Essays in Quantitative Economics

by

Zhao Liu

Department of Economics
Duke University

Date: _____

Approved:

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A. Craig Burnside, Advisor

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Mariano Massimiliano Croce

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Lukas Schmid

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Federico A. Bugni

Dissertation submitted in partial fulfillment of the
requirements for the degree of Doctor of Philosophy
in the Department of Economics
in the Graduate School of
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ABSTRACT

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Abstract

This thesis contains essays on quantitative economics. It focuses on understanding capital markets and macroeconomics through general equilibrium models and econometric tools. In the second chapter, I propose a two-sector production-based dynamic stochastic general equilibrium model to study the interaction between R&D activities and firm heterogeneity. I argue that the different business risks faced by R&D and non-R&D firms, could be an important source of heterogeneity in asset prices between R&D and non-R&D firms. In the third chapter, co-authored with Riccardo Colacito and Mariano Massimiliano Croce, we characterize the equilibrium of a complete market economy with multiple agents featuring a preference for the timing of the resolution of uncertainty. We provide conditions under which the solution of the planner's problem exists, and it features a nondegenerate invariant distribution of Pareto weights. In the fourth chapter, I define a first-order good uncertainty measure. I then incorporate it into the DSGE model to evaluate the aggregate effects of both good and bad uncertainty. In the final chapter, I propose a buffered double autoregressive (BDAR) time series model to depict the buffering phenomenon of conditional mean and conditional variance in time series. I first prove strict stationarity and geometric ergodicity of the BDAR model under several sufficient conditions. I then propose a quasi-maximum likelihood estimation procedure and study its nontrivial asymptotic property. Furthermore, a model selection criteria and its asymptotic property have been established. I evaluate the model's performance, using both simulated and real data.

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Chapter 1

Introduction

This thesis focuses on building off cutting-edge models to enhance the understanding of contemporary economic issues and financial markets. Empirical evidence suggests that R&D-intensive firms experience higher expected equity returns, yet lower leverage, default rates, and credit spreads, than R&D-nonintensive firms. To provide a unified explanation for these cross-sectional differences, in the second chapter, "R&D, Risk Premia, and Credit Spreads", I propose a two-sector production-based dynamic stochastic general equilibrium model in which R&D firms are highly exposed to innovation risks. Bondholders impose more stringent borrowing limits over R&D firms than non-R&D firms. Hence, R&D firms tend to show low leverage, lowering both default rates and credit spreads; nevertheless, the inherent riskiness of R&D activities dominates, generating high equity returns. The model generates sizable heterogeneity in the quantities of interest between R&D and non-R&D firms, as in the data, and fits the aggregate macroeconomic and asset pricing moments reasonably well.

In the third chapter, "Recursive Allocations and Wealth Distribution with Multiple Goods: Existence, Survivorship, and Dynamics" (initially published at Quantitative Economics (Volume 10, Issue 1, Pages 311-351)), Professor Riccardo Colacito, Professor Mariano Croce and I write about international macroeconomics, specifically, international risk-sharing. A natural question emerges. Will the risk-sharing mechanism guarantee each country participating in this mechanism to survive in the long run? If so, under which conditions? We answer these questions by providing sufficient conditions for the existence of a well-behaved equilibrium in an interna-

tional Epstein-Zin DSGE model. Our findings thus provide a general framework from which many models of international macroeconomics with rich dynamics could be developed.

Since the great recession, the impacts of uncertainty shock over the macroeconomy and financial markets have attracted lots of attention from policymakers and academic researchers. While most of the studies focus on the negative effects of uncertainty over the macroeconomy, there are certain channels from which the uncertainty could have a positive impact. Thus, the aggregate effects of the uncertainty is worth a further investigation. In the fourth chapter,"Good Uncertainty, Bad Uncertainty, and the Aggregate Effects", I define a first-order good uncertainty through the investment option channel and incorporate it into the standard DSGE model to evaluate the aggregate effects of both good and bad uncertainty.

I am, furthermore, applying econometrics tools to the study of the dynamics of financial markets. In the final chapter, "On Buffered Double Autoregressive Time Series Models", I propose a time series model with heteroscedasticity and novel regime-switching mechanism which can better capture the dynamics of financial markets. Formal proofs suggest strict stationarity and geometric ergodicity of the BDAR model under several sufficient conditions. I then propose a quasi-maximum likelihood estimation procedure and study its nontrivial asymptotic property. Furthermore, a model selection criteria and its asymptotic property have been established. I evaluate the model's performance, using both simulated and real data.

Chapter 2

R&D, Risk Premia, and Credit Spreads

2.1 Introduction

Research and development (R&D) has continued to play an essential role in the long-run success of the U.S. economy in the 21st century. Total US R&D expenditure by sectors reached nearly 500 billion in 2017 in constant 2012 US dollars (see Figure 2.1). More importantly, from 2000 to 2017, 60%–70% of R&D funding came from firms in business sector. As firms tend to finance their innovation activities by equity and debt, R&D activities naturally affect firms' choices of capital structures. Moreover, such an impact is further synchronized on the stock and bond market and reflected upon both the equity returns and credit spreads.

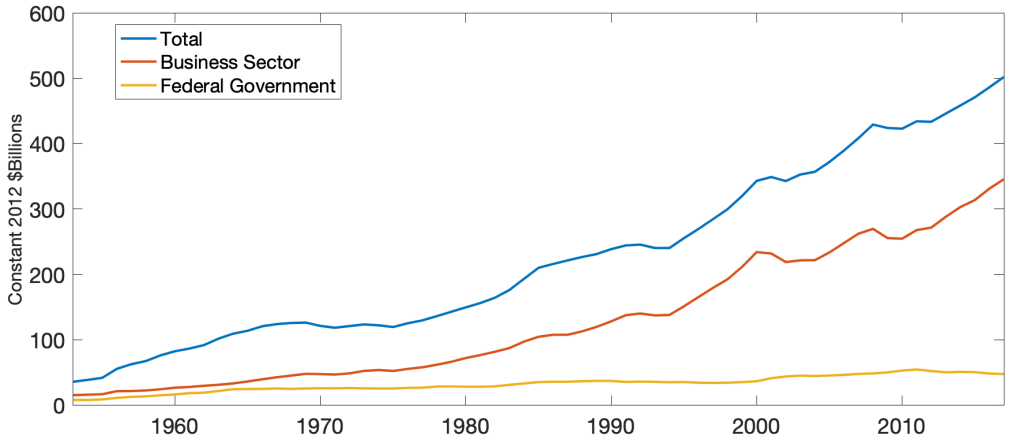

Figure 2.1: U.S. R&D expenditure by sectors

Data Source(s): National Science Foundation, National Center for Science and Engineering Statistics, National Patterns of R&D Resources (Annual Series).

In this paper I investigate, both theoretically and empirically, the impact of firms' R&D activities on cross-sectional heterogeneity in capital structures, default risks, and asset prices. Sorting firms into portfolios based on their R&D intensity, I find simultaneously significant differences in equity returns, leverage, default probability, and credit spreads between the top quintile, which I label R&D-intensive firms, and the bottom quintile, labeled as R&D non-intensive firms. Specifically, compared to R&D-nonintensive firms, R&D-intensive firms earn a 9.28% higher expected stock return; use less debt, with 0.12 less in book leverage ratio; default less, based on two default measures; and pay 94 bps less on the bonds issued. Such cross-sectional differences are statistically significant and robust when firm characteristics are controlled.

Why does such heterogeneity exist between the two types of firms? To answer this question, I develop a two-sector production-based dynamic stochastic general equilibrium (DSGE) model that links the endogenous growth via innovation to firms' capital structure choice and asset prices. In this model, a parent firm in the R&D sector has two subsidiaries. One subsidiary is responsible for technology adoption, that is, transforming raw ideas into patents in production and selling them on the open market. Firms in the other subsidiary purchase these patents and use them to produce sub-intermediary goods, which are then sold to firms in the non-R&D sector. Together with capital and labor, the bundle of sub-intermediary goods is exploited by firms in the non-R&D sector to produce intermediary goods. These goods are then further bundled by a representative final good producer. In each period, firms in the R&D and non-R&D sectors choose capital structures through evaluating tax benefits and potential bankruptcy costs, under the borrowing limits imposed by the bondholders. On top of that, the representative household, which is equipped with the Epstein-Zin preference, has access to stocks and corporate and government bonds

to smooth its consumption.

The underlying risk in technology adoption induces a high equity risk premium for R&D firms. Under a persistent positive innovation shock, the increase in raw technology largely drives up the value of R&D firms' growth options, which then further and strongly boosts the R&D firms' value. At the same time, once adopted, the greater number of technological ideas can persistently improve productivity growth in the economy. Such an improvement induces a steep decline in the stochastic discount factor (SDF) of the household with Epstein-Zin preference. The increasing firm value and decreasing SDF together exhibit a strong negative comovement. As a compensation, a high equity premium is required by shareholders of R&D firms. In contrast, in the absence of technology adoption, non-R&D firms are less exposed to the innovation shock and thus show only a mild increase of firm value and a weaker negative comovement with SDF. Consequently, a lower equity premium is observed in non-R&D firms.

In reality, debt holders are usually more reluctant to lend money to the R&D firms. It could be attributed to the unstable cash flows induced by the high business risks of R&D firms. It could also reflect the market failures such as asymmetric information and moral hazards, due to the high uncertainty in the innovation activities. Hence, in the model, I assume bond holders impose more stringent borrowing limits for the R&D firms than the non-R&D firms. When facing severe borrowing constraints, R&D firms tend to maintain a low amount of debt. This further signals a low ex-post default probability for R&D firms, reflected as low credit spreads in the bond market.

It is commonly believed that a high equity return is supposed to be linked with a high leverage, since a high debt level amplifies a firm's exposure to systematic risk, for which a high stock return is required as a compensation. However, contradicting this view, this paper provides evidence for the coexistence of high equity return

and low leverage in R&D firms. This can be explained by noting that a firm's equity risk (its equity beta) can be affected by both its leverage and its underlying business risk (its asset beta). Thus, although the low leverage of R&D firms alleviates their exposure to the systematic risks, the high business risk inherent in technology adoption dominates the low leverage and leads to overall high equity returns for R&D firms. To substantiate this idea, I test the asset betas of firms in R&D-intensive and R&D-nonintensive portfolios in the data. The results show that the asset betas of R&D-intensive firms are significantly higher that those of the R&D-nonintensive firms, consistent with the model's mechanism.

2.2 Literature Review

My study is related to the literature which ties R&D investment to firms' cost of capital. Lev and Sougiannis (1996) are among the first to have reported that the R&D intensity of US public firms predicts the cross-sectional equity return. Since then, researchers have proposed several possible explanations to rationalize such return predictability of the R&D intensity (e.g.Chan et al. (2001), Penman and Zhang (2002), Chambers et al. (2002), Kothari et al. (2002), Eberhart et al. (2004, 2008), Berk et al. (2004), Lev et al. (2005), Ciftci et al. (2011), Li (2011), Lin (2012), Hou et al. (2016)). On the other hand, there is a debate in the extant empirical literature about the relation between R&D investment and welfare of debt holders (e.g., Shi (2003), Eberhart et al. (2008)), and there is very little recent work that offers theoretical guidance to interpret this relationship. Moreover, the above mentioned studies do not explicitly consider the link between R&D and asset prices via firms' capital structure choice, a natural channel through which the effect of R&D can be transmitted to the asset market. The present paper contributes to this literature

by investigating the interactions of R&D, capital structure, default decision, and asset prices in a joint theoretical framework, providing a potential explanation for the cross-sectional heterogeneity observed in the data. In addition, the empirical study in this paper takes into account the recent development in data collection, especially in the corporate bond market (TRACE), and uses more robust econometric methods (Gow et al. (2010)) to provide further support for the previous empirical findings in the literature.

The current study is built on the recent progress on the endogenous growth model in macroeconomics. Comin and Gertler (2006) focuses on the medium-term business cycles, Comin et al. (2009) highlights the innovation-driven fluctuations. Other important work includes but not limited to Kung and Schmid (2015), Corhay et al. (2017), Croce et al. (2019), and Bianchi et al. (2019). My paper is closely related to Liao and Schmid (2017), who show that the endogenous supply of collateral through technology adoption is an important driver of aggregate levered return and credit spread. My work differs from theirs by modeling cross-sectional capital structure and debt financing and quantifying the cross section of levered equity returns between R&D and non-R&D firms. Innovation activities, effectively provide a source of low-frequency risk, as noted by Bansal and Yaron (2004). More broadly, this paper is linked to the literature about production-based asset pricing. Important work includes Jermann (1998), Boldrin et al. (2001), Kaltenbrunner and Lochstoer (2010), Croce (2014), Petrosky-Nadeau et al. (2018) and Ai et al. (2018), among others.

The present study is also linked to the pricing of corporate securities in economics and finance (e.g. Leland et al. (1994), Hackbarth et al. (2006), Chen et al. (2008), Bhamra et al. (2010), and Chen (2010), among others). This literature succeeds in creating a realistic large credit spread in spite of a low default probability.

In this paper, I follow Gomes et al. (2016) to model the default decision and

capital structures, who investigate the effects of nominal rigidity on real quantities through the debt deflation channel. More generally, my work could be attributed to the general equilibrium production-based models with defaultable corporate debt. Important work includes Miao and Wang (2010), who study the amplification of business cycle fluctuations through the long term real corporate debt; Gourio (2013), who focuses on disaster risk; and Corhay (2017), who studies the relation between industry competition and credit spread. Gomes and Schmid (2017) explore the mechanism for the propagation of fluctuation in corporate leverage into the real economy through the lens of a model with heterogeneous firms, while Favilukis et al. (2019) emphasize the importance of the interaction between labor and financial leverage for the understanding of credit markets. I contribute to this strand of literature by investigating the interaction of firms' innovation activities and their cross-sectional capital structures and credit spreads.

2.3 Empirical Analysis

In the present section, empirical investigation is conducted on the relation between R&D intensity, equity return, leverage, default rate and credit spread. After describing the data sample, I construct portfolios sorted by firms' R&D intensity. The results of univariate analysis reveal significant differences in the firm characteristics among portfolios, highlighting the important role played by R&D. To check the robustness of the results, I implement a number of panel regressions and show that the effects of R&D cannot be explained by other firm characteristics.

2.3.1 Data

I retrieve the data for my sample from the dataset of the Center for Research in Security Prices (CRSP) and Compustat and use these to construct my equity and firm-level fundamental accounting sample from 1980 through 2017. Observations before 1980 are dropped, consistent with the practice of the Financial Accounting Standards Board (FASB), which asks for expensing the R&D expenditures since 1975. The bond sample is built by merging the transaction data from the Trade Reporting and Compliance Engine (TRACE) with bond issue and issuer characteristics from the Mergent Fixed Income Securities Database (FISD). The sample starts from 2002, when TRACE was first launched.

The bond default rate is measured in two ways. First, the default rate is expressed using Moody's ratings. Specifically, I use integer values 1 through 5 to transform Aaa, Aa, A, Baa, and Ba and below into numerical values. Second, following Vassalou and Xing (2004), I compute the expected default frequency (EDF) as a measure of the default rate using the model developed in Merton (1974).

Next, I compute the credit spread of corporate bonds, which is defined as the difference of yield-to-maturity between the corporate bond and its benchmark Treasury bond. Only bonds with fixed or zero coupons are retained in the sample; the bonds which are callable, putable, exchangeable, asset backed, or convertible are dropped from the sample. The yield-to-maturity of the corporate bond is computed by equating the present value of cash flows and the dirty price, which is defined as the sum of the clean bond price and accrued interest. I calculate the yield of the benchmark Treasury by computing the price of a synthetic Treasury bond, specifically, by adding up the cash flows discounted by the Treasury yield curve retrieved from the US Department of the Treasury, interpolated if necessary. With a match of

duration between the corporate and Treasury bonds, this method provides a better benchmark than a simple maturity match. Next, the yield of the Treasury bond is derived by equating the price of the synthetic bond and the present value of cash flows. The credit spread is then given by the yield difference between the two types of bonds. Finally, like Gilchrist and Zakrajšek (2012), I drop the bonds with credit spreads below 5bps or above 3500bps and those with a maturity of more than 30 years.

2.3.2 Results

To run the univariate analysis, I construct the portfolios sorted by firms' R&D intensity, which is defined as the ratio of R&D expenses to total assets. Specifically, at the July of year t, all firms are grouped into five quintiles based on their R&D intensity in year $t-1$. This guarantees that the R&D information is absorbed in equity and bond transactions. I label the top quintile as the R&D-intensive portfolio and the bottom quintile as the R&D-nonintensive portfolio. Table 2.1 lists the representative industries as well as their proportions in R&D-intensive and R&D-nonintensive portfolios.

From the July in year t (i.e. the formation date of the portfolios) to June in year $t+1$ (i.e. the end date of the portfolios), the equally-weighted stock returns, credit spreads, and default rates of each portfolio are computed at a monthly frequency. In addition, I calculate the leverages at the end of the fiscal year t and equally weight them in portfolios. The results of the univariate analysis of the R&D effect indicate that R&D intensity is related to the cross-sectional heterogeneity of asset prices and capital structure (see Table 2.2). Specifically, firms in the R&D-intensive portfolio obtain a 9.28% higher stock return than those in the R&D-nonintensive portfolios. Also, R&D-intensive firms tend to use less debt, with a 0.12 lower leverage ratio.

Moreover, a statistically significant difference obtains for the default rate measures as well as the credit spreads of the two portfolios, implying that the R&D-intensive portfolio faces less credit risk than the nonintensive portfolio. As a robustness check, I further corroborate that the differences above are not driven by other firm characteristics by means of the following panel regressions:

$$Y_{it} = \beta R\&D_{i,t-1} + \gamma X_{i,t-1} + FE_{ind,year} + \epsilon_{i,t}.$$

Y_{it} represents the firm-level variables of interest, namely, equity return, default rate, and credit spread from July in year t to June in year $t+1$, and leverage in fiscal year t. It is linked with $R\&D_{i,t-1}$, firm-level $R\&D$ intensity in fiscal year $t-1$, and $X_{i,t-1}$, the standard firm-level controls. The regressions are run, controlling for fixed effects of industry and year. The standard errors are clustered by industry and year. The results, reported in Table 2.3, support the findings of the univariate analysis that R&D-intensive firms have significantly higher equity returns, and significantly lower leverages, default rates, and credit spreads. Specifically, the coefficients of R&D intensity in the regressions of equity return and leverage are 0.02 and -0.27, respectively. These numbers imply that a one-standard-deviation increment of R&D intensity boosts the annual equity return by 0.456% and decreases leverage by 0.005. As for the default rate measured by the numerical Moody rating and EDF, the coefficients of R&D intensity are -1.06 and -0.05, respectively, implying that a higher R&D intensity is associated with a lower credit risk. Consistent with this finding, the credit spread also exhibits a negative correlation with R&D intensity. The regression results indicate that a one-standard-deviation increment of R&D intensity will lead to a 7.79 bps decrease in credit spread.

Overall, firms with a higher R&D intensity tend to obtain a higher expected return

in the stock market. On the other hand, given their lower debt level, they tend to exhibit less credit risk in the bond market. These empirical findings motivate my model, which rationalizes the heterogeneity of cross-sectional asset prices and firm dynamics driven by R&D intensity.

2.4 Model

This section presents a two-sector DSGE model with endogenous technology adoption and optimal default and quantitatively investigate R&D and the cross-section heterogeneity. There are a measure of one firms in each of the two sectors of the economy, the R&D sector and the non-R&D sector. Firms in both sectors can finance with equity and defaultable corporate bonds. Within the non-R&D sector, firms strategically compete in the product market and produce intermediate goods, which are used as inputs by a representative final good producer in this economy. Firms in the R&D sector, however, adopt and commercialize raw ideas which arrive at the economy exogenously. The transformation of these ideas generates endogenous productivity growth in the economy. I start by specifying the final good producer and then describe the non-R&D and the R&D sectors. Finally, I describe the representative household's behavior.

2.4.1 Final Good Producer

The representative final good producer generates the final goods and behaves competitively in the economy. It is a bundle of non-R&D sector intermediate goods, using the CES technology

$$Y_t = \left(\int_0^1 Y_{iN,t}^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}} \tag{2.1}$$

The elasticity of substitution is controlled by ϵ. Profit maximization by the final firm yields a downward-sloping demand curve for each intermediate firm

$$Y_{iN,t} = \left(\frac{P_{iN,t}}{P_t}\right)^{-\epsilon} Y_t \tag{2.2}$$

and the associated price index

$$P_t = \left(\int_0^1 P_{iN,t}^{1-\epsilon} di\right)^{\frac{1}{1-\epsilon}} \tag{2.3}$$

2.4.2 Non-R&D Sector

There is a continuum of intermediate good firms $i \in [0, 1]$ in non-R&D sector, each producing a differentiated good, denoted as $Y_{iN,t}$. Each intermediate good firm i is equipped with the following production technology

$$Y_{iN,t} = \left(K_{iN,t}^{\alpha}(\Omega_t L_{iN,t})^{1-\alpha}\right)^{1-\xi} G_{i,t}^{\xi} \tag{2.4}$$

where $G_{i,t}$ is a composite of sub-intermediate goods that are produced by R&D sector and is defined as

$$G_{i,t} = \left[\int_0^{A_t} X_{ij,t}^{\nu} dj\right]^{\frac{1}{\nu}} \tag{2.5}$$

$X_{ij,t}$ denotes intermediate good firm i's demand of sub-intermediate good $j \in [0, A_t]$, and the total mass of sub-intermediate good varieties is denoted by A_t. Since each sub-intermediate good requires a patent to produce, it is natural to interpret A_t as the amount of adopted raw ideas currently available. The physical capital share is denoted by α, and the share of the bundled sub-intermediate goods is denoted by ξ. The elasticity of substitution among sub-intermediate goods is $\frac{1}{1-\nu}$. The stationary process $\Omega_t = e^{a_t}$ constitutes the exogenous component of the measured productivity.

I assume a_t follows an AR(1) process

$$a_t = \rho a_{t-1} + \epsilon_t \tag{2.6}$$

where $\epsilon_t \sim N(0, \sigma_a)$. As we will see later, the measured productivity also contains an endogenous element which is linked to the adopted technologies A_t.

Operating profits of the intermediate good firm. Intermediate good firm i's operating profits can be written as

$$\Pi_{iN,t} = P_{iN,t}Y_{iN,t} - W_{N,t}L_{iN,t} - \int_0^{A_t} P_{ij,t}X_{ij,t}dj - z_{iN,t}\overline{K}_{N,t} \tag{2.7}$$

where $W_{N,t}$ is the wage rate and the term $\int_0^{A_t} P_{ij,t}X_{ij,t}dj$ captures the expense associated with using sub-intermediate goods as inputs. Moreover, following Gomes et al. (2016), I introduce an i.i.d. idiosyncratic shock $z_{iN,t}$ with zero mean. The overall impact of this shock is $z_{iN,t}\overline{K}_{N,t}$, where $\overline{K}_{N,t}$ is non-R&D-sector average capital stock and is needed to guarantee that this cost shock does not diminish along the balanced growth path. This cash flow shock can be thought of as a random fixed cost incurred by the firm each period. It provides a parsimonious summary of the firm-specific component of overall business risk. The reason to introduce such costs is to create a continuous distribution of intermediate good firm values such that some of the firms will default after being hit by a large realization of such shock. Moreover, the i.i.d. nature of the shock and the way it enters the firm's problem simplify aggregation in a tractable way.

In the discussion below $\Phi_N(\cdot)$ and $\phi_N(\cdot)$ will be used to represent the cumulative distribution function (c.d.f.) and the probability density function (p.d.f.) of the idiosyncratic shock $z_{iN,t}$, whose support is $[\underline{z}, \overline{z}]$.

Debt Financing. Intermediate good firms can issue corporate bonds to exploit the tax shield of US tax code, since the interest payments of corporate bonds are tax-deductible. However, issuing such bonds also exposes firms to expensive default. Therefore, there is a trade-off of tax benefits and default costs that firms must take into account when choosing their optimal capital structure. Furthermore, I assume there is an upper bound which limits the total amount of corporate bonds a firm can issue. Each period, equity holders own the option to deny injecting additional funds into the firm and declare bankruptcy. Once the firm goes bankrupt, shareholders leave with zero payoff due to limited liability. On the other hand, creditors go through the restruction process by incurring a one-time expenses and take charge of the restructured firm.

If default doesn't happen, the firm is required to pay back the debt obligation:

$$((1-\tau)C+1)B_{iN,t}$$

where C is coupon rate, τ is tax rate, and $B_{iN,t}$ is the principal. The fund raised by issuing new corporate bonds is

$$Q_{iN,t}B_{iN,t+1}$$

where $Q_{iN,t}$ denotes the price of one unit bond and $B_{iN,t+1}$ is the amount of one period bond firm i chooses to issue at time t. Moreover, each firm i faces a borrowing limit imposed by the bond holders

$$B_{iN,t+1} \leq \lambda_{N,t+1}\overline{K}_{N,t+1} \tag{2.8}$$

Note that this constraint does not guarantee that the debt is risk-free. As illustrated later, default is induced by a sufficiently large idiosyncratic cash flow shock. Once

default happens, the total enterprise value will be smaller than the total debt claim. Hence, corporate bonds are risky assets in the model. The net cash flow generated by issuing corporate bonds is

$$Q_{iN,t}B_{iN,t+1} - ((1-\tau)C + 1)B_{iN,t} \tag{2.9}$$

Investment The intermediate good firm i accumulates physical capital for next period's production by investment $I_{iN,t}$. The physical capital obeys the following evolution:

$$K_{iN,t+1} = (1-\delta)K_{iN,t} + \Gamma\left(\frac{I_{iN,t}}{K_{iN,t}}\right)K_{iN,t} \tag{2.10}$$

The wear and tear is captured by parameter δ. There exists frictions when transforming the investment into physical capital, which is captured by the adjustment cost function Γ. The net cash flow is

$$-I_{iN,t} + \tau\delta K_{iN,t} \tag{2.11}$$

where the second component captures the tax deductibility of physical capital depreciation.

Intermediate Good Firm's Problem. The dividend of intermediate good firm consists of firm's after-tax operating profit as well as the net cash flows from firm's debt financing and investment:

$$D_{iN,t} = (1-\tau)\Pi_{iN,t} - I_{iN,t} + \tau\delta K_{iN,t} + Q_{iN,t}B_{iN,t+1} - ((1-\tau)C + 1)B_{iN,t} \tag{2.12}$$

The intermediate good firm's objective is to make the hiring, purchasing, investment and financing decisions in order to maximize its value for the shareholders:

$$V^E_{iN,t}(K_{iN,t}, B_{iN,t}, z_{iN,t}, S_t) = \max\left(0, V_{iN,t}(K_{iN,t}, B_{iN,t}, z_{iN,t}, S_t)\right) \tag{2.13}$$

where $V_{iN,t}$ is firm's continuation value, that is, the value of equity without the option to default at time t, which is the sum of the discounted value of future dividend streams. The aggregate states that firms take exogenously are collected into S_t. The limited liability of shareholders is implied by the maximization operator. Intuitively, shareholders find it optimal to keep operating the firm when its value remains positive. When the firm's value becomes negative, it is optimal for shareholders to declare bankruptcy. Under this situation, due to the limited liability, shareholders' payoff is zero.

Default Decision. The default decision in this model is essentially determined by the idiosyncratic shock $z_{iN,t}$ and obeys a threshold rule. Denoting by $z^\star_{N,t}$ the level of idiosyncratic shock at which the intermediate good firm's continuation value is exactly zero, i.e. $V_{iN,t}(K_{iN,t}, B_{iN,t}, z^\star_{N,t}, S_t) = 0$. It is clear that the firm becomes insolvent whenever the realized cash flow shock $z_{iN,t} > z^\star_{N,t}$, i.e. the negative cash flow shock is adequately large. Otherwise the firm keeps operating. Note that this setting implicitly links the default threshold to the firm's continuation value and thus the aggregate economic conditions, making the countercyclical default probability likely.

Debt Value In default, the bond holder goes through the restructuring process and acquires the firm after incurring a one-time loss accounting for a proportion Ξ_N of the firm value. The loss may reflect the various frictions in the restructuring process. Intuitively, the bond price is affected by the default probability and the cost

of bankruptcy. This is reflected in the following no-arbitrage bond pricing formula:

$$Q_{iN,t}B_{iN,t+1} = E_t M_{t+1}\left(\Phi_N(z^{\star}_{N,t+1})(C+1)B_{iN,t+1} + (1-\Xi_N)\int_{z^{\star}_{N,t+1}}^{\overline{z}} V_{iN,t}(K_{iN,t+1},0,z_{iN,t+1},S_{t+1})d\Phi_N\right) \tag{2.14}$$

The LHS is the cost of the bond holders at time t. The RHS reflects the risk-adjusted expected payoff at time $t+1$, which consists of two terms. The first term corresponds to the surviving situation where the bond holder receives the full repayment. The second term corresponds to the default situation where only partial repayment could be retrieved. Furthermore, since the cash flow shocks $z_{iN,t}$ are i.i.d. and enter the firms' problems in an additive way, all intermediate good firms in default will make identical decisions as those non-defaulting ones after restructuring. Therefore, every intermediate good firm in the non-R&D sector looks ex ante identical at the start of each period.

Intermediate Good Firm's Optimal Decision. Here I inspect optimality conditions related to the intermediate good firm's problem and discuss the corresponding economic intuitions.

The optimal investment decision corresponds to the following investment Euler equation:

$$\Lambda_{iN,t} = \frac{\partial Q_{iN,t}}{\partial K_{iN,t+1}}B_{iN,t+1} + E_t\left[M_{t+1}\int_{\underline{z}}^{z^{\star}_{N,t+1}}\frac{\partial V_{iN,t+1}}{\partial K_{iN,t+1}}d\Phi\right] \tag{2.15}$$

The left-hand side of the equation is the marginal cost of investing in additional unit physical capital today. Increasing physical capital facilitates the issuance of the corporate debt, since it increases the firm value and the bond price per unit, as reflected in the two terms on the right-hand side of the equation. In contrast, the

equilibrium condition for capital structure is

$$Q_{iN,t} = -\frac{\partial Q_{iN,t}}{\partial B_{iN,t+1}} B_{iN,t+1} - E_t \left[M_{t+1} \int_{\underline{z}}^{z^\star_{N,t+1}} \frac{\partial V_{iN,t+1}}{\partial B_{iN,t+1}} d\Phi \right] + \psi_{iN,t} \tag{2.16}$$

The LHS is the cash flow collected by issuing one unit of bond today, which is also equal to the bond price $Q_{iN,t}$. However, issuing more debt also increases the default probability and decreases bond price, which is recognized by the first term of the right-hand side. Moreover, issuing more debt also leads to a reduction in the expected future equity value, as reflected by the second term. The third term is the Lagrange multiplier associated with the borrowing limit.

The demand of intermediate good firm i for the sub-intermediate good j is given as

$$P_{ij,t} = P_{iN,t} \left(1 - \frac{1}{\epsilon} \right) \left(K^{\alpha}_{iN,t} (\Omega_t L_{iN,t})^{1-\alpha} \right)^{1-\xi} \xi \left[\int_0^{A_t} X^{\nu}_{ij,t} dj \right]^{\frac{\xi}{\nu}-1} X^{\nu-1}_{ij,t} \tag{2.17}$$

where the price $P_{ij,t}$ is set by the R&D sector.

Output and Balanced Growth. Using equilibrium conditions and a parametric restriction $\alpha + \frac{\xi-\xi}{1-\xi} = 1$, imposed to guarantee balanced growth path of the economy, the equilibrium production function equation (2.4) can be rewritten as

$$Y_{iN,t} = K^{\alpha}_{iN,t} (\overline{A} A_t \Omega_t L_{iN,t})^{1-\alpha} \tag{2.18}$$

We can see that the measured productivity process contains two dynamic components, one of which is exogenous Ω_t and the other endogenous A_t. Importantly, Ω_t is stationary while A_t, the stock of adopted patents in this economy, grows at an endogenous rate through the endogenous technology adoption mechanism. In other words, it is the endogenous component of the productivity process that generates sustained growth of the economy in the model.

2.4.3 R&D Sector

There is a continuum of parent firms $i \in$[0,1] in the R&D sector. Within each parent firm, there are two additional subsectors: the sub-intermediate good subsector and the adoption subsector. The sub-intermediate good subsector of parent firm i further contains a continuum of firms $j \in [0, A_t]$ competing monopolistically, each of which produces a differentiated sub-intermediate good. There is one representative firm in the adoption subsector which tries to adopt the raw ideas into usable technologies and behaves competitively.

Sub-intermediate Good Subsector. Sub-intermediate goods are generated by sub-intermediate good firms with monopoly power. Each sub-intermediate good firm $j \in [0, A_t]$ can generate one unit of its own product by purchasing one unit of consumption good, under the authorization of a specific patent. Taking the demand schedule set by the intermediate good firm in the non-R&D sector as given, each sub-intermediate good firm j aims to maximize its own profit through solving the following simple problem:

$$\Pi_{ij,t} = \max_{P_{ij,t}} P_{ijt} \cdot X_{ij,t}(P_{ij,t}) - X_{ij,t}(P_{ij,t}) \tag{2.19}$$

The firm's value is thus the sum of the maximized profit and the continuation value:

$$v_{ij,t} = \pi_{ij,t} + \phi E_t \left[M_{t+1} v_{ij,t+1} \right] \tag{2.20}$$

Note that the survival probability of a patent ϕ shows up because each sub-intermediate good requires a patent to produce. In this sense, the sub-intermediate good firm's value essentially represents the value of of holding a specific patent. Here, the positive patent value is an important feature of the endogenous growth model, because it provides rents to the firm in the adoption subsector and incentivizes it to adopt the raw ideas into usable technologies.

In the symmetric equilibrium this paper considers, each sub-intermediate good producer makes the same decision, leading to

$$P_{ij,t} = \frac{1}{\nu} \tag{2.21}$$

$$X_{ij,t} \equiv X_t \tag{2.22}$$

$$v_{ij,t} \equiv v_t \tag{2.23}$$

$$\pi_{ij,t} \equiv \pi_t \tag{2.24}$$

Adoption Subsector. Each period, there will be an exogenous arrival of raw ideas into the economy. These raw ideas cannot be immediately used for production, but must first be transformed by the representative firm in the adoption subsector. The process of transforming raw ideas into patents with additional resource inputs is referred to technology adoption. Each raw idea can only be successfully adopted at certain probability that is endogenously determined by the resource inputs. Hence, for the representative firm in the adoption subsector, technology adoption is a time-consuming, uncertain, and costly process.

Denote by Z_t the amount of raw ideas at time t. In line with Comin et al. (2009), it is assumed to obey the following evolution:

$$Z_{t+1} = (\overline{\chi}\chi_t + \phi)Z_t \tag{2.25}$$

Note that Z_t can also be interpreted as the technological frontier of the economy. ϕ is the survival rate of a raw idea. The growth rate of this frontier is driven by the stochastic process χ_t, which is assumed to follow an AR(1) process:

$$\log \chi_t = \rho_\chi \log \chi_{t-1} + \epsilon_t^\chi \tag{2.26}$$

where ρ_χ is persistence parameter and $\epsilon_t^\chi \sim N(0, \sigma_\chi)$ is a normally distributed shock. ϵ_t^χ is labeled as the innovation shock in the model since it drives the movement of the technology frontier in the economy. The parameter $\overline{\chi}$ further affects the impact of the innovation shock on the stock of raw ideas in the economy.

Importantly, through the technology adoption, the future growth prospect of the economy links tightly to the exogenously evolving technological frontier. In this sense, the innovation shock ϵ_t^χ is connected to the conventional news shock described in the literature, since it provides information about the future path of technological frontier, and thus the growth capacity of the economy. Unlike the conventional news shocks which directly boosts the productivity of the economy next period, a positive innovation shock here does not improve productivity unless the associated newly arrived raw ideas are successfully adopted. Moreover, since the propagation of the impact of innovation shock into the economy is through the endogenous technology adoption, R&D sector has direct exposure to this shock. In other words, the model setting naturally generates heterogeneous exposure of the innovation shock between R&D sector and non-R&D sector.

Regarding technology adoption, the representative firm in the adoption subsector employs resources in an attempt to make unexplored ideas productive and behaves competitively. Each period, the firm needs to choose the optimal amount of expenditure H_t inputted for technology adoption by solving the following Bellman equation

$$J_{i,t} = \max_{H_{i,t}} \{-H_{i,t} + \phi E_t[M_{t+1}(\lambda(H_{i,t})v_{t+1} + (1 - \lambda(H_{i,t}))J_{i,t+1})]\} \tag{2.27}$$

where $J_{i,t}$ is the value of a raw idea. The value contains two parts. The first part $-H_{i,t}$ is the amount of resources inputted for the current period. And the second part is related to the continuation value. With the endogenous probability $\lambda(H_{i,t})$ the raw idea is successfully transformed into a patent and the firm can sell it on the open market at the price v_{t+1}. On the other hand, if the technology adoption is not successful, which has probability $1 - \lambda(H_{i,t})$, the raw idea will be kept into the next

period. Note that the continuation value is further multiplied by ϕ, the survival rate of a patent or a raw idea.

Given the optimal resource inputted for technology adoption, the stock of adopted patents $A_{i,t+1}$ obeys the following evolution

$$A_{i,t+1} = \lambda(H_{i,t})\phi[Z_t - A_{i,t}] + \phi A_{i,t} \tag{2.28}$$

It contains two parts. The first part is the newly adopted patents that will survive into the next period, and the second part represents the old patents that will survive into the next period.

Operating Profits of Parent Firm. Parent firm i's aggregate before-tax operating profits are the sum of the profits generated by the sub-intermediate good subsector and the adoption subsector, subtracted by the realization of an i.i.d. idiosyncratic shock $z_{iR,t}$ with zero mean:

$$\Pi_{iR,t} = \pi_t A_{i,t} - H_{i,t}(Z_t - A_{i,t}) - z_{iR,t}\overline{A}_t \tag{2.29}$$

where $\overline{A}_t$ denotes the average stock of adopted patents in the R&D sector. The idiosyncratic shock is located on the support $[\underline{z}, \overline{z}]$, and has a c.d.f. and p.d.f. of $\Phi_R(\cdot)$ and $\phi_R(\cdot)$ respectively. I assume that the idiosyncratic shocks in the R&D and non-R&D sectors follow the same distribution, specifically $\Phi_R(\cdot) = \Phi_N(\cdot)$ and $\phi_R(\cdot) = \phi_N(\cdot)$. Again, the introduction of the idiosyncratic shock into R&D sector is to generate variations among the parent firms' values so that some parents firms default while others keep operating.

Debt Financing of Parent Firm. As is the case in the non-R&D sector, each parent firm in the R&D sector can issue defaultable corporate bonds. The firm's net

operating cash flow from issuing corporate bonds is

$$Q_{iR,t}B_{iR,t+1} - ((1-\tau)C+1)B_{iR,t} \tag{2.30}$$

that is, it collects funds raised $Q_{iR,t}B_{iR,t+1}$ and pays for the debt obligation including coupon and principal. The term $(1-\tau)C$ indicates that interest payments are tax deductible according to the US tax code. Firms in the model issue corporate debts to exploit such tax benefits.

Similar to the non-R&D sector, each parent firm in the R&D sector face the following borrowing limit imposed by the creditors

$$B_{iR,t+1} \leq \lambda_{R,t+1}\overline{A}_{t+1} \tag{2.31}$$

Note this constraint does not guarantee that the debt is risk-free. As in the non-R&D sector, default is triggered by a sufficiently large idiosyncratic cash flow shock in the R&D sector. Once default happens, the total enterprise value will be smaller than the total debt claim, making the corporate bonds risky.

