Contact Process on Random Graphs and Trees

by

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Dissertation submitted in partial fulfillment of the
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ABSTRACT

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Abstract

We study the contact process on random graphs and trees.

In Chapter 2 we study the asymptotics for the critical values λ_1 and λ_2 on a general class of periodic trees. A little over 25 years ago Pemantle [32] pioneered the study of the contact process on trees, and showed that on homogeneous trees the critical values λ_1 and λ_2 for global and local survival were different. He also considered trees with periodic degree sequences, and Galton-Watson trees. Here, we will consider periodic trees in which the number of children in successive generations is $(n, a_1, \ldots, a_k)$ with $\max_i a_i \le Cn^{1-\delta}$ and $\log(a_1 \cdots a_k)/\log n \to b$ as $n \to \infty$. We show that the critical value for local survival is asymptotically $\sqrt{c(\log n)/n}$ where $c = (k-b)/2$. This supports Pemantle's claim that the critical value is largely determined by the maximum degree, but it also shows that the smaller degrees can make a significant contribution to the answer.

In Chapter 3 we study the contact process on Galton-Watson trees and configuration models. The key to our investigation is an improved (and in a sense sharp) understanding of the survival time of the contact process on star graphs. Using these results, we show that for the contact process on Galton-Watson trees, when the offspring distribution (i) is subexponential the critical value for local survival $\lambda_2 = 0$ and (ii) when it is Geometric(p) we have $\lambda_2 \le C_p$, where the C_p are much smaller than previous estimates. We also study the critical value $\lambda_c(n)$ for "prolonged persistence" on graphs with n vertices generated by the configuration model. In the case of power law and stretched exponential distributions where it is known $\lambda_c(n) \to 0$ we give estimates on the rate of convergence. It was predicted in physics papers that $\lambda_c(n) \sim 1/\Lambda(n)$ where $\Lambda(n)$ is the maximum eigenvalue of the adjacency matrix. Our results show that this is accurate for graphs with power-law degree distributions, but

not for stretched exponentials.

In Chapter 4 we study the supercritical contact process on Galton-Watson trees and periodic trees. We prove that if the contact process survives weakly then it dominates a supercritical Crump-Mode-Jagers branching process. Hence the number of infected sites grows exponentially fast. As a consequence we conclude that the contact process dies out at the critical value λ_1 for weak survival, and the survival probability $p(\lambda)$ is continuous with respect to the infection rate λ. Applying this fact, we show the contact process on a general periodic tree experiences two phase transitions in the sense that $\lambda_1 < \lambda_2$, which confirms a conjecture of Stacey's [34]. We also prove that if the contact process survives strongly at λ then it survives strongly at a $\lambda' < \lambda$, which implies that the process does not survive strongly at the critical value λ_2 for strong survival.

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Chapter 1

Introduction

The contact process, also known as the Susceptible-Infection-Susceptible (SIS) model, is a fundamental model that describes the spread of epidemics on networks. It was first introduced by Harris [15] on $\mathbb{Z}^d$ and later studied on more general graphs. Vertices of the graph represent individuals and the states 0 and 1 indicate that an individual is *healthy* or *infected*. On a given graph $G = (\mathbb{V}, \mathbb{E})$, the contact process $\xi_t : \mathbb{V} \to \{0, 1\}$ evolves as follows:

- Each infected vertex becomes healthy independently at rate 1;
- Each healthy vertex becomes infected at rate λ times the number of its infected neighbors.

Let ξ_t^0 be the process starting from only the origin occupied and let ξ_t^1 be the process starting from all sites occupied. Harris introduced the critical value

$$\lambda_c = \inf\{\lambda : P(\xi_t^0 \neq \varnothing \text{ for all } t) > 0\}$$

and proved that on $\mathbb{Z}^d$ we have $0 < \lambda_c < \infty$. He also showed that for $\lambda > \lambda_c$, ξ_t^1 converges to a limit that is a nontrivial stationary distribution. A rich theory has been developed for the contact process on $\mathbb{Z}^d$, see Liggetts 1999 book [22] for a summary of much of what is known.

Pemantle [32] began the study of the contact process on the tree $\mathbb{T}^d$ in which each vertex has degree $d+1$. Here, and in what follows, we assume $d \geq 2$ since $\mathbb{T}^1 = \mathbb{Z}$. Pemantle found that the contact process on $\mathbb{T}^d$ has two critical values

$$\lambda_1 = \inf\{\lambda : P(\xi_t^0 \neq \varnothing \text{ for all } t) > 0\}$$

$$\lambda_2 = \inf\{\lambda : \liminf_{t\to\infty} P(0 \in \xi_t^0) > 0\},$$

where ξ_t^0 denotes the contact process on the tree starting from only the root infected. The contact process is said to survive *weakly* if the process survives but the root 0 is infected for finitely many times almost surely, and survive *strongly* if the root 0 is infected for infinitely many times with positive probability. By deriving bounds on the critical values, Pemantle showed that $\lambda_1 < \lambda_2$ when $d \geq 3$. Liggett [20] settled the case $d = 2$ by showing $\lambda_1 < 0.605 < 0.6609 < \lambda_2$. At about the same time, Stacey [34] gave a proof that $\lambda_1 < \lambda_2$ that did not rely on bounds. The stationary distributions and limiting behavior of the contact process on trees is an interesting subject that has been extensively studied. See Liggetts book [22] for an account of the results.

So far the contact process is well understood on homogeneous graphs such as $\mathbb{Z}^d$, regular trees and random regular graphs, while much less is known on more general networks. Some of these random networks are locally tree-like, so it is also of interest to study the process on more general trees such as Galton-Watson trees and periodic trees. The next four sections are devoted to understanding various aspects of the contact process on different graphs and seeing how the underlying spatial structure would influence the behavior of the process.

1.1 Contact process on periodic trees

Throughout this section we will denote by $(a_1, a_2, \ldots, a_k)$ a periodic tree in which the number of children in successive generations is given by the sequence $a_1, a_2, \ldots, a_k$.

In a 1992 paper [32] Pemantle studied the contact process on periodic trees. In the special case of a (a, b) tree, he showed $1/(\sqrt{a} + \sqrt{b}) \leq \lambda_2$. He also showed that for a general period k tree with degree sequence $(a_1, a_2, \ldots, a_k)$ the critical value $\lambda_2 \leq C/(a_1 \cdots a_k)^{1/2k}$ for some constant $C > 0$. When $k = 2$ the bound is $C/(ab)^{1/4}$. When $a = b$, the upper and lower bounds differ by a factor of 2. However when $a = 1$ and $b = n$

$$1/(\sqrt{a} + \sqrt{b}) = 1/(1 + \sqrt{n}) \qquad C/(ab)^{1/4} = C/n^{1/4}. \tag{1.1}$$

As Pemantle pointed out the upper and lower bounds are different orders of magnitude, which leads to the natural question on what the sharp asymptotics for λ_2 is. Pemantle wrote in [32] "The somewhat surprising answer is that the lower bound is sharp even though the geometric mean $(a_1 \cdots a_k)^{1/k}$ is clearly a better representative for the growth rate of the tree."

In the paper [18] with my advisor Rick Durrett, we show that the lower bound in (1.1) is more accurate than the upper bound, but it is not quite sharp. The next theorem gives the sharp asymptotics for λ_2.

Theorem 1.1.1. *On the $(1, n)$ tree, as $n \to \infty$ the critical value*

$$\lambda_2 \sim \sqrt{c_2(\log n)/n} \quad \textit{where } c_2 = 1/2.$$

The notation "$\sim$" means that for any $\varepsilon > 0$ we have

$$\sqrt{(1-\varepsilon)c_2(\log n)/n} \le \lambda_2 \le \sqrt{(1+\varepsilon)c_2(\log n)/n}$$

when n is sufficiently large.

On page 2103 [32], Pemantle says that "for reasonably regular non-homogeneous trees the critical value is determined by M the maximum number of children and is at most $rM^{-1/2}$ where r is a logarithmic measure of how far apart vertices with M children are from each other." To investigate the role played by the maximum degree vertices, we consider the periodic tree $(n, a_1, a_2, \ldots, a_k)$ where the a_i's are of smaller order than n. It turns out that our result confirms his intuition about the importance of the maximum degree but also shows that the lower degree vertices can have a significant influence on λ_2.

Theorem 1.1.2. *Consider the $(n, a_1, a_2, \ldots, a_k)$ tree where k is a fixed integer and a_i can depend on n. Suppose $\max_i a_i \le Cn^{1-\delta}$ for some positive C, δ and*

$$b = \lim_{n\to\infty} \frac{\log(a_1 a_2 \cdots a_k)}{\log n}.$$

As $n \to \infty$ the critical value $\lambda_2 \sim \sqrt{c_k \log n/n}$ where $c_k = (k-b)/2$.

Theorem 1.1.1 is the special case where $k = 1$, $a_1 = 1$, so it suffices to prove Theorem 1.1.2.

Using the methods developed to study λ_2 we can derive asymptotics for λ_1.

Theorem 1.1.3. *Let* $c_k' \equiv (k+1)/2 - (b+1)$. *Under the assumptions of Theorem 1.1.2,*

(i) If $c_k' > 0$ *then as* $n \to \infty$ *the critical value* $\lambda_1 \sim \sqrt{c_k' \log n/n}$.

(ii) If $c_k' < 0$ *then* $(\log \lambda_1)/\log n \to -(b+1)/(k+1)$.

When $b = k$ the critical value is of order $1/n$, which is the correct order of magnitude value for the homogeneous tree in which all vertices have degree n. The difference between (i) and (ii) is that in the second case $\lambda_1 \ll n^{-1/2}$ so the contact process does not survive very long on stars, which refers to the graph with a central vertex and many leaf vertices.

1.1.1 The survival time of the contact process on finite trees

The key idea in studying the aymptotics for λ_1 and λ_2 is to understand the behavior of the contact process on subgraphs of the periodic tree $(n, a_1, a_2, \ldots, a_k)$. Let ρ be the root of the periodic tree. Truncating this tree at height k gives a subgraph

$$S_k = \{x : d(\rho, x) \le k\},$$

where $d(\rho, x)$ is the distance between ρ and x on the tree. Note that S_k itself is a finite tree where the number of children in successive generations is $n, a_1, a_2, \ldots, a_k$. We will obtain an upper bound on the survival time of the contact process on S_k.

Theorem 1.1.4. *Suppose* $\lambda = \sqrt{c(\log n)/n}$ *where* $c > 0$. *Let* τ_k *be the survival time of contact process on* $S_k = (n, a_1, \ldots, a_k)$ *starting from all sites occupied where* $\max_i a_i \le Cn^{1-\delta}$ *for some positive* C *and* δ. *For any* $\varepsilon > 0$, *when* n *is sufficiently large*

$$E\tau_k \le C_0(\log n)e^{(1+\varepsilon)\lambda^2 n} = C_0(\log n)n^{c(1+\varepsilon)},$$

where C_0 *is some positive constant depending on* k *but not on* C, δ.

When $k = 1$, S_k reduces to the star graph. In this case the result holds for all $\lambda > 0$. Theorem 1.1.4 gives the only upper bound we know of for the survival time for the contact process on a star graph.

There are many lower bounds for the survival time on stars. See Theorem 4.1 in [32], Lemma 5.3 in [1], and Lemma 1.1 in [6]. These bounds can be used to show that the critical value for prolonged survival of the contact process on some random graphs is 0, but to identify the asymptotics for the critical value on the $(n, a_1, \ldots, a_k)$ tree, we need a more precise result on the survival time on the star graph with n leaves. We denote the state of the star by (i, j) where i is the number of occupied leaves and $j = 1, 0$ when the center is occupied, vacant. Let $P_{i,j}$ be the law of the process starting at state (i, j) and let $T_{0,0}$ be the time to hit $(0, 0)$.

Theorem 1.1.5. *Let $L = (1 - 4\delta)\lambda n$ with $\delta > 0$. If $\eta > 0$ is small then*

$$P_{L,1}\left(T_{0,0} \geq e^{(1-\eta)\lambda^2 n}\right) \to 1 \qquad \text{as } n \to \infty.$$

With Theorem 1.1.4 and 1.1.5 we are able to construct good upper and lower bounding processes for the contact process on periodic trees and thus obtain sharp asymtotics for the critical values.

1.2 Contact process on Galton-Watson trees

In contrast to $0 < \lambda_1 < \lambda_2$ for the contact process on regular trees, Pemantle [32] proved that both critical values $\lambda_1 = \lambda_2 = 0$ on Galton-Watson trees whose offspring distribution D satisfies

$$p_k \equiv P(D = k) \leq C \exp(-k^{\alpha}) \quad \text{for some } 0 < \alpha < 1 \text{ and } C > 0.$$

In Chapter 3 we extend Pemantle's result to Galton-Watson trees with *subexponential* tail distribution, i.e.,

$$\limsup_{k \to \infty} \frac{\log p_k}{k} = 0.$$

To prove this we improved Pemantles method for estimating λ_2. Pemantle [32] has given a formula that gives an upper bound for λ_2, in which the constants are difficult to track down. Hence the formula does not immediately yield an explicit bound.

Theorem 3.2 in [32]. *There are constants c_2 and c_3 so that if μ is the mean of the offspring distribution, then for any $k > 1$, if we let $r_k = \max\{2, c_2 \log(1/kp_k)/\mu\}$.*

$$\lambda_2 < c_3\sqrt{r_k \log r_k \log(k)/k}. \tag{1.2}$$

In a joint work [16] with my advisor Rick Durrett, we traced through his calculations and are convinced that his bound is worse than the following:

Theorem 1.2.1. *If $p_k = 2^{-k}$ for $k \geq 1$, then $\lambda_2 \leq 2.5$.*

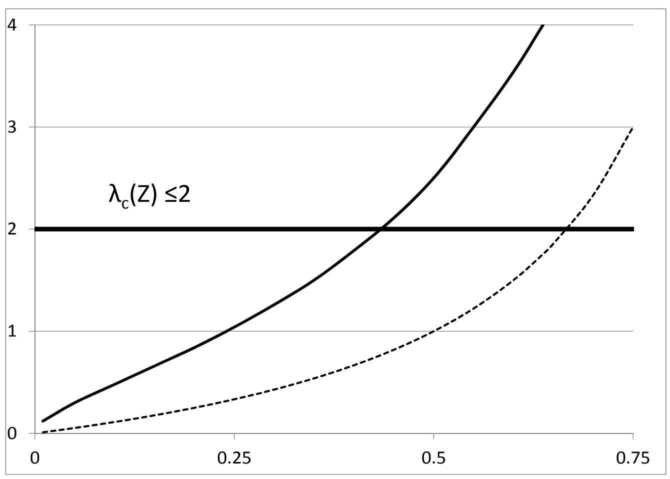

Figure 1.1: Upper bounds on λ_2 (solid line) and λ_1 (dotted line) as a function of p for the geometric degree distribution. $\lambda_2 \leq \lambda_c(\mathbb{Z}) \leq 2$ is a trivial upper bound since $\mathbb{Z}$ is embedded in the Galton-Watson tree.

The proof of Theorem 1.2.1 works for a general geometric $p_k = (1-p)^{k-1}p$, $k \geq 1$. We cannot get a nice formula for the upper bound as a function of p but the upper bounds can easily be computed numerically and graphed. These upper bounds are only interesting for small p. A Galton-Watson tree with $p_0 = 0$ and $p_1 < 1$ contains a copy of $\mathbb{Z}$ (start with a vertex with two children and follow their descendants) so using Liggett's bound on $\lambda_c(\mathbb{Z})$ proved in [20] we conclude $\lambda_2 \leq 2$ for all $0 < p < 1$.

In addition, we can give an improvement of Pemantle's result for stretched exponential distributions.

Theorem 1.2.2. *If the offspring distribution p_k for a Galton-Watson tree is subexponential and has mean $\mu > 1$ then $\lambda_2 = 0$.*

To give some intuition why $\lambda_2 = 0$ we discuss briefly the case where the offspring distribution satisfies $P(D \geq k) = \exp(-k^\alpha)$ for all $k \geq 0$ where $\alpha < 1$. Within distance k of the root there are roughly μ^k vertices where μ is the mean offspring number. Hence the vertex with the largest degree within distance k has degree $\approx (k \log \mu)^{1/\alpha}$. A vertex with degree n along with its neighbors is called a star graph with degree n. The contact process on a star graph with degree n can survive for time $\exp(C\lambda^2 n)$ for some $C > 0$ when n is sufficiently large (see [6]). Here and in what follows C is a positive constant whose value can change. While surviving on a star graph within distance k of the root, the contact process can try to push the infection back to the root with success probability $(\lambda/(1+\lambda))^k$ for each attempt.

Since the process can survive for time $\exp(C\lambda^2(k \log \mu)^{1/\alpha})$ on a star graph with degree $(k \log \mu)^{1/\alpha}$, the probability that it fails to push the infection back to the root is (while omitting some details)

$$\leq \left(1 - (\lambda/(1+\lambda))^k\right)^{\exp(C\lambda^2(k\log\mu)^{1/\alpha})} \leq \exp\left(-\left(\frac{\lambda}{1+\lambda}\right)^k \exp(C\lambda^2(k\log\mu)^{1/\alpha})\right). \quad (1.3)$$

Since the first term in the exponent decays exponentially while the second grows superexponentially, the failure probability goes to 0 as $k \to \infty$ for any positive values of λ. That is, the contact process can successfully push the infection back to the root and thus survive strongly for any $\lambda > 0$.

It turns out that Theorem 1.2.2 is sharp. The critical value is positive on Galton-Watson trees with tails thinner than the subexponential distribution. In 2019 Shankar Bhamidi, Danny Nam, Oanh Nguyen and Allan Sly [3] proved that

Theorem 1.2.3. *Consider the contact process on the Galton-Watson tree with offspring distribution D. If $E(\exp(cD)) < \infty$ for some $c > 0$, then $\lambda_1 > 0$.*

An offspring distribution with this property is said to have an *exponential tail*, which is thinner than the tail of the subexponential distribution. For example, suppose $P(D \geq k) = \exp(-ck)$ for $k \geq 0$ and some $c > 0$. Following the previous heuristics the vertex with the largest degree within distance k of the root has degree $\approx (k \log \mu)/c$. Again the contact process can survival for time $\exp(C\lambda^2 k)$ on a star graph with degree $(k \log \mu)/c$ for some $C > 0$, and can push the infection back to the root with failure probability

$$\left(1 - (\lambda/(1+\lambda))^k\right)^{\exp(C\lambda^2 k)} \approx \exp\left(-\left(\frac{\lambda}{1+\lambda}\right)^k \exp(C\lambda^2 k)\right) \tag{1.4}$$

when λ is sufficiently small. As $\log(\lambda/(1+\lambda)) + C\lambda^2 < 0$ when λ is small, (1.4) does not go to 0 when k goes to infinity. This provides some intuition why λ_1 is strictly positive when the offspring distribution has an exponential tail. Comparing (1.3) and (1.4), we can see the difference between the subexpnential distribution and the exponential-tail distribution is that in the first case the largest degree within distance k (i.e., $(k \log \mu)^{1/\alpha}$) grows super-linearly in k and overwhelms the success probability for pushing the infection back to the root.

1.3 Contact process on configurations models

To construct a random graph G_n on the vertex set $\{1, 2, \ldots, n\}$ having a specified degree distribution, we use the *configuration model*. Let $d_1, \ldots, d_n$ be independent and have the distribution $P(d_i = k) = p_k$. In order to have a valid degree sequence, we condition on the event $E_n = \{d_1 + \cdots + d_n \text{ is even}\}$. Since $P(E_n) \to 1/2$ as $n \to \infty$, the conditioning will have a little effect on the distribution of d_i's. Having chosen the degree sequence $(d_1, d_2, \ldots, d_n)$, we attach d_i half-edges to the vertex i, and then pair these half-edges at random. This procedure may produce a graph with self-loops or parallel edges, but we will ignore that problem for the moment.

1.3.1 Some known results

In the early 2000's physicists studied the contact process on a random graphs with a power-law degree distribution, i.e., the degree of each vertex is k with probability

$$p_k \sim Ck^{-\alpha} \quad \text{as } k \to \infty.$$

Pastor-Satorras and Vespignani [28, 29, 30] have made an extensive study of this model using mean-field methods. Their nonrigorous computations suggest the following conjectures about λ_c, the threshold for "prolonged persistence" of the contact process, and the critical exponent β, that controls the rate at which the equilibrium density of occupied sites $\rho(\lambda)$ goes to 0, i.e., $\rho(\lambda) \sim C(\lambda - \lambda_c)^\beta$.

- If $\alpha \leq 3$, then $\lambda_c = 0$. If $\alpha < 3$ then $\beta = 1/(3-\alpha)$.
- If $3 < \alpha \leq 4$, then $\lambda_c > 0$ and $\beta = 1/(\alpha - 3) > 1$.
- If $\alpha > 4$, then $\lambda_c > 0$ and $\beta = 1$.

See also Section V of [31]. The values of β quoted above are given in formula (29) of [31].

Chatterjee and Durrett [6] showed in 2009 that $\lambda_c > 0$ is not correct when $\alpha > 3$ and $P(d_i \leq 2) = 0$. The last condition guarantees that the graph is connected and that random walks on the graph have good mixing properties. They proved survival for time $\exp(O(n^{1-\varepsilon}))$, which was improved to $\exp(O(n))$ three years later by Mountford, Mourrat, Valesin, and Yao [24].

Chatterjee and Durrett [6] also obtained bounds on the critical exponent β. In 2013 Mountford, Mourrat, Valesin, and Yao [23] extended the results of [6] to include $2 < \alpha \leq 3$ and proved upper and lower bounds that had the same dependence on λ but different constants, showing that

$$\rho(\lambda) \sim \begin{cases} \lambda^{1/(3-\alpha)} & 2 < a \leq 5/2, \\ \lambda^{2\alpha-3}\log^{2-\alpha}(1/\lambda) & 5/2 < \alpha \leq 3, \\ \lambda^{2\alpha-3}\log^{4-2\alpha}(1/\lambda) & 3 < \alpha. \end{cases}$$

The result for $2 < \alpha \le 5/2$ agrees with the mean-field calculations quoted above but that formula is claimed to hold for $2 < \alpha < 3$.

1.3.2 Critical value asymptotics when $\lambda_c = 0$

While the results cited above show that the mean-field calculations are not correct, physicists have never said they were wrong. Indeed, a 2010 paper Castellano and Pastor-Satorras [4] claims they knew the right answer all along. "Already in 2003, Wang et al [36] argued that the SIS epidemic threshold on any graph is set by the largest eigenvalue of the adjacency matrix, Λ

$$\lambda_c(n) = 1/\Lambda(n).'' \tag{1.5}$$

Two years earlier Pemantle snd Stacey [33] proved that $1/\Lambda(n)$ is the critical value of branching random walk on the graph. To be precise they showed in Lemma 3.1 that

Lemma 3.1 in [33]. *Let $G = (V, E)$ be a graph and let $M(v, 2n)$ be the number of paths with $2n$ steps that begin and end at v. Let*

$$M = \lim_{n\to\infty} M(v, 2n)^{1/2n} = \sup_n M(v, 2n)^{1/2n}.$$

The limit exists by supermultipicativity and is independent of v. The critical probability for local survival of the branching random walk is given by $1/M$.

However, it is far from obvious why this should also be the critical value for the contact process. For example on $\mathbb{Z}$, the critical value λ for branching random walk is $1/2$ while for the contact process $\lambda_c \approx 0.82$.

The first question that needs to be addressed before (1.5) can become a theorem is the definition of λ_c. According to page 942 of the 2015 survey paper in *Reviews of Modern Physics* [31] "Above the epidemic threshold, the activity must be *endemic*, so that the average time to absorption is $O(e^{cn})$." To make it clear that they wanted to insist on this standard we note that the discussion continued with

> "Chatterjee and Durrett proved that in graphs with power law degree distribution $ET > \exp(O(N^{1-\delta}))$ for any $\delta > 0$. This result pointed to a vanishing threshold but still left the possibility for nonendemic long-lived metastable states."

Survival for time e^{cn} is certainly the gold standard for prolonged persistence, but following the footsteps of Ganesh, Masoulie, and Towsley [14], we will accept survival for time $\exp(O(n^{\varepsilon}))$ for some $\varepsilon > 0$ as evidence that $\lambda > \lambda_c$.

Wang et al [36] use (1.5) to conclude that the critical value for the contact process on a star graph with n leaves is $1/\sqrt{n}$. This claim is contradicted by Theorem (1.1.4), which still holds when $\lambda = \alpha/\sqrt{n}$. When n is sufficiently large, the survival time τ satisfies

$$E\tau \leq C_0(\log n)e^{(1+\varepsilon)\lambda^2 n} = C_0(\log n)e^{2\alpha^2}.$$

Thus the survival time is $O(\log n)$ which is much smaller than the $O(e^{cn})$ that [31] demands. Since the results in Theorem 1.1.4 and 1.1.5 show that the survival time is $\exp(O(\lambda^2 n))$, we would have to take $\lambda > 0$ independent of n for the contact process on the star graph to survive for this long. This serves as a counterexample to the claim of (1.5).

Returning to the implications of (1.5) for the contact process, the maximum eigenvalue of the adjacency matrix of a random graph is trivially $\geq d_{max}^{1/2}$ (generated by paths going back and forth between a vertex with degree d_{max} and its neighbors). Using results of Chung, Lu, and Vu [7] for the maximum eigenvalue for random graphs the authors of [4] concluded that the critical value for power law random graphs satisfies

$$\lambda_c \sim \begin{cases} \langle d \rangle / \langle d^2 \rangle & 2 < \alpha < 5/2, \\ 1/\sqrt{d_{max}} & 5/2 < \alpha, \end{cases}$$

where d_{max} is the maximum degree in the graph, and $\langle d \rangle$, $\langle d^2 \rangle$ are the average values of $d(x)$ and $d(x)^2$ for the graph. More concretely

$$\lambda_c(n) \sim \begin{cases} n^{(\alpha-3)/(\alpha-1)} & 2 < \alpha < 5/2, \\ n^{-1/2(\alpha-1)} & 5/2 < \alpha. \end{cases}$$

In the joint work [16] with my advisor, we prove an upper bound on λ_c that supports this prediction when $\alpha > 3$. Here we set $a = \alpha - 1$.

Theorem 1.3.1. *Suppose that the degree distribution has*

$$P(d(x) \geq k) = 3^a k^{-a} \quad \textit{for } k \geq 3.$$

We assume $a > 2$ so that $Ed(x)^2 < \infty$. Let $\lambda = n^{-(1-2\eta)/2a}$ and $\eta > 0$. If we start from all 1's then there is an $\varepsilon > 0$ so that the system survives for time $\exp(O(n^\varepsilon))$ with high probability.

Combining this result with the fact that $1/\Lambda$ gives the critical value for branching random walk and hence a lower bound on the critical value for the contact process we have

$$\lambda_c(n) = n^{-(1+o(1))/2a}. \tag{1.6}$$

Next we consider the stretched exponential degree distribution

$$P(d(x) \geq k) = \exp(-x^{1/b} + 3^{1/b}) \quad \text{for } k \geq 3 \text{ and } b > 1.$$

In this case, the maximum degree vertex on a graph with n vertices is $\sim \log^b n$, so the maximum eigenvalue $\Lambda \sim \log^{b/2} n$ and the formula in (1.5) predicts that $\lambda_c \approx \log^{-b/2} n$ but results of [3] show that this cannot be correct for $b \leq 1$. In that case the moment generating function of the degree distribution is finite for some positive θ so $\lambda_c(n)$ converges to a positive limit.

Theorem 1.3.2. *Suppose $\lambda_n = \log^{(1-\eta)(1-b)/2} n$. If we start from all 1's then for any $\varepsilon > 0$ the system survives for time $\exp(O(n^{1-\varepsilon}))$ with high probability.*

We believe that the last result gives the right answer.

