A Study in Laterally Restrained Buckled Beams for the use in a Vertical Isolation System Department of Civil & Environmental Engineering Duke University

by

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Thesis submitted in partial fulfillment of the requirements for the degree of Master

of Science in the Department of Civil and Environmental Engineering in the

Graduate School of Duke University.

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ABSTRACT

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Abstract

Linear vibration isolation systems, used to reduce the transmissibility of vertical vibration, requires a vertical static displacement that increases with the square of the natural period of the isolation system. The static displacement of a vertical isolation system with a one second natural period is 0.25 m. The nonlinear stiffness of buckled beams loaded in the transverse direction can be designed to reduce the vertical static displacement requirement of vertical systems. This study presents an analysis of large displacement mechanics of slender beams that buckle against a constraint, and extracts the transverse constraint force via the Lagrange multiplier enforcing the constraint. The constraint prescribes a maximum allowable lateral displacement along the length of the beam and a specified longitudinal displacement at the mid-span of the beam. No small curvature assumption is involved. Lateral and longitudinal displacements are parameterized in terms of Fourier coefficients. Coefficient values for constrained equilibria are found by minimizing the bending strain energy such that lateral and longitudinal constraints are satisfied. Because the full expression for curvature is used, this is a nonlinear constrained optimization problem.

Edge and mid-point horizontal constraint positions are varied to gain a better understanding of the constraint forces at each position. This modeling approach is then used to design a system of post-buckled leaf springs in order to meet vibration isolation requirements without over-stressing the springs. This process is discussed in detail along with the process and challenges associated with the physical model. Theoretical predictions are compared to laboratory scale measurements. Experimental results from the physical model are compared to the theoretical and numerical simulation results. The potential for rocking responses of the vertical isolation system are quantified via the modeling of the nonlinear dynamics of a platform supported by a system of springs and carrying a mass concentrated above the platform.

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Chapter 1

Introduction

1.1 Motivation

Vertical isolation is <u>not</u> often regarded as a major consideration in the development of isolation systems for seismic design of structures, as seismic input is predominantly in the form of horizontal ground motion. However, it has been shown that vertical floor vibrations can be significant in moderate to strong earthquakes with peak ground acceleration (PGA) $\approx 0.2g$, where g represents the acceleration due to gravity. The Wieser Technical Report Assessment of Floor Accelerations in Yielding Buildings, shows that the median ratio of peak vertical floor accelerations to peak horizontal ground acceleration is about one-half. For peak ground accelerations of 0.2 g, peak vertical accelerations would be large enough to damage fragile nonstructural components and equipment situated on the floor. Financial losses to building contents can account for approximately 75% of the incurred losses.¹ Additionally, and perhaps surprisingly, vertical floor vibrations are stronger in shorter structures than in taller ones as shown in the report's spectra, seen in figures 1.2 through 1.5, comparing the the normalized vertical acceleration at various relative elevations within buildings. These

¹Wieser, Joseph D., Pekhan, Gokhan, Zaghi, Arash E., Itani, Ahmad E., Maragakis, Emmanuel "Manos," Assessment of Floor Accelerations in Yielding Buildings, Technical Report MCEER-12-0008, October 5, 2012.

significant floor vibrations can often be attributed to the fact that beam design is almost always dictated by beam deflection requirements for steel beams, $\Delta_{max} \approx \frac{L}{200}$ to $\frac{L}{400}$, as shown below in figure 1.1. Resonance between seismic activity, or even vertical motion induced by the horizontal isolation system, and the floor beams of a building can amplify the deflections and cause serious and costly damage.



Figure 1.1: Example of simply supported beam deflection under distributed load



Figure 6-3 Influence of Relative Height on Normalized Vertical Acceleration Spectra [B3]

Figure 1.2: Median spectra of the ratio of vertical floor acceleration to horizontal peak ground acceleration for a three story building at column and open bay

 $\frac{1}{2}$



Figure 6-4 Influence of Relative Height on Normalized Vertical Acceleration Spectra [H3]

Figure 1.3: Median spectra of the ratio of vertical floor acceleration to horizontal peak ground acceleration for a three story hospital building at column and open bay locations 2

²Wieser, Joseph D., Pekhan, Gokhan, Zaghi, Arash E., Itani, Ahmad E., Maragakis, Emmanuel "Manos," Assessment of Floor Accelerations in Yielding Buildings, Technical Report MCEER-12-0008, October 5, 2012.



Figure 6-5 Influence of Relative Height on Normalized Vertical Acceleration Spectra [B9]

Figure 1.4: Median spectra of the ratio of vertical floor acceleration to horizontal peak ground acceleration for a nine story hospital building at column and open bay locations

3

4.0 4.0Column Open-Bay 3.5 3.5 3.0 3.0 FSA_v/PGA_h 2.5 2.5 2.0 2.0 1.5 1.5 1.01.0 0.5 0.5 0.0 0.0 0.6 0.0 0.3 0.9 1.2 1.5 0.00.3 0.6 0.9 1.2 1.5 Component Period [T_p] (sec) Component Period [T_p] (sec) •••••• z/H = 0.31 - - z/H = 0.66- z/H = 1.00

Figure 6-6 Influence of Relative Height on Normalized Vertical Acceleration Spectra [B20]

Figure 1.5: Median spectra of the ratio of vertical floor acceleration to horizontal peak ground acceleration for a twenty story hospital building at column and open bay locations

³Wieser, Joseph D., Pekhan, Gokhan, Zaghi, Arash E., Itani, Ahmad E., Maragakis, Emmanuel "Manos," Assessment of Floor Accelerations in Yielding Buildings, Technical Report MCEER-12-0008, October 5, 2012.

