha^3\wedge\alpha^1}\\ \overline{\alpha^1\wedge\alpha^2}\end{pmatrix}. \tag{4.55}$$

Moreover, there exists a nonvanishing complex function F on U such that

$$\varphi + \mathrm{i}\,{*\varphi} = F^{-1}\,\alpha^1 \wedge \alpha^2 \wedge \alpha^3, \tag{4.56}$$

implying that

$$\pi = \mathrm{d}^{-1,2}(\varphi + \mathrm{i}\,{*\varphi}) = F^{-1}\,\begin{pmatrix}\alpha^2\wedge\alpha^3 & \alpha^3\wedge\alpha^1 & \alpha^1\wedge\alpha^2\end{pmatrix} N(\alpha) \begin{pmatrix}\overline{\alpha^2\wedge\alpha^3}\\ \overline{\alpha^3\wedge\alpha^1}\\ \overline{\alpha^1\wedge\alpha^2}\end{pmatrix}. \tag{4.57}$$

Comparison with (4.55) now yields that $N(\alpha) = F\,I_3$. In particular, $N(\alpha)$ is a nonzero multiple of the identity, which is what needed to be shown.

The proof in the positive indefinite case is completely similar. One merely has to note that the formulae become

$$\eta_\varphi = \tfrac{\mathrm{i}}{2}\left(\alpha^1 \wedge \overline{\alpha^1} - \alpha^2 \wedge \overline{\alpha^2} - \alpha^3 \wedge \overline{\alpha^3}\right) \tag{4.58}$$

so that

$$\pi = 2\,{\eta_\varphi}^2 = \begin{pmatrix}\alpha^2\wedge\alpha^3 & -\alpha^3\wedge\alpha^1 & -\alpha^1\wedge\alpha^2\end{pmatrix} \wedge \begin{pmatrix}\overline{\alpha^2\wedge\alpha^3}\\ \overline{\alpha^3\wedge\alpha^1}\\ \overline{\alpha^1\wedge\alpha^2}\end{pmatrix}. \tag{4.59}$$

The rest of the proof is entirely similar to the positive definite case. □

Remark 30 (Further information and a characterization). The calculations made in the course of the proof can be carried a little further: Assuming that U is simply connected, one can, by multiplying α by a unimodular complex function, arrange that F is real and positive.

For example, imposing this condition in the positive definite case specifies α up to replacement of the form $\alpha \mapsto g\alpha$ where $g : U \to \mathrm{SU}(3)$ is smooth. Since $N(\alpha) = F\,I_3$, it follows from the definitions of ω and of ψ that $\omega = F^2\,\eta_\varphi$ and that $\psi = F^3\,\alpha^1{\wedge}\alpha^2{\wedge}\alpha^3 = F^4\,(\varphi + \mathrm{i}\,{*}\varphi)$.

Using this information, it is easy to see that the strictly quasi-integrable structures $\bigl(J,\eta\bigr)$ constructed in Proposition 11 are characterized by the condition that the canonical conformal curvature form $\mathrm{d}\rho$ vanishes identically, or, equivalently, in terms of the almost complex structure J, that there be a positive function F on M such that $F^{-4}\,\omega(J)^2$ is closed. (Such an F, if it exists, is clearly unique up to constant multiples.)

Remark 31 (Generality). In Cartan's sense of generality, the closed 3-forms in dimension 6 depend on 10 functions of 6 variables. It is an open condition (of order 0) on a 3-form φ that it lie in $\mathcal{A}^3_+(M)$ and it is a further open condition (of order 1) on φ that $\mathrm{d}\varphi$ lie in $\mathcal{H}^4(M)^+ \cup \mathcal{H}^4(M)^{1,2}$. Thus, one can say that the local positive (definite or indefinite) 3-forms $\phi \in \mathcal{A}^3_+(M)$ satisfying $\mathrm{d}({*}\phi) = 0$ depend on 10 functions of 6 variables.

These conditions are obviously invariant under diffeomorphisms in dimension 6, so it makes sense to say that germs of solutions modulo diffeomorphism depend on $10 - 6 = 4$ arbitrary functions of 6 variables. (Strictly speaking, in order to make this count work, one needs to observe that the group of symmetries of any particular ϕ satisfying these conditions is the group of symmetries of the associated $\mathrm{SU}(3)$- or $\mathrm{SU}(1,2)$-structure and hence is finite dimensional.) This count can be made more rigorous (and verified) by appealing to Cartan's theory of generality, but I will not do this analysis here.

