

Modifying the Uniform-Price Auction to Eliminate ‘Collusive-Seeming Equilibria’

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Abstract

The uniform-price auction is used in many regional electricity procurement auctions and its “collusive-seeming equilibria” have been linked to potential exercise of market power. Such equilibria do not exist, however, if a small amount of cash is split among rationed bidders. To shed light on what drives this result, I also examine variations in which the auctioneer is able to increase and/or decrease quantity after receiving the bids. “Inceasable demand” also eliminates all collusive-seeming equilibria. These results suggest ways to modify the uniform-price auction in order to reduce the potential exercise of market power.

1 Introduction

The U.S. Treasury recently switched to the uniform-price auction from the more traditional discriminatory (or multiple-price) auction to issue all government debt. Proponents of the uniform-price format argue that the switch will improve the efficiency of market operations, reduce the cost of financing the Federal debt, and broaden participation in Treasury auctions. Such

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claims are based on the intuitive notion that “auction participants will bid more aggressively in uniform-price auctions since successful bidders in uniform-price auctions pay only the price of the lowest accepted bid, rather than the actual price they bid, as in the multiple-price approach”.¹

This intuition is drawn from an apparent analogy between the uniform-price auction and the second-price auction. But as Ausubel and Cramton (1998) have argued, this analogy is faulty in the multi-unit setting: In the uniform-price auction each bidder generally shades down his bid on all but the first unit for which he bids and the resulting allocation is inefficient. This bid shading follows from the fundamental linkage between payments for marginal and inframarginal units in the uniform-price auction. If a bidder suspects that his bid for an k th unit will set the price, then he has an incentive to lower his bid on that unit, since then he will pay less on all $k - 1$ inframarginal units. This logic does not apply to the *first* unit on which he bids, of course, and it is a weakly dominant strategy to bid one’s true valuation for the first unit.

This linkage between marginal and inframarginal units also explains why the uniform-price auction often possesses multiple equilibria which produce very undesirable outcomes from the auctioneer’s point of view. For example, consider a simplified model of electricity procurement with two bidders / suppliers. (This example and others in which bidders have private information will be discussed in Section 1.1.) A “bid” by $i = 1, 2$ is a supply schedule $S_i(\cdot)$. At time t , realized electricity demand is $D(t)$ and perfectly inelastic. Given the bids and demand at time t , each bidder receives price p^* for all quantity q_i^* that he provides, where p^* is the market-clearing price, $D(t) = S_1(p^*) + S_2(p^*)$, and q_i^* his market-clearing quantity, $q_i^* = S_i(p^*)$. The bidders have constant marginal costs $c_1 \leq c_2$ and know each other’s costs. Finally, demand is never less than $\min D$ at all times with probability one. Then for every $\gamma > 0$ such that $c_1 + \frac{\min D}{\gamma} \geq c_2$, an equilibrium exists in which bidders submit supply schedules

$$S_i(p) = \gamma(p - c_i)$$

and the realized price and quantities are

$$p^*(t) = \frac{c_1 + c_2}{2} + \frac{D(t)}{2\gamma}, \quad q_i^*(t) = \frac{D(t)}{2} + D(t)\gamma \frac{c_i - c_j}{2}.$$

¹Lawrence Summers, Deputy Secretary of the Treasury, in the foreword to the Treasury’s report, “Uniform-Price Auctions: Update of the Treasury Experience” (Malvey and Archibald, October 1998).

All of these equilibria share the peculiar feature that the realized price is sometimes greater than an upper bound (c_2) on all bidders' marginal costs. Actually, with the exception of the “best” of these equilibria (corresponding to $\gamma = \frac{\min D}{c_2 - c_1}$) in the one event that demand equals $\min D$, the realized price *always* strictly exceeds c_2 .

This sort of equilibrium clearly *seems* collusive, since neither bidder bids to win more units, despite the fact that his marginal cost is strictly less than the price. Returning to the terminology in which bidders are buyers, let us say that an equilibrium is *clearly collusive-seeming* if the price is always strictly greater than an upper bound \bar{c} on bidders' marginal costs. That is to say, for every possible allocation \vec{q} in every possible state of the world \vec{s} , all bidders have marginal cost $c_i(q_i; q_{-i}, \vec{s}) \leq \bar{c}$. (In my model, bidder i 's marginal cost c_i may depend on others' quantities q_{-i} .) An equilibrium is “*clearly collusive-seeming*” iff the realized price is always strictly greater than this upper bound: $\inf_{\vec{s}} p^*(\vec{s}) \equiv \underline{p} > \bar{c}$.

In this paper, I focus on showing how certain modifications to the rules of the uniform-price auction can eliminate not only these clearly collusive-seeming equilibria but also a broader class of equilibria, some of which may or may not strike the reader as seeming collusive. Nonetheless, I call this broader class collusive-seeming equilibria. Let \bar{p} be the highest price that is realized in a given equilibrium and let $\bar{c}_i(\bar{p})$ be bidder i 's highest realized marginal cost when price \bar{p} is realized. By definition of \bar{c} , $\bar{c}_i(\bar{p}) \leq \bar{c}$ for all bidders i in any equilibrium.

Definition (Collusive-seeming equilibrium). An equilibrium is *collusive-seeming* iff $\#\{i : \bar{c}_i(\bar{p}) < \bar{p}\} > 1$.²

See Section 2.6 for the formal definition of a collusive-seeming equilibrium. For instance, in the simplified electricity procurement example on page 3, $\bar{p} = \frac{c_1 + c_2}{2} + \frac{\max D}{2\gamma}$ and $\bar{c}_i(\bar{p}) = c_i$ for $i = 1, 2$. (The bidders have constant marginal cost by assumption.) Thus, *every* equilibrium described there (i.e. for all $\gamma > 0$ satisfying $c_1 + \frac{\min D}{\gamma} \geq c_2$) is collusive-seeming because $\bar{p} > c_2$.

What does it mean for an equilibrium *not* to be collusive-seeming? Consider the event in which the price achieves its upper bound. In an equilibrium which is not collusive-seeming, at most one bidder has marginal cost in

²Any clearly collusive-seeming equilibrium is collusive-seeming. In a clearly collusive-seeming equilibrium, by definition, $\bar{c} < \underline{p}$. Since $\bar{c}_i(\bar{p}) \leq \bar{c}$ for all i and $\underline{p} \leq \bar{p}$, we have that $\#\{i : \bar{c}_i(\bar{p}) < \bar{p}\} = n$, where n is the number of bidders.

this event which is always less than the realized price, i.e. with probability one conditional on that event. In the electricity procurement context, consider the times of peak demand, unexpected plant outages, and transmission grid congestion in which price achieves its maximum. In a non-collusive-seeming equilibrium, at most one electricity generator has marginal cost always strictly less than the price in this event. (Any capacity constrained generator has infinite marginal cost.) Also, it is clear why I can not rule out the possibility that *one* bidder may withhold supply: if all but one of the generators are capacity constrained, then the remaining generator will monopsonize electricity demand.

Much academic and policy debate surrounds the question of auction design, in particular whether a uniform-price rule is superior to a discriminatory rule, a Vickrey rule, etc.. Each of the auction formats that are most commonly championed have their weaknesses and it seems unlikely that a clear, unequivocal superior will emerge.³ I do not intend to enter this debate here, except so far as my results weaken one of the arguments against the uniform-price format, that it is susceptible to collusive-seeming equilibria. My view is rather that, for a variety of reasons, uniform-price auctions are important in applications and are likely to remain so for the foreseeable future. This point of view motivates the primary concern of this paper: While maintaining the uniform-price rule, is it possible to otherwise modify the rules of the uniform-price auction so as to at least eliminate collusive-seeming outcomes?

Empirical research suggests potential exercise of market power in electricity procurement auctions in California and other regions. Furthermore, the apparent extent of market power exercise is greatest (in both absolute and percentage terms) in the periods of peak demand in which price is also the highest. For instance, Joskow and Kahn (2001) examine data from California electricity auctions in the summer of 2000 and conclude that marginal cost increases can not fully explain the high prices observed during that period. Indeed, actual price far exceeded estimated marginal costs and especially so

³Among the chief weaknesses of these other auction forms: In the discriminatory auction, to maximize their payment on inframarginal units, generators will tend to submit flat supply curves at their price “guess”, potentially leading to very inefficient generation if low cost generators overestimate the realized price. In the Vickrey auction, payments in the event of tight supply may need to be enormous (equal to the opportunity cost associated with brownouts or rolling blackouts) and bidders may collude more easily since they are nearly indifferent to raising their extramarginal bids and thereby increasing others’ payments.

in months of peak demand. The percentage increase in price that can not be attributed to marginal costs and that they ascribe to exercise of market power: .4% in May 2000, 92.0% in June, 55.5% in July, 36.8% in August, and 10.1% in September. While Joskow and Kahn (2001) only had access to monthly data, Borenstein, Bushnell, and Wolak (2000) were able to analyze market performance on an hour-by-hour basis during the period July 1998 - September 1999. They ascribe a 16% price increase⁴ over competitive levels due to market power exercise, averaged over the entire 15 month sample, and observe that the extent of market power exercise is greatest in high demand months and in high demand hours of the day. Even in the summer of 1998 when average prices were relatively low, the daily early-afternoon price spikes were remarkable. From BBW (2000)'s estimate of the time-path of true marginal cost, it appears that while true marginal cost may have spiked from about \$25/MWh up to about \$40/MWh, the price often spiked from about \$25/MWh up to \$100/MWh or more. As I explain below in Section 1.1, these are exactly the periods in which existing theory suggests that relatively severe collusive-seeming equilibria are most likely to exist, when many bidders are capacity constrained and/or the transmission grid is congested. By definition, however, in any non-collusive-seeming equilibrium, at most one bidder exercises any market power (in the sense of withholding supplies that could be produced at a marginal cost below price) when prices peak.

In this paper, I show that collusive-seeming equilibria do not exist under various modifications of the rules which encourage more aggressive bidding for marginal units. Broadly speaking, collusive-seeming equilibria exist because the uniform-price rule allows bidders to discourage others from bidding more aggressively on marginal units by submitting very inelastic demand: The auctioneer is unable to demand an additional marginal unit, so any given bidder must outbid someone else to supply that unit. Since payments are determined by a uniform-price rule, he must therefore receive a lower price on all inframarginal units. To achieve higher prices in equilibrium, each bidder submits more inelastic demand, further increasing others' incentive to submit more inelastic demand.⁵ Indeed, in a sense the uniform-price auction resem-

⁴After accounting for the cost of tradable NOx emissions permits, Joskow and Kahn (2001) revise down these estimates of BBW by less than 3%.

⁵This intuitive discussion can be formalized as follows: Consider a restricted game in which there are n bidders, each of whom has constant marginal cost c_i and submits a supply schedule from the one-dimensional parametrized family, $S_i(p; \gamma_i) = (p - c_i)\gamma_i$. Then each bidder's payoff is log-supermodular in $\vec{\gamma}$. I thank Paul Milgrom for suggesting this

bles a coordination game with multiple equilibria, in which all bidders prefer to coordinate upon an equilibrium with more inelastic announced demand and a higher price.

