Exact Lagrangian Fillings of Legendrian Links and Weinstein 4-manifolds

by

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Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
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Abstract

One approach to studying symplectic manifolds with contact boundary is to consider Lagrangian submanifolds with Legendrian boundary; in particular one can study exact Lagrangian fillings of Legendrian links. There are still many open questions on the spaces of exact Lagrangian fillings of Legendrian links in the standard contact 3-sphere, and one can use Floer theoretic invariants to study such fillings. In this thesis we prove that a family of oriented Legendrian links has infinitely many distinct exact orientable Lagrangian fillings which are smoothly isotopic but not smoothly isotopic. To distinguish these fillings we use Floer theoretic techniques developed by Casals and Ng. We provide one of the first examples of a Legendrian link that admits infinitely many planar exact Lagrangian fillings. As part of a collaboration, we also explore obstructions to the existence of exact Lagrangian cobordisms between Legendrian links that can be applied to obstructing certain immersed exact Lagrangian fillings.

Weinstein domains are examples of a symplectic manifold with contact boundary that have a handle decomposition compatible with the symplectic structure of the manifold. Weinstein 4-dimensional domains can be represented with Weinstein handlebody diagrams of Legendrian links in $(\#^m(S^1 \times S^2), \xi_{std})$ or (S^3, ξ_{std}). Studying the symplectic topology of Weinstein domains has allowed for new perspectives when studying various manifolds including complex affine varieties. We study the Milnor fibers M_f of isolated unimodular singularities. Keating constructed an exact Lagrangian torus in M_f. We show that there are exact infinitely many Hamiltonian non-isotopic Lagrangian tori in M_f using Weinstein handlebody diagrams and exact Lagrangian fillings of Legendrian links. We also show that M_f contains a new infinite set of symplectically knotted Lagrangian spheres. Additionally, we provide a generalization of a criterion for when the symplectic homology of a Weinstein 4 manifold is non-vanishing given a Weinstein handlebody diagram. Finally, we provide a summary of a second collaboration which studies the complement of smoothings of toric divisors in toric 4-manifolds. We show that for certain smoothings, these complements have a particular Weinstein structure, and we provide an algorithm to construct the Weinstein handlebody diagram of such complements.

To my mother, Catherine Searle,

I love you to the moon and back uncountably many times

and in memory of my grandmother Harriette Burns Myer Searle

and my nonna Anna Maria Orsola Chiariglione Capovilla.

Contents

List of Figures

Acknowledgements

I would first like to thank my family for all their love and support. A special thank you to my mother, Catherine Searle for all of the thousand ways that she has supported me and encouraged me.

I would like to thank Lenhard Ng for his guidance and support throughout my time at Duke. In particular, for introducing me to various interesting topics, illuminating conversations and for his patience and encouragement.

I would like to thank Bahar Acu, Roger Casals, Agnès Gadbled, Ailsa Keating, Noémie Legout, Adam Levine, Caitlin Leverson, Chester Lian, Maÿlis Limouzineau, Aleksandra Marinković, Chindu Mohanakumar, Ziva Myers, Emmy Murphy, Seppo Niemi-Colvin, Yu Pan, Greg Parker, Aarthi Sridhar, Laura Starkston, Lisa Traynor, and Angela Wu for useful discussions. I am also grateful to Lisa Traynor for introducing me to research as an undergraduate and to the topic of Legendrian links. I would also like to thank Roger Casals for suggesting I look for infinitely many Lagrangian tori in the Weinstein handlebody diagrams of the Milnor fibers I was studying.

I would like to thank my collaborators Bahar Acu, Agnes Gadbled, Daniel Irvine, Noemie Legout, Maÿlis Limouzineau, Aleksandra Markinković, Emmy Murphy, Ziva Myers, Laura Starkston, Lisa Traynor, and Angela Wu. These collaborations have been a delight.

I am grateful to the National Science Foundation which supported this work through the NSF Graduate Research Fellowship under grant $DGE-1644868$ and also partially through NSF grant $DMS - 1707652$. I am also grateful to Duke University for the Deans graduate fellowship and to University Scholars program. I would like to thank the Mellon Mays Foundation and my Mellon cohort for supporting me as an undergraduate student, and helping me work towards an academic career.

I want to thank my friends and peers in the Duke mathematics department without whom I would not have made it through this graduate program. I especially would like to thank my cohort, Seppo Niemi-Colvin, Ran Huo, Ruby Kim, Chester Lian, Caitlin Leverson, Chindu Mohanakumar, Rob Ravier, and Yu Pan. I would also like to thank the EDGE foundation, and my EDGE 2015 cohort and instructors. Finally, I would like to thank my friends who encouraged and supported me in Durham and from all over. I am incredibly lucky to have such amazing friends, and to have met lifelong friends during my time in Durham.

1

Introduction

Symplectic topology arose from the study of Hamiltonian dynamics and has emerged as an important field that has contributed to key developments in low-dimensional topology, geometry and mathematical physics. A **symplectic manifold** (X, ω) is a smooth $2n$-dimensional manifold X, equipped with a closed non-degenerate bilinear form ω. One way to study a symplectic manifold is through its **Lagrangian submanifolds**, $L \subset (X, \omega)$, n-dimensional submanifolds on which the symplectic form vanishes, i.e. $\omega|_{TL} = 0$. A cousin of symplectic topology is contact topology which studies **contact manifolds** (Y, ξ), smooth $(2n-1)$-dimensional manifolds equipped with a **contact structure**, ξ. A contact structure ξ is a maximally non-integrable hyperplane field. Locally, a contact structure is the kernel of a 1-form α such that $\alpha \wedge (d\alpha)^n \neq 0$. A submanifold $\Lambda \subset Y$ can have its tangent space contained in the hyperplanes of the contact structure ξ only if $dim(\Lambda) \leqslant n$. **Legendrian submanifolds** of Y are maximal submanifolds of dimension n such that $T_p\Lambda \subset \xi_p$ for all $p \in \Lambda$. One example of a contact 3 manifold is $\mathbb{R}^3$ with the standard contact structure given by $ker(dz - ydx)$, which we will denote by $\mathbb{R}^3_{std}$. Legendrian submanifolds in $\mathbb{R}^3_{std}$ are topologically links. Both symplectic and contact manifolds have no local invariants thanks to the Darboux Theorems, but one can define global invariants using Lagrangian submanifolds, and Legendrian submanifolds respectively. This approach to studying symplectic and contact manifolds still presents various challenges as Lagrangian submanifolds are not classified in general, and there are many open questions on Legendrian submanifolds.

The geometries of contact and symplectic manifolds are closely related. For instance, the symplectic and contact conditions are compatible enough that one can consider symplectic manifolds whose boundary is a contact manifold. Weinstein domains, certain symplectic manifolds with a handlebody decomposition that is compatible with the symplectic structure, are examples of symplectic manifolds with a contact boundary. Lagrangian cobordisms with Legendrian boundary are the relative version of closed Lagrangians, and are a natural object to consider in symplectic manifolds with contact boundary. In the case of Weinstein domains, their the symplectic topology is encoded by Legendrian submanifolds. Therefore, studying Legendrian submanifolds provides a new perspective on high dimensional symplectic manifolds. In this thesis we study Lagrangian cobordisms of Legendrian links $\Lambda \subset \mathbb{R}^3_{std}$ using Legendrian invariants, and construct new examples of **symplectically knotted** Lagrangian surfaces with Legendrian boundary, that is, Lagrangian surfaces that are smoothly isotopic but not Hamiltonian isotopic. We will also apply our results to construct symplectically knotted closed Lagrangian surfaces in 4-dimensional Milnor fibers of isolated hypersurface singularities. We will also summarize two collaborations that further explore questions on Lagrangian cobordisms and Weinstein 4-manifolds, demonstrating additional aspects of the rich interplay between Legendrian submanifolds and symplectic 4-manifolds.

There are two classical Legendrian link invariants, the Thurston-Bennequin number and rotation number, which can distinguish infinitely many distinct Legendrian links up to Legendrian isotopy for every topological smooth link type $\mathcal{K}$.

However, these classical Legendrian link invariants do not distinguish all Legendrian links. In the last two decades new Legendrian link invariants have been defined using $\mathcal{J}$-holomorphic curves. First introduced by Gromov [44], $\mathcal{J}$-holomorphic curves in symplectic manifolds have played a pivotal role in the development of symplectic and contact topology. By considering the symplectization of contact manifolds one can define Legendrian invariants with $\mathcal{J}$-holomorphic curves such as the **Legendrian contact homology Differential Graded Algebra** (DGA), $(\mathcal{A}(\Lambda; R), \partial)$ [19, 31]. This invariant has a combinatorial definition for Legendrian links in $\mathbb{R}^3_{std}$ and $(\#^k(S^1 \times S^2), \xi_{std})$ [30, 35, 58].

Consider two Legendrian links $\Lambda_\pm \subset \mathbb{R}^3_{std}$. An **exact Lagrangian cobordism**, Σ, from Λ_- to Λ_+ is an embedded Lagrangian surface in the symplectization $(\mathbb{R}_t \times \mathbb{R}^3, d(e^t(dz - ydx)))$ that is cylindrical on the ends, and such that the 1-form $e^t(dz - ydx)$ is exact on Σ. A cobordism from the empty set to Λ is called a **filling** of Λ. There is a natural Topological Quantum Field Theory-like structure, in the spirit of Symplectic Field Theory [32], given by a category whose objects are Legendrian links and whose morphisms are exact Lagrangian cobordisms. This category behaves functorially with respect to $\mathcal{J}$-holomorphic curve invariants. In particular, exact Lagrangian cobordisms from Λ_- to Λ_+ induce a DGA map $\mathcal{A}(\Lambda_+; R) \to \mathcal{A}(\Lambda_-; R)$ [28]. One can also obtain new Legendrian link invariants by considering **augmentations** ϵ of $\mathcal{A}(\Lambda)$, these are DGA maps $\epsilon : (\mathcal{A}(\Lambda; R), \partial) \to (R, 0)$. By functoriality of the DGA, embedded exact Lagrangian fillings of Λ produce augmentations of $(\mathcal{A}(\Lambda; R), \partial)$. Although not all augmentations come from such embedded fillings, they do come from immersed exact Lagrangian fillings [63].

The space of exact Lagrangian fillings for a Legendrian link Λ is still not understood for many Legendrian links. In comparison to the case of smooth cobordisms which give an equivalence relation for the set of smooth knots, the relation between two Legendrians given by the existence of an exact Lagrangian cobordism is transitive but not symmetric. Another example of the rigidity of exact Lagrangian cobordisms is that all exact Lagrangian fillings Σ of a Legendrian Λ realize the smooth embedded 4-ball genus of Λ [15]. Given a Legendrian link, one can construct an exact Lagrangian cobordism using Legendrian isotopy, and pinch moves (saddle moves) [28]. Such exact Lagrangian cobordisms are called **decomposable** and it is still unknown whether all exact Lagrangian cobordisms in the symplectization of $\mathbb{R}^3_{std}$ are decomposable or not. Exact Lagrangian cobordisms have a **Maslov grading** associated to them; if they are Maslov 0 then they are orientable. For the purposes of this thesis, all Lagrangian cobordisms and fillings are assumed to be exact and Maslov 0 unless otherwise stated. Additionally, exact Lagrangian fillings are equivalent if there exists a Hamiltonian isotopy between them which fixes the ends. We will study families of Legendrian links and show that these Legendrian links have infinitely many distinct exact Maslov 0 Lagrangian fillings up to Hamiltonian isotopy.

It has been remarkably difficult to classify exact Lagrangian fillings of Legendrian links in $\mathbb{R}^3_{std}$. Indeed, the only Legendrian link for which we have a complete classification is the maximal-tb Legendrian unknot which has a unique exact Lagrangian disk filling [33]. One of the few other Legendrian links for which there is an approximate count of exact Lagrangian fillings are the maximal-tb Legendrian $(2, n)$ torus links. These Legendrian links have at least $C_n = \frac{1}{n+1}\binom{2n}{n}$ exact Lagrangian fillings which were first constructed with pinch moves [28], and then shown to be distinct using augmentations by Pan [62]. Shende, Treumann, Williams and Zaslow have also constructed C_n distinct exact Lagrangian fillings of the Legendrian $(2, n)$ torus links using cluster varieties [70]. Until 2020 it was not known whether there exists a Legendrian link that has infinitely many distinct exact Lagrangian fillings up to Hamiltonian isotopy. In [11] Casals and Gao showed using the theory of microlocal sheaves that there are Legendrian links with infinitely many distinct exact Lagrangian fillings up to Hamiltonian isotopy. That same year Gao, Shen, and Weng [37, 38], Casals and Zaslow [14], Casals and Ng [13] found more such examples. Casals and Zaslow also produced examples of Legendrian 2-dimensional spheres in $\mathbb{R}^5_{std}$ which have infinitely many Lagrangian fillings [14]. See Chapter 3 for a summary of all currently known counts of exact Maslov 0 Lagrangian fillings of Legendrian links $\Lambda \subset \mathbb{R}^3_{std}$. Many of the examples that have been constructed are also examples of symplectically knotted surfaces with fixed Legendrian boundary.

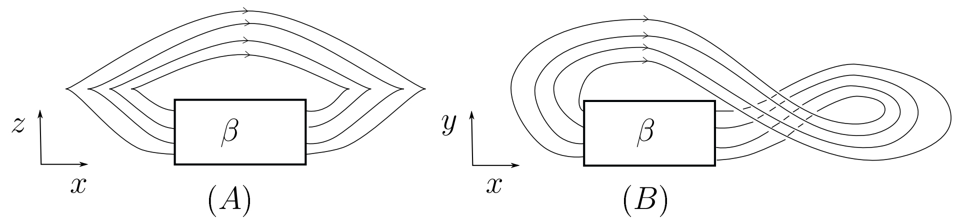


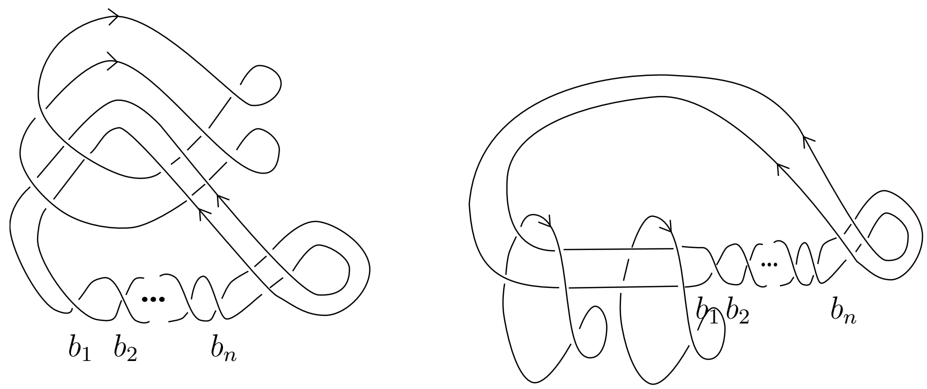
FIGURE 1.1: (A) The front projection of the rainbow closure of a positive braid β. (B) The Lagrangian projection of the (-1) closure of a positive braid β.

Up until now the Legendrian links with infinitely many distinct Lagrangian fillings have been either the rainbow or (-1) closure of a positive braid. The rainbow closure of a positive braid $\beta \in Br_n^+$ has the front projection shown in the leftmost diagram in Figure 1.1. The (-1) closures of a positive braids $\beta \in Br_n^+$ is obtained by placing β in a standard contact neighborhood of the standard maximal-tb Legendrian unknot in $\mathbb{R}^3_{std}$, as shown in the rightmost diagram in Figure 1.1. The cluster algebra and microlocal sheaf techniques used to differentiate exact Lagrangian fillings in [11, 14, 37, 38] are constrained to studying rainbow closures of positive braids. In contrast, the Floer theoretic methods developed by Casals and Ng [13] are not constrained to this class of Legendrians. They focus on (-1)-closures of positive braids because some of the Floer theoretic computations can be streamlined for this class of Legendrians. We consider a family of oriented Legendrian links that are not rainbow or (-1) closures of positive braids and show they admit infinitely many distinct exact Lagrangian fillings. We distinguish these fillings using augmentations following the techniques developed in [13].

Theorem 22, see page 48.. *The oriented Legendrian links $\Lambda_n \in \mathbb{R}^3_{std}$ shown in the rightmost diagram of Figure 1.2 have infinitely many exact Maslov* 0 *Lagrangian fillings up to Hamiltonian isotopy for $n \geqslant 1$ that are all smoothly isotopic.*

The Legendrian links Λ_1 and Λ_2 have genus 0 exact Lagrangian fillings, and are some of the first examples of a Legendrian link admitting infinitely many distinct planar exact Lagrangian fillings. This makes progress on a question of Casals and Ng, who asked whether there exists a Legendrian knot with infinitely many planar exact Maslov 0 Lagrangian fillings. The Legendrian link Λ_1 is also the smallest known Legendrian with infinitely many fillings, in terms of its Thurston-Bennequin number, its number of link components and the genus of the fillings. We also show that the Legendrian links Λ_1 has an infinite order Legendrian isotopy loop. A Legendrian isotopy loop is a loop $\Phi : (S^1, pt) \to (\mathcal{L}(\Lambda), \Lambda)$, where $\mathcal{L}(\Lambda)$ denotes the space of Legendrian links Legendrian isotopic to Λ with a basepoint given by an arbitrary Legendrian representative of Λ. Another implication of Theorem 22 is that the Legendrian knot $12n_{293}$ has infinitely many exact Lagrangian fillings. We will also be considering the Legendrian links Λ'_m for $m \geqslant 1$ shown to the right in Figure 1.2. These Legendrian links Λ'_m were first studied by Casals and Ng who proved that they had infinitely many distinct exact Lagrangian fillings for $m \geqslant 1$.

Exact Lagrangian fillings of a Legendrian Λ allow us to construct closed, exact Lagrangian surfaces in Weinstein 4-manifolds. **Weinstein manifolds** are Liouville manifolds equipped with a Morse function that is compatible with the Liouville structure. See Section 2.4 for more precise definitions. A Liouville domain $(X, \omega = d\alpha)$ with boundary has a Liouville vector field Z that is transverse to the boundary $\partial X = Y$ and whose flow exponentially expands ω ($\mathcal{L}_Z\omega = \omega$), so ∂X has a contact structure induced by Z. A Weinstein domain is a Liouville domain with a Morse function ϕ that is gradient-like with respect to Z. Weinstein domains can be com-



(A) (B)

Figure 1.2: Legendrian links (A) Λ_n, and (B) Λ'_n that both contain sublinks that are the (-1) closure of $\sigma_1^n \in Br_2^+$ for $n \geqslant 1$.

pleted to a Weinstein manifold by adding a cylindrical end $\partial X \times \mathbb{R}_+$ to the boundary. Cotangent bundles of closed smooth manifolds, and Stein manifolds such as complex affine varieties, are examples of Weinstein manifolds. The handlebody of a Weinstein manifold (X, ω, Z, ϕ), has handles of index $k \leqslant \dim(X)/2$ with a symplectic structure and partial contact boundary [74]. Each symplectic k-handle is attached along isotropic spheres. If $k = \dim(X)/2$, the k-handle is called a *critical* handle and is attached along a Legendrian sphere. Thus, Weinstein handlebody decompositions can be represented by the projections of the Legendrian attaching spheres. We call these Weinstein handlebody diagrams. It is difficult to produce explicit handlebody diagrams for Weinstein manifolds, and the collection of Weinstein manifolds where one can do this is currently somewhat limited. One can compute various symplectic and topological invariants from the Weinstein handlebody diagrams that may be otherwise more difficult to compute, such as the fundamental group, homology, the intersection form of closed 4-manifold, symplectic homology and the Wrapped Fukaya category. Weinstein handlebody diagrams also give access to sometimes unexpected equivalences using Legendrian Kirby calculus.

Explicit Weinstein handlebody diagrams can be used to construct closed exact

Lagrangian submanifolds. Let X_Λ be a Weinstein 4-manifold constructed with a 2-handle attached along a Legendrian link Λ in (S^3, ξ_{std}) or $(\#^m(S^1 \times S^2), \xi_{std})$. Then, the union of the Lagrangian core disk of the 2-handle and an exact Lagrangian filling Σ of Λ is a closed exact Lagrangian surface in $\bar{\Sigma} \subset X_\Lambda$. Casals and Ng proved (see Proposition 27 on page 53) that if two fillings Σ_1 and Σ_2 of Λ have distinct induced **restricted** augmentations, then $\overline{\Sigma_1}$ is not Hamiltonian isotopic to $\overline{\Sigma_2}$ in X_Λ. Restricted augmentations are augmentations satisfying an additional algebraic condition, see Definition 25. In Corollary 31 we show that both Λ_n and Λ'_m for $n, m \geqslant 1$ have infinitely many distinct restricted augmentations induced by fillings. Thus, we arrive at the following Corollary:

Corollary 32, see page 69.. *Let X be a Weinstein manifold that has a Weinstein handlebody diagram containing either of the Legendrian sublinks Λ_n or Λ'_m for $n, m \geqslant 1$. Then X contains infinitely many Hamiltonian non-isotopic Lagrangian surfaces that are all smoothly isotopic and which intersect the Lagrangian co-core disks of the 2-handles corresponding to the sublink once.*

We will apply Corollary 32 to **Milnor fibers of isolated hypersurface singularities** which are Weinstein manifolds that have been extensively studied in homological mirror symmetry, singularity theory and low-dimensional topology. An isolated hypersurface singularity is the equivalence class of the germ of a holomorphic function, $f : \mathbb{C}^{n+1} \to \mathbb{C}$, whose differential has an isolated zero at the origin. The Milnor fiber of such a singularity is the smooth manifold $M_f = f^{-1}(\epsilon_\delta) \cap B_\delta(0)$ for suitably small ϵ and δ [57]. Isolated hypersurface singularities of modality 0 and 1 have been classified [5]. The modality of a singularity f can be thought of as the dimension of a parameter space covering a neighborhood of f in the space of singularities after the appropriate holomorphic reparametrization. The only possible exact Lagrangian submanifolds of 4-dimensional Milnor fibers of modality 0 singularities are spheres [2, 48]. In contrast, Keating [51] showed that there exists an exact Lagrangian torus in the 4 dimensional Milnor fiber of any unimodular singularity. In the process of constructing this Lagrangian torus, she provides explicit Lefschetz fibrations of the 4-dimensional affine varieties $T_{p,q,r} = \{(x, y, z) \in \mathbb{C}^3 \mid x^p + y^q + z^r + xyz = 1\}$. For $p, q, r \in \mathbb{Z}_{\geqslant 1}$ such that $p, q, r \in \mathbb{Z}_{\geqslant 0}$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leqslant 1$, Keating showed that $T_{p,q,r}$ is symplectomorphic to the 4-dimensional Milnor fiber of a hyperbolic singularity.

Recent work of Shende, Treumann and Williams [69], outlines a proof that there are in fact infinitely many distinct Lagrangian tori that are not Hamiltonian isotopic in $T_{p,q,r}$ for $p, q, r \geqslant 1$. These tori, distinguished by cluster algebra techniques, are constructed from fillings of Legendrians in distinct Weinstein handlebodies of $T_{p,q,r}$ that are symplectomorphic but not yet shown to be Weinstein homotopic. This raises the question of whether one can find infinitely many distinct Lagrangian tori via a single Weinstein handle decomposition of $T_{p,q,r}$ that has an attaching Legendrian link with infinitely many distinct Lagrangian genus 1 fillings. The first step towards answering this question is to produce a Weinstein handlebody diagram of $T_{p,q,r}$.

One useful algorithm that produces Weinstein handlebody diagrams is Casals and Murphy's [12] affine dictionary. This algorithm allows one to obtain the Weinstein handlebody diagram of a Weinstein manifold from particular Lefschetz fibrations. Thus, one can apply the affine dictionary to the Lefschetz fibration of $T_{p,q,r}$ given by Keating to obtain a Weinstein handlebody diagram of $T_{p,q,r}$. This was first done in [12], but an issue in the translation of this Lefschetz fibration resulted in an incorrect handlebody diagram, and we provide the correction here.

Proposition 39, see page 80.. *The Weinstein 4-dimensional domains*

$$T_{p,q,r} = \{(x, y, z) \in \mathbb{C}^3 \mid x^p + y^q + z^r + xyz = 1\}$$

have the Weinstein handlebody diagram shown in Figure 1 for $p, q, r \geqslant 0$.

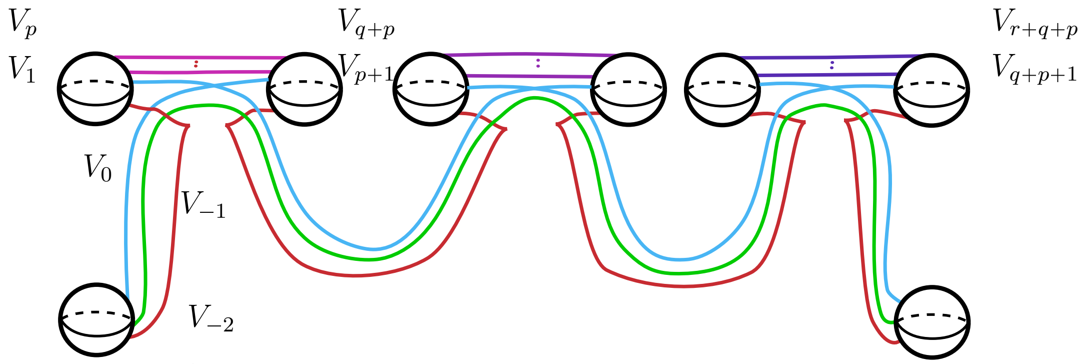


FIGURE 1.3: A Weinstein handlebody diagram of the Milnor fiber of a $T_{p,q,r}$-singularity where $p, q, r \geqslant 0$.

We apply Legendrian Kirby calculus moves to the Weinstein handlebody diagram shown in Figure 1 to find another Weinstein handlebody diagram of $T_{p,q,r}$ for $p, q \geqslant 1$ and $r \geqslant 3$ that contains the Legendrian sublink Λ_1'. This implies that $T_{p,q,r}$ has infinitely many distinct Lagrangian tori since Λ_1' has infinitely many distinct exact genus 1 Lagrangian fillings which are smoothly isotopic. We also find another Weinstein handlebody diagram of $T_{p,q,r}$ for $p, q, r \geqslant 2$ with Λ_2 as a sublink. As a result we deduce:

Theorem 40, see page 83.. *For any $p, r \geqslant 1$, and $q \geqslant 3$, the Weinstein 4-manifold $T_{p,q,r}$ contains infinitely many Hamiltonian non-isotopic Lagrangian tori that are all smoothly isotopic. For any $p, q, r \geqslant 2$, the Weinstein 4-manifold $T_{p,q,r}$ contains infinitely many Hamiltonian non-isotopic Lagrangian spheres that are all smoothly isotopic and intersect the Lagrangian co-cores once.*

Note that by our construction, the Milnor fiber $T_{p,q,r}$ contains infinitely many distinct exact Lagrangian spheres and tori which intersect the co-cores of the critical handles at most once. It was already known that these Milnor fibers contain infinitely many distinct exact Lagrangian spheres that are smoothly isotopic. Let (M, ω) be a compact symplectic manifold with contact boundary such that $[\omega] = 0$ and $2c_1(M, \omega) = 0$ that contains an A_3 configuration of Lagrangian spheres. Then, Seidel [67] proved that there are infinitely many symplectically knotted Lagrangian spheres in (M, ω). Three Lagrangian spheres L_0, L_1, L_2 are in an A_3 configuration if $|L_i \cap L_j| = 1$ if $i - j = \pm 1$ and $|L_0 \cap L_2| = 0$. Seidel constructs these symplectically knotted Lagrangian spheres by taking generalized Dehn twists and distinguishes them by their Floer homology grading. For the Milnor fiber $T_{p,q,r}$ with $p \geqslant 3$, one can easily find the A_3 configuration of Lagrangian spheres as follows. The Weinstein handlebody diagram of $T_{p,q,r}$ shown in Figure 4.6 has a 3 component Legendrian sublink of max-tb unknots $\Lambda^{(0)} \cup \Lambda^{(1)} \cup \Lambda^{(2)}$ if p, q, or r are greater than or equal to 2. The union of the Lagrangian cores of these link components and their unique exact Lagrangian disk fillings is a set of three Lagrangian spheres L_0, L_1, L_2 in an A_3 configuration. Then, thanks to Seidel, we know that $\tau_{L_2}^{2k}(L_1)$ are distinct Lagrangian spheres up to Hamiltonian isotopy. By definition of a generalized Dehn twist, these Lagrangian spheres intersect the co-cores of the critical attaching handles $2k$-times.

Finally, applying properties of Milnor fibers of adjacent singularities to Theorem 40 we arrive at the following more general result:

Corollary 41, see page 87.. *Suppose that M_f is the Milnor fiber of a positive modality isolated hypersurface singularity f, then M_f contains infinitely many Hamiltonian non-isotopic exact Maslov 0 Lagrangian tori that are smoothly isotopic and infinitely many Hamiltonian non-isotopic exact Lagrangian spheres that are smoothly isotopic.*

An important invariant of compact symplectic manifolds (X, ω) is the Fukaya category. The objects of the Fukaya category are exact Lagrangian submanifolds, the morphisms are intersections of said Lagrangian submanifolds and the higher morphisms are given by counts of moduli spaces of $\mathcal{J}$-holomorphic curves. For non-compact symplectic manifolds we also have the Wrapped Fukaya category, denoted by $\mathcal{WF}(X)$. The objects of $\mathcal{WF}(X)$ are compact Lagrangian submanifolds, and non-compact Lagrangian manifolds that are conical at infinity. For a Weinstein 4-

manifold X_Λ obtained by attaching a 2-handle along a Legendrian link $\Lambda \subset (\#^m(S^1 \times S^2), \xi_{std})$, $\mathcal{WF}(X_\Lambda)$ corresponds to modules over the DGA $(\mathcal{A}(\Lambda; R), \partial)$ [9, 25, 29]. Keatings result, [51] implies that there are Milnor fibers whose Fukaya category is not generated by any collection of vanishing cycles. Theorem 41 implies there are Milnor fibers M_f such that the Fukaya category of M_f potentially have an infinite number of generators which are not generated by the vanishing cycles. In comparison, Seidel showed that the Fukaya category of the Milnor fiber of simple singularities and certain singularities given by weighted polynomials is generated by the vanishing cycles [67, 68]

One can also compute the **Symplectic homology** of a Weinstein manifold X_Λ using the Legendrian contact homology DGA $\mathcal{A}(\Lambda; \partial)$ [9, 25]. We extended the criterion first given by Leverson for determining when the symplectic homology of a Weinstein manifold, denoted by $S\mathbf{H}$, does not vanish [55]. Symplectic homology is a difficult invariant to compute, and this extension gives an easy way to check whether it vanishes for a slightly broader class of Weinstein manifolds.

Theorem 43 see page 43. *Let X_Λ be the Weinstein 4 manifold resulting from attaching 2 handles along a Legendrian link $\Lambda = \Lambda^{(1)} \cup \cdots \cup \Lambda^{(n)} \subset (\#^m(S^1 \times S^2), \xi_{std})$. If there is any sublink $\Lambda^{(1)} \cup \cdots \cup \Lambda^{(l)}$ for $l < n$, such that its differential graded algebra has a representation*

$$\rho : (\mathcal{A}(\Lambda^{(1)} \cup \cdots \cup \Lambda^{(l)}; \mathbb{Z}[t_1^{\pm 1} \ldots, t_l^{\pm 1}]), \partial) \to End(V)$$

where V is a vector space over $\mathbb{Q}$ and $\rho(t_k) = -Id$ for $k = 1, \ldots, l$, then $S\mathbf{H}(X_\Lambda) \neq 0$.

An important theme in symplectic and contact topology is understanding flexibility phenomena which refers to when an object or situation is governed only by homotopy data. Flexible Weinstein manifolds abide by an h-principle [20] and it is often difficult to determine whether a Weinstein manifold is flexible or not. Since a flexible Weinstein manifold has vanishing symplectic homology, Theorem 43 also gives a criterion for when a Weinstein manifold is flexible.

Finally, in the last two chapters we provide summaries of the results from two collaborations that took place during the authors graduate studies and which also investigated questions on exact Lagrangian fillings and Weinstein 4-manifolds. In a collaboration with Legout, Limouzineau, Murphy, Pan, and Traynor we studied obstructions to the existence of exact Maslov 0 Lagrangian cobordisms of Legendrian links in $\mathbb{R}^3_{std}$ using augmentations. An important category associated to augmentations is the **Augmentation Category** $Aug_+(\Lambda; R)$, a unital A_∞ category defined in [59] over a unital ring R. The objects of $Aug_+(\Lambda; R)$ are augmentations of Λ over R. There exists a bijection from $Aug_+(\Lambda; R)$ to a category of constructible sheaves defined by [71]. The DGA map $f : \mathcal{A}(\Lambda_+; R) \to \mathcal{A}(\Lambda_-; R)$ induced by an exact Lagrangian cobordism from Λ_- to Λ_+ can be composed with an augmentation ϵ_- of Λ_- to obtain an augmentation ϵ_+ of Λ_+. Therefore, there is a map between the appropriate set of augmentations $f^* : Aug(\Lambda_-; R) \to Aug(\Lambda_+; R)$. When $\Lambda_\pm$ are single component knots, Pan showed that f^* is an injective category map [61]. We extended Pan's result on the injectivity of f^* to Legendrian links using classical Floer techniques based on work of [17] and [54].

Theorem 48, see page 103. (Capovilla-Searle, Legout, Limouzineau, Murphy, Pan, Traynor [10])**.** *Let $\Lambda_\pm$ be Legendrian links in $\mathbb{R}^3_{std}$, and let $R = \mathbb{Z}_2$ denote a field. If there exists an exact, embedded, Maslov 0 Lagrangian cobordism from Λ_- to Λ_+, then the induced map $\mathcal{F} : Aug(\Lambda_-; R) \to Aug(\Lambda_+; R)$ is injective.*

We then obtained as a corollary obstructions in terms of normalized counts of augmentations (denoted by $\#Aug_+$), linearized contact homology, normal ruling polynomials, and other invariants related to the Augmentation Category.

Corollary 50, see page 104. (Capovilla-Searle, Legout, Limouzineau, Murphy, Pan, Traynor [10]). *Let $\Lambda_\pm$ be Legendrian links in $\mathbb{R}^3_{std}$, and let $R = \mathbb{Z}_2$. If there exists an exact, embedded, Maslov 0 Lagrangian cobordism from Λ_- to Λ_+, then*

$$\# Aug_+(\Lambda_+; R) \geqslant \# Aug_+(\Lambda_-; R).$$

We also showed that any immersed exact Lagrangian Maslov 0 filling of Λ with k vanishing action double points implies the existence of an exact Lagrangian cobordism from k disjoint copies of the positive Hopf link to Λ. Therefore Corollary 50 can be applied to obstruct specific immersed exact Lagrangian fillings of Λ:

Theorem 46 see page 100. (Capovilla-Searle, Legout, Limouzineau, Murphy, Pan, Traynor [10]). *Given any $g \in \mathbb{Z}^+$ and $p \in \mathbb{Z}_{\geqslant 0}$, there is a Legendrian link in $\mathbb{R}^3_{std}$ which has an immersed, oriented, Maslov 0 filling Σ with genus g and p double points but no exact Maslov 0 Lagrangian filling with genus $g-1$ and $p+1$ double points such that one of the double points has vanishing index and action.*

Lagrangian surgery on a double point gives a way to replace a double points with a Lagrangian handle [53, 65]. Theorem 46 then gives a negative answer to the question of whether all embedded exact oriented Lagrangian fillings are obtained from surgery on an immersed Lagrangian filling. Furthermore, we find an example of a Legendrian link that has a smooth disk filling with two double points, but no exact Lagrangian disk filling with two double points. We also find families of Legendrian links that provide candidates of topological knots whose 4-ball genus is strictly less than the number of double points of any smooth immersed disk filling of the knot.

