

Combined Deterministic-Stochastic Identification
with Application to Control of Wave Energy
Harvesting Systems

by

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Thesis submitted in partial fulfillment of the requirements for the degree of
Master of Science in the Department of Civil and Environmental Engineering
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ABSTRACT

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Abstract

This thesis proposes an integrated procedure for identifying the nominal models of the deterministic part and the stochastic part of a system, as well as their model error bounds in different uncertainty structures (e.g. \mathcal{H}_2 -norm and \mathcal{H}_∞ -norm) based on the measurement data. In particular, the deterministic part of a system is firstly identified by closed-loop instrumental variable method in which a known external signal sequence uncorrelated with the system noises is injected in the control input for the identifiability of the system in closed loop. By exploiting the second-order statistics of the noise-driven output components, the stochastic part of a system is identified by the improved subspace approach in which a new and straightforward linear-matrix-inequality-based optimization is proposed to obtain a valid model even under insufficient measurement data.

To derive an explicit model error bound on the identification model, we investigate a complete asymptotic analysis for identification of the stochastic part of the system. We first derive the asymptotically normal distributions of the empirical sample covariance and block-Hankel matrix of the outputs. Thanks to these asymptotic distributions and the perturbation analysis of singular value decomposition and discrete algebraic Riccati equation, several central limit theorems for the identified controllability matrix, observability matrix, and the state-space matrices in the associated covariance model are derived, as well as the norm bounds of Kalman gain and the innovations covariance matrix in the innovations model. By combining these

asymptotic results, the explicit \mathcal{H}_2 -norm and \mathcal{H}_∞ -norm bounds of the model error are identified with a given confidence level.

Practical applicability of the proposed combined deterministic-stochastic identification procedure is illustrated by the application to indirect adaptive control of a multi-generator wave energy harvesting system.

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Introduction

1.1 Background and Motivation

There are many dynamical systems, such as physical systems, biological systems, and economic systems, whose behaviors and performance attract much attention from industry and academia. However, in practice, these dynamical systems often demonstrate unsatisfactory performance or unstable behaviors. To compensate for these behaviors, a controller designed based on the system model given by physical modelling or system identification, is often implemented on these systems. Due to modelling difficulties resulting from high system complexity and lack of complete knowledge for the governing physical laws, system identification in many cases is the only approach to derive a feasible system model. This thesis is concerned with system identification.

The focus of the previous system identification research has been primarily on the identification of the deterministic part of a system. In this area, many consistent and efficient methods are developed, such as prediction error (PE) methods (Ljung (1997)), instrumental variable (IV) methods (Soderstrom and Stoica (1983)), and

subspace methods (Van Overschee and De Moor (1996)), both in open loop and closed loop. However, in order to precisely estimate the system state and design the optimal feedback controllers (Kailath and Sayed (2000); Zhou and Doyle (1997); Dullerud and Paganini (2000)), identification of the stochastic part of a system attracts an increasing attention in recent years (Katayama et al. (2006); Petersen et al. (2008); Goethals et al. (2003); Mari et al. (2000)). This thesis focuses on the identification of the stochastic part of a system represented by vector autoregressive, moving-average (ARMA) process. One notorious problem associated with identifying ARMA processes is on the solvability of the associated discrete algebraic Riccati equation (DARE) especially when the measurement data is insufficient and the system poles are close to the unit circle (Mari et al. (2000); Goethals et al. (2003)). To identify the deterministic part driven by control input, as well as the stochastic part of a system driven by system noises in closed loop, this thesis considers a two-step approach, i.e. firstly identify the deterministic part of a system by injecting known external signals uncorrelated with system noises, and then identify the stochastic part of a system from the noise-driven output components obtained by subtracting the input-driven output components from the whole outputs.

After identifying a system model, a critical question appears, i.e. how much uncertainty it contains. For the deterministic part, two different approaches have emerged which provide promising solutions to this problem. The first approach (Hakvoort (1994)) gives an upper bound on the \mathcal{H}_∞ norm of the weighted additive model error under several simplified assumptions, such as a bound on the markov parameters of the system and uncorrelation between control input and system noises which limits its application for the closed loop case. The second approach (Bombois and Gevers (1999)) quantifies the parametric uncertainty region by a ellipsoid for single input single output (SISO) case with a given confidence level. For the stochastic part, corresponding quantification of the model error are not thoroughly studied. One main

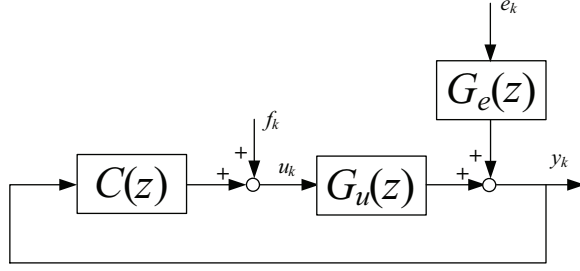


FIGURE 1.1: Closed-loop MIMO system configuration where u_k is the control input, f_k is the injected external signal, e_k is the innovations, and y_k is the output.

reason for it might concern the solvability of DARE mentioned before.

With a nominal model from identification and a model error bound, various robust control design and analysis methods (Zhou and Doyle (1997); Dullerud and Paganini (2000); Dahleh (1996)) can be used to derive a controller which exhibits the satisfactory performance both for the nominal model and any perturbed model within the error bound.

In summary, the focus of this thesis will be on the identification of the deterministic part of a multiple-input multiple-output (MIMO) system, as well as its stochastic part in closed loop. After identifying a nominal system model, the \mathcal{H}_2 and \mathcal{H}_∞ norm bounds of the model error are also derived with a given confidence level.

1.2 System Identification

1.2.1 Identification of the Deterministic Part of a System in Closed Loop

Due to the restrictions arising from safety consideration and operation conditions, identification experiments for many dynamical systems can only be operated in closed loop, as shown in Fig. 1.1. In this situation, the main difficulty to identify the system model is due to the correlation between control input and system noises. To identify $G_u(z)$ in closed loop under this difficulty, much improvement has been achieved for traditional identification methods, such as IV method (Gilson and Van den Hof

(2005)), PE methods (Van den Hof (1998)) and subspace methods (Van Overschee and De Moor (1997); Ljung and McKelvey (1996); Qin and Ljung (2003)). Among these methods, IV method seems to be rather attractive due to its advantages of closed-loop identification consistency even without the knowledge of the noise model, and the computational efficiency due to its employment of simple linear regression. Also, IV method can be easily extended to practical cases involving non-linear and time-varying controllers. Due to the attractive merits of IV method, this thesis employs it to identify the deterministic part of a system in closed loop.

1.2.2 Identification of the Stochastic Part of a System

Identifying the stochastic part of a system, i.e. $G_e(z)$ in Fig. 1.1, is undoubtedly crucial to estimating the system state and designing optimal feedback controllers (Kailath and Sayed (2000); Zhou and Doyle (1997); Dullerud and Paganini (2000)). This problem has received significant attention from many researchers and engineers for more than fifty years, during which a number of identification methods have been proposed. A class of methods (Hannan and Deistler (1988); Glover and Willems (1974); Ober (1991); Fuchs (1990)) require the canonical parameterization of the ARMA model which is computationally burdensome especially for the multivariate case and often encounters ill-conditioning problems when the system poles are very close to the unit circle. Several methods (Van Overschee and De Moor (1994, 1996)) may encounter a failure mode, i.e. these methods may obtain an invalid model outside the permissible model set or terminate without identifying any model. The reason for this failure mode is verified by experimental evidence in Dahlén et al. (1998). Maximum-likelihood methods (Stoica et al. (1987)) may terminate at a local minimum, or may diverge. Subspace methods, proposed first by Faurre (1976) and then followed by the study of several other researchers (Akaike (1976); Larimore (1983); Van Overschee and De Moor (1991)), also may lead to the identified system poles

outside of the unit circle, or the corresponding DARE being unsolvable. To overcome these difficulties, Mari et al. (2000) proposed an improved version of subspace approach for stochastic system identification in which the semidefinite programming (SDP) is used to constrain the identified system poles inside the unit circle by matrix Schur restabilizing procedures, and also used for multivariate covariance fitting, guaranteeing a positive real identified covariance model and thus the solvability of DARE. Although promising, one drawback of this approach is its use of coprime factorization, the robustness of which is not well studied for large-dimensional MIMO systems. Goethals et al. (2003) added a regularization term to a least squares cost function in the subspace identification algorithm for imposing positive realness on the associated covariance model. The solvability of the corresponding DARE and thus the feasibility of a valid model are satisfied, although at the cost of introducing a small bias on the identified model. Different from these approaches, this thesis first establishes an equivalence between the solvability of DARE and the nonemptiness of a convex set, and then proposes a new and straightforward approximation approach based on linear matrix inequalities (LMI) which is computationally attractive and guarantees a valid model to be returned.

1.2.3 Quantification of Model Error

Due to the finite sample and influence from system noises, the identified model is not exactly the same as the original model, and a model error therefore exists. To handle model error and guarantee a good performance, robust control theory is often adopted, in which an upper bound on the \mathcal{H}_∞ norm of the model error is required. A tight \mathcal{H}_∞ norm model error bound, like \mathcal{H}_∞ optimal model reduction problem, requires solution of a nonconvex optimization whose global minimum remains an open problem. To derive an \mathcal{H}_∞ norm model error bound for the deterministic part of a system, various identification methods are proposed by Goodwin et al. (1990);

Ninness and Goodwin (1992); De Vries and Van den Hof (1993, 1995); Hakvoort (1994). In Hakvoort (1994), an upper \mathcal{H}_∞ norm bound of model error is derived by a frequency response curve fitting procedure which minimizes a maximum amplitude criterion and guarantees the stability of the identified model. In this procedure, both the linear and nonlinear programming techniques are used. Another approach to quantify a model error is proposed by Bombois (2000) which constructs a framework connecting PE methods with robust control theory. In this framework, the tools of PE methods are used to quantify an uncertainty region to which robustness tools are conveniently adapted such that robustness analysis of a controller and the quality assessment of the uncertainty region are easily carried out.

For the stochastic part of a system, the quantification of model error is rarely reported. A new contribution of this thesis is to derive a model error bound in terms of \mathcal{H}_2 and \mathcal{H}_∞ norm with a confidence level given by asymptotic analysis of vector ARMA identification.

1.3 Contributions of this Thesis

In this thesis, a two-step combined deterministic-stochastic approach is proposed which enables identification of a nominal system model including $G_u(z)$ and $G_e(z)$, as shown in Fig. 1.1, in closed loop, as well as their model error bounds in terms of \mathcal{H}_2 norm and \mathcal{H}_∞ norm with a given confidence level.

The first contribution of this thesis is to propose a new and straightforward LMI-based optimization method which adjusts the associated covariance model and guarantees the solvability of DARE. The intuition of this method comes from the equivalence between the solvability of DARE and the positive realness of the covariance model associated with the innovations model which represents the stochastic part of a system. After imposing the positive realness on the covariance model by the LMI-based optimization method, no failure mode will appear in the identification of

the stochastic part of a system. This method possesses several merits, such as no need for canonical parameterization, no failure mode, high computational efficiency of convex optimization, feasibility even under the insufficient measurement data case, and superior performance in the large-dimensional MIMO case.

The second contribution of this thesis is to investigate the asymptotic analysis of vector ARMA identification in the frame of a subspace approach. In this analysis, we derive the asymptotic distributions of the state-space matrices in the associated covariance model and the norm bounds of the Kalman gain and innovations covariance matrix in innovations model. With these asymptotic properties, we solve a critical problem of how much data is required to identify the stochastic part of a system given a model error bound with a confidence level.

The third contribution of this thesis is to derive the \mathcal{H}_2 and \mathcal{H}_∞ norm bounds of the model error for the identified system. The explicit expressions for the \mathcal{H}_2 and \mathcal{H}_∞ norm error bounds are derived in terms of the Frobenius norm (F-norm) error bounds of the state-space matrices or the empirical output covariance matrix. We also propose an LMI-based approach to computing the \mathcal{H}_∞ norm error bound.

Finally, the fourth contribution of this thesis is to propose a new system identification algorithm which enables recursive identification of the deterministic part and stochastic part of a large-scale MIMO linear system in closed loop. This algorithm, illustrated by a simulation example, was successfully applied to the identification-based linear quadratic gaussian (LQG) control of a wave energy converter in a stochastic environment.

Combined Deterministic-Stochastic Identification in Closed Loop

In this chapter, a combined deterministic-stochastic identification procedure is developed for closed-loop systems, in which a known external signal uncorrelated with system noise, is injected to the system for the identifiability in closed loop. In this procedure, the deterministic part of a system is first identified by the closed-loop IV method in which a noise-free IV is constructed but requiring the precise knowledge of the deterministic part of a system. To overcome this difficulty, a bootstrap method is used to estimate the plant parameters for the deterministic part of a system.

After the deterministic part of a system is identified by the closed-loop IV method, the identified model for this part is used to compute the output components driven by the inputs. Subtracting the input-driven output components from the complete outputs, gives the noise-driven output components from which we then propose an improved subspace approach for identifying the stochastic part of a system. In this approach, we address the failure mode of stochastic subspace identification methods reported by Dahlén et al. (1998), i.e. the associated covariance model is not positive

real. To impose the positive realness for a valid model returned, a new and straightforward approximation method enlightened by Real Positive Lemma is proposed in which we can use 2-norm or F-norm error minimization to express the method as an LMI optimization problem.

2.1 Identifying $G_u(z)$ in Closed Loop by IV Method

In the stochastic autoregressive moving-average model with auxiliary input (ARMAX), the MIMO system is expressed by the following stochastic difference equation:

$$A(z^{-1})y_k = B(z^{-1})u_k + C(z^{-1})e_k \quad (2.1)$$

where $\{y_k\}$, $\{u_k\}$ and $\{e_k\}$ represent the output, the control input and the innovations, respectively; the matrix polynomials in forward-shift operator z

$$\begin{aligned} A(z^{-1}) &= I + A_1z^{-1} + \cdots + A_pz^{-p} \\ B(z^{-1}) &= B_1z^{-1} + \cdots + B_qz^{-q} \\ C(z^{-1}) &= I + C_1z^{-1} + \cdots + C_rz^{-r} \end{aligned} \quad (2.2)$$

are of the orders p , q and r , respectively. For brevity, we denote the moving-average (MA) part as a whole

$$\eta_k = C(z^{-1})e_k = e_k + C_1e_{k-1} + \cdots + C_re_{k-r} \quad (2.3)$$

In closed loop, $\{\eta_k\}$ is an unknown colored noise sequence which is not necessarily uncorrelated with the system inputs and outputs. Transform the ARMAX model to its counterpart in linear regression model

$$y_k = \theta^T \phi_k + \eta_k \quad (2.4)$$

where

$$\theta = [A_1, \cdots, A_p, B_1, \cdots, B_q]^T, \quad \phi_k = [-y_{k-1}; \cdots; -y_{k-p}; u_{k-1}; \cdots; u_{k-q}] \quad (2.5)$$

In (2.4), we apply Least-Squares (LS) algorithm to identify the plant parameter θ which converges to

$$\begin{aligned}
\theta_N^{LS} &= \left[\frac{1}{N} \sum_{k=1}^N \phi_k \phi_k^T \right]^{-1} \frac{1}{N} \sum_{k=1}^N \phi_k y_k^T \\
&= \left[\frac{1}{N} \sum_{k=1}^N \phi_k \phi_k^T \right]^{-1} \frac{1}{N} \sum_{k=1}^N \phi_k (\theta^T \phi_k + \eta_k)^T \\
&= \theta + \left[\frac{1}{N} \sum_{k=1}^N \phi_k \phi_k^T \right]^{-1} \frac{1}{N} \sum_{k=1}^N \phi_k \eta_k^T
\end{aligned} \tag{2.6}$$

Since in closed loop η_k is correlated with ϕ_k , the second term of (2.6) does not converge to 0. Thus, directly applying LS algorithm leads to a biased estimation. To eliminate the biased estimation, substituting ϕ_k in (2.6) with an instrumental variable ζ_k yields

$$\begin{aligned}
\tilde{\theta} &= \left[\frac{1}{N} \sum_{k=1}^N \zeta_k \phi_k^T \right]^{-1} \frac{1}{N} \sum_{k=1}^N \zeta_k y_k^T \\
&= \left[\frac{1}{N} \sum_{k=1}^N \zeta_k \phi_k^T \right]^{-1} \frac{1}{N} \sum_{k=1}^N \zeta_k (\theta^T \phi_k + \eta_k)^T \\
&= \theta + \left[\frac{1}{N} \sum_{k=1}^N \zeta_k \phi_k^T \right]^{-1} \frac{1}{N} \sum_{k=1}^N \zeta_k \eta_k^T
\end{aligned} \tag{2.7}$$

In (2.7), the instrumental variable ζ_k is required to satisfy the following two conditions for the consistency of the identified parameter θ .

1. $\frac{1}{N} \sum_{k=1}^N \zeta_k \phi_k^T$ is nonsingular as $N \rightarrow \infty$.
2. $\frac{1}{N} \sum_{k=1}^N \zeta_k \eta_k^T \rightarrow 0$ as $N \rightarrow \infty$.

To satisfy these two conditions in closed loop where control input and output are correlated with system noises, we inject a known external signal $\{f_k\}$ to the control

inputs $\{u_k\}$. $\{f_k\}$ is assumed to be uncorrelated with the system noises $\{e_k\}$ and satisfies the persistent excitation condition. Consider the control input

$$u_k = \bar{u}_k + f_k \quad (2.8)$$

where \bar{u}_k is the deterministic control and f_k is a known external signal. \bar{u}_k is $\{\mathcal{Y}_k, \mathcal{X}_0\}$ -measurable where $\{\mathcal{Y}_k\}$ is the collection $\{y_k, \dots, y_1\}$ and \mathcal{X}_0 is the initial condition. Assume the control law $\bar{u}_k = \Phi(\mathcal{Y}_k, \mathcal{X}_0)$ is linear with respect to \mathcal{Y}_k and \mathcal{X}_0 (later we will show it is true for wave energy harvesting applications). Define the innovations-driven output component $y_{k,e}$ and the external signal driven output component $y_{k,f}$ as

$$A(z^{-1})y_{k,e} = B(z^{-1})\bar{u}_{k,e} + C(z^{-1})e_k \quad (2.9)$$

$$A(z^{-1})y_{k,f} = B(z^{-1})(\bar{u}_{k,f} + f_k) \quad (2.10)$$

where in closed-loop systems we have that

$$\bar{u}_{k,e} = \Phi(\mathcal{Y}_{k,e}, \mathcal{X}_0) \quad (2.11)$$

$$\bar{u}_{k,f} = \Phi(\mathcal{Y}_{k,f}, \mathbf{0}) \quad (2.12)$$

where $\{\mathcal{Y}_{k,e}\} = \{y_{k,e}, \dots, y_{1,e}\}$ and $\{\mathcal{Y}_{k,f}\} = \{y_{k,f}, \dots, y_{1,f}\}$. Thus, \bar{u}_k consists of two components, i.e. $\bar{u}_{k,f}$ and $\bar{u}_{k,e}$ which are respectively $\{\mathcal{Y}_{k,f}\}$ -measurable and $\{\mathcal{Y}_{k,e}, \mathcal{X}_0\}$ -measurable, and $\mathcal{Y}_{k,f}$ and $\mathcal{Y}_{k,e}$ are the output components driven by $\{f_{k-1}, \dots, f_1, \mathbf{0}\}$ and $\{e_{k-1}, \dots, e_1, \mathcal{X}_0\}$, respectively. As a result, control input can be rewritten as

$$u_k = \bar{u}_{k,e} + \bar{u}_{k,f} + f_k \quad (2.13)$$

Correspondingly, we can split the output y_k into two components respectively driven by the collections $\{f_{k-1}, \dots, f_1\}$ and $\{e_{k-1}, \dots, e_1, \mathcal{X}_0\}$

$$y_k = y_{k,f} + y_{k,e} \quad (2.14)$$

Since $\{y_{k,f}\}$, $\{\bar{u}_{k,f}\}$ and $\{f_k\}$ are uncorrelated with the innovations $\{e_k\}$, we can construct a new instrumental variable

$$\zeta_k = [-y_{k-1,f}; \cdots; -y_{k-p,f}; \bar{u}_{k-1,f} + f_{k-1}; \cdots; \bar{u}_{k-q,f} + f_{k-q}] \quad (2.15)$$

which is the noise-free part of ϕ_k satisfying two conditions for the consistency of system identification. With ζ_k in (2.15) as the instrumental variable, the recursive IV algorithm, like the recursive LS, is given by

$$\theta_k = \theta_{k-1} + a_{k-1} P_{k-1} \zeta_k (y_k^T - \phi_k^T \theta_{k-1}) \quad (2.16)$$

$$P_k = P_{k-1} - a_{k-1} P_{k-1} \zeta_k \phi_k^T P_{k-1}, \quad a_{k-1} = (1 + \phi_k^T P_{k-1} \zeta_k)^{-1} \quad (2.17)$$

$$\phi_k = [-y_{k-1}; \cdots; -y_{k-p}; u_{k-1}; \cdots; u_{k-q}] \quad (2.18)$$

$$\zeta_k = [-y_{k-1,f}; \cdots; -y_{k-p,f}; \bar{u}_{k-1,f} + f_{k-1}; \cdots; \bar{u}_{k-q,f} + f_{k-q}] \quad (2.19)$$

Since ζ_k in (2.15) requires the knowledge of θ , we thus use a bootstrap method to estimate θ

$$\tilde{\theta}^{h+1} = \theta + \left[\frac{1}{N} \sum_{k=1}^N \zeta_k(\tilde{\theta}^h) \phi_k^T \right]^{-1} \frac{1}{N} \sum_{k=1}^N \zeta_k(\tilde{\theta}^h) \eta_k^T \quad (2.20)$$

where the instrumental variable $\zeta_k(\tilde{\theta}^h)$, although without the precise value of the true θ , is estimated using the plant parameter $\tilde{\theta}^h$ identified in the h th iteration. The validity of this bootstrap method is verified by theorem 4.5 in Soderstrom and Stoica (1983) which proves the nonsingularity of $\frac{1}{N} \sum_{k=1}^N \zeta_k(\hat{\theta}^h) \phi_k^T$, and that $\tilde{\theta}^h$ converges to the true θ .

