Complex and Lagrangian Engel Structures

by

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Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
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Abstract

In this dissertation, we study the geometry of Engel structures, which are 2-plane fields on 4-manifolds satisfying a generic condition, that are compatible with other geometric structures. A *complex* Engel structure is an Engel 2-plane field on a complex surface for which the 2-planes are complex lines. A *Lagrangian* Engel structure is an Engel 2-plane field on a symplectic 4-manifold for which the 2-planes are Lagrangian with respect to the symplectic structure. We solve the equivalence problems for complex Engel structures and Lagrangian Engel structures and use the resulting structure equations to classify homogeneous complex Engel structures and homogeneous Lagrangian Engel structures. This allows us to determine all compact, homogeneous examples.

For complex Engel structures, compact manifolds that support homogeneous complex Engel structures are diffeomorphic to $S^1 \times SU(2)$ or quotients of $\mathbb{C}^2$, $S^1 \times SU(2)$, $S^1 \times G$ or H by co-compact lattices, where G is the connected and simply-connected Lie group with Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ and H is a solvable Lie group. For Lagrangian Engel structures, compact manifolds that support homogeneous Lagrangian Engel structures are diffeomorphic to quotients of one of a determined list of nilpotent or solvable 4-dimensional Lie groups by co-compact lattices.

To my parents

Contents

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1

Introduction

A *distribution* is a subbundle $D \subset TM$ of the tangent bundle of a manifold M. We will consider certain distributions with special properties, for example, distributions with some integrability conditions. Given a distribution of rank n on a manifold M^m, if, for each point of M, there exists a coordinate neighborhood U and local coordinates $x_1, x_2, \cdots, x_m$ such that $\frac{\partial}{\partial x_i}$, $i = 1, \cdots, n$ forms a local basis for the distribution on U, then the distribution is said to be *completely integrable.* To check complete integrability, we have the Frobenius Theorem, which can be found in [Boothby (1986)].

Theorem 1.0.1 (Frobenius Theorem)**.** *If a distribution D is involutive, i.e., for all vector fields X, Y that are sections of D, we have $[X, Y]$ is a section of D, then D is completely integrable.*

Besides complete integrability, one can consider partially integrable distributions. An extreme condition is to be nowhere integrable, i.e., for every $p \in M$, there exist X, Y, sections of D, such that $[X, Y]_p$ is not a section of D. For example, contact structures are nowhere integrable distributions on odd dimensional manifolds. The study of contact structures [Geiges (2006)] usually involves interplay between geometry, topology and dynamics. Contact structures play an important role in the study of low-dimensional topology.

We will study *Engel structures*, which are certain non-integrable distributions defined on 4-manifolds. We will see that, locally, all Engel structures are isomorphic but the global theory of Engel structures is not trivial. A 4-manifold can carry many nonisomorphic Engel structures [Gershkovich (1995)]. There are relations between contact structures on 3-dimensional manfiolds and Engel structures. For example, V. Gershkovich [Gershkovich (1995)] proved that each Engel manifold carries a canonical one-dimensional foliation and an Engel structure defines a contact distribution on any three-dimensional submanifold transversal to the canonical foliation.

In this dissertation, we solve the equivalence problem for complex and Lagrangian Engel structures and present the classification of compact quotients of homogeneous complex Engel structures and homogeneous Lagrangian Engel structures.

Before discussing the classification of Engel structures, we need to know the definition of an Engel structure. We will first introduce some basic concepts that will be used to characterize Engel structures.

Definition 1.0.1. *A* Pfaffian system *on M^n is a sub-bundle $I \subset T^*M$. The space of sections of I, $\Gamma(I)$, is a $C^\infty(M)$-submodule of $\Omega^1(TM)$, the* 1*-forms on M. We let $I^\perp \subset TM$ be the annihilator of $I \subset T^*M$.*

Engel structures (to be defined below) can be characterized in terms of the derived system construction.

Proposition 1.0.2 (Bryant et al. (1991))**.** *Given a Pfaffian system I, there exists a bundle map*

$$\delta : I \longrightarrow \Lambda^2(T^*M/I)$$

that satisfies $\delta\omega \equiv d\omega \mod (I)$ for all $\omega \in \Gamma(I)$.

Definition 1.0.2. *By Proposition 1.0.2, we have a bundle map δ. Set*

$$I^{(1)} = \ker \delta$$

and call $I^{(1)}$ the first derived system*. Continuing with this construction, we can get a filtration*

$$I^{(k)} \subset \cdots \subset I^{(2)} \subset I^{(1)} \subset I^{(0)} = I,$$

defined inductively by

$$I^{(k+1)} = (I^{(k)})^{(1)}\,.$$

$I^{(k)}$ *is called* the kth derived system*.*

Now we present the definition and characterization of Engel structures.

Definition 1.0.3 (Engel Structure)**.** *Given a 4-manifold M and a Pfaffian system $I \subset T^*M$, an* Engel structure *is a sub-bundle $D = I^{\perp}$ of the tangent bundle of M that satisfies*

1. *I is of rank 2*
2. *$I^{(1)}$ is of rank 1*
3. *$I^{(2)} = 0$*

A manifold endowed with an Engel structure $D = I^{\perp}$ is called an Engel manifold*.*

Definition 1.0.4. *Given an ideal $\mathcal{I}$ generated by a Pffafian system I, a vector field ξ is called a* Cauchy characteristic vector field *of $\mathcal{I}$ if $\xi \lrcorner \mathcal{I} \subset \mathcal{I}$. At a point $x \in M$, the set of Cauchy characteristic vector fields is*

$$A(I)_x = \{\xi_x \in T_xM | \xi_x \lrcorner \mathcal{I}_x \subset \mathcal{I}_x\} \subset I^{\perp}$$

and the retracting space *or* Cartan system *is defined to be*

$$C(I)_x = A(I)_x^{\perp} \subset T_x^*M\,.$$

By the definition of Engel structure $I^{\perp}$, there is a canonical flag of sub-bundles

$$0 \subset I^{(1)} \subset I \subset C(I^{(1)}) \subset T^*M\,.$$

V. Gershkovich [Gershkovich (1995)] proved the following theorem which can also be found in [Kazarian et al. (1997)].

Theorem 1.0.3. *If an orientable 4-manifold admits an orientable Engel structure, then it has trivial tangent bundle.*

T. Vogel [Vogel (2009)] proved the converse of the above theorem:

Theorem 1.0.4. *Every parallelizable 4-manifold admits an orientable Engel structure.*

Thus for an orientable 4-manifold, parallelizability is equivalent to the existence of an orientable Engel structure. This is a global characterization of manifolds that support orientable Engel structures. Locally, we have the following *Engel normal form* [Bryant et al. (1991)], which implies that there is no local invariant for Engel structures, i.e., all Engel structures are locally equivalent.

Theorem 1.0.5 (Engel normal form)**.** *Let I be a Pfaffian system on M^4 such that $I^{\perp} \subset TM$ is an Engel structure. Then every point of M has an open neighborhood U on which there exists local coordinates $(x, y_0, y_1, y_2) : U \to \mathbb{R}^4$ such that*

$$I|_U = \{dy_0 - y_1 dx, dy_1 - y_2 dx\}\,.$$

By Theorem 1.0.5, all Engel structures are locally equivalent.

1.1 Problem description

In this dissertation, we will consider two extra structures, complex structures and symplectic structures, that are compatible with the underlying Engel structure. We will define the meaning of *compatibility* as follows.

Definition 1.1.1. *Given a complex manifold* (M, J, I) *with an Engel structure* $D = I^{\perp}$*, if* $I^{\perp} \subset TM$ *is a complex line field, the Engel structure is called a* complex Engel structure.

Remark 1.1.1. *Note the definition of complex Engel structure is different from that of holomorphic Engel structure. A holomorphic Engel structure [Presas and Solá Conde (2014)] on a complex manifold* M *of complex dimension 4 is a holomorphic sub-bundle* $D \subset TM$ *of complex rank 2 which is maximal non-integrable.*

Definition 1.1.2. *A* Lagrangian Engel structure (M, Ω, D) *is a 4-manifold* M *endowed with* a symplectic form Ω and an Engel 2-plane field D that is Lagrangian for Ω. *If we let* $I = D^{\perp} \subset T^{*}M$ *denote the annihilator, then* $\Omega \in \langle I \rangle$.

We will consider the following problems:

1. We will calculate the structure equation for complex Engel structures and classify homogeneous complex Engel structures and compact quotients that support homogeneous complex Engel structures.

2. We will derive the structure equation for Lagrangian Engel structures and classify homogeneous Lagrangian Engel structures and compact quotients that support homogeneous Lagrangian Engel structures.

In Section 1, we introduce the basic definitions and theorems related to Engel structures. In Section 2, we classify homogeneous complex Engel structures and compact homogeneous complex Engel structures. In Section 3, we classify homogeneous Lagrangian Engel structures and compact homogeneous Lagrangian Engel structures.

1.2 Fundamental concepts

In this section, we will introduce some concepts that are related to Engel structures and the equivalence problems for Engel structures.

We use the definition of coframe in our analysis of Engel structures.

Definition 1.2.1. *Let M be a smooth n-manifold. A* coframe *at $x \in M$ is a linear isomorphism $u : T_xM \to \mathbb{R}^n$. The set of coframes based at x will be denoted by $F^*_x(M)$.*

Definition 1.2.2. *For any open set $U \subset M$, a* coframing *of U is a choice $\eta = (\eta^i)$ where $1 \leqslant i \leqslant n$ and the η^i are n 1-forms on U that are everywhere linearly independent.*

Given a coframing $\eta : TU \to \mathbb{R}^n$, there is a map $H : U \times GL(n, \mathbb{R}) \to F^*(U)$ defined by the formula

$$H(x, A) = A^{-1}\eta_x.$$

This map respects the right action by $GL(n, \mathbb{R})$ on $F^*(U)$, i.e.

$$H(x, AB) = B^{-1}H(x, A) = H(x, A) \cdot B.$$

Definition 1.2.3 (G-structure)**.** *Let G be a Lie subgroup of $GL(n, \mathbb{R})$. A* G-structure *on an n-manifold M is simply a smooth G-sub-bundle of $F^* = F^*(M)$, i.e., a smooth submanifold $B \subset F^*$ such that the restricted basepoint mapping $\pi : B \to M$ is a surjective submersion whose fibers $B_x = B \cap F^*_x$ are G-orbits. If G is the trivial group containing only the identity, the G-structure is called an* e-structure*.*

Complex Engel structures and Lagrangian Engel structures can be effectively described in terms of G-structures as will be shown below. The classification of homogeneous complex or Lagrangian Engel structures will be reduced to the equivalence problem of the corresponding G-structures. This equivalence problem of G-structures can be solved by the equivalence method of É. Cartan [Gardner (1989)]. In Sections 2 and 3, we will discuss how to formulate the classification problems as equivalence problems and solve these problems via the equivalence method.

2

Complex Engel Structures

In this section, we will discuss the classification of homogeneous complex Engel structures and compact quotients that support homogeneous complex Engel structures. Then we discuss characteristic fields of complex Engel structures and the first order variation of complex Engel structures.

2.1 Geometry of Complex Engel Structures

Let J be the complex structure on the underlying manifold M. Choose a local J-complex coframing (ω_1, ω_2) on an open set $U \subset M$ such that

1. ω_1, ω_2 are of J-type (1,0)
2. $\omega_2 = 0$ defines $I^{\perp}$ on U

Then (ω_1, ω_2) is called a *0-adapted coframing* for the Engel structure.

It is easy to see that

Lemma 2.1.1. *The* 0*-adapted coframings are the sections of a* G*-structure on* M*,* *where* $G \subset GL(2, \mathbb{C})$ *is the 3-dimensional complex subgroup*

$$G = \left\{ \left(\begin{array}{cc} a & b \\ 0 & c \end{array} \right) \middle| \, a, b, c \in \mathbb{C} \text{ and } a, c \neq 0 \right\} .$$

In the following analysis, we will denote the conjugate of ω_1, ω_2 by $\bar{\omega}_1, \bar{\omega}_2$ instead of $\overline{\omega_1}, \overline{\omega_2}$.

Theorem 2.1.2. *A complex Engel structure has a canonical coframing* $(\omega_1,\ \omega_2)$ *(i.e., an e-structure) such that*

$$\begin{aligned} d\omega_2 &\equiv \omega_1 \wedge \bar{\omega}_1 && \mod && \omega_2 \, , \\ d(\omega_2 + \bar{\omega}_2) &\equiv -\frac{1}{2}(\omega_1 - \bar{\omega}_1) \wedge (\omega_2 - \bar{\omega}_2) && \mod && \omega_2 + \bar{\omega}_2 \, . \end{aligned} \tag{2.1}$$

Proof. Since $\omega_2 = 0$ defines the complex Engel structure, a 0-adapted coframing (ω_1, ω_2) is defined on an open set $U \subset M$ up to the following change of coframing:

$$\left(\begin{array}{c} \hat{\omega}_1 \\ \hat{\omega}_2 \end{array} \right) = \left(\begin{array}{cc} a & b \\ 0 & c \end{array} \right) \left(\begin{array}{c} \omega_1 \\ \omega_2 \end{array} \right),$$

where a, b, c are complex functions on U and $ac \neq 0$ on U.

The Engel condition implies that

$$d\omega_2 \equiv A\,\omega_1 \wedge \bar{\omega}_1 \qquad \mod \quad \omega_2, \bar{\omega}_2 \, ,$$

where $A \neq 0$. Define $\hat{\omega}_2 = \frac{1}{A}\omega_2$, then

$$\begin{aligned} d\hat{\omega}_2 &= d\left(\tfrac{1}{A}\right) \wedge \omega_2 + \tfrac{1}{A} d\omega_2 \\ &\equiv \tfrac{1}{A} A \omega_1 \wedge \bar{\omega}_1 \qquad \mod \quad \omega_2, \bar{\omega}_2 \\ &\equiv \omega_1 \wedge \bar{\omega}_1 \qquad \mod \quad \omega_2, \bar{\omega}_2 \, . \end{aligned} \tag{2.2}$$

Thus, after rescaling ω_2, we can arrange that $A = 1$. Then

$$d\omega_2 \equiv \omega_1 \wedge \bar{\omega}_1 \qquad \mod \quad \omega_2, \bar{\omega}_2 \, . \tag{2.3}$$

Such coframings will be said to be 1-adapted. They are the sections of a G_1-structure, where $G_1 \subset G$ is defined by

$$c = a\bar{a}\,.$$

The change of the coframing that preserves (2.3) is reduced to

$$\begin{pmatrix} \hat{\omega}_1 \\ \hat{\omega}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & a\bar{a} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}. \tag{2.4}$$

Now ω_2 is defined up to a real multiple, so the real and imaginary parts of ω_2 are uniquely defined up to a real multiple. Suppose the real part of ω_2 spans $I^{(1)}$, the first derived system.

By (2.3), the real part of ω_2 spans $I^{(1)}$, the first derived system, i.e.,

$$d(\omega_2 + \bar{\omega}_2) \equiv 0 \qquad \text{mod} \quad \omega_2, \bar{\omega}_2\,.$$

Since $d(\omega_2 + \bar{\omega}_2)$ is a real 2-form, there must be a complex function p_1 such that

$$d(\omega_2 + \bar{\omega}_2) \equiv (p_1\omega_1 - \bar{p}_1\bar{\omega}_1) \wedge (\omega_2 - \bar{\omega}_2) \qquad \text{mod} \quad \omega_2 + \bar{\omega}_2\,.$$

Because $I^{(2)} = (0)$, $d(\omega_2 + \bar{\omega}_2) \wedge (\omega_2 + \bar{\omega}_2) \not\equiv 0$ implies that $p_1 \neq 0$. By replacing (ω_1, ω_2) by $\left(-\frac{1}{2p_1}\omega_1, \frac{1}{4p_1\bar{p}_1}\omega_2\right)$, we can arrange $p_1 = -\frac{1}{2}$. After this arrangement, a is fixed to be 1 in the transformation. Now the coframing satisfies

$$d(\omega_2 + \bar{\omega}_2) \equiv -\tfrac{1}{2}(\omega_1 - \bar{\omega}_1) \wedge (\omega_2 - \bar{\omega}_2) \qquad \text{mod} \quad \omega_2 + \bar{\omega}_2\,. \tag{2.5}$$

Such coframings will be said to be 2-adapted. They are the sections of a G_2-structure, where $G_2 \subset G_1$ is defined by $a = 1$.

After setting $a = 1$, by (2.4), ω_2 is unique and ω_1 is unique modulo ω_2, i.e. a change of coframing that preserves (2.3) and (2.5) is reduced to

$$\begin{pmatrix} \hat{\omega}_1 \\ \hat{\omega}_2 \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

Because ω_2 is uniquely defined now, we can write

$$d\omega_2 \equiv \omega_1 \wedge \bar{\omega}_1 + f\omega_1 \wedge \bar{\omega}_2 \qquad \mod \quad \omega_2$$

for some function f. Note that there is no $\bar{\omega}_1 \wedge \bar{\omega}_2$ term since we assumed that (ω_1, ω_2) be of type (1,0), and the underlying almost complex structure is integrable.

By adding a multiple of ω_2 to ω_1, we can arrange $f = 0$.

$$d\omega_2 \equiv \omega_1 \wedge \bar{\omega}_1 \qquad \mod \quad \omega_2. \tag{2.6}$$

After arranging this modification, the coframing satisfying (2.5) and (2.6) is completely determined: the original G-structure defines a canonical sub e-structure. $\square$

Thus by a Theorem of Kobayashi [Kobayashi (1954)],

Corollary 2.1.3. *The symmetry group of a complex Engel structure acts freely on the underlying connected manifold.*

Theorem 2.1.4. *The canonical coframing of a complex Engel structure satisfies*

$$\begin{aligned} d\omega_1 &= -(p_1\omega_1 + p_2\omega_2 + \bar{q}_1\bar{\omega}_1 + \bar{q}_2\bar{\omega}_2) \wedge \omega_1 - (q_2\omega_1 + \bar{r}_1\bar{\omega}_1 + \bar{r}_2\bar{\omega}_2) \wedge \omega_2 \\ d\omega_2 &= (\omega_1 - \omega_2) \wedge \bar{\omega}_1 - (p_1\omega_1 + p_2\omega_2 + \bar{p}_1\bar{\omega}_1 + \bar{p}_2\bar{\omega}_2) \wedge \omega_2 \end{aligned} \tag{2.7}$$

where $p_1, p_2, q_1, q_2, r_1, r_2$ are complex functions. Thus, a complex Engel structure has 6 *fundamental functional invariants.*

Proof. From Theorem 2.1.2, the structure equation can be writen as

$$d\omega_2 = \omega_1 \wedge \bar{\omega}_1 + (p\omega_1 + q\bar{\omega}_1 + r\bar{\omega}_2) \wedge \omega_2$$

for some functions p, q, r.

Therefore

$$\begin{aligned} d(\omega_2 + \bar{\omega}_2) &\equiv (p\omega_1 + q\bar{\omega}_1 + r\bar{\omega}_2) \wedge \omega_2 + (\bar{p}\bar{\omega}_1 + \bar{q}\omega_1 + \bar{r}\omega_2) \wedge \bar{\omega}_2 \\ &\equiv ((p - \bar{q})\omega_1 + (q - \bar{p})\bar{\omega}_1) \wedge \omega_2 \qquad \mod \quad (\omega_2 + \bar{\omega}_2)\,. \end{aligned} \tag{2.8}$$

But according to (2.5)

$$\begin{aligned} d(\omega_2 + \bar{\omega}_2) &\equiv -\tfrac{1}{2}(\omega_1 - \bar{\omega}_1) \wedge (\omega_2 - \bar{\omega}_2) \\ &\equiv -(\omega_1 - \bar{\omega}_1) \wedge \omega_2 \qquad \mod \quad (\omega_2 + \bar{\omega}_2)\,. \end{aligned} \tag{2.9}$$

By comparing (2.8) and (2.9), we find

$$q = \bar{p} + 1\,.$$

Thus

$$d\omega_2 = \omega_1 \wedge \bar{\omega}_1 + \bar{\omega}_1 \wedge \omega_2 + (p\omega_1 + \bar{p}\bar{\omega}_1 + r\bar{\omega}_2) \wedge \omega_2\,.$$

Let $\alpha = -(p\omega_1 + \bar{r}\omega_2)$, then

$$d\omega_2 = \omega_1 \wedge \bar{\omega}_1 + \bar{\omega}_1 \wedge \omega_2 - (\alpha + \bar{\alpha}) \wedge \omega_2\,, \tag{2.10}$$

where α is a $(1,0)$-form, uniquely defined by (2.10).

Let γ be a $(1,0)-$form and β be any 1-form. The structure equation can be written as

$$\begin{aligned} d\omega_1 &= -(\alpha + \bar{\gamma}) \wedge \omega_1 - \beta \wedge \omega_2\,, \\ d\omega_2 &= \omega_1 \wedge \bar{\omega}_1 + \bar{\omega}_1 \wedge \omega_2 - (\alpha + \bar{\alpha}) \wedge \omega_2\,. \end{aligned} \tag{2.11}$$

Taking the exterior derivative of $d\omega_2$ then yields

$$-\bar{\gamma} \wedge \omega_1 \wedge \bar{\omega}_1 + \omega_1 \wedge \bar{\beta} \wedge \bar{\omega}_2 + \omega_1 \wedge \gamma \wedge \bar{\omega}_1 \equiv 0 \qquad \mod \omega_2\,.$$

Recall that γ is a $(1,0)$-form, so

$$\bar{\gamma} \wedge \bar{\omega}_1 + \bar{\beta} \wedge \bar{\omega}_2 \equiv 0 \qquad \mod \quad \omega_1,\ \omega_2\,.$$

Let $\gamma = q_1\omega_1 + q_2\omega_2$, then

$$(\bar{\beta} - \bar{q}_2\bar{\omega}_1) \wedge \bar{\omega}_2 \equiv 0 \qquad \mod \quad \omega_1,\ \omega_2,$$

so

$$\beta \equiv q_2\omega_1 \qquad \mod \quad \bar{\omega}_1,\ \omega_2,\ \bar{\omega}_2\,.$$

Thus, the final structure equation of a complex Engel structure is

$$\begin{aligned} d\omega_1 &= -(p_1\omega_1 + p_2\omega_2 + \bar{q}_1\bar{\omega}_1 + \bar{q}_2\bar{\omega}_2) \wedge \omega_1 - (q_2\omega_1 + \bar{r}_1\bar{\omega}_1 + \bar{r}_2\bar{\omega}_2) \wedge \omega_2 \,, \\ d\omega_2 &= (\omega_1 - \omega_2) \wedge \bar{\omega}_1 - (p_1\omega_1 + p_2\omega_2 + \bar{p}_1\bar{\omega}_1 + \bar{p}_2\bar{\omega}_2) \wedge \omega_2 \,. \end{aligned} \tag{2.12}$$

□

Remark 2.1.1. *We can take exterior derivatives of (2.12), and see that there are no further relations on $p_1, p_2, q_1, q_2, r_1, r_2$. All differential invariants of complex Engel structures are these six or their derivatives with respective to the canonical coframing.*

2.2 Homogeneous Complex Engel Structures and Characteristic Line Fields

2.2.1 Homogeneous Complex Engel Structures

In this section, we will classify homogeneous complex Engel structures. The group of diffeomorphisms preserving a complex Engel structure also preserves its canonical coframing and hence preserves its fundamental invariants. Thus, if it is homogeneous, then the invariants must be constant.

