Essays on Dynamic Tournaments

by

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Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in Business Administration
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Abstract

This dissertation studies dynamic tournaments and their economic and managerial implications from two different perspectives. The first half focuses on the optimal timing of information release when a tournament uses a feedback scheme, while the other half investigates the impact of the use of a mercy rule in a dynamic tournament on the economic output and other system wide characteristics.

In Chapter 2, we study dynamic tournaments in which time is modeled explicitly, as opposed to with the abstract notion of “periods.” By doing so, we characterize the effects of the ex-ante-designated timing of an interim progress report. Whether or not a policy of reporting increases total expected effort does not depend on the release time of the report, however the magnitude of the effect does. We demonstrate that total expected effort is single-peaked or single-troughed in the report's release time depending on parameters, with the peak/tough located at a time strictly more than halfway through the tournament. However, a policy of releasing information always harms the expected utility of the tournament's participants. Implications for tournament design are discussed.

Chaper 3 explores dynamic tournaments in a continuous space and continuous time framework, in which contestants can observe their opponents' progresses in real time and have the opportunity to end the contest early when one's lead over the other is larger than some pre-determined threshold (a.k.a a mercy rule). We first show that the game has a unique equilibrium, then characterize the equilibrium numerically, and investigate the impacts of mercy rules on tournament design. By doing so, we find that there exists an optimal mercy rule that induces the best economic output, even though players always prefer a tournament without a mercy rule. Depending on the cost and noises parameters, a non-monotonic mercy rule may perform better. We also consider the scenario in which players prefer to end the game early because of outside options and have the choice to drop out. Given an exogenous mercy rule, this drop-out option endogenizes another boundary. And surprisingly, the endogenous mercy rule is not always dominated by the exogenous rule in terms of inducing efforts.

To my beloved Mom and Dad,

who offered me continuous support and unreserved love all these years

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List of Abbreviations and Symbols

Symbols

Notation for Chapter 2

e_t^i	Player i's effort at time t
$c(\cdot)$	Cost function
a	The power parameter for the cost function
b	The scaling paramter for the cost function
x_t^i	The scoring process
Z_t	A standard Brownian Motion
σ	The volatility of the diffusion process
X_t	The relative difference of two players' scores at time t
t	Time
τ	Feedback release time
d	The scaling parameter for the second noise term
Y	The relative difference of two players' final scores
ϵ	Random Noise

Notation for Chapter 3

R_t^i	Player i's Economic output at time t
e_t^i	Player i's effort at time t
X_t	The relative difference of two players' scores at time t
B_t	A standard Brownian Motion
σ	The volatility of the diffusion process
d	The threshold for the lottery zone
Ω	A mercy rule
t	Time
τ	A stopping time
V	Player 1's value function
U	Player 2's value function
P	The sum of two players' value function
Q	The difference of two players' value function

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1

Introduction

Tournaments are widely used as a mechanism to motivate agents to make more efforts in an organization or an activity. The winner of a tournament gets a prize for his or her hardworking. Prominent examples in reality include: extra bonus for the auto-salesman who makes the most car sales; a job promotion for the employee who gets the best performance review in law firms and consulting companies; and a large cash award for the winner in an innovation contest. Since such tournaments usually last over weeks or months even years, they are dynamically evolved over time - the contestants need to decide how much effort to devote to the contest in the future based on their current information.

Depending on the specific setting and goal of a tournament, we can observe different information structures in actuality. When the economic output of a contest is not directly observable by the contestants, they are often left in the dark. For example, in an office competition for a job promotion, the employees know how hard they worked in the past, but they do not know their real "scores" or progress in the contest since they cannot observe how their manager perceive or judge them. In these situations, the principal (a.k.a the tournament designer or organizer) usually has the option to provide updates regarding the contestants' and their competitors' progress. For example, mid-year performance review is widely utilized in big corporations as a feedback scheme to let a worker be aware of his real progress in the office, as well as an important measure of a worker's overall performance. Various interesting economic questions related to the above mentioned feedback scheme can be asked.

In Chapter 2, entitled "When to release feedback in a dynamic tournament", we ask the following questions: (1) under what conditions should the principal release feedback, and (2) what is the optimal timing for the principal to do so if he opts for a feedback scheme. To answer these questions, we set up a game theory model in which two players select instantaneous effort levels, incur associated flow costs, at all moments during the tournament. These effort levels positively correlate to their stochastic output/score progresses, and a prize with a fixed value is allocated to the player with higher ending score. If the principal decides to use a feedback scheme, at some pre-determined time during the contest, a feedback will be released by the principal, which makes the relative difference of the players' score public. And based on this new update, players can revise their strategies accordingly. The principal's problem is to determine whether to release an information update and to choose a release time in order to induce an optimal total economic output. We find surprising robust results through this model. Whether or not a policy reporting increases total expected effort does not depend on the release time of the report, but on the concavity/convexity of the marginal cost of effort function. However, the magnitude of the effect does depend on timing, and is single-peaked/single-troughed in the report's release time. Intuitively speaking, a release time very close to the beginning of the contest will have little impact since the report contains very little new information, whereas a feedback very close to the end of the contest, though very informative, leaves too little time for the players to react effectively. Then it appears a release time exactly halfway through the contest would be optimal. After all, the feedback schemes observed in actuality usually default into "mid-term" reviews. For instance, the mid-year performance review in big companies literally mean a six-month performance review. However, we prove in this essay that if it is indeed more desirable to release a feedback, then it is always optimal to release the feedback strictly more than halfway through the tournament (usually when 60% to 80% of time has passed).

Since there is only one feedback release, the model in Chapter 2 essentially has only two periods, a pre-report period and a post-report period, even though time is modeled continuously. A natural question to ask following the model shown in Chapter 2 is what happens if there are multiple feedbacks. Admittedly, the same methodology (i.e. discrete-time dynamic programming and backward induction) used in Chapter 2 can be easily extended to study a tournament with multiple feedbacks. However, the algebra will become much more complicated and quite possibly intractable when the number of periods gets larger. Besides, the purpose of exploring a multi-period model is to see how contestants adjust their strategy when more information is available. A more efficient way to overcome this technical difficulty and to achieve our research goal is to assume contestants have the ability to observe each other's progress in real time, i.e. infinite many feedbacks and time periods. By doing so, we can apply a continuous-time methodology as shown in Chapter 3, which allows us to characterize the whole equilibrium readily. Moreover, from the perspective of modeling real life, as mentioned at the outset of this chapter, when the economic output is observable, the contestants actually are aware of their scores in continuous time. For example, in a competition for a bonus among salesman, how much sales a salesman have realized could be observable or verifiable by others.

In Chapter 3, entitled "Show no mercy? When to end a tournament", we study a continuous-state and continuous-time model, in which the score difference between contestants is public knowledge at all moments during the tournament. The principal's goal is still to induce the maximal possible economic output, however, the tool he uses to achieve this goal will be implementing a mercy rule, instead of providing new information. The use of mercy rules in reality can be often observed in sports, especially in youth sports league as a spare of further humiliation for the losing team, but the economic impacts of mercy rules are not well known. Our paper appears to be the first one that explicitly explores the effects of mercy rules in a dynamic tournament framework. While the contestants' equilibrium strategies are highly dynamic depending on parameters (such as the cost parameters and the noise parameters), we manage to show that there exists a unique equilibrium, in which: (1) the leader always works harder; (2) the effort choice is positively correlated to the size of the score difference, i.e. a large score difference discourages contestants to make efforts in the future, while a head-to-head situation forces them to both work harder; (3) large background noises reduce the total economic output. More importantly, regarding the principal's problem under the setting of a constant winning threshold, we find there exists an optimal mercy rule which induces the maximal economic output, even though contestants prefer a tournament without a mercy rule. The intuition of this result is given as follows: starting from a mercy rule with a very small winning threshold, by increasing the threshold, the contestants are forced to stay in the competition longer (a positive effect for inducing more efforts) but also makes less effort on average (a negative effect for inducing more efforts) since it is getting harder to cross the winning threshold and invoke the mercy rule. The positive effect outweighs the negative effect in the beginning as the expected duration of the tournament increases very fast as the threshold increases. However, the negative effects eventually dominates the positive counterpart because the maximal possible length of a tournament is bounded, resulting in a single-peaked shape of the total cumulative effort as a function of the winning threshold. We further investigate mercy rules with a more general linear form (i.e. the winning threshold can be changing in time according to a linear function of time), and find that depending on the parameters the optimal mercy rule may be increasing or decreasing.

2

When to Release Feedback in a Dynamic Tournament

2.1 Motivation

Starting with the seminal work of Lazear and Rosen (1981), the study of rank-order tournaments as simultaneous-moves games has received much attention, especially regarding the design and use of tournaments as a means of providing incentives to agents (e.g., employees, researchers, students, etc.).[1] More recent, but growing, is a body of work on dynamic tournaments, modeled as games with multiple periods (commonly, two). This short paper seeks to add to the literature on dynamic tournaments in two ways. First, by developing a tractable framework in which time

[1] Prominent examples include (but are not limited to) Green and Stokey (1983); Singh and Wittman (1988); Moldovanu and Sela (2001); Gaba et al. (2004); Orrison et al. (2004); Kilgour and Gerchak (2004); Johnstone (2007); Chen et al. (2011); Charness et al. (2014). See Konrad (2009) for an extensive survey of both theory and applications. Notably, we demonstrate that Lazear and Rosen's (1981) model and equilibrium are equivalent to our dynamic ones under the specification that there is no information released to the tournament participants.

is explicitly incorporated into the model. That is, there are no exogenously given "periods" or "stages." Second, by using the framework to investigate the optimal timing of an interim feedback report.

The lead example in mind is the classic example of Lazear and Rosen (1981): two workers in an organization exert effort with the aim of being the one who lands the coveted promotion, as is the primary reward structure in so-called "up-or-out" organizations like law firms, consultant companies, and (to an extent) academia.[2] In actuality, such competitions are inherently dynamic—each day (over a course of years), the workers decide how much effort to expend toward their goal. Hence, the question of whether or not to have an interim progress report is not sufficiently well-posed: it does not address *when* the report in question is to be released.

In our model, participants select instantaneous effort levels, and incur associated flow costs, at all moments during the tournament. These effort levels influence their stochastic output/score processes, and the prize is allocated to the player with highest end-of-tournament score. Naturally, players' effort choices will be responsive to their information about relative position. However, the inclusion of true time into the model links the timing of any progress report to the amounts of effort and degrees of randomness both before and after its release.

A handful of recent papers (see below) have investigated under what conditions on the tournament setting should an interim progress report be released between the exogenously given periods in their models. We use our formulation to answer the natural analog of this question: *when* should a progress report be released? This

[2] See, again, Konrad (2009) for additional examples, and Ederer (2010) for discussion of feedback policies in applications from education, sports, and politics.

statement of the question (and the title of the paper) is intentionally a bit imprecise, for the purpose of juxtaposition. In particular, we are using "when" in the literal sense (i.e., at what time), not the more colloquial sense of "under what conditions." Precisely, we investigate the optimal timing of a progress report that reveals the current state of the competition, if the goal is maximizing expected cumulative effort and the release time is specified ex ante.[3]

We find surprisingly robust answers. Whether or not a policy of reporting increases total expected effort does not depend on the release time of the report, but on the concavity/convexity of the marginal cost of effort function (similar to Aoyagi (2010); Ederer (2010), discussed below). However, the *magnitude* of the effect does depend on timing, and is single-peaked in the report's release time. Intuitively, if the release time is very near the beginning of the contest, it will have little effect on average as there will be little information to report. Conversely, a report very near the end of the contest will reveal quite a lot of information, but with too little time remaining for players to react in a way that meaningfully alters cumulative efforts.

Intuition might then suggest that the report would have maximal impact if released halfway through the tournament—indeed, in many settings principals appear to default into "mid-term" reviews (Murphy and Cleveland, 1991)—or perhaps that whether an earlier or later release will be more impactful would depend on parameters. However, we demonstrate that a report has maximal impact on expected cumulative efforts when it is released strictly more than halfway through the tournament. To fix ideas, consider the case in which having a report increases expected

[3] Since releasing the report at the very beginning of the tournament provides no additional information (so is equivalent to not having a report), this question strictly nests the question of whether or not to have the report at all.

effort. The only force in favor of a later report is the following: in the event that the competition is still "close," with less time until the end, there is less randomness ahead of the players, which induces greater instantaneous efforts. There are several countervailing forces (as we describe in Section 2.4.1), but our result shows that this first force has sufficient power to always keep the optimal release time more than half-way through the tournament.

Even if knowing the exact convexity of the cost function is only an abstraction, one can still take away the following. If having a report is expected to increase expected effort, it is never optimal to have the release time before the half-way point in the tournament. If having a report is expected to decrease expected effort, but there is some (for example, legal) requirement to have one sometime in the interval $[t_1, t_2]$, then either t_1 or t_2 is the optimal time for the release (because expected effort is single-troughed in the release time).

However, we also show that a policy of releasing information always harms the expected utility of the tournament's participants. The reasoning here is that the anticipation of information release creates uncertainty about a player's own future effort provision. Because the cost of effort is convex, players are averse to this risk.

Finally, we hope that our model's incorporation of explicit time will prove useful for future research, as we discuss in the concluding remarks.

2.1.1 Related Literature

As mentioned at the outset, the literature on tournaments is, by now, far too large to be adequately summarized here. Narrowing more on the interest of the present paper, several recent papers have extended the analysis of tournaments into dynamic settings (Yildirim, 2005; Casas-Arce and Martnez-Jerez, 2009; Gershkov and Perry, 2009; Zhang and Wang, 2009; Aoyagi, 2010; Ederer, 2010; Goltsman and Mukherjee, 2011; Kuhnen and Tymula, 2012; Ridlon and Shin, 2013). Once dynamic, the information released to tournament participants plays a critical role in determining their behavior. For example, Casas-Arce and Martnez-Jerez (2009) document that when players have full information, early-period performance differences demotivate effort in later periods. Our findings align with this result, but we also highlight the inverse: knowledge that early performance differences are negligible motivates substantially higher effort levels near the tournament's end. Further, we show that this force pushes the optimal release time more than halfway through the tournament. Though based on a different analysis, Gershkov and Perry (2009) and Ridlon and Shin (2013) also find it may be optimal to treat early and late effort differently.[4]

Unlike ours, all of the above papers use exogenously given "periods" or "stages." In contrast, we build a framework in which the timing of information release is a meaningful consideration. The two papers most closely related to ours are Aoyagi (2010) and Ederer (2010) in that our focus on explicit timing is the primary difference. On a more technical note, though we believe our specification to be quite natural, for any given release time our model does not satisfy the conditions for equilibrium existence given in either paper, so we provide our own. Like them, we assume that players privately know their past effort choices, but that their performances/scores can only be revealed by the principal. In the context of within-organization worker competition (e.g., for a promotion), Ederer justifies this assumption,

[4] Marinovic (2015) builds on this literature by introducing the possibility that the principal may not need to deliver truthful feedback.

> "[W]orkers' performance is generally private information of the manager or the firm's personnel division. Workers, on the other hand, cannot fully observe their own performance because output can often only be measured by a combination of factors such as social skills, originality, team-working ability and because there may be a significant subjective element in the evaluation of performance. Furthermore, organizations usually have much more experience in assessing the contributions of an individual worker who tends to have little experience with the tasks he is performing." (Ederer, 2010, p. 736)

The assumption is also useful in that it implies that our tournament without feedback is equivalent to a standard, static tournament, facilitating comparisons. Like we do, Aoyagi (2010) and Ederer (2010) find that the desirability of feedback for inducing effort hinges on the convexity/concavity of the marginal cost of effort.[5] In addition to studying timing, we also include analysis of player welfare not found in these papers.

Our notion of a tournament entails an explicit end time, regardless of how the participants' outputs stand in either relative or absolute terms when that time is reached. There is, however, a considerable literature on dynamic competitions with different rules governing when the prize is to be allocated. For example, Harris and Vickers (1987); Moscarini and Smith (2011); Cao (2014) study a "tug-of-war" style competition: the prize is awarded to the first player to build a pre-determined output lead over his opponent, regardless of the time at which it occurs. In these models, players have constant feedback (they know their relative position—the only state variable that matters—at every point in time). Since complete information is

[5] Consistent with this result, Liden and Mitchell (1985); Podsakoff and Farh (1989) find evidence that instituting interim-feedback policies has lead to both higher and lower performance outcomes in practice.

assumed, the design questions instead look at optimal dynamic handicapping, which can be thought of as a tax (subsidy) for the current leader (trailer). Because the problem is stationary, in the continuous-time version equilibrium is characterized by ODEs. In contrast, having constant feedback in a model with a fixed tournament end-date means that equilibrium will be characterized by more complex PDEs, with both relative position and time being relevant components of the state, which we are attempting to investigate in follow-up work.[6]

The other main form of dynamic competition can be referred to as a "race": the prize is allocated to the first player whose cumulative output hits a pre-determined absolute standard, again independent of the time at which this event occurs. This form is commonly used in models of R&D/innovation competition, where the prize (patent, market leadership) is conferred on the first participant to develop a viable (patentable, marketable) product (Choi, 1991; Malueg and Tsutsui, 1997; Halac et al., 2016). Within this domain, Bimpikis et al. (2016) study a similar question to ours, asking if it is useful for the principal to (solicit and) release information about participants' interim progress. They find it is optimal to induce "silent" periods in which contestants receive no information from the principal, followed by information release. Their paper also introduces the novel apsect that there is aggregate uncertainty about whether the prize will ever be awarded (i.e., the innovation sought may not even be possible), which brings additional considerations for the design.

[6] In the Management literature, Beer (1987); Gibbs (1991); Murphy and Cleveland (1991), argue that complete transparency is unlikely to be optimal, and that managers should thoughtfully release/withhold information to improve worker performance. This recommendation is in line with the theoretical work of Bimpikis et al. (2016) on innovation races.

2.2 The Model and Preliminaries

There are two symmetric players, 1 and 2, competing in a tournament for a single prize, with a normalized common value of 1. Time is continuous from 0 to 1. At every moment in time, t, each player, i decides how much effort to exert, denoted $e_t^i \in \mathbb{R}_+$, and incurs a flow cost of $c(e_t^i) = b \cdot (e_t^i)^a$, where $a > 1$ and $b > 0$.

For each player i define the process $\{x_t^i : 0 \leqslant t \leqslant 1\}$, sometimes referred to as the player's *score*, as $x_0^i = 0$ with the evolution $dx_t^i = e_t^i dt + \frac{\sigma}{2} dZ_t^i$, where $\sigma > 0$ and Z^1, Z^2 are mutually independent standard Brownian motions. It is convenient to define $X \equiv x^1 - x^2$. Hence, $X_0 = 0$, and $dX_t = (e_1^t - e_2^t)dt + \sigma dZ_t$, where $Z \equiv Z^1 - Z^2$. This formulation in which an agent incurs cost to control the drift of a Brownian diffusion process is common in both stochastic single decision-maker problems (see Harrison (2013)) and in stochastic games (see, for example, Budd et al. (1993)). In addition, we allow that perhaps not all tournament-relevant uncertainty is resolved during the contest, but might only realize at time $t = 1$ or later. To capture this, let ϵ^1, ϵ^2 be i.i.d. $\mathcal{N}(0, \frac{d \cdot \sigma^2}{2})$, where $d > 0$ (but can be arbitrarily small), and $y^i = x_1^i + \epsilon^i$.[7]

The prize is allocated to player 1 if $y^1 \geqslant y^2$ and to player 2 otherwise—the specification that ties are broken in favor of player 1 will have no effect on results because ties will have zero probability, but eases exposition. Again, it is convenient to define $Y \equiv y^1 - y^2$, meaning the prize is awarded to player 1 (respectively, 2) if

[7] This latent uncertainty, even if arbitrarily small, also aids in the existence of the pure-strategy equilibrium in the case where the report's release time, τ, is arbitrarily close to 1 (i.e., the end of the tournament)—see the proof of Lemma 2. Alternatively, one could place a bound $\overline{\tau} < 1$, and constrain τ to $[0, \overline{\tau}]$.

$Y \geqslant 0$ ($Y < 0$). Player 1's payoff is

$$\mathbb{I}_{\{Y \geqslant 0\}} - \int_0^1 c(e_t^1)dt,$$

where $\mathbb{I}$ is the indicator function. Player 2's payoff is the analogous, opposing expression. Both players are expected-utility maximizers.[8]

The last element to specify is players' information. At each time t, each player privately knows his history of past efforts, $(e_{t'}^i)_{t' \leqslant t}$. In addition, at a pre-specified time $\tau \in [0, 1)$, a report is issued that makes X_τ public knowledge. All aspects of the environment are common knowledge at the outset of the game. Note that setting $\tau = 0$ is akin to revealing no information (since $X_0 = 0$ is already common knowledge).

Our solution concept is pure-strategy Perfect Bayesian Equilibrium, henceforth simply *equilibrium*.[9] Our interest is in characterizing the symmetric equilibrium of the game.

2.2.1 Preliminaries

While each player exerts effort in continuous time, they have no new information between times $t = 0$ and τ and between τ and $t = 1$. Hence, it is without loss to consider each player selecting their effort processes $\{e_t^i : 0 \leqslant t < \tau\}$ at the outset and $\{e_t^i : \tau \leqslant t \leqslant 1\}$ following the report at time τ. In addition, because the flow

[8] Players do not discount time. This both simplifies analytic expressions and, more importantly, facilities comparison to the canonical static model of Lazear and Rosen (1981) as a special case of ours.

[9] While players have private information regarding their own past action choices, the specification of equilibrium beliefs is trivial, and therefore not a point of focus below.

cost of effort is convex, players prefer to "smooth" their efforts, summarized in the following lemma (all proofs are found in the Appendix).

Lemma 1. *Fix any $\tau \in [0,1)$. For $i = 1,2$, in any equilibrium, e_t^i is constant for almost all $t \in [0,\tau)$ (i.e., $e_t^i = \frac{1}{\tau}\int_0^\tau e_s^i ds$ for almost all $t \in [0,\tau)$). Likewise, e_t^i is constant for almost all $t \in [\tau,1]$ (i.e., $e_t^i = \frac{1}{1-\tau}\int_\tau^1 e_s^i ds$ for almost all $t \in [\tau,1]$).*

Hence, in the subsequent analysis of the game, we simplify by identifying each player's choices of cumulative pre- and post-report efforts, denoted $e_0^i \cdot \tau$ and $e_\tau^i(1-\tau)$ (respectively), which carry cumulative costs $c(e_0^i)\tau$ and $c(e_\tau^i)(1-\tau)$.