Parent Firm's Problem. The parent firm i's dividend is the sum of its after-tax operating profit and the cash flow from the debt financing:

$$D_{iR,t} = (1-\tau)\Pi_{iR,t} + Q_{iR,t}B_{iR,t+1} - ((1-\tau)C+1)B_{iR,t} \tag{2.32}$$

Parent firm's objective is to maximize its firm value for the shareholders:

$$V^E_{iR,t}(B_{iR,t}, z_{iR,t}, S_t) = \max\left(0, V_{iR,t}(B_{iR,t}, z_{iR,t}, S_t)\right) \tag{2.33}$$

If default happens, creditors will take charge of the firm after incurring a deadweight loss and shareholders will leave with zero payoff, which is captured by the first term on the right-hand of the equation. If default doesn't happen, the positive firm value

will be captured by the second term of the equation $V_{iR,t}(B_{iR,t}, z_{iR,t}, S_t)$, which is the firm's continuation value, that is, the discounted future dividend streams in the non-default states.

Default Decision. As in the situation encountered by firms in the non-R&D sector, the default decision is essentially a threshold rule, which is determined by the idiosyncratic shock $z_{iR,t}$. Whenever the realized idiosyncratic cash flow shock $z_{iR,t}$ is so severe that its magnitude is larger than a certain level $z^{\star}_{R,t}$, shareholders will execute the default option and declare bankruptcy. The threshold is pinned down by solving $V_{iR,t}(B_{iR,t}, z^{\star}_{R,t}, S_t) = 0$. It is the level of the idiosyncratic shock at which shareholders find it indifferent to default or keep operating.

Debt Value. Corporate bond value $Q_{iR,t}$ depends on the default probability and the cost of default. Its price satisfies the following equation:

$$Q_{iR,t}B_{iR,t+1} = E_t M_{t+1}\left(\Phi_R(z^{\star}_{R,t+1})(C+1)B_{iR,t} + (1-\Xi_R)\int_{z^{\star}_{R,t+1}}^{\overline{z}} V_{R,t}(0, z_{iR,t+1}, S_{t+1})d\Phi_R\right) \tag{2.34}$$

The LHS is the cost of the bond holders at time t. The RHS reflects the risk-adjusted expected payoff at time $t+1$, which contains two terms. The first term in the brackets is the payoff outside of default, multiplied by the survival probability. The payoff consists of principal and coupon payment. The second term, which is the payment, shows that creditors recover $1-\Xi_R$ proportion of firm value at default.

Parent Firm's Optimal Decision. The equilibrium condition for the bond financing is

$$Q_{iR,t} = -\frac{\partial Q_{iR,t}}{\partial B_{iR,t+1}}B_{iR,t+1} - E_t\left[M_{t+1}\int_{\underline{z}}^{z^{\star}_{R,t+1}}\frac{\partial V_{iR,t+1}}{\partial B_{iR,t+1}}d\Phi\right] + \psi_{iR,t} \tag{2.35}$$

The cash flow collected by issuing one unit of bond today is $Q_{iR,t}$, which is offset by the decrease in the bond price and future equity value. The first two terms on the RHS indicate that shareholders rationally take into account their choices on the bond price and expected future equity value. The third term on the RHS is the Lagrange multiplier associated with the borrowing limit.

2.4.4 Households

The representative household equipped with Epstein-Zin preferences tries to maximize its utility

$$U_t = \left[(1-\beta)\tilde{C}_t^{1-\frac{1}{\psi}} + \beta(E_t[U_{t+1}^{1-\gamma}])^{\frac{1-\frac{1}{\psi}}{1-\gamma}}\right]^{\frac{1}{1-\frac{1}{\psi}}} \tag{2.36}$$

Here I use parameter γ to denote the relative risk aversion. Besides, the parameter ψ controls the household's incentive to smooth consumption (E.I.S). When $\gamma = \frac{1}{\psi}$, the preference collapses to the standard time-additive preference and agents will not care about the news about long run growth prospects. Following long run risk literature, I assume $\gamma > \frac{1}{\psi}$, reflecting a preference for early resolution of uncertainty of the household. Hence, any low frequency movement of the future growth rate will be disliked. The consumption bundle $\tilde{C}_t$ is defined as

$$\tilde{C}_t = C_t^{\varphi}[A_t(3-L_t)]^{1-\varphi} \tag{2.37}$$

where C_t denotes the consumption of the household and L_t is the labor supply.
The representative household supplies labor and participates in the financial market to support its consumption stream. Specifically, it collects wages $W_{R,t}L_t$ from non-R&D sector. In addition, it invests in R&D-sector and non-R&D-sector stocks and in R&D-sector, non-R&D-sector, and government bonds. Accordingly, the budget

constraint can be written as follows:

$$\begin{aligned} & C_t + T_t + \mathcal{B}_{R,t+1} + \mathcal{B}_{N,t+1} + \mathcal{B}_{G,t+1} + \mathcal{V}_{R,t} + \mathcal{V}_{N,t} \\ = & W_{N,t}L_t + R_{BR,t}\mathcal{B}_{R,t} + R_{BN,t}\mathcal{B}_{N,t} + R_{f,t}\mathcal{B}_{G,t} + R_{R,t}\mathcal{V}_{R,t-1} + R_{N,t}\mathcal{V}_{N,t-1} \end{aligned} \tag{2.38}$$

where the total positions in R&D-sector equity and non-R&D-sector equity are denoted by $\mathcal{V}_{R,t}$ and $\mathcal{V}_{N,t}$, respectively, while $\mathcal{B}_{R,t+1}$, $\mathcal{B}_{N,t+1}$, and $\mathcal{B}_{G,t+1}$ respectively denote the total amount invested in R&D-sector corporate bond, non-R&D-sector corporate bonds, and government bonds. $R_{f,t}$ is the risk-free rate. Corporate bond returns and equity returns for both the R&D and non-R&D-sectors are also shown in the budget constraint and are defined in section 2.4.6 below. T_t is lump-sum government taxes.

The stochastic discount factor is standard:

$$M_{t,t+1} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-1} \left(\frac{\tilde{C}_{t+1}}{\tilde{C}_t}\right)^{1-\frac{1}{\psi}} \left(\frac{U_{t+1}}{E_t[U_{t+1}^{1-\gamma}]^{\frac{1}{1-\gamma}}}\right)^{\frac{1}{\psi}-\gamma} \tag{2.39}$$

where the third part seizes the continuation utility risk, the key feature distinguishing Epstein-Zin from standard time additive preference. In addition, the wage rate is determined as

$$W_{N,t} = \left(\frac{1}{\varphi} - 1\right) \frac{C_t}{(3 - L_t)} \tag{2.40}$$

2.4.5 Resource Constraint

I consider a symmetric equilibrium in which within each specific sector, firms are ex-ante identical. The final good output is used in the calculation of consumption, physical investment in the non-R&D sector, sub-intermediate goods production in the R&D sector, adoptions in the R&D sector, and resources lost incurred during the

restruction process:

$$Y_t = C_t + I_{N,t} + H_t(Z_t - A_t) + A_t X_t + \Xi_R \int_{z^\star_{R,t}}^{\overline{z}} V_{R,t}(0, z_{iR,t}, S_t) d\Phi_R + \Xi_N \int_{z^\star_{N,t}}^{\overline{z}} V_{N,t}(K_{iN,t}, 0, z_{iN,t}, S_t) d\Phi_N \tag{2.41}$$

where $I_{N,t} = \int_0^1 I_{iN,t} di$, $X_t = \int_0^1 X_{i,t} di$, $H_t = \int_0^1 H_{it} di$.
Furthermore, GDP can be defined as:

$$GDP_t = C_t + I_{N,t} + H_t(Z_t - A_t) \tag{2.42}$$

2.4.6 Equity Values and Asset Returns

Given symmetric equilibrium, the ex-dividend equity value of the intermediate good firm in non-R&D sector is defined as

$$V^{ex}_{N,t} = V_{N,t} - D_{N,t} \tag{2.43}$$

This equity value captures the market value of physical capital installed in the firm, as well as the expected discounted value of future monopoly profits due to monopolistic competition. Moreover, it contains the value brought by the tax shield, and the value destruction through the default.
In contrast, the ex-dividend equity value for the parent firm in the R&D sector can be decomposed into two components:

$$V^{ex}_{R,t} = V_{R,t} - D_{R,t} = V^{ex}_{R1,t} + (1 - \tau) V^{ex}_{R2,t} \tag{2.44}$$

where $V_{R2,t}^{ex}$ is used to denote the value created by the R&D activities. Based on a similar argument as in Comin et al. (2009), we can further decompose $V_{R2,t}^{ex}$ as follows:

$$V_{R2,t}^{ex} = (\tilde{V}_t - \Pi_t)A_t + (\tilde{J}_t + H_t)(Z_t - A_t) + \tilde{\xi}_t \tag{2.45}$$

where

$$\tilde{V}_t = \Pi_t + \phi E_t[M_{t+1}\Phi_R(z^{\star}_{R,t+1})\tilde{V}_{t+1}] \tag{2.46}$$

$$\tilde{J}_t = -H_t + E_t[M_{t+1}\Phi_R(z^{\star}_{R,t+1})(\lambda(H_t)\tilde{V}_{t+1} + (1-\lambda(H_t))\tilde{J}_{t+1})] \tag{2.47}$$

$$\tilde{\xi}_t = E_t[M_{t+1}\Phi_R(z^{\star}_{R,t+1})\tilde{J}_{t+1}(Z_{t+1} - \phi Z_t) + \tilde{\xi}_{t+1}] \tag{2.48}$$

The first term on the right-hand side of equation (2.45) captures the value of adopted raw ideas that are currently in use. This term effectively serves as the asset-in-place component. The second term is liked to the market value of existing raw ideas not transformed and is naturally interpreted as the growth option currently available. The last part denotes the value of raw ideas which will come in the following periods. The appearance of $\Phi_R(z^{\star}_{R,t+1})$ in equations (2.46)-(2.48) indicates that these values can only be captured when firms do not default.

The returns on equity in the R&D sector and in the non-R&D sector take into consideration the proportion of firms that default and are defined as follows:

$$R_{R,t} = \frac{\int_{\underline{z}}^{z^{\star}_{R,t}}[D_{R,t} + V_{R,t}^{ex}]d\Phi_R(z)}{V_{R,t-1}^{ex}} \tag{2.49}$$

$$R_{N,t} = \frac{\int_{\underline{z}}^{z^{\star}_{N,t}}[D_{N,t} + V_{N,t}^{ex}]d\Phi_N(z)}{V_{N,t-1}^{ex}} \tag{2.50}$$

Corporate bond returns in the two sectors can be defined as follows and also account for the firms that default each period:

$$R_{BR,t} = \frac{\Phi_R(z^{\star}_{R,t})(1+C)B_{R,t} + (1-\Xi_R)\int_{z^{\star}_{R,t}}^{\overline{z}} V_{R,t}(0,z,S_t)d\Phi_R(z)}{Q_{R,t-1}B_{R,t}} \tag{2.51}$$

$$R_{BN,t} = \frac{\Phi_N(z^\star_{N,t})(1+C)B_{N,t} + (1-\Xi_N)\int_{z^\star_{N,t}}^{\overline{z}} V_{N,t}(K_{N,t},0,z,S_t)d\Phi_N(z)}{Q_{N,t-1}B_{N,t}} \tag{2.52}$$

2.5 Quantitative Analysis

This section presents the model's quantitative performance based on simulations. I first describe the functional forms of adjustment cost, borrowing limits, and adoption probability, followed by a discussion of parameter choices. I then evaluate the model's ability to generate reasonable aggregate macroeconomic and asset pricing implications. Furthermore, I investigate if the model could deliver sensible cross-sectional heterogeneity between R&D sector and non-R&D sector.

2.5.1 Functional Forms

Based on Jermann (1998), the adjustment cost function of physical capital is given as:

$$\Gamma(x) = \frac{\alpha_1}{1-\frac{1}{\zeta_k}} x^{1-\frac{1}{\zeta_k}} + \alpha_2 \tag{2.53}$$

where ζ_k measures the curvature of the adjustment cost function, and α_1 and α_2 are chosen so that at the deterministic steady state, no investment friction exists.

The borrowing limits in R&D and non-R&D sectors are assumed to have the following law of motions:

$$\begin{aligned} \log\lambda_{R,t+1} - \log\lambda_{R,ss} &= \rho_l(\log\lambda_{R,t} - \log\lambda_{R,ss}) + \chi_l(g_{t+1} - g_{ss}) \\ \log\lambda_{N,t+1} - \log\lambda_{N,ss} &= \rho_l(\log\lambda_{N,t} - \log\lambda_{N,ss}) + \chi_l(g_{t+1} - g_{ss}) \end{aligned} \tag{2.54}$$

where $g_{t+1} = \frac{A_{t+1}}{A_t}$, and g_{ss} is the corresponding steady state. The adoption probability is specified as

$$\lambda(H_t) = \Lambda H_t^\kappa \tag{2.55}$$

where Λ is a scale parameter. Moreover, the sensitivity of adoption success probability

to adoption expenditure H_t is captured by the parameter κ.

2.5.2 Calibration

Table 4.1 presents the quarterly calibration of the baseline model. The model is solved by perturbation methods. Most of the parameters are chosen in a standard way. Consistent with the long run risk literature, I choose the elasticity of intertemporal substitution (EIS) to be 2. The subjective discount factor parameter β is 0.99, suggesting 1% risk-free rate per quarter at the steady state. The relative preference over consumption good φ in the consumption bundle is chosen to make the hours worked at the steady state equal to one third of the time endowment.

With respect to the firms in the non-R&D sector sector, the share of physical capital α is 0.35, the wear and tear parameter of physical capital δ is 0.02 per quarter, and the elasticity of substitution ϵ among intermediate goods in the non-R&D sector is 10. All of these are standard settings in the macroeconomic literature. The curvature of the adjustment cost function ζ_k is selected to bring about a volatile physical investment, as in the data. The parameters that determine the innovation activities are set in line with Comin and Gertler (2006) and Kung and Schmid (2015). More specifically, based on the BEA's estimation of annual depreciate rate of R&D capital, the non-obsolescent rate of a patent ϕ to set as 0.96 per quarter. The adoption scale parameter Λ controls the speed at which raw ideas are transformed into productive patents. The parameter choice implies that it on average takes five years to adopt an idea, consistent with the estimation in Comin and Gertler (2006). The markup related parameter ν is determined by the balanced growth path restriction, imposed as in Kung and Schmid (2015).

As mentioned above, I assume there is no heterogeneity in the cross-sectional idiosyncratic shock distribution, that is, $\Phi_R(\cdot) = \Phi_N(\cdot)$ and $\phi_R(\cdot) = \phi_N(\cdot)$. The idiosyncratic cash flow shock volatility $\sigma_R = \sigma_N$ is chosen to target Moody's average annual default probability of 1%. Following Chen (2010), I choose the bankruptcy cost Ξ to yield an

aggregate mean recovery rate of 40%. The tax rate is set to 0.36, consistent with the US tax code. The parameters related to borrowing limits are set to generate realistic leverage dynamics. Specifically, I set the steady state values of the borrowing limit of the R&D sector and the non-R&D sector to be 0.6 and 0.7 respectively, generating a leverage difference which is consistent with the data. Finally, the parameters related to productivity are calibrated in line with Comin and Gertler (2006) and Kung and Schmid (2015).

2.5.3 Results

This section first evaluates whether the model can generate reasonable aggregate macroeconomic, asset pricing, and credit moments. I then focus on the model implied cross-sectional heterogeneity in equity returns, leverage, default, and credit spreads between R&D and non-R&D firms, motivated by the empirical evidence.

Panel A of Table 4.2 documents the aggregate macroeconomic moments. Overall, the model does reasonably well in matching the aggregate macroeconomic moments. Specifically, the annual mean consumption growth is 2.18% and the average investment-to-GDP ratio of 20% in the model, both of which are close to the data. In addition, the fluctuations generated by the model, such as the fluctuation in consumption growth, GDP growth, and physical and R&D investment growth, are all consistent with those observed in the data. Moreover, the model also generates a realistic countercyclical cost of debt, captured by the negative correlation coefficient between GDP growth and aggregate credit spread.

In Figure 2.2, I illustrate the aggregate dynamics of the model, which investigates the impacts of the standard neutral technology shock ϵ_t and the innovation shock ϵ_t^{χ} on the quantities of interest through the lens of impulse response analysis. From the figure we can see that both neutral technology and innovation shock increase the productivity and thus the GDP growth of the economy. Although smaller in magnitude, the innovation shock's effect is much more persistent, reflecting the slowness

of the successful adoption of raw technological innovations. The higher persistence brought about by the innovation shock also leads to a more pronounced decrease in the stochastic discount factor, since the representative agent in this model is assumed to have recursive utility and to be averse to the uncertainty of future growth prospects. In other words, it is the innovation shock rather than the technology shock that generates a volatile stochastic discount factor. Moreover, the high productivity generated by either shock implies that firms have better profit opportunities, which boosts aggregate market value and leads to a positive realization of the market return. At the same time, the high market value decreases the default rates and also increases bond valuation. As a result, firms issue more bonds to exploit the tax benefits.

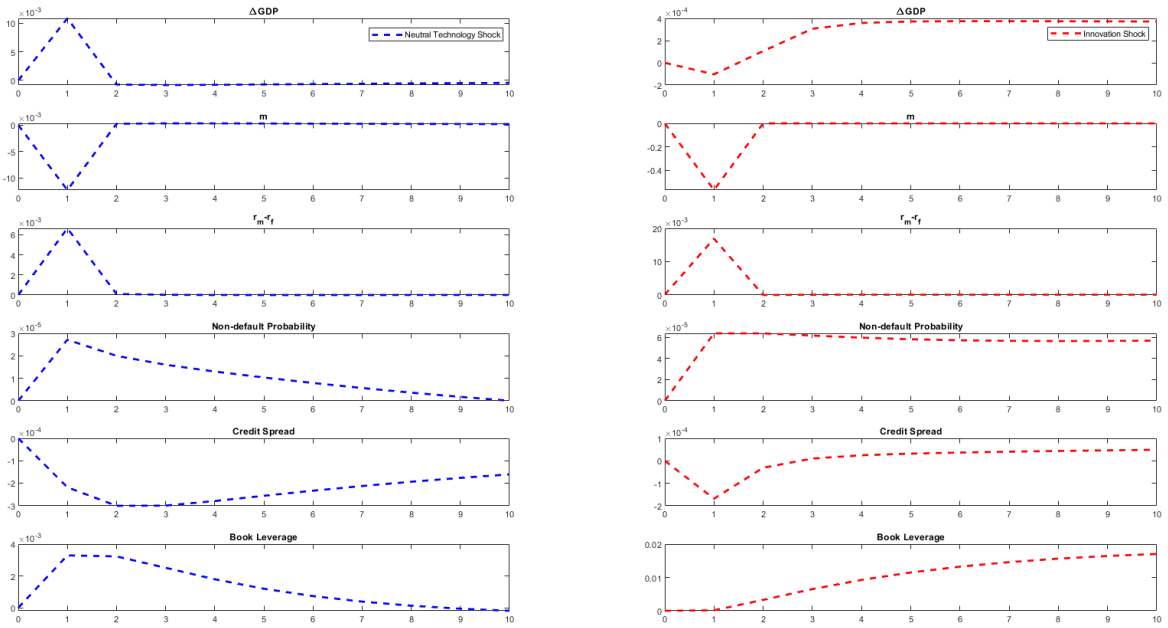

Figure 2.2: Impulse responses

Notes: The blue dashed line corresponds to the neutral technology shock and the red dashed line corresponds to the innovation shock. Lines are about the percentage deviation from steady state.

Panel B of Table 4.2 investigates the fit of the model around the aggregate asset

pricing and credit moments. The model performs reasonably well in quantifying the basic patterns of the asset market, generating a sizable equity risk premium of 3.90% per annum. The model underestimates the volatility of the excess equity return; however, this is not uncommon in the production-based general equilibrium asset pricing literature. The model also generates a stable risk-free rate consistent with the data. As for the aggregate credit moments, the model generates sizable aggregate credit spread with sizable variation. The model also matches the average annual default rate and the average bond recovery rate.

Next, I investigate the role of R&D in the cross-sectional heterogeneity of asset prices and credit moments (see Table 2.6). By comparing the simulated moments to the data, we can see that the model captures the heterogeneity observed in the data. First, the model generates a sizable difference in the equity risk premium between the R&D and non-R&D sectors, consistent with the data. This can be explained as follows. As discussed above, through the endogenous technology adoption mechanism, the innovation shock generates low-frequency movement in future growth prospects, that is, long run risk, which leads to a very volatile stochastic discount factor under the setting of Epstein-Zin preferences. Meanwhile, firms in the R&D sector inherently have high exposure to innovation shocks due to their direct participation in technology adoption. Specifically, innovation shocks will greatly affect the growth options of R&D firms, and hence these shocks naturally link the movement in the SDF and the value of the R&D sector, giving rise to a strong negative comovement between them. This strong negative comovement further leads to a high equity premium for firms in the R&D sector. On the other hand, firms in the non-R&D sector do not participate in technology adoption and have low exposure to the innovation shock. As a result, they tend to obtain a low equity premium.

Second, in Table 2.6 we note that in the model, R&D firms show lower leverage, default probability, and credit spreads than non-R&D firms, consistent with the data. This can be explained as follows. In the model, firms in the R&D sector face more

stringent borrowing limits imposed by the bond holders, consistent with reality. The more severe constraints faced by R&D firms could be due to the unstable cash flows induced by the high innovation risks. It may also reflect the market failures of R&D firms such as asymmetric information and moral hazards, due to the high uncertainty in the innovation activities. As a result, R&D firms tend to maintain a low amount of debt, which suppresses the default rates and credit spreads.

So far we have seen that the model generates cross-sectional heterogeneity in capital structures and asset prices consistent with our observations in the data. However, the coexistence of high equity returns and low leverage in R&D firms may still seem a little surprising, since a high equity return might be expected to be accompanied by a high leverage, the high return compensating for the firms' increased exposure to systematic risk from higher levels of leverage. Note that firms' equity risk can be affected both by their leverage and by their underlying business risks, that is, their asset betas. Hence, even as the low leverage of an R&D firm alleviates its exposure to systematic risk, the high level of business risk inherent in its innovation activities dominates over the low leverage and leads to an overall high equity return for the firm.

This idea is further substantiated by the following empirical test on the portfolio level. I compute the asset beta of R&D-intensive and R&D-nonintensive portfolios using regressions with two-year rolling windows. The results are reported in the first row of Table 2.7. The asset beta of the R&D intensive portfolio is almost twice as high as that of the R&D-nonintensive portfolio, and the difference is extremely significant. The second panel of Table 2.7 shows that similar results obtain in regressions that use the full sample. This finding implies that the level of business risk may dominate the low leverage in R&D firms' equity returns. Generally speaking, the innovation risk incurred by R&D activities plays the crucial role in shaping R&D firms' characteristics.

2.6 Concluding Remarks

This paper introduces a two-sector production-based dynamic stochastic general equilibrium model to study the interaction between R&D activities and firm heterogeneity. Specifically, it rationalizes the empirical finding that R&D-intensive firms tend to show higher expected equity returns, yet lower leverage, default rates, and credit spreads, than the R&D-nonintensive firms.

The model also explains why low leverage coexists with high equity returns in firms with high R&D intensity. Although the low leverage signals a low risk in the equity market, the strong uncertainty underlying the R&D activities plays a dominant role in pricing and is considered by the household with Epstein-zin preferences as a large risk that needs to be compensated by a high equity return.

Generally speaking, the model generates sizable heterogeneity as in the data and fits the aggregate moments reasonably well. However, the model is silent on other related factors such as liquidity risk and taxation. Future studies could examine the impact of such factors, and in so doing further elucidate the effects of R&D activities.

Table 2.1: Top 10 Industries in R&D Sorted Portfolios

Low R&D Industries		
SIC	Category	Percentage (%)
5812	Eating Places	10.13
5411	Grocery Stores	3.40
5331	Variety Stores	3.08
7990	Miscellaneous Amusement and Recreation	2.97
5311	Department Stores	2.68
5651	Family Clothing Stores	2.68
5621	Women's Clothing Stores	2.60
5961	Catalog and Mail-Order Houses	2.50
7011	Hotels and Motels	2.04
5940	Miscellaneous Shopping Goods Stores	1.98
	Total	34.06

High R&D Industries		
SIC	Category	Percentage (%)
2836	Biological Products, except Diagnostic Substances	12.49
2834	Pharmaceutical Preparations	11.96
7372	Prepackaged Software	11.56
3674	Semiconductors and Related Devices	5.38
2835	In Vitro and In Vivo Diagnostic Substances	3.36
3845	Electromedical and Electrotherapeutic Apparatus	3.10
3576	Computer Communications Equipment	2.70
3661	Telephone and Telegraph Apparatus	2.63
3841	Surgical and Medical Instruments and Apparatus	2.53
7373	Computer Integrated Systems Design	2.40
	Total	58.11

Notes: This table shows the top-10 industries in the baseline high and low R&D-sorted portfolios. I count SIC codes across time and firms in each portfolio and report the most frequent industries within each portfolio.

Table 2.2: R&D Sorted Portfolio Summary Statistics

Variable	High-R&D	Low-R&D	HML-R&D
Equity Return (%)	24.57	15.29	9.28^{**}
Leverage	0.14	0.26	-0.12^{***}
Default Rate (Moody)	3.33	4.08	-0.76^{***}
Default Rate (EDF)	0.07	0.16	-0.10^{***}
Credit Spread (bps)	251.71	347.14	-95.44^{***}

Notes: This table shows the summary statistics in the high and low R&D-intensive portfolios and the differences between these two portfolios. Equity returns are in annualized percentage. Book leverage is defined as the ratio of debt to total asset. The first measure of default rate is Moody's rating, converted into numerical values. The second measure of default rate is expected default frequency (EDF), developed first by Merton (1974). Credit spread is in annualized basis point, defined as the difference of yield to maturity between the corporate bond and the corresponding treasury bond. *, **, and *** stands for statistical significance at the 10%, 5%, and 1% level, respectively.

Table 2.3: Panel Regressions

	Equity Return	Leverage	Default Rate (Moody)	Default Rate (EDF)	Credit Spread
$\beta_{R\&D}$	0.02^{**}	-0.27^{***}	-1.06^{**}	-0.05^{***}	-4.10^{**}
	(2.09)	(-4.50)	(-2.55)	(-3.01)	(-2.26)
N	579335	40484	266379	441947	291108
R^2	0.01	0.19	0.59	0.27	0.41
Controls	Yes	Yes	Yes	Yes	Yes
Time FE	Yes	Yes	Yes	Yes	Yes
Industry FE	Yes	Yes	Yes	Yes	Yes

Notes: This table documents the firm level panel regressions to study the linkage between R&D intensity and equity return, leverage, measures of default rates, credit spread respectively. All standard errors are two way clustered by industry and year. The t-stats are reported in the parentheses below the coefficient estimates. *, **, and *** stands for statistical significance at the 10%, 5%, and 1% level, respectively.

Table 2.4: Benchmark Calibration

Preferences		
Intertemporal Elasticity of Subsitution	ψ	2
Discount Factor	β	0.99
Relative Risk Aversion	γ	20
Innovation		
Technological Innovation Scale Parameter	$\bar{\chi}$	0.045
Scale Parameter	Λ	3.268
Adoption Success Elasticity	κ	0.80
Patent Survival Rate	ϕ	0.96
Production		
Elasticity of Intermediate Goods	ϵ	10
Depreciation	δ	0.02
Adjustment Cost	ζ_k	10
Markup Inverse	ν	0.616
Capital Share	α	0.35
Patent Share	ξ	0.51
Productivity		
Neutral Technology Volatility	σ_a	0.012
Innovation Volatility	σ_χ	0.012
Neutral Technology Persistence	ρ	$0.6561^{1/4}$
Innovation Persistence	ρ_χ	$0.9606^{1/4}$
Finance		
Tax Rate	τ	0.36
Coupon Rate	C	0.07/4
Borrowing Limit R&D Sector	$\lambda_{R,ss}$	0.6
Borrowing Limit Non-R&D Sector	$\lambda_{N,ss}$	0.7
Borrowing Limit Persistence	ρ_l	0.8
Borrowing Limit Sensitivity	χ_l	8
Bankruptcy Loss	Ξ	0.20
Idiosyncratic Shock Volatility	$\sigma_R = \sigma_N$	1.345

Notes: This table summarizes the calibration at a quarterly frequency

Table 2.5: Aggregate Moments

Variable	Data	Model
A. Macro Moments		
$E(\Delta C)$ (%)	2.00	2.18
$\sigma(\Delta C)$ (%)	1.66	1.67
$E[I/GDP]$	0.20	0.20
$\sigma(\Delta C)/\sigma(\Delta GDP)$	0.64	0.68
$\sigma(\Delta I_N)/\sigma(\Delta C)$	4.38	4.49
$\sigma(\Delta I_R)/\sigma(\Delta C)$	1.87	1.85
$ACF(\Delta C)$	0.32	0.38
$\rho(\Delta C, \Delta I)$	0.39	0.85
$\rho(cs, \Delta GDP)$	-0.36	-0.43
B. Asset Pricing and Credit Moments		
$E[R_M - R_f]$ (%)	5.84	3.90
$\sigma[R_M - R_f]$ (%)	17.87	3.91
$\sigma[R_f]$ (%)	0.67	0.60
$E(cs)$ (bps)	90	61
$\sigma(cs)$ (bps)	42	25
Default Probability (%)	1	1.04
Average Bond Recovery Rate (%)	40	39.66

Notes: This table documents the annualized aggregate macroeconomic, asset pricing and credit moments.

Table 2.6: Cross Sectional Moments

Variable	R&D	Non-R&D	HML-R&D	
			Data	Model
Excess Equity Return (%)	4.33	3.49	9.28 (4.03)	0.84
Leverage	0.62	0.72	-0.12 (0.014)	-0.10
Default Probability (%)	0.47	1.61	-0.59 (0.14)	-1.14
Credit Spread (bps)	24.10	77.92	-95.44 (15.31)	-53.82

Notes: This table documents the model simulated moments of excess equity return, leverage, default probability, credit spread and excess bond return of R&D sector and non-R&D sector. HML-R&D Model is the difference of first two columns while the HML-R&D Data corresponds to the difference of R&D-intensive portfolio and nonintensive portfolio in the univariate portfolio sorting. Numbers in the brackets are standard errors. Returns and default probability are in annualized percentages while credit spread is in annualized base points.

Table 2.7: Asset Beta

	High R&D	Low R&D	HML-R&D	P Value of Diff
$\beta_{rolling}$	1.94^{***}	1.03^{***}	0.91	0.00
$\alpha_{rolling}$	0.0037	-0.0003	0.0040	0.22
β_{all}	1.87^{***}	1.00^{***}	0.87	-
α_{all}	0.0032	-0.0002	0.0035	-

Notes: This table reports the estimates of the asset beta of R&D-intensive and R&D-nonintensive portfolios as well as their differences. The results in the first and second rows are estimated by regressions using two-year rolling window. The results in the third and fourth rows are estimated by regressions using the full sample directly.

Chapter 3

Recursive Allocations and Wealth Distribution with Multiple Goods: Existence, Survivorship, and Dynamics

R. Colacito, M. M. Croce, Zhao Liu[1]
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3.1 Introduction

In the context of single-agent economies, recursive preferences have become increasingly relevant for the analysis of issues at the forefront of the macro-finance agenda (see, among others, Hansen and Sargent (1995), Tallarini (2000), Bansal and Yaron (2004), and Backus et al. (2005)). In models populated by multiple agents, in contrast, the adoption of recursive preferences is less common, as it produces a key challenge in the characterization of the risk-sharing dynamics.

With recursive preferences, in fact, optimal allocations are a function not only of aggregate endowment, but also of a possibly time-varying distribution of wealth. As documented by Anderson (2005), in a one-good economy in which agents have risk-sensitive preferences there is typically a tension between ensuring that a non-degenerate distribution of wealth exists and having interesting heterogeneity across agents.The same paper documents that this tension can be relaxed if multiple preference parameters are changed simultaneously and agents have power-reward functions with risk aversion between zero and one.

In this paper, we overcome these challenges by focusing on an economy with mul-

[1]In this joint work, I mainly contribute to the partial proof of the main result, the analysis of the recursive risk-sharing in a two-period model, and the higher-order approximations of the model in more general environments.

tiple goods. We show that rich dynamics of Pareto optimal allocations are obtained even in the case in which all agents share the same risk-sensitivity parameter and have logarithmic period reward functions, provided that they feature a different degree of preference for one of the two goods. Furthermore, we provide conditions under which a nondegenerate ergodic distribution of Pareto weights exists. This is the case in which every agent in the economy has a strictly positive wealth in the long run.

An agent with recursive preferences is willing to trade off expected utility for higher conditional moments of future utility. In a world with a Cobb-Douglas aggregate over multiple goods, the intensity of this trade-off is stronger for agents that consume a large share of aggregate resources, that is, agents with high Pareto weights. For example, as shown in a simple two-period model, when agent 1 has a high initial share of resources, she will have a strong incentive to buy insurance from agent 2 to mitigate future utility uncertainty.

In equilibrium, under a preference for early resolution of uncertainty, agent 1 accepts a reduction in her expected average share of resources (i.e., her Pareto weight is expected to decline) in exchange for a reduction of future utility variance. This trade-off between expected utility and conditional volatility of future utility results in a well-defined invariant distribution of Pareto weights.

Several authors have documented the theoretical properties of one-good versions of the economy analyzed in this paper (Lucas and Stokey (1984), Ma (1993), and Kan (1995)). In particular, Anderson (2005) shows that in an economy with heterogenous agents and recursive preferences it is difficult to ensure the existence of a nondegenerate ergodic distribution of wealth, unless very extreme forms of heterogeneity are considered (see, for example, Backus et al. (2008)).

Our focus on the case of a consumption aggregate of multiple goods resolves these problems, and it is important in many economic applications. In a closed economy, we may think of agents featuring different propensities across commodities produced by different firms or sectors. In an open economy, it is common to assume

that consumers located in different countries are biased toward the consumption of the domestically produced good (see, for example, Tretvoll (2018)). The economy analyzed in this paper provides the foundations for the international macro-finance model in Colacito and Croce (2013). In related work Backus et al. (2016) show that the endogenous variation in Pareto weights in the type of economies that we consider can be interpreted as wedges from the perspective of a frictionless model with additive preferences.

Colacito and Croce (2012) apply the results in this manuscript to the heterogeneous-beliefs literature (among others, see Kubler and Schmedders (2012) and Tsyrennikov (2012)).[2] They show that consumption home bias is isomorphic to endogenous disagreement about the fundamentals of the economy. Under the conditions explored in our paper, the ergodic distribution of wealth is nondegenerate, despite the existence of *endogenous* heterogenous beliefs. For a detailed study of both the survivorship and risk sharing in economies populated by recursive agents with *exogenous* heterogeneous beliefs see Borovička (2018).

From a computational point of view, the characterization of the risk-sharing arrangement with recursive preferences poses additional challenges, as the state space includes the relative wealth of the agents, which in turn, depends on the continuation utilities. We compare a global method that uses value-function iterations and a perturbation-based approach and document that a first-order Taylor expansion about the stochastic steady state of the economy is not appropriate for capturing the dynamics of the economy. This approximation severely deteriorates in regions distant from the steady state, and it produces a counterfactual limiting wealth distribution in which either agent may find herself with zero wealth with probability one. Higher-order approximations are necessary not only to provide a better period-by-period

[2]Specifically, Colacito and Croce (2012) interpret the preferences used in this manuscript as describing a concern for model misspecification, according to the definition of Hansen and Sargent (2008). This results in agent-specific distorted conditional distributions of the endowment processes. Since each probability depends on the utility of a specific agent, when preferences feature heterogenous bias across goods, agents disagree on the transition probabilities across states of the world.

characterization of the dynamics of the model, but also to capture the stationarity of the model. These findings are consistent with the analysis of Anderson et al. (2012) and Pohl et al. (2017).

Additionally, we show that our setting produces endogenous time variation in higher-order conditional moments of consumption, and hence it offers general equilibrium foundations for the analyses of Bansal et al. (2014), Kuehn and Boguth (2013), Colacito et al. (2016), and Segal et al. (2015).

We also study important extensions of our benchmark model by considering the case in which agents have intertemporal elasticities of substitution different from 1 and endowment shocks that are persistent. This means that our analysis can be informative for the growing body of the literature that has explored the macro-finance implications of Epstein and Zin (1989) preferences (see, for example, Bansal et al. (2014)). Furthermore, we show that the introduction of a moderate degree of heterogeneity in the calibration of the two countries may still result in a well-defined ergodic distribution of wealth in equilibrium. This is relevant for the application of our study to economies in which, for example, investors in different countries face a heterogeneous degree of fundamental risk in their endowments or productivities.

Baker and Routledge (2017) study an economy similar to the one analyzed in this paper. Like us, they consider the risk-sharing arrangement between two agents with recursive preferences defined over a Cobb-Douglas aggregate of two goods: oil and a general consumption good. Since the main focus of their paper is matching the price of oil and related futures contracts, they rely on asymmetric calibrations of the two agents. This choice typically results in the survivorship of only one of the two agents in the economy. In this respect, the results that we provide in section 3.5, in which we relax the symmetry of the calibration, are informative for the general class of model that they consider.

Our paper is organized as follows. In section 3.2 we provide the setup of our benchmark economy, featuring unit intertemporal elasticity of substitution and $i.i.d.$

shocks. In section 3.3 we discuss the main intuitions of our framework in the context of a simple two periods model and provide the set of conditions under which a nondegenerate limiting distribution of Pareto weights exists in the infinite horizon setting. In section 3.4 we compare a numerical solution of the model obtained via value function iteration with first and higher order approximations. In section 3.5 we present the results of several generalized versions of our baseline setup. Section 3.6 concludes the paper.

3.2 Setup of the Economy

In this section, we describe the assumptions that we use in the benchmark version of our model. For the purpose of simplifying the analytical proofs and the intuitions of the model, in our benchmark we assume that the intertemporal elasticity of substitution is equal to one and that the two countries share a symmetrical calibration. In section 3.5, we use simulations to show that our results apply to more general settings.

The following three assumptions about preferences, consumption, and endowments will be retained throughout the rest of the paper.

Assumption 1 (Preferences)**.** *Let there be two agents, indexed by 1 and 2, whose preferences are recursively defined as*

$$U_i(c_i, q_i) \quad = \quad (1-\delta_i)\log c_i + \delta_i\theta_i \log \sum_{z'} \pi(z') \exp\left\{\frac{q_i(z')}{\theta_i}\right\}, \quad \forall i \in \{1,2\}, (3.1)$$

where $q_i(z')$ gives the utility remaining from the next period on when next-period's state is z'. For each agent i, $\theta_i < 0$.

This class of preferences can be interpreted in several ways. First, they correspond to the case of risk-sensitive preferences studied by, among others, Hansen and Sargent (1995), Tallarini (2000), and Anderson (2005). Second, they coincide with

a log-transformation of Epstein and Zin (1989) preferences in the case in which the intertemporal elasticity of substitution is equal to one. In this case, the risk-sensitive parameter, θ, is related to risk aversion, γ, by imposing

$$\theta = \frac{1}{1-\gamma}.$$

In this paper, we focus on a discrete time setting as opposed to the continuous time approach of Epstein (1987), Duffie et al. (1994), Geoffard (1996), and Dumas et al. (2000).