Thus, assume that the functions be constant and take exterior derivatives of $d\omega_1$, $d\bar{\omega}_1$, $d\omega_2$ and $d\bar{\omega}_2$, and set all of these to be zero. This will yield quadratic equations on $p,\ q,\ r$. After solving these equations, which was done with the help of MAPLE, we arrive at the following theorem:

Theorem 2.2.1 (Classification of Homogeneous Complex Engel Structures)**.** *There are six distinct two-parameter families of homogeneous structure equations of complex Engel structures. The constants $(p_1, p_2, q_1, q_2, r_1, r_2)$ in equation (2.7) are listed as follows for the six cases:*

- *Case C1:* $(a + ib,\ 0,\ 0,\ 0,\ 0,\ 0)$
- *Case C2:* $(\frac{1}{2} + ib,\ 0,\ 2ia,\ 0,\ 0,\ 0)$

- *Case C3:* $(\frac{1}{2} - ib,\ 0,\ 2ib,\ \frac{1}{2}(2b+i)(2ia-b),\ 2b^2 - ib,\ (b^2+\frac{1}{4})(2a+ib))$

- *Case C4:* $(0,\ a-ib,\ 2ib,\ a-ib,\ -a-ib,\ 0)$

- *Case C5:* $(a(1-2ib),\ \frac{1}{4}(2a-1)(2b+i)^2,\ 2ib,\ -\frac{1}{4}(2b+i)^2,$
  $-\frac{1}{4}(2b+4ab+i(2a-1))(-2b+i),\ \frac{1}{4}a(-1+2bi)(1+4b^2))$

- *Case C6:* $(\frac{1}{2}(\cos a + i\sin a)(2b+i),\ \frac{1}{4}(-\sin a + 2b\cos a - 1)(2b+i)^2,\ 2ib,$
  $-\frac{1}{8}(2ib\sin a + i\cos a - 2b\cos a - 2ib + \sin a + 1)(2b+i)^2,$
  $\frac{1}{4}(1+2bi)(2ib\sin a + i\cos a + 2b\cos a - 2ib - \sin a - 1),$
  $\frac{1}{16}(1+4b^2)(-1+2ib)(2ib\sin a + i\cos a + 2b\cos a - 2ib - \sin a - 1))$

where a and b are real constants.

Proof. First take the exterior derivatives of $d\omega_2$ and $d\bar{\omega}_2$, which yield

$$\begin{aligned} r_2 &= p_1 q_2 + p_2 - q_2\,, \\ \bar{r}_2 &= \bar{p}_1 \bar{q}_2 + \bar{p}_2 - \bar{q}_2\,. \end{aligned} \tag{2.13}$$

Substituting these into exterior derivatives of $d\omega_2$ and $d\bar{\omega}_2$ yields

$$q_1 + \bar{q}_1 = 0\,.$$

So q_1 is pure imaginary. Set

$$q_1 = 2iq_0\,,\ \bar{q}_1 = -2iq_0\,. \tag{2.14}$$

Substituting these relations into $d\omega_2$ and $d\bar{\omega}_2$ yields

$$\Im p_2 = q_0(p_1 + \bar{p}_1 - 1)\,,$$

where $\Im p_2$ means the imaginary part of p_2.

Now take the exterior derivatives of $d\omega_1$ and $d\bar{\omega}_1$ and set these to be zero. The equations are quadratic expressions in the coefficients of $d\omega_1, d\omega_2$. Then solve these quadratic equations. We get the six different 2-parameter solutions, listed in the Theorem. □

2.2.2 Characteristic Line Fields

Before classifying compact cases for homogeneous complex Engel structures, we will first discuss Engel line fields. As we mentioned in the Introduction, an Engel structure generates contact distributions on 3-dimensional submanifolds transversal to the canonical foliation of the Engel line field. Thus Engel line fields play an important role in the study of Engel structures.

As explained in the Introduction, an Engel distribution D determines a flag

$$L \subset D \subset E \subset TM$$

of subbundles of the tangent bundle. Here $L = \ker(C(I^{(1)}))$ is a line bundle and $E = [D, D]$ is a bundle of rank three.

Definition 2.2.1. *The line field $L \subset D$ is defined by the condition*

$$[L, E] \subset E$$

This line field L is called the characteristic line field, or Engel line field*. The integral curves of the line field are called* characteristic leaves*.*

In terms of a canonical coframing $(\omega_1,\ \omega_2)$, the Engel distribution is

$$D = \ker(\omega_2)\,.$$

The line field is

$$L = \ker(\omega_2,\ d(\omega_2 + \bar{\omega}_2))\,.$$

The distribution is

$$E = \ker(\omega_2 + \bar{\omega}_2)\,.$$

Since

$$\begin{aligned} d(\omega_2 + \bar{\omega}_2) &\equiv (\omega_1 - \omega_2) \wedge \bar{\omega}_1 + (\bar{\omega}_1 - \bar{\omega}_2) \wedge \omega_1 \\ &\equiv (\bar{\omega}_1 - \omega_1) \wedge \omega_2 \qquad \mod \quad (\omega_2 + \bar{\omega}_2)\,, \end{aligned}$$

the characteristic line field is defined as

$$L = \ker(\omega_2,\ \bar{\omega}_1 - \omega_1)\,.$$

2.3 Compact Homogeneous Complex Engel Structures

We have proved the classification result for homogeneous complex Engel structures. Now we can classify compact homogeneous complex Engel structures. We will prove the following theorem:

Theorem 2.3.1 (Classification of Compact Homogeneous Complex Engel Structures)**.** *Let $\mathfrak{g}$ be the 4-dimensional Lie algebra of symmetry vector fields of a complex Engel structure. For the six distinct 2-parameter families of homogeneous complex Engel structures listed in Theorem 2.2.1, the results about compactness in each case are listed as follows:*

- *Case C1: If $a = \frac{1}{2}$ and $b = 0$, the Lie algebra is a 4-dimensional solvable Lie algebra. There exists a compact quotient that supports a homogeneous complex Engel structure if and only if $a = \frac{1}{2}$ and $b = 0$.*

- *Case C2: If there exists a complex number λ and a matrix $A \in SL_3(\mathbb{Z})$ such that*

  1. $b = -a$
  2. $|\lambda| \neq 1$
  3. *the eigenvalues of A are* $(\lambda\bar{\lambda})^{-1}, \lambda, \bar{\lambda}$
  4. *there exists $k \in \mathbb{Z}$ such that* $-\frac{1}{2a}\log(|\lambda|) = \arg\lambda + 2k\pi$

  then there exists a co-compact lattice Γ such that G/Γ supports a homogeneous complex Engel structure.

- *Case C3:*

  1. *If* $a = -\frac{1}{4}$, *the Lie algebra is a solvable Lie algebra, and there exists a compact quotient.*
  2. *If* $a = b^2 \neq 0$, *the Lie algebra is a solvable Lie algebra, but there does not exist a compact quotient.*
  3. *If* $(a < -\frac{1}{4})$ *or* $(0 \leqslant a < b^2)$ *or* $(b = 0 \quad \text{and} \quad a < 0)$, *the Lie algebra is* $\mathbb{R} \times \mathfrak{sl}(2, \mathbb{R})$. *There exists a compact quotient.*
  4. *If* $(-\frac{1}{4} < a < 0)$ *or* $(a > b^2)$ *or* $(b = 0 \quad \text{and} \quad a > 0)$, *the Lie algebra is* $\mathbb{R} \times \mathfrak{su}(2)$. *There exists a compact quotient.*

- *Case C4: There is no compact quotient that supports a homogeneous complex Engel structure.*

- *Case C5: If* $a = \frac{1}{2}$, *there exists a co-compact lattice* Γ *of a solvable Lie group* G *that* G/Γ *supports a homogeneous complex Engel structure.*

- *Case C6: There is no compact quotients that supports a homogeneous complex Engel structure unless* $a = -\frac{\pi}{2} + 2k\pi,\ k \in \mathbb{Z}$ *and* $b = 0$. *Under this condition, this is a special case of case* $C1$.

In summary, compact quotients that support homogeneous complex Engel structures can occur in case $C1$, *case* $C2$, *case* $C3$, *and case* $C5$.

We will prove the theorem in the following section by analyzing the structure equation for each case. If the structure equation is solvable, we will also provide a local coordinate system expression of the coframing.

2.4 Proof of Theorem 2.3.1

2.4.1 Homogeneous Case C1

In this case, $(p_1, p_2, q_1, q_2, r_1, r_2) = (a+ib,\ 0,\ 0,\ 0,\ 0,\ 0)$. The structure equation is

$$\begin{aligned} d\omega_1 &= 0\,, \\ d\omega_2 &= (\omega_1 - \omega_2) \wedge \bar{\omega}_1 - ((a+ib)\,\omega_1 + (a-ib)\,\bar{\omega}_1) \wedge \omega_2\,. \end{aligned} \tag{2.15}$$

By (2.15),

$$d(\omega_1 \wedge \omega_2 \wedge \bar{\omega}_2) = -(1 - 2a + 2bi)\omega_1 \wedge \bar{\omega}_1 \wedge \omega_2 \wedge \bar{\omega}_2\,. \tag{2.16}$$

By Stokes' Theorem, there is no compact example unless $a = \frac{1}{2}$ and $b = 0$.

By (2.15), $d\omega_1 = 0$. By the complex Poincaré Lemma, there exists a holomorphic function z locally on the manifold such that

$$\omega_1 = dz\,. \tag{2.17}$$

Thus

$$d\omega_2 = dz \wedge d\bar{z} + \left[-(a+ib)dz + (1-a+ib)d\bar{z}\right] \wedge \omega_2\,.$$

We will find a local coordinate system for the coframing (ω_1, ω_2) in order to explicitly describe its group of symmetries. Since the groups of symmetries are different for different a and b, we will consider two cases: $a + ib = 1$ (special case) and generic case.

2.4.1.0.1 Special Case

If $a + ib = 1$, i.e. $a = 1$ and $b = 0$,

$$d(\omega_2 + \bar{z}dz) = -dz \wedge (\omega_2 + \bar{z}dz)\,. \tag{2.18}$$

By the complex Frobenius Theorem, there exists a complex function f and a holomorphic function w locally on the manifold such that

$$\omega_2 + \bar{z}dz = f dw\,. \tag{2.19}$$

Since $(\omega_1,\ \omega_2)$ is a coframing, ω_1 and ω_2 are linearly independent. By comparing the local coordinate expressions (2.17) and (2.19), we know that f is nowhere zero on its defining domain. Substituting (2.19) into (2.18) yields

$$df \wedge dw = -f\ dz \wedge dw\,.$$

Setting $f = e^{-z}g$ for some function $g \neq 0$, we have $dg \wedge dw = 0$. So the function g is a function of w only.

$$\omega_2 + \bar{z}dz = e^{-z}g(w)dw\,.$$

By defining $\tilde{w} = \int g(w)dw$ and dropping the tilde in the local coordinate, we get

$$\omega_2 + \bar{z}dz = e^{-z}dw\,.$$

Since (ω_1, ω_2) is a $(1,0)$-coframing, (z, w) can serve as a local holomorphic coordinate system in a neighborhood of the manifold. In the coordinate system of (z, w), the coframing can be expressed as

$$\begin{aligned} \omega_1 &= dz\,. \\ \omega_2 &= -\bar{z}dz + e^{-z}dw\,. \end{aligned} \tag{2.20}$$

We consider the symmetry group of the coframing in these local coordinate. Let

$$\Gamma = \{k_1 + ik_2 | (z, w) \to (z, w + k_1 + ik_2),\ \text{where}\ \ k_1, k_2 \in \mathbb{Z}\}\,.$$

Γ acts freely and discontinuously on the coframing in the local coordinate. So we can take a global model for this complex Engel structure

$$M^4 = \mathbb{C}^2/\Gamma \cong \mathbb{R}^2 \times T^2\,.$$

2.4.1.0.2 Generic Case

If $a + ib \neq 1$, i.e. $a \neq 1$ or $b \neq 0$, by (2.15), we have

$$d\left(\omega_2 - \frac{dz}{1-a+ib}\right) = [-(a+ib)dz + (1-a+ib)d\bar{z}] \wedge \left(\omega_2 - \frac{dz}{1-a+ib}\right)\,. \tag{2.21}$$

By the complex Frobenius Theorem, there exists a complex function f and a holomorphic function w locally on the manifold such that

$$\omega_2 - \frac{dz}{1-a+ib} = f dw \,. \tag{2.22}$$

Substituting (2.22) into (2.21) yields

$$df \wedge dw = f\left[-(a+ib)dz + (1-a+ib)d\bar{z}\right] \wedge dw \,.$$

By defining $f = e^{-(a+ib)z+(1-a+ib)\bar{z}}g$ for some function $g \neq 0$, we get $dg \wedge dw = 0$. Thus

$$\omega_2 - \frac{dz}{1-a+ib} = e^{-(a+ib)z+(1-a+ib)\bar{z}}g(w)dw \,.$$

By defining $\tilde{w} = \int g(w)dw$ and dropping the tilde, we have

$$\omega_2 - \frac{dz}{1-a+ib} = e^{-(a+ib)z+(1-a+ib)\bar{z}}dw \,.$$

Since (ω_1, ω_2) is a coframing, (z, w) can serve as a local holomorphic coordinate system on a open set. The coframing can be written as

$$\begin{aligned} \omega_1 &= dz \,, \\ \omega_2 &= \frac{dz}{1-a+ib} + e^{-(a+ib)z+(1-a+ib)\bar{z}}dw \,. \end{aligned}$$

We will analyze the symmetry group of the coframing. Define

$$G_{a,b} = \{(\alpha,\ \beta) | -(a+ib)\alpha + (1-a+ib)\bar{\alpha} = 2k\pi i, \quad \text{where } \alpha,\ \beta \in \mathbb{C} \text{ and } k \in \mathbb{Z}\} \,,$$

where $G_{a,b}$ acts on the local coordinate as $(z,\ w) \to (z+\alpha,\ w+\beta\,)$. We will analyze the elements of $G_{a,b}$. Let $\alpha = \alpha_0 + i\alpha_1$, where $\alpha_0, \alpha_1 \in \mathbb{R}$. We have

$$\begin{aligned} &(1-2a)\alpha_0 + 2b\alpha_1 = 0 \,, \\ &\alpha_1 = 2k\pi \,, \end{aligned} \tag{2.23}$$

where $k \in \mathbb{Z}$.

For different $a,\ b$, there exist three families of solution for α:

1. if $a \neq \frac{1}{2}$, then $\alpha_0 = \frac{-4bk\pi}{1-2a},\ \alpha_1 = 2k\pi$. Define

   $$\Gamma_1 = \left\{ \left(\left(\frac{-4b\pi}{1-2a} + i2\pi \right) k, \beta_0 + i\beta_1 \right) \middle| k,\ \beta_0,\ \beta_1 \in \mathbb{Z} \right\} .$$

   We can get a non-compact quotient

   $$\mathbb{C}^2/\Gamma_1 \cong \mathbb{R} \times S^1 \times T^2$$

   that supports a homogeneous complex Engel structure.

2. if $a = \frac{1}{2},\ b = 0$, then $\alpha_1 = 2k\pi$. Define

   $$\Gamma_2 = \left\{ (\alpha_0 + i2k\pi, \beta_0 + i\beta_1) \middle| k,\ \alpha_0,\ \beta_0,\ \beta_1 \in \mathbb{Z} \right\} .$$

   We get a compact quotient $\mathbb{C}^2/\Gamma_2$ that supports a homogeneous complex Engel structure.

   In this case, we can define $\theta_1 = -\omega_1$ and $\theta_2 = \omega_1 - \frac{1}{2}\omega_2$. The structure equation is

   $$\begin{aligned} d\theta_1 &= 0\,, \\ d\theta_2 &= \tfrac{1}{2}\left(\theta_1 - \bar{\theta}_1\right) \wedge \theta_2\,. \end{aligned} \tag{2.24}$$

   Define $\theta_1 = \alpha + i\beta$ and $\theta_2 = \gamma + i\delta$ for real parts and imaginary parts decomposition. We have

   $$\begin{aligned} d\alpha &= 0\,, \\ d\beta &= 0\,, \\ d\gamma &= -\beta \wedge \delta\,, \\ d\delta &= \beta \wedge \gamma\,. \end{aligned} \tag{2.25}$$

Thus the Lie algebra is a 4-dimensional solvable Lie algebra $\mathfrak{g}$ with nontrivial brackets:

$$[X, Y] = Z,\ [X, Z] = -Y,$$

where $X, Y, Z \in \mathfrak{g}$. By the classification results in [Bock (2016)], the corresponding connected and simply-connected Lie group has a co-compact lattice.

3. if $a = \frac{1}{2},\ b \neq 0$, then $\alpha_1 = 0$. Define

$$\Gamma_3 = \{(\alpha_0, \beta_0 + i\beta_1) |\, \alpha_0, \beta_0,\ \beta_1 \in \mathbb{Z}\}\ .$$

Then we can get a non-compact quotient

$$\mathbb{C}^2/\Gamma_3 \cong \mathbb{R}^1 \times S^1 \times T^2\,.$$

In summary, there exists a compact quotient of type $C1$ that can support a homogeneous complex Engel structure if and only if $a = \frac{1}{2}$ and $b = 0$.

2.4.1.0.3 Characteristic Line Fields

The characteristic line field of the complex Engel structure is defined as

$$L = \ker(\omega_2,\ \bar{\omega}_1 - \omega_1)\,. \tag{2.26}$$

In the local coordinate, let the characteristic line field be

$$v = a_1\frac{\partial}{\partial z} + a_2\frac{\partial}{\partial w} + \bar{a}_1\frac{\partial}{\partial \bar{z}} + \bar{a}_2\frac{\partial}{\partial \bar{w}}\,. \tag{2.27}$$

We only consider the case that $a + ib \neq 1$. In this case, the coframing is

$$\begin{aligned} \omega_1 &= dz\,, \\ \omega_2 &= \tfrac{dz}{1-a+ib} + e^{-(a+ib)z+(1-a+ib)\bar{z}}dw\,. \end{aligned} \tag{2.28}$$

Substituting (2.27) into (2.26) and (2.28) yields

$$\omega_2(v) = (\bar{\omega}_1 - \omega_1)(v) = 0,$$

that is equivalent to

$$\begin{aligned} \bar{a}_1 - a_1 &= 0\,, \\ \tfrac{a_1}{1-a+ib} + e^{-(a+ib)z+(1-a+ib)\bar{z}}a_2 &= 0\,. \end{aligned}$$

Note a_1 cannot be zero. Normalize the vector field v such that $a_1 = 1$, then

$$a_2 = -\tfrac{1}{1-a+ib}e^{(a+ib)z-(1-a+ib)\bar{z}}\,.$$

Let $\gamma(t) = (z(t), w(t), \bar{z}(t), \bar{w}(t))$ be a characteristic leaf. Then

$$\begin{aligned} z'(t) &= 1\,, \\ w'(t) &= -\tfrac{1}{1-a+ib}e^{(a+ib)z-(1-a+ib)\bar{z}}\,. \end{aligned} \tag{2.29}$$

So there exists a constant $c_1 = z(0)$ such that $z(t) = t + c_1$. Substituting this into (2.29) yields

$$w'(t) = e^{(2a-1)t} \times e^{(a+ib)c_1-(1-a+ib)\bar{c}_1} \times \tfrac{-1}{1-a+ib}\,.$$

There exists a constant $c_2 = w(0)$ such that

$$w(t) = \begin{cases} -2e^{\frac{1}{2}(c_1-\bar{c}_1)}t + c_2, \text{ if } a = \frac{1}{2},\ b = 0 \\ e^{(a+ib)c_1-(1-a+ib)\bar{c}_1} \times \frac{-1}{(1-a+ib)(2a-1)} \times (e^{(2a-1)t} - 1) + c_2, \text{ if } a \neq \frac{1}{2} \end{cases} \tag{2.30}$$

So in both cases, the characteristic leaf depends on two complex numbers, that are the initial point of the integral curve.

If $a = \frac{1}{2},\ b = 0$, we can take a compact quotient and a characteristic leaf is closed if and only if

$$e^{\frac{1}{2}(c_1-\bar{c}_1)} = (\cos \Im c_1, \sin \Im c_1) \in \mathbb{Q} + i\mathbb{Q}\,.$$

Thus, the closedness of characteristic leaves is related to $SO(2, \mathbb{Q})$. In this case we have an infinite discrete family of closed integral curves that are Engel characteristic lines of the underlying complex Engel structure.

2.4.2 Homogeneous Case C2

Now, $(p_1, p_2, q_1, q_2, r_1, r_2) = (\frac{1}{2} + ib,\ 0,\ 2ia,\ 0,\ 0,\ 0)$. Assume $a \neq 0$, otherwise, it is a special case of $C1$. The structure equation is

$$\begin{aligned} d\omega_1 &= 2ia\bar{\omega}_1 \wedge \omega_1\,, \\ d\omega_2 &= (\omega_1 - \omega_2) \wedge \bar{\omega}_1 - \left[\left(\tfrac{1}{2} + ib\right)\omega_1 + \left(\tfrac{1}{2} - ib\right)\bar{\omega}_1\right] \wedge \omega_2\,. \end{aligned} \tag{2.31}$$

By (2.31)

$$d\left(\omega_1 + \left(-\tfrac{1}{2} + i\,(2a - b)\right)\omega_2\right) = \left(\tfrac{1}{2} + ib\right)(\bar{\omega}_1 - \omega_1) \wedge \left(\omega_1 + \left(-\tfrac{1}{2} + i\,(2a - b)\right)\omega_2\right)\ .$$

By the complex Frobenius Theorem, there exist complex functions p and q, and holomorphic functions z and w such that

$$\omega_1 = pdz\,,$$

$$\omega_1 + \Big(-\frac{1}{2} + i(2a - b)\Big)\omega_2 = qdw\,.$$

To calculate the function p, write $\omega_1 = \alpha + i\beta$, where α and β are the real and imaginary part of ω_1, respectively. From the structure equation

$$\begin{aligned} d\alpha &= -4a\alpha \wedge \beta\,, \\ d\beta &= 0\,. \end{aligned}$$

Thus there exist functions $f,\ x,\ y$ such that $\alpha = fdy, \beta = dx$ and $df \equiv 4afdx$ mod dy. After redefining y, we can write $\alpha = e^{4ax}dy$. Thus

$$\omega_1 = e^{4ax}d\left(y - \tfrac{i}{4a}e^{-4ax}\right)\ . \tag{2.32}$$

Define $z = y - \frac{i}{4a}e^{-4ax}$ as a local holomorphic coordinate (Note: $\Im(z) \neq 0$. So, according to the sign of a, we can restrict the definition of z to half of the complex plane), then

$$\omega_1 = \frac{-i}{2a(z - \bar{z})}dz\,.$$

From the structure equation, we get

$$dq \wedge dw = \Big(\frac{1}{2} + ib\Big)\frac{i}{2a(\bar{z} - z)}d(\bar{z} - z) \wedge q dw$$

Thus

$$dq \equiv \Big(\frac{1}{2} + ib\Big)\frac{i}{2a(\bar{z} - z)}d(\bar{z} - z)q \qquad \text{mod} \qquad dw$$

After redefining w, we can take $q = (\bar{z} - z)^{\frac{-2b+i}{4a}}$. So in the local holomorphic coordinate system $z,\ w$,

$$\begin{aligned}
\omega_1 &= \frac{-i}{2a(z - \bar{z})}dz\,, \\
\omega_2 &= \frac{2}{-1 + 2(2a - b)i}\left((\bar{z} - z)^{\frac{-2b+i}{4a}}dw + \frac{i}{2a(z - \bar{z})}dz\right)\,.
\end{aligned}$$

2.4.2.1 Compact case of case $C2$

By (2.31),

$$d(\omega_1 \wedge \bar{\omega}_2 \wedge \omega_2) = 2i(a + b)\omega_1 \wedge \bar{\omega}_1 \wedge \omega_2 \wedge \bar{\omega}_2\,. \tag{2.33}$$

If $a+b \neq 0$, the volume form is exact. By Stokes' Theorem, there cannot be a compact quotient that supports a homogeneous complex Engel structure when $a + b \neq 0$. In the following, only consider $a + b = 0$. Define $\theta_1 = \omega_1$. The structure equation is

$$\begin{aligned}
d\theta_1 &= 2ia\bar{\theta}_1 \wedge \theta_1\,, \\
d\theta_2 &= \left(\frac{1}{2} - ia\right)(\bar{\theta}_1 - \theta_1) \wedge \theta_2\,.
\end{aligned} \tag{2.34}$$

Let $\theta_1 = \alpha + i\beta\,, \theta_2 = \gamma + i\delta$ be real part and imaginary part decompositions. Then (2.34) is equivalent to

$$\begin{aligned} d\alpha &= -4a\alpha \wedge \beta\,, \\ d\beta &= 0\,, \\ d\gamma &= \beta \wedge \delta - 2a\beta \wedge \gamma\,, \\ d\delta &= -\beta \wedge \gamma - 2a\beta \wedge \delta\,. \end{aligned}$$

Let X_1, X_2, X_3, X_4 be left-invariant vector fields dual to the left-invariant forms $\alpha, \beta, \gamma, \delta$, respectively. Then the nontrivial brackets are

$$\begin{aligned} [X_2, X_1] &= 4aX_1\,, \\ [X_2, X_4] &= X_3 - 2aX_4\,, \\ [X_2, X_3] &= -2aX_3 - X_4\,. \end{aligned}$$

By [Bock (2016)], there exists a co-compact lattice for some a. We will calculate the conditions for the existence of a co-compact lattice.