2.3 Equilibrium

Given Lemma 1, we solve the game using backward induction.

2.3.1 After the Report

At time $t = \tau$ player 1 solves

$$\max_{e_\tau^1} F_{(1-\tau+d)}\left(X_\tau + \left(e_\tau^1 - e_\tau^2\right)(1-\tau)\right) - (1-\tau)\, c(e_\tau^1)$$

where F_Δ is the distribution function of a normal random variable with mean zero and variance $\Delta\sigma^2$, and let f_Δ be the associated density function. The first-order condition is

$$c'(e_\tau^1) = f_{(1-\tau+d)}\left(X_\tau + \left(e_\tau^1 - e_\tau^2\right)(1-\tau)\right). \tag{2.1}$$

Likewise, player 2 solves

$$\max_{e_\tau^2} \left(1 - F_{(1-\tau+d)}\left(X_\tau + \left(e_\tau^1 - e_\tau^2\right)(1-\tau)\right)\right) - (1-\tau)\, c(e_\tau^2),$$

which yields the first-order condition

$$c'(e_\tau^2) = f_{(1-\tau+d)}\left(X_\tau + \left(e_\tau^1 - e_\tau^2\right)(1-\tau)\right). \tag{2.2}$$

Noting that the right-hand-sides of (2.1) and (2.2) are identical yields that, if effort levels satisfy first-order conditions, then

$$c'(e_\tau^1) = c'(e_\tau^2) = f_{(1-\tau+d)}\left(X_\tau + \left(e_\tau^1 - e_\tau^2\right)(1-\tau)\right). \tag{2.3}$$

Since c' is strictly monotonic, $e_\tau^1 = e_\tau^2$, implying that $c'(e_\tau^1) = f_{(1-\tau+d)}(X_\tau) = c'(e_\tau^2)$. What is left is verifying that equilibrium effort levels are indeed governed by first-order conditions, which is done in the proof of the following lemma.

Lemma 2. *Fix any $\tau \in [0,1)$. There is a unique equilibrium candidate in the continuation game starting from time τ. In it, for all X_τ and almost all $t \geqslant \tau$,*

$$e_t^1 = e_t^2 = \left(\frac{f_{(1-\tau+d)}(X_\tau)}{ab}\right)^{\frac{1}{a-1}}. \tag{2.4}$$

The equilibrium exists if σ is large enough (all other parameters fixed), d is large enough (all other parameters fixed), or b is large enough (all other parameters fixed). An explicit sufficient condition is provided in Lemma A.0.1 in the Appendix.

Four observations are in order. First, notice that symmetry is not invoked as a condition, but is a result: (if it exists) the unique equilibrium of the continuation game *is* symmetric. Intuitively, by the zero-sum nature of the prize allocation, each player faces the same marginal incentive: a slight gain in the probability of winning the prize ($f_{(1-\tau+d)}(X_\tau)$), traded off against a slightly higher effort cost ($c'(e_\tau^i)$).

Second, immediately from the shape of the normal density function, post-report efforts are decreasing in $|X_\tau|$, the size of the lead one player enjoys over another—as the reported leader is more likely to ultimately win, both players have less incentive to exert effort. Third, while the "large enough" statements in the proposition may call to mind existence only as a limit result, this is not so since we supply an explicit condition on parameters for equilibrium existence.[10] Fourth, in the special case of $\tau = 0$, the distribution of X_τ is degenerate on zero, and (given Lemma 1) the game is equivalent to a static tournament, as first studied by Lazear and Rosen (1981). The equilibrium specified by Lemma 2 aligns with unique equilibrium of the corresponding static game.

2.3.2 Before the Report

If $\tau = 0$, no further analysis is needed. Suppose, instead, that $\tau > 0$. At time $t = 0$ player 1, solves

$$\max_{e_0^1} E\left[F_{(1-\tau+d)}\left(\underbrace{X_0 + \left(e_0^1 - e_0^2\right)\tau + \sigma Z_\tau}_{X_\tau} + \left(e_\tau^1 - e_\tau^2\right)(1-\tau)\right) - (1-\tau)c\left(e_\tau^1\right)\right] - \tau c(e_0^1). \tag{2.5}$$

The first-order condition is[11]

$$\begin{aligned}\tau c'(e_0^1) = &\int\left[f_{(1-\tau+d)}\left(\left(e_0^1 - e_0^2\right)\tau + z\right)\left(\tau + (1-\tau)\left(\frac{\partial e_\tau^1}{\partial e_0^1} - \frac{\partial e_\tau^2}{\partial e_0^1}\right)\right)\right] f_\tau(z)dz \\ &- \int\left[f_{(1-\tau+d)}\left(\left(e_0^1 - e_0^2\right)\tau + z\right)(1-\tau)c'\left(e_\tau^1\right)\frac{\partial e_\tau^1}{\partial e_0^1}\right] f_\tau(z)dz.\end{aligned}$$

---

[10] Note that we cannot rely on the sufficient condition provided in Aoyagi (2010), which in our notation is: $\sup_{z\in\mathbb{R}} f'_{(1-\tau+d)}(z) < \inf_{e\in\mathbb{R}_+}(1-\tau)c''(e)$. Clearly this is impossible since the first term is (strictly) positive and the second is zero. Hence, we present our own sufficient condition.

[11] Every integral is taken over the entire real line unless bounds are explicitly written.

Collecting the $\frac{\partial e_\tau^1}{\partial e_0^1}$-terms yields

$$\tau c'(e_0^1) = \int f_{(1-\tau+d)}\left(\left(e_0^1 - e_0^2\right)\tau + z\right)\left(\tau + (1-\tau)\left(-\frac{\partial e_\tau^2}{\partial e_0^1}\right)\right) f_\tau(z)dz$$

$$+ (1-\tau)\int\left[\left(f_{(1-\tau+d)}\underbrace{\left(\left(e_0^1 - e_0^2\right)\tau + z\right)}_{=X_\tau} - c'\left(e_\tau^1\right)\right)\frac{\partial e_\tau^1}{\partial e_0^1}\right] f_\tau(z)dz.$$

From Lemma 2, $c'(e_\tau^1) = f_{(1-\tau+d)}(X_\tau)$, realization-by-realization. Hence, the $\frac{\partial e_\tau^1}{\partial e_0^1}$-terms cancel, leaving[12]

$$c'(e_0^1) = \int f_{(1-\tau+d)}\left(\left(e_0^1 - e_0^2\right)\tau + z\right)\left(1 + \left(\frac{1-\tau}{\tau}\right)\left(-\frac{\partial e_\tau^2}{\partial e_0^1}\right)\right) f_\tau(z)dz.$$

$$= \int f_{(1-\tau+d)}\left(\left(e_0^1 - e_0^2\right)\tau + z\right) f_\tau(z)dz -$$

$$\left(\frac{1-\tau}{\tau}\right)\int f_{(1-\tau+d)}\left(\left(e_0^1 - e_0^2\right)\tau + z\right)\left(\frac{\partial e_\tau^2}{\partial e_0^1}\right) f_\tau(z)dz.$$

[12] The symmetry of post-report efforts in the unique continuation equilibrium (Lemma 2) makes the cancelation of the $\frac{\partial e_\tau^1}{\partial e_0^1}$-terms particularly easy to see. However, the cancelation would hold even if post-report efforts were asymmetric (due to some change in the model), as the result is an application of the Envelop Theorem.

By convolution[13], $\int f_{(1-\tau+d)}\left(\left(e_0^1-e_0^2\right)\tau+z\right)f_\tau(z)dz = f_{(1+d)}(\left(e_0^1-e_0^2\right)\tau)$. So the first-order condition further simplifies to

$$c'(e_0^1) = f_{(1+d)}\left(\left(e_0^1-e_0^2\right)\tau\right) - \left(\frac{1-\tau}{\tau}\right)\int f_{(1-\tau+d)}\left(\left(e_0^1-e_0^2\right)\tau+z\right)\left(\frac{\partial e_\tau^2}{\partial e_0^1}\right)f_\tau(z)dz. \tag{2.6}$$

Player 2's first-order condition is analogous:

$$c'(e_0^2) = f_{(1+d)}\left(\left(e_0^1-e_0^2\right)\tau\right) - \left(\frac{1-\tau}{\tau}\right)\int f_{(1-\tau+d)}\left(\left(e_0^1-e_0^2\right)\tau+z\right)\left(\frac{\partial e_\tau^1}{\partial e_0^2}\right)f_\tau(z)dz. \tag{2.7}$$

In a symmetric equilibrium, $e_0^1 = e_0^2$, giving,

$$c'(e_0^1) = f_{(1+d)}(0) - \left(\frac{1-\tau}{\tau}\right)\int f_{(1-\tau+d)}(z)\left(\frac{\partial e_\tau^2}{\partial e_0^1}\right)f_\tau(z)dz. \tag{2.8}$$

$$c'(e_0^2) = f_{(1+d)}(0) - \left(\frac{1-\tau}{\tau}\right)\int f_{(1-\tau+d)}(z)\left(\frac{\partial e_\tau^1}{\partial e_0^2}\right)f_\tau(z)dz. \tag{2.9}$$

Finally, direct evaluation of the integrals in (2.8) and (2.9) yields that both are equal to zero (see proof of Lemma 3 in the Appendix). Hence, if effort levels satisfy

[13] See, for example, Casella and Berger (2002, p. 215). Or directly evaluated,

$$\begin{aligned}
&\int f_{(1-\tau+d)}\left(\left(e_0^1-e_0^2\right)\tau+z\right)f_\tau(z)dz\\
&= \int \frac{1}{\sqrt{2\pi(1-\tau+d)}\sigma}\exp\left(-\frac{\left(\left(e_0^1-e_0^2\right)\tau+z\right)^2}{2(1-\tau+d)\sigma^2}\right)\frac{1}{\sqrt{2\pi\tau}\sigma}\exp\left(-\frac{z^2}{2\tau\sigma^2}\right)dz\\
&= \frac{1}{2\pi\sigma^2\sqrt{\tau(1-\tau+d)}}\int \exp\left(-\frac{\tau\left(\left(e_0^1-e_0^2\right)\tau+z\right)^2+(1-\tau+d)z^2}{2\tau(1-\tau+d)\sigma^2}\right)dz\\
&= \underbrace{\frac{1}{\sqrt{2\pi(1+d)}\sigma}\exp\left[-\frac{\left(e_0^1-e_0^2\right)^2\tau^2}{2(1+d)\sigma^2}\right]}_{=f_{(1+d)}((e_0^1-e_0^2)\tau)}\underbrace{\int\frac{1}{\sqrt{2\pi}\sigma'}\exp\left[-\frac{(z-\mu')^2}{2\sigma'^2}\right]dz}_{=1,\text{ since integrating a normal pdf}},
\end{aligned}$$

where $\mu' = \frac{-\left(e_0^1-e_0^2\right)\tau^2}{1+d}$ and $\sigma'^2 = \sqrt{\frac{(1-\tau+d)\tau\sigma^2}{1+d}}$.

first-order conditions, then $c'(e_0^1) = c'(e_0^2) = f_{(1+d)}(0)$, which has unique solution since c' is strictly monotone. Again, what is left to verify is that equilibrium efforts are governed by first-order conditions, which is done in the proof of the following lemma.

Lemma 3. *Fix any $\tau \in (0,1)$. There is a unique symmetric equilibrium candidate. In it, for all $t < \tau$,*

$$e_t^1 = e_t^2 = \left(\frac{f_{(1+d)}(0)}{ab} \right)^{\frac{1}{a-1}}, \tag{2.10}$$

and continuation play for $t \geqslant \tau$ is as described in Lemma 2. The equilibrium exists if σ is large enough (all other parameters fixed), d is large enough (all other parameters fixed), or b is large enough (all other parameters fixed). An explicit sufficient condition is provided in Lemma A.0.2 in the Appendix.

Three observations are in order. First, as in Aoyagi (2010), the pre-report effort levels are exactly the effort levels the players would select in the unique equilibrium of a static/no-report environment. Second, as we also noted following Lemma 2, we provide an explicit sufficient condition for existence (so our result is not merely a limit result).[14] Third, the rationale for this result is precisely captured by the presented analysis. If a player marginally increases his pre-report effort level (e_0^i), he both influences the lead (X_τ) and incurs a higher cost ($c'(e_0^i)$). The change in the lead affects his effort level and cost after the report ($\frac{\partial e_\tau^i}{\partial e_0^i}$ and $c'(e_\tau^i)\frac{\partial e_\tau^i}{\partial e_0^i}$), but from the Envelope Theorem, these effects negate one another. He also affects his

[14] Again, we cannot rely on the sufficient condition provided in Aoyagi (2010)—this time Assumption 1 therein—as it is even stronger than the condition seen in footnote 10, and therefore impossible to satisfy in our model.

opponent's post-report effort ($\frac{\partial e_\tau^j}{\partial e_0^i}$). However, in equilibrium, this effect is zero on average. Why? The symmetries of both the continuation equilibrium (Lemma 2) and the normal density function imply that if player i increases his pre-report effort from equilibrium slightly, for any realization of the pre-report noise (Z_τ) for which this results in j increasing his post-report effort by ε, there is an equally likely realization of the pre-report noise for which the deviation results in j decreasing his post-report efforts by ε. Thus, the expected effect on the opponent's future action is zero. This implies that players are only trading off the benefit of increasing the initial lead and the cost of doing so (as seen in (2.8) and (2.9)). This is exactly the same tradeoff they face in a static tournament. Therefore, they choose the same effort level in both situations.

2.4 Total Expected Effort

Before analyzing the optimal timing of the report, we collect two results about players' expected total efforts and payoffs in equilibrium. Let $W(\tau) \equiv E[\int_0^1 e_t^i dt|\tau]$ denote the *ex-ante* expected cumulative effort per player in equilibrium, given the report will be released at time τ. As in Aoyagi (2010) and Ederer (2010), whether a report increases total effort depends on the convexity/concavity of the marginal cost of effort.

Proposition 4. *For any* $\tau \in (0,1)$*,* $sign(W(\tau) - W(0)) = sign(2 - a)$*.*

Informally, the result follows from two facts. First, from Lemma 3, the pre-report effort levels are identical to what they would be with no report. Hence, any difference must come from the expected post-report effort levels. Second, from Lemma 2, post-report efforts are governed by the marginal cost of effort function. That the value of a report is positive/negative hinges on the concavity/convexity of the marginal cost follows then from Jensen's inequality.

Now, let $S(\tau)$ denote the *ex-ante* expected equilibrium payoff to a player given the report will be released at time τ. We obtain a less nuanced result.

Proposition 5. *For any* $\tau \in (0, 1)$*,* $S(\tau) < S(0)$*.*

Hence, contestants strictly prefer tournaments without information reports—even if a report would decrease their expected effort. While we (like many others) have taken participation as granted, this result could be important in a richer model wherein agents choose among different available tournaments, or simply to opt out, implying a participation (or individual-rationality) constraint that lower bounds their required payoff.

An intuition for Proposition 5 is that, although players are risk neutral with respect to lotteries over the prize, the convexity of the cost function builds in a risk aversion with respect to future effort. In tournaments without information arrivals this is not a concern since a player has no uncertainty about his effort level at any future date. If a player expects to receive information, he expects to have to alter his action. The convexity of the cost function implies that this uncertainty is harmful to the agent *ex ante*.

Customarily, the assumption that the cost of effort is convex is made without any thought to risk aversion. It is usually intended to capture the notion that work is increasingly difficult the more one does of it (and also to ensure optimization problems are well posed with interior solutions). Though this may not have been our intention, once we assume players are expected utility maximizers, it is inescapable that they will be risk averse with respect to future effort costs.

2.4.1 Optimally Timing the Report

Because time is an explicit feature of the model, we investigate not only whether or not having a report increases expected effort (Proposition 4), but *when* is the effort-maximizing release time. Of course, since $\tau = 0$ is feasible and is equivalent to having no report, the question of optimal timing strictly nests the question of whether to have a report at all.

Consider the case of $a < 2$, so having an interim report ($\tau \in (0,1)$) increases expected effort relative to having no report ($\tau = 0$). Recall from the proof of Proposition 4 that

$$W(\tau) = \tau W(0) + (1-\tau)E[e^i_\tau] = \underbrace{\tau}_{(1)} W(0) + \underbrace{(1-\tau)}_{(1)} E[h(\underbrace{f_{(1-\tau+d)}}_{(3)}(\underbrace{Z_\tau}_{(2)}))]. \qquad (2.11)$$

Increasing τ has three effects on $W(\tau)$, which can be isolated as follows (and are identified by the underbraces in (2.11)).

1. ***Duration effect***: Increasing τ means more of the tournament is spent in the pre-report phase. By Proposition 4, for any $\tau > 0$, the pre-report effort level is lower than the expected post-report effort level (i.e., $W(0) < E[e^i_\tau]$, when $a < 2$). As seen in (2.11), a later report time increases the weight (i.e., τ) on $W(0)$ and decreases the weight (i.e., $(1-\tau)$) on $E[e^i_\tau]$. All else equal, clearly, this has a negative effect on W.

2. ***Dispersion effect***: Increasing τ increases the variance of Z_τ, meaning the size of the lead revealed by the report, $|X_\tau|$, is higher on average. To see this

recall that, in equilibrium, the players' pre-report efforts cancel out, and the difference in their scores at τ, X_τ, is determined by noise, Z_τ. Since, post-report efforts, e^i_τ, decrease in $|X_\tau|$ (see Lemma 2), this too has a negative effect on W (all else equal).

3. ***Incentive effect***: Increasing τ decreases the variance of the post-report noise (i.e., the variance of $Z_1 - Z_\tau + \epsilon^1 - \epsilon^2$). From Lemma 2, we can observe that there is a threshold k such that this decrease in noise increases e^i_τ if $|X_\tau| < k$, but decreases e^i_τ if $|X_\tau| > k$. That is, if the report reveals that "the lead is small," post-report efforts are higher with larger τ, but also the reverse if the report reveals "the lead is large." Figure 2.1 illustrates.

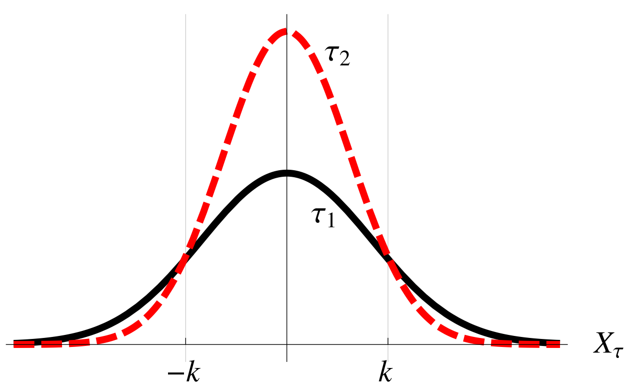


FIGURE 2.1: Example comparison of post-report effort levels as they depend on X_τ, for an earlier release time (τ_1, solid black curve) and a later release time ($\tau_2 > \tau_1$, dashed red curve).

It might appear that things are piling up against having a "late report" (i.e., a large value of τ), given that (1) and (2) are both unambiguously negative, and (3) has both a negative and positive component. In addition, (2) exacerbates the negative aspect of (3) by making larger values of X_τ more likely. Nevertheless, we establish the following, meaning the sole positive force, as described in (3), has quite a lot of power.

Proposition 6. *If $a < 2$, $W(\tau)$ is single-peaked and maximized at unique $\tau_a^* > \frac{1}{2}$.*

Figure 2.2 illustrates the result. To do so, we define the percentage increase in expected effort relative to the no-report/static/$\tau = 0$ benchmark,

$$P(\tau) \equiv \frac{W(\tau) - W(0)}{W(0)}.$$

Primarily this facilitates having different a-values on same graph, but also has the appealing feature that it is independent of σ and b (see the proof of Proposition 6.) The heavy dots indicate the maximums.

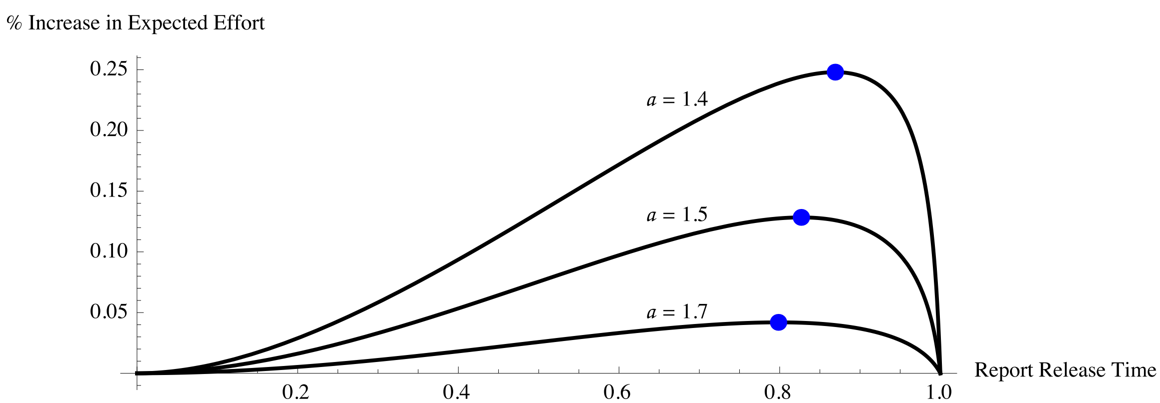


FIGURE 2.2: Percentage increase in expected cumulative effort relative to having no interim report, as it depends on the report's release time (i.e, $P(\tau)$), for $a = 1.4, 1.5, 1.7$ and $d = .01$.

Figure 2.2 suggests two additional features. First, that, for any $\tau > 0$, the percentage increase in expected effort increases as c becomes less convex (i.e., as a decreases from 2 to 1).[15] Second that τ_a^* increases as c becomes less convex. Intuitively, the less convex is c, the stronger is the *incentive effect* described above (i.e., the more pronounced is the difference between the two curves in Figure 2.1). For the second observation, things are slightly more subtle, but we demonstrate that it holds when latent uncertainty, d, is small.

