Persistent Cohomology Operations

by

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Department of Mathematics
Duke University

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Sayan Mukherjee

Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
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Abstract
([Mathematics])

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Abstract

The work presented in this dissertation includes the study of cohomology and cohomological operations within the framework of Persistence. Although Persistence was originally defined for homology, recent research has developed persistent approaches to other algebraic topology invariants. The work in this document extends the field of persistence to include cohomology classes, cohomology operations and characteristic classes.

By starting with presenting a combinatorial formula to compute the Stiefel-Whitney homology class, we set up the groundwork for Persistent Characteristic Classes. To discuss persistence for the more general cohomology classes, we construct an algorithm that allows us to find the Poincaré Dual to a homology class. Then, we develop two algorithms that compute persistent cohomology, the general case and one for a specific cohomology class. We follow this with defining and composing an algorithm for extended persistent cohomology.

In addition, we construct an algorithm for determining when a cohomology class is decomposible and compose it in the context of persistence. Lastly, we provide a proof for a concise formula for the first Steenrod Square of a given cohomology class and then develop an algorithm to determine when a cohomology class is a Steenrod Square of a lower dimensional cohomology class.

To all of us who are described as persistent.

Contents

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List of Abbreviations and Symbols

Symbols

Put general notes about symbol usage in text here. Notice this text is double-spaced, as required.

$\mathcal{X}$	A topological space.
$\mathcal{K}$	A simplicial complex.
$\mathcal{M}$	A manifold.
$\mathrm{Sd}(K)$	The first barycentric subdivision of K.
$St(\sigma)$	The star of a simplex, σ.
$Vert(\sigma)$	The vertex set of a simplex, σ.
$b(\sigma)$	The barycentre of a simplex, σ.
∂	The boundary operator.
$\mathcal{C}_p$	A p dimensional chain group.
$\mathcal{C}^q$	A q dimensional cochain group.
H_p	The p dimensional homology group.
H^q	The q dimensional cohomology group.
γ	A homology class for a given dimension H_p.
Γ	A cohomology class for a given dimension H^q.
σ^*	The unique cochain that evaluates one on σ and zero elsewhere.
$\sigma * \tau$	The join of two simplices, σ and τ.

$\sigma \cap \tau$	The intersection of two simplices, σ and τ.
$\Gamma * \Lambda$	The cup product of two cohomology classes, Γ and Λ.
$Sq^q(\Gamma)$	The q dimensional Steenrod Square of Γ.
$\mathbb{RP}^2$	The real projective space of dimension 2.
T	The 2 dimensional torus.
$low(j)$	The index of the row that contains the lowest one of column j.

Acknowledgements

At each stage of this process, I have been incredibly grateful to a lot of people. I was brought up on the principle that nobody has to do anything for anyone, so always remember your pleases and thank yous. Since this is my one chance to formally thank everyone, if I forgot anyone please accept my sincerest apologies and know that I am very grateful to you.

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1

Introduction

In 2002, Edelsbrunner, Letscher, and Zomorodian first developed the area of Computational Topology known as Persistent Homology. In the last 9 years, Persistent Homology has been used in many different applications such as protein docking (AEHW06), image compression (CI06), speech pattern analysis (BK), MRI analysis (C^+08), and satellite network coverage (dSG07), to name a few. With each new application, the mathematical and computational tools have evolved to include other distances (M0́9), intersection homology (Ben08), and now cohomology.

The idea behind Persistent Homology is simple, yet the details of computing Persistent Homology requires some background in algebraic topology. We save the details and definitions for Chapter 2, however we hope to give the intuition here. If we were to look at a circle, take a sampling and add noise, then we would get something that looks like Figure 1.1.

Our eyes can tell that these points might be a sampling of a circle. What if the circle were embedded in 15 dimensional space? While we could not picture it, Persistent Homology tells us it is there. By using a chosen metric, we connect points that are within a given distance of each other and form higher dimensional simplices.

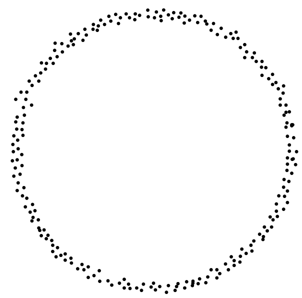

FIGURE 1.1: A circle in $\mathbb{R}^2$ that has been sampled with noise added in

At each distance, we calculate the homology of the resulting space, which equates to counting the number of holes of each dimension. The zeroth homology class refers to the number of components.

As the distance increases, the homology changes. Persistence measures these changes. It will take a relatively short distance to connect all the points and form a circle, or a 1 dimensional hole. Similarly, it will require a relatively large distance to close the hole. Persistence tells us that the 1 dimensional hole exists for a large range of distances, thus providing us with the structure of our data.

The idea of tracking how something changes throughout distances can be applied to many different algebraic topological invariants. This dissertation is similar to that of Bendich in that we applied a persistent approach to significant concepts in the pure study of Algebraic Topology. Bendich developed persistent versions of local and intersection Homologies. This dissertation develops persistent approaches to cohomology classes, characteristic classes, and cohomology operations, specifically Steenrod Squares. Moving away from homology and applying persistence to cohomology allows us to harness the power of cohomology operations. As written in (MT68):

> In the past two decades, cohomology operations have been the center of a major area of activity in algebraic topology. This technique for supplementing and enriching the algebraic structure of the cohomology

> ring led to important progress, both in general homotopy theory and in specific geometric applications. For both theoretical and practical reasons, much analysis has been made of the formal properties of families of operations.

Creating a persistent approach to some of these operations allows us to begin integrating the major developments in cohomology operations to the future of data analysis.

This document begins with the necessary background of Persistent Homology in Chapter 2. Following this, Chapter 3 gives the necessary background for Cohomology and provides a combinatorial formula for the Stiefel-Whitney Homology class of an Euler mod 2 Space. Chapter 4 begins the contributions to this field by defining a procedure for finding the Poincaré Dual of any homology class. Chapter 5 then defines CoPersistence and develops two algorithms; one computes the persistent cohomology of a given simplicial complex, while the other finds the point in the filtration where a specific cohomology class dies. Chapter 6 extends these definitions much in the same way as Extended Persistence extends the definitions of Persistence. Chapter 7 constructs an algorithm for computing when a cohomology class is decomposable as a cup product of two lower dimensional cohomology classes and discusses it in the context of persistence. Chapter 8 develops an algorithm for determining when a cohomology class is the Steenrod Square of a lower dimensional cohomology class and provides a concise formula for Sq^1 of a cohomology class. The necessary background is given within each section.

Because we chose to work with Stiefel-Whitney classes, which are defined for $\mathbb{Z}_2$ coefficients, all homology and cohomology is defined with $\mathbb{Z}_2$ coefficients. Since both persistence homology and cohomology operations can be calculated over any finite field, we leave a short discussion of the general case for the Appendix.

2

Persistent Homology

To understand fully the context in which this new work adds to the current research in Computational Topology, I will present a brief background of the major advances in Persistent Homology. This chapter is divided into four major sections: Homology on Simplicial Complexes, Persistent Homology for manifolds, Extended Persistent Homology, and Persistence Algorithm. The mathematical objects necessary to develop these sections will be defined and where necessary illustrations will be provided to reinforce the intuition. Because homology of simplicial complexes is more computable by nature, we begin by defining it so that we may discuss Persistent Homology for Manifolds and Extended Persistence. We conclude this chapter by returning to the Persistence Homology algorithm calculated on simplicial complexes.

2.1 Homology on Simplicial Complexes

There are several ways to calculate the homology of a topological space. Due to the computational nature of the Persistent Homology algorithm, we define homology on Simplicial complexes.

Definition 1. *A **simplicial complex**, K, in $\mathbb{R}^N$ is a collection of simplices in $\mathbb{R}^N$ such that every face of a simplex of K is in K and the intersection of any two simplicies of K is empty or a face of each of them.*

A **n-simplex**, σ, spanned by a geometrically independent set $\{a_0, a_1, \ldots, a_n\}$ in $\mathbb{R}^N$, is the set of all points x of $\mathbb{R}^N$ such that

$$x = \sum_{i=0}^{n} t_i a_i, \text{ where } \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i.$$

To denote an n-simplex spanned by the vertices $\{v_0, v_1, \ldots, v_n\}$, we write $\langle v_0, v_1, \ldots, v_n \rangle$.

For each n, we define $\mathcal{C}_n$ to be the $\mathbb{Z}_2-$Vector space, whose generators are the n simplices of K. There is a boundary map $\partial_n : \mathcal{C}_n \to \mathcal{C}_{n-1}$ defined as follows. Let σ be the n simplex, $\langle v_0, v_1, \ldots, v_n \rangle$. The boundary map on σ is:

$$\partial_n(\langle v_0, v_1, \ldots, v_n \rangle) = \sum_{i=0}^{n} \langle v_0, v_1, \ldots, \hat{v}_i, \ldots, v_n \rangle.$$

The boundary map extends linearly to all of $\mathcal{C}_n$. An easy calculation shows that $\partial_n(\partial_{n+1}(\sigma)) = 0$ for any simplex and therefore on all of $\mathcal{C}_n$. The sequence of chain groups on our d-dimensional simplicial complex K is then:

$$\mathcal{C}_d \xrightarrow{\partial_d} \mathcal{C}_{d-1} \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_2} \mathcal{C}_1 \xrightarrow{\partial_1} \mathcal{C}_0 \xrightarrow{\partial_0} 0$$

Definition 2. *Elements of $\mathcal{C}_n$ are called the n-chains. The kernel of $\partial_n : \mathcal{C}_n \to \mathcal{C}_{n-1}$ is called the vector space of n-**cycles**, denoted $\mathcal{Z}_n$. The image of $\partial_{n+1} : \mathcal{C}_{n+1} \to \mathcal{C}_n$ is called the vector space of n-**boundaries**, denoted by $\mathcal{B}_n$. Since, $\partial_n \circ \partial_{n+1} = 0$, each boundary of a $n+1$ chain, is automatically called a cycle, i.e. $\mathcal{B}_n \subset \mathcal{Z}_n$. The nth **homology group** of K is defined to be*

$$H_n(K) = \mathcal{Z}_n / \mathcal{B}_n$$

In the next couple of sections we discuss homology in the context of smooth manifolds and Morse functions. Since every smooth manifold can be triangulated into a simplicial complex, and homology is independent of triangulations, defining homology for simplicial complexes will suffice. Futhermore, to develop the intuitive understanding of Persistent Homology for Manifolds and Extended Persistence, it is helpful to consider the notion that homology counts the number of holes in each dimension.

2.2 Persistent Homology for Manifolds

At the fundamental basis of the theory for Persistence on Manifolds, we have two crucial concepts, a smooth manifold and a Morse function. We assume the reader is familiar with manifolds and Morse functions. A good reference for these are in (Mil63). In most cases, when computing Persistent homology on manifolds, the Morse function chosen is a height function along a given normal vector. This gives a nice geometric way of seeing persistence as a shape measure on the manifold by looking at the change of homology over a sequence of sublevel sets.

Suppose we have a smooth, closed d dimensional manifold, $\mathcal{M}$. Let f be a Morse function on $\mathcal{M}$, such that it has n distinct critical values, $a_1 < a_2 < \cdots < a_n$. We can interleave real numbers between these critical values. That is,

$$a_1 < b_1 < a_2 < \cdots < a_n < b_n \quad \text{where} \quad b_i = a_i + \epsilon$$

for a sufficiently small ϵ.

Definition 3. *The* ***sublevel set*** $\mathcal{M}_i = f^{-1}(-\infty, b_i]$*. It is a submanifold of* M *with boundary equal to the level set* $f^{-1}(b_i)$*.*

Since $\mathcal{M}_i \subset \mathcal{M}_{i+1}$, the sequence of sublevel sets form a filtration of submanifolds of $\mathcal{M}$. The main theorem attributed to Morse states that if $f : \mathcal{M} \to \mathbb{R}$ is a smooth function with only a single, nondegenerate critical point x of index λ located between the sublevel sets $\mathcal{M}_c$ and $\mathcal{M}_{c+1}$, then $\mathcal{M}_{c+1}$ has the homotopy type of $\mathcal{M}_c$ with a λ-cell attached. (Mil63) Thus, in the sequence of homology groups along with the maps induced by inclusion

$$0 = H_p(\mathcal{M}_0) \to H_p(\mathcal{M}_1) \to \cdots \to H_p(\mathcal{M}_n) = H_p(\mathcal{M})$$

only a single change in homology can occur at each step. This is either an increase in rank by 1, requiring $\lambda = p$, a decrease in rank by 1, requiring $\lambda = p+1$, or no change if $\lambda \neq p, p+1$. Persistence watches how the homology classes change throughout each sequence for each dimension, p. To do this, we examine the images of the homology classes through the maps of the sequence.

Definition 4. *Let $\phi_p^{i,j} : H_p(\mathcal{M}_i) \to H_p(\mathcal{M}_j)$ be induced by the inclusion $\mathcal{M}_i \subset \mathcal{M}_j$, $i \leq j$. We say a p-dimensional homolog class, α, is* ***born*** *at $\mathcal{M}_i$ if $\alpha \in H_p(\mathcal{M}_i)$ but $\alpha \notin Im(\phi_p^{i-1,i})$. We say that this homology class, α,* ***dies*** *entering $\mathcal{M}_j$ if j is the lowest index in which $\exists$ a $\beta \in H_p(\mathcal{M}_{i-1})$ such that $\phi_p^{i-1,j}(\beta) = \phi_p^{i,j}(\alpha) \in H_p(\mathcal{M}_j)$. The* ***persistence*** *of α, $pers(\alpha) = j - i$ or $b_j - b_i$.*

When a homology class α dies entering $\mathcal{M}_j$, every class in the entire coset $[\alpha]$ also dies entering $\mathcal{M}_j$. The number of p dimensional birth events at $\mathcal{M}_i$ which die entering $\mathcal{M}_j$ is the rank of

$$P_p^{i,j}(\mathcal{M}) = \frac{Im(\phi_p^{i,j-1}) \cap Ker(\phi_p^{j-1,j})}{Im(\phi_p^{i-1,j-1}) \cap Ker(\phi_p^{j-1,j})}$$

$P_p^{i,j}$ is called the *pair group* for a given manifold, $\mathcal{M}$ and filtration. (Ben08)

We define the **persistence pair** to be (i, j) or (b_i, b_j) if and only if there exists a p dimensional homology class, α, such that α is born at $\mathcal{M}_i$ and dies at $\mathcal{M}_j$. We plot the persistence values of a certain dimension, p, in a persistence diagram of matching dimension to obtain a graphical representation of the persistence of our manifold through the chosen filtration. For each nonzero pair group, we graph a point of multiplicity rank($P_p^{i,j}$) in the x, y plane. Since every homology class must die after it is born, all persistence pairs are necessarily plotted in the first quadrant above the $y = x$ line. The further away from this diagonal the points are, the longer that respective homology class persisted through the filtration. Every point graphed as (i, j) corresponds to an *inessential* homology class, while points graphed as (i, ∞) correspond to an *essential* homology class.

Actually, there are two ways of plotting a persistence diagram. The first is to have a unique diagram for every dimension $0 \leq p \leq d$. The second is to plot the persistence pairs of all dimensions on the same plot, distinguishing between dimensions via different markers or colors. We demonstrate both of these plots, the first in this section, and the second in the Extended Persistence section.

2.2.1 Example 1

Suppose our manifold, $\mathcal{M} \subset \mathbb{R}^2$ is the graph of a Morse function, $f(x)$. An example that has 6 critical points is illustrated in Figure 2.1.

We interleave regular values between the critical values of f. Using these values, we define the sublevel sets of $\mathcal{M}$ at each point. This is illustrated below with lines indicating the real values. Everything in $\mathcal{M}$ below the line at $y = b_i$ is the sublevel set $\mathcal{M}_i$.

Below the line at b_0, the sublevel set $\mathcal{M}_0$ consists of a single connected component, so its homology is $H_0(\mathcal{M}_0) = \mathbb{Z}_2$. Because there are no 1-dimensional cycles, i.e. parts of $\mathcal{M}$ homotopic to S^1, nor is the dimension of $f(x)$ greater than 1, all i $H_p(\mathcal{M}_i) = 0$, $p \geq 1$. In other words, its betti numbers are $\beta_0 = 1$ and $\beta_p = 0$ for $p \geq 1$. When we pass the critical value a_1, $\mathcal{M}_1$ has two connected components and $\beta_0 = 2$. When we get to height b_2, just above a_2, we have 3 connected components. Our $\beta_0 = 3$. Passing a_3, the sublevel set $\mathcal{M}_3$ consists of only two components so now

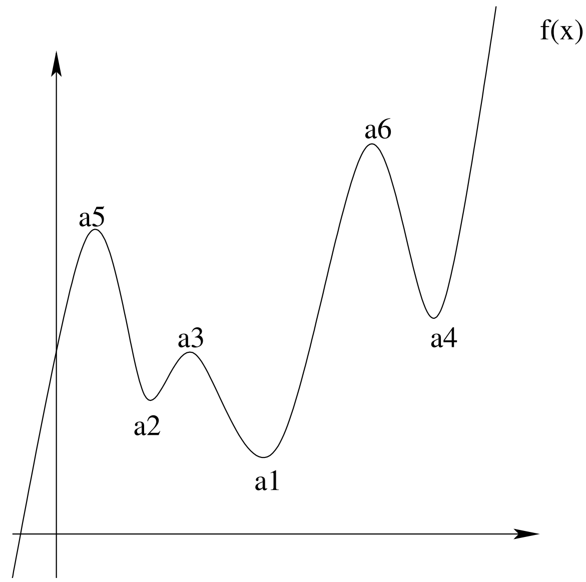

FIGURE 2.1: A graph of a Morse function with 6 nondegenerate critical points.

$\beta_0 = 2$. The component that was born at a_2 has merged with the component born at a_1. As a rule, we choose the elder of the two as the homology representative. In terms of persistence, the 0 dimensional homology class born at a_2 is in the image of the 0 dimensional homology class born at a_1 in $H_0(\mathcal{M}_3)$. Thus, we say that homology class was born at a_2 and died at a_3, and we have the persistence of a_2 being $pers(a_2) = a_3 - a_2$ and a persistence pair (a_2, a_3). We plot that persistence pair in our diagram.

At $\mathcal{M}_4$ there are 3 connected components, so $\beta_0 = 3$ and β_i remains 0 for $i \geq 1$. A component is born at a_4. When we pass a_5, however, we again merge two components. The component that was born at a_1 merges into the component that had existed in $\mathcal{M}_0$. The homology class born at a_0 gets mapped into the image of the homology class born at $-\infty$, so that homology class, represented by a_1 dies at a_5. Its persistence, $pers(a_1) = a_5 - a_1$, and we plot the point (a_1, a_5). When we pass a_6, the component born at a_4 merges with the component born at $-\infty$, thus, the homology class born at a_4 dies at a_6. Its persistence is $pers(a_4) = a_6 - a_4$ and we plot

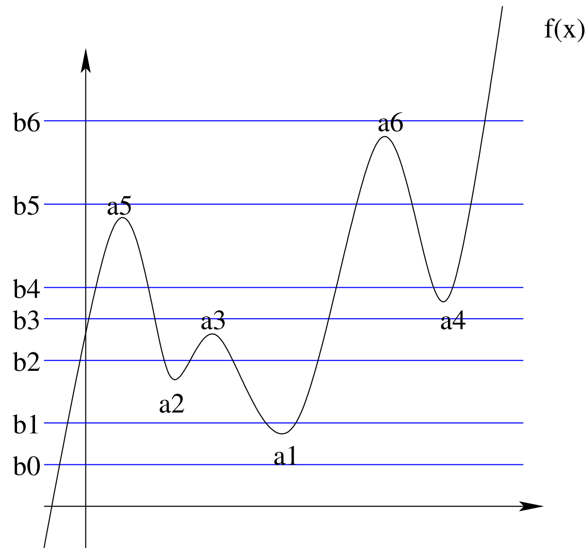

FIGURE 2.2: The Morse function with horizontal lines drawn at each regular value b_i. The union of the parts of $\mathcal{M}$ that are below each line is the corresponding sublevel set $\mathcal{M}_i$.

a persistence pair (a_4, a_6). As one can easily see by this example, persistence pairs are not necessarily comprised of adjacent critical points. The resulting 0 dimensional persistence diagram for this example is given in Figure 2.4.

The point on the y-axis of the diagram above $a6$ symbolizes a point at infinity, representing the persistence of the 0 dimensional homology class that is born at $-\infty$ and never dies, therefore is never paired.

2.3 Extended Persistence

We continue our discussion of Persistent Homology by looking closely at Extended Persistent Homology. One of the main reasons that Extended Persistence was developed is to pair all critical points; this is useful for certain applications.

For Extended Persistence, we have a closed manifold $\mathcal{M}^d$ and a Morse function $f : \mathcal{M}^d \to \mathbb{R}$. As before, since f is Morse, it has n non-degenerate critical points

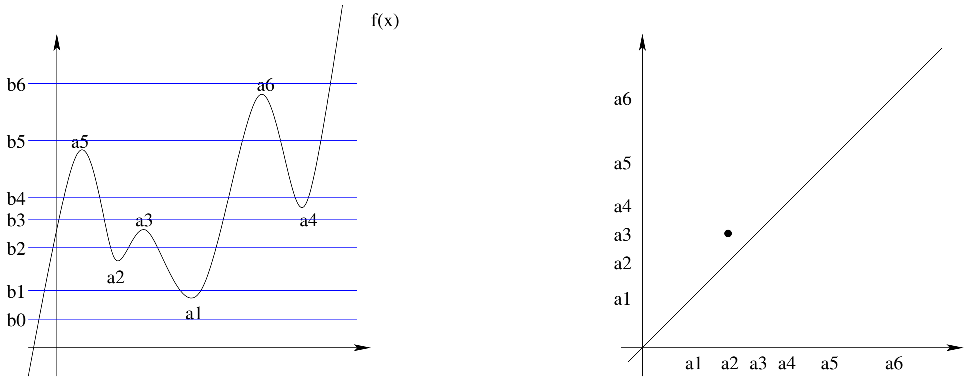

FIGURE 2.3: For the homology class born at a_2 which dies at a_3, we plot the corresponding point into the 0-dimensional persistence diagram.

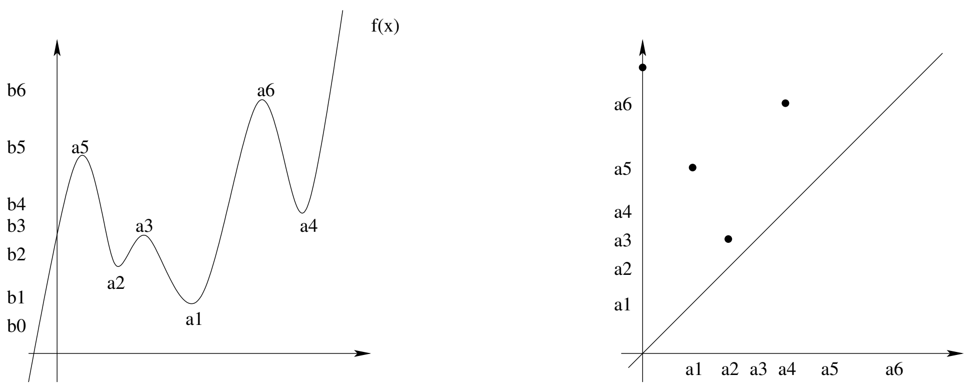

FIGURE 2.4: Side by side, the morse function with 6 nondegenerate critical points and its corresponding final 0 dimensional Persistence Diagram.

$a_1 < a_2 < \cdots < a_n$ for some n. We again interleave real numbers

$$a_1 < b_1 < \cdots < a_n < b_n \quad \text{such that} \quad b_i = a_i + \epsilon$$

for a sufficiently small ϵ, and define the sublevel sets $\mathcal{M}_i = f^{-1}(-\infty, b_i]$ for each b_i. In addition, we define superlevel sets.

Definition 5. *For a Morse function $f : \mathcal{M} \to \mathbb{R}$, the* ***superlevel set****, $\mathcal{M}^i := f^{-1}[b_i, \infty)$. Like the sublevel set, it is a manifold with boundary. In fact, $\partial\mathcal{M}_i = \partial\mathcal{M}^i = f^{-1}(b_i)$, the level set at b_i.*

We again have a filtration of manifolds, $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_n$ that gives us the sequence $0 = H_p(\mathcal{M}_0) \to H_p(\mathcal{M}_1) \to \cdots \to H_p(\mathcal{M}_n) = H_p(\mathcal{M})$ for all p. Since $\mathcal{M}$ is compact, we can use Poincaré Duality, i.e. $H_p(\mathcal{M}) \cong H^{d-p}(\mathcal{M})$, to get an extended sequence with maps induced by inclusion:

$$
\begin{aligned}
0 = H_p(\mathcal{M}_0) \to H_p(\mathcal{M}_1) \to \cdots \to H_p(\mathcal{M}_n) = H_p(\mathcal{M}) \cong H^{d-p}(\mathcal{M}) = \\
= H^{d-p}(\mathcal{M}_n) \to H^{d-p}(\mathcal{M}_{n-1}) \to \cdots \to H^{d-p}(\mathcal{M}_1) \to H^{d-p}(\mathcal{M}_0) = 0
\end{aligned}
$$

When we think about the geometry, $M_i = \overline{M - M^i}$, so in reality, we have the sequence:

$$
\begin{aligned}
0 = H_p(\mathcal{M}_0) \to H_p(\mathcal{M}_1) \to \cdots \to H_p(\mathcal{M}_n) = H_p(\mathcal{M}) \cong H^{d-p}(\mathcal{M}) = \\
= H^{d-p}(\overline{\mathcal{M} - \mathcal{M}^n}) \to H^{d-p}(\overline{\mathcal{M} - \mathcal{M}^{n-1}}) \to \cdots \\
\cdots \to H^{d-p}(\overline{\mathcal{M} - \mathcal{M}^1}) \to H^{d-p}(\overline{\mathcal{M} - \mathcal{M}^0}) = 0
\end{aligned}
$$

Since Lefschetz Duality states $H^{d-p}(\overline{\mathcal{M} - \mathcal{A}}) \cong H_p(\mathcal{M}, \mathcal{A})$, for $\mathcal{A}$ a sub-manifold of $\mathcal{M}$, we obtain the sequence.

$$
\begin{aligned}
0 = H_p(\mathcal{M}_0) \to H_p(\mathcal{M}_1) \to \cdots \to H_p(\mathcal{M}_n) \to \\
H_p(\mathcal{M}, \mathcal{M}^n) \to H_p(\mathcal{M}, \mathcal{M}^{n-1}) \to \cdots \to H_p(\mathcal{M}, \mathcal{M}^1) \to H_p(\mathcal{M}, \mathcal{M}^0) = 0
\end{aligned}
$$

We define extended persistence using this sequence much in the same way as we defined regular persistence. However, it is important to recognize that a regular homology class born at a sublevel might die entering a relative homology group. The other definitions, such as persistence pairs and the persistence diagram are also defined as they were in Persistence for Manifolds. Since there are apparent symmetries in the construction of Extended Persistence, we have the following symmetry formulas (EH09) for our persistence diagram.

$$\mathrm{Ord}_p(f) = \mathrm{Rel}^{\top}_{d-p}(f)$$

$$\mathrm{Ext}_p(f) = \mathrm{Ext}^{\top}_{d-p}(f)$$

$$\mathrm{Rel}_p(f) = \mathrm{Ord}^{\top}_{d-p}(f)$$

where $^{\top}$ means reflection across the $y = x$ axis.(EH09) Ord stands for the set of ordinary persistence pairs, those that are born and die in the first half of the sequence. Ext stands for the set of extended persistence pairs, those that are born in the first half of the sequence and die as a relative homology class in the second half of the sequence. Lastly, Rel stands for the set of the relative homology classes that are born and die in the second half of the sequence. Because of the obvious construction and correlation between the sub- and super- level sets of $\mathcal{M}$, we have further symmetry formulas that relate f and $-f$.

$$\mathrm{Ord}_p(f) = \mathrm{Ord}^{\mathrm{R}}_{d-p-1}(-f)$$

$$\mathrm{Ext}_p(f) = \mathrm{Ext}^{\mathrm{O}}_{d-p}(-f)$$

$$\mathrm{Rel}_p(f) = \mathrm{Rel}^{\mathrm{R}}_{d-p+1}(-f)$$

where $^{\mathrm{R}}$ is the reflection across the $y = -x$ axis, and $^{\mathrm{O}}$ is the central reflection by $180°$ which takes $(a, b) \mapsto (-a, -b)$. (EH09)

The symmetries given in the above formulas were key in an application that modeled protein docking. Specifically, they were used to measure local protrusions or cavities in a protein. In addition, these symmetries have been an important part of defining topological feature size.