The way to break this logic is to *reward* more aggressive bidding on marginal units. A natural and intuitive way to do this is to set demand after receiving the bids. For example, underwriters to an initial public offering in the United States are routinely granted the Green Shoe option to increase the number of shares issued by up to 15%. More aggressive bidding is rewarded by increased supply of shares. Each bidder does not necessarily have to pay a higher price to receive an additional unit. The auctioneer will produce that unit for him, a reward for more aggressive bidding that becomes all the greater the less aggressively that others bid. Of course, this logic applies in the procurement auction setting as well. The fact that the auctioneer will increase demand in response to more aggressive bidding disconnects the price that a bidder receives on marginal units from that which he receives on inframarginal units. In the “adjustable demand auction”, the auctioneer sets demand so as to maximize (ex post) surplus given announced supply and some non-increasing marginal value function. Similarly, in the “increasable demand auction” demand is set to maximize surplus subject to the constraint of demanding at least \hat{D} . In Section 3, I prove that no collusive-seeming equilibria exist in the adjustable- or increasable demand auction in an illustrative simple model.

A more direct way to reward more aggressive bidding on marginal units is to pay them for such behavior. In a “ration-reward perturbation” of the uniform-price auction, an amount ψ of money is split evenly among all bidders who are rationed. That is to say, if $p^*(\vec{s})$ is the market-clearing price and $\vec{q}^*(\vec{s})$ is the allocation in state \vec{s} , then i qualifies to share the reward iff $q_i^*(\vec{s}) < \max S_i(p^*(\vec{s}))$. Any rationed bidder has, in a sense, bid aggressively for marginal units: he is willing to supply more quantity without demanding a higher price. Furthermore, since this reward is *shared*, each bidder has an incentive to slightly underbid others, so as to get all of it for himself. This is the key feature that drives the result: In Section 3 and the Appendix, I prove that no collusive-seeming equilibria exist in any such perturbation, no matter how small ψ may be.

line of reasoning. (But once the bid space is expanded to include shifts, i.e. $S_i(p; \gamma_i, \alpha_i) = (p - c_i)\gamma_i + \alpha_i$, then this property is lost. In fact, even single-crossing in own bid and others’ bids fails. See McAdams (2001).)

The discussion thus far has revolved about how to ensure that collusive-seeming equilibria do not exist by changing the rules of the uniform-price auction. But such a result has limited relevance unless the first three of four related concerns can be satisfied:

1. Is existence of collusive-seeming equilibria a real concern in the unmodified uniform-price auction? If not, then any non-existence result is of little beyond theoretical interest.
2. Does the fact that collusive-seeming equilibria do not exist in the ration-reward perturbations of the uniform-price auction depend on the fact that there is a continuum of prices and quantities?
3. Does any (non-collusive-seeming) equilibrium exist in these perturbed versions of the uniform-price auction? If not, the fact that collusive-seeming equilibria do not exist is of little interest.
4. If so, are the equilibria in these perturbed versions close to equilibria in the unmodified uniform-price auction? When / if this is the case, we can conclude that the unmodified uniform-price auction possesses non-collusive-seeming equilibria, a new result.

[1] Yes, in some but not all settings. Although the point of this paper is that collusive-seeming equilibria are not robust to certain small rules modifications, I do *not* conclude that collusive-seeming equilibria are unstable or less likely to occur in practice than previously believed. Others have identified structural features which tend to make collusive-seeming equilibria *less* of a concern: (i) greater uncertainty about the number of bidders, (ii) greater uncertainty about the quantity to be demanded, and/or (iii) more bidders decreases the severity of possible collusive-seeming outcomes when (iv) demand is price elastic (Klemperer and Meyer (1989)); (v) the absence of capacity constraints decreases the severity of possible collusive-seeming outcomes (Green and Newbery (1992)); and (vi) lack of cheap talk opportunities among the bidders decreases the probability and severity of collusive-seeming outcomes in an experimental setting (Goswami, Noe, and Rebello (1996)).

My reading of the literature is that (i) is typically fatal to collusive-seeming equilibria whereas (ii – vi) typically are not fatal but reduce the severity (the markup of price over true marginal cost) in any collusive-seeming equilibria that do exist. The models of these papers do not capture

the full complexity of Treasury and electricity auctions since they do not capture their repeated nature nor the fact that bidding in these auctions is only one part of a larger game. Treasury auctions of new bond issues are preceded by a pre-market and followed an after-market. In electricity procurement, supply arrangements can be made with different lead-times and there is more than one sort of product that is needed. (In addition to generation, the system operator needs to procure spinning reserves and other capacity that can be used to balance the system on a moment-by-moment basis.) Nonetheless, it is useful to see what the theory of one-shot single-product-type auctions suggests. When the transmission system is not congested, conditions (i,v) and possibly (vi) hold in electricity auctions making existence of any collusive-seeming equilibrium unlikely. (Also, demand may be slightly elastic and some bidders such as hydro power generators may have continuous marginal costs, so (iii,iv) hold to a limited extent.) Similarly, in Treasury auctions conditions (i,iv,v) and possibly (vi) hold since institutional bidders face an uncertain number of small “market bidders”. This fits with the Treasury’s own assessment of its experiment with the uniform-price format, Malvey and Archibald (1998). When the grid is congested, however, generators in a region face no uncertainty about who they are bidding against (to supply marginal demand) and so none of these conditions except possibly (vi) hold. On this basis I feel that collusive-seeming equilibria are, from a theoretical point of view, sometimes a real potential concern in electricity procurement auctions. Indeed, such equilibria are of concern precisely in those critical times when the market system is otherwise stressed by high demand and transmission constraints. This concern is bolstered by the empirical inference (though not proof) of strategic withholding of generation capacity in California in the Summer of 2000 discussed earlier.

[2] No, the non-existence result does not depend on a continuum of prices and quantities. As I prove in the Appendix, when there is a finite grid of prices with fineness Δ_p and a grid of quantities with fineness Δ_q , $\bar{c}_i(\bar{p}) \geq \bar{p} + \Delta_p(\bar{D} - \underline{D})/\beta$ for all but at most one bidder in any equilibrium of the (ψ, β) ration-reward auction whenever $\psi > 2\Delta_p(\bar{D} - \underline{D})$ and $\beta \geq 2\Delta_q$. (A bidder gets a share of the reward ψ only if he is rationed by at least β . See Section 2.7.2.) Thus, for any given (ψ, β) -perturbation of the uniform-price auction, all equilibria are “almost non-collusive-seeming” as long as the grid of prices and quantities is fine enough.

[3] Yes, equilibria do exist in these perturbed versions of the uniform-price

auction, at least when there is a finite grid of both prices and quantities.⁶ In this case, standard results such as Theorem 1 of Milgrom and Weber (1985) imply that a mixed strategy equilibrium exists in every (ψ, β) perturbation of the uniform-price auction. So, an equilibrium does exist in the same class of finite-grid models in which I prove that all equilibria must be almost non-collusive-seeming.

[4] Furthermore, given a finite set of types and a (fixed) finite grid of prices and quantities, straightforward arguments imply that any sequence of mixed strategy equilibria in (ψ, β) -perturbations has a subsequence that converges to an equilibrium in the standard uniform-price auction. I conclude that collusive-seeming equilibria in the uniform-price auction are not robust to small modifications of the rules of the auction. Rewarding bidders for aggressive bidding – whether by being able to increase demand as in the adjustable demand auction or by paying cash as in the perturbations – is the key to eliminating such equilibria. Furthermore, *the standard uniform-price auction has an equilibrium that is almost non-collusive-seeming.*

1.1 Collusive-Seeming Equilibria

In any equilibrium of the uniform-price auction, each bidder will not announce his true supply. (To avoid confusion, I will consistently use the terminology in which bidders are sellers in this paper, even when I discuss papers that study auctions in which bidders are buyers.) Since it is possible that his bid for a k th unit will set the price, he has an incentive to raise that bid since then he receives more on all $k - 1$ inframarginal units. Still, it is natural to suspect that this effect should disappear as the number of bidders goes to infinity. After all, it would seem, the probability that one of my bids will set the price goes to zero as more bidders enter. This intuition is valid enough when there are a fixed number of indivisible units at auction, but it fails when either the auctioned good is perfectly divisible or when the number of goods increases proportionately with the number of bidders.

For a simple example, suppose that there are N bidders, $50N + 1$ indivisible units demanded and each bidder has per unit cost of \$0 for up to a capacity of 100 units. (For concreteness, say also that each bidder has

⁶Given a continuum of prices and any finite grid of prices, Jackson, Simon, Swinkels, and Zame (2001)'s approach also can be applied to conclude that a mixed strategy equilibrium exists with respect to some allocation rule.

marginal cost of \$100 for more than 100 units.) Then it is a Nash equilibrium for each bidder to announce a bid of \$0 for units one through 49, bid \$25 for a 50th unit, and \$50 for units 51 through 100, and bid \$100 for all additional units. Why? The realized price is \$50, $N - 1$ bidders receive 50 units, and one randomly selected bidder supplies 51 units. Thus, each bidder views himself as winning $50 + \frac{1}{N}$ units for a total surplus of $2500 + \frac{50}{N}$ if he follows his equilibrium strategy. He can guarantee that he will win 51 units if he bids \$51 for the 51st unit, but then his surplus is only $51 * 49 < 2500$. Similarly, he can guarantee any quantity up to $51 + N$ units by bidding \$76, but then his surplus is at most $24 * 100 < 2500$.

Wilson (1979) provided the first example of a collusive-seeming equilibrium in the uniform-price auction in a simple setting with fixed demand, 2 bidders, and a common and constant marginal cost c which is common knowledge among the bidders. Wang and Zender (1998) show how similar equilibria can exist given any number of bidders and random demand with any support, as long as demand is perfectly inelastic. The equilibrium described on page 3 is an example. I illustrate that two bidder equilibrium below in Figure 1. Each bidder has a constant marginal cost c_i which is commonly known. Given that bidder 2 submits supply $S_2(p) = \gamma(p - c_2)$ and that demand D is random, bidder 1 maximizes his surplus after every realization of demand by submitting supply $S_1(p) = \gamma(p - c_1)$.

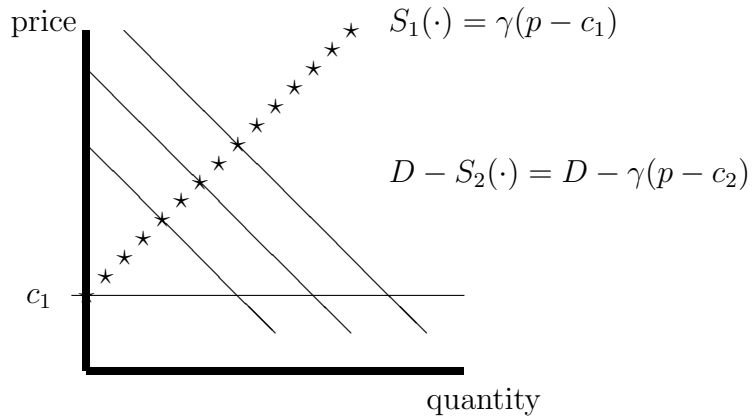


Figure 1: Illustration of a 2-bidder equilibrium with random demand

When demand is somewhat price elastic, arbitrarily low prices can not be

realized in equilibrium as in Wang and Zender (1998). Even so, Klemperer and Meyer (1989) identify a continuum of collusive-seeming equilibria in differentiable schedules when bidders have marginal cost functions $c_i(\cdot)$ which are not private information. Green and Newbery (1992) expanded upon this analysis, proving that collusive-seeming outcomes in equilibrium may be more severe when bidders face capacity constraints. Finally, in a broad class of settings in which bidders have private information and there is fixed demand, Back and Zender (1993) construct collusive-seeming equilibrium in which bidders submit kinked supply schedules.