Let $f = -2a + i$, $V = X_3 + iX_4$ and $W = X_3 - iX_4$. The nontrivial brackets are

$$\begin{aligned} [X_2, X_1] &= -(f + \bar{f})X_1\,, \\ [X_2, V] &= f\,V\,, \\ [X_2, W] &= \bar{f}\,W\,. \end{aligned}$$

Since center of the Lie algebra $\mathfrak{g}$ is trivial, we have an exact sequence

$$0 \to \mathfrak{g} \xrightarrow{ad} End(\mathfrak{g}),$$

where $\mathfrak{g} \xrightarrow{ad} End(\mathfrak{g})$ is the adjoint representation. Let $X = xX_1 + yX_2 + zV + \bar{z}W$ be an element of $\mathfrak{g}$. Then

$$ad(X)(X_1, V, W, X_2) = (X_1, V, W, X_2) \begin{bmatrix} -(f+\bar{f})y & 0 & 0 & (f+\bar{f})x \\ 0 & fy & 0 & -fz \\ 0 & 0 & \bar{f}y & -\bar{f}\bar{z} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The connected and simply-connected Lie group corresponding to the Lie algebra $\mathfrak{g}$ is

$$G \cong \left\{ \left[\begin{array}{cccc} y^{-(f+\bar{f})} & 0 & 0 & x \\ 0 & y^f & 0 & z \\ 0 & 0 & y^{\bar{f}} & \bar{z} \\ 0 & 0 & 0 & 1 \end{array} \right] \middle| x, y \in \mathbb{R}, y > 0, z \in \mathbb{C} \right\}.$$

Proposition 2.4.1. *If there exists a complex number λ and a matrix $A \in SL_3(\mathbb{Z})$ such that*

1. $|\lambda| \neq 1$
2. *the eigenvalues of A are $(\lambda\bar{\lambda})^{-1}, \lambda, \bar{\lambda}$*
3. *there exists $k \in \mathbb{Z}$ such that $-\frac{1}{2a}\log(|\lambda|) = \arg\lambda + 2k\pi$*

then there exists a co-compact lattice Γ such that G/Γ supports a homogeneous complex Engel structure.

Proof. Let $N \subset G$ be the subgroup

$$N = \left\{ \left[\begin{array}{cccc} 1 & 0 & 0 & r \\ 0 & 1 & 0 & z \\ 0 & 0 & 1 & \bar{z} \\ 0 & 0 & 0 & 1 \end{array} \right] \middle| r \in \mathbb{R}, z \in \mathbb{C} \right\}. \tag{2.35}$$

It is easy to verify that N is a normal subgroup of G. Thus

$$G/N \cong \left\{ \left[\begin{array}{cccc} y^{-(f+\bar{f})} & 0 & 0 & 0 \\ 0 & y^f & 0 & 0 \\ 0 & 0 & y^{\bar{f}} & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] \middle| y \in \mathbb{R}, y > 0 \right\}$$

is a quotient group. Let $L_1 = \langle \vec{v}_1, \vec{v}_2, \vec{v}_3 \rangle$ be a lattice of the normal subgroup N, to be determined later. We need to find a lattice L_2 of G/N such that the lattice of the group G is

$$L = \left\{ \left[\begin{array}{cc} \gamma & \vec{v} \\ 0 & 1 \end{array} \right] \middle| \text{ where } \gamma \in L_2, \vec{v} \in L_1 \right\}.$$

By the multiplication rule of the group G, this is equivalent to $\gamma\vec{v} \in L_1$ for any $\gamma \in L_2$ and any $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in L_1$. Hence we need to find $a_{ij} \in \mathbb{Z}$ and $c > 0$ and $c \neq 1$ such that

$$\begin{aligned}
\gamma_c\vec{v}_1 &= a_{11}\vec{v}_1 + a_{21}\vec{v}_2 + a_{31}\vec{v}_3\,, \\
\gamma_c\vec{v}_2 &= a_{12}\vec{v}_1 + a_{22}\vec{v}_2 + a_{32}\vec{v}_3\,, \qquad (2.36) \\
\gamma_c\vec{v}_3 &= a_{13}\vec{v}_1 + a_{23}\vec{v}_2 + a_{33}\vec{v}_3\,,
\end{aligned}$$

where γ_c is the linear transform with transformation matrix $\begin{bmatrix} c^{-(f+\bar{f})} & 0 & 0 \\ 0 & c^f & 0 \\ 0 & 0 & c^{\bar{f}} \end{bmatrix}$. Since $\langle \gamma_c\vec{v}_1, \gamma_c\vec{v}_2, \gamma_c\vec{v}_3 \rangle$ will be a new basis for the lattice L_1, then

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in SL_3(\mathbb{Z}).$$

(2.36) is equivalent to

$$\begin{bmatrix} c^{-(f+\bar{f})} & 0 & 0 \\ 0 & c^f & 0 \\ 0 & 0 & c^{\bar{f}} \end{bmatrix} = (\vec{v}_1, \vec{v}_2, \vec{v}_3) \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} (\vec{v}_1, \vec{v}_2, \vec{v}_3)^{-1} \qquad (2.37)$$

The eigenvalues of the matrix A should be $c^{-(f+\bar{f})}$, c^f and $c^{\bar{f}}$ for some $c > 0$. Assume the eigenvalues are $(\lambda\bar{\lambda})^{-1}, \lambda, \bar{\lambda}$ and fix

$$c^{-2a+i} = \lambda\,. \qquad (2.38)$$

Thus

$$c = |\lambda|^{-\frac{1}{2a}}\,. \qquad (2.39)$$

By (2.38),

$$|\lambda| \times |\lambda|^{-\frac{1}{2a}i} = |\lambda| \times e^{i \arg \lambda}\,. \qquad (2.40)$$

Thus the eigenvalue λ and the parameter a satisfy

$$-\tfrac{1}{2a}\log(|\lambda|) = \arg\lambda + 2k\pi \tag{2.41}$$

for certain $k \in \mathbb{Z}$. $\square$

We will calculate an explicit condition on the existence of co-compact lattice. Since the eigenvalues are $(\lambda\bar{\lambda})^{-1}, \lambda, \bar{\lambda}$, the characteristic polynomial of the matrix A is

$$\begin{aligned}&(x-(\lambda\bar{\lambda})^{-1})(x-\lambda)(x-\bar{\lambda})\\ &= x^3 - \left(\left(\lambda\bar{\lambda}\right)^{-1} + \lambda + \bar{\lambda}\right)x^2 + \left(\lambda\left(\lambda\bar{\lambda}\right)^{-1} + \bar{\lambda}\left(\lambda\bar{\lambda}\right)^{-1} + \lambda\bar{\lambda}\right)x - 1\,.\end{aligned} \tag{2.42}$$

Since $A \in SL_3(\mathbb{Z})$, there exist $m, n \in \mathbb{Z}$ such that

$$\begin{aligned}\left(\lambda\bar{\lambda}\right)^{-1} + \lambda + \bar{\lambda} &= m\,,\\ \lambda\left(\lambda\bar{\lambda}\right)^{-1} + \bar{\lambda}\left(\lambda\bar{\lambda}\right)^{-1} + \lambda\bar{\lambda} &= n\,.\end{aligned} \tag{2.43}$$

Note if (2.43) satisfies, we can choose $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -n \\ 0 & 1 & m \end{bmatrix} \in SL_3(\mathbb{Z})$.

Define $p = \lambda + \bar{\lambda}$ and $q = \left(\lambda\bar{\lambda}\right)^{-1}$. It is easy to verify that p and q are real numbers and $\frac{4}{q} \geqslant p^2$. To be a co-compact lattice, $0 < q < 1$. Then by (2.43),

$$\begin{aligned}p + q &= m\,,\\ pq + \frac{1}{q} &= n\,.\end{aligned} \tag{2.44}$$

Then the eigenvalue is $\lambda = \frac{p}{2} + i\sqrt{\frac{1}{q} - \frac{p^2}{4}}$.

Remark 2.4.1. *There exist countably infinite families of solutions for p and q, that yield infinitely many families of co-compact lattices. The co-compact lattices can be derived from solutions of (2.44).*

Now we give an example for some a such that there exists a compact quotient.

Example 2.4.1. *Let* $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \in SL_3(\mathbb{Z})$. *Define* $s = (108 + 12\sqrt{69})^{\frac{1}{3}}$, *then*

$$\lambda = -\frac{s}{12} - \frac{1}{s} + \frac{\sqrt{3}\left(\frac{s}{6} - \frac{2}{s}\right)}{2} i \,.$$

We can calculate a *by (2.41) and* c *by (2.39).*

2.4.3 Homogeneous Case C3

Now

$$(p_1, p_2, q_1, q_2, r_1, r_2) = \left(\tfrac{1}{2} - ib,\ 0,\ 2ib,\ \tfrac{1}{2}(2b + i)(2ia - b),\ 2b^2 - ib,\ (b^2 + \tfrac{1}{4})(2a + ib)\right)$$

Assume at least one of a or b is nonzero. Otherwise, it will be a special case of $C1$ with $a = \frac{1}{2}$ and $b = 0$. The structure equation is

$$\begin{aligned} d\omega_1 &= -[-2ib\bar{\omega}_1 + \frac{1}{2}(2b - i)(-2ia - b)\bar{\omega}_2] \wedge \omega_1 \\ &\quad - \left[\frac{1}{2}(2b + i)(2ia - b)\omega_1 + (2b^2 + ib)\bar{\omega}_1 + \left(b^2 + \frac{1}{4}\right)(2a - ib)\bar{\omega}_2\right] \wedge \omega_2 \,, \\ d\omega_2 &= \omega_1 \wedge \bar{\omega}_1 - \left(\frac{1}{2} - ib\right)(\omega_1 - \bar{\omega}_1) \wedge \omega_2 \,. \end{aligned} \tag{2.45}$$

Define $\theta = \omega_1 + \left(-\frac{1}{2} + ib\right)\omega_2$. By (2.45), we have

$$d\theta = \left[\left(\tfrac{1}{2} + ib\right)(\bar{\omega}_1 + (2a - ib)\,\bar{\omega}_2) - \left(\tfrac{1}{2} + 2a\right)\omega_1\right] \wedge \theta \,. \tag{2.46}$$

Since the symmetry groups are different for different parameters, we will consider the structure equation with different parameters:

1. $a = -\frac{1}{4}$

2. $a = b^2 \neq 0$

3. $a \neq -\frac{1}{4}$ and $a \neq b^2$

2.4.3.0.1 $a = -\frac{1}{4}$

The structure equation (2.46) reduces to

$$
\begin{aligned}
d\Big(\omega_1 + \Big(-\frac{1}{2} + ib\Big)\omega_2\Big) = \Big(\frac{1}{2} + ib\Big)\overline{\Big(\omega_1 + \Big(-\frac{1}{2} + ib\Big)\omega_2\Big)} \\
\wedge \Big(\omega_1 + \Big(-\frac{1}{2} + ib\Big)\omega_2\Big)\,.
\end{aligned}
\tag{2.47}
$$

Let $\omega_1 + \Big(-\frac{1}{2} + ib\Big)\omega_2 = \alpha + i\beta$, where α and β are real 1-forms. Then

$$
\begin{aligned}
d\alpha &= -2b\alpha \wedge \beta\,, \\
d\beta &= \alpha \wedge \beta\,.
\end{aligned}
\tag{2.48}
$$

Since $d(\alpha + 2b\beta) = 0$, there exists function x such that $\alpha + 2b\beta = dx$. By the structure equation, we have $d\beta = dx \wedge \beta$, that implies the existence of a function y such that $\beta = e^x dy$. Thus in terms of local coordinate $x,\ y$,

$$
\begin{aligned}
\alpha &= dx - 2be^x dy\,, \\
\beta &= e^x dy\,.
\end{aligned}
$$

So

$$\omega_1 + \left(-\frac{1}{2} + ib\right)\omega_2 = e^x d(-e^{-x} - 2by + iy)\,.$$

Let $z = -e^{-x} - 2by + iy$. Then

$$\omega_1 + \Big(-\frac{1}{2} + ib\Big)\omega_2 = \frac{dz}{\Big(-\frac{1}{2} + ib\Big)z + \overline{\Big(-\frac{1}{2} + ib\Big)z}}\,.$$

By (2.45), we have

$$\begin{aligned} d\omega_2 &= \omega_1 \wedge \bar{\omega}_1 - \left(\frac{1}{2} - ib\right)(\omega_1 - \bar{\omega}_1) \wedge \omega_2 \\ &= \frac{\Big(-\frac{1}{2} + ib\Big) dz \wedge \omega_2}{\Big(-\frac{1}{2} + ib\Big) z + \overline{\Big(-\frac{1}{2} + ib\Big) z}} + \frac{-\overline{\Big(-\frac{1}{2} + ib\Big)} dz \wedge \bar{\omega}_2}{\Big(-\frac{1}{2} + ib\Big) z + \overline{\Big(-\frac{1}{2} + ib\Big) z}} \\ &\quad + \frac{dz \wedge d\bar{z}}{\bigg(\Big(-\frac{1}{2} + ib\Big) z + \overline{\Big(-\frac{1}{2} + ib\Big) z}\bigg)^2} . \end{aligned} \tag{2.49}$$

Lemma 2.4.2. *Assume $\langle dz, \omega_2 \rangle$ forms a Frobenius system. Then there exists a function f and a holomorphic function w such that*

$$\omega_2 = dw + f dz$$

Proof. Since $\langle dz, \omega_2 \rangle$ forms a Frobenius system of rank 2, there exist functions $u,\ v,\ a,\ b$ such that $dz = adu + bdv$ and $du,\ dv$ are linearly independent . Since $dz \neq 0$, at least one of $a,\ b$ is nonzero. Without loss of generality, assume $b \neq 0$. So $dv = \frac{1}{b}(dz - adu)$. So v is a function of z and x and locally we take z and u, instead of v and u, as local coordinates.

By the complex Frobenius Theorem, there exist functions $r(z, u)$ and $s(z, u)$ such that

$$\begin{aligned} \omega_2 &= r(z, u) du + s(z, u) dz \\ &= d\Big(\int r(z, u)\ du\Big) - \Big(\int \frac{\partial r(z, u)}{\partial z} du\Big) dz + s(z, u) dz \\ &= d\Big(\int r(z, u)\ du\Big) + \Big[s(z, u) - \Big(\int \frac{\partial r(z, u)}{\partial z} du\Big)\Big] dz . \end{aligned}$$

Since ω_2 and dz are linearly independent, $d\Big(\int r(z, u)\ du\Big) \neq 0$. Let $w = \Big(\int r(z, u)\ du\Big)$

and $f = s(z,u) - \left(\int \frac{\partial r(z,u)}{\partial z} du\right)$, then

$$\omega_2 = dw + f dz\,.$$

$\square$

Let $D = -\frac{1}{2} + ib$. By (2.49) and Lemma 2.4.2, we get

$$df \wedge dz = \frac{dz}{Dz + \bar{D}\bar{z}} \wedge \left(D dw - \bar{D} d\bar{w} + \frac{d\bar{z}}{Dz + \bar{D}\bar{z}} - \bar{D}\bar{f} d\bar{z}\right)\,.$$

Thus

$$\begin{aligned} \frac{\partial f}{\partial w} &= \frac{-D}{Dz + \bar{D}\bar{z}}\,, \\ \frac{\partial f}{\partial \bar{w}} &= \frac{\bar{D}}{Dz + \bar{D}\bar{z}}\,, \\ \frac{\partial f}{\partial \bar{z}} &= \frac{\bar{D}\bar{f} - \frac{1}{Dz + \bar{D}\bar{z}}}{Dz + \bar{D}\bar{z}}\,. \end{aligned} \tag{2.50}$$

Let $f = f_1 + i f_2$, where f_1 and f_2 are real and imaginary parts of f, respectively. Let $z = x + iy$ and $w = u + iv$, then (2.50) is equivalent to

$$\begin{aligned} \frac{\partial f_1}{\partial u} &= 0\,, \\ \frac{\partial f_1}{\partial v} &= 0\,, \\ \frac{\partial f_2}{\partial u} &= \frac{2b}{x + 2by}\,, \\ \frac{\partial f_2}{\partial v} &= -\frac{1}{x + 2by}\,, \\ \frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial y} &= \frac{f_1 + 2b f_2 - \frac{2}{x+2by}}{x + 2by}\,, \\ \frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial x} &= \frac{2b f_1 - f_2}{x + 2by}\,. \end{aligned} \tag{2.51}$$

So $f_1 = f_1(x,y), f_2 = g(x,y) + \frac{2bu}{x+2by} - \frac{v}{x+2by}$. The equation is equivalent to

$$\begin{aligned} \frac{\partial f_1}{\partial x} - \frac{\partial g}{\partial y} &= \frac{f_1 + 2bg - \frac{2}{x+2by}}{x+2by}\,, \\ \frac{\partial f_1}{\partial y} + \frac{\partial g}{\partial x} &= \frac{2bf_1 - g}{x+2by}\,. \end{aligned} \tag{2.52}$$

Consider the differential ideal $I = \langle \theta_1,\ \theta_2 \rangle$, where

$$\begin{aligned} \theta_1 &= df_1 - pdx - qdy\,, \\ \theta_2 &= dg + \left(q - \frac{2bf_1 - g}{x+2by}\right)dx - \left(p - \frac{f_1 + 2bg - \frac{2}{x+2by}}{x+2by}\right)dy\,. \end{aligned} \tag{2.53}$$

and

$$\begin{aligned} d\theta_1 &= -\pi_1 \wedge dx - \pi_2 \wedge dy\,, \\ d\theta_2 &= \pi_2 \wedge dx - \pi_1 \wedge dy\,, \end{aligned} \tag{2.54}$$

where $\pi_1 \equiv dp \mod (dx)$ and $\pi_2 \equiv dq \mod (dy)$. This system is involutive and its Cartan characters are $(s_1, s_2) = (2, 0)$. So the solution depends on 2 functions of 1 variable.

We will calculate the coframing in local coordinate for $b = 0$. Let $F_1 = F(-iz)$, $F_2 = G(i\bar{z})$ be two functions of one variable z and C be a constant. The general solution is of the following form

$$\begin{aligned} f_1 &= F_1' + F_2' + \left(\frac{4}{(z+\bar{z})^2} + C\right)\frac{z+\bar{z}}{2}\,, \\ f_2 &= iF_2' - iF_1' - \frac{2}{z+\bar{z}}\left(F_1 + F_2 + \frac{w-\bar{w}}{2i}\right)\,. \end{aligned}$$

Thus in the case $b = 0$, the coframing is

$$\begin{aligned} \omega_1 &= \frac{1}{2}(dw + fdz) - \frac{2dz}{z+\bar{z}}\,, \\ \omega_2 &= dw + fdz\,. \end{aligned}$$

In our original parametrization, $z = -e^{-x} - 2by + iy$. So $z + \bar{z} < 0$. Take a special form $F_1 = 0$ and $F_2 = 0$. Then

$$f(z, w) = \frac{2 - (w - \bar{w})}{z + \bar{z}}\,.$$

The coframing can be written as

$$\omega_1 = \frac{1}{2}\left(dw - \frac{2 + (w - \bar{w})}{z + \bar{z}}dz\right)\,,$$

$$\omega_2 = dw + \frac{2 - (w - \bar{w})}{z + \bar{z}}dz\,.$$

We will prove that there exist compact quotients that support homogeneous complex Engel structures. Before proving this, we need to know the Lie algebra of the homogeneous complex Engel structures.

Proposition 2.4.3. *There exists a basis* (X_1, X_2, X_3, X_4) *such that the nontrivial brackets of the Lie algebra are*

$$[X_2, X_3] = X_1,\ [X_2, X_4] = X_2,\ [X_3, X_4] = -X_3. \tag{2.55}$$

Proof. Recall that

$$\begin{aligned} d\alpha &= -2b\alpha \wedge \beta\,, \\ d\beta &= \alpha \wedge \beta\,. \end{aligned} \tag{2.56}$$

Define $\omega_1 = \gamma + i\delta$.

$$\begin{aligned} d\gamma &= -2b\alpha \wedge \beta - \delta \wedge \beta + 2b\delta \wedge \alpha\,, \\ d\delta &= \alpha \wedge \beta - \alpha \wedge \delta - 2b\beta \wedge \delta\,. \end{aligned} \tag{2.57}$$

Let e_1, e_2, e_3, e_4 be left-invariant vector fields dual to the left-invariant forms $\alpha, \beta, \gamma, \delta$, respectively. Then by (2.57), the nontrivial brackets are

$$\begin{aligned} [e_1, e_2] &= -2be_1 + e_2 - 2be_3 + e_4 \\ [e_1, e_4] &= -2be_3 - e_4 \\ [e_2, e_4] &= e_3 - 2be_4\,. \end{aligned} \tag{2.58}$$

We will consider 2 separate cases:

1. $b = 0$

2. $b \neq 0$

If $b = 0$, the nontrivial brackets are

$$\begin{aligned}
[e_1, e_2] &= e_2 + e_4 \\
[e_1, e_4] &= -e_4 \\
[e_2, e_4] &= e_3\,.
\end{aligned}$$

Define $X_1 = -2e_3, X_2 = e_4, X_3 = 2e_2 + e_4, X_4 = e_1$. Then

$$[X_2, X_3] = X_1,\ [X_2, X_4] = X_2,\ [X_3, X_4] = -X_3\,. \tag{2.59}$$

If $b \neq 0$, define $\tilde{e}_1 = e_1 - \frac{1}{2b}e_2 - \frac{1}{2b}\left(e_4 - \frac{1}{2b}e_3\right), \tilde{e}_4 = e_4 - \frac{1}{2b}e_3$. The nontrivial brackets are

$$\begin{aligned}
[\tilde{e}_1, e_2] &= -2b\tilde{e}_1 + \tilde{e}_4 + \left(\tfrac{1}{2b} - 2b\right)e_3 \\
[\tilde{e}_1, \tilde{e}_4] &= -\left(2b + \tfrac{1}{2b}\right)e_3 \\
[e_2, \tilde{e}_4] &= -2b\tilde{e}_4
\end{aligned}$$

Define $\tilde{\tilde{e}}_1 = \tilde{e}_1 - \frac{1}{4b}\tilde{e}_4$. Then the nontrivial brackets are

$$\begin{aligned}
[\tilde{\tilde{e}}_1, e_2] &= -2b\tilde{\tilde{e}}_1 + \left(\tfrac{1}{2b} - 2b\right)e_3 \\
[\tilde{\tilde{e}}_1, \tilde{e}_4] &= -\left(2b + \tfrac{1}{2b}\right)e_3 \\
[e_2, \tilde{e}_4] &= -2b\tilde{e}_4
\end{aligned}$$

Define $X_1 = -\frac{4b^2+1}{2b}e_3, X_2 = \tilde{\tilde{e}}_1, X_3 = \tilde{e}_4, X_4 = \frac{1}{4b^2+1}\left(\left(-2b - \frac{1}{2b}\right)e_2 - \left(-2b + \frac{1}{2b}\right)e_4\right)$. Then

$$[X_2, X_3] = X_1,\ [X_2, X_4] = X_2,\ [X_3, X_4] = -X_3\,. \tag{2.60}$$

Thus the theorem is true for both cases. □

By [Bock (2016)], there exists a co-compact lattice when $a = -\frac{1}{4}$. Thus there exists a compact quotient of type $C3$ when $a = -\frac{1}{4}$.