Example

Suppose our compact manifold is the 2 dimensional torus standing upright in $\mathbb{R}^3$ as depicted in Figure 2.5. Let f be our height function along the z direction. f has 6 critical points, labeled as $a1, a2, a3, a4, a5$ and $a6$ in Figure 2.5. The regular values have been labeled and the level sets have been indicated at heights $b1, b2, b3, b4$, and $b5$. $b6$ is not illustrated in the Figure as it is above the torus. The sublevel $\mathcal{M}_{b1}$ is the part of the torus below the first level set at $b1$ and the corresponding superlevel set $\mathcal{M}^{b1}$ is everything above that first level set. The sublevel set $\mathcal{M}_{b2}$ is everything below the second level set while the superlevel set $\mathcal{M}^{b2}$ is everything above it. The sublevel sets for $\mathcal{M}_{b3}$, $\mathcal{M}_{b4}$, $\mathcal{M}_{b5}$ and the superlevel sets $\mathcal{M}^{b3}$, $\mathcal{M}^{b4}$, and $\mathcal{M}^{b5}$ are defined similarly. The sublevel set associated to $b6$ is the entire torus, while the superlevel set is the empty set.

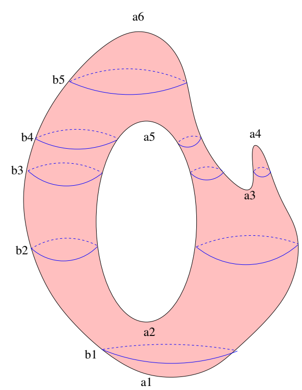

FIGURE 2.5: Two dimensional torus in $\mathbb{R}^3$. The critical points for the Morse function are labeled $a1, a2, a3, a4, a5$, and $a6$. The level sets at the regular values are indicated with horizontal circular cross-sections and labeled with $b1, b2, b3, b4$ and $b5$. $b6$ is not shown as it is above the torus.

We begin computing Extended Persistence. Immediately, when we pass the critical point $a1$, a 0-dimensional homology class is born. When we pass the critical point $a2$, a 1 dimensional homology class is born indicated in Figure 2.6 with a line circumscribing the torus through the critical points $a1$ and $a2$. At $a3$ another 1 dimensional homology class is born illustrated in Figure 2.6 with the line circumscribing the torus connecting $a1$ and $a3$. When we pass $a4$ the 1 dimensional cycle born at $a3$ becomes contractable to a point, thus dies. After the height $a5$ a 1 dimensional homology class is born illustrated by the path around the hole in the center of the torus. When we pass $a5$ a 2 dimensional homology class is born. If we were computing regular Persistent Homology, we would stop here with an inessential persistence pair, $(a_3, a4)$ and the four essential homology classes still present at the end of our filtration.

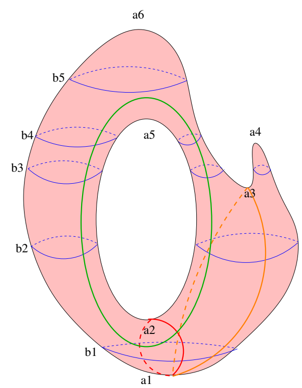

FIGURE 2.6: The one dimensional essential homology classes are indicated with the lines circumscribing 1 cycles of the Torus.

For this example, we will plot all diagrams on the same plot to highlight the symmetry formulas from above. When we continue with extended persistence, $\mathcal{M} - \mathcal{M}^{b6} = \mathcal{M} - \emptyset$, so $(\mathcal{M}, \mathcal{M}_{b6}) = \mathcal{M}$. Continuing down the torus, since $\mathcal{M}^{b5}$ is homotopically equivalent to a point, when we pass to $(\mathcal{M}, \mathcal{M}_{b5})$, the 0 dimensional homology class born at $a1$ dies as a relative homology class here. Thus, we pair $a1$ with $a4$, and plot the persistence pair $(a1, a4)$ in our persistence diagram, as a 0 dimensional homology class indexed with a square plotted point. At $(\mathcal{M}, \mathcal{M}_{b4})$ we have passed $a5$ and our space is homotopically equivalent to a circle. The 1 dimensional homology class that was born at $a2$ dies at $a5$. Specifically, the 1-dimensional homology class that was born at $a2$ is equivalent to the boundary of $(\mathcal{M}, \mathcal{M}_{b4})$, so it is in the kernel of the map from $H(\mathcal{M}, \mathcal{M}_{b5})$ to $H(\mathcal{M}, \mathcal{M}_{b4})$. We plot the pair $(a2, a5)$ into the combined persistence diagram as a one dimensional extended persistence pair indexed with a circular point. When we get to $(\mathcal{M}, \mathcal{M}_{b3})$, a 1 dimensional relative cycle is born. Once we pass to $a3$ on the way down, that one dimensional relative cycle dies. We plot the relative persistence pair $(a4, a3)$ in our persistence diagram as a diamond point. When we pass $a2$, we have lost the other 1 dimensional homology class in the relative homology of this space. Thus, $a5$ dies at $a2$ and we graph the pair $(a5, a2)$ into our joint persistence diagram. Lastly, when we get to a height of the Morse function below the critical point $a1$, we are essentially looking at the relative pair $(\mathcal{M}, \mathcal{M}) = \emptyset$, so our 2 dimensional cohomology class that was born at $a6$ dies at $a1$. We graph the point $(a6, a1)$ with a triangle to indicate the 2 dimensional class. Our extended persistence diagram is illustrated in Figure 2.7.

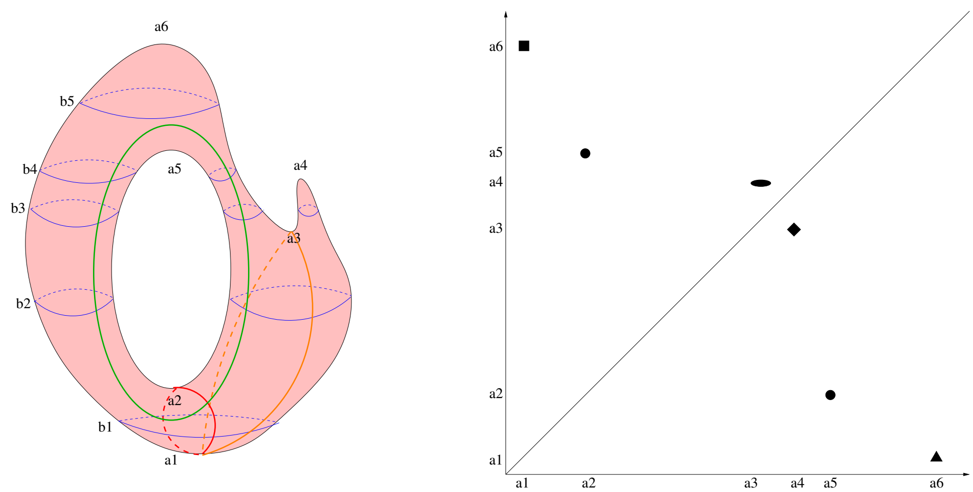

FIGURE 2.7: Side by side, the torus on which we compute Extended Persistence and the final persistence diagram. The single graph has all three persistence diagrams together. The oval indicates the ordinary persistence pair. The square is to index the extended 0 dimensional diagram, the circles are for the extended 1 dimensional diagram, and the triangle represents the extended 2 dimensional diagram. The diamond represents the 1 dimensional relative pair.

2.4 Persistence Algorithm

We continue our discussion by transitioning to the algorithm that computes Persistent Homology for Simplicial Complexes. Suppose we have a simplicial complex K of dimension d. Suppose further that we have a filtration on K, that is a sequence of subcomplexes of K such that

$$\emptyset \subset K_0 \subset K_1 \subset \cdots \subset K_n = K$$

where each K_i is a proper subcomplex. This means that if K_i is a subcollection of a simplicial complex, K, that contains all faces of its simplices, then K_i is a *subcomplex* of K. The filtration places a natural partial ordering on the simplices of K. A total ordering can be obtained by choosing an ordering on the simplicies that are present in K_i but not in K_{i-1}. The persistence algorithm now goes as follows. Given a total ordering of the simplices of K, we create the **Incidence matrix**, I, defined as

$$I(i,j) = \begin{cases} 1 & \text{if the } i\text{th simplex is a codimension 1 face of the } j\text{th simplex} \\ 0 & \text{otherwise} \end{cases}$$

The algorithm is restricted entirely to column operations of the Incidence matrix. Let $low(j)$ be the row number of the lowest non-zero entry in column j and if the entire column is zero, we set $low(j) = 0$. We call the Incidence matrix *reduced* if each row has at most a single entry that corresponds to the lowest 1 of a column. In order to reduce I, we perform column operations in $\mathbb{Z}_2$ arithmetic from left to right according to the following algorithm

```
for  j = 1  to  n  do
    while  ∃j' < j with low(j') = low(j) ≠ 0  do
        add column j' to column j
    endwhile
```

```
endfor.
```

Because adding column j' to j decreases $low(j)$ this process will eventually terminate.

We can read the persistence homology pairs and Betti numbers off of the reduced matrix, R. For each dimension n, we scan the columns associated to n-simplices to find their lowest ones. Suppose j is such a column and suppose further $i = low(j)$. Then, we pair i with j since the homology class born at the level that the simplex associated to row i enters in the filtration, σ_i, dies at the level of the filtration at which σ_j enters. If j does not have a lowest one, i.e. it corresponds to a zero column, then the chain of simplices associated to column j of the reduced matrix, R, is positive, meaning that a homology class is born with the addition of σ_j into the filtration. We scan the row j. If there exists a column, k, such that $j = low(\sigma_k)$, the class born at σ_j dies with the chain comprising the k^{th} column of R; we pair (j, k). If, on the other hand, row j does not contain the lowest one of some column, then an essential homology class is born at the level of the filtration associated to σ_j. If we want the full representation of our persistent homology class, we keep track of which columns have been added to which other columns to obtain column j in the reduced matrix. The chain of n simplicies corresponding to the columns added to column j is the full representation of our persistent homology class. The Pairing lemma (EH10), given below, shows that the specific reduction process is not important for calculating persistence.

Lemma 6 (Pairing Lemma)**.** *The pairing between rows and columns defined by the lowest ones in the reduced matrix does not depend on the Incidence matrix.*

2.4.1 Example

Suppose our simplicial complex, K, and its filtration is given in Figure 2.8:

Given the partial ordering defined from the filtration, we choose a total ordering. The corresponding Incidence matrix is:

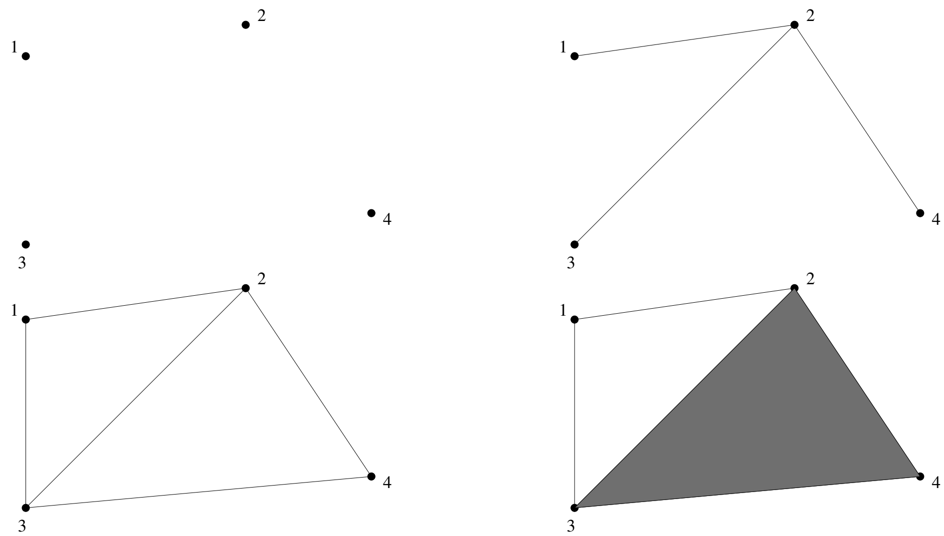

FIGURE 2.8: The four pictures depict four stages of the filtration of the final simplicial complex. The first stage of the filtration is the union of four vertices. The second level of the filtration is the union of the four vertices and 3 one dimensional simplices. The next level of the filtration is the union of the four vertices and 4 one simplices. The final simplical complex is the union of the 4 vertices, 4 one-simplices and a single 2 simplex.

	1	2	3	4	1, 2	2, 3	2, 4	1, 3	3, 4	2, 3, 4
1					1			1		
2					1	1	1			
3						1		1	1	
4							1		1	
1, 2										
2, 3										1
2, 4										1
1, 3										
3, 4										1
2, 3, 4										

We begin the algorithm by finding the lowest ones of each column. For the zero dimensional simplices, $low(\langle 1\rangle) = low(\langle 2\rangle) = low(\langle 3\rangle) = low(\langle 4\rangle) = 0$ since they are all zero columns. For the columns associated to one dimensional simplices, $low(\langle 1,2\rangle) = 2$, $low(\langle 2,3\rangle) = 3$, $low(\langle 2,4\rangle) = 4$, $low(\langle 1,3\rangle) = 3$, and $low(\langle 3,4\rangle) = 4$. Since $low(\langle 2,3\rangle) = 3 = low(\langle 1,3\rangle)$ and $low(\langle 2,4\rangle) = 4 = low(\langle 3,4\rangle)$, we add column

$\langle 2,3\rangle$ to column $\langle 1,3\rangle$ and column $\langle 2,4\rangle$ to column $\langle 3,4\rangle$ with modulo 2 arithmetic. Now, $low(\langle 1,3\rangle + \langle 2,3\rangle) = 2 = low(\langle 1,2\rangle)$, we add column $\langle 1,2\rangle$ to $(\langle 1,3\rangle + \langle 2,3\rangle)$. Similarly, $low(\langle 3,4\rangle + \langle 2,4\rangle) = 3 = low(\langle 2,3\rangle)$, so we add $\langle 2,3\rangle$ to $(\langle 3,4\rangle + \langle 2,4\rangle)$. Since $\langle 2,3,4\rangle$ corresponds to the only 2 simplex and $low(\langle 2,3,4\rangle)$ is in the row associated to $\langle 3,4\rangle$, there are no additional column operations. The reduced matrix, R, becomes

	1	2	3	4	1, 2	2, 3	2, 4	1, 3	3, 4	2, 3, 4
1					1					
2					1	1	1			
3						1				
4							1			
1, 2										
2, 3										1
2, 4										1
1, 3										
3, 4										1
2, 3, 4										

Beginning with the columns associated to the zero simplices, since each of these columns are all zero, we continue by scanning the associated rows. There is no column whose lowest one is in row $\langle 1\rangle$, so the simplex $\langle 1\rangle$ represents the 0 dimensional homology class of K. When we scan $\langle 2\rangle$, we see that $low(\langle 1,2\rangle) = 2$, so $\langle 2\rangle$ pairs with $\langle 1,2\rangle$. This coincides with the fact that the 0 dimensional class given by the vertex $\langle 2\rangle$ dies when the 1 simplex that connects $\langle 1\rangle$ to $\langle 2\rangle$ enters the filtration. Scanning $\langle 3\rangle$, we see $low(\langle 2,3\rangle) = 3$, so $\langle 3\rangle$ pairs with $\langle 2,3\rangle$. Because $low(\langle 2,4\rangle) = 4$, we have the persistence pair $(\langle 4\rangle, \langle 2,4\rangle)$.

Since we have paired the 1 simplices $\langle 1,2\rangle$, $\langle 2,3\rangle$ and $\langle 2,4\rangle$, we continue by scanning the column $\langle 1,3\rangle$. $low(\langle 1,3\rangle) = 0$, we scan the row associated to $\langle 1,3\rangle$. There is no column whose lowest one is in the row associated to $\langle 1,3\rangle$. Therefore, this column is associated to an essential homology class. The representative for this homology class is $\langle 1,3\rangle + \langle 2,3\rangle + \langle 1,2\rangle$, since that was the set of column operations required to get the corresponding column in the reduced matrix. Since $low(\langle 3,4\rangle =$ 0, we scan the row associated to $\langle 3,4\rangle$. Finding that $low(\langle 2,3,4\rangle)$ is in the row associated $\langle 3,4\rangle$. Therefore we have the pair $(\langle 3,4\rangle, \langle 2,3,4\rangle)$. Since this column in the reduced matrix is actually $\langle 3,4\rangle + \langle 2,4\rangle + \langle 2,3\rangle$. Pairing $\langle 3,4\rangle$ corresponds to the cycle $\langle 3,4\rangle + \langle 2,4\rangle + \langle 2,3\rangle$ that was present until we add the 2 simplex whose boundary is this cycle.

Our simplicial complex K has the nonzero homology groups $H_0(K) = \mathbb{Z}_2$ and $H_1(K) = \mathbb{Z}_2$. The above algorithm for this example found the representatives for these two homology groups. $\langle 1\rangle$ represents the generator for the 0-dimensional homology class. $\langle 1,2\rangle + \langle 2,3\rangle + \langle 1,3\rangle$ represents the generator for the 1 dimensional homology class generating $H_1(K)$.

3

Stiefel-Whitney Classes

Most current research in persistence focuses on homology. Cohomology groups, however, carry with them a ring structure, making cohomology more powerful than homology. In the beginning, cohomology was calculated to be the algebraic dual to the homology classes on an underlying topological space, K. However, further research demonstrated that cohomology was useful to answer questions in obstruction theory. It then became practice to take the cohomology of a vector bundle associated to that topological space. An appropriate choice of vector bundle gave rise to more information about the underlying space, and thus a special type of cohomology classes, the characteristic classes. Although there are also Chern and Pontryagin classes, we will work only with Stiefel-Whitney classes since they are defined over $\mathbb{Z}_2$.

In this chapter, we will review the main concepts of cohomology, the general form of a Stiefel-Whitney characteristic class, and give a combinatorial formula for computing the Stiefel-Whitney characteristic homology class of a Euler mod-2 space. In Chapter 5 we will add in persistence, developing a theory of Persistent Cohomology and Persistent Stiefel-Whitney classes.

3.1 Cohomology

In its natural construction, cohomology is dual to homology. Recall from Chapter 2 that the p-dimensional homology group of a space K, $H_p(K)$, is an abelian group. We define Hom$(\mathcal{C}_p, G)$ as the group of all homomorphisms of $\mathcal{C}_p$ into $\mathbb{Z}_2$; it is also an abelian group. It follows directly that if we let K be a simplicial complex, we can easily define the p-dimensional cochains of K.

Definition 7. *The group of p* ***dimensional cochains*** *of K, with coefficients in $\mathbb{Z}_2$, is the group $\mathcal{C}^p(K) = Hom(\mathcal{C}_p(K), \mathbb{Z}_2)$. An element $\Gamma \in C^q(K)$ specifies an element of $\mathbb{Z}_2$ (either 0 or 1) for every p-simplex $\sigma \in K$.*

The **coboundary operator**, δ, is defined to be the dual of the boundary operator $\partial : \mathcal{C}_{p+1}(K) \to \mathcal{C}_p(K)$ given by $\delta(\Gamma)(\gamma) = \Gamma(\partial\gamma)$ for every $\gamma \in C_{p+1}(K)$. Therefore we have the following maps between cochain groups:

$$\mathcal{C}^{p+1}(K) \xleftarrow{\delta} \mathcal{C}^p(K)$$

Similar to homology theory, we define the group of **cocycles**, denoted $\mathcal{Z}^p(K)$, to be the kernel of the δ homomorphism and the group of **coboundaries**, denoted $\mathcal{B}^{p+1}(K)$, to be the image of the coboundary map. Since $\partial^2 = 0$, and δ is dual to ∂, it follows that $\delta^2 = 0$ also.

Definition 8. *The p dimensional* ***cohomology group*** *for a simplicial complex, K is*

$$H^p(K) = \mathcal{Z}^p(K)/\mathcal{B}^p(K).$$

$\mathcal{C}^p(K)$ has a natural basis of elements, σ^*, whose value is 1 on the p-dimensional simplex σ and zero elsewhere. Thus, any cochain, $\gamma \in \mathcal{C}^p(K)$ can be represented by $\gamma = \sum g_i\sigma_i^*$, where g_i are elements of $\mathbb{Z}_2$. The coboundary operator commutes with addition, so $\delta\gamma = \sum g_i\delta\sigma_i^*$. Sometimes this representation of the cochain in elementary basis form will be denoted by $\gamma = \sigma^*_{i_1} + \sigma^*_{i_2} + \cdots + \sigma^*_{i_m}$ where $\sigma^*_{i_j}(\sigma_{i_j}) = 1$ and $\sigma^*_{i_j}(\sigma_{i_k}) = 0$ for $\sigma_{i_j} \neq \sigma_{i_k}$.

As previously mentioned, cohomology carries with it a ring structure that homology does not. There are significant operations that can be done on cohomology classes, such as the cup product and Steenrod squares. In this work, we have developed a persistent approach to both of these operations; their definitions and expositions will appear later, in Chapters 7 and 8.

3.2 Characteristic Classes

Before describing in full detail the Stiefel-Whitney characteristic class, we need to lay the foundation.

Definition 9. *A real **vector bundle**, ξ, is defined to be the triple $\xi = (E, B, \pi)$, where B is a topological space called the base space, $E = E(\xi)$ is a topological space called the total space, $\pi : E \to B$ is a continuous map called the projection map such that for every point $b \in B$, there exists the structure of a vector space over the real numbers in the space $\pi^{-1}(b)$. The three parts of a vector bundle, must satisfy the local triviality condition. That is, for every point $b \in B$, there exists a neighborhood, $U \subset B$ of b, an integer $n \geq 0$, and a homeomorphism*

$$h : U \times \mathbb{R}^n \to \pi^{-1}(b)$$

where for every $b \in B$, the map $x \mapsto h(b, x)$ defines an isomorphism between the vector space $\mathbb{R}^n$ and the vector space $\pi^{-1}(b)$.

We call the pair (U, h) a local coordinate system for ξ at $b \in B$. If it is possible to choose $U = B$, then ξ is a trivial bundle. If there are two bundles, ξ and ν, then a *bundle map* from ξ to ν is a continuous function $g : E(\xi) \to E(\nu)$ which carries each vector space $F_b(\xi)$ isomorphically onto one of the vector spaces $F_{b'}(\nu)$. When

$\bar{g}(b) = b'$, the function $\bar{g} : B(\xi) \to B(\nu)$ is a continuous function. There exists an important, canonical vector bundle called the *universal bundle*, γ^n. It is defined with $E(\gamma^n)$ as the set of all pairs (n planes in $\mathbb{R}^{n+k}$, a vector in that n plane), where an n plane is a linear subspace of dimension n. Its projection map $\pi(X, x) = X$ and its vector space structure over a fibre is defined by $t_1(X, x_1)+t_2(X, x_2) = (X, t_1x_1+t_2x_2)$.

To define characteristic classes, it is important to look at a specific manifold that gives rise to relationships between it and real bundles, the Grassmann manifold. But first, we define an n**-frame** of $\mathbb{R}^{n+k}$ to be an n-tuple of linearly independent vectors of $\mathbb{R}^{n+k}$. The Stiefel manifold, $\mathcal{V}_n(\mathbb{R}^{n+k})$, is an open subset of the n-fold cartesian product, $\mathbb{R}^{n+k} \times \cdots \times \mathbb{R}^{n+k}$, defined to be the set of all n-frames of $\mathbb{R}^{n+k}$.

Definition 10. *The* ***Grassmann manifold****, denoted $\mathcal{G}_n(\mathbb{R}^{n+k})$ is the set of n-dimensional planes through the origin of the coordinate space $\mathbb{R}^{n+k}$. It is topologized as a quotient space of the* ***Stiefel manifold****, $\mathcal{V}_n(\mathbb{R}^{n+k})$.*

Based solely on definitions, there is a canonical map

$$q : \mathcal{V}_n(\mathbb{R}^{n+k}) \to \mathcal{G}_n(\mathbb{R}^{n+k})$$

mapping each n-frame to the n plane its vectors span. The quotient topology for $\mathcal{G}_n(\mathbb{R}^{n+k})$ is then defined by the rule that a subset $U \subset \mathcal{G}_n(\mathbb{R}^{n+k})$ is open if and only if its inverse image $q^{-1}(U) \subset \mathcal{V}_n(R^{n+k})$ is open.

Definition 11. *The* ***infinite Grassmann manifold****, $\mathcal{G}_n = \mathcal{G}_n(\mathbb{R}^\infty)$, is the set of all n-dimensional linear subspaces of $\mathbb{R}^\infty$ topologized as follows: A subset of $\mathcal{G}_n$ is open (or closed) if and only if its intersection with $\mathcal{G}_n(\mathbb{R}^{n+k})$ is open (or closed) as a subset of $\mathcal{G}_n(\mathbb{R}^{n+k})$ for each k.*

It can be shown that any $\mathbb{R}^n$-bundle, ξ, over a paracompact space B determines a unique homotopy class of maps $\bar{f}_\xi : B \to \mathcal{G}_n$. This is because we can define $\bar{f}_\xi$ to be the pullback of the universal bundle over $\mathcal{G}_n$. Specifically, given a bundle map $\pi : E \to B$, and a map $f : A \to B$ of base spaces, there is a *pullback* $\bar{f}(E) \to A$ obtained by $\bar{f}(E) = \{(a, e) \in A \times E | f(a) = \pi(e)\}$

It follows that if we have any cohomology class $c \in H^i(\mathcal{G}_n; G)$, where G is any coefficient group or ring, then ξ and c determine a cohomology class together, denoted $c(\xi)$. That is

$$c(\xi) = \bar{f}_\xi{}^* c \in H^i(B; G)$$

Definition 12. $c(\xi)$ *is called the* ***characteristic cohomology class*** *of the bundle* ξ *determined by* c.

Although this definition may be used to define each type of characteristic class, as previously mentioned our focus is on the Stiefel-Whitney Classes. The structure of $\mathcal{G}_\infty$ has a single cell in every dimension. Thus, $H^*(\mathcal{G}_\infty; \mathbb{Z}_2)$ is a polynomial algebra with a single nonzero generator $\lambda_i \in H^i(\mathcal{G}; \mathbb{Z}_2)$. Then,

Definition 13. *The* i^{th} ***Stiefel-Whitney cohomology class*** *is* $w_i = f^*(\lambda_i)$.