Multi-unit demand is essential to the existence of collusive-seeming equilibria. Indeed, if bidders have unit demand and private costs, it is obvious that in the “ $k + 1$ -st price” auction of k units each bidder’s weakly dominant strategy is to bid his own valuation. (In the $k + 1$ -st price procurement auction, each bidder who supplies a unit receives the lowest losing bid.) And in much more general settings which allow for interdependency of bidders’ marginal costs on each other’s private information, Pesendorfer and Swinkels (2000) have proven that as the number of unit-supply bidders goes to infinity, the uniform-price auction is asymptotically efficient.

I have discussed how higher prices in collusive-seeming equilibria are linked to more inelastic announced demand. But what of the auctioneer? If high equilibrium prices can be realized, but only when bidders announce very inelastic demand, it would like to be able to decrease demand after receiving the bids. For example, the Mexican and Swiss Treasuries reserve the right to decrease the quantity of issued bonds (Umlauf (1993) and Heller and Lengwiler (1998)). The option to reduce demand does limit the extent to which low prices can be sustained in equilibrium. Since bidders know that the auctioneer has the option to decrease demand they will moderate the inelasticity of their announced demand. This observation applies equally in the procurement setting: In the simplest setting, I show that only prices in the range $p \in \left[c, \frac{n+1}{n}c + \frac{v(\hat{D})}{n} \right]$ can be realized in Nash equilibria of the decreaseable demand auction in which the auctioneer will reduce demand from a maximum \hat{D} if doing so increases surplus with respect to some continuous marginal value function $v(\cdot)$. (See Section 3.4. Back and Zender (1999) prove a similar result in the context in which buyers are bidders, specialized to the case in which the auctioneer decreases supply if doing so will increase revenues.)

Similarly, setting a maximum price does not eliminate the existence of

clearly collusive-seeming equilibria, unless it is low enough, $p^{max} \leq \bar{p}$. In many applications, of course, setting such a maximum price may be infeasible if the auctioneer has imperfect information about the structure of bidder payoffs. Furthermore, since my goal is to show that modifications in the rules of the uniform-price auction are in themselves sufficient to eliminate *all* collusive-seeming equilibria (not just the clearly collusive-seeming ones), it serves my purpose to consider environments in which the auctioneer sets a very high maximum price.

Adjustable demand, increasable demand, or splitting an amount ψ of cash among rationed bidders ultimately serve to break the linkage between payments for marginal and inframarginal units. This disconnect is also at play in the discriminatory auction since each bidder receives the sum of his marginal bids on the quantity that he wins, explaining why the discriminatory has no collusive-seeming equilibria (Back and Zender (1993)).

2 Models

I illustrate the intuition behind the paper’s main result in a simple model with no uncertainty and a continuum grid of prices and quantities which I call “the simplest setting”. In this model, I show why collusive-seeming equilibria do not exist given adjustable demand, increasable demand, or any cash amount $\psi > 0$ split among rationed bidders, whereas some collusive-seeming equilibria do exist in the decreasable demand auction. The main result that no collusive-seeming equilibria exist in any of my perturbations of the uniform-price auction, however, is proved in the context of what I call “the general model”.⁷ I lay out the assumptions of the general model here for the case of a continuum price and quantity grid. I treat the case of a discrete price and quantity grid in the Appendix. Maintained model assumptions are displayed as bulleted • text. Additional conditions *not required* by the paper’s results are displayed as open-bulleted text ◦ (##) and discussed on page ##. Prominent assumptions *not* made are indented without any bullet.

⁷In a previous version of this paper, McAdams (2000), I also prove that the adjustable demand auction has no collusive-seeming equilibria in an augmented version of this model in which each bidder’s cost may depend on others’ private information.

2.1 Types and Payoffs

2.1.1 “General Model”

Each bidder $i = 1, \dots, n$ has a *type* $t_i \in T_i \subset [0, 1]^{k_i}$ which is private information.

- (9) T_i is finite.

If $|T_i| = 1$, then bidder i has no private information.

- (22) Bidder types t_i are independent.

Bidder i 's *cost* to supply q_i shares is

$$C_i(q_i, q_{-i}; \vec{t}) \equiv \int_0^{q_i} c_i(x; q_{-i}, \vec{t}) dx,$$

where $c_i(q_i; q_{-i}, \vec{t})$ is his marginal cost of q_i shares when others receive quantities q_{-i} and the vector of types is \vec{t} .

- c_i is non-decreasing in \vec{q} and non-increasing in t_i .
- (22) c_i has non-decreasing differences in (q_{-i}, t_i) .

I make no other assumptions about bidder costs, notably:

Continuity of c_i in \vec{q} is not assumed.

For example, mine would include standard electricity procurement models in which each generator has marginal costs that are (i) non-decreasing in own quantity, (ii) non-increasing in own type, and (iii) do not depend on others' quantities, (iv) with discontinuities in own quantity corresponding to generation capacity constraints. (Even when each generators' operating costs are commonly known, as is usually assumed, a generator has cost-relevant private information such as that regarding the opportunity cost of not shutting down to perform maintenance and repair.)

Marginal costs c_i may depend on the allocation q_{-i} to others, as long as c_i is non-decreasing in q_{-i} and, possibly additionally, has non-decreasing differences in (q_{-i}, t_i) .

What do these conditions mean? One way in which marginal costs may vary with others' output is if others' output increases the cost of a scarce common input. (NOx emissions permits provide an example in California electricity production.) In this case, if a producer's type captures the extent to which it uses the scarce input, then higher total production will lead to a higher input cost and therefore *increasing* differences in type and others' quantities. If own costs do not depend on others' output, of course, then these requirements are automatically satisfied. Similarly, the non-decreasing differences requirement will automatically be satisfied if a given generator's type does not interact with the effect of others' quantities on cost. (Then each generator's cost would exhibit constant differences in (q_{-i}, t_i) .) This would be the case, for example, if a generator's private type captured the extent of its need for maintenance but if the generator's usage rate of the scarce input – when operating – was commonly known.

As an example in which bidders are buyers, consider a stock IPO in which the value of each share is decreasing in the total number of shares: $v_i(\vec{q}, \vec{t}) = \frac{Z(\vec{t})}{\sum_j q_j}$, where $Z(\vec{t})$ is the total equity value of the offering. In this setting, my requirements translate as that marginal value must be non-increasing in \vec{q} and non-decreasing in t_i with non-increasing differences in (q_{-i}, t_i) . These conditions are satisfied since marginal values are decreasing in q_{-i} and the *change* in marginal values due to more positive information decreases as there are more shares. ($(Z(t'_i, t_{-i}) - Z(t_i, t_{-i})) / \sum_j q_j$ is decreasing in q_{-i} for any $t'_i > t_i$.)

The *state* $\vec{s} \equiv (\vec{t}, \tau, \rho, \vec{\gamma})$, , $\gamma_i \in [0, 1]$ is bidder i 's “randomization variable” to be discussed in Section 2.3, $\tau \in [0, 1]$ is a variable relevant to the auctioneer's demand to be discussed in Sections 2.4.1 and 2.4.2, and $\rho \in [0, 1]$ is a “tie-breaking (or rationing) variable” to be discussed in Section 2.5. Each bidder seeks to maximize his *expected utility*.

- Each bidder's utility function $u_i(\cdot)$ is a strictly increasing function of his surplus, the difference between the amount he is paid and his cost of production.
- (22) Bidders are risk neutral.

2.1.2 “Simplest Setting”

Same as general model except that (\vec{t}, τ) is common knowledge among the bidders, there is perfectly inelastic demand, $D(p; \tau) = D$ for all p , and there

is a common and constant marginal cost $c_i(q_i; q_{-i}, \vec{t}) = c$ for all i , \vec{q} , and \vec{t} . In this setting, I will simplify notation by dropping reference to the types \vec{t} and to τ . The tie-breaking variable ρ and the randomizations $\vec{\gamma}$, however, are not known to the bidders (except that γ_i is known to i).

2.2 Bids and Grids

There is a *grid* of permissible prices \mathbf{p} and permissible quantities \mathbf{q} . In most of this paper, each of these grids consists of a continuum (along with “low and high null prices”). There is also some analysis of models with a discrete price and quantity grid in the Appendix, where those sorts of grids will be defined. All other definitions and assumptions from the continuum case modelled here carry over to that case as well and apply to both the general model and the simplest setting.

- $\mathbf{p} = \{\emptyset\} \cup [p^{min}, p^{max}] \cup \infty$
- $\mathbf{q} = [0, q^{max}] \cup \infty$.

To avoid confusion, please note that $\{\emptyset\} \neq \emptyset$. The latter is the emptyset. The former is “the low null price”. Similarly, ∞ is “the high null price”. (See below for a discussion of the null prices and the null quantity ∞ .)

A *bid* $S_i(\cdot) : \mathbf{p} \rightarrow \mathcal{P}(\mathbf{q})$ is a supply correspondence that satisfies the following requirements:

- (I) : $\forall p \in \mathbf{p}, S_i(p) \neq \emptyset$
- (II) : $\forall q \in \mathbf{q} \exists p \in \mathbf{p} : q \in S_i(p)$
- (III) : $\forall p \in \mathbf{p}$, if $\{q', q''\} \subset S_i(p)$ and $q \in \mathbf{q} \cap (q', q'')$, then $q \in S_i(p)$
- (IV) : $\forall k = 1, \dots, |\mathbf{p}| - 1, \min S_i(p^{k+1}) = \max S_i(p^k)$

Roughly speaking, these requirements ensure that $S_i(\cdot)$ is non-decreasing with a “connected graph”. The null prices are meant to allow bidders (and the auctioneer) to express “requirements”. For instance, if a bidder is not willing to receive even the maximum permissible price p^{max} for more than q units, then it can bid $S_i(\infty) = [q, \max \mathbf{q}]$ to require that it never wins more than q units in the auction. Similarly, a bidder may require that it supplies no less than q by bidding $S_i(\{\emptyset\}) = [0, q]$. Finally, a bidder may require that he be rationed when the price p is realized by including $\infty \in S(p)$. If there is no way to satisfy all of these requirements, i.e. if the only market-clearing

price is $\{\emptyset\}$ or ∞ , then the auction will be void. (See below. If there is a non-void m-c price, then there is always a non-void m-c allocation under my requirements on supply and demand schedules.)

Any supply correspondence $S_i(\cdot)$ is equivalent to an inverse supply correspondence $P_i(\cdot)$: $p \in P_i(q)$ iff $q \in S_i(p)$. I will use both supply and inverse supply correspondence notation for bids, as convenient. For each subset of bidders $I \in \mathcal{P}(\{1, \dots, n\})$, define the announced aggregate supply of I as $S_I(\cdot) = \sum_{j \in I} S_j(\cdot)$. That is to say, $Q \in S_I(p)$ iff there exists $\{q_j\}_{j \in I}$ such that $\sum_{j \in I} q_j = Q$ and each $q_j \in \mathfrak{q}$. Let $P_I(\cdot)$ be the corresponding inverse supply correspondence.