2.4.3.0.2 $a = b^2 \neq 0$

Define

$$\theta_1 = \omega_1 + (2b^2 + ib)\omega_2 + \frac{1}{4b^2+1}((4b^2-1) - 4bi)\bar{\omega}_1 + \frac{b}{4b^2+1}(2b(4b^2-3) - (12b^2-1)i)\bar{\omega}_2 \tag{2.61}$$

Then the coframing (θ, θ_1) satisfies

$$\begin{aligned} d\theta_1 &= 0\,, \\ d\theta &= \left(-\tfrac{1}{2} + ib\right)\theta_1 \wedge \theta\,. \end{aligned}$$

Thus there exists a holomorphic coordinate system (z, w) such that θ_1 and θ can be written as linear combinations of $(dw,\ d\bar{w})$ and dz, respectively. There exists a function $f(w, \bar{w})$ such that

$$\begin{aligned} \theta_1 &= df\,, \\ \theta &= e^{(-\frac{1}{2}+ib)f} dz\,. \end{aligned}$$

Thus

$$\omega_1 = e^{\left(-\frac{1}{2}+ib\right)f} dz - \left(-\tfrac{1}{2} + ib\right)\omega_2\,.$$

By (2.61), we have

$$\begin{aligned} &\left(2b^2 - \frac{1}{2} - 2ib\right)\bar{\omega}_2 + \left(2b^2 + \frac{1}{2}\right)\omega_2 \\ &= df - e^{(-\frac{1}{2}+ib)f} dz - \tfrac{1}{4b^2+1}\Big((4b^2-1) - 4bi\Big) e^{(-\frac{1}{2}-ib)\bar{f}} d\bar{z}\,. \end{aligned}$$

Let $\omega_2 = h dz + g dw$, where h and g are functions. By scaling, fix $g = 1$. Take $f = \left(2b^2 + \frac{1}{2}\right)w + \left(2b^2 - \frac{1}{2} - 2ib\right)\bar{w}$, then

$$h = -\frac{2e^{(-\frac{1}{2}+ib)\left[\left(2b^2+\frac{1}{2}\right)w + \left(2b^2-\frac{1}{2}-2ib\right)\bar{w}\right]}}{4b^2+1}\,.$$

So

$$\omega_2 = -\frac{2e^{\left(-\frac{1}{2}+ib\right)\left[\left(2b^2+\frac{1}{2}\right)w+\left(2b^2-\frac{1}{2}-2ib\right)\bar{w}\right]}}{4b^2+1}dz + dw$$

and

$$\omega_1 = e^{\left(-\frac{1}{2}+ib\right)\left[\left(2b^2+\frac{1}{2}\right)w+\left(2b^2-\frac{1}{2}-2ib\right)\bar{w}\right]}dz - \left(-\frac{1}{2}+ib\right)\omega_2\,.$$

Proposition 2.4.4. *There does not exist a compact quotient that supports a homogeneous complex Engel structure when* $a = b^2 \neq 0$*.*

Proof. After changing θ_1 to $\frac{1}{\left(-\frac{1}{2}+ib\right)}\theta_1$, the structure equation is

$$\begin{aligned} d\theta_1 &= 0\,, \\ d\theta &= \theta_1 \wedge \theta\,. \end{aligned} \tag{2.62}$$

Define $\theta_1 = \alpha + i\beta, \theta = \gamma + i\delta$ as the real and imaginary parts decompositions. Then (2.62) is equivalent to

$$\begin{aligned} d\alpha &= 0\,, \\ d\beta &= 0\,, \\ d\gamma &= \alpha \wedge \gamma - \beta \wedge \delta\,, \\ d\delta &= \alpha \wedge \delta + \beta \wedge \gamma\,. \end{aligned} \tag{2.63}$$

Let $-X_3, X_4, X_1, X_2$ be left-invariant vector fields dual to the left-invariant forms $\alpha, \beta, \gamma, \delta$, respectively. Then the nontrivial brackets are

$$\begin{aligned} [X_1, X_3] &= X_1\,, \\ [X_2, X_3] &= X_2\,, \\ [X_1, X_4] &= -X_2\,, \\ [X_2, X_4] &= X_1\,. \end{aligned}$$

The Lie algebra is a solvable Lie algebra, denoted by $\mathfrak{g}_{4,10}$ in [Mubarakzjanov (1963)]. According to the classification results of the existence of co-compact lattices for 4-dimensional solvable Lie groups in [Bock (2016)], the connected and simply-connected Lie group corresponding to $\mathfrak{g}_{4,10}$ does not have a co-compact lattice. Therefore, there does not exist a compact quotient that supports a homogeneous complex Engel structure in this case. □

2.4.3.0.3 $a \neq -\frac{1}{4}$ and $a \neq b^2$

Let $\omega_1 = \alpha + i\beta$ and $\omega_2 = \gamma + i\delta$. From the structure equation (2.45), we have

$$\begin{aligned} d\alpha &= -4b\alpha \wedge \beta - 2b^2\alpha \wedge \gamma + b\alpha \wedge \delta + (4ab - 2b)\beta \wedge \gamma - (2a + 4b^2)\beta \wedge \delta \\ &\quad - 2b\left(\frac{1}{4} + b^2\right)\gamma \wedge \delta\,, \\ d\beta &= -4ab\alpha \wedge \gamma + 2a\alpha \wedge \delta + 2b^2\beta \wedge \gamma - b\beta \wedge \delta - 4a\left(\frac{1}{4} + b^2\right)\gamma \wedge \delta\,, \\ d\gamma &= \beta \wedge (\delta - 2b\gamma)\,, \\ d\delta &= \beta \wedge (2\alpha - \gamma - 2b\delta)\,. \end{aligned} \tag{2.64}$$

Define $\tilde{\alpha} = -2a\alpha + b\beta - (4a^2 - b^2)\gamma + 4ab\delta$. Since at least one of a or b is not zero, $\tilde{\alpha} \neq 0$ and $d\tilde{\alpha} = 0$. Let $e_1,\ e_2,\ e_3,\ e_4$ be the dual vector fields of the 1-forms $\alpha, \beta, \gamma, \delta$, respectively.

Remark 2.4.2. *From Lie theory, there is a split exact sequence [Fulton and Harris (1991)]*

$$0 \to Rad(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}/Rad(\mathfrak{g}) \to 0\,,$$

where $Rad(\mathfrak{g})$ is the radical ideal of $\mathfrak{g}$. Thus

$$\mathfrak{g} \cong Rad(\mathfrak{g}) \oplus \mathfrak{g}/Rad(\mathfrak{g})\,.$$

Denote $\mathfrak{g}_1 = \mathfrak{g}/Rad(\mathfrak{g})$. We will prove the following theorem:

Theorem 2.4.5. *For the Lie algebra* $\mathfrak{g}$ *corresponding to the structure equation(2.64),* $Rad(\mathfrak{g})$ *is 1-dimensional and* $\mathfrak{g}_1$ *is simple 3-dimensional Lie algebra. Specially,*

- $(a < -\frac{1}{4})$ *or* $(0 \leqslant a < b^2)$ *or* $(b = 0$ *and* $a < 0)$*, the Lie algebra is* $\mathbb{R} \times \mathfrak{sl}(2, \mathbb{R})$*.*
- $(-\frac{1}{4} < a < 0)$ *or* $(a > b^2)$ *or* $(b = 0$ *and* $a > 0)$*, the Lie algebra is* $\mathbb{R} \times \mathfrak{su}(2)$*.*

Proof. We will prove this theorem by analyzing the result for the following 3 cases:

- 1. $b = 0$

  In this case $\tilde{\alpha} = \alpha + 2a\gamma$. Define $\tilde{\beta} = \beta,\ \tilde{\gamma} = -2\alpha + \gamma,\ \tilde{\delta} = \delta$, then

$$\begin{aligned} d\tilde{\beta} &= -a\tilde{\gamma} \wedge \tilde{\delta}\,, \\ d\tilde{\gamma} &= -(1+4a)\tilde{\delta} \wedge \tilde{\beta}\,, \\ d\tilde{\delta} &= -\tilde{\beta} \wedge \tilde{\gamma}\,. \end{aligned} \tag{2.65}$$

  If $a > 0$, we define $\tilde{\tilde{\beta}} = \frac{1}{\sqrt{1+4a}}\tilde{\beta}, \tilde{\tilde{\gamma}} = \frac{1}{\sqrt{a}}\tilde{\gamma}, \tilde{\tilde{\delta}} = \frac{1}{\sqrt{a(1+4a)}}\tilde{\delta}$. (2.65) is equivalent to

$$\begin{aligned} d\tilde{\tilde{\beta}} &= \tilde{\tilde{\gamma}} \wedge \tilde{\tilde{\delta}}\,, \\ d\tilde{\tilde{\gamma}} &= \tilde{\tilde{\delta}} \wedge \tilde{\tilde{\beta}}\,, \\ d\tilde{\tilde{\delta}} &= \tilde{\tilde{\beta}} \wedge \tilde{\tilde{\gamma}}\,. \end{aligned}$$

  Thus $\tilde{\tilde{\beta}},\ \tilde{\tilde{\gamma}},\ \tilde{\tilde{\delta}}$ are left-invariant forms of the Lie group $SU(2)$. So if $a > 0$, the manifold can be taken as $S^1 \times SU(2)$.

  If $a < 0$, $Rad(\mathfrak{g}) = \{e_1 + 2e_3\}$ and $\mathfrak{g}/Rad(\mathfrak{g}) = \{u = e_1 - \frac{1}{2a}e_3, v = \ e_2, w = \ e_4\}$. Define

$$\begin{aligned} H &= \frac{-4\sqrt{-a}}{1+4a}u\,, \\ X &= \sqrt{-a}v + w\,, \\ Y &= \frac{\sqrt{-a}}{a(1+4a)}v - \frac{1}{a(1+4a)}w\,. \end{aligned}$$

The nontrivial brackets are

$$[H,X] = 2X\,,$$

$$[H,Y] = -2Y\,,$$

$$[X,Y] = H\,.$$

(H,X,Y) forms a canonical basis for the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. Since $SL_2(\mathbb{R})$ has co-compact lattices [Benoist (2009)], in this case, there exists a compact quotient that supports homogeneous complex Engel structure.

- 2. $a = 0$

By our assumption, $b \neq 0$. The radical ideal is $Rad(\mathfrak{g}) = \left\{e_1 + \frac{2e_3+4be_4}{1+4b^2}\right\}$ and $\mathfrak{g}/Rad(\mathfrak{g}) = \{u = e_1, v = \ e_2 - \frac{e_3}{b}, w = \ e_4\}$. The nontrivial brackets are

$$[u,v] = 2bu + 2w$$

$$[u,w] = -bu$$

$$[v,w] = (2b^2 - \frac{1}{2})u + bv + 2bw$$

Define

$$\begin{cases} A = \frac{2b-1}{4b^2},\ B = \frac{1}{2b},\ C = \frac{1}{2b^2}, D = 2b+1,\ E = -2b, F = 2, s = 2, & \text{if } b \neq -\frac{1}{2}\,, \\ A = \frac{2b+1}{4b^2},\ B = -\frac{1}{2b},\ C = \frac{1}{2b^2}, D = 2b-1,\ E = 2b, F = 2, s = -2, & \text{if } b \neq \frac{1}{2}\,. \end{cases}$$

and

$$H = sv\,,$$

$$X = Au + Bv + Cw\,,$$

$$Y = Du + Ev + Fw\,.$$

Then (H,X,Y) forms a canonical basis for the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ such that

$$[H,X] = 2X\,,$$

$$[H,Y] = -2Y\,,$$

$$[X,Y] = H\ .$$

Since $SL_2(\mathbb{R})$ has co-compact lattices [Benoist (2009)], in this case there exists a compact quotient that supports a homogeneous complex Engel structure.

- 3. $a \neq 0, a \neq -\frac{1}{4}$ and $b \neq 0$

The radical ideal is $Rad(\mathfrak{g}) = \left\{e_1 + \frac{2e_3+4be_4}{1+4b^2}\right\}$ and

$$\mathfrak{g}/Rad(\mathfrak{g}) = \left\{u = e_1 + \frac{e_4}{2b}, v = \ e_2 - \frac{e_4}{4a}, w = \ e_3 + \frac{4a^2-b^2}{4ab}e_4\right\}.$$

The nontrivial brackets are

$$\begin{aligned}
[u,v] &= -\frac{-8ab^2+4a^2-b^2}{4ab}u + \frac{1}{2b}w\,, \\
[u,w] &= -(-4ab^2+4a^2-b^2+a)\left(\frac{1}{4a}u + \frac{1}{2b}v\right)\,, \\
[v,w] &= \frac{-4b^4+16a^3+12ab^2+b^2}{8ab}u + \frac{-4ab^2+4a^2-b^2+a}{4a}v \\
&\quad - \frac{-8ab^2+4a^2-b^2}{4ab}w\,.
\end{aligned}$$

After the proof of the following proposition, we will finish the proof of the theorem.

Proposition 2.4.6. *If $a \neq 0, a \neq -\frac{1}{4}$ and $b \neq 0$, there exists a compact quotient that supports a homogeneous complex Engel structure.*

Proof. 2.4.3.0.4 $a > b^2$

Define

$$U = \sqrt{\frac{1}{-4ab^2+4a^2-b^2+a}}\,,$$

$$V = \sqrt{\frac{a}{2(-4ab^2+4a^2-b^2+a)}}$$

and

$$\begin{aligned}
A &= \frac{V}{2a}(-8ab^2U + 4a^2U - b^2U + b),\ B = V,\ C = -UV,\\
D &= -\frac{V}{2a}(-8ab^2U + 4a^2U - b^2U - b),\ E = V, F = UV,\\
s &= 2bU\,.
\end{aligned}$$

Then define

$$\begin{aligned}
H &= sv\,,\\
X &= Au + Bv + Cw\,, \qquad (2.66)\\
Y &= Du + Ev + Fw\,.
\end{aligned}$$

Then (H, X, Y) forms a canonical basis for the Lie algebra $\mathfrak{su}(2)$ with the following nontrivial brackets

$$\begin{aligned}
[H, X] &= Y\,,\\
[H, Y] &= -X\,,\\
[X, Y] &= H\,.
\end{aligned}$$

Since $SU(2)$ has co-compact lattices, there exists a compact quotient that supports a homogeneous complex Engel structure.

2.4.3.0.5 $0 < a < b^2$

Define

$$\begin{aligned}
U &= \sqrt{\frac{-1}{-4ab^2 + 4a^2 - b^2 + a}}\,,\\
V &= \sqrt{\frac{-a}{-4ab^2 + 4a^2 - b^2 + a}}
\end{aligned}$$

and

$$\begin{aligned}
A &= -\frac{V}{2a}(-8ab^2U + 4a^2U - b^2U - b),\ B = V,\ C = UV, \\
D &= \frac{V}{2a}(-8ab^2U + 4a^2U - b^2U + b),\ E = V, F = -UV, \\
s &= 4bU.
\end{aligned}$$

Then define

$$\begin{aligned}
H &= sv\,, \\
X &= Au + Bv + Cw\,, \\
Y &= Du + Ev + Fw\,.
\end{aligned}$$

Then (H, X, Y) forms a canonical basis for the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ with non-trivial brackets

$$\begin{aligned}
[H, X] &= 2X\,, \\
[H, Y] &= -2Y\,, \\
[X, Y] &= H\,.
\end{aligned}$$

Since $SL_2(\mathbb{R})$ has co-compact lattices [Benoist (2009)], there exists a compact quotient that supports a homogeneous complex Engel structure.

2.4.3.0.6 $-\frac{1}{4} < a < 0$

Define

$$\begin{aligned}
U &= \sqrt{\frac{1}{-4ab^2 + 4a^2 - b^2 + a}}\,, \\
V &= \sqrt{\frac{a}{2(-4ab^2 + 4a^2 - b^2 + a)}}
\end{aligned}$$

and

$$
\begin{aligned}
A &= \frac{V}{2a}(-8ab^2U + 4a^2U - b^2U + b),\ B = V,\ C = -UV, \\
D &= -\frac{V}{2a}(-8ab^2U + 4a^2U - b^2U - b),\ E = V, F = UV, \\
s &= 2bU\,.
\end{aligned}
$$

Then define

$$
\begin{aligned}
H &= sv\,, \\
X &= Au + Bv + Cw\,, \\
Y &= Du + Ev + Fw\,.
\end{aligned}
$$

Note $Y = \overline{X}$. (H, X, Y) forms a canonical basis of the Lie algebra $\mathfrak{su}(2)$ with nontrivial brackets

$$
\begin{aligned}
[H, X] &= \overline{X}\,, \\
[H, \overline{X}] &= -X\,, \\
[X, \overline{X}] &= H\,.
\end{aligned}
$$

Since $SU(2)$ has co-compact lattices, there exists a compact quotient that supports a homogeneous complex Engel structure.

2.4.3.0.7 $a < -\frac{1}{4}$

Define

$$
\begin{aligned}
U &= \sqrt{\frac{1}{-4ab^2 + 4a^2 - b^2 + a}}\,, \\
V &= \sqrt{\frac{-a}{2(-4ab^2 + 4a^2 - b^2 + a)}}
\end{aligned}
$$

and

$$\begin{aligned}
A &= \frac{V}{2a}(-8ab^2U + 4a^2U - b^2U + b),\ B = V,\ C = -UV,\\
D &= -\frac{V}{2a}(-8ab^2U + 4a^2U - b^2U - b),\ E = V, F = UV,\\
s &= 2bU.
\end{aligned}$$

Then define

$$\begin{aligned}
H &= sv\,,\\
X &= Au + Bv + Cw\,,\\
Y &= Du + Ev + Fw\,.
\end{aligned}$$

(H, X, Y) forms a canonical basis of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ with nontrivial brackets

$$\begin{aligned}
[H, X] &= Y\,,\\
[H, Y] &= -X\,,\\
[X, Y] &= -H\,.
\end{aligned}$$

Since $SL_2(\mathbb{R})$ has co-compact lattices [Benoist (2009))], there exists a compact quotient that supports a homogeneous complex Engel structure. □

Since there exists a co-compact lattice for each case, we have proved the theorem.

□

2.4.3.0.8 Characteristic line field of the Engel structure

In this section, we only consider characteristic line field on $S^1 \times SU(2)$. Let

$$\mathfrak{su}(2) = \left\{ g^{-1}dg = \left[\begin{array}{cc} -\frac{i}{2}\alpha_1 & -\frac{1}{2}(\alpha_2 + i\alpha_3) \\ -\frac{1}{2}(-\alpha_2 + i\alpha_3) & \frac{i}{2}\alpha_1 \end{array} \right] \middle| \alpha_1, \alpha_2, \alpha_3 \text{ are real 1-forms} \right\} .$$

The left-invariant 1-forms satisfy

$$d\alpha_1 = \alpha_2 \wedge \alpha_3 \,,$$

$$d\alpha_2 = \alpha_3 \wedge \alpha_1 \,,$$

$$d\alpha_3 = \alpha_1 \wedge \alpha_2 \,.$$

Let X_1, X_2, X_3 be the dual vectors of $\alpha_1, \alpha_2, \alpha_3$, respectively. We first calculate integral curves along left-invariant vector fields.

1. Consider the vector field X_1 satisfying

   $$g^{-1}dg(X_1) = \begin{bmatrix} -\frac{i}{2} & 0 \\ 0 & \frac{i}{2} \end{bmatrix},$$

   so

   $$\mathcal{L}_{X_1} g = g \begin{bmatrix} -\frac{i}{2} & 0 \\ 0 & \frac{i}{2} \end{bmatrix}.$$

   Thus

   $$g(\exp_{tX_1} \cdot x) = g(x) \begin{bmatrix} e^{-\frac{i}{2}t} & 0 \\ 0 & e^{\frac{i}{2}t} \end{bmatrix}.$$

   The period of the integral curve along X_1 is 4π.

2. Consider the vector field X_2 satisfying

   $$\mathcal{L}_{X_2} g = g \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}.$$

   Thus

   $$g(\exp_{tX_2} \cdot x) = g(x) \cdot \exp\left(t \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \right).$$

   Since the eigenvalues of the matrix $\begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$ are $-\frac{i}{2}$ and $\frac{i}{2}$, the period of the integral curve along X_2 is 4π.

3. Consider a general left invariant vector field $X = c_1X_1 + c_2X_2 + c_3X_3$ satisfying

$$\mathcal{L}_X g = g \begin{bmatrix} -\frac{i}{2}c_1 & -\frac{1}{2}(c_2 + ic_3) \\ -\frac{1}{2}(-c_2 + ic_3) & \frac{i}{2}c_1 \end{bmatrix}.$$

Thus

$$g(\exp_{tX} \cdot x) = g(x) \cdot \exp\left(t \begin{bmatrix} -\frac{i}{2}c_1 & -\frac{1}{2}(c_2 + ic_3) \\ -\frac{1}{2}(-c_2 + ic_3) & \frac{i}{2}c_1 \end{bmatrix} \right).$$

Since the eigenvalues of the matrix

$$\begin{bmatrix} -\frac{i}{2}c_1 & -\frac{1}{2}(c_2 + ic_3) \\ -\frac{1}{2}(-c_2 + ic_3) & \frac{i}{2}c_1 \end{bmatrix}$$

are $-\frac{i}{2}\sqrt{c_1^2 + c_2^2 + c_3^2}$ and $\frac{i}{2}\sqrt{c_1^2 + c_2^2 + c_3^2}$, the period of the integral curve along X is $\frac{4\pi}{\sqrt{c_1^2+c_2^2+c_3^2}}$.

Now we calculate characteristic line fields for the Engel structure. We only derive the expression for $b = 0$; the calculations of other cases are similar. If $b = 0$, the Engel line field is defined as

$$\begin{aligned} L &= \ker(\omega_2, \bar{\omega}_1 - \omega_1) \\ &= \ker(\gamma, \delta, \beta) \qquad (2.67) \\ &= \ker(\tilde{\beta}, \tilde{\delta}, \tilde{\gamma} + 2\tilde{\alpha}). \end{aligned}$$

Let X_0, X_1, X_2, X_3 be left-invariant vector fields dual to $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$, respectively. By (2.67), $L = \langle X_0 - 2X_2 \rangle$. Thus the integral curve starting at $(0, g)$ is

$$\left(t, g \cdot \exp\left(t \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \right) \right).$$

2.4.4 Homogeneous Case C4

Now $(p_1, p_2, q_1, q_2, r_1, r_2) = (0,\ a-ib,\ 2ib,\ a-ib,\ -a-ib,\ 0)$. The structure equation is

$$\begin{aligned} d\omega_1 &= 2ib\bar{\omega}_1 \wedge \omega_1 - (a+ib)\bar{\omega}_2 \wedge \omega_1 + (a-ib)\bar{\omega}_1 \wedge \omega_2, \\ d\omega_2 &= (\omega_1 - \omega_2) \wedge \bar{\omega}_1 - (a+ib)\bar{\omega}_2 \wedge \omega_2. \end{aligned} \tag{2.68}$$

It is easy to verify that

$$d(\omega_1 \wedge \bar{\omega}_2 \wedge \omega_2) = (1+2bi)\omega_1 \wedge \bar{\omega}_1 \wedge \omega_2 \wedge \bar{\omega}_2 \neq 0.$$

By Stokes' Theorem, there is no compact quotient of type $C4$ that supports a homogeneous complex Engel structure.