If ξ is an n-plane bundle, then $w_0(\xi) = 1 \in H^0(B(\xi); \mathbb{Z}_2)$ and $w_i(\xi) = 0$ for $i > n$. In addition to this definition, the Stiefel-Whitney classes must satisfy naturality. Formally, if $f : B(\xi) \to B(\eta)$ is covered by a bundle map from ξ to η, then $w_i(\xi)$ is isomorphic $f^* w_i(\eta)$. The diagram to consider here is:

$$\begin{array}{ccccc} f^* w_i(\eta) \cong w_i(\xi) \in & H^i(B(\xi)) & \xleftarrow{f^*} & H^i(B(\eta)) \\ & \downarrow & & \downarrow \\ & B(\xi) & \xrightarrow{f} & B(\eta) \end{array}$$

In addition to naturality, the *Whitney Product Theorem* states that the Stiefel-Whitney classes also satisfy a product formula. If ξ and η are vector bundles over the same base space, we define their *Whitney sum*, $\xi \oplus \eta$ to be the bundle where each fiber $\pi_b^{-1}(\xi \oplus \eta) = \pi_b^{-1}(\xi) \oplus \pi_n^{-1}(\eta)$, the direct sum of the respective fibers. Then,

$$w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_i(\xi) \cup w_{k-i}(\eta)$$

where $\cup$ is the cup product of two cohomology classes, $\alpha \in H^p(X)$ and $\beta \in H^q(X)$ given by

$$(\alpha \cup \beta)(\langle v_0, v_1, \ldots, v_{p+q}\rangle) = (-1)^{p+q}\alpha(\langle v_0, v_1, \ldots, v_p\rangle) \cdot \beta(\langle v_p, v_{p+1}, \ldots, v_{p+q}\rangle)$$

see Chapter 7 and (Mun96).

In general a nontrivial 1st Stiefel-Whitney class indicates whether a manifold has a nonorientable bundle, which answers questions about embeddability and classifies the manifold. For example, in $\mathbb{R}^3$, there exists a line bundle, ξ_1^1, over the circle S^1, and the Stiefel-Whitney class $w_1(\xi_1^1)$ is nonzero. This classifies the S^1 from the annulus as no such nonorientable bundle exists for the annulus.

There are immediate consequences of these properties. First, if ξ is isomorphic to η, then their Stiefel-Whitney Classes are equal. i.e. $w_i(\xi) = w_i(\eta)$. Second, if ξ is a trivial bundle, then $w_i(\xi) = 0$ for all $i \geq 0$. Third, if ξ is trivial, then $w_i(\xi \oplus \eta) = w_i(\eta)$. Lastly, if ξ is an $\mathbb{R}^n$ bundle with Euclidean metric which has a nowhere zero cross-section, then $w_n(\xi) = 0$. What follows from this last statement is if ξ possesses k nowhere linearly dependent cross sections, then $w_{n-k+1}(\xi) = w_{n-k+2}(\xi) = \cdots w_n(\xi) = 0$. In this case, we actually have a splitting of ξ as a Whitney sum where $\xi = \varepsilon \oplus \varepsilon^\perp$ where ε is trivial and $\varepsilon^\perp$ has dimension $n - k$.

As one can see, it is helpful to look at each of the Stiefel-Whitney classes all together. To do this, we define the **total Stiefel-Whitney class** of an n-plane bundle ξ over B to be the element

$$w(\xi) = 1 + w_1(\xi) + w_2(\xi) + \cdots + w_n(\xi).$$

As previously mentioned, $H^*(\mathcal{G}_\infty, \mathbb{Z}_2)$ is a polynomial algebra with a single element in each dimension. These elements are added together to form the total Stiefel-Whitney class of a bundle.

In order to merge the power of Stiefel-Whitney cohomology classes with persistence, we need a way to compute these classes from simplicial complexes. The next section provides a combinatorial formula for computing the Stiefel-Whitney *homology class* found in (GT76).

3.3 Stiefel-Whitney Homology Classes

Characteristic homology classes were defined for any bundle, however here we focus on the tangent bundle of a smooth manifold. In (GT76), Goldstein and Turner gave the following combinatorial definition for the Stiefel-Whitney homology class $u_p(K)$ of an Euler mod 2 space, K, of dimension d was presented.

Let K be a finite-dimensional, locally finite simplicial complex. K is said to be a mod 2 Euler space if the link of every simplex in K has even Euler characteristic. If K triangulates a manifold without boundary, then it is truly an Euler mod 2 space. In this case, the Stiefel-Whitney homology class is Poincaré dual to the Stiefel-Whitney cohomology class. The pth Stiefel-Whitney class of K, denoted $\omega_p(K)$, is the p-dimensional mod 2 homology class which has as a representation the p-dimensional chain consisting of all p-simplexes in the first barycentric subdivision of K. This chain is a cycle for each p iff K is a mod 2 Euler space.

Assume that the vertices of the simplicial complex, K, are given a total ordering so that each simplex has a unique representation by writing its vertices in increasing order. Let $s = \langle v_0, v_1, \ldots, v_p \rangle$ be a p-simplex in K and let t be another simplex which has s as a face; i.e., $s \subset t$ (s may be equal to t). Let

$$B_{-1} = \{ \text{ vertices of } t < v_0 \ \},$$

$$B_m = \{ \text{ vertices of } t \text{ such that } v_m < t < v_{m+1} \ \}, 0 \leq m \leq p-1,$$

$$B_p = \{ \text{ vertices of } t > v_p \ \}.$$

We say s is *regular* in t if $|B_m| = 0$ for every odd m. Given t whose dimension is $\geq p$, let $\mathcal{D}_p(t)$ denote the mod 2 chain which consists of all p-dimensional simplices $s \subset t$ such that s is regular in t. Note that $\dim(t)$ can be anything larger than $\dim(s)$.

Theorem 14. $\displaystyle\sum_{\dim(t)\geq p} \mathcal{D}_p(t)$ *is a chain which represents* $u_p(K)$.

Again, in the case where K is manifold, then $u_p(K) \cong w_{d-p}(\tau_K)$. That is, $u_p(K)$, the Stiefel-Whitney homology class is dual to its Poincaré dual, $w_{d-p}(\tau_K)$, the $d-p$ dimensional Stiefel-Whitney cohomology class of the tangent bundle, τ_K.

3.3.1 Example

To truly appreciate the formula, it is best seen through an example. Suppose our space is $\mathbb{RP}^2$ with the triangulation given in Figure 3.1.

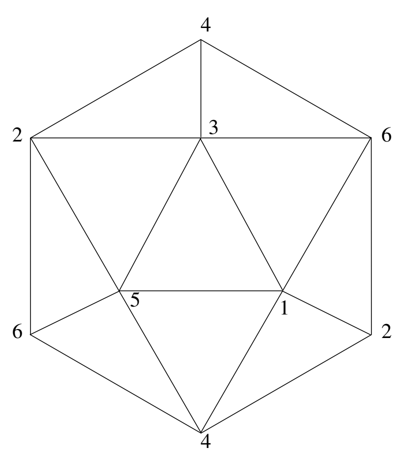

FIGURE 3.1: A standard triangulation of $\mathbb{RP}^2$.

To compute the Stiefel-Whitney class of dimension 0 we need to consider $\mathcal{D}_0(t)$ for all possible t of higher dimension. For a 2-simplex of the form $\langle a, b, c\rangle$, only the lowest vertex $\langle a\rangle$ is a regular 0-simplex. In this case, $B_{-1} = \emptyset$ while $B_0 = \{b, c\}$ and the conditions for regularity hold. Similarly, for the 1-dimensional simplices, the lowest vertex is the only regular vertex. Since each 0-simplex is regular in itself, we see that each vertex is weighted by the number of simplices for which it is regular, i.e. it is the lowest vertex. To compute u_0, we have the following:

$$u_0 = \Big(11\langle 1\rangle + 8\langle 2\rangle + 5\langle 3\rangle + 4\langle 4\rangle + 2\langle 5\rangle + \langle 6\rangle\Big) \mod 2 = \langle 1\rangle + \langle 3\rangle + \langle 6\rangle$$

The coefficient of $\langle 1\rangle$ is 11 because $\langle 1\rangle$ is regular in the 11 simplices:
$\{\langle 1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 1,5\rangle, \langle 1,6\rangle, \langle 1,2,4\rangle, \langle 1,3,5\rangle, \langle 1,4,5\rangle, \langle 1,2,6\rangle, \langle 1,3,6\rangle\}$.

To compute u_1, we need to recognize that each 1 simplex is regular in itself while the only 1 simplex that is a regular face in the 2-simplex, $\langle a, b, c\rangle$ is $\langle a, c\rangle$. From this, we get $u_1 = \langle 1,2\rangle + \langle 1,3\rangle + \langle 1,5\rangle + \langle 1,6\rangle + \langle 2,3\rangle + \langle 3,4\rangle + \langle 3,5\rangle + \langle 4,5\rangle + \langle 5,6\rangle$. As written, this class does not seem to represent the expected nonzero homology class of $\mathbb{RP}^2$. In fact, geometrically, it looks like the curve drawn Figure 3.2:

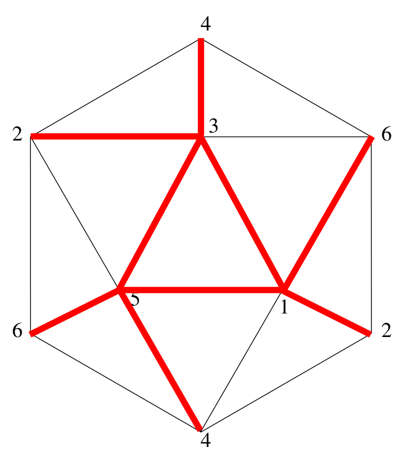

Figure 3.2: The triangulation of $\mathbb{RP}^2$ with the 1st Stiefel-Whitney homology, $u_1(\mathbb{RP}^2)$, highlighted.

However, after using the relationships that the union of two codimension 1 faces of a 2-simplex is homologous to the third, we can reduce this representation of u_1 to the simpler form:

$$u_1 = \langle 1, 4\rangle + \langle 1, 3\rangle + \langle 4, 3\rangle.$$

u_2 is simply the top dimensional homology class, which is the sum of all the 2-

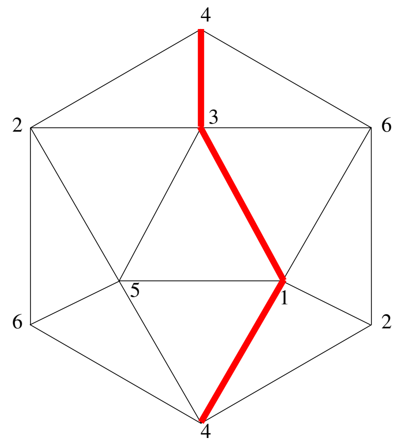

FIGURE 3.3: The triangulation of $\mathbb{RP}^2$ with the reduced 1st Stiefel-Whitney homology, $\tilde{u}_1(\mathbb{RP}^2)$, highlighted.

simplices of $\mathbb{RP}^2$. This is because every 2-simplex is a regular face of itself and there are no 3 dimensional simplices.

The benefit of having a combinatorial formula for the Stiefel-Whitney homology class is that we can define the notion of a persistent Stiefel-Whitney cohomology class. Recall, that the 1-dimensional Stiefel-Whitney cohomology class determines whether a manifold is orientable. Therefore, if we can use this combinatorial formula for each dimensional Stiefel-Whitney homology class, and find their Poincaré duals, we can find a representations for each Stiefel-Whitney cohomology class. Then, we can develop the theory of Persistent Cohomology to define the persistent Stiefel-Whitney Class of a simplicial complex. The following chapters do just that.

4

Finding the Dual

4.1 Finding the Poincaré Dual

The last chapter included a formula to compute the Stiefel-Whitney homology class of an mod 2 Euler space. However, as previously mentioned, these are Poincaré dual to the Stiefel-Whitney cohomology classes which are normally thought of as bundle invariants. This inspired us to consider extending the study of persistence homology to include characteristics classes and, more generally, CoPersistence.

In preparation for our definition and algorithms for CoPersistence in Chapter 5, this chapter develops an algorithm for converting a representative of a homology class to its Poincaré dual. To that end, let K be a d-dimensional simplicial complex that triangulates a manifold without boundary and suppose we have a homology class $\gamma \in H_p(K)$. This algorithm finds the Poincaré dual, $\Gamma \in H^q(K)$, where $p + q = d$.

Let $\mathrm{Sd}(K)$ be the first barycentric subdivision of K. Let γ be a p-dimensional cycle that is a representative for a generator of $H_p(K; \mathbb{Z}_2)$. We can write the cycle $\gamma = \sum_i g_i \sigma_i$ where the σ_i are the p-dimensional simplices in K and $g_i \in \mathbb{Z}_2$.

The purpose of this algorithm is to find the Poincaré dual of γ. We want to map γ to a p-chain in Sd(K) that is homologous to γ and transverse to the original simplices of K. When we have done so, there will be a set of q dimensional simplices of K that intersect this new p-chain transversely. This set of q simplices will be the exact set of q simplices on which the Poincaré dual evaluates 1 and 0 elsewhere.

To do this, we will construct a set of $p+1$ maps recursively in steps; each determines the image of simplicies of a given dimension in γ from their upper link in the image of the previous map. Each map is defined for all simplices of certain dimension in γ, so they are defined on the entire chain. The construction of these maps was inspired by the fact that for any simplex, σ_i, $\sigma_i = b(\sigma_i) * \partial(\sigma_i)$, i.e. σ_i equals the join of its barycentre and its boundary. The last of the $p+1$ inductive maps will complete the push-off of γ into the first barycentric subdivision. Its image will involve a set of barycentres of q dimensional simplices. The q simplices in K, whose barycentres are in the image of the last map, are transverse to the image of γ under the composition of all of our maps, and therefore comprise a representation of the Poincaré dual to γ. While the construction of the algorithm is based in the structure of simplicies within the first barycentric subdivision and the principles behind Poincaré duality, this is the first time that this process has been detailed and outlined in a combinatorial fashion.

The first step in our algorithm is to define the map Φ_1. We start by finding the top dimensional simplices of K, such that σ_i is a face. This is done by creating and storing the star of σ_i for each i, St(σ_i)= $\{\lambda | \lambda \in K$ such that Vert$(\sigma_i) \subset$ Vert(λ) $\}$, where Vert(σ) is the vertex set of σ. We store that information in a set of St(σ_i)-matrices, each denoted S_{σ_i}. Since $\gamma = \sigma_1 + \cdots + \sigma_n$, we will have n star matrices in total. Each S_{σ_i} is the incidence matrix of the subset of the simplices of dimension $\geq p$ that are in the star of σ_i. That is, each simplex associated to rows and columns in S_{σ_i} has σ_i as a face. σ_i is also in S_{σ_i} since σ_i is a face of itself. Similar to the Incidence matrix, $S_{\sigma_i}(i,j) = 1$ if the simplex corresponding to row i is a codimension

1 face of the simplex corresponding to column j and 0 otherwise.

For each $1 \leq i \leq n$, we choose a d-dimensional simplex, $\Delta_i^d \in \mathrm{St}(\sigma_i)$. Without loss of generality, any d-dimensional simplex in $\mathrm{St}(\sigma_i)$ can be chosen. Different choices of Δ_i^d may result in different representatives for Γ, the cohomology class that is Poincaré dual to γ. This choice of Δ_i^d for each i is the end result for defining the first map of the algorithm. It is possible to choose the same d-dimensional simplices for two separate σ_i's. Our first map is defined to be the continuous map, $\Phi_1 : b(\sigma_i) * \partial(\sigma_i) \to b(\Delta_i^d) * \partial(\sigma_i)$, for each σ_i, a p-dimensional simplex in γ, and the identity otherwise. An illustration of this is in Figure 4.1.

FIGURE 4.1: The first illustration depicts a part of the the simplicial complex, K, with part of the original γ as a bold line. The second depicts the image of this part of γ under the first map, Φ_1.

In general, to define Φ_s, for each $2 \leq s \leq p+1$, we take into consideration $b(\tau) * \partial(\tau) * lk(\tau \subset \gamma)$ for each $\tau \in \partial(\sigma_i)$. To map the lower dimensional simplices to the barycentric subdivision continuously, we not only look at τ, but the link of τ in γ under the image of Φ_{s-1}. In other words, we observe how the higher dimensional simplices have already been pushed into $Sd(K)$ to understand the image of Φ_s.

Φ_2 maps the $p-1$ dimensional simplices in the boundary of each p simplex, σ_i. The link of each $p-1$ simplex in γ is a $d-p-1$ sphere. It bounds a $d-p$ disk. The image of τ under Φ_2 is this disk. The disk depends on the image of the σ_i's under Φ_1. Suppose $\tau \in \sigma_i \cap \sigma_j$, such that Φ_1 maps the barycentre of σ_i to the barycentre of Δ_i^d and the barycentre of σ_j to the barycentre of Δ_j^d. If $\Delta_i^d \cap \Delta_j^d$ has a $d-1$ dimensional face of both, $\Delta_{i,j}^{d-1}$, then $\Phi_2(\tau * St(\tau \subset \Phi_1(\gamma))) = b(\Delta_{i,j}^{d-1}) * \partial(\tau) * St(\partial(\tau) \subset \Phi_2(\gamma))$.

An illustration of this is given in Figure 4.2.

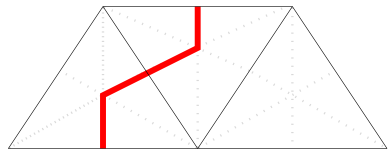

FIGURE 4.2: This is image of the part of a 1 dimensional cycle, γ given in Figure 4.1 under Φ_2. The d simplices whose barycentres are in the image of Φ_1 share a codimension 1 face. The image of $\tau \in \sigma_i \cap \sigma_j$ is the 1 dimensional disk connecting $b(\Delta_i^d)$ to $b(\Delta_j^d)$ through $b(\Delta_{i,j}^{d-1}$.

If $\tau \in \sigma_i \cap \sigma_j$, such that Φ_1 maps the barycentre of σ_i to the barycentre of Δ_i^d and the barycentre of σ_j to the barycentre of Δ_j^d and $\Delta_i^d \cap \Delta_j^d$ has no $d-1$ face of both, then we need to find a path through barycentres of $d-1$ and d dimensional simplices that connect $b(\Delta_i^d)$ to $b(\Delta_j^d)$. To do so, we use the star matrix of $\Delta_i^d \cap \Delta_j^d$, $S_{\Delta_i^d \cap \Delta_j^d}$. We initialize the set $V_{i,j}^{d-1} = \{\Delta_i^d\}$. Scanning the column associated to Δ_i^d, we find the set of $d-1$ dimensional simplices in its boundary. Without loss of generality, we choose any simplex associated to the row that has a nonzero entry in the column associated to Δ_i^d, and denote it Δ_{i,j_1}^{d-1}. A different choice will result in a different representation of the Poincaré dual, Γ. We add this simplex to the set $V_{i,j}^{d-1} = \{\Delta_i^d, \Delta_{i,j_1}^{d-1}\}$. We scan that row to find the another d dimensional simplex of which it is a face, say $\Delta_{k_1}^d$. Adding that simplex to the set, $V_{i,j}^{d-1}$ is now $\{\Delta_i^d, \Delta_{i,j_1}^{d-1}, \Delta_{k_1}^d\}$. We scan that column, and choose a row associated to a $d-1$ dimensional simplex that is a face of $\Delta_{k_1}^d$ not previously listed in $V_{i,j}^{d-1}$. Supposing that simplex is Δ_{i,j_2}^{d-1}, our $V_{i,j}^{d-1}$ gets modified to $\{\Delta_i^d, \Delta_{i,j_1}^{d-1}, \Delta_{k_1}^d, \Delta_{i,j_2}^{d-1}\}$.

For each new element of $V_{i,j}^{d-1}$, we either scan the row of the previously added $d-1$ dimensional simplex to find a d simplices that is not already in $V_{i,j}^{d-1}$ or a column of a d dimensional simplex to find a $d-1$ face that is not already in $V_{i,j}^{d-1}$. Because the $\mathrm{St}(\Delta_i^d \cap \Delta_j^d)$-matrix is finite, eventually we will reach a situation when we are scanning a the row associated to a $d-1$ dimensional simplex and one of the possible d simplices is Δ_j^d. When this happens, we choose it and add it to $V_{i,j}^{d-1}$. Here $V_{i,j} = \{\Delta_i^d, \Delta_{i,j_1}^{d-1}, \Delta_{k_1}^d, \Delta_{i,j_2}^{d-1}, \Delta_{k_2}^d, \ldots, \Delta_{i,j_{m-1}}^{d-1}, \Delta_{k_{m-1}}^d, \Delta_{i,j_m}^{d-1}, \Delta_j^d\}$. We have reached a path from $b(\Delta_i^d)$ to $b(\Delta_j^d)$ through the barycentred of $d-1$ and d dimensional simplices of K. In this case, we define $\Phi_2(b(\tau_{i,j}) * \partial(\tau_{i,j}) * lk(\tau \subset \gamma)) = [(b(\Delta_{i,j_1}^{d-1}) * b(\Delta_{k_1}^d)) \cup (b(\Delta_{k-1}^d) * b(\Delta_{i,j_2}^{d-1})) \cup (b(\Delta_{i,j_2}^{d-1} * b(\Delta_{k_2}^d)) \cup \cdots \cup (b(\Delta_{k_{m-1}}^d) * b(\Delta_{i,j_m}^{d-1}))] * \partial(\tau_{i,j}) * lk(\partial(\tau) \subset \gamma))$. An illustration of this case is given in Figure 4.3.

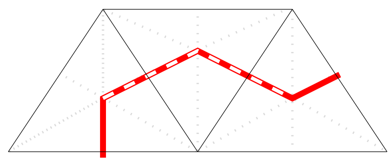

FIGURE 4.3: This is image of the part of a 1 dimensional cycle, γ given in Figure 4.1 under Φ_2. The d simplices whose barycentres are in the image of Φ_1 do not share a codimension 1 face. Therefore, the image of $\tau \in \sigma_i \cap \sigma_j$ is the 1 dimensional disk connecting $b(\Delta_i^d)$ to $b(\Delta_j^d)$ through a 1 dimensional path of barycentres depicted with the dashed line.

We can now define the general form of Φ_s for $2 < s \leq p+1$. For each simplex $\lambda \in \partial(\sigma_i)$ of dimension $p-s+1$ for some $i \in [1, \ldots, n]$, we regard λ as $b(\lambda) * \partial(\lambda)$ and consider $lk(\lambda \subset \gamma)$. Specifically, we define $\Phi_s : b(\lambda) * \partial(\tau) * lk(\lambda \subset \gamma) \to (\tilde{D}^{d-s+1}) * \partial(\lambda) * \phi_{s-1}(lk(\lambda \subset \gamma))$. Since $\phi_{s-1}(lk(\lambda \subset \gamma))$ lies in $lk(\lambda \subset Sd(K))$ and is a sphere of lower dimension, it bounds a disk in $Sd(K)$. That disk is represented by $\tilde{D}^{d-s+1}$ in the image of Φ_s.

Eventually our process will terminate when we find Φ_{p+1} since this will map the 0 dimensional simplices in γ each to $d-p$ dimensional disk in $Sd(K)$. This disk is the final image of Φ_{p+1} since the boundary of a vertex is empty. Furthermore, the $d-p$ disk is a $d-p=q$ dimensional cell in $Sd(K)$, one which transversely intersects the original expression of γ. The barycentres of the 0 simplices, the 0 simplices themselves, get mapped to barycentres of $d-p=q$ dimensional simplices. Let Q

denote the set of the q dimensional simplices whose barycentres are in the image of Φ_{p+1}. That is $Q = \{\varsigma_1, \varsigma_2, \ldots, \varsigma_l\}$. Define $\Gamma = \sum_{l_i} \varsigma_{l_i}^*$, for $1 \leq l_i \leq l$, where $\varsigma_{l_i}^*$ is the unique q cochain that evaluates 1 on ς_{l_i} and zero on all other q simplices. This Γ is the Poincaré dual to push-off of γ into the first barycentric subdivision, and thus γ, by construction.

4.2 Illustrated Example

Let us consider the above process to find the Poincaré dual to a particular homology class. Suppose K, our simplicial complex, is the following triangulation of $\mathbb{RP}^2$ with first barycentric subdivision illustrated in Figure 4.1. Let our homology class $\gamma \in H_1(K)$ be defined by $\gamma = \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4$, with $\sigma_1 = \langle 2, 3\rangle$, $\sigma_2 = \langle 3, 5\rangle$, $\sigma_3 = \langle 1, 5\rangle$ and $\sigma_4 = \langle 1, 2\rangle$. This is illustrated with the highlighted curve in Figure 4.4.

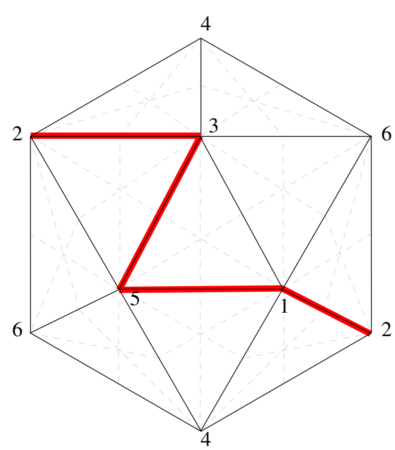

FIGURE 4.4: The triangulation of $\mathbb{RP}^2$ with first barycentric subdivision and 1 dimensional homology class, γ, highlighted.

We begin our process by assigning a 2 dimensional simplex to each σ_i for each $1 \leq i \leq 4$. To do so, we look at the stars and star matrices of each σ_i, we have the following:

$$\mathrm{St}(\langle 2,3\rangle) = \{\langle 2,3,4\rangle, \langle 2,3,5\rangle\} \qquad S_{\sigma_1=\langle 2,3\rangle} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$\mathrm{St}(\langle 3,5\rangle) = \{\langle 1,3,5\rangle, \langle 2,3,5\rangle\} \qquad S_{\sigma_2=\langle 3,5\rangle} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$\mathrm{St}(\langle 1,5\rangle) = \{\langle 1,3,5\rangle, \langle 1,4,5\rangle\} \qquad S_{\sigma_3=\langle 1,5\rangle} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$\mathrm{St}(\langle 1,2\rangle) = \{\langle 1,2,4\rangle, \langle 1,4,6\rangle\} \qquad S_{\sigma_4=\langle 1,2\rangle} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Suppose, for each i, we choose $\Delta_1^2 = \langle 2,3,4\rangle$ to $\langle 2,3\rangle$, $\Delta_2^2 = \langle 2,3,5\rangle$ to $\langle 3,5\rangle$, $\Delta_3^2 = \langle 1,3,5\rangle$ to $\langle 1,5\rangle$, and $\Delta_4^2 = \langle 1,2,4\rangle$ to $\langle 1,2\rangle$. Thus, we define Φ_1 to be the following:

$$\Phi_1(\gamma) = b(\langle 2,3,4\rangle) * \partial(\langle 2,3\rangle) \cup b(\langle 2,3,5\rangle) * \partial(\langle 3,5\rangle) \cup b(\langle 1,3,5\rangle) * \partial(\langle 1,5\rangle) \cup$$
$$\cup\, b(\langle 1,2,4\rangle) * \partial(\langle 1,2\rangle)$$

The dashed lines on the triangulation of $\mathbb{RP}^2$ illustrated in Figure 4.5 represent the image of Φ_1.

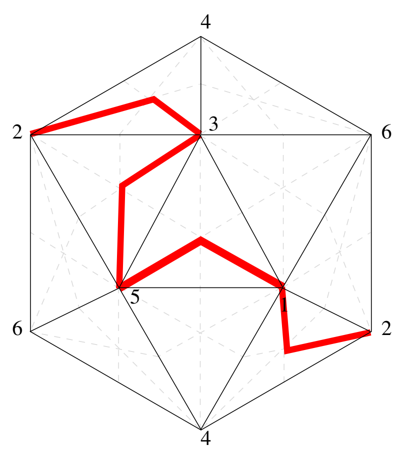

FIGURE 4.5: The triangulation of $\mathbb{RP}^2$ with the first step of the algorithm illustrated.