1.4 Exponential growth and continuous phase transitions on trees

The behavior of the contact process on a d-regular tree $\mathbb{T}^d$ is now well understood. Liggett [22] gives a very extensive account for most of the known results on regular trees. It is of

interest to see if these results can be extended to more general trees such as the Galton-Watson trees and periodic trees. For example, some of the interesting problems are:
(i) proving exponential growth rate for the process in supercritical regime ($\lambda > \lambda_1$),
(ii) understanding the behavior of the contact process at the critical values λ_1 and λ_2,
(iii) determining if $\lambda_1 < \lambda_2$ (the existence of an intermediate phase).

The difficulty in addressing these problems lies in the fact that the proofs on regular trees rely heavily on translation invariance, and special functions such as $w_\rho(\xi_t) = \sum_{x\in\xi_t} \rho^{\ell(x)}$, where ρ is some positive constant and $\ell(x)$ is a function from $\mathbb{T}^d$ to $\mathbb{Z}$ so that for each $x \in \mathbb{T}^d$, $\ell(y) = \ell(x) - 1$ for exactly one neighbor y of x and $\ell(y) = \ell(x) + 1$ for the other d neighbors y of x. The expectation of $w_\rho(\xi_t)$ satisfies a submultiplicative relation

$$Ew_\rho(\xi_{t+s}) \leq Ew_\rho(\xi_t)Ew_\rho(\xi_s)$$

which implies $Ew_\rho(\xi_t) \geq [\phi(\rho)]^t$ where $\phi(\rho) = \lim_{t\to\infty}[Ew_\rho(\xi_t)]^{1/t}$. Properties of $\phi(\rho)$ help us obtain a lot of detailed information about the behavior of the contact process on regular trees.

1.4.1 Galton-Watson trees

Turning to Galton-Watson trees, the analysis of the contact process becomes more complicated because of the spatial heterogeneity and randomness in the tree structure. Throughout the discussion we consider the contact process on Galton-Watson trees with an offspring distribution D that is a random variable on $\mathbb{N}$ that satisfies

$$P(D \geq 1) = 1 \text{ and } ED > 1. \tag{1.7}$$

Recall that a Galton-Watson tree is said to have a *subexponential* offspring distribution if its offspring distribution D satisfies

$$\limsup_{k\to\infty} \left(\log P(D = k)\right)/k = 0.$$

In this case Theorem 1.2.2 states that $\lambda_1 = \lambda_2 = 0$. Clearly when the offspring distribution is subexponential the contact process dies out at both critical values and there does not exist an intermediate phase. If the offspring distribution D has an *exponential tail*, i.e., if $E(\exp(cD)) < \infty$ for some $c > 0$, Shankar Bhamidi et al. [3] proved, as stated in Theorem 1.2.3, that $\lambda_1 > 0$. In this case questions (ii) and (iii) listed at the beginning of this section become non-trivial.

For the contact process on Galton-Watson trees we solve the problems (i) and (ii) by constructing a supercritical Crump-Mode-Jagers (CMJ) branching process that is dominated by the contact process. Let ξ_t^0 be the contact process on $\mathbf{GW}(D)$ with only the root infected initially.

Theorem 1.4.1. *Suppose the offspring distribution D satisfies (1.7). When $\lambda > \lambda_1(\boldsymbol{GW}(D))$, there is a positive constant c so that*

$$\liminf_{t\to\infty} |\xi_t^0|/e^{ct} > 0 \qquad P_\lambda\text{-}a.s.\ \ on\ \Omega_\infty \equiv \{\xi_t^0 \neq \varnothing\ \ \forall t \geq 0\}. \tag{1.8}$$

Here P_λ is the annealed measure which is defined rigorously in Section **XXX** and $\mathbf{GW}(D)$ denotes the set of all realizations of the Galton-Watson trees with offspring distribution D.

Suppose an individual in the comparison CMJ branching process gives birth only once and let τ denote that birth time. We can then look at the branching process in terms of time blocks of duration τ and roughly know the growth of this process by time τ. This suggests that a "block argument" should be a good strategy to prove the continuity of the phase transition at λ_1. Indeed, if the contact process ξ_t^λ with infection rate λ survives then it dominates a supercritical CMJ branching process. For a sufficiently small $\varepsilon > 0$, the contact process $\xi_t^{\lambda-\varepsilon}$ should behave very much the same as ξ_t^λ within a finite time block of length τ and thus also dominate a supercritical branching process. Hence we can show that

Theorem 1.4.2. *The contact process on $\boldsymbol{GW}(D)$ dies out at $\lambda_1(\boldsymbol{GW}(D))$ and the survival probability $p(\lambda)$ is continuous in $[0, \infty)$.*

Finally, we prove that

Theorem 1.4.3. *The contact process on $\boldsymbol{GW}(D)$ does not survive strongly at $\lambda_2(\boldsymbol{GW}(D))$.*

The proof of this result is more complicated than that of Theorem 1.4.2 but similar in spirit. We find the suitable "block events" and construct a comparison process based on the block events. If the contact process with infection rate λ survives strongly then it dominates a comparison process that guarantees strong survival, and hence the contact process survives strongly at $\lambda - \varepsilon$ for sufficiently small $\varepsilon > 0$.

1.4.2 Periodic trees

Let $\mathbb{T}_\kappa = (a_1, a_2, \dots, a_\kappa)$ be a periodic tree in which the number of children in successive generations is $a_1, a_2, \dots, a_\kappa$. We only assume $a_i \geq 1$ for $1 \leq i \leq \kappa$ and $a_1 \cdot a_2 \cdots a_k \neq 1$. In order for the graph to be quasi-transitive and thus simplify the computation, $\mathbb{T}_\kappa = (a_1, a_2, \dots, a_\kappa)$ is arranged in the way that every vertex x in $\mathbb{T}_\kappa$ has one neighbor above it and $d(x)$ neighbors below it, where $d(x) \in \{a_1, \dots, a_\kappa\}$ is the offspring number of x.

To describe $\mathbb{T}_\kappa$ more rigorously we will introduce a function $\ell : \mathbb{T}_\kappa \to \mathbb{Z}$ that assigns a *level* to every vertex in $\mathbb{T}_k$. Without loss of generality, a distinguished vertex o is chosen to be the root where $d(o)$ can be any value in $\{a_1, a_2, \dots, a_\kappa\}$. Now each vertex can be assigned a *level* according to their position relative to the root o. Specifically, we will define the function $\ell : \mathbb{T}_\kappa \to \mathbb{Z}$ so that (i) $\ell(o) = 0$ and (ii) for each $x \in \mathbb{T}_\kappa$, $\ell(y) = \ell(x) - 1$ for exactly one neighbor y of x, and $\ell(y) = \ell(x) + 1$ for the other $d(x)$ neighbors of x. For a vertex $x \in \mathbb{T}_\kappa$, $\ell(x)$ is said to be the level of x.

If we choose the root o to be a vertex with a_1 children, then the periodic tree $\mathbb{T}_\kappa = (a_1, a_2, \dots, a_\kappa)$ is such that a vertex on level ℓ of $\mathbb{T}_\kappa$ has $a_{(\ell \mod \kappa)+1}$ children.

The treatment for the contact process on Galton-Watson trees can be extended to prove analogous conclusions on periodic trees.

Theorem 1.4.4. *Let ξ_t^o denote the contact process on $\mathbb{T}_\kappa$ starting with vertex o infected.*
(i) When $\lambda > \lambda_1(\mathbb{T}_\kappa)$, there is a positive constant c so that

$$\liminf_{t \to \infty} |\xi_t^o|/e^{ct} > 0 \qquad P_\lambda\text{-a.s. on } \Omega_\infty \equiv \{\xi_t^o \neq \varnothing \ \forall t \geq 0\}. \tag{1.9}$$

(ii) The contact process ξ_t^o on the periodic tree T_κ dies out at $\lambda_1(\mathbb{T}_\kappa)$ and does not survive strongly at $\lambda_2(\mathbb{T}_\kappa)$.

We are also interested in determining if $\lambda_1 < \lambda_2$ on general periodic trees, which is believed to be true by Pemantle [32] and Stacey [34]. Stacey was able to prove $\lambda_1 < \lambda_2$ on what he called "isotropic block trees", which are to some extent similar to periodic trees but have more stringent assumptions on the structure. Intuitively, the assumptions require that there exist automorphisms on the tree that map distinguished vertices to one another. The rigorous definition of the isotropic block trees is a bit involved and hence will not be stated here. Curious readers are referred to page 1718 and 1719 in [34] for a full account.

To see the connection between isotropic block trees and periodic trees we will present two examples of the former.

Example 1. $G = (a_1, a_2, \ldots, a_\kappa)$ where $a_i = 1$ for $1 \le i \le \kappa - 1$ and $a_\kappa \ge 2$.

Example 2. $G = (a_1, a_2)$ where $a_1 a_2 \ne 1$, which is a general period-2 tree.

The above examples suggest that the isotropic block trees are, to some extent, similar to periodic trees. However, due to the somewhat stringent assumptions on the tree structure there are still many examples of periodic trees that fall outside this category. One of the simplest examples among them is the periodic (2,3,4) tree, where the the number of children in successive generations is 2,3,4.

The special structure of the isotropic blocks is a key ingredient that Stacey uses in his proof of

$$\exp(\alpha t) \le E(|\xi_t^A|) \le C \exp(\alpha t), \tag{1.10}$$

where C is a constant, α is a continuous function of λ and A is a set of vertices chosen specially, see Proposition 2.1 in [34]. For example, for a period-2 tree (a_1, a_2) the initially infected set A is chosen to be a finite tree of depth 2 where the first generation has a_1 offspring and each vertex in the second generation has a_2 offspring. The relation (1.10) implies that $\alpha = 0$ when $\lambda = \lambda_1$. Hence at λ_1 not only $E(|\xi_t^A|) \leq C$ for all $t \geq 0$ but also ξ_t^A dies out on an isotropic block tree. These facts are the key elements in Stacey's proof.

As we will see, the extinction at λ_1 is the real key to the proof of the existence of an intermediate phase. For this reason we can prove $\lambda_1 < \lambda_2$ for general periodic trees without trying to obtain the relation in (1.10), which is a nice relation but stronger than necessary for our purpose. We confirm Stacey's conjecture by showing

Theorem 1.4.5. *The contact process on the periodic tree T_κ has an intermediate phase in the sense that $\lambda_1 < \lambda_2$.*

The relation $\lambda_1 < \lambda_2$ is believed to hold for a much broader class of trees as discussed in [32] and [34]. It will be meaningful to prove

Conjecture 1.4.1. *There exists an intermediate phase $\lambda_1 < \lambda_2$ on Galton-Watson trees whose offspring distribution has an exponential tail.*

Part of the difficulty in proving this conjecture lies in the correlation between the state of a vertex and the geometry near this vertex. In an inhomogeneous graph such as a Galton-Watson tree, the fact that a vertex v is infected at a certain time would imply that it is more likely for v itself to have a large degree or to be connected to some vertices with large degrees. Our method requires a certain extent of independence between the state of a vertex and the geometry nearby, and hence does not extend to a highly inhomogeneous graph.

Chapter 2

The Contact Process on Periodic Trees

2.1 Introduction

This chapter is based on the joint work [18] with my advisor Rick Durrett.

Pemantle [32] considered periodic trees and Galton-Watson trees. In the special case that the number of children alternates between a and b he showed $1/(\sqrt{a}+\sqrt{b}) \le \lambda_2$. He did not give the details of the proof, but this can easily be proved using Lemma 3.1 in Pemantle and Stacey [33], which gives a formula for the critical value for local survival for branching random walk on a general graph. Pemantle also showed, see the first sentence after (7) on page 2103, that for a general period k tree with degree sequence $(a_1, a_2, \ldots, a_k)$ the critical value $\lambda_2 \le C/(a_1 \cdots a_k)^{1/2k}$. When $k = 2$ the bound is $C/(ab)^{1/4}$. When $a = b$, the upper and lower bounds differ by a factor of 2. However when $a = 1$ and $b = n$

$$1/(\sqrt{a}+\sqrt{b}) = 1/(1+\sqrt{n}) \qquad C/(ab)^{1/4} = C/n^{1/4}. \tag{2.1}$$

As Pemantle notes on page 2103, the upper and lower bounds are different orders of magnitude. He continues with "Which of these asymptotics for λ_2 is sharp if either? The somewhat surprising answer is that the lower bound is sharp even though the geometric mean $(a_1 \cdots a_k)^{1/k}$ is clearly a better representative for the growth rate of the tree." The next result shows that the lower bound is more accurate than the upper bound, but it is not quite sharp.

Theorem 1.1.1 *On the $(1, n)$ tree, as $n \to \infty$ the critical value*

$$\lambda_2 \sim \sqrt{c_2(\log n)/n} \quad \textit{where } c_2 = 1/2.$$

On page 2103, Pemantle says that “for reasonably regular non-homogeneous trees the critical value is determined by M the maximum number of children and is at most $rM^{-1/2}$ where r is a logarithmic measure of how far apart vertices with M children are from each other.” The next result confirms his intuition about the importance of the maximum degree but also shows that the lower degree vertices can have a significant influence on λ_2.

Theorem 1.1.2 *Consider the $(n, a_1, a_2, \ldots, a_k)$ tree where k is a fixed integer and a_i can depend on n. Suppose* $\max_i a_i \le Cn^{1-\delta}$ *for some positive C, δ and*

$$b = \lim_{n\to\infty} \frac{\log(a_1 a_2 \cdots a_k)}{\log n}.$$

As $n \to \infty$ the critical value $\lambda_2 \sim \sqrt{c_k \log n/n}$ where $c_k = (k-b)/2$.

Theorem 1.1.1 is the special case $k = 1$, $a_1 = 1$, so it suffices to prove Theorem 1.1.2.

Using the methods developed to study λ_2 we can derive asymptotics for λ_1

Theorem 1.1.3 *Under the assumptions of Theorem 1.1.2. (i) If $c'_k = (k+1)/2-(b+1) > 0$ then as $n \to \infty$ the critical value $\lambda_1 \sim \sqrt{c'_k \log n/n}$. (ii) If $(k+1)/2 - (b+1) < 0$ then $(\log \lambda_1)/\log n \to -(b+1)/(k+1)$.*

When $b = k$ the critical value is of order $1/n$, which is the correct order of magnitude value for the homogeneous tree in which all vertices have degree n. The difference between (i) and (ii) is that in the second case $\lambda_1 \ll n^{-1/2}$ so the contact process does not survive very long on stars.

2.1.1 The survival time of the contact process on finite trees

Let ρ be the root of the periodic tree $(n, a_1, \ldots, a_k)$. Truncating the periodic tree at height k gives a subgraph $S_k = \{x : d(\rho, x) \le k\}$, where d is the distance on the tree. A vertex $x \in S_k$ at distance i from the center ρ is said to be in the set L_i (“level i”). In S_k, vertices on level $1 \le i < k$ have a_i children while vertices on level k are leaves, i.e., they have no children. When the context is clear we also write $S_k = (n, a_1, \ldots, a_k)$, where the sequence corresponds to the offspring number on each level. When we delete the root from S_k we end up with n subtrees $\{T_{k,i}\}_{i=1}^n$ with $T_{k,i} = (a_1, \ldots, a_k)$.

The first step in the proof of Theorem 1.1.2 is to prove in Section 2 an upper bound on the survival time of contact process on S_k.

Theorem 1.1.4 *Suppose* $\lambda = \sqrt{c(\log n)/n}$ *where* $c > 0$. *Let* τ_k *be the survival time of contact process on* $S_k = (n, a_1, \dots, a_k)$ *starting from all sites infected where* $\max_i a_i \le Cn^{1-\delta}$ *for some positive* C *and* δ. *For any* $\varepsilon > 0$, *when* n *is sufficiently large*

$$E\tau_k \le C_0(\log n)e^{(1+\varepsilon)\lambda^2 n} = C_0(\log n)n^{c(1+\varepsilon)},$$

where C_0 *is some positive constant depending on* k *but not on* C, δ.

When $k = 1$, S_k reduces to the star graph. In this case the result holds for all λ (see [**?**] for details). Theorem 1.1.4 gives the only upper bound we know of for the survival time for the contact process on the star. There are many lower bounds for the survival time on stars. See Theorem 4.1 in [32], Lemma 5.3 in [1], and Lemma 1.1 in [6]. These bounds can be used to show that the critical value for prolonged survival of the contact process on some random graphs is 0, but to identify the asymptotics for the critical value on the $(n, a_1, \dots, a_k)$ tree, we need a more precise result on the survival time on the star graph with n leaves. We denote the state of the star by (i, j) where i is the number of infected leaves and $j = 1, 0$ when the center is infected, healthy. Let $P_{i,j}$ be the law of the process starting at state (i, j) and let $T_{0,0}$ be the time to hit $(0, 0)$.

Theorem 1.1.5 *Let* $L = (1 - 4\delta)\lambda n$ *with* $\delta > 0$. *If* $\eta > 0$ *is small then*

$$P_{L,1}\left(T_{0,0} \ge e^{(1-\eta)\lambda^2 n}\right) \to 1 \qquad \text{as } n \to \infty.$$

Combining this with Theorem 1.1.4 shows that if $\lambda^2 n \to \infty$ the survival time on the star is $\exp((1 + o(1))\lambda^2 n)$.

2.2 Upper bound on survival times on S_k

2.2.1 Equilibrium on T_k^*

To prepare for the proof of Theorem1.1.4, we first consider the contact process on S_k with the center ρ permanently infected. Let $T_k = (a_1, \dots, a_k)$ denote a generic subtree of ρ and let T_k^* be T_k with ρ attached to the root and with ρ permanently infected. Since the center ρ is always infected, the contact process on S_k can be simply treated as n independent contact processes on the $T_{k,i}^*$'s. As ρ is always infected, the contact process ξ_t on T_k^* has a stationary distribution ξ_∞.

We use x_i to denote a generic vertex in L_i. To compute the occupancy probability for a vertex $x_i \in L_i$ we need the notion of the dual process ζ_t of ξ_t. To construct ξ_t by graphical representation, we assign a Poisson process N_x of rate 1 to each vertex $x \in T_k^* \backslash \{\rho\}$ and a Poisson process $N_{(x,y)}$ of rate λ to each ordered pair of vertices that are joined by an edge of T_k^*. The dual process ζ_t is constructed by looking at the dual path on the same graphical representation we used to construct ξ_t. (See Liggett [22] for an account of graphical representation and duality.) It follows from duality that

$$P(x_i \in \xi_t) = P(\rho \in \zeta_s^{x_i} \text{ for some } s \le t) \le P(\rho \in \zeta_s^{x_i} \text{ for some } s \ge 0).$$

Letting $t \to \infty$ gives $P(x_i \in \xi_\infty) \le P(\rho \in \zeta_s^{x_i}$ for some $s \ge 0)$, that is, if $x_i \in \xi_\infty$ then the dual contact process $\zeta_t^{x_i}$ starting from x_i has to reach ρ at some time. If we have a dual path of length $i + 2m$ from x_i to ρ then $i + m$ steps will be toward ρ and m steps away. Let $(y_0, y_1, \dots, y_{i+2m})$ denote a path from x_i to ρ with $y_0 = x_i$ and $y_{i+2m} = \rho$. To produce a particle at ρ, we need a birth from y_j to y_{j+1} to occur before the particle at y_j dies for all $j = 0, \dots i + 2m - 1$. So the expected number of particles produced at ρ by this path is $(\lambda/(1+\lambda))^{i+2m} \le \lambda^{i+2m}$. If we let $d = Cn^{1-\delta}$ so that $a_i \le d$ for all $i = 1, \dots, k$, then the expected number of particles $N_{x_i,\rho}$ that reach ρ has

$$\begin{aligned} EN_{x_i,\rho} &\le \sum_{m=0}^{\infty} \binom{i+2m}{m} \lambda^{i+2m} d^m \le \lambda^i (1 + \sum_{m=1}^{\infty} 2^{i+2m} \lambda^{2m} d^m) \\ &= \lambda^i (1 + 2^i \sum_{m=1}^{\infty} (4\lambda^2 d)^m) \le (1+\eta)\lambda^i. \end{aligned} \tag{2.2}$$

Since $\lambda^2 d \to 0$ as $n \to \infty$, $\eta > 0$ can be arbitrarily small if n is large enough. It follows that

$$P(x_i \in \xi_\infty) \le P(N_{x_i,\rho} \ge 1) \le E(N_{x_i,\rho}) \le (1+\eta)\lambda^i. \tag{2.3}$$

2.2.2 Proof of Theorem 1.1.4

Proof. Starting with all sites infected on S_k, we will set the center ρ to be infected for an amount of time M chosen so that the distribution of the contact process on each $T^*_{k,i}$ becomes close to the equilibrium ξ_∞. After time M when the center ρ first becomes healthy we run a trial to see if the following event occurs:

$$G = \{\rho \text{ is not recolonized before the contact process dies out on } S_k\}.$$

If G occurs we terminate the process and obtain an upper bound on the survival time; if G does not occur then we set all of the sites in S_k to be 1, make ρ infected for the next M units of time, and begin the next try.

Our first step is to choose M. The contact process ξ_t on T^*_k is additive so we can write $\xi_t = \xi^\rho_t \cup \hat{\xi}^1_t$ where $\hat{\xi}^1_t$ is the contact process on T_k with initially all 1's, and ξ^ρ_t the contact process on T^*_k with ρ initially and permanently infected. By the time $\hat{\xi}^1_t$ dies out, we have $\xi_t = \xi^\rho_t$, whose distribution is stochastically dominated by the stationary distribution ξ_∞.

For our purpose M should be chosen to be roughly the extinction time of $\hat{\xi}^1_t$. To simplify notation we let $d = Cn^{1-\delta}$, the upper bound on $\max_i a_i$ and consider the contact process on the regular tree $\mathbb{T}_d$. Let A_t be the contact process on $\mathbb{T}_d$ with birth rate γ and death rate 1. Since $T_k \subset \mathbb{T}_d$, if we take $\gamma > \lambda$ the contact process ξ_t on T_k is stochastically dominated by A_t. Following the proof of Theorem 4.1 in part I of Liggett [22], we define

$w_\theta(A_t) = \sum_{y \in A_t} \theta^{\ell(y)}$. where $\ell(y)$ is a function from $\mathbb{T}^d$ to $\mathbb{Z}$ so that for each $y \in \mathbb{T}^d$, $\ell(z) = \ell(y) - 1$ for exactly one neighbor z of y and $\ell(z) = \ell(y) + 1$ for the other d neighbors z of y. Liggett shows that if $\theta = 1/\sqrt{d}$

$$\frac{d}{dt} E_A w_\theta(A_t)\bigg|_{t=0} \le [2\sqrt{d}\gamma - 1] w_\theta(A_0)$$

When n is sufficiently large and $\gamma = 1/(\log n)\sqrt{d} \approx (\log n)^{-1} n^{-(1-\delta)/2} \gg \lambda$, for any small $\eta > 0$ we have

$$E_{T_k} w_\theta(A_t) \le w_\theta(T_k) e^{-(1-\eta)t} \le (d\theta)^k e^{-(1-\eta)t}. \tag{2.4}$$

Markov's inequality implies that

$$d^{-k/2} P_{T_k}(\ell(x) \le k \text{ for some } x \in A_t) \le d^{k/2} e^{-(1-\eta)t} \le d^{k/2} e^{-t/2}.$$

It follows that

$$P(\hat{\xi}_t^1 \ne \emptyset) \le P_{T_k}(\ell(x) \le k \text{ for some } x \in A_t) \le d^k e^{-t/2}, \tag{2.5}$$

so the process dies out with high probability when $t \ge 4k \log d$. Since $d = Cn^{1-\delta}$ we can choose $M = 4k \log n$.

With the choice of M fixed, next we will estimate the probability of the event G and obtain an upper bound on the survival time. Now we start the contact process on T_k^* with all sites infected and set ρ to be infected for the first M units of time. After time M we allow ρ to become healthy at rate 1. For any $\eta > 0$ and $t \ge M$, when n is large enough

$$\begin{aligned} P(\xi_t(x_i) = 1) &\le P(\xi_t^\rho(x_i) = 1) + P(\hat{\xi}_t^1(x_i) = 1) \le P(x_i \in \xi_\infty) + P(\hat{\xi}_t^1 \ne \varnothing) \\ &\le (1+\eta)\lambda^i + n^{-(1-\delta)k} \le (1+2\eta)\lambda^i \qquad \text{by (2.3).} \end{aligned}$$

Call the subgraph consisting of ρ and all of its neighbors the central star. When the center ρ first becomes healthy (after time M), we use D_1 to denote the set of infected sites on the central star and D_2 the set of infected sites outside the central star. By the additivity of the process we observe that G is the intersection of the following two events:

$$G_1 = \{\rho \notin \xi_t^{D_1} \text{ for any } t\} \quad \text{and} \quad G_2 = \{\rho \notin \xi_t^{D_2} \text{ for any } t\},$$

where ξ_t^A is the contact process on S_k with an initial set A of infected sites. Since G_1 and G_2 are both decreasing events, i.e., having more births or fewer deaths is bad for them, by the Harris-FKG inequality

$$P(G) = P(G_1 \cap G_2) \geq P(G_1)P(G_2).$$

We begin by estimating $P(G_2)$. Note that the expected number of particles outside the central star at time $t \geq M$ is

$$\leq (1+2\eta)\left(na_1\lambda^2 + na_1a_2\lambda^3 + \dots na_1a_2\cdots a_{k-1}\lambda^k\right).$$

Since $\lambda = \sqrt{(c\log n)/n}$, if $a_i = Cn^{1-\delta}$ for all i with $\delta < 1/2$ this grows rapidly as $n \to \infty$.