In a collaboration with Acu, Gadbled, Marinković, Murphy, Starkston and Wu, [3, 4], we studied Weinstein complements of partially smoothed symplectic divisors in toric 4-manifolds. A **symplectic divisor** $D \subset (X, \omega)$ is a symplectic co-dimension two submanifold with controlled singularities. Donaldson showed that every symplectic manifold has a symplectic divisor and Giroux showed that one can always choose a symplectic divisor D such that the complement of a regular neighborhood of D is a Weinstein domain [22, 40]. There has been recent interest in studying Weinstein complements of symplectic divisors because of applications to homological mirror symmetry, see for example [7, 47]. However, the Weinstein Kirby diagrams of these complements are not often known even for basic examples. Toric manifolds have a natural symplectic divisor, which we will call a **toric divisor**, given by the preimage under the moment map of the facets of the Delzant polytope. The pair given by a toric manifold and the toric divisor is log Calabi-Yau, which also play an important role in homological mirror symmetry. We showed that certain smoothings of toric divisors have Weinstein complements.

Theorem 54, see page 109. (Acu, Capovilla-Searle, Gadbled, Marinković, Murphy, Starkston, Wu [3]). *Let (X, ω) be a $\{V_i\}_{i=1}^k$-centered toric 4-manifold with toric divisor D, and let (a_i, b_i) denote the difference of the inward normals of the edges adjacent to V_i. Then the smoothing $\tilde{D}$ of the divisor at the nodes $\{V_i\}_{i=1}^k$ has an arbitrarily small neighborhood whose complement supports a Weinstein structure obtained by attaching k Weinstein 2-handles to a Weinstein subdomain of T^*T^2 along Legendrians which are the co-oriented conormal lifts of k distinct oriented curves in T^2 of slopes $\{(a_i, b_i)\}_{i=1}^k$.*

See Definition 53 in Chapter 7 for a definition of a centered toric 4-manifold. This condition ensures that the complement of the smoothed toric divisor is a Weinstein domain, and there are examples of non-centered toric manifolds such that the complements of smoothed toric divisors do not admit a Weinstein structure. We also constructed an algorithm to produce a Weinstein handlebody diagram for the following Weinstein domains.

Theorem 59, see page 114. (Acu, Capovilla-Searle, Gadbled, Marinković, Murphy, Starkston, Wu [3]). *For any surface F and any 4 dimensional Weinstein mani-*

fold (X,ω) constructed by attaching k Weinstein 2-handles to a Weinstein subdomain of T^*F along Legendrians which are the conormal lifts of k distinct co-oriented curves in F, there exists an explicit Weinstein handlebody diagram of X.

Theorems 54 and 59 allowed us to obtain the Weinstein handlebody diagram of various Weinstein domains, including the complement of a smooth cubic in $\mathbb{C}P^2$.

1.1 Outline

In Chapter 2 we review the background material on Legendrian links in $\mathbb{R}^3_{std}$, the Legendrian contact homology DGA, exact Lagrangian cobordisms and Weinstein 4-manifolds. In Chapter 3 we provide the proofs of the results concerning Lagrangian fillings of the two families of Legendrian links we study. Chapter 4 focuses on results on Milnor fibers of unimodular singularities. In Chapter 5 we discuss the extended criterion for the non-vanishing of symplectic homology for Weinstein 4-manifolds. Finally, in Chapters 6 and 7 we provide summaries of the results obtained from the two collaborations.

2

Background

2.1 Legendrian Links in $\mathbb{R}^3_{std}$

A contact manifold (Y, ξ) has a contact structure ξ which is a maximally non-integrable field of hyperplanes. For any 1-form α such that $\alpha \wedge (d\alpha)^{n-1} \neq 0$, $ker(\alpha) = \xi$ is a contact structure. Given a contact manifold $(Y, \xi = ker(\alpha))$ of dimension $2n+1$, a submanifold $\Lambda \subset (Y, \xi)$ such that $T_p\Lambda \subset \xi_p$ for all $p \in \Lambda$ is called an **isotropic** submanifold. If $\dim(\Lambda) = n$, then we say it is a **Legendrian** submanifold. Two Legendrian submanifolds are Legendrian isotopic if there exists a smooth isotopy between them through Legendrian submanifolds.

We will be considering Legendrian links in $\mathbb{R}^3_{std} = (\mathbb{R}^3, ker(dz - ydx))$ and Legendrian links in (S^3, ξ_{std}), since one can obtain (S^3, ξ_{std}) from $\mathbb{R}^3_{std}$ by adding a point at infinity. Legendrian links in $\mathbb{R}^3_{std}$ have two useful projections to the plane. The **front projection** $\Pi_{x,z}(x, y, z) = (x, z)$ and the Lagrangian projection $\Pi_{x,y}(x, y, z) = (x, y)$. Figure 2.1 shows the front and Lagrangian projections of a Legendrian Hopf link. Note that for a Legendrian link Λ, one can recover the y-coordinate from the front

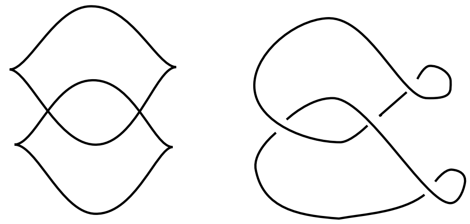


FIGURE 2.1: Front and Lagrangian projections of the maximal-tb Legendrian Hopf link.

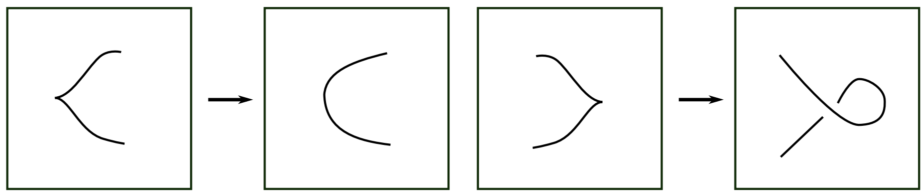


FIGURE 2.2: Ng's resolution of front projections to Lagrangian projections.

projection $\Pi_{x,z}(\Lambda)$ since $T_p\Lambda \subset \xi_p$ implies that $y = \frac{dx}{dz}$ for points $p = (x, y, z) \in \Lambda$. Both projections have their own advantages and fortunately one can translate from a front projection to a Lagrangian projection using Ng's resolution [58] where cusps are smoothed as shown in Figure 2.2.

Two front projections of Legendrian links in $\mathbb{R}^3_{std}$ represent Legendrian isotopic links if and only if the front projections are related by regular homotopy and the Legendrian Reidemeister moves $R1, R2$, and $R3$ shown in Figure 2.3, [72]. Two Lagrangian projections of Legendrian links represent equivalent Legendrian links if and only if they are related by ambient planar isotopies of the immersed curve subject to certain area constraints, and Legendrian Reidemeister moves $R2$, and $R3$ shown in Figure 2.6 and Figure 2.7. For Lagrangian projections of Legendrian links in $\mathbb{R}^3_{std}$ we do not have a Reidemeister $R1$ move.

We say that a Legendrian link has smooth type $\mathcal{K}$ if it smoothly a link $\mathcal{K}$. Thanks to the classical Legendrian invariants, the Thurston-Bennequin number and the rotation number, we know that for every smooth knot there are infinitely many distinct Legendrian links. Legendrian links Λ have a canonical framing of their normal bun-

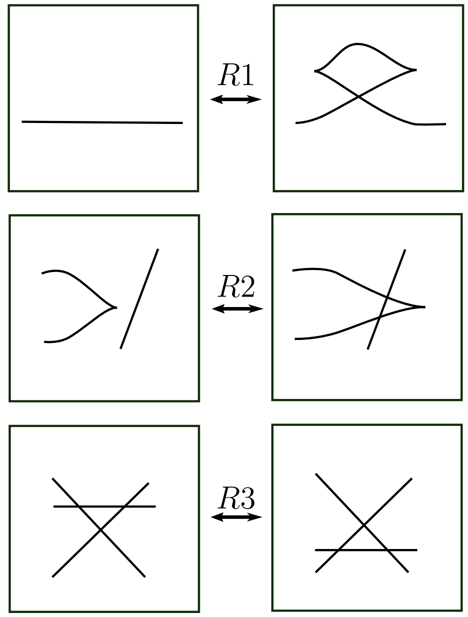


FIGURE 2.3: Legendrian Reidemeister moves in the front projection.

dle ν defined by the line bundle $l_p = \xi_p \cap \nu_p$. For a null-homologous link (one that bounds a Seifert surface), the **Thurston-Bennequin number** (tb-number) is the difference between the canonical framing and Seifert framing. Given a front projection of a Legendrian link Λ, the tb-number of Λ is given by

$$tb(\Lambda) = w(\Pi_{x,z}(\Lambda)) - \frac{1}{2}\#\{\text{cusps}\},$$

where $w(\Pi_{x,z}(\Lambda))$ is the writhe of the projection, that is a signed count of crossings. See Figure 2.5 for examples of positive and negative crossings. The **rotation number** is defined for oriented null homologous Legendrian links, and it measures the winding number of the tangent vector to Λ in a trivialization of the contact planes induced by the oriented surface Σ whose boundary is Λ. Given a front projection of Λ, the rotation number of Λ is given by

$$rot(\Lambda) = \frac{1}{2}(\#\{\text{down cusps}\} - \#\{\text{up cusps}\}),$$

where we count which cusps in $\Pi_{xy}(\Lambda)$ are traversed upwards or downwards. Starting with an oriented Legendrian link Λ one can find distinct oriented Legendrian links by positive or negative stabilization, $S_{\pm}(\Lambda)$. The link $S_+(\Lambda)$ ($S_-(\Lambda)$) is obtained by taking a front projection of Λ and replacing a small strand with a zig-zag with two down (up) cusps. Stabilization does not change the smooth knot type but does change the Thurston-Bennequin number by 1 and rotation number by ± 1. Therefore, for any smooth knot $\mathcal{K}$ one can easily construct infinitely many distinct Legendrian links of smooth type $\mathcal{K}$. If all Legendrian links of smooth type $\mathcal{K}$ are classified by the tb-number and rotation number we call $\mathcal{K}$ Legendrian simple, otherwise we call $\mathcal{K}$ Legendrian non-simple.

The first non-classical Legendrian link invariant that distinguished Legendrian non-simple links was **Legendrian contact homology DGA**, which was first introduced by Chekanov and Eliashberg [19, 31]. The Legendrian contact homology DGA is defined using Reeb trajectories of a Legendrian link, and $\mathcal{J}$-holomorphic curves in the symplectization of the contact manifold.

Definition 1. *The **Reeb vector field** in a contact manifold $(Y, \xi = ker(\alpha))$ is the unique vector field R_α such that $\alpha(R_\alpha) = 1$, and $d\alpha(R_\alpha, \cdot) = 0$. A **Reeb chord** of a Legendrian submanifold $\Lambda \subset (Y, \xi)$ is a trajectory of the Reeb vector field flow that begins and ends on Λ. Let $\mathcal{R}(\Lambda)$ denote the set of Reeb chords of a generic Legendrian link Λ.*

In the case of Legendrian links $\Lambda \subset \mathbb{R}^3_{std}$, the Reeb vector field is ∂_z. After a suitable perturbation, the double points in the Lagrangian front projection $\Pi_{xy}(\Lambda)$ are in one-to-one correspondence with the Reeb chords of Λ. The **symplectization** of a contact manifold $(Y, \xi = ker(\alpha)$, is the symplectic manifold $(\mathbb{R}_t \times Y, \omega = d(e^t\alpha))$. For $\mathbb{R}^3_{std}$, the symplectization is given by $(\mathbb{R}^4 = \mathbb{R}_t \times \mathbb{R}^3, d(e^t\alpha))$. Moreover, for any Legendrian $\Lambda \subset \mathbb{R}^3_{std}$, $\mathbb{R} \times \Lambda$ is an exact Lagrangian submanifold of $(\mathbb{R}^4 = \mathbb{R}_t \times \mathbb{R}^3, d(e^t\alpha))$.

2.2 The Legendrian contact homology differential graded algebra

The **Legendrian contact homology differential graded algebra** defined over a ring of coefficients R, also known as the **Chekanov Eliashberg DGA** is denoted by $(\mathcal{A}(\Lambda; R), \partial)$ and is a Legendrian link invariant up to stable tame isomorphisms [19, 31]. This DGA was first defined over $\mathbb{Z}/2$ by Chekanov, and later over $\mathbb{Z}[s_1^{\pm}, \ldots, s_k^{\pm}]$ by Etnyre, Ng and Sabloff [36]. There are now various surveys on this Legendrian contact DGA [30, 34].

Let $\Lambda \subset \mathbb{R}^3_{std}$ be an oriented Legendrian link with n components, $\Lambda = \Lambda^{(1)} \cup \cdots \cup \Lambda^{(n)}$ and with k basepoints such that every component of Λ has at least one basepoint. Each base point is decorated with a monomial $\pm s_i^{\pm}$. We will assume for simplicity that Λ has rotation number zero since all of the Legendrian links we will consider satisfy this condition. We will also assume Λ is generic enough that the Lagrangian projection of $\Pi_{xy}(\Lambda)$ in $\mathbb{R}^2$ is an immersed curve with transverse double points which are in one-to-one correspondence with the Reeb chords of Λ. For a Reeb chord $a \in \mathcal{R}(\Lambda)$, let a^+ and a^- denote the points of Λ at the end and beginning of the Reeb chord. Then, define $r(a), c(a) \in \{1, \ldots, l\}$ as follows: $r(a)$ ($c(a)$) is the number of the link component containing a^- (a^+). We say a is a **mixed Reeb chord** if $r(a) \neq c(a)$ and a **pure Reeb chord** otherwise. In particular, we denote by $\mathcal{R}_{i,j}(\Lambda)$ the set of Reeb chords a of Λ such that $c(a) = i$ and $r(a) = j$.

2.2.1 The Underlying algebra

The underlying algebra $\mathcal{A}(\Lambda; R)$ is a unital non-commutative graded algebra over the coefficient ring $R = \mathbb{Z}[s_1^{\pm 1}, \ldots, s_k^{\pm 1}]$ generated by Reeb chords $a \in \mathcal{R}(\Lambda)$, and basepoints labeled by monomials $\pm s_i^{\pm}$ for $1 \leqslant i \leqslant k$. There is a surjective map from $\mathbb{Z}[s_1^{\pm}, \ldots, s_k^{\pm}]$ to $\mathbb{Z}[H_1(\Lambda)]$ where each basepoint is mapped the homology class of the link component on which the basepoint lies. It will be convenient later on to choose a distinguished basepoint that we will denote by t_i on each link component $\Lambda^{(i)}$ that represents the homology class of the link component. See Remark 3 for how one can define the DGA over multiple basepoints and for why the DGA is well defined regardless of where we place the basepoints. We also assume that the basepoints $s_j^{\pm}$ commute with the Reeb chords but one could also consider the fully non-commutative DGA.

2.2.2 Grading

The algebra $\mathcal{A}(\Lambda; \mathbb{Z}[s_1^{\pm}, \ldots, s_k^{\pm}])$ is graded over $\mathbb{Z}$. If Λ is a knot the grading is well defined, but if $\Lambda = \Lambda^{(1)} \cup \cdots \cup \Lambda^{(n)}$ is a link with $n > 1$ link components, then we have to make extra choices to ensure that the grading is well defined. Assume for now that Λ is a knot. For a Reeb chord $a \in \mathcal{R}(\Lambda)$, the grading of a is given by

$$|a| = CZ(\gamma_a) - 1,$$

where γ_a is a capping path for a a path following the orientation of Λ starting at a^- and ending at a^+. The Conley Zehnder index $CZ(\gamma_a)$ is defined as the Maslov index of the smooth loop given by concatenating γ_a with a standard closure defined in [26]. In particular, if the a is a perpendicular double point in the Lagrangian projection of Λ, the grading can be computed explicitly from $|a| = \lfloor 2rot(\gamma_a) \rfloor$. For a knot Λ and any Reeb chord $a \in \mathcal{R}(\Lambda)$, there are two choices of capping paths for a, but the difference of gradings is given by twice the rotation number $2r(\Lambda)$. Since we have assumed that Λ has rotation number zero, the grading is well defined. Suppose that $\Lambda = \Lambda^{(1)} \cup \cdots \cup \Lambda^{(n)}$ is a link with more than one link component. Choose one basepoint t_i for each link component $\Lambda^{(i)}$ that does not include any endpoints of Reeb chords, and so that the tangent vectors of $\Lambda^{(i)}$ at t_i have the same slope for all $i = 1, \ldots, l$. Then, define a capping path γ_a along Λ as the concatenation of the path along Λ from a^- to $t_{c(a)}$, and the path from a^+ to $t_{r(a)}$. We can identify the tangent vectors of $t_{c(a)}$ and $t_{r(a)}$ and thus produce a smooth closed loop and take its Maslov index. Finally, using the fact that $r(\Lambda^{(i)}) = 0$, we define the grading of all basepoints to be zero, $|s_i| = 0$.

2.2.3 *The Differential*

The differential ∂ of the Legendrian contact homology DGA is defined by counting rigid $\mathcal{J}$-holomorphic disks in the symplectization $(\mathbb{R}^4, d(e^t\alpha))$ with boundary on $\mathbb{R}\times\Lambda$. For Reeb chords $a, b_1 \ldots, b_m$ of Λ, denote by $\mathcal{M}(a; b_1, \ldots, b_m)$ the moduli space of $\mathcal{J}$-holomorphic disks

$$u : (\mathbf{D}_{m+1}, \partial\mathbf{D}_{m+1}) \to (\mathbb{R}\times\mathbb{R}^3, \mathbb{R}\times\Lambda)$$

such that

- $\mathbf{D}_{m+1}$ is the disk with $m+1$ boundary points $p, q_1, \ldots, q_m$ removed and which are labeled in a counterclockwise order,
- u is asympototic to $[0,\infty)\times a$ at p
- u is asympototic to $(\infty, 0]\times b_i$ at q_i.

See Figure 2.4 for an example of such a disk. The quotient of $\mathcal{M}(a; b_1, \ldots, b_m)$ by a vertical translation of $\mathbb{R}$ is denoted by $\widetilde{\mathcal{M}}(a; b_1, \ldots, b_m)$. If $\dim(\widetilde{\mathcal{M}}(a; b_1, \ldots, b_m)) = 0$, the we say the disks $u \in \mathcal{M}(a; b_1, \ldots, b_m)$ are rigid. Furthermore, in this case

$$|a| - \sum_{i=1}^{m} |b_i| = 1.$$

One can count such rigid $\mathcal{J}$-holomorphic disks over the ring $\mathbb{Z}[H_1(\Lambda)]$ as follows. For such a disk u the boundary segment between q_i and q_{i+1} can be closed off on $\mathbb{R}\times\Lambda$ into a curve by concatenating with the capping paths from q_i^- to $t_{c(q_i)}$, and the capping path from $t_{r(q_{i+1})}$ to q_{i+1}^+ since $c(q_i) = r(q_{i+1})$. Let τ_i denote the homology

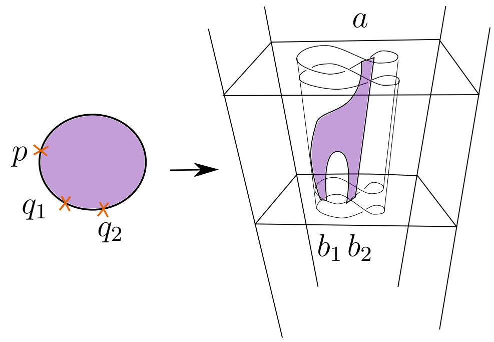


FIGURE 2.4: A rigid $\mathcal{J}$-holomorphic disk u with boundary on $\mathbb{R} \times \Lambda$ for Λ a Hopf link.

class of this curve. Then, $w(u)$ is the product of the Reeb chords and homology classes $\tau_0 \cdots \tau_m b_1 \cdots b_m$. Here we are setting $b_0 = a$.

If Λ is spin then the moduli spaces of $\mathcal{J}$-holomorphic curves we are considering admit a coherent orientation, and [27, 50], so one can assign a sign $sgn(u) \in \{-1, 1\}$ to each rigid $\mathcal{J}$-holomorphic disk. See Remark 4 for how the choice of spin structure of Λ affects the signs in the differential. For a Reeb chord $a \in \mathcal{R}(\Lambda)$, the differential is given by

$$\partial(a) = \sum_{\dim(\tilde{\mathcal{M}}(a;b_1,\ldots,b_m))=0} \sum_{u \in \mathcal{M}(a;b_1,\ldots,b_m)} sgn(u)w(u). \tag{2.1}$$

Let $\partial s_i^{\pm} = 0$ for any basepoint, and extend the differential to $\mathcal{A}$ using the graded Leibniz rule:

$$\partial(uv) = \partial(u)v + (-1)^{|u|}u\partial(v)$$

One can compute the differential ∂ explicitly from the Lagrangian projection of Λ as follows. Each double point in $\Pi_{xy}(\Lambda)$ is a Reeb chord of Λ. The four quadrants of every crossing in $\Pi_{xy}(\Lambda)$ are decorated with a Reeb sign, and an orientation sign as shown in Figure 2.5. For each crossing, two opposite quadrants have positive Reeb sign, and two have negative Reeb sign. If a crossing is positive then two adjacent quadrants have positive orientation sign and two adjacent quadrants have negative orientation sign. If a crossing is negative then all of the quadrants have positive

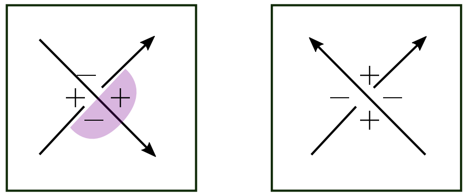


FIGURE 2.5: Reeb signs indicated for a positive crossing and a negative crossing. The quadrants with purple shading have negative orientation sign.

orientation sign. Each rigid $\mathcal{J}$-holomorphic disk with $m+1$ boundary punctures is projected to an immersed disk in $\mathbb{R}^2$,

$$\Delta : (\mathbf{D}_{m+1}, \partial \mathbf{D}_{m+1}) \to (\mathbb{R}^2, \Pi_{xy}(\Lambda))$$

such that

- a neighborhood of each puncture is mapped to one of the four quadrants of a crossing of $\Pi_{xy}(\Lambda)$,
- $\Delta(p) = a$, and the neighborhood of a is mapped to a quadrant of a labeled with a positive Reeb sign,
- For $i = 1, \ldots, m$, Δ sends q_i to b_i and a neighborhood of q_i is mapped to a quadrant of b_i labeled with a negative Reeb sign.

Let $\Delta(a)$ denote the moduli space of such rigid holomorphic disks Δ. For every such disk u, let $sgn(\Delta)$ be the product of the orientation signs at each corner of Δ, and the signs of any basepoints on the boundary of the disk. We also define $w(\Delta)$ to be the product of Reeb chords at the negative Reeb sign corners and basepoints on the boundary of the disk as we go around the disk in the counterclockwise direction starting and ending at the corner at a. A basepoint labeled $\pm s_i^{\pm 1}$ contributes $\pm s_i^{\pm}$ if it is traversed along the orientation of Λ and $\pm s_i^{\mp 1}$ otherwise. Then, the differential is given by:

$$\partial(a) = \sum_{\Delta \in \Delta(a)} sgn(\Delta) w(\Delta).$$

The Legendrian contact homology DGA $(\mathcal{A}(\Lambda;\mathbb{Z}[s_1^{\pm},\dots,s_k^{\pm}]),\partial)$ is only a Legendrian invariant up to stable tame isomorphism [19, 36]. In order to define stable tame isomorphisms we first define elementary automorphisms of DGAs. An **elementary automorphism** of a DGA $(\mathbb{Z}[s_1^{\pm 1},\dots,s_k^{\pm 1}]\langle a_1,\dots,a_r\rangle,\partial)$ is a chain map

$$\phi:\mathbb{Z}[s_1^{\pm 1},\dots,s_k^{\pm 1}]\langle a_1,\dots,a_r\rangle\to\mathbb{Z}[s_1^{\pm 1},\dots,s_k^{\pm 1}]\langle a_1,\dots,a_r\rangle$$

such that there exists a $1\leqslant j\leqslant n$ such that $\phi(a_j)=\pm s_i^{(n+m)}a_j+u$ for $u\in\mathbb{Z}[s_1^{\pm 1},\dots,s_k^{\pm 1}]\langle a_1,\dots a_{j-1},a_{j+1},\dots,a_r\rangle$, $n,m\in\mathbb{Z}$ and $1\leqslant i\leqslant k$. If $i\neq j$, then $\phi(a_i)=a_i$, and $\phi(s_i)=s_i$ for $i=1,\dots,k$. A **tame isomorphism** is a chain map

$$\psi:(\mathbb{Z}[s_1^{\pm 1},\dots,s_k^{\pm 1}]\langle a_1,\dots,a_r\rangle,\partial)\to(\mathbb{Z}[{s'}_1^{\pm 1},\dots,{s'}_k^{\pm 1}]\langle a_1',\dots,a_r'\rangle,\partial')$$

given by a composition of elementary automorphisms of $(\mathbb{Z}[s_1^{\pm 1},\dots,s_k^{\pm 1}]\langle a_1,\dots,a_r\rangle,\partial)$ and the algebra map sending $a_i\to a_i'$ and $s_i\to s_i'$.

A **grading k stabilization** of a DGA $(\mathbb{Z}[s_1^{\pm 1},\dots,s_k^{\pm 1}]\langle a_1,\dots,a_r\rangle,\partial)$ is the DGA $(\mathbb{Z}[s_1^{\pm 1},\dots,s_k^{\pm 1}]\langle f_k,f_{k-1},a_1,\dots,a_r\rangle,\partial')$ where $|f_k|=k$ and $|f_{k-1}|=k-1$ and the differential ∂' is given by $\partial'(a)=\partial(a)$ and $\partial'(f_k)=f_{k-1}$, $\partial(f_{k-1})=0$. One can quickly check that a Legendrian Reidemeister $R2$ on Λ move induces grading k stabilizations of $\mathcal{A}(\Lambda)$, and that a Legendrian Reidemeister $R3$ move on Λ induces an elementary automorphism of $\mathcal{A}(\Lambda)$. Two DGAs are **stable tame isomorphic** if after each is stabilized some number of times, they are equivalent under tame isomorphisms. In the case that Λ is a link we also consider link automorphisms which we will now define.

Definition 2 ([13])**.** *Let Λ be an m-component Legendrian link and $(\mathcal{A}(\Lambda;R);\partial)$ its Legendrian contact DGA. Given a choice of units $e_1,\dots,e_m$ in the coefficient ring R, a link automorphism of Λ is an algebra automorphism $\psi:\mathcal{A}(\Lambda;R)\to\mathcal{A}(\Lambda;R)$ such that for any Reeb chord a*

$$\psi(a)=e_{r(a)}e_{c(a)}^{-1}a.$$

Recall that $r(a)$ ($c(a)$) is the number of the link component containing a^- (a^+).

Remark 3 (**Multiple basepoints**)**.** *One can define the Legendrian contact homology DGA with as many basepoints as one prefers so long as there is at least one basepoint for each link component of Λ. Moreover, one can move a basepoint anywhere on Λ since moving a basepoint $\pm s_i^{\pm 1}$ through a crossing a (in the Lagrangian projection of Λ) corresponds to the automorphism that maps a to $(\pm s_i^{\pm 1})^{\pm 1}(a)$ and which is the identity on all other generators. The differential then changes by conjugation with this automorphism. Given k multiple basepoints on a link component, we can then move them all to the same segment on the Lagrangian projection of Λ, and then replace the k basepoints with a single basepoint labeled by the product of the k basepoints.*

Remark 4 (**Spin structures and signs** [27, 50])**.** *The signs in the differential of the Legendrian contact DGA over $\mathbb{Z}$ of a Legendrian link Λ depend on a choice of a spin structure on Λ [27]. Each link component of Λ, which is topologically an S^1, has two spin structures associated to it: the Lie group spin structure and the null-cobordant spin structure. The Lie group spin structure arises from the fact that S^1 is a Lie group. The null-cobordant spin structure instead is defined using the fact that S^1 bounds a disk $\mathbf{D}^2$ in $\mathbb{R}^3$, and is the spin structure induced by restriction to the boundary S^1 of the unique spin structure on $\mathbf{D}^2$. Consequently, the null-cobordant spin structure is more natural than the Lie group spin structure for Lagrangian cobordisms which we will introduce in Section 2.3.*

The choice of spin structure affects the grading of the DGA, and the signs of the rigid $\mathcal{J}$-holomorphic disks counted in the differential. However, if Λ is a Legendrian link in $\mathbb{R}^3_{std}$, the Legendrian contact DGAs of Λ over $\mathbb{Z}[H_1(\Lambda)]$ associated to any two spin structures are stable tame isomorphic [27]. In contrast, if we consider only the DGA of Λ over $\mathbb{Z}$, then there is a dependence on spin structures. Following [13, 27], *we will describe the stable tame isomorphisms between the DGAs over $\mathbb{Z}[H_1(\Lambda)]$ for different spin structures. The set of spin structures of a link Λ with n components is the affine space based over $\mathbb{Z}_2^n \cong H^1(\Lambda; \mathbb{Z}_2)$.*

We will denote the DGA obtained using the Lie spin structure on each link component by $(\mathcal{A}(\Lambda; \mathbb{Z}[H_1(\Lambda)]), \partial^{Lie})$ and the one obtained using the null-cobordant spin structure on each link component by $(\mathcal{A}(\Lambda; \mathbb{Z}[H_1(\Lambda)]), \partial^{NC})$. For the DGA we have defined $(\mathcal{A}(\Lambda; \mathbb{Z}[H_1(\Lambda)]), \partial)$ where t_i is a basepoint on the ith link component of Λ, the differential is ∂^{Lie}. The differential ∂^{NC} can be obtained from ∂^{Lie} by mapping $\phi(t_i) = -t_i$ for all i and fixing the Reeb chords a. If we had more basepoints than just one per link component, it suffices for each link component to move all of them to the same segment, treat them as a single basepoint and then apply an automorphism such that their product is mapped to its negative. More generally, one can change the DGA from a spin structure to another that differ by $(c_1, \ldots, c_n) \in \mathbb{Z}_2^n$ by mapping some subset of basepoints to their negatives so that for each link component $\Lambda^{(i)}$, the product of basepoints on the ith component changes by a factor of $(-1)^{c_i}$.

Since the Legendrian contact DGA $(\mathcal{A}(\Lambda; \mathbb{Z}[s_1^{\pm}, \ldots, s_k^{\pm}]), \partial)$ is a Legendrian invariant up to stable tame isomorphism it is often difficult to compare two DGAs of Legendrian links and show whether or not they are equivalent. However, there are various Legendrian link invariants one can obtain from this DGA which are both easier to compute and to compare. One of these is the set of augmentations of a Legendrian link.

Definition 5. *A* ***graded augmentation*** *of $(\mathcal{A}(\Lambda; R), \partial)$ to a ring of coefficients R supported in grading zero is a chain map $\epsilon : (\mathcal{A}(\Lambda; R), \partial) \to (R, 0)$ such that $\epsilon(1) = 1$, and for any element a of nonzero grading $\epsilon(a) = 0$.*

The set $\{\epsilon \mid \epsilon \text{ is an augmentation of } \Lambda \text{ over } R\}$ is a Legendrian link invariant. It is useful to know when two augmentations are equivalent:

Definition 6. *Let $\Lambda = \Lambda^{(1)} \cup \cdots \cup \Lambda^{(n)}$ be a Legendrian link. Two augmentations $\epsilon_1, \epsilon_2 : (\mathcal{A}(\Lambda; R), \partial) \to (R, 0)$ are* ***DGA homotopic*** *if there exists a chain operator $K : \mathcal{A} \to R$ such that*

1. *K has degree 1,*

2. *$K(x \cdot y) = K(x) \cdot \epsilon_2(y) + (-1)^{|x|} \epsilon_1(x) \cdot K(y)$ for all $x, y \in \mathcal{A}_1$,*

3. *For any Reeb chord $a \in \mathcal{R}_{ij}(\Lambda)$ that begins on Λ_i and ends on Λ_j,*

$$\alpha_i \epsilon_1(a) - \alpha_j \epsilon_2(a) = K\partial$$

   where $\alpha_i, \alpha_j \in R^*$.

Remark 7. *In the case that Λ only has non-negative graded Reeb chords, two augmentations ϵ_1 and ϵ_2 are DGA homotopic if and only if $\epsilon_1(a) = \epsilon_2(a)$ for all Reeb chords a of Λ. Note that chain homotopic augmentations give the same map on homology. Therefore, even if Λ is a link with negatively graded Reeb chords, if there is a zero graded Reeb chord a which is a cycle in the DGA (so $\partial a = 0$) then $\epsilon_1(a) \neq \pm \epsilon_2(a)$ implies that ϵ_1 and ϵ_2 are not DGA homotopic.*

There is another way in which two augmentations can be equivalent. The Augmentation Category, $Aug_+(\Lambda)$ defined in [59], is a unital A_∞ category where the objects are augmentations. In [59], the authors showed that two augmentations are equivalent in $Aug_+(\Lambda)$ if and only if they are DGA homotopic for single component Legendrian knots Λ. Definition 6 is usually stated with $\alpha_i, \alpha_j = 1$, but by allowing for $\alpha_i, \alpha_j \in R^*$ one can conclude the following.

Proposition 8 ([10, 59])**.** *Augmentations of Legendrian links $\Lambda \subset (\mathbb{R}^3, \xi_{std})$ are DGA homotopic if and only if they are equivalent in $Aug_+(\Lambda; R)$.*

2.3 Exact Lagrangian cobordisms and Augmentations

Instead of just considering the exact Lagrangian cylinders $\mathbb{R} \times \Lambda$ in the symplectization $(\mathbb{R}_t \times \mathbb{R}^3, d(e^t\alpha))$, where $\Lambda \subset]R^3_{std}$ is a Legendrian link, one can consider exact Lagrangian cobordisms with Legendrian ends.

Definition 9. *Let Λ_+ and Λ_- be Legendrian links in $\mathbb{R}^3_{std}$. An* **exact Lagrangian cobordism** *Σ from Λ_- to Λ_+ is an embedded, oriented Lagrangian surface in the symplectization $(\mathbb{R}_t \times \mathbb{R}^3, d(e^t\alpha))$ that has cylindrical ends and is exact in the following sense: for some $N > 0$,*

1. *$\Sigma \cap ([-N, N] \times \mathbb{R}^3)$ is compact,*
2. *$\Sigma \cap ((N, \infty) \times \mathbb{R}^3) = (N, \infty) \times \Lambda_+$,*
3. *$\Sigma \cap ((-\infty, -N) \times \mathbb{R}^3) = (-\infty, -N) \times \Lambda_-$, and*
4. *there exists a function $f : \Sigma \to \mathbb{R}$ and constants $\mathfrak{c}_\pm$ such that $e^t\alpha|_\Sigma = df$, where $f|_{(-\infty,-N)\times\Lambda_-} = \mathfrak{c}_-$, and $f|_{(N,\infty)\times\Lambda_+} = \mathfrak{c}_+$. We call f the primitive on Σ.*

An **exact Lagrangian filling** *Σ of a Legendrian link Λ is an exact oriented Lagrangian cobordism from to Λ.*

The **Maslov number** of an exact Lagrangian surface Σ is the greatest common divisor of the Maslov numbers of the closed loops in Σ. For a Legendrian Λ, the Maslov number of Λ is the Maslov number of the surface $\mathbb{R} \times \Lambda$. Maslov 0 surfaces are orientable. We assume that all Legendrians and Lagrangian cobordisms are Maslov 0 because we can then work over $\mathbb{Z}$. For any Legendrian link Λ with $rot(\Lambda) = 0$ that admits a Maslov 0 exact Lagrangian filling Σ, Chantraine [15] showed that the tb-number of Λ is related to the genus of Σ:

$$tb(\Lambda) = 2g(\Sigma) - 2c + 1$$

where c is the number of link components of Λ.