We now find Φ_2 for each τ a 0 dimensional simplices in each σ. For this example, we will denote this $\tau_{i,j} = \sigma_i \cap \sigma_j$. We get the set $\tau_{1,2} = \langle 3\rangle$, $\tau_{1,4} = \langle 2\rangle$, $\tau_{2,3} = \langle 5\rangle$,

$\tau_{3,4} = \langle 1 \rangle$. Since each $\tau_{i,j}$ are 0 dimensional simplices, our process will stop at the end of this step.

$$\tau_{1,2} = \langle 3 \rangle = \sigma_1 \cap \sigma_2 \quad \text{and} \quad \Delta_1^2 \cap \Delta_2^2 = \langle 2,3,4 \rangle \cap \langle 2,3,5 \rangle = \langle 2,3 \rangle$$

$$\tau_{2,3} = \langle 5 \rangle = \sigma_2 \cap \sigma_3 \quad \text{and} \quad \Delta_2^2 \cap \Delta_3^2 = \langle 2,3,5 \rangle \cap \langle 1,3,5 \rangle = \langle 3,5 \rangle$$

$$\tau_{3,4} = \langle 1 \rangle = \sigma_3 \cap \sigma_4 \quad \text{and} \quad \Delta_3^2 \cap \Delta_4^2 = \langle 1,3,5 \rangle \cap \langle 1,2,4 \rangle = \langle 1 \rangle$$

$$\tau_{1,4} = \langle 2 \rangle = \sigma_1 \cap \sigma_4 \quad \text{and} \quad \Delta_1^2 \cap \Delta_4^2 = \langle 2,3,4 \rangle \cap \langle 1,2,4 \rangle = \langle 2,4 \rangle$$

Since the intersections $\Delta_1^2 \cap \Delta_2^2, \Delta_2^2 \cap \Delta_3^2$, and $\Delta_1^2 \cap \Delta_4^2$ each have 1 dimensional simplices, we can easily define the image of these corresponding $\tau_{i,j}$'s under Φ_2. Our obvious choices are: $\Delta_{1,2}^1 = \langle 2,3 \rangle, \Delta_{2,3}^1 = \langle 3,5 \rangle$, and $\Delta_{1,4}^1 = \langle 2,4 \rangle$. To find the image of $\tau_{3,4}$, we need to find a path between 1 and 2 dimensional simplices connecting $\langle 1,3,5 \rangle$ to $\langle 1,2,4 \rangle$. We look at the Star of $\Delta_3^2 \cap \Delta_4^2 = \langle 1 \rangle$:

$$\mathrm{St}(\langle 1 \rangle) = \{\langle 1,2 \rangle, \langle 1,3 \rangle, \langle 1,4 \rangle, \langle 1,5 \rangle, \langle 1,6 \rangle, \langle 1,2,4 \rangle, \langle 1,3,5 \rangle, \langle 1,4,5 \rangle, \langle 1,2,6 \rangle, \langle 1,3,6 \rangle\}$$

Whose matrix is:

	$\langle 1,2 \rangle$	$\langle 1,3 \rangle$	$\langle 1,4 \rangle$	$\langle 1,5 \rangle$	$\langle 1,6 \rangle$	$\langle 1,2,4 \rangle$	$\langle 1,3,5 \rangle$	$\langle 1,4,5 \rangle$	$\langle 1,2,6 \rangle$	$\langle 1,3,6 \rangle$
$\langle 1 \rangle$	1	1	1	1	1					
$\langle 1,2 \rangle$						1			1	
$\langle 1,3 \rangle$							1			1
$\langle 1,4 \rangle$						1		1		
$\langle 1,5 \rangle$							1	1		
$\langle 1,6 \rangle$									1	1

In order to find the 1-dimensional path that will be associated to $\tau_{3,4}$, we begin with $V_{3,4}^1 = \{(\Delta_3^2 = \langle 1,3,5 \rangle)\}$. At each choice we will add to this set $V_{3,4}^1$. By scanning the column of the $\mathrm{St}(\langle 1 \rangle)$-matrix associated with $\langle 1,3,5 \rangle$ for the non-zero entries, we find the codimension-1 faces in $\partial(\langle 1,3,5 \rangle)$ that are also in $\mathrm{St}(\langle 1 \rangle)$. For this particular example, this set is $\{\langle 1,3 \rangle, \langle 1,5 \rangle\}$. We choose $\langle 1,5 \rangle$; choosing $\langle 1,3 \rangle$ will lead to another representation of the same cohomology class. Without loss of generality, say the choice is $\langle 1,5 \rangle$. Now $V_{3,4} = \{\langle 1,3,5 \rangle, \langle 1,5 \rangle\}$. By then scanning the row associated to $\langle 1,5\rangle$ for the non-zero entries, we find the simplices of which it is a co-dimension 1 face that are also in the $\mathrm{Lk}(\langle 1\rangle)$. This set is $\{\langle 1,3,5\rangle, \langle 1,4,5\rangle\}$. Since $V^1_{3,4}$ already has $\langle 1,3,5\rangle$, our choice is forced to be $\langle 1,4,5\rangle$. Now, $V^1_{3,4} = \{\langle 1,3,5\rangle, \langle 1,5\rangle, \langle 1,4,5\rangle\}$. Scanning the column associated to $\langle 1,4,5\rangle$ for non-zero entries produces the set $\{\langle 1,4\rangle, \langle 1,5\rangle\}$. Since our current $V^1_{3,4}$ has $\langle 1,5\rangle$, our choice becomes $\langle 1,4\rangle$. Now, $V^1_{3,4} = \{\langle 1,3,5\rangle, \langle 1,5\rangle, \langle 1,4,5\rangle, \langle 1,4\rangle\}$. Scanning the row associated to $\langle 1,4\rangle$, we find that its co-dimension faces that are also in the $\mathrm{St}(\langle 1\rangle)$ are $\{\langle 1,4,5\rangle, \langle 1,2,4\rangle\}$. Since at this point $V^1_{3,4}$ contains $\langle 1,4,5\rangle$, our choice is $\langle 1,2,4\rangle = \sigma^2_4$. $V^1_{3,4} = \{\langle 1,3,5\rangle, \langle 1,5\rangle, \langle 1,4,5\rangle, \langle 1,4\rangle, \langle 1,2,4\rangle\}$. Since we have a choice of a 2-dimensional simplex that is Δ^2_4, this process terminates. Our path is through the 1 and 2 dimensional simplices in $V^1_{3,4}$. The image of $\tau_{3,4}$ in Φ_2 is $b(\langle 1,5\rangle) * b(\langle 1,4,5\rangle) * b(\langle 1,4\rangle)$.

By concatenating the previous two maps, we get the following:

$$\begin{aligned}\Phi_2(\Phi_1(\sum_i b(\sigma^1_i) * \tau_{i,j})) = (b(\langle 2,3,4\rangle) * b(\langle 2,3\rangle))\cup \\ \cup\, (b(\langle 2,3\rangle) * b(\langle 2,3,5\rangle)) \cup (b(\langle 2,3,5\rangle) * b(\langle 3,5\rangle))\cup \\ \cup\, (b(\langle 3,5\rangle) * b(\langle 1,3,5\rangle)) \cup (b(\langle 1,3,5\rangle) * b(\langle 1,5\rangle))\cup \\ \cup\, (b(\langle 1,5\rangle) * b(\langle 1,4,5\rangle)) \cup (b(\langle 1,4,5\rangle) * b(\langle 1,4\rangle))\cup \\ \cup\, (b(\langle 1,4\rangle) * b(\langle 1,2,4\rangle)) \cup (b(\langle 1,2,4\rangle) * b(\langle 2,4\rangle))\end{aligned}$$

It is illustrated by the dashed bold line in Figure 4.5.

Let Q be the set of 1-dimensional simplices, whose barcyentre is in the image of Ψ. That is, $Q = \{\langle 2,3\rangle, \langle 3,5\rangle, \langle 1,5\rangle, \langle 1,4\rangle, \langle 2,4\rangle\}$. For each element $\varsigma_k \in Q$, define ς^*_k to be the co-chain that evaluates 1 on ς_k and 0 elsewhere. Our Poincaré dual to γ is $\Gamma = \sum_k \varsigma^*_k$. Pictorially, this means that the ς^*_k are defined as the cochains that evaluates 1 on the set of 1-simplices that the final dashed line intersects and 0 on all other 1-dimensional simplices. For this example, $\Gamma = \langle 2,3\rangle^* + \langle 3,5\rangle^* + \langle 1,5\rangle^* + \langle 1,4\rangle^* + \langle 2,4\rangle^*$.

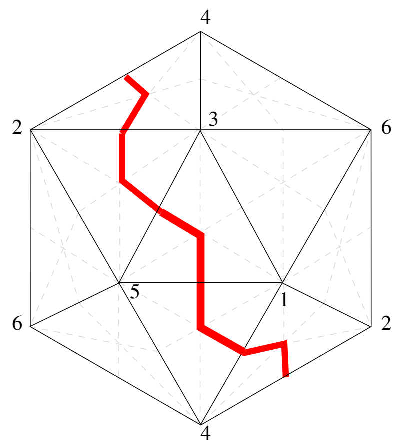

FIGURE 4.6: The triangulation of $\mathbb{RP}^2$ with image of γ pushed into the $\mathrm{Sd}(K)$.

5

Persistent Cohomology

In Chapter 4, we constructed a method for finding the Poincaré dual of an existing homology class. This section develops two distinct algorithms, one that computes Persistent Cohomology, or Copersistence, and one that finds when a given cohomology class dies in the filtration of a simplicial complex, K. There has been research that claims a persistent approach to cohomology (dSMVJ09), but this is the first publication that defines and gives an algorithm to compute Persistent Cohomology.

5.1 CoPersistence

Suppose we have a filtration of topological spaces.

$$\emptyset = X_0 \subset X_1 \subset \cdots \subset X_n = X$$

that is represented by a filtration of simplicial complexes

$$\emptyset = K_0 \subset K_1 \subset \cdots \subset K_n = K.$$

We assume for simplicity that $K_i = K_{i-1} \cup \sigma_i$; a single simplex is added at each level of the filtration. We define the birth and death of a cohomology class by proceeding backwards in the filtration. This brings us to the main definition in this section:

Definition 15. *Let $\psi^q_{j,i} : H^q(K_j) \to H^q(K_i)$ be the map induced by the inclusion $K_i \subset K_j$, $i \leq j$. We say a q dimensional cohomology class, Γ is* ***born*** *at K_j if $\Gamma \in H^q(K_j)$, but $\Gamma \notin Im(\psi^q_{j+1,j})$. We say that this cohomology class* ***dies*** *entering K_i if i is the largest index such that $\exists$ a $\Lambda \in H^q(K_{j+1})$ with $\psi^q_{j+1,i}(\Lambda) = \psi^q_{j,i}(\Gamma) \in H^q(K_i)$. The* ***copersistence*** *of Γ is $copers(\Gamma) = j - i$.*

In order to find the level K_j at which a cohomology class is "born" by the previous definition, we need a simplicial way of representing this cohomology class. Given $\Gamma \in \mathcal{C}^q(K_j)$, $j \geq i$ and $0 \leq q \leq d$, we can write $\Gamma = \sigma_1^* + \sigma_2^* + \cdots + \sigma_n^*$ for some n. To determine $\psi^q_{j,i}(\Gamma)$, our main tool in calculation is the q-dimensional CoIncidence matrix of K_j defined as follows:

Definition 16. *For a simplicial complex, K, the q****-dimensional CoIncidence Matrix****, CI_q is the matrix whose rows correspond to the ordered q-dimensional simplices of K, σ_i, and whose columns correspond to the ordered $q-1$ dimensional simplices of K, τ_j where:*

$$CI_q[i,j] = \begin{cases} 1 & \sigma_i \text{ is a coface of } \tau_j \\ 0 & \text{otherwise} \end{cases}$$

Here σ is a coface of τ means τ is a face of σ. The ordering on the simplicies of K in CI_q, give an ordering on the cochains $\sigma^* \in \mathcal{C}^*(K)$. For simplicity, we will often refer to row i by the q-simplex it represents and column j by the $q-1$ simplex it represents.

5.2 CoPersistence Algorithm

To calculate Persistent Cohomology, we follow an algorithm that is close to the algorithm for computing persistent homology. We reduce the total CoIncidence matrix, CI. CI is the matrix whose columns and rows correspond to every simplex in K

with the relationship that $C(i,j) = 1 \Leftrightarrow$ the simplex corresponding to column j is a codimension 1 face of the simplex corresponding to row i. This differs from the CI_q in that the rows and columns correspond to simplices of all dimensions, not just q and $q-1$. This algorithm differs from the Persistent Homology algorithm in that we also remove rows and columns from CI to move backwards through the filtration of K. Recall that in order to reduce CI, we perform column operations in $\mathbb{Z}_2$ arithmetic from left to right according to the following algorithm:

```
for  j = 1  to  n  do
    while  ∃j' < j with low(j') = low(j) ≠ 0  do
        add column j' to column j
    endwhile
endfor.
```

When we reduce CI there are some columns of zeros and others that have a lowest one in a unique row. Suppose in CI, there exists a $q-1$ dimensional column corresponding to σ that is all zeros. We look at the row corresponding to σ to see if there is a column whose lowest one is in that row. If there is such a column, the class represented by σ is trivial. If there does not exist a column, that class is a generator for $H^{q-1}(K)$.

This algorithm diverts from the persistent homology algorithm here. We continue by removing the bottom row and the right-most column of the total CoIncidence matrix. By removing a row and the right-most column, we are removing that simplex from the CI, and moving through backwards through the filtration.

Suppose K has a total of N simplices, and we have moved backwards through the filtration by removing m rows and columns from the reduced CI. If the last row of this new matrix, $\widetilde{CI}_{N-m}$, σ_{N-m} does not contain a lowest one of a column, then

$\widetilde{CI}_{N-m}$ does not need to be further reduced. As we move backwards through the filtration, a class is born or dies depends on when the last row of $\widetilde{CI}_{N-m}$, contains a lowest one of some column, τ. When we remove the last row of the matrix to obtain $\widetilde{CI}_{N-m}$, and its corresponding column, there are two primary cases:

1. A column, τ, in the $\widetilde{CI}_{N-m}$ contains a lowest one in row $N-m$. If necessary, we reduce the new matrix, CI and consider:
   (a) If after reducing the matrix, no new 0-columns are created in $\widetilde{CI}_{N-m}$, then the new lowest one in the column corresponding to τ exists when it did not earlier. Thus, it pairs with a 0 column (corresponding to a cocycle) that was not previously paired, so that cocycle "dies" entering K_{N-m}.
   (b) If after reducing the matrix, a new 0-column is created, then this cocycle corresponding to the cochain comprising that column is born at K_{N-m}.
2. A column, τ, becomes an all zero column in $\widetilde{CI}_{N-m}$, where it was not previously. Then the cocycle represented by τ is "born" entering K_{N-m}.

This first algorithm determines the copersistence of a topological space that is expressed as a simplicial complex. It gives generators for $H^p(K)$ at various levels of the filtration. In order to find the cochain representatives for these generators, knowing the column additions done in reducing the matrix CI gives the representative.

Proof of Correctness. To prove that this algorithm is correct, we follow the principles developed in persistent homology. In Persistent Homology, it was recognized that each cycle, γ, is the sum of negative simplices and one positive simplex. A negative simplex is one which, by adding it into the filtration, kills a one lower dimensional existing homology class. When adding into the filtration the positive simplex, γ is born. This relationship is embedded in the Inicdence matrix through the *low* function. The Pairing Uniqueness Lemma from (CSEM06) says that reducing the Incidence matrix does not affect the *low* function, thus will not change the basis of positive simplices that generate each cycle.

Since our algorithm for computing CoPersistence begins with reducing the transpose of the Incidence matrix, the CoIncidence matrix, it is in many respects dual to the algorithm for computing Persistent homology. When we compute the cohomology of K_n, the last level of the filtration, we have a basis for our cohomology classes. Each cocycle is represented by a basis of positive simplices. Each zero column in the reduced CI corresponds to the positive simplex that represent the cocycle. The rows that contain the lowest ones of some columns are negative simplices. We follow this basis through the algorithm and the filtration.

Suppose we are at the K_i-th level of the filtration. If we add a positive simplex, now we are at the K_{i+1}-st level and homologically we have created a cycle. Since this means that the map $H_p(K_i) \to H_p(K_{i+1})$ increases the rank by 1, the map $H^p(K_i) \leftarrow H^p(K_{i+1})$ decreases the rank by 1. Thus, removing the positive simplex to move backwards from K_{i+1} to K_i kills a cohomology class. On the other hand, if we add a negative simplex, going from K_i to K_{i+1}, homologically we kill a cycle. Thus, the map $H_p(K_i) \to H_p(K_{i+1})$ decreases the rank by one. Moving backwards through the filtration, removing this negative simplex from K_{i+1}, thus getting to K_i, means that the map $H^p(K_i) \leftarrow H^p(K_j)$ increases the rank by one. □

5.3 Persistent Cohomology Class Algorithm

This algorithm determines when a specifically given cohomology class, known to exist at some level of the filtration K_j, dies. This differs from the previous algorithm in that we begin with a given q dimensional cocycle $\Gamma \in \mathcal{C}^q(K_j)$. Suppose $\Gamma = \sigma^*_{i_1} + \sigma^*_{i_2} + \cdots \sigma^*_{i_n}$. The objective of this algorithm is to find when Γ first becomes a coboundary, or when there is a column in CI_q, such that after column addition the only nonzero entries in the column are in the rows corresponding to the σ_{i_j}. We begin with a Γ that is known to be nontrivial at K_j and start this algorithm with qth CoIncidence matrices, CI_q for K_j. As in the previous algorithm, removing rows from this CoIncidence matrix brings us to lower levels of the filtration. Determining when to remove rows depends on certain outcomes of column operations. After a number of row removals, there will be a column, say τ_*, such that the nonzero entries of τ_* are exactly the $\sigma_{i_j} \in \Gamma$ that are still present in CI_q. It is at this point that the remaining set of q simplices directly determine the filtration level at which Γ has become a coboundary and is thus trivial in $H^q(K_i)$ for some K_i. In order to discuss the specifics of this algorithm, we provide an example of it here. Suppose our simplicial complex, K, is given by the following triangulation of $\mathbb{RP}^2$:

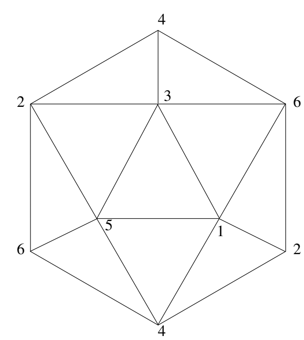

FIGURE 5.1: K is the simplicial complex associated to $\mathbb{RP}^2$ with standard triangulation.

We have that a given cohomology class, $\Gamma \in H^1(K)$ is nontrivial, so in this example $K_j = K$. One possible representation of $\Gamma = \sigma_{i_1}^* + \sigma_{i_2}^* + \cdots + \sigma_{i_n}^*$ is $\langle 2,3\rangle^* + \langle 1,4\rangle^* + \langle 2,4\rangle^* + \langle 1,5\rangle^* + \langle 3,5\rangle^*$. Since $\Gamma \in H^1(K)$, our algorithm uses the CI_1

matrix:

	$\langle 1\rangle$	$\langle 2\rangle$	$\langle 3\rangle$	$\langle 4\rangle$	$\langle 5\rangle$	$\langle 6\rangle$
$\langle 1,2\rangle$	1	1				
$\langle 1,3\rangle$	1		1			
$\langle 2,3\rangle$		1	1			
$\langle 1,4\rangle$	1			1		
$\langle 2,4\rangle$		1		1		
$\langle 3,4\rangle$			1	1		
$\langle 1,5\rangle$	1				1	
$\langle 2,5\rangle$		1			1	
$\langle 3,5\rangle$			1		1	
$\langle 4,5\rangle$				1	1	
$\langle 1,6\rangle$	1					1
$\langle 2,6\rangle$		1				1
$\langle 3,6\rangle$			1			1
$\langle 4,6\rangle$				1		1
$\langle 5,6\rangle$					1	1

The algorithm starts with σ_{i_n} in row i_n. In this example, $\sigma_{i_n} = \langle 3,5\rangle$. We scan row i_n and find its nonzero entries. We store the indexes, and therefore associated codimension-1 simplices, of the columns having a 1 in row associated to σ_{i_n}. Call this set $T = \{\tau_{c_1}, \tau_{c_2}, \ldots, \tau_{c_t}\}$. For this example, $T = \{\langle 3\rangle, \langle 5\rangle\}$. The next step of the algorithm is to scan each column in this set to see if it has a lowest 1 in row σ_{i_n}.

Suppose none of the columns (τ_{c_1} through τ_{c_t}) have their lowest one in row σ_{i_n}. This is the case in this example; $low(\langle 3\rangle) = \langle 3,6\rangle$ and $low(\langle 5\rangle) = \langle 5,6\rangle$. If this is the case, we remove the fewest number of rows from CI_q so that at least one τ_{c_s}, $1 \leq s \leq t$, has its lowest one in the row associated to σ_{i_n}. In our example, this would be removing the first 3 rows, so that $low(\langle 3\rangle) = \langle 3,5\rangle$. If, however, there is at least one τ_{c_s} such that $low(\tau_{c_s}) = \sigma_{i_n}$, then we do not remove any rows and continue to the next step in our algorithm.

Since for our example, we must remove 3 rows, the new CI_1 is:

	$\langle 1\rangle$	$\langle 2\rangle$	$\langle 3\rangle$	$\langle 4\rangle$	$\langle 5\rangle$	$\langle 6\rangle$
$\langle 1,2\rangle$	1	1				
$\langle 1,3\rangle$	1		1			
$\langle 2,3\rangle$		1	1			
$\langle 1,4\rangle$	1			1		
$\langle 2,4\rangle$		1		1		
$\langle 3,4\rangle$			1	1		
$\langle 1,5\rangle$	1				1	
$\langle 2,5\rangle$		1			1	
$\langle 3,5\rangle$			1		1	
$\langle 4,5\rangle$				1	1	
$\langle 1,6\rangle$	1					1
$\langle 2,6\rangle$		1				1

The last step of the algorithm guarantees that at least one of the $\tau_{c_s} \in T$ has a lowest one in the row associated to σ_{i_n}. We choose one such column, say τ_{c_s}. We that column τ_{c_s} now serves as our τ_*, and we scan it from the bottom upwards. For our example $\tau_* = \langle 3\rangle$. The next step of the algorithm depends on two things: whether the row directly above σ_{i_n}, σ_{i_n-1}, is in the expression of Γ and the value of the entry in τ_* in this row. For the two cases, where σ_{i_n-1} is not in the expression of Γ and the entry in τ_* is 0, or the case where σ_{i_n-1} is in the expression of Γ and the corresponding entry is 1 in τ_*, we do nothing. We continue up τ_* to the next row of CI_q. In our example, $CI_1(\langle 2,5\rangle, \langle 3\rangle) = 0$ and $\langle 2,5\rangle$ is not in the expression of Γ, it is one of these two cases. We continue to $CI_1(\langle 1,5\rangle, \langle 3\rangle)$.

If, on the other hand, we encounter the two cases where σ_{i_n-1} is in the expression of Γ and its entry in τ_* is 0, or σ_{i_n-1} is not in the expression of Γ and its entry in τ_* is 1, the next step of our algorithm becomes scanning the row associated to σ_{i_n-1}. We search for a lowest one in the row σ_{i_n-1} and if such a column exists, in either instance we add it to τ_*. Recall, we want a column τ_*, such that the only nonzero entries correspond to the σ_{i_j}. If τ_* has a 1 in the row associated to σ_{i_n-1} and σ_{i_n-1} is not in the expression of Γ, then adding a column who has a lowest one in σ_{i_n-1} with $\mathbb{Z}_2$ coefficients will remove the one, and τ_* will have its entries match the simplicies in Γ for σ_{i_n-1} and below. It is a similar result for the other situation. If no such column contains a lowest one in row σ_{i_n-1}, then we remove the fewest number of rows until row σ_{i_n-1} contains a lowest one of a column.

In our example, $CI_1(\langle 2,5\rangle, \langle 3\rangle)$ was not of either of the two latter cases. This would imply that we continue to $CI_1(\langle 1,5\rangle, \langle 3\rangle)$, which is one of these two later cases. Specifically, $CI_1(\langle 1,5\rangle, \langle 3\rangle) = 0$ however $\langle 1,5\rangle$ is in the expression of Γ. So, according to the last step in the algorithm, we would scan $\langle 1,5\rangle$ for its nonzero entries. These are in columns $\langle 1\rangle$ and $\langle 5\rangle$. Since $low(\langle 1\rangle) = \langle 1,6\rangle$ and $low(\langle 5\rangle) = \langle 4,5\rangle$, we are not in a situation where the lowest one of either is in $\langle 1,5\rangle$. Thus, we remove the least number of rows from CI_1 so that the lowest one of at least one column is in $\langle 1,5\rangle$. For this example, that would be 2 rows. Our new CI_1 is:

	$\langle 1\rangle$	$\langle 2\rangle$	$\langle 3\rangle$	$\langle 4\rangle$	$\langle 5\rangle$	$\langle 6\rangle$
$\langle 1,2\rangle$	1	1				
$\langle 1,3\rangle$	1		1			
$\langle 2,3\rangle$		1	1			
$\langle 1,4\rangle$	1			1		
$\langle 2,4\rangle$		1		1		
$\langle 3,4\rangle$			1	1		
$\langle 1,5\rangle$	1				1	
$\langle 2,5\rangle$		1			1	
$\langle 3,5\rangle$			1		1	
$\langle 4,5\rangle$				1	1	

The algorithm proceeds as above. As we move up the column τ_*, the next step of the algorithm is determined by the next entry of τ_*. Suppose we are at row σ_{i_j} in some reduced CI_q. When we look at the σ_{i_j} entry of τ_* we want $CI_q(\sigma_{i_j}, \tau_*) = 1$ iff σ_{i_j} is in the expression of Γ. If $\sigma_{i_j} \in \Gamma$ and $CI_q(\sigma_{i_j}, \tau_*) = 1$ or $\sigma_{i_j} \notin \Gamma$ and $CI_q(\sigma_{i_j}, \tau_*) = 0$, then we move up the column τ_* to the next row. If either of the two cases where $CI_q(\sigma_{i_j}, \tau_*) = 1$ but $\sigma_{i_j} \notin \Gamma$ or $CI_q(\sigma_{i_j}, \tau_*) = 0$ but $\sigma_{i_j} \in \Gamma$, the next step in our algorithm is ti scan the row associated to σ_{i_j} to find a column whose lowest one is in that row. If such a column does exist, we add it to τ_* and continue by looking at $CI_q(\sigma_{i_j-1}, \tau_*)$. If, on the other hand, no such column exists, we remove the least number of rows from the bottom of CI_q such that either (a) there is a lowest one in the row σ_{i_j} or (b) we remove the lowest remaining σ_{i_m} in the expression of Γ. Sometimes the lowest ones for columns which have nonzero entries in row σ_{i_j} might exist in rows with a lesser index than the lowest $\sigma_{i_n} \in \Gamma$. If this is the case, then we remove rows from CI_q up to and including σ_{i_n} and begin the algorithm again with $\sigma_{i_{n-1}}$, choosing a different τ_*, accordingly.