2.3 Strategies and Equilibrium

A *pure strategy* $S_i(\cdot; \cdot)$ maps types into permissible bids, with $S_i(\cdot; t_i)$ being the announced supply curve of type t_i . A *strategy* $S_i(\cdot; \cdot, \cdot)$ maps types into probability distributions over permissible bids. $S_i(\cdot; t_i, \gamma_i)$ is bidder i 's announced supply curve when his type is t_i and “randomization variable” equals $\gamma_i \in [0, 1]$. Unconditional on γ_{-i} , $\gamma_i \sim U[0, 1]$. Sometimes I will use the shorthand S_i for i 's bid $S_i(\cdot)$. Whenever I refer to a strategy, however, I will use the longer notation $S_i(\cdot; \cdot, \cdot)$.

A strategy profile $\vec{S}^*(\cdot; \cdot, \cdot)$ is an *equilibrium* iff for every $t_i \in T_i$ and $\gamma_i \in [0, 1]$, $S(\cdot; t_i, \gamma_i)$ maximizes i 's expected utility over the space of all possible permissible bids given t_i, γ_i and the profile of others' strategies $S_{-i}^*(\cdot; \cdot, \cdot)$. In a “mixed strategy equilibrium”, γ_i are independent. When I refer to an “equilibrium” in general, I mean to allow also for all possible correlated equilibria.

2.4 Supply and Market-Clearing Price

2.4.1 Demand Schedule Regime

Define $\mathcal{Q} \equiv \sum_{j=1}^n \mathfrak{q}$ to be the grid of all permissible aggregate quantities. The auctioneer commits to a *demand correspondence* $D(\cdot; \tau) : \mathfrak{p} \rightarrow \mathcal{P}(\mathcal{Q})$ that may depend on the variable τ . (If τ is unknown to the bidders, then so is the demand correspondence.)

- $D(\cdot; \cdot)$ is continuous and non-decreasing in τ . That is to say, $\min D(p; \tau)$ and $\max D(p; \tau)$ are continuous and non-decreasing in τ for all $p \in \mathfrak{p}$.

I also require that each $D(\cdot; \tau)$ meet requirements that are analogous to those placed on bids:

- (I) : $\forall p \in \mathbf{p}, D(p; \tau) \neq \emptyset$
- (II) : $\forall Q \in \mathcal{Q} \exists p \in \mathbf{p} : Q \in D(p; \tau)$
- (III) : $\forall p \in \mathbf{p}$, if $\{Q', Q''\} \subset D(p; \tau)$ and $Q \in \mathcal{Q} \cap (Q', Q'')$, then $Q \in D(p; \tau)$
- (IV) : $\forall k = 1, \dots, |\mathbf{p}| - 1, \min D(p^k; \tau) = \max D(p^{k+1}; \tau)$

Interpret $\min D(\infty; \tau)$ as the maximal quantity and $\max D(\{\emptyset\}; \tau)$ as the minimal quantity that the auctioneer is willing to purchase at any permissible price.

A *market-clearing price* $p^{mc}(\vec{S}; \tau)$ (p^{mc} for short) is a price such that

$$p^{mc} \in \{p \in \mathbf{p} : D(p; \tau) \cap S_{1, \dots, n}(p) \neq \emptyset\}$$

(Loosely speaking, the market-clearing prices are those that are no more than the lowest losing bid and no less than the highest winning bid.) A *market-clearing demand* $D^{mc}(\vec{S}; \tau)$ (D^{mc} for short), similarly, is a level of total demand that allows the market to clear at that price:

$$D^{mc} \in D(p, \tau) \cap S_{1, \dots, n}(p)$$

Given my restrictions on $\{S_i(\cdot)\}$ and on $D(\cdot; \tau)$, such a price always exists. Furthermore, it is easy to verify that (i) if there is more than one possible market-clearing price, then there is a unique market-clearing demand and that (ii) if there is more than one possible m-c demand, then there is a unique m-c price. In the event that there is more than one m-c price, the realized price $p^* = p^*(\vec{S}; \tau, \rho)$ depends on the “tie-breaking” variable ρ . Similarly, if there is more than one m-c demand, which one is realized depends on ρ : $D^* = D^*(\vec{S}; \tau, \rho)$.

2.4.2 Demand Rule Regime (or “Endogenous Demand”)

The auctioneer commits to a *demand rule* $D(\vec{S}(\cdot); \tau)$ and sets price so as to maximize consumer surplus given the announced aggregate supply with respect to some consumer surplus (or value) function. For each τ , I will represent total consumer value for quantity q as $V(q; \tau)$ and the marginal value as $v(q; \tau)$.

- $v(\cdot; \cdot)$ is continuous in q and τ and non-decreasing in τ .

The auctioneer sets quantity to maximize consumer surplus, i.e.

$$D^* \equiv D \left(\vec{S}(\cdot; \tau) \right) \equiv \arg \max_D (v(D; \tau) - D \max P_{1, \dots, n}(D))$$

$$p^* \equiv p \left(\vec{S}(\cdot) \right) \equiv \min P_{1, \dots, n}(D^*)$$

(For any given total demand, the auctioneer sets price equal to the lowest price that is consistent with supply for that quantity in order to maximize consumer surplus.)

Discussion. In the electricity procurement application, the assumption that marginal consumer value is continuous in quantity will typically not be satisfied. The system operator may have arrangements with some electricity users to reduce (or increase) their demand instantaneously given higher (or lower) spot prices and may also potentially store some amount of excess electricity generation for future dispatch, whether by pumping up water or by other means. But typically users will have some minimal power requirements and the amount of (or rate at which) power can be stored is limited, leading to a discontinuity in the marginal value of having more electricity supplied at some quantity level.

Before I discuss this issue further, however, please note that this continuity assumption is important for my analysis of the adjustable- and increasable demand auctions but plays no role whatsoever in my analysis of the ration-reward perturbations of the uniform-price auction. The main result of the paper, that no collusive-seeming equilibria exist in these auctions in the general model, holds no matter what the structure of demand, etc..

Even when the true marginal value for electricity is discontinuous, of course, the system operator may choose to act *as if* it is maximizing consumer surplus with respect to some other, continuous marginal values. Here is one way to do this: for each τ define $\hat{v}(\cdot; \tau)$ as follows:

$$\hat{v}(D; \tau) \equiv \frac{V(D) - V(D - \Delta)}{\Delta} \text{ for all } D > \Delta$$

$$\equiv \frac{V(D) - V(0)}{D} \text{ for all } D \leq \Delta$$

where $\Delta > 0$. In this case, $\hat{v}(\cdot; \tau)$ is continuous and non-increasing in quantity (and everywhere weakly greater than $v(\cdot; \tau)$) since $v(\cdot; \tau)$ is assumed to be non-increasing.

If the auctioneer is able to commit to some such value function, then all of my results regarding the adjustable demand auction carry through. One potential disadvantage is that the auctioneer may have difficulty committing to a demand rule based on \hat{v} , if its goal is to maximize consumer surplus. Ex post consumer surplus will be less under this demand rule than under one based on $v(\cdot; \tau)$. In the electricity application, for instance, the system operator would act as if the marginal value for electricity is everywhere higher than it really is. This could lead to somewhat lower consumer surplus and higher prices. On the other hand, for small Δ , this modification amounts to a reduction in the system operator’s exercise of its monopsony power, resulting in less demand reduction given the generators’ supply bids. Assuming no problems with commitment, then, consumers lose little when the system operator acts as if it is maximizing profits with respect to continuous marginal costs (and it is possible⁸ that the overall system will be more efficient).

2.5 Payment and Quantity Rationing

If no market-clearing price exists in the permissible range $[p^{min}, p^{max}]$ given the bids \vec{S} , then the auction is cancelled. Else each bidder supplies a *market-clearing quantity*, $q_i^{mc}(\vec{S}; \tau) \in S_i(p^*)$. If there is more than one possible m-c allocation, then each bidder’s realized quantity depends on ρ : $q_i^* = q_i^*(\vec{S}; \tau, \rho)$.

Discussion. It is again easy to verify that (i) if there is more than one possible market-clearing price, then there is a unique market-clearing allocation \vec{q}^* and that (ii) if there is more than one possible m-c allocation, then there is a unique m-c price p^* . There are several reasonable ways to specify which m-c price (p^*), allocation (\vec{q}^*), and demand ($D^* = \sum_i q_i^*$) will prevail when there is more than one m-c outcome of the auction. My analysis in this paper does not require a specific rule or method for determining exactly how prices and quantities are determined in these indeterminate cases, but I will make certain requirements on the “rationing rule”. (In my terminology, the rationing or tie-breaking rule determines not only the quantities but also the price.)

1. The allocation \vec{q}^* and price p^* must be market-clearing.

⁸If the generators bid their true supply curves, then the system would be more efficient the less that the auctioneer exercised its monopsony power on behalf of consumers. But since generators will shade their bids up, the effect on efficiency is ambiguous (although for small Δ the effect is certainly small).

2. The allocation \vec{q}^* depends only on expressed demand and demand at the realized price, i.e. only on $\vec{S}(p^*)$ and on $D(p^*)$
3. Bidder i 's quantity q_i^* is non-increasing and all others' quantities q_{-i}^* are non-decreasing in i 's bid. That is to say, $q_i^*(S'_i, S_{-i}) \leq q_i^*(S_i, S_{-i})$ and $q_j^*(S'_i, S_{-i}) \geq q_j^*(S_i, S_{-i})$ for all $j \neq i$ whenever $S'_i > S_i$. (Higher bids lead one to win less. It is easy to verify that this property always holds in cases in which the m-c allocation after both (S'_i, S_{-i}) and (S_i, S_{-i}) are unique.)

These requirements are satisfied in all rationing rules that I have seen which are based on a market-clearing condition. Kremer (1998) studies a variation of the uniform-price auction with an allocation rule in which bidders may not receive a market-clearing quantity, and shows that no collusive-seeming equilibria exist in the simplest setting. Kremer's allocation rule works as follows with fixed demand D : when the market-clearing price is p^* , each bidder $i = 1, \dots, n$ supplies $D / \sum_{i=1}^n \max S_i(p^*)$. Consider an example with $n = 2, D = 1, S_i(p) = p$, and $S_2(1/2) = [0, 1]$. $p^* = 1/2$ and bidder 2 receives $2/3$ of the quantity. Bidder 1 receives only $1/3$, although he is willing to supply up to $1/2$ at the price $1/2$. Adjustable-demand provides another example of an allocation rule that is not based on market-clearing: quantities are not determined by where announced supply crosses demand but rather what maximizes consumer surplus. Of course, these restrictions also rule out some possible (though strange) market-clearing allocation rules.

2.6 Definition: Collusive-Seeming Equilibrium

Bidder i 's "realized marginal cost" in state $\vec{s} = (\vec{t}, \tau, \rho, \vec{\gamma})$ given the allocation \vec{q} is the *right-limit* of his marginal cost,

$$c_i^+(q_i; q_{-i}, \vec{t}) \equiv \lim_{\varepsilon \rightarrow 0} c_i(q_i + \varepsilon; q_{-i}, \vec{t}),$$

his marginal cost of producing *more*. $(\tau, \rho, \vec{\gamma})$ affect bidder i 's costs only indirectly as they affect his quantity.)