Proposition 7. *Let $a < 2$, τ be an arbitrarily release time in $(0, 1)$, and τ_a^* be the optimal release time. Then, i) $P(\tau)$ is decreasing in a, and ii) for any d, there exists $\bar{a}_d$ such that τ_a^* is decreasing in a for $a < \bar{a}_d$. Further, $\bar{a}_d \to 2$ as $d \to 0$.*

In fact, for $d \approx 0$, $\lim_{a \uparrow 2} \tau_a^* \approx 0.8$. Hence, if latent uncertainty is negligible, the optimal release time always falls quite far toward the end of the tournament.

Finally, for the case where having no report/$\tau = 0$ is optimal (i.e., $a > 2$), we obtain the reciprocal analog of Proposition 6.[16]

Proposition 8. *If $a > 2$, $W(\tau)$ is single-troughed and minimized at unique $\tau_a^{\min} > \frac{1}{2}$.*

This result could be germane if there were an exogenous (e.g., legal) requirement to have a interim report, but with some limited flexibility in when to have it. For instance, if it were required to have a report during an interval $[t_1, t_2]$, the optimal timing of τ would always be one of the boundaries, t_1 or t_2.

[15] This, of course, implies that the percentage increase in expected effort induced by the *optimal* timing, $P(\tau_a^*)$, also increases as c becomes less convex.

[16] An analog to Proposition 7 can also be obtained, but is omitted for brevity.

2.5 Conclusion

We have proposed a model of tournaments in which true time plays an explicit role, and used the model to investigate an issue (the timing of the report) that is the natural extension of the question posed by several preceding two-period works, yet is unanswerable in those models. We view our main methodological contribution, then, as building a workable model that incorporates explicit time.[17] Looking forward, the incorporation of true time introduces the possibility for policies that have no analog in models with two exogenous periods.[18] As indicated in the Introduction, we hope this framework will aid in answering such further new inquires related to timing, information, effort, and tournament design.

[17] To do so, we use familiar Brownian diffusions to model the players' score processes. While such formulations have been widely employed in dynamic models, there is, naturally, some loss of generality in doing so. Nevertheless, we hope both our explicit model and general approach will prove useful for future research.

[18] For just one example, the report could be "triggered early" if the lead hits a certain threshold r before $t = \tau$: i.e., the, now random report time is $\tilde{\tau} = \inf\{t : |X_t| \geqslant r \text{ or } t \geqslant \tau\}$.

3

Show No Mercy? *When* to End a Tournament

3.1 Motivation

In reality, while the rule of a tournament is usually straightforward in reality, i.e. the winner takes it all, the principal still has freedom in defining what a winner is by introducing a mercy rule. More specifically, enforcing a mercy rule means in a two-player tournament with a fixed time horizon, if one player's lead over the other is larger than some pre-determined threshold at any time before the game ends, the player will win immediately instead of finishing the rest of the game. Even though we can often observe the use of mercy rules in sports contests, especially in youth sports league, they are purely served as a courtesy to spare further humiliation for the losing team. The economic impact of mercy rules on players' strategies and the total efforts from the perspective of game theory is not well studied in existing literature, especially in continuous-time dynamic settings. Based on the few papers in dynamic settings (Moscarini and Smith (2011); Cao (2014)) where time horizon can be infinite, it appears enforcing a mercy rule has only negative affects on inducing efforts, for the obvious reason that it ends a game prematurely. However, the assumption of the infinite time horizon seems quite unlikely in reality for most circumstances. Even based only on intuition, one can tell under some scenarios, mercy rules should have at least some positive influence on the contest. Consider the following tournament with a finite time horizon, if no contestant can be ahead of the other for a pre-fixed margin, they will enter a lottery at the end with a probability of winning of $\frac{1}{2}$. This is clearly a valid and reasonable setting. Now, increase the required margin of winning to infinity (i.e. no mercy rule), what will happen? Both contestants will make no efforts throughout the entire game and wait to enter the lottery, while they should make at least some efforts when there is a finite mercy rule.

To study the role of mercy rules in dynamic tournaments with pre-set time horizon, we propose a dynamic game model with a continuous state space and continuous time setting. In this model, participants select instantaneous effort levels and incur corresponding flow costs at all moments during the tournament. These effort levels influence their stochastic output/score processes, which can be observed by their opponents. The prize is allocated to the first player who achieves a score lead equal to some pre-determined threshold. If no player is able to hit this threshold before time runs out, they enter a lottery with a probability of winning the prize roughly proportional to their final score. The lottery is essentially a form of tie-breaking as often seen in literature on tournament games as well as in reality (e.g. the penalty kick in soccor to break tie at the end of a game). Since we treat time as an explicit state variable in this framework, the underlying stochastic control problem corresponds to a partial differential equation group (PDE) instead of an ordinary differential equation group (ODE), thus making it very challenging to find a closed-form expression for the equilibrium strategies. Therefore, instead of pursuing an analytical solution, we characterize the model using a numerical approximation method.

Following the existing literature on numerical analysis of PDE solutions, we have showed that the equilibrium we find through numerical computation is unique. Even though the equilibrium strategy is highly dynamic depending on the values of parameters, such as the parameter of the flow cost and the diffusion parameter of the stochastic process, we find the following results regarding the equilibrium strategy, which are consistent with precedent literature (Harris and Vickers (1987); Moscarini and Smith (2011); Cao (2014)): (1) the leader always works harder; (2) a large score difference discourages effort inputs; (3) large background noises reduce the total economic output. In addition to these well known results, we find robust results regarding the impacts from implementing a mercy rule. When restricting our scope only to constant threshold mercy rules (i.e. the threshold of winning is constant in time), the total expected economic output is single-peaked in the threshold of a mercy rule, which indicates there exists an optimal point for the threshold from the angle of mechanism design. However, the players always prefer a tournament without a mercy rule. Next, we focus on time-varying mercy rules within the linear function class. We find that when the overall effort levels are low, the optimal linear mercy rule is decreasing (i.e. the threshold of winning is decreasing in time); and when the overall effort levels are high, the optimal linear mercy rule is increasing. Finally, we investigate a model that considers players' preference to end the tournament early (we call this "time value" of a tournament). For example, when contestants have outside options with positive flow payoffs, the longer they stay in the tournament, the more opportunity costs they have to pay. To model this effect, we add an extra constant term to the players' cost function and give players the option to drop out of the contest. When agents have to pay to stay in the game, they no longer strictly prefer a tournament without mercy rules. Furthermore, for each mercy rule specified by the principal, the option to drop out endogenizes another boundary. Surprisingly, the total economic output corresponding to the exogenous rule does not always outperforms that corresponding to the endogenous rule.

3.1.1 Related Literature

Our model can be thought of as a combination of a dynamic tug-of-war and a dynamic tournament, so there are two branches of related literature. From the tounament-game side, starting with the pioneering work of Lazear and Rosen (1981), the study of rank-order tournaments as simultaneous-moves game has received much attention. Prominent examples include (but not limit to) Green and Stokey (1983); Nalebuff and Stiglitz (1983); Glazer and Hassin (1988). More recent literature, such as Yildirim (2005); Gershkov and Perry (2009); Zhang and Wang (2009); Aoyagi (2010); Ederer (2010) (and among many others), focus on tournaments in dynamic settings. They are all concerned with questions regarding interim information release, thus usually in a framework of a two-period model. Daley and Wang (2016) studies the timing of interim information release in a continuous-time setup, but the setup can be simplified to a two-period model due to only one information release. A natural question to ask following the current studies on dynamic tournament is how the equilibrium strategies and economic properties change when players can observe each others' progresses in real time during the entire tournament. It is very hard to address this question in a multi-period discrete-time setting because of the potential mathematical complexity. A continuous-time method is more desirable in this situation.

From the tug-of-war side, the analysis built on dynamic models is scarce. Harris and Vickers (1987) is widely regarded as the first paper that investigates a dynamic tournament in a discrete space and discrete time setting. Budd et al. (1993) extends Harris and Vickers (1987) to a continuous time and continuous space framework, but their method only works when the discount rate approaches infinity. Moscarini and Smith (2011); Cao (2014) are the papers most closely related to ours. Moscarini and Smith (2011) characterizes a dynamic model of tug-of-war with infinite time horizon in continuous space and continuous time, and finds closed-form solutions to equilibrium strategies and related economic properties. Cao (2014) proves that results in Moscarini and Smith (2011) still hold under a cost function of a more general form. As mentioned above, however, most tournaments in reality have a specific finite time bound, which makes some results in these papers appear to be unrealistic. For example, they show that as the threshold of winning goes to infinity, the total economic output also goes to infinity. In order to mimick and analyze a tournament that resembles reality, we incorporate time explicitly in our model.

3.2 A Continuous Tournament in Finite Horizon

3.2.1 Setup

Two contestants ($i = 1, 2$) compete for a fixed value prize, which has a normalized value of 1, in a finite horizon T. Each contestant needs to choose continuously his effort level e_t^i, which incurs a flow cost $b \cdot (e_t^i)^2$ (where $b > 0$ is a positive parameter), to generate flow output R_t^i. The flow output follows a drifted Brownian Motion with the effort e_t^i as the drift term, i.e.

$$dR_t^i = e_t^i dt + \frac{\sigma}{\sqrt{2}} dB_t^i,$$

where each B_t^i is a standard Brownian Motion and independent of each other.

Let $X_t = R_t^1 - R_t^2$ (called score difference) denote the difference of the progress between agent 1 and agent 2, then

$$\begin{aligned} dX_t &= (e_t^1 - e_t^2)dt + \frac{\sigma}{\sqrt{2}}(dB_t^1 - dB_t^2) \\ &= (e_t^1 - e_t^2)dt + \sigma dB_t, \end{aligned} \tag{3.1}$$

where B_t is a standard Brownian Motion.

The ending rule of the contest, which explicitly depends on both time t and the score difference X_t, is as follows. Fixing time horizon T, if the score difference X_t at time t for $0 \leqslant t < T$ crosses some pre-determined symmetric threshold $\Omega(t)$ (which is allowed to depend on time explicitly or not, e.g. $\Omega(t) = \pm\sqrt{T-t}$ or $\Omega(t) = \pm\omega$), the game ends and the prize is allocated to player 1 if and only if $X_t \geqslant |\Omega(t)|$ and to player 2 if and only if $X_t \leqslant -|\Omega(t)|$. If X_t never crosses the ending threshold before the game ends, the winner is determined as per the following. There exists a positive value $d \leqslant |\Omega(T)|$, called a lottery threshold, such that if $X_T \geqslant d$, player 1 wins; if $X_T \leqslant -d$, player 2 wins; and a lottery will be drawn to determine the winner if $X_T \in (-d, d)$. More specifically, if the final score difference falls in the lottery zone, an independent random noise term ε_T that follows a cumulative distribution function (CDF) of $f(x)$ will be drawn and the prize is allocated to player 1 if $X_T \geqslant \varepsilon_T$

and to player 2 otherwise. The CDF $f(x)$ is a quintic function (a.k.a a smoothstep function), which has a general form $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$ and whose parameters can be easily pinned down using the following "smoothness" conditions,

$$\begin{aligned} f(-d) &= 0 \text{ and } f(d) = 1; \\ f'(-d) &= 0 \text{ and } f'(d) = 0; \\ f''(-d) &= 0 \text{ and } f''(d) = 0. \end{aligned} \tag{3.2}$$

The above conditions guarantee the smoothness (i.e. second order derivative is continuous) on the whole game-ending boundary. Figure 1 shows an example of $f(x)$ with $\Omega(t) = \pm 1$ and $d = 0.7$. Figure 2 shows an example of the sample path of the score difference X_t with constant boundary condition $\Omega(t) = \pm 1$ and $d = 0.7$.

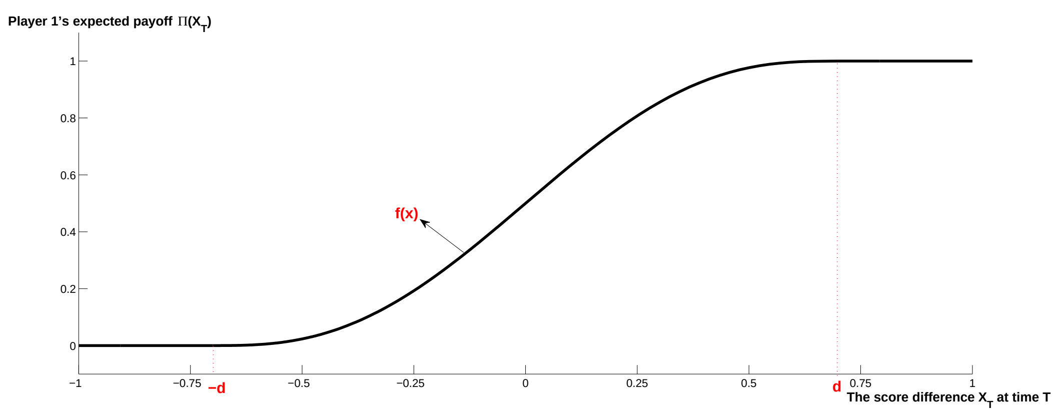


FIGURE 3.1: Player 1's expected payoff at time T, when $\Omega(T) = \pm 1$ and $d = 0.7$

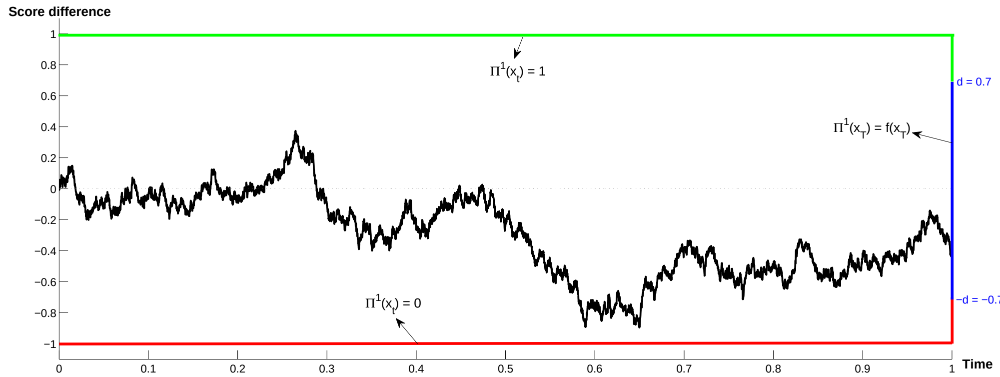


FIGURE 3.2: An example of how the score difference $X(t)$ evolves over time t, when $T = 1$, $\Omega(t) = \pm 1$, and $d = 0.7$.

In summary, when the game ends at time t, player 1's expected terminal payoff Π_t^1 (which depends on the realization x_t of the score difference X_t) can be expressed as

$$\begin{aligned} \Pi_t^1(x_t) &= 1 \text{ if } x_t \geqslant |\Omega(t)| \text{ and } \Pi_t^1(x_t) = 0 \text{ if } x_t \leqslant -|\Omega(t)| \text{ for } t < T; \quad (3.3) \\ \Pi_T^1(x_T) &= 1 \text{ if } x_T \geqslant d \text{ and } \Pi_T^1(x_T) = 0 \text{ if } x_T \leqslant -d; \\ \Pi_T^1(x_T) &= f(x_T) \text{ if } x_T \in (-d, d). \end{aligned}$$

Player 2's expected terminal payoff Π_t^2 is simply given by $\Pi_t^2 = 1 - \Pi_t^1$.

3.2.2 Players' decision making problem

Let τ denote the stopping time that X_t crosses $\Omega(t)$ either from below or above for the first time, i.e. $\tau := \inf\{t | \ |X_t| \geqslant |\Omega(t)|\}$. Taken e_t^2 as known and given that both players start at the same position, player 1's problem can be written as,

$$\max_{e_t^1 \in [0,\infty)} E\left[\int_0^{\tau \wedge T} -b(e_t^1)^2 dt + \Pi_{\tau\wedge T}^1(X_{\tau\wedge T}) \mid X_0 = 0\right]. \quad (3.4)$$

Similarly, player 2's problem is given by,

$$\max_{e_t^2 \in [0,\infty)} E\left[\int_0^{\tau\wedge T} -b(e_t^2)^2 dt + \Pi^2_{\tau\wedge T}(X_{\tau\wedge T}) \mid X_0 = 0\right]. \tag{3.5}$$

Note that e_t^i depend on both the score difference x and time t, so it is equivalent to writing it in the functional form $e^i(x,t)$.

3.2.3 HJB equations and the Symmetric Markov Perfect Equilibrium

Let $V(x,t)$ and $U(x,t)$ represent player 1 and player 2's value-to-go function respectively, i.e.

$$\begin{aligned} V(x,t) &= \max_{e_r^1 \in [0,\infty)} E\left[\int_t^s -b(e_r^1)^2 dr + V(X_s,s) \mid X_t = x\right], \qquad (3.6)\\ U(x,t) &= \max_{e_r^2 \in [0,\infty)} E\left[\int_t^s -b(e_r^2)^2 dr + U(X_s,s) \mid X_t = x\right]. \end{aligned}$$

Following the Dynamic Programming Principle for stochastic control problems (See Bardi and Capuzzo-Dolcetta (2008)), the HJB equations corresponding to the stochastic control problems are given by,

$$\begin{aligned} \frac{\partial V(x,t)}{\partial t} + \max_{e_1 \in [0,\infty)}\left[-b(e^1(x,t))^2 + (e^1(x,t) - e^2(x,t))\frac{\partial V(x,t)}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 V(x,t)}{\partial x^2}\right] &= 0, \qquad (3.7)\\ \frac{\partial U(x,t)}{\partial t} + \max_{e_2 \in [0,\infty)}\left[-b(e^2(x,t))^2 + (e^1(x,t) - e^2(x,t))\frac{\partial U(x,t)}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 U(x,t)}{\partial x^2}\right] &= 0, \end{aligned}$$

with terminal conditions

$$V(x,t) = \Pi^1(x,t) \text{ and } U(x,t) = \Pi^2(x,t) \text{ for } |x_t| \geqslant |\Omega(t)|.$$

First order conditions with respect to e suggest

$$e^{1*}(x,t) = \frac{1}{2b}\frac{\partial V(x,t)}{\partial x} \text{ and } e^{2*}(x,t) = -\frac{1}{2b}\frac{\partial U(x,t)}{\partial x}. \tag{3.8}$$

Lemma 9. *If $\frac{\partial V(x,t)}{\partial x} \geqslant 0$ and $\frac{\partial U(x,t)}{\partial x} \leqslant 0$, the first order conditions are sufficient to guarantee the optimality (equilibrium conditions).*

The proof is given in the appendix. Notice that $\frac{\partial V(x,t)}{\partial x} \geqslant 0$ and $\frac{\partial U(x,t)}{\partial x} \leqslant 0$ suggest that players are better off with a larger lead, which is the "right" economic intuition we would expect in a "reasonable" equilibrium. And we can always come back and check this sufficient condition after we fully characterize the equilibrium.

Substituting the first order conditions back to the HJB equations, we get the following equilibrium conditions

$$\frac{\partial V(x,t)}{\partial t} + \frac{1}{4b}\left(\frac{\partial V(x,t)}{\partial x}\right)^2 + \frac{1}{2b}\frac{\partial U(x,t)}{\partial x}\frac{\partial V(x,t)}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 V(x,t)}{\partial x^2} = 0 \tag{3.9a}$$

$$\frac{\partial U(x,t)}{\partial t} + \frac{1}{4b}\left(\frac{\partial U(x,t)}{\partial x}\right)^2 + \frac{1}{2b}\frac{\partial V(x,t)}{\partial x}\frac{\partial U(x,t)}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 U(x,t)}{\partial x^2} = 0 \tag{3.9b}$$

We restrict our focus on the Symmetric Markov Perfect Equilibrium (SMPE), in which the players' effort choices depend only on the score difference and time, and $V(x,t) = U(-x,t)$ for all x and t. Thus, in a SMPE, the equilibrium efforts can be rewritten as,

$$e^{1*}(x,t) = \frac{1}{2b}\frac{\partial V(x,t)}{\partial x} = -\frac{1}{2b}\frac{\partial U(-x,t)}{\partial x} \text{ and } e^{2*}(x,t) = -\frac{1}{2b}\frac{\partial U(x,t)}{\partial x} = \frac{1}{2b}\frac{\partial V(-x,t)}{\partial x}. \tag{3.10}$$

With the above expression of the equilibrium efforts, the SMPE can be formally expressed by the following conditions,

$$\frac{\partial V(x,t)}{\partial t}+\frac{1}{4b}\left(\frac{\partial V(x,t)}{\partial x}\right)^2-\frac{1}{2b}\frac{\partial V(-x,t)}{\partial x}\frac{\partial V(x,t)}{\partial x}+\frac{1}{2}\sigma^2\frac{\partial^2 V(x,t)}{\partial x^2}=0, \quad (3.11a)$$

$$\frac{\partial U(x,t)}{\partial t}+\frac{1}{4b}\left(\frac{\partial U(x,t)}{\partial x}\right)^2-\frac{1}{2b}\frac{\partial U(-x,t)}{\partial x}\frac{\partial U(x,t)}{\partial x}+\frac{1}{2}\sigma^2\frac{\partial^2 U(x,t)}{\partial x^2}=0, \quad (3.11b)$$

$$V(x,t)=U(-x,t), \quad (3.11c)$$

$$\frac{\partial V(x,t)}{\partial x}\geqslant 0 \text{ and } \frac{\partial U(x,t)}{\partial x}\leqslant 0. \quad (3.11d)$$

Ignoring condition 3.11d for now and verifying later, Conditions 3.11a - 3.11c form a Partial Functional Differential Equation (PFDE) group. To convert the PFDE into a regular PDE group, let $P(x,t)=V(x,t)+U(x,t)$ and $Q(x,t)=V(x,t)-U(x,t)$, then Equation (3.9a) + Equation (3.9b) and Equation (3.9a) - Equation (3.9b) give,

$$\frac{\partial P(x,t)}{\partial t}+\frac{3}{8b}\left(\frac{\partial P(x,t)}{\partial x}\right)^2-\frac{1}{8b}\left(\frac{\partial Q(x,t)}{\partial x}\right)^2+\frac{1}{2}\sigma^2\frac{\partial^2 P(x,t)}{\partial x^2}=0, \quad (3.12a)$$

$$\frac{\partial Q(x,t)}{\partial t}+\frac{1}{4b}\frac{\partial P(x,t)}{\partial x}\frac{\partial Q(x,t)}{\partial x}+\frac{1}{2}\sigma^2\frac{\partial^2 Q(x,t)}{\partial x^2}=0, \quad (3.12b)$$

with boundary conditions

$$P(x,t)=1 \text{ and } Q(x,t)=2\Pi^1(x,t)-1 \text{ for } |x_t|\geqslant|\Omega(t)|.$$

And the symmetry condition 3.11c is equivalent to that $P(x,t)$ is an even function and $Q(x,t)$ is an odd function with respect to x for any t.