Since these preferences depart from the expected utility case, higher moments of continuation utilities matter for the determination of optimal risk sharing. As as example, if continuation utilities $q_i(z')$ are normally distributed, the functional form in (3.1) can be written as

$$U_i(c_i, q_i) = (1-\delta_i)\log c_i + \delta_i E_i(q_i) + \frac{\delta_i}{2\theta_i} V_i(q_i), \quad \forall i \in \{1,2\}, \tag{3.2}$$

where

$$\begin{aligned} E_i(q_i) &= \sum_{z'} \pi(z') q_i(z') \\ V_i(q_i) &= \sum_{z'} \pi(z') \left[q_i(z') - \left(\sum_{z'} \pi(z') q_i(z') \right) \right]^2 \end{aligned}$$

are the conditional mean and variance of the continuation utility, respectively.

Although we will work with the general specification in (3.1), equation (3.2) is instructive, since it intuitively shows that when the parameters θ_i's are less than zero, the risk-sharing scheme must account for an efficient endogenous trade-off between utility level and utility variance.

As the dynamics of second-order moments are crucial for characterizing the equilibrium of the model, in section 3.4 we also assess the accuracy of several approxima-

tions based on how well they can account for the dynamics of volatilities.

Assumption 2 (Consumption bundles)**.** *Let consumption c_i be an aggregate of two goods, x_i and y_i. Specifically, let*

$$c_i \;=\; (x_i)^{\lambda_i}\,(y_i)^{1-\lambda_i} \tag{3.3}$$

be the consumption bundle, with $\lambda_1 > 1/2$ and $\lambda_2 < 1/2$ so that the two agents have a bias for different goods.

This assumption generalizes the one-good framework studied by Anderson (2005), which obtains as the special case in which $\lambda_i = 1/2$, $\forall i$, that is, the case in which there is no preference bias across goods and hence we are effectively in a one-good economy.

The next assumption pertains to the endowment process and is common to Anderson (2005):

Assumption 3 (Endowments)**.** *The endowment of the two goods follows a first-order time-homogenous Markov process $(z_0, z_1, ...)$ which takes values in a finite set $\mathcal{N} = \{1, ..., n\}$. The aggregate supply of the two goods at time t is such that $0 < X_t = X(z_t) < \infty$, and $0 < Y_t = Y(z_t) < \infty$.*

Finally, we need to make sure that the preference parameters are chosen so that the utility recursion converges:

Assumption 4 (Contraction)**.** *The parameters $\{\lambda_i, \gamma_i, \delta_i\}$ are such that the right-hand side of equation (3.1) has a modulus smaller than one, $\forall i = \{1, 2\}$.*

Recursive planner's problem. Let $\log W_i^*(z, c_i, \{q_{i,z'}\}_{z'})$ be the right-hand side of equation (3.1). Given the conditions specified by Ma (1993) and Ma (1996), the social planner's value function, denoted as $Q_p(z, \mu_1) : \mathcal{N} \times [0, 1] \to R$, satisfies the following functional equation proposed by Lucas and Stokey (1984) and Kan (1995):

$$Q_p(z, \mu_1) \;=\; \max_{\{x_i, y_i, q_{i,z'}\}_{i\in\{1,2\}, z'\in\mathcal{N}}} \sum_{i=1}^{2} \mu_i \log W_i^*(z, c_i, \{q_{i,z'}\}_{z'}) \tag{3.4}$$

subject to$\mu_2 = 1 - \mu_1$

$$0 \leq x_1 \leq X(z), \qquad 0 \leq x_2 \leq X(z) - x_1$$

$$0 \leq y_1 \leq Y(z), \qquad 0 \leq y_2 \leq Y(z) - y_1$$

$$c_i = (x_i)^{\lambda_i} (y_i)^{1-\lambda_i}, \quad \forall i = \{1, 2\}$$

$$0 \leq \min_{\mu_1(z') \in [0,1]} Q_p(z', \mu_1(z')) - \mu_1(z') q_{1,z'} - (1 - \mu_1(z')) q_{2,z'} \quad \forall z' \in \mathcal{N}. \quad (3.5)$$

Differentiability and first-order conditions. Let the ratio of the Pareto weights be defined as

$$S = \frac{\mu_1}{\mu_2} = \frac{\mu_1}{1 - \mu_1}.$$

Let $U_i(z, S), \quad i = 1, 2$, denote the utility function of agent i evaluated at the optimum when the exogenous state is z, and $S \in (0, \infty)$. On a consumption path that is bounded away from zero, $U_i(z, S)$ is differentiable (see Kan (1995) and Anderson (2005)) and

$$\frac{dU_i}{d\mu_i} > 0 \quad \mu_1 \in (0, 1), \mu_2 = 1 - \mu_1.$$

On a consumption path that is bounded away from zero for both agents, $Q_p(s, \mu_1)$ is also differentiable with respect to $\mu_1 \in (0, 1)$. The optimality condition in equation (3.5) implies that

$$\frac{dQ_p}{d\mu_1}(z, \mu_1) = U_1(z, S) - U_2(z, S), \qquad \mu_1 \in (0, 1) \quad (3.6)$$

$$\frac{d^2 Q_p}{d\mu_1^2}(z, \mu_1) = \frac{dU_1}{d\mu_1}(z, S) + \frac{dU_2}{d\mu_2}(z, S) > 0. \quad (3.7)$$

Therefore, $Q_p(s, \mu_1)$ is strictly convex with respect to μ_1, as in Lucas and Stokey (1984). This is relevant because it implies that the unique optimal policy of the planner can be characterized using first-order conditions.

According to the first-order conditions, for a given S, the optimal allocation of goods satisfies the following system of equations common to all static Pareto problems

with two goods and two agents:

$$\begin{aligned}(1-\delta_1)\frac{\partial \log c_1}{\partial x_1}\cdot S &= (1-\delta_2)\frac{\partial \log c_2}{\partial x_2} \\ (1-\delta_1)\frac{\partial \log c_1}{\partial y_1}\cdot S &= (1-\delta_2)\frac{\partial \log c_2}{\partial y_2} \\ X = x_1 + x_2, &\qquad Y = y_1 + y_2.\end{aligned} \tag{3.8}$$

The optimal dynamic adjustment of the ratio of the pseudo-Pareto weights is then given by

$$S' = S \cdot \mathcal{M}(z', S') \tag{3.9}$$

where

$$\mathcal{M}(z', S') \equiv \frac{\delta_1 \exp\{U_1(z', S')/\theta_1\}}{\sum_{z'} \pi(z') \exp\{U_1(z', S')/\theta_1\}} \cdot \frac{\sum_{z'} \pi(z') \exp\{U_2(z', S')/\theta_2\}}{\delta_2 \exp\{U_2(z', S')/\theta_2\}}.$$

Equation (3.9) determines the dynamics of the ratio of the Pareto weights and implicitly generates a continuous function that we denote by $f_S(z', \cdot) : [0, +\infty) \to [0, +\infty)$:

$$S' = f_S(z', S). \tag{3.10}$$

Characterizing the planner's problem through first-order conditions is useful because it allows us to represent the planner's problem in (3.4) as a simple system of first-order stochastic difference equations, namely (3.1), (3.3), (3.8), and (3.9). In the next section, we use perturbation methods to solve our dynamic system of equations.

Relative price of the two goods. The relative price of the two goods, p, is the equilibrium marginal rate of substitution across goods

$$p = \frac{(1-\lambda_1)}{\lambda_1} \cdot \frac{x_1}{y_1}.$$

Given the optimal allocations of x_1 and y_1, we can write the relative price as

$$p \;=\; \tilde{p} \cdot \frac{X}{Y},$$

where

$$\tilde{p} \equiv (1-\lambda_1)/\lambda_1 \cdot \left[1 + S \cdot \frac{(1-\lambda_1)}{(1-\lambda_2)}\right] \Big/ \left[1 + S \cdot \frac{\lambda_1}{\lambda_2}\right].$$

When the supply of good X relative to good Y is high, the price of good Y increases for two reasons. First, the last term (X/Y) directly affects the relative price. This channel would be at work even for $\lambda_1 = 1/2$, in which case $\tilde{p} = 1$, and $p = X/Y$. Second, since $\lambda_1 > \lambda_2$ it follows that

$$\frac{\partial \tilde{p}}{\partial S} = -\frac{1-\lambda_1}{\lambda_1} \cdot \frac{\lambda_2}{1-\lambda_2} \cdot \frac{(\lambda_1 - \lambda_2)}{(\lambda_2 + \lambda_1 \cdot S)^2} < 0.$$

This means that the price will further increase as long as the ratio of Pareto weights (S) declines when X/Y is large (we prove this statement formally in proposition 1). Equivalently, the price of good Y is large whenever its supply is low. This effect is magnified in the context of our model since $\lambda_1 > 1/2$, $\lambda_2 < 1/2$, and the ratio of Pareto weights can move away from a symmetric wealth distribution. This enhanced price adjustment allows agent 2 to purchase a larger share of resources whenever the supply of its most preferred good is low.

Share of world consumption (SWC). We note that under our Cobb-Douglas aggregator across goods, the relative share of world consumption of agent 1, SWC, evolves as follows:

$$SWC = \frac{x_1 + p y_1}{X + pY} = \frac{S}{1+S} = \mu_1. \tag{3.11}$$

According to equations (3.8) and (3.11), the Pareto weight of agent 1 has a simple economic interpretation, namely, the relative size of consumption allocated to agent 1.

Symmetry. So far, we have not imposed any specific assumptions on the conditional probability of the Markov chain governing the supply of the two goods, nor have we imposed any special restrictions on the preference parameters of our agents. In order to have a well-specified problem, all we need is that assumptions 1–4 hold. In what follows, however, we list further restrictions that are necessary to analytically characterize the main properties of the optimal risk-sharing policy of the planner. These conditions impose symmetry and are sufficient, but in many cases not necessary, for the existence of a stationary distribution. In the next section, we relax many of these assumptions.

Assumption 5 (Symmetrical preferences)**.** *Let the preference parameters δ_i and θ_i be identical $\forall i \in \{1, 2\}$. Let the consumption-bundle's parameters be symmetrical across agents, that is, $\lambda_1 = 1 - \lambda_2 > 1/2$.*

Assumption 6 (Balanced endowment space)**.** *Let the support of the endowment of good X be given by the vector $H = [h_1, h_2, ..., h_N]$. Let the support of the endowment of good Y be H as well. The endowments of the two goods take values in the finite set $\mathcal{N}$, given by all possible pairwise permutations of H. We refer to $\mathcal{N}$ as a balanced endowment space.*

Definition 1 (Symmetric states)**.** *Let the states $z_i, z_{-i} \in \mathcal{N}$ be such that $z_i = \{X_i = X(i), Y_i = Y(i)\}$ and $z_{-i} = \{X_{-i} = Y(i), Y_{-i} = X(i)\}$. Then z_i and z_{-i} are symmetric states.*

Finally, just to simplify our proof, we focus on a setting with *i.i.d.* shocks and on symmetric probability distributions.

Assumption 7 (*i.i.d.* case)**.** *Assume that $\pi(z'|z) = \pi(z') > 0 \quad \forall z, z' \in \mathcal{N}$.*

Assumption 8 (Symmetric probabilities)**.** *Let the states $z_i, z_{-i} \in \mathcal{N}$ be symmetric. Then $\pi(z_i) = \pi(z_{-i})$.*

3.3 Characterizing the Distribution of Pareto Weights

In this section, we show the main properties of the Pareto weights under our recursive scheme in two settings. We start with a simplified two-period model for which we have a closed-form approximate solution. We then provide results for our infinite-horizon economy.

3.3.1 Recursive Risk-sharing in a Two-period Model

The goal of this section is to illustrate the role of the income and substitution effects as a function of the initial Pareto weights. To highlight the key features of the model, we focus on the special case in which symmetry applies, consistent with the assumptions employed for our main propositions. In 3.7.1, we study a two-period version of the planner's problem detailed in the system of equations (3.4)–(3.5) and provide a general solution which allows for asymmetries across the two agents.

Specifically, consider the following Pareto problem:

$$\max_{\left\{X_1^1, X_1^2, Y_1^1, Y_1^2\right\}} \mu_0 \cdot \theta \log E_0 \left[exp\left\{ \frac{u_1^1}{\theta} \right\} \right] + (1-\mu_0) \cdot \theta \log E_0 \left[exp\left\{ \frac{u_1^2}{\theta} \right\} \right],$$

subject to the following conditions:

$$u_1^i = \log\left(C_1^i\right), \quad \forall i = \{1, 2\},$$

$$C_1^1 = \left(X_1^1\right)^{\lambda} \left(Y_1^1\right)^{(1-\lambda)}, \quad C_1^2 = \left(X_1^2\right)^{(1-\lambda)} \left(Y_1^2\right)^{\lambda}.$$

and the following resource constraints:

$$X_1^1 + X_1^2 = e^{\xi}, \quad Y_1^1 + Y_1^2 = e^{-\xi}, \quad \xi \sim N(0, \sigma^2),$$

where, for simplicity, we are assuming that the endowments of the two goods are perfectly negatively correlated. Let $\overline{s} = \log(\mu_0/(1-\mu_0))$ and let s be the log-ratio

of the pseudo-Pareto weights at time 1. We use this simple setup to illustrate three key features of this class of models.

Income and substitution effects. We show in 3.7.1 that the equilibrium utility functions can be written as

$$u_1^i = \bar{u}_1^i + \lambda_{u_1^i}^{\xi} \cdot \xi, \quad \forall i = \{1, 2\}, \tag{3.12}$$

where

$$\lambda_{u_1^1}^{\xi} = \underbrace{(2\lambda - 1)}_{>0} \left[1 + \underbrace{\frac{2\beta(\bar{s})}{\theta - \beta(\bar{s})(1 + e^{\bar{s}})}}_{<0} \right], \quad \textit{and} \quad \lambda_{u_1^2}^{\xi} = -\underbrace{(2\lambda - 1)}_{>0} \left[1 + \underbrace{\frac{2\beta(\bar{s})\, e^{\bar{s}}}{\theta - \beta(\bar{s})(1 + e^{\bar{s}})}}_{<0} \right]$$

with $\beta(\bar{s})$ defined as the following non-negative, and monotonically decreasing function of $\bar{s}$:

$$\beta(\bar{s}) = \left[\frac{\lambda(1-\lambda)}{(1-\lambda) + \lambda e^{\bar{s}}} + \frac{\lambda(1-\lambda)}{\lambda + (1-\lambda) e^{\bar{s}}} \right].$$

Without loss of generality, let us focus on the response of agent 1's utility to an endowment shock ξ, which is captured by the coefficient $\lambda_{u_1^1}^{\xi}$. A positive *endowment effect*, measured by $(2\lambda - 1)$, determines an increase in the utility that is proportional to the degree of preference for good X. In a single-agent economy, that is, for $\bar{s} \to +\infty$, this is the only effect relevant for the dynamics of future utilities. In a multiple-agent economy, an additional negative *redistribution effect* captures the reallocation of resources that takes place by virtue of risk sharing. This effect depends on $\beta(\bar{s})$, and hence it declines with the original ratio of Pareto weights, $\bar{s}$.

Redistribution of resources. Using the equilibrium utilities in (3.12), it is easy to show that the transition dynamics of the logarithm of the ratio of the Pareto

weights in (3.9) becomes

$$\begin{aligned} s-\bar{s} &= \frac{u_1^1}{\theta}-\frac{u_1^2}{\theta}-\left[\frac{E_0\left(u_1^1\right)}{\theta}+\frac{Var_0\left(u_1^1\right)}{2\theta^2}\right]+\left[\frac{E_0\left(u_1^2\right)}{\theta}+\frac{Var_0\left(u_1^2\right)}{2\theta^2}\right] \quad (3.13)\\ &= \left[\frac{Var_0\left(u_1^2\right)}{2\theta^2}-\frac{Var_0\left(u_1^1\right)}{2\theta^2}\right]+\lambda_s^{\xi}\cdot\xi, \end{aligned}$$

where

$$\lambda_s^{\xi}=\frac{\lambda_{u_1^1}^{\xi}-\lambda_{u_1^2}^{\xi}}{\theta}=\frac{2(2\lambda-1)}{\theta-\beta\left(\bar{s}\right)\left(1+e^{\bar{s}}\right)}$$

is the elasticity of the log-ratio of the Pareto weights with respect to the underlying shock.

When $\lambda>1/2$ and $\theta<0$, then $\lambda_s^{\xi}<0$. This means that when the supply of good X is relatively scarce (i.e., $\xi<0$), agent 1, whose preferences are relatively more tilted toward the consumption of this good, is compensated by means of a greater transfer of resources (i.e., s increases). If $\lambda=1/2$, then the reallocation is null ($\lambda_s^{\xi}=0$). This is a relevant case to consider because it corresponds to the situation in which the multiple-goods economy is equivalent to a one-good economy. As documented by Anderson (2005), in such an economy the distribution of resources is constant over time, unless preference heterogeneity is introduced.

Conditional expectation of $s-\overline{s}$. We can characterize the drift in the log-ratio of the Pareto weights in equation (3.13) as follows:

$$\begin{aligned} \left[\frac{Var_0\left(u_1^2\right)}{2\theta^2}-\frac{Var_0\left(u_1^1\right)}{2\theta^2}\right] &= \frac{\sigma^2}{2\theta^2}\left[\left(\lambda_{u_1^2}^{\xi}\right)^2-\left(\lambda_{u_1^1}^{\xi}\right)^2\right]\\ &= \underbrace{\frac{\sigma^2\beta\left(\bar{s}\right)(2\lambda-1)}{\theta-\beta\left(\bar{s}\right)\left(1+e^{\bar{s}}\right)}}_{<0}\cdot\underbrace{\left(\theta^{-1}\lambda_s^{\xi}\right)}_{>0}\cdot\left(e^{\bar{s}}-1\right). \end{aligned}$$

The value of the drift is pinned down by precautionary motives related to continuation utility variance. When the consumption share of agent 1 rises (i.e., $\bar{s}>0$), agent 1 wants to buy an increasing amount of insurance from agent 2. In equilibrium, agent 1 accepts a reduction in her expected average share of resources (i.e., the drift

in $(s - \overline{s})$ is negative) in exchange for a reduction in future utility variance. At the same time, agent 2 provides limited insurance at a higher price to agent 1 and expects to receive a greater future consumption share.[3] In 3.7.1, we quantify this intuition further and show that the expected growth of the agent with the smaller consumption share increases with a larger degree of preference for one of the two goods (λ), more fundamental risk (σ^2), and stronger risk sensitivity (θ).

3.3.2 Infinite-horizon Model

In this section, we prove that under symmetry a nondegenerate limiting distribution of Pareto weights exists. Equivalently, in the limit no agent receives a Pareto weight of zero with probability one. Furthermore, we characterize the adjustment of the Pareto weights as a function of the realization of the shocks, and the conditional expectation of the Pareto weights as a function of the current state of the economy.

An illustrative example. We introduce a simple example used in the subsequent sections to better illustrate the properties of the model. Endowments can take on one of the following four equally likely pairs of realizations:

$$\mathcal{N} = \{(X = 103, Y = 103), (X = 103, Y = 100), (X = 100, Y = 103), (X = 100, Y = 100)\}.$$

We assume that the coefficient $\lambda_1 = 1 - \lambda_2 = 0.97$ so that agent 1 enjoys a higher period utility when the supply of good X is large and agent 2 is better off when good Y is more abundant. The risk-sensitive parameter θ is set to $(1 - \gamma)^{-1}$ where $\gamma = 25$ in order to enhance the role of risk-sensitivity in this basic setup. We consider lower values of risk aversion in sections 3.4 and 3.5. The qualitative implications of the model are the same as long as $\gamma > 1$. For both agents, the subjective discount factor, δ, is set to 0.96 to ensure fast convergence of our algorithm.

Ranking of Pareto weights. We use the following proposition to characterize

[3]We report the budget constraints associated to the decentralized economy in 3.7.1.

the ranking of Pareto weights as a function of the state of the economy.

Proposition 1. *Let assumptions (1)–(8) hold. Let the events* $a, b \in \mathcal{N}$ *be such that*

$$X'(a)/Y'(a) > X'(b)/Y'(b).$$

Then the ratio of Pareto weights is such that

$$S'(a, S) \leq S'(b, S).$$

If $X'(a)/Y'(a) = X'(b)/Y'(b)$*, then* $S'(a, S) = S'(b, S)$*.*

Proof. See 3.7.2. □

The interpretation of proposition 1 is simple: whenever agent 1 receives a good shock to the endowment of the good that she likes the most, the social planner reduces her weight. This reallocation enables agents 1 and 2 to share part of the endowment risk of the economy, and it is consistent with what is shown in our two-period model, where we document that the elasticity $\lambda^{\xi}_{u_1^1}$ is negative. Furthermore, if the two goods are in identical supply, the optimal choice of Pareto weights is independent of the supply level.

Figure 3.1 documents this ranking by showing the optimal policies associated with our illustrative example. First, notice that the optimal policy is identical in the two states of equal supply of the two goods (see the top-left and the bottom-right panel). Second, notice that next-period's Pareto weight attached to agent 1 is lower when the supply of good X is relatively more abundant (top-right panel) than it is when the supply of good X is relatively more scarce (lower-left panel).

The distribution of the Pareto weights: ergodicity and mean reversion. We document that in the limit no agent receives a Pareto weight of zero with probability one. Furthermore, it is possible to demonstrate that in our economy the dynamics of Pareto weights are characterized by mean reversion. This means that when the

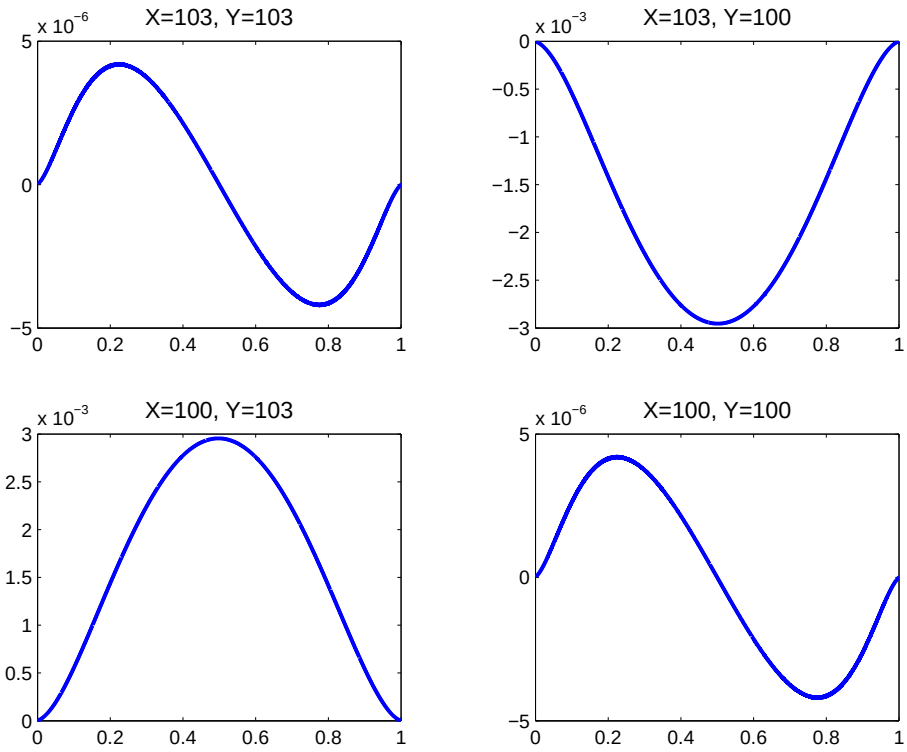


Figure 3.1: Phase diagrams of Pareto weights

Notes: Phase diagrams of Pareto weights. Each panel refers to a different realization of the endowment of the two goods at time $t+1$, $z_{t+1} = [X_{t+1} \quad Y_{t+1}]$. On the vertical axis, we depict the difference between the future Pareto weight for agent 1, $\mu_{1,t+1} = f_{\mu_1}(z_{t+1}, \mu_{1,t})$, and its current value, $\mu_{1,t}$. On the horizontal axis we have $\mu_{1,t}$.

Pareto weight of any agent is small (large), it is expected to increase (decrease) going forward. The following two propositions formalize these statements.

Proposition 2. *Let assumptions (1)–(8) hold. The stochastic processes* μ_1 *and* μ_2 *cannot converge to either* 0 *or* 1 *almost surely.*

Proof. See 3.7.2. □

Proposition 3. *Let assumptions (1)–(8) hold. The expectation of the next-period's Pareto weight on agent 1 conditional on the current Pareto weight is such that (i)* $E\left[\mu_1' | S\right] = \mu_1$, *if* $\mu_1 = \frac{1}{2}$; *(ii)* $E\left[\mu_1' | S\right] < \mu_1, \forall \mu_1 \in \left(\frac{1}{2}, 1\right)$; *and (iii)* $E\left[\mu_1' | S\right] > \mu_1, \forall \mu_1 \in \left(0, \frac{1}{2}\right)$.

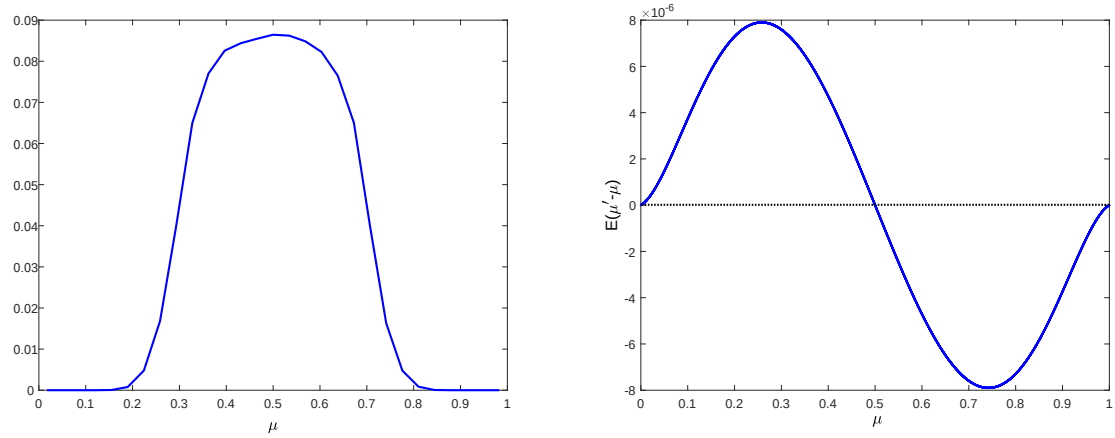


Figure 3.2: Survivorship

Notes: The left panel reports the invariant distribution of the Pareto weight of agent 1 (μ_1). The right panel shows the conditional expectation of the Pareto weight increment for agent 1, $E[\mu_1' - \mu_1|\mu_1]$, as a function of the current μ_1.

Proof. See 3.7.2. □

The content of propositions 2 and 3 is depicted in figure 3.2. The left panel of figure 3.2 shows that the invariant distribution of Pareto weights does not display any mass in the limiting cases of $\mu_1 = \{0, 1\}$. This means that in the long run both agents "survive," that is, they both consume a nonzero share of the aggregate resources.

The right panel of figure 3.2 shows that the conditional change in the Pareto weight of each agent is positive when the Pareto weight is small, and negative when the Pareto weight is large. Equivalently, the dynamics of the Pareto weight feature mean reversion, which is due to endogenous asymmetries in precautionary saving motives. Taken together, propositions 2 and 3 ensure the existence of a well-defined invariant distribution of Pareto weights.

As in our two-period economy, the substitution effect generated by the reallocation channel is size dependent due to the nonlinearity of the aggregator of the two goods. An agent with a large share of consumption benefits the least from the substitution effect and is willing to buy very expensive insurance from the other agent in order to reduce the conditional variance of her continuation utility. As a result, the agent

with a small consumption share is expected to receive a positive transfer of resources going forward, and her consumption share is expected to become larger.

3.4 Comparison of Approximations

In this section we investigate the ability of both first- and higher-order approximations to capture the short- and long-run characteristics of the model. In order to use common perturbation techniques, we assume that endowments are jointly log-normally distributed, with the following means and covariance matrix:

$$\begin{pmatrix} \log X \\ \log Y \end{pmatrix} \sim N\left(\begin{bmatrix} 4.62 \\ 4.62 \end{bmatrix}, \begin{bmatrix} 0.025^2 & 0.3 \cdot 0.025^2 \\ 0.3 \cdot 0.025^2 & 0.025^2 \end{bmatrix} \right).$$

This calibration captures the degree of correlation of output (Colacito and Croce (2013)).

In what follows we show that the results with Gaussian shocks are similar to those obtained with a finite discretized joint normal distribution. This constitutes a generalized setup relative to our earlier sections, which will prove important in allowing us to numerically analyze several interesting extensions of our benchmark model in section 3.5. Specifically, we discretize the distribution of the exogenous endowment shocks on a 21×21 grid of equally spaced nodes on the range

$$[\exp\{4.62 - 5 \times 0.025\}, \exp\{4.62 + 5 \times 0.025\}].$$

We set $\delta = 0.96$, $\lambda_1 = 0.97$, and $\gamma = 15$. This value of risk aversion is in line with those typically employed in the equity premium puzzle literature (see, among others, Tallarini (2000)) and can be lower than that used in section 3.3.2 because we adopt a richer set of exogenous states.

The curse of the linear approximation. A first-order Taylor approximation

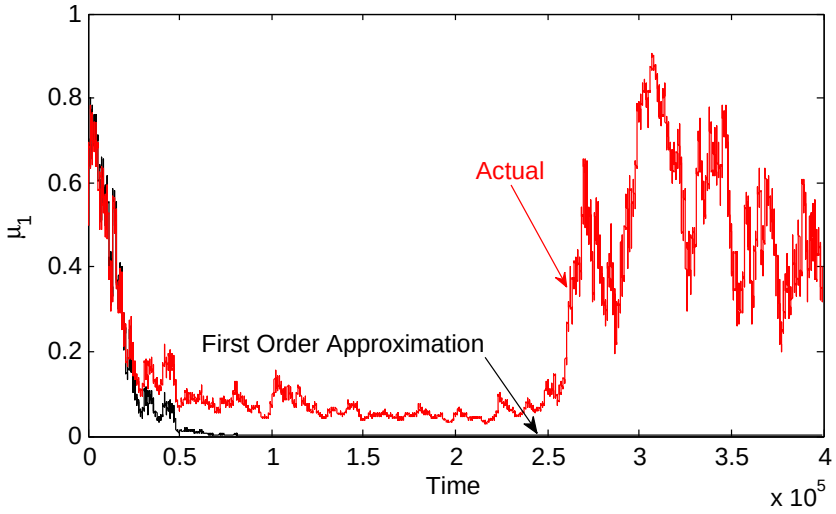


Figure 3.3: Comparison of the actual dynamics of Pareto weights and that obtained through a first-order Taylor expansion.

Notes: For the same sequence of shocks, the black line shows the actual path of the agent 1 Pareto weight, μ_1, while the red line shows the path obtained using a first-order approximation about the unconditional mean of μ_1, that is, $\overline{\mu}_1 = 1/2$.

about the unconditional mean of the ratio of Pareto weights fails to reproduce at least two crucial aspects of the economy. First, it provides a highly inaccurate description of the period-by-period dynamics of the model. Second, and most importantly, it does not capture the mean-reverting property of the model. This results in the possibility that one of the two agents eventually dies and is assigned a steady-state Pareto weight of zero.

In order to show these two facts, we proceed as follows. First, we solve the model numerically by value-function iteration (see 3.7.3) and obtain what we denote as the "actual" solution. Second, we compare the "actual" dynamics of Pareto weights with the dynamics computed using a first-order Taylor expansion about $\mu_1 = \mu_2 = 0.5$. Figure 3.3 reports this comparison for a simulation of $400,000$ periods.

For the first part of the simulation, the Pareto weights are in the relatively small neighborhood of 0.5. In this region, the first-order Taylor expansion does a good

job of approximating the actual dynamics of the economy. The approximation, however, starts deteriorating significantly as the economy departs from $\mu_1 = 0.5$. On this history, according to the first-order Taylor expansion, the Pareto weight of agent 1 should level off at zero, even though this is in sharp contrast to the actual dynamics of the model and the survivorship results explained in the previous sections. As a consequence, the long-run implications of the first-order Taylor expansion are unreliable.

In this clear-cut example, what may at first look like a small error results in an irreversible misrepresentation of the actual dynamics of the economy and its long-run moments. In this economy, higher-order approximations are needed not only to provide a more accurate description of the period-by-period dynamics, but also to preserve the existence of a well-defined ergodic distribution.

The case for higher-order approximations. The linear approximation does not accurately describe the dynamics of the Pareto weights because it impels a first-order integrated process. This is clearly depicted in the left panel of figure 3.4, in which we compare the expected growth of the ratio of Pareto weights as a function of the current ratio across different solution methods. The flat line for the first-order approximation suggests that the conditional expected change of the ratio of Pareto weights is identically zero, implying the lack of any kind of mean reversion.

The second-order approximation does capture some of the mean reversion, although not enough to be comparable to the actual solution of the model. Furthermore, by looking at the right panel of figure 3.4 we notice that the second-order approximation does not feature any time variation in the conditional variance of the ratio of Pareto weights. This is in stark contrast to the actual dynamics of the second moments of the actual solution. Equivalently, the second-order approximation completely misses the time variation in the last term of equation (3.2), that is, the key determinant of the risk-sharing motive of our agents.

In order to capture time-varying volatilities of both consumption and continuation

utilities, we implement a third-order approximation. In contrast to the second-order perturbation, the third-order approximation provides an extremely accurate representation of the dynamics of both the first and second conditional moments. This is certainly the case in a 99% confidence interval of the actual long-run distribution of the ratio of Pareto weights. As expected, the quality of the approximation deteriorates toward the tails of the distribution.

To summarize, this class of models produces rich dynamics for both the first and second conditional moments of the Pareto weights and, therefore, consumption shares across agents. To appropriately capture these dynamics, an approximation of at least the third order is required. In the next section, we use third-order approximations to study more general settings.

3.5 More General Environments

In this section we generalize our setting in two respects. First of all, we consider preferences defined as in Epstein and Zin (1989),

$$U_{i,t} = \left[(1-\delta) \cdot (C_{i,t})^{1-1/\psi} + \delta E_t \left[(U_{i,t+1})^{1-\gamma} \right]^{\frac{1-1/\psi}{1-\gamma}} \right]^{\frac{1}{1-1/\psi}}, \quad \forall i \in \{1,2\}, \quad (3.14)$$

where ψ denotes the IES and γ represents RRA. Second, we consider the following endowment process that allows persistence:

$$\begin{aligned} \log X_t &= \mu + \rho \log X_{t-1} - \tau \left[\log X_{t-1} - \log Y_{t-1} \right] + \varepsilon_t^X \quad (3.15) \\ \log Y_t &= \mu + \rho \log Y_{t-1} + \tau \left[\log X_{t-1} - \log Y_{t-1} \right] + \varepsilon_t^Y \\ \begin{bmatrix} \varepsilon_t^X \\ \varepsilon_t^Y \end{bmatrix} &\sim iidN \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^{X2} & \rho_{X,Y}\sigma^X\sigma^Y \\ \rho_{X,Y}\sigma^X\sigma^Y & \sigma^{Y2} \end{bmatrix} \right), \quad (3.16) \end{aligned}$$

where $\rho \in [0,1]$ and $\tau \in (0,1)$ determine the extent of cointegration when $\rho = 1$. Cointegration is required to have a well-defined ergodic distribution of the relative

supply of the two goods, but it plays a minor quantitative role in our analysis as we set it to a very small number.

Solving the planner's problem with global methods and multiple exogenous state variables goes beyond the scope of this manuscript. The reason is that properly capturing the mean reversion of the pseudo-Pareto weights requires a very thin grid, and it exposes us to the curse of dimensionality even with one extra state. Hence in this section we explore the generality of our results through simulations based on a third-order perturbation of our dynamic model, which we detail in 3.7.4.

Reference calibration. Our reference calibration features $\mu = 2\%$, $\rho = 0.90$, $\sigma^X = \sigma^Y = 1.87\%$, $\rho_{X,Y} = 0.35$, $\tau = 5.0E-04$, $\gamma = 5$, $\psi = 1$, $\delta = 0.96$, and $\lambda_1 = \lambda_2 = 0.97$. The parameters for the endowment processes are set in the spirit of Colacito and Croce (2013). In what follows, we first consider different endowment processes and different levels of the IES and RRA while preserving symmetry across agents and goods. We then explore the implications for a small degree of heterogeneity in preference for the two goods (λ_i) and in fundamental volatility across goods (σ^X and σ^Y).

3.5.1 Symmetric Environments

The role of persistence. We vary the persistence of our endowment shocks from zero to one. When $\rho = 0$, we have *i.i.d.* level shocks, as in the previous section. When $\rho = 1$, level shocks are permanent. We depict key features of the distribution of the log-ratio of the Pareto weights, s_t, in figure 3.5 and simulated moments in table 3.1.

We make several observations. First, as we increase ρ, the endowment shocks become more long-lasting and volatile. As a result, the endogenous process s_t becomes more volatile, as documented by its fatter tails (rightmost plot of figure 3.5, panel A) and the higher conditional volatility of Δs_t (center plot of figure 3.5).

Table 3.1: The Role of Persistence and Preferences

Parameter	Moments of SWC		Percentiles of Approx. Errors (%)				Cumul.
Value	Mean	Std.Dev.	50	75	90	95	ERR (%)
The role of persistence (ρ)							
0	0.5	0.18	2.0E-14	4.0E-14	7.0E-14	9.0E-14	8.0E-01
0.9	0.5	0.19	8.0E-15	2.0E-14	2.0E-14	4.0E-14	7.0E-04
1	0.5	0.2	2.0E-14	4.0E-14	6.0E-14	7.0E-14	5.0E-05
The role of RRA (γ)							
3	0.5	0.21	2.0E-14	3.0E-14	6.0E-14	9.0E-14	1.0E-02
5	0.5	0.2	8.0E-15	2.0E-14	2.0E-14	4.0E-14	7.0E-04
7	0.5	0.16	3.0E-14	6.0E-14	8.0E-14	9.0E-14	5.0E-04
The role of IES (ψ)							
2/3	0.5	0.19	4.0E-11	4.0E-11	4.0E-14	4.0E-14	1.0E-01
1	0.5	0.2	2.0E-14	4.0E-14	5.0E-14	7.0E-14	5.0E-05
1.5	0.5	0.21	4.0E-14	9.0E-14	2.0E-13	2.0E-13	5.0E-05

Note: Our model features Epstein and Zin (1989) preferences as specified in equation (3.14) and the endowments reported in equation (3.16). Our reference calibration is detailed in section 3.5. We vary one parameter at a time, leaving the others unchanged. Both the approximation errors and the cumulative approximation errors (Cumul. ERR) are defined in 3.7.4 and are multiplied by 100. All numbers are based on repetitions of long-sample simulations (at least 2 million periods) with different starting points for log ratio of the Pareto weights (s_t).

The unconditional volatility of s_t, however, increases just slightly, as documented in table 3.1. This is because as shocks become more long lasting the precautionary motives of these agents become more pronounced and more sensitive to size. As the exogenous endowment persistence (ρ) increases, the endogenous persistence of s_t declines, as captured by the more negative slope of $E_t[\Delta s_{t+1}]$ with respect to s_t (leftmost plot of figure 3.5, panel A).

Second, given the lower persistence of s_t, our cumulative error measure declines to very small numbers with higher values of ρ. This result is reassuring because in many realistic applications the endowment shocks are calibrated to be very persistent.