The solution to mitigating these damaging floor oscillations lies in gaining a better understanding of why they occur and what differing vertical isolation systems have to offer in terms of reducing transmissibility. Here, we consider the application of buckled beams instead of common coil springs for vertical isolation. The squashed beams offer larger contact surfaces and greater lateral stiffness than that of coil springs. This attributes to a more horizontally rigid mechanism with less of a tendency to tip.

1.2 Objectives

The objectives of this study are to:

- 1. Determine the deformed shape of a buckled beam with a constraint on the maximum transverse displacement and with potentially large curvatures.
- 2. Determine the deformed shape of a buckled beam with a constraint on both the maximum transverse displacement and a mid-span longitudinal displacement as a function of the longitudinal displacement.
- 3. Determine the nonlinear transverse force-displacement relationship (the stiffness) and use that relationship to assess the natural frequency of oscillation and relative displacement of the physical model's platform given a specified load.
- 4. Experimentally validate these results by comparing observations from a physical model to those predicted by the theoretical model and simulation.

1.3 Problem Description

In order to provide a thorough analysis of the squashed beam approach, we must first model the deformed configuration of the beams. One side of an eventual square platform using squashed beams for motion isolation will be analyzed computationally to find the optimum constraint positions. The ends of the squashed beam will be secured at each end to the lower platform, with the ability of adjustment, in predetermined increments, on one edge. In order to analyze the longitudinal and transverse stiffness of a buckled beam, the mid-point is constrained to a prescribed horizontal displacement and the transverse displacements are constrained to not exceed a prescribed limit. The equilibrium configuration of a transversely and longitudinally buckled beam corresponds to the minimum of the strain energy in the squashed beam. The result of the optimization problem provides values for the forces required at each of the constraint locations in the form of Lagrange multipliers. Following appropriate scaling of these Lagrange multipliers and experimentation with the physical model, a comparison will be made between these theoretical force values and experimental results.

Chapter 2 Method

2.1 Problem Formulation

The forces imposed on the squashed beam at the two ends and at the mid-point cause internal stresses and strains throughout the elastic solid. These strains are consistent with the displacements of the beam.⁴ These displacements can be predicted as the configuration that minimizes the total potential energy, $\bar{\Pi}$, given by the equation

$$\bar{\Pi} = \bar{\mathcal{U}} + \bar{\mathcal{V}} \tag{2.1}$$

where $\bar{\mathcal{U}}$ is the internal strain energy and $\bar{\mathcal{V}}$ is the potential energy function of external loads. For a beam with a cross section that is much broader than deep, the beam behavior resembles that of a plate in single curvature bending. The internal strain energy, $\bar{\mathcal{U}}$, is given by the equation

$$\bar{\mathcal{U}} = \frac{1}{2} \left(\frac{EI}{1 - \nu^2} \right) \int_{s=0}^{L} (\bar{\phi}(\bar{s}))^2 d\bar{s}, \qquad (2.2)$$

where E is the tensile modulus of elasticity, $I = bh^3/12$ is the section second moment of area, ν is Poisson's ratio, and the cross section dimension is $b \times h$ with $h \ll b$.

Here $\bar{\phi}(\bar{s})$ is the curvature in units of [1/L], E is the modulus of elasticity in units of [F/L²], I is the moment of inertia in units of [L⁴], and the domain \bar{s} represents the

⁴Gavin, H. P., *Minimum Total Potential Energy, Quadratic Programming and Lagrange Multipliers*, Department of Civil and Environmental Engineering, Duke University, 2020.

arc-length of the buckled beam in units of [L]. The potential energy of external forces $\bar{\mathcal{V}}$ is given by the equation

$$\bar{\mathcal{V}} = \int_{\bar{s}=0}^{L} \bar{f}(\bar{s})\bar{v}(\bar{s}) \ d\bar{s}$$
(2.3)

where $\bar{f}(\bar{s})$ represents a distributed external load in units of [F/L] and $\bar{v}(\bar{s})$ represents the set of displacements in units of [L].⁵

2.2 Constrained Minimization of Total Potential Energy

We seek to determine the displaced configuration of the beam by minimizing its total potential energy subject to constraints on the lateral position and the longitudinal position. To accomplish this, we must first define the positions on the beam using a convenient coordinate system. For a simple two-dimensional setting, \bar{x} and \bar{y} Cartesian coordinates are sufficient. For a beam of length L, we use the arc-length of the buckled beam, \bar{s} , as the independent variable, as shown in figures 2.1 and 2.2. Thus, we can represent the curvature of the buckled beam, $\bar{\phi}(\bar{s})$, using the first and second derivatives of $\bar{x}(\bar{s})$ and $\bar{y}(\bar{s})$. In equation form, the most general expression for curvature is

$$\bar{\phi}(s) = \frac{(\bar{x}'(\bar{s})\bar{y}''(\bar{s}) - \bar{y}'(\bar{s})\bar{x}''(\bar{s}))}{(\bar{x}'(\bar{s})^2 + \bar{y}'(\bar{s})^2)^{3/2}}$$
(2.4)

where $\bar{x}(\bar{s})$ and $\bar{y}(\bar{s})$ are the location of the beam in parametric form.