3.2.4 A numerical approximation

Even though the PFDE group can be reduced to a regular PDE group as shown in the previous section, the resulting PDEs (Equations 3.12a and 3.12b) are still second order nonlinear due to the presence of P_x^2, Q_x^2 and $P_x Q_x$, which is very difficult to solve analytically, if possible at all. To further reduce the complexity of the PDEs, we apply Cole-Hopf transform to Equation 3.12a.

Let $P = \frac{4b\sigma^2}{3} \ln Z$ (or $Z = \exp\left(\frac{3}{4b\sigma^2} P\right)$, The PDE group can be rewritten as,

$$Z_t + \frac{1}{2}\sigma^2 Z_{xx} - \frac{Z}{32b^2\sigma^2} Q_x^2 = 0 \tag{3.13a}$$

$$Q_t + \frac{\sigma^2 Z_x}{3Z} Q_x + \frac{1}{2}\sigma^2 Q_{xx} = 0. \tag{3.13b}$$

Given Q, Equation 3.13a has the form of as a nonhomogeneous heat equation; similarly, given Z, Equation 3.13b is also a nonhomogeneous heat equation. Even though it is impossible to separate Z and Q and find solutions in close form, the heat equation expression shown above suggests that we can find numerical approximation using traditional numerical methods, e.g. finite difference methods, finite element methods and etc.

Finite Difference Methods

Using a backward difference at time t_n and a second-order central difference for the space derivative at x_j, we get the following recurrence equations:

$$\frac{Z_j^{n+1}-Z_j^n}{\Delta t}+\frac{\sigma^2}{2}\frac{Z_{j+1}^{n+1}-2Z_j^{n+1}+Z_{j-1}^{n+1}}{(\Delta x)^2}-\frac{Z_j^n}{32b^2\sigma^2}\left(\frac{Q_{j+1}^{n+1}-Q_j^{n+1}}{\Delta x}\right)^2=0.$$

$$\frac{Q_j^{n+1}-Q_j^n}{\Delta t}+\frac{\sigma^2}{3Z_j^n}\left(\frac{Z_{j+1}^{n+1}-Z_j^{n+1}}{\Delta x}\right)\left(\frac{Q_{j+1}^{n+1}-Q_j^{n+1}}{\Delta x}\right)+\frac{\sigma^2}{2}\frac{Q_{j+1}^{n+1}-2Q_j^{n+1}+Q_{j-1}^{n+1}}{(\Delta x)^2}=0.$$

Rearranging the equations gives us an explicit scheme to compute Z and Q :

$$\begin{aligned}
Z_j^n &= \frac{1}{1+\frac{\Delta t}{32b^2\sigma^2}\left(\frac{Q_{j+1}^{n+1}-Q_j^{n+1}}{\Delta x}\right)^2}\left(\left(1-\frac{\Delta t\sigma^2}{(\Delta x)^2}\right)Z_j^{n+1}+\frac{\Delta t\sigma^2}{2\Delta x}\left(Z_{j+1}^{n+1}+Z_{j-1}^{n+1}\right)\right) \\
Q_j^n &= \left(1-\frac{\Delta t\sigma^2}{3(\Delta x)^2}\frac{Z_{j+1}^{n+1}-Z_j^{n+1}}{Z_j^n}-\frac{\Delta t\sigma^2}{(\Delta x)^2}\right)Q_j^{n+1} \\
&\quad+\left(\frac{\Delta t\sigma^2}{3(\Delta x)^2}\frac{Z_{j+1}^{n+1}-Z_j^{n+1}}{Z_j^n}+\frac{\Delta t\sigma^2}{2(\Delta x)^2}\right)Q_{j+1}^{n+1}+\frac{\Delta t\sigma^2}{2(\Delta x)^2}Q_{j-1}^{n+1} \qquad (3.14)
\end{aligned}$$

Note that the terminal condition is given at the end of the time horizon, even though this is a backward iteration, the above scheme is explicit, which implies it is thus easy and fast to implement. However, we need to control the ratio of $\Delta t/(\Delta x)^2$ in order to get a stable numerical solution.

Lemma 10. *There exists an $\varepsilon > 0$ such that when $\Delta t/(\Delta x)^2 < \varepsilon$ holds (the explicit expresseion of ε is given in the proof), the numerical scheme 3.14 converges uniformly to the viscosity solution of the Partial Differential Equation 3.13 (if it exists) as $\Delta x \to 0$ and $\Delta t \to 0$.*

The proof is given in the appendix. The above finite difference method is easy to implement, but may require large computation space since we need to choose a Δt

small enough to satisfy the Von-Neumann condition. In practice, we use $\Delta t = 10^{-6}$ and $\Delta x = 0.005$.

3.3 Equilibrium and Tournament Design

3.3.1 The principal's problem

Let $Y(x,t)$ denote the total expected cumulative efforts put into the game starting from time t when the score difference is x, i.e.

$$Y(x,t) = E\left[\int_t^T \big(e^1(x,s) + e^2(x,s)\big)\, ds\right].$$

Note that since $E\left[\int_t^T (R^1(x,s) + R^2(x,s))\, ds\right] = E\left[\int_t^T (e^1(x,s) + e^2(x,s))\, ds\right]$, $Y(x,t)$ also denotes the total expected output produced from time t to the end. The value function $Y(x,t)$ solves the following HJB equation

$$\frac{\partial Y(x,t)}{\partial t} + (e^1(x,t) + e^2(x,t)) + (e^1(x,t) - e^2(x,t))\frac{\partial Y(x,t)}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 Y(x,t)}{\partial x^2} = 0, \quad (3.15)$$

with terminal conditions

$$Y(x,t) = 0 \text{ for } |x_t| \geqslant |\Omega(t)| \text{ and } Y(x,T) = 0 \text{ for } \forall x.$$

Assuming the contestants start equally from the beginning, i.e. $x_0 = 0$, the principal's goal is to design an ending rule $\Omega(t)$ of the tournament to maximize the total expected cumulative efforts devoted into the game, which can be expressed by the following free boundary problem,

$$\max_{\Omega(t)} Y(0,0).$$

Ideally, if we know the analytical form of $e^i(x,t)$, Equation 3.15 is an inhomogeneous heat equation, which may have close form solutions. However, the inability to solve Equation 3.12 analytically also implies no close form solutions can be found for Equation 3.15. Nonetheless, the expression of Y and the associated HJB equation 3.15 enable us to compute the principal's problem more precisely and quickly than evaluating the total expected cumulative efforts in the forward time order.

To solve the free boundary problem, we need to compare all different possible ending rules $\Omega(t)$, which is impossible in practice. In the following sections, we restrict our attention to computationally affordable but still general ending rules. We will start with constant mercy rules and expand our horizon to general linear mercy rules.

3.3.2 Expected Duration

Let $D(x,t)$ denote the expected remaining time of the contest when the current score difference is x at time t. It can be easily shown based on the Dynkin's formula for time inhomogeneous diffusion process that $D(x,t)$ solves the following equation,

$$\frac{\partial D(x,t)}{\partial t} + (e^1(x,t) - e^2(x,t))\frac{\partial D(x,t)}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 D(x,t)}{\partial x^2} = -1,$$

with terminal conditions

$$D(x,t) = 0 \text{ for } |x_t| \geqslant |\Omega(t)| \text{ and } D(x,T) = 0 \text{ for } \forall x.$$

The expected duaration can be an useful tool to the average time of a contest and the relative effectiveness and efficiency of a specific mercy rule .

3.3.3 A Unique Equilibrium

Proposition 11. *Given a mercy rule $\Omega(t)$, there exists a unique Symmetric Perfect Markov equilibrium.*

Proof. According to existing theories regarding viscosity solutions (See Katzourakis (2014), page 86), Equation (3.9) can admit at most one continuous viscosity solution. To see this, it is straightforward to verify that the underlying stochastic process and the boundary conditions satisfy the regularity conditions in Katzourakis (2014), i.e. the drift term and the volatility term of the stochastic process as well as the boundary conditions should be lipschitz continuous given a control. And we know the value function defined in (3.6) is a solution to HJB Equation (3.9) in the viscosity sense (see Touzi (2002), Page 46). Therefore, the value function is the only viscosity solution and characterizes the unique SPME. $\square$

By Lemma 10, the numerical scheme mentioned in the previous section converges to the desired value function.

Proposition 12. $V(x,t) \geqslant 0$ *and* $U(x,t) \geqslant 0$ *for all possible* x *and* t*.*

Proof. Suppose there exists a pair (x_1, t_1) such that $V(x_1, t_1) < 0$. By the continuity of the viscosity solution and the fact that $V(x, T) \geqslant 0$ for all x, there must exist a $t_2 > t_1$ such that $V(x_2, t_2) < 0$ for some x_2 and $V(x, t_2 + \varepsilon) \geqslant 0$ for all x and $\varepsilon > 0$. If player 1 deviates the effort choice $e^1(x_2, t_2)$ to 0 at state (x_2, t_2), then $V(x_2, t_2) \geqslant 0$ by definition (3.6), which contradicts the fact that $V(x_2, t_2) < 0$. And $U(x,t) = V(-x,t)$ immediately implies that $U(x,t) \geqslant 0$. $\square$

Proposition 13. *Increasing (Decreasing) the value of the cost parameter b is equivalent to decreasing (increasing) the prize value.*

Proof. An alternative way to simplify the PDE group (3.11) is to apply Cole-Hopf transformation $V = 2b\sigma^2 \ln R$ and $U = 2b\sigma^2 \ln S$ directly, which yields

$$\begin{aligned} R_t + \sigma^2 \frac{R_x S_x}{S} + \frac{\sigma^2}{2} R_{xx} &= 0, \\ S_t + \sigma^2 \frac{R_x S_x}{R} + \frac{\sigma^2}{2} S_{xx} &= 0, \end{aligned}$$

with terminal conditions

$$R(x,t) = \exp\left(\frac{V(x,t)}{2b\sigma^2}\right) \text{ and } S(x,t) = \exp\left(\frac{U(x,t)}{2b\sigma^2}\right) \text{ for } |x_t| \geqslant |\Omega(t)|.$$

Notice that b only appears in the terminal conditions, which suggests doubling b and the prize value at the same time will result in the same R and S, and the resulting value function V and U will also be doubled. Therefore, to see how the game dynamics change in the prize value, we can simply change b, and scale the value functions accordingly. □

3.4 Numerical Results

3.4.1 The equilibrium strategy

Under a constant mercy rule (i.e. $\Omega(t) = \pm\omega$, where ω is a positive constant), the tournament ends once the absolute score difference $|x_t|$ is greater than the threshold ω and the prize is allocated to the winner. In the following, we will keep d as a fraction of ω, i.e. $d = \alpha \cdot \omega$, and use $\alpha \in (0, 1]$ as a parameter. The reasons we choose

α over d are: 1) we need $d \leqslant \omega$ to ensure continuity on the boundary; 2) α serves as a better measure of the intensity of the noise term at the end of the game, so it makes more sense to keep α unchanged when we identify the impact of ω on the principal's problem by comparing games with identical parameters but different ω.

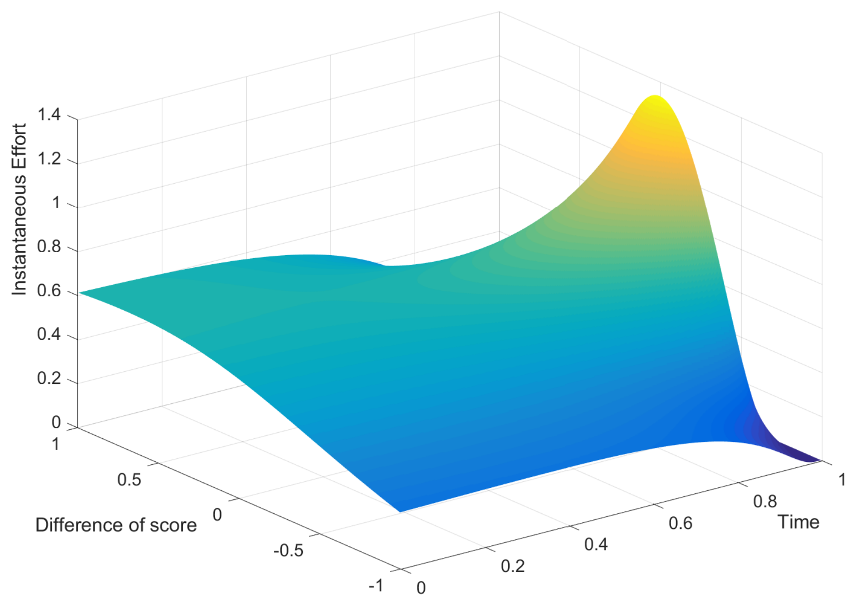
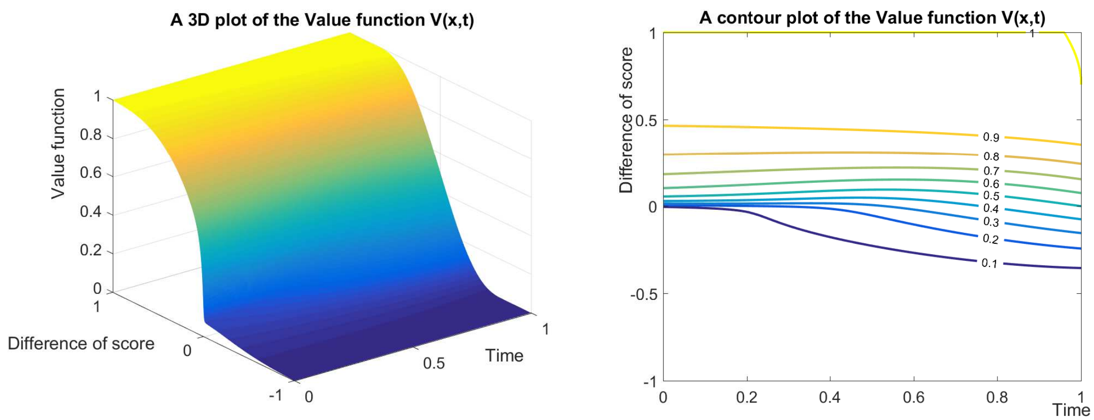


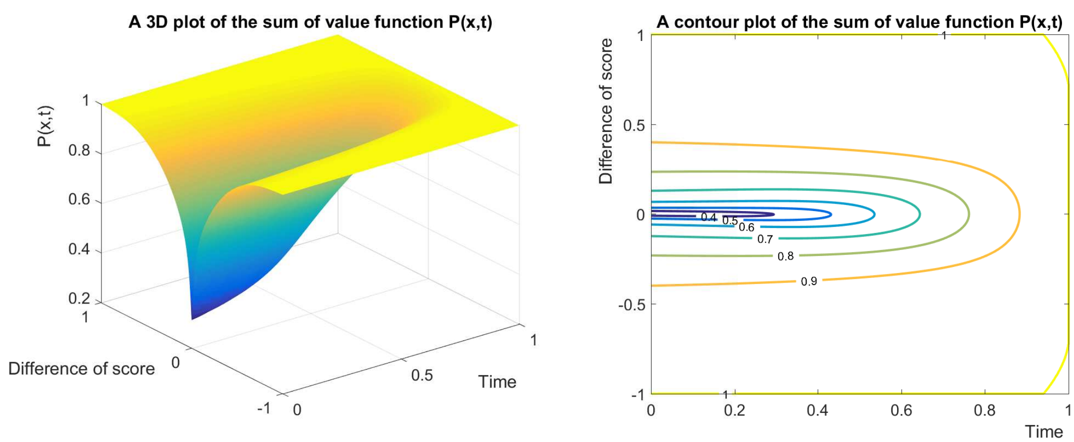
FIGURE 3.3: An example of the effort function that does not achieve maximum at $x = 0$ for all t

Although the dynamics of the equilibrium strategy and the value functions vary by different values of the parameters b, α, σ and ω, we find the following common trends, as illustrated in Figure 3.4- Figure 3.7: 1) The sum of the instantaneous effort $e^+(x,t) \equiv e^1(x,t) + e^2(x,t)$ has a peak at $x_t = 0$ for all t (but $e^i(x,t)$ does not always have a peak at $x_t = 0$ for all t, see Figure 3.3 for an example) and is strictly decreasing in the absolute difference of the score $|x_t|$; 2) The sum of both players' value function $P(x,t)$ has a symmetric U-shape with the trough point at $x_t = 0$ increasing over time; 3) The value function $V(x,t)$ is strictly increasing in x, and increases faster in $[0, \omega]$ than in $[-\omega, 0]$.

Example 1. *An example of the equilibrium strategies and game dynamics with a small diffusion term, where the parameter set is:* $b = 1$, $\sigma = 0.25$, $\omega = 1$, $d = 0.7$, *and* $T = 1$.

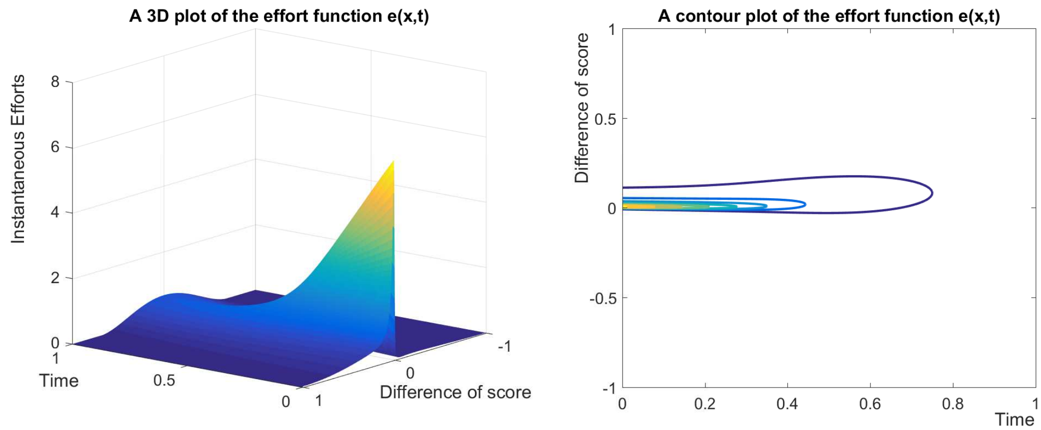


(a)

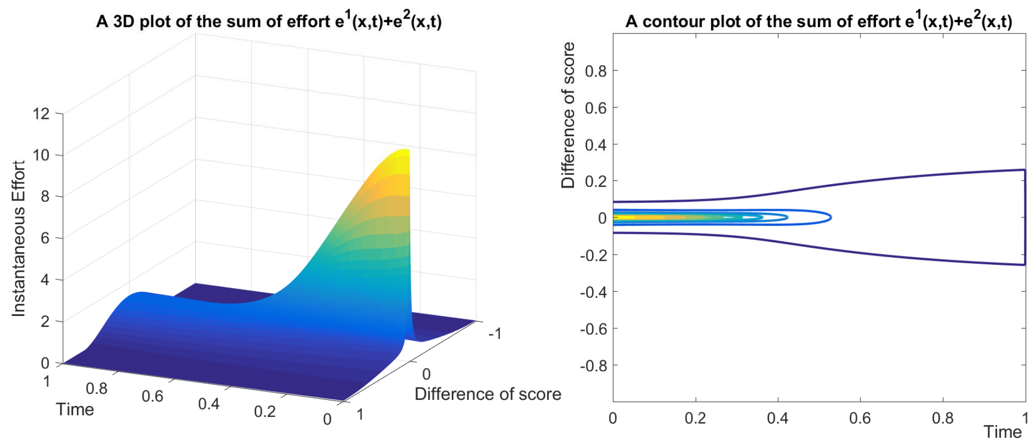


(b)

FIGURE 3.4: The game dynamics for Example 1: (a) Value funcrion; (b) Sum of value functions



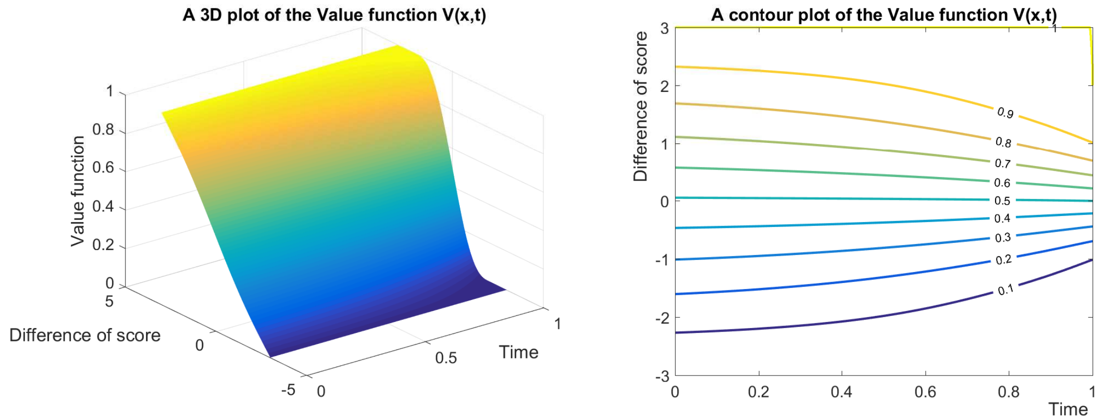
(a)



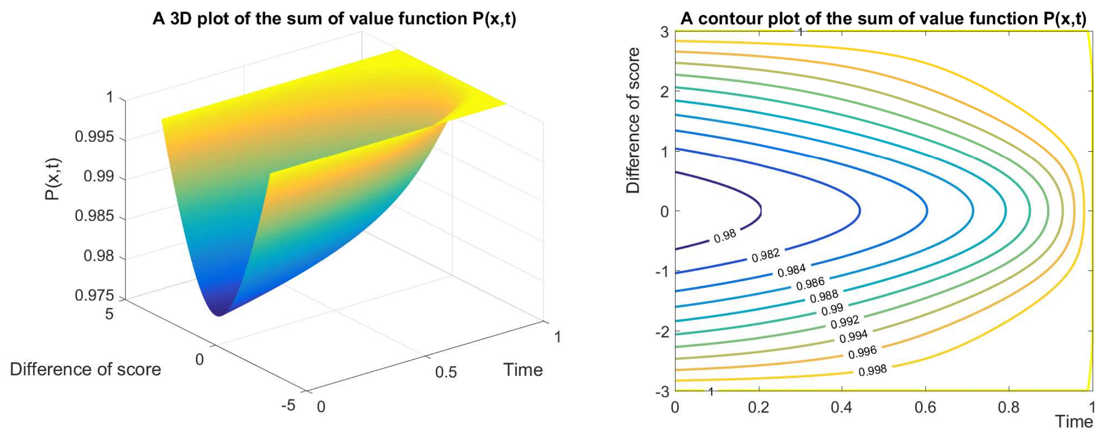
(b)

FIGURE 3.5: The game dynamics for Example 1 (continued): (a) Instantaneous Efforts; (b) Sum of instantaneous efforts

Example 2. *An example of the equilibrium strategies and game dynamics with a large diffusion term, where the parameter set is:* $b = 1$, $\sigma = 2$, $\omega = 3$, $d = 2$, *and* $T = 1$.