Appendix A. 3-forms in Dimension 6

In this appendix, I will supply a proof of the following normal form result, whose analog over the complex field is well-known and due to Reichel [17], but whose complete proof over the real field does not appear to be easy to locate in the literature.[15] For applications in this article, the important case is the second normal form.

Proposition 12 (Normal forms)**.** *Let V be a real vector space of dimension 6 over $\mathbb{R}$ and let $\phi \in \Lambda^3(V^*)$ be any element. Then there exists a basis $e^1, \dots, e^6$ of V^* so that ϕ is equal to one of the following*

(1) $e^1{\wedge}e^2{\wedge}e^3 + e^4{\wedge}e^5{\wedge}e^6$,
(2) $e^1{\wedge}e^3{\wedge}e^5 - e^1{\wedge}e^4{\wedge}e^6 - e^2{\wedge}e^3{\wedge}e^6 - e^2{\wedge}e^4{\wedge}e^5$,
(3) $e^1{\wedge}e^5{\wedge}e^6 + e^2{\wedge}e^6{\wedge}e^4 + e^3{\wedge}e^4{\wedge}e^5$,
(4) $e^1{\wedge}e^2{\wedge}e^5 + e^3{\wedge}e^4{\wedge}e^5$,
(5) $e^1{\wedge}e^2{\wedge}e^3$, *or*

[15] Probably, this is due to my inability to read Russian. If any reader can point me to a proof in the literature, I'll be grateful.

(6) 0.

Moreover, these six forms are mutually inequivalent. The $\mathrm{GL}(V)$*-orbits of the first two* 3*-forms are open in* $\Lambda^3(V^*)$*, the* $\mathrm{GL}(V)$*-orbit of the third* 3*-form is a hypersurface in* $\Lambda^3(V^*)$*, and the* $\mathrm{GL}(V)$*-orbits of the remaining forms are of higher codimension.*

Proof. The proof will be a series of steps, beginning with a sort of zeroth step to take care of certain special cases. A form $\phi \in \Lambda^3(V^*)$ will be said to be *degenerate* if there is a non-zero vector $v \in V$ that satisfies $v \lrcorner \phi = 0$. Equivalently, ϕ is degenerate if there exists a subspace $v^\perp \subset V^*$ of rank 5 so that ϕ lies in $\Lambda^3(v^\perp)$. In such a case, using the canonical isomorphism

$$\Lambda^3(v^\perp) = \Lambda^2\big((v^\perp)^*\big) \otimes \Lambda^5(v^\perp) \tag{A.1}$$

together with the well-known classification of 2-forms, it follows that either $\phi = 0$ or else is there is a basis $e_1, \ldots, e_5$ of $(v^\perp)^*$, with dual basis $e^1, \ldots, e^5$ of $v^\perp$, so that ϕ is either

$$(e_1 \wedge e_2 + e_3 \wedge e_4) \otimes e^1 \wedge e^2 \wedge e^3 \wedge e^4 \wedge e^5 = e^1 \wedge e^2 \wedge e^5 + e^3 \wedge e^4 \wedge e^5 \tag{A.2}$$

or

$$(e_4 \wedge e_5) \otimes e^1 \wedge e^2 \wedge e^3 \wedge e^4 \wedge e^5 = e^1 \wedge e^2 \wedge e^3. \tag{A.3}$$

Thus, the degenerate cases are accounted for by the last three types listed in the Proposition.

It is useful to note that ϕ is degenerate if and only if ϕ admits a linear factor, i.e., is of the form $\phi = \alpha{\wedge}\beta$ for some $\alpha \in V^*$ and $\beta \in \Lambda^2(V^*)$. To see this, note that each of the three degenerate types has at least one linear factor. Conversely, given the linear factor α, the 2-form β can actually be regarded as a well-defined element of $\Lambda^2\big(V^*/(\mathbb{R}\,\alpha)\big)$. Since the quotient space $V^*/(\mathbb{R}\,\alpha)$ has dimension 5, it follows that β is either zero, decomposable, or the sum of two decomposable terms, again yielding forms of the last three types listed in the Proposition.