After identifying the plant parameters θ , the deterministic part $G_u(z)$ of a system is estimated by:

$$G_u(z) = A (z^{-1})^{-1} B (z^{-1}) \quad (2.21)$$

$G_u(z)$, in state-space model, can be instead represented by:

$$G_u(z) = \left[\begin{array}{c|c} A_u & B_u \\ \hline C_u & 0 \end{array} \right] \quad (2.22)$$

where

$$A_u = \begin{bmatrix} -A_1 & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ & \mathbf{0} & \ddots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \mathbf{0} \\ -A_s & \mathbf{0} & & \cdots & \mathbf{0} \end{bmatrix}, \quad B_u = \begin{bmatrix} B_1 \\ \vdots \\ B_s \end{bmatrix}, \quad C_u = [\mathbf{I} \quad \mathbf{0} \quad \cdots \quad \mathbf{0}] \quad (2.23)$$

and

$$s = \max\{p, q\}, \quad A_i = \mathbf{0}, B_j = \mathbf{0} \quad \text{for any } p < i \leq s \quad \text{and} \quad q < j \leq s \quad (2.24)$$

2.2 Identifying $G_e(z)$ Using an Improved Subspace Approach

The system, represented by the innovations model (2.25), can be divided into two subsystems, illustrated in (2.26) for input-driven subsystem and (2.27) for innovations-driven subsystem.

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + Ke_k \\ y_k &= Cx_k + e_k \end{aligned} \quad (2.25)$$

$$\begin{aligned} x_{1,k+1} &= A_u x_{1,k} + B_u u_k \\ y_{1,k} &= C_u x_{1,k} \end{aligned} \quad (2.26)$$

$$\begin{aligned} x_{2,k+1} &= A_e x_{2,k} + KQ^{\frac{1}{2}} \bar{e}_k \\ y_{2,k} &= C_e x_{2,k} + Q^{\frac{1}{2}} \bar{e}_k \end{aligned} \quad (2.27)$$

where $x_{2,k} \in \mathcal{R}^{n_x \times 1}$; $y_{2,k} \in \mathcal{R}^{n_e \times 1}$; K is the Kalman gain; $Q = E[e_k e_k^T] \in \mathcal{R}^{n_e \times n_e}$ is the white innovations covariance matrix; $\bar{e}_k \in \mathcal{R}^{n_e \times 1}$ are the normalized white innovations with covariance matrix $E[\bar{e}_k \bar{e}_k^T]$ equal to the identity matrix. We assume zero-mean processes throughout.

From the last section, we have identified the deterministic part of a system. With this part known, (2.26) can be used to compute the input-driven output components

by which we subtract the whole output components for obtaining the innovations-driven output components. From their second-order statistics, we can identify the subsystem (2.27) driven by the innovations. With the output measurements y_k available, the output components driven by the innovations can be obtained by

$$y_{2,k} = y_k - y_{1,k} \quad (2.28)$$

Therefore, the problem is simplified to a stochastic identification problem in which A_e , C_e , K and Q are estimated from a vector ARMA process $\{y_{2,k}\}$.

A useful identification approach for a vector ARMA process is the subspace identification method proposed by Faure (1976). One notorious difficulty with it concerns the nonpositive realness of the associated covariance model which results in the unsolvability of DARE especially when the system poles are very close to the unit circle and the measurement data is insufficient. To overcome this difficulty and guarantee a valid model returned, several approximation methods are proposed to impose the positive realness (e.g. Mari et al. (2000); Goethals et al. (2003); Van Overschee and De Moor (1996); Vaccaro and Tomislav (1993)). Mari et al. (2000) used process covariance fitting (Stoica et al. (2000); McKelvey et al. (2000a)) to approximate a positive real covariance model. A key step of the method relies on coprime factorization for formulating an LMI problem. However, the reliability of coprime factorization, for the large-dimensional MIMO case, is not well studied. According to our simulation results, the reliability of coprime factorization, with the increase of the system dimension, tends to be weakened for the case where the system poles are close to unit circle. To overcome this drawback, a new and more straightforward LMI-based approximation approach derived from the Positive Real Lemma is proposed in this section.

2.2.1 Subspace Identification of Vector ARMA Process

In this subsection, we will recall the standard stochastic identification in Faurre (1976) and Mari et al. (2000). From the innovations model in (2.27), the following covariance relations can be derived:

$$P = A_e P A_e^T + K Q K^T \quad (2.29)$$

$$R_0 = C_e P C_e^T + Q \quad (2.30)$$

$$D_e = A_e P C_e^T + K Q \quad (2.31)$$

$$R_i = C_e A_e^{i-1} D_e \quad (2.32)$$

where

$$P = E\{x_{2,k} x_{2,k}^T\}, \quad R_i = E\{y_{2,k} y_{2,k-i}^T\}, \quad D_e = E\{x_{2,k+1} y_{2,k}^T\} \quad (2.33)$$

In practice, we estimate R_i by

$$\tilde{R}_i = \frac{1}{N-i} \sum_{k=i+1}^N y_{2,k} y_{2,k-i}^T \quad (2.34)$$

where \tilde{R}_i denotes the empirical estimate of R_i from finite data samples. \tilde{R}_i can be recursively updated by

$$\tilde{R}_{i,k+1} = \tilde{R}_{i,k} - \frac{1}{k+1} (\tilde{R}_{i,k} - y_{2,k+1} y_{2,k+1-i}^T) \quad (2.35)$$

Let the dimension of the innovations model in (2.27) be n_x and assume it to be a minimal realization. We have that the following observability matrix and controllability matrix are in full rank.

$$\Omega = \begin{bmatrix} C_e \\ C_e A_e \\ \vdots \\ C_e A_e^{m-1} \end{bmatrix}, \quad \Gamma = [D_e \quad A_e D_e \quad \cdots \quad A_e^{m-1} D_e], \quad m \geq n_x + 1 \quad (2.36)$$

Noting that the sequences $\{R_i\}$ are the Markov parameters of the system $\{A_e, D_e, C_e, R_0\}$, the following factorization of the block-Hankel matrix of the covariances holds:

$$H = \begin{bmatrix} R_1 & R_2 & \cdots & R_m \\ R_2 & R_3 & \cdots & R_{m+1} \\ \vdots & & \ddots & \vdots \\ R_m & R_{m+1} & \cdots & R_{2m-1} \end{bmatrix} = \Omega\Gamma \quad (2.37)$$

From (2.36) and (2.37), the rank properties of Ω and Γ imply that

$$\text{rank}(H) = n_x, \quad \text{for } m \geq n_x + 1 \quad (2.38)$$

Consider the singular value decomposition (SVD) of H :

$$\Omega\Gamma = H = U\Sigma V^T \quad (2.39)$$

From Σ , the dimension of the subsystem (2.27) can be determined by the number of singular values more than the prescribed tolerance. Setting the singular values less than the prescribed tolerance to zero, we can rewrite (2.39) as

$$\begin{aligned} \Omega\Gamma &= [U_1 \quad U_2] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} [V_1 \quad V_2]^T \\ &= U_1 \Sigma_1 V_1^T \end{aligned} \quad (2.40)$$

where Ω , Γ , U_1 , Σ_1 and V_1 all have the same rank n_x . Since Ω and $U_1 \Sigma_1^{\frac{1}{2}}$ span the same column space, and that Γ and $\Sigma_1^{\frac{1}{2}} V_1^T$ span the same row space, we have that

$$\begin{aligned} \Omega T &= U_1 \Sigma_1^{\frac{1}{2}} \\ T^{-1} \Gamma &= \Sigma_1^{\frac{1}{2}} V_1^T \end{aligned} \quad (2.41)$$

where $T \in R^{n \times n}$ is a non-singular matrix representing a similarity transformation. By similarity transformation, the subsystem driven by the innovations is transformed to a new subsystem:

$$\begin{aligned} \hat{x}_{2,k+1} &= \hat{A}_e \hat{x}_{2,k} + \hat{K} Q^{\frac{1}{2}} \bar{e}_k \\ y_{2,k} &= \hat{C}_e \hat{x}_{2,k} + Q^{\frac{1}{2}} \bar{e}_k \end{aligned} \quad (2.42)$$

where we denote

$$\hat{A}_e = T^{-1}A_eT, \quad \hat{K} = T^{-1}K, \quad \hat{C}_e = C_eT, \quad \hat{x}_{2,k} = Tx_{2,k} \quad (2.43)$$

Correspondingly, under the same similarity transformation, we denote the new observability matrix and controllability matrix as

$$\hat{\Omega} = \Omega T = \begin{bmatrix} \hat{C}_e \\ \hat{C}_e\hat{A}_e \\ \vdots \\ \hat{C}_e\hat{A}_e^{m-1} \end{bmatrix}, \quad \hat{\Gamma} = T^{-1}\Gamma = \begin{bmatrix} \hat{D}_e & \hat{A}_e\hat{D}_e & \cdots & \hat{A}_e^{m-1}\hat{D}_e \end{bmatrix}, \quad m \geq n_x+1 \quad (2.44)$$

This notation, together with (2.41), indicates that

$$\begin{aligned} \hat{C}_e &= \text{the first } n_e \text{ rows of } U_1\Sigma_1^{\frac{1}{2}} \\ \hat{D}_e &= \text{the first } n_e \text{ columns of } \Sigma_1^{\frac{1}{2}}V_1^T \end{aligned} \quad (2.45)$$

and that \hat{A}_e can be approximated by the following overdetermined linear equation:

$$\begin{bmatrix} \hat{C}_e \\ \hat{C}_e\hat{A}_e \\ \vdots \\ \hat{C}_e\hat{A}_e^{m-2} \end{bmatrix} \hat{A}_e = \begin{bmatrix} \hat{C}_e\hat{A}_e \\ \hat{C}_e\hat{A}_e^2 \\ \vdots \\ \hat{C}_e\hat{A}_e^{m-1} \end{bmatrix} \quad (2.46)$$

Before going on to the next step, the stability of \hat{A}_e should be checked. Once the unstable poles are detected, some care should be taken to stabilize \hat{A}_e . A cheap approach is to directly project the unstable eigenvalues into the unit circle (McKelvey et al. (2000b)). Another more accurate approach is to use the LMI-based optimization suggested by Mari et al. (2000):

$$\begin{aligned} \min_{\bar{A}_e, P} \quad & \|(\bar{A}_e - \hat{A}_e)P\|_F \\ \text{s.t.} \quad & P - \bar{A}_eP\bar{A}_e^T > 0 \\ & P > 0 \end{aligned} \quad (2.47)$$

where \hat{A}_e is obtained by (2.46), and $\|\cdot\|_F$ represents the F-norm. Substituting the estimated \hat{A}_e , \hat{C}_e and \hat{D}_e to (2.30) and (2.31) and then to (2.29) yields the DARE for P :

$$P = \hat{A}_e P \hat{A}_e^T + (\hat{D}_e - \hat{A}_e P \hat{C}_e^T)(R_0 - \hat{C}_e P \hat{C}_e^T)^{-1}(\hat{D}_e - \hat{A}_e P \hat{C}_e^T)^T \quad (2.48)$$

After this DARE is solved, \hat{K} and Q are then given by

$$\begin{aligned} Q &= R_0 - \hat{C}_e P \hat{C}_e^T \\ \hat{K} &= (\hat{D}_e - \hat{A}_e P \hat{C}_e^T) Q^{-1} \end{aligned} \quad (2.49)$$

2.2.2 A New Approach to Imposing the Positive Realness on the Covariance Model

By establishing an equivalence between the solvability of DARE and the positive realness of a covariance model, a new and more straightforward approximation approach based on LMI is proposed to impose the positive realness on the covariance model. Hereafter we refer the system $(A_e, D_e, C_e, R_0/2)$ as the covariance model associated with the innovations model (2.27).

Due to the insufficient data, the DARE often fails especially for the large dimensional system with system poles close to the unit circle. To illustrate the reason, we first define a set \mathcal{P} as

$$\mathcal{P} = \left\{ P \mid \begin{bmatrix} P - A_e P A_e^T & D_e - A_e P C_e^T \\ D_e^T - C_e P A_e^T & R_0 - C_e P C_e^T \end{bmatrix} \geq 0, \text{ and } P > 0 \right\} \quad (2.50)$$

According to Faurre (1976), each member in set \mathcal{P} represents a stochastic realization of the vector ARMA process $\{y_{2,k}\}$.

$$\begin{aligned} P - A_e P A_e^T &= Q \\ D_e - A_e P C_e^T &= S \\ R_0 - C_e P C_e^T &= R \\ \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} &\geq 0 \end{aligned} \quad (2.51)$$

The following theorem reveals the equivalence between the solvability of DARE and the positive realness of the covariance model.

Theorem 1. *Given a symmetric, positive definite matrix R_0 and a controllable and observable covariance model $(A_e, D_e, C_e, R_0/2)$, the associated Riccati equation (2.48) has a positive definite solution P with $R_0 - C_e P C_e' \geq 0$ if and only if the set \mathcal{P} is nonempty.*

Proof. Faurre (1976) and Vaccaro and Tomislav (1993) have shown that given a controllable and observable model, (2.48) has a positive definite solution P with $R_0 - C_e P C_e' \geq 0$ if and only if $C_e(zI - A_e)^{-1}D_e + R_0/2$ is positive real. Thus, we have $D_e^T(zI - A_e^T)^{-1}C_e^T + R_0/2$ is also positive real. Applying Positive Real Lemma obtains the conclusion. \square

Thus, the failure of DARE indicates that a null set \mathcal{P} resulting from the estimate $(\tilde{A}_e, \tilde{D}_e, \tilde{C}_e, \tilde{R}_0)$ where $(\tilde{\cdot})$ denotes the identified value of (\cdot) . This suggests that a feasible member in \mathcal{P} be approximated by solving an optimization

$$\begin{aligned} \left(P, \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^T & \Phi_{22} \end{bmatrix} \right) = \arg \min_{P, \Phi_{11}, \Phi_{12}, \Phi_{22}} & \left\| \begin{bmatrix} P - \tilde{A}_e P \tilde{A}_e^T & \tilde{D}_e - \tilde{A}_e P \tilde{C}_e^T \\ \tilde{D}_e^T - \tilde{C}_e P \tilde{A}_e^T & \tilde{R}_0 - \tilde{C}_e P \tilde{C}_e^T \end{bmatrix} - \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^T & \Phi_{22} \end{bmatrix} \right\| \\ \text{s.t. } & \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^T & \Phi_{22} \end{bmatrix} \geq 0 \\ & P > 0 \end{aligned} \tag{2.52}$$

in which we can use 2-norm or F-norm to express it as a semidefinite program. For

simplicity, choosing 2-norm in (2.52) gives the following LMI problem

$$\min \quad \lambda$$

$$\begin{bmatrix} \lambda \mathbf{I} & \mathbf{0} & P - \tilde{A}_e P \tilde{A}_e^T - \Phi_{11} & \tilde{D}_e - \tilde{A}_e P \tilde{C}_e^T - \Phi_{12} \\ & \lambda \mathbf{I} & \tilde{D}_e^T - \tilde{C}_e P \tilde{A}_e^T - \Phi_{12}^T & R_0 - \tilde{C}_e P \tilde{C}_e^T - \Phi_{22} \\ & & \mathbf{I} & \mathbf{0} \\ \text{sym} & & & \mathbf{I} \end{bmatrix} > 0 \quad (2.53)$$

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^T & \Phi_{22} \end{bmatrix} \geq 0 \quad (2.54)$$

$$P > 0 \quad (2.55)$$

where the variables are the positive definite matrix P , the nonnegative definite matrices Φ_{11} and Φ_{22} , the non-symmetric matrix Φ_{12} , and the scalar λ .

For convenience, we keep \tilde{A}_e and \tilde{C}_e estimated from (2.45) and (2.46) unchanged, and then adjust \tilde{D}_e and \tilde{R}_0 for guaranteeing that a positive definite solution to (2.48) exists. After solving (2.52), the new \hat{P} is updated by the discrete Lyapunov equation:

$$\hat{P} = \tilde{A}_e \hat{P} \tilde{A}_e^T + \Phi_{11} \quad (2.56)$$

and then the new $\hat{\tilde{D}}_e$ and $\hat{\tilde{R}}_0$ are adjusted by

$$\hat{\tilde{D}}_e = \tilde{A}_e \hat{P} \tilde{C}_e^T + \Phi_{12} \quad (2.57)$$

$$\hat{\tilde{R}}_0 = \tilde{C}_e \hat{P} \tilde{C}_e^T + \Phi_{22} \quad (2.58)$$

Substituting the adjusted $\hat{\tilde{D}}_e$ and $\hat{\tilde{R}}_0$ to DARE (2.48), it is guaranteed that DARE has a positive definite solution \tilde{P} , and finally, the estimate \tilde{K} and \tilde{Q} in (2.27) are updated by

$$\tilde{Q} = \hat{\tilde{R}}_0 - \tilde{C}_e \tilde{P} \tilde{C}_e^T \quad (2.59)$$

$$\tilde{K} = (\hat{\tilde{D}}_e - \tilde{A}_e \tilde{P} \tilde{C}_e^T) \tilde{Q}^{-1} \quad (2.60)$$

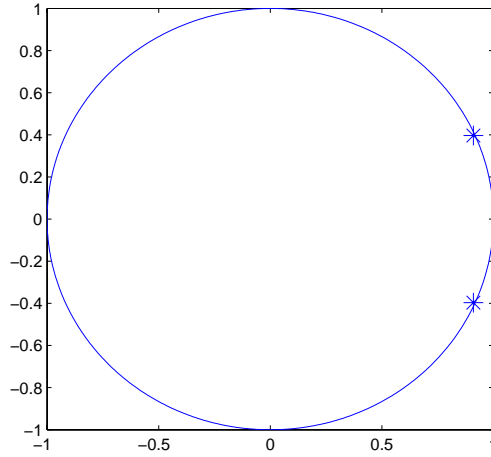


FIGURE 2.1: Poles (*) of the innovations model (2.62).

With the estimate \tilde{K} and \tilde{Q} , the stochastic part of a system $G_e(z)$ is represented by

$$G_e(z) = \left[\begin{array}{c|c} \tilde{A}_e & \tilde{K}\tilde{Q}^{\frac{1}{2}} \\ \hline \tilde{C}_e & \tilde{Q}^{\frac{1}{2}} \end{array} \right] \quad (2.61)$$

2.2.3 A Numerical Example

In this subsection, we will present a typical innovations model identification example (i.e., a example having poles close to the unit circle) in which the failure mode appears in the standard subspace stochastic identification.

The example corresponds to the following MIMO innovations model:

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 0.86 & 0.8 \\ -0.2 & 0.96 \end{bmatrix} x_k + \begin{bmatrix} 0.18 & 0.85 \\ -0.25 & -0.4 \end{bmatrix} e_k \\ y_k &= \begin{bmatrix} -0.3 & -0.65 \\ 0.76 & -1.1 \end{bmatrix} x_k + e_k \end{aligned} \quad (2.62)$$

with the covariance $Q = E[e_k e_k^T] = \begin{bmatrix} 0.075 & 0.037 \\ 0.037 & 0.068 \end{bmatrix}$. Its poles is shown in Figure 2.1.

We identify the simulated system (2.62) for 100 cases in each of which the size of measurement data $N = 2000$ and m in block-Hankel matrix (2.37) is chosen to be 4. For most of the identification cases, the standard stochastic subspace identification fails. The results shown afterwards are all from the proposed stochastic subspace identification. To reflect the performance of the proposed identification procedure, we define two indices E_1 and E_2

$$E_2 = \frac{\|G(\omega) - \tilde{G}(\omega)\|_{\mathcal{H}_2}}{\|G(\omega)\|_{\mathcal{H}_2}} \quad (2.63)$$

$$E_\infty = \frac{\|G(\omega) - \tilde{G}(\omega)\|_{\mathcal{H}_\infty}}{\|G(\omega)\|_{\mathcal{H}_\infty}} \quad (2.64)$$

where $G(\omega)$ is the transfer function from the normalized innovations with covariance equal to the identity matrix to the output, i.e.

$$G(\omega) = C_e(e^{i\omega T_s} I - A_e)^{-1} K Q^{1/2} + Q^{1/2} \quad (2.65)$$

where T_s is the sample period. Figures 2.2 and 2.3 show the two indices E_2 and E_∞ for 100 identifications, respectively. In particular, the expectations for E_2 and E_∞ are 0.2502 and 0.3349, respectively, and their standard deviation is 0.1045 and 0.1607, respectively. Figure 2.4 shows the frequency response for one identified $\tilde{G}(\omega)$ with $E_2 = 0.1022$ and $E_\infty = 0.1251$.

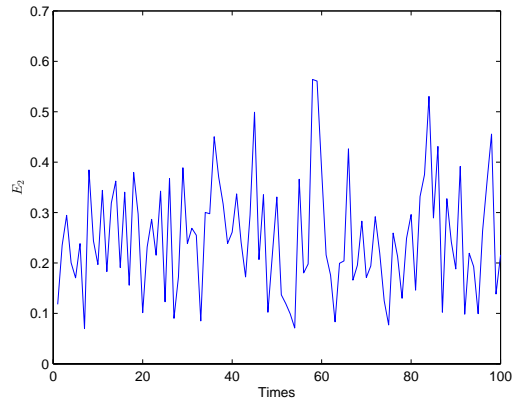


FIGURE 2.2: \mathcal{H}_2 norm relative error of the identified system for 100 identifications.