By induction on the rows of CI_q, we will reach a reduced CI_q where the nonzero entries of a given column τ_* are in only the rows on which Γ evaluates 1. It is this level of reduction of CI_q that gives us the level of the filtration of K where Γ "dies." When we have removed enough rows from CI_q where τ_* has its only nonzero entries in rows in the expression of Γ, the remaining q-simplices of CI_q and define the filtration level K_i. It is at this point that τ_* represents the coboundary of Γ, and thus, where Γ becomes trivial, or "dies." If you return the last rows removed, they would define the next level of the filtration, K_{i+1}, where the nonzero elements of τ_* are not in a one-to-one correspondence with the representation of Γ, thus Γ is not a coboundary. By moving from K_{i+1} to K_i, Γ becomes trivial in $H^q(K_i)$, and thus "dies."

Proof of Correctness. Because of the construction of the CoIncidence matrix, $CI_q(\sigma_i, \sigma_j) = 1$ means that σ_i is in the coboundary of σ_j. Thus, for any stage of the filtration K_i, if there exist a set of column operations so that the resulting column τ_*, is such that $CI_q(\sigma_{i_j}, \tau_*) = 1$ iff $\Gamma(\sigma_{i_j}) = 1$, then Γ is exactly the coboundary of the cochain comprising τ_* and is therefore trivial in $H^q(K_i)$.

Beginning the algorithm by ensuring there exists one such column that has its lowest one in the row associated to σ_{i_n}, it follows that $CI_q(\sigma_m, \tau_*)$ is zero for all $m > i_n$. Since the lowest index row in CI_q on which Γ evaluates 1 is σ_{i_n}, the nonzero entries of τ_* agree with Γ for all σ_m for $m \geq i_n$. As we scan up τ_* we only perform column addition when $CI_q(\sigma_j, \tau_*) \neq \Gamma(\sigma_j)$. This ensures that the nonzero entries of τ_* are in one-to-one correspondence with the simplicies in the expression of Γ for each σ_m for $m > j$. Whenever a column is added to τ_*, our goal is to preserve this one-to-one correspondence for each σ_m. This can only be done by requiring that the lowest one for the column being added to τ_* is above m, that is, its lowest one is in row σ_j.

Every time we remove rows from the bottom of CI_q, we are moving backwards through the filtration. If we cannot obtain τ_* through our method, then Γ is not a coboundary. By removing rows in the CoIncidence matrix, we begin the algorithm again until either discovering that we cannot create τ_* or a τ_* has been created. The column τ_* can only be created at a level of the filtration K_i, where Γ is trivial in $H^q(K_i)$, thus we find the level of the filtration at which Γ "dies" entering. □

5.4 Illustrated Example

Here we return to our example to and finish it off to find where Γ "dies" in our filtration of $\mathbb{RP}^2$. Recall, that we have removed the rows up to and including $\langle 1, 6 \rangle$ and our CI_1 is:

	$\langle 1\rangle$	$\langle 2\rangle$	$\langle 3\rangle$	$\langle 4\rangle$	$\langle 5\rangle$	$\langle 6\rangle$
$\langle 1,2\rangle$	1	1				
$\langle 1,3\rangle$	1		1			
$\langle 2,3\rangle$		1	1			
$\langle 1,4\rangle$	1			1		
$\langle 2,4\rangle$		1		1		
$\langle 3,4\rangle$			1	1		
$\langle 1,5\rangle$	1				1	
$\langle 2,5\rangle$		1			1	
$\langle 3,5\rangle$			1		1	
$\langle 4,5\rangle$				1	1	

For $\tau_* = \langle 3\rangle$, we have $CI_1(\langle 3,5\rangle, \langle 3\rangle) = 1, CI_1(\langle 2,5\rangle, \langle 3\rangle) = 0$, and $CI_1(\langle 1,5\rangle, \langle 3\rangle) = 0$, where $\langle 3,5\rangle$ and $\langle 1,5\rangle \in \Gamma$ but $\langle 2,5\rangle \notin \Gamma$. Scanning $\langle 1,5\rangle$, we have a lowest one in column $\langle 1\rangle$, so we add $\langle 1\rangle$ to $\langle 3\rangle$ and $\tau_* = (\langle 3\rangle + \langle 1\rangle)$. τ_* has non-zero entries in $\langle 3,5\rangle$, $\langle 1,5\rangle$, $\langle 3,4\rangle$, $\langle 1,4\rangle$, $\langle 2,3\rangle$ and $\langle 1,2\rangle$.

Scanning up $(\langle 3\rangle + \langle 1\rangle)$, $CI_1(\langle 3,4\rangle, (\langle 3\rangle + \langle 1\rangle)) = 1$ but $\Gamma(\langle 3,4\rangle) = 0$, we have a contradiction. We scan the row associated to $\langle 3,4\rangle$ and find that it is nonzero entries are in columns $\langle 3\rangle$ and $\langle 4\rangle$. $low(\langle 3\rangle) = \langle 3,5\rangle$ and $low(\langle 4\rangle) = \langle 4,5\rangle$. Thus, we remove all the rows up to and including the row associated to $\langle 4,5\rangle$ from our CoIncidence matrix. CI_1 becomes:

	$\langle 1\rangle$	$\langle 2\rangle$	$\langle 3\rangle$	$\langle 4\rangle$	$\langle 5\rangle$	$\langle 6\rangle$
$\langle 1,2\rangle$	1	1				
$\langle 1,3\rangle$	1		1			
$\langle 2,3\rangle$		1	1			
$\langle 1,4\rangle$	1			1		
$\langle 2,4\rangle$		1		1		
$\langle 3,4\rangle$			1	1		
$\langle 1,5\rangle$	1				1	
$\langle 2,5\rangle$		1			1	
$\langle 3,5\rangle$			1		1	

$\tau_* = (\langle 3\rangle + \langle 1\rangle)$ has non zero entries in $\langle 3,5\rangle$, $\langle 1,5\rangle$, $\langle 3,4\rangle$, $\langle 1,4\rangle$, $\langle 2,3\rangle$ and $\langle 1,2\rangle$.

Since $CI_1(\langle 3,5\rangle, \tau_*) = CI_1(\langle 1,5\rangle, \tau_*) = CI_1(\langle 3,4\rangle, \tau_*) = 1$, $CI_1(\langle 2,5\rangle, \tau_*) = 0$ and $\langle 3,5\rangle, \langle 1,5\rangle \in \Gamma$ while $\langle 2,5\rangle, \langle 3,4\rangle \notin \Gamma$, we scan the row associated to $\langle 3,4\rangle$ to find that its nonzero entries are in columns $\langle 3\rangle$ and $\langle 4\rangle$. Since $low(\langle 4\rangle)$ is now in $\langle 3,4\rangle$ and $low(\langle 3\rangle)$ is in $\langle 3,5\rangle$, we add $\langle 4\rangle$ to τ_*. Now, we continue scanning up column τ_*. $CI_1(\langle 2,4\rangle, \tau_*) = 1$ and $\Gamma(\langle 2,4\rangle) = 1$, so there is no contradiction. Continuing, $CI_1(\langle 1,4\rangle, \tau_*) = 0$ but $\Gamma(\langle 1,4\rangle) = 1$. We have a contradiction. Since $low(\langle 1\rangle) = \langle 1,5\rangle$ and $low(\langle 4\rangle) = \langle 3,4\rangle$ and $\sigma_{i_n} = \langle 3,5\rangle$, we cannot remove the contradiction through column operations and we begin our algorithm from the start.

Since the row associated to $\langle 3,5\rangle$ is the last row of CI_1, we could also have chosen column $\langle 5\rangle$ as τ_*. Scanning up column $\langle 5\rangle$, $CI_1(\langle 2,5\rangle, \tau_*) = 1$ and $\Gamma(\langle 2,5\rangle) = 0$. We scan row $\langle 2,5\rangle$ and find that its nonzero entries are in columns $\langle 2\rangle$ and $\langle 5\rangle$. Since $low(\langle 2\rangle) = \langle 2,5\rangle$, we add column $\langle 2\rangle$ to column $\langle 5\rangle$. Scanning from bottom to top, $CI_1(\langle 1,5\rangle, \tau_*) = 1$ and $\Gamma(\langle 1,5\rangle) = 1$, so we have no contradiction. Continuing, $CI_1(\langle 3,4\rangle, \tau_*) = 0$ and $\Gamma(\langle 3,4\rangle) = 0$ so we move up a row in column τ_*. Since $CI_1(\langle 2,4\rangle, \tau_*) = 1$ and $\Gamma(\langle 2,4\rangle) = 1$, there is no still no contradiction. Continuing, $CI_1(\langle 1,4\rangle, \tau_*) = 0$ but $\Gamma(\langle 1,4\rangle) = 1$, so the nonzero entries of τ_* at $\langle 1,4\rangle$ and below are not in one-to-one correspondence with Γ. Since $low(\langle 1\rangle) = \langle 1,5\rangle$ and $low(\langle 4\rangle) = \langle 3,4\rangle$, we cannot remove the contradiction through column operations. We have exhausted the possibilities of columns whose lowest ones are in row $\langle 3,5\rangle$.

We remove the rows up to and including $\langle 3,5\rangle$. Doing so reduces the expression of Γ to $\tilde{\Gamma} = \langle 2,3\rangle^* + \langle 1,4\rangle^* + \langle 2,4\rangle^* + \langle 1,5\rangle^*$. Furthermore, our CoIncidence matrix becomes:

	⟨1⟩	⟨2⟩	⟨3⟩	⟨4⟩	⟨5⟩	⟨6⟩
⟨1, 2⟩	1	1				
⟨1, 3⟩	1		1			
⟨2, 3⟩		1	1			
⟨1, 4⟩	1			1		
⟨2, 4⟩		1		1		
⟨3, 4⟩			1	1		
⟨1, 5⟩	1				1	
⟨2, 5⟩		1			1	

We begin the process again. Now, the lowest row corresponding to an single cochain in Γ is $\langle 1,5\rangle$. We scan the row associated to $\langle 1,5\rangle$ and discover that it has nonzero entries in columns $\langle 1\rangle$ and $\langle 5\rangle$. $low(\langle 1\rangle) = \langle 1,5\rangle$ and $low(\langle 5\rangle) = \langle 2,5\rangle$. Therefore, we set column $\langle 1\rangle = \tau_*$. $CI_1(\langle 3,4\rangle, \tau_*) = 0$ and $\Gamma(\langle 3,4\rangle) = 0$, so we continue up τ_*. Since $CI_1(\langle 2,4\rangle, \tau_*) = 0$ while $\Gamma(\langle 2,4\rangle) = 1$, we scan the row associated to $\langle 2,4\rangle$ and find that its nonzero entries are in columns $\langle 2\rangle$ and $\langle 4\rangle$. Since $low(\langle 2\rangle) = \langle 2,5\rangle$ and $low(\langle 4\rangle) = \langle 3,4\rangle$, we need to remove the simplices up to an including $\langle 2,5\rangle$ from both CI_1 and our expression of Γ. Our CoIncidence matrix is now:

	⟨1⟩	⟨2⟩	⟨3⟩	⟨4⟩	⟨5⟩	⟨6⟩
⟨1, 2⟩	1	1				
⟨1, 3⟩	1		1			
⟨2, 3⟩		1	1			
⟨1, 4⟩	1			1		
⟨2, 4⟩		1		1		
⟨3, 4⟩			1	1		
⟨1, 5⟩	1				1	

We begin our procedure again, noticing that both $\langle 1\rangle$ and $\langle 5\rangle$ are associated to columns with a lowest one in row $\langle 1,5\rangle$. Without loss of generality, we set τ_* to be column $\langle 1\rangle$. Since $CI_1(\langle 3,4\rangle, \tau_*) = 0$ and $\Gamma(\langle 3,4\rangle) = 0$, we move up a row and see

$CI(\langle 2,4\rangle, \tau_*) = 0$ while $\Gamma(\langle 2,4\rangle) = 1$. We scan the row associated to $\langle 2,4\rangle$ that it has nonzero entries in columns $\langle 2\rangle$ and $\langle 4\rangle$. Since $low(\langle 2\rangle) = \langle 2,4\rangle$, we add column $\langle 2\rangle$ to $\langle 1\rangle$ and $\tau_* = (\langle 2\rangle + \langle 2\rangle)$. Continuing scanning from bottom to top, $CI_1(\langle 1,4\rangle, \tau_*) = 1$ and $\Gamma(\langle 1,4\rangle) = 1$, therefore we have no contradiction and continue to scan up τ_*. Since $CI_1(\langle 2,3\rangle, \tau_*) = \Gamma(\langle 2,3\rangle) = 1$, we again have no contradiction. Moving up τ_*, $CI_1(\langle 1,3\rangle, \tau_*) = 1$, however $\Gamma(\langle 1,3\rangle) = 0$. Here, we have a contradiction. Since the nonzero entries in the row $\langle 1,3\rangle$ are in columns $\langle 1\rangle$ and $\langle 3\rangle$, where $low(\langle 1\rangle) = \langle 1,5\rangle$ and $low(\langle 3\rangle) = \langle 3,4\rangle$. We cannot remove this contradiction with merely a column operation.

Since column $\langle 5\rangle$ also had its lowest one in row $\langle 1,5\rangle$, we could also choose $\tau_* = \langle 5\rangle$. $CI(\langle 3,4\rangle, \tau_*) = 0$ and $\Gamma(\langle 3,4\rangle) = 0$ implies no contraction. Continuing, $CI(\langle 2,4\rangle, \tau_*) = 0$ while $\Gamma(\langle 2,4\rangle) = 1$, implies a contradiction. We scan the row associated to $\langle 2,4\rangle$ and find that its nonzero entries are in columns $\langle 2\rangle$ and $\langle 4\rangle$. Since $low(\langle 2\rangle) = \langle 2,4\rangle$, we add column $\langle 2\rangle$ to column $\langle 5\rangle$. τ_* is now $(\langle 5\rangle + \langle 2\rangle)$. Scanning from bottom to top, $CI_1(\langle 1,4\rangle, \tau_*) = 0$, while $\Gamma(\langle 1,4\rangle) = 1$, so we have a contradiction. Scanning the row associated to $\langle 1,4\rangle$, we find its nonzero entries in columns $\langle 1\rangle$ and $\langle 4\rangle$. Since $low(\langle 1\rangle)$ is in row $\langle 1,5\rangle$ and $low(\langle 2\rangle)$ is in the row associated to $\langle 3,4\rangle$; both of these are lower in CI_1 than $\langle 1,4\rangle$. Therefore we again have a contradiction that cannot be removed by a column operation. We remove all the simplices up to an including $\langle 1,5\rangle$ from both CI_1 and Γ. Our CoIncidence matrix becomes:

	$\langle 1\rangle$	$\langle 2\rangle$	$\langle 3\rangle$	$\langle 4\rangle$	$\langle 5\rangle$	$\langle 6\rangle$
$\langle 1,2\rangle$	1	1				
$\langle 1,3\rangle$	1		1			
$\langle 2,3\rangle$		1	1			
$\langle 1,4\rangle$	1			1		
$\langle 2,4\rangle$		1		1		
$\langle 3,4\rangle$			1	1		

Beginning again, we have the lowest simplex associated to Γ in CI_1 is now $\langle 2,4\rangle$. Scanning the row associated to $\langle 2,4\rangle$, we find its nonzero entries in columns $\langle 2\rangle$ and $\langle 4\rangle$. Since $low(\langle 2\rangle)$ is in $\langle 2,4\rangle$, we set $\tau_* = \langle 2\rangle$. $CI_1(\langle 1,4\rangle,\tau_*) = 0$ but $\Gamma(\langle 1,4\rangle) = 1$, so we have a contradiction. Scanning row $\langle 1,4\rangle$, we find its nonzero entries in columns $\langle 1\rangle$ and $\langle 4\rangle$. Since $low(\langle 1\rangle)$ is in the row associated to $\langle 1,4\rangle$, we add column $\langle 1\rangle$ to column $\langle 2\rangle$ making $\tau_* = \langle 2\rangle + \langle 1\rangle$. Continuing, $CI_1(\langle 2,3\rangle,\tau_*) = \Gamma(\langle 2,3\rangle) = 1$, we have no contradiction. $CI_1(\langle 1,3\rangle,\tau_*) = 1$ while $\Gamma(\langle 1,3\rangle) = 0$. We have a contradiction. Scanning the row $\langle 1,3\rangle$, we find its nonzero entries in columns $\langle 1\rangle$ and $\langle 3\rangle$. Since $low(\langle 1\rangle) = \langle 1,4\rangle$ and $low(\langle 3\rangle) = \langle 3,4\rangle$, we remove all simplices up to and including $\langle 3,4\rangle$ from the CoIncidence matrix. Now, CI_1 is:

	$\langle 1\rangle$	$\langle 2\rangle$	$\langle 3\rangle$	$\langle 4\rangle$	$\langle 5\rangle$	$\langle 6\rangle$
$\langle 1,2\rangle$	1	1				
$\langle 1,3\rangle$	1		1			
$\langle 2,3\rangle$		1	1			
$\langle 1,4\rangle$	1			1		
$\langle 2,4\rangle$		1		1		

Starting our procedure again, we have 2 nonzero entries in $\langle 2,4\rangle$, in columns $\langle 2\rangle$ and $\langle 4\rangle$, both of which have lowest ones in row $\langle 2,4\rangle$. Choosing to set $\tau_* = \langle 2\rangle$, $CI_1(\langle 1,4\rangle,\tau_*) = 0$ while $\Gamma(\langle 1,4\rangle) = 1$. Searching row $\langle 1,4\rangle$, we find its nonzero entries in columns $\langle 1\rangle$ and $\langle 4\rangle$. Since $low(\langle 1\rangle) = \langle 1,4\rangle$, we can add column $\langle 1\rangle$ to τ_*.

Continuing, $CI(\langle 2,3\rangle, \tau_*) = 1\Gamma(\langle 2,3\rangle) = 1$. $CI(\langle 1,3\rangle, \tau_*) = 1$ while $\Gamma(\langle 1,3\rangle) = 0$. Scanning row $\langle 1,3\rangle$ we find its nonzero elements are in columns $\langle 1\rangle$ and $\langle 3\rangle$. Since $low(\langle 1\rangle) = \langle 1,4\rangle$ and $low(\langle 3\rangle) = \langle 2,3\rangle$, we have a contradiction that cannot be removed by a column operation.

Choosing column $\tau_* + \langle 4\rangle$ instead and scanning it up, we see $CI(\langle 1,4\rangle,) = \Gamma(\langle 1,4\rangle) = 1$, thus, there is no contradiction. Continuing, $CI(\langle 2,3\rangle, \tau) = 0$ while $\Gamma(\langle 2,3\rangle) = 1$; we have a contradiction. Scanning $\langle 2,3\rangle$, we find its nonzero entries in columns $\langle 2\rangle$ and $\langle 3\rangle$. Since $low(\langle 3\rangle) = \langle 2,3\rangle$, we add column $\langle 3\rangle$ to column τ_* and continue scanning upwards. $CI_1(\langle 1,3\rangle, \tau_*) = 1$ while $\Gamma(\langle 1,3\rangle) = 0$. This is a contradiction, but since the two nonzero entries of $\langle 1,3\rangle$ are in columns whose lowest ones are greater than $\langle 1,3\rangle$, we cannot remove this contradiction with column addition. We remove all the simplices up to and including $\langle 2,4\rangle$ from both CI_1 and Γ. Our current CoIncidence matrix is :

	$\langle 1\rangle$	$\langle 2\rangle$	$\langle 3\rangle$	$\langle 4\rangle$	$\langle 5\rangle$	$\langle 6\rangle$
$\langle 1,2\rangle$	1	1				
$\langle 1,3\rangle$	1		1			
$\langle 2,3\rangle$		1	1			
$\langle 1,4\rangle$	1			1		

Beginning with the now lowest simplex associated to Γ, $\langle 1,4\rangle$. Searching the row $\langle 1,4\rangle$, we find that its nonzero entries are in columns $\langle 1\rangle$ and $\langle 4\rangle$, both of which have their lowest ones in the row associated to $\langle 1,4\rangle$. Without loss of generality, we choose $\tau_* = \langle 1\rangle$. Scanning from bottom to top, $CI_1(\langle 2,3\rangle, \langle 1\rangle) = 0$ while $\Gamma(\langle 2,3\rangle) = 1$. Searching row $\langle 2,3\rangle$, we find that it has two columns, both of which have its lowest one in row $\langle 2,3\rangle$. Without loss of generality, we choose column $\langle 2\rangle$ and add it to τ_*. $CI_1(\langle 1,3\rangle, \tau_*) = 1$ but $\Gamma(\langle 1,3\rangle) = 0$. Neither of the columns containing the nonzero entries of $\langle 1,3\rangle$ have a lowest one in that row, so we have reached a contradiction that cannot be removed by a column operation. If, however, we had chosen $\langle 3\rangle$ to add to column τ_*, we see $CI_1(\langle 1,3\rangle, \tau_*) = 0$ and $\Gamma(\langle 1,3\rangle) = 0$; no contradiction. However, $CI_1(\langle 1,2\rangle, \tau_*) = 1$ where $\Gamma(\langle 1,2\rangle) = 0$. Since none of the nonzero entries of $\langle 1,2\rangle$ have columns with lowest one in row $\langle 1,2\rangle$, we have reached a contradiction that cannot be removed with column operations.

On the other hand, suppose we set tau_* to equal column $\langle 4\rangle$ instead. $CI_1(\langle 2,3\rangle, \tau_*) = 0$ but $\Gamma(\langle 2,3\rangle) = 1$. Scanning the row associated to $\langle 2,3\rangle$, we find nonzero entries in columns $\langle 2\rangle$ and $\langle 3\rangle$. Since both of these columns have lowest ones in that row, we can choose, without loss of generality $\langle 2\rangle$. Adding column $\langle 2\rangle$ to column $\langle 4\rangle$, τ_* becomes $(\langle 4\rangle + \langle 2\rangle)$. $CI_1(\langle 2,3\rangle, \tau_*) = \Gamma(\langle 2,3\rangle) = 1$, so there is no contradiction. Continuing, $CI_1(\langle 1,3\rangle, \tau_*) = \Gamma(\langle 1,3\rangle) = 0$, again there is no contradiction. $CI_1(\langle 1,2\rangle, \tau_*) = 1$ while $\Gamma(\langle 1,2\rangle) = 0$. Since none of the columns containing the nonzero entries of $\langle 1,2\rangle$ have their lowest ones in that row, so there is a contradiction that cannot be removed with column operations. A similar situation would have occurred had we added column $\langle 3\rangle$ to $\langle 4\rangle$ instead. Therefore, we remove all simplices up to and including $\langle 1,4\rangle$ from CI_1 as well as Γ. Our CoIncidence matrix is now:

	$\langle 1\rangle$	$\langle 2\rangle$	$\langle 3\rangle$	$\langle 4\rangle$	$\langle 5\rangle$	$\langle 6\rangle$
$\langle 1,2\rangle$	1	1				
$\langle 1,3\rangle$	1		1			
$\langle 2,3\rangle$		1	1			

The lowest simplex in our Γ is now $\langle 2,3\rangle$. Scanning this row, we have nonzero elements in columns $\langle 2\rangle$ and $\langle 3\rangle$, both of which have their lowest ones in this row. Without loss of generality, if we choose to set $\tau_* = \langle 2\rangle$, $CI_1(\langle 1,3\rangle, \tau) = \Gamma(\langle 1,3\rangle) = 0$. However, $CI_1(\langle 1,2\rangle, \tau_*) = 1$, but $\Gamma(\langle 1,2\rangle) = 0$. Since the two columns that contain the nonzero elements of $\langle 1,2\rangle$ do not have their lowest ones in $\langle 1,2\rangle$, we have a contradiction that cannot be removed. If we had chosen to set $\tau_* = \langle 3\rangle$ instead,

$CI_1(\langle 1,3\rangle, \tau_*) = 1$ while $\Gamma(\langle 1,3\rangle) = 0$. Since column $\langle 1\rangle$ has a lowest one in $\langle 1,3\rangle$, we add it to τ_*. Continuing, $CI_1(\langle 1,2\rangle, \tau_*) = 1$ while $\Gamma(\langle 1,2\rangle) = 0$. However, since $\langle 1\rangle$ and $\langle 2\rangle$, have lowest ones below $\langle 1,2\rangle$, we have a contradiction that cannot be removed with adding columns.

At this point we would remove all the simplices up to and including $\langle 2,3\rangle$, which also removes the last simplex corresponding to Γ. At this point, Γ is now trivial and our coicidence matrix consists of two rows. Since Γ is now trivial, we have reached the level of the filtration directly prior to the point where Γ is born. Since the last set of simplicies removed before Γ became trivial was $\langle 2,3\rangle$, the level of the filtration at which Γ dies is the simplicial complex that is the union of the following simplices: $\langle 1\rangle, \langle 2\rangle, \langle 3\rangle, \langle 1,2\rangle, \langle 1,3\rangle$, and $\langle 2,3\rangle$.

6

Extended CoPersistence

When the study of Persistent Homology was being developed, one advancement was Extended Persistence. There are two major benefits to considering extended persistence over regular persistence. First, Extended Persistent Homology is able to pair all homology classes, essential absolute homology classes are paired with essential relative homology classes. The second benefit is that extended persistence provides a method for comparing two filtrations. As mentioned in Chapter 2, this helps define a measure of topological feature size, an important component in some modeling applications. Because of the close relation of homology and cohomology, the principles used for Extended Persistence can likewise be used to develop Extended CoPersistence. In this chapter, we create an algorithm to define and compute Extended Persistent Cohomology.

6.1 Principles of Extended Persistent Cohomology

As with Extended Persistent Homology, we again have two filtrations of our topological space, K, which we splice together to make a single sequence of cohomology groups. This allows us to compare the filtrations as well as define persistent cohomology for essential cohomology classes. First, however, let us develop the objects used in extended persistent cohomology as we did for extended persistent homology.

When defining CoPersistence, we began with a filtration

$$\emptyset = X_0 \subset X_1 \subset \cdots \subset X_n = X$$

of our topological space, X. We then considered the following sequence of Cohomology groups:

$$0 = H^q(X_0) \leftarrow H^q(X_1) \leftarrow \cdots \leftarrow H^q(X_n) \cong H^q(X)$$

with the maps induced by inclusions. In the case that X is a manifold, we can use Lefschetz duality to show that the above sequence is dual to:

$$0 \to H_p(X_0, \partial X_0) \to H_p(X_1, \partial X_1) \to \cdots \to H_p(X_n, \partial X_n) = H_p(X)$$

$(p + q = d)$ from which we define persistent homology. We could define Extended Persistent Cohomology as the dual to Extended Persistent Homology, but instead we are looking for a definition that works on any simplicial complex. So suppose we are given two filtrations of simplicial complexes that triangulate the same topological space, X:

$$\emptyset = K_0 \subset K_1 \subset \cdots \subset K_n = X \quad \text{and} \quad \emptyset = L_0 \subset L_1 \subset \cdots \subset L_m = X$$

The sequence used for extended persistent homology is

$$0 = H_p(K_0) \to H_p(K_1) \to \cdots \to H_p(K_n) =$$

$$= H_p(K, L_0) \to H_p(K, L_1) \to \cdots H_p(K_n, L_m) = 0$$

Similarly, we use the following for extended persistent cohomology:

$$0 = H^q(K_0) \leftarrow H^q(K_1) \leftarrow \cdots \leftarrow H^q(K_n) =$$

$$= H^q(K_n, L_0) \leftarrow H^q(K_n, L_1) \leftarrow \cdots \leftarrow H^q(K_n, L_m) = 0$$

The definitions of **birth** and **death** of a cohomology class in this sequence are the analogues of those in Chapter 5. That is, let $\psi_{j,i} : H^q(K_j) \to H^q(K_i)$ with $j \geq i$. A cohomology class, Γ, is said to be **born** at K_j, if $\Gamma \neq 0 \in H^q(K_j)$, and $\Gamma \notin Im(\psi_{j+1,j})$. This cohomology class, Γ is said to **die** entering K_i if there exists a $\Lambda \in H^q(K_{j+1})$ such that $\psi_{j+1,i}(\Lambda) = \psi_{j,i}(\Gamma) \in H^q(K_i)$, but $\psi_{j,i+1}(\Gamma) \notin Im(\psi_{j+1,i+1})$. We extend the definition to include relative classes and extended pairs.