Henceforth, it will be assumed that ϕ is nondegenerate, i.e., that $v \lrcorner \phi \neq 0$ for all nonzero $v \in V$ and also that ϕ has no linear divisors, i.e., that $\alpha{\wedge}\phi \neq 0$ for all nonzero $\alpha \in V$. It remains to show that a nondegenerate element of $\Lambda^3(V^*)$ can be put into one of the first three forms listed in the Proposition.

It is convenient to start with what is apparently a special case. Let $e_1, \ldots, e_6$ be a basis of V, with dual basis $e^1, \ldots, e^6$ of V^*. Set

$$\phi_0 = e^1 \wedge e^2 \wedge e^3 + e^4 \wedge e^5 \wedge e^6. \tag{A.4}$$

I claim that the $\mathrm{GL}(V)$-orbit of ϕ_0 is open in $\Lambda^3(V^*)$ and will establish this by showing that the dimension of the stabilizer

$$G_0 = \{\ g \in \mathrm{GL}(V) \mid g^*(\phi_0) = \phi_0\ \} \tag{A.5}$$

is 16, so that the orbit $\mathrm{GL}(V)\cdot\phi_0 = \mathrm{GL}(V)/G_0$ has dimension $36 - 16 = 20$, which is the dimension of $\Lambda^3(V^*)$.

Let $C_0 = \{\ v \in V \bigm| (v \lrcorner \phi)^2 = 0\ \}$. Computation shows that $C_0 = P_0^+ \cup P_0^-$ where $P_0^+ = \mathrm{span}\{e_1, e_2, e_3\}$ and $P_0^- = \mathrm{span}\{e_4, e_5, e_6\}$. Since the elements of G_0 must preserve C_0, it follows that they permute the subspaces $P_0^\pm$. Let $G_0' \subset G_0$ be the subgroup consisting of elements that preserve the subspaces P_0^+ and P_0^-. Since the linear transformation $c : V \to V$ defined by

$$c(e_i) = \begin{cases} e_{i+3} & i = 1,2,3, \\ e_{i-3} & i = 4,5,6. \end{cases} \tag{A.6}$$

lies in G_0 but not in G_0', it follows that $G_0 = G_0' \cup G_0'{\cdot}c$. Since ϕ_0 pulls back to each of $P_0^\pm$ to be a volume form, it follows that $G_0' = \mathrm{SL}(P_0^+) \times \mathrm{SL}(P_0^-) \simeq \mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$. Thus, $\dim G_0 = 16$, as desired.

Now, for any $\phi \in \Lambda^3(V)$ (degenerate or not), define the map

$$J_\phi : V \to \Lambda^5(V^*) = V \otimes \Lambda^6(V^*) \tag{A.7}$$

by $J_\phi(v) = (v \lrcorner \phi){\wedge}\phi$, and regard J_ϕ as an element of $\mathrm{End}(V) \otimes \Lambda^6(V^*)$. Then the map $\phi \mapsto J_\phi$ is a quadratic polynomial map from $\Lambda^3(V^*)$ to $\mathrm{End}(V) \otimes \Lambda^6(V^*)$ that is equivariant with respect to the natural actions of $\mathrm{GL}(V)$ on the domain and range.

For simplicity of notation, write J_0 for J_{ϕ_0}. Inspection shows that

$$\begin{aligned} J_0 = \bigl(e_1 \otimes e^1 + e_2 \otimes e^2 + e_3 \otimes e^3 - e_4 \otimes e^4 - e_5 \otimes e^5 - e_6 \otimes e^6\bigr) \\ \otimes e^1 \wedge e^2 \wedge e^3 \wedge e^4 \wedge e^5 \wedge e^6. \end{aligned} \tag{A.8}$$

In particular, it follows that $\operatorname{tr} J_0 = 0 \in \Lambda^6(V^*)$. Because of the $\mathrm{GL}(V)$-equivariance already mentioned, it follows that $\operatorname{tr} J_\phi = 0$ for all ϕ in the orbit $\mathrm{GL}(V){\cdot}\phi_0$. Since this latter orbit is open and since J is a polynomial mapping, it follows that $\operatorname{tr} J_\phi = 0$ for all $\phi \in \Lambda^3(V^*)$.