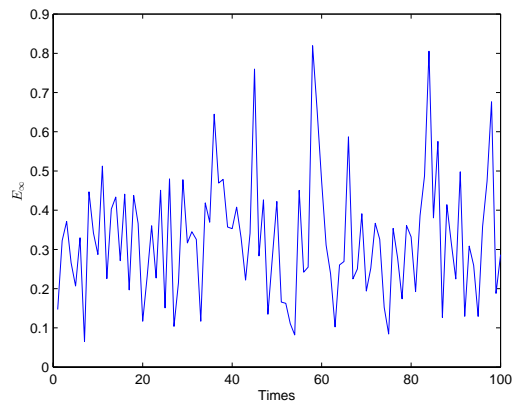


FIGURE 2.3: \mathcal{H}_∞ norm relative error of the identified system for 100 identifications.

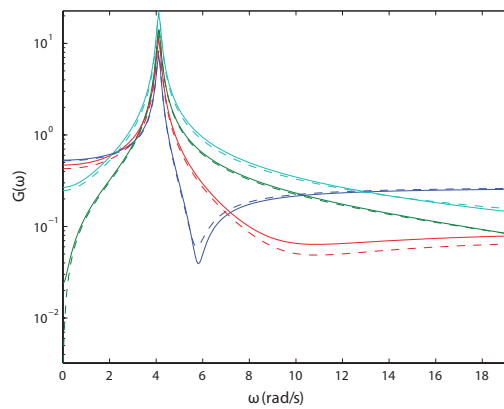


FIGURE 2.4: Frequency response for the original $G(\omega)$ (solid) and the identified $\tilde{G}(\omega)$ (dashed).

Uncertainty Quantification for Combined Deterministic-Stochastic Identification

3.1 Uncertainty Quantification for the Identification of the Deterministic Part

In this section, we will derive the asymptotic distribution of the plant parameter estimate $\tilde{\theta}$ from the identification of the deterministic part of a system, as well as the parametric uncertainty region for the transfer function $G_u(z)$ of the system deterministic part.

For brevity, hereafter in this chapter we use I or $I_n \in \mathcal{R}^{n \times n}$ to denote the identity matrix with compatible dimension, $\tilde{(\cdot)}$ to denote the estimated value of (\cdot) from the measurement data, $\delta(\cdot) = \tilde{(\cdot)} - (\cdot)$ to denote the perturbation of (\cdot) due to finite data samples, $o(\cdot)$ to denote the higher order perturbation satisfying $o(\cdot)/(\cdot) \rightarrow 0$ as $(\cdot) \rightarrow 0$, and $\dot{(\cdot)}$ to denote the first-order approximation due to the perturbation δH . $\text{vec}(\cdot)$, \otimes , and $\text{tr}(\cdot)$ represent the columnwise vectorization of a matrix, Kronecker product and the trace of a matrix, respectively. $(\cdot)^*$, $(\cdot)^T$ and $(\cdot)^H$ represent conjugate, transpose and conjugate transpose, respectively. $\|\cdot\|_F$ and $\|\cdot\|_2$ represent F-norm and 2-norm, respectively.

3.1.1 Asymptotic Distribution of the Estimated Parameters

The ARMAX model of MIMO system (2.1) is assumed to fulfill the following assumptions:

1. The system is asymptotically stable, i.e. the roots of $A(z^{-1}) = 0$ are inside the unit circle.
2. The injected external signal $\{f_k\}$ is a persistently exciting signal of a sufficiently high order.
3. The equality $A(z^{-1}, \theta_1)^{-1}B(z^{-1}, \theta_1) = A(z^{-1}, \theta_2)^{-1}B(z^{-1}, \theta_2)$ implies $\theta_1 = \theta_2$, i.e. the matrix fraction description of the system (2.1) is canonical.

To compute the covariance of the identified parameters $\tilde{\theta}$, we first give the following lemma which was proved by Stoica and Soderstrom (1983).

Lemma 2. *If the ARMAX system (2.1) satisfies the assumptions 1 ~ 3, the noise-free part ζ_k of ϕ_k obeys*

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{k=1}^N \zeta_k(\phi_k)^T = E[\zeta_k(\zeta_k)^T] > 0 \quad (3.1)$$

Theorem 3. *Consider the estimate $\tilde{\theta}$ given by (2.7). Assume that the system (2.1) satisfies the assumptions 1 ~ 3. Then $\tilde{\theta}$ is asymptotically Gaussian distributed*

$$\sqrt{N} \text{vec}(\tilde{\theta} - \theta) \rightarrow \mathcal{N}(0, P_\theta) \quad (3.2)$$

with the covariance matrix P_θ given by

$$P_\theta = (I_{n_y} \otimes R^{-1}) \frac{1}{2\pi} \int_{-\pi}^{\pi} S_\eta^*(\omega) \otimes S_\zeta(\omega) d\omega (I_{n_y} \otimes R^{-1}) \quad (3.3)$$

where $S_\zeta(\omega)$ is the power spectral density (PSD) of the instrumental variable ζ_k in (2.15); $S_\eta(\omega)$ is the PSD of the MA part η_k in (2.3) and given by

$$S_\eta(\omega) = A(e^{-i\omega}) G_e(e^{i\omega}) G_e(e^{-i\omega})^T A(e^{i\omega})^T \quad (3.4)$$

with the polynomial matrix function $A(z^{-1})$ given in (2.2) and $G_e(z)$ the innovations-output transfer function of system (2.27); $R = E[\zeta_k \zeta_k^T]$ is the variance of the instrumental variable ζ_k and given by

$$R = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{\zeta}(\omega) d\omega \quad (3.5)$$

Proof. According to Theorem 3.1 in Stoica and Soderstrom (1983), the estimated $\tilde{\theta}$ by multivariable IV methods is asymptotically normally distributed. Here we derive an explicit expression for its covariance matrix P_{θ} which can be estimated by the identified $\tilde{\theta}$ and $\tilde{G}_e(z)$. As $N \rightarrow \infty$, according to Lemma 2, we can rewrite (2.7) as

$$\sqrt{N}R(\tilde{\theta} - \theta) = \frac{1}{\sqrt{N}} \sum_{k=1}^N \zeta_k \eta_k^T \quad (3.6)$$

Vectorizing both sides of (3.6) yields

$$(I_{n_y} \otimes R) \sqrt{N} \text{vec}(\tilde{\theta} - \theta) = \frac{1}{\sqrt{N}} \sum_{k=1}^N \eta_k \otimes \zeta_k \quad (3.7)$$

from which we have the covariance matrix

$$P_{\theta} = (I_{n_y} \otimes R^{-1}) \frac{1}{N} \sum_{1 \leq k, s \leq N} E [(\eta_k \otimes \zeta_k)(\eta_s^T \otimes \zeta_s^T)] (I_{n_y} \otimes R^{-T}) \quad (3.8)$$

where as $N \rightarrow \infty$, we have that

$$\begin{aligned}
& \frac{1}{N} \sum_{1 \leq k, s \leq N} E [(\eta_k \otimes \zeta_k)(\eta_s^T \otimes \zeta_s^T)] \\
&= \frac{1}{N} \sum_{1 \leq k, s \leq N} E [(\eta_k \eta_s^T) \otimes (\zeta_k \zeta_s^T)] \\
&= \frac{1}{N} \sum_{1 \leq k, s \leq N} R_\eta(k-s) \otimes R_\zeta(k-s) \\
&= \sum_{\tau} R_\eta(\tau) \otimes R_\zeta(\tau) \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{\tau_1, \tau_2} R_\eta(\tau_1) \otimes R_\zeta(\tau_2) e^{-i\omega(\tau_2 - \tau_1)} d\omega \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{\tau_1} R_\eta(\tau_1) e^{i\omega\tau_1} \otimes \sum_{\tau_2} R_\zeta(\tau_2) e^{-i\omega\tau_2} d\omega \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} S_\eta^*(\omega) \otimes S_\zeta(\omega) d\omega \tag{3.9}
\end{aligned}$$

where $R_\eta(\tau)$ and $R_\zeta(\tau)$ are the covariance functions of η_k and ζ_k , respectively. Substituting (3.9) to (3.8) gives (3.3). With the knowledge of $G_e(z)$, we have the transfer function from the innovations \bar{e}_k to the MA part η_k

$$\bar{C}(z) = A(z^{-1})G_e(z) \tag{3.10}$$

Noting that the innovations \bar{e}_k is a vector Gaussian variable with its covariance matrix equal to identity matrix, we have the PSD $S_\eta(\omega)$ of $\{\eta_k\}$ in (3.4). \square

Remark 1: suppose $\mathcal{K}(z) : f_k \rightarrow \bar{u}_{k,f} + f_k$ is the mapping from the injected external signal $\{f_k\}$ to the control input component $\{\bar{u}_{k,f} + f_k\}$ contributed by the injected external signal $\{f_k\}$. We have the injected external signal driven output component

$$\begin{aligned}
y_{k,f} &= G_u(z)(\bar{u}_{k,f} + f_k) \\
&= G_u(z)\mathcal{K}(z)f_k \tag{3.11}
\end{aligned}$$

Denote $G_f(z) = G_u(z)\mathcal{K}(z)$ as the transfer function from f_k to $y_{k,f}$. Then we can rewrite ζ_k in terms of f_k as

$$\zeta_k = \mathcal{Q}(z)f_k \quad (3.12)$$

where

$$\mathcal{Q}(z) = \begin{bmatrix} -G_f(z)z^{-1} \\ \vdots \\ -G_f(z)z^{-p} \\ \mathcal{K}(z)z^{-1} \\ \vdots \\ \mathcal{K}(z)z^{-q} \end{bmatrix} \quad (3.13)$$

From (3.12), we have the PSD of $\{\zeta_k\}$

$$S_\zeta(\omega) = \mathcal{Q}(e^{i\omega})S_f(\omega)\mathcal{Q}(e^{-i\omega})^T \quad (3.14)$$

where $S_f(\omega)$ is the PSD of $\{f_k\}$.

Remark 2: from (3.3), with more external signal f_k injection, the matrix covariance P_θ decreases and thus the closed-loop IV method is able to identify the system with less measurement data. For instance, when the PSD $S_f(\omega)$ is doubled, the matrix covariance P_θ , as a result, decreases by one half.

3.1.2 Parametric Uncertainty Region of $G_u(z)$

In this subsection, we extend the quantification of the parametric uncertainty region of $G_u(z)$ from SISO case (Bombois and Gevers (1999)) to MIMO case.

To briefly quantify the uncertainty regions delivered by closed-loop IV identification, we assume that an unbiased model set is used. We also assume that the true system is linear and time-invariant, with a rational input-output transfer function $G_u(z)$ and a rational innovations-output transfer function $G_e(z)$:

$$y_k = G_u(z)u_k + G_e(z)\bar{e}_k \quad (3.15)$$

Associated with MIMO IV identification, we consider a stable model set \mathcal{M}_{G_u} with the following structure:

$$\begin{aligned}\mathcal{M}_{G_u} &= \{G_u(z, \theta^v) | G_u(z, \theta^v) = (I + A_1 z^{-1} + \dots + A_p z^{-p})^{-1} (B_1 z^{-1} + \dots + B_q z^{-q}) \\ &= (I + \theta^T Z_D)^{-1} (\theta^T Z_N)\}\end{aligned}\quad (3.16)$$

where the plant parameter $\theta = [A_1, \dots, A_p, B_1, \dots, B_q]^T$, and

$$\begin{aligned}Z_D &= [z^{-1} I_{n_y} \dots z^{-p} I_{n_y} \ 0 \dots 0]^T \in R^{(n_y p + n_u q) \times n_y} \\ Z_N &= [0 \dots 0 \ z^{-1} I_{n_u} \dots z^{-q} I_{n_u}]^T \in R^{(n_y p + n_u q) \times n_y}\end{aligned}\quad (3.17)$$

where n_u and n_y are the dimensions of the input u_k and the output y_k , respectively. To avoid the biased case, we assume that the true $G_u(z) \in \mathcal{M}_{G_u}$, i.e. \exists a $\theta_0 \in R^{(n_y p + n_u q) \times n_y}$ such that $G_u(z) = G_u(z, \theta_0)$. From the measurement data, a nominal model $G_{nom} = G_u(z, \tilde{\theta}) \in \mathcal{M}_{G_u}$ is then identified, together with a covariance estimation P_θ as in (3.3). The true θ_0 lies with probability $\alpha(n_y(n_y p + n_u q), \chi_\alpha^2)$ in the ellipsoidal uncertainty region

$$\mathcal{U}_{G_u} = \left\{ \theta \mid \text{vec}(\theta - \tilde{\theta})^T N P_\theta^{-1} \text{vec}(\theta - \tilde{\theta}) < \chi_\alpha^2 \right\} \quad (3.18)$$

where $\alpha(n_y(n_y p + n_u q), \chi_\alpha^2)$ is the chi-square probability with $n_y(n_y p + n_u q)$ parameters and χ_α^2 is a scalar bound. This uncertainty region \mathcal{U}_{G_u} corresponds to an uncertainty region in the space of transfer function which we denote \mathcal{D}_{G_u} :

$$\mathcal{D}_{G_u} = \{G_u(z, \theta) \mid G_u(z, \theta) \in \mathcal{M}_{G_u}, \text{ and } \theta \in \mathcal{U}_{G_u}\} \quad (3.19)$$

Then the true $G_u(z)$ is in \mathcal{D}_{G_u} with the probability $\alpha(n_y(n_y p + n_u q), \chi_\alpha^2)$.

3.2 Uncertainty Quantification for the Identification of the Stochastic Part

In this section, we will derive the asymptotic distributions of the controllability matrix, observability matrix (2.44), and the state-space matrices of the associated

covariance model. Then, we apply the perturbation analysis of DARE for deriving the F-norm bounds of Kalman gain and the innovations covariance matrix in the innovations model. By combining these asymptotic results, the \mathcal{H}_2 norm bound, as well as the \mathcal{H}_∞ norm bound of the model error, is derived with a given confidence level.

3.2.1 Influence of the Deterministic Part Identification on the Covariance Matrix Estimation

In this subsection, the perturbation analysis is used to explore the propagation of the error $\delta\theta$ from $G_u(z)$ identification. In this analysis, matrix Taylor series expansion using Vetter Calculus (Vetter (1973)) is used to analyze the error impact of $\tilde{\theta} - \theta_*$ where we denote $\tilde{\theta}$ as the estimated deterministic plant parameters and θ_* as the true deterministic plant parameters. The noise-driven output components derived from the previously estimated deterministic plant parameters are given by

$$\begin{aligned}\tilde{y}_{2,k}(\tilde{\theta}) &= y_k - \tilde{y}_{1,k}(\tilde{\theta}) \\ &= \theta_*^T \phi_k(\theta_*) + \eta_k - \tilde{\theta}^T \tilde{\phi}_k(\tilde{\theta})\end{aligned}\tag{3.20}$$

where $\tilde{\phi}_k(\tilde{\theta}) = [-\tilde{y}_{1,k-1}(\tilde{\theta}); \dots; -\tilde{y}_{1,k-p}(\tilde{\theta}); u_{k-1}; \dots; u_{k-q}]$. In (3.20), applying matrix Taylor expansion of $\tilde{\theta}^T \tilde{\phi}_k(\tilde{\theta})$ on $\tilde{\theta} = \theta_*$ in Vetter's notation yields

$$\begin{aligned}\tilde{y}_{2,k}(\tilde{\theta}) &= \theta_*^T \phi_k(\theta_*) + \eta_k - \tilde{\theta}^T \tilde{\phi}_k(\tilde{\theta}) \\ &= \theta_*^T \phi_k(\theta_*) + \eta_k - (\theta_*^T \tilde{\phi}_k(\theta_*) + d(\tilde{\theta}^T \tilde{\phi}_k(\tilde{\theta}))) + o(\|\tilde{\theta} - \theta_*\|) \\ &= (\theta_*^T \phi_k(\theta_*) - \theta_*^T \tilde{\phi}_k(\theta_*)) + \eta_k - d(\tilde{\theta}^T \tilde{\phi}_k(\tilde{\theta})) + o(\|\tilde{\theta} - \theta_*\|) \\ &= y_{2,k} - d(\tilde{\theta}^T \tilde{\phi}_k(\tilde{\theta})) + o(\|\tilde{\theta} - \theta_*\|) \\ &= y_{2,k} - \left[\mathcal{D}_{\text{rs}\theta}(\theta^T \tilde{\phi}_k(\theta)) \right]_{\theta=\theta_*} \left[\text{rs}(\tilde{\theta} - \theta_*) \right]^T + o(\|\tilde{\theta} - \theta_*\|) \\ &= - \left[\mathcal{D}_{\text{rs}\theta}(\theta^T) \right]_{\theta=\theta_*} (I_{n_y^2(p+q)} \otimes \tilde{\phi}_k(\theta_*)) + \theta_*^T (\mathcal{D}_{\text{rs}\theta} \tilde{\phi}_k(\theta))|_{\theta=\theta_*} \left[\text{rs}(\tilde{\theta} - \theta_*) \right]^T \\ &\quad + y_{2,k} + o(\|\tilde{\theta} - \theta_*\|)\end{aligned}\tag{3.21}$$

where $d(\cdot)$ is the differential of (\cdot) ; $\text{rs}(\cdot)$ is the rowwise vectorization of (\cdot) ; $\mathcal{D}_{\mathbf{M}}(\mathbf{N})$ is the directional derivation of \mathbf{N} with respect to $\mathbf{M} \in \mathcal{R}^{n \times p}$

$$\mathcal{D}_{\mathbf{M}}(\mathbf{N}) = \begin{bmatrix} \frac{\partial \mathbf{N}}{\partial m_{11}} & \frac{\partial \mathbf{N}}{\partial m_{12}} & \cdots & \frac{\partial \mathbf{N}}{\partial m_{1p}} \\ \frac{\partial \mathbf{N}}{\partial m_{21}} & \frac{\partial \mathbf{N}}{\partial m_{22}} & \cdots & \cdot \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mathbf{N}}{\partial m_{n1}} & \frac{\partial \mathbf{N}}{\partial m_{n2}} & \cdots & \frac{\partial \mathbf{N}}{\partial m_{np}} \end{bmatrix} \quad (3.22)$$

where m_{ij} is the (i,j) element of matrix \mathbf{M} . For brevity, denote

$$\begin{aligned} \delta\theta &= \left[\text{rs}(\tilde{\theta} - \theta_*) \right]^T \\ \mathcal{K}_k &= \mathcal{D}_{rs\theta}(\theta^T)|_{\theta=\theta_*} (I_{n_y^2(p+q)} \otimes \tilde{\phi}_k(\theta_*)) + \theta_*^T (\mathcal{D}_{rs\theta} \tilde{\phi}_k(\theta))|_{\theta=\theta_*} \end{aligned} \quad (3.23)$$

then we have the estimated covariance matrices of vector ARMA process $\{y_{2,k}\}$

$$\begin{aligned} \tilde{R}_0(\tilde{\theta}) &= \frac{1}{N} \sum_{k=1}^N \left[\tilde{y}_{2,k}(\tilde{\theta}) \tilde{y}_{2,k}^T(\tilde{\theta}) \right] \\ &= \frac{1}{N} \sum_{k=1}^N \left[\left(y_{2,k} - \mathcal{K}\delta\theta + o(\|\tilde{\theta} - \theta_*\|) \right) \left(y_{2,k} - \mathcal{K}\delta\theta + o(\|\tilde{\theta} - \theta_*\|) \right)^T \right] \\ &= \frac{1}{N} \sum_{k=1}^N \left[y_{2,k} y_{2,k}^T - y_{2,k} \delta\theta^T \mathcal{K}_k^T - \mathcal{K}_k \delta\theta y_{2,k}^T + o(\|\tilde{\theta} - \theta_*\|) \right] \\ &\xrightarrow{N \rightarrow \infty} \begin{cases} R_0(\theta_*) + o(\|\tilde{\theta} - \theta_*\|), & \text{if in open loop} \\ R_0(\theta_*) - E \left[y_{2,k} \delta\theta^T \mathcal{K}_k^T + \mathcal{K}_k \delta\theta y_{2,k}^T \right] + o(\|\tilde{\theta} - \theta_*\|), & \text{if in closed loop} \end{cases} \end{aligned} \quad (3.24)$$

Since in closed loop the control inputs which appear in \mathcal{K}_k are correlated with the output component $y_{2,k}$ driven by the innovations $\{\bar{e}_k\}$, the term $E \left[y_{2,k} \delta\theta^T \mathcal{K}_k^T + \mathcal{K}_k \delta\theta y_{2,k}^T \right]$ does not approach to 0 as $N \rightarrow \infty$. This term, while for the open loop case, approaches to 0 as $N \rightarrow \infty$ due to the uncorrelation of the control inputs with innova-

tions. Likewise, for $\tilde{R}_i(\tilde{\theta})$, we have

$$\begin{aligned} \tilde{R}_i(\tilde{\theta}) &= \frac{1}{N-i} \sum_{k=i+1}^N \left[\tilde{y}_{2,k}(\tilde{\theta}) \tilde{y}_{2,k-i}^T(\tilde{\theta}) \right] \\ &\xrightarrow{N \rightarrow \infty} \begin{cases} R_i(\theta_*) + o(\|\tilde{\theta} - \theta_*\|), & \text{if in open loop} \\ R_i(\theta_*) - E \left[y_{2,k} \delta \theta^T \mathcal{K}_{k-i}^T + \mathcal{K}_{k-i} \delta \theta y_{2,k-i}^T \right] + o(\|\tilde{\theta} - \theta_*\|), & \text{if in closed loop} \end{cases} \end{aligned} \quad (3.25)$$

It is noticed that due to the perturbation $\delta\theta$, the difference of $R_i(\tilde{\theta}) = E[\tilde{R}_i(\tilde{\theta})]$ ($i = 0, 1, \dots, 2m-1$) from $R_i(\theta_*)$ is $O(\delta\theta)$ for the closed-loop case while for the open loop case it is $o(\delta\theta)$. Next, we will study the asymptotic performance of stochastic subspace identification. To avoid the contamination from $\delta\theta$, we assume $\tilde{\theta} = \theta_*$.