Non-dimensionalizing all length variables by L and all force variables by $(EI)/((1-\nu^2)L^2)$, we obtain variables in dimensionless form, $\bar{x} = Lx$, $\bar{y} = Ly$, $\bar{s} = Ls$, $d\bar{x} = L dx$, $d\bar{y} = L dy$, $d\bar{s} = L ds$, $\bar{x}' = x'$, $\bar{y}' = y'$, $\bar{s}' = s'$, $\bar{x}'' = x''/L$, $\bar{y}'' = y''/L$, $\bar{s}'' = s''/L$, $\bar{\phi}(\bar{s}) = \phi(s)/L$, and $\bar{P} = (EI)/(((1-\nu^2)L^2)P)$, and thus

$$\mathcal{U} = \frac{1}{2} \int_{s=0}^{1} (\phi(s))^2 \, ds. \tag{2.5}$$

⁵Gavin, H. P., *Minimum Total Potential Energy, Quadratic Programming and Lagrange Multipliers*, Department of Civil and Environmental Engineering, Duke University, 2020.

Approximating x(s) and y(s) as Fourier series

$$x(s; \boldsymbol{a}) = s(1 - (\frac{D}{L})) + \sum_{n} a_n \sin(n\pi s)$$
(2.6)

and

$$y(s; \boldsymbol{b}) = \sum_{m} b_m \sin(m\pi s), \qquad (2.7)$$

where a_n and b_m are the coefficients used to parameterize the the configuration of the buckled beam and its strain energy, \mathcal{U} . Given sets of coefficients \boldsymbol{a} and \boldsymbol{b} , x'(s), x''(s), y'(s) and y''(s) can be found analytically. Combining equation 2.2 and equation 2.4, we have an expression for the strain energy in terms of the dimensionless arc-length, s.

$$\mathcal{U}(\boldsymbol{a}, \boldsymbol{b}) = \frac{1}{2} \int_{s=0}^{1} \left(\frac{(x'(s; \boldsymbol{a})y''(s; \boldsymbol{b}) - y'(s; \boldsymbol{b})x''(s; \boldsymbol{a}))}{(x'(s; \boldsymbol{a})^2 + y'(s; \boldsymbol{b})^2)^{1.5}} \right)^2 ds$$
(2.8)

The non-dimensional potential energy is minimized with respect to the coefficients a and b such that the following conditions are satisfied:

$$\max_{s}(y(s; \boldsymbol{b})) \leq \frac{Y}{L} \leq \frac{2}{\pi} \sqrt{\frac{D}{L}}$$
(2.9)

$$\min_{s}(y(s; \boldsymbol{a})) \geq 0 \tag{2.10}$$

$$\max_{s}(x(s; \boldsymbol{b})) \leq 1 - \frac{D}{L}$$
(2.11)

$$x(1/2; \boldsymbol{b}) \leq \frac{1}{2} \left(1 - \frac{D}{L}\right) + \frac{X}{L}$$
 (2.12)

$$\int_0^1 \left((x'(s; \boldsymbol{a}))^2 + (y'(s; \boldsymbol{b}))^2 \right) ds \ge 1$$
(2.13)

This constrained minimization is solved numerically using Sequential Quadratic Programming (SQP). In the SQP method constraints are enforced using Lagrange multipliers λ by solving

$$\max_{\boldsymbol{\lambda}} \min_{\boldsymbol{a}, \boldsymbol{b}} \left[U(\boldsymbol{a}; \boldsymbol{b}) + \sum_{i=1}^{4} \lambda_i g_i(\boldsymbol{a}, \boldsymbol{b}) \right].$$
(2.14)

The solution to this constrained minimization provides values for the coefficients a and b, along with the addition of Lagrange multipliers, λ , representing the forces

experienced at the constraint locations.⁶ The Lagrange multiplier associated with $g_3 = \max(x(s; \mathbf{b})) - 1 + D/L \leq 0$ corresponds to the force in the *x* direction required to maintain a position at $x(1; \mathbf{b}) = 1 - D/L$. The Lagrange multiplier associated with $g_1 = \max(y(s; b)) - Y/L$ corresponds to the force in the *y* direction required to prevent y(s) from exceeding Y/L. In dimensional form the Lagrange multipliers, in units of [F], are

$$\bar{\boldsymbol{\lambda}} = \frac{EI}{(1-\nu^2)L^2} \,\boldsymbol{\lambda} \,. \tag{2.15}$$

MATLAB[®] programming is used to carry out the optimization calculation, producing values for the minimum strain energy and constraint forces. These results are covered in detail in the following chapter.



Figure 2.1: Example of buckled arch showing use of arc-length



Figure 2.2: Example of compressed buckled arch

⁶Gavin, H. P., Scruggs, J. T., *Constrained Optimization using Lagrange Multipliers*, Department of Civil and Environmental Engineering, Duke University, 2020.