Now consider ${J_\phi}^2 \in \mathrm{End}(V) \otimes S^2\bigl(\Lambda^6(V^*)\bigr)$. By the above formula,

$$\begin{aligned} {J_0}^2 = \bigl(e_1 \otimes e^1 + e_2 \otimes e^2 + e_3 \otimes e^3 + e_4 \otimes e^4 + e_5 \otimes e^5 + e_6 \otimes e^6\bigr) \\ \otimes \bigl(e^1 \wedge e^2 \wedge e^3 \wedge e^4 \wedge e^5 \wedge e^6\bigr)^2. \end{aligned} \tag{A.9}$$

In particular, $\operatorname{tr} {J_0}^2 = 6\ \bigl(e^1{\wedge}e^2{\wedge}e^3{\wedge}e^4{\wedge}e^5{\wedge}e^6\bigr)^2$, so that

$${J_0}^2 - \tfrac{1}{6}\ \mathrm{id}_V \otimes \operatorname{tr}\bigl({J_0}^2\bigr) = 0. \tag{A.10}$$

This implies that the $\mathrm{GL}(V)$-equivariant quartic polynomial map from $\Lambda^3(V^*)$ to $\mathrm{End}(V) \otimes S^2\bigl(\Lambda^6(V^*)\bigr)$ defined by

$$\phi \mapsto {J_\phi}^2 - \tfrac{1}{6}\ \mathrm{id}_V \otimes \operatorname{tr}\bigl({J_\phi}^2\bigr) \tag{A.11}$$

vanishes on the $\mathrm{GL}(V)$-orbit of ϕ_0, which is open. Thus, it follows that the identity

$${J_\phi}^2 = \tfrac{1}{6}\ \mathrm{id}_V \otimes \operatorname{tr}\bigl({J_\phi}^2\bigr) \tag{A.12}$$

holds for all $\phi \in \Lambda^3(V^*)$.

The remainder of the proof will divide into cases according to the $\mathrm{GL}(V)$-orbit of $\operatorname{tr}\bigl({J_\phi}^2\bigr)$ in the 1-dimensional vector space $S^2\bigl(\Lambda^6(V^*)\bigr)$. There are three such orbits and they can naturally be designated as ‘positive’, ‘negative’, and ‘zero’.

Suppose first that $\operatorname{tr}\bigl({J_\phi}^2\bigr)$ be ‘positive’, i.e., that it can be written in the form $6\,\Omega^2$ for some $\Omega \in \Lambda^6(V^*)$, which is unique up to sign. By the above identity, it then follows that ${J_\phi}^2(v) = v \otimes \Omega^2$ for all $v \in V$. Define two ‘eigenspaces’ of J_ϕ by

$$P_\phi^\pm = \{\ v \in V \mid J_\phi(v) = \pm v \lrcorner \Omega = \pm v \otimes \Omega\ \}. \tag{A.13}$$

Because of the identity ${J_\phi}^2(v) = v\otimes\Omega^2$, it follows that $V = P^-_\phi \oplus P^+_\phi$. Since $\operatorname{tr} J_\phi = 0$, it follows that $\dim P^-_\phi = \dim P^+_\phi = 3$. Note that replacing Ω by $-\Omega$ will simply switch these two spaces.

Now, this splitting of V implies a splitting of $\Lambda^3(V^*)$ as

$$\text{(A.14)}\qquad \begin{aligned}\Lambda^3(V^*) = \Lambda^3\big((P^+_\phi)^*\big) &\oplus \Lambda^2\big((P^+_\phi)^*\big)\wedge(P^-_\phi)^*\\ &\oplus (P^+_\phi)^*\wedge\Lambda^2\big((P^-_\phi)^*\big)\oplus\Lambda^3\big((P^-_\phi)^*\big)\end{aligned}$$

and hence a corresponding splitting of ϕ into four terms as

$$\text{(A.15)}\qquad \phi = \phi^{+++} + \phi^{++-} + \phi^{+--} + \phi^{---}.$$