3.2.2 Asymptotic Distribution of the Empirical Sample Covariance

To facilitate the deduction of asymptotic distributions of the empirical sample estimates of variances and covariances of vector ARMA process $\{y_{2,k}\}$, we introduce a Lemma which illustrates that the expectation of the product of four scalar or vector random variables, which are jointly Gaussian distributed, can be expressed in terms of first- and second-order moments.

Lemma 4. *If $x_1, x_2 \in \mathcal{R}$ and $X_3, X_4 \in \mathcal{R}^{n \times 1}$ have jointly gaussian distributions, then*

$$\begin{aligned} E[x_1 x_2 X_3 X_4^T] &= E[x_1 x_2] E[X_3 X_4^T] + E[x_1 X_3] E[x_2 X_4^T] + E[x_2 X_3] E[x_1 X_4^T] \\ &\quad - 2E[x_1] E[x_2] E[X_3] E[X_4^T] \end{aligned} \quad (3.26)$$

Proof. We note that for $1 \leq i \leq n$ and $1 \leq j \leq n$, the (i, j) entry of $E[x_1x_2X_3X_4^T]$,

$$\begin{aligned}
& (E[x_1x_2X_3X_4^T])_{i,j} \\
&= \mathbf{e}_i^T E[x_1x_2X_3X_4^T] \mathbf{e}_j = E[x_1x_2\mathbf{e}_i^T X_3X_4^T \mathbf{e}_j] \\
&= E[x_1x_2]E[\mathbf{e}_i^T X_3X_4^T \mathbf{e}_j] + E[x_1\mathbf{e}_i^T X_3]E[x_2X_4^T \mathbf{e}_j] + E[x_1X_4^T \mathbf{e}_j]E[x_2\mathbf{e}_i^T X_3] \\
&\quad - 2E[x_1]E[x_2]E[\mathbf{e}_i^T X_3]E[X_4^T \mathbf{e}_j] \\
&= \mathbf{e}_i^T (E[x_1x_2]E[X_3X_4^T] + E[x_1X_3]E[x_2X_4^T] + E[x_2X_3]E[x_1X_4^T] \\
&\quad - 2E[x_1]E[x_2]E[X_3]E[X_4^T]) \mathbf{e}_j \tag{3.27}
\end{aligned}$$

where from the second line to the third line in the equation above we use the fact (Bär and Dittrich (1971)) that for four real scalar random variables $\{x_i\}$, $i = 1, 2, 3, 4$ which are jointly Gaussian distributed, we have that

$$\begin{aligned}
E(x_1x_2x_3x_4) &= E(x_1x_2)E(x_3x_4) + E(x_1x_3)E(x_2x_4) + E(x_1x_4)E(x_2x_3) \\
&\quad - 2E(x_1)E(x_2)E(x_3)E(x_4) \tag{3.28}
\end{aligned}$$

Then the Lemma is concluded. \square

We can now propose a theorem which reveals the asymptotic distributions of the empirical sample estimates of variances and covariances of the vector ARMA process $\{y_{2,k}\}$.

Theorem 5. *Consider the empirical sample covariance \tilde{R}_0 for the vector ARMA process $\{y_{2,k}\}$ given by (2.27) and the empirical Hankel matrix \tilde{H} given by substituting each R_k in (2.37) by \tilde{R}_k in (2.34). Then \tilde{R}_0 and \tilde{H} are both asymptotically normally distributed*

$$\sqrt{N}\text{vec}(\tilde{R}_0 - R_0) \longrightarrow \mathcal{N}(0, P_{R_0}) \tag{3.29}$$

$$\sqrt{N}\text{vec}(\tilde{H} - H) \longrightarrow \mathcal{N}(0, P_H) \tag{3.30}$$

with the covariance matrices P_{R_0} and P_H given by

$$P_{R_0} = (I_{n_e^2} + K_{n_e}) \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{y_2}^*(\omega) \otimes S_{y_2}(\omega) d\omega \quad (3.31)$$

$$P_H = (I_{n_e^2 m^2} + K_{n_e m}) \frac{1}{2\pi} \int_{-\pi}^{\pi} ((E_2 E_2^H) \otimes S_{y_2}(\omega))^* \otimes ((E_1 E_1^H) \otimes S_{y_2}(\omega)) d\omega \quad (3.32)$$

where $S_{y_2}(\omega)$ is the PSD of the vector ARMA process $\{y_{2,k}\}$; K_n is a permutation matrix satisfying

$$K_n(\mathbf{e}_i \otimes \mathbf{e}_j) = \mathbf{e}_j \otimes \mathbf{e}_i \quad (3.33)$$

; $E_1 = [1 \ e^{i\omega} \ \dots \ e^{i\omega(m-1)}]^T$; $E_2 = [e^{-i\omega} \ e^{-2i\omega} \ \dots \ e^{-i\omega m}]^T$; \mathbf{e}_i is a n -dimensional unit vector with the i th element be 1 and others 0. n_e is the dimension of subsystem (2.27).

Proof. We first proof (3.31), and then extend the proof to obtain (3.32). We have the covariance of $\text{vec}(\tilde{R}_0 - R_0)$

$$\begin{aligned} & \text{Cov}(\text{vec}(\tilde{R}_0 - R_0), \text{vec}(\tilde{R}_0 - R_0)) \\ &= E \left[\frac{1}{N} \sum_{k=1}^N (y_{2,k} \otimes y_{2,k} - \text{vec}(R_0)) \frac{1}{N} \sum_{h=1}^N (y_{2,h} \otimes y_{2,h} - \text{vec}(R_0))^T \right] \\ &= \frac{1}{N^2} \sum_{1 \leq k, h \leq N} E \left[(y_{2,k} \otimes y_{2,k} - \text{vec}(R_0)) (y_{2,h} \otimes y_{2,h} - \text{vec}(R_0))^T \right] \end{aligned} \quad (3.34)$$

In the second line, we use the fact $\text{vec}(y_{2,k} y_{2,k}^T) = y_{2,k} \otimes y_{2,k}$. Each entry in the sum

is

$$\begin{aligned}
& E \left[(y_{2,k} \otimes y_{2,k} - \text{vec}(R_0)) (y_{2,h} \otimes y_{2,h} - \text{vec}(R_0))^T \right] \\
&= E \left[(y_{2,k} \otimes y_{2,k}) (y_{2,h} \otimes y_{2,h})^T \right] - \text{vec}(R_0) \text{vec}(R_0)^T \\
&= E \left[(y_{2,k} y_{2,h}^T) \otimes (y_{2,k} y_{2,h}^T) \right] - \text{vec}(R_0) \text{vec}(R_0)^T \\
&= E \left[\left(\sum_{i,j} y_{2,k}^{(i)} y_{2,h}^{(j)} \mathbf{e}_i \mathbf{e}_j^T \right) \otimes (y_{2,k} y_{2,h}^T) \right] - \text{vec}(R_0) \text{vec}(R_0)^T \\
&= \sum_{i,j} \mathbf{e}_i \mathbf{e}_j^T \otimes E \left(y_{2,k}^{(i)} y_{2,h}^{(j)} y_{2,k} y_{2,h}^T \right) - \text{vec}(R_0) \text{vec}(R_0)^T \\
&= \sum_{i,j} \mathbf{e}_i \mathbf{e}_j^T \otimes \left(E(y_{2,k}^{(i)} y_{2,h}^{(j)}) E(y_{2,k} y_{2,h}^T) + E(y_{2,k}^{(i)} y_{2,k}) E(y_{2,h}^{(j)} y_{2,h}^T) + E(y_{2,h}^{(j)} y_{2,k}) E(y_{2,k}^{(i)} y_{2,h}^T) \right) \\
&\quad - \text{vec}(R_0) \text{vec}(R_0)^T \\
&= A_1 + A_2 + A_3 - \text{vec}(R_0) \text{vec}(R_0)^T \tag{3.35}
\end{aligned}$$

where we use Lemma 4 to derive the sixth line from the fifth line, and we denote

$$A_1 = \sum_{i,j} \mathbf{e}_i \mathbf{e}_j^T \otimes E(y_{2,k}^{(i)} y_{2,h}^{(j)}) E(y_{2,k} y_{2,h}^T) \tag{3.36}$$

$$A_2 = \sum_{i,j} \mathbf{e}_i \mathbf{e}_j^T \otimes E(y_{2,k}^{(i)} y_{2,k}) E(y_{2,h}^{(j)} y_{2,h}^T) \tag{3.37}$$

$$A_3 = \sum_{i,j} \mathbf{e}_i \mathbf{e}_j^T \otimes E(y_{2,h}^{(j)} y_{2,k}) E(y_{2,k}^{(i)} y_{2,h}^T) \tag{3.38}$$

For A_1 , we have

$$\begin{aligned}
A_1 &= \sum_{i,j} \mathbf{e}_i \mathbf{e}_j^T \otimes E(y_{2,k}^{(i)} y_{2,h}^{(j)}) E(y_{2,k} y_{2,h}^T) \\
&= \sum_{i,j,m,n} (\mathbf{e}_i \mathbf{e}_j^T) \otimes (\mathbf{e}_m \mathbf{e}_n^T) E(y_{2,k}^{(i)} y_{2,h}^{(j)}) E(y_{2,k}^{(m)} y_{2,h}^{(n)}) \\
&= \sum_{i,j,m,n} (\mathbf{e}_i \otimes \mathbf{e}_m) (\mathbf{e}_j^T \otimes \mathbf{e}_n^T) E(y_{2,k}^{(i)} y_{2,h}^{(j)}) E(y_{2,k}^{(m)} y_{2,h}^{(n)}) \tag{3.39}
\end{aligned}$$

For A_2 , we have

$$\begin{aligned}
A_2 &= \sum_{i,j} \mathbf{e}_i \mathbf{e}_j^T \otimes (E(y_{2,k}^{(i)} y_{2,k}) E(y_{2,h}^{(j)} y_{2,h}^T)) \\
&= \sum_{i,j} \left(\mathbf{e}_i \otimes E(y_{2,k}^{(i)} y_{2,k}) \right) \left(\mathbf{e}_j \otimes E(y_{2,h}^{(j)} y_{2,h}^T) \right)^T \\
&= \sum_{i,j} \text{vec}(E(y_{2,k}^{(i)} y_{2,k}) \mathbf{e}_i^T) \text{vec}^T(E(y_{2,h}^{(j)} y_{2,h}^T) \mathbf{e}_j^T) \\
&= \text{vec}(E(y_{2,k} y_{2,k}^T)) \text{vec}^T(E(y_{2,h} y_{2,h}^T)) \\
&= \text{vec}(R_0) \text{vec}(R_0)^T
\end{aligned} \tag{3.40}$$

For A_3 , we have

$$\begin{aligned}
A_3 &= \sum_{i,j} \mathbf{e}_i \mathbf{e}_j^T \otimes E(y_{2,h}^{(j)} y_{2,k}) E(y_{2,k}^{(i)} y_{2,h}^T) \\
&= \sum_{i,j} (\mathbf{e}_i \mathbf{e}_j^T) \otimes \left(\sum_{m,n} \mathbf{e}_m \mathbf{e}_n^T E(y_{2,k}^{(m)} y_{2,h}^{(j)}) E(y_{2,k}^{(i)} y_{2,h}^{(n)}) \right) \\
&= \sum_{i,j,m,n} ((\mathbf{e}_i \mathbf{e}_j^T) \otimes (\mathbf{e}_m \mathbf{e}_n^T)) E(y_{2,k}^{(m)} y_{2,h}^{(j)}) E(y_{2,k}^{(i)} y_{2,h}^{(n)}) \\
&= \sum_{i,j,m,n} ((\mathbf{e}_m \mathbf{e}_j^T) \otimes (\mathbf{e}_i \mathbf{e}_n^T)) E(y_{2,k}^{(i)} y_{2,h}^{(j)}) E(y_{2,k}^{(m)} y_{2,h}^{(n)}) \\
&= \sum_{i,j,m,n} (\mathbf{e}_m \otimes \mathbf{e}_i) (\mathbf{e}_j^T \otimes \mathbf{e}_n^T) E(y_{2,k}^{(i)} y_{2,h}^{(j)}) E(y_{2,k}^{(m)} y_{2,h}^{(n)}) \\
&= \sum_{i,j,m,n} K_{n_e} (\mathbf{e}_i \otimes \mathbf{e}_m) (\mathbf{e}_j^T \otimes \mathbf{e}_n^T) E(y_{2,k}^{(i)} y_{2,h}^{(j)}) E(y_{2,k}^{(m)} y_{2,h}^{(n)}) \\
&= K_{n_e} A_1
\end{aligned} \tag{3.41}$$

where we switch the indices i and m to derive the fourth line from the third line.

Substituting (3.39) ~ (3.41) to (3.35), we have

$$\begin{aligned}
& E \left[(y_{2,k} \otimes y_{2,k} - \text{vec}(R_0)) (y_{2,h} \otimes y_{2,h} - \text{vec}(R_0))^T \right] \\
&= (I_{n_e^2} + K_{n_e}) \sum_{i,j,m,n} (\mathbf{e}_i \otimes \mathbf{e}_m) (\mathbf{e}_j^T \otimes \mathbf{e}_n^T) E(y_{2,k}^{(i)} y_{2,h}^{(j)}) E(y_{2,k}^{(m)} y_{2,h}^{(n)}) \\
&= (I_{n_e^2} + K_{n_e}) E(y_{2,k} y_{2,h}^T) \otimes E(y_{2,k} y_{2,h}^T) \\
&= (I_{n_e^2} + K_{n_e}) R_{k-h} \otimes R_{k-h} \tag{3.42}
\end{aligned}$$

Substituting (3.42) to (3.34), we have

$$\begin{aligned}
& \text{Cov}(\text{vec}(\tilde{R}_0 - R_0), \text{vec}(\tilde{R}_0 - R_0)) \\
&= \frac{1}{N^2} (I_{n_e^2} + K_{n_e}) \sum_{1 \leq k, h \leq N} R_{k-h} \otimes R_{k-h} \\
&= \frac{1}{N} (I_{n_e^2} + K_{n_e}) \sum_{\tau} R(\tau) \otimes R(\tau) \\
&= \frac{1}{N} (I_{n_e^2} + K_{n_e}) \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{\tau_1, \tau_2} R(\tau_1) \otimes R(\tau_2) e^{-i\omega(\tau_2 - \tau_1)} d\omega \\
&= \frac{1}{N} (I_{n_e^2} + K_{n_e}) \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{\tau_1} R(\tau_1) e^{i\omega\tau_1} \otimes \sum_{\tau_2} R(\tau_2) e^{-i\omega\tau_2} d\omega \\
&= \frac{1}{N} (I_{n_e^2} + K_{n_e}) \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{y_2}^*(\omega) \otimes S_{y_2}(\omega) d\omega
\end{aligned}$$

where $S_{y_2}(\omega)$ denotes the PSD of the vector ARMA process $\{y_{2,k}\}$. Thus, (3.31) is concluded. For H in (2.37), following the same procedure we have

$$\begin{aligned}
& \text{Cov} \left(\text{vec}(\tilde{H} - H), \text{vec}(\tilde{H} - H) \right) \\
&= \text{Cov} \left(\text{vec} \left(\frac{1}{N} \sum_{k=1}^N V_{1,k} V_{2,k}^T - H \right), \text{vec} \left(\frac{1}{N} \sum_{k=1}^N V_{1,k} V_{2,k}^T - H \right) \right) \\
&= \frac{1}{N} (I_{n_e^2 m^2} + K_{n_e m}) \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{V_2}^*(\omega) \otimes S_{V_1}(\omega) d\omega \tag{3.43}
\end{aligned}$$

where $V_{1,k} = [y_{2,k}^T \ y_{2,k+1}^T \ \dots \ y_{2,k+m-1}^T]^T$ and $V_{2,k} = [y_{2,k-1}^T \ y_{2,k-2}^T \ \dots \ y_{2,k-m}^T]^T$; $S_{V_1}(\omega)$ and $S_{V_2}(\omega)$ denotes the spectral densities of $V_{1,k}$ and $V_{2,k}$ respectively, and can be obtained by

$$\begin{aligned}
S_{V_1}(\omega) &= \left(\begin{bmatrix} 1 \\ e^{i\omega} \\ \vdots \\ e^{i\omega(m-1)} \end{bmatrix} \otimes I_{n_e} \right) S_{y_2}(\omega) \left(\begin{bmatrix} 1 \\ e^{i\omega} \\ \vdots \\ e^{i\omega(m-1)} \end{bmatrix} \otimes I_{n_e} \right)^H \\
&= \left(\begin{bmatrix} 1 \\ e^{i\omega} \\ \vdots \\ e^{i\omega(m-1)} \end{bmatrix} \begin{bmatrix} 1 \\ e^{i\omega} \\ \vdots \\ e^{i\omega(m-1)} \end{bmatrix}^H \right) \otimes S_{y_2}(\omega) \tag{3.44}
\end{aligned}$$

Likewise, we have

$$\begin{aligned}
S_{V_2}(\omega) &= \left(\begin{bmatrix} e^{-i\omega} \\ e^{-2i\omega} \\ \vdots \\ e^{-i\omega m} \end{bmatrix} \otimes I_{n_e} \right) S_{y_2}(\omega) \left(\begin{bmatrix} e^{-i\omega} \\ e^{-2i\omega} \\ \vdots \\ e^{-i\omega m} \end{bmatrix} \otimes I_{n_e} \right)^H \\
&= \left(\begin{bmatrix} e^{-i\omega} \\ e^{-2i\omega} \\ \vdots \\ e^{-i\omega m} \end{bmatrix} \begin{bmatrix} e^{-i\omega} \\ e^{-2i\omega} \\ \vdots \\ e^{-i\omega m} \end{bmatrix}^H \right) \otimes S_{y_2}(\omega) \tag{3.45}
\end{aligned}$$

Substituting (3.44) and (3.45) to (3.43) gives (3.32). Proposition 7.3.2 ~ 7.3.4 in Brockwell and Davis (1991), as well as theorem 1 in Delmas and Meurisse (2000), claim that the sample covariances \tilde{R}_k in (2.34) are normally distributed as the measurement data size N goes to infinity. Thus, (3.29) and (3.30) are concluded. \square

Remark 1: the PSD $S_{y_2}(\omega)$ of vector ARMA process $\{y_{2,k}\}$ can be computed by the transfer function from the normalized innovations to output, i.e.

$$S_{y_2}(\omega) = G_e(e^{i\omega})G_e^T(e^{-i\omega}) \tag{3.46}$$

3.2.3 Asymptotic Distributions of the State Space Matrices in the Covariance Model

Based on the asymptotic distributions of the estimated covariances of vector ARMA process $\{y_{2,k}\}$, perturbation analysis of SVD is firstly applied to derive the asymptotic distributions of the controllability and observability matrices, from which we then derive the asymptotic distributions of the state space matrices in the covariance model.