Fortunately, if we start a contact process on S_k from a site on level i and freeze any particle that reaches the center ρ, then the expected number of such particles is $\leq (1+\eta)\lambda^i$ by the same dual path argument as in (2.2). Therefore the expected number N_ρ of particles reaching the center ρ is

$$\leq (1+\eta)(1+2\eta)\left(na_1\lambda^4 + na_1a_2\lambda^6 + \dots na_1a_2\cdots a_{k-1}\lambda^{2k+2}\right) \leq n^{-\delta/2}.$$

Hence when n is large

$$P(G_2^c) \leq P(N_\rho \geq 1) \leq E(N_\rho) \leq n^{-\delta/2} \leq \eta.$$

Now we turn to event G_1. Let $\zeta_t^{x_1}$ and $\tilde{\zeta}_t^{x_1}$ denote the contact process on T_k and T_k^*, respectively, with the site $x_1 \in L_1$ initially infected. We need to estimate the probability of event $B = \{\tilde{\zeta}_t^{x_1}$ dies out before giving birth to $\rho\}$. Let σ be the amount of time 0 is infected in $\zeta_t^{x_1}$. Using the definition of w_θ and a calculation similar to (2.4)

$$E\sigma \leq \int_0^\infty E_{x_1}w_\theta(A_t)dt \leq \int_0^\infty e^{-(1-\eta)t}dt \leq 1+2\eta.$$

when η is small. Note that on event B, $\zeta_t^{x_1}$ and $\tilde{\zeta}_t^{x_1}$ can be coupled together exactly. Hence event B occurs if there is no arrival of infection arrows pointing from x_1 to ρ during the survival time σ of $\zeta_t^{x_1}$, i.e.,

$$P(B) \geq P(Poisson(\lambda\sigma) = 0) = Ee^{-\lambda\sigma} \geq e^{-\lambda E\sigma} \geq e^{-(1+2\eta)\lambda}.$$

If there are m infected neighbors of ρ when it becomes healthy then we need event B to occur on m subtrees to ensure the occurrence of G_1. Again by the Harris-FKG inequality $P(G_1) \geq P(B)^m \geq e^{-(1+2\eta)m\lambda}$. Since the occupancy of sites adjacent to the root are independent events, (2.3) and the law of large numbers implies that for sufficiently large n we have $P(m \leq (1+3\eta)\lambda n) \geq 1-\eta$. From this it follows that for small η and large n

$$P(G_1) \geq (1-\eta)e^{-(1+2\eta)(1+3\eta)\lambda^2 n} \geq (1-\eta)e^{-(1+6\eta)\lambda^2 n}. \tag{2.6}$$

Therefore, $P(G) \geq P(G_1)P(G_2) \geq (1-\eta)^2 e^{-(1+6\eta)\lambda^2 n}$. That is, in expectation we need to try $(1-\eta)^{-2}e^{(1+6\eta)\lambda^2 n}$ times to have a success. In each trial, it takes an expected amount of time $M+1$ for ρ to become healthy. Let $\{X_i\}_{i=1}^n$ be a set of i.i.d. random variables representing the survival time of the contact process $\hat{\xi}_t^1$ on $\{T_{k,i}\}_{i=1}^n$, respectively. We observe that it takes time $\leq E(\max_{1\leq i\leq n} X_i)$ to determine if G occurs. Since

$$\begin{aligned} E(\max_{1\leq i\leq n} X_i) &\leq 2(k+1)\log n + \int_{2(k+1)\log n}^{\infty} P(\max_{1\leq i\leq n} X_i > t)\, dt \\ &\leq 2(k+1)\log n + \int_{2(k+1)\log n}^{\infty} n\cdot (Cn^{1-\delta})^k e^{-t/2}\, dt \leq 4k\log n \qquad \text{by (2.5)}, \end{aligned}$$

each round takes at most $M+1+4k\log n \leq 9k\log n$ units of time in expectation. It follows that

$$E\tau_k \leq (9k\log n)(1-\eta)^{-2}e^{(1+6\eta)\lambda^2 n} \leq C_0(\log n)e^{(1+6\eta)\lambda^2 n}$$

for some $C_0 > 0$ and sufficiently large n. □

2.3 Lower bound on λ_2

Lemma 2.3.1. *Let $c_k = (k-b)/2$ and $\epsilon > 0$. When n is sufficiently large, the critical value λ_2 of the contact process on the $(n, a_1, a_2, \ldots, a_k)$ tree in Theorem 1.1.2 satisfies*

$$\lambda_2 \geq \sqrt{\frac{c_k \log n}{(1+\varepsilon)n}}.$$

Proof. Let $S(\rho) \equiv S_k \cup L_{k+1}$, where we recall that $L_{k+1} = \{y : d(y,\rho) = k+1\}$. We start the contact process on $S(\rho)$. When a site in S_k gives birth onto a site on L_{k+1} we freeze the particle at y_{k+1} We begin with only ρ infected and run the process until there are no particles on S_k. These particles will be the descendants of the initial particle ρ in a branching random walk that we use to dominate the contact process on the periodic tree. When the contact process on S_k dies out we are left with frozen particles in L_{k+1}. Each frozen particle at y_{k+1} starts a new contact process on a subgraph $S(y_{k+1}) \equiv \{z : |z - y_{k+1}| \le k+1\}$ which is isomorphic to $S(\rho)$ and has center y_{k+1}. If there are several frozen particles at the same site they start independent contact processes. Then we freeze every particle that escapes from $S(y_{k+1})$, and so on.

Let $B(S(\rho), y_{k+1})$ be the total number of particles frozen at $y_{k+1} \in L_{k+1}$ in the contact process on $S(\rho)$. Let y_1 denote the neighbor of ρ that is at distance k to y_{k+1}. When the center ρ is infected, it gives birth to a particle at y_1 at rate λ. By the same reasoning as in (2.2), starting from a particle at y_1, to produce a particle at y_{k+1} we need a path from y_1 to y_{k+1} where in each step a birth occurs before death. If we set the center ρ to be always infected, then we can ignore the paths from y_1 to y_{k+1} that go through ρ. Therefore, starting from a particle at y_1 the expected number of particles reaching y_{k+1} is $\le (1+\eta)\lambda^k$ by the same computation as (2.2).

By Theorem 1.1.4 the expected survival time on S_k is $\le C(\log n)e^{(1+\eta)\lambda^2 n}$. If during this whole time the center ρ is infected and pushing particles to y_1 at rate λ, then there are an expected number of $\le \lambda \cdot C(\log n)e^{(1+\eta)\lambda^2 n}$ times we start a process from a particle at y_1 to produce particles at y_{k+1}. Hence

$$EB(S(\rho), y_{k+1}) \le \lambda \cdot C(\log n)e^{(1+\eta)\lambda^2 n} \cdot (1+\eta)\lambda^k \le \lambda^{k+1} n^{(1+2\eta)c}. \tag{2.7}$$

To bound the number of particles on the periodic tree that reach the root ρ, we consider a tree consisting of the vertices of degree n in which each vertex is connected to the others vertices of degree n at distance $k+1$. This is a N-regular tree with $N = n(a_1 a_2 \cdots a_k)$. Starting from the root, there are

$$\le \binom{2m}{m} N^m \cdot 1^m \le 2^{2m} N^m$$

paths of length $2m$ that returns to it. So the expected number of particles returning to the root is

$$\leq 2^{2m} N^m (\lambda^{k+1} n^{(1+2\eta)c})^{2m} = (2N^{1/2} \lambda^{k+1} n^{(1+2\eta)c})^{2m} \tag{2.8}$$

We have $\log(\lambda)/\log n \to -1/2$ and $(\log N)/\log n \to 1+b$ so

$$\lim_{n\to\infty} \frac{\log(2\lambda^{k+1} N^{1/2} n^{(1+\varepsilon)c})}{\log n} = -\frac{k+1}{2} + \frac{1+b}{2} + (1+2\eta)c < 0$$

if $c < (k-b)/2(1+2\eta)$. In this case, the expected number of particles that return to the origin is finite, which means the process does not survive locally. Taking $\varepsilon = 2\eta$ completes the proof. □

2.4 Lower bound on survival time on stars

Here, following the approach of Chatterjee and Durrett [6], we will reduce the contact process on a star to a one dimensional chain. We denote the state of the star by (j, k) where j is the number of infected leaves and $k = 1, 0$ when the center is infected, healthy. We will only look at times when the center is infected. When the center is healthy and there are j infected leaves, the next event will occur after exponential time with mean $1/(j\lambda + j)$. The probability that it will be a birth at the center is $\lambda/(\lambda+1)$. The probability it will be the death of a leaf particle is $1/(\lambda+1)$. Thus, the number of leaf particles Z that will be lost while the center is healthy has a shifted geometric distribution with success probability $\lambda/(\lambda+1)$, i.e.,

$$P(Z = j) = \left(\frac{1}{\lambda+1}\right)^j \cdot \frac{\lambda}{\lambda+1} \quad \text{for } j \geq 0. \tag{2.9}$$

Note that $EZ = 1/\lambda$. Since we are interested in a lower bound on the survival time, we can simply ignore the time spent when the center is healthy. Here we will construct a process X_t that gives a lower bound on the number of infected leaves in the contact process.

Let $\delta > 0$ and $L = (1-4\delta)\lambda n$. When there are $k \leq L$ infected leaves and the center is infected, new leaves become infected at rate $\lambda(n-k) \geq \lambda(n-\lambda n) \geq \lambda(1-\delta)n$ for sufficiently large n since $\lambda = \sqrt{c \log n/n} \to 0$ as $n \to \infty$. Let X_t have the following transition rates:

jump	at rate
$X_t \to X_t - 1$	L
$X_t \to \min\{X_t + 1, L\}$	$(1-\delta)\lambda n$
$X_t \to X_t - Z$	1

Here Z is independent of X_t and has the distribution given in (2.9).

Lemma 2.4.1. *Let $\delta > 0$. Suppose $\lambda = \sqrt{c(\log n)/n}$ and let*

$$\theta = \frac{1}{\lambda + 1}\left(\lambda - \frac{1}{\delta\lambda n}\right)$$

If n is large then $h(X_t) \equiv (1-\theta)^{X_t}$ is a supermartingale when $X_t < L$.

Proof. Suppose the current value is $V = (1-\theta)^{X_t}$ where $X_t \le L = (1-4\delta)\lambda n$. We have: $V \to V/(1-\theta)$ t rate $\le L$, $V \to V(1-\theta)$ at rate $\ge (1-\delta)\lambda n$, and $V \to V(1-\theta)^{-Z}$at rate 1. The changes in value due to the first two transitions are, if θ is small,

$$V\left(\frac{1}{1-\theta} - 1\right) \le (1-\delta)^{-1}\theta V \quad \text{at rate} \le L$$
$$V[(1-\theta) - 1] = -\theta V \quad \text{at rate} \ge (1-\delta)\lambda n$$

We have $L = (1-4\delta)\lambda n < (1-\delta)(1-3\delta)\lambda n$, so the first two types of jumps have a net drift

$$\left((1-\delta)^{-1}L - (1-\delta)\lambda n\right)\theta V \le -(2\delta\lambda n)\theta V. \tag{2.10}$$

In the third case, ignoring the fact that the number of infected leaves cannot drop below 0, we have

$$\begin{aligned} E(1-\theta)^{-Z} &\le \sum_{k=0}^{\infty}\left(\frac{1}{1+\lambda}\right)^k \frac{\lambda}{1+\lambda}\cdot(1-\theta)^{-k} = \frac{\lambda}{1+\lambda}\sum_{k=0}^{\infty}\left(\frac{1}{(1+\lambda)(1-\theta)}\right)^k \\ &= \frac{\lambda}{1+\lambda}\cdot\frac{1}{1-1/(1+\lambda)(1-\theta)} = \frac{\lambda(1-\theta)}{\lambda-\theta-\theta\lambda} \end{aligned}$$

so we have $V(E(1-\theta)^{-Z} - 1) = \frac{\theta V}{\lambda - \theta(1+\lambda)} = (\delta\lambda n)\theta V$ for the chosen value of θ. Combining this with (4.22) gives that for any $\delta > 0$, $h(X_t)$ is a supermartingale for large n. $\square$

We use P_i to denote the law of the process X_t starting with $X_0 = i$. Since X_t omits some time intervals from the contact process on the star, the next result implies Theorem 1.1.5.

Lemma 2.4.2. *Let $L = (1-4\delta)\lambda n$ and $T_a^- = \inf\{t : X_t < a\}$. If $\eta > 0$ is small then*

$$P_{L-1}\left(T_{\eta L}^- \geq \frac{1}{\lambda^2 n} e^{(1-4\eta)\lambda^2 n}\right) \to 1 \qquad \text{as } n \to \infty.$$

Proof. Suppose $a < x < b$ are integers. Let $T_b = \inf\{t : X_t = b\}$ and note that $X(T_b) = b$ while $X(T_a^-) \leq a - 1$. Since $h(X_t)$ is a supermartingale and h is decreasing

$$h(x) \geq h(a-1)P_x(T_a^- < T_b) + h(b)[1 - P_x(T_a^- < T_b)]$$

Rearranging we have

$$P_x(T_a^- < T_b) \leq \frac{h(x) - h(b)}{h(a-1) - h(b)}.$$

When $x = b - 1$ this implies

$$P_x(T_a^- < T_b) \leq \frac{h(b-1) - (1-\theta)h(b-1)}{h(a-1) - h(b-1)} = \frac{\theta h(b-1)/h(a-1)}{1 - h(b-1)/h(a-1)}$$

Let $\eta > 0$. We will apply this result with $b = L = (1-4\delta)\lambda n$ and $a = \eta b$. If δ is small $b \geq (1-\eta)\lambda n$. If λ is small then $1 - \theta < 1 - (1-\eta)\lambda$. With these choices

$$h(b-1)/h(a-1) = (1-\theta)^{b-a} < (1-(1-\eta)\lambda)^{(1-2\eta)\lambda n} \leq \exp\left(-(1-3\eta)\lambda^2 n\right).$$

If n is large,

$$P_{b-1}(T_a^- < T_b) \leq 2\lambda \exp(-(1-3\eta)\lambda^2 n). \tag{2.11}$$

Let $G_L = \{X_t \text{ returns } (1/2\lambda)e^{(1-4\eta)\lambda^2 n} \text{ times to } L \text{ before going } < \eta L\}$. It follows from (2.11) that $P(G_L) \geq 1 - e^{-\eta\lambda^2 n}$. In order to return to L we have to jump from $L-1$ to L, a time that dominates an exponential random variable with parameter $\lambda n/2$ so the law of large numbers tells us that the total amount of time before $X_t < \eta L$ is $\geq \frac{1}{\lambda^2 n} e^{(1-4\eta)\lambda^2 n}$ on G_L which completes the proof. $\square$

2.5 Ignition on a star graph

In this section we will describe a mechanism for the contact process on the period tree $(n, a_1, \dots, a_k)$ to survive, which basically relies on the dynamics on the star graphs of degree n embedded in tree. Vertices with degree n will be called hubs.

There are three ingredients in the proof of the upper bounds on λ_2

1. survival of the process on the star graph containing a hub for a long time,

2. pushing particles from one hub to other hubs at distance $k+1$,

3. "ignition", which refers to increasing the number of infected leaves at the new hub to L.

The first point was taken care of in the previous section. The third is covered in this one. Starting from only the central vertex of a degree n hub infected, we need to increase the number of infected leaves to $L = (1-4\delta)\lambda n$ by time $n^c/4$, which is referred to as the ignition of a hub. We treat L and K in the following lemma as integers for simplicity.

Lemma 2.5.1. *Suppose $\lambda = \sqrt{c_0(\log n)/n}$. Let $T_{0,0}$ be the first time the star is healthy and T_i be the first time the star has i infected leaves. For any small $\delta > 0$ if $K = \lambda n/\sqrt{\log n}$ and $L = (1-4\delta)\lambda n$, then for large n*

$$(i) P_{0,1}(T_K > T_{0,0}) \le 3/\sqrt{\log n},$$

$$(ii) P_{K,1}(T_{0,0} < T_L) \le 2\exp(-(c_0/3)\sqrt{\log n})$$

$$(iii) E_{0,1}\min\{T_{0,0}, T_L\} \le (1+\log n)/2\delta$$

Proof. Let $p_0(t)$ be the probability a leaf is infected at time t when there are no infected leaves at time 0 and the central vertex has been infected for all $s \le t$. $p_0(0) = 0$ and

$$\frac{dp_0(t)}{dt} = -p_0(t) + \lambda(1 - p_0(t)) = \lambda - (\lambda+1)p_0(t)$$

Solving gives $p_0(t) = \lambda(1 - e^{-(\lambda+1)t})/(\lambda+1)$. As $t \to 0$, $p_0(t) \sim \lambda t$ so if t is small $p_0(t) \ge \lambda t/2$ Taking $t = 2/\sqrt{\log n}$ it follows that if $B = \text{Binomial}(n, \lambda/\sqrt{\log n})$

$$P_{0,1}(T_K < T_{0,0}) \ge P(B > K)\exp(-2/\sqrt{\log n})$$

The second factor is the probability that the center stays infected until time $2/\sqrt{\log n}$, and $\exp(-2/\sqrt{\log n}) \geq 1 - 2/\sqrt{\log n}$. B has mean $\lambda n/\sqrt{\log n}$ and variance $\leq \lambda n/\sqrt{\log n}$ so Chebyshev's inequality implies

$$P(B < \lambda n/(2\sqrt{\log n})) \leq \frac{\lambda n/\sqrt{\log n}}{(\lambda n/(2\sqrt{\log n}))^2} \leq \frac{4\sqrt{\log n}}{\lambda n} \leq \frac{1}{\sqrt{\log n}}.$$

For (ii) we use the supermartingale $h(X_t)$ from Lemma 3.11.

$$\begin{aligned} P_{K,1}(T_{0,0} < T_L) &\leq 2(1-\lambda/3)^{\lambda n/\sqrt{\log n}} \\ &\leq 2\exp(-\lambda^2 n/3\sqrt{\log n}) = 2\exp(-(c_0/3)\sqrt{\log n}). \end{aligned}$$

For (iii) we compare with the process X_t in which we ignore the time spent when the center is healthy. To bound the time for the process X_t to reach L or die out we note that $EZ = (\lambda+1)/\lambda - 1 = 1/\lambda$ so when n is large

$$\mu = (1-\delta)\lambda n - (1-4\delta)\lambda n - 1/\lambda = 3\delta\lambda n - 1/\lambda \geq 2\delta\lambda n$$

gives a lower bound on the drift. Let $\hat{T}_{0,0}$ be the first time X_t hits 0 and $\hat{T}_L$ be the first time X_t hits L. $X_t - \mu t$ is a submartingale before time $V_L = \hat{T}_{0,} \wedge \hat{T}_L$. Stopping the submartingale $X_t - \mu t$ at the bounded stopping time $V_L \wedge s$

$$EX(V_L \wedge s) - \mu E(V_L \wedge s) \geq EX_0 = 0.$$

Since $EX(V_L \wedge s) \leq L$, it follows that $E(V_L \wedge s) \leq L/\mu$.

Letting $s \to \infty$ we have $EV_L \leq L/\mu \leq 1/2\delta$ since $L = (1-4\delta)\lambda n$ and $\mu \geq 2\delta\lambda n$. Note that the above calculation is for X_t which ignores the time when the center is healthy. To bound the time when the center is healthy, we note that the most extreme excursion that starts at n and goes to 0 takes a time with mean $(\log n)/(1+\lambda)$. During time $[0, V_L]$ the excursions occur at rate 1, that is, $E_{0,1}\min\{T_{0,0}, T_L\} \leq (1+\log n)EV_L \leq (1+\log n)/2\delta$. $\square$

2.6 Upper bound on λ_2

We will prove the result for $(n, a_1, \ldots, a_k)$ tree. Suppose that $\lambda = \sqrt{c(\log n)/n}$ with $c = c_k + \varepsilon$ where $c_k \equiv (k-b)/2$. We select one hub to call the root.

Step 1. Pushing the particles out to distance $(k+1)m$. Lemma 2.4.2 implies that if there at least L infected leaves before time $n^c/4$ then with high probability we have

$$G_0 = \{\text{there will be at least } \eta L \text{ infected leaves during } [n^c/4, 3n^c/4]\}.$$

During this time interval the hub will try to push particles to hubs at distance $k+1$. The first step is to show that the center is never healthy for very long.

Lemma 2.6.1. *Suppose that the number of infected leaves is always $\geq \eta L$ on $I = [n^c/4, 3n^c/4]$. Let $t_0 = 2/(1-4\delta)\eta$ and $G_1 = \{$there is no interval of length $\geq t_0$ in I during which the center is always healthy$\}$. As $n \to \infty$, $P(G_1) \to 1$.*

Proof. The center becomes healthy at times of a Poisson process with rate 1. Using large deviations results for the Poisson process, the probability there are more than n^c arrivals in an interval of length $n^c/2$ is $\leq \exp(-\gamma n^c)$. Suppose the center is healthy and let R be the time needed until it becomes infected.

$$P(R > t_0) \leq \exp(-t_0 \lambda \eta L) = \exp(-2c \log n) = n^{-2c}$$

Hence $P(G_1^c) \leq \exp(-\gamma n^c) + n^c n^{-2c} \to 0$. □

When the center is infected there is probability $e^{-1}(1-e^{-\lambda})$ that it will stay infected for time 1 and give birth onto a given leaf within time 1. With probability e^{-1} that leaf will stay infected until time 1. Doing this for $k+1$ times the probability of passing a particle to a given hub at distance $k+1$ is

$$\geq \left[e^{-1}(1-e^{-\lambda})e^{-1}\right]^{k+1} \geq \lambda^{k+1}/(2e^2)^{k+1} \geq C_1 \left(\frac{\log n}{n}\right)^{(k+1)/2} \tag{2.12}$$

where $C_1 = (\sqrt{c}/2e^2)^{k+1}$. Since our cycle takes time $t_1 \equiv t_0 + 2$, we have n^c/t_1 chances to do this during $[n^c/4, 3n^c/4]$. The probability that all attempts fail is

$$\leq \left(1 - C_1 \left(\frac{\log n}{n}\right)^{(k+1)/2}\right)^{n^c/t_1} \leq \exp\left(-C_2 n^{c-(k+1)/2} (\log n)^{(k+1)/2}\right) \tag{2.13}$$

where $C_2 = C_1/t_1$.

It follows from (4.11) that the probability of a successful push in $[n^c/4, 3n^c/4]$ is

$$\geq 1 - \exp\left(-C_2 n^{c-(k+1)/2} (\log n)^{(k+1)/2}\right) \geq n^{c-(k+1)/2} \quad \text{when } n \text{ is large.} \tag{2.14}$$

We say that a hub at distance $(k+1)m$ that is a descendant of the root is wet if it has $\geq \eta L$ infected leaves at time mn^c. Starting with a wet hub at time 0, the center of the hub will become infected at rate at least ηL and then Lemma 3.2.5 implies that the hub will be ignited within time $n^c/4$ with high probability. When the hub successfully pushes a particle to an adjacent hub during time $[n^c/4, 3n^c/4]$, that adjacent hub can ignite within the next $n^c/4$ units of time with high probability and hence be wet at time n^c. Therefore a wet hub can make an adjacent hub wet with probability $\geq n^{c-(k+1)/2}$ when n is large. The pushing events for different neighbors are not independent, but we can estimate the expected number Z_m of hubs that become wet at distance $(k+1)m$ at time mn^c. Let $N \equiv n(a_1 \cdots a_k)$. Then $EZ_m = (Nn^{c-(k+1)/2})^m$.

Step 2. Bringing a particle back to the root. To simplify notation we will write $N \equiv n(a_1 \cdots a_k)$. Let $\mathbb{T}_N$ denote an N-regular tree. We will compare with an oriented percolation in $\mathbb{T}_N \times \mathbb{Z}_+$, where the probability for successfully pushing a particle to a neighbor is $p \equiv n^{c-(k+1)/2}$ so that it is dominated by the contact process (see (2.14)). The mean number of paths N_m that go out a distance $(k+1)m$ and lead back to the origin is $EN_m = N^m p^{2m}$.

Note that the pushing events to different neighbors in the oriented percolation are not independent due to the underlying contact process. To estimate the second moment EN_m^2 we need to control the correlation between different paths back to the root. For simplicity in notation we write $a_0 = n$. For some $0 \leq l < m$ and $0 \leq r < k+1$, the number of pairs of paths from distance $(k+1)m$ back to the root that agree in the last $l(k+1)+r$ steps is

$$N^l \left(\prod_{i=0}^{r-1} a_i\right) \left(N^{m-l-1} \prod_{i=r}^{k} a_i\right)^2 \leq N^l N^{2(m-l-1)} N^2 = N^{2m-l}.$$

Since the two paths merge at distance $l(k+1)+r$ from the root, the corresponding step in the oriented percolation involves two dependent pushing events from two vertices to their common neighbor. The probability of this event is trivially upper bounded by p. Hence the probability that all the edges in the combined path are successful pushes is $\leq p^l(p^{m-l-1})^2 p = p^{2m-l-1}$. Thus the second moment of the number of successful paths out and back is

$$EN_m^2 \leq \sum_{l=0}^{m} N^{2m-l}(p^{2m-l-1})^2 \leq (Np^2)^{2m} p^{-2} \left(\sum_{l=0}^{m} (Np^2)^{-l} \right).$$

Since $c = c_k + \varepsilon$, $Np^2 = n^{2(c_k+\varepsilon)-(k-b)} > 1$ when n is sufficiently large, which then gives $EN_m^2 \leq Cp^{-2}(Np^2)^{2m}$ for some constant $C > 0$.

The Cauchy-Schwarz inequality implies that $E(N_m 1_{\{N_m>0\}})^2 \leq E(N_m^2)P(N_m > 0)$. Rearranging we conclude that

$$P(N_m > 0) \geq \frac{(EN_m)^2}{E(N_m^2)} \geq \frac{(Np^2)^{2m}}{Cp^{-2}(Np^2)^{2m}} = \frac{p^2}{C} = n^{2c-(k+1)}/C.$$

Since m is arbitrary we have that a particle returns to the root at arbitrarily large times in the oriented percolation. This implies that the dominating contact process survives strongly with probability $\geq n^{2c-(k+1)}/C > 0$ for any sufficiently large n. That is, $\lambda_2 \leq \sqrt{c(\log n)/n}$ when n is large enough.

2.7 Proof of Theorem 1.1.3

Case 1: $c'_k = (k+1)/2 - (b+1) > 0$.

We begin with the lower bound, which is proved using results from Section 3. Let $\alpha = c'_k - \varepsilon$ and $\lambda = \sqrt{\alpha(\log n)/n}$. Using (2.7) it follows that the total number of frozen particles on neighboring hubs is $\leq C_0(\log n)n^{(1+\eta)\alpha}n^{1+b}\lambda^{k+1}$ for some constant C_0. From this we see that if η is small then the expected number of particles that escape from S_k is < 1 and comparing with a branching process implies that the process dies out.

Turning to the upper bound, let $\beta = c'_k + \varepsilon$ and $\lambda = \sqrt{\beta(\log n)/n}$. Let $L = (1-4\delta)\lambda n$ with $\delta > 0$. Theorem 1.1.5 implies that starting from a "wet hub" (that has L infected neighbors) then with high probability (i) the infection on the associated star survives for at least $\exp((1-\eta)\lambda^2 n) = n^c$ where $c = (1-\eta)\beta$, and (ii) has ηL infected neighbors during this time. Computations in Step 3 of Section 6 imply that the probability of successfully pushing the infection to a neighboring hub during $[n^c/4, 3n^c/4]$ is $\geq n^{c-(k+1)/2}$ when n is large. The ignition result, Lemma 3.2.5, implies that at time n^c the new hub will have at least L infected neighbors with high probability. The expected number of new wet neighboring hubs is $\geq n^{c+b+1-(k+1)/2}$. If η is small then under our choice of β the number of wet hubs dominates a supercritical branching process.

Case 2: $(k+1)/2 - (b+1) < 0$**.**

Again we begin with the lower bound. Suppose $\lambda = n^{-\alpha}$ with $\alpha > 1/2$. In this case, Theorem 1.1.4 implies that the contact process survives for $O(\log n)$ on the graph S_k. Using (2.7) again, the expected number of particles that escaped from S_k is $\leq C(\log n) n^{b+1}\lambda^{k+1} = C(\log n)n^{b+1-\alpha(k+1)}$ for some positive constant C. If $\alpha > \frac{b+1}{k+1}$ the above is < 1 when n is large. Comparing with a branching process implies the the process dies out.

Starting with the center of S_k infected, the probability that it successfully pushes the infection to a neighboring hub is $\geq C_1\lambda^{k+1}$ by (2.12). If we only use neighboring hubs that are further from the root then we can compare with a branching process whose expected number of offspring is $\geq C_1\lambda^{k+1}n^{1+b} = C_1 n^{-\alpha(k+1)+1+b}$. Hence when $\alpha < \frac{b+1}{k+1}$ the contact process dominates a supercritical branching process, which implies $\lambda > \lambda_1$.

Chapter 3

The Contact Process on Random Graphs and Galton-Watson Trees

3.1 Introduction

This chapter is based on the joint work [16] with my advisor Rick Durrett.

3.1.1 Results for star graphs

Let G_k be a star graph with center 0 and leaves $1, 2, \ldots, k$ and let ξ_t be the set of vertices infected in the contact process at time t. Write the state ξ_t as (i, j) where i is the number of infected leaves and $j = 1$ if the center is infected and $j = 0$ otherwise. We write $P_{i,j}$ for the law of the process starting from (i, j). Pemantle [32] was the first to study the persistence time of the contact process on stars. See his Section 4. He did his analysis on the "ladder graph" $\{0, \ldots, n\} \times \{0, 1\}$ so he ended up with a very approximate superharmonic function $W(\xi)$. Let i be the number of infected leaves, and let $I(\xi) = 1$ if the root is infected and $= 0$ otherwise.

$$W(\xi) = e^{-\lambda i/10} \left(1 - I(\xi) \frac{(e^{\lambda/10} - 1)}{\lambda} \right).$$

To make the connection change Pemantle's n (the number of leaves) to our k and note that his birth rate $\lambda = \alpha/\sqrt{n}$. Pemantle has an interesting heuristic discussion on pages 2015–2016 that explains why this form is reasonable. However the 10's that are thrown in to make it is easier to prove it is superharmonic ruin its accuracy.