(a)



(b)

FIGURE 3.6: The game dynamics for Example 2: (a) Value funcrion; (b) Sum of value functions;

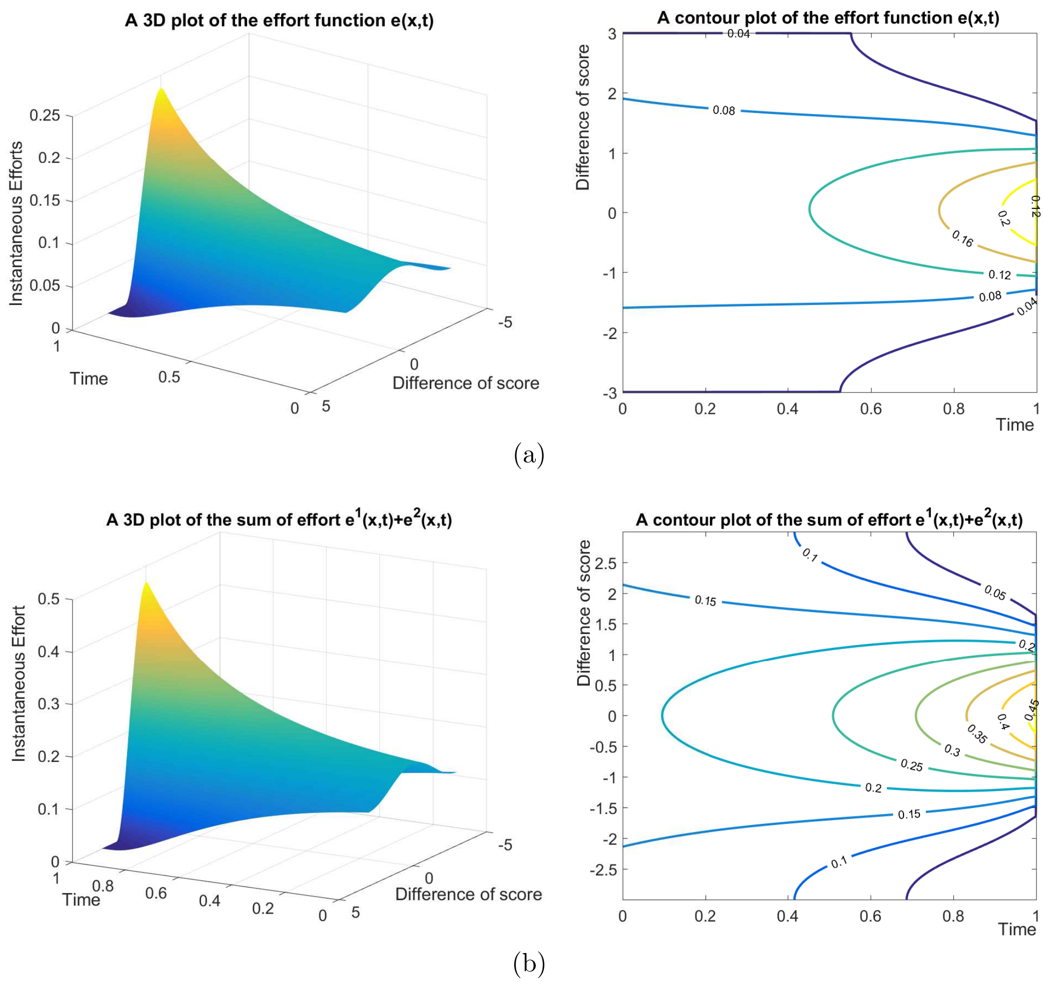


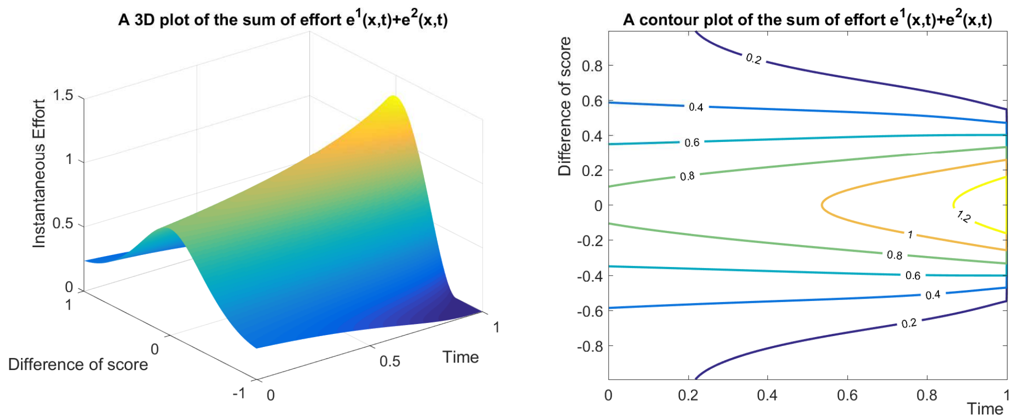
FIGURE 3.7: The game dynamics for Example 2 (continued): (a) Instantaneous Efforts; (b) Sum of instantaneous efforts

Result 14. *Leader always works harder.*

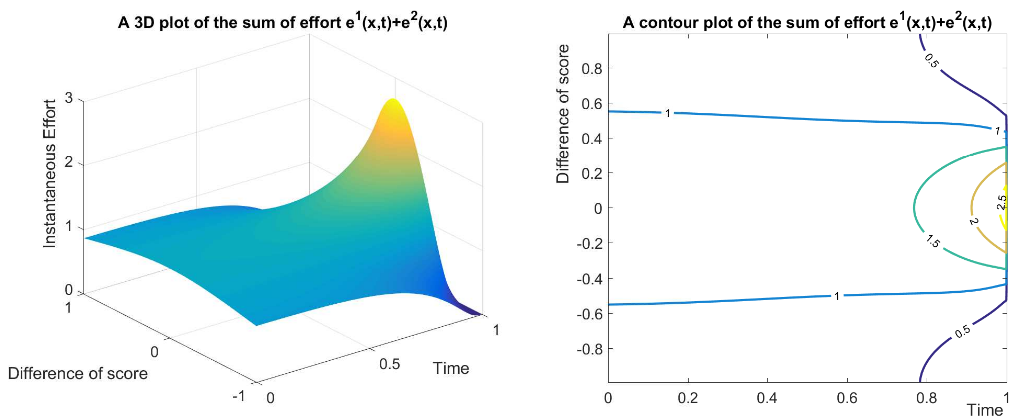
As mentioned above, $V(x,t)$ increases faster when $x > 0$, which can be roughly justified by the fact that $V(\omega,t) = 1$ and $V(-\omega,t) = 0$, but $V(0,t) < \frac{1}{2}(V(\omega,t) + V(-\omega,t)) = \frac{1}{2}$. On the other hand, notice that the difference of the equilibrium efforts $e^1(x,t) - e^2(x,t) = \frac{1}{2b}\left(\frac{\partial V(x,t)}{\partial x} + \frac{\partial U(x,t)}{\partial x}\right) = \frac{1}{2b}\frac{\partial P(x,t)}{\partial x}$ and $P(x,t)$ is (weakly) increasing in $|x|$ for any t (As shown in Figure 3.4 and 3.6), which is compatible with the intuition that a more competitive game should be more costly to the players, so it is not surprised to see that $e^1(x,t) - e^2(x,t) > 0$ for $x > 0$ and $e^1(x,t) - e^2(x,t) < 0$ for $x < 0$. And we notice that when b or σ is smaller, the leader-works-harder effect is more stronger (i.e. the difference between two players' efforts is larger).

Result 15. *Fixing all other parameters (α and ω), when b or σ is large, $e_t^+(x)$ has an increasing trend in t; When b and σ are small enough, $e_t^+(x)$ has a roughly decreasing trend in t. (As shown in Figure 3.8)*

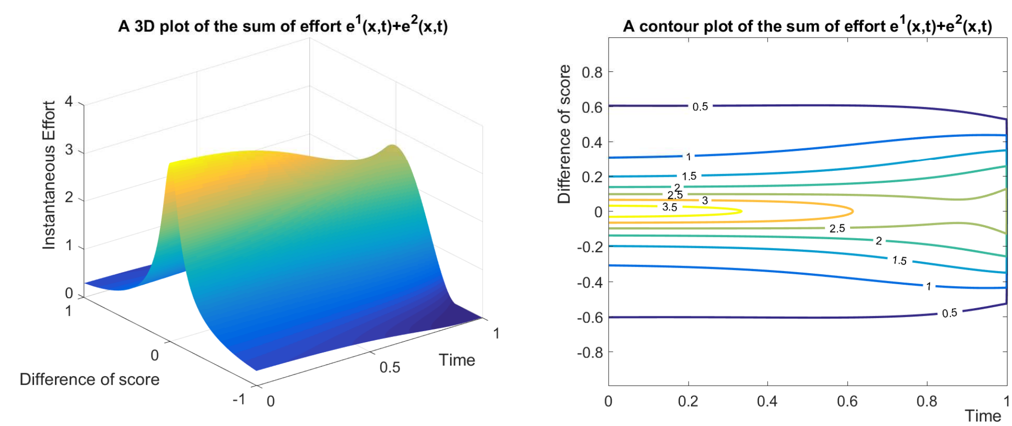
This result indicates that the game can have drastically different strategies depending on the diffusion and cost parameter (or the value of the prize equivalently). When b is large (or prize value is small), it is more costly for the players to make efforts; and when σ is large, high uncertainty drives down efforts. In either scenario, the overall instantaneous efforts are at a low level; the players are more reserved to make efforts in the beginning in a neck-to-neck game (i.e. x close to 0) because noises from the diffusion process play a big role in this situation. As time goes, the remaining uncertainty of the game decreases, which in turn boosts the efforts in a neck-to-neck game (Same intuition as in Daley and Wang (2016)). However, when b and σ are both small enough (or prize value is large), the overall instantaneous efforts are at a high level; both players start strong due to the fact that it is more important for the players to occupy a leading position in the beginning because it is very hard for the follower to catch up with the leader as a result of the stronger leader-works-harder effect as well as a relatively weaker diffusion effect. As time goes, the leader-works-harder effect gets weaker, a leading position has relatively less value, the efforts in a neck-to-neck game decreases until some time close to T, when the boosting effect from the decreasing diffusion noises dominates the game.



(a)



(b)



(c)

FIGURE 3.8: Three examples of $e^{+}(x,t)$ for Result 15: (a)$b = 1$, $\sigma = 0.5$, $\omega = 1$ and $\alpha = 0.7$; (b) $b = 0.5$, $\sigma = 1$, $\omega = 1$ and $\alpha = 0.7$; (c) $b = 0.5$, $\sigma = 0.5$, $\omega = 1$ and $\alpha = 0.7$

Result 16. *Fixing all other parameters, increasing σ will always hurt $Y(0,0)$ and shrinks $D(0,0)$.*

As mentioned above, increasing σ will bring more uncertainty to the tournament, which is adverse to the principal as players choose to make less efforts throughout the tournament. Despite of the decreased instantaneous efforts, higher volatility gives players higher probabilities to cross $\Omega(t)$ and end the contest early.

Result 17. *Fixing all other parameters, if the allowed time horizon for the game is T, it is never optimal for the principal to specify a tournament with a time horizon $T' < T$ (i.e. $Y_{T'}(0,0) \leqslant Y_T(0,0)$).*

The result shows that if both player can commit a length of time T to the tournament, the principal should always take advantage of the full time horizon. While this result seems intuitive, it is not necessarily obvious. Given two different time horizons $T_1 < T_2$, even though from the aspect of backward induction, at time $t_1 = T_1 - t$ and $t_2 = T_2 - t$ for any $t < \min\{T_1, T_2\}$, $e^i_{T_1}(x, t_1) = e^i_{T_2}(x, t_2)$, where $e^i_{T_j}(x, t)$ denotes player i's effort at (x, t) in the game with time horizon T_j, $E_{t_1}\left[e^i_{T_1}(x, t_1)\right] \neq E_{t_2}\left[e^i_{T_2}(x, t_2)\right]$ due to the different distribution of state x in different games. In the following, we will assume T has a normalized value of 1 without loss of generality.

Result 18. *Fixing all other parameters (b, σ and ω), $Y(0,0)$ is not necessarily monotonically decreasing in α. (As shown in Figure 3.9)*

This result seems counter-intuitive at first glance because α is a measure of the second noise term, and a stronger noise usually reduces the overall effort level. However, when α is small (e.g. less than 15%), increasing α actually may boost the total cumulative expected effort $Y(0,0)$ because the incremental noise region actually gives the follower a better chance to enter the lottery zone and secure a larger expected payoff, thus inducing more efforts from both players. But when α is already large, further increasing α will bring too much uncertainty, which lowers the effort level.

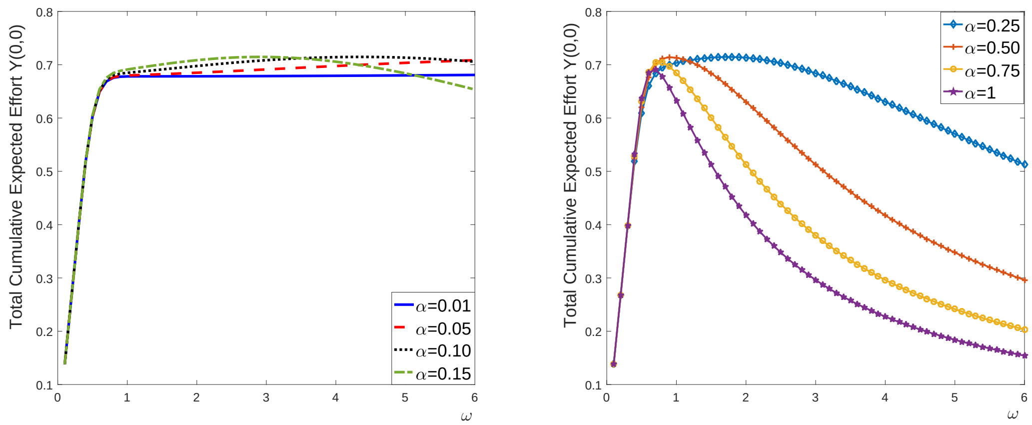


FIGURE 3.9: An example of the total cumulative expected effort as a function of ω. The parameter set is given by, $b = 1$, $\sigma = 0.5$, $T = 1$. The left plot shows $Y(0,0)$ increases in some region as α increases, while the right plot shows a decreasing trend of $Y(0,0)$ as α further increases.

3.4.2 An Optimal constant mercy rule

Before discussing the main result regarding the constant mercy rule, we first identify two effects of increasing ω on $Y(0,0)$ (See Figure 3.10 for an illustration).

1. **Incentive effect**: Fixing all other parameters, increasing ω will reduce instantaneous efforts in most regions of (x,t). Mathematically, let $e^i_\omega(x,t)$ and $e^i_{\omega'}(x,t)$ denote the equilibrium instantaneous effort at (x,t) associated with a game with different mercy rule, if $x < \omega < \omega'$, then the inequality $e^i_\omega(x,t) >$

$e^i_{\omega'}(x,t)$ holds in most regions of (x,t). This says when it is harder to hit the winning threshold, the players tend to make less efforts (though there may exist a small region of exception).

2. **Duration effect:** A higher threshold keeps the players in the tournament longer and enforce them to make efforts in the incremental state space.

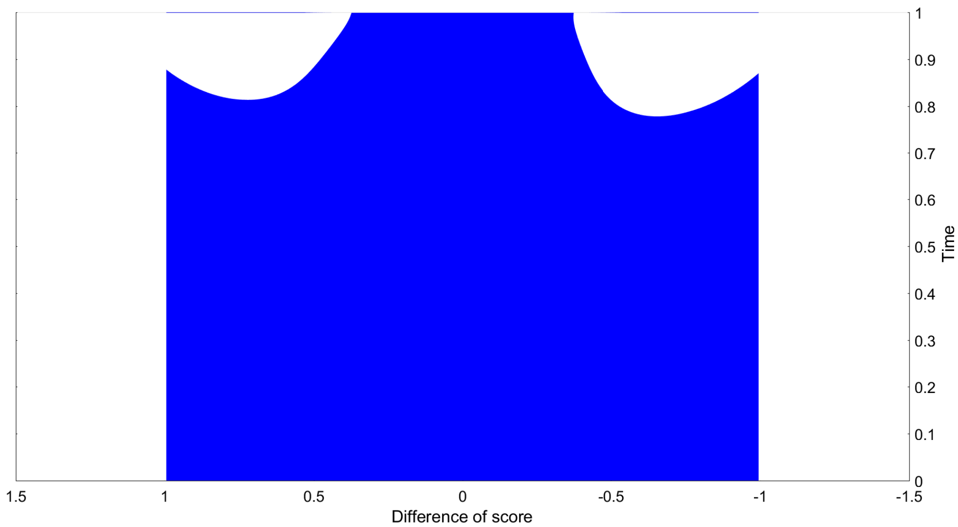


FIGURE 3.10: An illustration of the two effects: keeping all other parameters fixed, when ω is increased from 1 to 1.5 , the dark region is where $e^1_{\omega=1} > e^1_{\omega=1.5}$. The parameter set used is: $b = 1$, $\sigma = 1$, and $\alpha = 0.7$.

Result 19. *Fixing all other parameters (b, α and σ), there exists an optimal mercy rule $\omega^*(\alpha)$ such that $Y(0,0)$ is maximized at $\omega = \omega^*(\alpha)$. More specifically, $Y(0,0)$ is increasing in ω for $\omega < \omega^*(\alpha)$ and decreasing in ω for $\omega > \omega^*(\alpha)$, and $\omega^*(\alpha)$ is (weakly) decreasing in α.*

This shows that when ω is small, the duration effect dominates the incentive effect, which can be substantiated by the fact that the expected duration $D(0,0)$ increases very fast in ω when ω is small (See Figure 3.11). However, when ω is large, the impact from the incentive effect is more influential as the expected duration moves very slowly, thus resulting in a decreasing total cumulative expected effort $Y(0,0)$ as ω further increases. Another interesting observation is that the expected durations associated with ω^* are not close to 1 (usually in the range from 0.7 to 0.9), which suggests a longer duration does not necessarily induces higher efforts.

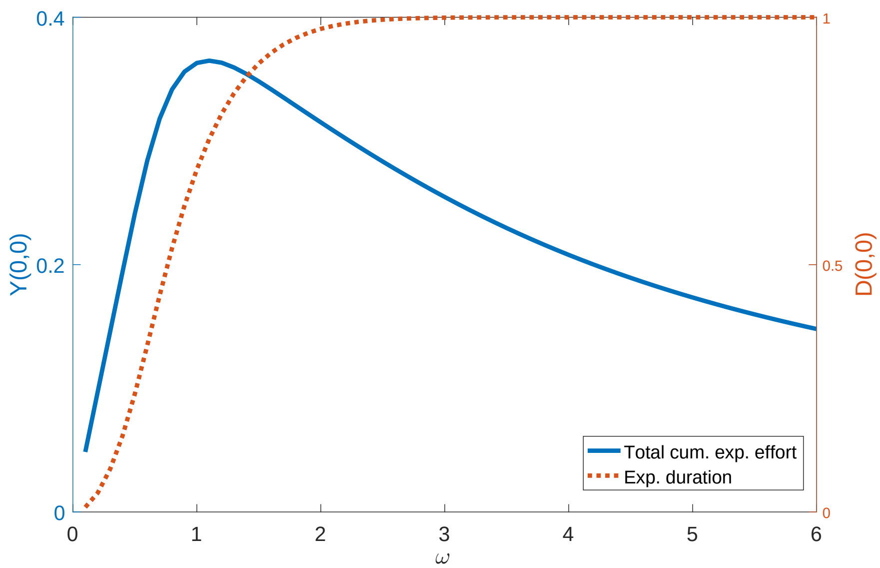


FIGURE 3.11: An illustration of Result 19: $Y(0,0)$ is single-peaked in ω and $D(0,0)$ increases faster in ω when ω is small.

3.4.3 Contestants' welfare

Let $\Phi(\omega)$ denote contestant i's welfare under a specific constant mercy rule $\Omega(t) = \pm\omega$. At the beginning of the game, $\Phi(\omega) = \frac{1}{2} - E\left[\int_0^{\tau \wedge T} b(e_t^1)^2 dt\right]$.

Result 20. *Contestants prefer a tournament without a mercy rule over any constant mercy rule. More specifically,* $\lim_{\omega \to \infty} \Phi(\omega) \geqslant \Phi(\omega')$ *for any* $\omega' \geqslant 0$.

In a tournament without any mercy rule (i.e. $\omega \to \infty$), the players' effort level will remain very low until sometime close to the end of the game since it is not possible to end the game before T. Therefore, players save cost in the sense that they make little efforts for most part of the contest, just wait till the end and enter the lottery phase. It is also worth pointing out that $\Phi(\omega)$ is not monotonically decreasing in ω. Under some parameter sets, we find $\Phi(\omega)$ first increasing and then decreasing in ω.

3.4.4 A general linear mercy rule

A general linear mercy rule has the form $\Omega(t) = \pm(\lambda \cdot t + \beta)$, where λ and β are constant. Depending on the sign of λ and β, the threshold of ending the contest $|\Omega(t)|$ can be increasing, decreasing or constant. Let Ω^L represent the best mercy rule among the linear structure.