In a given equilibrium $(S_1(\cdot; \cdot, \cdot), \dots, S_n(\cdot; \cdot, \cdot)) = \vec{S}(\cdot; \cdot, \cdot)$, define

$$\begin{aligned} \bar{c}_i([p_1, p_2]) &\equiv \min \left\{ c : \Pr [c \leq c_i^+(q_i^*(\vec{s}), q_{-i}^*(\vec{s}), \vec{s}) \mid p^*(\vec{s}) \in [p_1, p_2]] = 1 \right\} \\ \bar{p} &\equiv \min \{ p < \infty : \Pr [p^*(\vec{s}) \leq p] = 1 \} \\ \bar{c}_i(\bar{p}) &\equiv \lim_{\varepsilon \searrow 0} \bar{c}_i([\bar{p} - \varepsilon, \bar{p} + \varepsilon]) \end{aligned}$$

where $p^*(\vec{s})$ is the price realized in state \vec{s} . (See the Appendix for more complete definitions.) \bar{p} and $\bar{c}_i(\bar{p})$ are always well-defined; $\bar{c}_i([p_1, p_2])$ when $\Pr[p_1 \leq p^*(\vec{s}) \leq p_2] > 0$,

Loosely speaking, \bar{p} is the highest non-null price that is realized in the equilibrium. $\bar{c}_i(\bar{p})$, similarly, is the highest that bidder i 's marginal cost is in the equilibrium when the realized price is \bar{p} . So, $\bar{c}_i(\bar{p}) < \bar{p}$ means that bidder i always has marginal cost strictly less than \bar{p} when this price is realized. (This is loosely speaking since the price \bar{p} may not be realized but only almost realized.)

$\vec{S}(\cdot; \cdot, \cdot)$ is *collusive-seeming* iff

$$|\{i : \bar{p} > \bar{c}_i(\bar{p})\}| > 1$$

Discussion. To fix ideas, consider the example in the introduction of electricity procurement with known marginal costs and uncertain demand. A pure strategy equilibrium is collusive-seeming iff the realized price of electricity in the event of peak demand, $\max D$, is greater than the marginal cost of two or more generators in that state. (A generator which is capacity constrained has infinite marginal cost.) Similarly, in a mixed strategy equilibrium, suppose that there is a unique state (i.e. demand realization and profile of bidder randomizations) in which the highest equilibrium price \bar{p} is realized. Then in this state at most one bidder has marginal cost less than this price.

Why is it significant whether $\bar{c}_i(\bar{p}) \geq \bar{p}$? If $\bar{c}_i(\bar{p}) < \bar{p}$, then this means that bidder i always strategically withholds some supply that could be produced at marginal cost below the price, when prices peak. Given Ausubel and Cramton (1998)'s general proposition that bidders will shade their bids in the uniform-price auction, this may not necessarily seem collusive. The point of this analysis, however, is that in fact there are equilibria in which bidders do not shade down their bids "at the top of one's supply schedule". And that by slightly perturbing the rules of the uniform-price auction, we can

guarantee that these are the only equilibria that exist. In such a “non-collusive-seeming equilibrium”, peak prices can be attributed to high true marginal costs (modulo the possible strategic withholding of one supplier).

The additional assumptions that (i) bidders are risk-neutral with (ii) independent types and (iii) marginal costs having non-decreasing differences in (q_{-i}, t_i) are not required by any results in this paper. Nonetheless, it is worth noting that when they are also satisfied McAdams (2001) implies that the uniform-price auction possesses an isotone mixed strategy equilibrium given any finite grid of prices and quantities. Loosely speaking, this means that the “lowest” bid ever made by a higher type must be greater than or equal to the “highest” bid ever made by a lower type. Or, formally, for all $p \in \mathbf{p}$ and all $t' > t$,

$$\begin{aligned} \inf_{\gamma_i} \min S_i(p; t', \gamma_i) &\geq \sup_{\gamma_i} \min S_i(p; t; \gamma_i) \\ \inf_{\gamma_i} \max S_i(p; t', \gamma_i) &\geq \sup_{\gamma_i} \max S_i(p; t; \gamma_i) \end{aligned}$$

Furthermore, if each bidder’s marginal costs have strictly increasing differences in (q_{-i}, t_i) , then every equilibrium is in such isotone strategies. Having such a (natural) monotone structure to equilibria is useful for interpreting the event in which peak prices are realized: The maximal price will be realized when (i) each bidder receives his lowest type, \underline{t}_i , and (ii) demand is at its highest, $\bar{\tau}$.⁹

2.7 Auction Rules

2.7.1 Standard Uniform-Price Auction

See Sections 2.2, 2.4.1, and 2.5.

2.7.2 (ψ, β) Ration-Reward Auctions

Same as the standard uniform-price auction, except that the payment rule differs in that the ration reward ψ is split evenly among each bidder i for

⁹More exactly, the set of (type vector, demand) pairs for which the price \bar{p} is realized for some randomization profile is a “box” that contains $(\underline{\vec{t}}, \bar{\tau})$ of the basic form $\prod_{i=1}^n \prod_{j=1}^{k_i} [\underline{t}_i^j, \tilde{t}_i^j] \times [\bar{\tau}, \bar{\tau}]$. (The actual set may differ slightly in that the lower boundary or the box may or may not be included). Furthermore, except on the lower boundary of this box, \bar{p} is realized after every randomization profile.

whom

$$q_i^*(\vec{s}) + \beta < \max S_i(p^*(\vec{s})) \text{ or } \infty \in S_i(p^*(\vec{s}))$$

In other words, all bidders who are rationed by more than β share an amount ψ of cash. Most of my analysis will focus on the case in which $\beta = 0$, but in the Appendix I also examine versions in which bidders must be rationed by some minimal amount in order to receive the reward.

2.7.3 Adjustable-Demand Auction

Sections 2.2, 2.4.2, and 2.5.

2.7.4 Infeasible-Demand Auction

Same as the adjustable demand auction, except that demand is constrained to be at least \hat{D} .

2.7.5 Decreaseable-Demand Auction

Same as the adjustable demand auction, except that demand is constrained to be at most \hat{D} .

3 Collusive-Seeming Equilibria

In this section I show that, in the simplest setting, no *pure strategy* collusive-seeming equilibria exist in the adjustable demand, increaseable demand, and in the $(\psi, 0)$ ration-reward perturbations of the uniform-price auction but they do exist in the decreaseable demand auction. I restrict attention to pure strategies purely to simplify the exposition. In the Appendix, the formal proofs allow for any equilibria with randomized strategies. My goal is to illustrate how *rewarding* bidders for submitting more elastic demand is the key to eliminating collusive-seeming equilibria. In the adjustable demand and increaseable demand auctions, bidders are rewarded by the fact that the auctioneer chooses to demand more; in the $(\psi, 0)$ perturbations, they receive a portion (or greater discount) of the reward ψ . In the decreaseable demand auction, however, bidders are not rewarded for bidding more aggressively but rather punished for bidding less aggressively; and some collusive-seeming equilibria remain.

3.1 Ration-reward Perturbations

Theorem 1. *In the simplest setting, $p^* \leq c$ in any pure strategy equilibrium of the $(\psi, 0)$ ration-reward perturbation of the uniform-price auction in which $\psi > 0$.*

Theorem 2. *In the general model, $\#\{i : \bar{c}_i(\bar{p}) < \bar{p}\} \leq 1$ in any equilibrium in the $(\psi, 0)$ ration-reward perturbation of the uniform-price auction in which $\psi > 0$.*

Theorem 3. *Suppose that $\psi > 2\Delta_p(\bar{D} - \underline{D})$, $\beta \geq 2\Delta_q$ and that $\Delta_p, \Delta_q > 0$. Then*

$$\bar{p} - \bar{c}_i(\bar{p})(\beta) \leq \Delta_p + \frac{\Delta_p(\bar{D} - \underline{D})}{\beta}$$


in any equilibrium of the (ψ, β) -perturbation, for all but at most one bidder.

The notation used in the statement of Theorem 3 is defined in the Appendix, where Theorems 2 and 3 are proven. In words, Theorem 3 implies that, for any given $\psi, \beta > 0$ and any very fine grid of prices and quantities, all equilibria in the (ψ, β) -perturbation of the uniform-price auction are “almost non-collusive-seeming”. ($\bar{c}_i(\bar{p})(\beta)$ is bidder i ’s maximal marginal cost for β more than he ever provides in the equilibrium in the event that the price is \bar{p} . Thus, the bound on marginal costs is less tight for larger β given that marginal costs are non-decreasing in quantity.)

PROOF OUTLINE OF THEOREM 1:

$$[\text{RD1}] : p^* > c \Rightarrow \max S_i^*(p^*) > q_i^* \text{ for all } i$$

$$[\text{RD2}] : \max S_i^*(p^*) > q_i^* \text{ for 2 or more bidders} \Rightarrow p^* \leq c$$

In words: [RD1] All bidders must be rationed (and hence evenly split the reward) in any collusive-seeming equilibrium. [RD2] If any bidders split the reward, then the realized price must be no more than c . Obviously, this is a contradiction. 

STEP [RD1]: Suppose that $q_i^* = \max S_i^*(p^*)$. In this case, i does not win any of the reward. Consider the deviation

$$\begin{aligned} \tilde{S}_i(p) &= S_i^*(p) \text{ for all } p < p^* \\ \tilde{S}_i(p^*) &= [\min S_i^*(p^*), D] \\ \tilde{S}_i(p) &= D \text{ for all } p > p^* \end{aligned}$$

By submitting $\tilde{S}_i(\cdot) < S_i^*(\cdot)$, the price stays the same, bidder i supplies no less quantity than before and guarantees that he receives a portion of the reward. (Here I use the assumption about the rationing rule that each bidder's quantity is non-increasing in his own bid.) Since $p^* > c$, he does better if he does win more quantity.

STEP [RD2]: So suppose that i and at least one other bidder shares the reward. Now consider the deviation

$$\begin{aligned}\hat{S}_i^\varepsilon(p) &= S_i^*(p) \text{ for all } p < p^* - \varepsilon \\ \hat{S}_i^\varepsilon(p^* - \varepsilon) &= [\min S_i^*(p^* - \varepsilon), D] \\ \hat{S}_i^\varepsilon(p) &= D \text{ for all } p > p^* - \varepsilon\end{aligned}$$

where ε is chosen so that $S_j^*(p^* - \varepsilon)$ is a singleton for all $j \neq i$. (An arbitrarily small ε satisfying this property exists since any non-decreasing supply correspondence must be single-valued at all but countably many prices.) Thus, bidder i wins the entire reward after this deviation. Overall, he receives $q_i^* \varepsilon$ less in payment on his original quantity q_i^* , wins no less quantity than before at price $p^* - \varepsilon$, and receives all of the ψ reward instead of only at most $\frac{\psi}{2}$. When ε is small enough so that $q_i^* \varepsilon < \frac{\psi}{2}$, $\hat{S}_i^\varepsilon(\cdot)$ is a profitable deviation. ♣

3.2 Adjustable-Demand Auction

Theorem 4. *In the simplest setting, $p^* \leq c$ in any pure strategy equilibrium of the adjustable demand auction.*

PROOF OUTLINE: I break the proof down into four steps:

$$\begin{aligned}\text{[A1]} & : P_{2,\dots,n}^{-'}(D^* - q_1^*) = 0 \implies p^* \leq c \\ \text{[A2]} & : p^* > c \implies p^* < v(D^*) \\ \text{[A3]} & : v(D^*) > p^* > c \implies \\ & \implies [P_{2,\dots,n}^-(D^* - q_1^*) = P_{2,\dots,n}^+(D^* - q_1^*) \text{ and } P_{2,\dots,n}^{+'}(D^* - q_1^*) = 0] \\ \text{[A4]} & : P_{2,\dots,n}^{-'}(D^* - q_1^*) \leq P_{2,\dots,n}^{+'}(D^* - q_1^*)\end{aligned}$$

In words: [A1] No collusive-seeming equilibria exist in which $P_{2,\dots,n}$ is locally flat to the left at $D^* - q_1^*$; [A2] $p^* < v(D^*)$ in any collusive-seeming equilibrium; [A3] $P_{2,\dots,n}$ must be both continuous and locally flat to the right

at $D^* - q_1^*$ in any collusive-seeming equilibrium; [A4] $P_{2,\dots,n}$ can not have a downward kink at $D^* - q_1^*$. These steps are sufficient to complete the proof. [A2], [A3], [A4], and the fact that each bidder's schedule must be non-increasing together imply that $P_{2,\dots,n}'(D^* - q_1^*) = 0$ whenever $p^* > c$. But then $p^* > c$ leads to a contradiction by [A1]. ♣

3.2.1 Proof intuition

[A1] intuition: If $P_{2,\dots,n}$ is flat to the left, then bidder 1's residual supply curve must be flat to the right. That is to say, bidder 1 is a local price taker: he can supply more quantity without lowering the price. Thus, he will change his bid so as to receive more quantity if price exceeded his true marginal cost. The proof of [A1] applies to Nash equilibria of the fixed, adjustable, increasable, and decreasable demand uniform-price auctions. Thus, this step captures a basic property of Nash equilibria in any uniform-price auction.