Here, following the approach in [6], we will reduce to a discrete time one dimensional chain, we will only look at times when $j = 1$. When the state is $(i, 0)$ with $i > 0$, the next event will occur after exponential time with mean $1/(i\lambda + i)$. The probability that it will be the reinfection of the center is $\lambda/(\lambda+1)$. The probability it will be the healing of a leaf is $1/(\lambda+1)$. Thus, the number of leaf infections N that will be lost while the center is healthy has a shifted geometric distribution with success probability $\lambda/(\lambda+1)$, i.e.,

$$P(N = j) = \left(\frac{1}{\lambda+1}\right)^j \cdot \frac{\lambda}{\lambda+1} \quad \text{for } j \geq 0.$$

Note that

$$EN = \frac{\lambda+1}{\lambda} - 1 = \frac{1}{\lambda}.$$

The next step is to modify the chain so that the infection rate is 0 when the number of infected leaves is at least

$$L = pk \quad \text{where} \quad p = \lambda/(1+2\lambda). \tag{3.1}$$

(The reader will see the reasons that underlie this choice later.) Note that for the modified chain the number of infected leaves is always $\leq pk$ and the number of uninfected leaves is $\geq (1-p)k$. Thus if we look at the embedded discrete time process for the contact process on the star and only look at times when the center is infected, the process dominates Y_n where

jump	with prob
$Y_n \to Y_n - 1$	pk/D
$Y_n \to \min\{Y_n + 1, pk\}$	$\lambda(1-p)k/D$
$Y_n \to Y_n - N$	$1/D$

Here N is independent of Y_n and the denominator

$$D = pk + \lambda(1-p)k + 1 \leq k + \lambda k + 1 \leq (2+\lambda)k. \tag{3.2}$$

The fact that Y_n is a reflecting random walk will simplify computations. We will use the process to lower bound survival times. Before the infection on the star graph goes extinct it will spend most of its time near pk, (i) this does not lose much compared to the more accurate birth and death chain, which uses the actual number of infected leaves not just a bound, and (ii) we make only a small error when we return to continuous time by assuming that jumps happen at the maximum rate. In [6] it is shown, see Lemma 2.2 on page 2339, that

Lemma 3.1.1. *Suppose* $\lambda \le 1$ *and* $\lambda^2 k \ge 50$. *Let* $L_0 = \lambda k/4$ *and* $S_0 = \frac{1}{4L_0}\exp(k\lambda^2/80)$. *Then*

$$P_{L,i}\left(\inf_{t\le S_0}|\xi_t| \le 0.4L_0\right) \le 7e^{-\lambda^2 L_0/80} \qquad \textit{for } i = 0, 1.$$

In contrast our Lemma 3.2.4 will show that if $L = \lambda k/(1+2\lambda)$ and $b = \varepsilon L$

$$P_{L,1}\left(\inf_{t\le S}|\xi_t| \le bL\right) \le (3+\lambda)(1+\lambda/2)^{-\varepsilon L} \quad \text{where} \quad S = \frac{1}{(2+\lambda)2k}(1+\lambda/2)^{L(1-2\varepsilon)}. \tag{3.3}$$

Part of the improvement comes from simply replacing L_0 by L and 0.4 by ε, but the most important change is to construct a more accurate superharmonic function. If one is proving that a critical value is 0, as [32] and [6] were, then it is not harmful to be off by a large constant factor, but if we are trying to get a good positive upper bound we need to be accurate. An easy corollary of Theorem 1.1.4 gives

Lemma 3.1.2. *Let* $K = \lambda n/(n+1)$. *For any* $\epsilon > 0$, *the contact process on the star graph has*

$$E_{K,1}T_{0,0} \le (\log n)e^{(1+\epsilon)\lambda^2 n}. \tag{3.4}$$

when n *is sufficiently large.*

See also Lemma 4 in [19] for a full proof. Hence we have shown that the improved lower bound is sharp.

If $\lambda^2 n \to \infty$ then the $\log n$ prefactor can be absorbed by changing ε however it is important if $\lambda = O(1/\sqrt{n})$, since in this case the exponential is $O(1)$. In contrast, the lower bound time T from (3.3), ignoring the prefactor, is

$$(1+\lambda/2)^{L(1-2\varepsilon)} \approx \exp\left((1-2\varepsilon)\frac{\lambda^2 k}{2(1+2\lambda)}\right).$$

If λ is small then the term in the exponential is about 1/2 the one in (3.4). Strictly speaking these results are not sharp (on the exponential scale) but a factor of 2 is much better than the factor of 80 that appears in [6]. It is not clear which result gives the right answer. The result in (3.4) is proved by looking at the first time the center becomes healthy and then all of the leaves become healthy before the center is reinfected. At first sight this bound seems crazy, but the calculations above show that it is fairly accurate. We have not been able to finding a good subharmonic function for Y_n to find a better upper bound so we leave it to a clever reader to determine the nature of the large deviation event that wipes out the infection on the star.

3.1.2 Galton-Watson trees

Given an offspring distribution p_k, we construct a Galton-Watson trees as follows. Starting with the root, each individual has k children with probability p_k. Pemantle has shown that

Theorem 3.2 in [32]. *There are constants c_2 and c_3 so that if μ is the mean of the offspring distribution, then for any $k > 1$, if we let $r_k = \max\{2, c_2 \log(1/kp_k)/\mu\}$.*

$$\lambda_2 < c_3 \sqrt{r_k \log r_k \log(k)/k}. \tag{3.5}$$

If the offspring distribution in the Galton-Watson tree is a stretched exponential $p_k = c_\gamma \exp(-k^\gamma)$ with $\gamma < 1$ then $\log(1/kp_k) \sim k^\gamma$ and hence $\lambda_2 = 0$.

Given this result, it is natural to ask about the critical values λ_1 and λ_2 when degrees have a geometric distribution. $p_k = (1-p)^{k-1}p$ for $k \geq 1$. The most interesting problem is to prove $\lambda_1 > 0$. Here, we prove upper bounds.

Theorem 3.1.1. *$\lambda_1 \leq p/(1-p)$.*

Proof. Modify the contact process so that births from a site can only occur on sites further from the root. Each vertex x will be occupied at most once. If x is occupied then it will give birth with probability $\lambda/(\lambda+1)$ onto each neighbor y. The birth events are not independent but that is not important. If we let Z_n be the number of sites at distance n

that are ever occupied, Z_n is a branching process in which the offspring distribution has mean $\lambda/((\lambda+1)\cdot p)$ which is > 1 if $\lambda > p/(1-p)$. $\square$

When $p_k = (1-p)^{k-1}p$, $\log(1/kp_k) \sim c_p k$, so (1.2) gives a finite upper bound on λ_2. It is difficult to trace through all the calculations to get an explicit lower bound. However, Pemantle uses $e^{-1}/5 = 0.0735$ as the lower bound for the probability of long time survival starting with only the center of a large degree star graph occupied, while Lemma 3.2.5 gives $1 - 3k^{-1/3}$ when the degree is k. This probability $e^{-1}/5$ appears cubed near the end of his proof, so we think that his bound is much worse than the following:

Theorem 1.2.1 *If $p_k = 2^{-k}$ for $k \geq 1$, then $\lambda_2 \leq 2.5$.*

This result is proved by combining our new estimates for the contact process on stars with the mysterious Lemma 2.4 in Pemantle's paper [32] (see Lemma 3.3.3 below). The proof works for a general geometric $p_k = (1-p)^{k-1}p$, $k \geq 1$. We cannot get a nice formula for the upper bound as a function of p but the upper bounds can easily be computed numerically and graphed. These upper bounds are only interesting for small p. A Galton-Watson tree with $p_0 = 0$ and $p_1 < 1$ contains a copy of $\mathbb{Z}$ (start with a vertex with two children and follow their descendants) so using Liggett's bound on $\lambda_c(\mathbb{Z})$ proved in [20] we conclude $\lambda_2 \leq 2$ for all $0 < p < 1$.

In addition, the proof of Theorem 1.2.1 yields an improvement of Pemantle's result for stretched exponential distributions. We say that p_k is subexponential if

$$\limsup_{k\to\infty}(1/k)\log p_k = 0.$$

Theorem 1.2.2 *If the offspring distribution p_k for a Galton-Watson tree is subexponential and has mean $\mu > 1$ then $\lambda_2 = 0$.*

Note that $\lambda_2 = 0$ implies $\lambda_1 = 0$. It turns out that the result in Theorem 1.2.2 is sharp, which has been proved by

Bhamidi, Nam, Nguyen, and Sly [3] *Consider the contact process on the Galton-Watson tree with offspring distribution ζ, and suppose that only the root of the tree is*

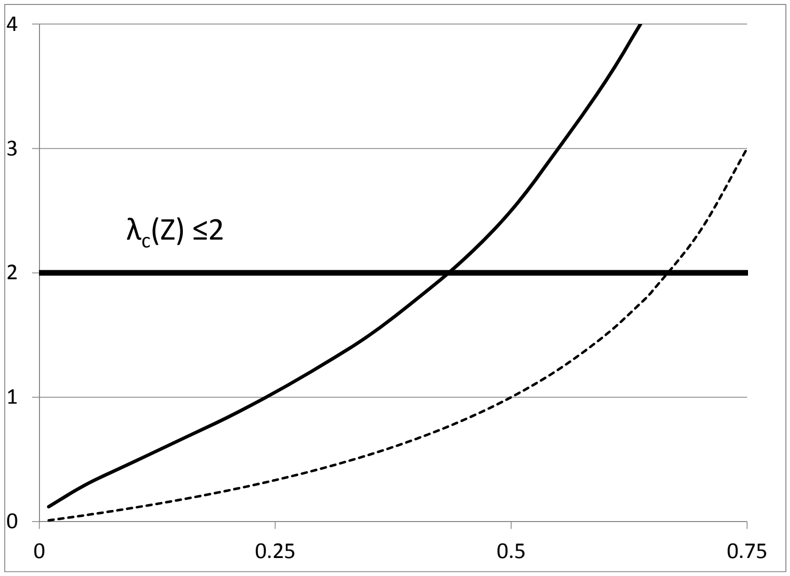

Figure 3.1: Upper bounds on λ_2 (solid line) and λ_1 (dotted line) as a function of p for the geometric degree distribution. The graph is computed by using (3.13).

initially infected. If $E(\exp(c\zeta)) < \infty$ *for some* $c > 0$*, then* $\lambda_1 > 0$*.*

They also prove results for random graphs. See [3] for more details.

3.1.3 Finite graphs

Consider the contact process on $\{-n, \dots n\}$ starting from all sites occupied and let $\tau_n = \inf\{t : \xi_t = \emptyset\}$. Combining results of Durrett and Liu [12] and Durrett and Schonmann [13] gives the following results

(i) If $\lambda < \lambda_c$ then there is a constant $\gamma_1(\lambda)$ so that

$$\tau_n / \log n \to \gamma_1(\lambda) \quad \text{in probability.}$$

(ii) If $\lambda > \lambda_c$ then there is a constant $\gamma_2(\lambda)$ so that

$$(\log \tau_n)/n \to \gamma_2(\lambda) \quad \text{in probability.}$$

(iii) When $\lambda > \lambda_c$ there is "metastability":

$$\tau_n / E\tau_n \Rightarrow \text{exponential}(1)$$

where $\Rightarrow$ means convergence in distribution. Intuitively, the process on the interval stays exponentially long in a state that looks like the stationary distribution for the process on $\mathbb{Z}$, and then suddenly dies out.

Results on $\mathbb{Z}^d$ with $d > 1$ had to wait for the work of Bezuidenhout and Grimmett [2], who showed that in $d > 1$ the contact process dies out at the critical value and in doing so introduced a block construction that can be used to study the supercritical process. Mountford [25] proved the metastability result in 1993 and that $(\log \tau_n)/n^d \to \gamma(\lambda)$ in 1999, see [26].

Stacey [35] studied the contact process on a tree truncated at height ℓ, $\mathbb{T}^d_\ell$. To be precise, the root has degree d, vertices at distance $0 < k < \ell$ from the root have degree $d+1$, while those at distance ℓ have degree 1. Cranston, Mountford, Mourrat, and Valesin improved Stacey's result to establish that the time to extinction starting from all sites occupied τ^d_ℓ satisfies

Theorem 3.1.2. *[8] (a) For any $0 < \lambda < \lambda_2(\mathbb{T}^d)$ there is a $c \in (0,\infty)$ so that as $\ell \to \infty$*

$$\tau^d_\ell / \log |\mathbb{T}^d_\ell| \to c \quad \textit{in probability.}$$

(b) For any $\lambda_2(\mathbb{T}^d) < \lambda < \infty$ there is a $c \in (0,\infty)$ so that as $\ell \to \infty$

$$\log(\tau^d_\ell)/|\mathbb{T}^d_\ell| \to c \quad \textit{in probability.}$$

Moreover $\tau^d_\ell / E\tau^d_\ell$ converges to a mean one exponential.

When a tree is truncated at a finite distance, a positive fraction of the sites are on the boundary. A more natural finite version of a tree is a random regular graph in which all vertices have degree $d+1$. In this case there is no boundary and the graph has the same distribution viewed from any point. If there are n vertices, the graph looks like $\mathbb{T}_d$ in neighborhoods of a point that have $\leq n^{1/3}$ vertices. Mourrat and Valesin have shown for a random regular graph, the time to extinction starting from all sites occupied τ_n satisfies:

Theorem 3.1.3. *[27] (a) For any $0 < \lambda < \lambda_1(\mathbb{T}^d)$ there is a $C < \infty$ so that as $n \to \infty$*

$$P(\tau_n < C \log n) \to 1,$$

(b) For any $\lambda_1(\mathbb{T}^d) < \lambda < \infty$ there is a $c > 0$ so that as $n \to \infty$

$$P(\tau_n > e^{cn}) \to c.$$

Notice that the threshold in the second result comes at λ_1, while the one in Stacey's result comes at λ_2. The difference is that when $\lambda \in (\lambda_1, \lambda_2)$ on the infinite tree the origin is in the middle of linearly growing vacant region. On the truncated tree the system dies out when the vacant region is large enough. However, on the random regular graph the occupied sites will later return to the origin. Durrett and Jung [11] investigated the qualitative differences between $\lambda \in (\lambda_1, \lambda_2)$ and $\lambda > \lambda_2$ on the small world graph.

To construct a random graph G_n on the vertex set $\{1, 2, \ldots, n\}$ having a specified degree distribution, we use the *configuration model.* Let $d_1, \ldots, d_n$ be independent and have the distribution $P(d_i = k) = p_k$. In order to have a valid degree sequence, we condition on the event $E_n = \{d_1 + \cdots + d_n \text{ is even}\}$. Since $P(E_n) \to 1/2$ as $n \to \infty$, the conditioning will have a little effect on the distribution of d_i's. Having chosen the degree sequence $(d_1, d_2, \ldots, d_n)$, we attach d_i half-edges to the vertex i, and then pair these half-edges at random. This procedure may produce a graph with self-loops or parallel edges, but we will ignore that problem for the moment.

In the early 2000's physicists studied the contact process on a random graphs with a power-law degree distribution, i.e., the degree of each vertex is k with probability

$$p_k \sim Ck^{-\alpha} \quad \text{as } k \to \infty.$$

Pastor-Satorras and Vespignani [28, 29, 30] have made an extensive study of this model using mean-field methods. Their nonrigorous computations suggest the following conjectures about λ_c, the threshold for "prolonged persistence" of the contact process, and the critical exponent β, that controls the rate at which the equilibrium density of occupied sites $\rho(\lambda)$ goes to 0, i.e., $\rho(\lambda) \sim C(\lambda - \lambda_c)^\beta$.

- If $\alpha \leq 3$, then $\lambda_c = 0$. If $\alpha < 3$ then $\beta = 1/(3 - \alpha)$.
- If $3 < \alpha \leq 4$, then $\lambda_c > 0$ and $\beta = 1/(\alpha - 3) > 1$.

- If $\alpha > 4$, then $\lambda_c > 0$ and $\beta = 1$.

See also Section V of [31]. The values of β quoted above are given in formula (29) of [31].

Chatterjee and Durrett [6] showed in 2009 that $\lambda_c > 0$ is not correct when $\alpha > 3$ and $P(d_i \leq 2) = 0$. The last condition guarantees that the graph is connected and that random walks on the graph have good mixing properties. They only proved survival for time $\exp(O(n^{1-\varepsilon}))$ but they obtained bounds on the critical exponent β.

In 2013 Mountford, Mourrat, Valesin, and Yao [23] extended the results of [6] to include $2 < \alpha \leq 3$ and proved upper and lower bounds that had the same dependence on λ but different constants, showing that

$$\rho(\lambda) \sim \begin{cases} \lambda^{1/(3-\alpha)} & 2 < a \leq 5/2 \\ \lambda^{2\alpha-3}\log^{2-\alpha}(1/\lambda) & 5/2 < \alpha \leq 3 \\ \lambda^{2\alpha-3}\log^{4-2\alpha}(1/\lambda) & 3 < \alpha \end{cases}$$

The result for $2 < \alpha \leq 5/2$ agrees with the mean-field calculations quoted above but that formula is claimed to hold for $2 < \alpha < 3$. Figure 2 gives a visual comparison of the mean-field and rigorous resultls for critical exponents. For more about why the change occurs at 5/2 see the next section and [23]. Three years later, Mountford, Mourrat, Valesin, and Yao [24] showed that for all $\lambda > 0$, there is a $c(\lambda) > 0$ so that the survival time $\geq e^{cn}$ with high probability.

3.1.4 Critical value asymptotics when $\lambda_c = 0$

While the results cited above show that the mean-field calculations are not correct, physicists have never said they were wrong. Indeed, a 2010 paper Castellano and Pastor-Satorras [4] claims they knew the right answer all along. "Already in 2003, Wang et al [36] argued that the SIS epidemic threshold on any graph is set by the largest eigenvalue of the adjacency matrix, Λ

$$\lambda_c(n) = 1/\Lambda(n)." \tag{3.6}$$

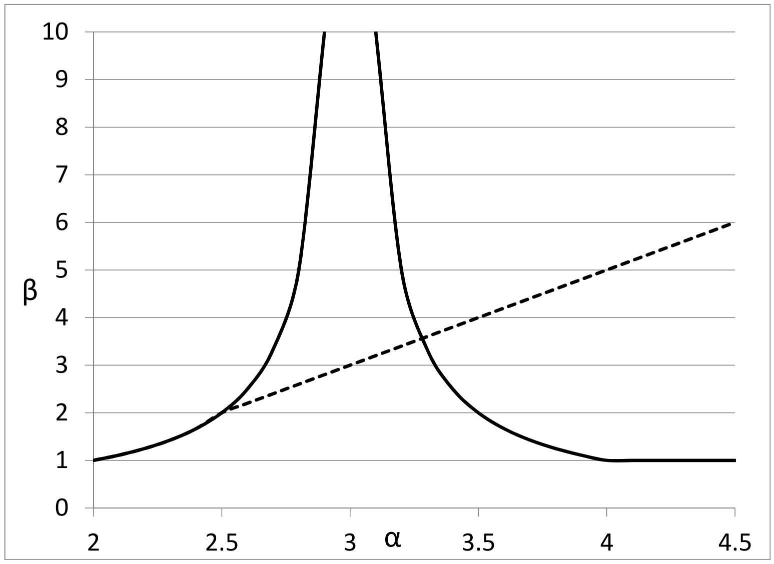

Figure 3.2: Mean field critical exponents (solid line) versus rigorous results (dashed line) as α varies from 2 to 4.5.

Two years earlier Pemantle snd Stacey [33] proved that $1/\Lambda(n)$ is the critical value of branching random walk on the graph. To be precise they showed in Lemma 3.1 that

Theorem. *Let $G = (V, E)$ be a graph and let $M(v, 2n)$ be the number of paths with $2n$ steps that begin and end at v. Let*

$$M = \lim_{n\to\infty} M(v, 2n)^{1/2n} = \sup_n M(v, 2n)^{1/2n}.$$

The limit exists by supermultipicativity and is independent of v. The critical probability for local survival of the branching random walk is given by $1/M$.

However, it is far from obvious why this should also be the critical value for the contact process. For example on $\mathbb{Z}$, the critical value λ for branching random walk is $1/2$ while for the contact process $\lambda_c \approx 0.82$.

The first question that needs to be addressed before (1.5) can become a theorem is the definition of λ_c. According to page 942 of the 2015 survey paper in *Reviews of Modern Physics* [31] "Above the epidemic threshold, the activity must be *endemic*, so that the average time to absorption is $O(e^{cn})$." To make it clear that they wanted to insist on this standard we note that the discussion continued with

> "Chatterjee and Durrett proved that in graphs with power law degree distribution $ET > \exp(O(N^{1-\delta}))$ for any $\delta > 0$. This result pointed to a vanishing threshold but still left the possibility for nonendemic long-lived metastable states."

Survival for time e^{cn} is certainly the gold standard for prolonged persistence, but following the footsteps of Ganesh, Masoulie, and Towsley [14], we will accept survival for time $\exp(O(n^\varepsilon))$ for some $\varepsilon > 0$ as evidence that $\lambda > \lambda_c$.

The proofs of (1.5) in [36] and [5] do not provide a lower bound on survival time. They let $n \to \infty$ to obtain a nonlinear dynamical system (NLDS). To explain, note that if we let $p_{i,t}$ be the probability i is infected at time t and let $\zeta_{j,t}$ be the probability j does not receive infection at time t then

$$\zeta_{i,t} = \prod_{j:j\sim i} (1 - \beta p_{j,t-1})$$

$$1 - p_{i,t} = (1 - p_{i,t-1})\zeta_{i,t} + \delta p_{i,t-1}\zeta_{i,t}$$

Then they argue that if $\lambda > \Lambda^{-1}$ then one of the eigenvalues of the linearization of the NLDS around 0 is > 1, see the Appendix of [5]. It is not clear what the last conclusion implies in terms of persistence. Wang et al [36] use (1.5) to conclude that the critical value for the contact process on a star graph with n leaves is $1/\sqrt{n}$.

The results discussed in Section 1.1 show that the survival time on the star graph increase dramatically when λ changes from $O(1/\sqrt{n})$ to $\gg 1/\sqrt{n}$. However, the claim that critical value on a star graph is $1/\Lambda(n)$ is contradicted by (3.4) which shows that if $\lambda = \alpha/\sqrt{n}$ then for large n

$$E_{K,1}T_{0,0} \le e^{2\alpha^2} \log n$$

where $K = \lambda n/(\lambda + 1)$. It is not hard to show that the time needed to go from n to K is $O(\log n)$. Thus the survival time is $O(\log n)$ which is much smaller than the $O(e^{cn})$ that [31] demands. Since the results in Section 1.1 show that the survival time is $\exp(O(\lambda^2 n))$, we would have to take $\lambda > 0$ independent of n for the contact process on the start to survive for this long.

Returning to the implications of (1.5) for the contact process, the maximum eigenvalue of the adjacency matrix of a random graph is trivially $\geq d_{max}^{1/2}$ (generated by paths going back and forth between a vertex with degree d_{max} and its neighbors). Using results of Chung, Lu, and Vu [7] for the maximum eigenvalue for random graphs the authors of [4] concluded that the critical value for power law random graphs satisfies

$$\lambda_c \sim \begin{cases} \langle d \rangle / \langle d^2 \rangle & 2 < \alpha < 5/2 \\ 1/\sqrt{d_{max}} & 5/2 < \alpha \end{cases}$$

where d_{max} is the maximum degree in the graph, and $\langle d \rangle$, $\langle d^2 \rangle$ are the average values of $d(x)$ and $d(x)^2$ for the graph. More concretely

$$\lambda_c(n) \sim \begin{cases} n^{(\alpha-3)/(\alpha-1)} & 2 < \alpha < 5/2 \\ n^{-1/2(\alpha-1)} & 5/2 < \alpha \end{cases}$$

Using our results we can prove an upper bound on λ_c that supports this prediction when $\alpha > 3$. Here $a = \alpha - 1$.

Theorem 3.1.4. *Suppose that the degree distribution has*

$$P(d(x) \geq k) = 3^a k^{-a} \quad \textit{for } k \geq 3.$$

We assume $a > 2$ so that $Ed(x)^2 < \infty$. Let $\lambda = n^{-(1-2\eta)/2a}$ and $\eta > 0$. If we start from all 1's then there is an $\varepsilon > 0$ so that the system survives for time $\exp(O(n^{\varepsilon}))$ with high probability.

Combining this result with the fact that $1/\Lambda$ gives the critical value for branching random walk and hence a lower bound on the critical value for the contact process we have

$$\lambda_c(n) = n^{-(1+o(1))/2a}. \tag{3.7}$$

Next we consider the stretched exponential

$$P(d(x) \geq k) = \exp(-x^{1/b} + 3^{1/b}) \quad \text{for } k \geq 3.$$

where $b > 1$. In this case, the maximum degree vertex on a graph with n vertices is $\sim \log^b n$, so the maximum eigenvalue $\Lambda \sim \log^{b/2} n$ and the formula in (1.5) predicts that $\lambda_c \approx \log^{-b/2} n$ but results of [3] show that this cannot be correct for $b \leq 1$. In that case the moment generating function of the degree distribution is finite for some positive θ so $\lambda_c(n)$ converges to a positive limit.

Theorem 3.1.5. *Suppose $\lambda_n = \log^{(1-\eta)(1-b)/2} n$. If we start from all 1's then for any $\varepsilon > 0$ the system survives for time $\exp(O(n^{1-\varepsilon}))$ with high probability.*

We believe that the last result gives the right answer.

Conjecture 3.1.1. *Suppose $\lambda_n = \log^{-a/2} n$ where $a > b - 1$. If we start from all 1's then for any $\varepsilon > 0$ the system dies out by time $\exp(O(n^{\varepsilon}))$ with high probability.*

The remainder of the paper is devoted to proofs. Section 2 gives our results for the star graph. Section 3 proves our results for Galton-Watson trees. Section 4 gives the asymptotics for $\lambda_c(n)$.

3.2 Results for the star graph

Recall from (1) that we set

$$L = pk \quad \text{where} \quad p = \lambda/(1+2\lambda).$$

The definition of Y_n is givn right after that formula.

Lemma 3.2.1. *Let $e^\theta = 1/(1+\lambda/2)$. If k is large enough $e^{\theta Y_n}$ is a supermartingale while $Y_n \in (0, pk)$.*

Proof. We begin by noting that

$$\begin{aligned} E(\exp(\theta Y_{n+1}) - \exp(\theta Y_n) | Y_n = y) &= e^{\theta y}(e^\theta - 1)\lambda(1-p)k/D \\ + e^{\theta y}(e^{-\theta} - 1)pk/D + \frac{e^{\theta y}}{D} &\left[\sum_{j=0}^{\infty} \left(\frac{e^{-\theta}}{1+\lambda} \right)^j \left(\frac{\lambda}{1+\lambda} \right) - 1 \right]. \end{aligned} \tag{3.8}$$

The term in square brackets is

$$\frac{1}{1 - e^{-\theta}/(1+\lambda)} \cdot \frac{\lambda}{1+\lambda} - 1 = \frac{\lambda}{1+\lambda-e^{-\theta}} - 1 = \frac{e^{-\theta}-1}{1+\lambda-e^{-\theta}} \geq 0.$$

Note that $\theta < 0$ so the last inequality implies that we must take $e^{-\theta} < 1+\lambda$.