Remark 10. *A* ***local system*** *of an exact Lagrangian cobordism* Σ *is a representation* $\pi_1(\Sigma) \to GL(n, \mathbb{F})$ *where* $\mathbb{F}$ *is a field. Local systems allow one to enrich various invariants by recording homotopic data, analogously to how we can enrich the Legendrian contact DGA by tracking* $H_1(\Lambda)$*. By abuse of notation we refer to representations* $\pi_1(\Sigma) \to GL(1, \mathbb{Z}) = \{+1, -1\}$ *for orientable surfaces* Σ *as rank-*1 *local systems over* Σ*. Furthermore, since* $GL(1, \mathbb{Z})$ *is abelian, it suffices to consider the maps* $H_1(\Sigma) \to GL(1, \mathbb{Z})$*.*

An important property of the Legendrian contact homology DGA is that it is functorial over the category whose objects are Legendrian links and whose morphisms are Maslov 0 exact Lagrangian cobordisms [28, 50]. For a Legendrian Λ with an embedded, Maslov 0 exact Lagrangian filling Σ, we can define its DGA over $\mathbb{Z}[H_1(\Sigma)]$ by following the construction of $(\mathcal{A}(\Lambda; H_1[(\Lambda)]), \partial)$ and using the fact that the inclusion $\Lambda_\pm \hookrightarrow \Sigma$ induces a map $\mathbb{Z}[H_1[(\Lambda_\pm)] \to \mathbb{Z}[H_1(\Sigma)]$ of the rings of coefficients.

Suppose Σ_0 is an oriented, embedded, Maslov 0 exact Lagrangian cobordism from Λ_- to another Legendrian link Λ_+, and that Σ is a Maslov 0 exact Lagrangian filling of Λ_-. Let Σ_+ denote the filling of Λ_+ given by concatenating Σ and Σ_0. Then, there exists an induced DGA map between the DGA of Λ_+ and Λ_-:

$$f : (\mathcal{A}(\Lambda_+; \mathbb{Z}[H_1(\Sigma_+)]), \partial) \to (\mathcal{A}(\Lambda_-; \mathbb{Z}[H_1(\Sigma)]), \partial)$$

where $\mathbb{Z}[H_1(\Sigma)]$ is in grading 0. The differential is given by counting rigid $\mathcal{J}$-holomorphic disks u with boundary on Σ,

$$u : (D_{m+1}, \partial D_{m+1}) \to (\mathbb{R} \times \mathbb{R}^3, \Sigma)$$

such that

1. u is asymptotic to $[0, \infty) \times a$ for $a \in \mathcal{A}(\Lambda_+)$

2. u is asymptotic to $(-\infty, 0] \times b_i$ for $b_i \in \mathcal{A}(\Lambda_-)$ and $i = 1, \ldots, m$.

Remark 11. *The functorial behavior of the Legendrian contact DGA over Lagrangian cobordisms can be extended to other Legendrian link invariants such as the Augmentation Category $Aug_+(\Lambda)$. In a collaboration with Legout, Limouzineau, Murphy, Pan, and Traynor, we extend a result of Pan to Legendrian links using classical Floer techniques based on work of [17] and [54]. See Chapter 6 for this result and an overview of the results obtained from this collaboration.*

One natural consequence of the fact that the DGA is functorial over Lagrangian cobordisms is that exact Lagrangian fillings induce augmentations.

Theorem 12 (Ekholm, Honda, Kalmán, Karlsson [28, 50])**.** *Suppose that Σ is an oriented, embedded, Maslov 0 exact Lagrangian filling of the Legendrian link $\Lambda \subset (\mathbb{R}^3, \xi_{std})$. Then Σ induces a DGA map*

$$\epsilon_\Sigma : (\mathcal{A}(\Lambda; \mathbb{Z}[H_1(\Sigma)]), \partial) \to (\mathbb{Z}[H_1(\Sigma)], 0)$$

where $\mathbb{Z}[H_1(\Sigma)]$ is in grading 0. If Σ and Σ' are Lagrangian fillings of Λ such that there exists a Hamiltonian isotopy from Σ_1 to Σ_2 through exact Lagrangian fillings of Λ that fixes the boundary, then the corresponding augmentations ϵ_Σ and $\epsilon_{\Sigma'}$ are chain homotopic maps.

It is important to note that not all augmentations are induced by embedded Maslov 0 exact Lagrangian fillings. However, they are induced from immersed exact Lagrangian fillings [63].

For an exact Lagrangian filling Σ, the induced augmentation ϵ_Σ over $\mathbb{Z}[H_1(\Sigma)]$ is really a family of augmentations over $\mathbb{Z}$ where each augmentation over $\mathbb{Z}$ corresponds to a choice of local system of Σ:

$$\epsilon_\Sigma : \mathcal{A}(\Lambda; \mathbb{Z}[H_1(\Sigma)]), \partial) \to (\mathbb{Z}[H_1(\Sigma)], 0) \to (\mathbb{Z}, 0).$$

One can also enlarge the coefficient ring $\mathbb{Z}[H_1(\Sigma)]$ to include link automorphisms. In particular, if Λ is a link with n components we want to consider the coefficient ring $\mathbb{Z}[H_1(\Sigma)] \oplus \mathbb{Z}[s_1^{\pm}, \ldots, s_{n-1}^{\pm}]$ where we set $s_i = e_i/e_n$ for $1 \leqslant i < n$. Then an augmentation over $\mathbb{Z}[H_1(\Sigma)]$ lifts to an augmentation over $\mathbb{Z}[H_1(\Sigma) \oplus \mathbb{Z}^{n-1}]$ by mapping $\tilde{\epsilon}(a) = e_{r(a)} e_{c(a)}^{-1} \epsilon(a)$. Casals and Ng then give the following definition which allows one to work with such augmentations whose coefficient rings keep track of $H_1(\Sigma)$ and link automorphisms.

Definition 13 (Casals, Ng [13])**.** *A k-**system of augmentations** of a Legendrian link Λ is an algebra map*

$$\epsilon : \mathcal{A}(\Lambda) \to \mathbb{Z}[s_1^{\pm 1}, \ldots, s_k^{\pm 1}]$$

such that $\epsilon \circ \partial = 0$, $\epsilon(1) = 1$, and $\epsilon(a) = 0$ for any $a \in \mathcal{A}(\Lambda)$ with nonzero grading. Two k-systems of augmentations

$$\epsilon : \mathcal{A}(\Lambda) \to \mathbb{Z}[s_1^{\pm 1}, \ldots, s_k^{\pm 1}] \text{ and } \epsilon' : \mathcal{A}(\Lambda) \to \mathbb{Z}[{s'}_1^{\pm 1}, \ldots, {s'}_k^{\pm 1}]$$

are equivalent if there exists a $\mathbb{Z}$-algebra isomorphism

$$\phi : \mathbb{Z}[s_1^{\pm 1}, \ldots, s_k^{\pm 1}] \to \mathbb{Z}[{s'}_1^{\pm 1}, \ldots, {s'}_k^{\pm 1}]$$

such that $\epsilon' = \phi \circ \epsilon$. The space of such isomorphisms is parametrized by $\mathbb{Z}_2^k \times GL_k(\mathbb{Z})$.

Theorem 12 can then be restated as follows:

Proposition 14 ([13])**.** *Let Λ be an n-component Legendrian link. Let Σ be a connected, orientable exact Lagrangian filling of Λ of genus g and Maslov number 0. Then Σ gives rise to a $(2g + 2n - 2)$-system of augmentations of Λ, and this system is well-defined, independent of choices, up to equivalence. Furthermore, if all Reeb chords of Λ have non-negative degree, then isotopic fillings of Λ give rise to equivalent systems of augmentations.*

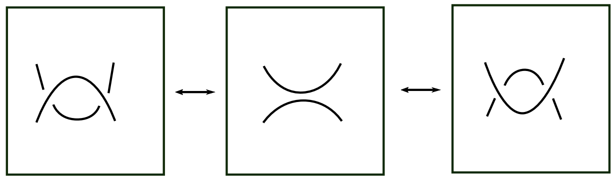


FIGURE 2.6: Reidemeister $R2$ move in the Lagrangian projection.

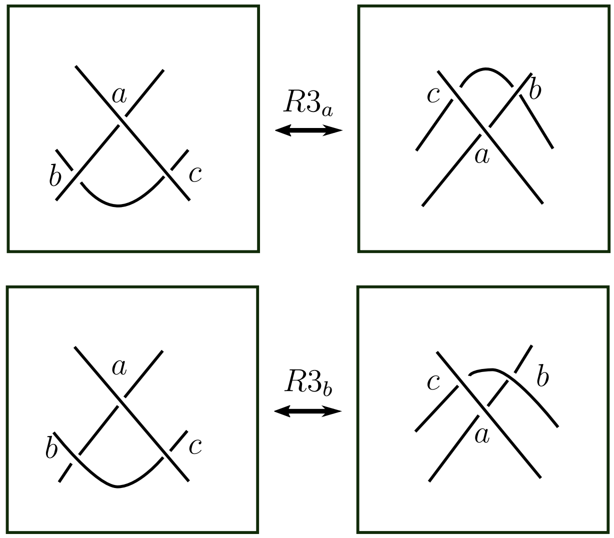


FIGURE 2.7: Two Reidemeister $R3$ moves in the Lagrangian projection.

2.3.1 DGA maps induced by Legendrian Reidemeister moves and pinching

If two Legendrian links Λ_- and Λ_+ are related by Legendrian isotopy or pinch moves then there exists an exact Lagrangian cobordism Σ from Λ_- to Λ_+ [28]. Moreover, there is a unique exact Lagrangian filling of the maximal-tb Legendrian unknot [33], which we call a minimum cobordism. **Decomposable** Lagrangian cobordisms are those which can be constructed using only pinch moves, Legendrian isotopy and minimum cobordisms. All of the Lagrangian fillings that we will construct are decomposable.

We will now summarize the DGA chain maps induced by Legendrian Reidemeister moves and pinching [13, 19, 28, 49, 62]. Although we can use these moves in the front projection of a Legendrian link, in order to write the induced chain maps on the DGA explicitly we need to use the Lagrangian projection of the Legendrian link.

Both Legendrian Reidemeister moves $R3$, and $R2$ shown in Figures 2.62.7 induce an identity map on the local systems of the Lagrangian fillings, and that there are two types of $R3$ moves to consider when computing the induced chain map.

If one performs a Legendrian Reidemeister move $R3_a$, then

$$a \to a,\ b \to b,\ c \to c,\ x \to x.$$

If one performs a Legendrian Reidemeister move $R3_b$ then

$$b \to b,\ c \to c,\ x \to x,\ \text{and } a \to a + \sigma cb,$$

where σ is the count of shaded quadrants of the disk with 3 punctures mapped to corners of a, b and c before performing the Reidemeister move. These chain maps can be reformulated using braids following [13]. Consider a braid of n strands labeled $\{1, \ldots, n\}$ from the bottom strand to the top one, σ_k is the positive crossing between the kth strand and the $(k+1)$ strand. Note that a Legendrian $R3_b$ Reidemeister move takes $\sigma_1\sigma_2\sigma_1 \to \sigma_2\sigma_1\sigma_2$. If we identify a, c with c, a respectively after performing the Reidemeister move then we find that

$$\begin{pmatrix} c \\ a \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ 1 & b \end{pmatrix} \begin{pmatrix} c \\ a \end{pmatrix}$$

A Legendrian $R3_b$ Reidemeister move takes $\sigma_2\sigma_1\sigma_2 \to \sigma_1\sigma_2\sigma_1$ and after identifying a, c with c, a respectively, the chain map is given by

$$\begin{pmatrix} a \\ c \end{pmatrix} \to \begin{pmatrix} -b & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix}.$$

A Legendrian Reidemeister $R2$ move that introduces two Reeb chords of grading k and $k-1$ induces a k-graded stabilization on the DGA. Now suppose we perform a Reidemeister $R2$ move on Λ that cancels out two Reeb chords a and b of grading k and $k-1$ such that $\partial a = b + v$ where v has no terms containing a or b. Then, the induced DGA chain map maps $a \to 0$ and $b \to v$ and fixes all other Reeb chords.

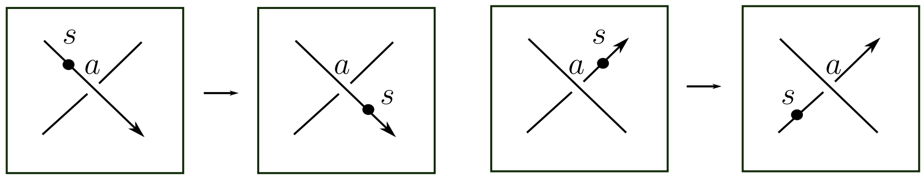


FIGURE 2.8: Two base point moves.

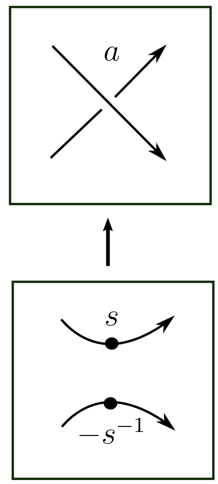


FIGURE 2.9: A pinch move at a chord a in the Lagrangian projection.

Moving a basepoint s over or under a crossing induces a DGA map as follows. Suppose $a \in \mathcal{R}(\Lambda)$ is the Reeb chord that corresponds to the crossing in question. If we move s over a crossing as shown in the leftmost diagram in Figure 2.8, then $a \to sa$. If we move s over a crossing as shown in the rightmost diagram in Figure 2.8, then $a \to as^{-1}$. Both of these DGA maps are isomorphisms and the reverse of either of these DGA maps is their inverse.

Figure 2.9 shows a pinch move in the Lagrangian projection of a Legendrian where the pinch is performed at a **contractible** Reeb chord a.

Definition 15 ([28])**.** *Let $\Lambda \subset \mathbb{R}^3_{std}$ be a Legendrian link, we call a Reeb chord a of Λ contractible if there exists a homotopy Λ_τ for $\tau \in [0,1]$ of Legendrian immersions such that $\Lambda_0 = \Lambda$, and the Lagrangian projection of Λ_τ only has transverse double points for all $\tau \in [0,1]$. Let a_τ denote the Reeb chord in Λ_τ corresponding to a. Then, we also require that Λ_1 has a transverse self-intersection given by contracting* *the length of the Reeb chord a_τ to zero as $\tau \to 1$.*

Ekholm, Honda and Kálmán gave an explicit description of the chain map over $\mathbb{Z}_2[H_1(\Sigma)]$ induced by pinching at a contractible simple Reeb chord. Casals and Ng have generalized this result and provide the explicit chain map over $\mathbb{Z}[H_1(\Sigma)]$ induced by pinching at a more general class of contractible Reeb chords which they call proper.

Definition 16 (Casals, Ng [13])**.** *A contractible Reeb chord $a \in \mathcal{R}(\Lambda)$ is proper if for any immersed disk Δ with possibly concave corners:*

- *Δ has a convex corner at some Reeb chord besides a labeled with a positive Reeb sign,*
- *Δ has at least one convex corner at a labeled with a positive Reeb sign,*
- *All other convex corners of Δ are labeled with negative Reeb signs, and*
- *if Δ has concave corners they are concave corners at a labeled with a positive Reeb sign.*

Observe that the Lagrangian cobordism between a pair of links related by a pinch move is a saddle cobordism. In order to describe the DGA chain map induced by pinching we will introduce oriented arcs and circles on the cobordism so that we can write this DGA map with coefficients over $H_1(\Sigma)$. We will view the Lagrangian cobordism from top to bottom and denote by Λ_+ the Legendrian link before pinching and Λ_- the Legendrian after pinching. There is a natural oriented arc in a saddle cobordism which intersects Λ_- in two points that we label by s_i and $-s_i^{-1}$. Furthermore, the basepoints t_i on each component of Λ_+ are joined to the corresponding basepoints t_i on Λ_- by arcs on the Lagrangian cobordism.

We will first define the following immersed disks in Λ_+ which will contribute to the pinching DGA map. Suppose that a_i is a Reeb chord in Λ_+ and $a_i \neq a$. Then $\Delta_a^{\leftarrow}(a_i)$ and $\Delta_a^{\rightarrow}(a_i)$ are the set of immersed disks Δ of Λ_+ such that

- Δ only has convex corners, only two of which are positive at a_i and of a;
- We say $\Delta \in \Delta_a^{\leftarrow}(a_i)$, if at the corner at a, the orientation of Λ_+ points towards the disk Δ.
- We say $\Delta \in \Delta_a^{\rightarrow}(a_i)$, at the corner at a, the orientation of Λ_+ points away from the disk Δ.

Then, for any immersed disk $\Delta \in \Delta_a^{\leftarrow}(a_i) \cup \Delta_a^{\rightarrow}(a_i)$ we define the following three quantities: $sgn(\Delta)$, and $\omega_j(\Delta)$ for $j = 1, 2$. The sign associated to the disk $sgn(\Delta) \in \{\pm 1\}$ is the product of the orientation signs associated to Λ_+ at each corner of Δ multiplied by the signs of any basepoints on the boundary of the disk. The word $\omega_1(\Delta)$ is the product of all the Reeb corners of Δ with negative Reeb sign and basepoints that we encounter if we travel along the boundary of Δ from a_i to a in the counterclockwise direction. The word $\omega_2(\Delta)$ is the product of all the Reeb corners of Δ with negative Reeb sign and basepoints that we encounter if we travel along the boundary of Δ from a to a_i in the counterclockwise direction. Note that $\omega_1(\Delta)\omega_2(\Delta)$ gives us the product of all of the Reeb chords with negative sign encountered as we travel along Δ in the clockwise direction starting and ending at a_i. See Figure 2.10 for examples of disks in $\Delta_a^{\leftarrow}(a_i) \cup \Delta_a^{\rightarrow}(a_i)$. On the left of Figure 2.10 we show a holomorphic disk $\Delta \in \Delta_a^{\leftarrow}(a_3)$, where $\omega_1(\Delta) = a_2 s a_1$, $\omega_2(\Delta) = 1$ and $sgn(\Delta) = +1$. On the right we show a holomorphic disk $\Delta' \in \Delta_a^{\leftarrow}(a_1)$, where $\omega_1(\Delta') = s$, and $\omega_2(\Delta') = s a_2$, and $sgn(\Delta') = -1$.

The DGA map $\Phi_a^{\text{comb}} : \mathcal{A}(\Lambda_+; \mathbb{Z}[s_1^{\pm}, \dots, s_k^{\pm}]) \to \mathcal{A}(\Lambda_-; \mathbb{Z}[s_1^{\pm}, \dots, s_k^{\pm}])$ induced by pinching at a proper Reeb chord a in Λ_+ is given by the composition $\Phi_a :=$

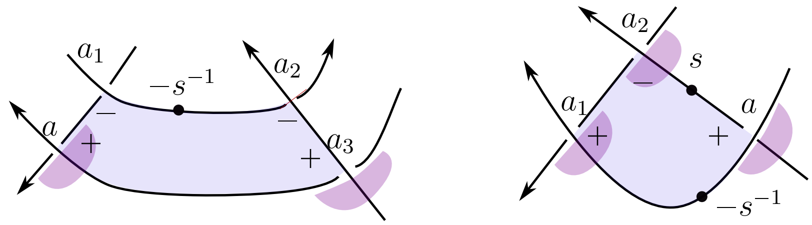


FIGURE 2.10: Two holomorphic disks $\Delta \in \Delta_a^{\leftarrow}(a_3)$ and $\Delta' \in \Delta_a^{\leftarrow}(a_1)$.

$\Phi^{\leftarrow} \circ \Phi^{\rightarrow} \circ \Phi^0$ where $\Phi_0(a) = s$ and for Reeb chords $a_i \neq a$

$$\Phi_0(a_i) := a_i$$

$$\Phi^{\rightarrow}(a_i) := a_i + \sum_{\Delta \in \Delta_a^{\rightarrow}(a_i)} (-1)^{|\omega_1(\Delta)|} sgn(\Delta)\Phi^{\rightarrow}(\omega_1(\Delta))s^{-1}\omega_2(\Delta)$$

$$\Phi^{\leftarrow}(a_i) := a_i + \sum_{\Delta \in \Delta_a^{\leftarrow}(a_i)} (-1)^{|\omega_1(\Delta)|} sgn(\Delta)\Phi^{\leftarrow}(\omega_1(\Delta))s^{-1}\omega_2(\Delta)$$

Theorem 17 (Casals, Ng [13])**.** *If $a \in \mathcal{R}(\Lambda)$ is a proper contractible Reeb chord, then the cobordism map $\Phi_a : \mathcal{A}(\Lambda_+; \mathbb{Z}[s_1^{\pm}, \ldots, s_k^{\pm}]) \to \mathcal{A}(\Lambda_-; \mathbb{Z}[s_1^{\pm}, \ldots, s_k^{\pm}])$ is equal to the combinatorial map $\Phi_{L_a}^{comb}$ up to a link automorphism of Λ_-.*

2.4 Weinstein 4-manifolds

A **Stein manifold** is a properly embedded complex submanifold of $\mathbb{C}^n$. Stein manifolds are necessarily noncompact and have a natural symplectic form given by restricting ω_{st} on $\mathbb{C}^n$. Smooth affine varieties are examples of Stein manifolds, as are complements of hyperplanes in closed complex projective manifolds. Grauert gave an intrinsic definition of a Stein manifold as follows [43]. For a complex manifold (X, J), any function $\phi : X \to \mathbb{R}$ has an associated 1-form $d^{\mathbb{C}}\phi := d\phi \circ J$ and an associated 2-form $\omega_\phi = dd^{\mathbb{C}}\phi$. A function ϕ is J-convex if $g_\phi(v, w) := \omega_\phi(v, Jw)$ defines a Riemannian metric, and exhausting if its proper and bounded from below. A complex manifold (V, J) that admits an exhausting J-convex function is Stein.

One way to study Stein manifolds, and more specifically to study their symplectic topology is to study Weinstein manifolds. One can show that every Stein manifold is also a Weinstein manifold, and that any Weinstein manifold is Weinstein homotopic to a Stein manifold. Weinstein manifolds have been a useful tool for understanding h-principles for Stein manifolds. Before defining Weinstein manifolds we give the definitions of a Liouville domain, and of a Lyapunov function.

Definition 18. *Let (X, ω) be a symplectic manifold with non-empty boundary, a 1-form is a **Liouville 1-form** if $d\lambda = \omega$. Then, the vector field Z that is ω-dual to λ ($\iota_Z \omega = \lambda$) is called the **Liouville vector field**. A domain $(X, \omega = d\lambda, Z)$ is a **Liouville domain** if the Liouville vector field is complete and transverse to the boundary, and points outward along the boundary. A **Liouville cobordism** $(W, \omega = d\lambda, Z)$ is a compact cobordism such that Z points outwards along $\partial_+ W$ and inwards along $\partial_- W$. A Liouville cobordism with $\partial_- W = \oslash$ is called a **Liouville domain**.*

For a Liouville cobordism $(W, \omega = d\lambda, Z)$, the condition that the Liouville vector field Z points outwards along the boundary $\partial_+ W$ implies that there is an induced contact structure on $\partial_+ W$ given by $ker(\lambda|_{\partial_+ W})$. Such a boundary is called **convex**. The boundary $\partial_- W$ along which Z points is **concave**. If (X, λ, Z) is a Liouville domain, its **completion** is the non-compact manifold constructed from X by attaching the symplectization of the boundary, $X \cup_\partial (\mathbb{R} \times \partial(X))$. A simple Liouville homotopy is a family of Liouville manifolds (X, λ_s, Z_s), $s \in [0, 1]$ such that there exists a smooth family of exhaustions $X = \cup_{k=1}^{\infty} X_s^k$ by compact domains $X_s^k \subset X$ with smooth boundaries along which Z_s is outward pointing. Two Liouville manifolds are **Liouville homotopic** if there exists a composition of simple Liouville homotopies taking one to the other

Definition 19. *A smooth function $\phi : X \to \mathbb{R}$ is a Lyapunov function for the vector* *field Z on X, and Z is gradient-like for ϕ, if*

$$Z \cdot \phi \leqslant |d\phi| \leqslant \frac{1}{\delta}(|X|^2 + |d\phi|^2) \tag{2.2}$$

for some $\delta > 0$, where $|Z|$ is the norm with respect to some Riemannian metric on V and $|d\phi|$ is the dual norm.

By the Cauchy-Schwarz inequality, if Z is gradient-like with respect to ϕ then

$$\delta|Z| \leqslant |d\phi| \leqslant \frac{1}{\delta}|Z|, \tag{2.3}$$

and the zeroes of Z are critical points of ϕ.

Definition 20. *A Weinstein manifold (X, ω, Z, ϕ) is an exact symplectic manifold (X, ω) with a complete Liouville field Z which is gradient-like for an exhausting Morse function $\phi : X \to \mathbb{R}$. A Weinstein cobordism (W, ω, Z, ϕ) is a Liouville cobordism (W, ω, Z), whose Liouville field Z is gradient-like for a Morse function $\phi : W \to \mathbb{R}$ which is constant on the boundary. In both cases the triple (ω, Z, ϕ) is called a Weinstein structure on X or W. A Weinstein cobordism with $\partial_- W = \oslash$ is called a Weinstein domain.*

As a basic example consider that $\mathbb{C}^n$ carries the canonical Weinstein structure given by

$$\omega_{st} = \sum_{j=1}^{n} dx_j \wedge dy_j \quad X_{st} = \frac{1}{2}(x_j \partial_{x_j} + y_j \partial_{y_j}), \quad \phi_{st} = \frac{1}{4}\sum_{j=1}^{n}({x_j}^2 + {y_j}^2).$$

The cotangent bundles of closed manifolds are also basic examples of Weinstein manifolds. In Chapter 7 we will explore the canonical Weinstein structure on the cotangent bundle of closed surfaces. A Weinstein domain (W, ω, Z, ϕ) can be completed into a Weinstein manifold by attaching to the boundary $\mathbb{R} \times \partial W$, the symplectization of the contact boundary. Although by definition every Weinstein manifold is Liouville, not every Liouville manifold is diffeomorphic to a Weinstein manifold.

Definition 21. *A **Weinstein homotopy** on a Weinstein cobordism or manifold is a smooth family of Weinstein structures* $(X_t, \lambda_t, Z_t, \phi_t)$*,* $t \in [0,1]$ *where we allow birth-death type degenerations such that the associated Liouville structures* (X_t, λ_t, Z_t) *form a Liouville homotopy. A **Stein homotopy** is a smooth family of Stein structures* (J_t, ϕ_t)*, where we allow birth-death type degenerations such that the associated Weinstein structures* $\mathbb{W}(J_t, \phi_t)$ *form a Weinstein homotopy.*

Note that if two Weinstein manifolds $W_0 = (V, \omega_0, X_0, \phi_0)$, and $W_1 = (V, \omega_1, X_1, \phi_1)$ are Weinstein homotopic they are symplectomorphic. Morse functions $\phi : M \to \mathbb{R}$ on smooth compact manifolds with no boundary encode the topology of the manifold. Each critical point p of ϕ has an index k associated to it, where k is the dimension of the maximal subspace of the tangent space T_pM where the Hessian is negative definite. Furthermore, if ϕ is a self-indexing Morse function (if p_1, p_2 are critical points of index k_1 and k_2 respectively and $\phi(p_1) < \phi(p_2)$ then $k_1 \leqslant k_2$), then it provides a handle decomposition of M. Recall that a handle decomposition of M is a thickened cellular decomposition:

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_m = M$$

where $m = \dim(M)$ and each M_k is obtained from M_{k-1} by attaching a k-handle. A k-handle, $\mathbf{D}^k \times \mathbf{D}^{m-k}$, where the index k critical point is at the origin, has the following components:

- The *core of the handle*, $\mathbf{D}^k \times \{0\}$, is the stable manifold of flow lines of $-\nabla\phi$ that limit positively towards zero at the origin.

- The *co-core of the handle*, $\{0\} \times \mathbf{D}^{m-k}$, is the unstable manifold of flow-lines of $-\nabla\phi$ that limit negatively towards the zero at the origin.

- The *attaching sphere* is the boundary of the core, $S^{k-1} \times \{0\}$.

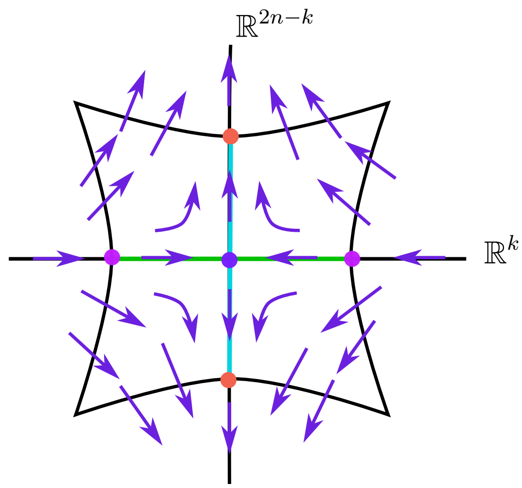


FIGURE 2.11: A Weinstein k-handle where the flow of the Liouville vector field is depicted by the purple arrows.

- The *attaching region* is a neighborhood of the $S^{k-1} \times \mathbf{D}^{m-k}$. An embedding $h : S^{k-1} \times \mathbf{D}^{m-k} \to M_{k-1}$ contains all of the attaching information of the k-handle to M_{k-1}.
- The *belt sphere* is the boundary of the co-core $\{0\} \times S^{m-k-1}$.

Weinstein domains were first introduced by Weinstein who gave a concrete model of a Weinstein k-handle shown in Figure 2.11. The model Weinstein handle H of index k in dimension $2n$ for $k \leqslant n$ is a subset of $\mathbb{R}^{2n}$ with coordinates $(x_1, y_1, \cdots, x_n, y_n)$, with the standard symplectic structure $\omega = \sum_j dx_j \wedge dy_j$ and Liouville vector field

$$Z_k = \sum_{j=1}^{k} \left(-x_j\partial_{x_j} + 2y_j\partial_{y_j}\right) + \sum_{j=k+1}^{n} \left(\frac{1}{2}x_j\partial_{x_j} + \frac{1}{2}y_j\partial_{y_j}\right).$$

This Liouville vector field is *gradient like* for the Morse function

$$\phi_k = \sum_{j=1}^{k} \left(-\frac{1}{2}x_j^2 + y_j^2\right) + \sum_{j=k+1}^{n} \left(\frac{1}{4}x_j^2 + \frac{1}{4}y_j^2\right).$$

The Liouville vector field is transverse to the boundary of $\mathcal{H}$. Indeed, $\mathcal{H}$ has boundary convex boundary $S^{k-1} \times \mathbf{D}^{m-k}$ (Z_k points outward to the handle), and concave boundary $\mathbf{D}^k \times S^{m-k-1}$ (Z_k points inward to the handle). Note that then Z_k has a dual 1-form λ_k which induces a contact structure on the convex boundary of $\mathcal{H}$, $(S^{k-1} \times \mathbb{D}^{m-k}, ker(\lambda_k))$. The components of the Weinstein k-handle have the following additional properties:

- The *core of the handle*, $\mathbf{D}^k \times \{0\}$ is a Lagrangian submanifold of $\mathcal{H}$
- The *co-core of the handle* $\{0\} \times \mathbf{D}^{m-k}$ is also a Lagrangian submanifold of $\mathcal{H}$.
- The *attaching sphere* $S^{k-1} \times \{0\}$ is an isotropic sphere in $(S^{k-1} \times \mathbf{D}^{m-k}, ker(\lambda_k))$.

A Weinstein handlebody decomposition can then be given explicity by attaching Weinstein handles each with a locally defined Morse functions which one can assemble into a global Morse function. Weinstein handlebody decompositions only contain Weinstein k-handles for $k \leqslant n$. In particular, if a Weinstein k-handle is of index $k < n$ we say it is a **subcritical handle** and if $k = n$ we say it is a **critical handle**. The attaching sphere of a critical handle is a Legendrian submanifold in the contact boundary. Thus, the Weinstein handle decomposition of any $2n$-dimensional Weinstein manifold (W, ω, Z, ϕ), is given by the Legendrian submanifold $\Lambda \subset (\partial W_0, \omega, Z, \phi)$ corresponding to the attaching spheres of the critical handles, where (W_0, ω, Z, ϕ) is the Weinstein subcritical domain given by attaching handles of index strictly less than n. One can then find exact Lagrangian surfaces using the Weinstein handle decomposition as follows. If $\Sigma \subset (W_0, \omega, Z, \phi)$ is an exact Lagrangian filling of Λ, then the union of Σ with the Lagrangian cores of the handles attached to Λ is a closed exact Lagrangian surface $\bar{\Sigma}$. See [12] for some combinatorial constructions of exact Lagrangian fillings for high dimensional Weinstein manifolds.

Weinstein 4-manifolds only admit handle decompositions with $0, 1$ and 2-handles. Attaching m Weinstein 1-handles to a Weinstein 0-handle (which is just $(\mathbf{D}^4, \omega_{std})$) gives the 4-manifold whose boundary is $(\#^m(S^1 \times S^2), \xi_{std})$. The attaching region of a 1-handle is $S^0 \times \mathbf{D}^3$ and it attaches to the boundary of the 0-handle, (S^3, ξ_{std}), the 3-sphere with the contact structure induced by the Liouville structure on $\mathbf{D}^4$. Diagrammatically, we consider S^3 as $\mathbb{R}^3$ with a point at infinity and draw the attaching region as a pair of 3-balls which are identified with a reflection thanks to the 1-handle which is not pictured. The contact manifold $(\#^m(S^1 \times S^2), \xi_{std})$ can be obtained from (S^3, ξ_{std}) by removing the interior of the 3-balls and gluing the boundaries by a contactomorphism which is explained in more detail in [42]. A 2-handle can now be attached and its attaching sphere is a Legendrian knot $\Lambda \subset (\#^m(S^1 \times S^2), \xi_{std})$. When attaching a 4-dimensional 2-handle it is not enough to know Λ, one must also know the framing of Λ. The framing of the 2-handle is an identification of the normal bundle of Λ with $S^1 \times \mathbb{R}^1$ and it is encoded by an integer $\mathbb{Z}$. In the Weinstein case, 2-handles have a canonical framing given by the contact structure. In particular, for a 4-dimensional Weinstein 2-handle attached along a null-homologous Legendrian link $\Lambda \subset (\#^m(S^1 \times S^2), \xi_{std})$ the framing is given by $tb(\Lambda) - 1$. Although the Thurston-Bennequin number is not invariant under all Legendrian isotopy in $(\#^m(S^1 \times S^2), \xi_{std})$, the canonical framing of the 2-handle is invariant under a contactomorphism.