The role of preferences. When we vary the subjective discount factor, we do not find significant changes in the dynamics of the log-ratio of the Pareto weights. For this reason, we focus only on the role of risk aversion and IES. Increasing risk

aversion amplifies the sensitivity of continuation utility to shocks, and hence it makes the redistribution channel stronger. This intuition is confirmed in panel (b) of figure 3.5, where we show that the conditional variation of Δs_{t+1} increases with higher values of γ and, at the same time, the mean reversion of the Pareto weights speeds up.

Since the endogenous change in mean reversion dominates quantitatively, as γ increases more mass is concentrated in the center of the probability distribution function of s_t, implying that the unconditional volatility of this process declines. Together, the faster speed of mean reversion and the lower unconditional volatility of s_t imply lower levels of approximation errors.

We conclude this analysis by examining the case in which we vary the IES. The effect of this parameter on the conditional volatility of the ratio of Pareto weights is almost negligible (center plot of panel (c), figure 3.5). The impact on the endogenous persistence of s_t is a bit more pronounced, but still moderate compared to the case in which we change risk aversion. Qualitatively, agents with a higher IES are more willing to accept fluctuations of consumption over time and hence are more willing to accept very long-lasting reallocations, that is, slower mean reversion in s_t (leftmost plot in panel (c), figure 3.5). Most importantly, we note that in the long-run risk literature the IES is set to values larger than or equal to one. For these values, the approximation errors are small, meaning that when the curvature of the utility function with respect to intertemporal aggregation is moderate ($\psi \geq 1$), the quality of our approximation is good.

3.5.2 Asymmetric Environments

Asymmetry preferences for the two goods. We lower the degree of preference for good Y of agent 2, λ_2, from 0.97 to 0.90, thus increasing the ability of agent 2 to smooth fluctuations in her consumption bundle by trading the two goods. All other parameters are unchanged. We depict key features of the distribution of the log-ratio

Table 3.2: Asymmetric Cases

Parameter value	Moments of SWC			Cumul ERR (%)
	Mean	Std	Skew	
Agent 2 Bias (λ_2) - EZ Case				
0.97	0.5	0.19	0	7.0E-04
0.95	0.47	0.18	0.22	2.6E-03
0.90	0.46	0.17	0.33	2.2E+00
Agent 2 Bias (λ_2) - CRRA Case ($\gamma = \psi^{-1} = 5$)				
0.95	0.47	0.02	0.03	5.0E-06
Y-Good Volatility (σ^Y)				
$1.00 \cdot \sigma^X$	0.5	0.19	0	7.0E-04
$1.05 \cdot \sigma^X$	0.58	0.19	-0.27	9.0E-04
$1.10 \cdot \sigma^X$	0.56	0.22	-0.27	1.3E-01

Our model features Epstein and Zin (1989) preferences as specified in equation (3.14) and the endowments reported in equation (3.16). Our reference calibration is detailed in section 3.5. We vary one parameter at the time, leaving the others unchanged. The cumulative approximation errors (Cumul ERR) are defined in 3.7.4 and are multiplied by 100. All numbers are based on repetitions of long sample simulations (at least 2 million periods) with different starting points for log ratio of the Pareto weights (s_t).

of the Pareto weights, s_t, in figure 3.6 and simulated moments in table 3.2.

Since agent 2 faces less consumption uncertainty than agent 1, her demand of insurance is moderate compared to that of agent 1. As a result, the average share of resources allocated to agent 2 increases and the curves depicted in the top-left panel of figure 3.6 shift to the left. In the appendix, we show that this intuition is also present in the simple two-period model (see figure 3.8).

We note that the distribution of the log-ratio of the Pareto weights does not shift to the left in a parallel way. As documented in the top portion of table 3.2, under the optimal risk-sharing scheme, agent 1 accepts a lower average level of resources in exchange for both a reduction in future utility uncertainty and positive skewness of its share of world consumption. That is, agent 1 benefits from a sizeable positive redistribution of resources along histories with a severe downside of the relative supply

of good X.

Rabitsch et al. (2015) point out that a global approximation is required when countries are subject to asymmetric constraints, such as a borrowing limit, and when their wealth distribution is nonstationary. Since we have a frictionless model with complete markets and a well-defined ergodic distribution of wealth, a perturbation approach provides a good approximation of the equilibrium. Consistent with the findings in Rabitsch et al. (2015), our cumulative errors increase as we make the two agents more asymmetric, but our errors remain as low as 2.2%.

CRRA and heterogeneous degree of preference for the two goods. It is useful to study this asymmetric scenario under time-additive CRRA preferences. We choose the intermediate case $\lambda_1 = 0.97$, $\lambda_2 = 0.95$, and set $\psi^{-1} = \gamma = 5$.[4] We fix the initial ratio of the Pareto weights to a value that delivers an average SWC of 0.47, as in the case with recursive preferences. Panel (b) of figure 3.6 and table 3.2 confirm that under recursive preferences, the reallocation channel is very pronounced and long-lasting, and it prescribes a significant amount of positive skewness for agent 1, as she is facing more consumption risk because of a higher degree of preference for good X.

Heterogeneous volatility. In many applications, the properties of the goods traded are asymmetric. In international finance, for example, different countries may be subject to productivity shocks with different volatilities. In the presence of the same degree of preference for the two goods, the agent which prefers the more volatile good faces more consumption and hence future utility uncertainty. As suggested by our two-period model (see figure 3.8), this agent should be willing to accept a low average SWC in exchange for insurance. Panel (c) of figure 3.6 and table 3.2 confirm this finding as we increase σ^Y.

[4]Note that since $\gamma > 1$, the ratio of Pareto weights is no longer constant over time despite the adoption of time additivity of preferences (Cole and Obstfeld (1991)). The implied variation in s_t, however, is very limited as highlighted in panel (b) of figure 3.6.

In this case, the average share of resources of agent 1 increases. The volatility of the SWC increases as well, as a direct result of the higher standard deviation of good Y. Agent 1 is willing to insure agent 2 against downside risk, and agent 1 accepts negative skewness in her own share of resources. Across all cases, the associated cumulative approximation errors are smaller than 0.14%.

Heterogenous RRA. A well-known result with multiple agents with risk-sensitive preferences in a one-good economy with growth is that only the agent with the lowest risk aversion remains wealthy in the long run, ceteris paribus (see Anderson (2005)). We confirm this finding in our setting by depicting in figure 3.7 the expected growth rate of the log-ratio of Pareto weights for the case $\lambda_1 = \lambda_2$ and $\gamma_2 > \gamma_1$. Since the expected growth is positive for all values of s_t, all resources are allocated to agent 1 in the limit.

Our risk-sharing mechanism, however, suggests that as we make the degree of preference for the two goods more asymmetric, size matters progressively more for the intensity of the reallocation channel. Hence as the high-risk-aversion agent receives a smaller share of world resources, her willingness to buy further insurance should decline, which in turn results in survivorship. Our simulations confirm this intuition and suggest that there exist regions of the parameter space in which survivorship is also possible with asymmetric risk aversion, provided that multiple parameters are simultaneously adjusted, in the spirit of Anderson (2005).

3.6 Concluding Remarks

We have characterized the solution of a planner's problem with multiple agents, multiple goods, and recursive preferences. The introduction of multiple goods substantially changes the dynamics of Pareto optimal allocations. Future research should extend our theoretical results to continuous time and continuous shocks. Asset pricing implications may be particularly appealing, due to the ability of this class of models to

endogenously produce time-varying second moments without requiring market frictions. Since the main features of the risk-sharing scheme can be accurately captured through a third-order approximation, this class of models could be easily extended to an international business cycle setting as well.

3.7 Additional Materials

3.7.1. Two-Period Model.

Environment. In this section, we present a simplified two-period version of our model in order to provide intuition on the reallocation motives induced by recursive preferences. Specifically, at time $t = 1$ agents receive news ξ about their time-1 endowment of goods. At time $t = 0$, that is, before the arrival of the shock, agents $i \in \{1, 2\}$ exchange a complete set of ξ-contingent securities to maximize their time-0 utility, given their initial wealth (reflected in their time-0 Pareto weights).

Utility and technology. In what follows, we take advantage of lognormality wherever possible. Up to a log linearization of the allocation shares, this modeling strategy enables us to obtain a simple closed-form solution. In this spirit, we start by assuming that agents have an IES equal to 1, that is, their preferences can be expressed as follows:

$$u_0^i = \begin{cases} \delta\theta_i \log E_0\left[exp\left\{\frac{u_1^i}{\theta_i}\right\}\right] & 0 > \theta_i > -\infty \\ \delta E_0[u_1^i] & \theta_i \to -\infty \end{cases} \tag{3.17}$$

where

$$u_1^i = \log\left(C_1^i\right)$$

and

$$C_1^1 = \left(X_1^1\right)^{\lambda_1}\left(Y_1^1\right)^{(1-\lambda_1)}, \quad C_1^2 = \left(X_1^2\right)^{(1-\lambda_2)}\left(Y_1^2\right)^{\lambda_2}. \tag{3.18}$$

The resource constraints are specified as follows:

$$X_1^1 + X_1^2 = e^{\xi}, \quad Y_1^1 + Y_1^2 = e^{-l\xi}, \quad \xi \sim N(0, \sigma^2), \tag{3.19}$$

where the parameter l determines whether agent 2 is more ($l > 1$) or less ($0 < l < 1$) exposed to the shock ξ than agent 1. We assume that ξ affects both goods to preserve

symmetry in our equations. In what follows, we show that the results are driven only by the relative supply of the goods $(1+l)\xi$.

Pareto problem. Under complete markets, the allocation can be recovered by solving the following Pareto problem:

$$\max_{\left\{X_1^1, X_1^2, Y_1^1, Y_1^2\right\}} \mu_0 \cdot u_0^1 + (1-\mu_0) \cdot u_0^2,$$

subject to the constraints specified in (3.18)–(3.19). Let $S_0 \equiv \mu_0/(1-\mu_0)$; after simplifying common coefficients, the optimality condition for the allocation of good X_1 is

$$S_1(\xi)\frac{\partial \log C_1^1}{\partial X_1} = \frac{\partial \log C_1^2}{\partial X_1} \tag{3.20}$$

with

$$S_1(\xi) = \begin{cases} S_0, & \theta_i \to -\infty, \\ S_0 \cdot \dfrac{exp\left\{\frac{u_1^1}{\theta_1}\right\}}{E_0\left[exp\left\{\frac{u_1^1}{\theta_1}\right\}\right]} \Big/ \dfrac{exp\left\{\frac{u_1^2}{\theta_2}\right\}}{E_0\left[exp\left\{\frac{u_1^2}{\theta_2}\right\}\right]} & 0 > \theta_i > -\infty. \end{cases} \tag{3.21}$$

Equation (3.20) establishes that the optimal allocation can be found as in a regular static problem, for a given value of S_1. Equation (3.21) pins down S_1 and yields two important results. First, in the time-additive case, the share of resources is time invariant, that is, it is not affected by the actual realization of ξ. This is consistent with the special log case considered by Cole and Obstfeld (1991).

Second, with recursive preferences, agents have a preference for the variance of their future utility, and hence their time-1 marginal utility depends on the time-1 level of their utility. If agents prefer early resolution of uncertainty ($0 > \theta^i > -\infty$), a higher relative future utility implies a lower relative marginal utility and hence a lower share of allocated resources ($u_1^1(\xi) > u_1^2(\xi) \to S_1(\xi) < S_0$). Because of

the dependence of S_1 on future utility levels, $u_1^1(\xi) - u_1^2(\xi)$, the shock ξ prompts a reallocation at time 1.

Approximate solution. Define $\frac{\lambda_1}{1-\lambda_2} = \kappa_1$ and $\frac{\lambda_2}{1-\lambda_1} = \kappa_2$. At time 1, ξ is known and the optimal allocations are

$$X_1^1 = \frac{S_1\kappa_1}{1+S_1\kappa_1}e^{\xi}, \quad X_1^2 = \frac{1}{1+S_1\kappa_1}e^{\xi}$$

and

$$Y_1 = \frac{\frac{S_1}{\kappa_2}}{1+\frac{S_1}{\kappa_2}}e^{-l\xi}, \quad Y_1^2 = \frac{1}{1+\frac{S_1}{\kappa_2}}e^{-l\xi}.$$

Since the allocations are a nonlinear function of $S_1(\xi)$, we log linearize them with respect to $s \equiv \log S_1$ around $\bar{s} = s_0$ and obtain

$$\begin{aligned}
\log\left(\frac{\kappa_1 e^s}{1+\kappa_1 e^s}\right) &= \log\left(\frac{\kappa_1 e^{\bar{s}}}{1+\kappa_1 e^{\bar{s}}}\right) + \frac{1}{1+\kappa_1 e^{\bar{s}}}(s-\bar{s}) \\
\log\left(\frac{1}{1+\kappa_1 e^s}\right) &= \log\left(\frac{1}{1+\kappa_1 e^{\bar{s}}}\right) - \frac{\kappa_1 e^{\bar{s}}}{1+\kappa_1 e^{\bar{s}}}(s-\bar{s}) \\
\log\left(\frac{\frac{e^s}{\kappa_2}}{1+\frac{e^s}{\kappa_2}}\right) &= \log\left(\frac{\frac{e^{\bar{s}}}{\kappa_2}}{1+\frac{e^{\bar{s}}}{\kappa_2}}\right) + \frac{1}{1+\frac{e^{\bar{s}}}{\kappa_2}}(s-\bar{s}) \\
\log\left(\frac{1}{1+\frac{e^s}{\kappa_2}}\right) &= \log\left(\frac{1}{1+\frac{e^{\bar{s}}}{\kappa_2}}\right) - \frac{\frac{e^{\bar{s}}}{\kappa_2}}{1+\frac{e^{\bar{s}}}{\kappa_2}}(s-\bar{s})
\end{aligned}$$

and hence

$$\begin{aligned}
\log C_1^1 &= \lambda_1\left[\log\left(\frac{\kappa_1 e^{\bar{s}}}{1+\kappa_1 e^{\bar{s}}}\right) + \frac{1}{1+\kappa_1 e^{\bar{s}}}(s-\bar{s}) + \xi\right] \\
&\quad + (1-\lambda_1)\left[\log\left(\frac{\frac{e^{\bar{s}}}{\kappa_2}}{1+\frac{e^{\bar{s}}}{\kappa_2}}\right) + \frac{1}{1+\frac{e^{\bar{s}}}{\kappa_2}}(s-\bar{s}) - l\xi\right] \\
\log C_1^2 &= (1-\lambda_2)\left[\log\left(\frac{1}{1+\kappa_1 e^{\bar{s}}}\right) - \frac{\kappa_1 e^{\bar{s}}}{1+\kappa_1 e^{\bar{s}}}(s-\bar{s}) + \xi\right] \\
&\quad + \lambda_2\left[\log\left(\frac{1}{1+\frac{e^{\bar{s}}}{\kappa_2}}\right) - \frac{\frac{e^{\bar{s}}}{\kappa_2}}{1+\frac{e^{\bar{s}}}{\kappa_2}}(s-\bar{s}) - l\xi\right].
\end{aligned}$$

We can now write

$$u_1^1 = \log C_1^1 = \bar{u}_1^1 + \lambda_{u_1^1}^{\xi}\xi \tag{3.22}$$

$$u_1^2 = \log C_1^2 = \bar{u}_1^2 + \lambda_{u_1^2}^{\xi}\xi, \tag{3.23}$$

where

$$\begin{aligned}
\lambda_{u_1^1}^{\xi} &\equiv \left[\frac{\lambda_1}{1+\kappa_1 e^{\bar{s}}} + \frac{1-\lambda_1}{1+\frac{e^{\bar{s}}}{\kappa_2}}\right]\lambda_s^{\xi} + \lambda_1 - l + \lambda_1 l \\
\lambda_{u_1^2}^{\xi} &\equiv \left[-(1-\lambda_2)\frac{\kappa_1 e^{\bar{s}}}{1+\kappa_1 e^{\bar{s}}} - \lambda_2\frac{\frac{e^{\bar{s}}}{\kappa_2}}{1+\frac{e^{\bar{s}}}{\kappa_2}}\right]\lambda_s^{\xi} + (1-\lambda_2) - \lambda_2 l,
\end{aligned} \tag{3.24}$$

and

$$s - \bar{s} = const(\bar{s}) + \lambda_s^{\xi}\cdot\xi.$$

The elasticity λ_s^{ξ} can be found taking into account equation (3.21):

$$\lambda_s^{\xi} = \frac{\lambda_{u_1^1}^{\xi}}{\theta_1} - \frac{\lambda_{u_1^2}^{\xi}}{\theta_2}. \tag{3.25}$$

Specifically, by combining equations (3.24)–(3.25), we have

$$\lambda_s^{\xi} = \frac{\frac{1}{\theta_1}\left[(1+l)\lambda_1 - l\right] + \frac{1}{\theta_2}\left[(1+l)\lambda_2 - 1\right]}{1 - \frac{1}{\theta_1}\left[\frac{\lambda_1}{1+\kappa_1 e^{\bar{s}}} + \frac{1-\lambda_1}{1+\frac{e^{\bar{s}}}{\kappa_2}}\right] - \frac{1}{\theta_2}\left[(1-\lambda_2)\frac{\kappa_1 e^{\bar{s}}}{1+\kappa_1 e^{\bar{s}}} + \lambda_2\frac{\frac{e^{\bar{s}}}{\kappa_2}}{1+\frac{e^{\bar{s}}}{\kappa_2}}\right]} \tag{3.26}$$

The constant can be recovered again from equation (3.21) accounting for second-order terms:

$$const(\bar{s}) = \frac{1}{2}\sigma^2\left[\left(\frac{\lambda_{u_1^2}^{\xi}}{\theta_2}\right)^2 - \left(\frac{\lambda_{u_1^1}^{\xi}}{\theta_1}\right)^2\right]. \tag{3.27}$$

Decentralization. Let the budget constraints of agent 1 and agent 2 be

$$X_1^1 + p \cdot Y_1^1 + \int_\xi Q(\xi) A_2^1(\xi) = e^\xi + A_1^1,$$
$$X_1^2 + p \cdot Y_1^2 + \int_\xi Q(\xi) A_2^2(\xi) = e^{-l\xi} + A_1^2,$$

where p is the price of good Y relative to good X, $Q(\xi)$ is the price of an Arrow-Debreu security that pays one unit of good X (the numeraire) contingent on the realization of ξ at date 2, and A_i^j are the holdings of such security of agent j at date i. By market clearing, $A_i^1 + A_i^2 = 0$, $\forall i \in \{1, 2\}$.

Define $\overline{\xi}$ as the specific value of the state in which there is no reallocation effect, that is, $s = \overline{s}$. Under our approximation,

$$\overline{\xi} = -\frac{const(\overline{s})}{\lambda_s^\xi}.$$

This value is an important threshold for the portfolio allocation of our two agents in the companion decentralized economy. In particular, at time 0, agent 1 buys Arrow-Debreu securities paying a positive payoff for $\xi < \overline{\xi}$, and sells Arrow-Debreu securities paying when $\xi > \overline{\xi}$. Since these assets are available in zero net supply, the exact opposite holds for agent 2. The probability of agent 1 receiving a positive transfer is $F(\overline{\xi})$, where F denotes the cumulative distribution function of ξ. When $\overline{\xi} > 0$, agent 1 has a higher probability of being a net receiver of resources than agent 2.

Special cases. In the special case in which both agents have the same risk-sensitivity parameter, $\theta_1 = \theta_2 = \theta$, and the same exposure to the underlying shock, $l = 1$, we obtain

$$\lambda_s^\xi = \frac{\frac{2}{\theta}(\lambda_1 + \lambda_2 - 1)}{1 - \frac{1}{\theta}\left[\frac{\lambda_1}{1+\kappa_1 e^{\overline{s}}} + \frac{1-\lambda_1}{1+\frac{e^{\overline{s}}}{\kappa_2}} + (1-\lambda_2)\frac{\kappa_1 e^{\overline{s}}}{1+\kappa_1 e^{\overline{s}}} + \lambda_2 \frac{\frac{e^{\overline{s}}}{\kappa_2}}{1+\frac{e^{\overline{s}}}{\kappa_2}}\right]} \tag{3.28}$$

and $\lambda_s^\xi < 0$ if $\theta < -\infty$ and $\lambda_1 + \lambda_2 > 1$, i.e., the ratio of the pseudo-Pareto weights is 'countercyclical', meaning that a smaller share of resources is assigned to agent 1 when there is relative abundance of good 1. In this case,

$$const(\bar{s}) = \frac{\sigma^2}{2\theta^2}\left[(\lambda_{u_1^2}^\xi)^2 - (\lambda_{u_1^1}^\xi)^2\right]. \tag{3.29}$$

Let us also assume that $\lambda_1 = \lambda_2 = \lambda > 1/2$. In this case, we can derive the following results:

$$\lim_{\bar{s}\to-\infty} \lambda_s^\xi = 4\frac{(2\lambda-1)}{\theta} < 0 \tag{3.30}$$

$$\lim_{\bar{s}\to-\infty} const(\bar{s}) = \frac{\sigma^2}{2\theta^2}\lambda_s^\xi\left[\lambda_s^\xi - 2(2\lambda-1)\right] > 0 \tag{3.31}$$

$$\lim_{\bar{s}\to-\infty} \overline{\xi} = (2\lambda-1)\frac{\sigma^2}{\theta(\theta-1)} > 0. \tag{3.32}$$

Let $F(\xi)$ be the cumulative distribution function of ξ. Conditional on $\overline{s} \to -\infty$, the probability of country 1 to receive a higher share of resources in period 1 is $F(\overline{\xi})$, and hence it increases with (i) a stronger degree of preference for one of the two goods (λ), (ii) larger fundamental risk (σ^2), and (iii) stronger risk sensitivity (θ).

Figure 3.8, depicts $const(\bar{s})$ (defined in equation 3.27) as a function of $\overline{s}$ in two scenarios with asymmetric calibrations.

3.7.2. Proofs

Proof of proposition 1

Proof of proposition 1. Define

$$\begin{aligned}
\varphi(z,S) &= \frac{1}{\theta}\log c_1(z,S) - \frac{1}{\theta}\log c_2(z,S) \\
\Phi(z,S) &= \frac{1}{\theta}U_1(z,S) - \frac{1}{\theta}U_2(z,S) \\
F(z,S) &= \delta\log\sum_{z'}\pi(z')\exp\left\{\frac{U_1(z',S')}{\theta}\right\} - \delta\log\sum_{z'}\pi(z')\exp\left\{\frac{U_2(z',S')}{\theta}\right\},
\end{aligned}$$

Note that $\Phi(z,S) = \varphi(z,S) + F(z,S)$. Let $k = \lambda_1/(1-\lambda_1)$ and let $X(z)$ and $Y(z)$ be the endowments of the two goods in state s. Then, by the symmetry of the preferences, the optimal allocations imply

$$\varphi(z,S) \;=\; \frac{1}{\theta}\left[(2\lambda_1 - 1)\log\left(\frac{X(z)}{Y(z)}\right) + \log S + (1-2\lambda_1)\log(1+kS) - (1-2\lambda_1)\log(k+S)\right].$$

Since $\theta < 0$, if $\lambda_1 > 1/2$ it follows that $\varphi(z,S)$ is always: 1) decreasing in $X(z)/Y(z)$ and 2) decreasing in S. Furthermore, $F(s,S)$ is decreasing in S, since $U_1(z,S)$ ($U_2(z,S)$) is increasing (decreasing) in S, by the optimality of the social planner problem. Therefore, it has to be the case that $\Phi(z,S)$ is decreasing in S.

Take two states, a and b, such that $X'(a)/Y'(a) < X'(b)/Y'(b)$, and let $S'_a = f_S(a,S)$ and $S'_b = f_S(b,S)$ be their respective ratios of Pareto weights. It is possible to characterize the ratio:

$$\frac{S'_b}{S'_a} \;=\; \exp\left\{\Phi\left(b,S'_b\right) - \Phi\left(a,S'_a\right)\right\}.$$

Since $\Phi\left(z,S\right)$ is decreasing in z holding S fixed, it follows $\exp\left\{\Phi\left(b,S'_a\right) - \Phi\left(a,S'_a\right)\right\} < 1$. Hence, if $X'(a)/Y'(a) < X'(b)/Y'(b)$, it is never optimal to set $S'_a = S'_b$.

Additionally, the fact that $\Phi(s,S)$ is decreasing in S implies that

$$1 > \exp\left\{\Phi\left(b,S'_a\right) - \Phi\left(a,S'_a\right)\right\} > \left\{\Phi\left(b,S'_b\right) - \Phi\left(a,S'_a\right)\right\}.$$

It follows that $S'_b < S'_a$. □

Proof of proposition 2

We proceed in steps to provide a proof to roposition 2. Specifically, we state and prove four propositions, which, combined, yield the proof of proposition 2. Throughout our exposition of these propositions, we utilize the example proposed in main text, in order to better illustrate the content of each statement.

We start by stating the following decomposition of the Pareto weights, the proof for which follows directly from Anderson (2005). The rest of this section will consist of propositions aimed at signing the covariance that appears on the right-hand side of this decomposition. This will enable us to establish the relevant properties concerning the ergodicity and mean reversion of the distribution of Pareto weights.

Proposition 4. *The ratio of Pareto weights can be decomposed as*

$$E\left[S'|S\right] \;=\; S - \frac{cov\left[\exp\left\{U_2'/\theta\right\}, S'|S\right]}{E\left[\exp\left\{U_2'/\theta\right\}|S\right]}, \tag{3.33}$$

where $E\left[S'|S\right]$ *denotes the expectation of* S' *conditional on* s *being the current state, and*

$$cov\left[\exp\left\{U_2'/\theta\right\}, S'|S\right]$$

denotes the covariance between $\exp\left\{U_2'/\theta\right\}$ *and* S' *conditional on the current* S*.*

Proof. See Anderson (2005). □

The next proposition draws a sharp contrast between a representative-agent economy and our setting with multiple agents and goods. In a representative-agent economy, the consumer enjoys a higher utility in the states of the world in which the supply of the most-preferred good is more abundant. In the two agent economy that we consider in this paper, for any two symmetric states there exists a finite ratio of Pareto weights below which the ranking of the future utility functions across states is reversed.

Proposition 5. *For any two symmetric states* z_i *and* z_{-i}*, such that* $X(z_i) > Y(z_i)$*, there exists a finite* $\widetilde{S}_1^i < 1$ *such that* $U_1\left(z_i, f_S\left(z_i, \widetilde{S}_1^i\right)\right) = U_1\left(z_{-i}, f_S\left(z_{-i}, \widetilde{S}_1^i\right)\right)$*, where* $f_S\left(\cdot,\cdot\right)$ *is defined in (3.10), and* $U_1\left(z_i, f_S\left(z_i, S_1^i\right)\right) < U_1\left(z_{-i}, f_S\left(z_{-i}, S_1^i\right)\right)$*,* $\forall S_1^i < \widetilde{S}_1^i$*.*

Proof. Let $S_i' = f_S(z_i, S)$ and $S_{-i}' = f_S(z_{-i}, S)$ be the ratios of Pareto weights when tomorrow's symmetric states are z_i and z_{-i}, respectively. Using equation (3.9), the

ratio between S_i' and S_{-i}' is

$$\frac{S_i'}{S_{-i}'} = \exp\left\{\frac{U_1\left(z_i', S_i'\right) - U_1\left(z_{-i}', S_{-i}'\right)}{\theta}\right\} \Bigg/ \exp\left\{\frac{U_2\left(z_i', S_i'\right) - U_2\left(z_{-i}', S_{-i}'\right)}{\theta}\right\}.$$

Rearranging, we have

$$U_1\left(z_i', S_i'\right) - U_1\left(z_{-i}', S_{-i}'\right) = U_2\left(s_i', S_i'\right) - U_2\left(z_{-i}', S_{-i}'\right) + \theta[\log S_i' - \log S_{-i}']. \quad (3.34)$$

We shall characterize the limit of the left-hand side of equation (3.34) for μ_1 that tends to zero. First, notice that consumption bias ($\lambda_1 > 1/2$) implies that

$$\lim_{\mu_1 \to 0} U_2\left(z_i', S_i'\right) - U_2\left(z_{-i}', S_{-i}'\right) < 0. \quad (3.35)$$

Then, notice that equation (3.51) and the fact that the planner's problem is twice continuously differentiable imply the continuity of the f_S function with respect to S. Also, at $\mu_1 = 0$, $S_i' = S_{-i}' = 0$. Hence

$$\lim_{\mu_1 \to 0} S_i' = S_i'|_{\mu_1 = 0} = S_{-i}'\big|_{\mu_1 = 0} = \lim_{\mu_1 \to 0} S_{-i}',$$

which implies that

$$\lim_{\mu_1 \to 0} \log\left(S_i' - \log S_{-i}'\right) = 0. \quad (3.36)$$

Combining (3.35) and (3.36) into (3.34), we obtain that

$$\lim_{\mu_1 \to 0} U_1\left(z_i', S_i'\right) - U_1\left(z_{-i}', S_{-i}'\right) < 0. \quad (3.37)$$

Since $\lambda_1 > 1/2$, it follows that

$$\lim_{\mu_1 \to 1} U_1\left(z_i, S'\right) - U_1\left(z_{-i}, S'\right) > 0. \quad (3.38)$$

Combining (3.37) and (3.38) concludes the proof. □

Corollary 1. *For any two symmetric states z_i and z_{-i}, such that $X(z_i) > Y(z_i)$, there exists a finite $\widetilde{S}_2^i > 1$ such that $U_2\left(z_i, f_S\left(z_i, \widetilde{S}_2^i\right)\right) = U_2\left(z_{-i}, f_S\left(z_{-i}, \widetilde{S}_2^i\right)\right)$, where $f_S(\cdot,\cdot)$ is defined in (3.10), and $U_2\left(z_i, f_S\left(z_i, S_2^i\right)\right) > U_2\left(z_{-i}, f_S\left(z_{-i}, S_2^i\right)\right)$, $\forall S_2^i > \widetilde{S}_2^i$.*

We illustrate the content of the preceding proposition and corollary in figure 3.9, which depicts the differences of the continuation utilities in the two states of unequal supply of the two goods for the example discussed in section 3.3.2 of the main text.

When the Pareto weight attached to agent 1 (agent 2) is approaching 1, the continuation utility for the state of abundant supply of good X (good Y) is higher than the continuation utility for the state of scarce supply of good X. However, as suggested by proposition 5, there exists a $\widetilde{\mu}_1 < 1/2$ ($1 - \widetilde{\mu}_1 < 1/2$) past which the ranking of the continuation utilities is reversed.

We are now ready to characterize the sign of the covariance term in (3.33). The following definition of conditional covariance in symmetric states is useful in establishing an upper bound on the last term of equation (3.33).

Definition 2 (Covariance of symmetric states)**.** *Let $z_i, z_{-i} \in \mathcal{N}$ be symmetric states. The conditional covariance between two random variables h and g valued on $\{z_i, z_{-i}\}$ is*

$$cov_{i,-i}\left[h, g|S\right] = \sum_{l=\{i,-i\}} p(z_l)h(z_l)g(z_l) - \left(\sum_{l=\{i,-i\}} p(z_l)h(z_l)\right)\left(\sum_{l=\{i,-i\}} p(z_l)g(l)\right),$$

where $p(z_l) \equiv \pi(z_l)/(\pi(z_i) + \pi(z_{-i}))$.

Proposition 6. *Let $\overline{cov}$ be the sum of the conditional covariances between $\exp\{U_2'/\theta\}$ and S' across all symmetric states:*

$$\overline{cov} = \sum_i cov_{i,-i}\left[\exp\{U_2'/\theta\}, S'|S\right].$$

If $\overline{cov} < 0$*, then*

$$cov\left[\exp\left\{U_2'/\theta\right\}, S'|S\right] \;<\; 0. \tag{3.39}$$

Proof. By the definition of conditional covariance,

$$\begin{aligned} cov\left[\exp\left\{U_2'/\theta\right\}, S'|S\right] &= \sum_{j=1}^{n} \pi(j)\exp\left\{U_2'(j)/\theta\right\} S'(j) - \left[\sum_{j=1}^{n} \pi(j)\exp\left\{U_2'(j)/\theta\right\}\right]\left[\sum_{j=1}^{n} \pi(j)S'(j)\right] \\ &= \overline{\pi}\sum_{i} cov_{i,-i}\left[\exp\left\{U_2'/\theta\right\}, S'|S\right] + cov_{eq}\left[\exp\left\{U_2'/\theta\right\}, S'|S\right] \\ &\quad - \left(\sum_{i} \widetilde{\pi}(i)\exp\left\{U_2'/\theta\right\}\right)\left(\sum_{i} \widetilde{\widetilde{\pi}}(i)S'\right), \end{aligned} \tag{3.40}$$

where $\overline{\pi}$ is a non-negative scalar smaller than one; $\sum_i cov_{i,-i}\left[\exp\left\{U_2'/\theta\right\}, S'|S\right]$ is the conditional covariance across all symmetric states; $cov_{eq}\left[\exp\left\{U_2'/\theta\right\}, S'|S\right]$ is the conditional covariance across all states of equal supply; and $\{\widetilde{\pi}(i)\}_{i=1}^{n}$ and $\left\{\widetilde{\widetilde{\pi}}(i)\right\}_{i=1}^{n}$ are non-negative sequences of scalars.

We know from proposition 1 that the optimal choice of Pareto weights is identical across all states in which the supply of the two goods is the same. This implies that

$$cov_{eq}\left[\exp\left\{U_2'/\theta\right\}, S'|S\right] = 0.$$

We can also conclude that the last term in equation 3.40 is non-negative, being the product of the sums of non-negative terms.

Therefore, we can state that

$$cov\left[\exp\left\{U_2'/\theta\right\}, S'|S\right] \leq \overline{\pi}\sum_{i} cov_{i,-i}\left[\exp\left\{U_2'/\theta\right\}, S'|S\right],$$

which concludes the proof. □

The sign of $\overline{cov}$ is key. Suppose that the current ratio of Pareto weights is in the region between zero and $\min_i\{\widetilde{S}_2^i\}$ as defined in corollary 1. For any two symmetric states, the utility of agent 2 is going to be larger when the supply of good Y is

relatively more abundant. Since the parameter θ is negative, the ranking of U_2/θ is reversed. Proposition 1 suggests that the ratio of Pareto weights is larger when the supply of good Y is larger. Therefore we can conclude that, in this region, $\overline{cov}$, and hence the covariance term in (3.33), are negative. This argument is summarized in figure 3.10.

The next proposition plays a crucial role in our proof for the existence of a well-defined ergodic distribution of the Pareto weights. We focus on the properties of the left tail of the stochastic process $S' = \frac{\mu_1'}{1-\mu_1'} = f_S(z', S)$ and base our proof on all the other propositions developed so far.

Proposition 7. *The stochastic process* $\widehat{S}(S) := \min\left(S, \min_i\{\widetilde{S}_2^i\}\right)$, *where* $\widetilde{S}_2^i$ *is defined as in corollary 1, is a submartingale.*

Proof. Follows directly from all the propositions in this appendix. □

We conclude this section with the proof of proposition 2.

Proof of Proposition 2. We know from Proposition 7 that the stochastic process

$$\widehat{S}(S) = \min\left(S, \min_i\{\widetilde{S}_2^i\}\right)$$

is a submartingale. Since a bounded submartingale cannot converge almost surely to its lower bound (see Sciubba (2005)), then $\widehat{S}(S)$ cannot converge almost surely to 0.

We prove the rest of the proposition by contradiction. Denote by $\Omega = \times_{t=1}^{\infty} z_t$ the set of sample paths of endowments, with representative element $\omega = (z_1, z_2, ..., z_t, ...)$. Given the initial Pareto weight ratio $S_0 = 1$ and the transition dynamics $S' = f_S(z', S)$, the ratio of Pareto weights at time t can be written as a function of the sample path: $S_t = S_t(\omega)$.

Suppose Proposition 2 does not hold. Then, there must exist a set of ω such that

$$Pr\left\{\omega|\lim_{t\to\infty} S_t(\omega) = 0\right\} > 0 \tag{3.41}$$

Denote by $Q = \{\omega|\lim_{t\to\infty} S_t(\omega) = 0\}$ and consider the natural filtration on Q: $Q_1 \subseteq Q_2 \subseteq ... \subseteq Q_t \subseteq ... \subseteq Q$. Recall that $\{z_t\}_{t=1}^{\infty}$ is *i.i.d.* As in Sciubba (2005), Q_{t-1} does not restrict the original probability distribution of z_t, and hence we have that $Pr\{z_t = z(i)|Q_{t-1}\} = \pi(z(i))$. As a consequence, all the results that we have proved so far in this section apply with respect to the conditioning set Q_{t-1}. This means that S_t is a bounded submartingale with respect to the conditioning information set Q_{t-1}. Since a bounded submartingale cannot converge almost surely to its lower bound, then S_t cannot converge almost surely to 0 on Q. This contradicts the definition of Q. Hence, S_t cannot converge to 0 with probability 1. Equivalently, the probability that S_t converges to 0 is null.

Note that $\mu_1\left(\widehat{S}\right) = \frac{\widehat{S}}{1+\widehat{S}}$, that is, $\mu_1\left(\widehat{S}\right)$ is a non-negative, monotonically increasing function of $\widehat{S}$. It follows that μ_1 cannot also converge to 0 almost surely. By repeating all the proofs in the paper for $\mu_2 = 1 - \mu_1$, it follows by symmetry that μ_2 cannot converge to 0 almost surely. Equivalently, μ_1 cannot converge to 1 almost surely either. □

Proof of proposition 3

This section reports the proof of the proposition on the mean-reversion property of the Pareto weight.

Proof of proposition 3. We prove the three statements in proposition 3 in the same order in which they appear in the proposition's claim.

1. We show that only $\mu_1 = 1/2$ is such that $E[\mu_1'|S] = \mu_1$.
   After imposing a symmetrical calibration, the recursive definition of the ratio

of Pareto weights derived in the paper becomes

$$\frac{\mu_1'}{1-\mu_1'} = \frac{\mu_1}{1-\mu_1}\cdot\frac{\exp\{U_1(z',S')/\theta\}}{\sum_{z'}\pi(z')\exp\{U_1(z',S')/\theta\}}\frac{\sum_{z'}\pi(z')\exp\{U_2(z',S')/\theta\}}{\exp\{U_2(z',S')/\theta\}}.$$

By rearranging and taking the conditional expectation operator, we obtain

$$E[\mu_1'|S] = \mu_1 - \mu_1\cdot\frac{cov[\mu_1',\exp\{U_1'/\theta\}|S]}{E[\exp\{U_1'/\theta\}|S]} - (1-\mu_1)\cdot\frac{cov[\mu_1',\exp\{U_2'/\theta\}|S]}{E[\exp\{U_2'/\theta\}|S]} \quad (3.42)$$

For each $\mu_1 = \mu_1^*$ such that $E[\mu_1'|S] = \mu_1^*$, equation (3.42) implies that

$$\mu_1^* = \frac{cov[\mu_1',\exp\{U_2'/\theta\}|S]}{E[\exp\{U_2'/\theta\}|S]}\Bigg/\left(\frac{cov[\mu_1',\exp\{U_2'/\theta\}|S]}{E[\exp\{U_2'/\theta\}|S]} - \frac{cov[\mu_1',\exp\{U_1'/\theta\}|S]}{E[\exp\{U_1'/\theta\}|S]}\right) \quad (3.43)$$

By symmetry, for each $\mu_1^* < 1/2$, there must exist a $\mu_1^{**} = 1-\mu_1^*$ such that $E[\mu_1'|S] = \mu_1^{**}$. At $\mu_1^{**} = 1-\mu_1^*$, equation (3.42) implies that

$$\mu_1^* = \frac{cov[\mu_1',\exp\{U_1'/\theta\}|S]}{E[\exp\{U_1'/\theta\}|S]}\Bigg/\left(\frac{cov[\mu_1',\exp\{U_1'/\theta\}|S]}{E[\exp\{U_1'/\theta\}|S]} - \frac{cov[\mu_1',\exp\{U_2'/\theta\}|S]}{E[\exp\{U_2'/\theta\}|S]}\right) \quad (3.44)$$

Combining equations (3.43) and (3.44), we obtain that at μ_1^* the following condition must hold:

$$\frac{cov[\mu_1',\exp\{U_1'/\theta\}|S]}{E[\exp\{U_1'/\theta\}|S]} = -\frac{cov[\mu_1',\exp\{U_2'/\theta\}|S]}{E[\exp\{U_2'/\theta\}|S]}. \quad (3.45)$$

By plugging condition (3.45) into equation (3.43) or (3.44), we conclude that the only $\mu_1^* \in (0,1)$ such that $E[\mu_1'|S] = \mu_1^*$ is $\mu_1^* = 1/2$.