Result 21. *Fixing* b, σ *and* α, *when* b *or* σ *are large enough,* Ω^L *is (weakly) decreasing (i.e.* $\left|\Omega^L(t)\right|$ *is decreasing in* t*). As shown in the following figure,* $\left|\Omega^L\right|$ *starts far from* 0 *and decreases as time flows. Moreover, the slope of the linear mercy rule (i.e.* $|\lambda|$*) gets steeper as* α *increases.*

As mentioned in Result 15, when b or σ are large, the instantaneous effort has a roughly increasing trend in time, so a good mercy rule in this situation should try to keep the players in the tournament and induce them to compete intensively at the ending part of the game. Compared to the optimal constant mercy rule ω^*, we find the expectation duration associated with Ω^L is always larger than that associated with ω^*, which indicates that Ω^L reduces the probability to exit the game early. Since the effort levels are much higher in the tail part of the game, Ω^L clearly induces a higher cumulative effort in the latter part of the tournament. Moreover, Ω^L also mitigates the negative impact from the ending noise term since players are more and more possible to cross the winning threshold $\left|\Omega^L(t)\right|$ as t goes to T and avoid the lottery region, which also explains the observation that $|\lambda|$ gets steeper as α increases, since a steeper $|\lambda|$ reduces the probability for the players of staying till the very end.

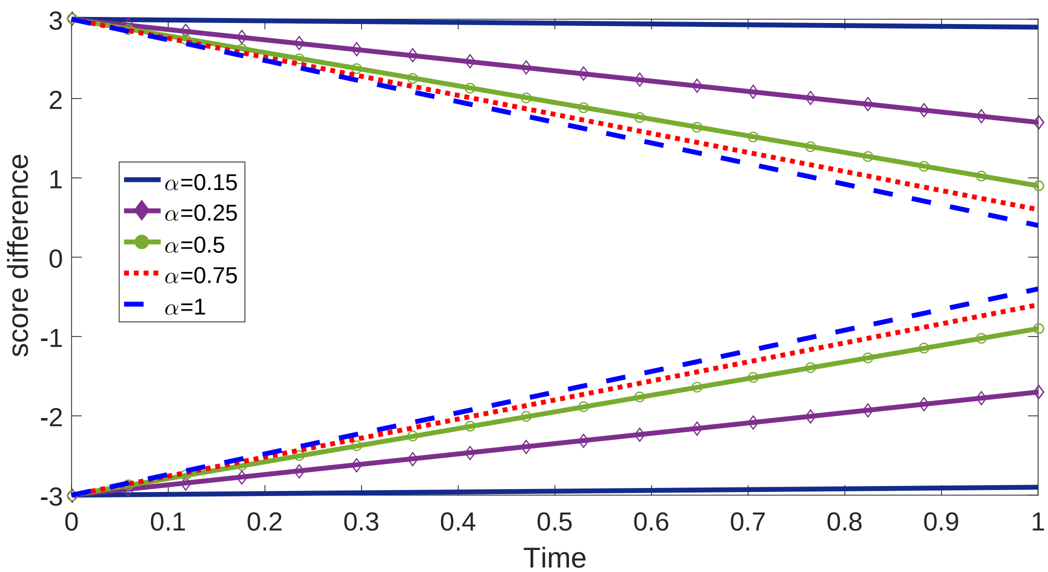


FIGURE 3.12: An illustration of Result 21, where the parameter set is given by $b = 1$, $\sigma = 0.5$.

Result 22. *Fixing b, σ and α, when both b and σ are small enough, Ω^L is (weakly) increasing.*

Similarly, as mentioned in Result 15, when both b and σ are small, the instantaneous effort has a decreasing trend until sometime very close to T, so a good mercy rule in this situation should take advantage of the higher effort values in the beginning of the tournament. An increasing mercy rule start with a relatively small $\left|\Omega^L(0)\right|$ outperforms a constant or decreasing mercy rule as it does a better job balancing the incentive effect and the duration effect.

3.5 The model with an additional constant cost

When the overall duration of the tournament T is large, it is worth considering the value of the time spent in the tournament, i.e. the longer the players stay in the tournament, the more cost they have to pay (or equivalently the less they value the prize). For example, when players have outside options that can bring in positive payoffs, a longer tournament essentially incurs higher opportunity costs. This time value gives players incentive to end the tournament early. To incorporate this effect, we consider an alternative flow cost of the following form, $b \cdot (e_t^i)^2 + K$, where K is a positive constant measuring the magnitude of the time value. With the new flow cost, player i's decision making problem becomes

$$\max_{e_t^i \in [0,\infty)} E\left[\int_0^{\tau \wedge T} -b(e_t^i)^2 - K + \Pi_t^i(x_t)dt \mid x_0 = 0\right].$$

And the corresponding HJB equations are given by,

$$\begin{aligned}
\frac{\partial V(x,t)}{\partial t} + \max_{e_1 \in [0,\infty)} \left[-(e^1(x,t))^2 + (e^1(x,t) - e^2(x,t))\frac{\partial V(x,t)}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 V(x,t)}{\partial x^2}\right] &= K, \\
\frac{\partial U(x,t)}{\partial t} + \max_{e_2 \in [0,\infty)} \left[-(e^2(x,t))^2 + (e^1(x,t) - e^2(x,t))\frac{\partial U(x,t)}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 U(x,t)}{\partial x^2}\right] &= K,
\end{aligned}$$

with terminal conditions

$$V(x,t) = \Pi^1(x,t) \text{ and } U(x,t) = \Pi^2(x,t) \text{ for } |x_t| \geqslant |\Omega(t)|.$$

Note that even though the time value described above sounds like discounting, the underlying model is very different from the one with a discounting factor (i.e. multiplying the value functions and cost terms with $\exp(-\delta t)$, where $\delta > 0$) in the following two aspects. First, the value functions in the time-value model could be negative at some pairs of (x,t) while those in the discounting model will always be non-negative which can be shown similarly as in Proposition 12. Second, the equilibrium strategy and the optimal mercy rules in the time-value model are exactly the same as those in the model in the previous section (i.e. $K = 0$) except that players pay an additional cost which equals to $K \cdot D(0,0)$. In the discounting model, however, the equilibrium strategy is different from a no-discounting model and the total cumulative expected effort as well as the total cumulative expected cost will be less than those in the model without discounting due to the decreasing value of the prize.

Result 23. *Contestants' utility function $\Phi(\omega)$ can be non-monotonic and negative (As shown in Figure 3.13).*

Note that with the additional constant flow cost, we can write $\Phi(\omega)$ as $\Phi(\omega) = \frac{1}{2} - \left(E\left[\int_0^{\tau\wedge T} b(e_t^1)^2 dt\right] + KD(0,0)\right)$. As ω gets larger, $D(0,0)$ also gets larger, but $E\left[\int_0^{\tau\wedge T} b(e_t^1)^2 dt\right]$ has a roughly decreasing trend (i.e it can be either strictly decreasing or firstly increasing and then decreasing). Depending on the relatively difference of the two terms, $\Phi(\omega)$ can behave differently. More specifically, when the overall effort level is low (e.g. when σ or b is large), the affect from the constant cost term tends to dominate that from the effort cost term, and $\Phi(\omega)$ is strictly decreasing in ω; however, when the overall effort level is high (e.g. when σ and b are small), the reverse is true, and $\Phi(\omega)$ can be non-monotonic with a negative minimum.

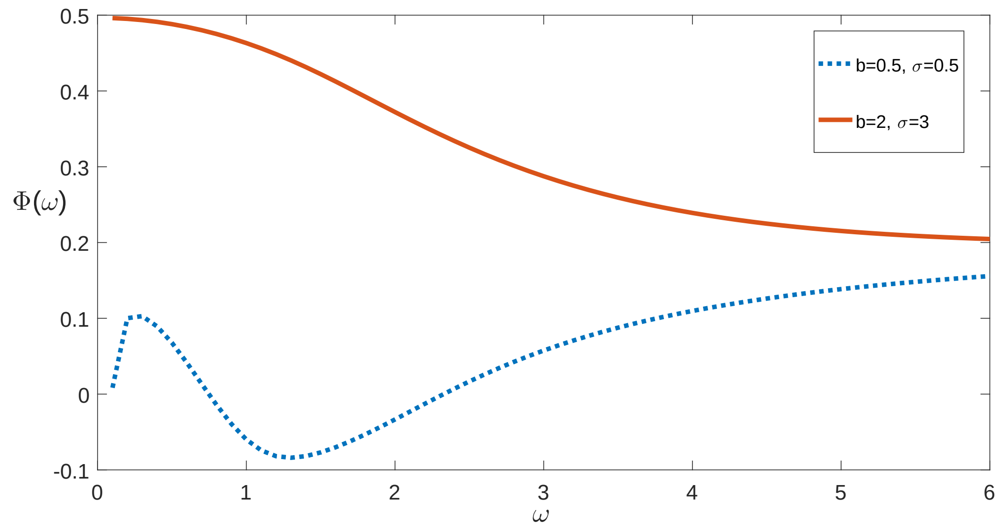


FIGURE 3.13: An example (Result 23) of the contestants' expected utility as a function of ω. The parameter set is given by, $K = 0.3$ and $\alpha = 0.5$. The solid line represents the low effort level scenario($b = 2$ and $\sigma = 3$), while the dotted line shows the high effort level scenario ($b = 0.5$ and $\sigma = 0.5$).

In the following part of this section, since expected payoffs could be negative, we give players the option to drop out of the contest, that is whenever the players have a negative expected payoff, they will stop making efforts and quit the tournament to ensure a zero payoff at that point and the opponent will win and claim the prize. Note that we can employ the algorithm of pricing an American Option in this situation. More specifically, starting with the backward induction on value functions, the negative values are replaced by zeros. Therefore, any exogenously given mercy rule actually induces an endogenous boundary which specifies an exit zone. We report results for the endogenous boundary when the exogenous mercy rule $|\omega^{ex}|$ is constant in the following.

Result 24. *Fixing all other parameters, the endogenous mercy rule $|\Omega^{en}(t)|$ can be increasing, decreasing or even not monotonic depending on the values of σ and b; and $|\Omega^{en}(T)| = d$.*

The result is mainly driven by the intensity of the leader-works-harder effect and the volatility of the underlying diffusion process. Consider two possible scenarios (x_1, t_1) and (x_2, t_2) for player 1, where $x_1 = x_2 < 0$, $t_2 = t_1 + \epsilon$, $x_2 = |\Omega_1^{en}(t_2)|$ and $|x_1| = |x_2| < |\omega^{ex}|$. If player 1 chooses to continue at $(x_{1,}t_1)$, he has to pay a flow cost of $K \cdot \epsilon + b\left(e^1(x_1, t_1)\right)^2 \cdot \epsilon$, and his expected payoff at time t_2 is

$$\begin{aligned} E\left[V(x_2', t_2)\right] &= \Pr(x_2' \leqslant x_2) E\left[V(x_2', t_2) \middle| \, x_2' \leqslant x_2\right] + \Pr(x_2' > x_2) E\left[V(x_2', t_2) \middle| \, x_2' > x_2\right] \\ &= \Pr(x_2' > x_2) E\left[V(x_2', t_2) \middle| \, x_2' > x_2\right]. \end{aligned}$$

Due to the losing position at (x_1, t_1) for player 1, we know $e^1(x_1, t_1) < e^2(x_2, t_2)$, and player 1 will move to a $x_2' < x_2$ at t_2 in expectation (i.e. $\Pr(x_2' > x_2) < 0.5$). Especially when σ is small, the overall instantaneous efforts are at a high level, $e^1(x_1, t_1)$ becomes larger, and $|e^1(x_1, t_1) - e^2(x_1, t_1)|$ also becomes larger due to the stronger leader-works-harder effect. On the other hand, $\Pr(x_2' > x_2)$ decreases because of a larger drift term and a smaller volatility of the underlying Brownian Motion, and $E\left[V(x_2', t_2) \middle| \, x_2' > x_2\right]$ also becomes smaller because of the higher total cumulative costs paid up to t_1. In the extreme scenario that $\sigma \to 0$, $E\left[V(x_2', t_2)\right] \to 0$

because $\Pr(x_2' > x_2) \to 0$. So it is not surprising that we find $|x_1| \geqslant |\Omega_1^{en}(t_1)|$ when σ is relatively small.

However, when σ is large enough, it is possible to have a decreasing endogenous exit boundary. Furthermore, depending on the value of b, a non-monotonic boundary may exist as well. More specifically, we find the endogenous boundary could be increasing first and decreasing, which can be explained by the complicated role of b on the players' decision making process. On one hand, when b increases, $\Pr(x_2' > x_2)$ gets larger because of a weaker leader-works-harder effect, and $E\left[V(x_2', t_2)\middle|\, x_2' > x_2\right]$ also gets boosted because of the lower total cumulative costs paid up to t_1; on the other hand, a larger b may or may not increase the flow cost. Therefore, depending on the overall dynamics and the exact location of (x_1, t_1), whether the player should continue at (x_1, t_1) can go either way.

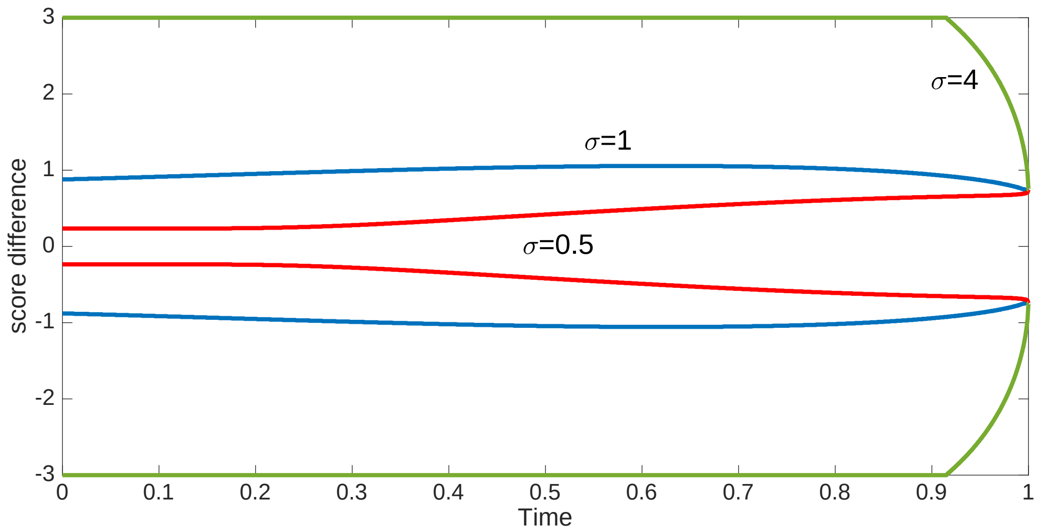


FIGURE 3.14: An illustration of Result 24, where the parameter set is given by $b = 1$, $K = 0.3$, $\omega^{ex} = 3$ and $\alpha = 0.25$. As σ increases, the endogenous boundary changes from being (weakly) increasing to non-monotonic to (weakly) decreasing.

Let Y_ω^{en} and Y_ω^{ex} denote the total expected cumulative effort associated with the endogenous boundary and the exogenous mercy rule ω respectively. And let $Y_{\omega *}^{en} \equiv \max_\omega Y_\omega^{en}$ and $Y_{\omega *}^{ex} \equiv \max_\omega Y_\omega^{ex}$.

Result 25. *Even though $Y_{\omega *}^{en} < Y_{\omega *}^{ex}$, Y_ω^{ex} does not dominate $Y_{\omega *}^{en}$ for all ω. Especially when ω is large, the endogenous mercy rule outperforms its exogenous counterpart in some region.*

The reason that the endogenous mercy rule is more effective in inducing effects when ω is large because it suffers less from the incentive effect compared to the exogenous rule as it is always smaller (i.e. $|\Omega^{en}(t)| \leqslant \omega$ for $\forall \omega$).

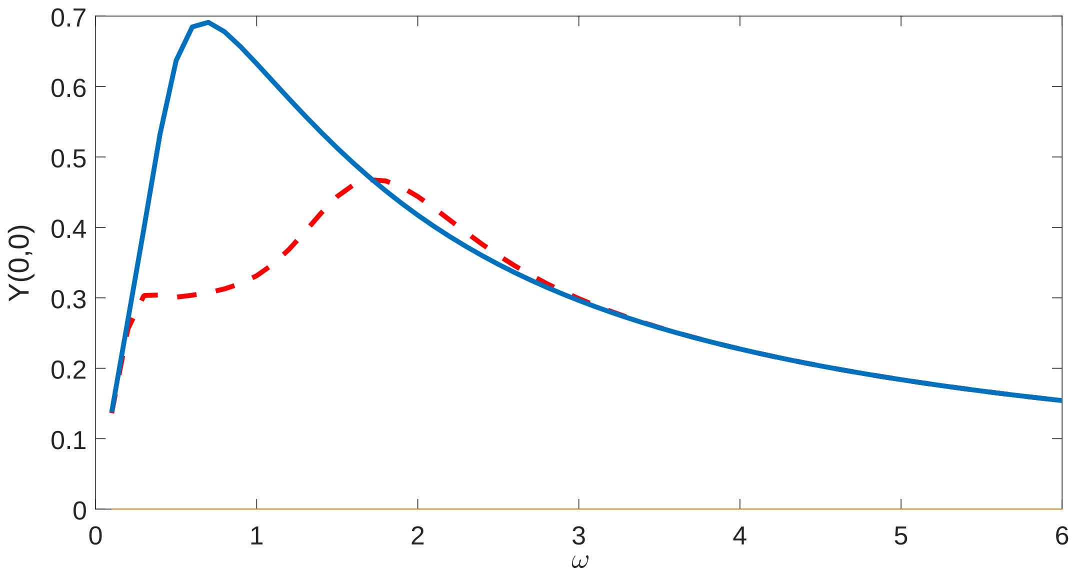


FIGURE 3.15: An illustration of Result 25: a comparison between the endogenous mercy rule (dashed line) and the exogenous mercy rule (solid line). The parameter set used is $b = 1$, $\sigma = 0.5$, $K = 0.3$ and $\alpha = 1$.

3.6 Conclusion

We have proposed a continuous-time stochastic model on dynamic tournaments in which players can observe each other's progress and adjust their strategies in real time. After showing that the equilibrium is unique in the weak sense, we have fully characterized the equilibrium strategy using numerical approximation. As expected, the results are highly dynamic depending on key parameters, such as the cost parameter of the players' cost function and the diffusion parameter of the underlying stochastic process.

Our main objective is to use the model to investigate the role of mercy rule played in tournaments and answer the following questions "Do mercy rules drive up or down total economic output?" and " What is the best mercy rule?". For tractability and realisticity of the model, we have focused on mercy rules within the linear function class, but our results are mostly likely robust to other monotonic mercy rules, such as $\Omega(t) = \sqrt{t}$ or $\Phi(t) = t^2$. In order to incorporate "time value", i.e. players tend to end the contest sooner to reduce opportunity cost from outside options, we study a modified model in which players are facing a constant flow cost in order to stay in the tournament and have options to drop out at any time. Surprisingly, we find endogenous mercy rule may even outperform exogenous mercy rule under some circumstances. We hope this framework will help provide intuitions on game dynamics and tournament design, and prompt further inquiries.

Appendix A

Proofs in Chapter 2

Proof of Lemma 1. Fix any $\tau \in [0, 1)$ and equilibrium, and without loss, consider player 1. Let $e^* = \frac{1}{\tau}\int_0^\tau e_s^1 ds$, and $\{q_t : 0 \leqslant t < \tau\}$ be any process such that $\frac{1}{\tau}\int_0^\tau q_s ds = e^*$ as well. Note that if $\int_0^\tau c(q_s)ds < \int_0^\tau c(e_s^1)ds$, then player 1 can profitably deviate to $e_t^i = q_t$ for all $t \in [0, \tau)$ (while leaving e_t^i for $t \geqslant \tau$ unchanged) thereby lowering total effort costs without altering his probability of winning. By Jensen's inequality, for all q, $\int_0^\tau c(q_s)ds \geqslant \tau c(e^*)$, with the inequality strict if q is not constant almost everywhere, implying the first claim. The proof of the second claim, for $t \in [\tau, 1]$, is analogous. □

Proof of Lemma 2. Clearly, for either player, $e_\tau^i > 0$ in any equilibrium because $c'(0) = 0$, and $f_{1-\tau+d}(\alpha) > 0$ for any $\alpha \in \mathbb{R}$, so a player would always gain by deviating from $e_\tau^i = 0$ to a slightly higher effort level. Hence, the first-order conditions are necessary for optimality, and (2.4) characterizes the unique equilibrium candidate by the analysis in Section 2.3.1. The proof of the equilibrium existence claims follows in Lemma A.0.1. $\square$

Lemma A.0.1. *The equilibrium described in Lemma 2 exists if* $\sigma^a > \frac{\exp\left(-\frac{(a-1)}{2}\right)}{\sqrt{2\pi}ab(a-1)^{\frac{a-1}{2}}(1-\tau+d)^{\frac{a}{2}}}$. *The condition is satisfied if σ is large enough (all other parameters fixed), d is large enough (all other parameters fixed), or b is large enough (all other parameters fixed).*

Proof. Fix both X_τ and the post-report of player 2, $e_\tau^2 = \bar{e}_\tau \equiv \left(\frac{f_{(1-\tau+d)}(X_\tau)}{ab}\right)^{\frac{1}{a-1}}$. Denote player 1's objective function as $V(e_\tau^1) \equiv F_{(1-\tau+d)}\left(X_\tau + \left(e_\tau^1 - \bar{e}_\tau\right)(1-\tau)\right) - (1-\tau)\,c(e_\tau^1)$. Note that V is differentiable, and because the prize is valued at 1, but c is unboundedly increasing, there exists $M > 0$ such that $V(e_\tau^1), V'(e_\tau^1) < 0$ for all $e_\tau^1 \geqslant M$. As a result, any maximizer for V has to be in $[0, M]$. By the Extreme Value Theorem, V attains its maximum at some $e_\tau^{1*} \in [0, M]$. However, because $c'(0) = 0$, $V'(0) > 0$, so $e_\tau^{1*} \neq 0$. Further, $V(M) < 0 < V(0)$, so $e_\tau^{1*} \neq M$. So, e_τ^{1*} must be a solution to the first-order condition (2.1). Therefore, if $e_\tau^1 = \bar{e}_\tau$ is the unique solution to (2.1) with $e_\tau^2 = \bar{e}_\tau$, then player 1's best response to $\bar{e}_\tau$ is $\bar{e}_\tau$. Since player 2's objective is symmetric, $e_\tau^1 = e_\tau^2 = \bar{e}_\tau$ constitutes an equilibrium.