I will now argue that $\phi^{++-} = \phi^{+--} = 0$, which will show that ϕ has the form (1), since neither ϕ^{+++} nor ϕ^{---} can vanish for such a ϕ if it be nondegenerate. To see this, let $v^+ \in P^+_\phi$ and $v^- \in P^-_\phi$ be arbitrary. By definition, $(v^+ \lrcorner\, \phi)\wedge\phi = v^+ \lrcorner\, \Omega$ while $(v^- \lrcorner\, \phi)\wedge\phi = -v^- \lrcorner\, \Omega$. Taking left hooks then yields the identities

$$\text{(A.16)}\qquad \begin{aligned} v^- \lrcorner\,(v^+ \lrcorner\,\Omega) &= \big(v^- \lrcorner\,(v^+ \lrcorner\,\phi)\big)\wedge\phi + (v^+ \lrcorner\,\phi)\wedge(v^- \lrcorner\,\phi)\\ -v^+ \lrcorner\,(v^- \lrcorner\,\Omega) &= \big(v^+ \lrcorner\,(v^- \lrcorner\,\phi)\big)\wedge\phi + (v^- \lrcorner\,\phi)\wedge(v^+ \lrcorner\,\phi)\end{aligned}$$

Now, subtracting these identities yields $2\big(v^-\lrcorner(v^+\lrcorner\phi)\big)\wedge\phi = 0$. If there did exist v^+ and v^- so that $v^- \lrcorner\,(v^+ \lrcorner\,\phi)$ were nonzero, then this equation would imply that ϕ had a linear divisor, contradicting the nondegeneracy assumption. Thus, it must be true that $v^- \lrcorner\,(v^+ \lrcorner\,\phi) = 0$ for all $v^+ \in P^+_\phi$ and $v^- \in P^-_\phi$. However, this is equivalent to the condition that $\phi^{++-} = \phi^{+--} = 0$, as desired. It follows that there is a basis $e^1,\dots,e^6$ of V^* so that $\phi^{+++} = e^1\wedge e^2\wedge e^3$ while $\phi^{---} = e^4\wedge e^5\wedge e^6$. Thus, ϕ is of the form (1).

Next, suppose that $\operatorname{tr}\big({J_\phi}^2\big)$ be 'negative', i.e., that it can be written in the form $-6\,\Omega^2$ for some $\Omega \in \Lambda^6(V^*)$, which is unique up to sign. By the above identity, it then follows that ${J_\phi}^2(v) = -v\otimes\Omega^2$ for all $v \in V$. Define two 'complex eigenspaces' of J_ϕ by

$$\text{(A.17)}\qquad P^\pm_\phi = \left\{\ v \in V^{\mathbb{C}}\ \middle|\ J_\phi(v) = \pm i\, v \lrcorner\,\Omega = \pm i\, v\otimes\Omega\ \right\}.$$

Note that since Ω is real, these two subspaces of $V^{\mathbb{C}} = V\otimes\mathbb{C}$ are conjugate. Reasoning as in the positive case now yields that there must exist a basis $e^1,\dots,e^6$ of V^* so that $\phi = \phi^{+++} + \phi^{---}$ where $\phi^{+++} = \frac{1}{2}\,(e^1 + \mathrm{i}\,e^2)\wedge(e^3 + \mathrm{i}\,e^4)\wedge(e^5 + \mathrm{i}\,e^6)$ while $\phi^{---} = \overline{\phi^{+++}} = \frac{1}{2}\,(e^1 - \mathrm{i}\,e^2)\wedge(e^3 - \mathrm{i}\,e^4)\wedge(e^5 - \mathrm{i}\,e^6)$. This yields the form (2).

Finally, suppose that $\operatorname{tr}\big({J_\phi}^2\big) = 0$, so that, by the identity above, ${J_\phi}^2 = 0$. Let $K_\phi \subset V$ be the kernel of J_ϕ and let $I_\phi \subset V$ be such that the image of J_ϕ is $I_\phi\otimes\Lambda^6(V^*) \subset V\otimes\Lambda^6(V^*) = \Lambda^5(V^*)$. Then $\dim K_\phi + \dim I_\phi = 6$ and $I_\phi \subset K_\phi$ (since ${J_\phi}^2 = 0$), so $\dim K_\phi \geq 3$.