Chapter 3 Simulation Results

Simulations were carried out to determine the relationship between the transverse "squashing" displacement $(2/\pi)\sqrt{D/L} - Y/L$ and its associated constraint force. These relationships are presented as the constraint force, $\bar{\lambda}$, normalized by the elastic buckling force

$$P_{\rm cr} = \frac{\pi^2 EI}{(1-\nu^2)L^2} \tag{3.1}$$

3.1 Lateral Displacement Constraint

The computational process began by varying the lateral displacement constraint, Y/L from $(2/\pi)\sqrt{D/L}$ to $0.5(2/\pi)\sqrt{D/L}$ in thirty steps, while keeping the end constraint D/L fixed at a prescribed value and the mid-point constraint X/L fixed at zero. The BuckSquashPP_opt.m MATLAB[®] program was used to analyze the configuration of the squashed beam and to minimize the strain energy with respect to the Fourier coefficients, \boldsymbol{a} and \boldsymbol{b} . An example of the equilibrium configuration of the constrained buckled configuration for D/L = 0.1 including the configuration coordinates, x(s) and y(s), and their derivatives x'(s), y'(s), x''(s), and y''(s), are shown in Figures 3.1 to 3.6. These figures show that the configuration smoothly conforms to the constraints, and when Y/L is less than about $(2/3)(2/\pi)\sqrt{D/L}$ a snap-though buckling effect is observed. Note also that for the longitudinal midpoint displacement constraint X/L fixed at zero, x(s) increases linearly from x(0) = 0 to x(1) = 1 - D/L in all cases. Cases of $X/L \neq 0$ are considered in the next section.



Figure 3.1: Equilibrium configuration of the laterally constrained buckled beam, and the associated longitudinal x(s) and lateral y(s) displacements, and their derivatives. D/L = 0.10 and $(2/\pi)\sqrt{D/L} - Y/L = 0.016$.



Figure 3.2: Equilibrium configuration of the laterally constrained buckled beam, and the associated longitudinal x(s) and lateral y(s) displacements, and their derivatives. D/L = 0.10 and $(2/\pi)\sqrt{D/L} - Y/L = 0.033$.



Figure 3.3: Equilibrium configuration of the laterally constrained buckled beam, and the associated longitudinal x(s) and lateral y(s) displacements, and their derivatives. D/L = 0.10 and $(2/\pi)\sqrt{D/L} - Y/L = 0.050$.



Figure 3.4: Equilibrium configuration of the laterally constrained buckled beam, and the associated longitudinal x(s) and lateral y(s) displacements, and their derivatives. D/L = 0.10 and $(2/\pi)\sqrt{D/L} - Y/L = 0.067$.



Figure 3.5: Equilibrium configuration of the laterally constrained buckled beam, and the associated longitudinal x(s) and lateral y(s) displacements, and their derivatives. D/L = 0.10 and $(2/\pi)\sqrt{D/L} - Y/L = 0.084$.



Figure 3.6: Equilibrium configuration of the laterally constrained buckled beam, and the associated longitudinal x(s) and lateral y(s) displacements, and their derivatives. D/L = 0.10 and $(2/\pi)\sqrt{D/L} - Y/L = 0.101$.

Repeating this process for D/L = 0.02, 0.05, 0.1, 0.15, 0.20, and 0.30. The scaled Lagrange multipliers for the lateral and longitudinal constraints, normalized by $P_{\rm cr}$ are plotted with respect to the lateral displacement constraint, $(2/\pi)\sqrt{D/L}-Y/L$. These results are shown in figures 3.7 through 3.12. These figures show that constraint forces increase monotonically and non-linearly up to the onset of snap-through buckling, and that beyond the snap-through buckling, the constraint forces drop sharply. This indicates an instability in the force-displacement behavior.

For values of D/L less than about 0.2, the relationship between the transverse constraint displacement and the transverse force has an inflection point at relatively low values of force. This inflection point is most pronounced for $D/L \approx 0.15$. Beyond this inflection point, the transverse force increases roughly quadratically with transverse displacement. Both of these behaviors are desirable for vertical vibration isolation systems; The inflection in behavior reduces the required static displacement by up to fifty percent for the D/L = 0.15 case. The quadratic increase in force with larger values of Y/L makes the natural frequency of the vertical isolation system roughly insensitive to the mass of the isolated object over a certain range of masses.

As the longitudinal displacement constraint D/L increases, both longitudinal constraint forces and lateral constraint forces increase. The lateral constraint forces (and lateral stiffness) are more sensitive to D/L than are the longitudinal constraint forces.

At the largest values of D/L considered, the sequential quadratic programming method as implemented in this study did not converge to a globally optimum solution for every value of Y/L. In these cases there are jagged irregularities or missing data in the force-displacement relationships.



Figure 3.7: Relationship between the squashing constraint position Y/L to the horizontal end constraint force and the vertical squashing constraint force for D/L = 0.02.



Figure 3.8: Relationship between the squashing constraint position Y/L to the horizontal end constraint force and the vertical squashing constraint force for D/L = 0.05.



Figure 3.9: Relationship between the squashing constraint position Y/L to the horizontal end constraint force and the vertical squashing constraint force for D/L = 0.10



Figure 3.10: Relationship between the squashing constraint position Y/L to the horizontal end constraint force and the vertical squashing constraint force for D/L = 0.15.