The first two terms are

$$\frac{e^{\theta y}k}{D}\left((e^{\theta}-1)\lambda(1-p) + (e^{-\theta}-1)p\right),$$

so we begin by solving

$$(e^{\theta}-1)\lambda(1-p) + (e^{-\theta}-1)p = 0.$$

Rearranging and setting $x = e^{\theta}$ we want

$$x^2\lambda(1-p) - [\lambda(1-p)+p]x + p = 0.$$

Factoring we have

$$(\lambda(1-p)x - p)(x-1) = 0.$$

Since $p = \lambda/(1+2\lambda)$ the smaller root is

$$\frac{p}{\lambda(1-p)} = \frac{\lambda/(1+2\lambda)}{\lambda(1+\lambda)/(1+2\lambda)} = \frac{1}{1+\lambda}.$$

We let $e^{\theta} = 1/(1+\lambda/2) \in (1/(1+\lambda), 1)$ so that there is a $\delta > 0$ with

$$e^{\theta}\lambda(1-p) + e^{-\theta}p = [\lambda(1-p)+p] - \delta$$

and hence

$$(e^{\theta}-1)\lambda(1-p)k + (e^{-\theta}-1)pk + \frac{e^{-\theta}-1}{1+\lambda-e^{-\theta}} = -\delta k + \frac{e^{-\theta}-1}{1+\lambda-e^{-\theta}}.$$

From this we see that if k is large enough $e^{\theta Y_n}$ is a supermartingale while $Y_n \in (0, pk)$. The reason we restricted Y_n to $(0, pk)$ is that when $Y_n \leq pk$, the number of infected leaves tends to grow, which makes it possible to construct a supermartingale $e^{\theta Y_n}$ with $\theta < 0$. Note that when Y_n is small the number of infected leaves may become 0 before the center is reinfected but in this case the number of lost infections N is truncated.

$\square$

Let $T_\ell^- = \inf\{n : Y_n \le \ell\}$ and let $T_m^+ = \inf\{n : Y_n \ge m\}$. We write P_i for the law of the process Y_n starting with $Y_0 = i$.

Lemma 3.2.2. *Let $a, b \in (0, L)$. If $b < a$ then*

$$P_a(T_b^- < T_L^+) \le (1 + \lambda/2)^{b-a}.$$

Proof. To estimate the hitting probability let $\phi(x) = \exp(\theta x)$ where we take $e^\theta = 1/(1 + \lambda/2)$ and note that if $\tau = T_b^- \wedge T_L^+$ then $\phi(Y(t \wedge \tau))$ is a supermartingale. Let $q = P_a(T_b^- < T_L^+)$. Using the optional stopping theorem we have

$$q\phi(Y(T_b^-)) + (1 - q)\phi(Y(T_L^+)) \le \phi(a).$$

It is possible that $Y(T_b^-) < b$. Note that since $\theta < 0$, we have $\phi(x) \ge \phi(b)$ for $x \le b$. Hence,

$$q\phi(b) + (1 - q)\phi(L) \le \phi(a).$$

Dropping the second term on the left, $q \le \phi(a)/\phi(b) = (1 + \lambda/2)^{b-a}$, which completes the proof. □

Lemma 3.2.3. *If $R_L = \inf\{n > T_{L-1}^- : Y_n = L\}$ and $b \in [0, L)$ then for sufficiently large k*

$$P_L(T_b^- < R_L) \le (2 + \lambda)(1 + \lambda/2)^{b-L}.$$

Remark. Here, and in later lemmas, the computation of explicit constants is somewhat annoying. However, when we consider asymptotics for critical values, λ will go to 0, so we will need to know how the constants depend on λ.

Proof. To compute the left-hand side we break things down according to the first jump. The definition of R_L allows us to ignore the attempted upward jumps that do nothing. Recall that $L = pk$. The jump is to $L - 1$ with probability $pk/(pk + 1)$ and to $L - j$ with probability $\frac{\lambda}{(1+\lambda)^{j+1}} \cdot \frac{1}{1+pk}$. In the first case the probability of going below b before returning to L is

$$\le (1 + \lambda/2)^{b-(L-1)} = (1 + \lambda/2) \cdot (1 + \lambda/2)^{b-L}.$$

In the second case we have to sum over the possible values of $L - j$. Using Lemma 3.2.2

$$\begin{aligned}
&\le (1+\lambda/2)^{b-L} \sum_{j=1}^{\infty} \frac{\lambda}{(1+\lambda)^{j+1}} (1+\lambda/2)^j + \frac{\lambda}{1+\lambda} P_L(T_b^- < R_L) \\
&\le (1+\lambda/2)^{b-L} \frac{\lambda}{\lambda+1} \cdot \sum_{j=0}^{\infty} \left(\frac{1+\lambda/2}{1+\lambda}\right)^j + \frac{\lambda}{1+\lambda} P_L(T_b^- < R_L) \\
&= 2(1+\lambda/2)^{b-L} + \frac{\lambda}{1+\lambda} P_L(T_b^- < R_L).
\end{aligned}$$

Noting that $\max\{2, 1+\lambda/2\} \le 2(1+\lambda/2) - \delta$ for some small $\delta < \lambda$, we have the following relation,

$$P_L(T_b^- < R_L) \le \frac{\lambda}{(1+\lambda)(1+pk)} P_L(T_b^- < R_L) + (2+\lambda-\delta)(1+\lambda/2)^{b-L}.$$

Hence for k sufficiently large, we have $P_L(T_b^- < R_L) \le (2+\lambda)(1+\lambda/2)^{b-L}$. $\square$

Recall that ξ_t denotes the original contact process on the star graph with k leaves.

Lemma 3.2.4. *Let $b = \varepsilon L$ and $S = \frac{1}{(2+\lambda)2k}(1+\lambda/2)^{L(1-2\varepsilon)}$*

$$P_{L,1}\left(\inf_{t \le S} |\xi_t| \le b\right) \le (3+\lambda)(1+\lambda/2)^{-L\varepsilon}.$$

We have returned to unmodified process so $(L, 1)$ means L leaves are infected and the center is as well. Again when we write the state as a subscript we drop the parentheses.

Proof. Let $M = (1+\lambda/2)^{L(1-2\varepsilon)}$. By Lemma 3.2.3 the probability that the chain fails to return M times to L before going below εL is

$$\le (2+\lambda)(1+\lambda/2)^{-L\varepsilon}.$$

Using Chebyshev's inequality on the sum S_M of M exponentials with mean 1 (and hence variance 1),

$$P(S_M < M/2) \le 4/M.$$

When the number of infected leaves is $\le L$ maximum jump rate is $D \le (2+\lambda)k$ so

$$P\left(\frac{S_M}{(2+\lambda)k} \le \frac{(1+\lambda/2)^{L(1-2\varepsilon)}}{2(2+\lambda)k}\right) \le 4(1+\lambda/2)^{-L(1-2\varepsilon)} \le (1+\lambda/2)^{-L\varepsilon}.$$

for large L. Adding up the error probabilities gives

$$P_{L,1}\left(\inf_{t\le S}|\xi_t| \le b\right) \le (3+\lambda)(1+\lambda/2)^{-L\varepsilon}$$

and completes the proof. $\square$

Up to this point we have shown that if a star has L infected leaves it will remain infected for a long time. To make this useful, we need estimates about what happens when the star starts with only the center infected. Let $T_{0,0}$ be the first time the star is healthy. We use the pair (n,i) to denote the state of the star graph, where n is the number of infected leaves and i indicates the state of the center ($i=1$ means the center is infected).

Lemma 3.2.5. *Let $\lambda > 0$ be fixed and $K = \lambda k^{1/3}$. Then for large k*

$$P_{0,1}(T_K^+ > T_{0,0}) \le 2\lambda k^{-1/3},$$
$$P_{K,1}(T_{0,0} < T_L^+) \le k^{-1/3},$$
$$E_{0,1}(T_L^+ | T_L^+ < T_{0,0}) \le 2/\lambda.$$

Proof. Clearly

$$P_{0,1}(T_K^+ < T_{0,0}) \ge \prod_{j=0}^{K-1} \frac{(k-j)\lambda}{1+(k-j)\lambda+j}$$

so subtracting the last inequality from $1 = \prod_{j=0}^{K-1} 1$ and using Lemma 3.4.3 from [10]

$$P_{0,1}(T_K^+ > T_{0,0}) \le \sum_{j=0}^{K-1} \frac{1+j}{(k-j)\lambda} \le \frac{\lambda^2 k^{2/3}}{(k-\lambda k^{1/3})\lambda} \le 2\lambda k^{-1/3}.$$

For the second result we use the supermartingale $e^{\theta Y_n}$ from Lemma 3.11. If $q = P_{K,1}(T_{0,0} < T_L^+)$, using optional stopping theorem we have

$$q\cdot 1 + (1-q)e^{\theta L} \le e^{\theta K}.$$

Dropping the second term on the left,

$$q \le e^{\theta K} = (1+\lambda/2)^{-K} \le k^{-1/3}.$$

To bound the time we return to continuous time

jump	at rate
$Y_t \to Y_t - 1$	pk
$Y_t \to \min\{Y_t + 1, pk\}$	$\lambda(1-p)k$
$Y_t \to Y_t - N$	1

Before time $V_L = T_{0,0} \wedge T_L^+$ the drift of Y_t is at least

$$\mu = \lambda(1-p)k - pk - 1/\lambda = \lambda pk - 1/\lambda \tag{3.9}$$

so $Y_t - \mu t$ is a submartingale. Stopping this martingale at the bounded stopping time $V_L \wedge t$

$$EY(V_L \wedge t) - \mu E(V_L \wedge t) \geq EY_0 \geq 0.$$

Since $EY(V_L \wedge t) \leq L$, it follows that

$$E(V_L \wedge t) \leq \frac{L}{\mu} = \frac{pk}{\lambda pk - 1/\lambda},$$

where $p = \lambda/(1+2\lambda)$, so if λ is fixed and k is large

$$E(V_L \wedge t) \leq 2/\lambda,$$

which completes the proof. $\square$

Combining Lemmas 3.2.4 and 3.2.5 gives the following. When G occurs, we say the star at 0 is good.

Lemma 3.2.6. *Let A_t denote the number of infected leaves at time t and take S as in Lemma 3.2.4. Define $G = \{\inf_{k^{2/3} \leq t \leq S} |A_t| \geq \varepsilon L\}$. If $\lambda > 0$ is fixed and k is large then*

$$P_{0,1}(G) \geq 1 - (2 + 2\lambda)k^{-1/3} \tag{3.10}$$

Proof. Lemma 3.2.5 implies

$$\begin{aligned} P_{0,1}(T_L^+ < k^{2/3}) &\geq P_{0,1}(T_L^+ < k^{2/3} | T_L^+ < T_{0,0}) P_{0,1}(T_L^+ < T_{0,0}) \\ &\geq (1 - (2/\lambda)k^{-2/3})(1 - (1+2\lambda)k^{-1/3}) \geq 1 - (\frac{3}{2} + 2\lambda)k^{-1/3} \end{aligned}$$

for large k. By the definition in (3.1), $L = pk$ where $p = \lambda/(1+2\lambda)$. Lemma 3.2.4 tells us that

$$P_{L,1}\left(\inf_{t\le S} |\xi_t| \le \varepsilon L\right) \le (3+\lambda)(1+\lambda/2)^{-L\varepsilon}.$$

Since λ is fixed the right-hand side is $\le k^{-1/3}/2$ for large k. Adding up the error probabilities completes the proof. $\square$

3.3 Proofs of results for Galton-Watson trees

In the previous section we developed estimates for the contact process on stars. The next step is to obtain estimates on the probability of "pushing an infection from one star to another." When $\lambda > 0$ is fixed we have to be careful not to lose too much.

Lemma 3.3.1. *Let $v_0, v_1, \ldots v_r$ be a path in a graph and suppose that v_0 is infected at time 0. Then there is a $\gamma > 0$ so that the probability that v_r will become infected by time $2r$ is*

$$\ge \left(\frac{\lambda}{\lambda+1}\right)^r (1-\exp(-\gamma r)).$$

If $\varepsilon > 0$ and we let $\hat\lambda = (1-\varepsilon)\lambda/(\lambda+1)$ then for large r this probability is $\ge \hat\lambda^r$.

Proof. The probability that v_{i-1} infects v_i before it is cured is $\lambda/(1+\lambda)$. When this transfer of infection occurs the amount of time is t_i exponential with rate $1+\lambda$. By large deviations for the exponential distribution $P(t_1 + \cdots + t_r > 2r) \le e^{-\gamma r}$ for some $\gamma > 0$. $\square$

We say a star is nice if starting from L infected leaves, the event in Lemma 3.2.4 occurs. Recall that $S = \frac{1}{(2+\lambda)2k}(1+\lambda/2)^{L(1-2\varepsilon)}$ as in Lemma 3.2.4.

Lemma 3.3.2. *Run the contact process on a graph consisting of a star of size k to which there has been added a single chain $v_1, \ldots v_r$ of length r where v_1 is a neighbor of 0, the center of the star. Suppose that at time 0 there are L infected leaves and the star with center 0 is nice.*

For large r and k, the probability that v_r will not be infected before time $T = m(2r+1)$ for some $m \le S/(2r+1)$ is

$$\le (1-\hat\lambda^r)^m.$$

Proof. Consider a sequence of times $t_i = (2r+1)i$ for $i \geq 1$. The center 0 may not be infected at time t_i but since the number of infected neighbors is $\geq \varepsilon L$ the center will be infected by time $t_i + 1$ with probability at least $1 - e^{-\lambda\varepsilon L}$. By Lemma 3.3.1 the probability v_r is successfully infected in $[t_i, t_{i+1})$ is

$$\geq (1 - e^{-\lambda\varepsilon L})\left(\frac{\lambda}{\lambda+1}\right)^r (1 - \exp(-\gamma r)) \geq \hat{\lambda}^r$$

for sufficiently large r and k. The desired result follows. □

Remark. Due to the way the proof is done, if we condition on 0 being good then successes on two different chains are independent events.

To prepare for the proof of the main results we need the next lemma, which is Lemma 2.4 from [32]. Let $\varphi(x) = \sum_{n=0}^{\infty} p_n x^n$ be the generating function of the Galton-Watson tree. We will apply Lemma 3.3.3 to

$$f(t) = P(0 \in \xi_t^0) \geq p_k P(0 \in \xi_t^0 |\ 0 \text{ has at least } k \text{ children}).$$

Lemma 3.3.3. *Let H be any nondecreasing function on the nonnegative reals with $H(x) \geq x$ when $x \in [0, x_0]$. If f satisfies (i) $\inf_{0\leq t\leq L} f(t) > 0$ and (ii) $f(t) \geq H(\inf_{0\leq s\leq t-L} f(s))$ for $t \geq L$ some $L > 0$ then $\liminf_{t\to\infty} f(t) > 0$.*

Proof. For any t_0 and $\varepsilon > 0$, (ii) implies that there is a decreasing sequence t_i with $t_{i+1} \leq t_i - L$ and $t_k < L$ for some k

$$f(t_i) \geq H(f(t_{i+1})) - \varepsilon 2^{-i}.$$

If $f(t_i) < x_0$ for all $1 \leq i \leq k$ then

$$f(t_i) \geq f(t_{i+1}) - \varepsilon 2^{-i}$$

and summing gives $f(t_0) > f(t_k) - \varepsilon$ which gives the desired result. Suppose now that j is the smallest index with $f(t_j) > x_0$. If $j = 0$ we have $f(t_0) > x_0$. If $j = 1$ we have $f(t_0) \geq H(x_0)$. If $j \geq 2$ we have

$$f(t_0) \geq f(t_{j-1}) - \varepsilon \geq H(x_0) - \varepsilon$$

so in all cases we get the desired conclusion. □

Proof for $p_n = 2^{-n}$, $n \geq 1$. Our proof follows the outline of the proof of Theorem 3.2 in [32], see pages 2109–2110. We can suppose without loss of generality that the root has degree k. Otherwise examine the children of the root until we find one with degree k and apply the argument to the children of this vertex. There are two steps in the proof.

1. Push the infection to vertices at a distance $r = k$ that have degree k.

2. Bring the infection back to the root at time t using Lemma 3.3.3.

Step 1. The mean of the offspring distribution 2. Let Z_r be the number of vertices at distance r from 0 and let $v_r^1, \ldots v_r^J$ be the subset of those that have exactly k children, where J is a random variable that represents the number of such vertices.

Since the root has degree k and $p_k = 2^{-k}$ if we set $r = k$

$$EJ \geq k\mu^{r-1} p_k = k/2,$$

where $\mu = 2$ is the mean offspring number.

If we condition on the value of $W = Z_r/(k\mu^{r-1})$ and let $\bar{J} = (J|W)$ be the conditional distribution of J given W then

$$\bar{J} = \text{Binomial}(k2^{r-1}W, 2^{-k}).$$

Let M be the random number of vertices among $v_r^1, \ldots v_r^J$ that are infected before time

$$S = \frac{1}{2k(2+\lambda)}(1+\lambda/2)^{L(1-2\epsilon)}$$

defined in Lemma 3.2.4. The event $G = \{\inf_{k^{2/3} \leq t \leq S} |A_t| \geq \varepsilon L\}$ in Lemma 4.7 occurs with high probability. By Lemma 3.3.2, conditioning on G the probability a given vertex will not become infected by time S is

$$p_{noi} \leq (1-\hat{\lambda}^k)^m \quad \text{where} \quad \hat{\lambda} = (1-\varepsilon)\frac{\lambda}{\lambda+1} \quad \text{and}$$
$$m = \frac{S-k^{2/3}}{2k+1} \geq \frac{(1+\lambda/2)^{L(1-2\varepsilon)}}{4k(2k+1)(2+\lambda)} \quad \text{with} \quad L = \frac{\lambda k}{1+2\lambda}.$$

Combining the definitions and using $(1-x) \le e^{-x}$ we have

$$p_{noi} \le \exp\left(-\frac{\Gamma^k}{4k(2k+1)(2+\lambda)}\right) \quad \text{where} \quad \Gamma = \hat{\lambda}(1+\lambda/2)^{(1-2\varepsilon)\lambda/(1+2\lambda)}.$$

When $\lambda = 2.5$

$$\frac{\lambda}{\lambda+1}(1+\lambda/2)^{\lambda/(1+2\lambda)} = 1.0014 > 1, \tag{3.11}$$

so $\Gamma > 1$ when ε is small and $p_{noi} \to 0$ as $k \to \infty$. From this we see that if $\delta > 0$ then for large k

$$EM \ge (1-\delta)EJ.$$

The remark after Lemma 3.3.2 implies that if we condition on the value of W and let $\bar{M} = (M|W)$ then

$$\bar{M} \ge \text{Binomial}(k2^{r-1}W, 2^{-k}(1-\delta)).$$

To prepare for the following two generalizations of the result for Geometric(1/2) offspring distribution we ask the reader to verify that in Step 2, all we use is the fact that (3.11) implies the bounds on EM and $\bar{M}$.

Step 2. Let $H_1(t) = P(v_r^i \in \xi_{t-S}$ for some $1 \le i \le J)$ and

$$H_2(t) = P(0 \in \xi_t | v_r^i \in \xi_{t-S} \text{ for some } 1 \le i \le J),$$

so that $f(t) \ge H_1(t)H_2(t)$. Fix $t > 2S$ and let

$$\chi(t) = \inf\{f(s) : s \le t - S\}.$$

Since t is fixed, we simplify the notation and write $\chi(t)$ as χ.

Ignore all but the first infection of each v_r^i by its parent. any of these will evolve independently from the time $s < S$ it is first infected, and will be infected at time $t - S$ with probability at least χ. Thus given M the number of infected at time $t-S$ will dominate $N = \text{binomial}(M, \chi)$. If we let $\bar{N} = \text{binomial}(\bar{M}, \chi)$ and let $\delta > 0$, then by Lemma 2.3 in [32] we see that there exists a $\varepsilon > 0$ such that

$$P(\bar{N} \ge 1) \ge (1-\delta)\chi EM \wedge \varepsilon$$

Therefore $H_1(t) \geq (1-\delta)\chi EM \wedge \varepsilon$ when $t > 2S$.

Finally, if some v_r^i is infected at time $t-S$ then the probability of finding 0 infected at time t is bounded below by $\rho_1\rho_2$ where

- ρ_1 is the probability that the contact process starting with only v_r^i infected at time $t-S$ infects 0 at some time s with $t-S \leq s \leq t$. By Lemmas 3.2.5, 4.7, and 3.3.2, $\rho_1 \geq 1-\delta$.

- ρ_2 is the probability 0 is infected at time t given the infection of 0 at such a time s. For any $\varepsilon > 0$, by Lemma 3.3.2 the probability that 0 have not been infected by time $S/2$ is less than ε when k is sufficiently large. By Lemma 4.7, with probability $\geq 1-(2+2\lambda)k^{-1/3}$ there should be at least εL infected leaves at time $t-\varepsilon$. Hence 0 is infected at t with probability at least $(1-e^{-\lambda\varepsilon^2 L})e^{-\varepsilon}$, where the second term guarantees that the root is infected at time t. Choosing ε is sufficiently small and k sufficiently large gives $\rho_2 \geq 1-\delta$.

Thus

$$f(t) \geq \begin{cases} \chi(t)EM(1-\delta)^3 \wedge \varepsilon & t > 2S, \\ \inf_{0\leq s\leq 2S} f(s) & S \leq t \leq 2S. \end{cases}$$

We can take $\varepsilon < \inf_{0\leq s\leq 2S} f(s)$ so that $f(t) \geq \chi(t)EM(1-\delta)^3 \wedge \varepsilon$ for all $t \geq S$. The result now follows from Lemma 3.3.3 with $L = S$ and $H(x) = (1-\delta)^3(EM)x \wedge \varepsilon$.

Proof for $p_n = (1-p)^{n-1}p$. It is now straightforward to replace 1/2 by p. We only have to pick k and r so that we can prove the analogue of (3.11). The mean of the offspring distribution is $1/p$. Let Z_r be the number of vertices at distance r from 0 and let $v_r^1, \dots v_r^J$ be those that have exactly k children. Since the root has degree k and $p_k = (1-p)^{k-1}p$

$$EJ \geq k(1/p)^{r-1}(1-p)^{k-1}p. \tag{3.12}$$

In this case we want to pick r so that $(1/p)^r(1-p)^k \approx 1$. Hence EJ can be large when k is large. Ignoring the fact that r and k must be integers this means

$$r/k = \log(1-p)/\log p.$$

Let M be the random number of vertices among $v_r^1, \dots v_r^J$ that are infected before time S. By Lemma 3.3.2 the probability a given vertex will not become infected is

$$\le (1 - \hat{\lambda}^r)^{\lceil S/(2r+1) \rceil} \le \exp\left(- \frac{\Gamma^k}{4k(2r+1)(2+\lambda)} \right)$$

where $\Gamma = \hat{\lambda}^{r/k}(1 + \lambda/2)^{(1-2\varepsilon)\lambda/(1+2\lambda)}$. That is, if we choose λ such that

$$\left(\frac{\lambda}{\lambda+1} \right)^{r/k} \cdot (1 + \lambda/2)^{\lambda/(1+2\lambda)} > 1 \tag{3.13}$$

then we have $\Gamma > 1$ for large k. By the same reasoning as before this choice of λ gives an upper bound on λ_2.

If we want to graph the bound as a function of p it is better to work backwards. Given λ the second factor is > 1 so we can easily find the value of r/k that makes this 1. Having done this we can easily compute the value of p for which λ gives the upper bound on λ_2.

Proof for subexponential distributions. We suppose that the mean of the offspring distribution is $\mu > 1$. If p_k is subexponential, i.e.,

$$\limsup_{k \to \infty} (1/k) \log p_k = 0,$$

then for any δ there is a k with $p_k \ge (1-\delta)^k$. It follows from the same reasoning as in (3.12) that we can take r such that

$$\frac{r}{k} = -\frac{\log(1-\delta)}{\log \mu}.$$

Given any $\lambda > 0$, (3.13) will hold if δ is small enough, which implies local survival of the process. Therefore $\lambda_2 = 0$.

3.4 Asymptotics for λ_c

We begin with some general computations and then consider our two examples: power laws and stretched exponential.

Survival on star graph. Our first step is to adapt Lemma 3.2.4 to the situation in which $\lambda \to 0$. For reasons that will become clear when we prove Lemma 3.4.3 we have to modify the definition of p:

$$p = (1-\varepsilon)\frac{\lambda}{1+\lambda}, \qquad L = pk, \quad \text{and} \quad b = \varepsilon L.$$

Defining Y_n as before

Lemma 3.4.1. *Let $\varepsilon > 0$. If $\lambda/(1+2\lambda) < \varepsilon$ then $(1+\lambda/2)^{-Y_n}$ is a supermartingale.*

Proof. $(1-p) = (1+\lambda\varepsilon)/(\lambda+1)$ so we have

$$\frac{p}{\lambda(1-p)} = \frac{1-\varepsilon}{1+\lambda\varepsilon}.$$

The right-hand side is $< 1/(1+\lambda)$ when

$$1 + \lambda - \varepsilon - \varepsilon\lambda < 1 + \lambda\varepsilon,$$

which holds if $\lambda/(1+2\lambda) < \varepsilon$, so the desired result follows from the proof of Lemma 3.11. $\square$

Lemma 3.4.2. *Let $\varepsilon > 0$ be fixed $T = \exp((1-4\varepsilon)\lambda^2 k/4)$. If λ is small then for large k*

$$P_{L,1}\left(\inf_{t\le T} |\xi_t| \le b\right) \le 4\exp(-(1-3\varepsilon)\lambda^2 k/4),$$

Proof. It follows from Lemma 3.2.4 that if $S = (1/2k(2+\lambda))(1+\lambda/2)^{L(1-2\varepsilon)}$ then

$$P_{L,1}\left(\inf_{t\le S} |\xi_t| \le b\right) \le (3+\lambda)(1+\lambda/2)^{-L(1-2\varepsilon)},$$

but now

$$(1-\varepsilon)L = (1-\varepsilon)^2\lambda k/(\lambda+1) > (1-2\varepsilon)\lambda k/(\lambda+1).$$

Expanding $\log(1+x) = x - x^2/2 + x^3/3 - \ldots$ and noting that if $x < 1$ then the right-hand side is an alternating series with decreasing terms

$$\begin{aligned}(1+\lambda/2)^{-(1-2\varepsilon)\lambda k/(1+\lambda)} &= \exp\left(-(1-2\varepsilon)\frac{\lambda k}{1+\lambda}\log(1+\lambda/2)\right)\\ &\le \exp\left(-(1-2\varepsilon)\frac{\lambda k}{1+\lambda}\left[\frac{\lambda}{2} - \frac{\lambda^2}{8}\right]\right)\\ &\le \exp(-(1-3\varepsilon)\lambda^2 k/4),\end{aligned}$$

when λ is small. To convert the formula for S we note that

$$\begin{aligned}(1/2k(2+\lambda))(1+\lambda/2)^{L(1-2\varepsilon)} &= \frac{1}{2k(2+\lambda)} \exp\left(\frac{(1-\varepsilon)(1-2\varepsilon)\lambda k}{(\lambda+1)} \cdot \log(1+\lambda/2)\right) \\ &\geq \frac{1}{6k} \exp\left(\frac{(1-3\varepsilon)\lambda k}{(\lambda+1)} \cdot \left[\frac{\lambda}{2} - \frac{\lambda^2}{8}\right]\right) \\ &\geq \exp((1-4\varepsilon)\lambda^2 k/4),\end{aligned}$$

when λ is small, which completes the proof. $\square$

Push. Now we work with the configuration model. Let $p_k = P(d(x) = k)$ and suppose that

(i) $\sum_k k^2 p_k < \infty$,

(ii) $P(d(x) = k) = 0$ for $k \leq 2$.