2. We show that if $\mu_1 > 1/2$, then $E[\mu_1'|S] < \mu_1$.
We know from proposition 7 that the stochastic process $\widehat{S}(S) = min\left(S, \min_i\{\widetilde{S}_2^i\}\right)$, where $\min_i\{\widetilde{S}_2\} > 1$ is a submartingale. Since, by definition, $\widehat{S}(S) = \frac{\mu_1(\widehat{S})}{1-\mu_1(\widehat{S})}$, and the stochastic process μ_1 cannot converge to 1 (see proposition 2), there must exist a $\mu_1^+ > \frac{\min_i \widetilde{S}_2^i}{1+\min_i \widetilde{S}_2^i} > 1/2$ such that $E[S'|S^+] = S^+$, where $S^+ = \frac{\mu_1^+}{1-\mu_1^+}$.

Since $S = \frac{\mu_1}{1-\mu_1}$ is a convex function in μ_1,

$$\begin{aligned}
\frac{\mu_1^+}{1-\mu_1^+} &= E\left[S'|S^+\right] = E\left[\frac{\mu_1'}{1-\mu_1'}\middle| S^+\right] \\
&= cov\left[\mu_1', \frac{1}{1-\mu_1'}\middle| S^+\right] + E\left[\mu_1'|S^+\right] E\left[\frac{1}{1-\mu_1'}\middle| S^+\right] \\
&>= cov\left[\mu_1', \frac{1}{1-\mu_1'}\middle| S^+\right] + E\left[\mu_1'|S^+\right] \frac{1}{1-E\left[\mu_1'|S^+\right]}.
\end{aligned}$$

Since μ_1' and $\frac{1}{1-\mu_1'}$ are both monotonically increasing functions of μ_1', we have $cov\left[\mu_1', \frac{1}{1-\mu_1'}\middle| S^+\right] > 0$. This implies that

$$\begin{aligned}
\frac{\mu_1^+}{1-\mu_1^+} &>= cov\left[\mu_1', \frac{1}{1-\mu_1'}\middle| S^+\right] + E\left[\mu_1'|S^+\right] \frac{1}{1-E\left[\mu_1'|S^+\right]} \\
&> E\left[\mu_1'|S^+\right] \frac{1}{1-E\left[\mu_1'|S^+\right]}
\end{aligned}$$

from which we obtain

$$E\left[\mu_1'|S^+\right] < \mu_1^+.$$

Since we know that there exists only one $\mu_1^* = 1/2$ such that $E\left[\mu_1'|\frac{\mu_1^*}{1-\mu_1^*}\right] = \mu_1^*$, by the continuity of f_S with respect to S, it must be the case that $E\left[\mu_1'|S\right] < \mu_1$, $\forall \mu_1 \in (1/2, 1)$.

3. We show that if $\mu_1 < 1/2$, then $E\left[\mu_1'|S\right] > \mu_1$.
   This proof mirrors the one that we just provided for $\mu_1 > 1/2$, and thus for parsimony we omit it here.

□

3.7.3. Recursive Method and Global Solution

Given the conditions in Ma (1993) and Ma (1996), the social planner's value function, denoted as $Q_p(z, \mu_1) : \mathcal{N} \times [0, 1] \to R$, satisfies the following functional equation

proposed by Lucas and Stokey (1984), and Kan (1995):

$$Q_p(z,\mu_1) = \max_{\{x_i,y_i,q_{i,z'}\}_{i\in\{1,2\},z'\in\mathcal{N}}} \sum_{i=1}^{2} \mu_i \log W_i^*(z,c_i,\{q_{i,z'}\}_{z'}) \tag{3.46}$$

subject to

$$\begin{aligned}
&\mu_2 = 1-\mu_1 \\
&0 \le x_1 \le X(z), \qquad 0 \le x_2 \le X(z) - x_1 \\
&0 \le y_1 \le Y(z), \qquad 0 \le y_2 \le Y(z) - y_1 \\
&c_i = (x_i)^{\lambda_i} (y_i)^{1-\lambda_i}, \quad \forall i = \{1,2\} \\
&0 \le \min_{\mu_1(z')\in[0,1]} Q_p(z',\mu_1(z')) - \mu_1(z')q_{1,z'} - (1-\mu_1(z'))q_{2,z'} \quad \forall z' \in \mathcal{N}
\end{aligned} \tag{3.47}$$

Remark 3.7.1. *Let assumption (1)–(4) hold. When $\mu_i = 0$, we can interpret that $\mu_i \log W_i^*(z,c_i,\{q_{i,z'}\}_{z'}) = 0$. Since $0 < X(z) < \infty$ and $0 < Y(z) < \infty$, $\forall z \in \mathcal{N}$, it can be proved that there exists a unique bounded and continuous solution to (3.46)–(3.47). A feasible allocation is Pareto optimal if and only if it is generated recursively from the solution to (3.46)–(3.47).*

Assumptions (1)–(4) are sufficient to apply the contraction mapping theorem to the planner's problem described by the system of equations (3.4)-(3.5). This result is parallel to those proposed by Lucas and Stokey (1984), Kan (1995) and Anderson (2005).

IES=1: a two-step approach. It is convenient to use the definition of risk sensitive preferences in equation (3.1) to notice that the recursive planner's problem can be decomposed in two parts:

$$Q_p(s,\mu_1) = Q_p^A(s,\mu_1) + Q_p^B(s,\mu_1)$$

where $Q_p^A(s,\mu_1)$ is the static social planning problem

$$Q_p^A(s,\mu_1) \equiv \max_{\{x_i\geq 0, y_i\geq 0\}_{i\in\{1,2\}}} \mu_1(1-\delta_1)\log c_1 + (1-\mu_1)(1-\delta_2)\log c_2 \quad (3.48)$$

subject to:

$$\begin{aligned} X(z) &\geq x_1 + x_2, \qquad & Y(z) &\geq y_1 + y_2, \\ c_i &= (x_i)^{\lambda_i}(y_i)^{1-\lambda_i} \quad & i &= 1,2; \end{aligned}$$

and $Q_p^B(s,\mu_1)$ is related to the allocation of future utilities

$$\begin{aligned} Q_p^B(z,\mu_1) \equiv \max_{D_{z'}} \mu_1\delta_1 \log\left[\sum_{z'} \exp\{(1-\gamma_1)(q_{2,z'}+D_{z'})\}\pi_s\right] + \\ (1-\mu_1)\delta_2 \log\left[\sum_{z'} \exp\{(1-\gamma_2)q_{2,z'}\}\pi_s\right] \end{aligned} \quad (3.49)$$

subject to

$$q_{2,z'} = Q_p\left(z', \frac{dQ_p}{d\mu_1}^{-1}(z', D_{z'})\right) - D_{z'}\frac{dQ_p}{d\mu_1}^{-1}(z', D_{z'}) \quad \forall z' \in \mathcal{N}, \quad (3.50)$$

where $D_{z'}$ is defined as the difference of the future utilities in state z': $D_{z'} \equiv q_{1,z'} - q_{2,z'}$. Equation (3.50) defines the Pareto frontier and is obtained using equations (3.5)–(3.7). By equation (3.7), the first derivative of Q_p is strictly increasing, hence, invertible. Equation (3.6) implies that:

$$\mu_1(z) = \frac{dQ_p}{d\mu_1}^{-1}(z, q_{1,z} - q_{2,z}). \quad (3.51)$$

3.7.4. Allocation as a Function of Pareto Weights

Let $W_t^i = W(C_t^i, U_{t+1}^i)$ be the right-hand side of equation (3.14). If we denote the partial derivatives of the aggregator W^i as

$$W_{1,t}^i := \frac{\partial W_t^i}{\partial C_t^i}, \quad W_{2,t}^i := \frac{\partial W_t^i}{\partial U_{t+1}^i},$$

then the stochastic discount factor is equal to

$$M_{t+1}^i = \frac{W_{2,t}^i W_{1,t+1}^i}{W_{1,t}^i} \quad \forall i = \{1, 2\}. \tag{3.52}$$

The optimality condition for the allocation of good X_t for $t = 1, 2, ...$ in each possible state is

$$\mu_0^h \cdot \left(\prod_{j=0}^{t-1} W_{2,j}^1 \right) \cdot W_{1,t}^1 C_t^1 \frac{\lambda_1}{x_t^1} = \frac{(1-\lambda_2)}{x_t^2} C_t^2 W_{1,t}^2 \cdot \left(\prod_{j=0}^{t-1} W_{2,j}^2 \right) \cdot \mu_0^2. \tag{3.53}$$

Define the date t Pareto weights as

$$\begin{aligned} \mu_t^i &= \mu_0^i \cdot \left(\prod_{j=0}^{t-1} W_{2,j}^i \right) \cdot W_{1,t}^i C_t^i \\ &= \mu_{t-1}^i \cdot W_{2,t-1}^i \cdot \frac{W_{1,t}^i}{W_{1,t-1}^i} \cdot \frac{C_t^i}{C_{t-1}^i} = \mu_{t-1}^i \cdot M_t^i \cdot \exp\left\{\Delta c_t^i\right\}, \quad \forall i \in \{h, f\}. \end{aligned}$$

It follows that equation (3.53) can be rewritten as

$$\mu_t^1 \cdot \frac{\lambda_1}{x_t^1} = \frac{(1-\lambda_2)}{x_t^2} \cdot \mu_t^2. \tag{3.54}$$

Let $S_t := \mu_t^1/\mu_t^2$. Then the optimality condition in equation (3.54) can be represented by the following system of recursive equations:

$$\begin{aligned} S_t \frac{\lambda_1}{x_t^1} &= \frac{(1-\lambda_2)}{x_t^2} \\ S_t &= S_{t-1} \frac{M_t^1 e^{\Delta c_t^1}}{M_t^2 e^{\Delta c_t^2}}, \end{aligned} \tag{3.55}$$

where

$$M_{i,t+1} = \delta \left(\frac{C_{i,t+1}}{C_{i,t}} \right)^{-\frac{1}{\psi}} \left(\frac{U_{i,t+1}^{1-\gamma}}{E_t \left[U_{i,t+1}^{1-\gamma} \right]} \right)^{\frac{1/\psi-\gamma}{1-\gamma}}. \tag{3.56}$$

A similar first-order condition applies with respect to good Y.

Approximation methods and errors. We use perturbation methods to solve our system of equations and compute our policy functions using the `dynare++4.2.1` package. All variables are expressed in log-units.

In tables 3.1 and 3.2, we report statistics regarding the maximum cumulative approximation error (Cumul. ERR) for our relative pseudo-Pareto weights, that is, the key drivers of both consumption shares. The maximum is taken across agents, and we account for the fact that the relative Pareto weights are persistent and errors can accumulate over longer simulations.

More specifically, we construct recursively the following processes:

$$\tilde{S}_{j,t} = \tilde{S}_{j,t-1} \cdot \frac{\hat{M}_{j,t}}{\hat{M}_{1,t}} \cdot \left(\frac{\hat{C}_{j,t}/\hat{C}_{j,t-1}}{\hat{C}_{1,t}/\hat{C}_{1,t-1}} \right), \quad \forall t \geq 1, \tag{3.57}$$

where we adopt the "$\hat{\cdot}$" notation to indicate approximated variables. We initialize the recursion at different starting points, $\tilde{S}_{j,0}$ and we then define the error for agent j as follows:

$$err_t^j = \frac{1}{t} \sum_{\tau=1}^{t} \left| \frac{\tilde{S}_{j,\tau}}{\hat{S}_{j,\tau}} - 1 \right| \cdot 100.$$

$\widehat{S}_j$ is the approximated relative pseudo-Pareto weight that we obtain directly from our perturbation method. $\tilde{S}_j$ is constructed by dynamically updating our equilibrium

recursion (i.e., an exact dynamic equation) for $S_{j,t}$ using our approximated forcing elements, $\hat{M}_{j,t}$, $\hat{M}_{1,t}$, $\hat{C}_{j,t}/\hat{C}_{j,t-1}$, and $\hat{C}_{1,t}/\hat{C}_{1,t-1}$. Examination of the discrepancy between $\tilde{S}_j$ and $\hat{S}_j$ constitutes a valid metric for assessing the quality of the approximation, as this discrepancy accounts for the persistent impact that the errors on our forcing terms can have on the relative pseudo-Pareto weights.

We also show percentiles of the one-period-ahead errors, which tend to be many orders of magnitude smaller. Across all cases, the approximation errors are small and comparable to those obtained under regular time-additive preferences and thus are negligible for all practical purposes.

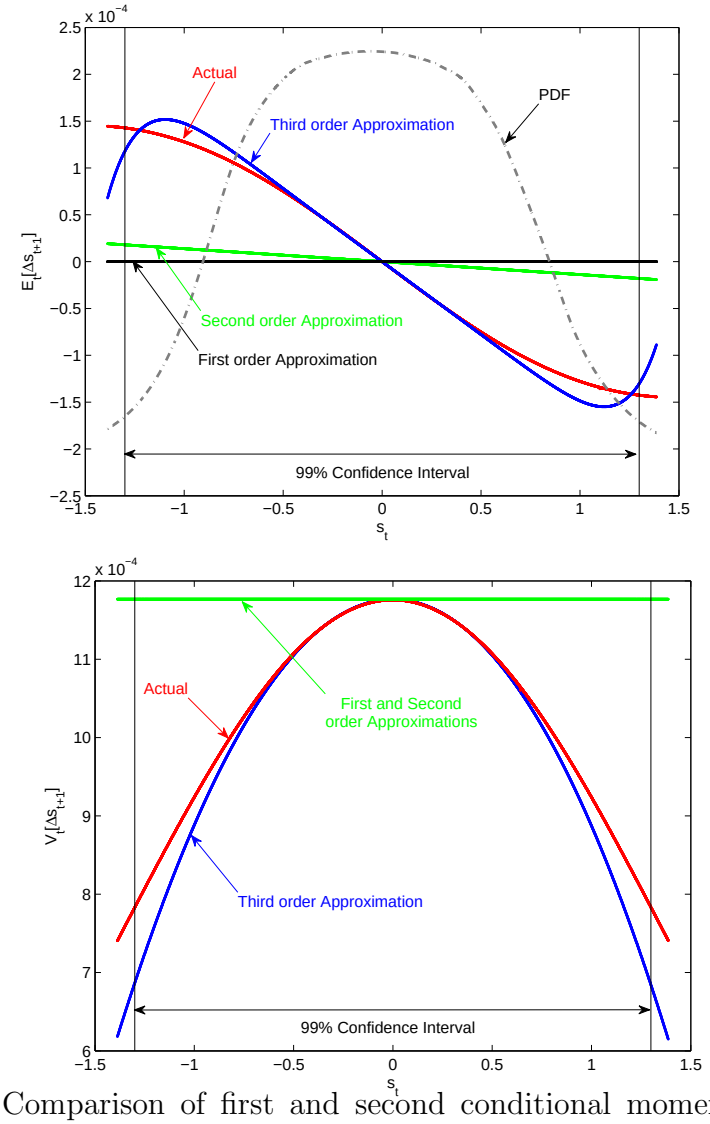
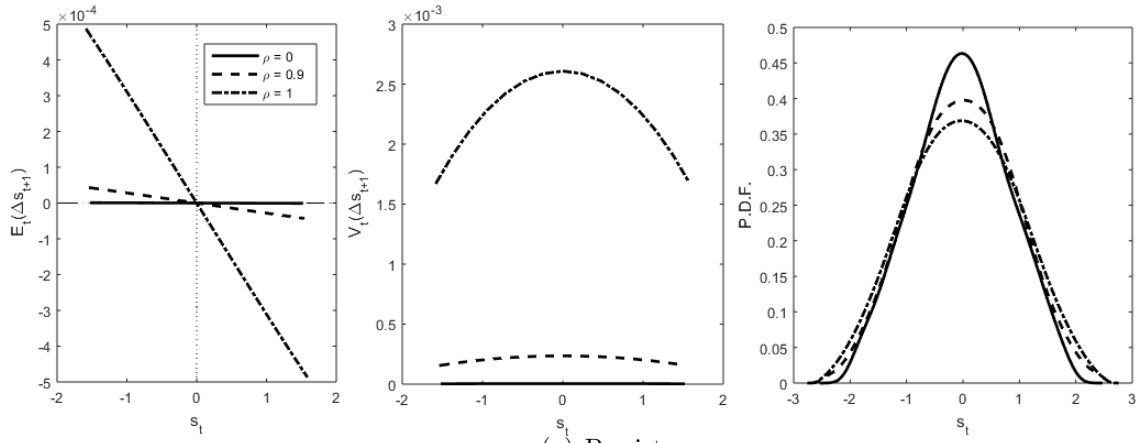
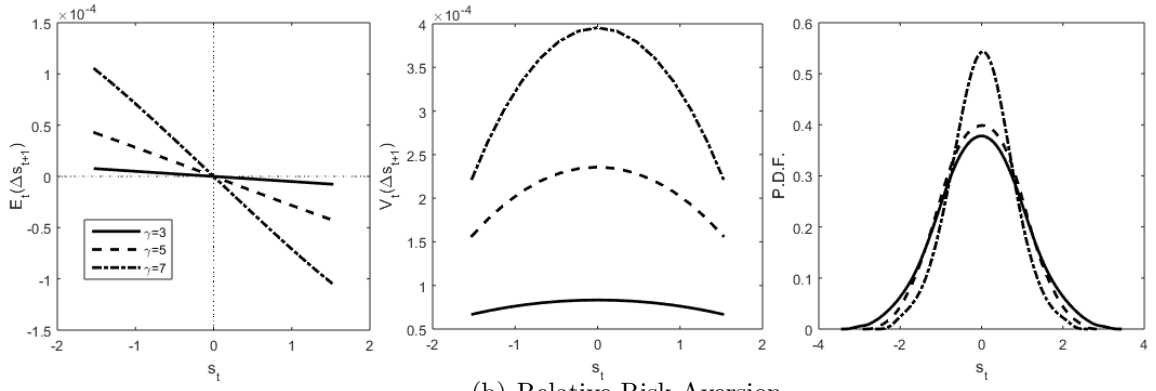


Figure 3.4: Comparison of first and second conditional moments across several solution methods.

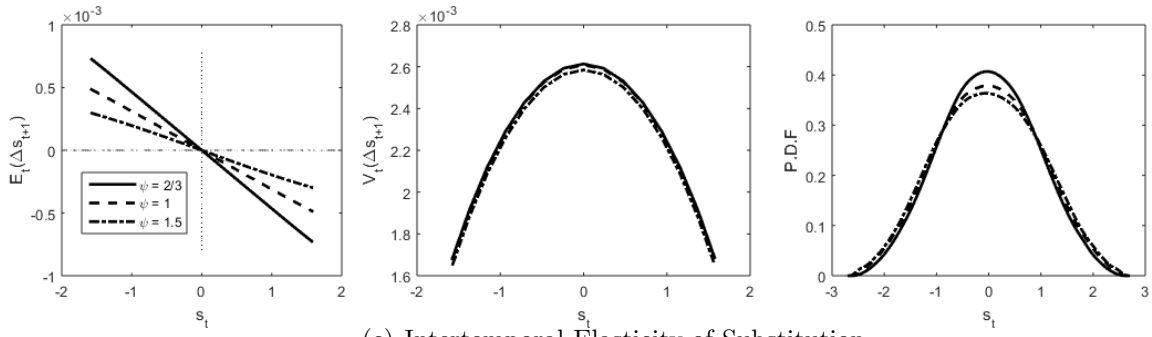
Notes: Comparison of first and second conditional moments across several solution methods. We denote the log of the ratio of the Pareto weights by $s_t \equiv \log(S_t)$ and its growth by $\Delta s_{t+1} = s_{t+1} - s_t$, respectively. The left panel reports the conditional mean $E[\Delta s_{t+1}|s_t]$ as a function of s_t. The right panel reports the conditional variance of Δs_{t+1} with respect to s_t. In each panel, the curve label "Actual" refers to the solution obtained through value-function iterations. The other three curves are based on Taylor approximations of higher order. In the left panel, the probability density function (PDF) is computed using the actual policy.



(a) Persistence



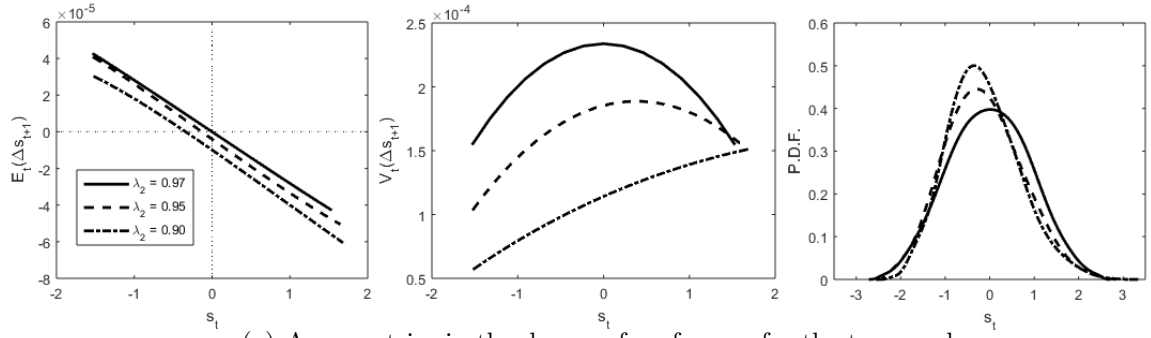
(b) Relative Risk Aversion



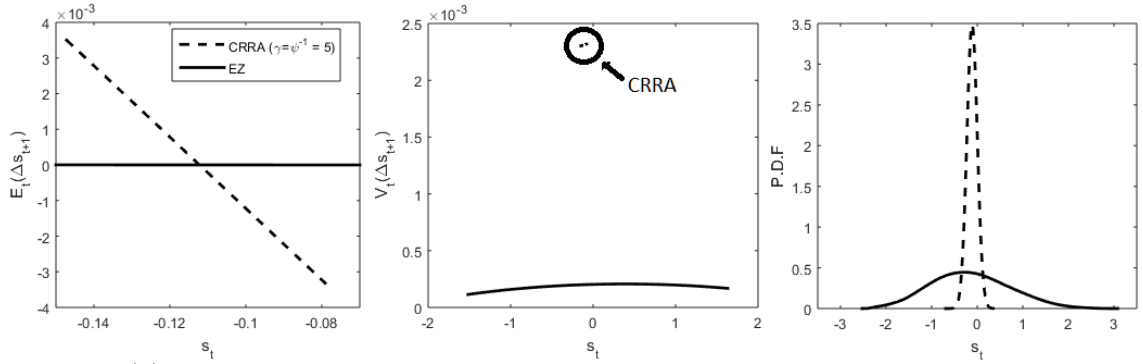
(c) Intertemporal Elasticity of Substitution

Figure 3.5: The role of persistence and preferences

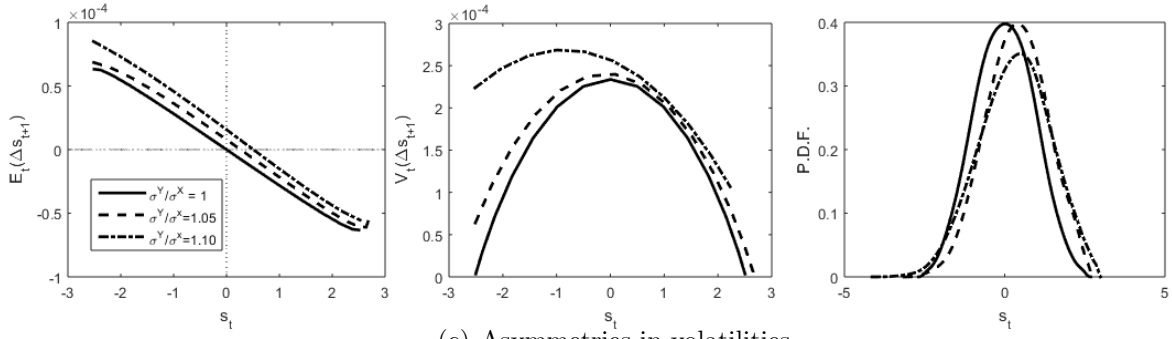
Notes: The role of persistence and preferences. Our model features Epstein and Zin (1989) preferences as specified in equation (3.14) and the endowments reported in equation (3.16). Our reference calibration is detailed in section 3.5. We vary one parameter at a time for both countries, leaving the other parameters unchanged.



(a) Asymmetries in the degree of preference for the two goods



(b) Asymmetries in the degree of preference for the two goods - CRRA versus EZ



(c) Asymmetries in volatilities

Figure 3.6: Asymmetric calibrations

Notes: Our model features Epstein and Zin (1989) preferences as specified in equation (3.14) and the endowments reported in equation (3.16). Our reference calibration is detailed in section 3.5. We vary one parameter at a time, leaving the others unchanged.

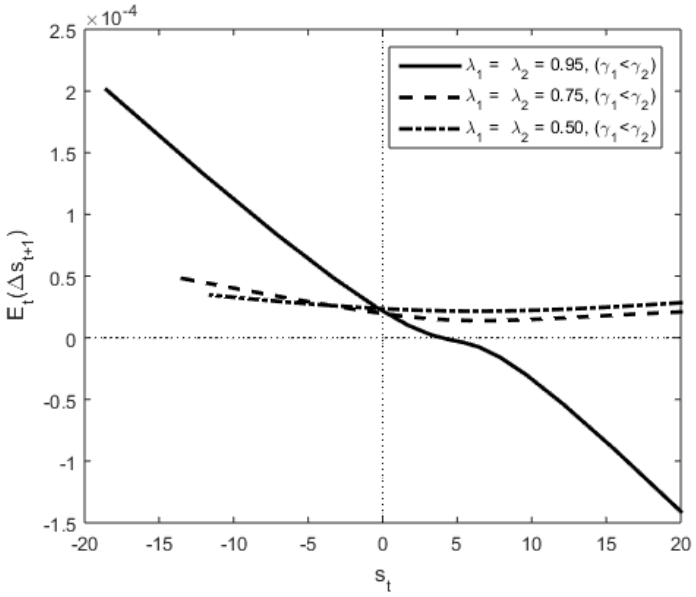
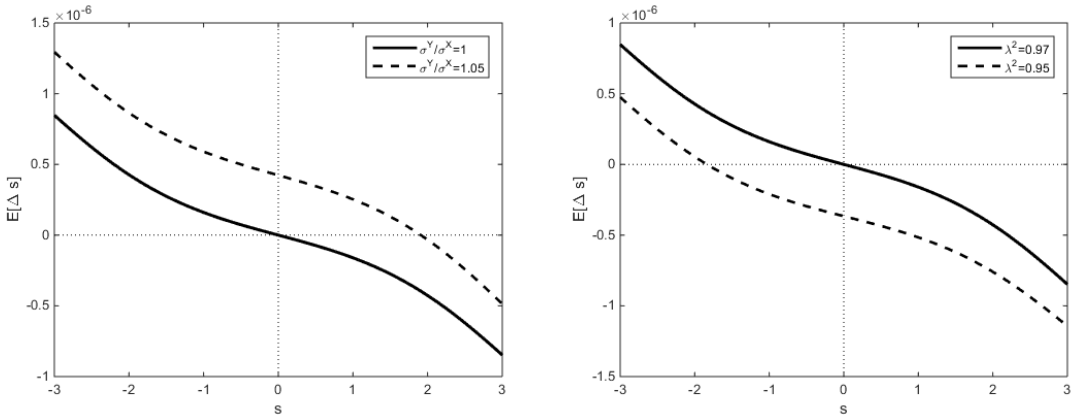


Figure 3.7: Asymmetric risk aversion

Notes: Our model features Epstein and Zin (1989) preferences as specified in equation (3.14) and the endowments reported in equation (3.16). Our reference calibration is detailed in section 3.5. We set $\gamma_2 = 7$ and $\gamma_1 = 5$ and vary simultaneously λ_1 and λ_2.



(a) Heterogenous volatility

(b) Heterogenous degree of preference for the two goods

Figure 3.8: Asymmetric calibrations in the 2-period model

Notes: All results are based on the two-period model developed in the Appendix. All common parameters are set as in our reference calibration detailed in section 3.5. We vary one parameter at a time, leaving the others unchanged.

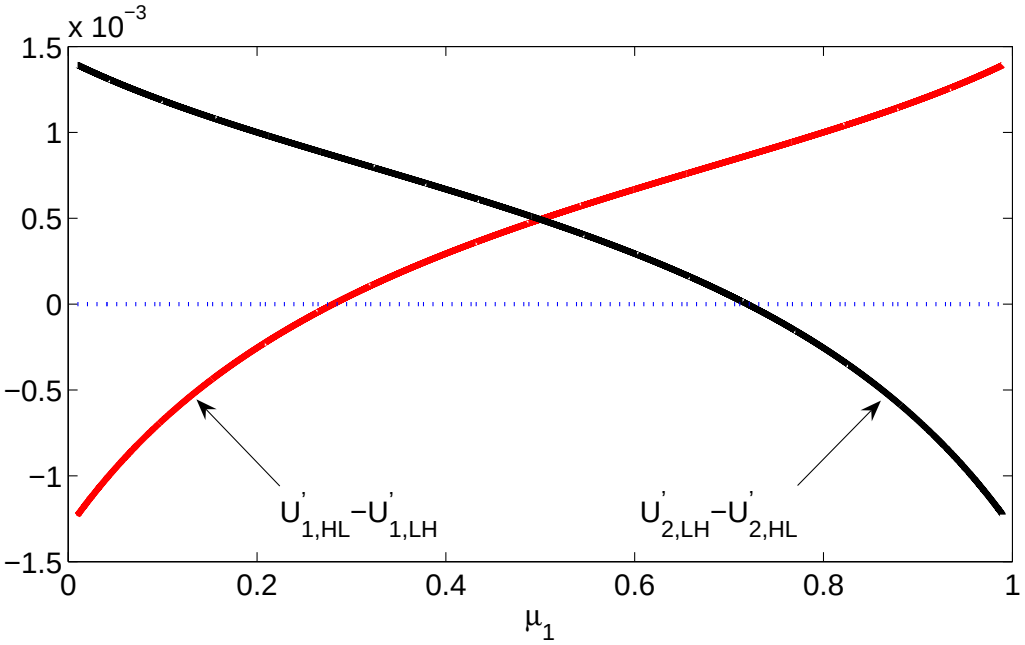


Figure 3.9: Difference of continuation utilities in symmetric states

Notes: This figure depicts the difference of the continuation utilities in the states of high supply of good X–low supply of good Y, $U'_{i,HL}$, and low supply of good X–high supply of good Y, $U'_{i,LH}$, for $i = 1, 2$. The red line refers to agent 1, the black line to agent 2.

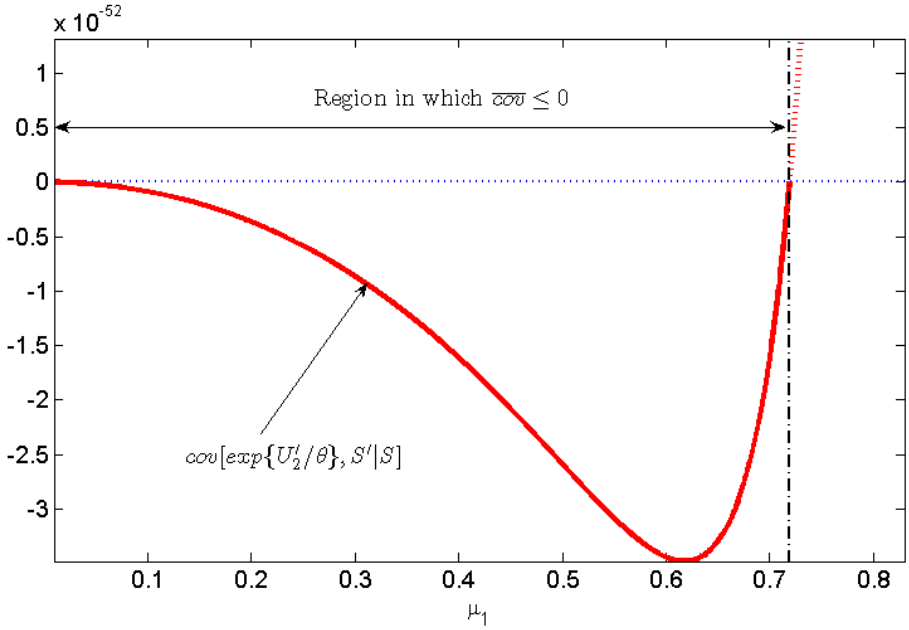


Figure 3.10: Upper bound on the conditional covariance

Notes: Upper bound on the conditional covariance between $\exp\{U_2'/\theta\}$ and S'as a function of the current Pareto weight.

Chapter 4

Good Uncertainty, Bad Uncertainty, and the Aggregated Effects

4.1 Introduction

Since the great recession, the impacts of uncertainty shock over the aggregate economy and financial market have been investigated by both policymakers and academic researchers. Seminal work includes Bloom (2009), Bloom et al. (2018), Gilchrist et al. (2014), Christiano et al. (2014), and Basu and Bundick (2017), among others. However, almost all these academic literature focus on the adverse effects of uncertainty shock over the aggregate economy. While it's conventional wisdom that uncertainty is usually bad, there are certain channels through which uncertainty shock can positively influence the economy. Such channels include the growth option channel (see, e.g., Jovanovic (2009)), the Oi (1961)-Hartman (1972)-Abel (1983)-Caballero (1991) channel, and the learning-by-doing channel. Since different theories emphasize different channels, the aggregate impact of uncertainty is worth a further investigation.

In this chapter, I aim to evaluate the aggregate effects of both good and bad uncertainty shocks over the aggregate macroeconomic quantities through the lens of DSGE model. Explicitly, I define a first-order good uncertainty shock through the investment option channel. The good uncertainty shock can be easily incorporated into the standard medium size DSGE model and thus facilitates evaluating the positive effect of the uncertainty in such a model. I find that good uncertainty shock can be an essential driver of economic fluctuations and generate the comovement of macroeconomic quantities. Moreover, through the impulse response analysis, I illustrate that the positive impact of good uncertainty could outweigh the negative impact of bad uncertainty. Last but not least, I also show the link between the defined good

uncertainty and the marginal efficiency of investment (M.E.I.) shock widely used in the literature (Justiniano et al. (2011)).

The chapter is organized as follows. Section 4.2 discusses the construction of good uncertainty shock and its properties. Section 4.3 outlines the baseline DSGE model that I use in the analysis of the aggregate effects of both good and bad uncertainty shocks. Section 4.4 presents the calibration and evaluates the model's performance. Section 4.5 concludes.

4.2 The Good Uncertainty

In this section, I illustrate how to define the first-order good uncertainty through investment option channel, based on the framework of Wang and Wen (2012).

Suppose there exists a continuum of competitive sub-intermediate good firms indexed by $i \in [0, 1]$ in the economy. Each firm has the following standard production function:

$$Y_{it} = (U_{it}K_{it})^{\alpha}(Z_t N_{it})^{1-\alpha} - \tilde{\Phi} \tag{4.1}$$

where U_{it} is the physical capital utilization rate. K_{it} and N_{it} are respectively the firm-level physical capital and employment. Z_t is the aggregate labor-augmenting technology and $\tilde{\Phi}$ is the fixed cost of production. The law of motion for accumulating physical capital of firm i is written as

$$K_{i,t+1} = (1 - \delta(U_{it}))K_{it} + \epsilon_{it}I_{it} \tag{4.2}$$

where $\delta(U_{it})$ is the depreciation rate which depends on the intensity of utilization. I_{it} is the investment expenditure. The transformation of the investment expenditure to physical capital at time t is subject to an i.i.d. idiosyncratic shock, which I assume follows a normal distribution: $\epsilon_{it} \sim N(0, \sigma_t)$. The idiosyncratic shock can be interpreted as the quality of the firm's investment opportunity. Clearly, the volatility σ_t

controls the dispersion of the idiosyncratic distribution and thus is a natural measure of the uncertainty of the investment opportunity. As we will see later, σ_t is the good uncertainty defined in my model, in the sense that an increase in σ_t boosts the economy. Note that since σ_t is time varying, ϵ_{it} is assumed to be independent but may not have the identical distribution across time.

Sub-intermediate good firm i's problem is to maximize its discounted real dividend streams $D_{it} = P_{mt}Y_{it} - W_t N_{it} - I_{it}$, where P_{mt} is the selling price. Similar to Wang and Wen (2012), I make the following two additional assumptions:

(i) Firm utilizes credit to finance its investment expenditure and faces the following borrowing constraint:

$$I_{it} \leq \theta K_{it} \tag{4.3}$$

where the parameter $\theta > 0$. It says the firm's investment expenditure could not exceed a fraction θ of its total capital stock.

(ii) Firm-level investment is totally irreversible:

$$I_{it} \geq 0 \tag{4.4}$$

Then the firm's problem can be written as

$$\max E_0 \sum_{t=0}^{\infty} M_t D_{it} \tag{4.5}$$

subject to the equations (2), (3), and (4). Equip these constraints with Lagrangian multipliers $\{\lambda_{it}, \mu_{it}, \pi_{it}\}$ and follow a similar argument as in Wang and Wen (2012), we can obtain the following results:

$$U_{it} = U_t \tag{4.6}$$

$$\lambda_{it} = \lambda_t = \frac{1}{\epsilon_t^\star} \tag{4.7}$$

$$I_{it} = \begin{cases} \theta K_{it} & \epsilon_{it} \geq \epsilon_t^\star \\ 0 & \epsilon_{it} < \epsilon_t^\star \end{cases} \tag{4.8}$$

$$\frac{1}{\epsilon_t^\star} = E_t M_{t+1} \left\{ U_{t+1} R_{t+1}^k + (1 - \delta(U_{t+1})) \frac{1}{\epsilon_{t+1}^\star} + F(\epsilon_{t+1}^\star) \right\} \tag{4.9}$$

Equation (4.6) says that every firm chooses the same utilization rate at time t, irrespective of the size of its capital stock. Equation (4.7) implies that each firm faces the same marginal cost of installing one unit of capital. Moreover, the marginal cost links to the optimal cutoff $\epsilon_t^\star$, a specific level of realization of the idiosyncratic shock.

Equation (4.8) illustrates the option feature of the idiosyncratic shock. Intuitively, when an firm's idiosyncratic shock realization is larger than the optimal cutoff $\epsilon_t^\star$, the efficiency of the transformation of investment into physical capital is high, the firm finds it optimal to invest up to its borrowing limit. Otherwise, the investment opportunity is not good enough. The firm abandons the option and stays inactive for time period t.