Figure 3.11: Relationship between the squashing constraint position Y/L to the horizontal end constraint force and the vertical squashing constraint force for D/L = 0.20.



Figure 3.12: Relationship between the squashing constraint position Y/L to the horizontal end constraint force and the vertical squashing constraint force for D/L = 0.30.

3.2 Horizontal Mid-span Displacement Constraint

We now vary the mid-point constraint, while keeping both the transverse constraint and the longitudinal end constraint fixed. The longitudinal end constraint of $\frac{D}{L} = 0.07$ was chosen due to the quality of simulation results provided at that location. Figures 3.13 through 3.15 show the results graphically. Lagrange multiplier values are recorded in table 3.1. Note here that for $X/L \neq 0$, x(s) is not linear in s (Figure 3.14) thus giving rise to an asymmetric configuration of (x(s), y(s)) (Figure 3.13). For -0.05 < X/L < +0.05 the vertical and lateral constraint forces are not significantly sensitive to X/L. This interesting preliminary result merits further investigation.



Figure 3.13: Horizontal mid-point constraint variation with edge constraint and lateral constraint fixed at $\frac{D}{L} = 0.07$.



Figure 3.14: x and y coordinates along the beam as a function of the arc-length, x(s) and y(s), and their derivatives $\frac{dx(s)}{ds}$, $\frac{dy(s)}{ds}$, $\frac{d^2x(s)}{ds^2}$, and $\frac{d^2y(s)}{ds^2}$.

Table 3.1: Table of Lagrange multiplier values or constraint forces (vertical constraint $\frac{Y}{L} = 0.12$, edge constraint $\frac{D}{L} = 0.07$).

Lagrange multiplier values, $\boldsymbol{\lambda}$				
D/L λ_1 λ_3 λ_4				
D/L = 0.07	7.4571	1.2831	9.2211	



Figure 3.15: Constraint position vs constraint forces, $\frac{D}{L} = 0.07$.

Chapter 4

Physical Model and Experimental Validation

A fully three-dimensional vertical isolation system was designed, built, and experimentally assessed in order to validate the model for the nonlinear elastic vertical isolation system analyzed in the previous sections.

4.1 Design and Construction of the three-dimensional Vertical Isolation System

Transitioning from the theoretical model to the physical model presented a number of challenges to be overcome. The two most obvious of these were the difference in size of the physical model and the addition of one more spatial dimension. The model needed to be easily movable and manageable by a single person. To make the isolation system more compact, the buckled arches were arranged as half-arches in such a manner that would facilitate small adjustments at the constraint locations. This allowed for each half-arch isolation spring to be two feet long, taking up far less space than a single four-foot long full arch. The half-arches were fixed against displacement and rotation at the top and pined against displacement at the bottom, thereby making the system invulnerable to the snap-through instability analyzed in the full-arch system.

The construction of the physical model required carpentry skills and common ma-

terials. A support table was fabricated using wooden 2x4s and $\frac{3}{4}$ inch thick plywood cut to 36 inches by 36 inches. The top platform, to be supported by the half-arch springs, was cut to the same 36 inch by 36 inch dimensions. The constraint positions were fabricated from aluminum angle sections that are $1x1x\frac{1}{8}$ inch for the bottom constraints and $\frac{3}{4}x\frac{3}{4}x\frac{1}{16}$ inch for the top constraints. The half-arches were purchased pre-cut to two inches wide by two feet long by 0.048 inch thick galvanized steel strips. These were initially used as the spring material, as shown in figure 4.1. Upon initial observations of the spring behavior, plastic deformation was evident with relatively small displacements of the platform. This prompted a change in material to spring steel strips with nearly the same dimensions, increasing only the thickness slightly to 0.05 inch. Eight strips of two foot length were cut from a 25 foot long roll of spring steel using an angle grinder. After ensuring that the strips measured the same length and that edges were squared, the new springs were installed on the table as shown in figure 4.2.

During this process, guide rods were also added to help prevent rocking, or outof-plane rotation, of the top platform due to small inconsistencies in the applied load, accidental eccentricities in mass and stiffness, and other incidental imperfections in isolation system. Guide rods were fashioned from $\frac{3}{4}$ inch partially threaded steel bolts and used linear sleeve bearings at the pass-through of the top platform for smooth movement, as shown in figures 4.2 and 4.3. The guide rods improved the overall stability of the platform, but presented new challenges with the potential for binding. This binding occurred if the platform was forced to rotate from a level position, causing one or more of the guide rods to bind. To mitigate this issue, graphite lubricant was added to the rods where the sleeve bearings made contact. Additionally, the bolts were loosened one quarter turn from their fixed position on the bottom platform to allow for slight flexibility and account for small errors in the construction process. Future modifications to the table may include the replacement of the linear sleeve bearings with linear ball bearings, capable of slight misalignment correction and smoother operation.



Figure 4.1: Vertical Isolation Table - galvanized springs



Figure 4.2: Vertical Isolation Table - spring steel springs with guide rods



Figure 4.3: Vertical Isolation Table - spring arrangement

4.2 Experimentation

Following the results from MATLAB[®] simulation, we now take the idea of the mid-point constraint and apply this to a physical model using a half-arch configuration. What previously served as a midpoint constraint now serves as an edge constraint and allows for adjustment, while preventing snap through behavior, seen in the plots depicting the buckled beams from figures 3.1 through 3.12. By eliminating the possibility for this behavior, we can assume that the force response of the system will be more predictable as the load and platform displacement increase.