Gompf showed that Legendrian links in $(\#^m(S^1 \times S^2), \xi_{std})$ can be isotoped into a standard form where the attaching spheres of the 1-handles are all vertically stacked and the Legendrian links can run through the 1-handles but are otherwise contained to an area between the pairs of 3-balls representing the 1-handles. Within this area the contact structure, thanks to a contactomorphism is $(\mathbb{R}^3, ker(dz - ydx)$, and we can therefore use the Lagrangian and front projections as before within this region. The Thurston-Bennequin number of a Legendrian $\Lambda \subset (\#^m(S^1 \times S^2), \xi_{std})$ can be computed from the front projection as in the case of a Legendrian link $\Lambda \subset \mathbb{R}^3_{std}$. Two Legendrian links in $(\#^m(S^1 \times S^2), \xi_{std})$ are related by Legendrian isotopy if and only if their front projections are related by a set of 6 moves [42], three of which are

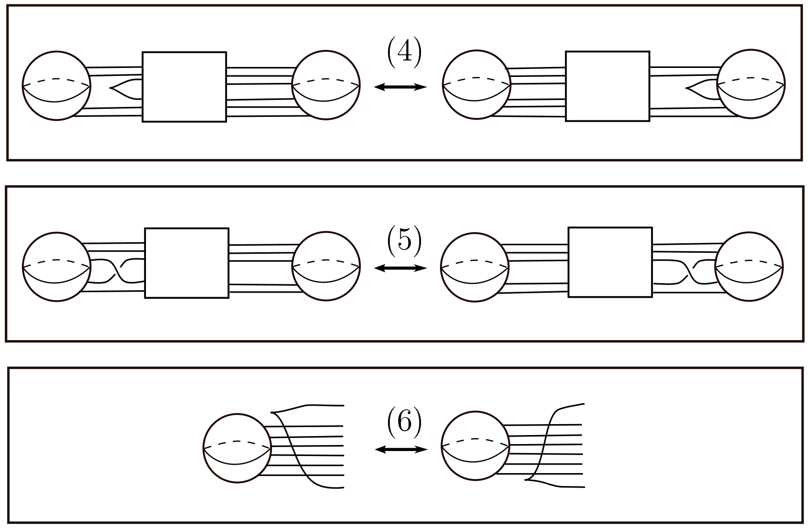


FIGURE 2.12: Gompf moves for Weinstein 4-manifolds.

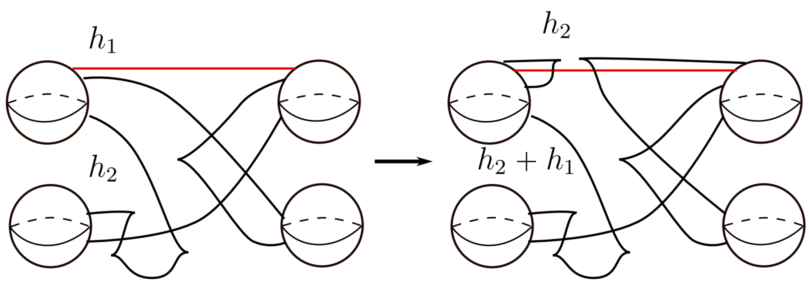


FIGURE 2.13: A Legendrian handle slide of h_2 over h_1.

the usual Legendrian Reidemeister moves and three that we will refer to as Gompf moves and are shown in Figure 2.12. Gompf move (4) and (5) corresponds to passing cusps and crossings through 1-handles respectively. Gompf move (6) corresponds to moving a Legendrian strand past the attaching region of a 1-handle. It is important to note that the Gompf move 6 does not preserve the tb-number of a Legendrian.

Two Weinstein in handlebody diagrams in Gompf standard form are equivalent as Weinstein domains if they are related by the following moves: isotropic or Legendrian isotopies of the attaching spheres, handle slides, handle cancellations and handle additions. If one has two Weinstein k-handles h_1 and h_2, a **handle slide** of h_1 over h_2 is given by isotoping the attaching sphere of h_1 and pushing it through the belt

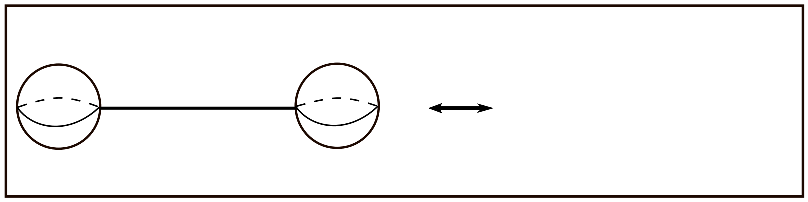


FIGURE 2.14: A canceling pair of 4-dimensional 1 and 2-handles.

sphere of h_2 where the isotopy is an isotropic or Legendrian isotopy. In the case of a handle slide for a 4-dimensional 1-handle, one drags one of the 3-balls in the attaching region of h_1, over h_2 and back following a Legendrian path. If h_1 and h_2 are 4-dimensional Weinstein handles, with attaching spheres $\Lambda_1, \Lambda_2 \subset (\#^m(S^1 \times S^2), \xi)$, then the Weinstein handlebody diagram after sliding h_1 over h_2 has attaching spheres Λ_1' and Λ_2 where Λ_1' has a front projection given by taking a connect sum with a pushoff of Λ_2 [21]. Figure 2.13 shows an example of a handle slide. One can add or cancel a pair of Weinstein handles h_1 and h_2 of index k and $k-1$ respectively if the attaching sphere of h_2 intersects the belt sphere of h_1 transversely at one point [12]. We call such a pair of handles a **canceling pair**, see Figure 2.14 for an example of a canceling pair.

3

Infinitely non-Hamiltonian isotopic Exact Lagrangian fillings of a link Λ

3.1 Distinguishing Lagrangian fillings with augmentations

In Section 3.2 of this Chapter we will prove the following Theorem and explore some its implications to Weinstein 4-manifolds.

Theorem 22. *The oriented Legendrian links* $\Lambda_n \subset \mathbb{R}^3_{std}$ *shown in Figure 3.1 have infinitely many exact Maslov* 0 *Lagrangian fillings up to Hamiltonian isotopy for* $n \geqslant 1$ *that are all smoothly isotopic.*

We prove Theorem 22 following the Floer theoretic techniques developed by Casals and Ng [13] which makes use of the fact that exact Maslov 0 Lagrangian fillings of a Legendrian link Λ can be distinguished by their induced systems of augmentations. Our construction also guarantees that all of the fillings are smoothly isotopic.

Definition 23 ([13])**.** *A Legendrian link* Λ *is* ***aug-infinite*** *if the collection of all* $\mathbb{Z}$ *valued augmentations* $(\mathcal{A}(\Lambda;\mathbb{Z}),\partial) \to (\mathbb{Z},0)$ *induced by Maslov* 0 *exact Lagrangian fillings of* Λ*, ranging over all such fillings, is infinite.*

Then, since each exact Lagrangian filling of a Legendrian link induces finitely many $\mathbb{Z}$ valued augmentations one can conclude the following:

Proposition 24 ([13]). *If Λ is aug-infinite then it has infinitely many exact Lagrangian fillings.*

Before 2020 there where limited results on the counts of exact Lagrangian fillings of Legendrian links, and there were no known examples of aug-infinite Legendrian links. In the past year, significant progress has been made on this problem and there are now many examples of aug-infinite Legendrian links. In contrast to the majority of Legendrian links that are now known to be aug-infinite, the Legendrian links Λ_n have negatively graded Reeb chords, and are not rainbow or (-1) closures of a positive braid. We now summarize all currently known counts of exact Maslov 0 Lagrangian fillings of Legendrian links $\Lambda \subset (S^3, \xi_{std})$ up to Hamiltonian isotopy.

- The max-tb Legendrian $(2, n)$ torus link have at least C_n Lagrangian fillings [28, 62, 70]

- The max-tb Legendrian (n, m) torus link $(n \geqslant 3, m \geqslant 6)$ have infinitely many Lagrangian fillings [11, 13, 14]

- The max-tb Legendrian (n, m) torus link $(n, m) = (4, 4), (4, 5)$ have infinitely many Lagrangian fillings [11, 13]

- The rainbow closure of the braid $\sigma_2\sigma_1\sigma_3\sigma_2^2\sigma_1\sigma_3\sigma_2\sigma_1^{n-4}\Delta^2 \in Br_4^+$ for $n \geqslant 4$ and $\Delta = \sigma_1(\sigma_2\sigma_1)\sigma_3\sigma_2\sigma_1$ have infinitely many Lagrangian fillings and are called Legendrian links of affine D_n type [13].

- The (-1) closure of the braid $(\sigma_2\sigma_1\sigma_1\sigma_2)^3\sigma_1^n \in Br_3^+$ for $n \geqslant 1$ have infinitely many Lagrangian fillings [13].

- The (-1) closures of the braids $\beta_{ab} \in Br_4^+$ where $\beta_{ab} = (\sigma_2\sigma_1\sigma_3\sigma_2)^4\sigma_3^a\sigma_1^b$ for $a, b \in \{1, 2\}$ have infinitely many Lagrangian fillings [13].

- The rainbow closures of the braids $\beta_{p,q} \in Br_3^+$ have infinitely many Lagrangian fillings, where $\beta_{p,q} = (\sigma_1^3\sigma_2)(\sigma_1^3\sigma_2^2)^p\sigma_1^3\sigma_2(\sigma_2\sigma_1^3)(\sigma_2^{q+1}\sigma_1^2\sigma_2^{p+2})$ for $p, q \in \mathbb{Z}_{\geqslant 1}$ [14].

- If Λ is the rainbow closure of a positive braid β such that Λ is not the split union of unknots and connect sums of standard ADE links, then it has infinitely many Lagrangian fillings [37]. The ADE Legendrian links are the rainbow closures of the following braids: $\sigma_1^{n+1} \in Br_2^+$ (A_n), $\sigma_1^{n-2}\sigma_2\sigma_1^2\sigma_2 \in Br_3^+$ (D_n), $\sigma_1^3\sigma_2\sigma_1^3\sigma_2 \in Br_3^+$ (E_6), $\sigma_1^4\sigma_2\sigma_1^3\sigma_2 \in Br_3^+$ (E_7), and $\sigma_1^5\sigma_2\sigma_1^3\sigma_2 \in Br_3^+$ (E_8).

- For any Legendrian links Λ_+ and Λ_- with rotation number 0 such that Λ_- has infinitely many Lagrangian fillings and the Lagrangian projection of Λ_- is obtained from the Lagrangian projection of Λ_+ by a sequence of saddle cobordisms at proper contractible Reeb chords of Λ_+ of degree 0, and all Reeb chords of Λ_+ have non-negative grading, then Λ_+ has infinitely many Lagrangian fillings [13]

We will also be considering the Legendrian links Λ'_m for $m \geqslant 1$ shown in Figure 3.2. These Legendrian links Λ'_m were first studied by Casals and Ng. Denote the (-1) closure of the braids $\beta_{ab} = (\sigma_2\sigma_1\sigma_3\sigma_2)^4\sigma_3^a\sigma_1^b$ for $a, b \in \mathbb{Z}_{\geqslant 1}$ by $\Lambda(\beta_{ab})$. The Legendrian links $\Lambda(\beta_{1b})$ are Legendrian isotopic to the Legendrian links Λ'_m for $b = m \in \mathbb{Z}_{\geqslant 1}$. Casals and Ng showed that $\Lambda(\beta_{ab})$ are aug-infinite for $a, b \geqslant 1$.

The primary motivation for studying these two families of Legendrian links is that we will find Weinstein handlebody decompositions of the Milnor fiber of a $T_{p,q,r}$ singularity that contains the sublinks Λ_2 and Λ'_2 for certain $p, q, r \in \mathbb{Z}_{\geqslant 0}$ in Chapter 4. We will then conclude that the Milnor fiber of any unimodular hypersurface singularity has infinitely many distinct exact Lagrangian tori as will be discussed in Chapter 4. To arrive at this result we have to consider a subset of augmentations

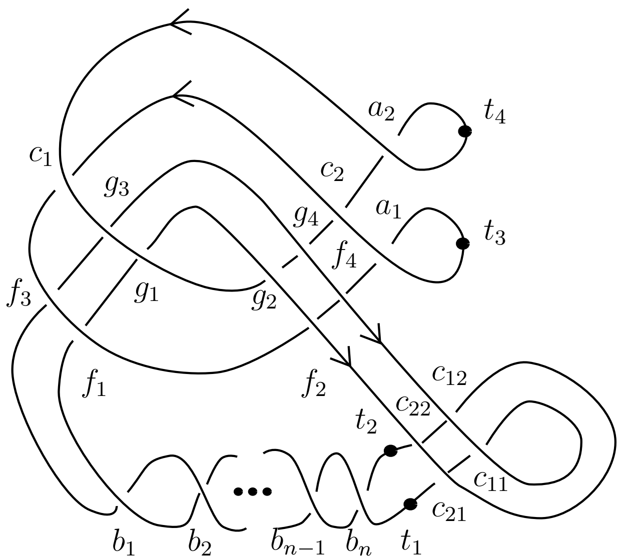


FIGURE 3.1: A family of Legendrian links Λ_n where one of the sublinks is the (-1) closure of $\sigma_1^n \in Br_2^+$ for $n \geqslant 1$.

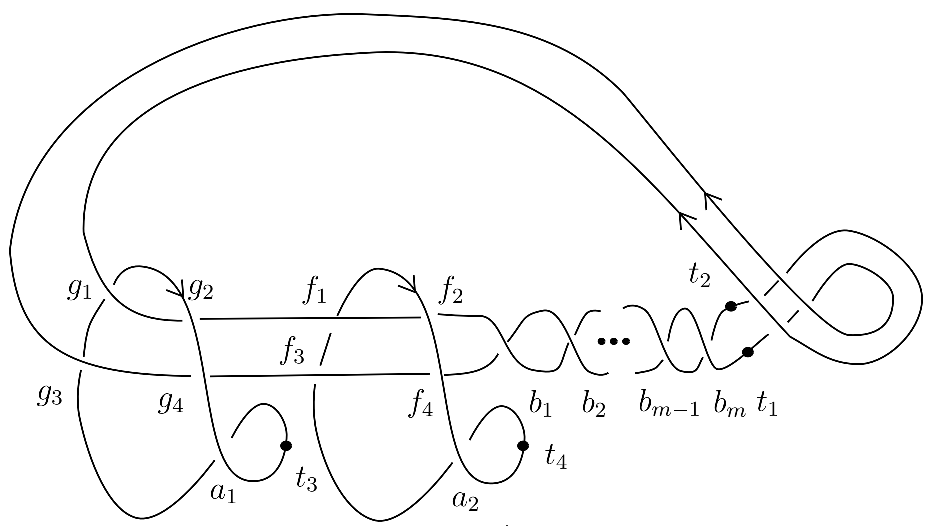


FIGURE 3.2: A family of Legendrian links Λ'_m where one of the sublinks is the (-1) closure of $\sigma_1^m \in Br_2^+$ for $m \geqslant 1$.

called **restricted augmentations** introduced by Casals and Ng. These will play an important role when considering closed Lagrangian surfaces in Weinstein 4-manifolds built from Lagrangian fillings.

Let X_Λ denote the Weinstein 4-manifold given by attaching 2-handles to a Legendrian link $\Lambda \subset (\#^k(S^1 \times S^2), \xi_{std})$. Then, the union of any exact Lagrangian filling Σ

of Λ' and the disjoint union of the Lagrangian cores of the 2-handles attached along Λ, is a closed, exact Maslov 0 Lagrangian surface which we will denote as $\bar{\Sigma}$.

Recall that for every Maslov 0 exact Lagrangian filling Σ of Λ there is an induced system of augmentations

$$\epsilon_\Sigma : (\mathcal{A}(\Lambda; \mathbb{Z}[H_1(\Sigma)]), \partial) \to (\mathbb{Z}[H_1(\Sigma)], 0).$$

One can compose this induced system of augmentations with a local system on $H_1(\Sigma)$ and obtain a family of $\mathbb{Z}$ valued augmentations. Such a composition is well behaved under equivalence of local systems and systems of augmentations. Since $H_1(\Sigma)$ and $H_1(\bar{\Sigma})$ are not in general isomorphic (namely, if Λ is a link with more than one component), then not all local systems on Σ extend to local systems on $\bar{\Sigma}$. We will now define a system of augmentations over $H_1(\Sigma)$ such that local systems on Σ do in fact extend to local systems on $\bar{\Sigma}$ (see Definition 7.9 in [13]).

Definition 25 (Casals, Ng [13])**.** *Suppose that Σ is an exact Lagrangian filling of a Legendrian link $\Lambda \subset (\#^k(S^1 \times S^2), \xi_{std})$ that induces a system of augmentations $\epsilon_\Sigma : (\mathcal{A}(\Lambda; \mathbb{Z}[H_1(\Sigma)]), \partial) \to (\mathbb{Z}[H_1(\Sigma)], 0)$, where Λ is equipped with the null-cobordant spin structure. The* ***restricted system of augmentations*** *associated to Σ is the composition*

$$\epsilon_{\bar{\Sigma}} : (\mathcal{A}(\Lambda; \mathbb{Z}[H_1(\Sigma)]), \partial) \to (\mathbb{Z}[H_1(\Sigma)], 0) \to (\mathbb{Z}[H_1(\bar{\Sigma})], 0)$$

where the second map is induced by the quotient map $H_1(\Sigma) \to H_1(\bar{\Sigma})$

Suppose one starts with the DGA of Λ $(\mathcal{A}(\Lambda; \mathbb{Z}[H_1(\Lambda)]), \partial)$ where Λ has the Lie group spin structure. To translate to the DGA where Λ has a null-cobordant spin structure we map $t_i \to -t_i$ where t_i is a basepoint on the link component Λ_i which represents $H_1(\Lambda_i)$. Since each homology component of Λ is null-homologous in $\bar{\Sigma}$, then any restricted augmentation defined over the Lie group spin structure must map $t_i \to -1$. In the case that Λ is a single-component Legendrian link, Leverson showed that any augmentation must map t to -1 [55]. This implies that for Legendrian knots, the restricted system of a filling is the same as the system of augmentations of the filling. It is also a natural geometric consequence of the fact that $H_1(\Sigma)$ is isomorphic to $H_1(\bar{\Sigma})$ for null-homologous knots Λ. It is also worth remarking that if two systems of restricted augmentations $\bar{\epsilon_1}, \bar{\epsilon_2}$ are equivalent as systems of augmentations over $\mathbb{Z}[s_1^{\pm}, \ldots, s_k^{\pm}]$, the system of augmentations ϵ_1, ϵ_2 are equivalent over $\mathbb{Z}[t_1^{\pm}, \ldots, t_n^{\pm}, s_1^{\pm}, \ldots, s_k^{\pm 1}]$. Thus, we can consider the subset of restricted augmentations within the set of augmentations of a Legendrian link, and count the number of restricted augmentations induced by exact Lagrangian fillings.

Definition 26. *A Legendrian link Λ is* ***restricted aug-infinite*** *if the collection of all $\mathbb{Z}$ valued restricted augmentations $(\mathcal{A}(\Lambda;\mathbb{Z}), \partial) \to (\mathbb{Z}, 0)$ induced by Maslov 0 exact Lagrangian fillings of Λ, ranging over all such fillings, is infinite.*

The following Proposition (see Proposition 7.11 in [13]) proves that Weinstein 4-manifolds constructed by attaching 2-handles along a restricted aug-infinite Legendrian link Λ to $(\mathbf{B^4}, \omega_{std})$ have infinitely many distinct exact Lagrangian surfaces.

Proposition 27 (Casals, Ng[13])**.** *Let $\Lambda \subset (S^3, \xi_{std})$ be a Legendrian link and $\Sigma_1, \Sigma_2 \subset (\mathbf{D^4}, \omega_{std})$ two exact Lagrangian fillings of Λ. Suppose that the two restricted systems of augmentations*

$$\epsilon_{\bar{\Sigma}_1} : \mathcal{A}(\Lambda) \to \mathbb{Z}[H_1(\overline{\Sigma_1})], \quad \epsilon_{\bar{\Sigma}_2} : \mathcal{A}(\Lambda) \to \mathbb{Z}[H_1(\overline{\Sigma_2})],$$

are not DGA homotopic. Then, the exact Lagrangian surfaces $\overline{\Sigma_1}, \overline{\Sigma_2} \subset W(\Lambda)$ are not Hamiltonian isotopic in the Weinstein 4-manifold $W(\Lambda)$.

The proof of this proposition relies on the the fact that for such a Weinstein 4-manifold, X_Λ, the Wrapped Fukaya Category of X_Λ, $\mathcal{W}(X_\Lambda)$ is generated by the union of co-cores of the Weinstein 2-handles of X_Λ [18]. Then, to argue that two fillings Σ_1, Σ_2 of Λ with distinct induced restricted augmentations, they show $C\mathcal{W}(C, \overline{\Sigma_1})$, and $C\mathcal{W}(C, \overline{\Sigma_2})$ are distinct as $C\mathcal{W}(C)$-modules.

3.2 Restricted aug-infinite Legendrian links in $\mathbb{R}^3_{std}$

In this section we will show that the Legendrian link Λ_n and Λ'_m are restricted aug-infinite for $n, m \geqslant 1$. Before proving Theorem 22, we first prove a generalization of Theorem 7.5 in [13]. Proposition 28 will allow us to prove that Λ_n and Λ_m are restricted aug-infinite for $n, m > 1$ from the fact that Λ_1 and Λ'_1 are restricted aug-infinite.

Proposition 28. *Let Λ_- be an aug-infinite Legendrian link and let Λ_+ be a Legendrian link such that $rot(\Lambda_\pm) = 0$ and there exists a decomposable exact Maslov 0 Lagrangian cobordism Σ from Λ_- to Λ_+. Suppose there exists a cycle $x \in (\mathcal{A}(\Lambda_+; R), \partial)$ such that $\Phi(x)$ is mapped to a cycle $a \in (\mathcal{A}(\Lambda_-; R), \partial)$ where Φ is the DGA map induced by Σ, and a is mapped to infinitely many distinct values in $\mathbb{Z}$ by the augmentations of Λ_-. Then Λ_+ is also aug-infinite. If we also impose the condition that Λ_- is restricted aug-infinite, then Λ_+ is restricted aug-infinite.*

Proof. First observe that we can concatenate Σ with any of the fillings Σ^i_- of Λ_- and obtain infinitely many exact Lagrangian fillings $\Sigma^i_+ = \Sigma \cup \Sigma^i_-$ of Λ_+. We will show that the systems of augmentations on Λ_+ induced by these fillings are pairwise distinct.

We now recall the following argument from [13], and make one modification since we do not assume that the Legendrian links $\Lambda_\pm$ only have Reeb chords of non-negative degree. Without loss of generality we can assume that the two Legendrian links Λ_- and Λ_+ are related by one pinch move at a proper contractible Reeb chord $a \in \mathcal{R}(\Lambda_+)$. Then, Λ_- has one more or less link component than Λ_+. If Λ_+ has m components, and one basepoint on each link component, each of these basepoints t_i are also on Λ_-. When we pinch at a, we use a basepoint s to track the saddle cobordism. The cobordism between Λ_+ and Λ_- induces a map $\Phi : (\mathcal{A}(\Lambda_+; \mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}, s^{\pm 1}]), \partial) \to (\mathcal{A}(\Lambda_-; \mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}, s^{\pm 1}]), \partial)$. If Λ_- has one more or one fewer link component, then we either add a basepoint or we consider the product of two basepoints as a single basepoint.

We have assumed that $(\mathcal{A}(\Lambda_-; \mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}, s^{\pm 1}]), \partial)$ has infinitely many distinct augmentations over $\mathbb{Z}$ induced by exact Lagrangian fillings. If we compose $\Phi$with the map taking t_i, s to ± 1, we obtain a map $\Phi^{\mathbb{Z}} : (\mathcal{A}(\Lambda_+; \mathbb{Z}, \partial) \to (\mathcal{A}(\Lambda_-; \mathbb{Z}), \partial)$ which satisfies $\Phi^{\mathbb{Z}}(a) = \pm 1$, and for all other Reeb chords of Λ_+, $\Phi^{\mathbb{Z}}(a_i) = a_i + f(a_i)$ where $f(a_i) \in \mathcal{R}(\Lambda_+)$. Since $\Phi^{\mathbb{Z}}$ respects the height filtration, it is a surjective map. When Λ_- has infinite many augmentations from fillings over $\mathbb{Z}$ then there exists a Reeb chord $a_i \in \mathcal{R}(\Lambda_-)$ sent to infinitely many values in $\mathbb{Z}$ under these augmentations. By the surjectivity of $\Phi^{\mathbb{Z}}$, there exists an $x \in (\mathcal{A}(\Lambda_+; \mathbb{Z}), \partial)$ such that $\Phi^{\mathbb{Z}}(x) = a_i$, therefore there is a Reeb chord x in Λ_+ that is mapped to infinitely many values in $\mathbb{Z}$ by the augmentations of induced by the fillings Σ_n of Λ_+. Moreover, we have assumed that there exists such an x that is also a cycle in $(\mathcal{A}(\Lambda_+; \mathbb{Z}), \partial)$. This implies that Λ_+ has a cycle x which is mapped to infinitely many values in $\mathbb{Z}$ by the augmentations induced by the fillings $\Sigma \circ \Sigma_-^i$ of Λ_+. Therefore, Λ_+ is aug-infinite.

Now suppose that Λ_- is also a restricted aug-infinite Legendrian link. We will show that restricted augmentations on $\Lambda-$ lift to restricted augmentations on Λ_+. Let ϵ_- denote a restricted augmentation induced by a filling on Λ_- and ϵ_+ the augmentation on Λ_+ induced by concatenating with the Lagrangian cobordism Σ. Then, for some i, j we have that

$$\epsilon_+(t_i) = \epsilon_-(t_i)s \quad \text{and} \quad \epsilon_+(t_j) = -\epsilon_-(t_j)s^{-1}.$$

If Λ_+ has one fewer link component than Λ_-, then $t_i t_j$ is a distinguished basepoint on Λ_+. Since Λ_- is assumed to be restricted aug-infinite, $\epsilon_-(t_i) = \epsilon_-(t_j) = -1$. Therefore $\epsilon_+(t_i t_j) = -\epsilon_-(t_i)\,\epsilon_-(t_j) s s^{-1} = -1$. If Λ_+ has one less link component than Λ_-, then $\epsilon_-(t_i t_j) = -1$. Choose a local system such that $\eta(s) = -\epsilon_-(t_i)$. Then, $\eta \circ \epsilon_+(t_i) = \epsilon_-(t_i)\eta(s) = -1$, and $\eta \circ \epsilon_+(t_j) = -\epsilon_-(t_j)\eta(s^{-1}) = -1$. Thus, a restricted augmentation on Λ_- lifts to a restricted augmentation on Λ_+. □

One way to construct different Lagrangian fillings of Legendrian links is through **Legendrian isotopy loops**. For any Legendrian link $\Lambda \subset \mathbb{R}^3_{std}$, consider the space of Legendrian links Legendrian isotopic to Λ, with basepoint an arbitrary but fixed Legendrian representative which we will denote by $\mathcal{L}(\Lambda)$. A Legendrian isotopy loop in $\mathcal{L}(\Lambda)$ is a map $\tilde{\Phi} : (S^1, pt) \to (\mathcal{L}(\Lambda), \Lambda)$. Any such loop $\tilde{\Phi}$ produces an exact Lagrangian concordance of Λ, which induces a DGA map

$$\Phi : \mathcal{A}(\Lambda_2; \mathbb{Z}[s_1^{\pm}, \ldots, s_k^{\pm}]) \to \mathcal{A}(\Lambda_2; \mathbb{Z}[s_1^{\pm}, \ldots, s_k^{\pm}]).$$

One can then construct an infinite family of Lagrangian fillings Σ^i by composing the i concordances produced by $\tilde{\Phi}^i$ and a specified exact Maslov 0 Lagrangian filling Σ for $i \in \mathbf{Z}_{\geqslant 0}$. Recall that exact Lagrangian filling Σ induces an augmentation $\epsilon : \mathcal{A}(\Lambda) \to (\mathbb{Z}[H_1(\Sigma)], 0)$. The exact Lagrangian filling Σ^i by construction induces an augmentation $\epsilon \circ \Phi^i : \mathcal{A}(\Lambda) \to (\mathbb{Z}[H_1(\Sigma^i)], 0)$. It is a natural question to ask whether the systems of augmentations $\epsilon \circ \Phi^i$ are distinct for different $i \in \mathbb{Z}_{\geqslant 0}$. This construction of exact Lagrangian fillings is first proposed in [11], and further developed in [13] where θ-loops were introduced.

To show that the family of augmentations $\epsilon \circ \Phi^i$ are pairwise distinct as systems of augmentations it is useful to consider the following function defined in [13]:

$$E(i, a) := \max_{\eta : \mathbb{Z}[s_1^{\pm 1}, \ldots, s_k^{\pm}] \to \mathbb{Z}} \left| \eta \circ \epsilon \circ \Phi^i \right|$$

for $i \in \mathbb{Z}_{\geqslant 0}$ and $a \in \mathcal{R}(\Lambda)$. Observe that the unital ring isomorphisms $\eta : \mathbb{Z}[s_1^{\pm 1}, \ldots, s_k^{\pm}] \to \mathbb{Z}$ which map s_i to ± 1 are in one-to-one correspondence with the composition of the

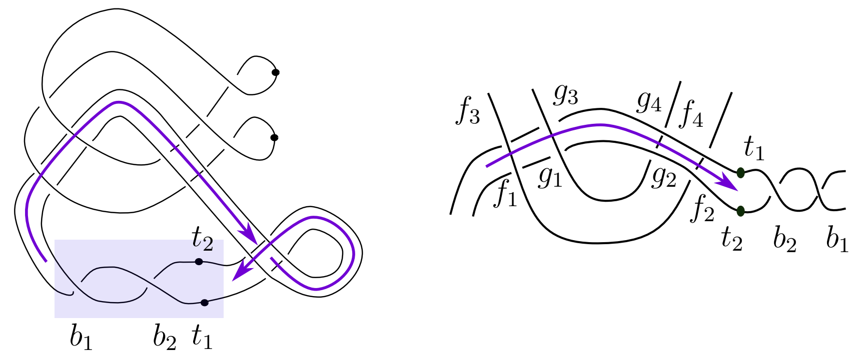


FIGURE 3.3: The purple arrow indicates the Legendrian isotopy of Λ_2 which moves the b_1, b_2 Reeb chords and t_1, t_2 basepoints clockwise.

augmentation $\epsilon \circ \Phi^n$ with a local system of the exact Lagrangian filling Σ^i. If for distinct non-negative integers i, and j, $E(i,a) \neq \pm E(j,a)$ for a Reeb chord a of Λ that a cycle in the DGA, one can conclude that $\epsilon \circ \Phi^i \neq \epsilon \circ \Phi^j$, as argued in Remark 7. Thus if $E(i,a)$ is a strictly increasing function over i for a Reeb chord a of Λ that is a cycle in the DGA $\mathcal{A}(\Lambda)$, then Λ is aug-infinite. To show that a Legendrian is restricted aug-infinite we consider instead the following invariant:

$$E_r(i,a) := \{max_{\eta:R\to\mathbb{Z}}|(\eta \circ \epsilon_\Sigma \circ \Phi^i)(a)| \mid \epsilon_\Sigma \text{ is a restricted augmentation}\}$$

Again, if $E_r(i,a)$ is a strictly increasing function over k for a Reeb chord a that is a cycle in the DGA of Λ, then Λ has infinitely distinct $\mathbb{Z}$ valued restricted augmentations induced by Lagrangian fillings.

This proof strategy to show that a Legendrian link Λ is aug-infinite or restricted aug infinite has further implications. That is, if $E(k,a)$ is a strictly increasing function over k for a Reeb chord a that is also a cycle, then the Legendrian isotopy loop $\Phi : (S^1, pt) \to (\mathcal{L}(\Lambda), \Lambda)$ is an loop of infinite order on $\mathcal{L}(\Lambda)$ whose induced monodromy action on the Legendrian contact homology DGA is of infinite order. Observe that we cannot only consider restricted augmentations to come to this conclusion. Moreover, in contrast to the property of being aug-infinite or restricted aug-infinite, the existence of a Legendrian isotopy loop of infinite order cannot be shown for a

Legendrian by building a Lagrangian cobordism from a Legendrian link that has an infinite order Legendrian loop. We prove that Λ_1 has a Legendrian loops of infinite order. The existence of a Legendrian isotopy loop of infinite order is also known for Λ_1' by work of Casals and Ng, and expected for Λ_n and Λ_m' for $n, m > 1$.

The Legendrian isotopy loops that we consider for Λ_1 consists of taking the Legendrian sublink that is a (-1) closure of $\sigma_1^n \in Br_2^+$ and moving the n positive crossings to the left and around the neighborhood of the max-tb Legendrian unknot as shown in Figure 3.2. Such a Legendrian loop involves only Reidemeister $R3_b$ moves and basepoint moves. In fact, we can write down the chain maps using braids as described in Section 2.3.1. We also provide an explicit proof of Λ_2 being restricted aug-infinite, even though it will also follow from the fact that Λ_1 is restricted aug-infinite as will be argued in Corollary 31.