Equation (4.9) is the traditional Euler equation for the firm-level investment. The LHS is the marginal cost of capital, which equals the marginal benefit on the RHS.

Now we are able to introduce the definition of the good uncertainty. Using the law of large numbers, we can obtain the following two equations characterizing the aggregate level investment and physical capital:

$$\frac{I_t}{K_t} = \theta[1 - \Phi(\epsilon_t^\star; \sigma_t)] \tag{4.10}$$

$$K_{t+1} = (1 - \delta(U_t))K_t + \theta K_t \int_{\epsilon \geq \epsilon_t^\star} \epsilon d\Phi(\epsilon; \sigma_t) \tag{4.11}$$

Note that the term $[1 - \Phi(\epsilon_t^\star; \sigma_t)]$ in Equation (4.10) and the term $\int_{\epsilon \geq \epsilon_t^\star} \epsilon d\Phi(\epsilon; \sigma_t)$ in Equation (4.11) are both increasing functions of σ_t. Therefore, both I_t and K_t will increase after a positive innovation to σ_t. Intuitively, when σ_t increases, there are more high-quality investment opportunities in the economy, which incentivizes more firms to invest to their borrowing limit, leading to an investment boom. Therefore,

it's natural to interpret σ_t as the first-order good uncertainty. Later in the numerical analysis, it will be shown that a higher level of σ_t also increases the labor supply and the aggregate output in the economy, making it a potential driver of the business cycle fluctuation.

Next we show the connection between the defined good uncertainty and the M.E.I. shock in the macroeconomic literature. Suppose there is a representative firm whose physical capital obeys the following evolution:

$$K_{t+1} = (1 - \delta(U_t))K_t + \tilde{\sigma}_t \Gamma(\frac{I_t}{K_t})K_t$$

Here $\tilde{\sigma}_t$ denotes the M.E.I. shock. In the current model setting, by combining Equation (4.10) and Equation (4.11) we can get the following expression

$$K_{t+1} = (1 - \delta(U_t))K_t + \tilde{\Gamma}(\frac{I_t}{K_t}, \sigma_t)K_t$$

where $\tilde{\Gamma}$ can be interpreted as a pseudo adjustment cost function. By comparing the above two expressions, we have the following observation: if we assume $\tilde{\Gamma}$ has a multiplicative structure such that $\tilde{\Gamma}(\frac{I_t}{K_t}, \sigma_t) = \sigma_t \bar{\Gamma}(\frac{I_t}{K_t})$, then the good uncertainty σ_t in the current setting can also be thought of as a M.E.I. shock.

4.3 A Full DSGE Model

Since the goal of this chapter is to evaluate the aggregate effect of both good and bad uncertainty, I need to specify the source of bad uncertainty as well. Here I adopt the notion of bad uncertainty from Basu and Bundick (2017), which is a second moment shock to the household's discount rate. This section outlines the baseline DSGE model for the analysis of good and bad uncertainty. The model features a representative household, a representative final good producer, and bundles of intermediate good producers and sub-intermediate good producers. A simple Taylor

rule is employed by the monetary authority. Prices are sticky followingRotemberg (1982).

There are a continuum of $[0,1]$ competitive firms producing the sub-intermediate goods in this economy. They are described in Section 2. These sub-intermediate goods are purchased as inputs by a continuum of $[0,1]$ intermediate good firms, which are monopolistic competitors. Each intermediate good firm i produces one specific type of intermediate good $\tilde{Y}_{it}$ and faces a quadratic adjustment cost ψ_p of changing its nominal price P_{it}. The firm's problem is to choose the nominal price P_{it} to maximize the discounted real cash flow $\tilde{D}_{it}$, given the aggregate demand Y_t and price P_t of the final good producer:

$$\tilde{D}_{it} = \left(\frac{P_{it}}{P_t}\right)^{1-\theta_\mu} Y_t - P_{mt}\left(\frac{P_{it}}{P_t}\right)^{-\theta_\mu} Y_t - \frac{\psi_p}{2}\left(\frac{P_{it}}{\Pi P_{i,t-1}} - 1\right)^2 Y_t \tag{4.12}$$

where P_{mt} is the purchase price of the sub-intermediate goods and Π is the steady state inflation rate.

The representative final good producer generates the final goods and behaves competitively in the economy. It is a bundle of non-R&D sector intermediate goods, using the CES technology

$$Y_t = \left(\int_0^1 \tilde{Y}_{it}^{\frac{\epsilon-1}{\epsilon}} di\right)^{\frac{\epsilon}{\epsilon-1}} \tag{4.13}$$

The elasticity of substitution is controlled by ϵ. Profit maximization by the final firm yields a downward-sloping demand curve for each intermediate firm

$$\tilde{Y}_{it} = \left(\frac{P_{it}}{P_t}\right)^{-\epsilon} Y_t \tag{4.14}$$

and the associated price index

$$P_t = \left(\int_0^1 P_{it}^{1-\epsilon} di\right)^{\frac{1}{1-\epsilon}} \tag{4.15}$$

In the symmetric equilibrium, all firms have the identical problems and the het-

erogeneity of the intermediate good firms collapse. Specifically, the price set by each firm is equal: $P_{it} = P_t$. This facilitates the definition of the gross inflation $\Pi_t = \frac{P_t}{P_{t-1}}$.

The simple Taylor rule employed by the central bank targets both inflation and output growth. Specifically,

$$r_t = r + \rho_\pi(\pi_t - \pi) + \rho_y \Delta y_t \tag{4.16}$$

where $r_t = \ln(R_t), \pi_t = \ln(\Pi_t)$, and $\Delta_y = \ln(Y_t/Y_{t-1})$. The Euler equation for the nominal interest rate is standard

$$E_t\left(M_{t+1}\frac{R_t}{\Pi_{t+1}}\right) = 1 \tag{4.17}$$

The representative household features the Epstein-Zin preference

$$V_t = \left[a_t \tilde{C}_t^{1-\frac{1}{\psi}} + \beta(E_t[V_{t+1}^{1-\gamma}])^{\frac{1-\frac{1}{\psi}}{1-\gamma}}\right]^{\frac{1}{1-\frac{1}{\psi}}} \tag{4.18}$$

Here I use parameter γ to denote the relative risk aversion. Besides, the parameter ψ controls the household's incentive to smooth consumption (E.I.S). When $\gamma = \frac{1}{\psi}$, the preference collapses to the standard time-additive preference. The discount rate β is subject to a shock process a_t, which is described later. The utility defines over the consumption bundle $\tilde{C}_t$

$$\tilde{C}_t = C_t^{\varphi}(1 - L_t)^{1-\varphi} \tag{4.19}$$

where C_t denotes the consumption of the household and L_t is the labor supply.
The budget constraint of the representative household can be written as follows:

$$C_t + \frac{P_t^E}{P_t}S_{t+1} + \frac{1}{R_{ft}}B_{t+1} \leq W_t L_t + (\frac{D_t^E}{P_t} + \frac{P_t^E}{P_t})S_t + B_t \tag{4.20}$$

where $R_{f,t}$ denotes the real risk-free rate and B_{t+1} represents the one-period riskless bonds. Equity shares have a nominal price of P_t^E and payout the nominal dividend

D_t^E. And the real wage is denoted by $W_t L_t$.

The stochastic discount factor is standard:

$$M_{t,t+1} = \beta \left(\frac{a_{t+1}}{a_t}\right) \left(\frac{C_{t+1}}{C_t}\right)^{-1} \left(\frac{\tilde{C}_{t+1}}{\tilde{C}_t}\right)^{1-\frac{1}{\psi}} \left(\frac{V_{t+1}}{E_t[V_{t+1}^{1-\gamma}]^{\frac{1}{1-\gamma}}}\right)^{\frac{1}{\psi}-\gamma} \tag{4.21}$$

where the third term captures the aversion to continuation utility risk. In addition, the household labor supply condition is

$$W_t = \left(\frac{1}{\varphi} - 1\right) \frac{C_t}{(1 - L_t)} \tag{4.22}$$

There are four exogenous processes in the model, which are parameterized as follows:

$$a_t = (1 - \rho_a)a + \rho_a a_{t-1} + \sigma_{t-1}^a \epsilon_t^a$$

$$\sigma_t^a = (1 - \rho_{\sigma^a})\sigma^a + \rho_{\sigma^a}\sigma_{t-1}^a + \sigma^{\sigma^a}\epsilon_t^{\sigma^a}$$

$$Z_t = (1 - \rho_Z)Z + \rho_Z Z_{t-1} + \sigma^Z \epsilon_t^Z$$

$$\sigma_t = (1 - \rho_\sigma)\sigma + \rho_\sigma \sigma_{t-1} + \sigma^\sigma \epsilon_t^\sigma$$

Recall that a_t is the shock to the household preference. σ_t^a is the second moment shock to a_t, which affects the demand uncertainty of the household. Following Basu and Bundick (2017), I label σ_t^a as the measure of bad uncertainty. Z_t is the neutral technology process to the production function of the sub-intermediate good firms. Finally, σ_t controls the dispersion of the investment opportunity of the sub-intermediate good firms. Following the previous analysis, I label it as the good uncertainty.

4.4 Quantitative Analysis

This section further brings the model discussed in the previous section to the data, and analyze the sign and magnitude of an increase in either the good or the bad

uncertainty. Table 4.1 lists the model's quarterly calibration, where the parameters in bold are parameters estimated using SMM, and the rest of the parameters are calibrated following Basu and Bundick (2017). Specifically, I use the unconditional volatility of output, consumption, investment, and hours worked to discipline the two exogenous processes related to good and bad uncertainty. The estimates imply that when compared to the bad uncertainty process, the good uncertainty process is less persistent yet has larger volatility. The IES is set to be 0.95, which is slightly less than 1 and is based on Basu and Bundick (2017). In the model, the household prefers the late resolution of the uncertainty and thus dislikes any short-run risk. The risk aversion is calibrated as 80 following Rudebusch and Swanson (2012). The calibration of δ_0 implies a 10 percent annual depreciate rate at the steady state. δ_1 is derived from the equilibrium condition, while δ_2 directly follows Basu and Bundick (2017). The parameters governing the Taylor rule, as well as the price adjustment parameters, are also standard as in the literature. The parameters related to the neutral technology process and the household preference process are based on the estimations of Basu and Bundick (2017). The parameter θ, which governs the borrowing limits of sub-intermediate good firms, is set to be 0.5.

Regarding the model performance, as a first check, Panel A of Table 4.2 reports the unconditional volatility of macroeconomic aggregates. Except for the volatility of the hours worked, the model-implied unconditional volatility stays close to the counterparts in the data, which indicates the SMM method performs reasonably well. Panel B of Table 4.2 displays the stochastic volatility of macroeconomic aggregates. Although the model generates sizable conditional volatility, the magnitude is only roughly half of that in the data.

I further investigate the macroeconomic comovement induced by uncertainty shocks. Figure 4.1 plots the impulse responses of one standard-deviation increment in the good uncertainty. It indicates that the innovation to the good uncertainty could act as a driver of the aggregate macro quantities. A heightened good uncertainty

implies that there are more high-quality investment options in this economy, which incentivizes more sub-intermediate good firms to execute the investment options, generating an investment boom. The higher stock of capital increases the productivity of the labor, so the household is willing to work longer. Higher capital stock and more extended hours worked lead to higher output, which can be further used for consumption. Higher demand for the production inputs pushes up the marginal cost, leading to a lower markup when the price is sticky.

Figure 4.2, on the contrary, plots the impulse responses to the heightened bad uncertainty. As in the Basu and Bundick (2017), higher bad uncertainty reduces the household's demand for the consumption goods. As there is less demand for the final good in the economy, it's not optimal for the sub-intermediate good firms to hold a large amount of capital stock. Hence, they reduce the capital stock by executing fewer investment options. The lower marginal product of labor due to the lower capital stock implies that fewer workers are hired by these firms, reducing the total amount of hours worked. Hence, the bad uncertainty shock from the demand side could also drive the comovement of the aggregate quantities.

Figure 4.1 and Figure 4.2 also shed some light on the aggregate impacts of the good uncertainty and the bad uncertainty. A one standard-deviation increment in the good uncertainty is associated with a roughly 0.8 percent increase in output, a 0.2 percent increase in consumption, a 3.5 percent increase in investment, and a 0.6 percent increase in the hours worked. These effects are larger in magnitude than those triggered by a one standard-deviation increment in the bad uncertainty, which generates roughly a 0.3 percent decrease in output, a 0.2 percent decrease in consumption, a 0.5 percent decrease in the investment, and a 0.2 percent decrease in the hours worked.

4.5 Concluding Remarks

This chapter introduces a first-order good uncertainty through the investment option channel, which is shown to be able to generate comovement of macro quantities. There is also a link of the good uncertainty to the marginal efficiency of investment (M.E.I.) shock used in the literature. Incorporating such type of good uncertainty into a standard DSGE model with bad uncertainty from demand side, I illustrate using impulse response analysis that the positive effects induced by the good uncertainty could outweigh the adverse effects triggered by the bad uncertainty.

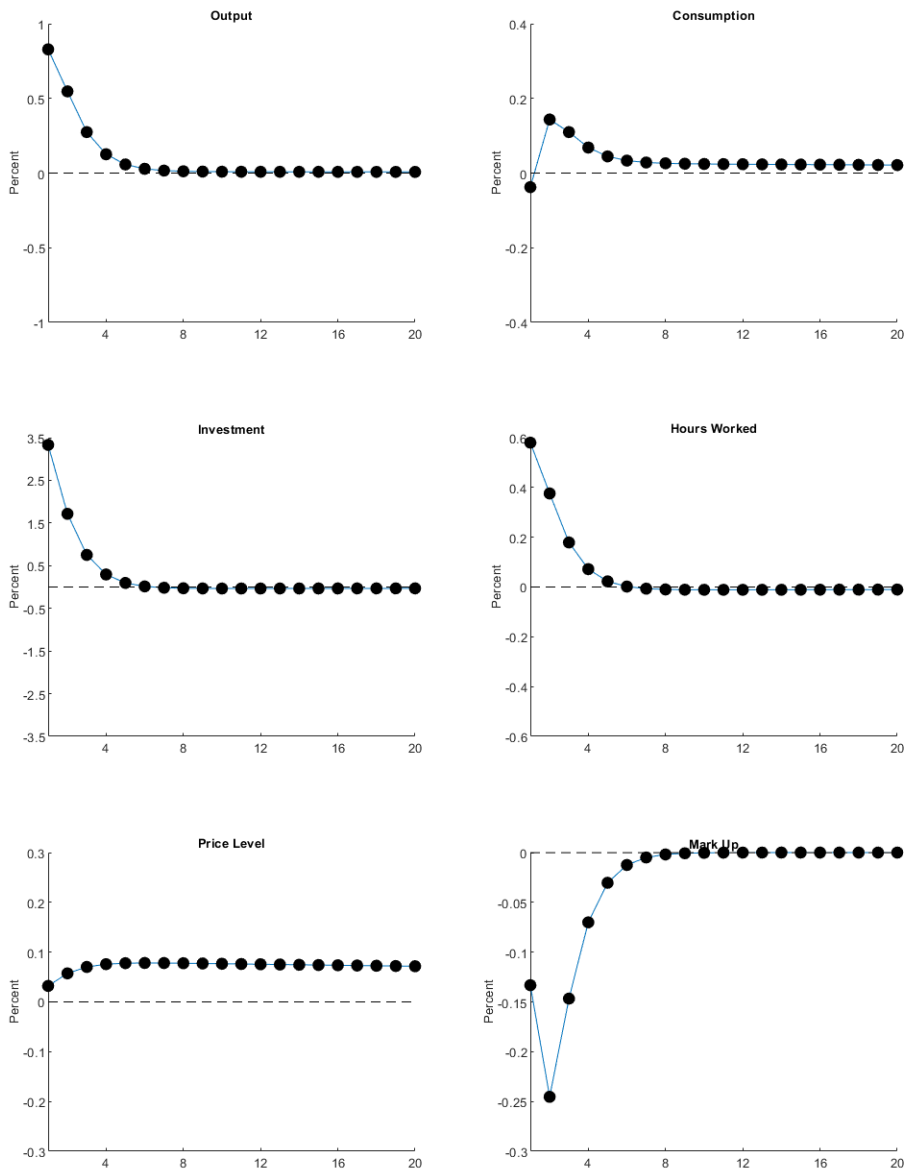


Figure 4.1: Impulse response of increase in good uncertainty

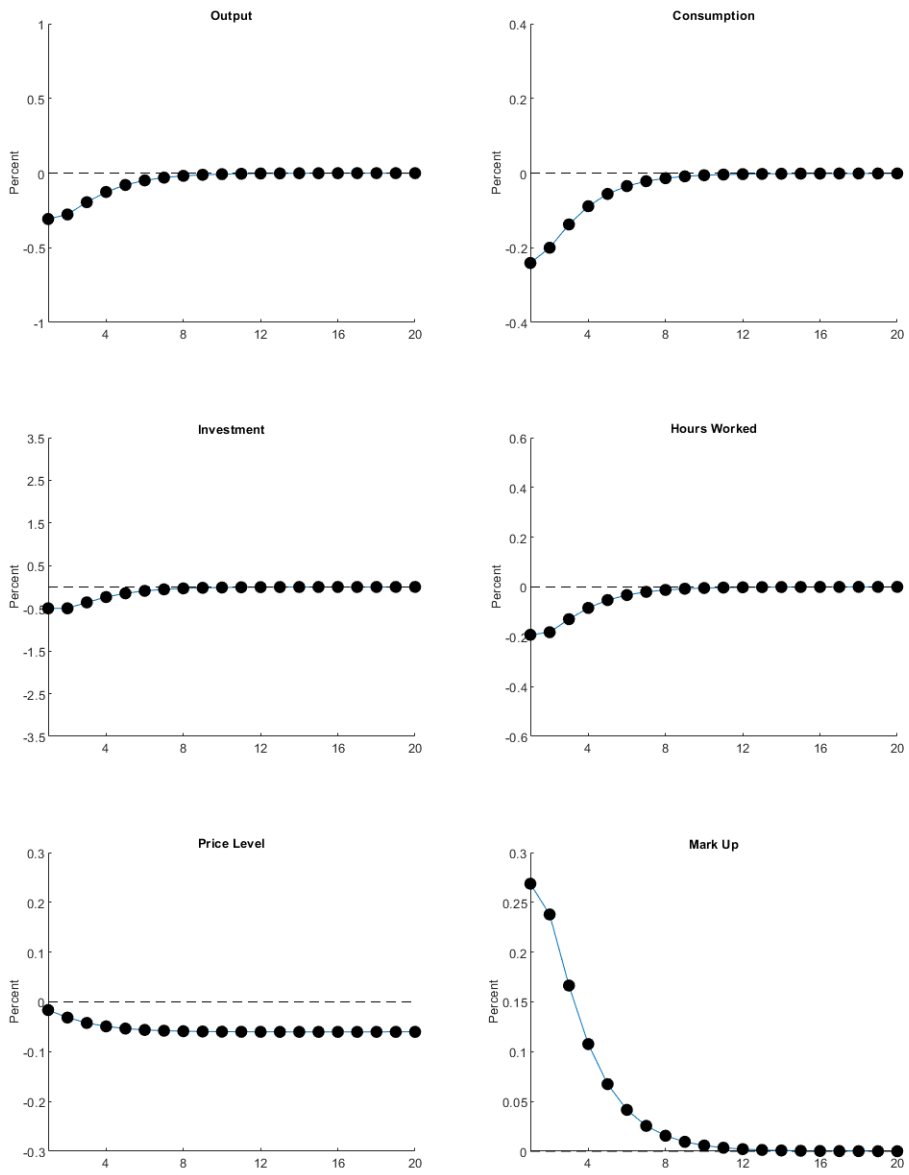


Figure 4.2: Impulse response of increase in bad uncertainty

Table 4.1: Model Parameters

Household		
Relative Risk Aversion	γ	80
Intertemporal Elasticity of Subsitution	ψ	0.95
Discount Factor	β	0.994
Consumption Share	φ	0.35
Firm		
Capital Share	α	0.333
Depreciation Related Parameter	δ_0	0.025
Depreciation Related Parameter	δ_1	0.026
Depreciation Related Parameter	δ_2	0.0003
Elasticity of Intermediate Goods	ϵ	6
Price Adjustment	ψ_p	100
Borrowing Limit	θ	0.5
Policy		
Inflation	Π	1.005
Taylor Rule	ρ_π	1.5
Taylor Rule	ρ_y	0.2
Shocks		
Persistence of Neutral Technology	ρ_z	0.9879
Volatility of Neutral Technology	σ_z	0.0013
Persistence of Preference	ρ_a	0.9356
Volatility of Preference	σ^a	0.0026
Persistence of Bad Uncertainty	ρ_{σ^a}	**0.6090**
Volatility of Bad Uncertainty	σ^{σ^a}	**0.0034**
Persistence of Good Uncertainty	ρ_σ	**0.3636**
Volatility of Good Uncertainty	σ^σ	**0.0056**

Notes: Parameters in bold are estimated by the SMM method. The calibrated parameters are mostly based on Basu and Bundick (2017)

Table 4.2: Aggregate Moments

Variable	Data	Model
A. Unconditional Volatility		
Consumption (%)	0.7	0.7
Output (%)	1.1	1.2
Investment (%)	3.8	4.2
Hours Worked (%)	1.4	0.8
B. Stochastic Volatility		
Consumption (%)	0.2	0.14
Output (%)	0.4	0.2
Investment (%)	1.6	0.7
Hours Worked (%)	0.5	0.15

Notes: This table reports both the unconditional and stochastic volatility of macroeconomic aggregates from the data and the model. The data column is based on the calculation in Basu and Bundick (2017).

Chapter 5

On Buffered Double Autoregressive Time Series Models

5.1 Introduction

A buffering phenomenon refers to the variable of interest sticks in the previous state when the variable of signal stays in a certain uninformative region. Consider, for example, a decision making of treatment plan for patients with diabetes. The up-crossing of blood sugar level over a certain high level r_U will intrigue an inpatient treatment, while a blood sugar level lower than r_L will evoke outpatient treatment. A doctor may easily choose between two treatment plans when a patient's blood sugar level is too high ($> r_U$) or too low ($< r_L$). However, he/she may become very hesitant to make the decision when the sugar level dwells in $[r_L, r_U]$, an interval without any other explicit benchmarks or information. This chaotic interval $[r_L, r_U]$ is exactly a typical buffer zone, where a retrospection of the historical data is needed to judge the status of the variable of interest. Therefore, in this case, a wise doctor might read the patient's previous record of blood sugar level. If the blood sugar level swings around r_U within the last week, the patient will be hospitalized overnight. Otherwise, the doctor will only give a prescription and require the patient to have at-home treatment.

Such regime switching mechanism was first formulated by Li, Guan, Li, and Yu (2015b), in an effort to extend the traditional threshold AR model. Only focusing on the conditional mean function, Li, Guan, Li, and Yu (2015b) has not considered conditional heteroscedasticity, which is, however, an essential part in the models of financial time series. Therefore, Lo et al. (2016) applied the similar idea to the conditional volatility process and proposed the buffered pure GARCH process in the

study of exchange rate. Zhu et al. (2017) then introduced the concept of buffered autoregressive model with generalized autoregressive conditional heteroscedasticity (BAR-GARCH). They empirically compared the performance of BAR-GARCH with AR-GARCH and Threshold-GARCH model by an analysis of exchange rate. However, the theoretical property of BAR-GARCH model remained unexplored, for which no statistical inference could be substantiated. Besides, the volatility could not be explicitly computed by directly using observations in the BAR-GARCH model.

Statistical test for the existence of threshold and model selection criteria are also crucial in the construction of buffered models. A quasi-likelihood ratio (LR) test for the thresholds in the buffered autoregressive (BAR) model has been established by Zhu et al. (2014), yet similar tests for a more generalized model such as BAR-GARCH model seems not readily available. Moreover, the application of model selection criteria (AIC, AIC_C and BIC) to the order selection in BAR-GARCH model lacks necessary simulation tests as well as theoretical support.

This paper proposes a buffered double AR (BDAR) model that captures the stickiness through applying a novel regime switching mechanism established by Li, Guan, Li, and Yu (2015b) to double AR (DAR) model (see Ling (2004, 2007), Zhu and Ling (2013)). The theoretical properties of the proposed model are thoroughly investigated. Unlike the BAR-GARCH model, the conditional variance of each regime in the BDAR model is governed directly by past observations which allows a visible dynamic behavior of conditional volatility.

The remainder of the paper is organized as follows. Section 5.2 proposes the buffered double autoregressive (BDAR) model. Formal proofs suggest strict stationarity and geometric ergodicity of the newly defined BDAR process under several sufficient conditions. A Gaussian quasi-maximum likelihood estimation (QMLE) is employed to compute the estimators in this model. The strict proof reveals the asymptotic normality of the estimated coefficients governing the conditional mean and variance of each regime in the BDAR model. More importantly, it has been demonstrated that

the estimated thresholds of the BDAR model is super consistent. Moreover, the limiting distribution of each of the estimated thresholds is derived. Our results include the buffered AR (BAR) model (Li, Guan, Li, and Yu (2015b)), threshold double AR (TDAR) model (Li, Ling, and Zhang (2016)) and traditional threshold AR (TAR) model (Tong (1990)) as special cases. A Bayesian-type information criteria (BIC), as well as its asymptotic property, are formally derived in section 5.3 for selecting orders of DAR models embedded in the BDAR model. Section 5.4 conducts several Monte Carlo simulations to illustrate the performance of estimation procedure and model selection criteria. Finally, the model is applied to Hang Seng Index (HSI) in Section 5.5 to show the asymmetry of investors' preference over losses and gains as well as the leverage effect of the stock market.

5.2 Buffered Double Autoregressive Models

Consider the following buffered double autoregressive (BDAR) model:

$$y_t = \begin{cases} \phi_{10} + \sum_{i=1}^{p} \phi_{1i} y_{t-i} + \varepsilon_t \sqrt{\alpha_{10} + \sum_{j=1}^{p} \alpha_{1j} y_{t-j}^2} & \text{if } R_t = 1 \\ \phi_{20} + \sum_{i=1}^{p} \phi_{2i} y_{t-i} + \varepsilon_t \sqrt{\alpha_{20} + \sum_{j=1}^{p} \alpha_{2j} y_{t-j}^2} & \text{if } R_t = 0 \end{cases} \tag{5.1}$$

$$R_t = \begin{pmatrix} 1 & \text{if } y_{t-d} \leq r_L \\ 0 & \text{if } y_{t-d} > r_U \\ R_{t-1} & \text{otherwise} \end{pmatrix}$$

Where $\alpha_{k0} > 0$ with $\alpha_{kj} \geq 0$ for $k = 1, 2$ and $j = 1, \ldots, p$. And $\{\varepsilon_t\}$ are $i.i.d.$ with E[ε_t]=0 and Var[ε_t]=1. The threshold indicator is denoted as y_{t-d} with $d > 0$. And the buffer zone is the interval $[r_L, r_U]$, where $r_L \leq r_U$. We denote the model by BDAR(p) for simplicity. Note (5.1) includes the BAR model proposed by Li, Guan, Li, and Yu (2015b) as a special case. When $r_L = r_U$, (5.1), the model becomes the threshold double AR model proposed by Li et al. (2016).

Denote $Y_t = (y_t, y_{t-1}, \ldots, y_{t-p+1}, R_t)'$, $m_{it} = \phi_{1i} I(A_t) + \phi_{2i} I(A_t^c)$, where A_t is the event $\{y_{t-d} \leq r_L\} \bigcup \{r_L < y_{t-d} \leq r_U, R_{t-1} = 1\}$ and A_t^c is its complement. After introducing notations M_{0t}, M_{1t}, $g_1(Y_{t-1})$ and $g_2(Y_{t-1})$ as follows, we can show that $Y_t = g_1(Y_{t-1}) + \epsilon_t g_2(Y_{t-1})$ forms a Markov Chain.

$$M_{0t} = \begin{bmatrix} \phi_{10} I(A_t) + \phi_{20} I(A_t^c) \\ 0 \\ \vdots \\ 0 \\ I(y_{t-d} \leqslant r_L) \end{bmatrix} \qquad M_{1t} = \begin{bmatrix} m_{1t} & m_{2t} & \ldots & m_{pt} & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \ldots & I(r_L < y_{t-d} \leqslant r_U) \end{bmatrix}$$

$$g_1(Y_{t-1}) = M_{0t} + M_{1t} Y_{t-1}$$

$$g_2(Y_{t-1}) = \begin{bmatrix} \sqrt{(\alpha_{10} + \sum_{j=1}^{p} \alpha_{1j} y_{t-j}^2) I(A_t) + (\alpha_{20} + \sum_{j=1}^{p} \alpha_{2j} y_{t-j}^2) I(A_t^c)} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

By applying a method similar to Lee (2006), several sufficient conditions are obtained for the geometric ergodicity of $\{Y_t\}$.

Theorem 5.2.1. *Suppose the distribution of ε_t has finite moment $E|\varepsilon_t|^s < \infty$ for a positive s. The probability density function f is positive over the real line, locally bounded away from zero, and decreases relatively fast in the sense that* $\sup_{x \in \mathbb{R}} \{(1 +$

$|x|)f(x)\} < \infty$. *Moreover, suppose one of the following conditions is satisfied:*
(i).$\sum_{j=1}^{p}\left(\sup_{1\le i\le 2}|\phi_{ij}|^r + \sup_{1\le i\le 2}(\alpha_{ij}^{r/2})E|\varepsilon_t|^r\right) < 1$, $r \in (0,1]$.
(ii).$\left(\sum_{j=1}^{p}\sup_{1\le i\le 2}|\phi_{ij}|\right)^r + \sum_{j=1}^{p}\sup_{1\le i\le 2}(\alpha_{ij}^{r/2})E|\varepsilon_t|^r < 1$, $r \in (1,2]$ *and* f *symmetric.*
(iii).$(1+3E(\epsilon_t^2))(\sum_{j=1}^{p}\sup_{1\le i\le 2}|\phi_{ij}|)^4 + (E(\epsilon_t^4)+3E(\epsilon_t^2))(\sum_{j=1}^{p}\sup_{1\le i\le 2}\alpha_{ij})^2 < 1$, $r = 4$

Then the process $\{Y_t\}$ *is geometrically ergodic and model* (5.1) *admits a strictly stationary and geometrically ergodic solution.*

Such conditions are easy to check but may be restrictive. Obtaining more general sufficient and necessary stability conditions is left for future research.

5.3 The QML Estimator of BDAR(p) Model

5.3.1 Estimation Procedure

This section considers the Gaussian QMLE of the BDAR model specified in (5.1). Denote by $\boldsymbol{\theta} = (\boldsymbol{\lambda}', r_L, r_U, d)'$ the parameter vector of model (5.1), $\boldsymbol{\lambda} = (\boldsymbol{\phi_1'}, \boldsymbol{\alpha_1'}, \boldsymbol{\phi_2'}, \boldsymbol{\alpha_2'})'$ with $\boldsymbol{\phi_i'} = (\phi_{i0}, \phi_{i1}, \ldots, \phi_{ip})$ and $\boldsymbol{\alpha_i'} = (\alpha_{i0}, \alpha_{i1}, \ldots, \alpha_{ip})$ and

$$\begin{aligned}\boldsymbol{\mu_t(\theta)} &= (\boldsymbol{\phi_1'Y_{1,t-1}})R_t(r_L, r_U, d) + (\boldsymbol{\phi_2'Y_{2,t-1}})(1 - R_t(r_L, r_U, d))\\ \boldsymbol{h_t(\theta)} &= (\boldsymbol{\alpha_1'X_{1,t-1}})R_t(r_L, r_U, d) + (\boldsymbol{\alpha_2'X_{2,t-1}})(1 - R_t(r_L, r_U, d))\end{aligned} \tag{5.2}$$

with $\boldsymbol{Y_{1,t-1}} = \boldsymbol{Y_{2,t-1}} = \boldsymbol{Y_{t-1}} = (1, y_{t-1}, \ldots, y_{t-p})'$ and $\boldsymbol{X_{1,t-1}} = \boldsymbol{X_{2,t-1}} = \boldsymbol{X_{t-1}} = (1, y_{t-1}^2, \ldots, y_{t-p}^2)'$. The conditional log-likelihood function (after multiplying -2 and omitting some constant) is

$$\boldsymbol{L_n(\theta)} = \sum_{t=1}^{n}\boldsymbol{l_t(\theta)}, \quad \text{where} \quad \boldsymbol{l_t(\theta)} = \log\boldsymbol{h_t(\theta)} + \frac{(y_t - \boldsymbol{\mu_t(\theta)})^2}{\boldsymbol{h_t(\theta)}}$$

Let $\boldsymbol{\Lambda}$ be a compact subspace of $\mathbb{R}^{4p+4}$ and $D = \{1, \ldots, d_{max}\}$ a discrete set. Assume $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$,$a \le r_L \le r_U \le b$ and $d \in D$. Furthermore, $\boldsymbol{\theta_0} = (\boldsymbol{\lambda_0}', r_{0L}, r_{0U}, d_0)'$ and

$R_{t0} = R_t(r_{0L}, r_{0U}, d_0)$ are respectively the true parameter vector and true regime indicator function.

As discussed in Li, Guan, Li, and Yu (2015b), the regime of the first t_0 observations may not be identified due to the possibility that the threshold variable y_{t-d} is located within the buffer zone. However, this is not a problem when we consider the large sample properties of the QMLE. Hence, we can assume that the first t_0 observations belong to either the regime 0 or regime 1 and define the regime indicator as $\widetilde{R}_t(r_L, r_U, d)$. It is clear $\widetilde{R}_t(r_L, r_U, d) = R_t(r_L, r_U, d)$ when $t_0 < t \leq n$.

Denote by $\widetilde{\boldsymbol{l}}_{\boldsymbol{t}}(\boldsymbol{\theta})$ and $\widetilde{\boldsymbol{L}}_{\boldsymbol{n}}(\boldsymbol{\theta})$ the corresponding functions with $R_t(r_L, r_U, d)$ replaced by $\widetilde{R}_t(r_L, r_U, d)$. We can then define the Gaussian QMLE as $\hat{\boldsymbol{\theta}}_{\boldsymbol{n}} = \arg\min_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \widetilde{\boldsymbol{L}}_{\boldsymbol{n}}(\boldsymbol{\theta})$

.

One can take two steps to search $\hat{\boldsymbol{\theta}}_{\boldsymbol{n}}$:

- First minimize $\hat{\boldsymbol{\theta}}_{\boldsymbol{n}}$ for each fixed (r_L, r_U, d), i.e.

$$\hat{\boldsymbol{\lambda}}_{\boldsymbol{n}}(r_L, r_U, d) = \arg\min_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}} \widetilde{\boldsymbol{L}}_{\boldsymbol{n}}(\boldsymbol{\lambda}, r_L, r_U, d)$$

- Search for the estimators of (r_L, r_U, d) by

$$(\hat{r}_L, \hat{r}_U, \hat{d}) = \arg\min_{d\in D,\ a\leq r_L\leq r_U\leq b} \widetilde{\boldsymbol{L}}_{\boldsymbol{n}}(\hat{\boldsymbol{\lambda}}_{\boldsymbol{n}}, (r_L, r_U, d), r_L, r_U, d)$$

Then we obtain $\hat{\boldsymbol{\theta}}_{\boldsymbol{n}} = (\hat{\boldsymbol{\lambda}}'_{\boldsymbol{n}}, \hat{r}_L, \hat{r}_U, \hat{d})'$.

5.3.2 Asymptotic Results

Assumption 1 $\text{pr}(y_t \in [a\ b]) < 1$, ε_t is *i.i.d.* with E[ε_t]=0 and Var[ε_t]=1. Moreover, its probability density function f is continuous and positive over the real line.

Assumption 2 Define the full parameter space as
$\boldsymbol{\Lambda} = \{\boldsymbol{\lambda} \in \mathbb{R}^{4p+4} : \boldsymbol{\phi_1} \neq \boldsymbol{\phi_2} \text{ and } \boldsymbol{\alpha_1} \neq \boldsymbol{\alpha_2}\}$ and assume it is compact. Moreover, each element in $\boldsymbol{\alpha_1}$ or $\boldsymbol{\alpha_2}$ is positive.

Theorem 5.3.1. *Suppose Assumptions 1 and 2 hold and* $\{y_t\}$ *has* $E[y_t^2] < \infty$*. Moreover, the strict stationarity and ergodicity of the BDAR process are satisfied. Then* $\hat{\boldsymbol{\theta}}_n \to \boldsymbol{\theta}_0$ *a.s..*

Theorem 5.3.1 establishes the strong consistency. From this theorem, we can conclude that that the convergence rate of the parameter d is much faster than the remaining parameters since it only take integer value. Hence we can simply assume that we know the true value of the parameter d for the asymptotic analysis and remove it from the parameter vector $\boldsymbol{\theta}$ and corresponding functions.

Assumption 3 $\kappa_4 \equiv E(\varepsilon_t^4) < \infty$ and $E(y_t^4) < \infty$.

Assumption 4 $\boldsymbol{\mu_t(\theta)}$ and $\boldsymbol{h_t(\theta)}$ in (5.2) are discontinuous in the buffer zone $[r_{0L}, r_{0U}]$, i.e. there are constants $z_{p-1},\ldots,z_{p-d+1},z_{p-d-1},\ldots,z_0$ satisfying

$$\{(\boldsymbol{\phi_{10}} - \boldsymbol{\phi_{20}})'\boldsymbol{z}\}^2 + \{(\boldsymbol{\alpha_{10}} - \boldsymbol{\alpha_{20}})'\boldsymbol{Z}\}^2 > 0$$

for all $z_{p-d} \in [r_{0L}, r_{0U}]$ where $\boldsymbol{z} = (1, z_{p-1}, \ldots, z_0)'$ and $\boldsymbol{Z} = (1, z_{p-1}^2, \ldots, z_0^2)'$.

Recall $Y_t = (y_t, \ldots, y_{t-p+1}, R_t)'$ is the defined Markov chain in previous section. We further use the notation $P^l(x, A)$ to represent its l-step transition probability, where $x \in \mathbb{R}^p \times \{0, 1\}$, $A \in \mathcal{B}_p \times \mathcal{U}$.

Assumption 5 The multivariate process is V-uniformly ergodic in the sense that $\|P^l(x, \cdot) - \pi(\cdot)\|_v \leq C(1 + \|x\|^2)\rho^l$, where $C > 0$, $0 < \rho_l < 1$ and $\pi(\cdot)$ is the unique invariant measure.

Theorem 5.3.2. *Given Assumptions 1 to 5 are satisfied and* $\boldsymbol{\theta_0}$ *is not a boundary point of the parameter space* $\boldsymbol{\Theta}$*. Then:*

(i). $n(\hat{r}_L - r_{0L}) = O_p(1)$ *and* $n(\hat{r}_U - r_{0U}) = O_p(1)$

(ii). $\sqrt{n}\sup_{n(|r_U - r_{0U}|+|r_L - r_{0L}|)\leq K} \|\hat{\boldsymbol{\lambda}}_{\boldsymbol{n}}(r_L, r_U) - \hat{\boldsymbol{\lambda}}_{\boldsymbol{n}}(r_{0L}, r_{0U})\| = o_p(1)$ $\forall K > 0$*, where* $\hat{\boldsymbol{\lambda}}_{\boldsymbol{n}}(r_L, r_U)$ *is the QMLE given* r_L, r_U *are known. Furthermore,*

$\sqrt{n}(\hat{\boldsymbol{\lambda}}_{\boldsymbol{n}} - \boldsymbol{\lambda_0}) = \sqrt{n}(\hat{\boldsymbol{\lambda}}_{\boldsymbol{n}}(r_{0L}, r_{0U}) - \boldsymbol{\lambda_0}) + o_p(1) \Rightarrow N(\boldsymbol{0}, \boldsymbol{A^{-1}BA^{-1}})$*, as* $n \to \infty$.