4.2.1 Setup

To measure the platform's displacement under load, fixed loads were added to the top and center of the platform in five pound increments from zero to 100 pounds. Measurements were made at the corners of the platform on each side and values were averaged together if differences existed. The numbers were subtracted from the zero position to obtain the displacement for each load, as seen in table 4.1. The force values were also adjusted to account for the platform weight of approximately 27 pounds. Images of the platform loaded at 50 pounds and 100 pounds are shown in figures 4.8 through 4.11.

For the frequency of oscillation of the platform, results were gathered under loadings of 50 pounds and 100 pounds. The platform was depressed $\frac{1}{2}$ inch and released as a timer was started. Cycles were counted as the platform returned to the bottom position each time for approximately five seconds. This process was repeated three times for each loading configuration and results were averaged.

4.2.2 Results

The force and displacement values from the model were non-dimensionalized, similar to the simulation results, for easier comparison. The force values were nondimensionalized by dividing the value by the critical force required to buckle the beams. This P_{cr} value was calculated as

$$P_{\rm cr} = \frac{\pi^2 E I}{(1 - \nu^2) L^2}, \text{ with } I = \frac{bh^3}{12},$$
 (4.1)

where E = 29,000,000psi is the modulus of elasticity for steel, $\nu = 0.3$ is Poisson's ratio for steel, $I = (8)(0.05)^3/12 = 8.33 \times 10^{-5}$ in⁴ is the moment of inertia, and L = 48 in is the length of the full beam. Here we use b = 8 in to represent the combined width of four full arches to be consistent with the calculations completed for the simulation. This gives us a $P_{\rm cr} = 11.4$ lbs. Initial model configuration results for the force values and displacements, with D/L = 0.0833, can be found in table 4.1 with the values for non-dimensionalized force versus displacement in table 4.2 and corresponding plot in figure 4.4. The plot shows how the spring response begins to demonstrate slightly non-linear behavior as the load increases and smaller displacements are observed. Now, we can compare the experimental data to the simulation data from previous sections. First, we must ensure that the beam edge constraint displacement for the simulation D/L matches the initial configuration of the physical model, D/L = 0.0833. This is because the behavior of the springs depends on the constraint displacement D/L as seen in the analyses of the previous section. Plotting the new data against the adjusted simulation data we can compare the results. This is shown in figure 4.5. Multiple iterations can be seen in this plot denoted by the different colored markers.

As the platform was loaded and after weight was removed, careful observations were made to note changes in the spring behavior and condition. Though the half-arch configuration does show an ability to support higher loads, some buckling away from the underside of the top platform was observed as the superimposed load increased above 205 lbs in the initial configuration of D/L = 0.0833, seen in figure 4.13. This buckling was expected due to the understanding of the full arch behavior and snapthrough observed in previous simulations. Regardless of the stresses placed on the springs during these experiments, no yielding of the spring steel was noticed after careful inspection following each experiment.

Taking the experiment a little further, we adjust the edge constraints by 1/2 inch, for D/L = 0.1042. A similar adjustment is made in the MATLAB simulation to match. The plot of this data against the adjusted simulation data is shown in figure 4.7. Comparing the results from the two constraint locations, we see that the experimental results and simulation results correlate very closely in both cases. The accurate prediction of the nonlinear elastic behavior of this system validates the nonlinear elastic model derived and analyzed in this thesis. This validation allows us to proceed with further studies using the simulation methods, having added confidence in our results.

Results for the frequency of oscillation can be found in table 4.3. The averaged values for the two loading configurations show virtually no difference in the frequency

of oscillation given the difference in load and therefore, the same period $T \approx 0.38$ sec. This is good news, as the oscillation behavior of such a vertical isolation system will remain constant if the mass and loads remain within a specified design range.

Displacement Values				
Force (lbs)	Measurement (in)	Displacement (in)		
27	8	0		
32	7.6875	0.3125		
37	7.40625	0.59375		
42	7.25	0.75		
47	7.0625	0.9375		
52	6.875	1.125		
57	6.75	1.25		
62	6.625	1.375		
67	6.53125	1.46875		
72	6.4375	1.5625		
77	6.375	1.625		
82	6.28125	1.71875		
87	6.21875	1.78125		
92	6.125	1.875		
97	6.0625	1.9375		
102	6	2		
107	5.96875	2.03125		
112	5.9375	2.0625		
117	5.875	2.125		
122	5.8125	2.1875		

Table 4.1: Table of displacement under loading - 5 lb increments, including an
adjustment of 27 lbs for the top platform

Continuation of Table 4.1				
127	5.78125	2.21875		
132	5.75	2.25		
137	5.6875	2.3125		
142	5.625	2.375		
147	5.59375	2.40625		
152	5.53125	2.46875		
157	5.5	2.5		
162	5.4375	2.5625		
167	5.40625	2.59375		
172	5.375	2.625		
177	5.34375	2.65625		
182	5.3125	2.6875		
187	5.28125	2.71875		
192	5.25	2.75		
197	5.234375	2.765625		
202	5.21875	2.78125		
207	5.203125	2.796875		
212	5.1875	2.8125		
217	5.15625	2.84375		
222	5.125	2.875		
227	5.109375	2.890625		
232	5.07812	2.921875		
237	5.0625	2.9375		
242	5.03125	2.96875		
247	5.015625	2.984375		
252	4.984375	3.015625		