Assume the true covariance matrix H has SVD

$$H = U_s \Lambda_s V_s^T + U_n \Lambda_n V_n^T \quad (3.47)$$

where the diagonal matrix $\Lambda_s \in \mathcal{R}^{n_e \times n_e}$ and $\Lambda_n = \mathbf{0}$. Due to finite data samples, there exists a perturbation δH in \tilde{H}

$$\tilde{H} = H + \delta H \quad (3.48)$$

which results in the corresponding perturbations of subspaces and singular values. Xu (2002) developed the perturbation analysis of SVD to the second order. We will apply its main theorem to analyze how δH influences the SVD of \tilde{H} . Accordingly, the SVD on \tilde{H} gives

$$\tilde{H} = \tilde{U}_s \tilde{\Lambda}_s \tilde{V}_s^T + \tilde{U}_n \tilde{\Lambda}_n \tilde{V}_n^T \quad (3.49)$$

Due to δH , all the terms on the right hand side of (3.49) may differ from those on the right hand side of (3.47). First we define the terms below

$$\begin{aligned} E_{ss} &= U_s^T \delta H V_s & E_{sn} &= U_s^T \delta H V_n \\ E_{ns} &= U_n^T \delta H V_s & E_{nn} &= U_n^T \delta H V_n \end{aligned} \quad (3.50)$$

According to the main theorem in Xu (2002), we have that

$$\begin{aligned}
\tilde{\Lambda}_s &= \Lambda_s + U_s^T \delta H V_s + o(\delta H), \quad \tilde{\Lambda}_n = \Lambda_n + U_n^T \delta H V_n + o(\delta H) \\
\tilde{U}_s &= U_s + U_n U_n^T \delta H V_s \Lambda_s^{-1} + o(\delta H) \\
\tilde{V}_s &= V_s + V_n V_n^T \delta H^T U_s \Lambda_s^{-1} + o(\delta H) \\
\tilde{U}_n &= U_n - U_s \Lambda_s^{-1} V_s^T \delta H^T U_n + o(\delta H) \\
\tilde{V}_n &= V_n - V_s \Lambda_s^{-1} U_s^T \delta H V_n + o(\delta H)
\end{aligned} \tag{3.51}$$

To evaluate $\tilde{U}_s \tilde{\Lambda}_s^{\frac{1}{2}}$ and $\tilde{\Lambda}_s^{\frac{1}{2}} \tilde{V}_s^T$, we need to first get the perturbed term of $\tilde{\Lambda}_s^{\frac{1}{2}}$. Suppose its first-order perturbation approximation as

$$\tilde{\Lambda}_s^{\frac{1}{2}} = \Lambda_s^{\frac{1}{2}} + \delta \Lambda_{sq} + o(\delta \Lambda_{sq}) \tag{3.52}$$

Substituting it to (3.51) obtains

$$\Lambda_s + U_s^T \delta H V_s + o(\delta H) = \left(\Lambda_s^{\frac{1}{2}} + \delta \Lambda_{sq} + o(\delta \Lambda_{sq}) \right) \left(\Lambda_s^{\frac{1}{2}} + \delta \Lambda_{sq} + o(\delta \Lambda_{sq}) \right) \tag{3.53}$$

Applying dominant balance yields

$$U_s^T \delta H V_s = \Lambda_s^{\frac{1}{2}} \delta \Lambda_{sq} + \delta \Lambda_{sq} \Lambda_s^{\frac{1}{2}} \tag{3.54}$$

Since $\Lambda_s^{\frac{1}{2}}$ is a diagonal matrix with each diagonal entry positive, $I_{n_e} \otimes \Lambda_s^{\frac{1}{2}} + \Lambda_s^{\frac{1}{2}} \otimes I_{n_e}$ is nonsingular. There exists a unique solution to (3.54)

$$\begin{aligned}
\text{vec}(\delta \Lambda_{sq}) &= \left(I_{n_e} \otimes \Lambda_s^{\frac{1}{2}} + \Lambda_s^{\frac{1}{2}} \otimes I_{n_e} \right)^{-1} \text{vec}(U_s^T \delta H V_s) \\
&= \left(I_{n_e} \otimes \Lambda_s^{\frac{1}{2}} + \Lambda_s^{\frac{1}{2}} \otimes I_{n_e} \right)^{-1} (V_s^T \otimes U_s^T) \text{vec}(\delta H)
\end{aligned} \tag{3.55}$$

Then we have

$$\begin{aligned}
\tilde{U}_s \tilde{\Lambda}_s^{\frac{1}{2}} &= (U_s + U_n U_n^T \delta H V_s \Lambda_s^{-1} + o(\delta H)) \left(\Lambda_s^{\frac{1}{2}} + \delta \Lambda_{sq} + o(\delta \Lambda_{sq}) \right) \\
&= U_s \Lambda_s^{\frac{1}{2}} + U_s \delta \Lambda_{sq} + U_n U_n^T \delta H V_s \Lambda_s^{-\frac{1}{2}} + o(\delta H)
\end{aligned} \tag{3.56}$$

Vectorizing both sides of the equation above gives

$$\begin{aligned}
& \text{vec} \left(\tilde{U}_s \tilde{\Lambda}_s^{\frac{1}{2}} - U_s \Lambda_s^{\frac{1}{2}} \right) \\
&= \text{vec} \left(U_s \delta \Lambda_{sq} + U_n U_n^T \delta H V_s \Lambda_s^{-\frac{1}{2}} \right) + o(\delta H) \\
&= (I_{n_e} \otimes U_s) \text{vec}(\delta \Lambda_{sq}) + \left(\left(\Lambda_s^{-\frac{1}{2}} V_s^T \right) \otimes (U_n U_n^T) \right) \text{vec}(\delta H) + o(\delta H) \quad (3.57)
\end{aligned}$$

Substituting (3.55) to the equation above gives

$$\begin{aligned}
& \text{vec} \left(\tilde{U}_s \tilde{\Lambda}_s^{\frac{1}{2}} - U_s \Lambda_s^{\frac{1}{2}} \right) \\
&= \left[(I_{n_e} \otimes U_s) \left(I_{n_e} \otimes \Lambda_s^{\frac{1}{2}} + \Lambda_s^{\frac{1}{2}} \otimes I_{n_e} \right)^{-1} (V_s^T \otimes U_s^T) + \left(\Lambda_s^{-\frac{1}{2}} V_s^T \right) \otimes (U_n U_n^T) \right] \text{vec}(\delta H) + o(\delta H) \quad (3.58)
\end{aligned}$$

Likewise, we have

$$\begin{aligned}
& \text{vec} \left(\tilde{\Lambda}_s^{\frac{1}{2}} \tilde{V}_s^T - \Lambda_s^{\frac{1}{2}} V_s^T \right) \\
&= \left[(V_s \otimes I_{n_e}) \left(I_{n_e} \otimes \Lambda_s^{\frac{1}{2}} + \Lambda_s^{\frac{1}{2}} \otimes I_{n_e} \right)^{-1} (V_s^T \otimes U_s^T) + (V_n V_n^T) \otimes \left(\Lambda_s^{-\frac{1}{2}} U_s^T \right) \right] \text{vec}(\delta H) + o(\delta H) \quad (3.59)
\end{aligned}$$

From theorem 5 and (3.58) ~ (3.59), we have the following theorem which reveals the asymptotic distributions of the observability and controllability matrices in the covariance model.

Theorem 6. *Consider the observability matrix $\tilde{\Omega}$ and controllability matrix $\tilde{\Gamma}$ estimated by (2.44) for $T = I_{n_x}$. They are asymptotically normally distributed*

$$\begin{aligned}
\sqrt{N} \text{vec}(\tilde{\Omega} - \Omega) &\longrightarrow \mathcal{N}(0, P_\Omega) \\
\sqrt{N} \text{vec}(\tilde{\Gamma} - \Gamma) &\longrightarrow \mathcal{N}(0, P_\Gamma) \quad (3.60)
\end{aligned}$$

The asymptotic covariance matrices are

$$P_\Omega = \Pi_1 P_H \Pi_1^T \quad (3.61)$$

$$P_\Gamma = \Pi_2 P_H \Pi_2^T \quad (3.62)$$

where Π_1 and Π_2 are given as

$$\begin{aligned}\Pi_1 &= \left[(I_{n_e} \otimes U_s) \left(I_{n_e} \otimes \Lambda_s^{\frac{1}{2}} + \Lambda_s^{\frac{1}{2}} \otimes I_{n_e} \right)^{-1} (V_s^T \otimes U_s^T) + \left(\Lambda_s^{-\frac{1}{2}} V_s^T \right) \otimes (U_n U_n^T) \right] \\ \Pi_2 &= \left[(V_s \otimes I_{n_e}) \left(I_{n_e} \otimes \Lambda_s^{\frac{1}{2}} + \Lambda_s^{\frac{1}{2}} \otimes I_{n_e} \right)^{-1} (V_s^T \otimes U_s^T) + (V_n V_n^T) \otimes \left(\Lambda_s^{-\frac{1}{2}} U_s^T \right) \right] \quad (3.63)\end{aligned}$$

From the asymptotical distributions of observability and controllability matrices, we have the following theorem for the asymptotical distributions of state space matrices in covariance model.

Theorem 7. *Assume $m \geq n+1$. The state space matrices \tilde{A}_e , \tilde{C}_e , and \tilde{D}_e estimated from (2.45) and (2.46) are asymptotically normally distributed*

$$\begin{aligned}\sqrt{N} \text{vec}(\tilde{A}_e - A_e) &\longrightarrow \mathcal{N}(0, P_A) \\ \sqrt{N} \text{vec}(\tilde{C}_e - C_e) &\longrightarrow \mathcal{N}(0, P_C) \\ \sqrt{N} \text{vec}(\tilde{D}_e - D_e) &\longrightarrow \mathcal{N}(0, P_D)\end{aligned} \quad (3.64)$$

where

$$\begin{aligned}P_A &= \Xi P_\Omega \Xi^T, \quad P_C = (I_{n_x} \otimes \Phi_3) P_\Omega (I_{n_x} \otimes \Phi_3)^T \\ P_D &= (\Phi_4^T \otimes I_{n_x}) P_\Gamma (\Phi_4^T \otimes I_{n_x})^T\end{aligned} \quad (3.65)$$

and

$$\begin{aligned}\Phi_1 &= [I, 0] \in \mathcal{R}^{(m-1)n_e \times mn_e}, \quad \Phi_2 = [0, I] \in \mathcal{R}^{(m-1)n_e \times mn_e} \\ \Phi_3 &= [I, 0] \in \mathcal{R}^{n_e \times mn_e}, \quad \Phi_4 = [I, 0]^T \in \mathcal{R}^{mn_e \times n_e} \\ \Psi_1 &= (\Omega^T \Phi_1^T \Phi_1 \Omega)^{-1}, \quad \Psi_2 = \Phi_1^T \Phi_2 \Omega, \quad \Psi_3 = \Omega^T \Phi_1^T \Phi_2, \quad \Psi_4 = \Omega^T \Phi_1^T \Phi_1 \\ \Xi &= (\Psi_2^T \otimes \Psi_1) K_{n_x, n_e m} + I_{n_x} \otimes (\Psi_1 \Psi_3) - (\Psi_1 \Psi_3 \Omega)^T \otimes (\Psi_1 \Psi_4) \\ &\quad - ((\Psi_4^T \Psi_1 \Psi_3 \Omega)^T \otimes \Psi_1) K_{n_x, n_e m}\end{aligned} \quad (3.66)$$

where $K_{n_x, n_e m}$ is a permutation matrix satisfying

$$\text{vec}(\delta \Omega^T) = K_{n_x, n_e m} \text{vec}(\delta \Omega)$$

Proof. From the structure of (2.44), we have

$$\begin{aligned}\text{vec}(\tilde{C}_e - C_e) &= \text{vec}(\Phi_3 \delta \Omega) = (I_{n_x} \otimes \Phi_3) \text{vec}(\delta \Omega) \\ \text{vec}(\tilde{D}_e - D_e) &= \text{vec}(\delta \Gamma \Phi_4) = (\Phi_4^T \otimes I_{n_x}) \text{vec}(\delta \Gamma)\end{aligned}\quad (3.67)$$

Applying (3.60) yields

$$\sqrt{N} \text{vec}(\tilde{C}_e - C_e) \longrightarrow \mathcal{N}(\mathbf{0}, (I_{n_x} \otimes \Phi_3) P_\Omega (I_{n_x} \otimes \Phi_3)^T) \quad (3.68)$$

$$\sqrt{N} \text{vec}(\tilde{D}_e - D_e) \longrightarrow \mathcal{N}(\mathbf{0}, (\Phi_4^T \otimes I_{n_x}) P_\Gamma (\Phi_4^T \otimes I_{n_x})^T) \quad (3.69)$$

From (2.46), the least-square solution of the overdetermined linear equation for \tilde{A}_e is

$$\tilde{A}_e = \left[(\Phi_1 \tilde{\Omega})^T \Phi_1 \tilde{\Omega} \right]^{-1} (\Phi_1 \tilde{\Omega})^T \Phi_2 \tilde{\Omega} \quad (3.70)$$

where the inverse of the perturbed term $(\Phi_1 \tilde{\Omega})^T \Phi_1 \tilde{\Omega}$ gives

$$\begin{aligned}& \left((\Phi_1 \tilde{\Omega})^T \Phi_1 \tilde{\Omega} \right)^{-1} \\ &= \left((\Phi_1 \Omega)^T \Phi_1 \Omega + \Omega^T \Phi_1^T \Phi_1 \delta \Omega + \delta \Omega^T \Phi_1^T \Phi_1 \Omega + o(\delta \Omega) \right)^{-1} \\ &= - \left((\Phi_1 \Omega)^T \Phi_1 \Omega \right)^{-1} \left(\Omega^T \Phi_1^T \Phi_1 \delta \Omega + \delta \Omega^T \Phi_1^T \Phi_1 \Omega \right) \left((\Phi_1 \Omega)^T \Phi_1 \Omega \right)^{-1} \\ & \quad + \left((\Phi_1 \Omega)^T \Phi_1 \Omega \right)^{-1} + o(\delta \Omega)\end{aligned}\quad (3.71)$$

Substituting it to \tilde{A}_e yields

$$\begin{aligned}\tilde{A}_e &= o(\delta \Omega) + A_e + \left(\Omega^T \Phi_1^T \Phi_1 \Omega \right)^{-1} \left(\delta \Omega^T \Phi_1^T \Phi_2 \Omega + \Omega^T \Phi_1^T \Phi_2 \delta \Omega \right) \\ & \quad - \left((\Phi_1 \Omega)^T \Phi_1 \Omega \right)^{-1} \left(\Omega^T \Phi_1^T \Phi_1 \delta \Omega + \delta \Omega^T \Phi_1^T \Phi_1 \Omega \right) \left((\Phi_1 \Omega)^T \Phi_1 \Omega \right)^{-1} \Omega^T \Phi_1^T \Phi_2 \Omega\end{aligned}\quad (3.72)$$

where we use the fact $A_e = \left(\Omega^T \Phi_1^T \Phi_1 \Omega \right)^{-1} \Omega^T \Phi_1^T \Phi_2 \Omega$. For brevity, substituting $\Psi_1 \sim \Psi_4$ given by (3.66) to the expression of \tilde{A}_e obtains

$$\tilde{A}_e - A_e = \Psi_1 \delta \Omega^T \Psi_2 + \Psi_1 \Psi_3 \delta \Omega - \Psi_1 \Psi_4 \delta \Omega \Psi_1 \Psi_3 \Omega - \Psi_1 \delta \Omega^T \Psi_4^T \Psi_1 \Psi_3 \Omega + o(\delta \Omega) \quad (3.73)$$

Vectorizing both sides of the equation above gives

$$\text{vec}(\tilde{A}_e - A_e) = \Xi \text{vec} \delta \Omega + o(\delta \Omega) \quad (3.74)$$

Likewise, applying (3.60) yields

$$\sqrt{N}\text{vec}(\tilde{A}_e - A_e) \longrightarrow \mathcal{N}(\mathbf{0}, \Xi P_\Omega \Xi^T) \quad (3.75)$$

□

3.2.4 Perturbation Analysis of DARE

In this subsection, we use perturbation analysis of DARE to derive the F-norm error bound of Kalman gain K and the innovations covariance Q .

Consider the general DARE

$$P = A_e P A_e^T + (D_e - A_e P C_e^T)(R_0 - C_e P C_e^T)^{-1}(D_e - A_e P C_e^T)^T \quad (3.76)$$

To facilitate its perturbation analysis, the general DARE is transformed to the standard DARE

$$P = A_d^T P A_d - A_d^T P C_d^T (C_d P C_d^T - R_0)^{-1} C_d P A_d + D_e R_0^{-1} D_e^T \quad (3.77)$$

where $A_d = A_e^T - C_e^T R_0^{-1} D_e^T$. Denote $B_d = (R_0^{-\frac{1}{2}} C_e)^T$, and $M_d = D_e R_0^{-1} D_e^T$, we have

$$P - A_d^T P A_d - A_d^T P B_d (I - B_d^T P B_d)^T B_d^T P A_d - M_d = 0 \quad (3.78)$$

Using the identity

$$(I - B_d B_d^T P)^{-1} = I + B_d (I - B_d^T P B_d)^{-1} B_d^T P \quad (3.79)$$

and denote $S_d = B_d B_d^T$, we can rewrite (3.78) in its equivalent form

$$P - A_d^T P (I - S_d P)^{-1} A_d - M_d = 0 \quad (3.80)$$

Assume under the perturbation of δA_d , δS_d , and δM_d , there exists a unique non-negative solution in (3.80). If this assumption is not satisfied, the LMI optimization (2.53) ~ (2.55) is used beforehand to guarantee the solvability of DARE. with perturbed terms on A_e , C_e , R_0 , and D_e , it is verified that

$$\begin{aligned} \delta A_d &= \delta A_e^T - \delta C_e^T R_0^{-1} D_e^T + C_e^T R_0^{-1} \delta R R_0^{-1} D_e^T - C_e^T R_0^{-1} \delta D_e^T + o(\delta A_d) \\ \delta M_d &= -D_e R_0^{-1} \delta R_0 R_0^{-1} D_e^T + D_e R_0^{-1} \delta D_e^T + \delta D_e R_0^{-1} D_e^T + o(\delta M_d) \\ \delta S_d &= -C_e^T R_0^{-1} \delta R_0 R_0^{-1} C_e + \delta C_e^T R_0^{-1} C_e + C_e^T R_0^{-1} \delta C_e + o(\delta S_d) \end{aligned} \quad (3.81)$$

Theorem 3.1 in Konstantinov et al. (1993) gives the solution perturbation in terms of δM_d , δA_d and δS_d . Applying it yields

$$\|\delta P\|_F \leq K_M \|\delta M_d\|_F + K_A \|\delta A_d\|_F + K_S \|\delta S_d\|_F + O(\|\Delta\|_F^2) \quad (3.82)$$

where $\Delta = (\|\delta M_d\|_F, \|\delta A_d\|_F, \|\delta S_d\|_F)^T$; K_M , K_A , and K_S are defined as

$$K_M = \|M_M\|_2, \quad K_A = \|M_A\|_2, \quad K_S = \|M_S\|_2 \quad (3.83)$$

where

$$\begin{aligned} M_M &= \Lambda^{-1}, \quad M_A = \Lambda^{-1}[I_{n_x} \otimes A_c^T P + (A_c^T P \otimes I_{n_x})K_{n_x}] \\ M_S &= \Lambda^{-1}(A_c^T P \otimes A_c^T P), \quad \Lambda = I - A_c^T \otimes A_c^T, \quad A_c = (I - S_d P)^{-1} A_d \end{aligned} \quad (3.84)$$

From (3.81), we obtain the F-norm error bound of A_d , M_d and S_d

$$\begin{aligned} \|\delta A_d\|_F &\leq \|\delta A_e\|_F + \Pi_1 \|\delta C_e\|_F + \Pi_2 \|\delta R_0\|_F + \Pi_3 \|\delta D_e\|_F + o(\|\Delta\|) \\ \|\delta M_d\|_F &\leq \Pi_4 \|\delta R_0\|_F + 2\Pi_1 \|\delta D_e\|_F + o(\|\Delta\|) \\ \|\delta S_d\|_F &\leq \Pi_5 \|\delta R_0\|_F + 2\Pi_3 \|\delta C_e\|_F + o(\|\Delta\|) \end{aligned} \quad (3.85)$$

where we use the fact $\|\text{vec}(\cdot)\|_2 = \|\cdot\|_F$ in the deduction, and

$$\begin{aligned} \Pi_1 &= \|D_e R_0^{-1} \otimes I\|_2, \quad \Pi_2 = \|D_e R_0^{-1} \otimes C_e^T R_0^{-1}\|_2, \quad \Pi_3 = \|I \otimes C_e^T R_0^{-1}\|_2 \\ \Pi_4 &= \|D_e R_0^{-1} \otimes D_e R_0^{-1}\|_2, \quad \Pi_5 = \|C_e^T R_0^{-1} \otimes C_e^T R_0^{-1}\|_2 \end{aligned} \quad (3.86)$$

Substituting (3.85) to (3.82) gives the F-norm error bound of the positive definite solution P to DARE.

$$\begin{aligned} \|\delta P\|_F &\leq K_A \|\delta A_e\|_F + (K_A \Pi_1 + 2K_S \Pi_3) \|\delta C_e\|_F + (K_M \Pi_4 + K_A \Pi_2 + K_S \Pi_5) \|\delta R_0\|_F \\ &\quad + (2K_M \Pi_1 + K_A \Pi_3) \|\delta D_e\|_F + o(\|\Delta\|_F) \end{aligned} \quad (3.87)$$

For $Q = R_0 - C_e P C_e^T$ and $K = (D_e - A_e P C_e^T) Q^{-1}$ with perturbed terms on R_0 , A_e , C_e , P , and D_e , it is verified that

$$\begin{aligned} \delta Q &= \delta R_0 - \delta C_e P C_e^T - C_e \delta P C_e^T - C_e P \delta C_e^T + o(\delta Q) \\ \delta K &= -K \delta R_0 Q^{-1} + K \delta C_e P C_e^T Q^{-1} + (K C_e P - A_e P) \delta C_e^T Q^{-1} \\ &\quad + (K C_e - A_e) \delta P C_e^T Q^{-1} + \delta D_e Q^{-1} - \delta A_e P C_e^T Q^{-1} + o(\delta K) \end{aligned} \quad (3.88)$$

In view of (3.88), we get

$$\begin{aligned}
\|\delta Q\|_F &\leq K_A \Pi_7 \|\delta A_e\|_F + (2\Pi_6 + K_A \Pi_1 \Pi_7 + 2K_S \Pi_3 \Pi_7) \|\delta C_e\|_F + (2K_M \Pi_1 \Pi_7 + K_A \Pi_3 \Pi_7) \|\delta D_e\|_F \\
&\quad + (1 + K_M \Pi_4 \Pi_7 + K_A \Pi_2 \Pi_7 + K_S \Pi_5 \Pi_7) \|\delta R_0\|_F + o(\|\Delta\|_F) \\
\|\delta K\|_F &\leq (\Pi_{13} + K_A \Pi_{11}) \|\delta A_e\|_F + (\Pi_9 + \Pi_{10} + K_A \Pi_1 \Pi_{11} + 2K_S \Pi_3 \Pi_{11}) \|\delta C_e\|_F \\
&\quad + (\Pi_{12} + 2K_M \Pi_1 \Pi_{11} + K_A \Pi_3 \Pi_{11}) \|\delta D_e\|_F \\
&\quad + (\Pi_8 + K_M \Pi_4 \Pi_{11} + K_A \Pi_2 \Pi_{11} + K_S \Pi_5 \Pi_{11}) \|\delta R_0\|_F + o(\|\Delta\|_F) \tag{3.89}
\end{aligned}$$

where

$$\begin{aligned}
\Pi_6 &= \|C_e P \otimes I\|_2, \quad \Pi_7 = \|C_e \otimes C_e\|_2, \quad \Pi_8 = \|Q^{-1} \otimes K\|_2, \quad \Pi_9 = \|Q^{-1} C_e P \otimes K\|_2 \\
\Pi_{10} &= \|Q^{-1} \otimes (K C_e - A_e) P\|_2, \quad \Pi_{11} = \|Q^{-1} C_e \otimes (K C_e - A_e)\|_2 \\
\Pi_{12} &= \|Q^{-1} \otimes I\|_2, \quad \Pi_{13} = \|Q^{-1} C_e P \otimes I\|_2 \tag{3.90}
\end{aligned}$$

Remark: provided that (3.77) fails due to insufficient data, (2.52), as well as (2.57) and (2.58), is required to adjust \tilde{D}_e and \tilde{R}_0 for a valid model. In this case, we shall estimate the F-norm error bounds of D_e and R_0 by triangle inequality.

$$\begin{aligned}
\|\delta D_e\|_F &= \|\hat{\tilde{D}}_e - \tilde{D}_e + \tilde{D}_e - D_e\|_F \leq \|\hat{\tilde{D}}_e - \tilde{D}_e\|_F + \|\tilde{D}_e - D_e\|_F \\
\|\delta R_0\|_F &= \|\hat{\tilde{R}}_0 - \tilde{R}_0 + \tilde{R}_0 - R_0\|_F \leq \|\hat{\tilde{R}}_0 - \tilde{R}_0\|_F + \|\tilde{R}_0 - R_0\|_F \tag{3.91}
\end{aligned}$$

where $\hat{\tilde{D}}_e - \tilde{D}_e$ and $\hat{\tilde{R}}_0 - \tilde{R}_0$ result from the LMI-based adjustments in (2.57) ~ (2.58), and $\tilde{D}_e - D_e$ and $\tilde{R}_0 - R_0$ are asymptotically normally distributed, as shown in (3.64) and (3.29).