Remark 29. *Any infinite family of Lagrangian fillings constructed by stacking concordances from as described above are smoothly isotopic. Therefore, if they are not Hamiltonian isotopic, they are examples of symplectically knotted surfaces with fixed boundary. Suppose that Λ_- is a Legendrian link with infinitely many distinct Lagrangian fillings Σ_-^i that are symplectically knotted, and Λ_+ is a Legendrian link such that there exists a Maslov 0 Lagrangian cobordism Σ from Λ_- to Λ_+. Then, $\Sigma \cup \Sigma_-^i$ are smoothly isotopic by construction. If they are also non-Hamiltonian isotopic Lagrangian fillings of Λ_+, then we can conclude that Λ_+ also bounds symplectically knotted surfaces.*

Proposition 30. *The Legendrian links Λ_1, and $\Lambda_2 \subset \mathbb{R}^3_{std}$ shown in Figure 3.1 are restricted aug-infinite. Moreover Λ_1 has an infinite order Legendrian isotopy loop.*

Proof. Let $\tilde{\Phi}_1 : (S^1, pt) \to (\mathcal{L}(\Lambda_1), \Lambda_1)$ be the Legendrian loop of the Legendrian link Λ_1 given by taking the crossings and basepoints labeled b_1, t_1, t_2 and moving them similarly to the Legendrian loop shown in Figure 3.2. This gives an automorphism

$\Phi_1 : \mathcal{A}(\Lambda_1; \mathbb{Z}[H_1(\Lambda_1)]) \to \mathcal{A}(\Lambda_1; \mathbb{Z}[H_1(\Lambda_1)])$ defined as follows. Let

$$M_1 := \begin{pmatrix} 0 & 1 \\ 1 & b_1 \end{pmatrix} \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} = \begin{pmatrix} 0 & t_2 \\ t_1 & t_2 b_1 \end{pmatrix},$$

$$\widetilde{M_1} := \begin{pmatrix} 0 & 1 \\ 1 & -b_1 \end{pmatrix} \begin{pmatrix} t_2 & 0 \\ 0 & t_1 \end{pmatrix} = \begin{pmatrix} 0 & t_1 \\ t_2 & -t_1 b_1 \end{pmatrix}.$$

Then,

$$\Phi_1(b_i) = b_i, \quad \Phi_1(t_i) = t_i \text{ for all } i, \quad \Phi_1(C) = M_1 C M_1^{-1}$$

$$\begin{pmatrix} \Phi_1(f_2) \\ \Phi_1(f_4) \end{pmatrix} = (\widetilde{M_1})^{-1} \begin{pmatrix} f_2 \\ f_4 \end{pmatrix}, \quad \begin{pmatrix} \Phi_1(g_2) \\ \Phi_1(g_4) \end{pmatrix} = (\widetilde{M_1})^{-1} \begin{pmatrix} g_2 \\ g_4 \end{pmatrix},$$

$$\begin{pmatrix} \Phi_1(g_3) \\ \Phi_1(g_1) \end{pmatrix} = (\widetilde{M_1})^{T} \begin{pmatrix} g_3 \\ g_1 \end{pmatrix}, \quad \begin{pmatrix} \Phi_1(f_3) \\ \Phi_1(f_1) \end{pmatrix} = (\widetilde{M_1})^{T} \begin{pmatrix} f_3 \\ f_1 \end{pmatrix},$$

where C is the matrix with entries c_{ij} for $i, j \in \{1, 2\}$. Consider the decomposable exact Maslov 0 Lagrangian filling Σ_1 given by first pinching the Reeb chords g_2, f_4, f_3, g_1, f_2 in order, then performing a Legendrian $R(2)$ move and finally composing with four minimum cobordisms. We will compute the DGA maps induced by Σ_1 for all degree-0 Reeb chords. We will use coefficients in $R_1 := \mathbb{Z}[s_1^{\pm 1}, \ldots, s_5^{\pm 1}]$ and set an isomorphism $R_1 \cong \mathbb{Z}[H_1(\Sigma)]$. The filling Σ_1 of Λ_1 constructed by this pinching sequence induces an augmentation ϵ_{Σ_1}. The specified isotopy loop $\tilde{\Phi}_1$, produces a DGA map $\Phi_1 : \mathcal{A}(\Lambda_1; R_1) \to \mathcal{A}(\Lambda_1; R_1)$, and by composing the chain map induced by the isotopy with the augmentation we obtain the Φ_1-orbit of ϵ_{Σ_1}:

$$\epsilon_{\Sigma_1} \circ \Phi_1^n : \mathcal{A}(\Lambda_1; R_1) \to R_1$$

We will start by describing the DGA map induced by exact Lagrangian filling Σ_1. Pinching g_2 has an effect on the Reeb chords g_1, f_4, and b_2. The pinch at f_4 has an effect only on the Reeb chords f_1, f_3, f_4, g_1, and g_3:

$$\begin{aligned}
f_1 &\to f_1 && \to f_1 + g_4 s_4^{-1} s_2^{-1}\\
f_2 &\to f_2 && \to f_2\\
f_3 &\to f_3 && \to f_3 - s_4^{-1}\\
f_4 &\to f_4 + g_4 f_2 s_2^{-1} && \to f_4 + g_4 f_2 s_2^{-1}\\
g_1 &\to g_1 - s_2^{-1} && \to g_1 - f_2 g_4 s_2^{-2} s_4^{-1} - s_2^{-1}\\
g_2 &\to s_2 && \to s_2\\
g_3 &\to g_3 && \to g_3 + f_2 s_2^{-1} s_4^{-1}\\
g_4 &\to g_4 && \to g_4\\
b_1 &\to b_1 - t_1^{-1} g_4 t_2 s_2^{-1} && \to b_2 - t_1^{-1} g_4 t_2 s_2^{-1}
\end{aligned}$$

Pinching at f_3 has an effect on f_2, f_3, g_1, and g_4.

$$\begin{aligned}
f_1 &\to f_1 + (g_4 - s_2 f_1 s_3^{-1}) s_4^{-1} s_2^{-1}\\
f_2 &\to f_2 - s_2 g_3 s_3^{-1}\\
f_3 &\to s_3 - s_4^{-1}\\
f_4 &\to s_4 + (g_4 - s_2 f_1 s_3^{-1})(f_2 - s_2 g_3 s_3^{-1}) s_2^{-1}\\
g_1 &\to g_1 - (f_2 - s_2 g_3 s_3^{-1})(g_4 - s_2 f_1 s_3^{-1}) s_2^{-2} s_4^{-1} - s_2^{-1} + g_3 f_1 s_3^{-1}\\
g_2 &\to s_2\\
g_3 &\to g_3 + (f_2 - s_2 g_3 s_3^{-1}) s_2^{-1} s_4^{-1}\\
g_4 &\to g_4 - s_2 f_1 s_3^{-1}\\
b_1 &\to b_1 - t_1^{-1}(g_4 - s_2 f_1 s_3^{-1}) t_2 s_2^{-1}
\end{aligned}$$

Pinching at g_1 has no effect on any Reeb chord except for g_1 which is sent to s_1.

Pinching at f_2 has an effect on f_1 and b_2

$$f_1 \to f_1 + (g_4 - s_2(f_1 - s_2 s_1 s_5^{-1}) s_3^{-1}) s_4^{-1} s_2^{-1} - s_2 s_1 s_5^{-1}$$

$$f_2 \to s_5 - s_2 g_3 s_3^{-1}$$

$$f_3 \to s_3 - s_4^{-1}$$

$$f_4 \to s_4 + (g_4 - s_2(f_1 - s_2 s_1 s_5^{-1}) s_3^{-1})(s_5 - s_2 g_3 s_3^{-1}) s_2^{-1}$$

$$g_1 \to s_1 - (s_5 - s_2 g_3 s_3^{-1})(g_4 - s_2(f_1 - s_2 s_1 s_5^{-1}) s_3^{-1}) s_2^{-2} s_4^{-1} - s_2^{-1} + g_3(f_1 - s_2 s_1 s_5^{-1}) s_3^{-1}$$

$$g_2 \to s_2$$

$$g_3 \to g_3 + (s_5 - s_2 g_3 s_3^{-1}) s_2^{-1} s_4^{-1}$$

$$g_4 \to g_4 - s_2(f_1 - s_2 s_1 s_5^{-1}) s_3^{-1}$$

$$b_1 \to b_1 - t_1^{-1}(g_4 - s_2(f_1 - s_2 s_1 s_5^{-1}) s_3^{-1}) t_2 s_2^{-1} - t_1^{-1} s_4 s_5^{-1} t_2$$

We then perform a Legendrian Reidemeister $R2$ whose induced DGA map maps c_1, g_3 to 0 and is the identity map on all other Reeb chords. Finally we compose with four minimum cobordisms and obtain the augmentation ϵ_{Σ_1} of Λ_1 induced by Σ_1.

$$\epsilon_{\Sigma_1}(f_1) = s_1 s_2 s_3^{-1} s_4^{-1} s_5^{-1} - s_2 s_1 s_5^{-1} \qquad \epsilon_{\Sigma_1}(g_1) = s_1 - s_1 s_3^{-1} s_4^{-1} - s_2^{-1}$$

$$\epsilon_{\Sigma_1}(f_2) = s_5 \qquad \epsilon_{\Sigma_1}(g_2) = s_2$$

$$\epsilon_{\Sigma_1}(f_3) = s_3 - s_4^{-1} \qquad \epsilon_{\Sigma_1}(g_3) = s_5 s_2^{-1} s_4^{-1}$$

$$\epsilon_{\Sigma_1}(f_4) = s_4 + s_2 s_1 s_3^{-1} \qquad \epsilon_{\Sigma_1}(g_4) = s_2^2 s_1 s_5^{-1} s_3^{-1}$$

$$\epsilon_{\Sigma_1}(t_1) = s_1 s_2 s_4 s_5^{-1} \qquad \epsilon_{\Sigma_1}(t_2) = -(s_3 s_5)^{-1}$$

$$\epsilon_{\Sigma_1}(t_3) = -s_3 s_4 \qquad \epsilon_{\Sigma_1}(t_4) = -s_1 s_2$$

$$\epsilon_{\Sigma_1}(b_1) = s_3 s_5^2 (s_1 s_2 s_4)^{-1} [s_3^{-1} s_2 s_1 s_5^{-1} + s_4 s_5^{-1}]$$

We now need to show that the loop $\tilde{\Phi}_1$ produces infinitely many distinct systems of augmentations of Λ_1. In particular, we will consider the Reeb chord f_2, which is a cycle and compute $\epsilon_{\Sigma_1} \circ \Phi_1^n(f_2)$. Recall that

$$\begin{pmatrix} \epsilon_{\Sigma_1} \circ \Phi_1^k(f_2) \\ \epsilon_{\Sigma_1} \circ \Phi_1^k(f_4) \end{pmatrix} = \epsilon_{\Sigma_1}((\widetilde{M_1}^{-1})^n) \begin{pmatrix} \epsilon_{\Sigma_1}(f_2) \\ \epsilon_{\Sigma_1}(f_4) \end{pmatrix} = \epsilon_{\Sigma_1}(\tilde{M}_1)^n \begin{pmatrix} \epsilon_{\Sigma_1}(f_2) \\ \epsilon_{\Sigma_1}(f_4) \end{pmatrix}.$$

For the fillings we have constructed:

$$\epsilon_{\Sigma_1}(\widetilde{M_1}^{-1}) = \epsilon_{\Sigma}(\begin{pmatrix} b_1 t_2^{-1} & t_2^{-1} \\ t_1^{-1} & 0 \end{pmatrix})$$

$$= \begin{pmatrix} -s_5(s_1s_2s_4)^{-1}[s_3^{-1}s_2s_1s_5^{-1} + s_4s_5^{-1}] & -s_3s_5 \\ (s_1s_2s_4)^{-1}s_5 & 0 \end{pmatrix},$$

and $\epsilon_{\Sigma_1}(f_2) = s_5$ and $\epsilon_{\Sigma_1}(f_4) = s_4 + s_2s_1s_3^{-1}$. For any choice of a local system η on these fillings, $\eta \circ \epsilon_{\Sigma_1}(b_1) \in \{0, \pm 2\}$. If η is a local system such that $\eta \circ \epsilon_{\Sigma_1}(b_1) = 0$, then $\eta \circ \epsilon_{\Sigma_1} \circ \Phi_1^k(f_2)(a) = \epsilon_{\Sigma_1}(a)$ for all k and any Reeb chord a of Λ_1. If instead, η is a local system such that $\eta \circ \epsilon_{\Sigma_1}(b_1) = \pm 2$, then we know that $\eta(s_3^{-1}s_2s_1) = \eta(s_4)$, and that $\eta \circ \epsilon_{\Sigma_1}(b_1) = 2\eta(s_5s_4^{-1})$. Therefore, there are four local systems which may maximize $\eta \circ \epsilon_{\Sigma_1} \circ \Phi_1^k(f_2)(f_2)$ or $\eta \circ \epsilon_{\Sigma_1} \circ \Phi_1^k(f_2)(f_4)$ which take the following values:

$$\begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}^k \begin{pmatrix} -1 \\ 2 \end{pmatrix} = (-1)^{k+1} \begin{pmatrix} 3k+1 \\ -2+3k \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}^k \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} -(1-\sqrt{2})^k(1+\sqrt{2}) - (1+\sqrt{2})^{k+1}(3+2\sqrt{2}) \\ -(1-\sqrt{2})^k(3+2\sqrt{2}) + (1+\sqrt{2})^k(3-2\sqrt{2}) \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^k \begin{pmatrix} -1 \\ -2 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} (1-\sqrt{2})^k(1-\sqrt{2}) + (1+\sqrt{2})^{k+1}(1+2\sqrt{2}) \\ (1-\sqrt{2})^k(-1+2\sqrt{2}) + (1+\sqrt{2})^k(-3+2\sqrt{2}) \end{pmatrix}$$

$$\begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix}^k \begin{pmatrix} 1 \\ 2 \end{pmatrix} = (-1)^k \begin{pmatrix} 3k+1 \\ -3k+1 \end{pmatrix}.$$

Therefore, $E(k, f_2) = \frac{1}{2\sqrt{2}}|-(1-\sqrt{2})^k(1+\sqrt{2}) - (1+\sqrt{2})^{k+1}(3+2\sqrt{2})|$, which is a strictly increasing function over k. Therefore, Λ_1 is aug-infinite, and $\tilde{\Phi_1}$ is an infinite order Legendrian isotopy loop. For any restricted augmentation $\eta \circ \epsilon_{\Sigma_1}(\widetilde{M}_1^{-1})$ is $\pm \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix}$ and $\eta \circ \epsilon_{\Sigma_1}(f_2) = 1, \eta \circ \epsilon_{\Sigma_1}(f_4) = 2$. So $E_r(k, f_2) = 1 + 3k$, which a strictly increasing function over k. Therefore, Λ_1 is restricted aug-infinite.

We will now proceed to show that Λ_2 is also restricted aug-infinite. Let $\tilde{\Phi}_2$ be the Legendrian loop of the Legendrian link Λ_2 given by taking the crossings and basepoints labeled b_1, b_2, t_1, t_2 and moving them to the left as shown in Figure 3.2. This gives an automorphism $\Phi_2 : \mathcal{A}(\Lambda_2; \mathbb{Z}[H_1(\Lambda_2)]) \to \mathcal{A}(\Lambda_2; \mathbb{Z}[H_1(\Lambda_2)])$ defined as follows. Let

$$M_2 := \begin{pmatrix} 0 & 1 \\ 1 & b_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & b_2 \end{pmatrix} \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} = \begin{pmatrix} t_1 & t_2 b_1 \\ t_1 b_2 & t_2 + t_2 b_1 b_2 \end{pmatrix},$$

$$\widetilde{M_2} := \begin{pmatrix} 0 & 1 \\ 1 & -b_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -b_2 \end{pmatrix} \begin{pmatrix} t_2 & 0 \\ 0 & t_1 \end{pmatrix} = \begin{pmatrix} t_2 & -t_1 b_1 \\ -t_2 b_2 & t_1 + t_1 b_1 b_2 \end{pmatrix}.$$

Then,

$$\Phi_2(b_i) = b_i \ \Phi_2(t_i) = t_i \text{ for all } i, \ \Phi_2(C) = M_2 C M_2^{-1}$$

$$\begin{pmatrix} \Phi_2(f_2) \\ \Phi_2(f_4) \end{pmatrix} = (\widetilde{M_2})^{-1} \begin{pmatrix} f_2 \\ f_4 \end{pmatrix}, \quad \begin{pmatrix} \Phi_2(g_2) \\ \Phi_2(g_4) \end{pmatrix} = (\widetilde{M_2})^{-1} \begin{pmatrix} g_2 \\ g_4 \end{pmatrix},$$

$$\begin{pmatrix} \Phi_2(g_3) \\ \Phi_2(g_1) \end{pmatrix} = (\widetilde{M_2})^{T} \begin{pmatrix} g_3 \\ g_1 \end{pmatrix}, \quad \begin{pmatrix} \Phi_2(f_3) \\ \Phi_2(f_1) \end{pmatrix} = (\widetilde{M_2})^{T} \begin{pmatrix} f_3 \\ f_1 \end{pmatrix}.$$

Consider the decomposable exact Maslov 0 Lagrangian filling Σ_2 given by first pinching the Reeb chords $g_2, f_4, f_3, g_1, f_2, f_1$ in order as shown in Figure 3.4, then performing a Legendrian $R2$ move and finally composing with four minimum cobordisms. We will use coefficients in $R_2 := \mathbb{Z}[s_1^{\pm 1}, \ldots, s_6^{\pm 1}] \cong \mathbb{Z}[H_1(\Sigma)]$. The filling Σ_2 of Λ_2 constructed by this pinching sequence induces an augmentation ϵ_{Σ_2}.

We will start by describing the DGA map induced by exact Lagrangian filling Σ_2 on degree 0 Reeb chords. The pinching sequence g_2, f_4, f_3 and g_1 on Λ_2 gives the

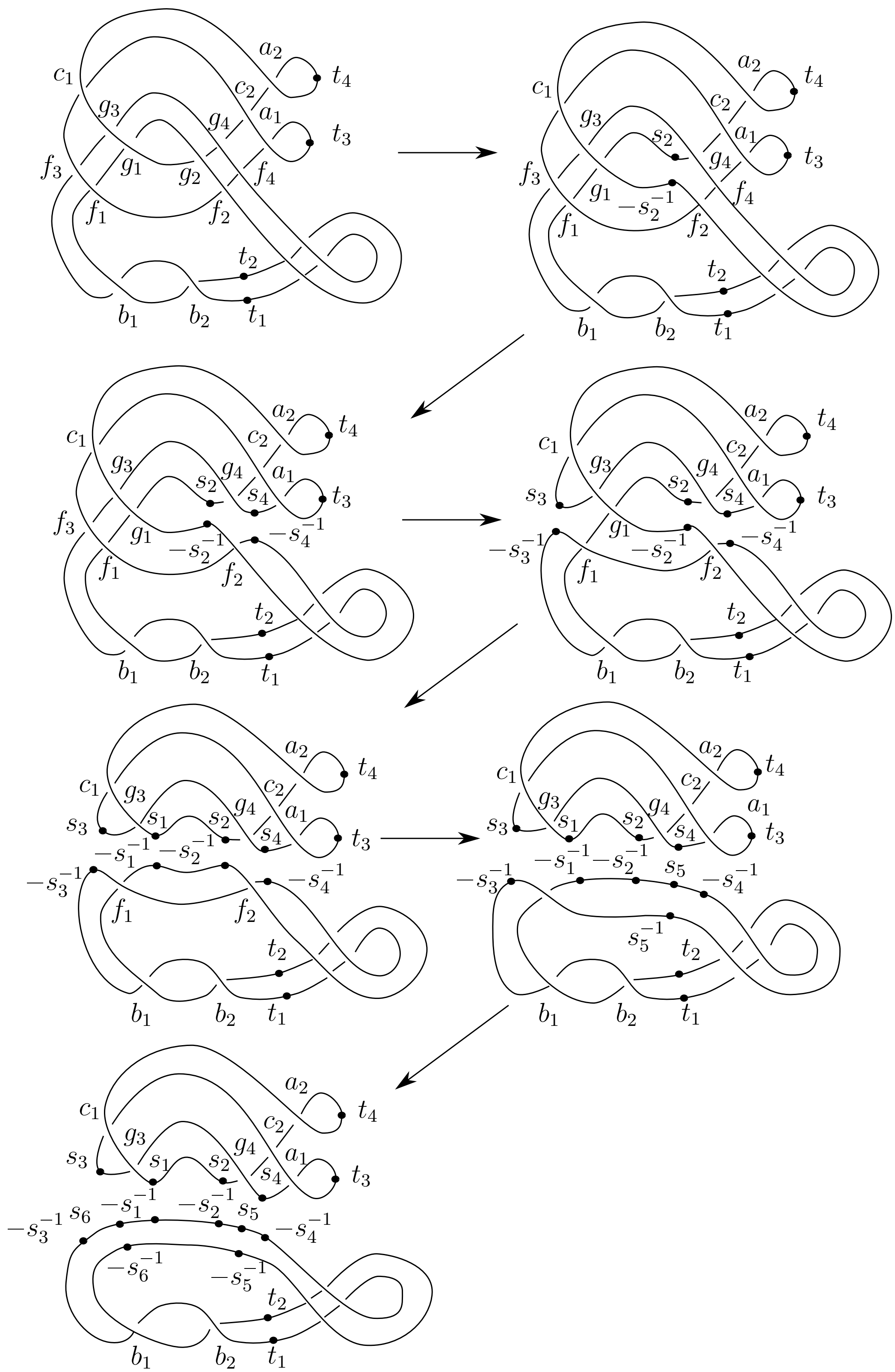


FIGURE 3.4: The exact Lagrangian filling Σ_2 of Λ given by pinching the Reeb chords g_2, f_4, f_3, g_1, f_1,and f_2.

following DGA map.

$$f_1 \to f_1 + (g_4 - s_2 f_1 s_3^{-1}) s_4^{-1} s_2^{-1}$$

$$f_2 \to f_2 - s_2 g_3 s_3^{-1}$$

$$f_3 \to s_3 - s_4^{-1}$$

$$f_4 \to s_4 + (g_4 - s_2 f_1 s_3^{-1})(f_2 - s_2 g_3 s_3^{-1}) s_2^{-1}$$

$$g_1 \to s_1 - (f_2 - s_2 g_3 s_3^{-1})(g_4 - s_2 f_1 s_3^{-1}) s_2^{-2} s_4^{-1} - s_2^{-1} + g_3 f_1 s_3^{-1}$$

$$g_2 \to s_2$$

$$g_3 \to g_3 + (f_2 - s_2 g_3 s_3^{-1}) s_2^{-1} s_4^{-1}$$

$$g_4 \to g_4 - s_2 f_1 s_3^{-1}$$

$$b_1 \to b_1$$

$$b_2 \to b_2 - t_1^{-1}(g_4 - s_2 f_1 s_3^{-1}) t_2 s_2^{-1}$$

Pinching at f_2 has an effect on f_1 and b_2

$$f_1 \to f_1 + (g_4 - s_2(f_1 - s_2 s_1 s_5^{-1}) s_3^{-1}) s_4^{-1} s_2^{-1} - s_2 s_1 s_5^{-1}$$

$$f_2 \to s_5 - s_2 g_3 s_3^{-1}$$

$$f_3 \to s_3 - s_4^{-1}$$

$$f_4 \to s_4 + (g_4 - s_2(f_1 - s_2 s_1 s_5^{-1}) s_3^{-1})(s_5 - s_2 g_3 s_3^{-1}) s_2^{-1}$$

$$g_1 \to g_1 - (s_5 - s_2 g_3 s_3^{-1})(g_4 - s_2(f_1 - s_2 s_1 s_5^{-1}) s_3^{-1}) s_2^{-2} s_4^{-1} - s_2^{-1} + g_3(f_1 - s_2 s_1 s_5^{-1}) s_3^{-1}$$

$$g_2 \to s_2$$

$$g_3 \to g_3 + (s_5 - s_2 g_3 s_3^{-1}) s_2^{-1} s_4^{-1}$$

$$g_4 \to g_4 - s_2(f_1 - s_2 s_1 s_5^{-1}) s_3^{-1}$$

$$b_1 \to b_1$$

$$b_2 \to b_2 - t_1^{-1}(g_4 - s_2(f_1 - s_2 s_1 s_5^{-1}) s_3^{-1}) t_2 s_2^{-1} - t_1^{-1} s_4 s_5^{-1} t_2$$

Pinching at f_1 has an effect on b_1 and b_2

$$f_1 \to s_6 + (g_4 - s_2(s_6 - s_2 s_1 s_5^{-1}) s_3^{-1}) s_4^{-1} s_2^{-1} - s_2 s_1 s_5^{-1}$$

$$f_2 \to s_5 - s_2 g_3 s_3^{-1}$$

$$f_3 \to s_3 - s_4^{-1}$$

$$f_4 \to s_4 + (g_4 - s_2(s_6 - s_2 s_1 s_5^{-1}) s_3^{-1})(s_5 - s_2 g_3 s_3^{-1}) s_2^{-1}$$

$$g_1 \to g_1 - (s_5 - s_2 g_3 s_3^{-1})(g_4 - s_2(s_6 - s_2 s_1 s_5^{-1}) s_3^{-1}) s_2^{-2} s_4^{-1} - s_2^{-1} + g_3(s_6 - s_2 s_1 s_5^{-1}) s_3^{-1}$$

$$g_2 \to s_2$$

$$g_3 \to g_3 + (s_5 - s_2 g_3 s_3^{-1}) s_2^{-1} s_4^{-1}$$

$$g_4 \to g_4 - s_2(s_6 - s_2 s_1 s_5^{-1}) s_3^{-1}$$

$$b_1 \to b_1 - s_3 s_6^{-1}$$

$$b_2 \to b_2 - t_1^{-1}(g_4 - s_2(s_6 - s_2 s_1 s_5^{-1}) s_3^{-1}) t_2 s_2^{-1} - t_1^{-1} s_4 s_5^{-1} t_2 + t_1^{-1} s_4 s_5^{-2} s_2 s_1 s_6^{-1} t_2$$

We then perform a Legendrian Reidemeister $R2$ move and obtain a new link Λ_2' whose induced DGA map induced maps $c_1, g_3 \to 0$ and is the identity map on all other Reeb chords. Then, composing with four minimum cobordisms, we finish constructing Σ_2. The augmentation ϵ_{Σ_2} of Λ_2 induced by Σ is given by

$$\epsilon_{\Sigma_2}(f_1) = s_6 - (s_6 - s_2 s_1 s_5^{-1}) s_3^{-1} s_4^{-1} - s_2 s_1 s_5^{-1} \qquad \epsilon_{\Sigma_2}(f_2) = s_5$$

$$\epsilon_{\Sigma_2}(f_3) = s_3 - s_4^{-1}\, \epsilon_{\Sigma_2} \qquad (f_4) = s_4 - (s_6 - s_2 s_1 s_5^{-1}) s_3^{-1} s_5$$

$$\epsilon_{\Sigma_2}(g_1) = s_2^{-1}[-1 + (s_6 - s_2 s_1 s_5^{-1}) s_3^{-1} s_4^{-1} s_5] \qquad \epsilon_{\Sigma_2}(g_2) = s_2$$

$$\epsilon_{\Sigma_2}(g_3) = +(s_5 - s_2 g_3 s_3^{-1}) s_2^{-1} s_4^{-1} \qquad \epsilon_{\Sigma_2}(g_4) = -s_2(s_6 - s_2 s_1 s_5^{-1}) s_3^{-1}$$

$$\epsilon_{\Sigma_2}(t_1) = -s_1 s_2 s_3 s_4 (s_5 s_6)^{-1} \qquad \epsilon_{\Sigma_2}(t_2) = -s_5 s_6$$

$$\epsilon_{\Sigma_2}(t_3) = -s_3 s_4 \qquad \epsilon_{\Sigma_2}(t_4) = -s_1 s_2$$

and

$$\epsilon_{\Sigma_2}(b_1) = -s_3 s_6^{-1}$$

$$\epsilon_{\Sigma_2}(b_2) = -\epsilon_{\Sigma_2}(t_1^{-1}t_2)(-s_6 + s_2 s_1 s_5^{-1} + s_4 s_5^{-1} - s_4 s_5^{-2} s_2 s_1 s_6^{-1})$$

We will now show that the loop $\tilde{\Phi}_2$ produces infinitely many distinct systems of augmentations of Λ_2. We will consider the Reeb chord f_2, which is a cycle and compute $\epsilon \circ \Phi_2^n(f_2)$:

$$\begin{pmatrix} \epsilon_{\Sigma_2} \circ \Phi_2^k(f_2) \\ \epsilon_{\Sigma_2} \circ \Phi_2^k(f_4) \end{pmatrix} = \epsilon_{\Sigma_2}((\widetilde{M_2}^{-1})^n) \begin{pmatrix} \epsilon_{\Sigma_2}(f_2) \\ \epsilon_{\Sigma_2}(f_4) \end{pmatrix}$$

where

$$\epsilon_{\Sigma_2}(\widetilde{M_2}^{-1}) = \begin{pmatrix} \epsilon_{\Sigma_2}(b_1)\epsilon_{\Sigma_2}(b_2)t_2^{-1} + t_2^{-1} & \epsilon_{\Sigma_2}(b_1)t_1^{-1} \\ \epsilon_{\Sigma_2}(b_2)t_1^{-1} & t_1^{-1} \end{pmatrix}$$

Since we know that for restricted augmentations t_i is mapped to -1 for $i = 1, 2, 3, 4$, then $\eta(s_5) = \eta(s_6), \eta(s_3) = \eta(s_4), \eta(s_1) = \eta(s_2)$. For any composition of a local system and a restricted augmentation, $\eta \circ \epsilon_{\Sigma_2}(b_2) = 0$, and $\eta \circ \epsilon_{\Sigma_2}(f_4) = \eta(s_4)$. It is now convenient to consider the matrix $N = \begin{pmatrix} \epsilon_{\Sigma_2}(f_2) & 1 \\ \epsilon_{\Sigma_2}(f_4) & 0 \end{pmatrix}$ since

$$\begin{pmatrix} \epsilon_{\Sigma_2} \circ \Phi_2^k(f_2) \\ \epsilon_{\Sigma_2} \circ \Phi_2^k(f_2) \end{pmatrix} = (\widetilde{M_2}^{-1})^k \begin{pmatrix} \epsilon_{\Sigma_2}(f_2) \\ \epsilon_{\Sigma_2}(f_4) \end{pmatrix} = (NN^{-1})(\widetilde{M_2}^{-1})^k (N \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix})$$

$$= N(N^{-1}(\widetilde{M_2}^{-1})N)^k \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = NL^k \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The matrix L has entries l_{ij} for $i, j \in \{1, 2\}$ which for a restricted augmentations map to

$$\eta(l_{11}) = \eta \circ \epsilon_{\Sigma_2}((t_2^{-1} f_4^{-1} b_2 f_2) + t_1^{-1}) = -1$$

$$\eta(l_{12}) = \eta \circ \epsilon_{\Sigma_2}(f_4^{-1} t_2^{-1} b_2) = 0$$

$$\eta(l_{21}) = \eta \circ \epsilon_{\Sigma_2}(t_2^{-1} f_2[1 + b_1 b_2 - f_4^{-1} f_2 b_2] + f_4 t_1^{-1}[b_1 - f_4^{-1} f_2])$$

$$\eta(-s_6^{-1} + s_4 s_5 s_6(s_3 s_6^{-1} - s_4^{-1} s_5) = -\eta(s_5)$$

$$\eta(l_{22}) = \eta \circ \epsilon_{\Sigma}(t_2^{-1}[1 + b_1 b_2 - f_4^{-1} f_2 b_2]) = -1$$

Therefore, $L \to \begin{pmatrix} -1 & 0 \\ -\eta(s_5) & -1 \end{pmatrix}$ and $L^k \to (-1)^k \begin{pmatrix} 1 & 0 \\ -k\eta(s_5) & 1 \end{pmatrix}$ for any restricted augmentation. Then,

$$\begin{pmatrix} \eta \circ \epsilon_{\Sigma_2} \circ \Phi_2^k(f_2) \\ \eta \circ \epsilon_{\Sigma_2} \circ \Phi_2^k(f_2) \end{pmatrix} = NL^k \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (-1)^k \begin{pmatrix} \eta(s_5) & 1 \\ \eta(s_4) & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -k\eta(s_5) & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= (-1)^k \begin{pmatrix} \eta(s_5)(1-k) \\ \eta(s_4) \end{pmatrix}$$

Note that for Σ_2, and the isotopy loop Φ_2 we have chosen, $E_r(k, f_2) = k - 1$, is a strictly increasing functions over k. Thus, we can conclude that Λ_2 is restricted aug-infinite. □

We can now move onto the more general result for the families of Legendrian links Λ_n, and Λ'_m.

Corollary 31. *The Legendrian links $\Lambda_n, \Lambda'_m \subset \mathbb{R}^3_{std}$ are restricted aug-infinite for $n, m \geqslant 1$.*

Proof. For $n > 2$, recall that there exists a Maslov 0 exact Lagrangian cobordism from Λ_2 to Λ_n given by pinching the Reeb chords $b_2, \ldots, b_{n-1}$ in order. The induced DGA map sends the Reeb chord f_2 to f_2. Similarly there exists a Maslov 0 exact Lagrangian cobordism from Λ'_1 and Λ'_m for $m > 1$. The Legendrian links Λ'_m have no negative Reeb chords. We showed in Proposition 30 that Λ_1 is restricted aug-infinite, and Casals and Ng showed that Λ'_1 is restricted aug-infinite [13]. By Proposition 28, we can then conclude that Λ_n, and Λ'_m are restricted aug-infinite for $n, m \geqslant 2$. □

Proposition 24 implies that restricted aug-infinite Legendrian links have infinitely many distinct exact Maslov 0 Lagrangian fillings, so Theorem 22 follows from 31. We can also apply Proposition 27 and Corollary 31 to conclude the following result for Weinstein manifolds constructed with critical handles attached along Legendrian links that we have just shown are restricted aug-infinite. The fact that the Lagrangian surfaces that we consider are symplectically knotted follows from our construction (see Remark 29).

Corollary 32. *Let X be a Weinstein manifold that has a Weinstein handlebody diagram containing either of the Legendrian sublinks Λ_n or Λ'_m for $n, m \geqslant 1$. Then X contains infinitely many Hamiltonian non-isotopic Lagrangian surfaces that are all smoothly isotopic and which intersect the Lagrangian co-core disks of the 2-handles corresponding to the sublink once.*

4

Milnor fibers of Arnold's exceptional singularities

4.1 Milnor fibers of isolated hypersurface singularities

We first review the definition of a Milnor fiber as a smooth manifold following [57], and as a symplectic manifold following [51].

Definition 33. *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a holomorphic function such that $f(0) = 0$, $df|_0 = 0$ and $df \neq 0$ on $B_r(0)\backslash\{0\}$ for a sufficiently small r. An isolated hypersurface singularity at 0 is the equivalence class of the germ of such a holomorphic function f, up to bi-holomorphic changes of coordinates that fix 0.*

Definition 34. *For an isolated hypersurface singularity f, the* ***Milnor fiber*** *of f is the smooth manifold*

$$M_f := f^{-1}(\epsilon_\delta) \cap B_\delta(0)$$

for suitable choices of δ, and ϵ_δ which depends on our choice of δ.

Milnor proved that M_f is independent of choices [57]. Start with the function $h : \mathbb{C}^{n+1} \to \mathbb{R}$ given by

$$h(x_0, \ldots, x_n) = |x_0|^2 + \ldots + |x_{n+1}|^2.$$

Then, $h : f^{-1}(0) \to \mathbb{R}$ is a real algebraic function and by the Curve Selection Lemma it has isolated critical values. We can choose δ_f to be the smallest critical value. Thus, for any $\delta < \delta_f$ and sufficiently small ϵ_δ,

$$f^{-1}(\epsilon_\delta) \pitchfork S_{\sqrt{\delta}}(0) = \partial M_f$$

where $S_{\sqrt{\delta}}(0)$ is the sphere of radius δ. Milnor also showed that the Milnor fiber is homotopy equivalent to a finite bouquet of half-dimensional spheres where the number of spheres is called the **Milnor number** of f [57]. Note that we do not have to restrict ourselves to only considering h, but could instead consider any real algebraic function $\tilde{h} : \mathbb{C}^{n+1} \to [0, \infty)$ such that $\tilde{h}^{-1}(0) = 0$. Again by the Curve Selection Lemma, the restriction $\tilde{h} : f^{-1}(0) \to [0, \infty)$ has isolated critical values. Thus, for sufficiently small δ, and ϵ_δ, then $f^{-1}(\epsilon_\delta) \pitchfork \{\tilde{h}(x_0, \ldots, x_n) = \delta\}$. Finally, $f^{-1}(\epsilon_\delta) \cap \{\tilde{h}(x_0, \ldots, x_n) \leqslant \delta\}$ are diffeomorphic to M_f since $h(1-t) + t\tilde{h}$ are real algebraic functions for which the Curve selection Lemma still applies.

In order to consider M_f as a symplectic manifold first note that as the intersection of a hypersurface with a ball, the Milnor fiber has an exact symplectic structure inherited from $(\mathbb{C}^{n+1}, d(\frac{i}{4}\sum_{i=0}^{n}(z_i d\bar{z}_i \wedge \bar{z}_i dz_i))$. Furthermore the negative Liouville flow given by is the gradient flow of $h(x)$ with respect to the standard Ḱahler metric. Choose a cut-off function

$$h_A(x) = ||Ax||^2$$

for some $A \in GL_{n+1}(\mathbb{C})$. Then,

$$f^{-1}(\epsilon_\delta) \cap \{h_A(x) \leqslant \delta\}$$

is an exact symplectic manifold with contact type boundary since the negative gradient flow of h points strictly inwards at any point of the hypersurface $||Ax||^2 = \delta$. Thus, we can describe the Milnor fiber as a Liouville domain which we can then complete by attaching cylindrical ends to obtain a Liouville manifold. Call this the completed Milnor fiber of f and denote it by M_f. Keating showed that M_f is independent of the choice of A, δ, and ϵ_δ up to exact symplectomorphisms. She also showed that M_f is independent of the choice of holomorphic representative of f.