Define the $\phi_{j,i} : H^q(K_n, L_j) \to H^q(K_n, L_i)$. We say that a *relative* cohomology class, Γ, is **born** at (K_n, L_j) if $\Gamma \neq 0 \in H^q(K_n, L_j)$ and $\Gamma \notin Im(\phi_{j+1,j})$. We say Γ **dies** entering (K_n, L_i) if there exists a $\Lambda \in H^q(K_n, L_{j+1})$ such that $\phi_{j,i}(\Gamma) \in Im(\phi_{j+1,i}(\Lambda))$ and $\phi_{j,i+1}(\Gamma) \notin Im(\phi_{j+1,i+1})$. In this case we define the copersistence, **copers**$(\Gamma) = j - i$.

We say that an *extended* cohomology class, Γ, is **born** at (K_n, L_j) if $\Gamma \neq 0 \in H^q(K_n, L_j)$ and $\Gamma \notin Im(\phi_{j+i,j})$. We say Γ **dies** entering K_i if there exists a $\Lambda \in H^q(K_n, L_{j+1})$ such that i is the largest index where $\psi_{n,i}(\phi_{j,0}(\Gamma)) = \psi_{n,i}(\phi_{j+1,0}(\Lambda)) \in H^q(K_i)$. In this case, we say that the copersistence, **copers**$(\Gamma) = j + (n - i)$. We can see from this last definition that a cohomology class, potentially the Stiefel-Whitney cohomology class could be born as a relative cohomology class, at some level of the filtration (K_n, L_j), and die as a cohomology class at some (K_i).

6.2 Extended Persistent Cohomology Algorithm

The main object used in computing extended persistent cohomology is the Extended CoIncidence matrix. In Chapter 5, we defined the q-dimensional CoIncidence matrix to the be the matrix whose rows correspond to the q-dimensional simplices and whose columns correspond to the $q-1$ dimensional simplices. It is just the transpose of the Incidence Matrix. Thus,$CI_q(i, j) = 1$ if and only if the q-simplex in row i has the $q-1$ simplex in column j as a codimension 1 face. When computing regular persistent cohomology, we could separate out each dimension easily, considering only the simplices of dimensions other than q and $q-1$. Now, however, it is easier to use the full CoIncidence matrix to calculate the persistence of all cohomology classes of all dimensions simultaneously.

For a given filtration of simplicial complexes,

$$\emptyset = K_0 \subset K_1 \subset \cdots \subset K_n = K$$

we create the total CoIncidence Matrix, TCI. Rows and columns of this matrix correspond to the simplices of K in their total order given by the partial ordering from the filtration and an arbitrary choice at each step of the filtration to complete a total ordering. Let σ_i be the simplex corresponding to row i of TCI. Then,

$$TCI(i,j) = \begin{cases} 1 & \text{if } \sigma_j \text{ is a codimension 1 face of } \sigma_i \\ 0 & \text{otherwise} \end{cases}$$

As previously mentioned, when discussing extended persistent cohomology, we have two filtrations,

$$\emptyset = K_0 \subset K_1 \subset \cdots \subset K_n = K \quad \text{and} \quad \emptyset = L_0 \subset L_1 \subset \cdots \subset L_m = K$$

each of which would, by itself, have their own total CoIncidence matrix. We define the Extended CoIncidence matrix as the block matrix

$$\begin{bmatrix} TCI_K & O \\ P & TCI_L \end{bmatrix}$$

where TCI_K is the total CoIncidence matrix for the filtration $K_0 \subset \cdots \subset K_n = K$ and TCI_L is the total CoIncidence matrix for the filtration $L_0 \subset \cdots \subset L_m = K$. P is the permutation matrix that permutes the order of the simplices associated to the filtration, $K_0 \subset K_1 \subset \cdots \subset K_n$, to the order of the simplices given by the $L_0 \subset L_1 \subset \cdots \subset L_m$ filtration. O is a block zero matrix.

We could compute the general extended persistent cohomology, which would be same as the first of our two algorithms presented in Chapter 5. That is, we could reduce the total CoIncidence matrix using column operations with modulo two arithmetic. Here, we focus on the algorithm for computing the extended death for a given cohomology class. Many of the details are similar to our second algorithm given in Chapter 5 which computes the death point of a specific cohomology class.

We work our way up from bottom to top of the matrix keeping note of which rows correspond to our cohomology classes. Suppose we want to compute the Extended Persistence of a q dimensional cohomology class, given a representation $\Gamma \in C^q(K_j)$. As detailed in Chapter 5, we tag the subset of q dimensional simplices on which our cohomology class evaluates one. However, unlike with regular coperistence, we will have a double set of rows corresponding to this set of q-simplices, one in the bottom half of the matrix, corresponding to the block TCI_L for the L-filtration and the block permutation matrix, and one in the top half of the matrix, corresponding to the block matrix associated to the TCI_K for the K-filtration. As with regular persistent cohomology, our goal is to have a column whose nonzero entries are solely in the rows that corresponds to the simplices on which our cohomology class, Γ, evaluates one.

Suppose $\Gamma = \sigma_1 * + \sigma_2^* + \cdots + \sigma_n^*$, where the order of the σ_k's in the representation of Γ are in such a way that indices of their corresponding rows are increasing. Since there is a repeat of the rows associated to Γ, as is a repeat of all the simplices, we choose the ordering based on the lower half of the matrix. We begin by scanning the lowest row associated to σ_n in our Extended CoIncidence matrix, to see if there is a column whose lowest one occurs in that row.

As detailed in Chapter 5, we remove rows from the bottom of our CoIncidence when we cannot perform an appropriate set of column operations to get a column whose non zero entries are in rows which are exactly the elements on which Γ evaluates one. In addition to removing rows, we also remove columns from the right side of our CoIncidence Matrix to likewise indicate that we are moving backwards through the filtrations. For every row that gets removed from the Extended CoIncidence matrix, we remove the corresponding column from the right. When we have removed enough rows and columns to obtain a column whose nonzero entries are in one-to-one correspondence with the remaining the cochains of Γ, our process terminates. We can say that this is the level at which Γ *dies*.

6.3 Illustrated Example

Suppose we have a simplicial complex, K, with 2 separate filtrations. The K-filtration is illustrated in Figure 6.1 and the L-filtration is illustrated in Figure 6.2.

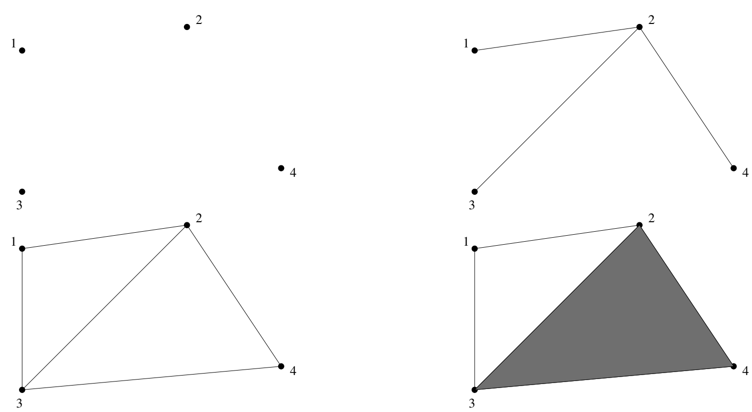

FIGURE 6.1: The four pictures depict stages of the K-filtration of our simplicial complex. The first stage of the filtration is the union of four vertices. The second level of the filtration is the union of the four vertices and 3 one dimensional simplices. The next level of the filtration is the union of the four vertices and 4 one simplices. The final simplical complex is the union of the 4 vertices, 4 one-simplices and a single 2 simplex.

Our Extended CoIncidence matrix becomes

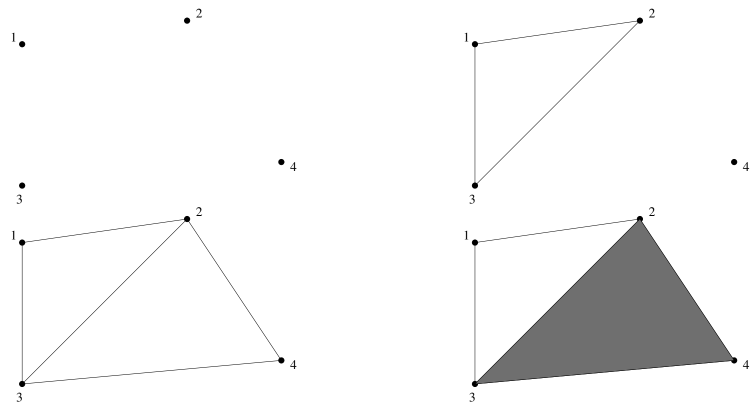

FIGURE 6.2: The four pictures give a representation of the L-filtration. Its first level is the union of four vertices. The second level is the union of 4 vertices and 3 different 1 simplices than those in level two of the K-filtration. The third level is the union of four vertices and four 1 simplices. The fourth level of the filtration is the full simplicial complex which is the union of 4 vertices, four 1 simplices and one 2 simplex.

	1	2	3	4	1, 2	2, 3	2, 4	1, 3	3, 4	2, 3, 4	1	2	3	4	1, 3	2, 3	2, 4	3, 4	2, 3, 4
1																			
2																			
3																			
4																			
1, 2	1	1																	
2, 3		1	1																
2, 4		1		1															
1, 3	1		1																
3, 4			1	1															
2, 3, 4						1	1		1										
1	1																		
2		1																	
3			1																
4				1															
1, 2					1						1	1							
1, 3								1			1		1						
2, 3						1						1	1						
2, 4							1					1		1					
3, 4									1				1	1					
2, 3, 4										1						1	1	1	

The representative for our cohomology class will be $\Gamma = \langle 1,2\rangle^* + \langle 1,3\rangle^* + \langle 2,3\rangle^*$. We begin by scanning the bottom most row corresponding to $\langle 2,3\rangle$ and find that it has nonzero entries in the columns associated to $\langle 2,3\rangle$, $\langle 2\rangle$, and $\langle 3\rangle$. Suppose we chose column $\langle 2,3\rangle$ since it is the leftmost column that has a lowest one in the row corresponding to $\langle 2,3\rangle$. It can easily be seen that adding columns $\langle 1,2\rangle$ and $\langle 1,3\rangle$ to this column would result in column $\langle 2,3\rangle$ having nonzero entries in rows $\langle 1,2\rangle$,

⟨1, 3⟩, ⟨2, 3⟩, and ⟨2, 3, 4⟩. This forms an unremovable contradiction since the row corresponding to ⟨2, 3, 4⟩ with smaller index does not contain the lowest one of any column. We choose instead column ⟨2⟩, without loss of generality.

Since low(⟨2⟩) = ⟨2, 4⟩, we scan that row. There are two columns with lowest ones in that row, ⟨2⟩ and ⟨2, 4⟩. We add column ⟨2, 4⟩ to column ⟨2⟩ and continue scanning up column ⟨2⟩ in the extended coincidence matrix. Since the nonzero entries of this column are in rows ⟨2, 3⟩, ⟨1, 2⟩ and ⟨2, 3, 4⟩, at the next row, ⟨1, 3⟩, we have reached a contradiction. Scanning the row corresponding to ⟨1, 3⟩, we find that its nonzero entries are in ⟨1, 3⟩, ⟨1⟩ and ⟨3⟩. Columns ⟨1, 3⟩ and ⟨1⟩ both have lowest ones in that row, so we arbitrarily choose to add ⟨1, 3⟩ to this column. Now, the nonzero entries are ⟨2, 3⟩, ⟨1, 3⟩, ⟨1, 2⟩, and ⟨2, 3, 4⟩. This yields a non-removable contradiction. If, however, we added column ⟨1⟩, we would have the nonzero entries of ⟨2, 3⟩, ⟨1, 3⟩, and ⟨2, 3, 4⟩. Although there is a contradiction at the next row of this column, ⟨1, 2⟩, we see that row ⟨1, 2⟩ is the row containing the lowest one for column ⟨1, 2⟩. When we add that column, we get the nonzero entries ⟨2, 3⟩, ⟨1, 3⟩, ⟨1, 2⟩ and ⟨2, 3, 4⟩. Again, we have a unremovable contradiction.

If we remove only the row associated to ⟨2, 3, 4⟩, we have not successfully removed any row that caused a contradiction. This fact is easy to see through example. We continue by removing the next row, ⟨3, 4⟩. Our Extended CoIncidence matrix is now:

	1	2	3	4	1,2	2,3	2,4	1,3	3,4	2,3,4	1	2	3	4	1,3	2,3	2,4	3,4	2,3,4
1																			
2																			
3																			
4																			
1,2	1	1																	
2,3		1	1																
2,4		1		1															
1,3	1		1																
3,4			1	1															
2,3,4						1	1		1										
1	1																		
2		1																	
3			1																
4				1															
1,2					1						1	1							
1,3								1			1		1						
2,3						1						1	1						
2,4							1					1		1					

By scanning the lowest row corresponding to a 1 dimensional simplex in our representative of our cohomology class, Γ, (i.e. $\langle 2,3\rangle$), we see again that it has nonzero entries in columns $\langle 2,3\rangle$, $\langle 2\rangle$ and $\langle 3\rangle$. Since columns $\langle 2,3\rangle$ and $\langle 2\rangle$ both have their lowest one in the row $\langle 2,3\rangle$, without loss of generality, we choose column $\langle 2,3\rangle$. Since the $\langle 1,3\rangle$ entry of column $\langle 2,3\rangle$ is zero, forming a contradiction to the one-to-one correspondence to Γ, we scan the row associated to $\langle 1,3\rangle$ and find that it has non zero entries in columns $\langle 1,3\rangle$, $\langle 1\rangle$ and $\langle 3\rangle$. Since columns $\langle 1,3\rangle$ and $\langle 1\rangle$ both have their lowest one in row $\langle 1,3\rangle$, we can choose column $\langle 1,3\rangle$. Adding this column to column, $\langle 2,3\rangle$, our nonzero entries are in the rows $\langle 2,3\rangle$, $\langle 1,3\rangle$, and $\langle 2,3,4\rangle$. Since the entry in $\langle 1,2\rangle$ is zero, which contradicts a one-to-one correspondence with Γ, we continue by scanning that row. The nonzero entries of row $\langle 1,2\rangle$ are in columns $\langle 1,2\rangle$, $\langle 1\rangle$ and $\langle 2\rangle$. The only column of this set whose lowest one is in row $\langle 1,2\rangle$, is column $\langle 1,2\rangle$, so we add column $\langle 1,2\rangle$ to the column $(\langle 2,3\rangle+\langle 1,3\rangle)$. Our nonzero entries are now in the rows $\langle 2,3\rangle$, $\langle 1,3\rangle$, $\langle 1,2\rangle$, and $\langle 2,3,4\rangle$, in decreasing order on the index of the rows. We still have a contradiction at row $\langle 2,3,4\rangle$. Scanning the row associated to $\langle 2,3,4\rangle$, we find that it has nonzero entries in the columns $(\langle 2,3\rangle+\langle 1,3\rangle+\langle 1,2\rangle)$, $\langle 2,4\rangle$, and $\langle 3,4\rangle$. The only one of these columns that has a lowest one in this row happens to be $\langle 3,4\rangle$. Therefore, we add the column associated to $\langle 3,4\rangle$, and our nonzero entries of the column $(\langle 2,3\rangle+\langle 1,3\rangle+\langle 1,2\rangle+\langle 3,4\rangle)$ are in the rows $\langle 2,3\rangle$, $\langle 1,3\rangle$ and $\langle 1,2\rangle$, from bottom to top of matrix. We have no further contradiction with the representative for Γ.

7

Decomposible CoPersistence

Chapters 4 through 6 have given us the appropriate background and development of cohomology in the context of persistence. We have seen that the benefit of working with cohomology is that it comes with operations which may provide more information about our topological spaces. In this chapter, we create an algorithmic and persistent approach to these operations so that we may capitalize on the additional information found there.

7.1 Cup Products

The cohomology of K is defined using the groups $\mathcal{C}^p(K) = \mathrm{Hom}(\mathcal{C}; G)$ where G is an abelian group. Since $H^{d-p}(K;\mathbb{Z}_2) = \ker(\delta : \mathcal{C}^{d-p-1}(K) \to \mathcal{C}^{d-p}(K))/\mathrm{Im}(\delta : \mathcal{C}^{d-p}(K) \to \mathcal{C}^{d-p+1}(K))$, a ring structure on the cohomology groups is defined by the *cup product.*

Definition 17. *The cup product is the homomorphism* $\cup : H^p(X) \otimes H^q(X) \to H^{p+q}(X)$*, defined by* $(\alpha^p \cup \beta^q)\langle v_0, v_1, \ldots, v_{p+q}\rangle = (-1)^{p+q}\alpha(\langle v_0, v_1, \ldots, v_p\rangle)\beta(\langle v_p, v_{p+1}, \ldots, v_{p+q}\rangle)$ *for* $\alpha \in \mathcal{C}^p(X)$ *and* $\beta \in \mathcal{C}^q(X)$*, which represent the elements of* H^p *and* H^q *respec-*

tively.

The cup product of two cohomology classes is combinatorial by its definition, so it is unnecessary to develop algorithms that compute the cup product of two cohomology classes directly. For example, suppose we have the two cohomology classes $\alpha = \langle 1,2\rangle^* + \langle 1,4\rangle^* + \langle 2,4\rangle^*$ and $\beta = \langle 2,3\rangle^* + \langle 3,4\rangle^* + \langle 2,4\rangle^*$. It is easy to determine on which, if any, simplicies $\alpha \cup \beta$ is nonzero, thus defining the cohomology class $\Gamma = \alpha \cup \beta$. It is a pairwise concatenation of the p simplices on which α evaluates one with the q simplex on which β evaluates 1, maintaining vertex ordering.

For this example, the only 2 simplices which are a vertex ordering preserving concatenation of the 1 simplices α and β both evaluate one on are $\langle 1,2,3\rangle$ and $\langle 1,2,4\rangle$. When we apply the cup product formula, we see $\alpha \cup \beta(\langle 1,2,3\rangle) = (-1)^{1+1}\alpha(\langle 1,2\rangle) \cdot \beta(\langle 2,3\rangle) = 1 \cdot 1 = 1$. Whereas, $\alpha \cup \beta(\langle 1,3,4\rangle) = \alpha(\langle 1,3\rangle) \cdot \beta(\langle 3,4\rangle) = 0 \cdot 1 = 0$. Therefore, if given two cohomology classes, it is easy to determine their cup product.

The ring structure of cohomology groups sometimes helps to distinguish between topological spaces where the homology groups, and thus cohomology groups, do not. For example, in $\mathbb{Z}_2$ coefficients, the torus, T^2, and Klein bottle, K^2, have the same homology and cohomology groups, i.e. $H_0(T^2) \cong H^0(T^2) = \mathbb{Z}_2 = H_0(K^2) \cong H^0(K^2)$, $H_1(T^2) \cong H^1(T^2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 = H_1(K^2) \cong H^1(K^2)$, and $H_2(T^2) \cong H^2(T^2) = \mathbb{Z}_2 = H_2(K^2) \cong H^2(K^2)$. However, the ring structures of their cohomology groups are not equal. Let α_1 and β_1 represent the 1 dimensional cohomology classes of the torus and α_2 and β_2 represent the 1 dimensional cohomology classes of the Klein bottle. Let Γ_1 be the 2 dimensional cohomology class of the torus, while Γ_2 is the 2 dimensional cohomology class of the Klein bottle. For the torus, it can be shown that $\alpha_1 \cup \beta_1 = \Gamma_1$, $\alpha_1 \cup \alpha_1 = 0$ and $\beta_1 \cup \beta_1 = 0$. However, for the Klein bottle, it can be shown $\alpha_2 \cup \beta_2 = \Gamma_2$, $\alpha_2 \cup \alpha_2 = 0$ but $\beta_2 \cup \beta_2 = \Gamma_2$. The differences in these ring structures establish a difference between the spaces that homology in $\mathbb{Z}_2$ coefficients would not have found.

One way of determining the ring structure of cohomology classes is to take all cup products of each representative of each cohomology class. This would help to find when the cup product of two cohomology classes results in a cohomology class. An alternative question that arises is when a cohomology class the cup product of two lower dimensional cohomology classes. This is decomposability.

Definition 18. *Suppose we have a cohomology class $\beta \in H^{p+q}(X)$. β is* ***decomposable*** *when there exists an $x \in H^p(X)$ and $y \in H^q(X)$, such that $x \cup y = \beta$.*

We give an algorithm to determine when a cohomology class is born as a decomposable class with a given cohomology class. This algorithm will then allow us to consider a persistent version of decomposability.

7.2 Decomposability Algorithm

Let K be a simplicial complex for which we have computed the cohomology groups using our Persistent Cohomology algorithm from Chapter 5. Our goal is to find when a cup product is born in the filtration for K.

Suppose we have a given p dimensional cohomology class $[x]$ that is born at K_s with $s > n$ and dies at K_t with $t < n - 1$ with the following representation $x = \sigma_1^* + \sigma_2^* + \cdots + \sigma_n^*$. Furthermore, suppose $K_n = K_{n-1} \cup \Delta$, where Δ is a $p+q$ dimensional simplex and through the Persistent Cohomology algorithm we find that $\Delta^* = \delta(y)$ at the level of the filtration K_n. By removing Δ and moving backwards to K_{n-1}, $\delta(y)$ becomes zero, and thus y is born as a cycle. These conditions are necessary to determine if this cycle is a cup product.

If by removing Δ, $\delta(y) = 0$, we can express this $[y] \in H^{p+q-1}(K_{n-1})$ as $y = \tau_1^* + \tau_2^* + \cdots + \tau_m^*$. We define a $D(y)-$matrix to be the CoIncidence submatrix whose columns are each $\tau_1, \tau_2, \ldots, \tau_m$, and whose rows are the codimension one cofaces of the τ_j's. This is a submatrix of the $p+q, p+q-1$ dimensional CoIncidence matrix, where the i,j entry is 1 when λ_k, the simplex associated to row i is a codimension one coface of τ_j. We look at each row λ_k. If there exists a $\sigma_i \in x$ such that σ_i is the front $r-$face of λ_k, we store λ_k in a list, denoted by β. After looking at each λ_k, β is finalized. If the size of β is even, then $\beta \in C^{p+q}$ is not a cup product with x. If, however, the size of β is odd, then β is born as a cup product with x. Thus, the birth of this cocycle, gives rise to the birth of a cup product. Where the birth of $[y]$ is a cup product, we can pair this $p+q$ dimensional class with the p dimensional class of which it is decomposable, $[x]$.

7.3 Illustrated Example

Suppose we have a two dimensional torus with a two cell glued in the interior, as shown in Figure 7.1.

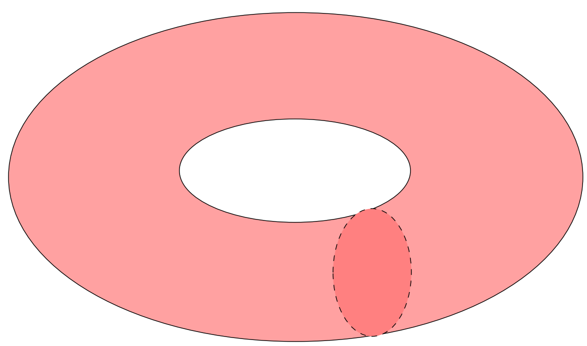

FIGURE 7.1: A 2 dimensional Torus with a 2 cell attached in its interior

This space could be built in the following fashion. The first level of our filtration could be homotopically equivalent to a point $\emptyset = K_0 \subset K_1 \simeq pt$. The second level of our filtration could be homotopically equivalent to the 1 dimensional sphere, S^1. Then, the next level of our filtration is the 2 dimensional torus. Lastly, the final stage of our filtration is the 2 dimensional torus with a two cell glued in the interior which is homologically equivalent to a pinched torus. Thus, $\emptyset = K_0 \subset K_1 \subset K_2 \subset$

$K_3 \subset K_4 = K$ is expressed as:

$$emptyset \subset pt \subset S^1 \subset T^2 \subset K$$

The homology for this space is $H_0 = \mathbb{Z}_2$, $H_1 = \mathbb{Z}_2$ and $H_2 = \mathbb{Z}_2$. If we were to remove the 2 cell in the interior, then we would have the two dimensional torus with homology $H_0 = \mathbb{Z}_2$, $H_1 = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $H_2 = \mathbb{Z}_2$. By removing the 2 cell, a one dimensional homology class is born. By Poincaré duality, this means a one dimensional cohomology class is born. Furthermore, we saw that the cup product of this one dimensional cohomology class and the previously existing one dimensional class equals the 2 dimensional class of the Torus. Therefore, we can say that the 2 dimensional cohomology class was not decomposable in the previous level of the filtration, but by removing the 2 cell and moving to the next level it suddenly becomes decomposable. In the language of persistence, we can say this is the *birth point* of its decomposability.

On the other hand, we would continue to the previous level of our filtration, S^1, by removing a 2 dimensional cell from the Torus, thus killing the 2 cohomology class. Seeing as how the 2 dimensional cohomology class becomes trivial here, it is no longer decomposable as 2 one dimensional cohomology classes. In the language of persistence, this would be the *death point* of both it as well as its decomposability. We now demonstrate how the algorithm shows these birth and death points when we have a filtered simplicial complex that represents this space.

Suppose K is the simplicial complex of the Torus with a 2 cell attached. This is given in Figure 7.2 with a common triangulation of the torus union a 2 cell $\langle 7, 8, 9 \rangle$.

Let $[\alpha]$ be the cohomology class represented by $\alpha = \langle 2, 3 \rangle^* + \langle 3, 5 \rangle^* + \langle 5, 6 \rangle^* + \langle 6, 8 \rangle^* + \langle 8, 9 \rangle^* + \langle 2, 9 \rangle^*$.

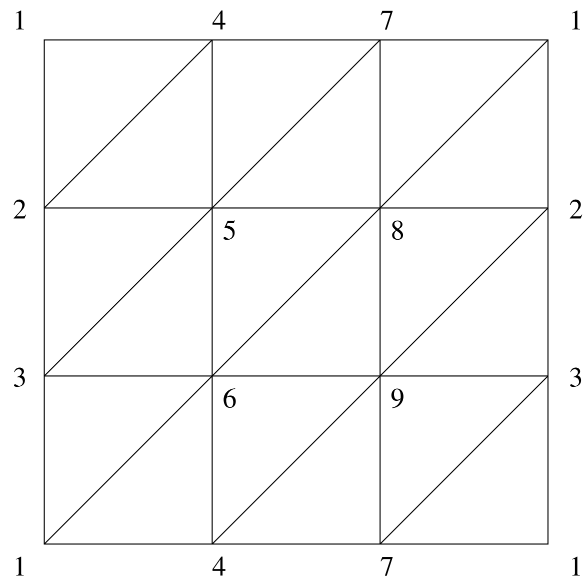

FIGURE 7.2: The figure gives a triangulation of the two Torus. To get the two Torus with a 2 cell attached, we union the above simplicial complex with the 2 simplex $\langle 7, 8, 9\rangle$.