To simplify notation, let $R(e_\tau^1) = f_{(1-\tau+d)}\left(X_\tau + \left(e_\tau^1 - \bar{e}_\tau\right)(1-\tau)\right)$. So at any solution to (2.1), $R(e_\tau^1) = c'(e_\tau^1)$. Note that at least one solution exists because *i*), both R and c' are continuous, *ii*) $R > 0$, whereas $c'(0) = 0$, and *iii*) R is uniformly bounded, whereas $\lim_{e\to\infty} c'(e) = \infty$. Further, at the minimum solution, e_m^*, $c''(e_m^*) \geqslant R'(e_m^*)$, and if $c''(e^*) > R'(e^*)$ at *all* solutions e^*, then (by continuity of both functions) there is only solution.

Suppose there exits a solution, e^*, with $c''(e^*) \leqslant R'(e^*)$. Then, $c'(e^*) = R(e^*)$

and

$$c'(e^*) = R(e^*) \implies ab(e^*)^{a-1} = \frac{\exp\left(-\frac{(X_\tau+(e^*-\bar{e}_\tau)(1-\tau))^2}{2(1-\tau+d)\sigma^2}\right)}{\sigma\sqrt{2\pi(1-\tau+d)}}$$

$$\textit{and} \quad \frac{c''(e^*)}{c'(e^*)} \leqslant \frac{R''(e^*)}{R'(e^*)} \implies \frac{a-1}{e^*} \leqslant -\frac{X_\tau + (e^* - \bar{e}_\tau)(1-\tau)}{(1-\tau+d)\sigma^2}.$$

Together, these imply

$$\sigma^a \leqslant \frac{\exp(-\frac{(a-1)^2(1-\tau+d)}{2z^2})}{z^{a-1}ab\sqrt{2\pi(1-\tau+d)}}$$

where $z^{a-1} = e^*/\sigma$. The expression on right is maximized at $z = \sqrt{(a-1)(1-\tau+d)}$. Hence, if

$$\sigma^a > \max_z \frac{\exp(-\frac{(a-1)^2(1-\tau+d)}{2z^2})}{z^{a-1}ab\sqrt{2\pi(1-\tau+d)}} = \frac{\exp\left(-\frac{(a-1)}{2}\right)}{\sqrt{2\pi}ab(a-1)^{\frac{a-1}{2}}(1-\tau+d)^{\frac{a}{2}}}$$

then, by contraposition, there is no solution e^* for which $c''(e^*) \leqslant R'(e^*)$, and is therefore a sufficient condition for the equilibrium existence. The claims regarding σ, d, or b large enough follow immediately. □

Proof of Lemma 3. To see that $e_0^1 = e_0^2 = 0$ cannot be part of an equilibrium, observe that $c'(0) = 0 < f_{(1+d)}(0)$, so a player would gain by deviating to a slightly higher effort level. Hence, $e_0^1, e_0^2 > 0$ and the first-order conditions are necessary for optimality. Therefore, the analysis in Section 2.3.2 characterizes the unique symmetric equilibrium candidate once we evaluate the integral: $\int f_{(1-\tau+d)}\left((e_0^1 - e_0^2)\tau + z\right)\left(\frac{\partial e_\tau^2}{\partial e_0^1}\right) f_\tau(z)dz =$

$$(e_0^1 - e_0^2)\exp\left(-\frac{a\tau^2\left(e_0^1 - e_0^2\right)^2}{2\sigma^2((a-1)(1+d)+\tau)}\right)\left(\frac{(2\pi)^{\frac{a}{2(1-a)}}\tau(1-\tau)\sqrt{a-1}}{\left(ab\sqrt{1-\tau+d}\right)^{\frac{1}{(a-1)}}((a-1)(1+d)+\tau)^{\frac{3}{2}}\sigma^{2+\frac{a}{a-1}}}\right),$$

which is 0 when $e_0^1 = e_0^2$. Hence, the unique symmetric equilibrium candidate is given by (2.10). The proof of the equilibrium existence claims follows in Lemma A.0.2. $\square$

Lemma A.0.2. *The equilibrium described in Lemma 3 exists if i) $a < 2$ and (A.5) holds, or ii) $a \geqslant 2$ and (A.7) holds. In either case, the conditions are satisfied if σ is large enough (all other parameters fixed), d is large enough (all other parameters fixed), or b is large enough (all other parameters fixed).*

Proof. Since the two players' best response functions are symmetric, we only need to show that given $e_0^2 = \left(\frac{f_{(1+d)}(0)}{ab}\right)^{\frac{1}{a-1}}$, $e_0^1 = \left(\frac{f_{(1+d)}(0)}{ab}\right)^{\frac{1}{a-1}}$ is the best response for player 1. The integral term in (2.6) can be directly evaluated, which allows us to rewrite the FOC for player 1 as,

$$c'(e_0^1) = f_{(1+d)}\left(\left(e_0^1 - e_0^2\right)\tau\right) + K\frac{1}{\sigma^{2+\frac{a}{a-1}}}\left(e_0^1 - e_0^2\right)\exp\left(-\frac{a\tau^2\left(e_0^1 - e_0^2\right)^2}{2\sigma^2((a-1)(1+d)+\tau)}\right) \tag{A.1}$$

$$= \frac{1}{\sqrt{2\pi(1+d)}\sigma}\exp\left(-\frac{\tau^2\left(e_0^1 - e_0^2\right)^2}{2\sigma^2(1+d)}\right) +$$

$$K\frac{1}{\sigma^{2+\frac{a}{a-1}}}\left(e_0^1 - e_0^2\right)\exp\left(-\frac{a\tau^2\left(e_0^1 - e_0^2\right)^2}{2\sigma^2((a-1)(1+d)+\tau)}\right),$$

where K is the positive constant:

$$K \equiv (2\pi)^{\frac{a}{2(1-a)}}\frac{\tau(1-\tau)\sqrt{a-1}}{\left(ab\sqrt{1-\tau+d}\right)^{\frac{1}{(a-1)}}((a-1)(1+d)+\tau)^{\frac{3}{2}}}.$$

Define the right hand side of the above expression as $u(e_0^1)$ and observe the following facts about $u(e_0^1)$. (1) The first component of $u(e_0^1)$, denote by $u_1(e_0^1) \equiv \frac{1}{\sqrt{2\pi(1+d)}\sigma} \exp\left(-\frac{\tau^2\left(e_0^1-e_0^2\right)^2}{2\sigma^2(1+d)}\right)$, is the density function for the normal distribution $N(e_0^2, (1+d)\sigma^2)$. So, $sign(u_1'(e_0^1)) = sign(e_0^2 - e_0^1)$. (2) The second component of $u(e_0^1)$, denote by $u_2(e_0^1) \equiv K\frac{1}{\sigma^{2+\frac{a}{a-1}}}(e_0^1 - e_0^2)\exp\left(-\frac{a\tau^2\left(e_0^1-e_0^2\right)^2}{2\sigma^2((a-1)(1+d)+\tau)}\right)$, is symmetric about $(e_0^2, u_2(e_0^2) = 0)$, negative for $e_0^1 < e_0^2$, and $u_2'(e_0^1)$ is maximized and positive at $e_0^1 = e_0^2$. Also, there exists a point $\overline{e_0^1}$ such that $u_2'(\overline{e_0^1}) = 0$ and $u_2'(e_0^1) < 0$ for all $e_0^1 > \overline{e_0^1}$. (3) As a result of (1) and (2), $u'(e_0^2) > u'(e_0^1)$ for all $e_0^1 > e_0^2$ and

$$\operatorname*{argmax}_{e_0^1 \in [0, e_0^2]} u(e_0^1) = e_0^2$$

$$\operatorname*{argmax}_{e_0^1} u(e_0^1) \in (e_0^2, \overline{e_0^1})$$

To show this FOC is sufficient for optimality, we first need to eliminate that player 1's maximization problem (2.5) is solved at the boundary: $e_0^1 = 0$. Clearly, this is eliminated if

$$u(0) > c'(0) = 0. \tag{A.2}$$

Moreover, note that in the objective function, the expectation term is bounded by 1 from above and the cost function is strictly increasing without bound. Therefore, there must exist a N such that if $e_0^1 \geqslant N$, the objective function is negative. Applying the Extreme Value Theorem on $[0, N]$ implies that the maximizer e_0^{1*} exists, and therefore if (A.2) holds, e_0^{1*} must solve (2.6). (Immediately $e_0^{1*} \neq N$ since the objective is negative at N.)

Given $e_0^2 = \left(\frac{f_{(1+d)}(0)}{ab}\right)^{\frac{1}{a-1}}$ (and some algebra), (A.2) becomes the following

$$\sigma^{2+\frac{2}{a-1}} \exp\left(\frac{\tau^2}{2\sigma^2}\left(\frac{1}{ab\sigma\sqrt{2\pi(1+d)}}\right)^{\frac{2}{a-1}}\underbrace{\left(\frac{a}{(a-1)(1+d)+\tau} - \frac{1}{1+d}\right)}_{>0}\right)$$

$$> \frac{K(\sqrt{2\pi(1+d)})^{\frac{a-2}{a-1}}}{(ab)^{\frac{1}{a-1}}}$$

The exponential term in the above expression is always greater than 1, thus a stronger condition for (A.2) is

$$\sigma^{2+\frac{2}{a-1}} > \frac{K(\sqrt{2\pi(1+d)})^{\frac{a-2}{a-1}}}{(ab)^{\frac{1}{a-1}}} \tag{A.3}$$

Hence, it is sufficient to find conditions under which $c'(e_0^1)$ and $u(e_0^1)$ only cross once for $e_0^1 \in [0, \infty)$. Since $c'(e_0^1)$ is always increasing, and we already know $c'(e_0^1)$ and $u(e_0^1)$ cross each other at $e_0^1 = e_0^2$, the desired single crossing property is equivalent to the following condition: $sign\,(c'(e_0^1) - u(e_0^1)) = sign(e_0^1 - e_0^2)$.

<u>If $a < 2$</u>: $c''(e_0^1)$ is strictly decreasing, but always positive, in e_0^1. Thus, a sufficient condition for the desired single cross property is

$$\sup_{e_0^1 \in [0, e_0^2]} u'(e_0^1) < c''(e_0^2) \text{ and } u'(e_0^2) < c''(\overline{e_0^1}), \tag{A.4}$$

where

$$u'(e_0^1) = -\frac{\tau^2\,(e_0^1 - e_0^2)}{\sqrt{2\pi}(1+d)^{3/2}\sigma^3} \exp\left(-\frac{\tau^2\,(e_0^1 - e_0^2)^2}{2\sigma^2(1+d)}\right) +$$

$$K\frac{1}{\sigma^{2+\frac{a}{a-1}}}\left(1 - \frac{a\tau^2\,(e_0^1 - e_0^2)^2}{\sigma^2((a-1)(1+d)+\tau)}\right)\exp\left(-\frac{a\tau^2\,(e_0^1 - e_0^2)^2}{2\sigma^2((a-1)(1+d)+\tau)}\right).$$

From observations (1)-(3) made about u, u_1, u_2 at the outset, the first inequality in (A.4) ensures that $c'(e_0^1) - u(e_0^1)$ strictly increasing in $e_0^1 \in [0, e_0^2]$, while the second condition guarantees the same monotonicity property in $e_0^1 \in [e_0^2, \overline{e_0^1}]$. And $c'(e_0^1) - u(e_0^1)$ is always positive in $e_0^1 > \overline{e_0^1}$. Thus, $c'(e_0^1) - u(e_0^1) > 0$ for all e_0^1, which implies the desired single crossing property.

Note that given $e_0^2 = \left(\frac{f_{(1+d)}(0)}{ab}\right)^{\frac{1}{a-1}} = \left(ab\sigma\sqrt{2\pi(1+d)}\right)^{-\frac{1}{a-1}}$, we have

$$\sup_{e_0^1} u(e_0^1) < \sup_{e_0^1}\left(-\frac{\tau^2\left(e_0^1 - e_0^2\right)}{\sqrt{2\pi}(1+d)^{3/2}\sigma^3}\exp\left(-\frac{\tau^2\left(e_0^1 - e_0^2\right)^2}{2\sigma^2(1+d)}\right)\right) +$$

$$\sup_{e_0^1}\left(K\frac{1}{\sigma^{2+\frac{a}{a-1}}}\left(1 - \frac{a\tau^2\left(e_0^1 - e_0^2\right)^2}{\sigma^2((a-1)(1+d)+\tau)}\right)\exp\left(-\frac{a\tau^2\left(e_0^1 - e_0^2\right)^2}{2\sigma^2((a-1)(1+d)+\tau)}\right)\right)$$

$$= \frac{\tau^2\exp(-\frac{1}{2})}{\sqrt{2\pi}(1+d)\sigma^2} + \frac{K}{\sigma^{2+\frac{a}{a-1}}},$$

and from $u_2'(\overline{e_0^1}) = 0$, we obtain $\overline{e_0^1} = e_0^2 + \sqrt{\frac{\sigma^2((a-1)(1+d)+\tau)}{a\tau^2}}$.

Therefore, the sufficient condition (A.4) can be explicitly given by

$$\frac{K}{\sigma^{2+\frac{a}{a-1}}} < \min\left\{a(a-1)b\left(ab\sigma\sqrt{2\pi(1+d)}\right)^{\frac{2-a}{a-1}} - \frac{\tau^2\exp(-\frac{1}{2})}{\sqrt{2\pi}(1+d)\sigma^2},\right.$$

$$\left. a(a-1)b\left(\left(ab\sigma\sqrt{2\pi(1+d)}\right)^{-\frac{1}{a-1}} + \sqrt{\frac{\sigma^2((a-1)(1+d)+\tau)}{a\tau^2}}\right)^{a-2}\right\} \tag{A.5}$$

Note that (A.5) actually implies (A.3), thus (A.5) is sufficient in isolation if $a < 2$. The claims regarding σ, d, or b large enough follow immediately.

<u>*If $a \geqslant 2$*</u>: c' is (weakly) convex, and the desired single crossing property is equivalent to: (i) $c''(e_0^2) > u'(e_0^2)$, which implies $c'(e_0^1) > u(e_0^1)$ for $e_0^1 > e_0^2$, and (ii) $c'(e_0^1) < u(e_0^1)$

for $e_0^1 \in [0, e_0^2]$. To achieve condition (ii), let

$$v(e_0^1) \equiv \frac{1}{\sqrt{2\pi(1+d)}\sigma} \exp\left(-\frac{\tau^2 \left(e_0^1 - e_0^2\right)^2}{2\sigma^2(1+d)}\right) + K \frac{1}{\sigma^{2+\frac{a}{a-1}}} \left(e_0^1 - e_0^2\right),$$

and see that $v(e_0^1) \leqslant u(e_0^1)$ for $e_0^1 \in [0, e_0^2]$. So, if $c'(e_0^1)$ and $v(e_0^1)$ cross only once in $[0, e_0^2]$, then $c'(e_0^1)$ and $u(e_0^1)$ can cross at most once in $[0, e_0^2]$. Note that $u'(e_0^2) = v'(e_0^2)$.

We now claim that if following is satisfied, then the desired single cross property holds (i.e., conditions (i) and (ii) are met):

$$v(0) > c'(0), e_0^2 - \frac{\sigma^2(1+d)}{\tau^2} \leqslant 0, \text{ and } c''(e_0^2) > v'(e_0^2), \tag{A.6}$$

First, condition (i) is implied by the third inequality in (A.6). Next, it is not hard to verify that for $e_0^1 \in \left[e_0^2 - \frac{\sigma^2(1+d)}{\tau^2}, e_0^2\right]$, $v'(e_0^1)$ strictly decreases, where $v'(e_0^1)$ is given by

$$v'(e_0^1) = -\frac{\tau^2 \left(e_0^1 - e_0^2\right)}{(1+d)\sqrt{2\pi(1+d)}\sigma^3} \exp\left(-\frac{\tau^2 \left(e_0^1 - e_0^2\right)^2}{2\sigma^2(1+d)}\right) + K \frac{1}{\sigma^{2+\frac{a}{a-1}}}.$$

If $e_0^2 - \frac{\sigma^2(1+d)}{\tau^2} \leqslant 0$, together with $c''(0) = 0 < v'(0)$ and $c''(e_0^2) > v'(e_0^2)$, we know $c''(e_0^1)$ and $v'(e_0^1)$ cross each other exactly once in $[0, e_0^2]$. More specifically, since $v(0) > c'(0)$, if $c''(e_0^1) < v'(e_0^1)$, $v(e_0^1) - c'(e_0^1)$ is increasing; however, if $c''(e_0^1) > v'(e_0^1)$, $v(e_0^1) - c'(e_0^1)$ begins to decrease and $c'(e_0^1)$ may surpass $v(e_0^1)$ at some point. Therefore, $c'(e_0^1)$ and $v(e_0^1)$ can cross at most once in $[0, e_0^2]$. Since we already know $c'(e_0^1)$ and $v(e_0^1)$ cross each other at $e_0^1 = e_0^2$, this immediately implies $c'(e_0^1) < v(e_0^1) \leqslant u(e_0^1)$ for $e_0^1 \in [0, e_0^2]$, and therefore condition (ii). An explicit expression of (A.6) is,

$$\sigma^{2+\frac{2}{a-1}} > \min\left\{ \frac{K(\sqrt{2\pi(1+d)})^{\frac{a-2}{a-1}}}{(ab)^{\frac{1}{a-1}} \exp\left(-\frac{\tau^2}{2(1+d)\sigma^2}\left(\frac{1}{ab\sigma\sqrt{2\pi(1+d)}}\right)^{\frac{2}{a-1}}\right)}, \frac{\tau^2}{(1+d)\left(ab\sqrt{2\pi(1+d)}\right)^{\frac{2}{a-1}}} \right\} \tag{A.7}$$

Note that it is easy to verify $v(0) > c'(0)$ actually implies $c''(e_0^2) > v'(e_0^2)$, so we suppress $c''(e_0^2) > v'(e_0^2)$ in (A.7). Further, $v(0) > c'(0)$ immediately implies (A.3), thus (A.7) is sufficient in isolation if $a \geqslant 2$. The claims regarding σ, d, or b large enough follow immediately. $\square$

Proof of Proposition 4. Let h denote the inverse of c'; so, $h(\alpha) = \left(\frac{\alpha}{ab}\right)^{\frac{1}{a-1}}$. Hence, if $a \leqslant 2$ (respectively, $a \geqslant 2$), then h is convex (concave). From Lemmas 1–3,

$$W(0) = \int_0^1 h(f_{(1+d)}(0))dt = h(f_{(1+d)}(0)),$$

and for $\tau > 0$,

$$\begin{aligned} W(\tau) &= \int_0^\tau h(f_{(1+d)}(0))dt + E\left[\int_\tau^1 h(f_{(1-\tau+d)}(Z_\tau))dt\right] \\ &= \tau W(0) + (1-\tau)E[h(f_{(1-\tau+d)}(Z_\tau))]. \end{aligned}$$

Therefore, $sign(W(\tau) - W(0)) = sign\big(E[h(f_{(1-\tau+d)}(Z_\tau))] - h(f_{(1+d)}(0))\big)$. Noting that, by convolution, $E[f_{(1-\tau+d)}(Z_\tau)] = f_{(1+d)}(0)$, the proposition follows from Jensen's inequality. $\square$

Proof of Proposition 5. Ex ante, each player's probability of winning is $\frac{1}{2}$. Hence, $S(\tau) = \frac{1}{2} - E[\int_0^1 c(e_t^i)dt|\tau]$. Noting that the composite function $c(h(\cdot))$ is strictly convex (for any $a > 1$), the proof is analogous to that given for Proposition 4. $\square$

Proof of Propositions 6 and 8. Recall from Lemmas 2 and 3 that

$$\begin{aligned} W(\tau) &= \tau h(f_{(1+d)}(0)) + (1-\tau)E[h(f_{(1-\tau+d)}(Z_\tau))] \\ &= \tau \left(\frac{f_{(1+d)}(0)}{ab}\right)^{\frac{1}{a-1}} + (1-\tau)E\left[\left(\frac{f_{(1-\tau+d)}(Z_\tau)}{ab}\right)^{\frac{1}{a-1}}\right]. \end{aligned}$$

Define $P(\tau) \equiv \frac{W(\tau)-W(0)}{W(0)}$ as the percentage change in expected effort relative to the no-report/static benchmark from having the release time be τ. Note that since this is only a normalization, τ maximizes (minimizes) W if and only if maximizes (minimizes) P and W is single-peaked (-troughed) if and only if P is single-peaked (-troughed).

$$P(\tau) = \frac{\tau\left(\frac{f_{(1+d)}(0)}{ab}\right)^{\frac{1}{a-1}} + (1-\tau)E\left[\left(\frac{f_{(1-\tau+d)}(Z_\tau)}{ab}\right)^{\frac{1}{a-1}}\right] - \left(\frac{f_{(1+d)}(0)}{ab}\right)^{\frac{1}{a-1}}}{\left(\frac{f_{(1+d)}(0)}{ab}\right)^{\frac{1}{a-1}}}$$

$$= (1-\tau)\frac{\int \left(f_{(1-\tau+d)}(z)\right)^{\frac{1}{a-1}} f_\tau(z)dz - \left(f_{(1+d)}(0)\right)^{\frac{1}{a-1}}}{\left(f_{(1+d)}(0)\right)^{\frac{1}{a-1}}}$$

$$= (1-\tau)\left(\sqrt{\frac{(1+d)^{1+w}}{(1+d+\tau w)(1+d-\tau)^w}} - 1\right) \tag{A.8}$$

where $w \equiv \frac{1}{a-1} - 1$. Consistent with Proposition 4, if $a = 2$, then $w = 0$, and $P(\tau) = 0$ for all τ. Now fix $a \in (1, 2)$, meaning $w > 0$ (the proofs for the $a > 2$ case are symmetric).