Isolated hypersurface singularities are equivalent to polynomials. This follows from the fact that the $(\mu + 1)$ jet of a function f at an isolated critical point with Milnor number μ is sufficient. That is, for any other function g with the same $(\mu + 1)$ jet, there exists a biholomorphic change of coordinates between f and g. One important property of a singularity is its modality. The group G of germs of diffeomorphisms $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ acts on the space of function germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. The modality of a singularity f is the least integer m such that a sufficiently small neighborhood of f is covered by a finite number of m-parameters of orbits. Moreover, this action of G on the space of function germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ induces an action on the k-jet space of these function germs. In fact, the **modality** of a singularity f is the modality of any of its k-jets for $k \geqslant \mu(f) + 1$ where μ is the Milnor number of f. Isolated hypersurface singularities of modality 0 and 1 are classified [5]. There are two infinite series A_k and D_k of modality 0 singularities and three exceptional modality 0 singularities E_6, E_7 and E_8:

$A_k,\ k \geqslant 1$	$D_k,\ k \geqslant 4$	E_6	E_7	E_8
x^{k+1}	$x^2y + y^{k-1}$	$x^3 + y^4$	$x^3 + xy^3$	$x^3 + y^5$

Modality 0 singularities are also called simple or elliptic singularities. There are three families of unimodular isolated singularities: three parabolic singularities, the hyperbolic series $T_{p,q,r}$ and 14 exceptional singularities. The three parabolic singularities are:

$$T_{3,3,3} : \ x^3 + y^3 + z^3 + axyz, \qquad \text{and } a \text{ such that } a^3 + 27 \neq 0,$$

$$T_{4,4,2} : \ x^4 + y^4 + z^2 + axyz, \qquad \text{and } a \text{ such that } a^2 - 9 \neq 0,$$

$$T_{6,3,2} : \ x^6 + y^3 + z^2 + axyz, \qquad \text{and } a \text{ such that } a^6 - 432 \neq 0.$$

The hyperbolic singularities $T_{p,q,r}$ are given by

$$x^p + y^q + z^r + axyz,$$

where $a \in \mathbb{C}^*$, $p, q, r \in \mathbb{Z}_{\geqslant 0}$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. Note that the three parabolic singularities are the $T_{p,q,r}$ singularities such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Finally, Arnold's 14 exceptional singularities are given in the following Table where $a \in \mathbb{C}$.

E_{12}	$x^3 + y^7 + axy^5 + z^2$	E_{13}	$x^3 + xy^5 + ay^8$
E_{14}	$x^3 + y^8 + axy^6$	Z_{11}	$x^3y + y^5 + axy^4$
Z_{12}	$x^3y + xy^4 + ax^2y^3$	Z_{13}	$x^3y + y^6 + axy^5$
W_{12}	$x^4 + y^5 + ax^2y^3$	W_{13}	$x^4 + xy^4 + ay^6$
Q_{10}	$x^3 + y^4 + yz^2 + axy^3$	Q_{11}	$x^3 + y^2z + xz^3 + az^5$
Q_{12}	$x^3 + y^5 + yz^2 + axy^4$	S_{11}	$x^4 + y^2z + xz^2 + ax^3z$
S_{12}	$x^2y + y^2z + xz^3 + az^5$	U_{12}	$x^3 + y^3 + z^4 + axyz^2$

If f is a weighted homogenous function with a single isolated singularity at 0, then M_f is exact symplectomorphic to any of the hypersurfaces $f^{-1}(\epsilon)$ for any $\epsilon \in \mathbb{C}^*$ equipped with the standard Ḱahler exact symplectic form. The $T_{p,q,r}$ singularities are not weighted homogeneous but Keating proved that they are independent of $a \in \mathbb{C}^*$, so we can set it to be $a = 1$. Arnold's 14 exceptional singularities are not weighted homogeneous unless $a = 0$, but they are also independent of the choice of $a \in \mathbb{C}$.

Lemma 35. *The Milnor fiber of each of Arnold's exceptional singularities is independent of $a \in \mathbb{C}$.*

Proof. Let f_a denote one of Arnold's exceptional singularities with complex parameter $a \in \mathbb{C}$. We want to show that for any two $a_1, a_2 \in \mathbb{C}$, $M_{f_{a_1}}$ is exact symplectomorphic to $M_{f_{a_2}}$, and we will do so by expressing them as the fibers of a Lefschetz fibration. Consider the map $F : \mathbb{C}^4 \to \mathbb{C}$ given by $F(x, y, z, a) = (f_a(x, y, z), a)$. The singular locus $\mathcal{C} \subset \mathbb{C}^2$ is algebraic so the complement in connected, indeed

$\mathcal{C} = F(\{(x,y,z,a) \in \mathbb{C}^4 \,|\partial_x f = 0, \partial_y f = 0, \partial_z f = 0\}$. The critical locus of F for all 14 exceptional singularities are all two or three curves in $\mathbb{C}^2$.

Let δ_1, δ_2 and ϵ_1, ϵ_2 be sufficiently small nonzero maps such that for $i = 1, 2$,

$$\{f_{a_i}(x,y,z) = \epsilon_i\} \pitchfork \{|x|^2 + |y|^2 + |z|^2 = \delta_i\}.$$

Then, choose a path $\gamma(t)$ in the smooth locus of F_f such that $\gamma(0) = (\epsilon_1, a_1)$ and $\gamma(1) = (\epsilon_2, a_2)$ which we can do since the smooth locus is path connected. Let $\delta(t) = \delta_1(1-t) + t\delta_2$, and let $h_A(||x||^2) = \delta(t)$. Then, over $\gamma(t)$, $F^{-1}(\gamma(t)) \cap \{|x|^2 + |y|^2 + |z|^2 \leqslant \delta(t)\}$ is a Lefschetz fibration and we can use symplectic parallel transport and Lemma 2.2 in [51] to conclude that $F^{-1}(\gamma(0))$ is exact symplectomorphic to $F^{-1}(\gamma(1))$. Since $F^{-1}(\gamma(0)) = M_{f_{a_1}}$ and $F^{-1}(\gamma(1)) = M_{f_{a_2}}$ we are done.

□

There are also important relations between singularities that one can consider, one of which is given by whether a singularity is adjacent to another. A singularity $[f]$ is **adjacent** to $[g]$ if there exists an arbitrarily small perturbation p such that $[f + p] = [g]$.

Lemma 36 (Keating [51, 52])**.** *Suppose $[f]$ and $[g]$ are singularities, such that $[f]$ is adjacent to $[g]$. Then there exists an exact symplectic embedding from a non-completed Milnor fiber of g into a completed Milnor fiber of f. Moreover, under such an embedding, vanishing cycles of g get mapped to (Hamiltonian displacements of) vanishing cycles for f.*

Keating proves this Lemma by considering a map $\mathbb{C}^{n+1} \to \mathbb{C}^2$ mapping (z,t) to $(f(z)+tp(z), t)$, intersecting the fibers with balls of sufficiently small radius and using a symplectic parallel transport argument. Moreover, to track the vanishing cycles under such an embedding, she notes that if $[f]$ is adjacent to $[g]$, so that $f + p = \tilde{g}$ where $\tilde{g}$ is a representative of $[g]$, then a morsification of $\tilde{g}$ in a neighborhood of

0 gives us a morsification of f. For adjacent singularities $[f]$ and $[g]$ one can also use Liouville flows to get an exact symplectic embedding of any compact subset of a completed Milnor fiber of g into the Milnor fiber of f. For the singularities we consider, Arnold's exceptional singularities are adjacent to one of the three parabolic singularities. For $p' \geqslant p, q' \geqslant q$, and $r' \geqslant r$, $T_{p',q',r'}$ is adjacent to $T_{p,q,r}$. Even more generally, Durfee showed that:

Theorem 37 (Durfee [24]). *Any positive modality hypersurface singularity is adjacent to a modality one hypersurface singularity.*

Therefore, any positive modality singularity is adjacent to a parabolic singularity. For adjacent singularities $[f]$ and $[g]$, one can at times realize the embedding described in Lemma 36 so that there is a Weinstein handlebody decomposition of M_g such that the Legendrian attaching spheres of the 2-handles of M_g is a sublink in the Weinstein handlebody diagram of M_f.

4.2 Using Lefschetz fibrations to obtain Weinstein handlebody decompositions

Lefschetz fibrations, the holomorphic analogue of Morse functions, are an important tool in algebraic geometry, and symplectic. In particular Lefschetz fibrations are holomorphic functions $\pi : X \to \mathbb{C}$ with finitely many non-denegerate critical points which locally around any critical point is give by $z_1^2 + \cdots + z_n^2$. Consider a Lefschetz fibration on an affine variety $\pi : W \to \mathbb{C}$, denote the regular fiber $\pi^{-1}(0)$ by F_π and critical values by $c_1, \ldots, c_s \in \mathbf{D}^2$. For every critical value c_i there is an associated **vanishing cycle** V_i, which is the boundary of an embedded Lagrangian disk $\Delta_i \subset W$. The image of the Lagrangian embedded disk $\pi(\Delta_i)$ is an embedded path $\gamma_i : [0, 1] \to \mathbf{D}^2$ with endpoints $\gamma_i(0) = 0$ and $\gamma_i(1) = c_i$. The vanishing cycle V_i is then $\Delta_i \cap F_\pi$, a Lagrangian sphere in the regular fiber. A Lefschetz fibration $\pi : X \to \mathbb{C}$ can be recuperated from the data of the regular fiber F_π and the ordered set of vanishing cycles, whose ordering is determined by the cyclic ordering of the vanishing paths.

Giroux and Pardon [41] proved that any Weinstein manifold (W, λ, ϕ) admits a Lefschetz fibration $\pi : (W, \lambda, \phi) \to \mathbb{C}$ with regular fiber $F_\pi = \pi^{-1}(0)$ and distinct critical values $c_1, \dots, c_n \in \mathbf{D}^2$. In the case that W is an affine variety it is simple to construct a Lefschetz fibration up to isotopy by using a generic hyperplane section [56] In fact, one can obtain a Weinstein handle-decomposition of W from data of the Lefschetz fibration, that is from the regular fiber F_π and the set of vanishing cycles $\{V_1, \dots, V_s\}$. Start with the Weinstein domain $F_\pi \times \mathbf{D}^2$ with contact boundary $F_\pi \times S^1 \subset \partial(F_\pi \times \mathbf{D}^2)$ and attach a Weinstein 2-handle long the Legendrian lift of the exact Lagrangian vanishing cycles $V_1, \dots, V_n$. The ordering of the vanishing cycles will determine the relative Reeb height of the Legendrian lifts. The disks Δ_i, often referred to as **vanishing thimbles** are the Lagrangian cores of the 2-handles. Thus, the data of a Lefschetz fibration, a regular fiber F and an ordered set of vanishing cycles $\{V_1, \dots, V_s\}$, determines a Weinstein domain which we will denote by $lf(F; V)$. Two Weinstein manifolds $lf(F; V)$, and $lf(F'; V')$ are Weinstein equivalent if F' is Weinstein equivalent to F and the vanishing cycles V' are Hamiltonian isotopic to those of V, up to a cyclic shift. Moreover, there are two Weinstein equivalences given by the following changes in the Lefschetz fibration:

1. (Hurwitz moves) One can modify the vanishing cycles by Dehn twists as follows and obtain equivalent Weinstein manifolds

   $$lf(F, \{V_1, \dots, V_i, V_{i+1}, \dots, V_s\}) = lf(F, \{V_1, \dots, V_{i+1}, \tau_{V_{i+1}}(V_i), \dots, V_s\}),$$
   $$lf(F, \{V_1, \dots, V_i, V_{i+1}, \dots, V_s\}) = lf(F, \{V_1, \dots, \tau_{V_i}^{-1}(V_{i+1}), V_i, \dots, V_s\})$$

   For an example of a Hurwitz move see Figure 4.1. Note that we are ordering the vanishing paths counterclockwise.

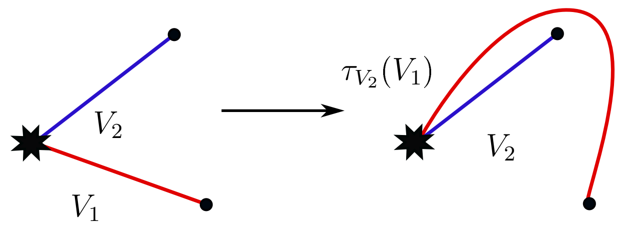


FIGURE 4.1: A Hurwitz move on vanishing paths.

2. (Stabilization) Given a Lagrangian disk $(D, \partial D) \subset (F, \partial F)$ with Legendrian boundary we can attach a critical handle along ∂D and obtain a new manifold F'. Then V' is the Lagrangian sphere given by gluing D with the core of the handle. We have introduced a pair of canceling handles of index $n-1$ and n, and therefore

$$lf(F; V) = lf(F'; V \cup V').$$

One can also sometimes replace a vanishing path γ with another vanishing path γ' if the corresponding Lagrangian thimbles Δ and Δ' are Hamiltonian isotopic. See [52, 66] for more such moves.

Given a Lefschetz fibration with regular fiber F and two critical points with Hamiltonian isotopic vanishing cycles $V_1 = V_2$, then the Lagrangian thimbles Δ_1 and Δ_2 can be glued together to obtain an exact Lagrangian sphere S. This sphere lies over the concatenation $\gamma = \gamma_1 \cup \gamma_2$, where γ_i is the vanishing path of V_i for $i = 1, 2$. We call such a sphere S a **matching cycle** and the path γ a **matching path**. For instance, given two matching paths α and γ that intersect at one point, then one can perform a half twist, a symplectomorphism τ_γ of the plane and map α to the curve $\tau_\gamma(\alpha)$ as shown in Figure 4.2. Then, the matching cycles $S_{\tau_\gamma(\alpha)}$ and $\tau_{S_\gamma}(S_\alpha)$ are Lagrangian isotopic [23].

Casals and Murphy give a dictionary to and produce the Weinstein handlebody decomposition of a complex affine variety using Lefschetz bifibrations. Lefschetz bifibrations, first defined by Seidel [68], are essentially Lefschetz fibrations where

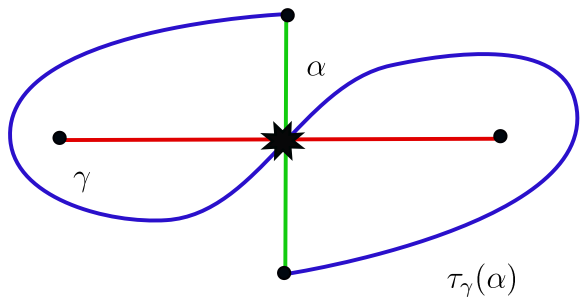


FIGURE 4.2: Matching paths α, γ and $\tau_\gamma(\alpha)$.

there is a compatible Lefschetz fibration on each fiber.

Definition 38. *A **Lefschetz bifibration** on an affine variety $X \subset \mathbb{C}^n$ is a pair of maps $X \xrightarrow{\rho} \mathbb{C} \times \mathbb{C} \xrightarrow{\eta} \mathbb{C}$ such that*

1. *η has no critical points and the composition $\pi := \rho \circ \eta$ is a Lefschetz fibration on X.*

2. *For the fibers $F_t := \pi^{-1}(t)$, and $C_t = \eta^{-1}(t)$, the restriction*

$$\rho|_t : F_t \to C_t$$

is a Lefschetz fibration. In particular for singular fibers of π, F_x this means holomorphic maps with isolated, non-degenerate critical points whose values are not $\rho(x)$.

3. *The derivative $D\rho \in Hom_{\mathbb{C}}(TX, \rho^* T\,\mathbb{C}^2)$ is transverse to the zero section and the rank 1 locus.*

4. *At critical points x of π, the Hessian $D^2\pi$ restricted to $ker(D(\rho) = T_x(crit(\rho))$ is non-denegerate.*

Note that in dimension 4 and for η a projection map onto the first factor, then $\rho = (\pi, \rho_t)$ where ρ_t is a branched cover on each fiber F_t. Moreover, the set of critical points of ρ_t is $crit(\rho) \cap F_t$ for each t. Around a critical point of π, a Lefschetz bifibration is given by $\rho(z_1, \ldots, z_n) = (z_1^2 + \cdots z_n^2, z_1)$ and η the projection map onto the first factor. Moreover, each vanishing cycle of π corresponds to a matching cycle of ρ. See [64] for a survey on Lefschetz fibrations and bifibrations. Casals and Murphy's algorithm, which we will refer to as the affine dictionary, consists of the following steps:

1. Choose a set of Lagrangian spheres $\mathbf{L} = \{L_1, \ldots, L_r\}$ in the fiber (F, λ, ϕ). Draw the Legendrian lift of $\mathbf{L}$ in front projection of the contact boundary $(\partial(F \times D^2), \lambda + \lambda_{st})$. Ideally, $\mathbf{L}$ is the Lagrangian skeleton of F, in which case we know what its Legendrian lift is.

2. Choose a Weinstein Lefschetz bifibration map on the fiber F, $\rho : (F, \lambda) \to \mathbb{C}$ and express the Lagrangian spheres in the set $\mathbf{L}$ as matching paths $\Gamma = \{\gamma_1, \ldots, \gamma_r\}$ of ρ.

3. Given a vanishing cycle $V_k \subseteq (F, \lambda)$, draw the embedded path $\rho(V_k) = \theta_i \subseteq \mathbb{C}$

4. Express each matching path θ_k as a word in half twists along the arcs in Γ.

5. Given θ_k expressed as a word in Dehn twists with the Lagrangian spheres in $\mathbf{L}$, apply Proposition 2.23 from [12] to draw the front projection of their Legendrian lifts $\Lambda^{(k)} \subseteq (\partial(F \times D^2), \lambda + \lambda_{st})$. See Figure 4.3 for examples of the Legendrian lifts of the Lagrangian spheres $S, L, \tau_S^{\pm 1}(L)$.

6. From the cyclically ordered set of vanishing cycles $\{V_1, \ldots, V_r\}$ obtain a well defined link $\Lambda = \Lambda^{(1)} \cup \cdots \cup \Lambda^{(r)}$. Each Legendrian component $\Lambda^{(k)}$ is pushed in the Reeb direction by the amount ϵk, for $\epsilon \in \mathbb{R}^+$ small enough.

7. Simplify the Legendrian front projection of $\cup_k \Lambda^{(k)}$ using allowed Reidemeister and Legendrian handle slides.

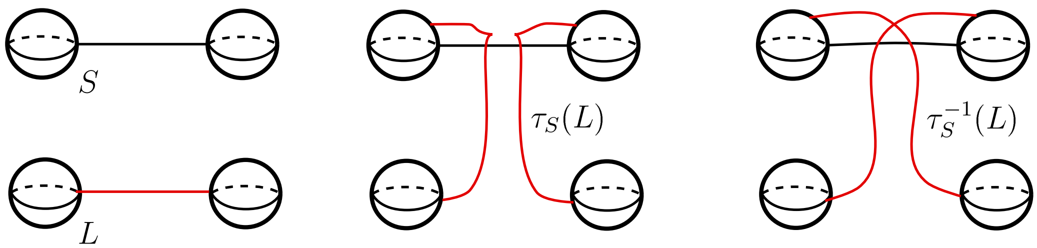


FIGURE 4.3: The Legendrian lift of $\tau_S(L)$ and $\tau_S^{-1}(L)$.

4.3 Milnor fiber of $T_{p,q,r}$ singularity

We will now employ the affine dictionary to find Weinstein handlebody diagrams of Milnor fibers of some isolated unimodular singularities.

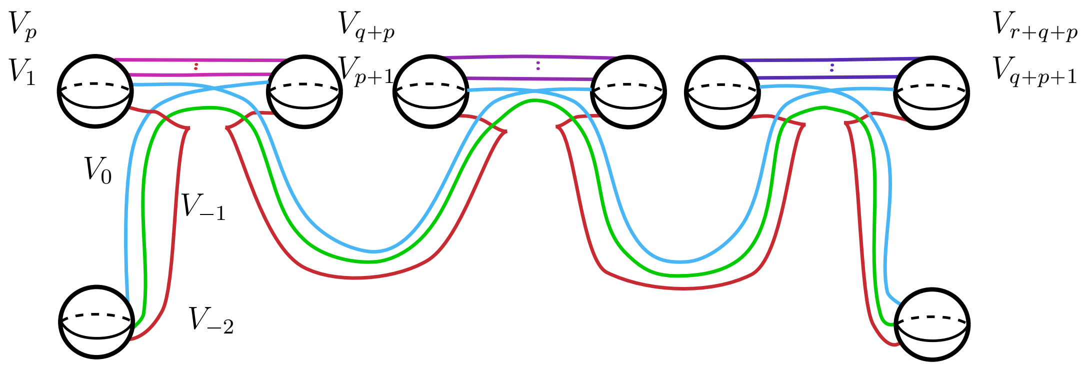


FIGURE 4.4: A Weinstein handlebody diagram of the Milnor fiber of a $T_{p,q,r}$-singularity where $p, q, r \geqslant 0$.

Proposition 39. *The Weinstein* 4*-dimensional domains*

$$T_{p,q,r} = \{(x, y, z) \in \mathbb{C}^3 \mid x^p + y^q + z^r + xyz = 1\}$$

have the Weinstein handlebody diagram shown in Figure 4.3 for $p, q, r \geqslant 0$.

Proof. In [51], Keating showed that the Milnor fiber of $T_{p,q,r}$ singularities are given by the affine varieties $T_{p,q,r} = \{(x, y, z) \in \mathbb{C}^3 \mid x^p + y^q + z^r + xyz = 1\} \subset \mathbb{C}^3$ for p, q, r such that $p, q, r \in \mathbb{Z}_{\geqslant 0}$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leqslant 1$. She also provided a Lefschetz fibration for the affine varieties $T_{p,q,r}$ for any $p, q, r \geqslant 0$ and used it to construct an exact

Lagrangian torus in these affine varieties. In [52] she used this Lefschetz fibration of $T_{p,q,r}$ and gave a new Lefschetz fibration $\Xi : T_{p,q,r} \to \mathbb{C}$ shown with regular fiber $\Xi^{-1}(0)$ symplectomorphic to a thrice punctured torus and $(p+q+r+3)$ distinct critical values. Casals and Murphy [12] use this Lefschetz fibration of $T_{p,q,r}$ and apply their affine dictionary to it to produce a Weinstein handlebody diagram of $T_{p,q,r}$, but an issue in the translation of the Lefschetz fibration of Ξ resulted in a handlebody with the wrong intersection form. We provide the correction here. Note that we are using the conventions for Lefschetz fibrations from [12] where we order vanishing cycles counterclockwise and also use their Hurwitz move convention where $\{V_i,\ V_{i+1}\} \to \{\tau_{V_i}^{-1} V_{i+1},\ V_i\}$.

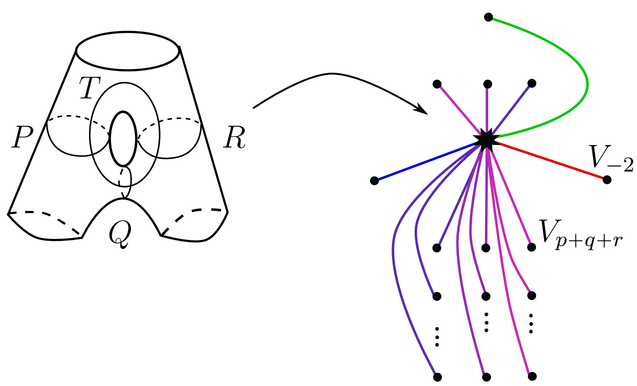


FIGURE 4.5: A Lefschetz fibration of $T_{p,q,r}$ whose vanishing cycles are Dehn twists of the four curves T, P, Q, R on the regular fiber.

We start with Keating's Lefschetz fibration $\Xi : T_{p,q,r} \to \mathbb{C}$ which has vanishing cycles that can be written as Dehn twists of the four curves four curves T, P, Q, R on the regular fiber $\Xi^{-1}(0)$ which are shown in Figure 4.5. These vanishing cycles are:

$$V_{-2} = \tau_P \tau_Q \tau_R T,\ V_{-1} = T,\ V_0 = P,\ V_1 = Q,\ V_2 = R,\ V_3 = T,$$
$$V_4 = P \ldots V_{p+3} = P, V_{p+4} = Q, \ldots, V_{p+q+6} = Q, V_{p+q+7} = R, \ldots, V_{p+q+r} = R.$$

We apply three Hurwitz moves and obtain the following collection of vanishing cycles:

$$V_{-2} = \tau_P\tau_Q\tau_R T,\ V_{-1} = T,\ V_0 = \tau_P^{-1}\tau_Q^{-1}\tau_R^{-1}T,$$

$$V_1 = P,\ \dots V_p = P, V_{p+1} = Q, \dots, V_{p+q} = Q, V_{p+q+1} = R, \dots, V_{p+q+r} = R.$$

Then we apply the affine dictionary and obtain the Weinstein handlebody diagram 4.3 as follows. First we use the Lefschetz fibration on the regular fiber $\Xi^{-1}(0)$ where the Lagrangian skeleton of consists of Lagrangian spheres $T, P, Q,$ and R. Since we already know how to write the vanishing cycles of Ξ as Dehn twists of these four curves T, P, Q, R, then we are in Step 5 of the affine dictionary and can use Proposition 2.23 to draw the front projection of their Legendrian lifts. As stated in Step 6 the order of the vanishing cycles determines the Reeb height of the Legendrian lifts. □

It will be convenient to simplify the Weinstein handlebody diagram of $T_{p,q,r}$ as shown in Figure 4.6: $(A) - (B)$ perform various handle-slides; (C) cancel out the 1-handles; $(D) - (E)$ perform Reidemeister moves; (F) use the fact that one can handle-slide an N-copy of the unknot to a chain of N unknots. As soon as we cancel the 1-handles we can check that $T_{p,q,r}$ has the correct intersection form for $p, q, r \geqslant 1$. Furthermore one can also check that $T_{0,0,0}$ is Weinstein equivalent to T^*T^2. We are now ready to use the Weinstein handlebody diagram of $T_{p,q,r}$ to find closed exact Lagrangian surfaces in $T_{p,q,r}$.

Theorem 40. *For any $p, r \geqslant 1$, and $q \geqslant 3$, the Weinstein 4-manifold $T_{p,q,r}$ contains infinitely many Hamiltonian non-isotopic Lagrangian tori that are all smoothly isotopic. For any $p, q, r \geqslant 2$, the Weinstein 4-manifold $T_{p,q,r}$ contains infinitely many Hamiltonian non-isotopic Lagrangian spheres that are all smoothly isotopic which intersect the Lagrangian co-cores once.*

Proof. We start with the Weinstein handlebody diagram of $T_{p,q,r}$ shown in Figure 4.6.

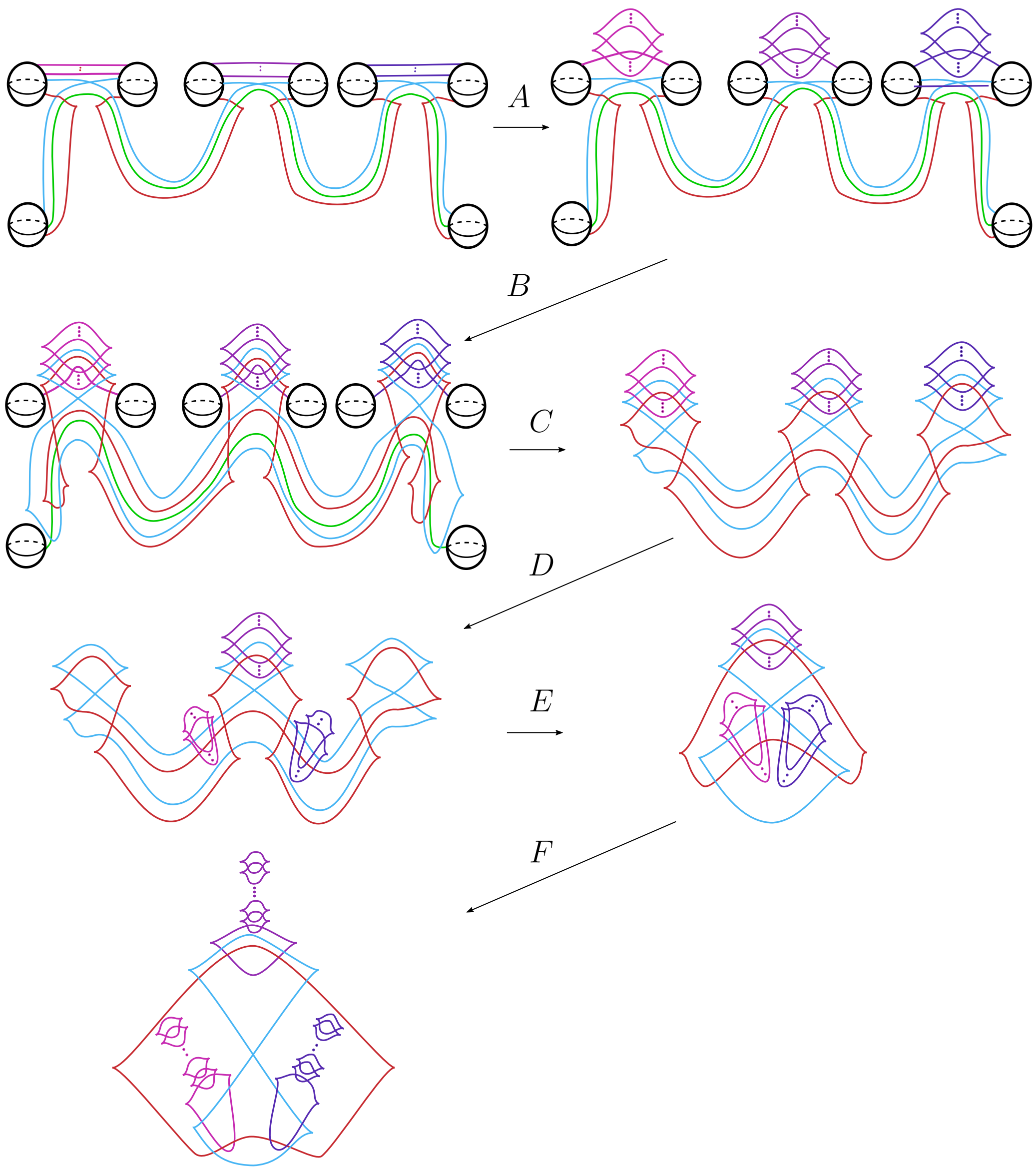


FIGURE 4.6: We simplify the Weinstein handlebody diagram of $T_{p,q,r}$ for $p, q, r \geqslant 1$.

After performing Legendrian isotopy, Legendrian handle slides and handle cancellation, as described in Figure 4.7 4.8 we find a Weinstein handlebody diagram of $T_{p,q,r}$ that contains a sublink that is isotopic to Λ_2' if $p, r \geqslant 1$, and $q \geqslant 3$. The oriented sublink Λ_2' has genus 1 exact Lagrangian fillings and so by Corollary 32, if $p, r \geqslant 1$, and $q \geqslant 3$ $T_{p,q,r}$ contains infinitely many Hamiltonian non-isotopic Lagrangian tori. Here we give a brief summary of the Legendrian handle-slides and cancellations em-

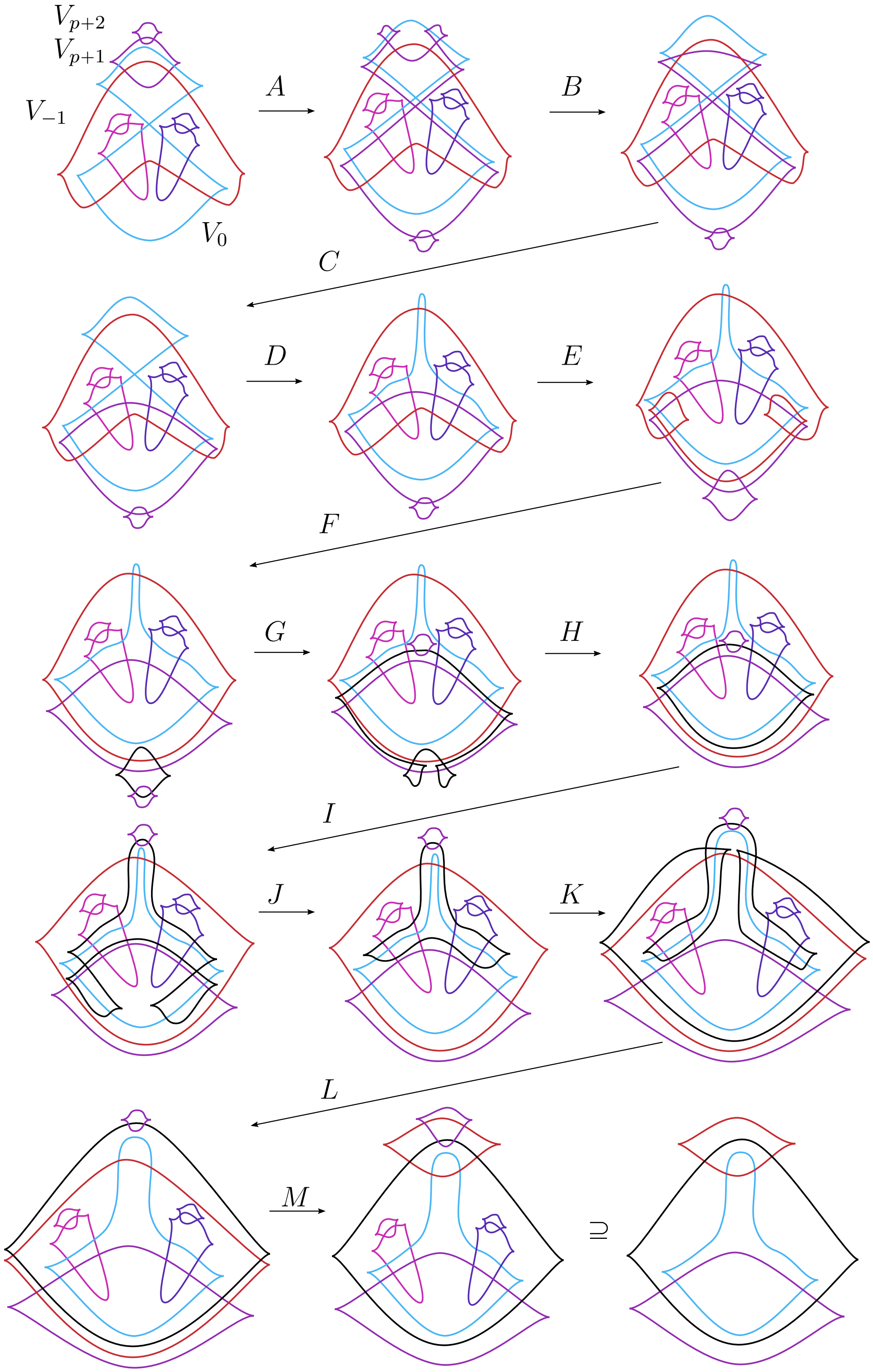


FIGURE 4.7: We find a Weinstein handlebody diagram of $T_{p,q,r}$ that contains as a sublink Λ_2' when $p, r \geqslant 1$ and $q \geqslant 3$.