	<1,2>	<1,3>	<2,3>	<1,4>	<2,4>	<2,5>	<3,5>	<4,5>	<1,6>	<3,6>	<4,6>	<5,6>	<1,7>	<3,7>	<4,7>	<5,7>	<1,8>	<2,8>	<5,8>	<6,8>	<7,8>	<2,9>	<3,9>	<4,9>	<6,9>	<7,9>	<8,9>
<1,2,4>	1			1	1																						
<2,3,5>			1			1	1																				
<2,4,5>					1	1		1																			
<1,3,6>		1							1	1																	
<1,4,6>				1					1		1																
<3,5,6>							1			1		1															
<1,3,7>		1											1	1													
<4,5,7>								1							1	1											
<1,2,8>	1																1	1									
<5,6,8>												1							1	1							
<1,7,8>													1				1				1						
<5,7,8>																1			1		1						
<2,3,9>			1																			1	1				
<4,6,9>											1													1	1		
<3,7,9>														1									1			1	
<4,7,9>															1									1		1	
<2,8,9>																		1				1					1
<6,8,9>																				1					1		1
<7,8,9>																					1					1	1

We note that $\langle 7,8,9\rangle^* = \delta(y)$ where $y = \langle 7,8\rangle^* + \langle 7,9\rangle^* + \langle 8,9\rangle^*)$. When we remove $\langle 7,8,9\rangle$, $\delta(y) = 0$, thus a cycle is born. The $D(y)$ matrix is the following:

	<7,8>	<7,9>	<8,9>
<1,7,8>	1		
<5,7,8>	1		
<3,7,9>		1	
<4,7,9>		1	
<2,8,9>			1
<6,8,9>			1

We now check the rows of $D(y)$. Since $\langle 1,7\rangle^*$, $\langle 5,7\rangle^*$, $\langle 3,7\rangle^*$, $\langle 4,7\rangle^*$, and $\langle 2,8\rangle^* \notin x$, the only λ_k that is placed in the set β is $\langle 6,8,9\rangle$. Since the size of β is $1 \mod 2 \neq 0$, we have that $\beta = \langle 6,8,9\rangle^*$. Thus, by removing $\langle 7,8,9\rangle$ and moving backwards in our filtration from K_n to K_{n-1}, we see that a 2 dimensional cocycle, $\langle 6,8,9\rangle^*$, is born as the cup product of x with another cycle. We have paired the birth of y with the birth of $\langle 6,8,9\rangle^*$.

8

Steenrod Squares

Steenrod Squares can be both enlightening about structure of our topological space and its bundles as well as useful when transitioning between different types of characteristic classes. We begin this chapter with background on Steenrod Squares, then develop an algorithm for computing when a higher dimensional cohomology class is the Steenrod Square of a lower dimensional cohomology class. We then construct a persistent approach to Steenrod Squares. We finish with proving that their exists a concise combinatorial formula for Sq^1.

8.1 Background of Steenrod Squares

Steenrod developed a set of additive homomorphisms, which were originally called "square upper i" and have since been referred to as Steenrod Squares. We give the axiomatic definition of these maps.

Definition 19. *The* ***Steenrod Squares*** *are cohomology operations that satisfy the following axioms:*

1. *For all integers $i \geq 0$ and $q \geq 0$, there is a natural transformation of functors*

which is a homomorphism $Sq^i : H^q(X,A) \to H^{q+i}(X,A)$*,* $q \geq 0$*.*

2. *Naturality. If* $f : (X,Y) \to (X',Y')$ *then* $Sq^i \circ f^* = f^* \circ Sq^i$*.*

3. $Sq^0 =$ *the identity.*

4. *If the dimension of* x *is* q*,* $Sq^q(x) = x \cup x$*.*

5. *If* i *is greater than the dimension of* x*,* $Sq^i(x) = 0$*.*

6. *The Cartan formula.* $Sq^k(\alpha \cup \beta) = \sum_{i+j=k} Sq^i(\alpha) \cup Sq^j(\beta)$*, whenever* $\alpha \cup \beta$ *is defined.*

From these axioms, it can be shown that if $\delta : H^q(X,A) \to H^{q+1}(X,A)$ is the coboundary map, then $\delta Sq^i = Sq^i \delta$.

In addition to being a unique cohomology operation, Steenrod Squares have been used to verify the existence of Stiefel-Whitney classes. Wu also used Steenrod Squares to give a formula for the Stiefel-Whitney classes of the tangent bundle to a manifold. A good reference for both of these is (MS74).

Steenrod Squares have also been defined as an extension of the cup-i products. One such example of constructing Steenrod Squares in this fashion can be found in (MT68). This construction was written more combinatorially by González-Díaz and Real in (GDR99b) and (GDR99a). In (GDR99a), they defined Steenrod Squares as the cup-n product of 2 cochains for a chosen ring R. We present this formula in this chapter in $\mathbb{Z}_2$ coefficients and refer the reader to Appendix A for the general form for any coefficient ring. The main proposition over $\mathbb{Z}_2$ is:

Proposition 20. *Let* $\mathbb{Z}_2$ *be the coefficient field and* K *a simplicial complex with a finite number vertices. If* $c \in \mathcal{C}^p(K)$ *and* $c' \in \mathcal{C}^q(K)$*, for all* n *with* $0 \leq n \leq p+q$*,* $c \cup_n c' \in \mathcal{C}^{p+q-n}(K)$ *is defined by the following formulae. Let* $m = p+q-n$ *and*

$\sigma = \langle v_0, v_1, \ldots, v_m\rangle \in \mathcal{C}^m(K)$, then if n is even,

$$c \cup_n c'(\sigma) =$$

$$\sum_{i_n=S(n)}^{m} \sum_{i_{n-1}=S(n-1)}^{i_n-1} \cdots \sum_{i_1=S(1)}^{i_2-1} c(\langle v_0, \ldots, v_{i_0}, v_{i_1}, \ldots, v_{i_2}, v_{i_3}, \ldots, v_{i_{n-2}}, v_{i_{n-1}}, \ldots, v_{i_n}\rangle)$$

$$\cdot\, c(\langle v_{i_0}, \ldots, v_{i_1}, v_{i_2}, \ldots, v_{i_3}, v_{i_4}, \ldots, v_{i_{n-1}}, v_{i_n}, \ldots, v_m\rangle)$$

and if n is odd, the formula is analogous. In these formulas $\cdot$ is the product in $\mathbb{Z}_2$, and

$$S(k) = i_{k+1} - i_{k+2} + \cdots + (-1)^{k+n-1} i_n + (-1)^{k+n} \left\lfloor \frac{m+1}{2} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor$$

for all $0 \leq k \leq n$, and $i_0 = S(0)$.

In order to convert these formulas for the cup-n product into the ith Steenrod Square formula, we need to recognize some things. First, for $\alpha \in \mathcal{C}^p(K)$, to get $Sq^i(\alpha)$, we would be taking the nth cup product of α with itself, $\alpha \cup_n \alpha$. Here $m = p+i$ since $Sq^i(\alpha) \in \mathcal{C}^{p+i}(K)$. However according to the formulas $m = p+p-n$, so $n = p-i$. Therefore, we have that $Sq^i(\alpha) = \alpha \cup_n \alpha$ when $n = p - i$.

8.1.1 example

Suppose we have $\alpha \in \mathcal{C}^4(K)$ for some simplicial complex K. Using this equation, we will calculate $Sq^1(\alpha) \in \mathcal{C}^5(K)$. In this example, $p = 4$, $i = 1$, $m = p + i = 5$, thus $n = p - i = 3$. Therefore, the formula becomes

$$Sq^1(\alpha)(\langle v_0, \ldots, v_5\rangle) = \alpha \cup_3 \alpha(\langle v_0, \ldots, v_5\rangle) =$$

$$\sum_{i_3=S(3)}^{5} \sum_{i_2=S(2)}^{i_3-1} \sum_{i_1=S(1)}^{i_2-1} \alpha(\langle v_0, \ldots, v_{i_0}, v_{i_1}, \ldots, v_{i_2}, v_{i_3}, \ldots, v_5\rangle)$$

$$\cdot\, \alpha(\langle v_{i_0}, \ldots, v_{i_1}, v_{i_2}, \ldots, v_{i_3}, v_m\rangle)$$

The first thing to do is to calculate the $S(k)$ for each $0 \leq k \leq n$ in decreasing order.

$$\begin{aligned}
i_3 =& S(3) = (-1)^{3+3}\left\lfloor \frac{5+1}{2} \right\rfloor + \left\lfloor \frac{3}{2} \right\rfloor = 4 \\
i_2 =& S(2) = (-1)^{2+3-1} i_3 + (-1)^{2+3}\left\lfloor \frac{5+1}{2} \right\rfloor + \left\lfloor \frac{2}{2} \right\rfloor = i_3 - 2 \\
i_1 =& S(1) = i_2 + (-1)^{1+3-1} i_3 + (-1)^{1+3}\left\lfloor \frac{5+1}{2} \right\rfloor + \left\lfloor \frac{3}{2} \right\rfloor = i_2 - i_3 + 3 \\
i_0 =& S(0) = i_1 - i_2 + (-1)^{0+3-1} i_3 + (-1)^{0+3}\left\lfloor \frac{5+1}{2} \right\rfloor + \left\lfloor \frac{3}{2} \right\rfloor = i_1 - i_2 + i_3 - 3
\end{aligned}$$

Therefore, we have 6 terms in our summands, each one corresponding to different indices. Let $\bar{i} = (i_0, i_1, i_2, i_3)$, then the full set of 6 distinct $\bar{i}$'s involved in this summands is: $\{(0,1,2,4),(0,2,3,4),(0,1,3,5),(1,2,3,5),(0,2,4,5),(1,3,4,5)\}$. Thus, on an arbitrary $\langle v_0, v_1, v_2, v_3, v_4, v_5\rangle \in \mathcal{C}_5$,

$$\begin{aligned}
&Sq^1(\alpha)(\langle v_0, \ldots, v_5\rangle) = \alpha \cup_3 \alpha(\langle v_0, \ldots, v_1) \\
&= \sum_{i_3=S(3)}^{5} \sum_{i_2=S(2)}^{i_3-1} \sum_{i_1=S(1)}^{i_2-1} \alpha(\langle v_0, \ldots, v_{i_0}, v_{i_1}, \ldots, v_{i_2}, v_{i_3}, \ldots, v_5\rangle) \\
&\qquad\qquad\qquad \cdot \alpha(\langle v_{i_0}, \ldots, v_{i_1}, v_{i_2}, \ldots, v_{i_3}, v_m\rangle) \\
&= \alpha(\langle v_0, v_1, v_2, v_4, v_5\rangle) \cdot \alpha(\langle v_0, v_1, v_2, v_3, v_4\rangle) + \alpha(\langle v_0, v_2, v_3, v_4, v_5\rangle) \cdot \alpha(\langle v_0, v_1, v_2, v_3, v_4\rangle) \\
&+ \alpha(\langle v_0, v_1, v_2, v_3, v_5\rangle) \cdot \alpha(\langle v_0, v_1, v_3, v_4, v_5\rangle) + \alpha(\langle v_0, v_1, v_2, v_3, v_5\rangle) \cdot \alpha(\langle v_1, v_2, v_3, v_4, v_5\rangle) \\
&+ \alpha(\langle v_0, v_2, v_3, v_4, v_5\rangle) \cdot \alpha(\langle v_0, v_1, v_2, v_4, v_5\rangle) + \alpha(\langle v_0, v_1, v_3, v_4, v_5\rangle) \cdot \alpha(\langle v_1, v_2, v_3, v_4, v_5\rangle)
\end{aligned}$$

Having this formula for computing the Steenrod Square allows us to reverse engineer an algorithm to determine when a higher dimensional cohomology class is a Steenrod Square of a lower dimensional class. The next section details that algorithm.

8.2 Algorithm for Steenrod Squares

In this section, we present an algorithm that determines whether a cohomology class is the Steenrod Square of a lower dimensional cohomology class.

Suppose the simplicial complex K is a triangulation of a manifold. Let $[\Gamma] \in H^{p+q}(K)$ can be represented by the cochain $\Gamma = \sigma_1^* + \sigma_2^* + \cdots + \sigma_s^*$. Without loss of generality, our goal is to determine if there exists an $\alpha \in C^p(x)$ such that $\Gamma = Sq^q(\alpha)$. We use the formula given in (GDR99b) with the conditions that $m = p + q$ and $n = p - q$. Recall this is:

$$\Gamma(\langle v_0, v_1, \ldots, v_m \rangle) = Sq^q(\alpha)(\langle v_0, v_1, \ldots, v_m \rangle) = \alpha \cup_n \alpha(\langle v_0, v_1, \ldots, v_m \rangle) =$$

$$\sum_{i_n=S(n)}^{m} \sum_{i_{n-1}=S(n-1)}^{i_n-1} \cdots \sum_{i_1=S(1)}^{i_2-1} \alpha(\langle v_0, \ldots, v_{i_0}, v_{i_1}, \ldots, v_{i_2}, v_{i_3}, \ldots, v_{i_{n-2}}, v_{i_{n-1}}, \ldots, v_{i_n} \rangle)$$

$$\cdot\, \alpha(\langle v_{i_0}, \ldots, v_{i_1}, v_{i_2}, \ldots, v_{i_3}, v_{i_4}, \ldots, v_{i_{n-1}}, v_{i_n}, \ldots, v_m \rangle),$$

where $S(k) = i_{k+1} - i_{k+2} + \cdots + (-1)^{n+k-1} i_n + (-1)^{n+k} \lfloor \frac{m+1}{2} \rfloor + \lfloor \frac{k}{2} \rfloor$. By writing $\bar{i} = \langle i_0, i_1, \ldots, i_n \rangle$, we can refer to each term of the summand by the unique $\bar{i}$ that defines it. Let I be the set of all possibilities of $\bar{i}$, given $m = p + q$ and $n = p - q$, the relationships $0 \leq i_0 < i_1 < \cdots < i_k < \cdots < i_N \leq m$, and the fact that each i_k can take values from $S(k)$ to $i_{k+1} - 1$. By construction, the $\bar{i}_l \in I$ have a canonical ordering and are universal for each σ_j on which Γ evaluates 1. The ordering is based on the sequence of corresponding terms in the summand of $\alpha \cup_n \alpha$. We can choose an arbitrary ordering for the σ_j in the expression of Γ, though in Section 8.3 this ordering is determined by the filtration.

Beginning at $\bar{i}_1 \in I$ and $\sigma_1 = \langle v_{1_0}, v_{1_1}, \ldots, v_{1_m} \rangle$, we define:

$$\lambda_{1_1, \bar{i}_1} = \langle v_{1_0}, \ldots, v_{1_{i_0}}, v_{1_{i_1}}, \ldots, v_{1_{i_2}}, v_{1_{i_3}}, \ldots, v_{1_{i_{n-2}}}, v_{1_{i_{n-1}}}, \ldots, v_{1_{i_n}} \rangle$$

$$\lambda_{1_2,\bar{i}_1} = \langle v_{1_{i_0}}, \ldots, v_{1_{i_1}}, v_{1_{i_2}}, \ldots, v_{1_{i_3}}, v_{1_{i_4}}, \ldots, v_{1_{i_{n-1}}}, v_{1_{i_n}}, \ldots, v_{1_m} \rangle$$

In the $p+1$ dimensional CoIncidence matrix, we add the column associated to $\lambda_{1_1,\bar{i}_1}$ to the column $\lambda_{1_2,\bar{i}_1}$. Performing column operations, as detailed in Chapter 7 for the decomposability algorithm, our goal is to obtain a column of zeros. If we can obtain a column of zeros, that column corresponds to a p dimensional part of the cocyle whose Sq is Γ. Two situations can occur:

1. We cannot obtain a column of zeros. We have reached a contradiction for $\bar{i}_1$ and we follow the above procedure on σ_1 with $\bar{i}_2$, the next element in I. Alternatively,

2. We can obtain a column of zeros and therefore have a cocycle, α_1. However, α_1 is the cycle associated only with the part of Γ that evaluates on σ_1. We continue by incorporating σ_2^* into our cycle. We replace the column associated to the co-chain α_1 with the cochain$= \alpha_1 + \lambda_{2_1,\bar{i}_1} + \lambda_{2_2,\bar{i}_1}$. Again, we perform column operations on this new column and two situations can occur

    (a) We cannot obtain a column of zeros. If this were to happen we continue computing on σ_2 with the next term in I, $\bar{i}_3$

    (b) We can obtain a column of zeros and call this cycle α_2. Then we continue computing on σ_3, starting with $\bar{i}_1$.

Sometimes we reach a contradiction straight away - i.e. we cannot find a cycle for σ_1 and $\bar{i}_1$. If this happens, we try σ_1 with $\bar{i}_2$ and higher, in increasing order until no contradiction occurs. If a contradiction occurs for all $\bar{i}_l \in I$, then we know that there does not exist an α such that $Sq^q(\alpha) = \Gamma$.

Assume, we can find a cycle for σ_1 and $\bar{i}_{l_1}$, for some $\bar{i}_{l_1} \in I$. Suppose further that for each σ_k, $1 < k < j$ for some $j < s$ we can find a cycle, α_{j-1} by the above method.

That is, α_{j-1} is the cycle associated to $\sigma_1^* + \sigma_2^* + \cdots + \sigma_{j-1}^*$. Then, two situations may occur:

1. There exists an extension cycle for σ_j corresponding to $\bar{i}_{l_j} \in I$, α_j. For this to occur, we move along to finding the extension cycle associated to $\alpha_{j-1} + \lambda_{j_1,\bar{i}_1} + \lambda_{j_2,\bar{i}_1}$. If $\bar{i}_1$ forms a contradiction, we move through the elements of I until we do not reach a contradiction. The assumption that there exists an extension cycle means that we will not reach a contradiction.

2. There does not exist an extension cycle for σ_j corresponding to any element of I. In this case, we begin the process all over again with σ_1 and $\bar{i}_{l_1+1}$ until no contradiction is reached.

We either finish the process with running out the possibilities, each ending in a contradiction, or obtaining the corresponding extension cycle for each σ_j in the expression of Γ. If each possibility leads to a contradiction, we conclude that there is no cycle, α, such that, $\Gamma = Sq^q(\alpha)$. Suppose however, we can get an extension cycle; we name it α. Suppose for this α, σ_1 was associated with $\lambda_{1_1,\bar{i}_l}$ and $\lambda_{1_2,\bar{i}_l}$ for some $\bar{i}_l < \bar{i}_s$. The last step is to verify that $\Gamma(\sigma_1+\sigma_2+\cdots+\sigma_s) = \alpha \cup_n \alpha(\sigma_1+\sigma_2+\cdots+\sigma_s)$. If the equality holds, then we have found an α such that $\Gamma = Sq^q(\alpha)$. On the other hand, if they are not equal, we begin the process again with σ_1 at $\bar{i}_{l+1}$.

The whole algorithm will either terminate with α such that $Sq^q(\alpha) = \Gamma$ or with every possibility reaching a contradiction. In the latter case, Γ is not the q square of a p dimensional cohomology class.

The outcome of the algorithm is dependent, in part, on the ordering of the σ_j's. The last thing to prove is that if we do get an extension cycle, α, from the above method, and there were to be another α' obtained from an alternative ordering, that they are cohomologous.

Proposition 21. *If α and α' are cocycles such that $\Gamma = Sq^q(\alpha) = Sq^q(\alpha')$, that is, both α and α' were found by the above algorithm with alternative orderings, they are cohomologous.*

Proof. Recall the property of Sq^q that $\delta Sq^q = Sq^q\delta$. It follows $\delta Sq^q(\alpha) = \delta\Gamma$ as well as $\delta Sq^q(\alpha') = \delta\Gamma$. Then, $\delta(Sq^q(\alpha - \alpha')) = \delta(Sq^q(\alpha) - Sq^q(\alpha')) = \delta(\Gamma - \Gamma) = 0$ because Steenrod Squares are additive. $0 = \delta Sq^q(\alpha - \alpha') = Sq^q(\delta(\alpha - \alpha'))$ Therefore, $\delta(\alpha - \alpha') = 0$, making α and α' are cohomologous. □

8.3 Persistent Steenrod Squares

Here we discuss the persistent approach to Steenrod Squares. As we move through a filtration of a space, there are levels of the filtration at which a cohomology class is a Steenrod Square of another, lower dimensional class and levels of the filtration where it no longer exists as the Steenrod Square of a lower class, but that lower class still exists. The most obvious example of this would be a situation where K triangulates a d dimensional manifold and $Sq^q(\alpha) = \Gamma \in H^d(K)$, for $\alpha \in H^p(K)$. Suppose K_j is the last level of our filtration without any $d-1$ simplicies, then Γ has certainly "died" by the time we enter K_j. However, it is possible that $\alpha \in Im(\psi_{n,j})$. While α does not die, Γ its qth Steenrod Square does.

Alternatively, as we move backwards through the levels of the filtration, the image of Γ through these inclusion maps may take on a different representation from $\Gamma \in H^d(K)$. As its representation changes, the simplicies on which it evaluates one also change. Supposing the image of α does not change as we move backwards through the filtration, the combinatorial formula and algorithm for computing when a class is a Steenrod Square of a lower dimensional class suggest that Γ may no longer be expressed as the qth Steenrod Square of α. Therefore, there could be a point in the filtration where Γ dies as a qth Steenrod Square of α, but does not itself die.

8.4 Square-1 Formula

It is sometimes necessary to calculate out the Steenrod Squares of various cohomology classes directly. To do so, one would want to use the formula from (GDR99b). Alternatively, if there were a simpler formula for $Sq^1(\alpha)$, then one could calculate the qth Steenrod Square iteratively through the Cartan Formula, albeit inefficiently. This section presents a simpler expression of the formula found in (GDR99b) and (GDR99a) for the Square 1 case. The proof can extend to the general group rings; it is presented in Appendix A.

Proposition 22. *If $\alpha \in \mathcal{C}^p(K)$, then*

$$Sq^1(\alpha)(\langle v_0, v_1, \ldots, v_{p+1}\rangle) =$$

$$\sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{j=0}^{i-1} \alpha(\langle v_0, \ldots, \widehat{v_{2i}}, \ldots, v_{p+1}\rangle) \cdot \alpha(\langle v_0, \ldots, \widehat{v_{2j}}, \ldots, v_{p+1}\rangle)$$

$$+ \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{j=0}^{i-1} \alpha(\langle v_0, \ldots, \widehat{v_{2i-1}}, \ldots, v_{p+1}\rangle) \cdot \alpha(\langle v_0, \ldots, \widehat{v_{2j-1}}, \ldots, v_{p+1}\rangle)$$

The above equation can be expressed as the sums of products of α evaluating on all combinations of removing 2 distinct even and odd vertices from a $p+1$ simplex. Recall, that the expression for the nth cup product is given in Proposition 21 in this chapter. To prove this result for $\mathbb{Z}_2$ coefficients, we will refer to that formula.

Proof. Since $\alpha \in \mathcal{C}^p(K)$, $Sq^1(\alpha) \in \mathcal{C}^{p+1}(K)$. In terms of the formula in Proposition 21, $m = p+1$ and $n = p-1$. Define $\bar{i} = (i_0, i_1, \ldots, i_n)$ for $i_j \in [0, p+1]$, each $n+1$ tuple $\bar{i}$ has only 2 omitted numbers from the $p+2$ tuple $(0, 1, \ldots, p+1)$. Therefore, we can write $\bar{i} = (i_0, i_1, \ldots, i_n) = (0, \ldots, \hat{a}, \ldots, \hat{b}, \ldots, p+1)$.

To see which two numbers are omitted from the $p+2$ tuple and how they relate, we calculate i_n. Recall

$$i_n = S(n) = (-1)^{n+n}\left\lfloor \frac{m+1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{p+1+1}{2} \right\rfloor + \left\lfloor \frac{p-1}{2} \right\rfloor$$

If n is even, p is odd and the above simplifies to

$$i_n = \frac{p+1}{2} + \frac{p-1}{2} = p$$

Similarly, if n is odd, p is even and the above simplifies to

$$i_n = \frac{p+1+1}{2} + \frac{p-1-1}{2} = p$$

This means that the first omitted number in the $p+2$ tuple is $p+1$. Continuing along, we calculate i_{n-1}.

$$i_{n-1} = i_n + (-1)^{n+(n-1)}\left\lfloor \frac{m+1}{2} \right\rfloor + \left\lfloor \frac{n-1}{2} \right\rfloor = p - \left\lfloor \frac{p+2}{2} \right\rfloor + \left\lfloor \frac{p-2}{2} \right\rfloor = p-2$$

$\bar{i}_1 = (0, 1, \ldots, p-3, p-2, p)$ and the only two omitted numbers are thus $p-1$ and $p+1$, which obviously have the same parity. However, recall that this would be the first term in the sum from the formula given earlier. The next term in the sum, corresponding to $\bar{i}_2$ would be obtained by forcing $i_{n-1} = p-1$. We calculate i_{n-2} and i_{n-3} to inspire the lemma that follows.

$$\begin{aligned} i_{n-2} = S(n-2) &= i_{n-1} - i_n + \left\lfloor \frac{m+1}{2} \right\rfloor + \left\lfloor \frac{n-2}{2} \right\rfloor \\ &= (p-1) - (p) + \left\lfloor \frac{p+2}{2} \right\rfloor + \left\lfloor \frac{p-3}{2} \right\rfloor = p-2 \\ i_{n-3} = S(n-3) &= i_{n-2} - i_{n-1} + i_n - \left\lfloor \frac{m+1}{2} \right\rfloor + \left\lfloor \frac{n-3}{2} \right\rfloor \\ &= (p-1) - (p+1) + \left\lfloor \frac{p+2}{2} \right\rfloor + \left\lfloor \frac{p-4}{2} \right\rfloor = p-4 \end{aligned}$$

Thus, for the next $\bar{i}_2 \in I$, the two omitted numbers from the $p+2$ tuple are $p+1$ and $p-3$, which obviously have the same parity. This leads us to the following lemma.

Lemma 23. *Suppose $i_{n-t} = i_{n-t+1} - 2$ for some t. If we replace i_{n-t} with $\tilde{i}_{n-t}$ such that $\tilde{i}_{n-t} = i_{n-t+1} - 1$, then i_{n-t-1} increases by 1 while i_{n-t-2} stays the same.*

Proof. By the formulas given,

$$i_{n-t-1} = i_{n-t} - i_{n-t+1} + i_{n-t+2} - \cdots + (-1)^{n-t-1+n}\left\lfloor\frac{p+1}{2}\right\rfloor + \left\lfloor\frac{p-1-t-1}{2}\right\rfloor$$

$$i_{n-t-2} = i_{n-t-1} - i_{n-t} + i_{n-t+1} - \cdots + (-1)^{n-t-2+n}\left\lfloor\frac{p+2}{2}\right\rfloor + \left\lfloor\frac{p-1-t-2}{2}\right\rfloor$$

When we replace i_{n-t} with $\tilde{i}_{n-t}$, we get the following, new i_{n-t-1} and i_{n-t-2} denoted $\tilde{i}_{n-t-1}$ and $\tilde{i}_{n-t-2}$:

$$\begin{aligned}\tilde{i}_{n-t-1} &= \tilde{i}_{n-t} - i_{n-t+1} + i_{n-t+2} - \cdots + (-1)^{2n-t-1}\left\lfloor\frac{p+2}{2}\right\rfloor + \left\lfloor\frac{p-2-t}{2}\right\rfloor \\ &= (i_{n-t}+1) - i_{n-t+1} + i_{n-t+2} - \cdots + (-1)^{2n-t-1}\left\lfloor\frac{p+2}{2}\right\rfloor + \left\lfloor\frac{p-t-2}{2}\right\rfloor \\ &= i_{n-t-1} + 1\end{aligned}$$

In order to calculate $\tilde{i}_{n-t-2}$, we must replace both i_{n-t} and i_{n-t-1} with $\tilde{i}_{n-t}$ and $\tilde{i}_{n-t-1}$, respectively.

$$\begin{aligned}\tilde{i}_{n-t-2} &= \tilde{i}_{n-t-1} - \tilde{i}_{n-t} + i_{n-t+1} - i_{t-n+2} + \cdots + (-1)^{n-t-2+n}\left\lfloor\frac{p+2}{2}\right\rfloor + \left\lfloor\frac{p-1-t-2}{2}\right\rfloor \\ &= (i_{n-t-1}+1) - (i_{n-t}+1) + i_{n-t+1} + \cdots + (-1)^{2n-t-2}\left\lfloor\frac{p+2}{2}\right\rfloor + \left\lfloor\frac{p-1-t-2}{2}\right\rfloor \\ &= i_{n-t-2}\end{aligned}$$

□

In the context of the larger proof, let us consider what this shows. Suppose $i_{n-t+1} = s$, for an integer $s \in [0, p+1]$. Originally $i_{n-t} = s - 2$, so for this particular $\bar{i}_l$, $s-1$ is omitted. Changing i_{n-t} to $s-1$ makes $i_{n-t-1} = s-2$ and $i_{n-t-2} = s-4$ and in the following $\bar{i}_{l+1}$ $s-3$ is omitted. Clearly, $s-1$ and $s-3$ have the same parity.