Continuation of Table 4.1			
257	4.96875	3.03125	
262	4.9375	3.0625	

Non-dimensionalized Force and Disp. Force $(\mathbf{P}/\mathbf{P_{cr}})$ Disp. (D/L + 0.015)2.370.015002.810.021513.250.027373.690.030634.130.034530.038444.575.010.041040.043655.455.890.045606.330.047556.770.048857.21 0.050817.650.052118.090.054068.530.055360.056678.979.410.057320.057979.85

Table 4.2: Table of non-dimensionalized values for force and displacement

Continuation of Table 4.2			
10.28	0.05927		
10.72	0.06057		
11.16	0.06122		
11.60	0.06188		
12.04	0.06318		
12.48	0.06448		
12.92	0.06513		
13.36	0.06643		
13.80	0.06708		
14.24	0.06839		
14.68	0.06904		
15.12	0.06969		
15.56	0.07034		
16.00	0.07099		
16.44	0.07164		
16.88	0.07229		
17.32	0.07262		
17.76	0.07294		
18.20	0.07327		
18.64	0.07359		
19.08	0.07424		
19.51	0.07490		
19.95	0.07522		
20.39	0.07587		
20.83	0.07620		
21.27	0.07685		

Continuation of Table 4.2		
21.71	0.07717	
22.15	0.07783	
22.59	0.07815	
23.03	0.07880	



Figure 4.4: Plot of D/L vs P/P_{cr} for 0-235 lbs of superimposed load - D/L=0.0833



Figure 4.5: Plot comparing results of experimental and simulation D/L vs P/P_{cr}



Figure 4.6: Plot of D/L vs P/P_{cr} for 0-235 lbs of superimposed load - D/L = 0.1042



Figure 4.7: Plot comparing results of experimental and simulation D/L vs P/P_{cr}

Averaged Oscillations in cycles per second (cps)				
Force(lbs)	Cycles	Time(sec)	Freq.(cps)	
75	12	4.50	2.67	
75	13	5.12	2.54	
75	14	5.17	2.71	
75AVG			2.64	
125	14	5.26	2.66	
125	15	5.65	2.65	
125	13	5.17	2.51	
125AVG		——	2.61	

Table 4.3: Table of oscillation frequencies



Figure 4.8: Vertical Isolation Table - 50lb load



Figure 4.9: Vertical Isolation Table - 50lb load spring view



Figure 4.10: Vertical Isolation Table - 100lb load



Figure 4.11: Vertical Isolation Table - 100lb load spring view



Figure 4.12: Vertical Isolation Table - 235lb load



Figure 4.13: Vertical Isolation Table - 235lb load spring view

Chapter 5

Coupled Vertical and Horizontal Isolation

5.1 Description using Lagrange's Equations

In order to better understand and describe the issues inherent in the vertical isolation system, a closer look at the dynamics is needed using mathematical theory. To do this, we assume that the vertical isolation system is fixed to a common horizontal isolation system and use Lagrange's equations to form the equations of motion for the combined system.

Using three degrees of freedom for a two-dimensional setting, we define the coordinates in terms of x(t) as the horizontal component, z(t) as the vertical component, and $\theta(t)$ as the rotational component for the top platform, as seen in figure 5.1. Input accelerations, possibly from seismic activity, are denoted in figure 5.1 as $a_x(t)$ and $a_z(t)$. Coil springs are depicted in the illustration to reaffirm the linearity of the vertical isolation system in question and to simplify the calculations. Assembling the energy equations, we determine the potential energy, V, to be

$$V = M_h g(d_z + 2h(x(t))) + M_v g(d_z + 2h(x(t)) + z(t)) + M_p g(d_z + 2h(x(t)) + z(t) + H\cos(\theta(t)))$$
(5.1)
$$+ \frac{1}{2} k(z(t) + bsin(\theta(t))^2 + \frac{1}{2} k(z(t) - bsin(\theta(t))^2$$

where M_h, M_v , and M_p represent the masses of the horizontal platform, the mass of the vertical platform, and the superimposed mass respectively. Additionally, krepresents the linear spring constant associated with the vertical isolation springs, hrepresents the vertical displacement of the horizontal isolation platform as a result of the opposing concave bearings, and H refers to the height of the superimposed mass above the vertical isolation platform of the system.