We will find local coordinate representations of ω_1 and ω_2 under different conditions for a and b. The symmetry groups of the coframing are different for different a and b. We define $A = 1$ and

$$B = \begin{cases} -\frac{1}{2} \pm c + ib, \text{ where } c = \sqrt{\frac{1}{4} - (b^2 + a)}, \text{ if } b^2 + a \leqslant \frac{1}{4} \\ -\frac{1}{2} \pm ic + ib, \text{ where } c = \sqrt{(b^2 + a) - \frac{1}{4}}, \text{ if } b^2 + a > \frac{1}{4} \end{cases} \tag{2.69}$$

By (2.68),

$$d(A\omega_1 + B\omega_2) \wedge (A\omega_1 + B\omega_2) = 0\,.$$

2.4.4.1 $a = b = 0$

The structure equation (2.68) reduces to

$$\begin{aligned} d\omega_1 &= 0\,, \\ d\omega_2 &= (\omega_1 - \omega_2) \wedge \bar{\omega}_1\,. \end{aligned}$$

Since $d\omega_1 = 0$, by the complex Poincaré Lemma, there exists a holomorphic function z such that $\omega_1 = dz$. So $d\omega_2 = (dz - \omega_2) \wedge d\bar{z}$, that is equivalent to $d(dz - \omega_2) =$

$-(dz-\omega_2)\wedge d\bar{z}$. By the complex Frobenius Theorem, there exists a function f and a holomorphic function w such that $dz-\omega_2 = f\ dw$ and $df \wedge dw = fd\bar{z}\wedge dw$. By modifying w, we can write $dz-\omega_2 = e^{\bar{z}}\ dw$. So in the local coordinate system $(z,\ w)$, the coframing is

$$\omega_1 = dz\,,$$

$$\omega_2 = -e^{\bar{z}}\ dw + dz\,.$$

Take a discrete symmetry group of the coframing

$$\Gamma = \{(2\alpha_0\pi i, \beta_0 + i\beta_1) | \alpha_0,\ \beta_0,\ \beta_1 \in \mathbb{Z}\} \cong \mathbb{Z}^3.$$

Γ acts on the local coordinate as $(z,\ w) \mapsto (z + 2\alpha_0\pi i,\ w+\beta_0+i\beta_1)$. This action keeps the coframing invariant. The quotient can be taken as

$$\mathbb{C}^2/\Gamma \cong \mathbb{R}^1 \times S^1 \times T^2\,.$$

2.4.4.2 $\quad b^2 + a < \frac{1}{4}$ *or* $b^2+a > \frac{1}{4}$

Since the calculations are similar for these two cases, without loss of generality, assume $b^2+a<\frac{1}{4}$. Define $U = -\frac{1}{2}+c+ib, V = -\frac{1}{2}-c+ib$ and $\theta_1 = \omega_1 + U\omega_2$, $\theta_2 = \omega_1 + V\omega_2$.

Then the structure equation is

$$d\theta_1 = (\frac{1}{2} - c + ib)\bar{\theta}_2 \wedge \theta_1\,,$$

$$d\theta_2 = (\frac{1}{2} + c + ib)\bar{\theta}_1 \wedge \theta_2\,.$$

By the complex Frobenius Theorem, there exist coordinates z and w and functions f and g, such that $\theta_1 = fdz,\ \theta_2 = \bar{g}d\bar{w}$. Since $d(\theta_1 \wedge \bar{\theta}_2) = d(fgdz\wedge dw) = 0$, so there exists a function $F(z,w)$ such that $fg = F(z,w)$. Since

$$d\theta_1 = df \wedge dz = \Big(\frac{1}{2} - c + ib\Big)F(z,w)dw\wedge dz\,,$$

$$d\bar{\theta}_2 = dg \wedge dw = \Big(\frac{1}{2} + c + ib\Big)F(z,w)dz\wedge dw\,,$$

f and g are both functions of z, w and

$$f_w = \Big(\frac{1}{2} - c + ib\Big) fg\,,$$

$$g_z = \Big(\frac{1}{2} + c + ib\Big) fg\,.$$

Thus

$$\begin{aligned}(\log fg)_{wz} &= \Big(\frac{f_w}{f}\Big)_z + \Big(\frac{g_w}{g}\Big)_z \\ &= \Big(\frac{1}{2} - c + ib\Big) g_z + \Big(\frac{g_w}{g}\Big)_z \\ &= \Big((\frac{1}{2} + ib)^2 - c^2\Big) fg + \Big(\frac{g_w}{g}\Big)_z\,,\end{aligned}$$

while

$$\begin{aligned}(\log fg)_{zw} &= \Big(\frac{f_z}{f}\Big)_w + \Big(\frac{g_z}{g}\Big)_w \\ &= \Big(\frac{f_z}{f}\Big)_w + \Big(\frac{1}{2} + c + ib\Big) f_w \\ &= \Big(\frac{f_z}{f}\Big)_w + \Big((\frac{1}{2} + ib)^2 - c^2\Big) fg.\end{aligned}$$

Since $(\log fg)_{wz} = (\log fg)_{zw}$,

$$\Big(\frac{f_z}{f}\Big)_w = \Big(\frac{g_w}{g}\Big)_z.$$

Since $(\log f)_{zw} = \Big(\frac{f_z}{f}\Big)_w$ and $(\log g)_{zw} = \Big(\frac{g_w}{g}\Big)_z$,

$$(\log f)_{zw} = (\log g)_{zw}.$$

So there exist functions $A(z)$ and $B(w)$ such that

$$\frac{f(z,w)}{g(z,w)} = \frac{A(z)}{B(w)}.$$

So there exists a function $G(z,w)$ such that

$$\theta_1 = A(z)G(z,w)dz, \quad \theta_2 = \overline{B(w)G(z,w)}d\bar{w}.$$

After redefining z and w and the function $f(z,w)$, we can write

$$\theta_1 = f(z,w)dz, \quad \theta_2 = \overline{f(z,w)}d\bar{w}$$

$$\begin{cases} f_w &= \left(\frac{1}{2} - c + ib\right)f^2 \\ f_z &= \left(\frac{1}{2} + c + ib\right)f^2 \end{cases}$$

Thus there exists a constant C such that

$$f(z,w) = \frac{1}{-\left(\frac{1}{2} + c + ib\right)z + \left(\frac{1}{2} - c + ib\right)w + C}.$$

After translating z or w,

$$\theta_1 = \frac{dz}{-\left(\frac{1}{2} + c + ib\right)z + \left(\frac{1}{2} - c + ib\right)w},$$

$$\theta_2 = \frac{d\bar{w}}{-\left(\frac{1}{2} + c + ib\right)\bar{z} + \left(\frac{1}{2} - c + ib\right)\bar{w}}.$$

Now change the notation from w to $\bar{w}$, then

$$\theta_1 = \frac{dz}{-\left(\frac{1}{2} + c + ib\right)z + \left(\frac{1}{2} - c + ib\right)\bar{w}},$$

$$\theta_2 = \frac{dw}{-\left(\frac{1}{2} + c + ib\right)\bar{z} + \left(\frac{1}{2} - c + ib\right)w}.$$

So the Engel structure can be defined on the complex 2-plane except two lines

$$\begin{aligned} -\left(\frac{1}{2} + c + ib\right)z + \left(\frac{1}{2} - c + ib\right)\bar{w} = 0, \\ -\left(\frac{1}{2} + c + ib\right)\bar{z} + \left(\frac{1}{2} - c + ib\right)w = 0. \end{aligned} \tag{2.70}$$

Now we can get the local coordinate representation of ω_1 and ω_2

$$\begin{aligned}\omega_1 &= \frac{1}{2c}\Big[(\frac{1}{2}+c+ib)\theta_1 + (-\frac{1}{2}+c+ib)\theta_2\Big],\\ \omega_2 &= \frac{1}{2c}(\theta_1-\theta_2).\end{aligned}$$

Let λ and μ be two complex constants. The symmetry group of the coframing is

$$(z,w)\to(\lambda z,\ \bar\lambda w)$$

and

$$(z,w)\to\left(z+\left(\frac{1}{2}-c+ib\right)\mu,\ w+\left(\frac{1}{2}+c+ib\right)\bar\mu\right).$$

2.4.4.3 $b^2+a=\frac{1}{4}$

The structure equation is

$$\begin{aligned}d\omega_1 &= 2ib\bar\omega_1\wedge\omega_1 - \left(\frac{1}{4}-b^2+ib\right)\bar\omega_2\wedge\omega_1 + \left(\frac{1}{4}-b^2-ib\right)\bar\omega_1\wedge\omega_2\,,\\ d\omega_2 &= (\omega_1-\omega_2)\wedge\bar\omega_1 - \left(\frac{1}{4}-b^2+ib\right)\bar\omega_2\wedge\omega_2\,.\end{aligned}$$

Define $\theta=\omega_1+\left(-\frac{1}{2}+ib\right)\omega_2$. Then

$$d\theta=\Big(\frac{1}{2}+ib\Big)\bar\theta\wedge\theta.$$

Let $\theta=\alpha+i\beta$ be the real part and imaginary part decomposition. Then

$$\begin{aligned}d\alpha &= -2b\alpha\wedge\beta\,,\\ d\beta &= \alpha\wedge\beta,\end{aligned}$$

so $d(\alpha+2b\beta)=0$. Thus there exists a real function x such that $\alpha+2b\beta=dx$. Thus, $d\beta=dx\wedge\beta$. This is equivalent to $d(e^{-x}\beta)=0$, that implies the existence of a real function y such that $\beta=e^x dy$. So

$$\theta=dx-2be^x dy+ie^x dy.$$

Let $z = -e^{-x} - 2by + iy$. Then

$$\theta = \frac{dz}{(-\frac{1}{2} + ib)z + (-\frac{1}{2} - ib)\bar{z}}.$$

Recall that the structure equation is

$$\begin{aligned} d\omega_1 &= \left(\frac{1}{2} + ib\right)\bar{\theta} \wedge \omega_1 + \left(-\frac{1}{2} + ib\right)\bar{\omega}_1 \wedge \theta, \\ d\omega_2 &= \theta \wedge \bar{\theta} + \left(\frac{1}{2} + ib\right)\left(\theta \wedge \bar{\omega}_2 + \bar{\theta} \wedge \omega_2\right). \end{aligned}$$

Let $\omega_1 = \gamma + i\delta$ be the real part and imaginary part decomposition. Since $\theta = \alpha + i\beta$, the exterior derivative of ω_1 can be written as

$$\begin{aligned} d\gamma &= \alpha \wedge \gamma + \beta \wedge \delta - 2b\alpha \wedge \delta + 2b\beta \wedge \gamma, \\ d\delta &= 0. \end{aligned}$$

So there exists a real function u such that $\delta = du$. Since $\langle \gamma, du \rangle$ forms a Frobenius system, there exist real functions $p,\ q,\ v$ such that $\gamma = pdv + qdu$. Write

$$\begin{aligned} dp &= p_x dx + p_y dy + p_u du + p_v dv\,, \\ dq &= q_x dx + q_y dy + q_u du + q_v dv\,. \end{aligned}$$

Then from the structure equation, the functions p and q satisfy

$$\begin{aligned} p_x &= \frac{p}{-x - 2by}, \\ p_y &= \frac{2bp}{-x - 2by}, \\ p_u &= q_v, \\ q_x &= \frac{q - 2b}{-x - 2by}, \\ q_y &= \frac{1 + 2bq}{-x - 2by}. \end{aligned}$$

From the first two equations, we know that there exists a function $C_1(u,v)$ such that $p = \frac{C_1(u,v)}{x+2by}$. From the last two equations, we know that there exists a function $C_2(u,v)$ such that $q = \frac{(2bx-y)+C_2(u,v)}{x+2by}$. From the third equation we know that $\frac{\partial C_1}{\partial u} = \frac{\partial C_2}{\partial v}$. Thus $d\Big(\int C_1(u,v)dv\Big) = C_1(u,v)dv + C_2(u,v)du$. Thus

$$\begin{aligned}
\gamma &= \frac{C_1(u,v)}{x+2by}dv + \frac{(2bx-y)+C_2(u,v)}{x+2by}du \\
&= \frac{C_1(u,v)dv + C_2(u,v)du}{x+2by} + \frac{2bx-y}{x+2by}du \\
&= \frac{d(\int C_1(u,v)dv)}{x+2by} + \frac{2bx-y}{x+2by}du.
\end{aligned}$$

Now define $(\int C_1(u,v)dv)$ as new v, then

$$\gamma = \frac{dv}{x+2by} + \frac{2bx-y}{x+2by}du.$$

So

$$\begin{aligned}
\omega_1 &= \gamma + i\delta \\
&= \frac{dv}{x+2by} + \frac{2bx-y}{x+2by}du + idu \\
&= \frac{1}{x+2by}\left[dv + (2b+i)(x+iy)du\right] \\
&= \frac{1}{x+2by}\left[d(v+(2b+i)(x+iy)u) - (2b+i)u\ d(x+iy)\right].
\end{aligned}$$

Define $w = v + (2b+i)(x+iy)u$ and redefine $z = x + iy$. The coframing can be written as

$$\begin{aligned}
\omega_1 &= \frac{1}{x+2by}[dw - (2b+i)u\ dz] \\
&= -\frac{1}{(-\frac{1}{2}+ib)z + (-\frac{1}{2}-ib)\bar{z}} \\
&\quad \times \left[dw + \left(\frac{1}{2} - ib\right) \frac{w-\bar{w}}{(-\frac{1}{2}+ib)z + (-\frac{1}{2}-ib)\bar{z}} dz \right], \\
\omega_2 &= \frac{1}{-\frac{1}{2}+ib}(\theta - \omega_1) \\
&= \frac{1}{(-\frac{1}{2}+ib)(-\frac{1}{2}+ib)z + (-\frac{1}{2}-ib)\bar{z}} \\
&\quad \times \left[dz + dw + \left(\frac{1}{2} - ib\right) \frac{w-\bar{w}}{(-\frac{1}{2}+ib)z + (-\frac{1}{2}-ib)\bar{z}} dz \right],
\end{aligned}$$

where (z, w) is a local holomorphic coordinate system on the manifold. And $\omega_2 = 0$ defines the complex Engel structure. Let $\lambda,\ s,\ t$ be any real constants. The coframing is invariant under the following local transformation

$$(z,\ w) \to \left(\lambda z + \left(b - \frac{1}{2}i\right)t,\ \lambda w + s\right).$$

We can take a discrete subgroup Γ such that M is locally biholomorphic to $\mathbb{C}^2/\Gamma \cong \mathbb{R}^2 \times T^2$.

2.4.5 Homogeneous Case C5

Assume $a \neq 0$. Otherwise, this is a special case of homogeneous case C4, with $a + b^2 = \frac{1}{4}$. The structure equation is

$$
\begin{aligned}
d\omega_1 = {} & \left(b^2 - \frac{1}{4} - ib\right)\bar{\omega}_2 \wedge \omega_1 + \left(\frac{1}{4}a + ab^2 + i\left(\frac{1}{2}ab + 2ab^3\right)\right)\bar{\omega}_2 \wedge \omega_2 + 2ib\bar{\omega}_1 \wedge \omega_1 \\
& + \left(\frac{1}{4} - \frac{1}{2}a - 2ab^2 - b^2 - ib\right)\bar{\omega}_1 \wedge \omega_2 + \left(\frac{1}{2}a - 2ab^2 - 2iab\right)\omega_2 \wedge \omega_1, \\
d\omega_2 = {} & \left(-\frac{1}{4} + \frac{1}{2}a - 2ab^2 + b^2 + i\left(2ab - b\right)\right)\bar{\omega}_2 \wedge \omega_2 + \omega_1 \wedge \bar{\omega}_1 \\
& + \left(-a + 2iab\right)\omega_1 \wedge \omega_2 + \left(-1 + a + 2iab\right)\omega_2 \wedge \bar{\omega}_1.
\end{aligned}
$$

Define $\theta = \omega_1 + \left(-\frac{1}{2} + ib\right)\omega_2$, that satisfies

$$
d\theta = \left(\frac{1}{2} + ib\right)\bar{\theta} \wedge \theta. \tag{2.71}
$$

This structure equation is same as that of the case C4, with $b^2 + a = \frac{1}{4}$. But in case C5, a and b do not have to satisfy this relation.

Let $\theta = \alpha + i\beta$ be the real and imaginary part decomposition. By (2.71),

$$
\begin{aligned}
d\alpha &= -2b\alpha \wedge \beta, \\
d\beta &= \alpha \wedge \beta.
\end{aligned}
$$

Since $d(\alpha + 2b\beta) = 0$, there exists a real function x such that $\alpha + 2b\beta = dx$. Thus $d\beta = dx \wedge \beta$. So there exists a real function y such that $\beta = e^x dy$, that yields

$$
\theta = dx - 2be^x dy + ie^x dy\,.
$$

Let $z = -e^{-x} - 2by + iy$. Then

$$
\theta = \frac{dz}{(-\frac{1}{2} + ib)z + (-\frac{1}{2} - ib)\bar{z}}\,.
$$

The symmetry groups of the coframing are different for different parameters a and b. In the following sections, we will consider the following cases:

1. $a \neq \pm\frac{1}{2}$
2. $a = \frac{1}{2}, b = 0$
3. $a = \frac{1}{2}, b \neq 0$
4. $a = -\frac{1}{2}$

2.4.5.1 $a \neq \pm\frac{1}{2}$

Define $A = r + is$, where

$$r = \frac{1}{\left(\frac{1}{2} + a\right)(1 + 4b^2)}$$

$$s = \frac{2b}{\left(\frac{1}{2} - a\right)(1 + 4b^2)}.$$

After defining $\theta_2 = \omega_2 + A\theta$, the structure equation reduces to

$$d\theta_2 = 2a\Big(-\frac{1}{2} + ib\Big)\theta \wedge \theta_2 + (2a - 1)\Big(\frac{1}{2} + ib\Big)\theta_2 \wedge \bar{\theta} + \Big(\frac{1}{2} + ib\Big)\theta \wedge \bar{\theta}_2.$$

Let $\theta_2 = \gamma + i\delta$ and $\theta = \frac{dx+idy}{A(z)}$, where $A(z) = (-\frac{1}{2} + ib)z + (-\frac{1}{2} - ib)\bar{z} = -x - 2by$ is a real function. Then

$$\begin{aligned} d\gamma &= \Big(-\frac{2a}{A(z)} + \frac{1}{A(z)}\Big)dx \wedge \gamma + \frac{1}{A(z)}dy \wedge \delta - \frac{4ab}{A(z)}dy \wedge \gamma \\ d\delta &= \frac{2b}{A(z)}dx \wedge \gamma + \Big(-\frac{4ab}{A(z)} + \frac{2b}{A(z)}\Big)dy \wedge \delta - \frac{2a}{A(z)}dx \wedge \delta. \end{aligned} \tag{2.72}$$

By (2.72), $\langle \gamma, dy \rangle$ and $\langle \delta, dx \rangle$ are two Frobenius systems, that implies the existence of functions $p,\ q,\ r,\ s$ and $u,\ v$ such that

$$\gamma = pdu + qdy,$$

$$\delta = rdv + sdx.$$

Write

$$dp = p_x dx + p_y dy + p_u du + p_v dv,$$

$$dq = q_x dx + q_y dy + q_u du + q_v dv,$$

$$dr = r_x dx + r_y dy + r_u du + r_v dv,$$

$$ds = s_x dx + s_y dy + s_u du + s_v dv.$$

By (2.72),

$$\begin{aligned} d\gamma &= dp \wedge du + dq \wedge dy \\ &= p_x dx \wedge du + p_y dy \wedge du + p_v dv \wedge du + q_x dx \wedge dy + q_u du \wedge dy + q_v dv \wedge dy \\ &= \frac{1}{-x-2by}[p(1-2a)dx \wedge du + (q(1-2a)-s)dx \wedge dy \\ &\quad + r dy \wedge dv - 4abp dy \wedge du]. \end{aligned}$$

Then the functions p and q must satisfy

$$\begin{aligned} p_y - q_u &= \frac{-4abp}{-x-2by}, \\ p_x &= \frac{1-2a}{-x-2by} p, \\ p_v &= 0, \\ q_v &= -\frac{r}{-x-2by}, \\ q_x &= \frac{q(1-2a)-s}{-x-2by}. \end{aligned}$$

Thus there exists a function $C_1(u, y)$ such that $p = (x + 2by)^{2a-1} C_1(u, y)$. After redefining u and q, we can assume $C_1(u, y) = 1$. Then

$$p = (x + 2by)^{2a-1}.$$

By (2.72),

$$\begin{aligned} d\delta &= dr \wedge dv + ds \wedge dx \\ &= r_x dx \wedge dv + r_y dy \wedge dv + r_u du \wedge dv + s_y dy \wedge dx + s_u du \wedge dx + s_v dv \wedge dx \\ &= \frac{1}{-x-2by}\Big[2bp dx \wedge du + (2bq + (4ab-2b)s) dx \wedge dy \\ &\quad + (-4ab+2b) r dy \wedge dv - 2ar dx \wedge dv \Big]. \end{aligned}$$

Then the functions r and s must satisfy

$$\begin{aligned} r_x - s_v &= -\frac{2ar}{-x-2by}, \\ r_y &= \frac{-4ab+2b}{-x-2by} r, \\ r_u &= 0, \\ s_y &= \frac{-2bq + (-4ab+2b)s}{-x-2by}, \\ s_u &= -\frac{2bp}{-x-2by}. \end{aligned} \tag{2.73}$$

So there exists a function $C_2(x, v)$ such that $r = (x+2by)^{2a-1} C_2(x, v)$. After redefining v and s, we can assume $C_2(x, v) = 1$. Then

$$r = (x + 2by)^{2a-1}.$$

Substituting this equation into (2.73), the equations are as follows:

$$\begin{aligned}
q_u &= -2b(x+2by)^{2a-2}, \\
q_v &= (x+2by)^{2a-2}, \\
q_x &= \frac{(2a-1)q+s}{x+2by}, \\
s_v &= -(x+2by)^{2a-2}, \\
s_u &= 2b(x+2by)^{2a-2}, \\
s_y &= \frac{2bq+(4ab-2b)s}{x+2by}.
\end{aligned}$$

This yields

$$\begin{aligned}
\theta_2 &= \gamma + i\delta \\
&= (x+2by)^{2a-1}(du+idv) + is\left(dx - i\frac{q}{s}dy\right).
\end{aligned} \tag{2.74}$$

To get a local holomorphic coordinate system, let $q = -s$. Then

$$\begin{aligned}
q_u &= -2b(x+2by)^{2a-2}, \\
q_v &= (x+2by)^{2a-2}, \\
q_x &= \frac{(2a-2)q}{x+2by}, \\
q_y &= -\frac{(-4ab+4b)q}{x+2by}.
\end{aligned}$$

From these equations, we get

$$q = (1-2bu+v)(x+2by)^{2a-2}$$

Substituting this equation into (2.74) yields

$$\begin{aligned}\theta_2 &= \gamma + i\delta \\ &= (pdu + qdy) + i(rdv + sdx) \\ &= ((x+2by)^{2a-1}du + (1-2bu+v)(x+2by)^{2a-2}dy) \\ &\quad + i((x+2by)^{2a-1}dv - (1-2bu+v)(x+2by)^{2a-2}dx) \\ &= (x+2by)^{2a-1}d(u+iv) - i(1-2bu+v)(x+2by)^{2a-2}d(x+iy).\end{aligned}$$

Define $w = u + iv$. We get the local coordinate representation of θ_2:

$$\theta_2 = (-A(z))^{2a-1}\left[dw + \left[i + \left(\frac{1}{2} - bi\right)w - \left(\frac{1}{2} + bi\right)\bar{w}\right]\frac{dz}{A(z)}\right].$$

Recall that

$$\theta = \omega_1 + \left(-\frac{1}{2} + ib\right)\omega_2,$$

$$\theta_2 = \omega_2 + \left[\frac{1}{\left(\frac{1}{2}+a\right)(1+4b^2)} + i\frac{2b}{\left(\frac{1}{2}-a\right)(1+4b^2)}\right]\theta.$$

Thus in the local holomorphic coordinate system $(z,\ w)$, the coframing can be written as

$$\omega_1 = \left\{1 + \left(-\frac{1}{2} + ib\right)\left[\frac{1}{\left(\frac{1}{2}+a\right)(1+4b^2)} + i\frac{2b}{\left(\frac{1}{2}-a\right)(1+4b^2)}\right]\right\}\theta - \left(-\frac{1}{2} + ib\right)\theta_2$$

$$\omega_2 = \theta_2 - \left[\frac{1}{\left(\frac{1}{2}+a\right)(1+4b^2)} + i\frac{2b}{\left(\frac{1}{2}-a\right)(1+4b^2)}\right]\theta.$$

2.4.5.2 $a = \frac{1}{2},\ b = 0$

In this case, we can choose (a, b) such that

$$\lim_{\substack{a\to\frac{1}{2} \\ b\to 0}} \frac{2b}{\left(\frac{1}{2}-a\right)(1+4b^2)} = 0$$

Then all the formula in the case $a \neq \frac{1}{2}$ applies to this case $a = \frac{1}{2},\ b = 0$. The local coordinate representation of the coframing is

$$\omega_1 = \left(\frac{1}{2} + ib\right)\theta - \left(-\frac{1}{2} + ib\right)\theta_2$$

$$\omega_2 = \theta_2 - \theta.$$

2.4.5.3 $\quad a = \frac{1}{2},\ b \neq 0$

The structure equation is

$$d\omega_2 = \omega_1 \wedge \bar{\omega}_1 + \left(-\frac{1}{2} + ib\right)\omega_1 \wedge \omega_2 + \left(-\frac{1}{2} + ib\right)\omega_2 \wedge \bar{\omega}_1. \tag{2.75}$$

Substituting $\theta = \omega_1 + (-\frac{1}{2} + ib)\omega_2$ into (2.75) yields

$$\begin{aligned} d\omega_2 &= \Big[\theta - \Big(-\frac{1}{2} + ib\Big)\omega_2\Big] \wedge \Big[\bar{\theta} - \Big(-\frac{1}{2} - ib\Big)\bar{\omega}_2\Big] \\ &\quad + \Big(\frac{1}{2} + ib\Big)\theta \wedge \omega_2 + \Big(-\frac{1}{2} + ib\Big)\omega_2 \wedge \Big[\bar{\theta} - \Big(-\frac{1}{2} - ib\Big)\bar{\omega}_2\Big] \\ &= \theta \wedge \bar{\theta} + \Big(\frac{1}{2} + ib\Big)\theta \wedge \bar{\omega}_2 + \Big(-\frac{1}{2} + ib\Big)\theta \wedge \omega_2. \end{aligned}$$

So $\langle \omega_2, \theta \rangle$ is a Frobenius system. Write $\omega_2 = \gamma + i\delta$. Then

$$d(\gamma + i\delta) = \frac{-2i}{A(z)^2} dx \wedge dy + \frac{i}{A(z)}(dx + idy) \wedge (-\delta + 2b\gamma).$$

Hence

$$d\gamma = -\frac{1}{A(z)} dy \wedge (-\delta + 2b\gamma)$$

$$d\delta = -\frac{2}{A(z)} dx \wedge dy + \frac{1}{A(z)} dx \wedge (-\delta + 2b\gamma).$$

So $\langle \gamma,\ dy \rangle$ and $\langle \delta,\ dx \rangle$ are two Frobenius systems. There exist functions $p,\ q,\ r,\ s$ and $u,\ v$ such that

$$\gamma = pdu + qdy$$

$$\delta = rdv + sdx.$$

From the structure equation,

$$
\begin{aligned}
d\gamma &= dp \wedge du + dq \wedge dy \\
&= p_x dx \wedge du + p_y dy \wedge du + p_v dv \wedge du + q_x dx \wedge dy + q_u du \wedge dy + q_v dv \wedge dy \\
&= -\frac{1}{A(z)} dy \wedge (-r dv - s dx + 2bp du).
\end{aligned}
$$

So

$$
\begin{aligned}
p_y - q_u &= \frac{2bp}{x+2by}, \\
p_x &= 0, \\
p_v &= 0, \\
q_v &= \frac{r}{x+2by}, \\
q_x &= \frac{s}{x+2by}.
\end{aligned} \tag{2.76}
$$

Since there are ambiguities for choosing p and u, by modifying u we can arrange $p = 1$.