The first assumption implies that the size biased degree distribution $q_{j-1} = jp_j/Ed(x)$ has finite mean ν. The second implies that the diameter of our graph $\sim (\log n)/\log \nu$.See Lemma 3.4.1 in [9]. Hence Lemma 3.3.1 implies that if $v_0, \ldots v_r$ is a path in the graph and v_0 is infected at time 0, then the probability v_r will be infected by time $2r$ is, for large r, $\geq (\lambda/2)^r$. Let

$$\kappa = n^{3\nu \log(2/\lambda)}.$$

If n is large then the distance between any two vertices is $\leq 2\nu \log n$ with high probability. Thus the probability that one star can transfer its infection to another before time $2r\kappa$ is

$$\begin{aligned}&\geq 1 - \left(1 - (\lambda/2)^{2\nu \log n}\right)^\kappa = 1 - \left(1 - n^{-2\nu \log(2/\lambda)}\right)^\kappa \\ &\geq 1 - \exp(-n^{\nu \log(2/\lambda)}).\end{aligned} \tag{3.14}$$

Ignition on star graph. We have more work to do this time. The proof of Lemma 3.2.5 requires that $K = \lambda k^{2/3} \to \infty$, and we need the new definition of L in part (iii).

Recall that $T_m^+ = \inf\{n : Y_n \geq m\}$.

Lemma 3.4.3. *Let $K = \lambda k/\sqrt{\log k}$. If $\lambda \to 0$ and $\lambda^2 k \to \infty$ then for large k*

$$(i) P_{0,1}(T_K^+ > T_{0,0}) \le 5/\sqrt{\log k},$$

$$(ii) P_{K,1}(T_{0,0} < T_L^+) \le \exp(-\lambda^2 k/2\sqrt{\log k}),$$

$$(iii) E_{0,1} \min\{T_{0,0}, T_L^+\} \le 2/\varepsilon.$$

Proof. Let $p_0(t)$ be the probability a leaf is infected at time t when there are no infected leaves at time 0 and the central vertex has been infected for all $s \le t$. $p_0(0) = 0$ and

$$\frac{dp_0(t)}{dt} = -p_0(t) + \lambda(1 - p_0(t)) = \lambda - (\lambda + 1)p_0(t).$$

Solving gives

$$p_0(t) = \frac{\lambda}{\lambda + 1}(1 - e^{-(\lambda+1)t}).$$

As $t \to 0$

$$\frac{1 - e^{-(\lambda+1)t}}{(\lambda + 1)t} \to 1,$$

so if t is small $p_0(t) \ge \lambda t/2$.

Taking $t = 4/\sqrt{\log k}$ it follows that the number of infected leaves at time t dominates $B = \text{Binomial}(k, 2\lambda/\sqrt{\log k})$

$$P_{0,1}(T_K^+ < T_{0,0}) \ge P(B > K) \exp(-4/\sqrt{\log k}).$$

The second factor is the probability that the center stays infected until time $4/\sqrt{\log k}$, and

$$\exp(-4/\sqrt{\log k}) \ge 1 - 4/\sqrt{\log k}.$$

B has mean $2\lambda k/\sqrt{\log k}$ and variance $\le 2\lambda k/\sqrt{\log k}$ so Chebyshev's inequality implies

$$P(B < \lambda k/\sqrt{\log k}) \le \frac{2\lambda k/\sqrt{\log k}}{(\lambda k/\sqrt{\log k})^2} \le \frac{2\sqrt{\log k}}{\lambda k} \le 1/\sqrt{\log k}$$

if k is large.

For (ii) we use the supermartingale from Lemma 3.4.1, which is the same as the one from Lemma 3.11, and simplify formulas as in the proof of Lemma 3.4.2. If $q = P_{K,1}(T_{0,0} < T_L^+)$ then for λ small optional stopping theorem gives

$$q \le (1 + \lambda/2)^{-\lambda k/\sqrt{\log k}} \le \exp(-\lambda^2 k/2\sqrt{\log k}).$$

For (iii), we follow the argument in Lemma 3.2.5. We return to continuous time and note that by (3.9) the drift is

$$\leq \mu = \lambda(1-p)k - pk - 1/\lambda$$

so $Y_t - \mu t$ is a submartingale before time $V_L = T_{0,0} \wedge T_L^+$. Using the optional stopping theorem as before we conclude

$$E(V_L) \leq \frac{L}{\mu} = \frac{(1-\varepsilon)\lambda k}{1+\lambda} \cdot \frac{1}{\lambda(1-p)k - pk - 1/\lambda}.$$

Recalling the definition of p

$$\begin{aligned}
\lambda(1-p)k - pk &= \lambda\left[k - \frac{(1-\varepsilon)\lambda k}{1+\lambda}\right] - \frac{(1-\varepsilon)\lambda k}{1+\lambda} - \frac{1}{\lambda} \\
&= \lambda\left[\frac{\varepsilon\lambda k}{1+\lambda}\right] + \frac{\varepsilon\lambda k}{1+\lambda} - \frac{1}{\lambda}.
\end{aligned}$$

The first term is much smaller than the second so multiplying by λ/λ

$$\frac{L}{\mu} \sim \frac{\lambda^2 k}{\varepsilon\lambda^2 k - (1+\lambda)} \sim 1/\varepsilon,$$

since $\lambda k^2 \to \infty$. $\square$

3.4.1 Power law graphs

Suppose $P(d(x) \geq m) = 3^a m^{-a}$ for $m \geq 3$, where $a > 2$ so that $Ed(x)^2 < \infty$. In this case, the maximum degree vertex on a graph with n vertices is $\sim n^{1/a}$, so the maximum eigenvalue $\Lambda \sim n^{1/2a}$ and the formula in (1.5) predicts that $\lambda_c \approx n^{-1/2a}$. To prove an upper bound on λ_c that is close to this, we suppose that $\lambda_0 = n^{-(1-2\eta)/2a}$.

If $d(x) \geq k = n^{(1-\eta)/a}$ we call the vertex x a star.

$$P(d(x) \geq n^{(1-\eta)/a}) = 3^a n^{-(1-\eta)}$$

so if n is large there are $\geq n^\eta$ stars with high probability. Now $\lambda_0^2 k = n^{\eta/a}$. By the estimate in the Lemma 3.4.2, each individual star survives for time

$$\geq \exp((1-4\eta)n^{\eta/a}/4). \tag{3.15}$$

with probability $\geq 1 - 7\exp(-(1-3\varepsilon)n^{\eta/a}/4)$. The time

$$2r\kappa \leq (4\nu \log n)\exp(O(\log^2 n))$$

so (3.14) implies that with high probability the chosen star will transfer its infection to its target by time $2r\kappa$ and we conclude that with high probability no lit star will die out during the process.

Combining these estimates shows that if n is large then the number of infected stars Y_k at time $2r\kappa k$ dominates a discrete time random walk that goes up by 1 with probability $p > e/(e+1)$ and down by 1 with probability $1-p$. Let $M \geq n^\eta$ be the number of stars. Recalling that $((1-p)/p)^x$ is harmonic function for a simple random walk that jumps up 1 with probability p, and down 1 with probability $1-p$ random walk, we see that $\exp(-Y_k)$ is a supermartingale while $Y_k \in (0, M)$, so

$$P_{0.9M}(T_0 < T_M) \leq e^{-0.9M}.$$

Since each cycle takes at least $0.1M(2r\kappa)$ units of time, we have survival for time $\exp(O(n^\varepsilon))$ for some $\varepsilon > 0$.

3.4.2 Stretched exponential

Suppose $P(d(x) \geq m) = \exp(-m^{1/b} + 3^{1/b})$ for $m \geq 3$, where $b > 1$. In this case, the maximum degree vertex on a graph with n vertices is $\sim \log^b n$, so the maximum eigenvalue $\Lambda \sim \log^{b/2} n$ and the formula in (1.5) predicts that $\lambda_c \approx \log^{-b/2} n$.

If $d(x) \geq k = \eta^b \log^b n$ we call the vertex x a star.

$$P(d(x) \geq \eta^b \log^b n) = \exp(3^{1/b})n^{-\eta},$$

so if n is large then the number of stars is $\geq n^{1-\eta}$ with high probability.

To see what value to take for λ in our lower bound, we set the survival time equal to 1 over the probability of a successful push, that is

$$\exp(\lambda^2 \log^b n) = (2/\lambda)^{2\nu \log n},$$

or taking logs and rearranging

$$\frac{\lambda^2}{\log(2/\lambda)} = 2\nu \log^{1-b} n.$$

This means that the best upper bound we can hope to get is $\lambda_0 = (\log n)^{(1-\eta)(1-b)/2}$ versus the predicted value of $\log^{-b/2} n$.

By our choices we have

$$\lambda_0^2 k = \eta^b (\log n)^{1+\eta(b-1)}$$

so Lemma 3.4.2 implies that the star survives for time

$$\geq \exp((1-4\eta)\eta^b(\log n)^{1+\eta(b-1)}/4)$$

with probability $\geq 1 - 7\exp(-(1-3\eta)\eta^b(\log n)^{1+\eta(b-1)}/4)$. The time

$$2r\kappa \leq (4\nu \log n)\exp(\log n \cdot O(\log\log n))$$

so (3.14) implies that with high probability the chosen star will transfer its infection to its target by time $2r\kappa$ and we conclude that with high probability no lit star will die out during the process. Comparing with random walk as in the previous proof, we have survival for time $\exp(O(n^{1-\varepsilon}))$ for any $\varepsilon > 0$.

Chapter 4

Exponential Growth and Continuous Phase Transitions on Trees

4.1 Introduction

Pemantle [32] began the study of the contact process on trees and found that there exist two critical values

$$\lambda_1 = \inf\{\lambda : P(\xi_t^0 \neq \varnothing \text{ for all } t) > 0\}$$
$$\lambda_2 = \inf\{\lambda : \liminf_{t\to\infty} P(0 \in \xi_t^0) > 0\},$$

where ξ_t^0 denotes the contact process on the tree starting from only the root infected. The contact process is said to survive weakly if the process survives but the root 0 is infected for finitely many times almost surely, and survive strongly if the root 0 is infected for infinitely many times with positive probability.

By deriving bounds on the critical values, Pemantle showed that $\lambda_1 < \lambda_2$ when $d \geq 3$. Liggett [20] settled the case $d = 2$ by showing $\lambda_1 < 0.605 < 0.6609 < \lambda_2$. At about the same time, Stacey [34] gave a proof that $\lambda_1 < \lambda_2$ that did not rely on bounds. It has also been shown, see [22], that the phase transitions on regular trees are continuous, that is, the contact process on $\mathbb{T}^d$ dies out at λ_1 and does not survive strongly almost surely at λ_2.

A special function

$$w_\rho(\xi_t) = \sum_{x\in\xi_t} \rho^{\ell(x)}$$

was introduced by Stacey [34] to study the contact process on regular trees, which exploits the translation invariance that is natural to regular spatial structures. ρ is some positive constant and $\ell(x)$ is a function from $\mathbb{T}^d$ to $\mathbb{Z}$ so that for each $x \in \mathbb{T}^d$, $\ell(y) = \ell(x) - 1$ for exactly one neighbor y of x and $\ell(y) = \ell(x) + 1$ for the other d neighbors y of x. The expectation of $w_\rho(\xi_t)$ satisfies a submultiplicative relation

$$Ew_\rho(\xi_{t+s}) \leq Ew_\rho(\xi_t)Ew_\rho(\xi_s)$$

which implies $Ew_\rho(\xi_t) \geq [\phi(\rho)]^t$ where $\phi(\rho) = \lim_{t\to\infty}[Ew_\rho(\xi_t)]^{1/t}$.

Properties of $\phi(\rho)$ can be used to obtain a lot of detailed information about the behavior of the contact process on regular trees. For example, it has been shown that

Proposition 4.27 in [22] *There is a constant $C(\rho) \leq \infty$ depending only on d and ρ so that*

$$[\phi(\rho)]^t \leq Ew_\rho(\xi_t^0) \leq C(\rho)[\phi(\rho)]^t, \quad t \geq 0,$$

where ξ_t^0 is the contact process with the root initially infected. One can take $C(\rho) < \infty$ if $\rho \neq 1/\sqrt{d}$.

For $\lambda > \lambda_1$, it is known that $\phi(1) > 1$. By Proposition 4.27 in [22] this leads to exponential growth for supercritical contact process. Moreover, $\phi(\rho)$ is also useful in terms of understanding the critical behavior of the contact process.

Proposition 4.39 in [22] *If $\lambda = \lambda_1$, then $\phi(1) = 1$ and hence*

$$\sup_{t>0} E|\xi_t^0| < \infty$$

and ξ_t^0 dies out.

In this chapter, we are interested in extending these results on regular trees to more general trees such as the Galton-Watson trees and periodic trees, and obtain answers to the following questions:

(i) proving exponential growth rate for the process in supercritical regime ($\lambda > \lambda_1$),

(ii) understanding the behavior of the contact process at the critical values λ_1 and λ_2,

(iii) determining if $\lambda_1 < \lambda_2$ (the existence of an intermediate phase).

4.2 Model definition

The set of all realizations of the Galton-Watson trees with offspring distribution D is denoted by $\mathbf{GW}(D)$ while a certain realization is denoted by $\mathbb{T} \in \mathbf{GW}(D)$. Condition (1.7) guarantees that $\mathbb{T}$ is infinite almost surely. Let $P_{\mathbb{T},\lambda}$ be the law of the contact process with infection rate λ on the tree $\mathbb{T}$, and let μ be the probability measure over the realizations in $\mathbf{GW}(D)$. We define the annealed measure P_λ to be

$$P_\lambda(\ \cdot\ ) = E_{\mathbb{T}\sim\mu}(P_{\mathbb{T},\lambda}(\ \cdot\ )) = \sum_{\mathbb{T}\in\mathbf{GW}(D)} P_{\mathbb{T},\lambda}(\ \cdot\ )\mu(\mathbb{T}).$$

To simplify notation we will drop the index λ when the context is clear and use $\mathbf{GW}(D)$ to also represent a Galton-Watson tree with offspring distribution D.

Let ξ_t^0 denote the contact process on $\mathbf{GW}(D)$ with initially only the root infected. The critical value for weak survival for the contact process on $\mathbb{T}$ is defined as

$$\lambda_1(\mathbb{T}) = \inf\{\lambda : P_{\mathbb{T},\lambda}(\xi_t^0 \neq \varnothing \text{ for all } t \geq 0) > 0\}.$$

In fact, $\lambda_1(\mathbb{T})$ does not depend on the specific realization $\mathbb{T} \in \mathbf{GW}(D)$. That is, $\lambda_1(\mathbb{T})$ is a constant μ-almost surely. Pemantle proved a slightly different version of this result in Proposition 3.1 in [32]. We will follow his proof to show

Proposition 4.2.1. *$\lambda_1(\mathbb{T})$ is a constant μ-almost surely.*

Proof. Let $q(\mathbb{T}) = P_{\mathbb{T},\lambda}(\xi_t^0 \neq \varnothing\ \forall t \geq 0)$ and $p_1 = P_\lambda(q = 0)$. If the root has n children then there are n independent identically distributed subtrees that are themselves Galton-Watson trees with offspring distribution D, which we denote by $\mathbb{T}_1, \dots, \mathbb{T}_n$. If $q(\mathbb{T}_i) > 0$ for some $1 \leq i \leq n$ then $q(\mathbb{T}) \geq \frac{\lambda}{1+\lambda} q(\mathbb{T}_i) > 0$, where the first term is the probability to push the infection to the root of $\mathbb{T}_i$. It follows that if the contact process dies out on $\mathbb{T}$ then it has to die out on all n subtrees $\mathbb{T}_1, \dots, \mathbb{T}_n$.

Let f be the generating function of D and let $a_k = P(D = k)$. By the independence of the structures of the subtrees,

$$p_1 = P_\lambda(q = 0) \leq \sum_n a_n (p_1)^n = f(p_1) \tag{4.1}$$

Note that $f'(1) = ED > 1$ by (1.7). (4.1) combined with $f'(1) > 1$ and $f(0) = 0$ implies either $p_1 = 1$ or p_1 is bounded above by the unique fixed point of f in $(0, 1)$, which then gives $p_1 = 0$. Hence $\lambda_1(\mathbb{T})$ is a constant μ-almost surely. $\square$

The critical value for strong survival is defined as

$$\lambda_2(\mathbb{T}) = \inf\{\lambda : P_{\mathbb{T},\lambda}(0 \in \xi_t^0 \ \text{ infinitely often}) > 0\}.$$

It follows from the same reasoning as in the proof of Proposition 4.2.1 that

Proposition 4.2.2. $\lambda_2(\mathbb{T})$ *is a constant* μ*-almost surely.*

From now on we write $\lambda_1(\mathbf{GW}(D))$ and $\lambda_2(\mathbf{GW}(D))$ for the critical values for the contact process on a Galton Watson tree with offspring distribution D.

4.3 Exponential growth

In this section we prove Theorem 1.4.1 by showing that the contact process dominates a supercritical Crump-Mode-Jagers branching process. On an inhomogeneous structure such as the Galton-Watson tree, the infection state of a vertex x is correlated with the tree structure near x. To address this issue we observe that starting with only the root infected, when a vertex x becomes infected for the first time no information is known about the structure of the subtree rooted at x (i.e., the subtree made of x and all of its descendants). Hence we can "decouple" the infection state of x and the tree structure below it.

For a vertex $x \in \mathbb{T}$ we define $S(x)$ to be the subtree with root x, that is, $S(x)$ contains x together with all the descendants of x on $\mathbb{T}$. For a finite set $A \subset \mathbb{T}$, we define its frontier $F(A)$ to be the set of points $x \in A$ for which at least one of its children, say x', has $S(x') \cap A = \varnothing$. Let A' denote the set of x' such that x' is the child of some $x \in A$ and $S(x') \cap A = \varnothing$.

Let $A_t = \cup_{s \le t} \ \xi_s^0$ be the set of vertices ever infected by time t. We are interested in the set of vertices that have never been infected and are accessible to the infection ξ_t^0 at time

t, i.e.,

$$B_t = \{x \in (A_t)' : \text{ the parent of } x \text{ is in } \xi_t^0 \cap F(A_t)\}.$$

Let $\tau_k = \inf\{t \geq 0 : |\xi_t^0 \cap F(A_t)| \geq k\}$. The fact that each vertex in $|\xi_t^0 \cap F(A_t)|$ corresponds to at least one vertex in B_t gives $|B_{\tau_k}| \geq k$, i.e., there are at least k unexplored subtrees accessible to the contact process $\xi_{\tau_k}^0$. At time τ_k, ξ_t^0 can try to start infections on each of the k subtrees, among which each infection could survive forever with a positive probability equal to $P_\lambda(\xi_t^0 \neq \varnothing \ \forall t \geq 0)$. When k is taken to be sufficiently large, at time τ_k the contact process ξ_t^0 gives birth to an expected number of more than one new process that lives forever, which make up an underlying branching process.

Recall that $\Omega_\infty = \{\xi_t^0 \neq \varnothing \ \forall t \geq 0\}$. We will begin the proof of Theorem 1.4.1 by showing

Lemma 4.3.1. *Suppose $\lambda > \lambda_1(\boldsymbol{GW}(D))$. If for every $k \in \mathbb{N}$, $\tau_k < \infty$ P_λ-almost surely on Ω_∞ then* (1.8) *holds.*

Proof. When $\lambda > \lambda_1(\mathbf{GW}(D))$ we have $\rho := P_\lambda(\xi_t^0 \neq \varnothing \ \forall t \geq 0) > 0$. Observe that with probability $\geq e^{-2}(1 - e^{-\lambda})$ an infected vertex $x \in \xi_t^0 \cap F(A_t)$ can infect its child $x' \in B_t$ within time 1 and x' will stay infected until time 1.

Since the subtree $S(x')$ is still unexplored when $x' \in B_t$ first becomes infected, we can first generate $S(x')$ according to the measure μ of $\mathbf{GW}(D)$ and then start a contact process on it. The probability that the process $\xi_t^{x'}$ restricted to $S(x')$ will survive is

$$E_{S(x') \sim \mu}(P_{S(x'),\lambda}(\xi_t^{x'} \neq t \text{ for all } t \geq 0)) = \rho.$$

If this infection on $S(x')$ survives, we say x' is a *particle with an infinite line of descendants*. Note that each particle with an infinite line of descendants will contribute at least size 1 to $|\xi_t^0|$ at all times. So it suffices to show the number of particles with an infinite line of descendants grows exponentially in time on the event Ω_∞.

We will construct a branching process Z_t that is dominated by the set of particles that has an infinite line of descendants. Let Z_0 contain only the root 0. Since $\{\tau_k < \infty\}$ almost surely on Ω_∞ we can choose M_1 large so that $P(\tau_k < M_1) \geq \rho/2$. If $\tau_k \geq M_1$ we discard the initial particle and there are no offspring. If $\tau_k < M_1$ then the initial particle gives birth to $X \sim Binomial(k, e^{-2}(1-e^{-\lambda}))$ offspring at time τ_k. This is because the birth events of different vertices in $\xi_t^0 \cap F(A_t)$ are independent, since they involve vertices that are disjoint in space. If k is chosen large enough then the number of offspring $\bar{X}$ that will start a new process on its subtree with $\tau_k < M_1$ has expectation

$$E\bar{X} \geq ke^{-2}(1-e^{-\lambda})\rho/2 > 1,$$

which implies that Z_t is supercritical.

The branching process is a Crump-Mode-Jagers (CMJ) branching process [**?**] because the initial particle gives birth at time τ_k. If $N(t)$ is the expected number of births by time t then the Malthusian parameter c is defined through

$$\int_0^\infty e^{-ct} dN(t) = 1.$$

Since our process has no births after time M_1 it is easy to see that it satisfies the conditions in Theorem 2 of [**?**], which gives

$$W(t) := \frac{Z_t}{e^{ct}} \to W \qquad \text{a.s. as } t \to \infty,$$

where W is a non-negative random variable with $P(W > 0) > 0$. To improve this to the desired conclusion we note that for any $m < \infty$, $\tau_m < \infty$ almost surely on Ω_∞. At time τ_m, each vertex in $\xi_t^0 \cap F(A_t)$ can start an independent branching process construction with probability at least $e^{-2}(1-e^{-\lambda})$ and each branching process will live forever with probability $P(W > 0)$. As long as one of the branching processes lives forever we will have the desired exponential growth of ξ_t^0. Hence

$$\begin{aligned} P\left(\liminf_{t\to\infty} |\xi_t^0|/e^{ct} > 0\right) &\geq P(\Omega_\infty) P(Binomial(m, e^{-2}(1-e^{-\lambda})P(W>0)) \geq 1) \\ &= P(\Omega_\infty)\left(1 - \left(1 - e^{-2}(1-e^{-\lambda})P(W>0)\right)^m\right) \end{aligned}$$

for any $m \in \mathbb{N}$. Letting $m \to \infty$ and noting that the limit can only be positive on Ω_∞ proves the desired result. □

Having established Lemma 4.3.1, to complete the proof of Theorem 1.4.1 it remains to show

Lemma 4.3.2. *For any $k \in \mathbb{N}$, $\tau_k < \infty$ almost surely on Ω_∞.*

Proof. In a contact process, an infected site is also referred to as a particle. The event that an infected site passes the infection to a susceptible site is also described as a particle gives birth to another particle. The recovery of an infected site is hence also referred to as the death of a particle. In this proof we adopt the narrative of the particles.

If some particle $x \in \xi_t^0 \cap F(A_t)$ gives birth to a particle $y \in B_t$, then at the moment of this birth y is added to $\xi_t^0 \cap F(A_t)$. In the worst case y is the last unexplored child of x, so x is removed from $\xi_t^0 \cap F(A_t)$ and the size of $\xi_t^0 \cap F(A_t)$ remains the same. Note that the size of $\xi_t^0 \cap F(A_t)$ will not decrease at a birth event.

When $y \in B_t$ becomes infected, we try to grow $|\xi_t^0 \cap F(A_t)|$ through consecutive birth events on the subtree $S(y)$. Since $S(y)$ has not been explored by the infection, every vertex in $S(y)$ still has all of its offspring unexplored. When a children of y (except the last one) is infected, the size of $\xi_t^0 \cap F(A_t)$ increases by 1. Hence if every particle gives birth onto a particle with offspring number at least 2, we need at most $2k$ consecutive birth events on $S(y)$ for $|\xi_t^0 \cap F(A_t)|$ to reach size k.

Taking account of the first birth event from x to y, we need $2k + 1$ consecutive birth events before any death occurs and each of them is onto a vertex with offspring number at least 2. Since each vertex in $S(y) \cap \xi_t^0 \cap F(A_t)$ is connected to at least one vertex outside, the probability that a birth occurs before any death in $S(y) \cap \xi_t^0 \cap F(A_t)$ has probability $\geq \lambda/(1 + \lambda)$.

We start the first trial at the first time when $|\xi_t^0 \cap F(A_t)| \geq 1$, i.e., when $t = 0$. A trial is said to have failed if a death occurs or a birth to a vertex with offspring number 1 occurs before we have $2k + 1$ consecutive birth events to vertices with offspring number at least 2. Define the running time S_i of the i-th trial to be the first time either this trial fails or succeeds. S_i is dominated by the sum of $2k + 1$ i.i.d. $Exp(1 + \lambda)$ random variables and hence is finite almost surely. Let $T_0 = 0$ and define

$$T_i = \inf\{t \geq T_{i-1} + S_{i-1} : |\xi_t^0 \cap F(A_t)| \geq 1\}$$

to be the time when the i-th trial starts. If $T_i = \infty$ for some $i \geq 1$ then set $T_j = \infty$ for all $j \geq i$. Note that with an independent probability

$$\geq \left(\frac{\lambda}{1+\lambda} \cdot P(D \geq 2)\right)^{2k+1} \equiv \alpha_k$$

a trial will succeed, so it takes finitely many attempts to have one success, which then gives $|\xi_t^0 \cap F(A_t)| \geq k$. If the i-th trial fails, either we still have $|\xi_t^0 \cap F(A_t)| \geq 1$, which means $T_{i+1} = T_i + S_i$, or $|\xi_t^0 \cap F(A_t)| = 0$ and we have to wait until $|\xi_t^0 \cap F(A_t)|$ becomes 1 before starting the next trial. On the event that ξ_t^0 survives, infinitely many sites will be infected, i.e.,

$$\lim_{t \to \infty} |A_t| = \infty \quad \text{ a.s. on } \Omega_\infty.$$

Since A_t cannot grow when $\xi_t^0 \cap F(A_t) = \varnothing$ we have $|\xi_t^0 \cap F(A_t)| \geq 1$ infinitely often on the event Ω_∞, meaning that for any $n \in \mathbb{N}$

$$\{T_n < \infty\} \text{ a.s. on } \Omega_\infty.$$

Hence we have infinitely many trials almost surely on the event of survival. Since the birth and death symbols involved in different trials come from regions disjoint in space and time, each trial succeeds with probability $\geq \alpha_k$ independent of the fates of previous trials. Therefore we will have a success in finite time almost surely on Ω_∞. The proof of Lemma 4.3.2 is complete and Theorem 1.4.1 has been established. $\square$

Remark 1. *Our method is robust in the sense that it works on a broad class of trees. For example, in order to prove exponential growth for the contact process on periodic trees it suffices to extend the proof of Lemma 4.3.2 to periodic trees. The key element in the proof of Lemma 4.3.2 is the expansion of the infected frontier $\xi_t^0 \cap F(A_t)$ through consecutive birth events, which holds true for periodic trees too. It is easy to see that if $a_i \geq 2$ for all $1 \leq i \leq \kappa$ in the periodic tree $\mathbb{T}_\kappa = (a_1, \ldots, a_\kappa)$, then each birth event in the infected* *frontier will cause the size $|\xi_t^0 \cap F(A_t)|$ to increase by 1. If we only assume $a_1 a_2 \cdots a_\kappa \neq 1$, then we need κ consecutive birth events in the frontier to ensure that the size $|\xi_t^0 \cap F(A_t)|$ increases by 1.*

4.4 Continuous survival probability

In the branching process Z_t in the proof of Lemma 4.3.1, a particle can only give birth within time M_1, which means we are only looking at a finite time "block" in the corresponding graphical representation. We drop the superscript 0 and let ξ_t^λ denote the contact process with infection rate λ and recovery rate 1 starting from the root infected. Within this "block" the process ξ_t^λ and $\xi_t^{\lambda'}$ should behave very much the same if λ' is sufficiently close to λ. If ξ_t^λ survives with positive probability, tuning down the infection rate λ slightly to λ' the resulting process $\xi_t^{\lambda'}$ should still behave similarly in the finite time "block", which then implies the survival of $\xi_t^{\lambda'}$. We will follow this idea to prove both results in this section. A superscript λ is used to denote the infection rate. For example, τ_k^λ is the corresponding hitting time of the event $\{|\xi_t^\lambda \cap F(A_t)| \geq k\}$.