Recall that the first $\bar{i}_1 \in I$ is $(0, 1, \ldots, p-3, p-2, p)$, so the first term in the summand of $Sq(\alpha)$ is

$$\alpha(\langle v_0, \ldots, v_{p-3}, v_{p-2}, v_{p-1}, v_p \rangle) \cdot \alpha(\langle v_0, \ldots, v_{p-2}, v_p, v_{p+1} \rangle)$$

Alternatively, one could write this term as:

$$\alpha(\langle v_0, \ldots, v_p, \hat{v}_{p+1} \rangle) \cdot \alpha(\langle v_0, \ldots, \hat{v}_{p-1}, \ldots, v_{p+1} \rangle)$$

In order to calculate the next $\bar{i}_2 \in I$, we change i_{n-1} to $p-1$ which by the lemma forces $i_{n-2} = p-2$ and $i_{n-3} = p-4$. $\bar{i}_2 = (0, 1, \ldots, p-4, p-2, p-1, p)$. $p+1$ and $p-3$ have the same parity and are both omitted from the $p+2$ tuple. To calculate the next $\bar{i}_3 \in I$, we increase i_{n-3} by one. $p+1$ is omitted and every time we increase a i_{n-t} by one we skip the next number with the same parity as the previously skipped number. Therefore, we remove each number that has the same parity as $p+1$ once. After that, our next choice is to increase i_n by one to $p+1$.

$$i_{n-1} = S(n-1) = i_n - \left\lfloor \frac{m+1}{2} \right\rfloor + \left\lfloor \frac{n-1}{2} \right\rfloor$$

$$= (p+1) - \left\lfloor \frac{p+2}{2} \right\rfloor + \left\lfloor \frac{p_2}{2} \right\rfloor = p-1$$

$$i_{n-2} = S(n-2) = i_{n-1} - i_+ \left\lfloor \frac{m+1}{2} \right\rfloor + \left\lfloor \frac{n-2}{2} \right\rfloor$$

$$= (p-1) - (p+1) + \left\lfloor \frac{p+2}{2} \right\rfloor + \left\lfloor \frac{p-3}{2} \right\rfloor = p-3$$

This removes p and $p-2$ from the p-tuple and we begin again, thus removing all numbers with the same pairity as p one at a time for each new $\bar{i}_l$. This process continues with each next $\bar{i}$ having two numbers of the same parity removed simultaneously. Thus, according to the equations given earlier, we can concisely express $Sq^1(\alpha)$ as:

$$
\begin{aligned}
&Sq^1(\alpha)(\langle v_0, v_1, \ldots, v_{p+1}\rangle) = \\
&\quad \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{j=0}^{i-1} \alpha(\langle v_0, \ldots, \widehat{v_{2i}}, \ldots, v_{p+1}\rangle) \cdot \alpha(\langle v_0, \ldots, \widehat{v_{2j}}, \ldots, v_{p+1}\rangle) \\
&\qquad + \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{j=0}^{i-1} \alpha(\langle v_0, \ldots, \widehat{v_{2i-1}}, \ldots, v_{p+1}\rangle) \cdot \alpha(\langle v_0, \ldots, \widehat{v_{2j-1}}, \ldots, v_{p+1}\rangle)
\end{aligned}
$$

□

9

Conclusion

As Meyer states in (Mey08), “Studying abstract topological spaces, with purely topological methods, is like playing a game of Marco Polo. You can’t really see what you are doing.” This is both the power and the problem in Persistent Homology. In this research defining Persistent Cohomology, it was our hope to add a little more sight to the Marco Polo game of data analysis.

When considering what was done to this effect, we remind the reader that we began with a combinatorial formula for the Stiefel-Whitney from (GT76). Because it is the Steifel-Whitney Cohomology class that determines an additional element of structure, orientability, in Chapter 4 we created a method of constructing the Poincaré Dual of a given homology class. If that homology class is the Stiefel-Whitney homology class,then we have constructed a way of finding the Stiefle-Whitney cohomology class of the tangent bundle, and thus have found a way to determine the orientability of a simplicial complex. Although we have not worked through the data ourselves, there is already an application for this.

In (CI08), Carlsson and Ishkhanov took a set of high contrast $4 \cdot 10^6$, ‘3 by 3’ patches of natural images and found that persistent homology indicated that there were a persistent 0 dimensional class, two persistent 1 dimensional classes, and 1 persistent 2 dimensional class. Blindly, both the Klein bottle and the torus share these homology classes and would both be considered possible candidates for the structure of the simplicial complex obtained from these points. Without knowing whether the first Stiefel-Whitney cohomology class was nontrivial, Carlsson and Ishkhanov found polynomial relations between the points that indicated the Klein Bottle was the appropriate fit. Had they been able to calculate the persistent Steifel-Whitney Cohomology class, they would not have needed the additional polynomial algebra.

In addition to orientability, there are many other geometric indicators that cohomology can disclose. Sometimes, bundle structures over topological spaces answer more questions than the topological spaces themselves. In Chapter 5, we defined CoPersistence through various stages of a filtration of simplicial complexes. We continued Chapter 5 with a separate algorithm that would allow us to compute the death point of a given cohomology class. While this algorithm was presented for any given cohomogy class, the above application indicated where this might be helpful if the given class was the Stiefel-Whitney cohomology class.

Chapter 6 detailed an Extended CoPersistence algorithm, which would measure how cohomology classes are born and die through two separate filtrations. Chapter 7 gave an algorithm that could determine when a given cohomology class is decomposable as the cup product of two lower dimensional classes. We ended this dissertation with Chapter 8 which disclosed a method to determine when a cohomology class is the Steenrod Square of another cohomology class. Chapters 5 through 8 provide a good start to the study of persistent cohomology which constructs a persistent approach to studying the bundle structure over a topological space. So far, there has not been a sufficient way to analyze stratified spaces through the lens of persistence. The research here could be applied to the bundles of stratified spaces, providing more analytical tools to data sets whose simplicial complexes do not represent a compact manifold.

A natural future direction for this research could be applying these algorithms to various data sets already analyzed using Persistent Homology to determine what new information can be acquired. Another place to develop persistent cohomology could be when developing a persistent approach to stratified spaces. In addition, future collaboration suggests that persistent cohomology operations could assist in research connecting Helmholtz-Hodge decomposition and Hodge decomposition with vector spaces; conversations on this topic have already begun.

Appendix A

Other Coefficient Fields

As previously stated in the introduction, we defined everything in the body of this document with $\mathbb{Z}_2$ coefficients. We made this decision because Stiefel-Whitney classes are defined in the cohomology ring with $\mathbb{Z}_2$ coefficients and Persistent Homology is typically calculated using $\mathbb{Z}_2$. When calculating Persistent Homology, the goal is to count and identify the homology classes, which can be done in any finite field. Using $\mathbb{Z}_2$ simplifies the algebra.

The other characteristic classes (Chern, Pontryagin, Euler classes) are defined in the Cohomology rings over the Complex numbers, Real numbers and Integers, respectively. In Milnor and Stasheff's "Characteristic Classes" (MS74), there are many defined relationships between the various characteristic classes, for example, the Wu Formula. While it is always a possibility to use the Universal Coefficient Theorem to convert our coefficient rings to $\mathbb{Z}_2$, it might be helpful to define a persistent approach to the study of other characteristic classes in another finite field $\mathbb{Z}_l$ for some odd prime l. In addition, it might be helpful to study how persistence is affected by a general cohomology operation of type (n, m, G_1, G_2), that is a set of maps, one for each cellular complex, Z, $\theta_Z : H^n(Z; G_1) \to H^m(Z; G_2)$.

One such cohomology operation that is defined for any coeffient group is the Steenrod Square. As we saw in Chapter 8, we can define a persistent approach to Steenrod Squares. In addition, we provided a general formula for Sq^1 in $\mathbb{Z}_2$ coefficients. This formula can be extended to a concise formula for general coefficients.

In their paper "Computing Cocycles on Simplicial Complexes" Díaz and Real defined the nth cup product in their Proposition 4.1 as the following:

Proposition 24. *Let R be the ground ring and K a simplicial complex with a finite number of vertices. If $c \in \mathcal{C}^p(K)$ and $c' \in \mathcal{C}^q(K)$, then for all nonnegative integers n, $c \cup_n c' \in \mathcal{C}^{p+q-n}(K)$ is defined by the following formulae. Let $m = p + q - n$ and $\sigma = \langle v_0, v_1, \ldots, v_m\rangle \in \mathcal{C}^m(K)$, then if n is even,*

$$c \cup_n c'(\sigma) = \sum_{i_n=S(n)}^{m} \sum_{i_{n-1}=S(n-1)}^{i_n-1} \cdots \sum_{i_1=S(1)}^{i_2-1} (-1)^{A(n)+B(n,m,\bar{i})+C(n,\bar{i})+D(n,m,\bar{i})}$$

$$c(\langle v_0, \ldots, v_{i_0}, v_{i_1}, \ldots, v_{i_2}, v_{i_3}, \ldots, v_{i_{n-2}}, v_{i_{n-1}}, \ldots, v_{i_n}\rangle)$$

$$\cdot\, c(\langle v_{i_0}, \ldots, v_{i_1}, v_{i_2}, \ldots, v_{i_3}, v_{i_4}, \ldots, v_{i_{n-1}}, v_{i_n}, \ldots, v_m\rangle)$$

and if n is odd, the formula is analogous. In these formulas, $\cdot$ is the product in R,

$$s(k) = i_{k+1} - i_{k+2} + \cdots + (-1)^{k+n-1} i_n + (-1)^{k+n} \left\lfloor \frac{m+1}{2} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor$$

for all $0 \le k \le n$, and $i_0 = S(0)$. Also,

$$A(n) = \begin{cases} 1 & \text{if } n \equiv 3, 4, 5, 6 \mod 8 \\ 0 & \text{otherwise} \end{cases} \tag{A.1}$$

$$B(n, m, \bar{i}) = \begin{cases} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} i_{2j} & \text{if } n \equiv 1, 2 \mod 4 \\ \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} i_{2j+1} + nm & \text{if } n \equiv 0, 3 \mod 4 \end{cases} \tag{A.2}$$

$$C(n,\bar{i}) = \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} (i_{2j} + i_{2j-1})(i_{2j-1} + \cdots + i_0) \tag{A.3}$$

$$D(n,m,\bar{i}) = \begin{cases} (m + i_n)(i_n + \cdots + i_0) & \textit{if } n \textit{ is odd} \\ 0 & \textit{if } n \textit{ is even} \end{cases} \tag{A.4}$$

In Chapter 8 we left out the $A(n), B(n,m,\bar{i}), C(n,\bar{i})$, and $D(n,m,\bar{i})$ formulae because we were only concerned with $\mathbb{Z}_2$ coefficients. Here, we provide the general form for any coeffient ring:

Proposition 25. *If $\alpha \in \mathcal{C}^p(X,G)$, then*

$$Sq^1(\alpha)(\langle v_0, v_1, \ldots, v_{p+1} \rangle) =$$

$$\sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{j=0}^{i-1} (-1)^{H(n)+K(s,t,n)} \alpha(\langle v_0, \ldots, \hat{v}_{2i}, \ldots, v_{p+1} \rangle) \cdot \alpha(\langle v_0, \ldots, \hat{v}_{2j}, \ldots, v_{p+1} \rangle)$$

$$+ \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{j=0}^{i-1} (-1)^{H(n)+K(s,t,n)} \alpha(\langle v_0, \ldots, \hat{v}_{2i-1}, \ldots, v_{p+1} \rangle) \cdot \alpha(\langle v_0, \ldots, \hat{v}_{2j-1}, \ldots, v_{p+1} \rangle)$$

where $(s,t) = (v_{2i}, v_{2j})$ or (v_{2i-1}, v_{2j-1}), respectively. And,

$$H(n) = \begin{cases} 1 & \textit{if } n \equiv 2,3,4,5 \mod 8 \\ 0 & \textit{otherwise} \end{cases} \tag{A.5}$$

$$K(s,t,n) = \begin{cases} \lfloor \frac{t+1}{2} \rfloor - \lfloor \frac{s+1}{2} \rfloor & \textit{if } n \equiv 1,2 \mod 4 \\ \lfloor \frac{s-1}{2} \rfloor + \lfloor \frac{p+1-t}{2} \rfloor & \textit{if } n \equiv 0 \mod 4 \\ \lfloor \frac{s-1}{2} \rfloor + \lfloor \frac{p+1-t}{2} + 1 \rfloor & \textit{if } n \equiv 3 \mod 4 \end{cases} \tag{A.6}$$

We already proved that the products of the α are across all combinations of removing two distinct even or odd vertices from a $p+1$ dimensional simplex in Chapter 8. What remains to prove is that $(-1)^{A(n)+B(n,m,\bar{i})+C(n,\bar{i})+D(n,m,\bar{i})} = (-1)^{H(n)+K(s,t,n)}$. Therefore,

Lemma 26. *For $\alpha \in C^p(X)$, with $m = p+1$ and $n = p-1$,*

$$\Big(A(n) + B(n,m,\bar{i}) + C(n,\bar{i}) + D(n,m,\bar{i}) = H(n) + K(s,t,n)\Big) \mod 2$$

Proof. Recall from Chapter 8 that $\bar{i} = (i_0, i_1, \ldots, i_n) = (0, 1, \ldots, \hat{s}, \ldots, \hat{t}, \ldots, p+1)$, where s and t have the same parity. The outline of this proof will go as follows, we will determine each evaluation of $A(n), B(n,m,\bar{i}), C(n,\bar{i})$, and $D(n,m,\bar{i})$ in $\mathbb{Z}_2$. It suffices to find their value in $\mathbb{Z}_2$ since it will determine if their sum is even or odd, i.e. if the coefficient is (1) or (−1), respectively. Then, we will combine those results to show that their sum equals $H(n) + K(s,t,n) \mod 2$.

We begin with $A(n)$, which is already reduced modulo 2. That is,

$$A(n) \mod 2 = \begin{cases} 1 & n \equiv 3,4,5,6 \mod 8 \\ 0 & \text{otherwise} \end{cases} \tag{A.7}$$

The next equation we reduce modulo 2 is $B(n,m,\bar{i})$. Recall,

$$B(n,m,\bar{i}) = \begin{cases} \displaystyle\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} i_{2j} & n \equiv 1,2 \mod 4 \\ \displaystyle\sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} i_{2j+1} + nm & n \equiv 0,3 \mod 4 \end{cases}$$

Recall, we can write $\bar{i} = (i_0, \ldots, i_n) = (0, 1, \ldots, \hat{s}, \ldots, \hat{t}, \ldots, p+1)$. In order to calculate $B(n,m,\bar{i}) \mod 2$, we evaluate the two cases for n.

If $n \equiv 1,2 \mod 4$: $B(n,m,\bar{i}) = \displaystyle\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} i_{2j}$.

This sum, modulo 2, depends on whether s and t are even or odd. Thus, we break down further to those two cases.

If s and t are even:

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} i_{2j} = 0{+}2{+}4{+}\cdots{+}(s{-}2){+}(s{+}1){+}(s{+}3){+}\cdots{+}(t{-}1){+}(t{+}2){+}\cdots{+}2\left\lfloor \frac{p+1}{2} \right\rfloor$$

If s and t are odd:

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} i_{2j} = 0{+}2{+}4{+}\cdots{+}(s{-}1){+}(s{+}2){+}(s{+}4){+}\cdots{+}(t{-}2){+}(t{+}1){+}\cdots{+}2\left\lfloor \frac{p+1}{2} \right\rfloor$$

Since all the numbers between 0 and s and t to $2\lfloor \frac{p+1}{2} \rfloor$ are even, determining $B(n, m, \bar{i}) \mod 2$ is based on the size of the set $|\{s+1, s+3, \ldots, t-1\}|$ or $|\{s+2, s+4, \ldots, t-2\}|$. In both cases, the size of the sets are $\lfloor \frac{t+1}{2} \rfloor - \lfloor \frac{s+1}{2} \rfloor$.

On the other hand, if $n \equiv 0, 3 \mod 4$: $B(n, m, \bar{i}) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} i_{2j+1} + nm$.

Again, this sum modulo 2 depends on whether s and t are even or odd.

If s and t are even:

$$\sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} i_{2j+1} + nm = 1{+}3{+}\cdots{+}(s{-}1){+}(s{+}2){+}\cdots{+}(t{-}2){+}(t{+}1){+}\cdots{+}(2\left\lfloor \frac{p}{2} \right\rfloor{+}1){+}nm$$

$B(n, m, \bar{i}) \mod 2$ is based on the number of odd numbers from 1 to $(s-1)$ and from $(t+1)$ to $(2\lfloor \frac{p}{2} \rfloor + 1)$.

If s and t are odd:

$$\sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} i_{2j+1} + nm = 1{+}3{+}\cdots{+}(s{-}2){+}(s{+}1){+}\cdots{+}(t{-}1){+}(t{+}2){+}\cdots{+}(2\left\lfloor \frac{p}{2} \right\rfloor{+}1){+}nm$$

$B(n,m,\bar{i}) \mod 2$ is based on the number of odd numbers from 1 to $(s-2)$ and from $(t+2)$ to $(2\lfloor \frac{p}{2} \rfloor + 1)$. The number of odd numbers for both of these cases is $(\lfloor \frac{s}{2} \rfloor + \lfloor \frac{p+1-t}{2} \rfloor)$. Lastly, $nm \equiv 0 \mod 2$ when n is even, and $nm \equiv 1 \mod 2$ when n is odd, so we can define $K(s,t,n)$ as:

$$B(n,m,\bar{i}) \mod 2 = K(s,t,n) = \begin{cases} \lfloor \frac{t+1}{2} \rfloor - \lfloor \frac{s+1}{2} \rfloor & n \equiv 1,2 \mod 4 \\ \lfloor \frac{s}{2} \rfloor + \lfloor \frac{p+1-t}{2} \rfloor & n \equiv 0 \mod 4 \\ \lfloor \frac{s}{2} \rfloor + \lfloor \frac{p+1-t}{2} \rfloor + 1 & n \equiv 3 \mod 4 \end{cases} \tag{A.8}$$

We now continue to reducing $C(n,\bar{i}) \mod 2$. Recall,

$$C(n,\bar{i}) = \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} (i_{2j} + i_{2j-1})(i_{2j-1} + \cdots + i_0)$$

Since this would be sums of the numbers 0 through $p+1$, except s and t are removed, we can split $C(n,\bar{i})$ into 3 separate sums:

$$C(n,\bar{i}) = \sum_{j=1}^{\lfloor \frac{s-1}{2} \rfloor} (i_{2j} + i_{2j-1}) \left(\sum_{k=0}^{2j-1} i_k \right) + \sum_{j=\lfloor \frac{s+2}{2} \rfloor}^{\lfloor \frac{t-1}{2} \rfloor} (i_{2j} + i_{2j-1}) \left(\sum_{k=0}^{2j-1} i_k \right) + \sum_{j=\lfloor \frac{t+2}{2} \rfloor}^{\lfloor \frac{n}{2} \rfloor} (i_{2j} + i_{2j-1}) \left(\sum_{k=0}^{2j-1} i_k \right)$$

We choose to make this split, because for $2j-1 < s$, each of the i_k for $k \leq 2j-1$ are the integers $[0, 2\lfloor \frac{s-1}{2} \rfloor - 1]$. In the second sum, $s < 2j-1 < t$, the i_k are the integers $[0, 2\lfloor \frac{t-1}{2} \rfloor - 1]$ with s skipped. Lastly in the third sum, $s < t < 2j-1$, the i_k are the integers $[0, 2\lfloor \frac{n}{2} \rfloor - 1]$. Since we are summing a interval of integers, we can use the identity $\sum_{i=1}^{n} i = \frac{(n+1)n}{2}$ to rewrite $C(n, \bar{i})$ as:

$$\sum_{j=1}^{\lfloor \frac{s-1}{2} \rfloor} (2j + 2j - 1)\Big(\frac{(2j-1+1)(2j-1)}{2}\Big) + \sum_{j=\lfloor \frac{s+2}{2} \rfloor}^{\lfloor \frac{t-1}{2} \rfloor} (2j + 2j - 1)\Big(\frac{(2j-1+1)(2j-1)}{2} - s\Big)$$

$$+ \sum_{j=\lfloor \frac{t-2}{2} \rfloor}^{\lfloor \frac{n}{2} \rfloor} (2j + 2j - 1)\Big(\frac{(2j-1+1)(2j-1)}{2} - s - t\Big)$$

$$= \sum_{j=1}^{\lfloor \frac{s-1}{2} \rfloor} (4j-1)j(2j-1) + \sum_{j=\lfloor \frac{s+2}{2} \rfloor}^{\lfloor \frac{t-1}{2} \rfloor} [(4j-1)j(2j-1) - s]$$

$$+ \sum_{j=\lfloor \frac{t-2}{2} \rfloor}^{\lfloor \frac{n}{2} \rfloor} [(4j-1)j(2j-1) - s - t]$$

When investigating the value of the first summand modulo 2, we recognize that $(4j-1)j(2j-1)$ is an even number when j is even, but odd when j is odd. Therefore, we need to count the number of odd js in the first summand. There are $\lfloor \frac{s}{2} \rfloor$ odd j's between $j = 1$ and $j = \lfloor \frac{s-1}{2} \rfloor$.

For the second summand, if s and t are even we count (as with the first summand) the number of odd js. On the other hand, if s and t are odd, we count the number of even js. There are $\left(\lfloor \frac{t}{2} \rfloor - \lfloor \frac{s}{2} \rfloor\right)$ even and odd integers (both) between $\lfloor \frac{s+2}{2} \rfloor$ and $\lfloor \frac{t-1}{2} \rfloor$.

Lastly, for the third summand, $-s-t$ is always an even number, so we count the number of odd js. There are $\left(\lfloor \frac{n}{2} \rfloor - \lfloor \frac{t}{2} \rfloor\right)$. It follows that

$$C(m, \bar{i}) \mod 2 \equiv \left(\left\lfloor \frac{s}{2} \right\rfloor + \left(\left\lfloor \frac{t}{2} \right\rfloor - \left\lfloor \frac{s}{2} \right\rfloor\right) + \left(\left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{t}{2} \right\rfloor\right)\right) \mod 2 = \left\lfloor \frac{n}{2} \right\rfloor \mod 2$$

This can be re-expressed as:

$$C(m, \bar{i}) \mod 2 = \begin{cases} 1 & n \equiv 2, 3 \mod 4 \\ 0 & n \equiv 0, 1 \mod 4 \end{cases} \tag{A.9}$$

We continue with $D(n,m,\bar{i})$, equation (A.4). Recall,

$$D(n,m,\bar{i}) = \begin{cases} (m+i_n)(i_n+\cdots+i_0) & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

Since if n is even, $D(n,m,\bar{i}) = 0 \equiv 0 \mod 2$, we focus on when n is odd. When n is odd, $n = p-1 \Rightarrow p$ is even, but $m = p+1$ is odd. As detailed in Chapter 8, the two possibilities for i_n are $p+1$ and p. If $i_n = p+1$, then the first term in the product is $(m+i_n) = 2(p+1)$, so $D(n,m,\bar{i}) = 2(p+1)(i_n+\cdots+i_0) \equiv 0 \mod 2$. If $i_n = p$, then the first term is $(m+i_n) = 2p+1$, and $D(n,m,\bar{i}) = (2p+1)(i_n+\cdots+i_0)$ which is even or odd depending on whether $(i_n+\cdots+i_0)$ is even or odd, respectively. Since $i_n = p$, $t = p+1$, an odd number. Since s has the same parity, it too is odd, which makes $\bar{i} = (0,1,\ldots,\hat{s},\ldots,2N)$, for $p = 2N$ for some number N. There are $N+1$ even numbers and $N-1$ odd numbers in $\bar{i}$. Obviously if $N-1$ is odd, then $D(n,m,\bar{i}) \equiv 1 \mod 2$, where if $N-1$ is even, then $D(n,m,\bar{i}) \equiv 0 \mod 2$.

We break down these cases based on $n \mod 8$, since $A(n)$ is based on the value of $n \mod 8$.

$$\text{if } n \equiv 1 \mod 8: \quad 8k+1 = 2N-1 \Rightarrow N-1 \equiv 0 \mod 4 \Rightarrow D(n,m,\bar{i}) \equiv 0 \mod 2$$

$$\text{if } n \equiv 3 \mod 8: \quad 8k+3 = 2N-1 \Rightarrow N-1 \equiv 1 \mod 4 \Rightarrow D(n,m,\bar{i}) \equiv 1 \mod 2$$

$$\text{if } n \equiv 5 \mod 8: \quad 8k+5 = 2N-1 \Rightarrow N-1 \equiv 2 \mod 4 \Rightarrow D(n,m,\bar{i}) \equiv 0 \mod 2$$

$$\text{if } n \equiv 7 \mod 8: \quad 8k+7 = 2N-1 \Rightarrow N-1 \equiv 3 \mod 4 \Rightarrow D(n,m,\bar{i}) \equiv 1 \mod 2$$

Thus, for $Sq^1(\alpha)$

$$D(m,n,\bar{i}) \mod 2 = \begin{cases} 1 & n \equiv 3,7 \mod 8 \\ 0 & \text{otherwise} \end{cases} \tag{A.10}$$

Now, if we combine equations (A.7), (A.9) and (A.10), we obtain:

$$\left(A(n) + C(n,\bar{i}) + D(m,n,\bar{i})\right) \mod 2 = H(n) = \begin{cases} 1 & n \equiv 2,3,4,5 \mod 8 \\ 0 & n \equiv 0,1,6,7 \mod 8 \end{cases}$$

□

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Biography

Aubrey HB was born as Aubrey Rae Hebda-Bolduc in Freehold, New Jersey on November 11, 1983. At three years of age, she grasped the concept of infinity and decided mathematics would be forever in her life. Aubrey transferred to Barnard College of Columbia University where she double majored in Mathematics and Theatre, both with honors. For theatre, she directed a full scale one act play and published "Exploring Richard Green's *The Author's Voice*." In mathematics, she won the Lucyle Hook grant that sponsored travel to the University of South Alabama to work with Drs. Williams and Silver. Her mathematics thesis entitled "An Exposition of a Knot Invariant Involving Symbolic Dynamics" earned her the first (and currently only) Math Senior Thesis Price. Aubrey HB graduated from Barnard College with a Bachelor of Arts degree in 2005. She also attended George Washington's Summer Program for Women in Mathematics in the summer between her Junior and Senior years.

While attending Duke University, Aubrey worked with Dr. Arlie Petters at the Petters' Research Institute in Dangriga, Belize and earned her Masters in December of 2006. During her time as a graduate student, she was funded under Darpa's TDA and Systems Biology programs. She has also receive travel grants to attend the Joint Mathematics Meetings in 2005, 2006, 2007, 2008, 2009 and 2010. The Duke University mathematics department offered her a visiting lecturer position to help her fund this last year.