From the structure equation,

$$
\begin{aligned}
d\delta &= dr \wedge dv + ds \wedge dx \\
&= r_x dx \wedge dv + r_y dy \wedge dv + r_u du \wedge dv + s_y dy \wedge dx + s_u du \wedge dx + s_v dv \wedge dx \\
&= -\frac{2}{A(z)^2} dx \wedge dy + \frac{1}{A(z)} dx \wedge (-r dv + 2bp du + 2bq dy).
\end{aligned}
$$

Then the functions r and s must satisfy

$$\begin{aligned} r_x - s_v &= -\frac{r}{-x-2by}, \\ r_y &= 0, \\ r_u &= 0, \\ s_y &= \frac{2}{(x+2by)^2} + \frac{2b}{x+2by}q, \\ s_u &= -\frac{2bp}{-x-2by}. \end{aligned} \tag{2.77}$$

Since there is ambiguity for choosing r and v, by modifying v we can arrange $r = 1$.

Thus (2.76) and (2.77) reduces to

$$\begin{aligned} q_u &= -\frac{2b}{x+2by}, \\ q_v &= \frac{1}{x+2by}, \\ q_x &= \frac{s}{x+2by}, \\ s_v &= -\frac{1}{x+2by}, \\ s_y &= \frac{2}{(x+2by)^2} + \frac{2b}{x+2by}q, \\ s_u &= \frac{2b}{x+2by}. \end{aligned}$$

Thus there exist functions $C_1(x, y)$ and $C_2(x, y)$ such that

$$q = \frac{1}{A(z)}(2bu - v) + C_1(x, y)$$

and

$$s = -\frac{1}{A(z)}(2bu - v) + C_2(x, y).$$

The functions $C_1(x, y)$ and $C_2(x, y)$ satisfy

$$\begin{aligned} C_{1x} &= \frac{1}{x+2by} C_2, \\ C_{2y} &= \frac{2b}{x+2by} C_1 + \frac{2}{(x+2by)^2}. \end{aligned}$$

Choose $C_2(x, y) = -C_1(x, y)$ such that z is a local holomorphic coordinate. Then

$$\begin{aligned} C_{1x} &= -\frac{1}{x+2by} C_1, \\ C_{1y} &= -\frac{2b}{x+2by} C_1 - \frac{2}{(x+2by)^2}. \end{aligned}$$

Then

$$C_1(x, y) = -\frac{\ln(x+2by)}{b(x+2by)}.$$

So

$$\begin{aligned} \omega_2 &= \gamma + i\delta \\ &= (pdu + qdy) + i(rdv + sdx) \\ &= d(u+iv) + \left(\frac{1}{A(z)}(2bu - v) - \frac{\ln(x+2by)}{b(x+2by)} \right) d(y - ix). \end{aligned}$$

Let $w = u + iv$. Then

$$\omega_2 = dw - i\frac{1}{A(z)} \left[\left(b + \frac{i}{2}\right) w + \left(b - \frac{i}{2}\right) \bar{w} + \frac{1}{b}\ln(-A(z)) \right] dz.$$

Recall that

$$\theta = \omega_1 + \left(-\frac{1}{2} + ib\right) \omega_2.$$

So we can write ω_1 as

$$\omega_1 = \frac{dz}{A(z)} - \left(-\frac{1}{2} + ib\right) \omega_2.$$

2.4.5.4 $a = -\frac{1}{2}$

The structure equation is

$$d\omega_2 = \Big(-\frac{1}{2} + 2b^2 - 2ib\Big)\bar{\omega}_2 \wedge \omega_2 + \omega_1 \wedge \bar{\omega}_1 + \Big(\frac{1}{2} - ib\Big)\omega_1 \wedge \omega_2 + \Big(-\frac{3}{2} - ib\Big)\omega_2 \wedge \bar{\omega}_1$$

$$= \theta \wedge \bar{\theta} + \Big(\frac{1}{2} + ib\Big)\theta \wedge \bar{\omega}_2 - 2\Big(\frac{1}{2} + ib\Big)\omega_2 \wedge \bar{\theta} + \Big(\frac{1}{2} - ib\Big)\theta \wedge \omega_2.$$

Let $\omega_2 = \gamma + i\delta$ be the real and imaginary part decomposition. Then

$$d\gamma = \frac{2}{A(z)}dx \wedge \gamma + \frac{1}{A(z)}dy \wedge \delta + \frac{2b}{A(z)}dy \wedge \gamma,$$

$$d\delta = -\frac{2}{A(z)^2}dx \wedge dy + \frac{1}{A(z)}dx \wedge \delta + \frac{2b}{A(z)}dx \wedge \gamma + \frac{4b}{A(z)}dy \wedge \delta.$$

So $\langle \gamma,\ dy\rangle$ and $\langle \delta,\ dx\rangle$ are two Frobenius systems. So there exist functions $p,\ q,\ r,\ s$ and $u,\ v$ such that

$$\gamma = pdu + qdy,$$

$$\delta = rdv + sdx.$$

From the structure equation,

$$d\gamma = dp \wedge du + dq \wedge dy$$

$$= p_x dx \wedge du + p_y dy \wedge du + p_v dv \wedge du + q_x dx \wedge dy + q_u du \wedge dy + q_v dv \wedge dy$$

$$= \frac{1}{A(z)}[2dx \wedge (pdu + qdy) + dy \wedge (rdv + sdx) + 2bdy \wedge (pdu + qdy)].$$

This yields

$$\begin{aligned} p_y - q_u &= \frac{2bp}{A(z)}, \\ p_x &= \frac{2p}{A(z)}, \\ p_v &= 0, \\ q_v &= -\frac{r}{A(z)}, \\ q_x &= \frac{2q-s}{A(z)}. \end{aligned} \tag{2.78}$$

Since there is ambiguity for choosing p and u, by modifying u we arrange $p = \frac{1}{A(z)^2}$.

From the structure equation,

$$\begin{aligned} d\delta &= dr \wedge dv + ds \wedge dx \\ &= r_x dx \wedge dv + r_y dy \wedge dv + r_u du \wedge dv + s_y dy \wedge dx + s_u du \wedge dx + s_v dv \wedge dx \\ &= -\frac{2}{A(z)^2} dx \wedge dy \\ &+ \frac{1}{A(z)}[dx \wedge (rdv + sdx) + 2bdx \wedge (pdu + qdy) + 4bdy \wedge (rdv + sdx)]. \end{aligned}$$

The functions r and s must satisfy

$$\begin{aligned} r_x - s_v &= \frac{r}{A(z)}, \\ r_y &= \frac{4br}{A(z)}, \\ r_u &= 0, \\ s_y &= \frac{2}{A(z)^2} + \frac{-2bq+4bs}{A(z)}, \\ s_u &= -\frac{2bp}{A(z)}. \end{aligned} \tag{2.79}$$

Since there is ambiguity for choosing r and v, by modifying v we arrange $r = \frac{1}{A(z)^2}$.

(2.78) and (2.80) reduce to

$$
\begin{aligned}
q_u &= -\frac{2b}{(x+2by)^3},\\
q_v &= \frac{1}{(x+2by)^3},\\
q_x &= \frac{2q-s}{A(z)},\\
s_v &= -\frac{1}{(x+2by)^3},\\
s_y &= \frac{2}{(x+2by)^2} + \frac{-2bq+4bs}{-(x+2by)},\\
s_u &= \frac{2b}{(x+2by)^3}.
\end{aligned}
$$

Then there are functions $C_1(x,y)$ and $C_2(x,y)$ such that

$$q = \frac{1}{A(z)^3}(2bu-v) + C_1(x,y) \tag{2.80}$$

and

$$s = -\frac{1}{A(z)^3}(2bu-v) + C_2(x,y). \tag{2.81}$$

The functions $C_1(x,y)$ and $C_2(x,y)$ satisfy

$$
\begin{aligned}
C_{1x} &= \frac{C_2-2C_1}{x+2by},\\
C_{2y} &= \frac{2bC_1-4bC_2}{x+2by} + \frac{2}{(x+2by)^2}.
\end{aligned}
$$

Choose $C_2(x,y) = -C_1(x,y)$ such that z is a local holomorphic coordinate. Then

$$
\begin{aligned}
C_{1x} &= -\frac{3}{x+2by}C_1,\\
C_{1y} &= -\frac{6b}{x+2by}C_1 - \frac{2}{(x+2by)^2}.
\end{aligned}
$$

We can solve $C_1(x,y)$

$$C_1(x,y) = -\frac{2y(x+by)}{(x+2by)^3}.$$

By (2.80), we get

$$q = \frac{1}{A(z)^3}(2bu - v) - 2\frac{y(x+by)}{(x+2by)^3}.$$

By (2.81), we get

$$s = -\frac{1}{A(z)^3}(2bu - v) + 2\frac{y(x+by)}{(x+2by)^3}.$$

So

$$\begin{aligned}\omega_2 &= \gamma + i\delta \\ &= (pdu + qdy) + i(rdv + sdx) \\ &= \frac{1}{A(z)^2}d(u+iv) + \Big(\frac{1}{A(z)^3}(2bu - v) - 2\frac{y(x+by)}{(x+2by)^3}\Big)d(y - ix).\end{aligned}$$

Let $w = u + iv$. Then

$$\omega_2 = \frac{1}{A(z)^2}dw - i\frac{1}{A(z)^3}\Big[\Big(b+\frac{i}{2}\Big)w + \Big(b-\frac{i}{2}\Big)\bar{w} - \frac{i}{2}(z-\bar{z})((1-ib)z + (1+ib)\bar{z})\Big]dz.$$

Recall that

$$\theta = \omega_1 + \left(-\frac{1}{2} + ib\right)\omega_2,$$

so we can write ω_1 such that

$$\omega_1 = \frac{dz}{A(z)} - (-\frac{1}{2} + ib)\omega_2$$

2.4.5.5 Compact case of type $C5$

From the structure equation,

$$d(\omega_1 \wedge \bar{\omega}_2 \wedge \omega_2) = (1-2a)(1+2bi)\omega_1 \wedge \bar{\omega}_1 \wedge \omega_2 \wedge \bar{\omega}_2 \neq 0.$$

If $a \neq \frac{1}{2}$, there is no compact quotient of type $C5$. In the following analysis, we assume $a = \frac{1}{2}$.

Recall that

$$\begin{aligned} d\theta &= \Big(\frac{1}{2} + ib\Big)\bar{\theta} \wedge \theta, \\ d\omega_2 &= \theta \wedge \bar{\theta} + \left(\frac{1}{2} + ib\right) \theta \wedge \bar{\omega}_2 + \left(-\frac{1}{2} + ib\right) \theta \wedge \omega_2. \end{aligned} \tag{2.82}$$

Let $\theta = \alpha + i\beta$ and $\omega_2 = \gamma + i\delta$ be the real and imaginary part decompositions. By (2.82),

$$\begin{aligned} d\alpha &= -2b\alpha \wedge \beta, \\ d\beta &= \alpha \wedge \beta, \\ d\gamma &= \beta \wedge \delta - 2b\beta \wedge \gamma, \\ d\delta &= -2\alpha \wedge \beta - \alpha \wedge \delta + 2b\alpha \wedge \gamma. \end{aligned}$$

Let e_1, e_2, e_3, e_4 be left-invariant vector fields dual to the left-invariant forms $\alpha, \beta, \gamma, \delta$, respectively. The nontrivial brackets are

$$\begin{aligned} [e_1, e_2] &= -2be_1 + e_2 - 2e_4, \\ [e_1, e_3] &= 2be_4, \\ [e_1, e_4] &= -e_4, \\ [e_2, e_3] &= -2be_3, \\ [e_2, e_4] &= e_3. \end{aligned}$$

Make a basis change: $\tilde{e}_3 = e_3 + 2be_4, \tilde{e}_2 = e_2 - 2be_1$. Then

$$\begin{aligned} [e_1, \tilde{e}_3] &= 0, \\ [e_2, \tilde{e}_3] &= 0. \end{aligned}$$

After dropping the tildes, the nontrivial brackets are

$$[e_1, e_2] = e_2 - 2e_4,$$

$$[e_1, e_4] = -e_4,$$

$$[e_2, e_4] = e_3.$$

Define $\tilde{e}_2 = e_2 - e_4$. Then

$$[e_1, \tilde{e}_2] = \tilde{e}_2,$$

$$[e_1, e_4] = -e_4,$$

$$[\tilde{e}_2, e_4] = e_3.$$

Then define $X_1 = -e_3, X_2 = e_4, X_3 = \tilde{e}_2, X_4 = e_1$. The nontrivial brackets are

$$[X_2, X_3] = X_1,$$

$$[X_2, X_4] = X_2,$$

$$[X_3, X_4] = -X_3.$$

By the classification result of solvmanifolds [Bock (2016)], there exists a co-compact lattice Γ such that G/Γ is compact and supports a homogeneous complex Engel structure.

2.4.6 Homogeneous Case C6

In this case, the symmetry groups of the coframing are different for different parameters a and b. We will consider the following 2 cases:

1. $a = -\frac{\pi}{2} + 2k\pi,\ k \in \mathbb{Z}$.

   In this case, the constants $(p_1, p_2, q_1, q_2, r_1, r_2)$ are $(\frac{1}{2} - ib,\ 0,\ 2bi,\ \frac{1}{2}ib(2b+i)^2,\ -b(-2b+i),\ -\frac{1}{4}b(-2b+i)(2b+i)^2)$. This is a special case of homogeneous case C3, with $a = b^2$. There is no compact quotient that can support a homogeneous complex Engel structure unless $b = 0$. Under this condition, this is a special case of case $C1$.

2. $a \neq -\frac{\pi}{2} + 2k\pi,\ k \in \mathbb{Z}$.

   By the structure equation, we get

   $d(\bar{\omega}_1 \wedge \omega_2 \wedge \bar{\omega}_2) = -2\left(\left(b + \frac{1}{2}i\right)\cos a + \left(-\frac{1}{2} + bi\right)(\sin a + 1)\right)\omega_1 \wedge \bar{\omega}_1 \wedge \omega_2 \wedge \bar{\omega}_2,$

   $d(\omega_1 \wedge \bar{\omega}_1 \wedge \bar{\omega}_2) = -\frac{1}{2}\left(-\sin a + 2b\cos a - 1\right)\left(4b^2 - 1 + 4bi\right)\omega_1 \wedge \bar{\omega}_1 \wedge \omega_2 \wedge \bar{\omega}_2.$

   It is easy to verify that $d(\bar{\omega}_1 \wedge \omega_2 \wedge \bar{\omega}_2) = d(\omega_1 \wedge \bar{\omega}_1 \wedge \bar{\omega}_2) = 0$ if and only if $a = -\frac{\pi}{2} + 2k\pi,\ k \in \mathbb{Z}$. But this is contradictory to our assumption. Thus the volume form is exact in this case. By Stokes' Theorem, there does not exist compact quotient that supports a homogeneous complex Engel structure.

2.5 Variation of Complex Engel Structures

In this section, we consider the variation of complex Engel structures with fixed underlying complex structure. We only consider case $C1$ with $a = \frac{1}{2}$ and $b = 0$. In this case, there exists a compact quotient that can support a homogeneous complex Engel structure. Recall that the structure equation is

$$d\omega_1 = 0\,,$$

$$d\omega_2 = (\omega_1 - \omega_2) \wedge \bar{\omega}_1 - \frac{1}{2}(\omega_1 + \bar{\omega}_1) \wedge \omega_2\,.$$

For any smooth complex function f, we want to consider the variation that is proportional to $\omega_2 + f\omega_1$.

Define $g = \overline{f}$ and $df = f_1\omega_1 + f_{-1}\omega_{-1} + f_2\omega_2 + f_{-2}\omega_{-2}$ and similar expression for the derivative of g.

Under the open condition $1 + \frac{1}{2}f - f_{-1} + gf_{-2} \neq 0$, define

$$\theta_1 = \omega_1\,,$$

$$\theta_2 = \frac{1}{1 + \frac{1}{2}f - f_{-1} + gf_{-2}}(\omega_2 + f\omega_1)\,.$$

Then

$$d\theta_2 \equiv \theta_1 \wedge \bar{\theta}_1 \qquad \text{mod} \quad \theta_2, \bar{\theta}\,.$$

The new Engel structure is defined by $\langle \theta_2 \rangle^{\perp}$.

Now consider the first order variation of the complex Engel structure. If we transform f to tf, we only consider the constant and first order term of t in all our expressions and drop all the higher order terms

$$\begin{aligned} d(\theta_2 + \bar{\theta}_2) &\equiv \tfrac{1}{2+B}[(A+2)\bar{\omega}_1 - (\bar{A}+2)\omega_1] \wedge \theta_2 \qquad \text{mod} \quad (\theta_2 + \bar{\theta}_2) \\ &\equiv (\tfrac{1}{2} - \tfrac{B}{4})[(A+2)\bar{\omega}_1 - (\bar{A}+2)\omega_1] \wedge \theta_2 \qquad \text{mod} \quad (\theta_2 + \bar{\theta}_2) \\ &\equiv \left(1 + \frac{A-B}{2}\right)\bar{\omega}_1 - \left(1 + \frac{\bar{A}-B}{2}\right)\omega_1 \qquad \text{mod} \quad (\theta_2 + \bar{\theta}_2), \end{aligned}$$

where

$$A = f + g - 3f_{-1} + 2f_{-1,-1} + 2g_{-2} + g_{-1} - 2g_1 - 2g_2 - 2g_{1,-1} + 2$$

and

$$B = f + g - 2(f_{-1} + g_1).$$

Let $X = xe_1 + \bar{x}\bar{e}_1 + ye_2 + \bar{y}\bar{e}_2$ be the characteristic vector field. Then

$$\begin{aligned} & y + xf = 0, \\ & (1 + \frac{A-B}{2})\bar{x} - (1 + \frac{\bar{A}-B}{2})x = 0. \end{aligned}$$

Only consider the first order term. Then

$$\begin{aligned} x &= 1 + \frac{A-B}{2}, \\ y &= -f. \end{aligned}$$

The characteristic vector field is

$$X = \left(1 + \frac{A-B}{2}\right)e_1 + \left(1 + \frac{\bar{A}-B}{2}\right)\bar{e}_1 - fe_2 - \bar{f}\bar{e}_2,$$

so the first variation of the characteristic vector field is

$$\delta X = \left(\frac{A-B}{2}\right) e_1 + \left(\frac{\bar{A}-B}{2}\right) \bar{e}_1 - f e_2 - \bar{f}\bar{e}_2$$

By the expression for δX, there exists an injection

$$0 \to C^\infty(M) \xrightarrow{\delta} \chi(M)$$

In local coordinates,

$$\omega_1 = dz,$$
$$\omega_2 = 2dz + e^{-\frac{1}{2}z+\frac{1}{2}\bar{z}} dw.$$

Write

$$\begin{aligned} df &= f_z dz + f_{\bar{z}} d\bar{z} + f_w dw + f_{\bar{w}} d\bar{w} \\ &= f_1\omega_1 + f_2\omega_2 + f_{\bar{1}}\bar{\omega}_1 + f_{\bar{2}}\bar{\omega}_2. \end{aligned}$$

Consider a special case $f_w = f_{\bar{w}} = 0$. Then $df = f_z\omega_1 + f_{\bar{z}}\bar{\omega}_1$. Then

$$A - B = \bar{f}_{\bar{z}} - f_{\bar{z}} + 2(f_{\bar{z}\bar{z}} - \bar{f}_{z\bar{z}}),$$
$$\bar{A} - B = A - B.$$

Let $\gamma(t) = (z(t), \bar{z}(t), w(t), \bar{w}(t))$ be the integral curve. Then

$$\dot{z}(t) = 1 + \frac{1}{2}(\bar{f}_{\bar{z}} - f_{\bar{z}}) + f_{\bar{z}\bar{z}} - \bar{f}_{z\bar{z}} \tag{2.83}$$

$$\dot{w}(t) = -f \tag{2.84}$$

To be a well-defined function on the compact manifold $T^2 \times T^2$, the variation function needs to satisfy the condition

$$f(z + k_0 + i2k_1\pi) = f(z), \qquad \forall k_0, k_1 \in \mathbb{Z}.$$

So the periodic function f can be expanded into Fourier series of Fourier basis. Take the following basis

$$e^{\frac{1}{2}(z-\bar{z})}, \qquad e^{i\pi(z+\bar{z})}$$

and their complex conjugates

$$e^{-\frac{1}{2}(z-\bar{z})}, \qquad e^{-i\pi(z+\bar{z})}.$$

In fact, if we write $z = x + iy$, then this basis is equivalent to the following basis

$$\sin x, \quad \cos x, \quad \sin y, \quad \cos y.$$

First take $f(z) = e^{\frac{1}{2}(z-\bar{z})}$, then $A - B = f + \bar{f}$. Equation (2.83) reduces to

$$\dot{z}(t) = 1 + \frac{1}{2}(f + \bar{f}), \tag{2.85}$$

$$\dot{w}(t) = -f. \tag{2.86}$$

We can solve $z(t)$

$$z(t) = \left(1 + \cos\frac{z_0 - \bar{z}_0}{2i}\right) t + z_0$$

From this expression and (2.85), we get

$$\dot{w}(t) = -e^{\frac{1}{2}(z_0-\bar{z}_0)}$$

so

$$w(t) = -e^{\frac{1}{2}(z_0-\bar{z}_0)}t + w_0$$

From the period analysis of the integral curve, the curve is closed if and only if

$$\cos\frac{z_0 - \bar{z}_0}{2i} \quad \text{and} \quad \sin\frac{z_0 - \bar{z}_0}{2i} \in \mathbb{Q}$$

So in this case we have an infinite discrete family of closed integral curves, which is one-to-one correspondence with the elements of $SO(2,\ \mathbb{Q})$.