3.2.5 \mathcal{H}_2 Norm Model Error Bound for $G_e(z)$

In this subsection, we will derive the \mathcal{H}_2 norm bound of the model error for the identified $\tilde{G}_e(z)$ as well as a confidence level.

For brevity, we denote $B_e = KQ^{\frac{1}{2}}$ and $F_e = Q^{\frac{1}{2}}$. Thus, we have the true innovations model transfer function and the identified one as

$$G_e(\omega) = \left[\begin{array}{c|c} A_e & B_e \\ \hline C_e & F_e \end{array} \right], \quad \tilde{G}_e(\omega) = \left[\begin{array}{c|c} \tilde{A}_e & \tilde{B}_e \\ \hline \tilde{C}_e & \tilde{F}_e \end{array} \right] \tag{3.92}$$

Theorem 8. Let (A_e, B_e, C_e, F_e) and $(\tilde{A}_e, \tilde{B}_e, \tilde{C}_e, \tilde{F}_e)$ be state-space representations of the original and identified systems, as shown in (3.92), such that

$$\begin{aligned}\delta A_e &= o(A_e), \quad \delta B_e = o(B_e) \\ \delta C_e &= o(C_e), \quad \delta F_e = o(F_e)\end{aligned}\tag{3.93}$$

Assume, without loss of generality, that the state matrices A_e and \tilde{A}_e are Hurwitz. Let $G_e(\omega)$ and $\tilde{G}_e(\omega)$ be the corresponding transfer functions. Then the \mathcal{H}_2 norm of the error system can be bounded by

$$\begin{aligned}\|G_e(\omega) - \tilde{G}_e(\omega)\|_{\mathcal{H}_2}^2 &\leq 4\|B_e B_e^T\|_F \|\Upsilon_1\|_2 \|\Upsilon_2\|_2 \|\delta C_e\|_F + 4\|B_e B_e^T\|_F \|\Upsilon_1\|_2 \|\Upsilon_3\|_2 \|\delta A_e\|_F \\ &\quad + \|\delta F_e\|_F^2 + o(\|\bar{\Delta}\|_F)\end{aligned}\tag{3.94}$$

where

$$\begin{aligned}\Upsilon_1 &= (A_e^T \otimes A_e^T - I_{n_x} \otimes I_{n_x})^{-1} \\ \Upsilon_2 &= I_{n_x} \otimes C_e^T, \quad \Upsilon_3 = I_{n_x} \otimes A_e^T P_{11} \\ \bar{\Delta} &= (\|\delta A_e\|_F, \|\delta C_e\|_F)^T\end{aligned}\tag{3.95}$$

and P_{11} is the positive definite solution of the following Lyapunov equation.

$$A_e^T P_{11} A_e - P_{11} + C_e^T C_e = 0\tag{3.96}$$

Proof. Consider the transfer function of the error system

$$G_e(\omega) - \tilde{G}_e(\omega) = \left[\begin{array}{cc|c} A_e & \mathbf{0} & B_e \\ \mathbf{0} & \tilde{A}_e & \tilde{B}_e \\ \hline C_e & -\tilde{C}_e & F_e - \tilde{F}_e \end{array} \right]\tag{3.97}$$

The \mathcal{H}_2 norm of $G_e(\omega) - \tilde{G}_e(\omega)$ is computed by an algebraic approach

$$\begin{aligned}\|G_e(\omega) - \tilde{G}_e(\omega)\|_{\mathcal{H}_2}^2 &= \text{tr} \left[(F_e - \tilde{F}_e)^T (F_e - \tilde{F}_e) + \begin{pmatrix} B_e \\ \tilde{B}_e \end{pmatrix}^T \begin{pmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{pmatrix} \begin{pmatrix} B_e \\ \tilde{B}_e \end{pmatrix} \right] \\ &= \|\delta F_e\|_F^2 + \text{tr} \left[\begin{pmatrix} B_e \\ \tilde{B}_e \end{pmatrix}^T \begin{pmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{pmatrix} \begin{pmatrix} B_e \\ \tilde{B}_e \end{pmatrix} \right]\end{aligned}\tag{3.98}$$

where P_{11} , P_{12} and P_{22} are the solutions to

$$A_e^T P_{11} A_e - P_{11} + C_e^T C_e = 0 \quad (3.99)$$

$$A_e^T P_{12} \tilde{A}_e - P_{12} - C_e^T \tilde{C}_e = 0 \quad (3.100)$$

$$\tilde{A}_e^T P_{22} \tilde{A}_e - P_{22} + \tilde{C}_e^T \tilde{C}_e = 0 \quad (3.101)$$

and $P_{11} > 0$ due to the fact that (A_e, C_e) is observable and A_e is Hurwitz. Perturbation analysis of (3.100) and (3.101) gives the perturbed solution

$$\begin{aligned} P_{12} &= -P_{11} + \bar{P}_{12} + o((\delta A_e, \delta C_e)^T) \\ P_{22} &= P_{11} + \bar{P}_{22} + o((\delta A_e, \delta C_e)^T) \end{aligned} \quad (3.102)$$

where \bar{P}_{12} and \bar{P}_{22} are respectively the first order terms in the asymptotic expansions of P_{12} and P_{22} . Dominant balance gives

$$\begin{aligned} A_e^T \bar{P}_{12} A_e - A_e^T P_{11} \delta A_e - \bar{P}_{12} - C_e^T \delta C_e &= 0 \\ \delta A_e^T P_{11} A_e + A_e^T \bar{P}_{22} A_e + A_e^T P_{11} \delta A_e - \bar{P}_{22} + \delta C_e^T C_e + C_e^T \delta C_e &= 0 \end{aligned} \quad (3.103)$$

Algebraic manipulation gives the F-norms of \bar{P}_{12} and \bar{P}_{22} bounded by

$$\begin{aligned} \|\bar{P}_{12}\|_F &\leq \|\Upsilon_1 \Upsilon_3\|_2 \|\delta A_e\|_F + \|\Upsilon_1 \Upsilon_2\|_2 \|\delta C_e\|_F \\ \|\bar{P}_{22}\|_F &\leq 2\|\Upsilon_1\|_2 \|\Upsilon_3\|_2 \|\delta A_e\|_F + 2\|\Upsilon_1\|_2 \|\Upsilon_2\|_2 \|\delta C_e\|_F \end{aligned} \quad (3.104)$$

Substituting (3.102) to the second term of (3.98) gives

$$\begin{aligned} &\text{tr} \left(\begin{pmatrix} B_e \\ \tilde{B}_e \end{pmatrix}^T \begin{pmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{pmatrix} \begin{pmatrix} B_e \\ \tilde{B}_e \end{pmatrix} \right) \\ &= \text{tr}(B_e^T (\bar{P}_{12}^T + \bar{P}_{12} + \bar{P}_{22}) B_e) + o((\delta A_e, \delta C_e)^T) \\ &= \text{tr}(B_e B_e^T (\bar{P}_{12}^T + \bar{P}_{12} + \bar{P}_{22})) + o((\delta A_e, \delta C_e)^T) \\ &= \text{vec}(B_e B_e^T)^T \text{vec}(\bar{P}_{12}^T + \bar{P}_{12} + \bar{P}_{22}) + o((\delta A_e, \delta C_e)^T) \\ &\leq \|\text{vec}(B_e B_e^T)^T\|_2 \|\text{vec}(\bar{P}_{12}^T + \bar{P}_{12} + \bar{P}_{22})\|_2 + o(\|\bar{\Delta}\|_F) \\ &\leq \|B_e B_e^T\|_F (2\|\bar{P}_{12}\|_F + \|\bar{P}_{22}\|_F) + o(\|\bar{\Delta}\|_F) \end{aligned} \quad (3.105)$$

where we use Hölder's inequality from the fourth line to the fifth line in the equation above. Combining (3.104), (3.105), and (3.98) gives the conclusion. \square

Likewise, we have the following corollary which gives the \mathcal{H}_2 norm error bound for the identified deterministic part of a system.

Corollary 9. *Let (A_u, B_u, C_u) and $(\tilde{A}_u, \tilde{B}_u, \tilde{C}_u)$ be the state-space representations of the original and identified deterministic part of a system, as shown in (2.26), such that*

$$\delta A_u = o(A_u), \quad \delta B_u = o(B_u), \quad \delta C_u = o(C_u) \quad (3.106)$$

Assume, without loss of generality, that the state matrices A_u and \tilde{A}_u are Hurwitz. Let $G_u(\omega)$ and $\tilde{G}_u(\omega)$ be the corresponding transfer functions. Then the \mathcal{H}_2 norm of the error system can be bounded by

$$\|G_u(\omega) - \tilde{G}_u(\omega)\|_{\mathcal{H}_2}^2 \leq o(\|\tilde{\Delta}\|_F) + 4\|B_u B_u^T\|_F \|\tilde{\Upsilon}_1\|_2 \|\tilde{\Upsilon}_2\|_2 \|\delta\theta_A\|_F \quad (3.107)$$

where

$$\begin{aligned} \tilde{\Upsilon}_1 &= (A_u^T \otimes A_u^T - I \otimes I)^{-1} \\ \tilde{\Upsilon}_2 &= I \otimes A_u^T \bar{P}_{11} \\ \tilde{\Delta} &= (\|\delta A_u\|_F)^T \\ \theta_A &= [A_1, \dots, A_p]^T \end{aligned} \quad (3.108)$$

and \bar{P}_{11} is the nonnegative definite solution of the following Lyapunov equation.

$$A_u^T \bar{P}_{11} A_u - \bar{P}_{11} + C_u^T C_u = 0 \quad (3.109)$$

With $F_e = Q^{\frac{1}{2}}$, we have the F-norm bound of its perturbation $\delta F_e = \tilde{Q}^{\frac{1}{2}} - Q^{\frac{1}{2}}$.

$$\|\delta F_e\|_F^2 \leq \|\Upsilon_4\|_2^2 \|\delta Q\|_F^2 \quad (3.110)$$

where

$$\Upsilon_4 = (I_{n_e} \otimes Q^{\frac{1}{2}} + Q^{\frac{1}{2}} \otimes I_{n_e})^{-1} \quad (3.111)$$

From (3.94), (3.110), as well as $B_e = KQ^{\frac{1}{2}}$, we obtain the \mathcal{H}_2 norm bound of the error system.

$$\begin{aligned} \|G_e(\omega) - \tilde{G}_e(\omega)\|_{\mathcal{H}_2}^2 &\leq \|\Upsilon_4\|_2^2 \|\delta Q\|_F^2 + 4\|KQK^T\|_F \|\Upsilon_1\|_2 \|\Upsilon_2\|_2 \|\delta C_e\|_F \\ &\quad + 4\|KQK^T\|_F \|\Upsilon_1\|_2 \|\Upsilon_3\|_2 \|\delta A_e\|_F + o(\|\bar{\Delta}\|_F) \end{aligned} \quad (3.112)$$

where the F-norm bound of δQ is given by (3.89).

Remark: for a special case where the poles of the error system (3.97) are all distinct, we can get an explicit \mathcal{H}_∞ norm error bound in terms of \mathcal{H}_2 norm error bound. Theorem 2 in Hara et al. (1997) gives

$$\|G_e(\omega) - \tilde{G}_e(\omega)\|_{\mathcal{H}_\infty}^2 \leq \left(1 + 2n_x \frac{1 + \rho}{1 - \rho}\right) \|G_e(\omega) - \tilde{G}_e(\omega)\|_{\mathcal{H}_2}^2 \quad (3.113)$$

where ρ is the maximum absolute value of the system poles of the error system.

The next theorem gives the confidence level associated with the \mathcal{H}_2 norm error bound for the identified $\tilde{G}_e(\omega)$.

Theorem 10. *Let ϵ be a small number such that*

$$\epsilon = O(\|\delta H\|_F)$$

Assume the data size N is sufficiently large such that (3.29) and (3.64) approximately hold. Then the \mathcal{H}_2 norm bound of the error system (3.97) is approximately given by

$$\begin{aligned} \|G_e(\omega) - \tilde{G}_e(\omega)\|_{\mathcal{H}_2}^2 &\leq 4\|KQK^T\|_F \|\Upsilon_1\|_2 \|\Upsilon_3\|_2 \|\Xi\Pi_1\|_2 \epsilon \\ &\quad + 4\|KQK^T\|_F \|\Upsilon_1\|_2 \|\Upsilon_2\|_2 \|(I_{n_x} \otimes \Phi_3)\Pi_1\|_2 \epsilon + o(\|\bar{\Delta}\|_F) \end{aligned} \quad (3.114)$$

with a confidence level more than $1 - \frac{\text{tr}(P_H)}{N\epsilon^2}$ where N is the data size, and $\bar{\Delta} = (\|\delta A_e\|_F, \|\delta C_e\|_F, \|\delta D_e\|_F, \|\delta R_0\|_F)^T$.

Proof. According to multivariable Chebyshev inequality, for a normally distributed random vector $\text{vec}(\delta H) \in \mathcal{R}^{mn_y \times mn_y}$ with covariance matrix P_H/N and cumulative distribution $F(\cdot)$, we have

$$P\{\|\text{vec}(\delta H)\|_2 \geq \epsilon\} \leq \frac{\text{Var}(\text{vec}(\delta H))}{\epsilon^2} \quad (3.115)$$

where $P(\cdot)$ is the probability, and

$$\text{Var}(\text{vec}(\delta H)) := \int_{V \in \mathbf{R}^{m^2 n_y^2}} \|V\|_2^2 dF(V) \quad (3.116)$$

With $\|\text{vec}(\cdot)\|_2 = \|\cdot\|_F$ and $\text{Var}(\text{vec}(\delta H)) = \text{tr}(\frac{P_H}{N})$, we have

$$P\{\|\delta H\|_F \leq \epsilon\} \geq 1 - \frac{\text{tr}(P_H)}{N\epsilon^2} \quad (3.117)$$

Combining (3.58), (3.59), (3.67) and (3.74) gives

$$\begin{aligned} \text{vec}(\delta A_e) &= \Xi \Pi_1 \text{vec}(\delta H) \\ \text{vec}(\delta C_e) &= (I_{n_x} \otimes \Phi_3) \Pi_1 \text{vec}(\delta H) \end{aligned} \quad (3.118)$$

Then we have

$$\begin{aligned} \|\delta A_e\|_F &\leq \|\Xi \Pi_1\|_2 \epsilon \\ \|\delta C_e\|_F &\leq \|(I_{n_x} \otimes \Phi_3) \Pi_1\|_2 \epsilon \end{aligned} \quad (3.119)$$

Also from (3.89), we have

$$\|\delta Q\|_F^2 = o(\|\bar{\Delta}\|_F) \quad (3.120)$$

Substituting (3.119) and (3.120) to (3.112) gives (3.114). \square

3.2.6 \mathcal{H}_∞ Norm Model Error Bound for $G_e(z)$

In this subsection, we will first quantify the uncertainty of the state space matrices in the system $G_e(z)$ in terms of F-norm with a confidence level, and then propose two approaches, based on perturbation analysis and LMI technique, respectively, to computing the \mathcal{H}_∞ norm model error bound for $G_e(z)$ with a given confidence level.

Since $\text{vec}(\delta R_0)$ and $\text{vec}(\delta H)$ are asymptotically normally distributed with the covariance matrices P_{R_0} and P_H , from multivariable Chebyshev inequality, we have that

$$P\left\{\left\|\begin{bmatrix} \text{vec}(\delta R_0) \\ \text{vec}(\delta H) \end{bmatrix}\right\|_2 \leq \epsilon\right\} \geq 1 - \frac{\text{tr}(P_{R_0}) + \text{tr}(P_H)}{N\epsilon^2} \quad (3.121)$$

Then with a confidence level more than $1 - \frac{\text{tr}(P_{R_0}) + \text{tr}(P_H)}{N\epsilon^2}$, we have that

$$\begin{aligned}\|\delta H\|_F &\leq \left\| \begin{bmatrix} \text{vec}(\delta R_0) \\ \text{vec}(\delta H) \end{bmatrix} \right\|_2 \leq \epsilon \\ \|\delta R_0\|_F &\leq \left\| \begin{bmatrix} \text{vec}(\delta R_0) \\ \text{vec}(\delta H) \end{bmatrix} \right\|_2 \leq \epsilon\end{aligned}\quad (3.122)$$

Then from (3.67), (3.74), (3.58),(3.59) and (3.63), we have the following F-norm error bounds of A_e , C_e , and D_e

$$\|\delta A_e\|_F \leq \Theta_1 \epsilon, \quad \|\delta C_e\|_F \leq \Theta_2 \epsilon, \quad \|\delta D_e\|_F \leq \Theta_3 \epsilon \quad (3.123)$$

where

$$\Theta_1 = \|\Xi \Pi_1\|_2, \quad \Theta_2 = \|(I_{n_x} \otimes \Phi_3) \Pi_1\|_2, \quad \Theta_3 = \|(\Phi_4^T \otimes I_{n_x}) \Pi_2\|_2 \quad (3.124)$$

It is also verified that

$$\begin{aligned}\|\delta F_e\|_F &\leq \Theta_4 \|\delta Q\|_F \\ \|\delta B_e\|_F &\leq \Theta_5 \|\delta Q\|_F + \Theta_6 \|\delta K\|_F\end{aligned}\quad (3.125)$$

where

$$\Theta_4 = \|\Upsilon_4\|_2, \quad \Theta_5 = \|I_{n_y} \otimes K\|_2 \|\Upsilon_4\|_2, \quad \Theta_6 = \|Q^{\frac{1}{2}} \otimes I_{n_x}\|_2 \quad (3.126)$$

Substituting (3.89) and (3.123) to (3.125) yields

$$\begin{aligned}\|\delta F_e\|_F &\leq \Theta_4 [K_A \Pi_7 \Theta_1 + (2\Pi_6 + K_A \Pi_1 \Pi_7 + 2K_S \Pi_3 \Pi_7) \Theta_2 + (2K_M \Pi_1 \Pi_7 + K_A \Pi_3 \Pi_7) \Theta_3 \\ &\quad + (1 + K_M \Pi_4 \Pi_7 + K_A \Pi_2 \Pi_7 + K_S \Pi_5 \Pi_7)] \epsilon + o(\|\Delta\|_F) \\ \|\delta B_e\|_F &\leq [\Theta_5 (K_A \Pi_7) \Theta_1 + \Theta_5 (2\Pi_6 + K_A \Pi_1 \Pi_7 + 2K_S \Pi_3 \Pi_7) \Theta_2 + \Theta_5 (2K_M \Pi_1 \Pi_7 + K_A \Pi_3 \Pi_7) \Theta_3 \\ &\quad + \Theta_5 (1 + K_M \Pi_4 \Pi_7 + K_A \Pi_2 \Pi_7 + K_S \Pi_5 \Pi_7) + \Theta_6 (\Pi_{13} + K_A \Pi_{11}) \Theta_1 \\ &\quad + \Theta_6 (\Pi_9 + \Pi_{10} + K_A \Pi_1 \Pi_{11} + 2K_S \Pi_3 \Pi_{11}) \Theta_2 + \Theta_6 (\Pi_{12} + 2K_M \Pi_1 \Pi_{11} + K_A \Pi_3 \Pi_{11}) \Theta_3 \\ &\quad + \Theta_6 (\Pi_8 + K_M \Pi_4 \Pi_{11} + K_A \Pi_2 \Pi_{11} + K_S \Pi_5 \Pi_{11})] \epsilon + o(\|\Delta\|_F)\end{aligned}\quad (3.127)$$

Thus, with a confidence level more than $1 - \frac{\text{tr}(P_{R_0}) + \text{tr}(P_H)}{N\epsilon^2}$, we have the F-norm error bounds of the state space matrices in the system $G_e(z)$, shown in (3.123) and (3.127).