[A2] intuition: By [A1] (and a few simple steps of logic), $p^* > c$ in equilibrium implies that the announced aggregate supply is not flat to the left at D^* . But then the auctioneer can certainly increase consumer surplus by lowering the total quantity if $p^* \geq v(D^*)$.

[A3] intuition: Otherwise each bidder can gain submitting supply that is flat to the right at his realized quantity at price p^* . ($v(D^*) > p^*$ implies that the auctioneer will demand more from this bidder when he announces willingness to supply more at p^* .)

[A4] intuition: As a monopsonist, the auctioneer will never set demand at a downward kink in aggregate supply.

3.2.2 Proof Steps

STEP [A1]: (*Relies on neither increasability nor decreasability of demand.*) First, suppose that $P_{2,\dots,n}$ is locally flat to the left at D^* in a Nash equilibrium of the fixed demand auction. In this case, bidder 1 faces residual supply $D^* - P_{2,\dots,n}$ that is locally flat to the right at quantity q_1^* . Thus, bidder 1 is a local price taker and $p^* \leq c$ for otherwise he would have acquired more

quantity. In particular, if $p^* > c$ then a profitable deviation for bidder 1 is to submit the flat schedule

$$\tilde{P}_1^\varepsilon(q) = p^* - \varepsilon^2 \text{ for all } q \in (0, \max \mathbf{q})$$

for small enough $\varepsilon > 0$. Since $D^* - P_{2,\dots,n}$ is locally flat to the right, bidder 1 will receive $\delta(\varepsilon)$ more quantity by deviating in this way, where

$$\lim_{\varepsilon \rightarrow 0} \frac{\delta(\varepsilon)}{\varepsilon^2} = \infty.$$

That is to say, bidder 1 can increase his quantity on the first-order while only decreasing the uniform price he receives on the second-order.

What if the auctioneer is able to increase and/or decrease demand from D^* ? Bidder 1 will never be worse off. Submitting \tilde{P}_1^ε instead of P_1 at least weakly decreases the auctioneer's marginal cost at all quantities. Thus, the auctioneer will never choose to demand less than before and bidder 1 can only receive more quantity at the same price as in the fixed demand auction.

What if there is a reward split among each bidder i who submits inverse demand such that $P_i^+(q_i^*) = 0$? (This is an equivalent way to saying that $\max D_i(p^*) > q_i^*$.) There must exist $\varepsilon \approx 0$ such that bidder 1 also wins (all of) the reward when he bids $\tilde{P}_1(\cdot)$. Thus, he can do no worse given the reward with the deviation.

STEP [A2]: *\langle Uses both increasability and decreasability of demand. \rangle* The auctioneer must not have an incentive to reduce quantity from the equilibrium level, i.e.

$$D^* \cdot P_{1,\dots,n}^{-'}(D^*) + (v(D^*) - p^*) \geq 0.$$

Since each bidder's schedule P_i must be non-decreasing, $P_{1,\dots,n}^{-'}(D^*) \geq 0$. Thus, this inequality can not be satisfied when $p^* > v(D^*)$. Furthermore, $p^* = v(D^*)$ is only possible when $P_{1,\dots,n}^{-'}(D^*) = 0$. Otherwise the auctioneer would certainly have demanded less than D^* given the announced aggregate supply $P_{1,\dots,n}$. Now, since all bidders' schedules are non-decreasing, $P_{1,\dots,n}^{-'}(D^*) = 0$ implies that some $P_i^{-'}(q_i^*) = 0$ for some bidder i . Without loss, I may suppose that $P_2^{-'}(q_2^*) = 0$, so that $P_{2,\dots,n}^{-'}(D^* - q_1^*) = 0$. By step [A1], then, $p^* \leq c$ and I have a contradiction.

STEP [A3]: *\langle Relies on increasability of demand only. \rangle* Suppose instead that $P_{2,\dots,n}$ is upward sloping or discontinuous up to the right at $D^* - q_1^*$ in a Nash equilibrium of the adjustable demand or the \hat{D} -increasable demand

auction. Then P_1 must be locally flat to the right at q_1^* , for otherwise bidder 1 could again get more quantity at price p^* by submitting the flat schedule $\widehat{P}_1(q) = p^*$. ($p^* > v(D^*)$ by assumption.) But then the realized demand would exceed D^* , a contradiction.

NOTE: If consumers' marginal value $v(\cdot)$ is not continuous, then this step fails. [A2] implies that $p^* > v^-(D^*)$ but if $p^* < v^+(D^*)$ then we can not conclude that some bidder has a profitable deviation as described in [A3], since the auctioneer will not, in fact, increase demand in response to the specified deviation.

STEP [A4]: *(Uses both increasability and decreasability of demand.)* In any Nash equilibrium of the adjustable demand auction, there can not be an downward kink in $P_{1,\dots,n}$ at D^* for the simple reason that the auctioneer, like any monopsonist, never maximizes profits by setting price at a downward kink of aggregate demand. Thus, there can not be an downward kink in $P_{2,\dots,n}$ at $D^* - q_1^*$.

3.3 Increasable-Demand Auction

In this section I show that no collusive-seeming equilibria exist in the \widehat{D} -increasable auction *but only if* the minimum price $p^{max} < v(\widehat{D})$.¹⁰

Theorem 5. $p^* \leq c$ in any pure strategy equilibrium of the \widehat{D} -increasable demand auction in which $v(\widehat{D}) > p^{min}$.

PROOF OUTLINE: I break the proof down into four steps:

$$\begin{aligned}
\text{[I0]} & : p^* > c \implies D^* = \widehat{D} \\
\text{[I1]} & : P_{2,\dots,n}^{-'}(\widehat{D} - q_1^*) = 0 \implies p^* \leq c \\
\text{[I3]} & : v(\widehat{D}) > p^* > c \implies \\
& \implies [P_{2,\dots,n}^-(\widehat{D} - q_1^*) = P_{2,\dots,n}^+(\widehat{D} - q_1^*) \text{ and } P_{2,\dots,n}^{+'}(\widehat{D} - q_1^*) = 0] \\
\text{[I4]} & : P_{1,\dots,n}^{+'}(D^* - q_1^*) = 0 \implies v(\widehat{D}) \leq p^*
\end{aligned}$$

¹⁰The same result holds if instead $p^{min} = v(\widehat{D})$, but the proof requires an additional assumption that all bidders submit differentiable schedules. (It is possible to prove that $v(\widehat{D}) = p^* > c \implies P_i'(q_i^*)$ does not exist for some i .)

To complete the proof from these steps: $D^* = \widehat{D}$ by [I0]. [I3] implies that $P_{2,\dots,n}$ must be locally flat to the right at $\widehat{D} - q_1^*$ and hence that $P_{1,\dots,n}$ must be locally flat to the right at \widehat{D} . But then [I4] implies that $p^* \leq c'(\widehat{D})$ which is a contradiction. ♣

Note that the proof of Steps [A1,A3] is the same as that of Steps [I1,I3]. The other steps are straightforward. [I0]: $D^* = \widehat{D}$ for otherwise demand is both decreasable and increasable from its equilibrium level and the proof of Theorem 4 implies that $p^* \leq c$. Step [I4]: the auctioneer will always increase demand further if its faces supply that is flat to the right at a price above its marginal cost.

Discussion. The requirement that $p^{max} < v(\widehat{D})$ is significant for several reasons. First of all, when bids at very high prices are allowed, collusive-seeming equilibria exist in the increasable demand auction. For example, suppose that $c = 0$, $\widehat{D} = 2$, $v(D) = 50 - 100(D - 2)$, and that there are two bidders. Let $P_i(q) = 100q$ for $i = 1, 2$. (P_1, P_2) is a Nash equilibrium in which $p^* = 100$ and $q_1^* = q_2^* = 1$. To see why, first note that the auctioneer will never demand more than 2 at any price greater than 50, and at price $p < 50$ will never demand more than $2 + (50 - p)/100$. Consider first outcomes in which the final price $p > 50$. The best of these for bidder 1 solves $\max_{p > 50} p(2 - p/100)$, i.e. to have $p^* = 100$ and $q_1^* = 1$ for profit of 100. Now consider outcomes in which $p^* \leq 50$. $p = 50$ solves $\max_{p \leq 50} p(2 - p/100 - (50 - p)/100)$, so that bidder 1's profits in such an outcome are bounded above by $50 * 3/2 < 100$. Thus, it is a best response for 1 to submit any schedule such as $P_i(q) = 100q$ so that the realized price is 100.

It does not seem natural to require that $p^{max} < v(\widehat{D})$. For this reason, it seems to me, Theorem 5 has limited practical relevance. The value of my analysis of the increasable demand auction is more that it sharpens intuition as to why collusive-seeming equilibria do not exist in either the ration-reward perturbations or in the adjustable demand auction: such equilibria can be broken by *rewarding* aggressive bidding on the margin for more quantity.

3.4 Decreasable Demand Auction

On the other hand, collusive-seeming equilibria always exist in the \widehat{D} -decreasable demand auction in any setting in which $v(\widehat{D}) > c$, i.e. in which the efficient

outcome is for \widehat{D} to be demanded.

Theorem 6. *When $v(\widehat{D}) > c$, Nash equilibria exist in the \widehat{D} -decreasable demand auction in which the realized price p ranges in $\left[c, \frac{n-1}{n}c + \frac{v(\widehat{D})}{n} \right]$.¹¹*

PROOF: In any such Nash equilibrium, it must be that $D^* = \widehat{D}$. Otherwise, Theorem 4 would apply. (All that the proof of Theorem 4 uses is the fact that the auctioneer is not constrained to increase or decrease demand from its equilibrium level. Thus, if the equilibrium demand is not equal to \widehat{D} , then $p^* = c$.) I will focus my equilibrium search on symmetric equilibria in which all bidders submit linear supply schedules and receive quantity $\frac{\widehat{D}}{n}$. Suppose that p^* is the realized price in such an equilibrium. Then each bidder's schedule must be of the form

$$P_i(q) = p^* + \gamma \left(q - \frac{\widehat{D}}{n} \right),$$

where $\gamma \geq 0$. In this case, the aggregate schedule $P_{1,\dots,n}(\cdot)$ has slope $\frac{\gamma}{n}$ and bidder 1's residual demand, $\widehat{D} - P_{2,\dots,n}(\cdot)$, has slope $-\frac{\gamma}{n-1}$. (Bidders may deviate from their equilibrium strategies in two ways, (1) so as to receive more or less quantity themselves without changing the total quantity demanded or (2) so as to receive less quantity and decrease the total quantity demanded. No bidder has an incentive to deviate in the second way. Thus, it suffices to consider deviations in which the bidders view the total demand as fixed at \widehat{D} in which "residual supply" is well-defined.)