If we take $f = e^{-\frac{1}{2}(z-\bar{z})}$, the calculation is similar.

$$\dot{z}(t) = 1 + \frac{1}{2}(f + \bar{f}),$$

$$\dot{w}(t) = -f.$$

Thus

$$z(t) = \left(1 + \cos\frac{z_0 - \bar{z}_0}{2i}\right) t + z_0,$$

and

$$w(t) = -e^{-\frac{1}{2}(z_0 - \bar{z}_0)} t + w_0,$$

Now if we take $f = e^{i\pi(z+\bar{z})}$,

$$\begin{aligned}\dot{z}(t) &= 1 + \left(\pi^2 - \frac{1}{2} i\pi\right)\bar{f} - \left(\pi^2 + \frac{1}{2} i\pi\right) f,\\ \dot{w}(t) &= -f,\end{aligned}$$

then

$$\begin{aligned}\dot{x}(t) &= 1\\ \dot{y}(t) &= -2\pi^2 \sin(2\pi x) - \pi\cos(2\pi x)\end{aligned}$$

Thus, there exist x_0 and y_0 such that

$$\begin{aligned}x(t) &= t + x_0\\ y(t) &= \pi\cos 2\pi(t + x_0) - \frac{1}{2}\sin 2\pi(t + x_0) - \pi\cos(2\pi x_0) + \frac{1}{2}\sin(2\pi x_0) + y_0\end{aligned}$$

Thus

$$z(t) = t + i\left(C(t) - C(0)\right) + z_0,$$

and

$$w(t) = \frac{i}{2\pi} e^{i\pi(2t + z_0 + \bar{z}_0)} - \frac{i}{2\pi} e^{i\pi(z_0 + \bar{z}_0)} + w_0,$$

where

$$C(t) = \pi\cos 2\pi\left(t + \frac{z_0 + \bar{z}_0}{2}\right) - \frac{1}{2}\sin 2\pi\left(t + \frac{z_0 + \bar{z}_0}{2}\right).$$

From the expression of $w(t)$, the integral curve is never closed on the manifold.

3

Lagrangian Engel Structures

In this section, we study the geometry of Lagrangian Engel structures. We will show that, after reduction of the structure group of Lagrangian Engel structures, we can get an e-structure in generic cases, which implies the existence of a canonical coframing of Lagrangian Engel structures. Using this canonical e-structure, we will classify homogeneous and compact Lagrangian Engel structures.

3.1 Geometry of Lagrangian Engel Structures

Recall that

Definition 3.1.1. *A* Lagrangian Engel structure (M, Ω, D) *is a 4-manifold* M *endowed with* a symplectic form Ω and an Engel 2-plane field D that is Lagrangian for Ω. *If we let* $I = D^{\perp} \subset T^*M$ *denote the annihilator, then* $\Omega \in \langle I \rangle$.

A coframing $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)$ such that the symplectic structure can be written as

$$\Omega = \omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_4$$

while $I = \langle \omega_1, \omega_2 \rangle$ and $I^{(1)} = \langle \omega_1 \rangle$ will be said to be *0-adapted* to (M, Ω, D).

Proposition 3.1.1. *The 0-adapted coframings are the sections of a G-structure on M where $G \subset GL(4, \mathbb{R})$ is the 6-dimensional subgroup*

$$G = \left\{ \begin{bmatrix} B_{11} & 0 \\ (B_{11}^T)^{-1}S & (B_{11}^T)^{-1} \end{bmatrix} \middle| B_{11} = \begin{bmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{bmatrix} \text{ and } S \in \mathbb{R}^{2\times 2}, S = S^T \right\} \tag{3.1}$$

Proof. Assume $(\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3, \tilde{\omega}_4)$ is a new coframing and the Engel structure in the new coframing is $I = \langle \tilde{\omega}_1, \tilde{\omega}_2 \rangle$ and $I^{(1)} = \langle \tilde{\omega}_1 \rangle$. According to the definition of a coframing on a manifold, there exists a matrix

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

such that

$$\begin{bmatrix} \tilde{\omega}_1 \\ \tilde{\omega}_2 \\ \tilde{\omega}_3 \\ \tilde{\omega}_4 \end{bmatrix} = B \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix},$$

where $B_{11}, B_{12}, B_{21}, B_{22}$ are 2×2 matrices. Since $I = \langle \tilde{\omega}_1, \tilde{\omega}_2 \rangle = \langle \omega_1, \omega_2 \rangle$, the block $B_{12} = 0$.

Let

$$J = \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix},$$

where I_2 is the identity matrix of dimension 2. To keep the symplectic structure invariant under the transformation, the matrices B_{ij} satisfy

$$\begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix}^T J \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix} = J.$$

Thus

$$\begin{aligned} B_{11}^T B_{21} &= B_{21}^T B_{11}\,, \\ B_{11}^T B_{22} &= I_2\,. \end{aligned} \tag{3.2}$$

Define $S = B_{11}^T B_{21}$. Then from (3.2), $S = S^T$. The element of the structure group can be written as

$$B = \begin{bmatrix} B_{11} & 0 \\ (B_{11}^T)^{-1} S & (B_{11}^T)^{-1} \end{bmatrix}.$$

Since $I^{(1)} = \langle \omega_1 \rangle = \langle \tilde{\omega}_1 \rangle$, B_{11} must be of the form $\begin{bmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{bmatrix}$, where b_{11}, b_{21}, b_{22} can be any functions.

Therefore, the structure group is of the form (3.1). $\square$

A Lagrangian Engel structure defines a G-structure, where G is defined by (3.1). We will prove that after reduction of the structure group, the manifold with a Lagrangian Engel structure belongs to at least one of the following categories:

1. the manifold is not compact
2. there exists a canonical coframing for the Lagrangian Engel structure on the manifold

Suppose $I = \langle \omega_1, \omega_2 \rangle$, $I^{(1)} = \langle \omega_1 \rangle$ and $I^{\perp}$ is an Engel structure, then

$$\begin{aligned} d\omega_1 &\not\equiv 0 && \mod \quad \omega_1\,, \\ d\omega_1 &\equiv 0 && \mod \quad \omega_1, \omega_2\,, \\ d\omega_2 &\not\equiv 0 && \mod \quad \omega_1, \omega_2\,. \end{aligned} \tag{3.3}$$

By (3.3), there exists a function $A \neq 0$ such that

$$d\omega_2 \equiv A\, \omega_3 \wedge \omega_4 \qquad \mod \quad \omega_1, \omega_2$$

We can arrange $A = 1$ by dividing ω_2 by A. Such coframings will be said to be *1-adapted.* They are the sections of a G_1-structure, where $G_1 \subset G$ is defined by

$$b_{11} b_{22}^2 = 1\,. \tag{3.4}$$

Now B_{11} is of the form $\begin{bmatrix} b_{22}^{-2} & 0 \\ b_{21} & b_{22} \end{bmatrix}$ and $B_{11}^{-1} = \begin{bmatrix} b_{22}^{2} & 0 \\ -b_{22}b_{21} & b_{22}^{-1} \end{bmatrix}$. After this arrangement,

$$d\omega_2 \equiv \omega_3 \wedge \omega_4 \qquad \mod \quad \omega_1, \omega_2 \,. \tag{3.5}$$

By (3.3), there exist functions p_3 and p_4 such that

$$d\omega_1 \equiv (p_3\omega_3 + p_4\omega_4) \wedge \omega_2 \qquad \mod \quad \omega_1 \tag{3.6}$$

and at least one of p_3 and p_4 is nonzero. Since we will mainly focus on the classification of homogeneous Lagrangian Engel structures, we will study the cases where either $p_3 \equiv 0$ or p_3 never vanishes.

Recall that the symplectic structure is $\Omega = \omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_4$. By (3.6),

$$\omega_1 \wedge d\omega_1 \wedge \omega_4 = \frac{p_3}{2} \Omega \wedge \Omega \,. \tag{3.7}$$

If the coframing is changed under the structure group G_1, the function p_3 is changed to $b_{22}^{-5} p_3$. Thus p_3 is well-defined up to scaling by b_{22}^{-5}. By (3.6), the Cartan system is $C(\langle\omega_1\rangle) = \langle\omega_1, \omega_2, (p_3\omega_3 + p_4\omega_4)\rangle$ and the symplectic complement of $\langle\omega_1\rangle$ is $\langle\omega_1\rangle^{\perp} = \langle\omega_1, \omega_2, \omega_4\rangle$. Generally, $C(\langle\omega_1\rangle) \neq \langle\omega_1\rangle^{\perp}$. If $C(\langle\omega_1\rangle) \neq \langle\omega_1\rangle^{\perp}$, this type of Lagrangian Engel structures is said to be *generic*. If $C(\langle\omega_1\rangle) = \langle\omega_1\rangle^{\perp}$, this type of Lagrangian Engel structures is said to be *non-generic*.

3.1.1 Geometry of Lagrangian Engel Structures in Generic Case

In the generic case, we have the following theorem:

Theorem 3.1.2 (Lagrangian Engel Structures in Generic Case). *Given a symplectic manifold (M, Ω, I) with a symplectic structure Ω and an Engel structure $D = I^{\perp}$. On the domain where $p_3 \neq 0$ in equation (3.7), there exists a unique 0-adapted coframing $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)$ satisfying*

$$d\omega_1 = \omega_3 \wedge \omega_2 + (a\omega_3 + b\omega_4) \wedge \omega_1 \,,$$

$$d\omega_2 = (c\omega_2 + e\omega_3 + f\omega_4) \wedge \omega_1 + \omega_3 \wedge \omega_4 \,,$$

where a, b, c, e, f are functions on M.

Proof. Under the transformation of the structure group, the structure equation is transformed to

$$d\omega_1 \equiv \big(p_3 b_{22}^5 \omega_3 + b_{22}^2(-p_3 b_{22}^2 b_{21} + p_4)\omega_4\big) \wedge \omega_2 \qquad \mod \quad \omega_1 \,. \tag{3.8}$$

By scaling ω_1 via b_{22}^5, we can arrange $p_3 = 1$. This fixes $b_{22} = 1$. By adding a multiple of ω_4 to ω_3, we can arrange $p_4 = 0$. This fixes $b_{21} = 0$. The structure equation is

$$d\omega_1 \equiv \omega_3 \wedge \omega_2 \qquad \mod \quad \omega_1 \,. \tag{3.9}$$

The element of the structure group reduces to the following form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ S_{11} & S_{12} & 1 & 0 \\ S_{12} & S_{22} & 0 & 1 \end{bmatrix} .$$

Recall that $d\omega_2 \equiv \omega_3 \wedge \omega_4 \mod \omega_1, \omega_2$. Thus there exist functions v_3 and v_4 such that

$$d\omega_2 \equiv (v_3\omega_3 + v_4\omega_4) \wedge \omega_2 + \omega_3 \wedge \omega_4 \qquad \mod \quad \omega_1. \tag{3.10}$$

By adding a multiple of ω_2 to ω_3, we can arrange $v_4 = 0$. This fixes $S_{12} = 0$. By adding a multiple of ω_2 to ω_4, we can arrange $v_3 = 0$. This fixes $S_{22} = 0$. Thus

$$d\omega_2 \equiv \omega_3 \wedge \omega_4 \qquad \mod \quad \omega_1. \tag{3.11}$$

From equation (3.9), there exist functions u_2, u_3, u_4 such that

$$d\omega_1 = \omega_3 \wedge \omega_2 + (u_2\omega_2 + u_3\omega_3 + u_4\omega_4) \wedge \omega_1 \,. \tag{3.12}$$

By adding a multiple of ω_1 to ω_3, we can arrange $u_2 = 0$. This yields $S_{11} = 0$. Now the structure group of the coframing contains only the identity element. We get an

e-structure. The structure equation is

$$\begin{aligned} d\omega_1 &= \omega_3 \wedge \omega_2 + (u_3\omega_3 + u_4\omega_4) \wedge \omega_1 \,, \\ d\omega_2 &\equiv \omega_3 \wedge \omega_4 \qquad\qquad \mod \quad \omega_1 \,. \end{aligned} \tag{3.13}$$

□

By a Theorem of Kobayashi [Kobayashi (1954)],

Corollary 3.1.3. *In generic case, the symmetry group of a Lagrangian Engel structure acts freely on the underlying connected manifold.*

3.1.2 Geometry of Lagrangian Engel Structures in Non-Generic Case

Now we will study the geometry of Lagrangian Engel structures in non-generic case. Since $p_3 \equiv 0$ and at least one of p_3 and p_4 is nonzero, then p_4 never vanishes. We can arrange $p_4 = \pm 1$ via dividing ω_1 by $|p_4|$. Then the entry b_{22} is fixed to be ± 1. From the expression of the symplectic structure, there is a transformation

$$\omega_2 \to -\omega_2, \qquad \omega_4 \to -\omega_4 \,.$$

that fixes the symplectic structure and Engel structure. We can fix $b_{22} = 1$ by this transformation.

Now

$$d\omega_1 \equiv \pm\omega_4 \wedge \omega_2 \qquad\qquad \mod \quad \omega_1 \,. \tag{3.14}$$

ω_1 is uniquely defined by equation (3.14).

By (3.14), there exist functions A_2, A_3 and A_4 such that

$$d\omega_1 = \pm\omega_4 \wedge \omega_2 + (A_2\omega_2 + A_3\omega_3 + A_4\omega_4) \wedge \omega_1 \,. \tag{3.15}$$

Under a change of adapted coframing, a new coframing $\tilde{\omega}$ satisfies

$$\begin{aligned} d\tilde{\omega}_1 = {} & \pm\tilde{\omega}_4 \wedge \tilde{\omega}_2 + A_3\tilde{\omega}_3 \wedge \tilde{\omega}_1 + (\pm b_{21} - b_{21}A_3 + A_4)\tilde{\omega}_4 \wedge \tilde{\omega}_1 \\ & + (\mp S_{12} \pm b_{21}S_{22} + A_2 + A_3(S_{12} - b_{21}S_{22}) + A_4S_{22})\,\tilde{\omega}_2 \wedge \tilde{\omega}_1 \,. \end{aligned} \tag{3.16}$$

By comparing (3.15) and (3.16), A_3 is an invariant of Lagrangian Engel structures in the case $p_3 \equiv 0$. And note that by (3.15),

$$d\omega_1 \wedge \omega_2 \wedge \omega_4 = -\tfrac{A_3}{2}\,\Omega \wedge \Omega\,. \tag{3.17}$$

If $A_3 \equiv \pm 1$, (3.16) is equivalent to

$$d\tilde{\omega}_1 = \pm\tilde{\omega}_4 \wedge \tilde{\omega}_2 \pm \tilde{\omega}_3 \wedge \tilde{\omega}_1 + A_4\tilde{\omega}_4 \wedge \tilde{\omega}_1 + (A_2 + A_4 S_{22})\,\tilde{\omega}_2 \wedge \tilde{\omega}_1\,. \tag{3.18}$$

Thus by comparing (3.15) and (3.18), A_4 is an invariant of Lagrangian Engel structures.

Based on the invariants A_3 and A_4 in (3.15), we will prove the following theorem:

Theorem 3.1.4 (Lagrangian Engel Structures in Non-Generic Case)**.**

1. *On the domain where* $(A_3 \equiv 0)$ *or* $(A_3 \neq \pm 1$ *and* $A_3 \neq 0)$ *in (3.17), we get an e-structure that whose defining conditions are*

$$\begin{aligned} d\omega_1 &= \pm\omega_4 \wedge \omega_2 + A_3\omega_3 \wedge \omega_1\,, \\ d\omega_2 &= (a\omega_1 + b\omega_4) \wedge \omega_2 + c\omega_3 \wedge \omega_1 + \omega_3 \wedge \omega_4\,, \end{aligned}$$

   where a, b, c are functions on M.

2. *On the domain where $A_3 \equiv \pm 1$ in equation (3.17), there are two cases depending on whether A_4 is 0:*

   (a) *On the domain where $A_4 \equiv 0$, the structure equation is*

$$\begin{aligned} d\omega_1 &= \pm\omega_4 \wedge \omega_2 \pm \omega_3 \wedge \omega_1 + A_2\,\omega_2 \wedge \omega_1\,, \\ d\omega_2 &\equiv \omega_3 \wedge \omega_4 \qquad \mod \quad \omega_1\,. \end{aligned} \tag{3.19}$$

   (b) *On the domain where $A_4 \neq 0$, the structure equation is*

$$\begin{aligned} d\omega_1 &= \pm\omega_4 \wedge \omega_2 \pm \omega_3 \wedge \omega_1 + A_4\,\omega_4 \wedge \omega_1\,, \\ d\omega_2 &\equiv q_3\omega_3 \wedge \omega_2 + \omega_3 \wedge \omega_4\,. \qquad \mod \quad \omega_1 \end{aligned} \tag{3.20}$$

Proof. We consider the following 3 sub-cases:

1. $A_3 \equiv 0$
2. $A_3 \equiv \pm 1$
3. $A_3 \neq 0$ and $A_3 \neq \pm 1$

3.1.2.1 case $A_3 \equiv 0$

If $A_3 \equiv 0$, (3.16) is equivalent to

$$d\tilde{\omega}_1 = \pm\tilde{\omega}_4 \wedge \tilde{\omega}_2 + (\pm b_{21} + A_4)\tilde{\omega}_4 \wedge \tilde{\omega}_1 + (\mp S_{12} \pm b_{21}S_{22} + A_2 + A_4 S_{22})\,\tilde{\omega}_2 \wedge \tilde{\omega}_1\,. \tag{3.21}$$

From the second term of (3.21), by adding a multiple of ω_1 into ω_2, we can arrange $A_4 = 0$. This yields $b_{21} = 0$. Now

$$B_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2\,.$$

Equation (3.21) is equivalent to

$$d\tilde{\omega}_1 = \pm\tilde{\omega}_4 \wedge \tilde{\omega}_2 + (\mp S_{12} + A_2)\,\tilde{\omega}_2 \wedge \tilde{\omega}_1\,. \tag{3.22}$$

From the second term of the right side of the above equation, by adding a multiple of ω_1 into ω_4, we can arrange $A_2 = 0$. This yields $S_{12} = 0$. Thus

$$d\omega_1 = \pm\omega_4 \wedge \omega_2\,. \tag{3.23}$$

And

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ S_{11} & 0 & 1 & 0 \\ 0 & S_{22} & 0 & 1 \end{bmatrix}.$$

By (3.5), there exist functions q_3 and q_4 such that

$$d\omega_2 \equiv (q_3\omega_3 + q_4\omega_4) \wedge \omega_1 + \omega_3 \wedge \omega_4 \qquad \mod \quad \omega_2\,.$$

By adding a multiple of ω_1 into ω_3, we can arrange $q_4 = 0$. This yields $S_{11} = 0$. Thus

$$d\omega_2 \equiv q_3\omega_3 \wedge \omega_1 + \omega_3 \wedge \omega_4 \qquad \mod \quad \omega_2 \,. \tag{3.24}$$

There exist functions r_1, r_3 and r_4 such that

$$d\omega_2 = (r_1\omega_1 + r_3\omega_3 + r_4\omega_4) \wedge \omega_2 + q_3\omega_3 \wedge \omega_1 + \omega_3 \wedge \omega_4 \,. \tag{3.25}$$

By adding a multiple of ω_2 into ω_4, we can arrange $r_3 = 0$. This yields $S_{22} = 0$.

Now the structure group contains only the identity element, i.e., we have found an e-structure. In this case, the structure equation is

$$\begin{aligned} d\omega_1 &= \pm\omega_4 \wedge \omega_2 \,, \\ d\omega_2 &= (r_1\omega_1 + r_4\omega_4) \wedge \omega_2 + q_3\omega_3 \wedge \omega_1 + \omega_3 \wedge \omega_4 \,. \end{aligned} \tag{3.26}$$

3.1.2.2 case $A_3 \equiv \pm 1$

(3.16) is equivalent to

$$d\tilde{\omega}_1 = \pm\tilde{\omega}_4 \wedge \tilde{\omega}_2 \pm \tilde{\omega}_3 \wedge \tilde{\omega}_1 + A_4\tilde{\omega}_4 \wedge \tilde{\omega}_1 + (A_2 + A_4S_{22})\,\tilde{\omega}_2 \wedge \tilde{\omega}_1, \tag{3.27}$$

where A_4 is an invariant of Lagrangian Engel structures in the case $A_3 \equiv \pm 1$.

We will consider the following 2 sub-cases:

1. $A_4 \equiv 0$
2. $A_4 \neq 0$

3.1.2.2.1 case : $A_4 \equiv 0$

$$d\omega_1 = \pm\omega_4 \wedge \omega_2 \pm \omega_3 \wedge \omega_1 + A_2\omega_2 \wedge \omega_1 \,. \tag{3.28}$$

Now A_2 is an invariant of Lagrangian Engel structures in this case. And

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ b_{21} & 1 & 0 & 0 \\ S_{11} - b_{21}S_{12} & S_{12} - b_{21}S_{22} & 1 & -b_{21} \\ S_{12} & S_{22} & 0 & 1 \end{bmatrix}$$

By (3.5), there exist functions q_3 and q_4 such that

$$d\omega_2 \equiv (q_3\omega_3 + q_4\omega_4) \wedge \omega_2 + \omega_3 \wedge \omega_4 \qquad \text{mod} \quad \omega_1 \,.$$

By adding a multiple of ω_2 to ω_4, we can arrange $q_3 = 0$. This yields $S_{22} = 0$.
By adding a multiple of ω_2 to ω_3, we can arrange $q_4 = 0$. This yields $S_{12} \pm b_{21} = 0$. And

$$d\omega_2 \equiv \omega_3 \wedge \omega_4 \qquad \text{mod} \quad \omega_1 \,. \tag{3.29}$$

There exist functions r_2, r_3 and r_4 such that

$$d\omega_2 = (r_2\omega_2 + r_3\omega_3 + r_4\omega_4) \wedge \omega_1 + \omega_3 \wedge \omega_4 \,.$$

The elements of the structure group are of the form

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ b_{21} & 1 & 0 & 0 \\ S_{11} \pm b_{21}^2 & \mp b_{21} & 1 & -b_{21} \\ \mp b_{21} & 0 & 0 & 1 \end{bmatrix} .$$

In this case, the structure group does not reduce to the trivial group. The structure group can be further reduced by considering the derivative of ω_3 and ω_4.

Now ω_1 is uniquely defined by (3.28) and ω_2 is uniquely defined up to an addition of a multiple of ω_1 by (3.29). Thus $\omega_1 \wedge \omega_2$ is uniquely defined by (3.28) and (3.29). Therefore,

$$\omega_1 \wedge \omega_2 \wedge d\omega_2 = \omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4 \tag{3.30}$$

is uniquely defined.

3.1.2.2.2 case : $A_4 \neq 0$

By (3.27), after adding a multiple of ω_2 into ω_4, we can arrange $A_2 = 0$. This yields $S_{22} = 0$. And

$$d\omega_1 = \pm\omega_4 \wedge \omega_2 \pm \omega_3 \wedge \omega_1 + A_4\omega_4 \wedge \omega_1 \,. \tag{3.31}$$

By (3.5), there exist functions q_3 and q_4 such that

$$d\omega_2 \equiv (q_3\omega_3 + q_4\omega_4) \wedge \omega_2 + \omega_3 \wedge \omega_4 \qquad \mod \quad \omega_1\,.$$

From this structure equation, q_3 is an invariant. By adding a multiple of ω_2 and ω_4 to ω_3, we can arrange $q_4 = 0$. This yields $S_{12} + b_{21}q_3 \pm b_{21} = 0$.

$$d\omega_2 \equiv q_3\omega_3 \wedge \omega_2 + \omega_3 \wedge \omega_4 \qquad \mod \quad \omega_1\,. \tag{3.32}$$

Now ω_1 is uniquely defined by (3.31) and ω_2 is uniquely defined up to an addition of a multiple of ω_1 by (3.32). Thus $\omega_1 \wedge \omega_2$ is uniquely defined by (3.31) and (3.32). Therefore,

$$\omega_1 \wedge \omega_2 \wedge d\omega_2 = \omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4 \tag{3.33}$$

is uniquely defined.