Similarly, the equation for the kinetic energy, T, is

$$T = \frac{1}{2} M_h ((V_X(t) + v_x(t))^2 + (V_Z(t) + 2\frac{d}{dt}h(x(t)))^2) + \frac{1}{2} M_v ((V_X(t) + v_x(t))^2 + (V_Z(t) + 2\frac{d}{dt}h(x(t)) + v_z(t))^2) + \frac{1}{2} J_v (v_\theta(t))^2$$
(5.2)
$$+ \frac{1}{2} M_p ((V_X(t) + v_x(t) - \frac{d}{dt}(Hsin(\theta(t))))^2 + (V_Z(t) + 2\frac{d}{dt}h(x(t)) + v_z(t) + \frac{d}{dt}(Hcos(\theta(t))))^2,$$

where $V_X(t)$ and $V_Z(t)$ represent the input velocities experienced by the system and J_v is the rotational inertia of the top platform. With these equations for the potential

and kinetic energy, we can then write the Lagrangian, $\mathcal{L} = T - V$, and use the Euler-Lagrange (EL) equation to derive the equations of motion. The EL equations for each degree of freedom are

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial z} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{z}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 0.$$
(5.3)

To assist with the algebra and limit possible errors, we turn to Maple[®] software to perform the derivations.



Figure 5.1: Example coupled horizontal and vertical isolation system

5.2 Equations of Motion

Following the assembly of the Lagrangian and subsequent use of the Euler Lagrange equations, we find the results for the equations of motion in for x(t), z(t), and $\theta(t)$, as seen in equations 5.4, 5.5, and 5.6. For motion in the horizontal direction,

$$\begin{aligned} x(t) &= 4\ddot{x}(M_v + M_h + M_p)(\frac{dh}{dx(t)})^2 \\ &+ (4\dot{x}^2(M_v + M_h + M_p)\frac{d^2h}{dx(t)^2} \\ &- 2H\dot{\theta}^2 cos(\theta(t))M_p - 2Hsin(\theta(t))\ddot{\theta}M_p \\ &+ (2M_v + 2M_h + 2M_p)\ddot{Z} + 2M_v\ddot{z} + 2g(M_h + M_p + M_v)) \\ &\frac{dh}{dx(t)} - HM_p\ddot{\theta}cos(\theta(t)) + HM_p\dot{\theta}^2sin(\theta(t)) \\ &+ (M_v + M_h + M_p)(\ddot{X} + \ddot{x}). \end{aligned}$$
(5.4)

Similarly, for motion in the vertical direction

$$z(t) = 2M_v \frac{d^2h}{dx(t)^2} \dot{x}^2 + 2M_v \ddot{x} \frac{dh}{dx(t)} + M_v \ddot{Z} + M_v \ddot{z} + 2z(t)k + g(M_p + M_v).$$
(5.5)

Finally, for the rotational or tipping motion of the vertical isolation platform

$$\theta(t) = -2M_p \frac{d^2h}{dx(t)^2} \dot{x}^2 sin(\theta)$$

$$- 2HM_p \ddot{x} \frac{dh}{dx(t)} sin(\theta) + 2cos(\theta) sin(\theta) b^2 k$$

$$+ H^2 M_p \ddot{\theta} - HM_p \ddot{X} cos(\theta) - HM_p \ddot{x} cos(\theta)$$

$$- HM_p \ddot{Z} sin(\theta) - M_p g H sin(\theta) + J_v \ddot{\theta}.$$
(5.6)

These results demonstrate a positive outcome for the tipping behavior of the platform. Using MATLAB[®] and the previously determined natural period of $T \approx$

0.38*sec*, we can simulate the motion of the system. Figures 5.2 and 5.3 from this simulation show only small responses in both displacement and rocking motion of the vertical isolation platform. This means that the isolation mechanism is effective at decoupling the input motion from the load supported by the system. Based on this preliminary analysis, the complexity of vertical guiding bearings may not be warranted.



Figure 5.2: Plot of Time vs Displacement using equations of motion



Figure 5.3: Plot of Time vs Rocking Response using equations of motion

Chapter 6

Conclusion

6.1 Current

In conclusion, the need for a feasible solution for vertical isolation in structural design is apparent, given the possibility of significant vertical motion caused by moderate to strong seismic activity. Though a number of techniques exist for this purpose, few can provide the lateral stability and dynamic response present in the buckled beam approach. Although further exploration and experimentation are needed, the results thus far provide a clear understanding of the benefits that such a system provides. With increased confidence from the closely correlated results in the experimental section, further investigation into the system's behavior will inform us on the feasibility of its use under a wider range of loading configurations. As we also saw in the final sections, decoupling the input motion from the supported load is dependent on careful analysis of the system's natural frequencies relative to the input frequencies. We must keep this in mind during further investigation to ensure that the system is properly tailored for the intended application. Furthermore, learning how the isolation system's response changes following large displacements, where yielding of the springs may occur, will be essential to understanding the performance limits for a given application.

6.2 Way Ahead

Future tasks may involve the items listed below, in an effort to better understand the strengths and weaknesses of the buckled beam isolation system.

- Conduct further experimentation and modification with the physical model to confirm or disprove assumptions made in this report, specifically focusing on the following:
 - (a) The frequency of oscillation for the physical model given a range of loading values and configurations using computer controlled sensors.
 - (b) The displacements tolerated by the physical model's springs in the linear range and how the isolation behavior changes as the springs enter the non-linear range.
 - (c) The advantages and disadvantages regarding the non-linear force response of the buckled beams when displaced beyond the linear range.
 - (d) A more complete spectrum of the isolation system's force responses given the displacement of the platform under a wider range of loads.
- 2. Determine the most effective method for maintaining alignment of the platforms under common loading and subjected to external input forces. This is necessary to reduce or eliminate unwanted damping and binding of the platform, which may affect performance.

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