$A = diag(\boldsymbol{M_1}, \boldsymbol{0.5N_1}, \boldsymbol{M_2}, \boldsymbol{0.5N_2}), \boldsymbol{B} = diag(\boldsymbol{B_1}, \boldsymbol{B_2})$ *with*

$$\boldsymbol{B_i} = \begin{pmatrix} \boldsymbol{M_i} & \frac{\kappa_3}{2}\boldsymbol{O_i} \\ \frac{\kappa_3}{2}\boldsymbol{O_i'} & \frac{\kappa_4-1}{4}\boldsymbol{N_i} \end{pmatrix},$$

where $\kappa_3 = E(\varepsilon_1^3)$

$$\boldsymbol{M_i} = E\left\{\frac{\boldsymbol{Y_{i,t-1}Y_{i,t-1}'}}{\boldsymbol{\alpha_{i0}'X_{i,t-1}}} g_i(r_{0L}, r_{0U})\right\} \boldsymbol{N_i} = E\left\{\frac{\boldsymbol{X_{i,t-1}X_{i,t-1}'}}{\boldsymbol{(\alpha_{i0}'X_{i,t-1})^2}} g_i(r_{0L}, r_{0U})\right\}$$

$$\boldsymbol{O_i} = E\left\{\frac{\boldsymbol{Y_{i,t-1}X_{i,t-1}'}}{\boldsymbol{(\alpha_{i0}'X_{i,t-1})^{3/2}}} g_i(r_{0L}, r_{0U})\right\}$$

with $g_1(r_{0L}, r_{0U}) = R_t(r_{0L}, r_{0U})$ *and* $g_2(r_{0L}, r_{0U}) = 1 - R_t(r_{0L}, r_{0U})$

To study the limiting distribution of $\hat{r}_L$ and $\hat{r}_U$, denote the $\xi_{1t} = \sum_{j=0}^{\infty} \zeta_{1,t+j} H_{t+j,j}$ and $\xi_{2t} = \sum_{j=0}^{\infty} \zeta_{2,t+j} H_{t+j,j}$, where

$$\zeta_{1t} = \log \frac{\boldsymbol{\alpha_{10}'X_{t-1}}}{\boldsymbol{\alpha_{20}'X_{t-1}}} + \frac{\{\boldsymbol{(\phi_{10} - \phi_{20})'Y_{t-1}} - \epsilon_t \sqrt{\boldsymbol{\alpha_{20}'X_{t-1}}}\}^2}{\boldsymbol{\alpha_{10}'X_{t-1}}} - \epsilon_t^2$$

$$\zeta_{2t} = \log \frac{\boldsymbol{\alpha_{20}'X_{t-1}}}{\boldsymbol{\alpha_{10}'X_{t-1}}} + \frac{\{\boldsymbol{(\phi_{10} - \phi_{20})'Y_{t-1}} + \epsilon_t \sqrt{\boldsymbol{\alpha_{10}'X_{t-1}}}\}^2}{\boldsymbol{\alpha_{20}'X_{t-1}}} - \epsilon_t^2$$

$$H_{t,j} = \prod_{l=1}^{j} 1(r_{0L} < y_{t-d+1-l} \leqslant r_{0U}) \text{ with convention } \prod_{l=1}^{0} = 1$$

Similar to Li, Guan, Li, and Yu (2015b), for $i = 1, 2$, the distribution of $\xi_{it}\{1 - R_{t-1}(r_0)\}$ and $\xi_{it} R_{t-1}(r_0)$ conditional on $y_{t-d} = y$ are written as $F_{i,L}(\cdot|y)$ and $F_{i,U}(\cdot|y)$ respectively, where $r_0 = (r_{0L}, r_{0U})'$. The probability density of y_t is denoted by $\pi(\cdot)$. Then define two processes:

$$\varphi_L(t) = 1(t \geqslant 0) \sum_{k=1}^{N_1^{(L)}(|t|)} \chi_k^{(1,L)} + 1(t < 0) \sum_{k=1}^{N_2^{(L)}(|t|)} \chi_k^{(2,L)}$$

$$\varphi_U(t) = 1(t \geqslant 0) \sum_{k=1}^{N_1^{(U)}(|t|)} \chi_k^{(1,U)} + 1(t < 0) \sum_{k=1}^{N_2^{(U)}(|t|)} \chi_k^{(2,U)}$$

where $\chi_k^{(i,L)}$ and $\chi_k^{(i,U)}$ have conditional distribution functions $F_{i,L}(\cdot|r_{0L})$ and $F_{i,U}(\cdot|r_{0U})$ respectively, $i = 1, 2$. $\{N_1^{(L)}(t), t \geqslant 0\}, \{N_2^{(L)}(t), t \geqslant 0\}, \{N_1^{(U)}(t), t \geqslant 0\}, \{N_2^{(U)}(t), t \geqslant$

0} are Poisson processes. $N_1^{(L)}(\cdot), N_2^{(L)}(\cdot)$ have jump rate $\pi(r_{0L})$, while $N_1^{(U)}(\cdot), N_2^{(U)}(\cdot)$ have jump rate $\pi(r_{0U})$. Moreover, $N_1^{(L)}(\cdot)$ and $N_1^{(U)}(\cdot)$ are right continuous. $N_2^{(L)}(\cdot)$ and $N_2^{(U)}(\cdot)$ are left continuous. Moreover, these Possion processes take value 0 when $t = 0$, and are mutually independent.

It is implied by Assumption 4 that $E[\chi_k^{(1,j)}]$ and $E[\chi_k^{(2,j)}] > 0$ for $j = L$ and U. Then $\varphi_j(t) \to \infty$ a.s. as $|t| \to \infty$. Then define $\varphi(z) = \varphi_L(z_L) + \varphi_U(z_U)$, where $z = (z_L, z_U) \in R^2$. As in Li, Guan, Li, and Yu (2015b) and Li, Ling, and Zakoian (2015a), $\varphi(z)$ attains the global minimum over $[M_-^{(L)}, M_+^{(L)}) \times [M_-^{(U)}, M_+^{(U)})$, where $[M_-^{(j)}, M_+^{(j)}) = \text{argmin}_{t \in R}\, \varphi_j(t)$.

The following theorem characterize the limiting distributions of $n(\hat{r}_L - r_{0L})$ and $n(\hat{r}_U - r_{0U})$

Theorem 5.3.3. *If Assumption 1 to 5 hold, then* $n(\hat{r}_L - r_{0L}) \to M_-^{(L)}$ *and* $n(\hat{r}_U - r_{0U}) \to M_-^{(U)}$. *Furthermore,* $n(\hat{r}_L - r_{0L}), n(\hat{r}_U - r_{0U})$ *and* $\sqrt{n}(\hat{\boldsymbol{\lambda}}_n - \boldsymbol{\lambda}_0)$ *are asymptotically independent.*

5.3.3 Model Selection

I investigate the performance of the Bayesian Information Criteria (BIC) for model selection, since BIC outperforms AIC for buffered type models, see Li, Guan, Li, and Yu (2015b).

Write for simplicity $\tilde{R}_t$ as the QMLE estimator of $\tilde{R}_t(\hat{r}_L, \hat{r}_U, \hat{d})$. $n_1 = \sum_{t=1}^{n} \tilde{R}_t$ and $n_2 = n - n_1$ Then define

$$\text{BIC(p)} = \sum_{t=1}^{n} \boldsymbol{l_t(\hat{\theta})} + (2p+2)\log n_1 + (2p+2)\log n_2$$

where $\boldsymbol{l_t(\theta)}$ is defined in Section 3.1. Let p_0 be the true order of the model and $p_{\max}$ be a predetermined large order. $\hat{p}_n = \text{argmin}_{0 \le p \le p_{\max}} \text{BIC(p)}$ Then,

Theorem 5.3.4. *If Assumption 1 to 5 hold, then* $Pr(\hat{p}_n = p_0) \to 1$ *as* $n \to \infty$.

5.4 Simulation Experiments

This subsection evaluates the effectiveness of the proposed QMLE for the BDAR model in the finite sample. For the sample size $n = 400, 800$, 500 replications are run for the following specification of the BDAR model

$$y_t = \begin{cases} -0.1 + 0.2y_{t-1} + 0.1y_{t-2} + \varepsilon_t\sqrt{0.1 + 0.3y_{t-1}^2 + 0.05y_{t-2}^2} & R_t = 1 \\ 0.1 - 0.2y_{t-1} + 0.3y_{t-2} + \varepsilon_t\sqrt{0.05 + 0.2y_{t-1}^2 + 0.1y_{t-2}^2} & R_t = 0 \end{cases} \tag{5.3}$$

with the regime indicator

$$R_t = \begin{pmatrix} 1 & \text{if } y_{t-4} \leq -0.1 \\ 0 & \text{if } y_{t-4} > 0.15 \\ R_{t-1} & \text{otherwise} \end{pmatrix}$$

where $\{\varepsilon_t\}$ are *i.i.d.* following standard normal distribution. The estimation procedure discussed in Section 3.1 is employed. The range of the boundary parameters r_L and r_U is set from 10th percentile to 90th percentile of each sample. And the maximum delay parameter d_{max} is set to be 6. For each sample size $n = 400, 800$, d can be correctly identified for all 500 replications respectively. Table 5.1 reports the results. As the sample size increases, we can observe the bias and the empirical standard deviation (ESD) decrease. Moreover, all the ESDs are close to corresponding asymptotic standard deviation (ASD), which are consistent with the asymptotic results derived in the previous theorems. Note the ASD of boundary parameters r_L and r_U cannot be estimated and only the ESD are included in Table 5.1. We observe the ESDs of r_L and r_U decrease roughly by one half when the sample size doubles, supporting the super-consistency of boundary parameters. For each sample size, the histograms of $n(\hat{r}_L - r_{0L})$ and $n(\hat{r}_U - r_{0U})$ are displayed in Figure 1, which share very similar shapes to those reported in Li et al. (2013) and Li, Guan, Li, and Yu (2015b) We now evaluate the effectiveness of the model selection criteria (BIC) in finite sam-

ple. The data generating process is the same as (5.3). The sample size is 800, and there are 150 replications. The maximum order p_{max} for each regime's conditional mean and conditional variance is set to be 6. And the maximum delay parameter d_{max} is still set to be 6. The result shows that all true orders are correctly identified by the model selection criteria at a rate of 100%.

5.5 Empirical Analysis

This subsection futher applies the proposed BDAR model to the real data. Specifically, I consider the Hang Seng Index of Hong Kong. And the time span is from Jan 2000 to Dec 2007. We focus on the log return in percentage $y_t = 100(\log P_t - \log P_{t-1})$, where P_t corresponds to the weekly closing price. There are 417 observations of $\{y_t\}$ in total. The plot of $\{P_t\}$ and $\{y_t\}$ are displayed in Figure 5.2. Li et al. (2016) have studied this series and they conclude the existence of nonlinearity, ARCH effect and threshold effect through Tsay's test (Tsay (1986)), Mcleod-Li's test (Mcleod and Li (1983)) and their score-based test respectively. The two-regime threshold double autoregressive model (TDAR) is employed to fit the data in Li et al. (2016). Since TDAR model can be considered as a special case of BDAR model, it motivates us to consider the proposed BDAR model to fit the percentage log return series.

The range of boundary parameters r_L and r_U are within the 10th and the 90th quantile of observations. d_{max} is set to be 6 and the maximum order p_{max} for each regime's conditional mean and conditional variance is set to be 5. Based on BIC, we have the following fitted model:

$$y_t = \begin{cases} 0.3937_{0.2621} + 0.0385_{0.1318} y_{t-1} + 0.2093_{0.0894} y_{t-2} + \varepsilon_t \sigma_t & R_t = 1 \\ -0.5992_{0.2665} + 0.2354_{0.0970} y_{t-1} + \varepsilon_t \sigma_t & R_t = 0 \end{cases} \tag{5.4}$$

where

$$\sigma_t^2 = \begin{cases} 5.3991_{0.9225} + 0.5432_{0.1583} y_{t-1}^2 + 0.1787_{0.1161} y_{t-2}^2 & R_t = 1 \\ 3.4473_{0.7086} + 0.0263_{0.0599} y_{t-1}^2 + 0.0678_{0.0578} y_{t-2}^2 + 0.1416_{0.0631} y_{t-3}^2 & R_t = 0 \end{cases}$$

with the regime indicator

$$R_t = \begin{cases} 1 & \text{if } y_{t-1} \leq -0.2048 \\ 0 & \text{if } y_{t-1} > 0.8770 \\ R_{t-1} & \text{otherwise} \end{cases}$$

where the subscript of each estimated parameter contains the corresponding estimated standard error. Figure 5.3 displays respectively the autocorrelation of the residuals and the squared residuals of the fitted model. Observe that they slightly go beyond the 95% confidence bands only at a few lags, which partially suggests the fit-adequacy. Furthermore, the Ljung-Box test Q_6 and Q_{12} show p-values 0.6590 and 0.3666 respectively. And the p-values of the McLeod-Li test $\widetilde{Q}_6$ and $\widetilde{Q}_{12}$ are 0.8200 and 0.4013. These results further suggests the fit-adequacy at 5% significance level. The asymmetry of the buffered zone around the average return 0.1417% might be interpreted as market participants are more sensitive to losses and gains. Based on the fitted model, the investors require the return of previous week to go above 0.8770% to confirm a good time, approximately three times larger in absolute value than the signal of a bad time. When the previous week's return lies in the buffer zone [-0.2048% 0.8770%], investors are not sure about the market condition, so they trace back to learn R_{t-1}, which may depend on returns several periods ago. Therefore, the buffered double autoregressive model will further rely on the information contained in the dynamic momentum structure, when the past week's return alone is not quite informative on the market conditions. The BDAR model provides a more realistic characterization on the regime switching mechanism than the traditional two-regime threshold type model. The sudden switch of the probabilistic structure as in the traditional threshold time series models can be delayed by the incorporation of the buffer zone.

Another feature of the fitted BDAR model is the asymmetry of volatility. The coefficients in the lower regime's conditional variance is much larger than their counterparts in the upper regime. This asymmetry can be interpenetrated as the stock market is more volatile during bad times.

5.6 Proofs of Theorems

5.6.1 Proof of Theorem 5.2.1

As in the main part of the paper, define

$$Y_t = \begin{bmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-p+1} \\ R_t \end{bmatrix}$$

$$M_{0t} = \begin{bmatrix} \phi_{10} I(A_t) + \phi_{20} I(A_t^c) \\ 0 \\ \vdots \\ 0 \\ I(y_{t-d} \leqslant r_L) \end{bmatrix} \qquad M_{1t} = \begin{bmatrix} m_{1t} & m_{2t} & \dots & m_{pt} & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & I(r_L < y_{t-d} \leqslant r_U) \end{bmatrix}$$

$$g_1(Y_{t-1}) = M_{0t} + M_{1t} Y_{t-1}$$

$$g_2(Y_{t-1}) = \begin{bmatrix} \sqrt{(\alpha_{10} + \sum_{j=1}^{p} \alpha_{1j} y_{t-j}^2) I(A_t) + (\alpha_{20} + \sum_{j=1}^{p} \alpha_{2j} y_{t-j}^2) I(A_t^c)} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$Y_t = g_1(Y_{t-1}) + \epsilon_t g_2(Y_{t-1}) \quad \text{forms a Markov Chain}$$

Let $\mathcal{B}^p$ be the class of Borel sets of $\mathcal{R}^p$, μ_{p+1} be the Lebesgue measure on $\mathcal{R}^{p+1}$ and $\mathcal{U} = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$

We denote its state space by $(\mathcal{R}^p \times \{0,1\}, \mathcal{B}^p \times \mathcal{U}, \mu_{p+1})$ and sets its transition probability function as:

$$P(x, A) = \int_{A_1} f_\epsilon(y)dy, \quad x \in \mathcal{R}^p \times \{0,1\} \text{ and } A \in \mathcal{B}^p \times \mathcal{U}$$

Where $A_1 = \{\epsilon : \epsilon g_2(x) + g_1(x) \in A\}$ and $f_\epsilon(\cdot)$ is the density of ϵ_t.

By a method similar to Lemma 1 and 2 of Lu (1998), it can be shown that the chain Y_t is aperiodic and μ_{p+1}-irreducible. Besides, non-null compact sets are small sets.

Before we proceed, we need to state one lemma and one theorem. The lemma is proved in Lee (2006) and the theorem comes from Tweedie (1983):

Lemma 1. (Lemma 2.2 in Lee (2006)) *Let* $V(z) = \sum_{i=1}^n \gamma_i |z_i|^r, z = (z_1, z_2, \ldots, z_n), n \in Z^+, r > 0$. *If* $\sum_{i=1}^n \xi_i < 1$ *with* $\xi_i \geqslant 0$, *we may choose* $\gamma_i > 0, i = 1, \ldots, n$ *so that for some positive constant* $\rho < 1, \gamma_1(\sum_{i=1}^n \xi_i |z_i|^r) + \sum_{i=2}^n \gamma_i |z_{i-1}|^r \leqslant \rho V(z)$.

Theorem 5.6.1. (Theorem 4 in Tweedie (1983)) *Suppose that the Markov process* $\{Y_t : t \geqslant 0\}$ *is aperiodic* μ*-irreducible and* $\mathcal{B}$ *is a small set. Suppose there are constants* $\rho < 1, \epsilon > 0$ *and a measurable function* $V \geqslant 1$ *such that*

$$E[V(Y_t)|Y_{t-1} = z] \leqslant \rho V(z) - \epsilon, z \in \mathcal{B}^c$$

$$\sup_{z \in \mathcal{B}} E[V(Y_t)|Y_{t-1} = z] < \infty$$

then the Markov process Y_t is geometrically ergodic.

We continue the proof. Let $b_j = \underset{1\leqslant i\leqslant 2}{max}|\phi_{ij}|, 0 \leqslant j \leqslant p, \quad a_j = \underset{1\leqslant i\leqslant 2}{max}\alpha_{ij}, 0 \leqslant j \leqslant p.$ Define a test function V by:

$$V(Y_t) = \sum_{i=1}^{p} \gamma_i |y_{t+1-i}|^r + 1$$

$|y_t| \leqslant b_0 + \sum_{j=1}^{p} b_j|y_{t-j}| + |\epsilon_t|(a_0 + \sum_{j=1}^{p} a_j y_{t-j}^2)^{\frac{1}{2}}$

Define:

$$s_1(y_{t-1}, \ldots, y_{t-p}) = b_0 + \sum_{j=1}^{p} |y_{t-j}|b_j$$

$$s_2(y_{t-1}, \ldots, y_{t-p}) = [a_0 + \sum_{j=1}^{p} y_{t-j}^2 a_j]^{\frac{1}{2}}$$

Similar to Lee (2006), we have:

$$s_1^r \leqslant \begin{cases} b_0^r + \sum b_j^r |y_{t-j}|^r, & 0 < r \leqslant 1 \\ (1+\epsilon)^r (\sum b_j)^{r-1} (\sum b_j |y_{t-j}|^r), & 1 < r \leqslant 2, \forall \epsilon > 0, \exists M(\epsilon) s.t. ||Y_{t-1}|| > M(\epsilon) \end{cases}$$

$$s_2^r \leqslant a_o^{\frac{r}{2}} + \sum a_j^{\frac{r}{2}} |y_{t-j}|^r, \quad 0 < r \leqslant 2$$

Moreover, by basic inequality, we have $E[|y_t|^r|Y_{t-1}] \leqslant s_1^r + s_2^r E|\epsilon_t|^r$, as long as $r \in (0, 1]$

When $1 < r \leqslant 2$, note ϵ_t is symmetric by assumption, by the following inequality:

$$(1+x)^r + (1-x)^r \leqslant 2(|x|^r + 1), -1 \leq x \leq 1$$

we can show that

$$E[|s_1 + s_2\epsilon_t|^r | Y_{t-1}] = \frac{1}{2}E[|s_1 + s_2\epsilon_t|^r + |s_1 - s_2\epsilon_t|^r | Y_{t-1}] \leqslant s_1^r + s_2^r E|\epsilon_t|^r$$

Therefore, $E[V(Y_t)|Y_{t-1}]$

$$\begin{aligned} &\leqslant \gamma_1(s_1^r + s_2^r E|\epsilon_t|^r) + \sum_{i=2}^{p} \gamma_i |y_{t+1-i}|^r + 1 \\ &\leqslant \gamma_1 \sum_{i=1}^{p} \xi_i |y_{t-i}|^r + \sum_{i=2}^{p} \gamma_i |y_{t+1-i}|^r + 1 + C_1, \end{aligned}$$

where C_j is a generic notation for positive constants

$$\sum_{i=1}^{p} \xi_i = \begin{cases} \sum_{j=1}^{p} b_j^r + \sum_{j=1}^{p} a_j^{\frac{r}{2}} E|\epsilon_t|^r, & 0 < r \leqslant 1 \\ (1+\epsilon)^r (\sum_{j=1}^{p} b_j)^r + \sum_{j=1}^{p} a_j^{\frac{r}{2}} E|\epsilon_t|^r, & 1 < r \leqslant 2, \forall \epsilon > 0, \exists M(\epsilon), s.t. ||Y_{t-1}|| > M(\epsilon) \end{cases}$$

By Lemma 1, we can set $\sum_{i=1}^{p} \xi_i < 1$ and find $\gamma_i > 0, i = 1, \ldots, p$ such that for $||z|| > M, \exists$ some constants $\rho < 1, E[V(Y_t)|Y_{t-1} = z] \leqslant \rho V(z) + C_2$. Thus, inequalities in Theorem 5.6.1 hold with some $\epsilon > 0$ and compact set $\mathcal{B} = \{||z|| \leqslant M\}$ for sufficiently large $M < \infty$, since $V(z)$ increases as $||z||$ increases. Hence, we obtain the geometric ergodicity and hence the strict stationarity.

To prove the third part of the theorem, define $V(z) = \sum_{i=1}^{p} \gamma_i z_i^4 + 1$ from $E(\epsilon_t) = E(\epsilon_t^3) = 0$ for $||z|| > M$.

$$E[(s_1+s_2\epsilon_t)^4|Y_{t-1}] \leqslant [(1+3E(\epsilon_t^2))s_1^4+(E\epsilon_t^4+3E\epsilon_t^2)s_2^4]$$

$$s_1^4 \leqslant (1+\epsilon)^4(\sum_{j=1}^{p} b_j y_{t-j}^4)(\sum_{j=1}^{p} b_j)^3$$

$$s_2^4 \leqslant (1+\epsilon)^2(\sum_{j=1}^{p} a_j y_{t-j}^4)(\sum_{j=1}^{p} a_j)$$

Therefore for $||z|| > M$

$$E[V(Y_t)|Y_{t-1}=z] \leqslant \gamma_1\{[1+3E(\epsilon_t^2)]s_1^4+[E(\epsilon_t^4)+3E(\epsilon_t^2)]s_2^4\}+\sum_{i=2}^{p}\gamma_i|y_{t+1-i}|^4+1$$

$$\leqslant \gamma_1\sum_{i=1}^{p}\xi_i|y_{t-i}|^4+\sum_{i=2}^{p}\gamma_i|y_{t+1-i}|^4+1$$

where $\sum_{i=1}^{p}\xi_i = [1+3E(\epsilon_t^2)](1+\epsilon)^4(\sum_{j=1}^{p} b_j)^4+[E(\epsilon_t^4)+3E(\epsilon_t^2)](1+\epsilon)^2(\sum_{j=1}^{p} a_j)^2$ By Lemma 1, we can set $\sum_{i=1}^{p}\xi_i < 1$ and obtain the geometric ergodicity as we do in proving the first two parts of the theorem.

5.6.2 Proof of Theorem 5.3.1

Before we prove Theorem 5.3.1, we first introduce some notations and a preliminary result.

Denote the parameter space as $\boldsymbol{\Theta} = \wedge \times [\boldsymbol{a},\boldsymbol{b}] \times [\boldsymbol{a},\boldsymbol{b}] \times \boldsymbol{D}$, where $\wedge = \wedge_{\boldsymbol{1}} \times \wedge_{\boldsymbol{2}}$. Parametre vector $\boldsymbol{\theta} = (\boldsymbol{\phi_1'}, \boldsymbol{\alpha_1'}, \boldsymbol{\phi_2'}, \boldsymbol{\alpha_2'}, \boldsymbol{r_L}, \boldsymbol{r_U}, \boldsymbol{d})' = (\boldsymbol{\lambda_1'}, \boldsymbol{\lambda_2'}, \boldsymbol{r_L}, \boldsymbol{r_U}, \boldsymbol{d})'$, where

$$\boldsymbol{\lambda_1} = \begin{bmatrix} \boldsymbol{\phi_1} \\ \boldsymbol{\alpha_1} \end{bmatrix} \qquad \boldsymbol{\lambda_2} = \begin{bmatrix} \boldsymbol{\phi_2} \\ \boldsymbol{\alpha_2} \end{bmatrix}$$

Recall in the main body of the paper, we define

$$\boldsymbol{l_t(\theta)} = \log \boldsymbol{h_t}(\theta) + \frac{\boldsymbol{u_t^2(\theta)}}{\boldsymbol{h_t(\theta)}}$$

$$= R_t\{\log \boldsymbol{h_{1t}(\theta)} + \frac{[y_t - \boldsymbol{\mu_{1t}(\theta)}]^2}{\boldsymbol{h_{1t}(\theta)}}\} + (1-R_t)\{\log \boldsymbol{h_{2t}(\theta)} + \frac{[y_t - \boldsymbol{\mu_{2t}(\theta)}]^2}{\boldsymbol{h_{2t}(\theta)}}\}$$

where for i=1,2

$$\boldsymbol{\mu_{it}(\theta)} = \phi_{i0} + \phi_{i1} y_{t-1} + \cdots + \phi_{ip} y_{t-p}$$
$$\boldsymbol{h_{it}(\theta)} = \alpha_{i0} + \alpha_{i1} y_{t-1}^2 + \cdots + \alpha_{ip} y_{t-p}^2$$

We further denote it as $\boldsymbol{l_t(\theta)} = R_t \boldsymbol{l_{1t}(\theta)} + (1 - R_t)\boldsymbol{l_{2t}(\theta)}$, with

$$\boldsymbol{l_{1t}(\theta)} = \log \boldsymbol{h_{1t}(\theta)} + \frac{[y_t - \boldsymbol{\mu_{1t}(\theta)}]^2}{\boldsymbol{h_{1t}(\theta)}}$$
$$\boldsymbol{l_{2t}(\theta)} = \log \boldsymbol{h_{2t}(\theta)} + \frac{[y_t - \boldsymbol{\mu_{2t}(\theta)}]^2}{\boldsymbol{h_{2t}(\theta)}}$$

If Assumption 2 and conditions in Theorem 5.3.1 hold, Lemma B.2 in Ling (2007) implies that

$$E[\sup_{\theta \in \Theta} |\boldsymbol{l_{it}(\theta)}|] < \infty \quad \text{i=1,2 and} \quad E[\sup_{\theta \in \Theta} |\boldsymbol{l_t(\theta)}|] < \infty \tag{5.5}$$

This preliminary result will be used to invoke dominance convergence theorem in the following. We are ready to prove Theorem 5.3.1 now.

Proof of Theorem 5.3.1

Based on the method in Huber(1967), strong consistency could be proved by verifying the following three claims:

- $S1$: $\sup_{\theta \in \Theta} \frac{1}{n} |\tilde{\boldsymbol{L}}_{\boldsymbol{n}}(\boldsymbol{\theta}) - \boldsymbol{L_n(\theta)}| \xrightarrow{a.s.} 0$, where $\tilde{\boldsymbol{L}}_{\boldsymbol{n}}(\boldsymbol{\theta})$ is the modified likelihood function defined in the paper.

- $S2$: $E[\boldsymbol{l_t(\theta)}]$ is uniquely minimized at $\boldsymbol{\theta = \theta_0 = (\phi'_{10}, \alpha'_{10}, \phi'_{20}, \alpha'_{20}, r_{0L}, r_{0U}, d_0)'}$

- $S3$: $E[\sup_{\boldsymbol{\theta^* \in U_\eta(\theta)}} |\boldsymbol{l_t(\theta) - l_t(\theta^*)}|] \xrightarrow{a.s.} 0$ as $\eta \to 0$, where $U_\eta(\boldsymbol{\theta}) = \{\boldsymbol{\theta^* \in \Theta, ||\theta^* - \theta||} < \eta\}$

Let us first show S1:

$$\frac{1}{n}|\tilde{\boldsymbol{L}}_{\boldsymbol{n}}(\boldsymbol{\theta}) - \boldsymbol{L}_{\boldsymbol{n}}(\boldsymbol{\theta})| = \frac{1}{n}|\sum_{t=1}^{k_n}(\tilde{\boldsymbol{l}}_{\boldsymbol{t}}(\boldsymbol{\theta}) - \boldsymbol{l}_{\boldsymbol{t}}(\boldsymbol{\theta}))|$$

$$= \{(1 - R_1)\prod_{t=1}^{k_n} 1(r_L < y_{t-d} \leqslant r_U)\}\{\frac{1}{n}\sum_{t=1}^{k_n}[\boldsymbol{l_{1t}}(\boldsymbol{\theta}) - \boldsymbol{l_{2t}}(\boldsymbol{\theta})]\}$$

$$\leqslant \prod_{t=1}^{k_n} 1(a \leqslant y_{t-d} \leqslant b)\{\frac{1}{n}\sum_{t=1}^{k_n}[\boldsymbol{l_{1t}}(\boldsymbol{\theta}) - \boldsymbol{l_{2t}}(\boldsymbol{\theta})]\}$$

If $\frac{k_n}{n} \to 0$, then clearly $\sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \frac{1}{n}|\tilde{\boldsymbol{L}}_{\boldsymbol{n}}(\boldsymbol{\theta}) - \boldsymbol{L}_{\boldsymbol{n}}(\boldsymbol{\theta})| \to 0$

If $\frac{k_n}{n} \to$ a positive number (constant), then by ergodic theorem, Assumption 1 in the paper and (5.5), we have:

$$\sup_{\theta\in\Theta}\tfrac{1}{k_n}\sum_{t=1}^{k_n}[\boldsymbol{l_{1t}}(\boldsymbol{\theta}) - \boldsymbol{l_{2t}}(\boldsymbol{\theta})] \xrightarrow{a.s.} E[\sup_{\theta\in\Theta}[\boldsymbol{l_{1t}}(\boldsymbol{\theta}) - \boldsymbol{l_{2t}}(\boldsymbol{\theta})] \leqslant E[\sup_{\theta\in\Theta}[\boldsymbol{l_{1t}}(\boldsymbol{\theta})] + E[\sup_{\theta\in\Theta}\boldsymbol{l_{2t}}(\boldsymbol{\theta})] < \infty$$

Moreover, by Assumption 1,

$$\tfrac{1}{k_n}\sum_{t=1}^{k_n} 1(y_{t-d} < a \text{ or } y_{t-d} > b) \to Pr(y_{t-d} < a \text{ or } y_{t-d} > b) > 0$$

$$Pr(\lim_{k_n\to\infty}\prod_{t=1}^{k_n} 1(a \leqslant y_{t-d} \leqslant b) = 1) = Pr(\lim_{k_n\to\infty}\tfrac{1}{k_n}\sum_{t=1}^{k_n} 1(y_{t-d} < a \text{ or } y_{t-d} > b) = 0) = 0$$

$$i.e., \lim_{k_n\to\infty}\prod_{t=1}^{k_n} 1(a \leqslant y_{t-d} \leqslant b) \to 0 \text{ as } k_n \to \infty.$$

Therefore, we still have $\sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \frac{1}{n}|\tilde{\boldsymbol{L}}_{\boldsymbol{n}}(\boldsymbol{\theta}) - \boldsymbol{L}_{\boldsymbol{n}}(\boldsymbol{\theta})| \to 0$. We finish the proof of $S1$.

Let us then show $S2$:

$$E_{\theta_0}\{\boldsymbol{l_t}(\boldsymbol{\theta})\} - E_{\theta_0}\{\boldsymbol{l_t}(\boldsymbol{\theta_0})\}$$

$$= E_{\theta_0}\{[\boldsymbol{l_{1t}}(\boldsymbol{\theta}) - \boldsymbol{l_{1t}}(\boldsymbol{\theta_0})]R_{0t}R_t\} + E_{\theta_0}\{[\boldsymbol{l_{2t}}(\boldsymbol{\theta}) - \boldsymbol{l_{2t}}(\boldsymbol{\theta_0})](1 - R_{0t})(1 - R_t)\}$$

$$+ E_{\theta_0}\{[\boldsymbol{l_{1t}}(\boldsymbol{\theta}) - \boldsymbol{l_{2t}}(\boldsymbol{\theta_0})](1 - R_{0t})R_t\} + E_{\theta_0}\{[\boldsymbol{l_{2t}}(\boldsymbol{\theta}) - \boldsymbol{l_{1t}}(\boldsymbol{\theta_0})]R_{0t}(1 - R_t)\}$$

Consider $E_{\theta_0}[(\boldsymbol{l_{1t}(\theta)} - \boldsymbol{l_{2t}(\theta_0)})R_{0t}R_t]$

$$= E_{\theta_0}\{(\log \frac{\boldsymbol{\alpha_1'X_{t-1}}}{\boldsymbol{\alpha_{10}'X_{t-1}}} + \frac{\boldsymbol{\alpha_{10}'X_{t-1}}}{\boldsymbol{\alpha_1'X_{t-1}}} - 1 + \frac{[(\boldsymbol{\phi_{10}} - \boldsymbol{\phi_1})'\boldsymbol{Y_{t-1}}]^2}{\boldsymbol{\alpha_1'X_{t-1}}})R_{0t}R_t\}$$

By $\log \frac{1}{x} + x - 1 \geqslant 0$, we know:

$$E_{\theta_0}\{[\boldsymbol{l_{1t}(\theta)} - \boldsymbol{l_{2t}(\theta)}]R_{0t}R_t\} \geqslant 0 \text{ with equality iff } \frac{\boldsymbol{\alpha_1'X_{t-1}}}{\boldsymbol{\alpha_{10}'X_{t-1}}} = 1 \text{ and}$$
$$(\boldsymbol{\phi_{10}} - \boldsymbol{\phi_1})'\boldsymbol{Y_{t-1}} = 0, i.e., \boldsymbol{\alpha_1} = \boldsymbol{\alpha_{10}}, \boldsymbol{\phi_1} = \boldsymbol{\phi_{10}}$$

Similarly, we have:

$$E_{\theta_0}\{[\boldsymbol{l_{2t}(\theta)} - \boldsymbol{l_{2t}(\theta_0)}](1 - R_{0t})(1 - R_t)\} \geqslant 0 \text{ with equality iff } \frac{\boldsymbol{\alpha_2'X_{t-1}}}{\boldsymbol{\alpha_{20}'X_{t-1}}} = 1 \text{ and}$$
$$(\boldsymbol{\phi_{20}} - \boldsymbol{\phi_2})'\boldsymbol{Y_{t-1}} = 0, i.e., \boldsymbol{\alpha_2} = \boldsymbol{\alpha_{20}}, \boldsymbol{\phi_2} = \boldsymbol{\phi_{20}}$$

For the third term, we have $E_{\theta_0}\{[\boldsymbol{l_{1t}(\theta)} - \boldsymbol{l_{2t}(\theta_0)}](1-R_{0t})R_t\} \geq 0$. Since we assume $\boldsymbol{\phi_{10}} \neq \boldsymbol{\phi_{20}}$ and $\boldsymbol{\alpha_{10}} \neq \boldsymbol{\alpha_{20}}$, $E_{\theta_0}\{[\boldsymbol{l_{1t}(\theta)} - \boldsymbol{l_{2t}(\theta_0)}](1-R_{0t})R_t\} = 0$ implies $E_{\theta_0}\{(1-R_{0t})R_t\} = 0$, from which it can be deduced, following the same argument in proving Theorem 2 in Li, Guan, Li, and Yu (2015b), that $d = d_0, r_L \geqslant r_{0L}, r_U \geqslant r_{0U}$. Similarly, from the last term $E_{\theta_0}\{[\boldsymbol{l_{2t}(\theta)} - \boldsymbol{l_{1t}(\theta_0)}]R_{0t}(1 - R_t)\} = 0$, we have $d = d_0, r_L \leqslant r_{0L}, r_U \leqslant r_{0U}$. We thus finish the proof of $S2$.

Let us further show $S3$:

$\boldsymbol{\theta^*} \in U_\eta(\boldsymbol{\theta})$

$$\begin{aligned}
\boldsymbol{l_t(\theta^*)} - \boldsymbol{l_t(\theta)} = &\{\boldsymbol{l_{1t}(\theta^*)} - \boldsymbol{l_{1t}(\theta)}\}R_t(r_L^*, r_U^*, d) \\
&+ \boldsymbol{l_{1t}(\theta)}[R_t(r_L^*, r_U^*, d) - R_t(r_L, r_U, d)] \\
&+ [\boldsymbol{l_{2t}(\theta^*)} - \boldsymbol{l_{2t}(\theta)}][1 - R_t(r_L^*, r_U^*, d)] \\
&+ \boldsymbol{l_{2t}(\theta)}[R_t(r_L, r_U, d) - R_t(r_L^*, r_U^*, d)]
\end{aligned} \tag{5.6}$$

$$\begin{aligned}\boldsymbol{l_{1t}(\theta^*) - l_{1t}(\theta)} &= \log \boldsymbol{\alpha_1^{*\prime}} X_{t-1} - \log \boldsymbol{\alpha_1' X_{t-1}} \\ &+ \frac{(y_t - \boldsymbol{\phi_1^{*\prime} Y_{t-1}})^2 - (y_t - \boldsymbol{\phi_1' Y_{t-1}})^2}{\boldsymbol{\alpha_1^{*\prime} X_{t-1}}} \\ &+ (y_t - \boldsymbol{\phi_1' Y_{t-1}})^2 [\frac{1}{\boldsymbol{\alpha_1^{*\prime} X_{t-1}}} - \frac{1}{\boldsymbol{\alpha_1' X_{t-1}}}]\end{aligned}$$

By a method similar to Zhu and Ling (2013), the following relations hold:

$$E[\sup_{\theta^* \in U_\eta(\theta)} |\log \boldsymbol{\alpha_1^{*\prime}} X_{t-1} - \log \boldsymbol{\alpha_1}' X_{t-1}|] \to 0 \quad \text{as} \quad \eta \to 0$$

$$E[\sup_{\theta^* \in U_\eta(\theta)} |\frac{(y_t - \boldsymbol{\phi_1^{*\prime} Y_{t-1}})^2 - (y_t - \boldsymbol{\phi_1}' \boldsymbol{Y_{t-1}})^2}{\boldsymbol{\alpha_1^{*\prime} X_{t-1}}}|] \to 0 \quad \text{as} \quad \eta \to 0$$

$$E[\sup_{\theta^* \in U_\eta(\theta)} |(y_t - \boldsymbol{\phi_1' Y_{t-1}})^2 (\frac{1}{\boldsymbol{\alpha_1^{*\prime} X_{t-1}}} - \frac{1}{\boldsymbol{\alpha_1' X_{t-1}}})|] \to 0 \quad \text{as} \quad \eta \to 0$$

Therefore, the first and third terms in expression (5.6) can be dealt with.

We now look at the second term in (5.6).