Lemma 4.4.1. *The contact process on $\boldsymbol{GW}(D)$ dies out at $\lambda_1(\boldsymbol{GW}(D))$.*

Proof. To prove the conclusion it suffices to show $P_\lambda(\xi_t^0 \neq \varnothing \text{ for all } t \geq 0) > 0$ implies $\lambda > \lambda_1$. Let $\rho := P_\lambda(\xi_t^0 \neq \varnothing \text{ for all } t \geq 0) > 0$. For reasons that will become clear later we choose k, M_1 so that

$$ke^{-2}(1 - e^{-\lambda})\rho/2 > 2 \quad \text{and} \quad P(\tau_k^\lambda < M_1) \geq \rho/2. \tag{4.2}$$

For a small $\delta > 0$ we have

$$P(\tau_k^{\lambda-\delta} < M_1) \geq P(\{\xi_t^\lambda = \xi_t^{\lambda-\delta} \text{ for all } t \leq \min\{\tau_k^\lambda, M_1\}\} \cap \{\tau_k^\lambda < M_1\}).$$

For any $\varepsilon > 0$, when δ is sufficiently small we have

$$P(\xi_t^\lambda \neq \xi_t^{\lambda-\delta} \text{ for some } t \leq \min\{\tau_k^\lambda, M_1\}) < \varepsilon.$$

Hence,

$$P(\tau_k^{\lambda-\delta} < M_1) \geq 1 - \varepsilon - (1 - \rho/2) = \rho/2 - \varepsilon.$$

Following the proof of Lemma 4.3.1 we can construct a branching process Z_t dominated by $\xi_t^{\lambda-\delta}$, where a particle gives birth to $X \sim Binomial(k, e^{-2}(1 - e^{-\lambda-\delta}))$ offspring if $\tau_k^{\lambda-\delta} < M_1$, and to no offspring if $\tau_k^{\lambda-\delta} \geq M_1$. When ε and δ are sufficiently small, by the choice of k in (4.2) the mean number of offspring is

$$EX \geq ke^{-2}(1 - e^{-\lambda-\delta})(\rho/2 - \varepsilon) > 1.$$

Hence Z_t is supercritical and survives with positive probability, which implies $\xi_t^{\lambda-\delta}$ survives with positive probability. So $\lambda > \lambda - \delta \geq \lambda_1$ and this completes the proof. □

Let $p(\lambda) := P_\lambda(\xi_t^0 \neq \varnothing \ \forall t \geq 0)$ be a function of λ that represents the survival probability of the contact process ξ_t^λ.

Lemma 4.4.2. *$p(\lambda)$ is continuous on $[0, \infty)$.*

Proof. The right continuity of $p(\lambda)$ is immediate since

$$P_\lambda(\xi_t^0 \neq \varnothing) \downarrow p(\lambda)$$

as $t \uparrow \infty$ and $P_\lambda(\xi_t^0 \neq \varnothing)$ is increasing and continuous in λ for each t.

We have proved that $p(\lambda) \equiv 0$ on $[0, \lambda_1]$, so it remains to prove the left continuity for $p(\lambda)$ on (λ_1, ∞). Take $\lambda > \lambda_1$, which trivially gives $p(\lambda) > 0$. We will show for any $\varepsilon > 0$, there exists $\lambda' < \lambda$ such that $p(\lambda') > p(\lambda) - \varepsilon$.

Since $\lambda > \lambda_1$ there exists λ_0 such that $\lambda > \lambda_0 > \lambda_1$. It follows from the same argument as in the proof of Lemma 4.4.1 that we can construct a supercritical branching process Z_t that is dominated by $\xi_t^{\lambda_0}$. Moreover, there exists some constant $c > 0$ and a non-negative random variable W_0 with $P(W_0 > 0) > 0$ such that

$$\frac{Z_t}{e^{ct}} \to W_0 \quad \text{as } t \to \infty.$$

To simplify notation we will write $\sigma = e^{-2}(1-e^{-\lambda_0})P(W_0 > 0)$. For a given $\delta > 0$ we will choose m so that $(1-\sigma)^m < \delta$. Since $\tau_m^\lambda < \infty$ almost surely on $\Omega_\infty \equiv \{\xi_t^\lambda \neq \varnothing \ \forall t \geq 0\}$, we can choose M_2 sufficiently large so that

$$P(\tau_m^\lambda \leq M_2) \geq (1-\delta)p(\lambda). \tag{4.3}$$

When $\lambda' < \lambda$ is sufficiently close to λ we also have

$$P(\xi_t^\lambda \neq \xi_t^{\lambda'} \text{ for some } t \leq M_2) < \delta. \tag{4.4}$$

Without loss of generality we can assume $\lambda' > \lambda_0$. Observe that each vertex in the frontier $\xi_t^{\lambda'} \cap F(A_t)$ can start an independent branching process with probability $\geq e^{-2}(1-e^{-\lambda_0})$ and each branching process lives forever with probability at least $P(W_0 > 0)$ since $\lambda' > \lambda_0$. On the event $\{\xi_t^\lambda = \xi_t^{\lambda'} \text{ for } t \leq M_2\} \cap \{\tau_m^\lambda \leq M_2\}$, the process $\xi_t^{\lambda'}$ can give birth to $X \sim Binomial(m,\sigma)$ branching processes that survive. If $X \geq 1$ then the process $\xi_t^{\lambda'}$ survives since it dominates a surviving branching process. Hence

$$p(\lambda') \geq P\left(\{\xi_t^\lambda = \xi_t^{\lambda'} \text{ for } t \leq M_2\} \cap \{\tau_m^\lambda \leq M_2\}\right) P(Binomial(m,\sigma) \geq 1) \tag{4.5}$$

It follows from (4.3) and (4.4) that when λ' is sufficiently close to λ

$$P\left(\{\xi_t^\lambda = \xi_t^{\lambda'} \text{ for } t \leq M_2\} \cap \{\tau_m^\lambda \leq M_2\}\right) \geq (1-\delta)p(\lambda) - \delta,$$

which then gives

$$p(\lambda') \geq ((1-\delta)p(\lambda)-\delta)(1-(1-\sigma)^m) \geq ((1-\delta)p(\lambda)-\delta)(1-\delta).$$

Therefore, for any given $\varepsilon > 0$ there exists δ sufficiently small and λ' sufficiently close to λ so that $p(\lambda') > p(\lambda) - \varepsilon$. $\square$

Remark 2. *The same construction can be carried out on periodic trees straightforwardly, thus proving the continuity of survival probability $p(\lambda)$ for the contact process on periodic trees. As a consequence, the contact process dies out at λ_1 on periodic trees.*

4.5 No strong survival at λ_2

We start by introducing some new notation. Let $\mathbf{GW}^+(D)$ be the set of all realizations of the "root-added" Galton-Watson tree in which the root 0 has offspring number 1 and its only offspring is the root of a Galton-Watson tree with offspring distribution D. A realization in $\mathbf{GW}^+(D)$ is denoted by $\mathbb{T}^+$ and the measure over the realizations by ν.

Define the measure $P_{\lambda,+}$ to be

$$P_{\lambda,+}(\,\cdot\,) = E_{\mathbb{T}^+\sim\nu}(P_{\mathbb{T}^+,\lambda}(\,\cdot\,)) = \sum_{\mathbb{T}^+\in\mathbf{GW}^+(D)} P_{\mathbb{T}^+,\lambda}(\,\cdot\,)\nu(\mathbb{T}^+).$$

We will drop the index λ when the context is clear. The contact process on $\mathbf{GW}^+(D)$ with with root 0 initially infected will be denoted by η_t^0 while the contact process on $\mathbf{GW}(D)$ will still be denoted by ξ_t^0.

From now on we set $A_t = \cup_{0\le s\le t}\ \eta_s^0$. Let $|v|$ denote the distance from vertex v to the root 0. For a vertex $x \in \eta_t^0 \cap F(A_t)$, define a branch of x to be

$$S^+(x) = \{x\} \cup S(x'),$$

where x' can be any child of x that has not been infected by time t. Observe that $S^+(x)$ follows the distribution of $\mathbf{GW}^+(D)$.

The strategy of proving Theorem 1.4.3 is to identify a tractable process embedded in the contact process that would infect the root infinitely often when the contact process survives strongly. To find this embedded process it helps to look at the the macroscopic behavior of the contact process when $\lambda > \lambda_2$.

Observe that for a sufficiently large j a lot of vertices on level j will be infected at a certain time jt_1 with probability δ. This observation is intuitive if we think about the contact process on regular trees and will be proved in Lemma 4.5.1. If there are sufficiently many of these vertices we should be able to use them to pass the infection back to the root with a sufficiently large probability q. Then we partition the Galton-Watson tree into blocks of depth j. The root of each block (since its first infection) will be infected after time $jt_1$with probability at least δq. The nice thing about trees is that at distance jn there will be $O(e^{cn})$ such blocks so that the success probability of pushing the infection from distance jn to $(j-1)n$ is sufficiently large. For this reason we can use a recursive relation to show that in the embedded process the probability that the root is infected infinitely often is positive.

Having identified the embedded process (which is essentially a block construction), we will slightly tune down the infection rate λ and show the embedded process with a reduced infection rate $\lambda-\epsilon$ still can infect the root infinitely often. The continuity of phase transition at λ_2 is thus established by a classical argument of contradiction.

The first step is to establish the observation that a lot of vertices on level j will be infected at a certain time jt_1 with probability δ. This will come in handy when we estimate the probability of successfully pushing the infection back to the root. Specifically, we consider the infected vertices in the frontier $F(A_t)$ in order to obtain the independence between the subtree structure and the infection state of its root.

Despite the seemingly complicated statement, the proof of Lemma 4.5.1 is standard calculation of Binomial random variable.

Lemma 4.5.1. *If $\lambda > \lambda_1$, there exists $\alpha, \beta, \delta > 0$ and t_1 such that for all $j \geq 1$,*

$$P_{\lambda,+}(|\eta^0_{jt_1} \cap F(A_{jt_1}) \cap \{v : j\alpha t_1 \leq |v| \leq j\beta t_1\}| \geq e^{jc_\lambda t_1}\}) \geq \delta,$$

where c_λ is a positive number which may depend on λ.

Proof. Let $\theta = P_+(\eta^0_t \neq \varnothing$ for all $t \geq 0)$. It follows from the proof of Lemma 4.3.1 that there exists $c > 0$ such that when t_1 is sufficiently large

$$P_+(|\eta^0_{t_1} \cap F(A_{t_1})| \geq e^{ct_1}) \geq \frac{3}{4}\theta.$$

For reasons that will become clear later, we choose t_1 large enough so that the following also holds

$$(\theta e^{ct_1})/8 > 1 \tag{4.6}$$

Given the choice of t_1 we can then choose α small and β large so that

$$P_+(|\eta^0_{t_1} \cap F(A_{t_1}) \cap \{v : \alpha t_1 \leq |v| \leq \beta t_1\}| \geq e^{ct_1}) \geq \frac{1}{2}\theta \equiv \delta. \tag{4.7}$$

A vertex x is said to be good if the event in (4.7) with probability δ occurs for the contact process on $S^+(x)$ with x initially infected. If the root 0 is good, we say the root is a good vertex at level 0. The good vertices in $\eta^0_{jt_1} \cap F(A_{jt_1}) \cap \{v : j\alpha t_1 \le |v| \le j\beta t_1\}$ are called the good vertices at level j. Let N_j be the number of good vertices at level j. Trivially we have

$$|\eta^0_{jt_1} \cap F(A_{jt_1}) \cap \{v : j\alpha t_1 \le |v| \le j\beta t_1\}| \ge N_j.$$

Hence it suffices to estimate N_j. Note that the good vertices in $\eta^0_{t_1} \cap F(A_{t_1}) \cap \{v : \alpha t_1 \le |v| \le \beta t_1\}$ correspond to disjoint branches, so N_1 dominates a $Binomial(e^{ct_1}, \delta)$ random variable if the root 0 is good.

For a random variable $X \sim Binomial(n, p)$, Chebyshev's inequality gives

$$P\left(X \le \frac{np}{2}\right) \le \frac{Var(X)}{(np/2)^2} \le \frac{4}{np}. \tag{4.8}$$

By (4.8) with $n = e^{ct_1}$ and $p = \delta$,

$$\begin{aligned} P\left(N_1 \le \frac{\delta e^{ct_1}}{2} \middle| N_0 = 1\right) &\le P\left(Binomial(e^{ct_1}, \delta) \le \frac{\delta e^{ct_1}}{2}\right) \\ &\le (4/\delta)e^{-ct_1}, \end{aligned}$$

where $(4/\delta)e^{-ct_1} < 1$ by (4.6). The branches of good vertices at level 1 are disjoint, so the random variable N_2 dominates $Binomial(N_1 e^{ct_1}, \delta)$, which gives

$$\begin{aligned} P\left(N_2 \le \left(\frac{\delta e^{ct_1}}{2}\right)^2 \middle| N_1 \ge \frac{\delta e^{ct_1}}{2}\right) &\le P\left(Binomial\left(\frac{\delta e^{2ct_1}}{2}, \delta\right) \le \left(\frac{\delta e^{ct_1}}{2}\right)^2\right) \\ &\le 2\left(\frac{2e^{-ct_1}}{\delta}\right)^2 \le \left((4/\delta)e^{-ct_1}\right)^2. \end{aligned}$$

It follows from the Markov property that N_{i+1} depends only on N_i for $i \ge 0$. Hence, for any $j \ge 1$

$$\begin{aligned} P\left(N_j \ge \left(\frac{\delta e^{ct_1}}{2}\right)^j\right) &\ge \delta P\left(N_1 \ge \frac{\delta e^{ct_1}}{2}\right) \prod_{i=1}^{j-1} P\left(N_{i+1} \ge \left(\frac{\delta e^{ct_1}}{2}\right)^{i+1} \middle| N_i \ge \left(\frac{\delta e^{ct_1}}{2}\right)^i\right) \\ &\ge \delta \prod_{i=1}^{j} \left(1 - \left((4/\delta)e^{-ct_1}\right)^i\right) \ge \delta\sigma \end{aligned}$$

where $\sigma \equiv \prod_{i=1}^{\infty}\left(1-((4/\delta)e^{-ct_1})^i\right) > 0$. Since $(\theta e^{ct_1})/8 > 1$ there exists $0 < c_\lambda < c$ such that

$$e^{c_\lambda t_1} < 2 < \frac{\theta e^{ct_1}}{4} = \frac{\delta e^{ct_1}}{2}.$$

The last equality follows from the definition $\delta = \theta/2$. Hence for any $j \geq 1$, $P(N_j \geq e^{jc_\lambda t_1}) \geq \delta\sigma > 0$. □

We are now ready to prove Theorem 1.4.3, i.e., the contact process on $\mathbf{GW}(D)$ does not survive strongly at $\lambda_2(\mathbf{GW}(D))$.

Proof of Theorem 1.4.3. Since $P_{\lambda_2}(0 \in \xi_t^0 \text{ i.o.}) = 0$ if and only if $P_{\lambda_2,+}(0 \in \eta_t^0 \text{ i.o.}) = 0$, it suffices to show the latter. Our goal is to show that $P_{\lambda,+}(0 \in \eta_t^0 \text{ i.o.}) > 0$ implies $\lambda > \lambda_2$. Now suppose $P_{\lambda,+}(0 \in \eta_t^0 \text{ i.o.}) > 0$. Theorem 4.4.1 implies that $\lambda > \lambda_1$, so there exists λ_0 such that $\lambda > \lambda_0 > \lambda_1$. Applying Lemma 4.5.1 to the contact process with infection rate λ_0, there exists positive c_{λ_0} so that for all $j \geq 1$,

$$P_{\lambda_0,+}(|\eta_{jt_1}^0 \cap F(A_{jt_1}) \cap \{v : j\alpha t_1 \leq |v| \leq j\beta t_1\}| \geq e^{jc_{\lambda_0} t_1}) \geq \delta \tag{4.9}$$

for some choice of α, β, δ and t_1.

Let $B_j = \eta_{jt_1}^0 \cap F(A_{jt_1}) \cap \{v : j\alpha t_1 \leq |v| \leq j\beta t_1\}$ and let x be a vertex in B_j. We use the notation $(x, 0) \to (0, t)$ to denote the event that running the contact process on the shortest path connecting x and 0 with x initially infected, 0 will be infected at time t. Trivially, if every particle in the direct path from $x \in B_j$ to 0 lives for at least $2/\alpha$ and pushes the infection toward the root 0 between time $[1/\alpha, 2/\alpha]$ then

$$P_{\lambda_0}((x,0) \to (0,t) \text{ for some } t \in [jt_1, (2\beta/\alpha)jt_1]) \geq \left(e^{-2/\alpha}(e^{-\lambda_0/\alpha} - e^{-2\lambda_0/\alpha})\right)^{j\beta t_1} \equiv (q_0)^{j\beta t_1}.$$

Fix $\varepsilon = (c_{\lambda_0}/2)$ and $\rho = \frac{1}{4}P_{\lambda,+}(0 \in \eta_t^0 \text{ i.o.})$. For reasons that will become clear later we choose j_0 such that

$$C_{j_0} \equiv \delta\rho e^{(c_{\lambda_0} - \varepsilon)j_0 t_1} > 1. \tag{4.10}$$

From now on we write $t_0 \equiv j_0 t_1$. We want to use the vertices in B_{j_0} to push the infection back to the root 0. For some vertex $x \in B_{j_0}$, let η_t^x denote the contact process restricted to the branch $S^+(x)$ starting with x infected. If $x \in \eta_t^x$ for many times, then the infection can try many times to push back to the root. Hence the probability of successfully pushing the infection from x back to 0 is increased if $x \in \eta_t^x$ for many times. To estimate the success probability let $\sigma_0^x = 0$ and define

$$\sigma_{i+1}^x = \inf\{t > \sigma_i + (2\beta/\alpha)t_0 : x \in \eta_t^x\}.$$

Choose m so that

$$1 - (1 - (q_0)^{\beta t_0})^m \geq e^{-\varepsilon t_0}, \tag{4.11}$$

which is a lower bound on the probability that x successfully pushes the infection to the root 0 within m independent trials. If the process η_t^x restricted to $S^+(x)$ reinfects x for infinitely many times then $\sigma_m^x < \infty$, i.e., $P_{\lambda,+}(\sigma_m^x < \infty) \geq P_{\lambda,+}(0 \in \eta_t^0 \text{ i.o.})$. So we can choose M_2 such that

$$P_{\lambda,+}(\sigma_m^x \leq M_2) \geq \frac{1}{2} P_{\lambda,+}(0 \in \eta_t^0 \text{ i.o.}) = 2\rho. \tag{4.12}$$

For any $\epsilon < \lambda - \lambda_0$, the estimates in (4.9) through (4.11) are still true for the contact process with infection rate $\lambda - \epsilon$ since $\lambda - \epsilon > \lambda_0$. By (4.12) there exists a sufficiently small $\epsilon > 0$ with $\epsilon < \lambda - \lambda_0$ that satisfies

$$P_{\lambda-\epsilon,+}(\sigma_m^x \leq M_2) \geq \rho.$$

A vertex x is said to be $(m, \lambda - \epsilon)$-recurrent if $\{\sigma_m^x \leq M_2\}$ occurs for the contact process η_t^x with infection rate $\lambda - \epsilon$. If x is $(m, \lambda - \epsilon)$-recurrent then with probability $\geq e^{-\varepsilon t_0}$ it can reinfect the root after time t_0. Each vertex in B_{j_0} is $(m, \lambda - \epsilon)$-recurrent independently with probability at least ρ.

Now we are ready to show $\lambda > \lambda_2$. The following argument is inspired by the proof of Proposition 4.57 in Liggett [22]. Let $r_i = P_{\lambda-\epsilon,+}(\exists\ t \geq 2it_0 \text{ such that } 0 \in \eta_t^0)$. To estimate r_{i+1} we first run the contact process up to time t_0. According to (4.9) at time t_0 the set of infected frontier B_{j_0} has size $\geq L_0 \equiv \exp(c_{\lambda_0} t_0)$ with probability δ. Recall that η_t^x is the contact process restricted to the branch $S^+(x)$ starting with x infected. Let

$$R_x = \{\exists\ t \geq 2it_0 \text{ such that } x \in \eta_t^x\} \quad \text{and} \quad G_x = \{x \text{ is } (m, \lambda - \epsilon)\text{-recurrent}\}.$$

If for some $x \in B_{j_0}$ the event $R_x \cap G_x$ occurs then x is reinfected after $(2i+1)t_0$ and can push the infection back to the root after $2(i+1)t_0$ with probability at least $e^{-\varepsilon t_0}$. It follows that

$$\begin{aligned} r_{i+1} &\geq P_{\lambda-\epsilon,+}\left(\exists x \in B_{j_0} \text{ such that } R_x \cap G_x \text{ occurs}\right) \cdot e^{-\varepsilon t_0} \\ &\geq P_{\lambda_0,+}\big(Binomial(|B_{j_0}|, P_{\lambda_0,+}(R_x \cap G_x)) \geq 1\big) \cdot e^{-\varepsilon t_0} \qquad (\text{since } \lambda_0 < \lambda - \epsilon). \quad (4.13) \end{aligned}$$

On a given tree $\mathbb{T}^+$, R_x and G_x are positively correlated, as they are both increasing functions of the infection events and decreasing functions of the recovery events (see Liggett [22] for an account of positive correlation). Thus

$$P_+(R_x \cap G_x) = E_{\mathbb{T}^+\sim\nu}(P_{\mathbb{T}^+}(R_x \cap G_x)) \geq E_{\mathbb{T}^+\sim\nu}(P_{\mathbb{T}^+}(R_x) \cdot P_{\mathbb{T}^+}(G_x)).$$

In order to estimate $E_{\mathbb{T}^+\sim\nu}(P_{\mathbb{T}^+}(R_x) \cdot P_{\mathbb{T}^+}(G_x))$ we will adopt a normal positive correlation argument. The reader can skip this part and jump straight to (4.14).

Now observe that the root-added Galton-Watson tree $\mathbb{T}^+$ can be represented by a collection of i.i.d. random variables. Let the random variables $Z_0 = 1$ and $Z_1 \sim D$ be the number of offspring of the added root and the root, respectively. Then we sample a series of i.i.d. random variables $Z_{1,1}, \dots, Z_{1,Z_1} \sim D$ to represent the number of offspring for each individual in the first generation. The offspring of each individual in the second generation is then given by the collection $\{Z_{1,i,1}, \dots, Z_{1,i,Z_{1,i}}\}_{i=1}^{Z_1}$, and so on.

To simplify notation we write the index $(i_1, i_2, \dots, i_n)$ as $\mathbf{i} = (i_1, i_2, \dots, i_n, 0, \dots) \in \mathbb{N}^{\mathbb{N}}$. Hence, given a collection $\vec{Z} = \{Z_{\mathbf{i}} : \mathbf{i} \in \mathbb{N}^{\mathbb{N}}\}$ of i.i.d. random variables we can construct a Galton-Watson tree accordingly. Note that different $\vec{Z}$'s can be mapped to the same $\mathbb{T}^+ \in \mathbf{GW}^+(D)$. For example, if $Z_1 = 1$ then we only use the element $Z_{1,1}$ to construct the next generation, ignoring the elements $Z_{1,2}, Z_{1,3}$ and so on. Let μ^* denote the measure over the realizations of $\vec{Z}$. A function h of $\vec{Z}$ is said to be increasing if for a pair $\vec{Z}^1, \vec{Z}^2$ with $Z^1_{\mathbf{i}} \leq Z^2_{\mathbf{i}}$ for all $\mathbf{i}$ we have $h(\vec{Z}^1) \leq h(\vec{Z}^2)$. It follows from the Harris inequality (see Section 2.2 in [**?**]) that μ^* has positive correlations, that is, $E_{\mu^*} fg \geq E_{\mu^*} f \cdot E_{\mu^*} g$ for all increasing functions f and g with finite second moments. Note that the measure ν on $\mathbf{GW}^+(D)$ can be obtained from μ^* by collapsing different $\vec{Z}$'s mapping to the same $\mathbb{T}^+$, which means, for example,

$$E_{\mathbb{T}^+\sim\nu}(P_{\mathbb{T}^+}(R_x)) = E_{\vec{Z}\sim\mu^*}(P_{\vec{Z}}(R_x))$$

Both $P_{\vec{Z}}(R_x)$ and $P_{\vec{Z}}(G_x)$ are increasing with respect to $\vec{Z}$, so it follows from the positive correlations of μ^* that

$$\begin{aligned} E_{\mathbb{T}^+\sim\nu}\left(P_{\mathbb{T}^+}(R_x)\cdot P_{\mathbb{T}^+}(G_x)\right) &= E_{\vec{Z}\sim\mu^*}(P_{\vec{Z}}(R_x)\cdot P_{\vec{Z}}(G_x)) \geq E_{\vec{Z}\sim\mu^*}(P_{\vec{Z}}(R_x))\cdot E_{\vec{Z}\sim\mu^*}(P_{\vec{Z}}(G_x)) \\ &= E_{\mathbb{T}^+\sim\nu}(P_{\mathbb{T}^+}(R_x))E_{\mathbb{T}^+\sim\nu}(P_{\mathbb{T}^+}(G_x)) = P_+(R_x)P_+(G_x) \end{aligned}$$

That is,

$$P_+(R_x\cap G_x) \geq P_+(R_x)P_+(G_x) \geq r_i\rho. \tag{4.14}$$

Now we can finish the calculation in (4.13)

$$r_{i+1} \geq P(|B_{j_0}| \geq L_0)P(Binomial(L_0, r_i\rho) \geq 1)e^{-\varepsilon t_0} \geq \delta e^{-\varepsilon t_0}(1-(1-r_i\rho)^{L_0})$$

where the term δ comes from (4.9). Set $f(r) = \delta e^{-\varepsilon t_0}(1-(1-r\rho)^{L_0})$. Then $f(0) = 0$ and $f(1) < 1$. Now the reason for our choice of j_0 in (4.10) has become clear. Under this choice of j_0 we have

$$f'(0) = \delta\rho e^{-\varepsilon t_0}L_0 = C_{j_0} > 1.$$

It follows that f has a fixed point $r^* \in (0,1)$, i.e., $f(r^*) = r^*$. Now we prove $r_i \geq r^*$ inductively for $i \geq 0$. First we have $r_0 = 1$. Suppose $r_i \geq r^*$, then the monotonicity of f implies that

$$r_{i+1} \geq f(r_i) \geq f(r^*) = r^*.$$

Thus we have proved $r_i \geq r^*$ for all $i \geq 0$, which implies that $P_{\lambda-\epsilon,+}(0\in\eta_t^0 \text{ i.o.}) > 0$ and thus $\lambda - \epsilon \geq \lambda_2$. Therefore we have shown $\lambda > \lambda_2$ and this completes the proof. □

4.6 The weak survival phase on periodic trees

In this section we consider the contact process on a general periodic tree $\mathbb{T}_\kappa$. We will prove (i) $\lambda_1 < \lambda_2$ (see Theorem 1.4.5) and (ii) the contact process does not survive strongly at λ_2

(see Theorem 1.4.4, (ii)). The rest of Theorem 1.4.4 can be proved by similar arguments to that of the Galton-Watson trees and hence is omitted here. The reader is referred to the remarks in previous sections for ideas of the proofs.