The auctioneer is willing to demand \widehat{D} only when reducing demand will not increase profits:

$$\widehat{D} \cdot \frac{\gamma}{n} \leq v(\widehat{D}) - p^*$$

Similarly, each bidder is willing to receive $\frac{\widehat{D}}{n}$ only when receiving more or less will not increase his surplus:

$$\frac{\widehat{D}}{n} \cdot \frac{\gamma}{n-1} = p^* - c.$$

¹¹Back and Zender (1999) have independently proven this result in the setting in which bidders are buyers and the auctioneer's goal is to maximize revenue. This is equivalent to the case in which bidders are sellers and the auctioneer's goal is to maximize consumer surplus with respect to some constant marginal values. Whereas my analysis allows for any continuous marginal values.

Putting these conditions together,

$$\frac{n(n-1)(p^* - c)}{\widehat{D}} = \gamma \leq \frac{n(v(\widehat{D}) - p^*)}{\widehat{D}}$$

Thus, a symmetric collusive-seeming equilibrium in linear strategies exists with realized price p^* whenever

$$(p^* - c)(n - 1) \leq v(\widehat{D}) - p^*$$

$$p^* \leq \frac{n-1}{n}c + \frac{v(\widehat{D})}{n}$$



The fact that collusive-seeming equilibria exist in the decreaseable demand auction is a consequence of the limited way in which the option to reduce demand alters bidders' incentives. Bidders know that if they submit demand schedules that are too inelastic, then the auctioneer will reduce demand. Thus, they don't submit such schedules. They still have an incentive to submit schedules that are as inelastic as possible, however, since doing so deters others from bidding more aggressively as much as possible.

4 Conclusion

This paper takes an unusual approach to comparing equilibria in different procurement auctions. A typical approach would be to solve for and compare the average performance of (say unique) equilibria in two different sorts of auctions in a range of environments. Whereas my analysis focuses entirely on the (possibly) low-probability event in which the highest price is realized. *Some* equilibria in the standard uniform-price procurement auction “perform very badly” in this event, in the sense that prices can far exceed true bidder marginal costs. On the other hand, *all* of the equilibria in the adjustable demand auction or in my ψ -perturbations of the uniform-price auction “perform well” in this event, in the sense that price can exceed the true marginal cost of at most one bidder. Such an approach has the advantage that equilibrium behavior in this event can be pinned down in relatively general environments in which it is challenging to calculate full equilibria.

The reason is that how one bids “at the top” of one’s supply schedule does not affect payoffs in the event that others do not submit their highest supply schedules. Furthermore, the performance of the uniform-price procurement auction in the periods of peaking prices continues to be in itself a vitally important matter in electricity markets in California and elsewhere.

Of course, when considering very different sorts of auctions, this sort of comparison is questionable. After all, perhaps one auction performs better most of the time while not performing well in the low-probability event. I avoid this problem by *perturbing* the rules of the uniform-price auction. For a given set of bids, each bidder’s payoff differs by at most ψ between the two auction formats. Furthermore, in a model with a finite grid of prices and quantities and finitely many types, it is straightforward to conclude that) any sequence of equilibria in ψ -perturbations, as $\psi \rightarrow 0$, has a subsequence that converges to an equilibrium of the standard uniform-price auction. In other words, the equilibrium set is upper hemi-continuous with respect to ψ and the standard uniform-price auction has an equilibrium in which at most one bidder has marginal cost exceeding price when price achieves its maximum in that equilibrium.

The system operator can guarantee that only such equilibria exist by modifying the rules so as to split a cash “ration-reward” ψ among all generators who are rationed, i.e. who do not supply all of the quantity that they announced a willingness to sell at the realized price. At least theoretically, the price in peak periods can not exceed all but at most one generator’s marginal cost, no matter how small ψ may be. Practically speaking, of course, this amount of money can not be extremely small¹².

Another way to guarantee that any equilibrium must not be collusive-seeming is to commit to an adjustable demand rule. For example, many system operators have arrangements with large electricity users to reduce their supply given higher prices and may also potentially store some amount of excess electricity generation for future dispatch (whether by pumping up water or by other means), in effect making the demand for electricity somewhat

¹²By submitting locally elastic supply for marginal units at a high price, a generator is providing evidence of a sort that it has withheld supply at lower price levels. This may have several negative ramifications for the generator. For one thing, it may lead regulators to focus more attention on that generator’s behavior. Any reward must be large enough to counteract this and other such effects. On the other hand, as my analysis makes clear, it only takes two bidders who find the reward to be an inducement to deviation to unravel a collusive-seeming equilibrium.

elastic. In this case, my results suggest that the system operator should determine its total dispatch quantity by an adjustable demand rule rather than by the standard market-clearing rule. This allows the system operator to exercise its monopsony power given the generators' bids on behalf of electricity consumers. More importantly, however, it reduces generators' incentives in equilibrium to submit bids that far exceed than their true marginal costs. On the other hand, of course, equilibria in the adjustable demand auction will not be "close" to equilibria in the standard uniform-price auction, and my analysis is not conclusive of a benefit to the adjustable demand rule. For any given set of others' strategies, each bidder's best response (the solution to his decision-theoretic problem) will generally involve less bid shading in an adjustable demand auction than in the standard uniform-price auction.¹³ On this basis, I conjecture that costs will generally be lower and efficiency greater in the adjustable demand auction, but this needs to be confirmed by future research. Unfortunately, the adjustable demand version of the uniform-price auction lacks even the weak ordinal structure that McAdams (2001) identifies in the standard uniform-price auction, so it seems likely that progress on this front may need to be through numerical simulation of equilibria.

Appendices

Discrete Grids and Special Notation

In Theorem 3, there is a finite *grid* of permissible prices \mathbf{p} and of permissible quantities \mathbf{q} :

$$\begin{aligned}\mathbf{p} &= \{\emptyset\} \cup \{p^2, \dots, p^{|\mathbf{p}|-1}\} \cup \infty \\ \mathbf{q} &= \{q^1 = 0, q^2, \dots, q^{|\mathbf{q}|-1}\} \cup \infty;\end{aligned}$$

¹³Sketch of the argument: Let $S_i^*(\cdot; t_i)$ be the supply schedule that is a best response for type t_i given others' strategies $S_{-i}(\cdot; \cdot)$ in the standard uniform-price auction. Now consider what happens if i slightly lowers its bid over a small quantity interval (keeping the slope constant). If demand were fixed, i would be indifferent to such a deviation. But in the adjustable demand auction, this lowering of prices will lead the auctioneer to demand more. Thus, i will supply more additional quantity than before and the price will fall less than before.

I assume that these grids have constant fineness:

$$\begin{aligned} p^{k+1} - p^k &= \Delta_p \text{ for } k = 2, \dots, |\mathbf{p}| - 2 \\ q^{l+1} - q^l &= \Delta_q \text{ for } l = 1, \dots, |\mathbf{q}| - 1 \end{aligned}$$

Other than that, all assumptions and framework of the general model are maintained. When there is a continuum grid of prices and quantities, I will say that $\Delta_p = \Delta_q = 0$.

The following notations will be used extensively in the proofs in the Appendix. Some have been defined earlier, but I group them here for convenience.

- ψ, β : Parameters of perturbations of the uniform-price auction. An amount of money ψ is divided evenly among all bidders who are rationed by at least β , i.e. for whom $\max S_i(p^*) \geq q_i^* + \beta$.
- $\vec{S}^*(\cdot; \cdot, \cdot)$: a profile of strategies that I assume, for the sake of contradiction, is a collusive-seeming equilibrium (in whatever specific sense is described in the statement of each Theorem).
- p^* : The realized price. When not otherwise specified,

$$p^*(\vec{s}) \equiv p^*(\vec{S}(\cdot; \vec{t}, \vec{\gamma}), \tau, \rho),$$

the price in state \vec{s} given strategies $\vec{S}^*(\cdot; \cdot, \cdot)$ and hence bids $\vec{S}^*(\cdot; \vec{t}, \vec{\gamma})$. (A different notation will be used to refer to prices that may result when bidder i deviates from his equilibrium bid.)

- $\Gamma([p_1, p_2]) \equiv \{\vec{s} : p^*(\vec{s}) = p\}$: The event in which the realized price is p in the equilibrium. If $p_1 = p_2 = p$, I will use the notation $\Gamma(p)$.
- $\Gamma_i([p_1, p_2]) \equiv \{t_i : \Pr[\vec{s} \in \Gamma([p_1, p_2]) | t_i] > 0\}$: Bidder i types who view a price in the range $[p_1, p_2]$ as likely to occur with positive probability given their own type. If $p_1 = p_2 = p$, I will use the notation $\Gamma_i(p)$.
- $\Gamma_i(t, \gamma) \equiv \{\vec{s} : t_i = t, \gamma_i = \gamma\}$: The event in which i has type t and randomization variable γ .
- $\bar{D} \equiv \sup_{\tau} \max D(p^{min}, \tau)$, $\underline{D} \equiv \inf_{\tau} \min D(p^{max}, \tau)$: Maximal and minimal possible demand.

- $RD_i^*(\cdot; \vec{s}) \equiv D(\cdot; \tau) - \sum_{j \neq i} S_j^*(\cdot; t_j, \gamma_j)$: Residual demand correspondence in the state \vec{s} in the equilibrium.
- q_i^* : Bidder i 's realized quantity. When not otherwise specified,

$$q_i^*(\vec{s}) \equiv q_i^*(\vec{S}(\cdot; \vec{t}, \vec{\gamma}), \tau, \rho),$$

i 's quantity in state \vec{s} given strategies $\vec{S}^*(\cdot; \cdot, \cdot)$.

- $\bar{p} \equiv \inf \{p \leq p^{max} : Pr(p^*(\vec{s}) \leq p) = 1\}$: The highest non-null price that is realized (or at least almost realized) in the equilibrium.
- $\bar{q}_i(t_i, \gamma_i) \equiv \inf \{q \leq q^{max} : Pr[q_i^*(\vec{s}) \leq q] = 1 | \Gamma_i(t_i, \gamma_i)\}$: Maximal quantity that bidder i supplies (or at least almost supplies) in the equilibrium when he has type t_i and randomization variable γ_i .
- $\bar{c}_i([p_1, p_2], \beta) \equiv \inf \{c : Pr[c_i^+(q_i^*(\vec{s}) + \beta, q_{-i}^*(\vec{s}), \vec{s}) \leq c | \Gamma([p_1, p_2])] = 1\}$: Upper bound on bidder i 's marginal cost for β more quantity than he supplies in the equilibrium, holding others' equilibrium quantities constant. (Since $c_i(\cdot)$ is non-decreasing in quantity, bidder i is willing to pay β times his marginal cost at $q_i^*(\vec{s}) + \beta$ to receive β more than $q_i^*(\vec{s})$.)
- $\bar{c}_i(p, \beta) \equiv \lim_{\varepsilon \rightarrow 0} \bar{c}_i([p - \varepsilon, p + \varepsilon], \beta)$

It is straightforward to show that \bar{p} and $\bar{q}_i(t_i)$ are always well-defined. Furthermore, $\bar{c}_i([p_1, p_2], \beta)$ is well-defined whenever $Pr[p^*(\vec{s}) \in [p_1, p_2]] > 0$, so that $\bar{c}(\bar{p}; \beta)$ is always well-defined.