A straightforward way to derive the \mathcal{H}_∞ -norm bound of the error system is by perturbation analysis, although a tight bound is not guaranteed. Asymptotic expansion of the transfer function $G_e(\omega)$ for the original system with respect to the parameters for the identified system $\tilde{G}_e(\omega)$ in (3.97) yields

$$\begin{aligned}
& G_e(\omega) - \tilde{G}_e(\omega) \\
&= (\tilde{C}_e - \delta C_e) \left(e^{i\omega} I - \tilde{A}_e + \delta A_e \right)^{-1} (\tilde{B}_e - \delta B_e) + \tilde{F}_e - \delta F_e - \tilde{C}_e \left(e^{i\omega} I - \tilde{A}_e \right)^{-1} \tilde{B}_e - \tilde{F}_e \\
&= -\delta C_e (e^{i\omega} I - \tilde{A}_e)^{-1} \tilde{B}_e - \tilde{C}_e (e^{i\omega} I - \tilde{A}_e)^{-1} \delta A_e (e^{i\omega} I - \tilde{A}_e)^{-1} \tilde{B}_e \\
&\quad - \tilde{C}_e (e^{i\omega} I - \tilde{A}_e)^{-1} \delta B_e - \delta F_e + o(\tilde{\Delta}) \tag{3.128}
\end{aligned}$$

where $\tilde{\Delta} = (\|\delta A_e\|_F, \|\delta B_e\|_F, \|\delta C_e\|_F, \|\delta F_e\|_F)^T$. Taking \mathcal{H}_∞ -norm in both sides of (3.128) and then applying triangle inequality and submultiplicative inequality (\mathcal{H}_∞ norm is defined on a closed Banach space) yields

$$\begin{aligned}
& \|G_e(\omega) - \tilde{G}_e(\omega)\|_\infty \\
&\leq \|\tilde{C}_e (e^{i\omega} I - \tilde{A}_e)^{-1}\|_\infty \|\delta A_e\|_2 \|(e^{i\omega} I - \tilde{A}_e)^{-1} \tilde{B}_e\|_\infty + \|\delta F_e\|_2 \\
&\quad + \|\delta C_e\|_2 \|(e^{i\omega} I - \tilde{A}_e)^{-1} \tilde{B}_e\|_\infty + \|\tilde{C}_e (e^{i\omega} I - \tilde{A}_e)^{-1}\|_\infty \|\delta B_e\|_2 + o(\|\tilde{\Delta}\|_2) \tag{3.129}
\end{aligned}$$

Using the fact $\|\cdot\|_2 \leq \|\cdot\|_F$ in (3.129), we have that

$$\begin{aligned}
& \|G_e(\omega) - \tilde{G}_e(\omega)\|_\infty \\
&\leq \|\tilde{C}_e (e^{i\omega} I - \tilde{A}_e)^{-1}\|_\infty \|\delta A_e\|_F \|(e^{i\omega} I - \tilde{A}_e)^{-1} \tilde{B}_e\|_\infty + \|\delta F_e\|_F \\
&\quad + \|\delta C_e\|_F \|(e^{i\omega} I - \tilde{A}_e)^{-1} \tilde{B}_e\|_\infty + \|\tilde{C}_e (e^{i\omega} I - \tilde{A}_e)^{-1}\|_\infty \|\delta B_e\|_F + o(\|\tilde{\Delta}\|_F) \tag{3.130}
\end{aligned}$$

With the identified state-space matrices \tilde{A}_e , \tilde{B}_e , \tilde{C}_e and \tilde{F}_e and their F-norm error bounds in (3.119) and (3.125), the \mathcal{H}_∞ -norm bound of the error system (3.97) can be explicitly derived with a confidence level more than $1 - \frac{\text{tr}(P_{R_0}) + \text{tr}(P_H)}{N\epsilon^2}$.

Next, we will introduce a lemma to address the uncertainty for the state-space matrices in the system $G_e(z)$. By this lemma, we propose an LMI approach for deriving an upper model error bound for $G_e(z)$.

Lemma 11. Let $A \in R^{n \times n}$, $Q = Q^T \in R^{n \times n}$, $H_k \in R^{n \times i}$, and $E_k \in R^{j \times n}$ ($k = 1, \dots, K$) be given matrices. If there exist K positive scalars $\mu_k > 0$ ($k = 1, \dots, K$) and a positive definite matrix $P > 0$ such that

$$\begin{bmatrix} -P & PA & PH_1 & PH_2 & \cdots & PH_K \\ Q + \sum_{k=1}^K \mu_k E_k^T E_k & 0 & 0 & 0 & \cdots & 0 \\ & -\mu_1 I & 0 & 0 & \cdots & 0 \\ & & -\mu_2 I & 0 & \cdots & 0 \\ & & & \ddots & \ddots & \vdots \\ & & & & & -\mu_K I \end{bmatrix} < 0 \quad (3.131)$$

(sym)

Then the following inequality holds

$$(A + \sum_{k=1}^K H_k F_k E_k)^T P (A + \sum_{k=1}^K H_k F_k E_k) + Q < 0 \quad (3.132)$$

for each F_k ($k = 1, \dots, K$) satisfying $\|F_k\|_2 \leq 1$.

Proof. From (3.131), applying Schur complement yields

$$\begin{bmatrix} -P + \sum_{k=1}^K \frac{1}{\mu_k} P H_k H_k^T P & PA \\ A^T P & Q + \sum_{k=1}^K \mu_k E_k^T E_k \end{bmatrix} < 0 \quad (3.133)$$

Multiplying (3.133) from the left by $\begin{bmatrix} P^{-1} & 0 \\ 0 & I \end{bmatrix}$ and from the right by $\begin{bmatrix} P^{-1} & 0 \\ 0 & I \end{bmatrix}$ yields

$$\begin{bmatrix} -P^{-1} + \sum_{k=1}^K \frac{1}{\mu_k} H_k H_k^T & A \\ A^T & Q + \sum_{k=1}^K \mu_k E_k^T E_k \end{bmatrix} < 0 \quad (3.134)$$

Also, for each k , we have that

$$\left(\frac{1}{\sqrt{\mu_k}} \begin{bmatrix} H_k \\ 0 \end{bmatrix} - \sqrt{\mu_k} \begin{bmatrix} 0 \\ E_k^T \end{bmatrix} F_k^T \right) \left(\frac{1}{\sqrt{\mu_k}} \begin{bmatrix} H_k \\ 0 \end{bmatrix} - \sqrt{\mu_k} \begin{bmatrix} 0 \\ E_k^T \end{bmatrix} F_k^T \right)^T \geq 0 \quad (3.135)$$

which can be simplified as

$$\begin{bmatrix} \frac{1}{\mu_k} H_k H_k^T & 0 \\ 0 & \mu_k E_k^T F_k^T F_k E_k \end{bmatrix} \geq \begin{bmatrix} 0 & H_k F_k E_k \\ E_k^T F_k^T H_k & 0 \end{bmatrix} \quad (3.136)$$

The condition $\|F_k\|_2 \leq 1$, is equivalent to

$$F_k^T F_k \leq I \quad (3.137)$$

From (3.136) and (3.137), in (3.134) we have that

$$\begin{bmatrix} -P^{-1} & A + \sum_{k=1}^K H_k F_k E_k \\ A^T + \sum_{k=1}^K E_k^T F_k^T H_k^T & Q \end{bmatrix} < 0 \quad (3.138)$$

which is equivalent to (3.132). \square

For the \mathcal{H}_∞ norm bound of the error system (3.97), we consider the following minimization problem

$$\begin{aligned} & \min_P \gamma^2 \\ \text{s.t.} & \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{F} \end{bmatrix}^T \begin{bmatrix} P & \\ & I \end{bmatrix} \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{F} \end{bmatrix} - \begin{bmatrix} P & \\ & \gamma^2 I \end{bmatrix} < 0 \\ & P > 0 \end{aligned} \quad (3.139)$$

where

$$\mathcal{A} = \begin{bmatrix} A_e & 0 \\ 0 & \tilde{A}_e \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} B_e \\ \tilde{B}_e \end{bmatrix}, \quad \mathcal{C} = [C_e \quad -\tilde{C}_e], \quad \mathcal{F} = [F_e \quad -\tilde{F}_e] \quad (3.140)$$

and

$$\|\delta A_e\|_F \leq \epsilon_1, \quad \|\delta B_e\|_F \leq \epsilon_2, \quad \|\delta C_e\|_F \leq \epsilon_3, \quad \|\delta F_e\|_F \leq \epsilon_4 \quad (3.141)$$

This LMI problem is equivalent to $\|G_e(\omega) - \tilde{G}_e(\omega)\|_{\mathcal{H}_\infty} \leq \gamma$ for any $\|\delta A_e\|_F \leq \epsilon_1$, $\|\delta B_e\|_F \leq \epsilon_2$, $\|\delta C_e\|_F \leq \epsilon_3$, $\|\delta F_e\|_F \leq \epsilon_4$ where ϵ_1 and ϵ_3 are given by (3.123), and ϵ_2 and ϵ_4 given by (3.127). Then we will use Lemma 11 to address the uncertainty of $A_e \sim F_e$ in (3.139). For brevity, we denote

$$\tilde{\mathcal{A}} = \begin{bmatrix} \tilde{A}_e & 0 \\ 0 & \tilde{A}_e \end{bmatrix}, \quad \tilde{\mathcal{B}} = \begin{bmatrix} \tilde{B}_e \\ \tilde{B}_e \end{bmatrix}, \quad \tilde{\mathcal{C}} = [\tilde{C}_e \quad -\tilde{C}_e] \quad (3.142)$$

and

$$\begin{aligned}
H_1 &= \begin{bmatrix} \sqrt{\epsilon_1}I \\ 0 \\ 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} \sqrt{\epsilon_2}I \\ 0 \\ 0 \end{bmatrix}, \quad H_3 = \begin{bmatrix} 0 \\ 0 \\ \sqrt{\epsilon_3}I \end{bmatrix}, \quad H_4 = \begin{bmatrix} 0 \\ 0 \\ \sqrt{\epsilon_4}I \end{bmatrix} \\
F_1 &= \frac{\delta A_e}{\epsilon_1}, \quad F_2 = \frac{\delta B_e}{\epsilon_2}, \quad F_3 = \frac{\delta C_e}{\epsilon_3}, \quad F_4 = \frac{\delta F_e}{\epsilon_4} \\
E_1 &= [\sqrt{\epsilon_1}I \ 0 \ 0], \quad E_2 = [0 \ 0 \ \sqrt{\epsilon_2}I], \quad E_3 = [\sqrt{\epsilon_3}I \ 0 \ 0], \quad E_4 = [0 \ 0 \ \sqrt{\epsilon_4}I] \quad (3.143)
\end{aligned}$$

Then the first LMI in (3.139) becomes

$$\left(\begin{bmatrix} \tilde{\mathcal{A}} & \tilde{\mathcal{B}} \\ \tilde{\mathcal{C}} & 0 \end{bmatrix} + \sum_{k=1}^4 H_k F_k E_k \right)^T \begin{bmatrix} P & \\ & I \end{bmatrix} \left(\begin{bmatrix} \tilde{\mathcal{A}} & \tilde{\mathcal{B}} \\ \tilde{\mathcal{C}} & 0 \end{bmatrix} + \sum_{k=1}^4 H_k F_k E_k \right) - \begin{bmatrix} P & \\ & \gamma^2 I \end{bmatrix} < 0 \quad (3.144)$$

where it is readily verified that $\|F_k\|_F \leq 1$ for $k = 1, 2, 3, 4$. Due to the fact that $\|\cdot\|_2 \leq \|\cdot\|_F$, we also have $\|F_k\|_2 \leq 1$ for $k = 1, 2, 3, 4$. By Lemma 11, we arrive at a suboptimal, convex minimization problem

$$\begin{aligned}
& \min_{P, \mu} \bar{\gamma}^2 \\
& s.t. \quad \begin{bmatrix} - \begin{bmatrix} P & \\ & I \end{bmatrix} & \begin{bmatrix} P & \\ & I \end{bmatrix} \begin{bmatrix} \tilde{\mathcal{A}} & \tilde{\mathcal{B}} \\ \tilde{\mathcal{C}} & 0 \end{bmatrix} & \begin{bmatrix} P & \\ & I \end{bmatrix} H_1 & \cdots & \begin{bmatrix} P & \\ & I \end{bmatrix} H_4 \\ & - \begin{bmatrix} P & \\ & \bar{\gamma}^2 I \end{bmatrix} + \sum_{k=1}^4 \mu_k E_k^T E_k & 0 & \cdots & 0 \\ & & -\mu_1 I & \cdots & 0 \\ & & & \ddots & \vdots \\ & & & & -\mu_4 I \end{bmatrix} < 0 \\
& P > 0 \\
& \mu_k > 0 \quad (k = 1, 2, 3, 4) \quad (3.145)
\end{aligned}$$

whose minimum $\bar{\gamma}^2$ is an upper bound of the γ^2 in (3.139) such that $\|G_e(\omega) - \tilde{G}_e(\omega)\|_{\mathcal{H}_\infty} \leq \bar{\gamma}$.

Application to Wave Energy Harvesting Systems

In this chapter, an application of the combined deterministic-stochastic identification to wave energy harvesting systems, is discussed. The objective is to design a controller for maximizing the energy harvested from ocean waves based on the identified model.

4.1 Wave Energy Harvester Description

The application of combined deterministic-stochastic identification under consideration is a wave energy harvester, as shown in Figure 4.1. In this harvester, the floating buoy, which is excited to vibration by the ocean waves, first drives the tethers, and then the tethers drive the pulleys to generate the electricity. In the circuit level, control theory can be employed to design the power electronics of the generator for the improvement in power generation. With the currents in the power generators as the control input, and the voltages as the output, a feedback system is shown in Figure 4.2. For the details of power electronics design in energy harvesters, the interested readers can refer to Scruggs and Li (2011).

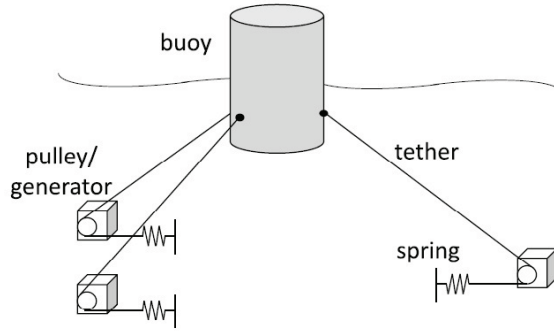


FIGURE 4.1: General diagram of a wave energy harvester (this picture has been created by Steven Lattanzio).

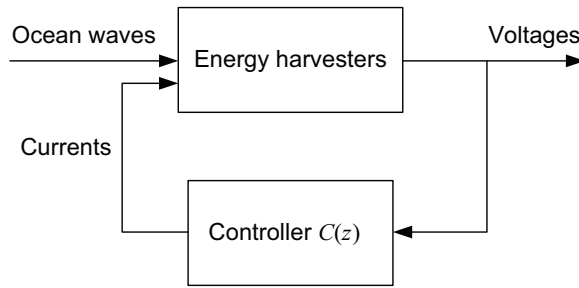


FIGURE 4.2: Conceptual diagram of an energy harvester feedback system.

4.2 Identification-based Control for Wave Energy Harvesting Systems

4.2.1 Optimal Controller Design for Wave Energy Harvesting Systems

Assume the wave energy harvesting systems are slowly time-varying such that in a short period (2 ~ 3 hours) the systems can be regarded as linear time invariant systems. Consider the energy harvesting systems admit a finite-dimensional state space innovations representation as follows:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + Ke_k \\ y_k &= Cx_k + e_k \end{aligned} \tag{4.1}$$

where u_k and y_k are the currents and voltages in the energy harvesters, respectively; e_k is the white innovations with the covariance matrix $S_e = E[e_k e_k^T]$. The construction of the state-space model was introduced in Scruggs and Lattanzio (2011). For

the consistency of identifying the systems in closed loop, the known external signal $\{f_k\}$ is injected in the optimal control input, as shown in (2.8). Substituting (2.8) to (4.1) obtains

$$\begin{aligned}x_{k+1} &= Ax_k + B\bar{u}_k + Bf_k + Ke_k \\y_k &= Cx_k + e_k\end{aligned}\tag{4.2}$$

where \bar{u}_k is the optimal control inputs.

To yield the most straight-forward analysis accounting for transmission dissipation, we assume that the transmission dissipation in the generators is modeled as a simple resistive loss associated with the currents, i.e. $u_k^T Ru_k$ where $R > 0$ is the resistance matrix for the stators of the generators. Our objective is to find the feedback law $C(z) : \{\mathcal{Y}_k, \mathcal{X}_0\} \rightarrow u_k$ such that the following harvested power is maximum.

$$\bar{P}_{gen} = E\{-u_k^T y_k - u_k^T R u_k\}\tag{4.3}$$

Substituting (4.1) to (4.3), we have an equivalent problem which is to find the feedback law $C(z)$ aimed at minimizing

$$J = -\bar{P}_{gen} = \frac{1}{2}E\left\{\begin{bmatrix} x_k \\ u_k \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & C^T \\ C & 2R \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}\right\}\tag{4.4}$$

which is a sign-indefinite \mathcal{H}_2 problem. Although the well-posedness of this problem is not guaranteed (i.e., J would have no minimum), for this specific case Scroggs (2010) proved that if the energy harvesters are Weakly-Strictly Positive Real then J has a unique, finite, and negative minimum. The following theorem gives the performance limit of energy harvesting systems (4.2) injected by the external signals $\{f_k\}$ with the covariance matrix $S_f = E[f_k f_k^T]$.

Theorem 12. *For any causal mapping $x \mapsto u$ (linear or non-linear), the following equality holds in stationary response:*

$$\bar{P}_{gen} = -\frac{1}{2}\text{tr}\{K^T Q K S_e\} - \text{tr}\left\{\left(R + \frac{B^T Q B}{2}\right) S_f\right\} - E\left\{(\bar{u}_k - \Omega x_k)^T \left(R + \frac{B^T Q B}{2}\right) (\bar{u}_k - \Omega x_k)\right\}\tag{4.5}$$

where $S_e = E[e_k e_k^T]$ and $S_f = E[f_k f_k^T]$ are respectively the covariance of innovations and injected external signals, and Ω is:

$$\Omega = -(2R + B^T Q B)^{-1} (C + B^T Q A) \quad (4.6)$$

where Q is solved by the following discrete-time Riccati Equation:

$$A^T Q A - Q - (C + B^T Q A)^T (2R + B^T Q B)^{-1} (C + B^T Q A) = 0 \quad (4.7)$$

Proof. Rearrange the objective as

$$-2\bar{P}_{gen} = E\{x_k^T C^T u_k + u_k^T C x_k + u_k^T 2R u_k\} \quad (4.8)$$

Suppose $u_k = \Omega x_k + r_k + f_k$ and substitute it to (4.8), we get

$$\begin{aligned} & E\{x_k^T C^T u_k + u_k^T C x_k + u_k^T 2R u_k\} \\ &= E\{x_k^T (C^T \Omega + \Omega^T C + \Omega^T 2R \Omega) x_k + r_k^T (2C + 4R \Omega) x_k + r_k^T 2R r_k + f_k^T 2R f_k\} \end{aligned} \quad (4.9)$$

From (4.6) and (4.7), we have

$$Q = C^T \Omega + \Omega^T C + \Omega^T 2R \Omega + (A + B \Omega)^T Q (A + B \Omega) \quad (4.10)$$

Thus,

$$C^T \Omega + \Omega^T C + \Omega^T 2R \Omega = Q - (A + B \Omega)^T Q (A + B \Omega) \quad (4.11)$$

Substituting (4.11) to (4.9) yields

$$\begin{aligned} & E\{x_k^T C^T u_k + u_k^T C x_k + u_k^T 2R u_k\} \\ &= E\{x_k^T (Q - (A + B \Omega)^T Q (A + B \Omega)) x_k + r_k^T (2C + 4R \Omega) x_k + r_k^T 2R r_k + f_k^T 2R f_k\} \end{aligned} \quad (4.12)$$

For (4.10), suppose we multiply both sides from the left by x_k^T and from the right by x_k and take the expectation for both sides, we get that

$$E\{x_k^T Q x_k\} = E\{x_k^T (C^T \Omega + \Omega^T C + \Omega^T 2R \Omega + (A + B \Omega)^T Q (A + B \Omega)) x_k\} \quad (4.13)$$

With stationarity assumption, we have

$$E\{x_k^T Q x_k\} = \text{tr}\{QE\{x_k x_k^T\}\} = \text{tr}\{QE\{x_{k+1} x_{k+1}^T\}\} \quad (4.14)$$

Note that

$$x(k+1) = Ax_k + B\bar{u}_k + Bf_k + Ke_k \quad (4.15)$$

we have

$$\begin{aligned} & \text{tr}\{QE\{x_{k+1} x_{k+1}^T\}\} \\ &= \text{tr}\{QE\{(Ax_k + B\Omega x_k + Br_k + Bf_k + Ke_k)(Ax_k + B\Omega x_k + Br_k + Bf_k + Ke_k)^T\}\} \end{aligned} \quad (4.16)$$

Likewise, the following equation holds

$$\begin{aligned} & E\{x_k^T Q x_k\} \\ &= \text{tr}\{QE\{(Ax_k + B\Omega x_k + Br_k + Bf_k + Ke_k)(Ax_k + B\Omega x_k + Bf_k + Ke_k)^T\}\} \\ &= E\{(Ax_k + B\Omega x_k + Br_k + Bf_k + Ke_k)^T Q (Ax_k + B\Omega x_k + Br_k + Bf_k + Ke_k)\} \end{aligned} \quad (4.17)$$

From the equation above, we get

$$\begin{aligned} & E\{x_k^T (Q - (A + B\Omega)^T Q (A + B\Omega)) x_k\} \\ &= E\{r_k^T (2B^T Q A + 2B^T Q B\Omega) x_k + e_k^T K^T Q K e_k + f_k^T B^T Q B f_k + r_k^T B^T Q B r_k\} \\ &= \text{tr}\{K^T Q K S_e\} + \text{tr}\{B^T Q B S_f\} + E\{r_k^T B^T Q B r_k\} + E\{r_k^T (2B^T Q A + 2B^T Q B\Omega) x_k\} \end{aligned} \quad (4.18)$$