Now, for $v \in K_\phi$, the identity $(v\lrcorner\phi)\wedge\phi = 0$ holds. It follows that, for $v_1, v_2 \in K_\phi$, the identity

$$\text{(A.18)}\qquad \big(v_1 \lrcorner\,(v_2 \lrcorner\,\phi)\big)\wedge\phi + (v_2 \lrcorner\,\phi)\wedge(v_1 \lrcorner\,\phi) = 0$$

must also hold. However, the first term in this expression is skewsymmetric in v_1, v_2 while the second term is symmetric in v_1, v_2. It follows that each term must vanish separately, i.e., that

$$\text{(A.19)}\qquad \big(v_1 \lrcorner\,(v_2 \lrcorner\,\phi)\big)\wedge\phi = (v_1 \lrcorner\,\phi)\wedge(v_2 \lrcorner\,\phi) = 0$$

for all $v_1, v_2 \in K_\phi$. Since ϕ is nondegenerate, it has no linear divisors, so the first of these equations implies that

$$v_1 \lrcorner (v_2 \lrcorner \phi) = 0 \tag{A.20}$$

for all $v_1, v_2 \in K_\phi$. This implies that ϕ must lie in the subspace $\Lambda^2({K_\phi}^\perp){\wedge}V^* \subset \Lambda^3(V^*)$. Now if $\dim K_\phi$ were greater than 4, then $\Lambda^2({K_\phi}^\perp)$ would vanish, which is absurd. If $\dim K_\phi$ were equal to 4, then $\Lambda^2({K_\phi}^\perp)$ would be spanned by a single decomposable 2-form, which would imply that ϕ has two linearly independent divisors, which is also impossible for a nondegenerate form. Thus $\dim K_\phi = 3$ is the only possibility. Let e^4, e^5, e^6 be a basis of ${K_\phi}^\perp$ and write

$$\phi = e^1 \wedge e^5 \wedge e^6 + e^2 \wedge e^6 \wedge e^4 + e^3 \wedge e^4 \wedge e^5 + c\, e^4 \wedge e^5 \wedge e^6 \tag{A.21}$$

for some $e^1, e^2, e^3 \in V^*$ and some constant $c \in \mathbb{R}$. The 1-forms $e^1, e^2, e^3, e^4, e^5, e^6$ must be linearly independent and hence a basis of V^* or else ϕ would be degenerate. Now replacing e^1 by $e^1 + c\, e^4$ in the above basis yields a basis in which

$$\phi = e^1 \wedge e^5 \wedge e^6 + e^2 \wedge e^6 \wedge e^4 + e^3 \wedge e^4 \wedge e^5\,, \tag{A.22}$$

showing that ϕ lies in the orbit of the form (3), as was desired. □

Remark 32 (The stabilizer groups). The proof of Proposition 12 indicates the stablizer subgroup $G_\phi \subset \mathrm{GL}(V)$ for each of the normal forms.

If ϕ is of type (1), then $G_\phi = G'_\phi \cup G'_\phi{\cdot}c$, where G'_ϕ is the index 2 subgroup of G_ϕ that preserves the two 3-planes $P^\pm_\phi$ and their volume forms while $c : V \to V$ is a linear map that exchanges these two planes and their volume forms. Thus, there is an exact sequence

$$0 \longrightarrow \mathrm{SL}(P^+_\phi) \times \mathrm{SL}(P^-_\phi) \longrightarrow G_\phi \longrightarrow \mathbb{Z}_2 \longrightarrow 0. \tag{A.23}$$

If ϕ is of type (2), then $G_\phi = G'_\phi \cup G'_\phi{\cdot}c$ where G'_ϕ is the index two subgroup that preserves a complex structure and complex volume form $\omega = 2\phi^{+++}$ on V (i.e., ω is a decomposable element of $\Lambda^3(V \otimes \mathbb{C})$), while $c : V \to V$ is a conjugate linear mapping of V to itself that takes ω to $\overline{\omega}$. In fact, ω can be chosen so that $\phi = \frac{1}{2}(\omega + \overline{\omega})$. Thus, there is an exact sequence

$$0 \longrightarrow \mathrm{SL}(V^{\mathbb{C}}) \longrightarrow G_\phi \longrightarrow \mathbb{Z}_2 \longrightarrow 0. \tag{A.24}$$

The important thing to note is that $G_\phi \cap \mathrm{GL}^+(V) = G'_\phi \simeq \mathrm{SL}(V^{\mathbb{C}})$. Thus, the orientation-preserving stabilizer of ϕ preserves a unique complex structure for which ϕ is the real part of a $(3,0)$-form.

The other stabilizers will not be important here, so we leave those to the reader.