Since $\boldsymbol{l_{1t}(\theta)}$ and $R_t(r_L^*, r_U^*, d) - R_t(r_L, r_U, d)$ have non-overlapping parameters,

$$E\{\sup_{\theta^* \in U_\eta(\theta)} \boldsymbol{l_{1t}(\theta)}[R_t(r_L^*, r_U^*, d) - R_t(r_L, r_U, d)]\} = E\{\sup_{\theta^* \in U_\eta(\theta)} \boldsymbol{l_{1t}(\theta)} \sup_{\theta^* \in U_\eta(\theta)} [R_t(r_L^*, r_U^*, d) - R_t(r_L, r_U, d)]\}$$

Li, Guan, Li, and Yu (2015b) shows that $E\{\sup_{\theta^* \in U_\eta(\theta)} |R_t(r_L^*, r_U^*, d) - R_t(r_L, r_U, d)|\} \to 0 \quad \text{as} \quad \eta \to 0$. Combining this with $E[\sup_{\theta^* \in U_\eta(\theta)} |\boldsymbol{l_{1t}(\theta)}|] < \infty$ and using dominated convergence theorem, we have:

$$E\{\sup_{\theta^* \in U_\eta(\theta)} \boldsymbol{l_{1t}(\theta)}[R_t(r_t^*, r_U^*, d) - R_t(r_t, r_U, d)]\} \to 0 \text{ as } \eta \to 0$$

The fourth term in (5.6) can be dealt with in the same way. We thus finish the proof of $S3$.

Based on $S1$-$S3$ and Huber (1967), Theorem 5.3.1 is proved.

5.6.3 Proof of Theorem 5.3.2

We first prove the super-consistency of the estimated boundary parameters at Claim (i). Note in Theorem 2, we know $\hat{\boldsymbol{\theta}}_{\boldsymbol{n}}$ is strong consistent. Therefore, it's enough for us to focus on the region close to the true parameter $\boldsymbol{\theta_0}$.

Define $w(\triangle) = \{\boldsymbol{\lambda} \in \wedge, a < r_L < r_U < b : ||\boldsymbol{\lambda} - \boldsymbol{\lambda_0}|| < \triangle, |r_L - r_{0L}| < \triangle, |r_U - r_{0U}| < \triangle\}$, where $0 < \triangle < \min\{1, \frac{r_{0U} - r_{0L}}{2}\}$ will be determined later. As in Chan (1993), it is sufficient to show for any $\epsilon > 0, \exists$ a positive K such that:

$$\Pr(\tilde{L}_n(\boldsymbol{\lambda}, r_{0L} + z_L, r_{0U} + z_U) - \tilde{L}_n(\boldsymbol{\lambda}, r_{0L}, r_{0U}) > 0) > 1 - \epsilon \tag{5.7}$$

where $\boldsymbol{\lambda} \in w(\triangle), |z_L| > \frac{K}{n}, |z_U| > \frac{K}{n}$.

We consider the case $p = d = 1$,$z_L < 0$ and $z_U > 0$. Denote the disjoint events:
$A_0 = \{r_{0U} < y_{t-1} \leqslant r_{0U} + z_U, R_{t-1} = 1\}$

$B_0 = \{r_{0L} + z_L < y_{t-1} \leqslant r_{0L}, R_{t-1} = 0\}$

$A_i = \{y_{t-1} \in (r_{0L}, r_{0U}], \ldots, y_{t-i} \in (r_{0L}, r_{0U}], r_{0U} < y_{t-i-1} \leqslant r_{0U} + z_U, R_{t-i-1} = 1\}, i \geqslant 1$

$B_i = \{y_{t-1} \in (r_{0L}, r_{0U}], \ldots, y_{t-i} \in (r_{0L}, r_{0U}], r_{0L} + z_L < y_{t-i-1} \leqslant r_{0L}, R_{t-i-1} = 0\}, i \geqslant 1$

where $R_t \equiv R_t(r_{0L} + z_L, r_{0U} + z_U)$. Denote $A_t(z_L, z_U) = \bigcup_{j=0}^{\infty} A_j, B_t(z_L, z_U) = \bigcup_{j=0}^{\infty} B_j$
As in Li, Guan, Li, and Yu (2015b), we have:

$$R_t(r_{0L} + z_L, r_{0U} + z_U) - R_t(r_{0L}, r_{0U}) = 1\{A_t(z_L, z_U)\} - 1\{B_t(z_L, z_U)\}$$

Moreover, $B_t(z_L, z_U) \subset \{R_t(r_{0L}, r_{0U}) = 1\}, A_t(z_L, z_U) \subset \{R_t(r_{0L}, r_{0U}) = 0\}$. As a result,

$$\begin{aligned}
&\boldsymbol{L_n}(\boldsymbol{\lambda}, r_{0L} + z_L, r_{0U} + z_U) - \boldsymbol{L_n}(\boldsymbol{\lambda}, r_{0L}, r_{0U}) \\
&= \sum_{t=1}^{n} (\boldsymbol{l_{1t}(\theta) - l_{2t}(\theta)})[1\{A_t(z_L, z_U)\} - 1\{B_t(z_L, z_U)\}] \\
&= \sum_{t=1}^{n} (\boldsymbol{l_{1t}(\theta) - l_{2t}(\theta)})1\{A_t(z_L, z_U)\} + \sum_{t=1}^{n} (\boldsymbol{l_{2t}(\theta) - l_{1t}(\theta)})1\{B_t(z_L, z_U)\} \\
&= \boldsymbol{L_{1n}}(z_L, z_U) + \boldsymbol{L_{2n}}(z_L, z_U)
\end{aligned}$$

Our next step is to show that

$$\Pr(\boldsymbol{L_{1n}}(z_L, z_U) > 0) > 1 - \epsilon \tag{5.8}$$

when $\boldsymbol{\lambda} \in w(\triangle), -\triangle < z_L < 0, \frac{K}{n} < z_U < \triangle$.

Define $Q_{z_L}(z_U) = E[1\{A_t(z_L, z_U)\}]$

$$Q_{n,z_L}(z_U) = \frac{1}{n}\sum_{t=1}^{n} 1\{A_t(z_L, z_U)\}$$

By a calculation, we can obtain

$$\begin{aligned}
&\frac{\boldsymbol{L_{1n}}(z_L, z_U)}{nQ_{z_L}(z_U)} \equiv \frac{\sum_{t=1}^{n}[\boldsymbol{l_{1t}(\theta) - l_{2t}(\theta)}][1\{A_t(z_L, z_U)\}]}{nQ_{z_L}(z_U)} \\
&= \frac{\sum_{t=1}^{n}[\log \frac{\boldsymbol{\alpha'_{10}X_{t-1}}}{\boldsymbol{\alpha'_{20}X_{t-1}}} + \frac{\boldsymbol{\alpha'_{20}X_{t-1}}}{\boldsymbol{\alpha'_{10}X_{t-1}}} - 1 + \frac{\{\boldsymbol{(\phi_{20} - \phi_{10})'Y_{t-1}}\}^2}{\boldsymbol{\alpha'_{10}X_{t-1}}}]1\{A_t(z_L, z_U)\}}{nQ_{n,z_L}(z_U)} \\
&+ \frac{\sum_{t=1}^{n} \frac{2\boldsymbol{(\phi_{20} - \phi_{10})'Y_{t-1}}\sqrt{\boldsymbol{\alpha'_{20}X_{t-1}}}}{\boldsymbol{\alpha'_{10}X_{t-1}}}\epsilon_t 1\{A_t(z_L, z_U)\}}{nQ_{n,z_L}(z_U)} \\
&+ \frac{\sum_{t=1}^{n} \frac{\boldsymbol{(\alpha_{20} - \alpha_{10})'X_{t-1}}}{\boldsymbol{\alpha'_{10}X_{t-1}}}(\epsilon_t^2 - 1)1\{A_t(z_L, z_U)\}}{nQ_{n,z_L}(z_U)} + Op(\triangle)
\end{aligned}$$

Note in the first term,

$$\log \frac{\boldsymbol{\alpha'_{10}X_{t-1}}}{\boldsymbol{\alpha'_{20}X_{t-1}}} + \frac{\boldsymbol{\alpha'_{20}X_{t-1}}}{\boldsymbol{\alpha'_{10}X_{t-1}}} - 1 + \frac{\{\boldsymbol{(\phi_{20} - \phi_{10})'Y_{t-1}}\}^2}{\boldsymbol{\alpha'_{10}X_{t-1}}} > 0$$

where $\boldsymbol{X_{t-1}} = (1, y_{t-1}^2), \boldsymbol{Y_{t-1}} = (1, y_{t-1})$

In the second term, $\sum_{t=1}^{n} \frac{2\boldsymbol{(\phi_{20} - \phi_{10})'Y_{t-1}}\sqrt{\boldsymbol{\alpha'_{20}X_{t-1}}}}{\boldsymbol{\alpha'_{10}X_{t-1}}}\epsilon_t 1\{A_t(z_L, z_U)\}$ is bounded in absolute value by $K_1|\sum_{t=1}^{n} \epsilon_t 1(A_t(z_L, z_U))|$

For the third term, the numerator is bounded in absolute value by $K_2|\sum_{t=1}^{n}(\epsilon_t^2 - 1)1(A_t(z_L, z_U))|$, where K_1 and K_2 are some constants independent of n.

Based on the method similar to Theorem 3 in Li, Guan, Li, and Yu (2015b), it's enough for us to verify the following three conditions for sufficiently large sample size

n: $\forall \epsilon > 0, \eta > 0, \exists$ a positive constant K

$$Pr[\sup_{\frac{k}{n}<z_U<\triangle,-\triangle<z_L\leqslant 0} |\frac{1}{nQ_{zL}(z_U)}\sum_{t=1}^{n} 1\{A_t(z_L,z_U)\} - 1| < \eta] > 1-\epsilon$$

$$Pr[\sup_{\frac{k}{n}<z_U<\triangle,-\triangle<z_L\leqslant 0} |\frac{1}{nQ_{zL}(z_U)}\sum_{t=1}^{n} \epsilon_t 1\{A_t(z_L,z_U)\}| < \eta] > 1-\epsilon$$

$$Pr[\sup_{\frac{k}{n}<z_U<\triangle,-\triangle<z_L\leqslant 0} |\frac{1}{nQ_{zL}(z_U)}\sum_{t=1}^{n} (\epsilon_t^2 - 1) 1\{A_t(z_L,z_U)\}| < \eta] > 1-\epsilon$$

Therefore, (5.8) is implied. We can further show a similar result for $\boldsymbol{L_{2n}}(z_L, z_U)$, and then that for $L_n(\boldsymbol{\lambda}, r_{0L}+z_L, r_{0U}+z_U) - L_n(\boldsymbol{\lambda}, r_{0L}, r_{0U})$. (5.7) is thus proved for the case that $p = d = 1$, $z_L < 0$ and $z_U > 0$.

The proof for other cases is similar and hence is skipped. Claim (ii) cab be proved by following similar arguments as in the Theorem 2.2 in Li, Ling, and Li (2013) and Theorem 3.2 in Li, Ling, and Zakoian (2015a), and is also omitted.

5.6.4 Proof of Theorem 5.3.3

Theorem 5.3.2 indicates that $n(\hat{r}_L - r_{0L}) = O_p(1)$ and $n(\hat{r}_U - r_{0U}) = O_p(1)$. To further characterize their limiting distributions, we construct a sequence of random functions as follows:

$$\psi_n(z) = \boldsymbol{L_n}(\tilde{\boldsymbol{\lambda}}_n(r_0 + \tfrac{z}{n}), r_0 + \tfrac{z}{n}) - \boldsymbol{L_n}(\tilde{\boldsymbol{\lambda}}_n(r_0), r_0)$$

where $r_0 = (r_{0L}, r_{0U})^T$ and $z = (z_L, z_U)^T$

$$\begin{aligned}
\text{Denote } \varphi_n(z) = &\sum_{t=1} \xi_{1t}\{1 - R_{t-1}(r_0)\} 1\{r_{0L} < y_{t-d} \leqslant r_{0L} + \frac{z_L}{n}\} 1(z_L \geqslant 0) \\
&+ \sum_{t=1} \xi_{2t}\{1 - R_{t-1}(r_0)\} 1\{r_{0L} + \frac{z_L}{n} < y_{t-d} \leqslant r_{0L}\} 1(z_L < 0) \\
&+ \sum_{t=1} \xi_{1t} R_{t-1}(r_0) 1\{r_{0U} < y_{t-d} \leqslant r_{0U} + \frac{z_L}{n}\} 1(z_U \geqslant 0) \\
&+ \sum_{t=1} \xi_{2t} R_{t-1}(r_0) 1\{r_{0U} + \frac{z_U}{n} < y_{t-d} \leqslant r_{0U}\} 1(z_U < 0)
\end{aligned}$$

where $\xi_{1t} = \sum_{j=0}^{\infty} \zeta_{1,t+j} H_{t+j,j}$

$$\xi_{2t} = \sum_{j=0}^{\infty} \zeta_{2,t+j} H_{t+j,j}$$

$$\zeta_{1t} = \log \frac{\boldsymbol{\alpha'_{10}} \boldsymbol{X_{t-1}}}{\boldsymbol{\alpha'_{20}} \boldsymbol{X_{t-1}}} + \frac{\{(\boldsymbol{\phi_{10}} - \boldsymbol{\phi_{20}})' \boldsymbol{Y_{t-1}} - \epsilon_t \sqrt{\boldsymbol{\alpha'_{20}} \boldsymbol{X_{t-1}}}\}^2}{\boldsymbol{\alpha'_{10}} \boldsymbol{X_{t-1}}} - \epsilon_t^2$$

$$\zeta_{2t} = \log \frac{\boldsymbol{\alpha'_{20}} \boldsymbol{X_{t-1}}}{\boldsymbol{\alpha'_{10}} \boldsymbol{X_{t-1}}} + \frac{\{(\boldsymbol{\phi_{10}} - \boldsymbol{\phi_{20}})' \boldsymbol{Y_{t-1}} + \epsilon_t \sqrt{\boldsymbol{\alpha'_{10}} \boldsymbol{X_{t-1}}}\}^2}{\boldsymbol{\alpha'_{20}} \boldsymbol{X_{t-1}}} - \epsilon_t^2$$

$$H_{t,j} = \prod_{l=1}^{j} 1(r_{0L} < y_{t-d+1-l} \leqslant r_{0U})$$

We first prove that, $\forall K > 0$,

$$\sup_{||z|| \leqslant K} |\psi_n(z) - \varphi_n(z)| = o_p(1) \tag{5.9}$$

It's enough to show:

$$\sup_{||z|| \leqslant K} |\psi_n(z) - \{\boldsymbol{L_n}(\boldsymbol{\lambda_0}, r_0 + \frac{z}{n}) - \boldsymbol{L_n}(\boldsymbol{\lambda_0}, r_0)\}| = o_p(1) \tag{5.10}$$

$$\sup_{||z|| \leqslant K} |\varphi_n(z) - \{\boldsymbol{L_n}(\boldsymbol{\lambda_0}, r_0 + \frac{z}{n}) - \boldsymbol{L_n}(\boldsymbol{\lambda_0}, r_0)\}| = o_p(1) \tag{5.11}$$

Following a similar strategy of proving Theorem 4 in Li, Guan, Li, and Yu (2015b), we can use Taylor's expansion and results of Theorem 5.3.2 in current paper to show that (5.10) holds. Below we will verify (5.11).

Denote $R_t(z) = R_t(r_0 + \frac{z}{n})$ for simplicity. As in Li, Guan, Li, and Yu (2015b), we have:

$$
\begin{aligned}
R_t(z) - R_t(0) = &\{R_{t-1}(z) - R_{t-1}(0)\}1(r_{0L} < y_{t-d} \leqslant r_{0U}) \\
&+ \{1 - R_{t-1}(z)\}1(r_{0L} < y_{t-d} \leqslant r_{0L} + \frac{z_L}{n})1(z_L \geqslant 0) \\
&- \{1 - R_{t-1}(z)\}1(r_{0L} + \frac{z_L}{n} < y_{t-d} \leqslant r_{0L})1(z_L < 0) \\
&+ R_{t-1}(z)1(r_{0U} < y_{t-d} \leqslant r_{0U} + \frac{z_U}{n})1(z_U \geqslant 0) \\
&- R_{t-1}(z)1(r_{0U} + \frac{z_U}{n} < y_{t-d} \leqslant r_{0U})1(z_U < 0)
\end{aligned}
$$

$$
\begin{aligned}
\text{and then } R_t(z) - R_t(0) = &\sum_{j=0}^{\infty} H_{t,j}\{1 - R_{t-1-j}(z)\}1(r_{0L} < y_{t-d-j} \leqslant r_{0L} + \frac{z_L}{n})1(z_L \geqslant 0) \\
&- \sum_{j=0}^{\infty} H_{t,j}\{1 - R_{t-1-j}(z)\}1(r_{0L} + \frac{z_L}{n} < y_{t-d-j} \leqslant r_{0L})1(z_L < 0) \\
&+ \sum_{j=0}^{\infty} H_{t,j}R_{t-1-j}(z)1(r_{0U} < y_{t-d-j} \leqslant r_{0U} + \frac{z_U}{n})1(z_U \geqslant 0) \\
&- \sum_{j=0}^{\infty} H_{t,j}R_{t-1-j}(z)1(r_{0U} + \frac{z_U}{n} < y_{t-d-j} \leqslant r_{0U})1(z_U < 0)
\end{aligned} \tag{5.12}
$$

where $H_{t,j} = \prod_{l=1}^{j} 1(r_{0L} < y_{t-d+1-l} \leqslant r_{0U})$

Denote $\zeta_t = \boldsymbol{l_{1t}(\theta_0) - l_{2t}(\theta_0)}$

$$
= \log \boldsymbol{\alpha'_{10}X_{t-1}} + \frac{(y_t - \boldsymbol{\phi'_{10}Y_{t-1}})^2}{\boldsymbol{\alpha'_{10}X_{t-1}}} - \log \boldsymbol{\alpha'_{20}X_{t-1}} - \frac{(y_t - \boldsymbol{\phi'_{20}Y_{t-1}})^2}{\boldsymbol{\alpha'_{20}X_{t-1}}}
$$

and it holds that $\boldsymbol{L_n}(\boldsymbol{\lambda_0}, r_0 + \frac{z}{n}) - \boldsymbol{L_n}(\boldsymbol{\lambda_0}, r_0) = \sum_{t=1}^{n} \zeta_t\{R_t(z) - R_t(0)\}$.

For the first term of (5.12), i.e., the case with $z_L \geqslant 0$, by a method similar to Li, Guan, Li, and Yu (2015b), it can be shown that:

$$
\sup_{||z|| \leqslant B} || \sum_{t=1}^{n} \zeta_t \sum_{j=0}^{\infty} H_{t,j}\{R_{t-1-j}(z) - R_{t-1-j}(0)\}1(r_{0L} < y_{t-d-j} \leqslant r_{0L} + \tfrac{z_L}{n})|| = o_p(1)
$$

Note $\zeta_t = \zeta_{1t}$ when $R_t(0) = 0$ and $\zeta_t = -\zeta_{2t}$ when $R_t(0) = 1$. Thus, we have

$$\sum_{t=1}^{n}\zeta_t\sum_{j=0}^{\infty}H_{t,j}\{1-R_{t-1-j}(0)\}1(r_{0L}<y_{t-d-j}\leqslant r_{0L}+\frac{z_L}{n})$$
$$=\sum_{j=0}^{\infty}\sum_{t=1-j}^{n-j}\zeta_{t+j}H_{t+j,j}\{1-R_{t-1}(0)\}1(r_{0L}<y_{t-d}\leqslant r_{0L}+\frac{z_L}{n})$$
$$=\sum_{j=0}^{\infty}(\sum_{t=1}^{n}+\sum_{t=1-j}^{0}-\sum_{t=n-j+1}^{n})\zeta_{t+j}H_{t+j,j}\{1-R_{t-1}(0)\}1(r_{0L}<y_{t-d}\leqslant r_{0L}+\frac{z_L}{n})$$
$$=\sum_{j=0}^{\infty}\sum_{t=1-j}^{n}\zeta_{t+j}H_{t+j,j}\{1-R_{t-1}(0)\}1(r_{0L}<y_{t-d}\leqslant r_{0L}+\frac{z_L}{n})+o_p(1)$$
$$=\sum_{t=1-j}^{n}\xi_{1t}\{1-R_{t-1}(0)\}1(r_{0L}<y_{t-d}\leqslant r_{0L}+\frac{z_L}{n})+o_p(1)$$

where the $o_p(1)$ is uniformly for $||z||\leqslant B$. See also the proof of Theorem 4 in Li, Guan, Li, and Yu (2015b) and Lemma 8.2 in Li, Ling, and Li (2013).

Similarly, we can handle the remaining terms in (5.12). Therefore, (5.11) is implied. With (5.10) and (5.11), it is clear that (5.9) holds.

For the space $\mathbb{D}(\mathbb{R}^2)$, we define the skorohod metric as

$$d(f_1,f_2)=\sum_{k=1}^{\infty}2^{-k}\min\{1,d_k(f_1,f_2)\}\quad\text{for } f_1,f_2\in\mathbb{D}(\mathbb{R}^2)$$

where $d_k(f_1,f_2)$ is the skorohod metric, see also Li et al. (2015b), Li and Ling (2012) and (16.4) in Billingsley (1999).

Based on the strategies of proving Theorem 3.3 in Li and Ling (2012) and Theorem 5 in Li, Ling, and Zhang (2016), together with Theorem 5.5 in Straf (1972), we are able to conclude that the limiting process $\{\varphi_n(z),z\in\mathbb{R}^2\}$ is $\{\varphi(z),z\in\mathbb{R}^2\}$.

Moreover, (5.9) implies that $d(\psi_n(z),\varphi_n(z))\to 0$ in probability as $n\to\infty$. Following Seijo and Sen (2011)'s Theorem 3.1, we can conclude $n(\hat{r}_L-r_{0L})\to M_-^{(L)}$ and $n(\hat{r}_U-r_{0U})\to M_-^{(U)}$ in distribution as $n\to\infty$ respectively. The rest of the theorem can be proved by following Theorem 2 in Chan (1993)).

5.6.5 Proof of Theorem 5.3.4

Based on Theorem 5.3.1 and 5.3.2, we know $\hat{r}_L$ and $\hat{r}_U$ are super-consistent, and $\hat{d}$ is consistent with integer values. Therefore, it can be assumed that the true values of (r_L, r_U, d) are known, indicating the true Regime indicators R_t are known.

By definition,

$$\begin{aligned}
\text{BIC}(p) &= \sum_{t=1}^{n} \boldsymbol{l_t}(\hat{\boldsymbol{\theta}}) + (2p+2)\log n_1 + (2p+2)\log n_2 \\
&= \sum_{t=1}^{n} \boldsymbol{l_{1t}}(\hat{\boldsymbol{\theta}})R_t + \sum_{t=1}^{n} \boldsymbol{l_{2t}}(\hat{\boldsymbol{\theta}})(1-R_t) + (2p+2)\log n_1 + (2p+2)\log n_2 \\
&= \sum_{t=1}^{n} \boldsymbol{l_{1t}}(\hat{\boldsymbol{\lambda}}_{\boldsymbol{1}})R_t + \sum_{t=1}^{n} \boldsymbol{l_{2t}}(\hat{\boldsymbol{\lambda}}_{\boldsymbol{2}})(1-R_t) + (2p+2)\log n_1 + (2p+2)\log n_2
\end{aligned}$$

where $\boldsymbol{l_{1t}}(\boldsymbol{\theta}) = \boldsymbol{l_{1t}}(\boldsymbol{\lambda_1})$ and $\boldsymbol{l_{2t}}(\boldsymbol{\theta}) = \boldsymbol{l_{2t}}(\boldsymbol{\lambda_2})$ are defined at the start of the proof of Theorem 5.3.1.

Below we introduce notations $\boldsymbol{\lambda}^p$,$\boldsymbol{\lambda}_0^p$ and $\hat{\boldsymbol{\lambda}}^p$ which depend on p. First, under the situation $p > p_0$, we have

$$\boldsymbol{l_{1t}}(\boldsymbol{\lambda}_{10}^p) = \boldsymbol{l_{1t}}(\boldsymbol{\lambda}_{10}^{p_0}), \boldsymbol{l_{2t}}(\boldsymbol{\lambda}_{20}^p) = \boldsymbol{l_{2t}}(\boldsymbol{\lambda}_{20}^{p_0})$$

By an intermediate result in deriving Theorem 4 in Li, Zhu, Liu, and Li (2017), we know

$$\begin{aligned}
&\sum_{t=1}^{n} \boldsymbol{l_{1t}}(\hat{\boldsymbol{\lambda}}_1^{p_0})R_t - \sum_{t=1}^{n} \boldsymbol{l_{1t}}(\boldsymbol{\lambda}_{10}^{p_0})R_t = O_p(1) \\
&\sum_{t=1}^{n} \boldsymbol{l_{1t}}(\hat{\boldsymbol{\lambda}}_1^{p})R_t - \sum_{t=1}^{n} \boldsymbol{l_{1t}}(\boldsymbol{\lambda}_{10}^{p})R_t = O_p(1) \\
&\sum_{t=1}^{n} \boldsymbol{l_{2t}}(\hat{\boldsymbol{\lambda}}_2^{p_0})(1-R_t) - \sum_{t=1}^{n} \boldsymbol{l_{2t}}(\boldsymbol{\lambda}_{20}^{p_0})(1-R_t) = O_p(1) \\
&\sum_{t=1}^{n} \boldsymbol{l_{2t}}(\hat{\boldsymbol{\lambda}}_2^{p})(1-R_t) - \sum_{t=1}^{n} \boldsymbol{l_{2t}}(\boldsymbol{\lambda}_{20}^{p})(1-R_t) = O_p(1)
\end{aligned}$$

As a result,

$$\begin{aligned}
\sum_{t=1}^{n} \boldsymbol{l_{1t}}(\hat{\boldsymbol{\lambda}}_1^{p}) R_t - \sum_{t=1}^{n} \boldsymbol{l_{1t}}(\hat{\boldsymbol{\lambda}}_1^{p_0}) R_t =& [\sum_{t=1}^{n} \boldsymbol{l_{1t}}(\hat{\boldsymbol{\lambda}}_1^{p}) R_t - \sum_{t=1}^{n} \boldsymbol{l_{1t}}(\boldsymbol{\lambda}_{10}^{p}) R_t] \\
& - [\sum_{t=1}^{n} \boldsymbol{l_{1t}}(\hat{\boldsymbol{\lambda}}_1^{p_0}) R_t - \sum_{t=1}^{n} \boldsymbol{l_{1t}}(\boldsymbol{\lambda}_{10}^{p_0}) R_t] \\
& + [\sum_{t=1}^{n} \boldsymbol{l_{1t}}(\boldsymbol{\lambda}_{10}^{p}) R_t] - \sum_{t=1}^{n} \boldsymbol{l_{1t}}(\boldsymbol{\lambda}_{10}^{p_0}) R_t] = O_p(1) \\
\sum_{t=1}^{n} \boldsymbol{l_{2t}}(\hat{\boldsymbol{\lambda}}_2^{p})(1 - R_t) - \sum_{t=1}^{n} \boldsymbol{l_{2t}}(\hat{\boldsymbol{\lambda}}_2^{p_0})(1 - R_t) =& [\sum_{t=1}^{n} \boldsymbol{l_{2t}}(\hat{\boldsymbol{\lambda}}_2^{p})(1 - R_t) - \sum_{t=1}^{n} \boldsymbol{l_{2t}}(\boldsymbol{\lambda}_{20}^{p})(1 - R_t)] \\
& - [\sum_{t=1}^{n} \boldsymbol{l_{2t}}(\hat{\boldsymbol{\lambda}}_2^{p_0})(1 - R_t) - \sum_{t=1}^{n} \boldsymbol{l_{2t}}(\boldsymbol{\lambda}_{20}^{p_0})(1 - R_t)] \\
& + [\sum_{t=1}^{n} \boldsymbol{l_{2t}}(\boldsymbol{\lambda}_{20}^{p})(1 - R_t) - \sum_{t=1}^{n} \boldsymbol{l_{2t}}(\boldsymbol{\lambda}_{20}^{p_0})(1 - R_t)] \\
& = O_p(1)
\end{aligned}$$

Therefore, $\mathrm{BIC}(p) - \mathrm{BIC}(p_0) = O_p(1) + 2(p - p_0) \log n_1 n_2 \to \infty$ as $n \to \infty$, since $\frac{n_1}{n} \to \Pr(R_t = 1) > 0$ and $\frac{n_2}{n} \to \Pr(R_t = 0) > 0$ as $n \to \infty$.

We then consider the case $p < p_0$. Again by an intermediate result in deriving Theorem 4 in Li, Zhu, Liu, and Li (2017), we know

$$\begin{aligned}
\sum_{t=1}^{n} \boldsymbol{l_{1t}}(\hat{\boldsymbol{\lambda}}_1^{p}) R_t - \sum_{t=1}^{n} \boldsymbol{l_{1t}}(\hat{\boldsymbol{\lambda}}_1^{\boldsymbol{p_0}}) R_t &= O_p(1) + o_p(n_1) + c_1 n_1 \\
\sum_{t=1}^{n} \boldsymbol{l_{2t}}(\hat{\boldsymbol{\lambda}}_2^{p})(1 - R_t) - \sum_{t=1}^{n} \boldsymbol{l_{2t}}(\hat{\boldsymbol{\lambda}}_2^{p_0})(1 - R_t) &= O_p(1) + o_p(n_2) + c_2 n_2
\end{aligned}$$

where c_1, c_2 are positive constants, defined in the same way as constant c in Li et al. (2017).

$$\begin{aligned}
\text{Therefore, } \mathrm{BIC}(p) - \mathrm{BIC}(p_0) &= c_1 n_1 + c_2 n_2 + O_p(1) + o_p(n_1) + o_p(n_2) + 2(p - p_0) \ln n_1 n_2 \\
&= c_3 n + O_p(1) + o_p(n) + O(\ln n) \\
&\to \infty \text{ as } n \to \infty
\end{aligned}$$

since constant $c_3 > 0$, $\frac{n_1}{n} \to \Pr(R_t = 1) > 0$ and $\frac{n_2}{n} \to \Pr(R_t = 0) > 0$ as $n \to \infty$.

Table 5.1: Simulation Results for Buffered Double Autoregressive Model

		1st regime							
n		ϕ_{10}	ϕ_{11}	ϕ_{12}	α_{10}	α_{11}	α_{12}	r_L	r_U
400	bias	0.0020	-0.0013	-0.0124	-0.0024	-0.0102	0.0054	0.0013	-0.0102
	empirical standard deviation	0.0288	0.0883	0.0765	0.0165	0.1161	0.0657	0.0421	0.0314
	asymptotic standard deviation	0.0255	0.0798	0.0723	0.0152	0.1057	0.0719		
800	bias	0.0028	-0.0025	-0.0030	-0.0005	-0.0037	-0.0030	0.0005	-0.0039
	empirical standard deviation	0.0188	0.0616	0.0529	0.0112	0.0855	0.0471	0.0173	0.0165
	asymptotic standard deviation	0.0184	0.0573	0.0518	0.0113	0.0786	0.0506		
		2nd regime							
n		ϕ_{20}	ϕ_{21}	ϕ_{22}	α_{20}	α_{21}	α_{22}		
400	bias	-0.0039	-0.0008	-0.0041	-0.0010	0.0010	-0.0045		
	empirical standard deviation	0.0225	0.0835	0.0680	0.0103	0.0959	0.0693		
	asymptotic standard deviation	0.0197	0.0681	0.0627	0.0086	0.0741	0.0571		
800	bias	-0.0004	-0.0055	-0.0020	-0.0005	0.0007	-0.0039		
	empirical standard deviation	0.0155	0.0503	0.0467	0.0068	0.0631	0.0494		
	asymptotic standard deviation	0.0142	0.0486	0.0448	0.0063	0.0551	0.0414		

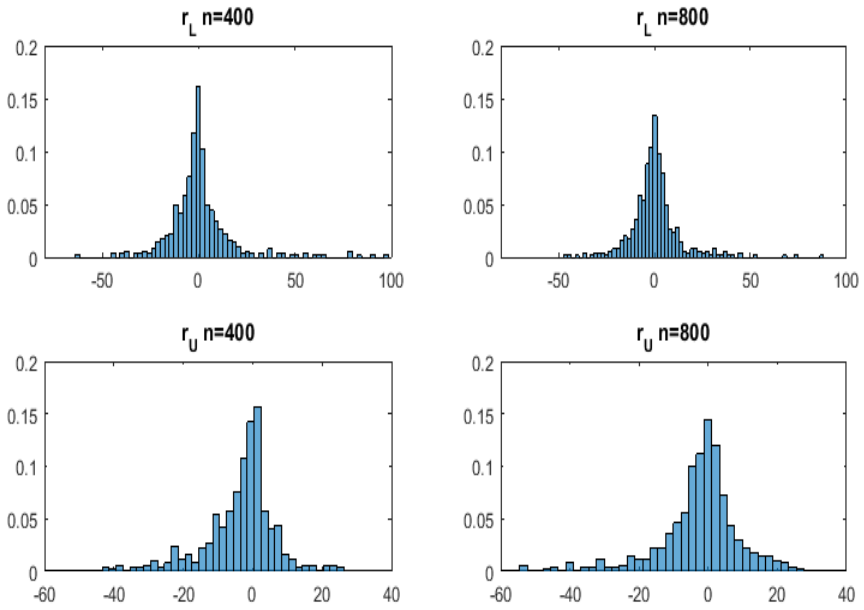


Figure 5.1: Histograms of random variables $n(\hat{r}_L - r_L)$(upper panels) and $n(\hat{r}_U - r_U)$(lower panels)

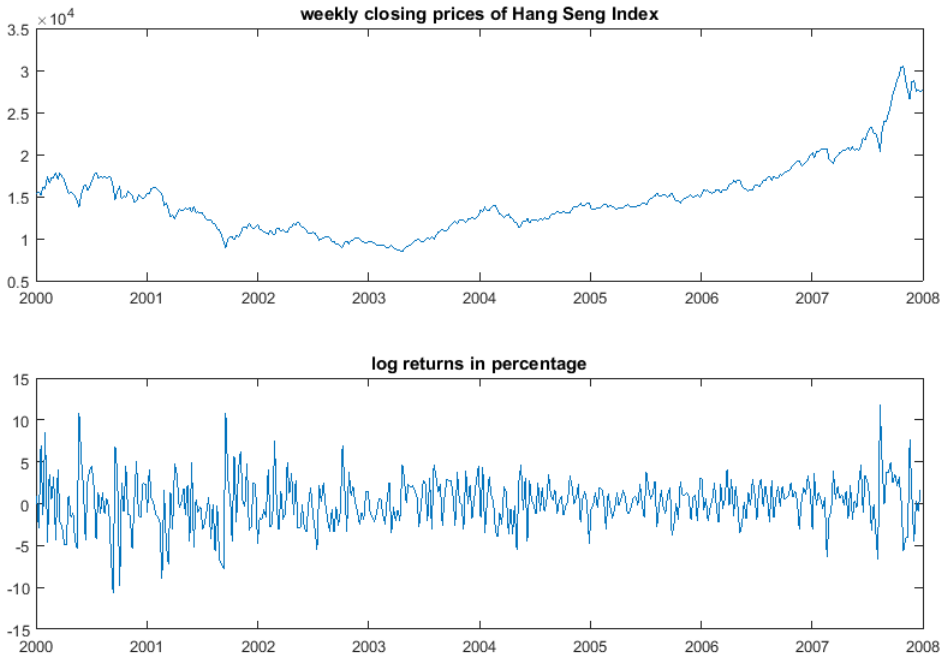


Figure 5.2: Time plots of Hang Seng Index (HSI) from Jan 2000 to Dec 2007(upper panel) and corresponding log retuns(lower panel) at a weekly frequency

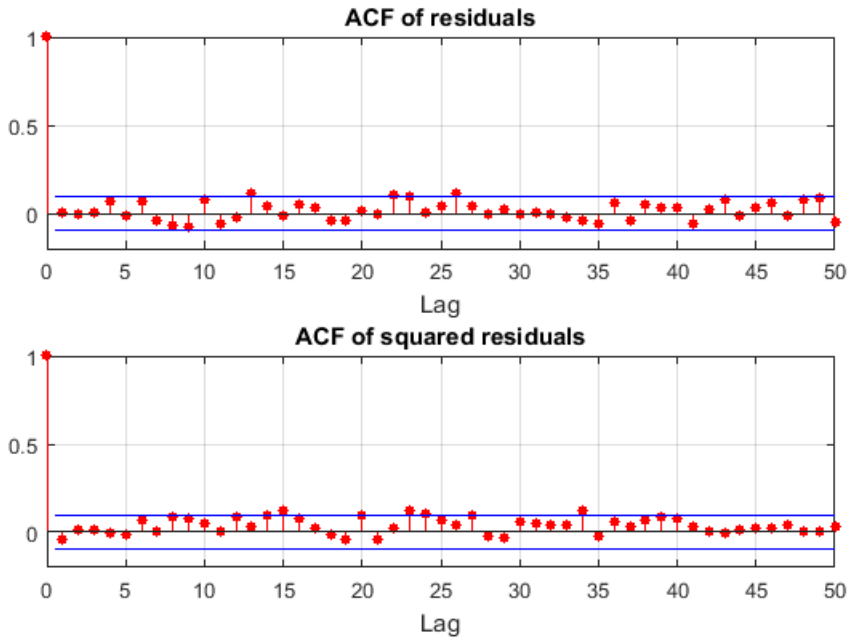


Figure 5.3: Sample ACFs of the fitted buffered double autoregressive model in empirical analysis

Chapter 6

Conclusion

In this dissertation, I focus on enhancing the understanding of contemporary economic issues and financial markets through the lens of general equilibrium models and econometric tools. In the first essay, I introduce a two-sector production-based dynamic stochastic general equilibrium model to study the interaction between R&D activities and firm heterogeneity. Specifically, it rationalizes the empirical finding that R&D-intensive firms tend to show higher expected equity returns, yet lower leverage, default rates, and credit spreads, than the R&D-nonintensive firms. Generally speaking, the model generates sizable heterogeneity as in the data and fits the aggregate moments reasonably well. However, the model is silent on other related factors such as liquidity risk and taxation. Future studies could examine the impact of such factors, and in so doing further elucidate the effects of R&D activities.

The second essay characterizes the solution of a planner's problem with multiple agents, multiple goods, and recursive preferences. The introduction of multiple goods substantially changes the dynamics of Pareto optimal allocations. Future research should extend our theoretical results to continuous time and continuous shocks. Asset pricing implications may be particularly appealing, due to the ability of this class of models to endogenously produce time-varying second moments without requiring market frictions. Since the main features of the risk-sharing scheme can be accurately captured through a third-order approximation, this class of models could be easily extended to an international business cycle setting as well.

The third essay proposes a first-order good uncertainty through the investment option channel in the DSGE model. When this uncertainty is high in the economy, there are more high-quality investment opportunities in the economy, which enhances the efficiency of producing physical capital. As an endogenous response, there are

more firms that choose to invest, which leads to a boom in this economy. I then incorporate it into the standard DSGE model with bad uncertainty from the demand side to evaluate the aggregate effects of the two types of uncertainty.

The fourth essay proposes a buffered double autoregressive (BDAR) time series model to depict the buffering phenomenon of conditional mean and conditional variance in time series. I investigate the probabilistic properties as well as the statistical inference. This type of model greatly extends some time series models in the literature and potentially has wide applications in modeling financial time series data.

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