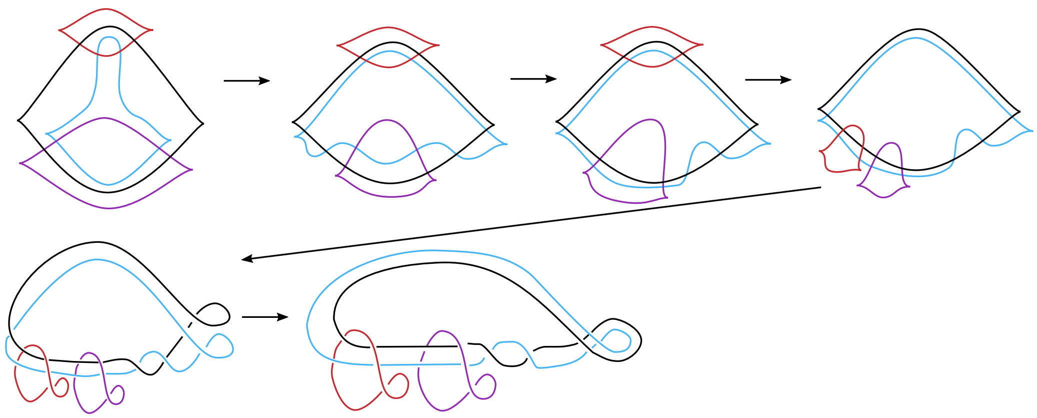


FIGURE 4.8: A sequence of Legendrian Reidemeister moves on the Legendrian link Λ_2'.

ployed in Figure 4.7. For simplicity of notation we do not change the notation for the attaching sphere of a 2-handle after handle slides, although they are distinct as Legendrians. We also do not include the vanishing cycles that are chains of unknots that will not play an active role in the computation. (A) Handle slide V_{p+1} under V_0; $(B)-(D)$ perform Reidemeister moves; (E) handle slide V_{-1} over V_{p+1}; (F) perform Reidemeister moves; (G) handle slide V_{p+2} under V_{p+1}; (H) perform Reidemeister moves; (I) handle slide V_{p+2} over V_0; (J) perform Reidemeister moves; (K) handle slide V_{p+2} over V_{-1}; (L) perform Reidemeister moves; (M) handle slide V_{-1} over V_{p+2}. In Figure 4.8 we perform a series of Reidemeister moves, use Ng's resolution to translate from the front projection to the Lagrangian link, and use Reidemeister moves once again to present Λ_2' as a link with a sublink that is the (-1) closure of the braid $\sigma_1^2 \in Br_2^+$.

We can also find a Weinstein handlebody diagram of $T_{p,q,r}$ that contains a sublink Λ_2 shown in Figure 3.1. The oriented sublink Λ_2 has genus 0 fillings, and so by Corollary 32, if $p, q, r \geqslant 2$ $T_{p,q,r}$ contains infinitely many Hamiltonian non-isotopic Lagrangian spheres. We will now describe the moves used in Figure 4.9. Note that we do not include the vanishing cycles $V_3, \ldots, V_p, V_{p+3}, V_{p+q}, V_{p+q+3}, \ldots, V_{p,q,r}$ as these are chains of unknots which will not play an active role in the computation. (A) han-

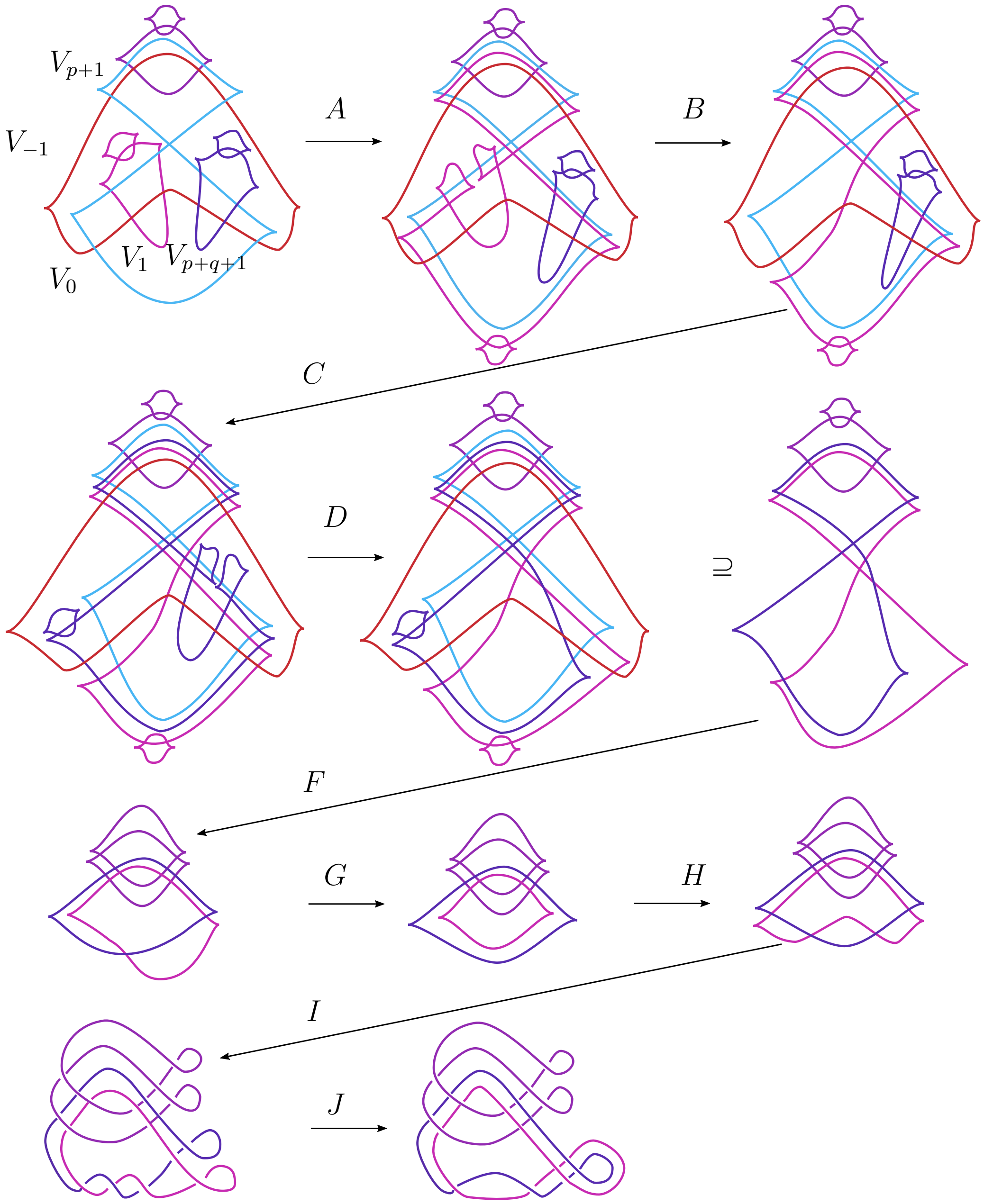


FIGURE 4.9: We use handle-slides and Legendrian isotopy to find a Weinstein handlebody diagram of $T_{p,q,r}$ that contains Λ_2 as a sublink when $p, q, r \geqslant 2$.

dle slide V_1 under V_0, (B) perform Reidemeister moves, (C) handle slide V_{p+q+1} under V_0, (D) perform Reidemeister moves and handle slide V_{p+2} under V_{p+1}. For steps $(F - G)$ we perform Legendrian Reidemeister moves to obtain the Legendrian link Λ_2 in the desired front projection. At step (I) we translate from the front projection to the Lagrangian projection using Ng's resolution, and (J) perform Reidemeister moves to obtain a projection of Λ_2 as a link with a sublink that is the (-1) closure of the braid $\sigma_1^2 \in Br_2^+$. □

Corollary 41. *Suppose that M_f is the Milnor fiber of a positive modality isolated hypersurface singularity f, then M_f contains infinitely many Hamiltonian non-isotopic exact Maslov 0 Lagrangian tori that are all smoothly isotopic and infinitely many Hamiltonian non-isotopic exact Lagrangian spheres that are all smoothly isotopic.*

Proof. By Theorem 40, the Milnor fiber of any of the three parabolic singularities, that is the $T_{p,q,r}$ singularity for $(p, q, r) = (3, 3, 3), (4, 4, 2), (6, 3, 2)$, has infinitely many Hamiltonian non-isotopic exact Maslov 0 Lagrangian tori and infinitely many non-isotopic exact Lagrangian spheres. All positive modality singularities f are adjacent to a unimodular singularity by Theorem 37, and all unimodular singularities are adjacent to at least one of the three parabolic singularities. By Lemma 36, there then exists an exact symplectic embedding of the non-completed Milnor fiber of such $T_{p,q,r}$ singularities into M_f. □

Keating proved that the Fukaya category of the Milnor fiber M_f for any unimodular isolated singularity was not generated by vanishing cycles [51]. She proved this by constructing an exact Maslov 0 Lagrangian torus and showing that it is not in the subcategory generated by the vanishing cycles. Corollary 41 gives an infinite family of Lagrangian tori which are potentially not generated by the vanishing cycles.

4.4 Examples of Handlebody descriptions of Arnold's exceptional singularities

We will use Lemma 35 and the affine dictionary to produce the Weinstein handlebody diagram of the Milnor fiber of some of Arnold's exceptional singularities. In particular, we find an example of two adjacent singularities Z_{11} and Z_{13}, where the

Weinstein handlebody diagram of the Milnor fiber of Z_{11} is a sublink in a Weinstein handlebody diagram of the Milnor fiber of Z_{13}.

Proposition 42. *Weinstein handlebody diagrams of the Milnor fiber of Arnold's exceptional singularities Z_{11} and Z_{13} are given in Figure 4.14 and 4.15. Moreover, there exists a Weinstein handlebody diagram of the Milnor fiber of Z_{13} which contains the Weinstein handlebody diagram of the Milnor fiber of Z_{11} as a sublink.*

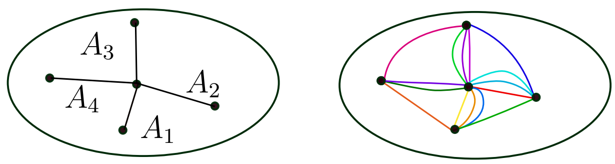


FIGURE 4.10: To the left we have a matching basis, and to the right we give the vanishing paths of Z_{11} as matching paths.

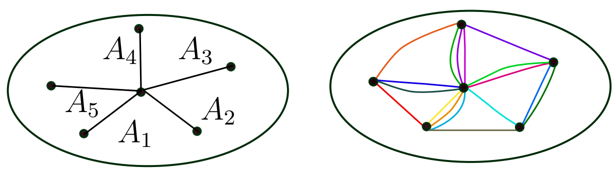


FIGURE 4.11: To the left we have a matching basis, and to the right we give the vanishing paths of Z_{13} as matching paths.

Proof. The Milnor fibers Z_{11} and Z_{13} are the complex affine varieties

$$\{(x,y,z) \in \mathbb{C}^3 \ \mid x^3y + y^p + z^2 = 1\}$$

for $p = 5, 6$ respectively. We will denote these Milnor fibers by Z_{11} and Z_{13} respectively. This family of affine varieties has a Lefschetz bifibration given by $F : \mathbb{C}^3 \to \mathbb{C}$ where $F(x,y,z) = x + \frac{p}{7}y$, and $\rho : \mathbb{C}^2 \to \mathbb{C}$, where $\rho : F^{-1}(a) \to \mathbb{C}$ is the projection map onto z. The regular fiber is symplectomorphic to $A^2_{p-1} = \{(x,z) \in \mathbb{C}^2 \,|x^p + z^2 = 1\}$, the linear plumbing of $p-1$ cotangent bundles of spheres, which has a natural basis of Lagrangian spheres. Figures 4.10 and 4.11 show the vanishing paths as matching paths for the Milnor fibers Z_{11} and Z_{13}. For Z_{11} the vanishing cycles are:

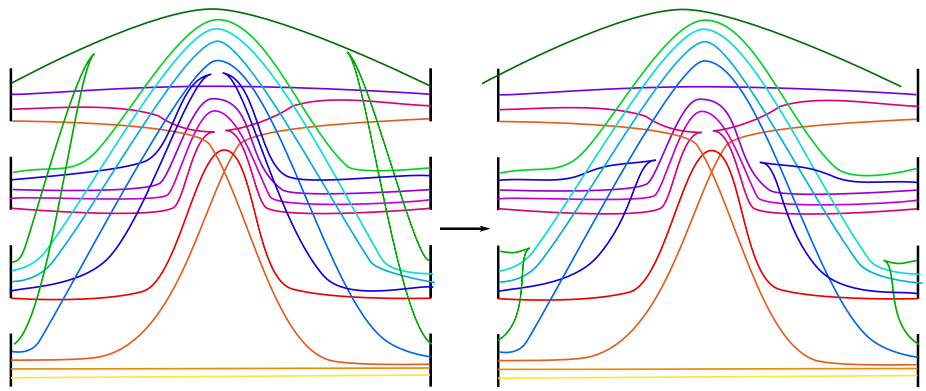


FIGURE 4.12: A Weinstein handlebody diagram of Z_{11}.

$$V_1 = A_1, V_2 = A_1, V_3 = \tau_{A_1} A_4, V_4 = A_2, V_5 = \tau_{A_4} A_3, V_6 = A_3, V_7 = A_3, V_8 = A_4,$$

$$V_9 = \tau_{A_3} A_2, V_{10} = A_1, V_{11} = A_2, V_{12} = A_2, V_{13} = A_3, V_{14} = \tau_{A_2} A_1, V_{15} = A_4.$$

For Z_{13} these are:

$$V_1 = A_1, V_2 = A_1, V_3 = A_2, V_4 = \tau_{A_1} A_5, V_5 = \tau_{A_5} A_4, V_6 = A_3, V_7 = A_4, V_8 = A_4,$$

$$V_9 = \tau_{A_4} A_3, V_{10} = A_5, V_{11} = \tau_{A_3} A_2, V_{12} = A_1, V_{13} = A_2, V_{14} = A_3, V_{15} = A_4,$$

$$V_{16} = \tau_{A_3} A_2, V_{17} = A_5, V_{18} = \tau_{A_2} A_1.$$

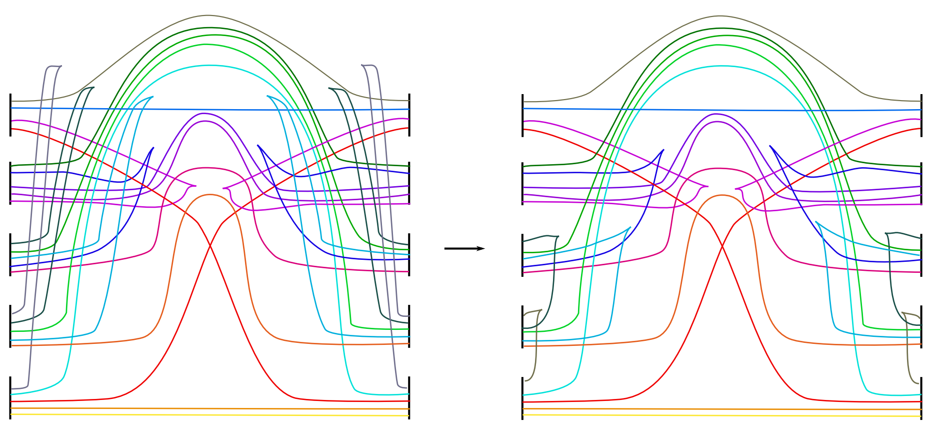


FIGURE 4.13: A Weinstein handlebody diagram of Z_{13}.

We can then apply the final steps in the affine dictionary to conclude that they have the Weinstein handlebodies shown in Figures 4.12, and 4.13. In particular, after performing handle-slides and handle cancellations we find two Weinstein handlebodies of $Z_{11} = X_{\Lambda(Z_{11})}$, and $Z_{13} = X_{\Lambda(Z_{13})}$ shown in Figures 4.14 and 4.15 for two Legendrian links $\Lambda(Z_{11}), \Lambda(Z_{13}) \subset \mathbb{R}^3_{std}$ where $\Lambda(Z_{11})$ is a sublink of $\Lambda(Z_{13})$.

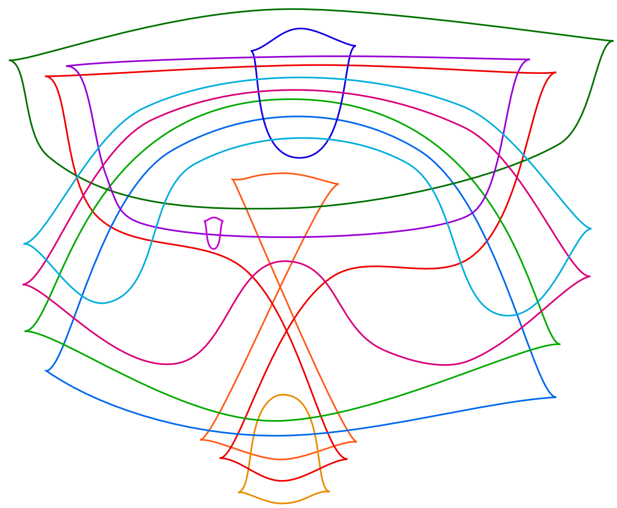


FIGURE 4.14: A Weinstein handlebody of Z_{11} with no 1-handles.

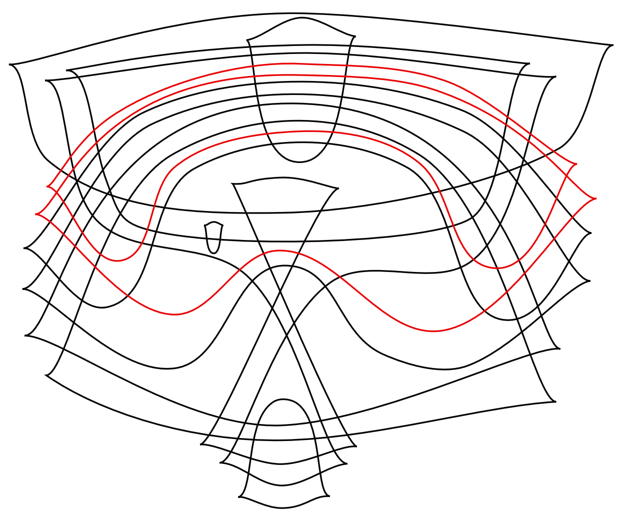


FIGURE 4.15: A simplified handlebody of Z_{13} where a Weinstein handlebody diagram of Z_{11} is a sublink.

□

5

Symplectic Homology Criterion

There are various topological and symplectic invariants that one can compute from a Weinstein handlebody diagram, one of which is symplectic homology. Let X_Λ denote the Weinstein 4-manifold obtained from attaching Weinstein 2-handles to a Legendrian link $\Lambda \subset (\#^m(S^1 \times S^2), \xi_{std})$. Bourgeois, Ekholm and Eliashberg in [9, 25] gave a relation between Legendrian invariants of the critical attaching Legendrian spheres of 2 handles and symplectic invariants of the resulting Weinstein manifold. In particular, they showed that

$$S\mathbf{H}(X_\Lambda) = L\mathbf{H}^{H_0}(\Lambda)$$

where $L\mathbf{H}^{H_0}(\Lambda)$ is the homology of Hochschild like complex associated to the Legendrian contact homology differential graded algebra over $\mathbb{Q}$. Leverson used this result to show that if Λ has a graded augmentation the symplectic homology $S\mathbf{H}(X_\Lambda)$ does not vanish. We provide a generalization of this non-vanishing criterion.

Theorem 43. *Let X_Λ be the Weinstein 4 manifold resulting from attaching 2 handles along a Legendrian link $\Lambda = \Lambda^{(1)} \cup \cdots \cup \Lambda^{(n)} \subset (\#^m(S^1 \times S^2), \xi_{std})$. If there is any sublink $\Lambda^{(1)} \cup \cdots \cup \Lambda^{(l)}$ for $l < n$, such that its differential graded algebra has a* *representation*

$$\rho : (\mathcal{A}(\Lambda^{(1)} \cup \cdots \cup \Lambda^{(l)}; \mathbb{Z}[t_1^{\pm 1} \ldots, t_l^{\pm 1}]), \partial) \to End(V)$$

where V is a vector space over $\mathbb{Q}$ and $\rho(t_k) = -Id$ for $k = 1, \ldots, l$, then $S\mathbf{H}(X_\Lambda) \neq 0$.

There is a fair amount of work that goes into defining a chain complex and differential of $L\mathbf{H}^{H_0}(\Lambda)$ in [9]. For the readers convenience we recall the relevant definitions and notation, with the slight modification that chain complexes are denoted with a C instead of CH. In Section 2.2 and Remark 4 we introduced the Legendrian contact homology differential graded algebra $\mathcal{A}(\Lambda; \mathbb{Z})$, with the null cobordant spin structure on each Λ_i, and generated by the Reeb chords of Λ. For the purposes of this chapter, we assume the DGA also had generators e_i, where e_i is an idempotent element associated to the link component $\Lambda^{(i)}$ and which we refer to as an empty Reeb chord. Moreover, we also require the empty Reeb chords to satisfy the following relations:

$$e_1 + \cdots + e_n = 1, \quad e_i \cdot e_j = \delta_{ij}$$

and

$$e_i \cdot c = \begin{cases} c & \text{If } c \in \mathcal{R}_{ij}(\Lambda) \text{ for some } j \\ 0 & \text{otherwise} \end{cases} \qquad c \cdot e_j = \begin{cases} c & \text{If } c \in \mathcal{R}_{ij}(\Lambda) \text{ for some } i \\ 0 & \text{otherwise} \end{cases}$$

The subalgebra $LCO(\Lambda) \subset \mathcal{A}(\lambda)$ is generated by cyclically composable monomials of Reeb chords, while $LCO^+(\Lambda) := LCO(\Lambda)/\langle e_1, \ldots, e_n \rangle$ is the subalgebra generated by non-trivial cyclically composable monomials of non-empty Reeb chords. Let $\widecheck{LCO}^+(\Lambda) := LCO^+(\Lambda)$ and $\widehat{LCO}^+(\Lambda) := LCO^+(\Lambda)[1]$. For any element $w = c_1 \cdots c_j \in LHO^+(\Lambda)$, write the corresponding elements in $\widecheck{LCO}^+(\Lambda)$ and $\widehat{LCO}^+(\Lambda)$ as follows $\check{w} = \check{c}_1 \cdots c_n$ and $\hat{w} = \hat{c}_1 \cdots c_n$. The chain complex is then

$$LC^{H_0}(\Lambda) = \widecheck{LCO}^+(\Lambda) \oplus \widehat{LCO}^+(\Lambda) \oplus \mathbb{Q}\langle \tau_1, \ldots, \tau_n \rangle$$

where each τ_i has grading 0. Each τ_i is in bijective correspondence with each e_i and thus with each Legendrian link component $\Lambda^{(i)}$. Given an element $(\check{w}, \hat{v}, \sum_{i=1}^{n} m_i\tau_i)$ in $LC^{H_0}(\Lambda)$, where $\hat{v} = \hat{c}_1 \cdots c_k$ and $m_i \in \mathbb{Q}$, the differential is given by

$$d_{H_0}(\check{w}, \hat{v}, \sum_{i=1}^{n} m_i\tau_i) = (\check{d}_{LCO^+}(\check{w}) + \check{c}_1 \cdots c_k - c_1 \cdots \check{c}_k, \hat{d}_{LCO^+}(\hat{v}), \delta_{H_0}(\check{w} + \hat{v})).$$

The differentials $\check{d}_{LCO^+}$ and $\hat{d}_{LCO^+}$ are the differential of $\mathcal{A}(\Lambda)$ restricted to $\widetilde{LCO}^+(\Lambda)$ and $\widehat{LCO}^+(\Lambda)$ respectively. The differential δ_{H_0} is defined as follows. If c is a non-empty pure Reeb chord beginning and ending on Λ_i, then $\delta_{H_0}(\check{c}) = n_{c_i}\tau_i$ where n_{c_i} is the count of the zero dimensional moduli space of holomorphic disks asymptotic at $+\infty$ to c. If w is a nonlinear monomial $\delta_{H_0}(\check{w}) = 0$, and for any element $\hat{v}$, $\delta_{H_0}(\hat{v}) = 0$.

Proof. Our goal is to construct a representation $\tilde{\rho} : (LC^{H_0}(\Lambda), d_{H_0}) \to (End(V^{\oplus n}), 0)$ such that $\tilde{\rho} \circ d_{H_0} = 0$. We will use such a representation $\tilde{\rho}$ to show that $d_{H_0}(\tau_k) = 0$ and $\tau_k \notin Im(d_{H_0})$ for $k = 1, \ldots, n$. We will then conclude that $LC\mathbf{H}^{H_0}(\Lambda) = S\mathbf{H}(X_\Lambda)$ is nonzero.

First define a graded map $\rho' : \mathcal{A}(\Lambda) \oplus \mathbb{Q}\langle\tau_1, \ldots, \tau_n\rangle \to End(V^{\oplus n})$ as follows. For $1 \leqslant k \leqslant l$

$$\rho'(t_k)_{ij} = \begin{cases} Id & i = j = k \\ 0 & \text{otherwise.} \end{cases}$$

For $k > l$,

$$\rho'(\tau_k)_{ij} = 0.$$

For $1 \leqslant k \leqslant l$

$$\rho'(\tau_k)_{ij} = \begin{cases} Id & i = j = k \\ 0 & \text{otherwise.} \end{cases}$$

If $c \in \mathcal{R}_{kr}(\Lambda)$ is a Reeb chord beginning on Λ_k and ending on Λ_r and $1 \leqslant k, r \leqslant l$

$$\rho'(c)_{ij} = \begin{cases} \rho(c) & i = k, \ j = r \\ 0 & \text{otherwise.} \end{cases}$$

If $c \in \mathcal{R}_{kr}(\Lambda)$ but $l < k$ or $l < r$, then

$$\rho'(c)_{ij} = 0.$$

Then, for any element $(\check{w}, \hat{v}, \sum_{i=1}^{n} m_i\tau_i) \in LC^{H_0}(\Lambda)$, where $\hat{v} = \hat{c_1} \cdots c_k$,and $m_i \in \mathbb{Q}$ define

$$\tilde{\rho}(\check{w}, \hat{v}, \sum_{i=1}^{n} m_i\tau_i) = \rho'(w) + \rho'(m_i\tau_i).$$

We need to check that $\tilde{\rho} \circ d_{H_0} = 0$:

$$\begin{aligned} \tilde{\rho}(d_{H_0}(\check{w}, \hat{v}, \sum_{i=1}^{n} m_i\tau_i)) &= \tilde{\rho}(dw + \check{c_1} \cdots c_j - c_1 \cdots \check{c_j}, \hat{d}_{LCO^+}(\hat{v}), \delta_{H_0}(\check{w} + \hat{v})) \\ &= \rho'(\check{d}_{LCO^+}(\check{w}) + c_1 \cdots c_k - c_1 \cdots c_k) + \rho'(\delta_{H_0}(\check{w} + \hat{v})) \\ &= \rho'(\check{d}_{LCO^+}(\check{w}) + \delta_{H_0}(\check{w})). \end{aligned}$$

By definition of δ_{H_0} there are two cases to consider. If w is a nonlinear monomial that begins and ends on the link component Λ_k for $1 \leqslant k \leqslant n$, then

$$\tilde{\rho}(d_{H_0}(\check{w}, \hat{v}, \sum_{i=1}^{n} m_i\tau_i)) = \rho'(\check{d}_{LCO^+}(\check{w}) + 0),$$

and by definition of ρ'

$$\rho'(\check{d}_{LCO^+}(\check{w}))_{ij} = \begin{cases} \rho(dw) & i = j = k \\ 0 & \text{otherwise} \end{cases}$$

since $d_{LCO^+} = d|^{+}_{LCO}$. If $w = c$ is a Reeb chord beginning and ending on the link component Λ_k for $1 \leqslant k \leqslant n$, and because we can identify τ_i with the empty Reeb word $e_i \in LCO(\Lambda)$,

$$\tilde{\rho}(d_{H_0}(\check{w}, \hat{v}, \sum_{i=1}^{n} m_i \tau_i)) = \epsilon'(\check{d}_{LCO^+}(\check{c}) + n_{c_k}\tau_k) = \rho'(d_{LCO}(c)).$$

Then,

$$\rho'(d_{LCO}(c))_{ij} = \begin{cases} \rho(d(c)) & i = j = k \\ 0 & \text{otherwise.} \end{cases}$$

Since we started with a representation ρ such that $\rho \circ \partial = 0$, we can conclude that $\tilde{\rho} \circ d_{H_0} = 0$. Finally, we observe that τ_k for $k = 1, \ldots, n$ is a cycle because $d_{H_0}(0, 0, \tau_k) = (0, 0, 0)$. Moreover, $\tilde{\rho}(\tau_k) = \rho(t_k) = -Id \neq 0$, and $\tilde{\rho} \circ d_{H_0} = 0$, so we can conclude that $\tau_k \notin Im(d_{H_0})$ for $k = 1, \ldots, n$.

□

Flexible Weinstein manifolds abide by an h-principle [20] and it is often difficult to determine whether a Weinstein manifold is flexible or not. A Weinstein manifold is said to be flexible if the critical handles are attached along loose Legendrian submanifold. Since flexible Weinstein manifolds have vanishing symplectic homology, we can then immediately conclude the following.

Corollary 44. *Let X_Λ be the Weinstein 4 manifold resulting from attaching 2 handles along a Legendrian link $\Lambda = \Lambda^{(1)} \cup \cdots \cup \Lambda^{(n)} \subset (\#^m(S^1 \times S^2), \xi_{std})$. If there is any sublink $\Lambda^{(1)} \cup \cdots \cup \Lambda^{(l)}$ for $l < n$, such that its differential graded algebra has a representation*

$$\rho : (\mathcal{A}(\Lambda^{(1)} \cup \cdots \cup \Lambda^{(l)}; \mathbb{Z}[t_1^{\pm 1} \ldots, t_l^{\pm 1}]), \partial) \to End(V)$$

where V is a vector space over $\mathbb{Q}$ and such that $\rho(t_k) = -Id$ for $k = 1, \ldots, l$. Then, X_Λ is not a flexible Weinstein manifold.

6

Obstructions to immersed exact Lagrangian fillings

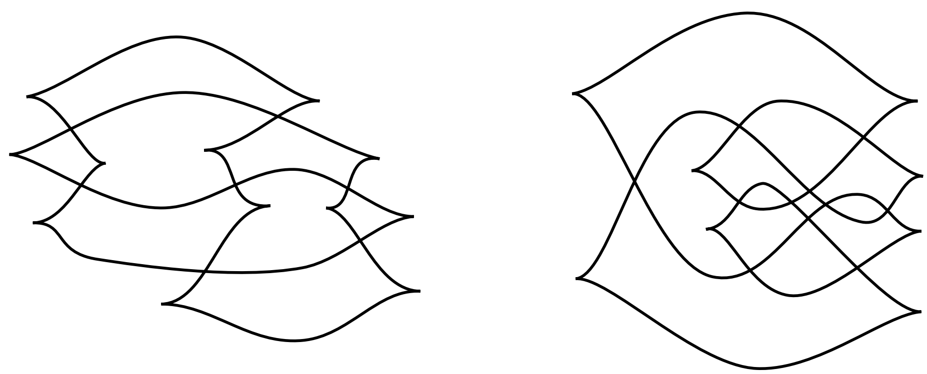
In this Chapter we will summarize the results obtained from a collaboration with Legout, Limouzineau, Murphy, Pan, and Traynor for more details and full proofs see [10].

There are various obstructions to the existence of an exact Lagrangian fillings of Legendrian links in $\mathbb{R}^3_{std}$ that arise from classical and Floer theoretic invariants of the Legendrian. Lagrangian fillings can exhibit more topological rigidity than smooth fillings, and it is interesting to understand when this occurs. For instance, an embedded, oriented Lagrangian filling of a Legendrian link Λ of smooth knot type $\mathcal{K}$ will always realize the smooth 4-ball genus of $\mathcal{K}$ [15]. Furthermore, we also have the following relation between the tb-number of Λ and the genus of its exact Maslov 0 Lagrangian filling:

$$tb(\Lambda) = g(\Sigma) - 2c + 1$$

where c is the number of components of Λ.

Smoothly, one can start with an immersed filling of a link with transverse double points and obtain an embedded filling by performing a surgery procedure at each double point that trades each double point for genus. Owens and Srle [60], have used Heegard Floer techniques to compute the minimal number of double points of a disk filling of $\mathcal{K}$, which they denote by $c^*(\mathcal{K})$. They have computed $c^*(\mathcal{K})$ for knots $\mathcal{K}$ with crossing number less than 12. They can then show that there exists knots $\mathcal{K}$ such that $c^*(\mathcal{K}) > g_4(\mathcal{K})$. For example, the knot 7_4, has $c^*(7_4) = 2$ and $g_4(7_4) = 1$. For such knots, one cannot trade genus for double points; that is, any embedded filling of $\mathcal{K}$ of genus $g_4(\mathcal{K})$ cannot be obtained by surgery from a disk filling of $\mathcal{K}$ with only double points.



(A) (B)

FIGURE 6.1: (A) the front projection of a Legendrian knot of type 7_4, (B) the front projection of a Legendrian of knot type 9_{48}.

One can ask the same question of whether genus can be traded for double points in the Lagrangian category. Lagrangian surgery was first defined for Lagrangian surfaces by Lalonde and Sikorav [53] and then generalized to higher dimensions by Polterovich [65]. Given a double point x on an immersed orientable Lagrangian one can perform Lagrangian surgery to remove a neighborhood of x and glue back in a Lagrangian handle. There are two ways to glue in the Lagrangian handle which yield an orientable immersed Lagrangian with one more genus and one less double point. Observe that Lagrangian surgery on any immersed exact Lagrangian filling does not necessarily result in an exact Maslov 0 Lagrangian filling. However we can characterize the immersed exact Lagrangian fillings on which Lagrangian surgery does produce an exact Maslov 0 Lagrangian filling.

Definition 45. *A* ***VIC*** *Lagrangian cobordism* Σ *between two Legendrian links* Λ_- *and* Λ_+ *is an exact Maslov* 0 *immersed Lagrangian cobordism between* Λ_- *and* Λ_+ *such that it is only immersed at double points and so that every double point has vanishing index and they are all simultaneously contractible.*

In order to define the index of a double point on a VIC Lagrangian we must first recall the definition of a Legendrian lift of a Lagrangian, following the convention of [17]. The **Legendrian lift** of an exact Lagrangian cobordism Σ with primitive f, is the Legendrian submanifold $\widetilde{\Sigma} = \{(q, -f(q) \mid q \in \Sigma\}$ in the contactization of $(\mathbb{R} \times \mathbb{R}^3, d(e^t(dz - ydx)))$ which is the contact manifold $((\mathbb{R} \times \mathbb{R}^3) \times \mathbb{R}_u), du + e^t(dz - ydx))$. The double points x of Σ corresponds to Reeb chords c_x of $\widetilde{\Sigma}$ since ∂_u is the Reeb vector field in $((\mathbb{R} \times \mathbb{R}^3) \times \mathbb{R}_u), du + e^t(dz - ydx))$. Now we can define the following properties associated to double points of immersed Lagrangian fillings.