Remark 33 (The symplectic version). Various authors [1, 15] have considered and solved a symplectic version of the above normal form problem:

Namely, consider a 6-dimensional real vector space V endowed with a symplectic (i.e., nondegenerate) 2-form $\omega \in \Lambda^2(V^*)$ and let $\mathrm{Sp}(\omega) \subset \mathrm{GL}(V)$ denote the stabilizer subgroup of ω.

As a representation of $\mathrm{Sp}(\omega)$, the vector space $\Lambda^3(V^*)$ is reducible, being expressible as a direct sum $\Lambda^3(V^*) = \omega{\wedge}V^* \oplus \Lambda^3_0(V^*)$, where $\Lambda^3_0(V^*)$ is the space of 3-forms ϕ on V that satisfy $\omega{\wedge}\phi = 0$. These two summands are irreducible $\mathrm{Sp}(\omega)$ modules and it is an interesting problem to classify the orbits of $\mathrm{Sp}(\omega)$ acting on $\Lambda^3_0(V^*)$.

Proposition 12 provides a simple solution to this classification problem, originally completed by Banos [1].

For example, suppose that $\phi \in \Lambda^3_0(V^*)$ has the normal form (1) and let $e^1, \dots, e^6$ be a basis of V^* such that

$$\phi = e^1 \wedge e^2 \wedge e^3 + e^4 \wedge e^5 \wedge e^6. \tag{A.25}$$

The condition $\omega \wedge \phi = 0$ is equivalent to the condition that there be a_{ij} such that

$$\omega = a_{ij}\, e^i \wedge e^{j+3} \tag{A.26}$$

(where i and j run from 1 to 3 in the summation). Since ω is non-degenerate, $\det(a)$ is nonvanishing. Thus, writing $a = \lambda\, b$ where $\det(b) = 1$ and $\lambda \neq 0$, one can make a unimodular basis change in e^1, e^2, e^3 so that

$$\begin{aligned} \omega &= \lambda\big(e^1 \wedge e^4 + e^2 \wedge e^5 + e^3 \wedge e^6\big) \\ \phi &= e^1 \wedge e^2 \wedge e^3 + e^4 \wedge e^5 \wedge e^6\,. \end{aligned} \tag{A.27}$$

Exchanging e^1, e^2, e^3 with e^4, e^5, e^6 if necessary, it can be assumed that λ is positive and then an overall scale change puts the pair (ω, ϕ) into the form

$$\begin{aligned} \omega &= e^1 \wedge e^4 + e^2 \wedge e^5 + e^3 \wedge e^6 \\ \phi &= \mu\big(e^1 \wedge e^2 \wedge e^3 + e^4 \wedge e^5 \wedge e^6\big) \end{aligned} \tag{A.28}$$

with $\mu > 0$. (Obviously, one cannot get rid of the factor μ entirely.)

Similarly, if ϕ is of type (2), then there are $\zeta^1, \zeta^2, \zeta^3 \in \mathbb{C} \otimes V^*$ such that

$$\phi = \tfrac{1}{2}\big(\zeta^1 \wedge \zeta^2 \wedge \zeta^3 + \overline{\zeta^1 \wedge \zeta^2 \wedge \zeta^3}\,\big)\,. \tag{A.29}$$

The condition that $\omega \wedge \phi = 0$ is then equivalent to the condition that

$$\omega = \tfrac{\mathrm{i}}{2} a_{i\bar{j}}\, \zeta^j \wedge \overline{\zeta^i} \tag{A.30}$$

for some Hermitian symmetric matrix $a = (a_{i\bar{j}}) = {}^t\bar{a}$ with nonvanishing determinant. Exchanging ζ^i for $\overline{\zeta^i}$ if necessary, it can be supposed that $\det(a) = \lambda^3 > 0$ for some $\lambda > 0$. By making a complex unimodular basis change in the ζ^i, it can be further supposed that $a = \lambda\,\mathrm{diag}(1, \pm1, \pm1)$, reducing (after scaling) to the two possible normal forms

$$\begin{aligned} \omega &= \tfrac{\mathrm{i}}{2}\big(\zeta^1 \wedge \overline{\zeta^1} \pm \zeta^2 \wedge \overline{\zeta^2} \pm \zeta^3 \wedge \overline{\zeta^3}\big) \\ \phi &= \mu\,\mathrm{Re}(\zeta^1 \wedge \zeta^2 \wedge \zeta^3) \end{aligned} \tag{A.31}$$