We first show that any maximizer of P, τ^*, is in $(\frac{1}{2}, 1)$. Because P is continuous in τ on $[0, 1]$, by the Extreme Value Theorem, there exits $\tau^* \in [0, 1]$ that maximizes P on the interval.[1] Further, $P(0) = P(1) = 0$, and by Proposition 4, $P(\tau) > 0$ for any τ in $(0, 1)$. Hence, any maximizer must be interior: $\tau^* \in (0, 1)$.

Let $g(\tau) \equiv \sqrt{\frac{(1+d)^{1+w}}{(1+d+\tau w)(1+d-\tau)^w}} - 1$. Then,

$$g'(\tau) = \frac{\tau w(w+1)\sqrt{\frac{(d+1)^{w+1}(d-\tau+1)^{-w}}{d+\tau w+1}}}{2(d-\tau+1)(d+\tau w+1)},$$

and

$$g''(\tau) = \frac{(d+1)^2 w(w+1)(1-\frac{\tau}{1+d})^{-2w}(2d^2+4d+\tau^2 w^2+3\tau^2 w+2)}{4(d-\tau+1)^2\left(\frac{(d+1)(1-\frac{\tau}{1+d})^{-w}}{d+\tau w+1}\right)^{\frac{3}{2}}(d+\tau w+1)^4}.$$

Thus, $g(0) = g'(0) = 0$, and, for $\tau \in (0, 1)$, $g'(\tau) > 0$ and $g''(\tau) > 0$. So g is convex in τ. Because g is continuous and differentiable, the Mean Value Theorem implies that for any $\tau \in (0, 1]$, there exists $x \in (0, \tau)$ such that $g(\tau) - g(0) = \tau g'(x)$. Since $g(0) = 0$, this reduces to $g(\tau) = \tau g'(x)$. Because g is convex, $g'(x) < g'(\tau)$, which

[1] While τ is restricted to $[0, 1)$ in the model, extending the domain of P to $[0, 1]$ only strengthens the result: if $\tau^* \in (0, 1)$ maximizes P on $[0, 1]$, then it also maximizes P on $[0, 1)$.

implies $g(\tau) = \tau g'(x) < \tau g'(\tau)$. Now consider $P(\tau) = (1 - \tau)g(\tau)$.

$$\begin{aligned}\frac{dP}{d\tau} &= -g(\tau) + (1-\tau)g'(\tau) \\ &> -\tau g'(\tau) + (1-\tau)g'(\tau) \\ &= (1-2\tau)g'(\tau).\end{aligned}$$

Therefore, for $\tau \in [0, \frac{1}{2}]$, $\frac{dP}{d\tau} > 0$. So P is (strictly) increasing on $[0, \frac{1}{2}]$, meaning any maximizer, τ^*, must be in $(\frac{1}{2}, 1)$.

Next, we show that P is single-peaked. To do so, we calculate

$$\frac{d^2P}{d\tau^2} = \left(\frac{-w(1+w)\sqrt{\frac{(1+d)^{1+w}(1+d-\tau)^{-w}}{1+d+\tau w}}}{4(1+d-\tau)^2(1+d+\tau w)^2} \right) h(\tau)$$

where, $h(\tau) \equiv w(-1+w)\tau^3 + (-4+4d(-1+w)+(1-w)w)\tau^2 + (6+12d+6d^2)\tau + (-2-4d-2d^2)$. So, the $sign(\frac{d^2P}{d\tau^2}) = -sign(h(\tau))$. At $\tau = \tau^*$, $\frac{dP}{d\tau}|_{\tau=\tau^*} = 0$ and $\frac{d^2P}{d\tau^2}|_{\tau=\tau^*} \leqslant 0$, which implies $h(\tau^*) \geqslant 0$. Notice that

$$\begin{aligned}h(0) &= -2-4d-2d^2 = -2(d+1)^2 < 0 \\ h(1) &= 4(d^2+d+dw) > 0.\end{aligned}$$

By the continuity of h, then, there exist $\tau \in (0, 1)$ such that $h(\tau) = 0$. It is sufficient to show that only one such root exists, because, if so, then for all $\tau > \tau^*$, $h(\tau) > 0$ and $\frac{d^2P}{d\tau^2} < 0$, implying there is only one local maximum in $[0, 1]$ and that P is single-peaked.

First, if $w = 1$, then $h(\tau) = -4\tau^2 + 6(d+1)^2\tau - 2(d+1)^2$, which has one root in $(0, 1)$. In this case, the claim is proved. Second, if $w \neq 1$, then h is a cubic polynomial with $h(0) < 0$ and $h(1) > 0$, implying there are either one or three roots in $[0,1]$. Suppose now that P has multiple peaks (i.e., local maxima) in $(\frac{1}{2},1]$, then between any two peaks, there must exist a local minimum. At any local maximum $\frac{d^2P}{d\tau^2} \leqslant 0$, and at any local minimum, $\frac{d^2P}{d\tau^2} \geqslant 0$. By the continuity of $\frac{d^2P}{d\tau^2}$, then h has two roots in $(\frac{1}{2},1]$. Let τ_1, τ_2, $\tau_3 \in (0,1)$ denote the roots of h. Then h can be written,

$$\begin{aligned} h(\tau) &= w(-1+w)(\tau-\tau_1)(\tau-\tau_2)(\tau-\tau_3) \\ &= w(-1+w)\left(\tau^3-(\tau_1+\tau_2+\tau_3)\tau^2+(\tau_1\tau_2+\tau_2\tau_3+\tau_1\tau_3)\tau-\tau_1\tau_2\tau_3\right). \end{aligned}$$

Comparing this expression with h's original, cubic-polynomial expression, see that

$$\tau_1\tau_2\tau_3 = \frac{2+4d+2d^2}{w(-1+w)} = \frac{2(d+1)^2}{w(-1+w)},$$

and

$$\begin{aligned} \tau_1+\tau_2+\tau_3 &= \frac{4-4d(-1+w)-(1-w)w}{w(-1+w)} \\ &= \frac{4-4d(-1+w)}{w(-1+w)}+1. \end{aligned}$$

Thus,

$$0<\tau_1\tau_2\tau_3<1 \implies 0<\frac{2(d+1)^2}{w(-1+w)}<1 \implies 2<w(-1+w) \implies 2<w$$

and

$$1<\tau_1+\tau_2+\tau_3<3 \implies 0<\frac{1-d(-1+w)}{w(-1+w)}<\frac{1}{2} \implies 0<d<\frac{1}{w-1}.$$

As a result,

$$-2<4d-2<-4+4d(-1+w)+(1-w)w=-w(-1+w)(\tau_1+\tau_2+\tau_3)<0.$$

It is then not hard to verify the discriminant, Δ, of the equation $h(\tau) = 0$ is always negative, which contradicts the assumption that h has three roots, and therefore completes the proof. To do so, first write $h(\tau)$ in the standard form, $h(\tau) = A\tau^3 + B\tau^2 + C\tau + D$. By the above,

$$\begin{aligned}
A &= w(-1 + w) > 2 \\
B &= (-4 + 4d(-1 + w) + (1 - w)w) \in (-2, 0) \\
C &= 6(d + 1)^2 > 6 \\
D &= -2(d + 1)^2 < -2
\end{aligned}$$

Thus,

$$\begin{aligned}
\Delta &= 18ABCD - 4B^3D + B^2C^2 - 4AC^3 - 27A^2D^2 \\
&= 3A(6BCD - C^3 - 9AD^2) + (B^2C^2 - AC^3 - 4B^3D),
\end{aligned}$$

where $6BCD - C^3 - 9AD^2 < 12 \cdot 6(d + 1)^2 \cdot 2(d + 1)^2 - 6^3(d + 1)^6 - 9AD^2 < 0$ and $B^2C^2 - AC^3 - 4B^3D < 4C^2 - 6AC^2 - 4B^3D < 0$. Therefore, $\Delta < 0$. $\square$

Proof of Proposition 7. Recalling that $w \equiv \frac{1}{a-1} - 1$, an equivalent restatement of claim (i) is $P(\tau)$ is increasing in w for all $\tau, w > 0$. Taking the derivative of (A.8) with respect to w yields

$$\frac{dP}{dw} = \frac{1 - \tau}{2(1 + d + \tau w)} \sqrt{\frac{(1 + d)^{1+w}}{(1 + d + \tau w)(1 - \tau + d)^w}} \left(-\tau + (1 + d + \tau w) \log \left(\frac{1 + d}{1 + d - \tau} \right) \right). \tag{A.9}$$

The first two terms are positive by inspection. Hence,

$$sign\left(\frac{dP}{dw}\right) = sign\left(-\tau + (1+d+\tau w)\log\left(\frac{1+d}{1+d-\tau}\right)\right) \tag{A.10}$$

$$= sign\left(-\tau + (1+d)\log\left(\frac{1+d}{1+d-\tau}\right) + \underbrace{\tau w\log\left(\frac{1+d}{1+d-\tau}\right)}_{\geqslant 0}\right). \tag{A.11}$$

Therefore, we need only show that $R(\tau, d) \equiv -\tau + (1+d)\log\left(\frac{1+d}{1+d-\tau}\right) > 0$. But this follows from $\frac{dR}{d\tau} = \frac{\tau}{1-\tau+d} > 0$ and $\lim_{\tau\to 0} R(\tau, d) = 0$, establishing the claim.

An equivalent restatement of claim (ii) is: fixing all other parameters, for any d, there exists $\underline{w}(d)$ such that τ^* is increasing in w for $w > \underline{w}(d)$. In addition, $\underline{w}(d) \to 0$ as $d \to 0$.

We start with the existence of such a $\underline{w}$. From the proof of Proposition 6, there is a unique solution, τ^*, to the first-order condition:

$$\frac{dP}{d\tau} = 1 - \sqrt{\frac{(1+d)^{1+w}}{(1+d+\tau w)(1+d-\tau)^w}} + (1-\tau)\frac{\tau w(w+1)\sqrt{\frac{(d+1)^{w+1}(d-\tau+1)^{-w}}{d+\tau w+1}}}{2(d-\tau+1)(d+\tau w+1)} = 0, \tag{A.12}$$

which algebra reveals is equivalent to

$$\underbrace{\frac{1-\tau}{2(1+d-\tau)}\frac{1+w}{\frac{1+d}{\tau w}+1} + \sqrt{1+\frac{\tau w}{1+d}}\left(1-\frac{\tau}{1+d}\right)^{\frac{w}{2}}}_{\equiv G(\tau|w)} = 1. \tag{A.13}$$

Further, $\tau^* \in (1/2, 1)$, and $sign(P'(\tau)) = sign(G(\tau|w) - 1) = sign(\tau^* - \tau)$. Next, note that there exists a threshold $\underline{w}(d)$ such that G is increasing in w if $w > \underline{w}(d)$: as w

grows large, the first term of G increases at a linear rate, while the second term tends to zero since $1 - \frac{\tau}{1+d} < 1$. Finally, τ^* is likewise increasing in w if $w > \underline{w}(d)$. To see this, let $G(\tau_1|w_1) = 1$ for $w_1 > \underline{w}(d)$. Then for $w_2 > w_1$, $G(\tau_1|w_2) > G(\tau_1|w_1) = 1$. From the proof of Proposition 6, $G(\tau|w)$ crosses 1 exactly once for $\tau \in (1/2, 1)$, doing so from above. So $G(\tau_2|w_2) = 1$, requires $\tau_2 > \tau_1$.

Next, we argue that $\underline{w}(d) \to 0$ as $d \to 0$. To do so compute,

$$\begin{aligned}\frac{\partial G(\tau, w)}{\partial w} = \frac{\tau(1+d)}{2(1+d+\tau w)^2}\left(1 - \frac{\tau}{1+d}\right)^{\frac{w}{2}}\left(1 + \frac{\tau w}{1+d}\right)^{\frac{3}{2}} \\ + \frac{\tau(1-\tau)(1+d+2dw+2w+\tau w^2)}{2(1+d-\tau)(1+d+\tau w)^2} \\ + \frac{1}{2}\left(1 - \frac{\tau}{1+d}\right)^{\frac{w}{2}}\sqrt{1 + \frac{\tau w}{1+d}}\log\left(1 - \frac{\tau}{1+d}\right).\end{aligned}$$

Then evaluate $\frac{\partial G(\tau,w)}{\partial w}$ at $G(\tau, w) = 1$, by replacing $\sqrt{1 + \frac{\tau w}{1+d}}\left(1 - \frac{\tau}{1+d}\right)^{\frac{w}{2}}$ with $1 - \frac{\tau w(1-\tau)(1+w)}{2(1+d-\tau)(1+d+\tau w)}$ in accordance with (A.13).

$$\frac{\partial G(\tau,w)}{\partial w}|_{G(\tau,w)=1} =$$

$$\frac{\tau(1+d)}{2(1+d+\tau w)^2}\left(1+\frac{\tau w}{1+d}\right)\left(1-\frac{\tau w(1-\tau)(1+w)}{2(1+d-\tau)(1+d+\tau w)}\right)$$

$$+\frac{\tau(1-\tau)(1+d+2dw+2w+\tau w^2)}{2(1+d-\tau)(1+d+\tau w)^2}$$

$$+\frac{1}{2}\left(1-\frac{\tau}{1+d}\right)^{\frac{w}{2}}\sqrt{1+\frac{\tau w}{1+d}}\log\left(1-\frac{\tau}{1+d}\right)$$

$$=\frac{4+\tau w}{2w(1+d+\tau w)}\left(1-\sqrt{1+\frac{\tau w}{1+d}}\left(1-\frac{\tau}{1+d}\right)^{\frac{w}{2}}\right)$$

$$+\frac{2\tau d^2+\tau d(6+4w-4\tau-2\tau w)}{4(1+d-\tau)(1+d+\tau w)^2}$$

$$+\frac{1}{2}\left(1-\frac{\tau}{1+d}\right)^{\frac{w}{2}}\sqrt{1+\frac{\tau w}{1+d}}\log\left(1-\frac{\tau}{1+d}\right).$$

Taking the limit as $d \to 0$,

$$\lim_{d\to 0}\left(\frac{\partial G(\tau,w)}{\partial w}|_{G(\tau,w)=1}\right) = \frac{1}{2}\frac{4+\tau w}{w(1+\tau w)}\left(1-(1-\tau)^{\frac{w}{2}}\sqrt{1+\tau w}\right)$$

$$+\frac{1}{2}(1-\tau)^{\frac{w}{2}}\sqrt{1+\tau w}\log(1-\tau)$$

Define $p(\tau, w)$ as the right hand side of the expression above. It is straightforward to verify that there exits a threshold $\underline{\tau} > 0$ such that $sign(p(\tau, w)) = sign\,(\tau - \underline{\tau})$, and that $\underline{\tau}$ is decreasing in w. Hence, it is sufficient to establish that $\lim_{w\to 0} p(\tau^*, w) \geqslant 0$, since if so the argument that τ^* is increasing in w, provided immediately following (A.13), again applies. So, finally, see that

$$\lim_{w \to 0} p(\tau, w) = -\tau - \frac{1}{2}\log(1-\tau) = 2 \lim_{w \to 0} \lim_{d \to 0} \frac{dP/d\tau}{w}$$

which is simply zero once evaluated at $\tau = \tau^*$ since the latter is the FOC (divided by w), completing the proof (note that exchanging the order of limits—or taking any relative rates of convergence to zero—in the last equality does not alter the result). □

Appendix B

Proofs in Chapter 3

Lemma B.0.1. *If* $\frac{\partial V(x,t)}{\partial x} \geqslant 0$ *and* $\frac{\partial U(x,t)}{\partial x} \leqslant 0$*, the first order conditions are sufficient to guarantee the optimality (equilibrium conditions).*

Proof. The optimization regarding e in the stochastic control problem is quadratic in e, so the first order conditions are sufficient to guarantee the optimality if $e^* \geqslant 0$. Since $e^{1*} = \frac{1}{2b}\frac{\partial V(x,t)}{\partial x}$ and $e^{2*}(x,t) = -\frac{1}{2b}\frac{\partial U(x,t)}{\partial x}$, if $\frac{\partial V(x,t)}{\partial x} \geqslant 0$ and $\frac{\partial U(x,t)}{\partial x} \leqslant 0$, we have $e^{1*} \geqslant 0$ and $e^{2*} \geqslant 0$. □

Lemma B.0.2. *There exists an* $\varepsilon > 0$ *such that when* $\Delta t/(\Delta x)^2 < \varepsilon$*, the parameters in the numerical recurrence equations 3.14 are always positive.*

Proof. For the Z_j^n equation, the claim is obvious since $\frac{\Delta t \sigma^2}{2\Delta x} > 0$ and we just need $\Delta t/(\Delta x)^2 < \sigma^{-2}$ so that $1 - \frac{\Delta t \sigma^2}{(\Delta x)^2} > 0$.

Note that since $1 + \frac{\Delta t}{32 b^2 \sigma^2}\left(\frac{Q_{j+1}^{n+1} - Q_j^{n+1}}{\Delta x}\right)^2 > 1$, we have $Z_j^n < \left(1 - \frac{\Delta t \sigma^2}{(\Delta x)^2}\right) Z_j^{n+1} +$

$\frac{\Delta t\sigma^2}{2\Delta x}\left(Z_{j+1}^{n+1}+Z_{j-1}^{n+1}\right)$, where the right hand side of the inequality is a convex combination between Z_j^{n+1}, Z_{j+1}^{n+1}, and Z_{j-1}^{n+1} when $\Delta t/(\Delta x)^2<\sigma^{-2}$. Therefore, $Z_j^n<\max\{Z_j^{n+1},Z_{j+1}^{n+1},Z_{j-1}^{n+1}\}$. Repeat the iteration in n, by the 1-to-1 correspondence between P and Z and the boundary condition of P, it is easy to show that $1<Z_j^n<\exp\left(\frac{3}{4b\sigma^2}\right)$ for all n and j. As a result, $\left|\frac{Z_{j+1}^{n+1}-Z_j^{n+1}}{Z_j^n}\right|<\exp\left(\frac{3}{4b\sigma^2}\right)-1\equiv\eta$.

For the Q_j^n equation, we rearrange it in the following way,

$$\begin{aligned}Q_j^n &= \left(1-\frac{\Delta t\sigma^2}{3(\Delta x)^2}\frac{Z_{j+1}^{n+1}-Z_j^{n+1}}{Z_j^n}-\frac{\Delta t\sigma^2}{(\Delta x)^2}\right)Q_j^{n+1}\\ &\quad+\frac{\Delta t\sigma^2}{6(\Delta x)^2Z_j^n}\left(2Z_{j+1}^{n+1}-2Z_j^{n+1}+3Z_j^n\right)Q_{j+1}^{n+1}+\frac{\Delta t\sigma^2}{2(\Delta x)^2}Q_{j-1}^{n+1}.\end{aligned}$$

Obviously, the parameter of Q_{j-1}^{n+1} is positive. And when

$$\frac{\Delta t}{(\Delta x)^2}<\frac{1}{\sigma^2\left(\frac{\eta}{3}+1\right)},$$

it is straightforward to verify that the parameter of Q_j^{n+1} is positive. As for the sign of the parameter of Q_{j+1}^{n+1}, i.e. $sign\left(2Z_{j+1}^{n+1}-2Z_j^{n+1}+3Z_j^n\right)$,

$$\begin{aligned}2Z_{j+1}^{n+1}-2Z_j^{n+1}+3Z_j^n &> 2Z_{j+1}^{n+1}-2Z_j^{n+1}+3\left(1-\frac{\Delta t\sigma^2}{(\Delta x)^2}\right)Z_j^{n+1}+\frac{3\Delta t\sigma^2}{2\Delta x}\left(Z_{j+1}^{n+1}+Z_{j-1}^{n+1}\right)\\ &= \left(2+\frac{3\Delta t\sigma^2}{2\Delta x}\right)Z_{j+1}^{n+1}+\left(1-\frac{3\Delta t\sigma^2}{(\Delta x)^2}\right)Z_j^{n+1}+\frac{3\Delta t\sigma^2}{2\Delta x}Z_{j-1}^{n+1}.\end{aligned}$$

Thus, $2Z_{j+1}^{n+1}-2Z_j^{n+1}+3Z_j^n>0$ when $\Delta t/(\Delta x)^2<\frac{1}{3}\sigma^{-2}$.

In summary, the claim is satisfied when

$$\frac{\Delta t}{(\Delta x)^2}<\min\{\frac{1}{\sigma^2\left(\frac{\eta}{3}+1\right)},\frac{1}{3}\sigma^{-2}\}\equiv\varepsilon. \tag{B.1}$$

$\square$

Lemma B.0.3. *When inequality B.1 holds, the numerical scheme 3.14 converges uniformly to the viscosity solution of the Partial Differential Equation 3.13 as* $\Delta x \to 0$ *and* $\Delta t \to 0$.

Proof. Per existing theories of viscosity solutions and numerical methods for nonlinear PDE (See Barles and Souganidis (1991); Barles (1997); Forsyth and Labahn (2007); Forsyth and Vetzal (2012)), when the numerical scheme satisfies the conditions of stability, consistency and monotonicity (see literature mentioned above for detailed definitions), the solution of the numerical scheme converges uniformly to the viscosity solution of the corresponding PDE. Intuitively speaking, stability means whatever the size of the grid (i.e. Δx and Δt) used in the numerical scheme, the resulting numerical solutions should be relatively close to each other; consistency requires the numerical solution converges to the viscosity solution of the PDE when $\Delta x \to 0$ and $\Delta t \to 0$; and monotonicity maintains the elliptic property of the PDE in the discrete numerical sense.

Mathematically, stability requires the numerical solution to be uniformly bounded by a bound independent of the choice of Δx and Δt. We have shown in the previous lemma that all Z_j^n are uniformly bounded by a constant. By a similar argument, since the right hand side of the Q_j^n equation in the numerical scheme is a convex combination between Q_j^{n+1}, Q_{j+1}^{n+1} and Q_{j-1}^{n+1}, we know $Q_j^n < \max\{Q_j^{n+1}, Q_{j+1}^{n+1}, Q_{j-1}^{n+1}\}$. Repeat this argument in n and j, it is easy to show all Q_j^n are also uniformly bounded by a constant.

The consistency condition is clearly satisfied because the truncation error is of order two in space and order one in time. A formal proof can be done using Taylor expansion.

Finally, as shown in the previous lemma, when inequality B.1 holds, all parameters in the numerical scheme are positive and the monotonicity condition follows immediately. □

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