Proof of Theorem 3

Suppose otherwise, that $\bar{p} - \bar{c}_i(\bar{p}, \beta) > \Delta_p + \Delta_p \frac{\bar{D} - D}{\beta}$ for $i \in \{i_1, i_2\}$. For the duration of the proof, I will refer to these two bidders as i_1, i_2 and to all other bidders generically as j . By definition of \bar{p} in the case of a discrete price grid, $Pr[p^*(\vec{s}) = \bar{p}] > 0$. Hence $\Gamma_j(\bar{p}) \neq \emptyset$ for all j . Let $(t_i, \gamma_i) \in \Gamma_i(\bar{p})$ for $i \in \{i_1, i_2\}$.

STEP 1: Both bidders i_1, i_2 must receive a portion of the reward with probability one conditional on the price \bar{p} being realized. PROOF: Suppose that bidder i_1 does not receive a portion of the reward with positive probability given (t_{i_1}, γ_{i_1}) and the realized equilibrium price is \bar{p} . Consider the

following deviation:

$$\begin{aligned}\tilde{S}_i(p; t_{i_1}, \gamma_{i_1}) &= S_i^*(p; t_{i_1}, \gamma_{i_1}) \text{ for all } p < \bar{p} \\ &= [\min S_i^*(\bar{p}; t_{i_1}, \gamma_{i_1}), \beta + \max S_i^*(\bar{p}; t_{i_1}, \gamma_{i_1})] \cap \mathbf{q} \text{ for } p = \bar{p} \\ &= \beta + \max S_i^*(\bar{p}; t_{i_1}, \gamma_{i_1}) \text{ for all } p > \bar{p}\end{aligned}$$

(If $\beta + \max S_i^*(\bar{p}; t_{i_1}, \gamma_{i_1}) > q^{max}$, then replace ∞ for $\beta + \max S_i^*(\bar{p}; t_{i_1}, \gamma_{i_1})$.) This deviation has no effect when $p^*(\vec{s}) < \bar{p}$. (It is straightforward to show that the realized price will be the same whether bidder i_1 submits $S_{i_1}^*(\cdot; t_{i_1}, \gamma_{i_1})$ or $\tilde{S}_{i_1}(\cdot; t_{i_1}, \gamma_{i_1})$. Furthermore, since by assumption rationed quantities can only depend on expressed demand at the market-clearing price, the allocation also does not change.) And by definition $p^*(\vec{s}) > \bar{p}$ with probability zero.

When $p^*(\vec{s}) = \bar{p}$, finally, this deviation is never worse and strictly better with positive probability for bidder i_1 . That it is never worse follows from the fact that (i) the realized price stays at \bar{p} after the deviation so there is no additional payment on inframarginal units, (ii) i_1 never wins less quantity and others never win more because by assumption each bidder's quantity is non-increasing in his own bid and non-decreasing in others' bids, (iii) i_1 never receives less of the reward.

Winning more (up to $\bar{q}_{i_1}(t_{i_1}, \gamma_{i_1})$) and having others win less will always be strictly better for i_1 : $\bar{c}_i(\bar{p}, \beta) < \bar{p} - \Delta_p$ implies that i_1 is certainly willing to receive price \bar{p} for up to $\bar{q}_{i_1}(t_{i_1}, \gamma_{i_1}) + \beta$. Finally, if i_1 wins more quantity with probability zero, then he must be rationed by at least β with probability one, making him strictly better off in any case.

STEP 2: When i_1 has type t_{i_1} and randomization variable γ_i , it is a profitable deviation for him to submit

$$\begin{aligned}\widehat{S}_{i_1}(p) &= S_{i_1}^*(p; t_{i_1}, \gamma_{i_1}) \text{ for all } p < \bar{p} - \Delta_p \in \mathbf{p} \\ &= [\min S_{i_1}^*(\bar{p} + \Delta_p; t_{i_1}, \gamma_{i_1}), \bar{q}_{i_1}(t_{i_1}, \gamma_{i_1}) + \beta] \cap \mathbf{q} \text{ for } p = \bar{p} - \Delta_p \\ &= \bar{q}_{i_1}(t_{i_1}, \gamma_{i_1}) + \beta \text{ for all } p > \bar{p} - \Delta_p \in \mathbf{p}\end{aligned}$$

(If $\beta + \max S_{i_1}^*(\bar{p}; t_{i_1}, \gamma_{i_1}) > q^{max}$, then replace ∞ for $\beta + \max S_{i_1}^*(\bar{p}; t_{i_1}, \gamma_{i_1})$.) I will show that this deviation makes bidder i_1 no worse off in every state and strictly better off with positive probability, conditional on t_{i_1} .

STEP 2A: Conditional on the realized price being less than $\bar{p} - \Delta_p$ in the equilibrium, bidder i_1 is always indifferent to the deviation since it does

not affect the price or allocation. And the price is greater than \bar{p} in the equilibrium with probability zero.

STEP 2B: Conditional on the realized price being equal to $\bar{p} - \Delta_p$, bidder i_1 is always no worse off after the deviation. The price remains $\bar{p} - \Delta_p$, and i_1 gets at least as much quantity and reward as in the equilibrium.

STEP 2C: Conditional on the realized price being equal to \bar{p} in the equilibrium, bidder i_1 is always strictly better off after the deviation. Suppose that $\vec{s} \in \Gamma_{i_1}(t_{i_1}, \gamma_{i_1}) \cap \Gamma_{i_1}(\bar{p})$. By Step 1, bidder i_1 receives at most $\frac{\psi}{2}$ of the reward and $\bar{q}_{i_1}(t_{i_1}, \gamma_{i_1}) \leq \max RD_{i_1}^*(\bar{p}; t_{-i_1}, \gamma_{-i_1}) = \min RD_{i_1}^*(\bar{p} - \Delta_p; t_{-i_1}, \gamma_{-i_1})$. (Bidder i_2 also shares the reward; I require in the model that bids have the structure $\min S_j(p) = \max S_j(p - \Delta_p)$ which implies that $\max RD_j(p) = \min RD_j(p - \Delta_p)$.)

After the deviation, the realized price will be $\bar{p} - \Delta_p$ since $\max \widehat{S}_{i_1}(\bar{p} - \Delta_p; t_{i_1}, \gamma_{i_1}) > \min RD_{i_1}(\bar{p} - \Delta_p; t_{-i_1}, \gamma_{-i_1})$. Thus, i_1 wins at least $\min RD_{i_1}(\bar{p} - \Delta_p; t_{-i_1}, \gamma_{-i_1}) = \max RD_{i_1}(\bar{p}; t_{-i_1}, \gamma_{-i_1})$, which is an upper bound on what he could have won in the equilibrium. Furthermore, by the definition of residual demand, if he wins $\min RD_{i_1}(\bar{p} - \Delta_p; t_{-i_1}, \gamma_{-i_1})$ then every other bidder must receive his maximal demand at $\bar{p} - \Delta_p$, implying that i_1 must receive *all* of the reward. Receiving at least $\psi/2$ more reward makes i_1 strictly better off since $\psi > 2\Delta_p(\bar{D} - \underline{D})$. (The additional $\psi/2$ cash reward more than compensates for the additional payments on inframarginal units by assumption of the theorem that $\psi > 2\Delta_p(\bar{D} - \underline{D})$.)

Indeed, if i_1 does not receive at least $\psi/2$ more reward, then it must be that he supplies at least β more quantity, for all of which (by assumption of the theorem) his marginal cost is less than $\bar{p} - \Delta_p$ by at least $\Delta_p(\bar{D} - \underline{D})/\beta$, implying that his gain from winning these additional units is greater than $\Delta_p(\bar{D} - \underline{D})$ and so exceeds his additional payments on inframarginal units. Thus, i_1 has a type with a profitable deviation. This contradicts the presumption that $\vec{S}^*(\cdot; \cdot, \cdot)$ is an equilibrium. \clubsuit

Proof of Theorem 2

[Very similar to step 2 in the proof of Theorem 3. Step 1 is not necessary given a continuum grid of prices and quantities.] Suppose otherwise. That is to say, $\bar{c}_i(\bar{p}, 0) < \bar{p}$ for $i \in \{i_1, i_2\}$. By definition of \bar{p} , $\Pr[p^*(\vec{s}) > \bar{p} - \varepsilon] > 0$

for all $\varepsilon > 0$. Hence $\Gamma_j([\bar{p} - \varepsilon, \bar{p}]) \neq \emptyset$ for all j . Let $(t_i, \gamma_i) \in \Gamma_i([\bar{p} - \varepsilon, \bar{p}])$ for $i \in \{i_1, i_2\}$.

It is a profitable deviation for one of the bidders $i \in \{i_1, i_2\}$ to submit

$$\begin{aligned}\widehat{S}_i(p; t_i, \gamma_i) &= S_i^*(p; t_i, \gamma_i) \text{ for all } p < \bar{p} - \varepsilon \in \mathbf{p} \\ &= [\min S_i^*(\bar{p} - \varepsilon; t_i, \gamma_i), \bar{q}_i(t_i, \gamma_i)] \cap \mathbf{q} \text{ for } p = \bar{p} - \varepsilon \\ &= \bar{q}_i(t_i, \gamma_i) - \varepsilon \text{ for all } p > \bar{p} - \varepsilon \in \mathbf{p}\end{aligned}$$

for some $\varepsilon > 0$. This deviation has no effect in states in which $p^*(\vec{s}) < \bar{p} - \varepsilon$, and $p^*(\vec{s}) > \bar{p}$ with probability zero.

Choose $\varepsilon > 0$ so that both bidders i_1 and i_2 would win all of the reward with probability one (conditional on t_i) if they deviated with the bids $\widehat{S}_{i_1}(\cdot; t_{i_1}, \gamma_{i_1})$ or $\widehat{S}_{i_2}(\cdot; t_{i_2}, \gamma_{i_2})$. An arbitrarily small ε exists with this property since there are at most countably many prices at which any bidder submits interval demand at that price. (If no other bidder submits interval demand at price $\bar{p} - \varepsilon$, then i wins all of the reward when he submits $\widehat{S}_i(\cdot; t_i, \gamma_i)$.) Now choose $i \in \{i_1, i_2\}$ so that, conditional on $\Gamma([\bar{p} - \varepsilon, \bar{p}])$, bidder i 's expected reward is no more than $\psi/2$ in the equilibrium. (Such a bidder exists since the total expected reward conditional on this event is no more than ψ .)

Thus, bidder i wins at least $\frac{\psi}{2}$ more of the reward, receives at most $\varepsilon \max S_i^*(\bar{p}; t_i)$ less on inframarginal units, and wins no less than by submitting $S_i^*(\cdot; t_i)$. His marginal cost for any additional quantity that he may win, furthermore, is bounded above by $\bar{c}_i(\bar{p}, 0) < \bar{p}$. For small enough ε , then, the benefit from receiving more of the reward (and more quantity) swamps any loss from lower inframarginal payments. \clubsuit