for some $\mu > 0$. Alternatively, writing $\zeta^1 = e^1 + \mathrm{i}e^4$, $\zeta^2 = e^2 \pm \mathrm{i}e^5$, and $\zeta^3 = e^3 \pm \mathrm{i}e^6$ for a real basis e^i of V^*, one can write this in the normal forms

$$\begin{aligned} \omega &= e^1 \wedge e^4 + e^2 \wedge e^5 + e^3 \wedge e^6 \\ \phi &= \mu\big(e^1 \wedge e^2 \wedge e^3 - e^1 \wedge e^5 \wedge e^6 \mp e^2 \wedge e^6 \wedge e^4 \mp e^3 \wedge e^4 \wedge e^5\big). \end{aligned} \tag{A.32}$$

If ϕ is of type (3), then there is a basis e^i of V^* such that

$$\phi = e^1 \wedge e^5 \wedge e^6 + e^2 \wedge e^6 \wedge e^4 + e^3 \wedge e^4 \wedge e^5. \tag{A.33}$$

The condition $\omega \wedge \phi = 0$ then implies that

$$\omega = a_{ij}\, e^i \wedge e^{j+3} + b_{ij}\, e^{i+3} \wedge e^{j+3} \tag{A.34}$$

where $a = {}^t a = (a_{ij})$ has $\det(a) \neq 0$ and $b = -{}^t b = (b_{ij})$ is arbitrary. Again, by making a basis change in the e^i that preserves the form of ϕ, one can suppose that $a = \mathrm{diag}(1, \pm1, \pm1)$. Then by replacing e^i by $e^i + s^i_j\, e^{j+3}$ for appropriate s^i_j satisfying $s^i_i = 0$, one can get rid of the b_{ij}, leaving the two normal forms

$$\begin{aligned} \omega &= e^1 \wedge e^4 \pm e^2 \wedge e^5 \pm e^3 \wedge e^6 \\ \phi &= e^1 \wedge e^5 \wedge e^6 + e^2 \wedge e^6 \wedge e^4 + e^3 \wedge e^4 \wedge e^5. \end{aligned} \tag{A.35}$$

or, alternatively,

$$\begin{aligned} \omega &= e^1 \wedge e^4 + e^2 \wedge e^5 + e^3 \wedge e^6 \\ \phi &= e^1 \wedge e^5 \wedge e^6 \pm e^2 \wedge e^6 \wedge e^4 \pm e^3 \wedge e^4 \wedge e^5. \end{aligned} \tag{A.36}$$

By entirely similar arguments, one sees that when ϕ is of type (4), one can find a basis e^i of V^* such that

$$\begin{aligned} \omega &= e^1 \wedge e^4 + e^2 \wedge e^5 + e^3 \wedge e^6 \\ \phi &= (e^1 \wedge e^2 \pm e^4 \wedge e^5) \wedge e^3, \end{aligned} \tag{A.37}$$

(the two signs give inequivalent normal forms); when ϕ is of type (5), one can arrange

$$\begin{aligned} \omega &= e^1 \wedge e^4 + e^2 \wedge e^5 + e^3 \wedge e^6 \\ \phi &= e^1 \wedge e^2 \wedge e^3; \end{aligned} \tag{A.38}$$

and when ϕ is of type (6), one can arrange

$$\begin{aligned} \omega &= e^1 \wedge e^4 + e^2 \wedge e^5 + e^3 \wedge e^6 \\ \phi &= 0. \end{aligned} \tag{A.39}$$

This completes the list of normal forms.

This analysis provides the stabilizer subgroups $G(\omega, \phi)$ of the various normal forms (ω, ϕ). For example, in the case of (A.28), the stabilizer subgroup is isomorphic to $\mathrm{SL}(3, \mathbb{R})$, in the case of (A.32) the stabilizer subgroup is isomorphic to $\mathrm{SU}(3)$ if the upper sign is taken and $\mathrm{SU}(1, 2)$ if the lower sign is taken, while in the case of (A.36), the stabilizer subgroup is isomorphic to the semidirect product of $\mathbb{R}^5$ with either $\mathrm{SO}(3)$ (upper sign) or $\mathrm{SO}(1, 2)$ (lower sign).

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