Substituting (4.18) to (4.12), we get that

$$\begin{aligned} & E\{x_k^T C^T u_k + u_k^T C x_k + u_k^T 2R u_k\} \\ &= \text{tr}\{K^T Q K S_e\} + \text{tr}\{B^T Q B S_f\} + E\{f_k^T 2R f_k\} + E\{r_k^T (B^T Q B + 2R) r_k\} \\ & \quad + 2E\{r_k^T (C + 2R\Omega + B^T Q A + B^T Q \bar{B}\Omega) x_k\} \end{aligned} \quad (4.19)$$

From (4.6), we have

$$C + 2R\Omega + B^T Q A + B^T Q \bar{B}\Omega = 0 \quad (4.20)$$

Substituting it into (4.8) gives

$$-2\bar{P}_{gen} = E\{K^T Q K S_e\} + \text{tr}\{B^T Q B S_f\} + \text{tr}\{2R S_f\} + E\{r_k^T (B^T Q B + 2R) r_k\} \quad (4.21)$$

which concludes the theorem. \square

From this theorem, we can construct the optimal controller as follows

$$\begin{aligned} x_{k+1} &= (A + B\Omega - KC)x_k + Bf_k + Ky_k \\ u_k &= \Omega x_k + f_k \end{aligned} \quad (4.22)$$

To implement this controller for adaptive control and eliminate the negative influence of the initial state on the estimation of the successive states after each controller update, we will transform the controller (4.22) to its equivalent form in terms of the past control inputs and outputs instead of the system states. By z-Transform, we rewrite the controller (4.22) as

$$u_k = \Omega(zI - (A + B\Omega - KC))^{-1}(Bf_k + Ky_k) + f_k \quad (4.23)$$

we perform the following left coprime factorization:

$$\Omega(zI - (A + B\Omega - KC))^{-1} = F^{-1}(z)G(z) \quad (4.24)$$

where $F(z)$ and $G(z)$ are polynomial matrices

$$\begin{aligned} F(z) &= F_l + F_{l-1}z + \cdots + F_0z^l \quad n_u \times n_u \\ G(z) &= G_l + G_{l-1}z + \cdots + G_0z^l \quad n_u \times n_x \end{aligned} \quad (4.25)$$

Here, we assume that $F(z)$ is row reduced defined in Definition A.3.E (Goodwin and Sin (1984)) and that the transfer function $\Omega(zI - (A + B\Omega - KC))^{-1}$ is proper. According to Lemma A.3.3 (Goodwin and Sin (1984)), the leading row coefficient matrix F_0 is nonsingular. Also, the properness of the transfer function gives $G_0 = \mathbf{0}$. Combining (4.23) and (4.24) yields

$$\begin{aligned} z^{-l}F(z)u_k &= z^{-l}G(z)(Bf_k + Ky_k) \\ (F_0 + F_1z^{-1} + \cdots + F_lz^{-l})u_k &= (G_0 + G_1z^{-1} + \cdots + G_lz^{-l})(Bf_k + Ky_k) \end{aligned} \quad (4.26)$$

Algebraic manipulation of the equation above gives the optimal controller in terms of u_{k-i} , f_{k-i+1} and y_{k-i} ($i = 1, 2, \dots, l + 1$).

$$\begin{aligned}
u_k = & [-F_0^{-1}F_1, -F_0^{-1}F_2, \dots, -F_0^{-1}F_l] \begin{bmatrix} u_{k-1} \\ u_{k-2} \\ \vdots \\ u_{k-l} \end{bmatrix} \\
& + [F_0^{-1}G_1B, F_0^{-1}G_2B, \dots, F_0^{-1}G_lB] \begin{bmatrix} f_{k-1} \\ f_{k-2} \\ \vdots \\ f_{k-l} \end{bmatrix} \\
& + [F_0^{-1}G_1K, F_0^{-1}G_2K, \dots, F_0^{-1}G_lK] \begin{bmatrix} y_{k-1} \\ y_{k-2} \\ \vdots \\ y_{k-l} \end{bmatrix} + f_k \quad (4.27)
\end{aligned}$$

Since the controller transform from (4.22) to (4.27) requires precise coprime factorization, the suitability of (4.27) will be decreased with the increase of system dimension. The system dimension ($n_x = 30 \sim 50$), for wave energy harvesting system with system poles very closed to the unit circle, is very large such that the precise coprime factorization is difficult to attain. Thus, the optimal controller we mention hereafter refers to (4.22).

4.2.2 Closed-loop Deterministic-Stochastic Identification Algorithm for Wave Energy Harvesting Systems

Combining the identification procedure in chapter 2 and the controller design in (4.22), the closed-loop deterministic-stochastic identification algorithm for wave energy harvesting systems is summarized as follows:

Algorithm:

init set N to be the time interval for system updating.

for $k = 1, 2, \dots$

input $y_k, \bar{u}_k, \bar{u}_{k,f}, f_k$

step 1: update IVM

$$\theta_k = \theta_{k-1} + a_{k-1}P_{k-1}\zeta_k(y_k^T - \phi_k^T\theta_{k-1}) \quad (4.28)$$

$$P_k = P_{k-1} - a_{k-1}P_{k-1}\zeta_k\phi_k^T P_{k-1}, \quad a_{k-1} = (1 + \phi_k^T P_{k-1}\zeta_k)^{-1} \quad (4.29)$$

$$\phi_k = [-y_{k-1}; \cdots; -yk - p; u_{k-1}; \cdots; u_{k-q}] \quad (4.30)$$

$$\zeta_k = [-y_{k-1,f}; \cdots; -y_{k-p,f}; \bar{u}_{k-1,f} + f_{k-1}; \cdots; \bar{u}_{k-q,f} + f_{k-q}] \quad (4.31)$$

step 2: compute the output components driven by injected external signals $\{f_k\}$ and by control input $\{u_k\}$ respectively

$$\begin{aligned} x_{k-1,f} &= A_u x_{k,f} + B_u(\bar{u}_{k,f} + f_k) \\ y_{k,f} &= C_u x_{k,f} \\ x_{k+1,d} &= A_u x_{k,d} + B_u u_k \\ y_{k,d} &= C_u x_{k,d} \end{aligned} \quad (4.32)$$

step 3: update the covariance sequence $\{R_i(k)\}$

$$\begin{aligned} y_{k,e} &= y_k - y_{k,d} \\ R_i(k) &= R_i(k-1) - \frac{1}{k}(R_i(k-1) - y_{k,e}y_{k-i,e}^T) \end{aligned} \quad (4.33)$$

step 4: update the Kalman state driven by $\{f_k\}$ and $\{e_k\}$, and then calculate \bar{u}_{k+1}

$$\begin{aligned} x_{k+1} &= (A + B\Omega - KC)x_k + Bf_k + Ky_k \\ \bar{u}_k &= \Omega x_k \end{aligned} \quad (4.34)$$

step 5: update the subsystem state driven by $\{f_k\}$ and calculate $\bar{u}_{k+1,f}$

$$\begin{aligned} \bar{x}_{k+1} &= (A + B\Omega)\bar{x}_k + Bf_k \\ \bar{u}_{k,f} &= \Omega\bar{x}_k \end{aligned} \quad (4.35)$$

step 6: update the system and calculate the control gain

if $\text{mod}(k, N) = 0$

compute the new $\{A_e, K_e, C_e\}$ from stochastic identification in Section 2.2 and also $\{A_u, B_u, C_u\}$ from (2.23).

update the system

$$A = \begin{bmatrix} A_u & 0 \\ 0 & A_e \end{bmatrix}, \quad B = \begin{bmatrix} B_u \\ 0 \end{bmatrix}, \quad K = \begin{bmatrix} 0 \\ K \end{bmatrix}, \quad C = [C_u \quad C_e] \quad (4.36)$$

model reduction obtains the renewed A, B, K, C .

update the control gain

$$\begin{aligned} A^T Q A - Q - (C + B^T Q A)^T (2R + B^T Q B)^{-1} (C + B^T Q A) &= 0 \\ \Omega &= -(2R + B^T Q B)^{-1} (C + B^T Q A) \end{aligned} \quad (4.37)$$

end

end

where mod represents Modulo operation.

4.3 Simulations

The cylindrical floating buoy in Figure 4.3 is considered as the wave energy harvester in the simulations. Its three tethers extend from points on the buoy 2.25m above the buoy's bottom rim and are separated by 120 degree angles in the horizontal plane with one tether extending along the positive x-axis. Three tethers all radiate from the centroid of the buoy. The JONSWAP spectrum is used to produce the wave excitation with a sharpness factor equal to 1 (corresponding to a fully-developed sea) and significant wave height of $H_{1/3} = 1m$.

The state space model for simulation is obtained by the finite dimensional approximation of wave energy harvesting systems from the frequency domain data (Scruggs and Lattanzio (2011)). The original system, with $G_u(e^{i\omega})$ and $G_e(e^{i\omega})$ illustrated in Figure 4.4 and Figure 4.5, respectively, is used to demonstrate the performance of the proposed combined deterministic-stochastic identification algorithm. Figure

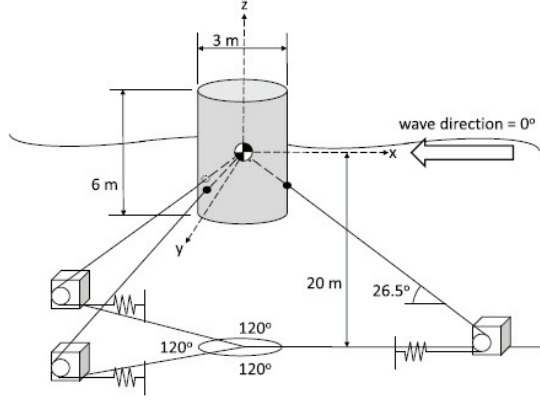


FIGURE 4.3: Wave energy harvester. Additional design parameters are $\mu = 5300kg$, $m = 50kg$, $c = 50N \cdot s/m$, $k = 50N/m$.

4.4 and Figure 4.5 clearly show there are three modes of resonance. The first is predominately surge motion, the second pitch, and the third heave. To reflect the estimation precision of the innovations covariance S_e for comparison, $G_e(\omega)$ is the transfer function from the normalized innovations with the covariance equal to the identity matrix to the output, i.e.

$$G_e(\omega) = C(e^{i\omega T_s} I - A)^{-1} K S_e^{1/2} + S_e^{1/2} \quad (4.38)$$

where T_s is the sample period. The process to be identified is described by (4.2) in which $\{f_k\}$ with its covariance matrix $S_f = 0.01I$ is a deterministic white noise signal sequence uncorrelated with the innovations $\{e_k\}$. In the identification of the process parameters in closed loop, the identification algorithm summarized in Section 4.2.2 is used in which the time interval for system updating is set to be 3.5×10^5 .

We define two indices E_u and E_e for measuring the performances of the combined deterministic-stochastic identification algorithm.

$$E_u = \frac{\|G_u(\omega) - \tilde{G}_u(\omega)\|_{\mathcal{H}_2}}{\|G_u(\omega)\|_{\mathcal{H}_2}} \quad (4.39)$$

$$E_e = \frac{\|G_e(\omega) - \tilde{G}_e(\omega)\|_{\mathcal{H}_2}}{\|G_e(\omega)\|_{\mathcal{H}_2}} \quad (4.40)$$

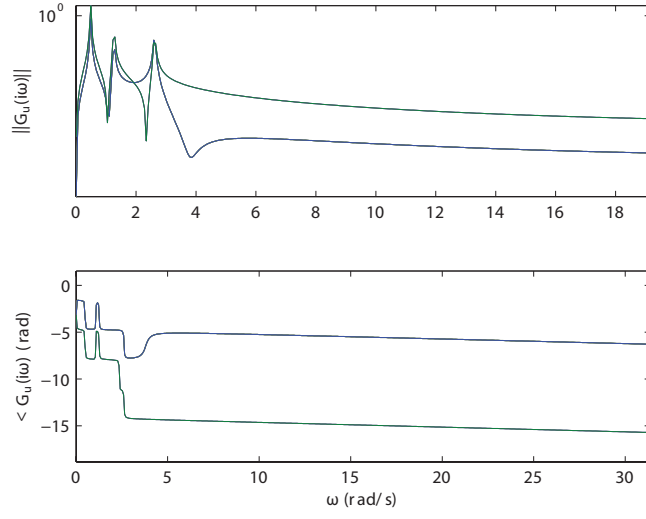


FIGURE 4.4: Frequency response of $G_u(\omega)$.

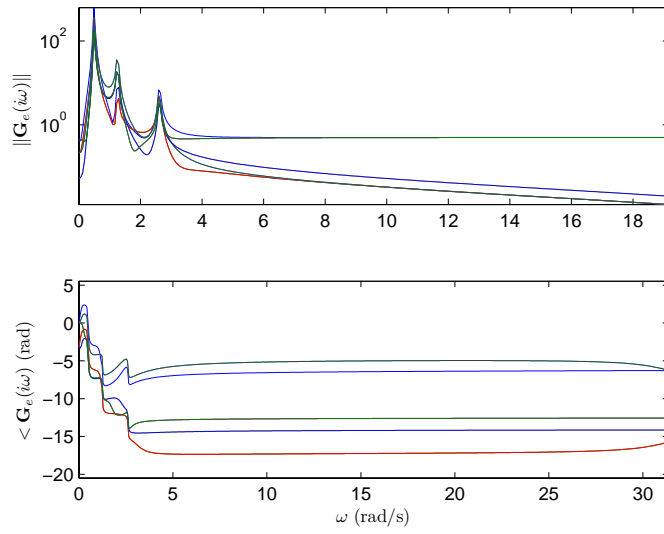


FIGURE 4.5: Frequency response of $G_e(\omega)$.

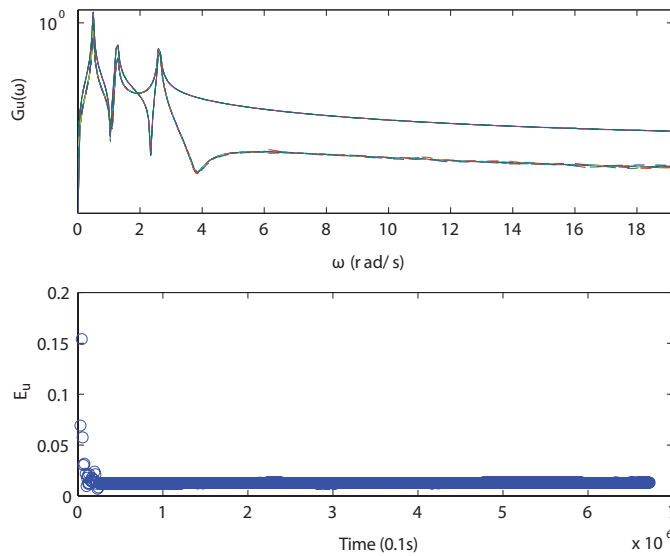


FIGURE 4.6: Frequency responses for the original $G_u(\omega)$ (solid) and the identified $\tilde{G}_u(\omega)$ (dashed).

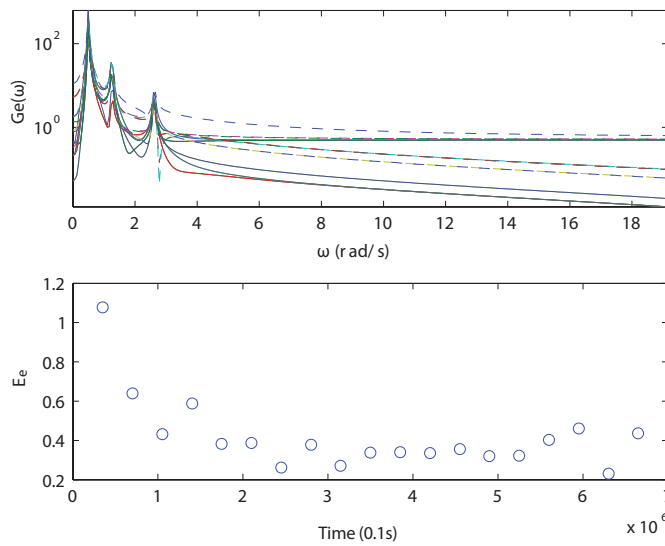


FIGURE 4.7: Frequency responses for the original $G_e(\omega)$ (solid) and the identified $\tilde{G}_e(\omega)$ (dashed).

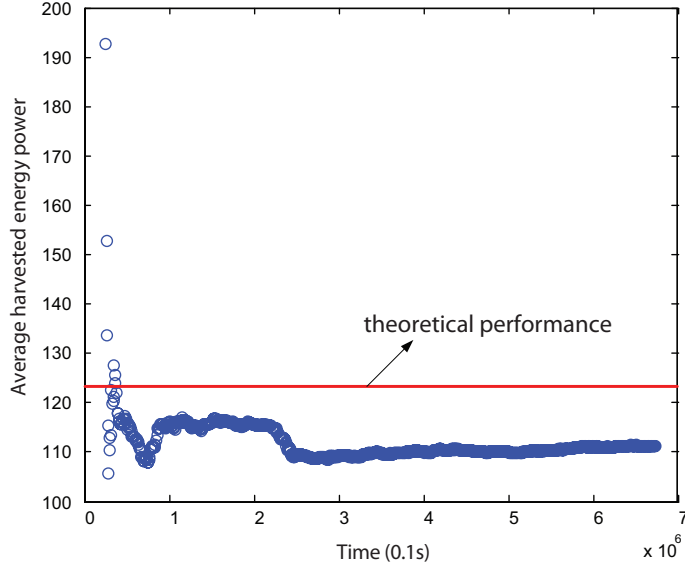


FIGURE 4.8: Average harvested energy power for a wave energy harvester.

where $\tilde{G}_u(\omega)$ and $\tilde{G}_e(\omega)$ corresponds to the identified systems; $\|G_u(\omega)\|_{\mathcal{H}_2}$ is the \mathcal{H}_2 norm of the transfer function $G_u(\omega)$, defined as

$$\|G_u(\omega)\|_{\mathcal{H}_2}^2 = \text{tr} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} G_u(e^{i\omega}) G_u(e^{i\omega})^H d\omega \right] \quad (4.41)$$

where $(\cdot)^H$ is the conjugate transpose of (\cdot) .

The comparison of the system deterministic part $G_u(\omega)$ and the stochastic part $G_e(\omega)$ with the identified ones in frequency domain, is shown in Figures 4.6 and 4.7. From Figure 4.6, it is shown that with the increase of data, the identified $\tilde{G}_u(\omega)$ approaches to the original one with gradually improved precision. From (3.25), we know that in closed loop the identified $\tilde{G}_e(\omega)$ is biased due to the biased estimation of the output covariance matrix R_0 and the block-Hankel matrix H . Since the bias of the asymptotic estimation of R_i is the same order as $\delta\theta$, the identified $\tilde{G}_e(\omega)$ will also be improved with the improved precision of θ estimation. Figure 4.8 shows the average harvested energy power with the solid line representing the theoretical performance computed from theorem 12 and the small circles representing the harvested power

when the identification-based controller is implemented. The initial optimal control is designed based on the system model identified at time point 2.5×10^5 and then implemented afterwards until the time point for the next system updating. Since the controller is designed based on the identified model with model error, there is a gap between the harvested power and its performance limit.

Summary and Perspectives

5.1 Summary

The main contribution of this thesis is the development of a combined deterministic-stochastic identification procedure. The identification procedure obtains an identified nominal model with an explicit model error bound suited for further applications in robust filtering and control.

In Chapter 2, we formulate a combined deterministic-stochastic identification procedure in closed loop by combining the IV method with the improved stochastic identification algorithm, which can be easily adjusted for the extension to on-line identification and adaptive control applications. In the improved stochastic identification algorithm, we illustrate that the identifiability of the stochastic system is equivalent to the positive realness of the associated covariance model by theorem 1. When the identifiability of the stochastic system under insufficient data is not satisfied, we propose a straightforward LMI-based approach for imposing the positive realness on the associated covariance model, guaranteeing a positive definite solution to the DARE and thus a valid innovations model returned. Although this approach

is proposed in state-space model, its extension to the estimation of ARMAX parameters in signal processing is easily reformulated.

In Chapter 3, we investigate the uncertainty quantification for the identified deterministic subsystem and stochastic subsystem by a complete asymptotic and perturbation analysis. For the deterministic subsystem, we quantify its uncertainty by an ellipsoidal uncertainty region and \mathcal{H}_2 -norm error bound of its transfer function, respectively. For the stochastic subsystem, thanks to the asymptotic normal distribution of the empirical block-Hankel matrix and perturbation analysis of SVD and DARE, several central limit theorems for the controllability matrix, observability matrix, and the state-space matrices of the associated covariance model are derived, as well as the norm bounds of Kalman gain and the innovations covariance matrix in the innovations model. By combining these asymptotic results, the \mathcal{H}_2 norm bound, as well as the \mathcal{H}_∞ norm bound, of the error system, is derived with a given confidence level.

Finally, Chapter 4 demonstrates the application of the combined deterministic-stochastic identification procedure to wave energy harvesting systems where an LQG controller is constructed for maximizing the harvested energy and meanwhile identifying the consistent system model in closed loop. In this application, we adjust the identification procedure proposed in Chapter 2 to an online system identification algorithm which enables to recursively identify the deterministic part and stochastic part of a large-scale MIMO energy harvesting system in closed loop. The simulation results illustrate the effectiveness of the proposed identification procedure.

5.2 Perspectives on Future Research

In the framework of the identification procedure proposed in this thesis, several topics deserve further investigation. One topic for future work would be the selection of the identified model uncertainty structures for the further design of robust controller.

For example, the weighted additive \mathcal{H}_∞ -norm model error bound appears suitable for modern robust control design.

Another topic would be the identification-based robust control. This concerns the integration of model error bound identification and robust controller design. The robust control design should make use of the identified nominal model and the associated model error bound with a given confidence level.

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