Recall that the periodic tree $\mathbb{T}_\kappa = (a_1, a_2, \ldots, a_\kappa)$ is arranged in the way that every vertex x in $\mathbb{T}_\kappa$ has one neighbor above it and $d(x)$ neighbors below it, where $d(x) \in \{a_1, \ldots, a_\kappa\}$ is the offspring number of x. To simplify notation we will let $\gamma = \prod_{i=1}^{\kappa} a_i$. Without loss of generality, a distinguished vertex o is chosen to be the root and is said to be on level 0. Recall that the function $\ell : \mathbb{T}_\kappa \to \mathbb{Z}$ assigns a level to each vertex in $\mathbb{T}_\kappa$, which is defined as follows: (i) $\ell(o) = 0$ and (ii) for each $x \in \mathbb{T}_\kappa$, $\ell(y) = \ell(x) - 1$ for exactly one neighbor y of x, and $\ell(y) = \ell(x) + 1$ for the other $d(x)$ neighbors of x.

4.6.1 The growth profile

We now follow Liggett [21] and Stacey [34] to define the weight of a vertex x by

$$w_\rho(x) = \rho^{\ell(x)},$$

where $\rho > 0$ is a constant to be specified later. The weight of a set A of vertices is defined to be

$$w_\rho(A) = \sum_{x \in A} \rho^{\ell(x)}.$$

Take $\{e_n : n \in \mathbb{Z}\}$ in $\mathbb{T}_\kappa$ such that $\ell(e_n) = n$ and $|e_n - e_{n+1}| = 1$, where e_0 is what we called the root o. As many of the things we will discuss are analogous to results in Liggett [22] for the homogeneous tree $\mathbb{T}^d$, we will use the same notation as in [22]. Throughout the discussion we will use ξ_t to denote the contact process on $\mathbb{T}_\kappa$ starting with only the root o infected.

Define

$$u(n) = P(e_n \in \xi_t \text{ for some } t).$$

In order for the infection ξ_t to reach $e_{(n+m)\kappa}$ it has to first reach $e_{n\kappa}$. Thus if we restart the contact process with only $e_{n\kappa}$ infected at the moment ξ_t reaches $e_{n\kappa}$, by the strong Markov property

$$u((m+n)\kappa) \geq u(m\kappa)u(n\kappa). \tag{4.15}$$

See (4.47) in Liggett [22] for a detailed argument. Hence

$$\beta(\lambda) := \lim_{n\to\infty} [u(n\kappa)]^{1/n}$$

exists and satisfies

$$\beta(\lambda) = \sup_n [u(n\kappa)]^{1/n}.$$

It is easy to show that $\beta(\lambda)$ is independent of the type of vertex that is initially infected.

Let $S(x)$ denote the subtree that contains vertex x and all of its descendants. We will use $\bar{\xi}_t$ to denote the contact process restricted to $S(o)$, starting with only the root o infected. The following can be derived from the proof of Lemma 4.53 in [22]. Hence we only give a sketch of proof here.

Lemma 4.6.1. *For any λ,*

$$\lim_{n\to\infty} \left(\sup_t P(e_{n\kappa} \in \bar{\xi}_t) \right)^{\frac{1}{n}} = \beta(\lambda).$$

Sketch of proof. Let $B(x,n) = \{y \in \mathbb{T}_\kappa : |y-x| \leq n\}$. Let $v_r(m)$ be the probability that ξ_t^o restricted to $B(o,r)$ infects e_m for some $t \leq r$. Then

$$\lim_{r\to\infty} v_r(m) = u(m).$$

Now take positive integers j, k, m, n satisfying

$$jm + r \leq n.$$

Let $x_0 = e_{(n-jm)\kappa}, x_1 = e_{(n-(j-1)m)\kappa}, \ldots, x_j = e_{n\kappa}$. In order to have an infection path from $(o,0)$ to $(e_{n\kappa}, t)$ for some $t \leq nr$, we consider the intersection of following events:

the infection starting from o reaches x_0 within time 1 without exiting $S(o)$,

the infection from x_0 reaches x_1 within time r without exiting $B(x_1, r)$,

$\ldots$

the infection from x_{j-1} reaches x_j within time r without exiting $B(x_j, r)$.

Therefore

$$P(e_{n\kappa} \in \bar{\xi}_t \text{ for some } t \leq nr) \geq P(e_{(n-jm)\kappa} \in \bar{\xi}_t \text{ for some } t \leq 1)[v_r(m\kappa)]^j$$

Observe that

$$P(e_{n\kappa} \in \bar{\xi}_{i+1}) \geq e^{-1} P(e_{n\kappa} \in \bar{\xi}_t \text{ for some } t \in [i, i+1]).$$

Hence

$$\begin{aligned}
\sup_t P(e_{n\kappa} \in \bar{\xi}_t) &\geq \max_{0 \leq i < nr} P(e_{n\kappa} \in \bar{\xi}_{i+1}) \\
&\geq e^{-1} \max_{0 \leq i < nr} P(e_{n\kappa} \in \bar{\xi}_t \text{ for some } t \in [i, i+1]) \\
&\geq \frac{e^{-1}}{nr} P(e_{n\kappa} \in \bar{\xi}_t \text{ for some } t \leq nr) \\
&\geq \frac{e^{-1}}{nr} P(e_{(n-jm)\kappa} \in \bar{\xi}_t \text{ for some } t \leq 1)[v_r(m\kappa)]^j.
\end{aligned}$$

Therefore

$$\liminf_{n \to \infty} \left(\sup_t P(e_{n\kappa} \in \bar{\xi}_t) \right)^{\frac{1}{n}} \geq [v_r(m\kappa)]^{1/m}.$$

Taking $r \to \infty$ and then $m \to \infty$ gives the result. □

To simplify notation, in this section we will write λ_1 for $\lambda_1(\mathbb{T}_\kappa)$ and λ_2 for $\lambda_2(\mathbb{T}_\kappa)$. The structure of the periodic trees are similar in a way to the homogeneous trees, which allows us to obtain many useful properties on $\beta(\lambda)$. Here we only list one property to be used later:

Lemma 4.6.2. $\left(\frac{\lambda}{1+\lambda}\right)^\kappa \leq \beta(\lambda_1) \leq 1/\gamma$.

Proof. When the vertex e_i is infected, the probability that it infects e_{i+1} before the infection recovers is $\lambda/(1+\lambda)$. If this event occurs for every vertex on the path from e_0 to $e_{n\kappa}$ then $e_{n\kappa} \in \xi_t^o$ for some t. That is,

$$u(n\kappa) = P(e_n \in \xi_t \text{ for some } t) \geq \left(\frac{\lambda}{1+\lambda}\right)^{n\kappa},$$

which implies that $\beta(\lambda) \geq \left(\frac{\lambda}{1+\lambda}\right)^\kappa$.

Suppose $\beta(\lambda) > 1/\gamma$. There exists some constant a so that $\beta(\lambda) > a > 1/\gamma$. By Lemma 4.6.1 there exists t_0 and n_0 such that

$$P(e_{n_0\kappa} \in \bar{\xi}_{t_0})^{1/n_0} \geq a.$$

For $n \in \mathbb{Z}$, we will use

$$\mathcal{L}_n = \{x \in \mathbb{T}_\kappa : \ell(x) = n\}$$

to denote the set of vertices on level n. Let $B_1 = \{x \in \bar{\xi}_{t_0} : x \in \mathcal{L}_{n_0\kappa}\}$. Then $E|B_1| \geq (a\gamma)^{n_0} > 1$. At time t_0, at each $x \in B_1$ we can start an independent contact process restricted to $S(x)$ and run them for time t_0. Continuing this construction gives a supercritical branching process. Therefore $\beta(\lambda) > 1/\gamma$ implies $\lambda > \lambda_1$, i.e., $\beta(\lambda_1) \leq 1/\gamma$.

□

4.6.2 Proof of Theorem 1.4.5

The above discussion completes our preparation for the proof. Now we will proceed to show the existence of an intermediate phase for the contact process on periodic trees.

There are κ types of vertices in the periodic tree T_κ categorized according to their offspring numbers and relative positions in the tree. A vertex that corresponds to a_{i+1} in the tree $T_\kappa = (a_1, a_2, \ldots, a_\kappa)$ is said to be of type i, where $i \in \{0, 1, \ldots, \kappa - 1\}$. Let $\xi_t^{i,\lambda}$ be the contact process on T_κ with infection rate λ, starting with an infected type i vertex that we denote by x_i. We will also set x_i to be the root on level 0 and define the function $\ell : T_\kappa \to \mathbb{Z}$ accordingly.

If we only look at levels $\{\mathcal{L}_{n\kappa}\}_{n\in\mathbb{Z}}$ of the tree $\mathbb{T}_\kappa$ then the structure resembles a homogeneous tree $\mathbb{T}^\gamma$. Here we state a Lemma in Liggett [22] that will be useful in our proof.

Lemma 4.26. (See [22] Part I) *Let T^d denote the homogeneous tree with degree $d+1$ and root o. Let*

$$\alpha_n(\rho) = \sum_{|x-o|=n} \rho^{\ell(x)}.$$

Then $\alpha_0(\rho) = 1$ *and for* $n \geq 1$

$$\alpha_n(\rho) = \begin{cases} \frac{d^{n-1}\rho^n(d^2\rho^2-1)+\rho^{-n}(\rho^2-1)}{d\rho^2-1}, & d\rho^2 \neq 1 \\ \rho^{-n}[(n+1)-(n-1)\rho^2], & d\rho^2 = 1. \end{cases}$$

It is known that for the contact process $\xi_t^{\lambda_1}$ on homogeneous trees, the expectation of its weight $Ew_\rho(\xi_t^{\lambda_1}) \to 0$ as $t \to \infty$, see [34]. This is a key ingredient in Stacey's proof in [34]. For a given $\varepsilon > 0$ and a sufficiently large t_0 we have $Ew_\rho(\xi_{t_0}^{\lambda_1}) \leq \varepsilon$. Hence for some λ slightly larger than λ_1 we have

$$Ew_\rho(\xi_{t_0}^{\lambda}) \leq 2\varepsilon.$$

Using this fact Stacey constructed a supermartingale $M_n := \frac{w_\rho(\xi_{nt_0}^{\lambda})}{(2\varepsilon)^n}$ and showed the contact process does not survive strongly at this λ.

We will carry out an analogous argument for the periodic trees. In Lemma 4.6.3 we estimate the weight function at λ_1 for the contact process on periodic trees and then in the proof of Theorem 1.4.5 the supermartingale is constructed. The new ingredient in our argument is the extinction of the contact process on periodic trees at λ_1, which is used in the proof of Lemma 4.6.3.

Lemma 4.6.3. *For any* $\varepsilon > 0$*, there exists* $t_0 > 0$ *so that*

$$\max_{i \in \{0,1,\ldots,\kappa-1\}} Ew_\rho(\xi_{t_0}^{i,\lambda_1}) \leq \varepsilon.$$

Proof. We will consider the contact process ξ_t^{i,λ_1} starting with a type i vertex x_i infected and set x_i to be the root of T_κ. Recall that $\mathcal{L}_n = \{x \in \mathbb{T}_\kappa : \ell(x) = n\}$ denotes the set of vertices on level n and $S(x)$ denotes the subtree that contains vertex x and all of its descendants. For a vertex $x \in \cup_{n\in\mathbb{Z}}\mathcal{L}_{n\kappa}$ we shall define a block with root x to be

$$S_\kappa^x = \{y \in S(x) : |y - x| \leq \kappa - 1\}.$$

Note that every vertex in $\cup_{n\in\mathbb{Z}}\mathcal{L}_{n\kappa}$ has type i. Define $\mathcal{D}_n(x_i) = \{x \in T_\kappa : |x - x_i| = n\}$ and

$$\mathcal{B}_n(x_i) = \{x \in T_\kappa : |x - x_i| = n\kappa, \ell(x) = 0 \pmod{\kappa}\}.$$

The tree T_κ can be partitioned into a disjoint union of blocks

$$T_\kappa = \cup_{x \in \cup_{m\in\mathbb{Z}} \mathcal{L}_{m\kappa}} S_\kappa^x.$$

The weight $Ew_\rho(\xi_t^{i,\lambda_1})$ can be split into two parts:

$$Ew_\rho(\xi_t^{i,\lambda_1}) \le \sum_{n=0}^{\kappa N-1} \sum_{x\in\mathcal{D}_n(x_i)} w_\rho(x) P(x \in \xi_t^{i,\lambda_1}) + \sum_{n=\kappa N}^{\infty} \sum_{x\in\mathcal{D}_n(x_i)} w_\rho(x) P(x \in \xi_t^{i,\lambda_1}) \tag{4.16}$$

To obtain an upper bound on the second term in (4.16), we count the weight of the whole block S_κ^x if some vertex in this block is infected at time t. Hence

$$\sum_{n=\kappa N}^{\infty} \sum_{x\in\mathcal{D}_n(x_i)} w_\rho(x) P(x \in \xi_t^{i,\lambda_1}) \le \sum_{n\ge N} \sum_{x\in\mathcal{B}_n(x_i)} w_\rho(S_\kappa^x) P(S_\kappa^x \cap \xi_t^{i,\lambda_1} \neq \varnothing) \tag{4.17}$$

Set $a_0 = 1$ and it is straightforward to compute

$$w_\rho(S_\kappa^x) = \sum_{m=0}^{\kappa-1} \left(\rho^{\ell(x)+m} \cdot \prod_{j=0}^{m} a_j \right) \le \gamma \rho^{\ell(x)} \left(\sum_{m=0}^{\kappa-1} \rho^m \right) \le \frac{\gamma}{1-\rho} \cdot \rho^{\ell(x)}. \tag{4.18}$$

Let $\Gamma_{x_i,x}$ denote the path on T_κ from x_i to x. If $\Gamma_{x_i,x} \cap S_\kappa^x \neq \{x\}$, then the first vertex in S_κ^x that will be reached by the infection is not x. In this case, let $x^* \in \Gamma_{x_i,x}$ denote the vertex that satisfies $|x - x^*| = \kappa$. That is, the infection starting at x_i would spread to S_κ^x through vertex $x^* \in S_\kappa^x$. Then we know

$$P(S_\kappa^x \cap \xi_t^{i,\lambda_1} \neq \varnothing) \le \begin{cases} P(x \in \xi_t^{i,\lambda_1} \text{ for some } t) & \text{if } \Gamma_{x_i,x} \cap S_\kappa^x = \{x\}, \\ P(x^* \in \xi_t^{i,\lambda_1} \text{ for some } t) & \text{otherwise.} \end{cases}$$

Therefore, for $x \in \mathcal{B}_n^i$,

$$P(S_\kappa^x \cap \xi_t^{i,\lambda_1} \neq \varnothing) \le \beta(\lambda_1)^{|x_i - x| - 1} = \beta(\lambda_1)^{n-1}. \tag{4.19}$$

Putting (4.18) and (4.19) together we have

$$(4.17) \le \sum_{n\ge N} \sum_{x\in\mathcal{B}_n(x_i)} \frac{\gamma}{1-\rho} \cdot \rho^{\ell(x)} \beta(\lambda_1)^{n-1}. \tag{4.20}$$

Define $\alpha_n(\rho) = \sum_{x\in\mathcal{B}_n(x_i)} \rho^{\ell(x)}$. As a simple corollary of Lemma 4.26 we have $\alpha_0(\rho) = 1$ and

$$\alpha_n(\rho) = \rho^{-n\kappa}[(n+1) - (n-1)\rho^{2\kappa}] \quad \text{when } \gamma\rho^{2\kappa} = 1. \tag{4.21}$$

It follows from (4.21) and $\beta(\lambda_1) \le 1/\gamma$ (see Lemma 4.6.2) that we can choose N sufficiently large so that

$$\sum_{n\ge N}\sum_{x\in\mathcal{B}_n(x_i)} \frac{\gamma}{1-\rho}\cdot\rho^{\ell(x)}\beta(\lambda_1)^{n-1} \le \frac{\gamma}{\beta(\lambda_1)(1-\rho)}\left(\sum_{n\ge N}\gamma^{-n/2}(n+1)\right) \le \varepsilon/2. \tag{4.22}$$

Notice that this choice of N is independent of the type of vertex that is initially infected as well as the time t. When N is fixed, since at λ_1 the contact process dies out

$$\lim_{t\to\infty}\sum_{n=0}^{\kappa N-1}\sum_{x\in\mathcal{D}_n^i} w_\rho(x)P(x\in\xi_t^{i,\lambda_1}) = 0$$

for any $i \in \{0,1,\dots,\kappa-1\}$. That is, there exists $M_i > 0$ so that when $t \ge M_i$,

$$\sum_{n=0}^{\kappa N-1}\sum_{x\in\mathcal{D}_n^i} w_\rho(x)P(x\in\xi_t^{i,\lambda_1}) \le \varepsilon/2. \tag{4.23}$$

Taking $t_0 = \max\{M_0, M_1, \dots, M_{\kappa-1}\}$ and adding up (4.22) and (4.23) proves the desired result. $\square$

Proof of Theorem 1.4.5. Given $\varepsilon < 1/4$, by Lemma 4.6.3 we can choose t_0 so that

$$\max_{i\in\{0,1,\dots,\kappa-1\}} E_\lambda w_\rho(\xi_{t_0}^{i,\lambda_1}) \le \varepsilon.$$

Since $\max_{i\in\{0,1,\dots,\kappa-1\}} E_\lambda w_\rho(\xi_{t_0}^{i,\lambda})$ is a continuous function with respect to λ, there exists $\lambda > \lambda_1$ so that

$$\max_{i\in\{0,1,\dots,\kappa-1\}} E_\lambda w_\rho(\xi_{t_0}^{i,\lambda}) < 2\varepsilon \equiv \delta.$$

Without loss of generality we consider the contact process $\xi_t^{0,\lambda}$ that starts with a type 0 vertex infected. Let $\mathcal{F}_t$ be the σ-algebra generated by $\xi_t^{0,\lambda}$ up to time t. We can show that

$$\begin{aligned}
E[w_\rho(\xi_{(n+1)t_0}^{0,\lambda})|\mathcal{F}_{nt_0}] &\le \sum_{x\in\xi_{nt_0}^{0,\lambda}}\rho^{\ell(x)}E\sum_{y\in\xi_{t_0}^{x,\lambda}}\rho^{\ell(y)-\ell(x)} \\
&\le \left(\sum_{x\in\xi_{nt_0}^{0,\lambda}}\rho^{\ell(x)}\right)\cdot\max_{i\in\{0,1,\dots,\kappa-1\}} Ew_\rho(\xi_{t_0}^{i,\lambda}) \le \delta\cdot w_\rho(\xi_{nt_0}^{0,\lambda}).
\end{aligned}$$

It follows that

$$M_n := \frac{w_\rho(\xi_{nt_0}^{0,\lambda})}{\delta^n}$$

is a non-negative supermartingale, which converges almost surely as n goes to infinity. Since $\delta = 2\varepsilon < 1/2$,

$$w_\rho(\xi_{nt_0}^{0,\lambda}) \to 0 \qquad \text{as } n \to \infty.$$

Intuitively this implies that the process $\xi_t^{0,\lambda}$ (where $\lambda > \lambda_1$) does not survive strongly. See Proposition 1.0 in [34] for a complete argument. Therefore $\lambda_1 < \lambda \leq \lambda_2$. □

4.6.3 Weak survival at λ_2

In this section we complete the proof of Theorem 1.4.4 by showing the contact process on $\mathbb{T}_\kappa$ does not survive strongly at λ_2. Since $\lambda_1(\mathbb{T}_\kappa) < \lambda_2(\mathbb{T}_\kappa)$ we know the process survives weakly at λ_2. The proof on periodic trees is very much similar to that of Galton-Watson trees. In this case, the estimations are simpler due to the more regular structure that comes from periodicity.

Without loss of generality, a vertex with offspring number a_1 is chosen to be the root o. Let $\bar{\xi}_t$ denote the contact process restricted to the subtree $S(o)$ starting with the root o infected.

Proof of Theorem 1.4.4 (ii). Our goal is to show that $\rho \equiv P(o \in \bar{\xi}_t^\lambda \text{ i.o.}) > 0$ implies $\lambda > \lambda_2$.

Suppose when the infection rate is λ we have $\rho > 0$. First observe that

$$\rho = P(o \in \bar{\xi}_t^\lambda \text{ for a sequence of times } t \uparrow \infty) \leq u(n)$$

for all n and $\lambda > 0$. (The reader is referred to Theorem 4.65 in [22] for more details.) Hence $\beta(\lambda) = \lim_{n\to\infty} u(n)^{1/n} \geq \lim_{n\to\infty} \rho^{1/n} = 1$. Lemma 4.6.1 then implies that for any $\delta > 0$ there exists n_0 and t_0 so that

$$P(e_{n_0\kappa} \in \bar{\xi}_{t_0}^\lambda) \geq (1-\delta)^{n_0}.$$

As $P(e_{n_0\kappa} \in \bar{\xi}_{t_0}^\lambda)$ is continuous with respect to λ, there exists $\varepsilon > 0$ so that

$$P(e_{n_0\kappa} \in \bar{\xi}_{t_0}^{\lambda-\varepsilon}) \geq (1-2\delta)^{n_0}.$$

Now we will estimate the size of the infection by comparing with a branching process. Define $\mathcal{L}_n = \{x \in S(o) : |x - o| = n\kappa\}$. Let $B_0 = \{o\}$ and $B_1 = \{x \in \bar{\xi}_{t_0}^{\lambda-\varepsilon} : x \in \mathcal{L}_{n_0}\}$. Then $E|B_1| \geq (\gamma(1-2\delta))^{n_0} > 1$ when we choose δ to be sufficiently small. Specifically we will choose δ so that $\gamma(1-2\delta)^2 > 1$ for reasons that will become clear later.

At time t_0, at each $x \in B_1$ we can start an independent contact process restricted to $S(x)$ and run it for time t_0. Continuing this construction gives a supercritical branching process $|B_m|$ with mean $E|B_1|$. It is clear from the construction that $B_m \subset \bar{\xi}_{mt_0}^{\lambda-\varepsilon} \cap \mathcal{L}_{mn_0}$. Since $|B_m|$ is a supercritical branching process,

$$\lim_{m\to\infty} \frac{|B_m|}{(E|B_1|)^m}$$

exists almost surely and is not trivially zero. Therefore, there exists a $\epsilon > 0$ such that

$$P(|B_m| \geq \epsilon(E|B_1|)^m) \geq \epsilon \tag{4.24}$$

for all sufficiently large m. For reasons that will become clear later we choose m_0 sufficiently large so that

$$P(|\bar{\xi}_{m_0t_0}^{\lambda-\varepsilon} \cap \mathcal{L}_{m_0n_0}| \geq \epsilon(\gamma(1-2\delta))^{m_0n_0}) \geq P(|B_{m_0}| \geq \epsilon(\gamma(1-2\delta))^{m_0n_0}) \geq \epsilon$$

and

$$\epsilon^2((1-2\delta)^2\gamma)^{m_0n_0} > 1.$$

Let $r_i = P(o \in \bar{\xi}_{2im_0t_0}^{\lambda-\varepsilon})$ and $M \equiv \epsilon(\gamma(1-2\delta))^{m_0}$. We can obtain the following recursive relation:

$$\begin{aligned} r_{i+1} &\geq P(Binomial(|\bar{\xi}_{m_0t_0}^{\lambda-\varepsilon} \cap \mathcal{L}_{m_0n_0}|, r_i) \geq 1)P(e_{m_0n_0\kappa} \in \bar{\xi}_{m_0t_0}^{\lambda-\varepsilon}) \\ &\geq P(|\bar{\xi}_{m_0t_0}^{\lambda-\varepsilon} \cap \mathcal{L}_{m_0n_0}| \geq M)(1-(1-r_i)^M)P(e_{n_0\kappa} \in \bar{\xi}_{t_0}^{\lambda-\varepsilon})^{m_0} \qquad (4.25) \\ &\geq \epsilon(1-2\delta)^{m_0n_0}(1-(1-r_i)^M). \qquad (4.26) \end{aligned}$$

Setting $f(r) = \epsilon(1-2\delta)^{m_0n_0}(1-(1-r)^M)$. By our choices of δ and m_0,

$$f'(0) = \epsilon^2((1-2\delta)^2\gamma)^{m_0n_0} > 1.$$

Following the same argument as in the proof of Theorem 1.4.3 we can show there exists $r^* > 0$ such that $r_i \geq r^*$ for all $i \geq 0$. That is, $\bar{\xi}_t^{\lambda-\varepsilon}$ survives strongly, which implies $\lambda > \lambda - \varepsilon \geq \lambda_2$. $\square$

Chapter 5

Conclusions

We study the contact process on random graphs and trees.

In Chapter 2 we study the asymptotics for the critical values λ_1 and λ_2 on a general class of periodic trees. A little over 25 years ago Pemantle [32] pioneered the study of the contact process on trees, and showed that on homogeneous trees the critical values λ_1 and λ_2 for global and local survival were different. He also considered trees with periodic degree sequences, and Galton-Watson trees. Here, we will consider periodic trees in which the number of children in successive generations is $(n, a_1, \dots, a_k)$ with $\max_i a_i \le Cn^{1-\delta}$ and $\log(a_1 \cdots a_k)/\log n \to b$ as $n \to \infty$. We show that the critical value for local survival is asymptotically $\sqrt{c(\log n)/n}$ where $c = (k-b)/2$. This supports Pemantle's claim that the critical value is largely determined by the maximum degree, but it also shows that the smaller degrees can make a significant contribution to the answer.

In Chapter 3 we study the contact process on Galton-Watson trees and configuration models. The key to our investigation is an improved (and in a sense sharp) understanding of the survival time of the contact process on star graphs. Using these results, we show that for the contact process on Galton-Watson trees, when the offspring distribution (i) is subexponential the critical value for local survival $\lambda_2 = 0$ and (ii) when it is Geometric(p) we have $\lambda_2 \le C_p$, where the C_p are much smaller than previous estimates. We also study the critical value $\lambda_c(n)$ for "prolonged persistence" on graphs with n vertices generated by the configuration model. In the case of power law and stretched exponential distributions where it is known $\lambda_c(n) \to 0$ we give estimates on the rate of convergence. It was predicted in physics papers that $\lambda_c(n) \sim 1/\Lambda(n)$ where $\Lambda(n)$ is the maximum eigenvalue of the adjacency matrix. Our results show that this is accurate for graphs with power-law degree distributions, but not for stretched exponentials.

In Chapter 4 we study the supercritical contact process on Galton-Watson trees and periodic trees. We prove that if the contact process survives weakly then it dominates a supercritical Crump-Mode-Jagers branching process. Hence the number of infected sites grows exponentially fast. As a consequence we conclude that the contact process dies out at the critical value λ_1 for weak survival, and the survival probability $p(\lambda)$ is continuous with respect to the infection rate λ. Applying this fact, we show the contact process on a general periodic tree experiences two phase transitions in the sense that $\lambda_1 < \lambda_2$, which confirms a conjecture of Stacey's [34]. We also prove that if the contact process survives strongly at λ then it survives strongly at a $\lambda' < \lambda$, which implies that the process does not survive strongly at the critical value λ_2 for strong survival.

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