- The **action** of a double point x of Σ is the length of the corresponding Reeb chord c_x in $\widetilde{\Sigma}$.
- A double point x of Σ is **contractible** if the corresponding Reeb chord c_x is contractible.
- The **index** of a double point x of Σ is $CZ(c_x) - 1$ where CZ is the Conley Zehnder index defined for Reeb chords as in Section 2.2. Recall that the Conley Zehnder index is well defined if Σ, the Lagrangian projection of $\widetilde{\Sigma}$, has Maslov class 0.

Lagrangian surgery on a vanishing action and index double point of an exact Maslov 0 Lagrangian filling results in an exact Maslov 0 Lagrangian filling. Definition 45 of a VIC filling is motivated by the fact that a VIC filling is Hamiltonian isotopic to an immersed exact Lagrangian filling with vanishing index and action double points. Additionally, if x is a double point of a VIC filling, then x is necessarily a positive double point. If the Lagrangian is 2-dimensional and x is a positive double point, the two possible Lagrangian surgeries on x are smoothly equivalent although not necessarily Hamiltonian isotopic. Thus, if Σ is a VIC filling with genus g and k double points, then one can perform Lagrangian surgery at any of the double points to obtain an embedded exact Maslov 0 Lagrangian filling of genus $g + l$ and $k - l$ double points for $0 < l \leqslant k$. Moreover, for any VIC filling Σ, a direct generalization of Chantraine's result [15] gives us that

$$tb(\Lambda) = 2(g(\Sigma) + p(\Sigma)) - 2c + 1$$

where p is the number of double points of Σ.

For pairs of non-negative integers g, p, let $G\tilde{amma}_{(g,p)}$ denote the set of Legendrian links $\Lambda \subset \mathbb{R}^3_{std}$ that have an immersed Lagrangian filling Σ with $g(\Sigma) = g$, $p(\Sigma) = p$, but do not have any immersed Maslov 0 Lagrangian fillings of genus $(g-1)$ and $(p + 1)$ double points, such that one of the double points has vanishing index and action. In particular, $\tilde{\Gamma}_{(g,p)}$ is the set of Legendrian links with a Lagrangian filling that has a filling which cannot be obtained from another immersed filling by Lagrangian surgery. Our first main result was to show that the sets $\tilde{\Gamma}(g, p)$ are not empty for any pairs of non-negative integers (g, p). We can then conclude that Lagrangian surgery is not always reversible.

Theorem 46 (Capovilla-Searle, Legout, Limouzineau, Murphy, Pan, Traynor [10])**.** *Given any $g \in \mathbb{Z}^+$ and $p \in \mathbb{Z}_{\geqslant 0}$, there is a Legendrian link $\Lambda \subset \mathbb{R}^3_{std}$ in the set $\tilde{\Gamma}_{(g,p)}$.*

For the proof of Theorem 46 we provide various infinite families of Legendrian links in $\tilde{\Gamma}_{(g,p)}$ for $g, p \in \mathbb{Z}_{\geqslant 0}$. One of our base examples is a Legendrian knot of type

7_4, Λ_{7_4}, shown in Figure 6.1. We give a reproof of the fact that Λ_{7_4} is contained in the set $\tilde{\Gamma}_{(1,0)}$ since it has a genus 1 exact Lagrangian filling, and no VIC disk filling with one double point. Recall that the disk filling with one double point is also obstructed in the smooth category [60]. However, it is not known if for every Legendrian link in the infinite families we construct, if there are also smooth obstructions to the existence of immersed fillings. It is also worth remarking that these Legendrian links can have large crossing numbers, and that these families contain Legendrians whose smooth knot type is that of an infinite knot.

We also provide an example of a Legendrian knot with an obstruction to a VIC filling in the Lagrangian category that is not obstructed in the smooth category. The Legendrian knot of smooth knot type 9_{48}, $\Lambda_{9_{48}}$, whose front projection is shown in Figure 6.1. This Legendrian link has a Lagrangian filling of genus 1 with one vanishing action double point of index -1. However, $\Lambda_{9_{48}}$ has no exact Maslov 0 Lagrangian disk filling with two vanishing action double points of index 0 and 1, so $\Lambda_{9_{48}} \subset \tilde{\Gamma}_{(1,1)}$. In contrast, the smooth knot 9_{48} has both a smooth disk filling with 2 double points, and a smooth disk filling with one double point. One can construct another family of Legendrian links which includes $\Lambda_{9_{48}}$, and which provide examples of Legendrian links in $\Lambda_{(g,p)}$. It would be interesting to understand whether any of the Legendrian links in this family provide examples of links which have Lagrangian obstructions but no smooth obstructions to the existence of immersed fillings.

We proved Theorem 46 by translating immersed exact Lagrangian to embedded exact Lagrangian cobordisms and applying new obstructions to the existence of embedded exact Lagrangian cobordisms. The translation from immersed fillings to embedded cobordisms is a generalization of Theorem 1.2 in [16]. Suppose Λ is a Legendrian link with an immersed exact Lagrangian filling Σ with p double points. A Darboux ball around every double point of Σ is Hamiltonian isotopic to a filling of a maximal-tb Legendrian Hopf link, so if one removes these Darboux balls one should obtain a Lagrangian cobordism from a disjoint union of p Hopf links to Λ. See Figure 2.1 for the front and Lagrangian projection of a maximal-tb Legendrian Hopf link, H_0. One of the main difficulties in proving the following Theorem was that we needed to produce not just an embedded Lagrangian cobordism from $\sqcup_p H_0$ to Λ, but one that is cylindrical, exact and with constant primitives on the boundaries.

Theorem 47 (Capovilla-Searle, Legout, Limouzineau, Murphy, Pan, Traynor [10])**.** *Suppose Σ is a VIC Lagrangian cobordism from Λ_- to Λ_+ with p double points, denoted by $x_1, \ldots, x_p$. Then, there exists a VIC cobordism with $p-1$ double points, $x_1, \ldots, x_{p-1}$ from $\Lambda_- \sqcup H_0$ to Λ_+, where H_0 is the maximal-tb Legendrian Hopf link. The index of the double point x_p determines the Maslov potential of the corresponding Hopf link.*

Our main obstruction tool is relies on the functorial behavior the Legendrian contact homology DGA over the category with objects Legendrian links and morphisms Maslov 0 exact Lagrangian cobordisms. When studying augmentations of a Legendrian link Λ one can work with the Augmentation Category $Aug_+(\Lambda; R)$, a unital A-infinity category defined in [59] over a unital ring R. This augmentation category followed the definition of a non-unital A-infinity category, Aug_- [8], and was shown to be equivalent to a category of constructible sheaves. The objects of $Aug_+(\Lambda; R)$ are augmentations of Λ over R. The morphism spaces of $Aug_+(\Lambda; R)$ are vector spaces generated by mixed Reeb chords in the the 2-copy of Λ, $\Lambda^{(1)} \cup \Lambda^{(2)}$; that is, $Hom_+(\epsilon^1, \epsilon^2) := \mathbb{F}\langle \mathcal{R}_{1,2}(\Lambda)\rangle[1]$. Although the morphism space $Hom_+(\epsilon^1, \epsilon^2)$ does not depend on the augmentations, the the A_∞ operators m_n^+ do. The A_∞ operators m_n^+ are defined using a generalization of linearized Legendrian contact homology and the n copy of Λ.

Given a Maslov 0 exact Lagrangian cobordism Σ from Λ_- to Λ_+, there is a map between the Augmentation categories $\mathcal{F} : Aug_+(\Lambda_-; R) \to Aug_+(\Lambda_+; R)$. In [61], Pan constructed such an induced category map f when $\Lambda_{\pm}$ are single component knots and showed it was injective. We extend Pan's result to Legendrian links using classical Floer techniques based on work of Chatraine, Dimitriglou-Rizell, Ghiggini, Golovko [17] and Legout [54].

Theorem 48 (Capovilla-Searle, Legout, Limouzineau, Murphy, Pan, Traynor[10])**.** *Let $\Lambda_{\pm}$ be Legendrian links in $(\mathbb{R}^3, \xi_{std})$ and let $R = \mathbb{Z}_2$. If there exists an exact, embedded, Maslov 0 Lagrangian cobordism from Λ_- to Λ_+, then the induced map*

$$\mathcal{F} : Aug(\Lambda_-; R) \to Aug(\Lambda_+; R)$$

is injective.

It is worth noting that the map $\mathcal{F}$ we construct is distinct from the category map f that Pan constructed for cobordisms between single component knots because f is not well-defined for cobordisms between links. We instead studied "wrong-way" maps $\iota : H^*Hom_+(\epsilon^1_+, \epsilon^2_+) \to H^*Hom_+(\epsilon^1_-, \epsilon^2_-)$ constructed from wrapped Floer theory [17]. These maps are dual to the natural maps given by restricting the augmentation functor to morphisms, $\mathcal{F} : H^*Hom_+(\epsilon^1_-, \epsilon^2_-) \to H^*Hom_+(\epsilon^1_+, \epsilon^2_+)$. We showed that $\mathcal{F}$ is injective by showing that the maps ι preserve the product structure and the unit. We considered both the product structure on the Augmentation category m_2^+, and the product structure on the floer complex associated to pairs of Lagrangian cobordisms defined by [54].

Remark 49. *The proofs of the results given in this collaboration assume that R is $\mathbb{Z}_2$. All of our results are expected to hold for $R = \mathbb{Z}$ given the additional assumption that the Legendrian submanifolds and Lagrangian cobordisms are relatively spin. What remains to be verified is that one can define coherent orientations for the relevant moduli spaces of $\mathcal{J}$-holomorphic curves that we use from [17, 54]. One could also generalize Theorem 48 for more general coefficient rings such as $R \cong \mathbb{Z}[H_1(\Lambda)]$, or $\mathbb{Z}[H_1(\Sigma)]$.*

From Theorem 48, one can find various relations between the normalized count of augmentations, linearized contact homology and normal ruling polynomials of Λ_- and Λ_+.

Corollary 50 (Capovilla-Searle, Legout, Limouzineau, Murphy, Pan, Traynor [10])**.** *Suppose $\Lambda_\pm$ are Legendrian links in $(\mathbb{R}^3, \xi_{std})$ and $R = \mathbb{Z}_2$. If there exists an exact, embedded, Maslov 0 Lagrangian cobordism from Λ_- to Λ_+, then*

$$\#Aug(\Lambda_+; R) \geqslant \#Aug(\Lambda_-; R),$$

where $\#Aug(\Lambda; R)$ denotes the number of augmentations of $\mathcal{A}(\Lambda)$ to R up to equivalence in the $Aug_+(\Lambda)$ category.

Theorem 47, and Corollary 50 together imply the following Corollary which allowed us to find Legendrian links in the sets $\tilde{\Gamma}_{(g,p)}$.

Corollary 51 (Capovilla-Searle, Legout, Limouzineau, Murphy, Pan, Traynor [10])**.** *Let $\Lambda \subset \mathbb{R}^3_{std}$ be a Legendrian link, and $R = \mathbb{Z}_2$. If Λ admits a genus g immersed exact Maslov 0 Lagrangian filling Σ, and $\#Aug(\Lambda; R) < \#Aug(\sqcup_{i=1}^{k} H_0; R)$ for $k \leqslant g$, then Λ does not admit any immersed exact Maslov 0 Lagrangian filling of genus $(g-k)$ and k simultaneously contractible double points of vanishing index.*

The Legendrian links Λ_{7_4} and $\Lambda_{9_{48}}$ both have one augmentation over $\mathbb{Z}_2$, while the Hopf link H_0 has three augmentations over $\mathbb{Z}_2$. Therefore, they do not admit any immersed exact Lagrangian disk fillings with any double points of vanishing action and index.

As noted in Remark 49 Theorem 48 is not yet proven for $R = \mathbb{Z}$, although it is expected to hold. One implication of showing Corollary 51 is true for more general unital rings such as $R = \mathbb{Z}$ is an improvement of Proposition 28. Namely, one could conclude that if $\Lambda_- \subset \mathbb{R}^3_{std}$ is an aug-infinite Legendrian link, and Λ_+ is a Legendrian link such that there exists a Maslov 0 exact Lagrangian cobordism from Λ_- to Λ_+, then Λ_+ is also aug-infinite.

Finally, we also show the following application of our obstructions to immersed Lagrangian fillings. Let $\Sigma \subset \mathbf{B}^4$ be an immersed exact Maslov 0 Lagrangian, and let $\gamma \subset \Sigma$ be an essential embedded curve in Σ. Suppose $\mathbf{D}^2 \subset \mathbf{B}^4$ is a Lagrangian disk with $\partial \mathbf{D}^2 = \gamma$, so that the interior of $\mathbf{D}^2$ is disjoint from Σ and $\mathbf{D}^2$ is never tangent to Σ along its boundary. Then, there exists another immersed exact Maslov 0 Lagrangian $\widetilde{\Sigma} \subset B^4$, contained in a small neighborhood of $\Sigma \cup \mathbf{D}^2$, so that $\widetilde{\Sigma}$ has one less genus and one more double point than Σ (informally, γ is collapsed to a point along $\mathbf{D}^2$). Since both Σ and $\mathbf{D}^2$ are Lagrangian a cylindrical neighborhood of γ in Σ is a Hopf link. Thus, obstructing immersed fillings of a Legendrian translates to obstructing such Lagrangian disks $\mathbf{D}^2$. .

Proposition 52 (Capovilla-Searle, Legout, Limouzineau, Murphy, Pan, Traynor [10])**.** *Let $\Sigma \subset (\mathbf{B}^4, \omega_{std})$ be an immersed exact Maslov 0 Lagrangian filling of genus g and k double points of a Legendrian link $\Lambda \subset \mathbb{R}^3_{std}$ contained in the set $\tilde{\Gamma}_{(g,p)}$. Then, there does not exist any Lagrangian disks $\mathbf{D}^2 \subset (\mathbf{B}^4, \omega_{std})$ such that $\partial \mathbf{D}^2 = \gamma \subset \Sigma$ is an essential curve so that the interior of $\mathbf{D}^2$ is disjoint from Σ and $\mathbf{D}^2$ is never tangent to Σ along its boundary.*

7

Weinstein handlebody diagrams of smoothed toric divisors

In this Chapter we will summarize the results obtained from a collaboration with Acu, Gadbled, Marinković, Murphy, Starkston and Wu. For more details and full proofs see [3].

Symplectic divisors are co-dimension 2 symplectic submanifolds $(D^{2n-2}, i^*\omega) \subset (M^{2n}, \omega)$, which can have controlled singularities. The existence of symplectic divisors was shown by Donaldson, who proved that every closed integral symplectic manifold (M, ω) has a smooth symplectic divisor that is Poincaré dual to $k[\omega]$ [22]. Giroux proved that from the set of such Donaldson divisors, there exists a divisor whose complement admits a Weinstein structure [39, 40]. We studied a particular class of symplectic divisors in toric 4-manifolds and gave an explicit construction of the Weinstein structure of the complements of these divisors. We also formulated an algorithm the Weinstein handlebody diagram of Weinstein 4-manifolds constructed by attaching 2-handles along co-normal lifts of curves on a closed surface F to D^*F. It is often difficult to construct explicit Weinstein handlebodies and many are not known for fundamental examples of divisors complements. Having an explicit Weinstein handlebody allows one to more easily compute various symplectic invariants, and can help one find Lagrangian submanifolds in the Weinstein manifold as described in Section 2.4.

A **toric manifold** is a symplectic manifold (M^{2n}, ω) with an effective Hamiltonian action of the torus of the maximal dimension. The Hamiltonian action induces a moment map $\Phi : M \to \mathbb{R}^n$ whose image is a convex polytope, denoted by Δ [6, 46]. Furthermore, Δ is a Delzant polytope which means that there are n edges meeting at every vertex, all inward normal vectors to the facts are primitive vectors in $\mathbb{Z}^n$, and all inward normal vectors of facets meeting at a single vertex form a $\mathbb{Z}^n$ basis. The Delzant classification tells us that every Delzant polytope is the moment map image of a unique toric manifold up to equivariant symplectomorphism. The preimage under Φ of any point in Δ is the torus modulo the stabilizer subgroup. The fixed points of the torus action map to the vertices of the Delzant polytope and are in one-to-one correspondence with the vertices of Δ. Every toric manifold (M, ω) has a natural symplectic divisor D with normal crossing singularities given by the preimage of the facets of the Delzant polytope. Since this symplectic divisor is also fixed under the torus action we call it the toric divisor of M. The complement of a neighborhood of a toric divisors is the disk cotangent cotangent bundle of the torus of maximal dimension, D^*T^n which has a canonical Weinstein structure.

A toric divisor and any of its smoothing is Poincaré dual to the anti-canonical class (the first Chern class) of (M, ω). Therefore, the pair given by a toric manifold (M, ω) and a toric divisor or any of its smoothings is **log Calabi-Yau**. Log Calabi-Yau manifolds have played an important role in homological mirror symmetry [1, 7, 45**?**]. If a divisor is both an anti-canonical divisor and a Donaldson divisor implies that $c_1(M, \omega)$ is a multiple of $[\omega]$, which means that the symplectic manifold is monotone. In the case that we smooth every singularity in the toric divisor, the complement may not be Weinstein unless the symplectic manifold is monotone. However, if we consider certain partial smoothings, this is no longer a requirement for the complement to carry a Weinstein structure. Some care is still needed since not all partially smoothed toric divisors have a complement that admits a Weinstein structure.

In the case of 4-dimensional toric manifolds, the toric divisors are singular surfaces whose normal crossing singularities are nodes. We will refer to both a node and its image vertex with the same notation. One can smooth any node and obtain a surface with one fewer singularity in the toric 4-manifold. Topologically smoothing a node corresponds to attaching a 2-handle to D^*T^2. One then needs to understand whether this handlebody supports a Weinstein structure, and if so what is the Legendrian attaching sphere of the 2-handle. The following condition allowed us to prove that the complements of certain partial smoothed toric divisors support a Weinstein structure.

Definition 53 (Acu, Capovilla-Searle, Gadbled, Marinković, Murphy, Starkston, Wu[3])**.** *For each vertex, V, of the Delzant polytope, we associate to it a ray R generated by the sum of the edge vectors of Δ adjacent to V and beginning at V. A toric manifold with a chosen subset $\{V_1, \dots, V_k\}$ of the vertices is called $\{V_1, \dots, V_k\}$-*centered *if the corresponding rays $R_1, \dots, R_k$ all intersect at a common single point in the interior of the Delzant polytope.*

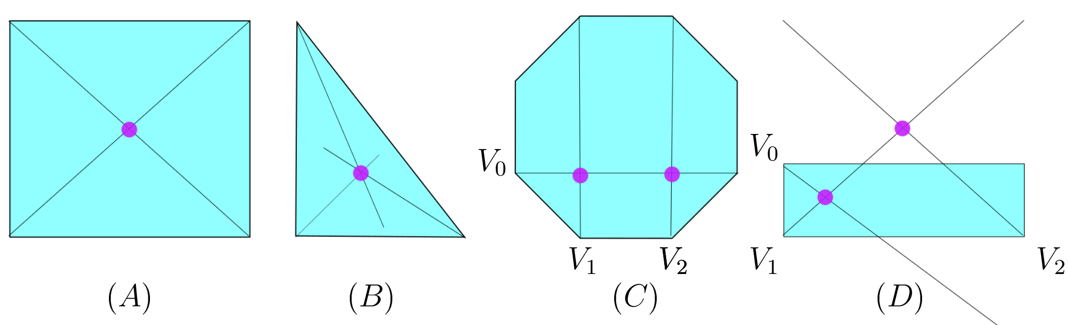


FIGURE 7.1: $(A) - (B)$ Examples of centered toric manifolds. $(C) - (D)$ Examples of only partially centered toric manifolds.

See Figure 7.1 for examples and non-examples of centered toric manifolds. Monotone toric manifolds such as $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $\mathbb{C}P^2$ shown to in the left of Figure 7.1 are centered with respect to every node. In contrast, $\mathbb{C}P^2\#5\overline{\mathbb{C}P^2}$ and a non-monotone $\mathbb{C}P^1 \times \mathbb{C}P^1$ shown to the right of Figure 7.1 are not $\{V_1, V_2\}$-centered, but are $\{V_0, V_1\}$-centered. The motivation for this definition is that, if for a $\{V_1, \ldots, V_k\}$-centered toric manifold, the complement of the toric divisor smoothed precisely at the nodes $V_1, \ldots, V_k$ admits a Weinstein structure.

Theorem 54 (Acu, Capovilla-Searle, Gadbled, Marinković, Murphy, Starkston, Wu [3])**.** *Let (M, ω) be a toric 4-manifold corresponding to Delzant polytope Δ which is $\{V_1, \ldots, V_k\}$-centered. Let D denote the divisor obtained by smoothing the toric divisor at the nodes $V_1, \ldots, V_k$. Then there exist arbitrarily small neighborhoods N of D such that $M \backslash N$ admits the structure of a Weinstein domain.*

Furthermore, $M \backslash N$ is Weinstein homotopic to the Weinstein domain obtained by attaching Weinstein 2-handles to the unit disk cotangent bundle of the torus D^*T^2, along the Legendrian co-normal lifts of co-oriented curves of slope $s(V_1), \ldots, s(V_k)$. Here $s(V_i)$ is equal to the difference of the inward normal vectors of the edges adjacent to V_i in Δ.

From now on, we will refer to the difference of the inward normals of the edges adjacent to a vertex V as the slope of the node, $s(V)$. The proof of Theorem 54 starts with the base case of smoothing a single singularity V. For any toric 4-manifold there exists an equivariant symplectomorphism mapping the Delzant polytope Δ to one where V and the facets meeting at V are mapped to the standard corner which is the moment map image of $(\mathbb{C}^2, \omega_{std})$. In particular, this symplectomorphism can be encoded as an $SL(2, \mathbb{Z})$ transformation on the pair of inward normals of the facets meeting at V. We then gave an explicit smoothing of a node, and find the Lagrangian core and co-core of the attaching 2-handle. We showed explicitly that the Legendrian attaching sphere is the co-normal lift of a co-oriented curve of slope $s(V) = (1, -1)$

in T^2. One can then check that using another $SL(2,\mathbb{Z})$ transformation that the attaching sphere is the co-normal lift of the curve of slope equal to the difference of inward normals at V. If we smooth multiple nodes $V_1,\ldots,V_k$ in a $\{V_1,\ldots,V_k\}$-centered toric manifold, we used the local model described above for the complement of each smoothed node V_i. Thanks to the centered condition we can patch these local models together and give a Weinstein structure on the complement of the smoothed toric divisors.

There is also a generalization of Theorem 54 for certain almost toric manifolds. **Almost toric manifolds** where first introduced by Symington [73], and are symplectic manifolds equipped with a Lagrangian fibration structure $\pi : (M,\omega) \to B$ over a surface B so that π locally has the structure of a moment map for an effective torus action or the Lagrangian analog of a Lefschetz fibration. The later means that singular Lagrangian torus fibers are allowed. One can translate from a toric manifold to an almost toric manifold via nodal trades where the neighborhood of a fixed point in the toric manifold is traded for a singular Lagrangian fibration. Note that the complement of the divisor in the almost toric setting after a nodal trade has one less singularity and their complement is D^*T^2 with a 2-handle attached to it. Furthermore, the slope of a node V is translated to an eigenray associated to the singular Lagrangian fiber. The centered condition can be adapted to almost toric manifolds as follows: we say an almost toric manifold is centered if the eigenrays of the singular fibers all intersect at a common single point in the interior of the polytope. Using nodal trades we can then conclude that for almost toric manifolds with this adapted centered condition, the complement of the toric divisor has a Weinstein structure given by attaching 2-handles to D^*T^2 along the Legendrian co-normal lift of the eigenrays associated to each singularity.

One can also ask how many distinct Weinstein 4-manifolds can be produced as smoothings or partial smoothings of toric divisors. There are only 5 monotone toric

4-manifolds, and therefore there are at least 5 Weinstein complements of smoothed divisors where every node has been smoothed. However, as soon as one allows for partial smoothings one can construct many Weinstein manifolds.

Theorem 55 (Acu, Capovilla-Searle, Gadbled, Marinković, Murphy, Starkston, Wu [3]). *There are infinitely many non-diffeomorphic Weinstein manifolds obtained by taking the completion of the complement of a neighborhood of a partially smoothed toric divisor in a toric 4-manifold.*

Given a set of primitive vectors $\{(a_1, b_1), \ldots, (a_k, b_k)\}$ we also explored the existence of a $\{V_1, \ldots, V_k\}$-centered toric 4-manifold where each V_i has slope $s_i = (a_i, b_i)$ and find that this is not always guaranteed.

Example 56 (Acu, Capovilla-Searle, Gadbled, Marinković, Murphy, Starkston, Wu [3]). *For any $K \geqslant 2$, there is no $\{V_1, V_2, V_3, V_4\}$-centered Delzant polytope where*

$$s(V_1) = (1, 1), s(V_2) = (1, 2), s(V_3) = (-K, -1), s(V_4) = (0, -1).$$

However, if one does not impose the partially-centered requirement we find that for any set of k primitive vectors there exists a toric manifold whose Delzant polytope has at least k nodes with the desired slopes.

Proposition 57 (Acu, Capovilla-Searle, Gadbled, Marinković, Murphy, Starkston, Wu[3]). *For any choice of primitive vectors $\{(a_1, b_1), \ldots, (a_k, b_k)\}$ there is a Delzant polytope with at least k edges such that there are k vertices whose slopes are precisely $\{(a_1, b_1), \ldots, (a_k, b_k)\}$.*

We provided examples to prove that in the non-centered case, the complement of a divisor does not support a Weinstein structure. In particular, we found some examples where the complement is not exact or does not have convex boundary.

Proposition 58 (Acu, Capovilla-Searle, Gadbled, Marinković, Murphy, Starkston, Wu [3])**.** *Let (M, ω) be a symplectic toric 4-manifold and Δ its Delzant polytope. Let $V_1, \ldots, V_k$ be a subsets of the vertices of Δ. Assume that M fails to be $\{V_1, \ldots, V_k\}$-centered because either*

(i) Two rays associated to two of the vertices $\{V_1, \ldots, V_k\}$ are parallel, or;

(ii) There exists three vertices, $\{V_{i_1}, V_{i_2}, V_{i_3}\}$ such that for the associated rays $R_{i_1}, R_{i_2}, R_{i_3}$, R_{i_1} intersects R_{i_2} at a point $c_1 \in \mathring{\Delta}$ in the interior of the polytope Δ that does not belong to R_{i_3}.

Then the complement of its toric divisor smoothed at $\{V_1, \ldots, V_k\}$ is not an exact symplectic manifold (and in particular cannot support a Weinstein handlebody structure).

The proof of Proposition 58 relies on constructing explicit 2-dimensional cycles with nonzero symplectic area in the complements that such smoothed divisors.

We also showed how to produce concrete Weinstein handlebody diagrams for the Weinstein 4-manifolds obtained as in Theorem 54. Let F be a surface and c a finite set of co-oriented curves in F. Then, the co-normal lifts of the curves in c are Legendrian curves in the cotangent sphere bundle $S^*F = \partial(D^*F)$ whose contact structure is induced by the canonical symplectic structure on D^*F. Let $\mathcal{W}_{F,c}$ denote the Weinstein 4-manifold obtained by attaching Weinstein 2-handles to D^*F along the Legendrian co-normal lifts of the curves in the set c.

Theorem 59 (Acu, Capovilla-Searle, Gadbled, Marinković, Murphy, Starkston, Wu [3])**.** *Let F be any surface and $\{\gamma_i\}_{i=1}^n$ a finite collection of co-oriented curves in F. Then the procedure of Section 8 in [3] produces a Weinstein handle diagram in standard form representing a Weinstein manifold $\mathcal{W}$ which is Weinstein homotopic to $\mathcal{W}_{F,c}$.*

In order to prove Theorem 59, we prove the base case for $c = \oslash$ and fill the following gap in the literature. Gompf [42] gave handlebody diagrams for D^*F for closed surfaces F that are diffeomorphic to D^*F. In the case of $F = T^2$, the Gompf handlebody diagram for D^*F is Weinstein homotopic to the canonical Weinstein structure on the cotangent bundles of a surface [75]. Ozbagci showed that contact structures on S^*F of the Gompf handlebody diagrams and from the canonical Weinstein structure on the cotangent bundle of F agree.

Theorem 60 (Acu, Capovilla-Searle, Gadbled, Marinković, Murphy, Starkston, Wu [3])**.** *The Gompf handlebody diagram for D^*F corresponds to a Weinstein structure which is Weinstein homotopic to the canonical Weinstein structure on the cotangent bundles of a surface.*

We also applied Theorem 43 to the Weinstein handlebodies constructed in Theorem 59 and concluded the following:

Proposition 61 (Acu, Capovilla-Searle, Gadbled, Marinković, Murphy, Starkston, Wu [3])**.** *Any Weinstein 4-manifold X constructed by attaching 1- or 2-handles to T^*F for $i = 1, \ldots, k$ for any orientable surface F, has nonvanishing symplectic homology.*

Corollary 62 (Acu, Capovilla-Searle, Gadbled, Marinković, Murphy, Starkston, Wu [3])**.** *Any Weinstein 4-manifold X constructed by attaching 1- or 2-handles to T^*F for $i = 1, \ldots, k$ for any orientable surface F, is not a flexible Weinstein manifold.*

Finally, we investigated the question of when Weinstein complements of different partially smoothed toric divisors are equivalent. We used the fact that if the slopes of two Delzant polytopes are related by an $SL(2, \mathbb{Z})$ transformation, then they are moment map images of equivariantly symplectomorphic toric 4-manifolds.

Proposition 63 (Acu, Capovilla-Searle, Gadbled, Marinković, Murphy, Starkston, Wu [3])**.** *Consider any* $\{V_1, \ldots V_k\}$*-centered toric* 4*-manifold where* V_i *has slope* (a_i, b_i), $i = 1, \ldots, k$*. If there is an* $SL(2, \mathbb{Z})$ *transformation mapping the slopes* $\{(a_1, b_1), \ldots, (a_k, b_k)\}$ *to* $\{(a'_1, b'_1), \ldots, (a'_k, b'_k)\}$ $k \geqslant 1$*, then the completions of* $\mathcal{W}_{T^2,\{(a_1,b_1),\ldots,(a_k,b_k)\}}$ *and* $\mathcal{W}_{T^2,\{(a'_1,b'_1),\ldots,(a'_k,b'_k)\}}$ *are symplectomorphic.*

Proposition 7 served as motivation to show the following Proposition which gives a geometric interpretation of such $SL(2, \mathbb{Z})$ transformations as Legendrian 1-handle slides of $\mathcal{W}_{T^2,c}$ where c is a set of slopes on T^2.

Proposition 64 (Acu, Capovilla-Searle, Gadbled, Marinković, Murphy, Starkston, Wu [3])**.** *Let* F *be an orientable surface. The group of orientation preserving homeomorphisms of* F *correspond to Legendrian* 1*-handle slide in the Gompf handlebody of* T^*F *and to* 1*-handle slides in the Weinstein handlebody of* $\mathcal{W}_{F,c}$ *where* c *is a set of co-oriented curves on* F*.*

The proof of Proposition 64 relies heavily on the algorithm constructed in Theorem 59. We also showed that the existence of an $SL(2, \mathbb{Z})$ transformation between sets of slopes is not a necessary condition.

Remark 65 ([3])**.** *The existence of an* $SL(2, \mathbb{Z})$ *transformation between sets of slopes* c *and* c' *is a sufficient condition to guarantee that* $\mathcal{W}_{T^2,c}$ *and* $\mathcal{W}_{T^2,c'}$ *are Weinstein homotopic, but it is not necessary (even when* $\mathcal{W}_{T^2,c}$ *and* $\mathcal{W}_{T^2,c'}$ *are both realizable as complements of partially smoothed toric divisors).*

Using Theorems 54, and 59 we constructed the Weinstein handlebody diagrams for various complements of partially smoothed toric divisors. These include the Weinstein handlebody of the complement of a smooth cubic shown in Figure 7 obtained by smoothing the three nodes in the toric divisor in $\mathbb{C}P^2$ which have slopes $(3, -1), (0, -1)$, and $(-3, 2)$.

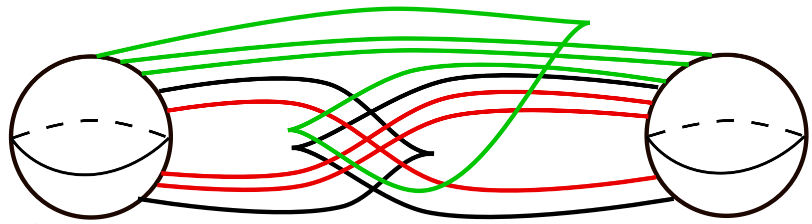


FIGURE 7.2: A Weinstein handlebody diagram of the complement of a smooth cubic in $\mathbb{C}P^2$.

8

Conclusions

We provide a brief summary of the main results of this dissertation. See the introduction for a more fleshed out overview. We found new examples of Legendrian links with infinitely many distinct exact orientable Lagrangian fillings:

Theorem 22. *The oriented Legendrian links $\Lambda_n \subset \mathbb{R}^3_{std}$ shown in Figure 3.1 have infinitely many exact Maslov* 0 *Lagrangian fillings up to Hamiltonian isotopy for* $n \geqslant 1$ *that are all smoothly isotopic.*

The Legendrian links Λ_1, Λ_2 are some of the first examples of Legendrian links with infinitely many exact, orientable planar fillings. We then apply this result to construct infinitely many closed exact Lagrangian tori and spheres in the Milnor fibers of isolated hypersurface singularities of positive modality.

Corollary 41. *Suppose that M_f is the Milnor fiber of a positive modality isolated hypersurface singularity f, then M_f contains infinitely many Hamiltonian non-isotopic exact Maslov* 0 *Lagrangian tori that are smoothly isotopic and infinitely many Hamiltonian non-isotopic exact Lagrangian spheres that are smoothly isotopic.*

Finally, we also provide a criterion for checking the non-vanishing of the symplectic homology of a Weinstein manifold given a Weinstein handlebody diagram.

Theorem 43. *Let X_Λ be the Weinstein 4 manifold resulting from attaching 2 handles along a Legendrian link $\Lambda = \Lambda^{(1)} \cup \cdots \cup \Lambda^{(n)} \subset (\#^m(S^1 \times S^2), \xi_{std})$. If there is any sublink $\Lambda^{(1)} \cup \cdots \cup \Lambda^{(l)}$ for $l < n$, such that its differential graded algebra has a representation*

$$\rho : (\mathcal{A}(\Lambda^{(1)} \cup \cdots \cup \Lambda^{(l)}; \mathbb{Z}[t_1^{\pm 1} \ldots, t_l^{\pm 1}]), \partial) \to End(V)$$

where V is a vector space over $\mathbb{Q}$ and $\rho(t_k) = -Id$ for $k = 1, \ldots, l$, then $S\mathbf{H}(X_\Lambda) \neq 0$.

We also provide summaries of two collaborations that took place during the authors graduate studies in Chapter 6 and Chapter 7.

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Biography

Orsola A. Capovilla-Searle earned a B.A from Bryn Mawr College where she majored in Mathematics and Physics. She was a Mellon Mays Undergraduate Fellow, and an NSF graduate fellow. In the Fall of 2015 she entered the graduate program of the Mathematics department at Duke University. During her time at Duke, Capovilla-Searle was a University scholars fellow and Deans Graduate fellow. Capovilla-Searle also participated in two research conferences, Women in Geometry II, and Women in Symplectic and Contact Topology, which resulted in two collaborations. Capovilla-Searle will continue her academic career as a NSF post-doctoral fellow at UC Davis, beginning in the summer of 2021.