3.1.2.3 case $A_3 \neq \pm 1$ and $A_3 \neq 0$

By (3.16), after adding a multiple of ω_4 to ω_3, we can arrange $A_4 = 0$. This yields $b_{21} = 0$. Then

$$d\omega_1 = \pm\omega_4 \wedge \omega_2 + A_3\omega_3 \wedge \omega_1 + (\mp S_{12} + A_2 + A_3 S_{12})\,\omega_2 \wedge \omega_1. \tag{3.34}$$

By adding a multiple of ω_2 to ω_3, we can arrange $A_2 = 0$. This yields $S_{12} = 0$. Then

$$d\omega_1 = \pm\omega_4 \wedge \omega_2 + A_3\omega_3 \wedge \omega_1. \tag{3.35}$$

By (3.5), there exist functions q_3 and q_4 such that

$$d\omega_2 \equiv (q_3\omega_3 + q_4\omega_4) \wedge \omega_1 + \omega_3 \wedge \omega_4 \qquad \mod \quad \omega_2.$$

By adding a multiple of ω_1 to ω_3, we can arrange $q_4 = 0$. This yields $S_{11} = 0$. Thus

$$d\omega_2 \equiv q_3\omega_3 \wedge \omega_1 + \omega_3 \wedge \omega_4 \qquad \mod \quad \omega_2. \tag{3.36}$$

There exist functions r_1, r_3 and r_4 such that

$$d\omega_2 = (r_1\omega_1 + r_3\omega_3 + r_4\omega_4) \wedge \omega_2 + q_3\omega_3 \wedge \omega_1 + \omega_3 \wedge \omega_4. \tag{3.37}$$

By adding a multiple of ω_2 to ω_4, we can arrange $r_3 = 0$. This yields $S_{22} = 0$.

Now the structure group contains only the identity element, i.e., we have found an e-structure. In this case, the structure equation is

$$\begin{aligned} d\omega_1 &= \pm\omega_4 \wedge \omega_2 + A_3\omega_3 \wedge \omega_1 \,, \\ d\omega_2 &= (r_1\omega_1 + r_4\omega_4) \wedge \omega_2 + q_3\omega_3 \wedge \omega_1 + \omega_3 \wedge \omega_4 \,. \end{aligned} \tag{3.38}$$

□

3.2 Classification of Homogeneous Lagrangian Engel Structures

In this section, we derive the structure equation of homogeneous Lagrangian Engel structures via equivalence method [Gardner (1989)].

Theorem 3.2.1 (Classification of Homogeneous Lagrangian Engel Structures)**.** *There are at most 6 distinct families of homogeneous Lagrangian Engel structures that can* have compact quotient manifolds. *These 6 families are listed as follows:*

1. *Case 1:*

$$d\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix} = \begin{bmatrix} \omega_2 \wedge \omega_3 + a\omega_1 \wedge \omega_3 \\ \omega_3 \wedge \omega_4 \\ 0 \\ b\omega_2 \wedge \omega_3 \end{bmatrix}$$

2. *Case 2:*

$$d\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix} = \begin{bmatrix} \omega_2 \wedge \omega_3 + a\omega_1 \wedge \omega_3 + b\omega_1 \wedge \omega_4 \\ b\omega_1 \wedge \omega_3 + \omega_3 \wedge \omega_4 + b\omega_2 \wedge \omega_4 \\ 0 \\ 0 \end{bmatrix}$$

3. *Case 3:*

$$d\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{bmatrix} = \begin{bmatrix} \omega_2\wedge\omega_3 + a\omega_1\wedge\omega_3 - \frac{1}{4}a^2\omega_1\wedge\omega_4 \\ -\frac{1}{4}a^2\omega_1\wedge\omega_3 + \omega_3\wedge\omega_4 - \frac{1}{4}a^2\omega_2\wedge\omega_4 \\ \frac{1}{2}a^2b(\omega_1\wedge\omega_3 - \omega_2\wedge\omega_4) + ab\omega_2\wedge\omega_3 - \frac{1}{4}a^3b\omega_1\wedge\omega_4 \\ ab(\omega_1\wedge\omega_3 - \omega_2\wedge\omega_4) + 2b\omega_2\wedge\omega_3 - \frac{1}{2}a^2b\omega_1\wedge\omega_4 \end{bmatrix}$$

4. *Case 4:*

$$d\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{bmatrix} = \begin{bmatrix} \omega_2\wedge\omega_3 + b\omega_1\wedge\omega_3 + a\omega_1\wedge\omega_4 \\ (a^2 + \frac{ab^2}{4})\omega_1\wedge\omega_2 + a\omega_1\wedge\omega_3 + \omega_3\wedge\omega_4 + a\omega_2\omega_4 \\ (2a^3 + \frac{1}{2}a^2b^2)\omega_{12} + \frac{ab^2}{2}\omega_{13} + b\omega_{23} + a^2b\omega_{14} + 2a^2\omega_{24} \\ ab(-a - \frac{1}{4})\omega_{12} + ab\omega_{13} + (a - \frac{1}{4})\omega_{23} + (a^2 - \frac{ab^2}{4})\omega_{14} - ab\omega_{24} \end{bmatrix}$$

5. *Case 5:*

$$d\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{bmatrix} = \begin{bmatrix} \omega_1\wedge\omega_3 + \omega_2\wedge\omega_4 \\ a\omega_1\wedge\omega_2 + \omega_3\wedge\omega_4 \\ a(\omega_1\wedge\omega_3 + \omega_2\wedge\omega_4) \\ -a\omega_1\wedge\omega_2 - \omega_3\wedge\omega_4 \end{bmatrix}$$

6. *Case 6:*

$$d\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{bmatrix} = \begin{bmatrix} \omega_1\wedge\omega_3 + \omega_2\wedge\omega_4 \\ \omega_3\wedge\omega_4 \\ 0 \\ a\omega_2\wedge\omega_3 - \omega_3\wedge\omega_4 \end{bmatrix}$$

where a and b are constants.

Proof. Since the structure group of Lagrangian Engel structures is of the form (3.1), there exists Lie algebra-valued differential form

$$\pi = \begin{bmatrix} \pi_1 & 0 & 0 & 0 \\ \pi_2 & \pi_3 & 0 & 0 \\ \pi_4 & \pi_5 & -\pi_1 & -\pi_2 \\ \pi_5 & \pi_6 & 0 & -\pi_3 \end{bmatrix} = (\pi_{ij})$$

such that the structure equation can be written as

$$d\omega_i = \sum \pi_{ij}\wedge\omega_j + \frac{1}{2}\sum \gamma_{ijk}\omega_j\wedge\omega_k\,, \tag{3.39}$$

where γ_{ijk} are torsion terms. By (3.3), there exist functions a_0, a_3, a_4 such that

$$d\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix} = -\begin{bmatrix}\pi_1 & 0 & 0 & 0\\ \pi_2 & \pi_3 & 0 & 0\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{bmatrix} + \begin{bmatrix}\omega_2\wedge(a_3\omega_3 + a_4\omega_4)\\ a_0\omega_3\wedge\omega_4\end{bmatrix}$$

and at least one of a_3 and a_4 is nonzero and $a_0 \neq 0$.

By modifying the Lie algebra valued 1-forms π_4, π_5, π_6 and absorption of torsions, there exist functions S_1 and S_2 and 1-form τ such that

$$d\begin{bmatrix}\omega_3\\ \omega_4\end{bmatrix} = -\begin{bmatrix}\pi_4 & \pi_5 & -\pi_1 & -\pi_2\\ \pi_5 & \pi_6 & 0 & -\pi_3\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{bmatrix} + \begin{bmatrix}S_1\omega_3\wedge\omega_4 + \tau\wedge\omega_2\\ S_2\omega_3\wedge\omega_4 - \tau\wedge\omega_1\end{bmatrix},$$

where $\tau = t_3\omega_3 + t_4\omega_4$ for some functions t_3 and t_4 . Thus after this absorption of torsions, we get 0-adapted coframing such that

$$d\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{bmatrix} = -\begin{bmatrix}\pi_1 & 0 & 0 & 0\\ \pi_2 & \pi_3 & 0 & 0\\ \pi_4 & \pi_5 & -\pi_1 & -\pi_2\\ \pi_5 & \pi_6 & 0 & -\pi_3\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{bmatrix} + \begin{bmatrix}\omega_2\wedge(a_3\omega_3 + a_4\omega_4)\\ a_0\omega_3\wedge\omega_4\\ S_1\omega_3\wedge\omega_4 + \tau\wedge\omega_2\\ S_2\omega_3\wedge\omega_4 - \tau\wedge\omega_1\end{bmatrix}. \quad (3.40)$$

Since Ω is a symplectic form, $d\Omega = 0$. By (3.40) and $d\Omega = 0$, we get $S_2 = -a_4$, $S_1 = 0$, $t_3 = 0$, $t_4 = 0$. Now (3.40) is transformed into

$$d\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{bmatrix} = -\begin{bmatrix}\pi_1 & 0 & 0 & 0\\ \pi_2 & \pi_3 & 0 & 0\\ \pi_4 & \pi_5 & -\pi_1 & -\pi_2\\ \pi_5 & \pi_6 & 0 & -\pi_3\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{bmatrix} + \begin{bmatrix}\omega_2\wedge(a_3\omega_3 + a_4\omega_4)\\ a_0\omega_3\wedge\omega_4\\ 0\\ -a_4\omega_3\wedge\omega_4\end{bmatrix}. \quad (3.41)$$

Now we calculate the reduction of the group using the equivalence method. Calculate $d^2\omega_1 = 0$ by (3.41)

$$\begin{aligned} da_4 + a_3\pi_2 + a_4\pi_1 &\equiv 0 \qquad \text{mod} \quad \omega_1, \omega_2, \omega_3, \omega_4\,,\\ da_3 + a_3(2\pi_1 - \pi_3) &\equiv 0 \qquad \text{mod} \quad \omega_1, \omega_2, \omega_3, \omega_4\,. \end{aligned} \quad (3.42)$$

Since at least one of a_3 and a_4 is nonzero, there are two cases: $a_3 \neq 0$ or $a_3 = 0$.

3.2.1 case $a_3 \neq 0$

We can scale $a_3 = 1$ and translate $a_4 = 0$. Then from (3.42)

$$\begin{aligned} \pi_2 &\equiv 0 \qquad \mod \omega_1, \omega_2, \omega_3, \omega_4\,, \\ 2\pi_1 - \pi_3 &\equiv 0 \qquad \mod \omega_1, \omega_2, \omega_3, \omega_4 \end{aligned} \tag{3.43}$$

This means π_2, $2\pi_1 - \pi_3$ are basic.

Calculating $d^2\omega_2 = 0$ from (3.41) yields

$$da_0 + a_0(\pi_1 + 2\pi_3) \equiv 0 \qquad \mod \omega_1, \omega_2, \omega_3, \omega_4\,.$$

Since $a_0 \neq 0$, we can scale $a_0 = 1$. Then

$$\pi_1 + 2\pi_3 \equiv 0 \qquad \mod \omega_1, \omega_2, \omega_3, \omega_4\,. \tag{3.44}$$

Thus from (3.43) and (3.44), we have

$$\pi_1 \equiv \pi_2 \equiv \pi_3 \equiv 0 \qquad \mod \omega_1, \omega_2, \omega_3, \omega_4\,.$$

Define $\pi_i = \sum_{j=1}^{4} a_{ij}\omega_j$, where $i = 1, 2, 3$. Calculate $d^2\omega_4 = 0$ from (3.41), then

$$da_{33} + \pi_6 \equiv 0 \qquad \mod \omega_1, \omega_2, \omega_3, \omega_4\,.$$

We can translate $a_{33} = 0$. Then

$$\pi_6 \equiv 0 \qquad \mod \omega_1, \omega_2, \omega_3, \omega_4\,.$$

Calculate $d^2\omega_3 = 0$ from (3.41), then

$$d(a_{23} - a_{14}) + \pi_5 \equiv 0 \qquad \mod \omega_1, \omega_2, \omega_3, \omega_4\,.$$

We can translate $a_{23} - a_{14} = 0$. Then

$$\pi_5 \equiv 0 \qquad \mod \omega_1, \omega_2, \omega_3, \omega_4\,.$$

Calculate $d^2\omega_1 = 0$ from (3.41), then

$$a_{34} = a_{14}$$

and

$$da_{12} + \pi_4 \equiv 0 \qquad \mod \quad \omega_1, \omega_2, \omega_3, \omega_4 \,.$$

We can translate $a_{12} = 0$. Then

$$\pi_4 \equiv 0 \qquad \mod \quad \omega_1, \omega_2, \omega_3, \omega_4 \,.$$

Thus

$$\pi_4 \equiv \pi_5 \equiv \pi_6 \equiv 0 \qquad \mod \quad \omega_1, \omega_2, \omega_3, \omega_4 \,.$$

Now we get a canonical coframing and the G-structure is reduced to an e-structure. The structure equation is

$$d \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix} = \begin{bmatrix} 0 & a_{13} & a_{14} & 1 & 0 & 0 \\ a_{22} - a_{31} & a_{14} & a_{24} & 0 & a_{14} & 1 \\ a_{42} - a_{51} & a_{11} + a_{43} & a_{44} + a_{21} & a_{53} & a_{54} + a_{22} & 0 \\ a_{52} - a_{61} & a_{53} & a_{54} + a_{31} & a_{63} & a_{64} + a_{32} & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \wedge \omega_2 \\ \omega_1 \wedge \omega_3 \\ \omega_1 \wedge \omega_4 \\ \omega_2 \wedge \omega_3 \\ \omega_2 \wedge \omega_4 \\ \omega_3 \wedge \omega_4 \end{bmatrix}, \tag{3.45}$$

where the nonzero terms of the right side of (3.45) represent intrinsic torsion of Lagrangian Engel structures. The coefficients of torsion terms are functional invariants of Lagrangian Engel structures. We have finished the analysis of the structure equation for the case $a_3 \neq 0$.

3.2.2 case $a_3 = 0$

We can scale $a_4 = 1$. Thus

$$\pi_1 \equiv 0 \qquad \mod \quad \omega_1, \omega_2, \omega_3, \omega_4 \,.$$

Calculate $d^2\omega_2 = 0$ from equation (3.41), then

$$da_0 + 2a_0\pi_3 \equiv 0 \qquad \mod \quad \omega_1, \omega_2, \omega_3, \omega_4 \,.$$

Since $a_0 \neq 0$, we can scale $a_0 = 1$. Then

$$\pi_3 \equiv 0 \qquad \mod \quad \omega_1, \omega_2, \omega_3, \omega_4 \,.$$

Calculate $d^2\omega_4 = 0$ from (3.41), then

$$da_{33} + \pi_6 \equiv 0 \qquad \mod \quad \omega_1, \omega_2, \omega_3, \omega_4 \,.$$

We can translate $a_{33} = 0$. Then

$$\pi_6 \equiv 0 \qquad \mod \quad \omega_1, \omega_2, \omega_3, \omega_4 \,.$$

Also from $d^2\omega_4 = 0$,

$$d(a_{32} + a_{64}) + 2\pi_5 + a_{63}\pi_2 \equiv 0 \qquad \mod \quad \omega_1, \omega_2, \omega_3, \omega_4 \,.$$

We can translate $a_{32} + a_{64} = 0$. Then

$$2\pi_5 + a_{63}\pi_2 \equiv 0 \qquad \mod \quad \omega_1, \omega_2, \omega_3, \omega_4 \,.$$

Calculate $d^2\omega_2 = 0$ from (3.41), then

$$da_{34} - \pi_5 + \pi_2 \equiv 0 \qquad \mod \quad \omega_1, \omega_2, \omega_3, \omega_4 \,.$$

There are 2 cases: $a_{63} \neq -2$ or $a_{63} \equiv -2$.

1. case 1. $a_{63} \neq -2$ We can translate $a_{34} = 0$. Then

$$\pi_5 \equiv \pi_2 \equiv 0 \qquad \mod \quad \omega_1, \omega_2, \omega_3, \omega_4 \,.$$

2. case 2. $a_{63} = -2$

   From $d^2\omega_1 = 0$,

$$da_{14} + (a_{13} - 1)\pi_2 \equiv 0 \qquad \mod \quad \omega_1, \omega_2, \omega_3, \omega_4$$

   and

$$da_{12} + (-a_{13} + 1)\pi_5 \equiv 0 \qquad \mod \quad \omega_1, \omega_2, \omega_3, \omega_4\,.$$

In summary, as long as $a_{13} \neq 1$ or $a_{63} \neq -2$,

$$\pi_5 \equiv \pi_2 \equiv 0 \qquad \mod \quad \omega_1, \omega_2, \omega_3, \omega_4\,.$$

3.2.2.1 $a_{13} \neq 1$ *or* $a_{63} \neq -2$

From $d^2\omega_2 = 0$,

$$da_{24} - \pi_4 \equiv 0 \qquad \mod \quad \omega_1, \omega_2, \omega_3, \omega_4\,.$$

We can translate $a_{24} = 0$. Then

$$\pi_4 \equiv 0 \qquad \mod \quad \omega_1, \omega_2, \omega_3, \omega_4\,.$$

So

$$\pi_1 \equiv \pi_2 \equiv \pi_3 \equiv \pi_4 \equiv \pi_5 \equiv \pi_6 \equiv 0 \qquad \mod \quad \omega_1, \omega_2, \omega_3, \omega_4\,.$$

Now we get an e-structure and a canonical coframing. In this case there are 2 different families of structure equations.

3.2.2.2 $a_{13} = 1$ *and* $a_{63} = -2$

In this case, we know that

$$\pi_1 \equiv \pi_3 \equiv \pi_6 \equiv \pi_2 - \pi_5 \equiv 0 \qquad \mod \quad \omega_1, \omega_2, \omega_3, \omega_4\,.$$

From $d^2\omega_1 = 0$,

$$da_{12} \equiv 0 \qquad \mod \quad \omega_1, \omega_2, \omega_3, \omega_4$$

and

$$da_{14} \equiv 0 \qquad \mod \quad \omega_1, \omega_2, \omega_3, \omega_4 \,.$$

I do not intend to finish the calculation of all invariants of Lagrangian Engel structures of this case. Since the goal is to classify compact quotients that support homogeneous Lagrangian Engel structures, I will prove that no compact quotients can support a homogeneous Lagrangian Engel structure of this case.

From the structure equation,

$$d(\omega_1 \wedge \omega_2 \wedge \omega_4) = -2\, \omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4 \,.$$

By Stokes's Theorem, there is no compact quotient that supports a homogeneous Lagrangian Engel structure when $a_{13} = 1$ and $a_{63} = -2$. In the following classification of compact homogeneous Lagrangian Engel structures, we will not consider this case any more.

Now we will classify homogeneous Lagrangian Engel structures. Assume all the coefficients in the structure equations are constants. By taking exterior derivative of the structure equation and setting all coefficients to zero, we can get quadratic equations of the constants. Via MAPLE, we can solve all the equations. The structure equations of homogeneous Lagrangian Engel structures are listed in the statement of the theorem.

For the case that $a_{13} = 1$ and $a_{63} = -2$, it remains to determine whether there exist homogeneous Lagrangian Engel structures. □

3.3 Classification of Compact Homogeneous Lagrangian Engel Structures

Theorem 3.3.1 (Classification of Compact Homogeneous Lagrangian Engel Structures)**.** *There is only a 1-parameter family of compact homogeneous Lagrangian Engel* *structures. There exists a canonical coframing* $(\omega_1, \omega_2, \omega_3, \omega_4)$ *such that*

$$d \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix} = \begin{bmatrix} \omega_2 \wedge \omega_3 \\ \omega_3 \wedge \omega_4 \\ 0 \\ b\omega_2 \wedge \omega_3 \end{bmatrix}$$

where $b \in \mathbb{R}$ *is a constant.*

Proof. We will prove this theorem by analyzing each homogeneous case in Theorem 3.2.1 and determining whether there exists a compact quotient that can support the corresponding homogeneous Lagrangian Engel structure of one particular case.

3.3.1 Analysis of Case 1

The structure equation is

$$d \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix} = \begin{bmatrix} \omega_2 \wedge \omega_3 + a\omega_1 \wedge \omega_3 \\ \omega_3 \wedge \omega_4 \\ 0 \\ b\omega_2 \wedge \omega_3 \end{bmatrix} \tag{3.46}$$

where a and b are constants. From the structure equation (3.46),

$$d(\omega_1 \wedge \omega_2 \wedge \omega_4) = -a\, \omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4 \,.$$

Thus if $a \neq 0$, there is no compact quotient that can support a homogeneous Lagrangian Engel structure of case 1.

In the following, we only consider $a = 0$. Since $d\omega_3 = 0$ and $d(\omega_4 - b\omega_1) = 0$, there exist functions x and y such that

$$\omega_3 = dy,$$

$$\omega_4 - b\omega_1 = dx.$$

Define $\tilde{\omega}_2 = \omega_2 + x dy$. Then

$$d \begin{bmatrix} \omega_1 \\ \tilde{\omega}_2 \end{bmatrix} = \begin{bmatrix} \tilde{\omega}_2 \wedge dy \\ b\, dy \wedge \omega_1 \end{bmatrix}. \tag{3.47}$$

Proposition 3.3.2. *If $a = 0$, there exists a compact quotient that supports a homogeneous Lagrangian Engel structure of Case1 for any b.*

We prove this proposition by considering different values for b.

3.3.1.1 $b = 0$

$$d\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} \omega_2 \wedge dy \\ dy \wedge dx \end{bmatrix}.$$

Thus there exist functions u and v such that

$$\omega_1 = udy + dv\,,$$

$$\omega_2 = -xdy + du\,.$$

Since

$$\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4 = dv \wedge du \wedge dy \wedge dx\,,$$

so x, y, u, v can be a local coordinate system for the homogeneous manifold.

Let

$$\omega = \begin{bmatrix} 0 & \omega_3 & -\omega_2 & 2\omega_1 \\ 0 & 0 & \omega_4 & \omega_2 \\ 0 & 0 & 0 & \omega_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.48}$$

be a matrix-valued 1-form. Then from (3.46), we have

$$d\omega = -\omega \wedge \omega \tag{3.49}$$

Thus ω is a left-invariant form of a Lie group G. The connected and simply-connected Lie group corresponding to the left-invariant form in (3.49) is isomorphic to

$$G = \left\{ \left. \begin{bmatrix} 1 & f & fe - c & d \\ 0 & 1 & 2e & fe + c \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{bmatrix} \right| \text{where } f, e, c, d \in \mathbb{R} \right\} \tag{3.50}$$

Note G is a nilpotent Lie group. In [Raghunathan (1972)], there is a theorem:

Theorem 3.3.3. *A simply-connected nilpotent Lie group G admits a lattice if and only if there exists a basis $(X_1, X_2, \cdots, X_n)$ of the Lie algebra $\mathfrak{g}$ of G such that the structure constants C_{ij}^k arising in the brackets*

$$[X_i, X_j] = \sum_k C_{ij}^k X_k \tag{3.51}$$

are rational numbers.

By the structure (3.49) and Theorem 3.3.3, there exists a co-compact lattice for the group G, and thus there exists a compact quotient that can support a homogeneous Lagrangian Engel structure. We will find an explicit co-compact lattice in this case. Take a discrete subgroup of Lie group G

$$\Gamma = \left\{ \begin{bmatrix} 1 & f & fe-c & d \\ 0 & 1 & 2e & fe+c \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{bmatrix} \middle| \text{ where } c, d, e, f \in \mathbb{Z} \right\}. \tag{3.52}$$

It is easy to verify that Γ is a subgroup of G and that $M = G/\Gamma$ is compact. So if $b = 0$, there exists a compact quotient, that supports a homogeneous Lagrangian Engel structure.

3.3.1.2 $\quad b < 0$

Set $b = -\beta^2$, where $\beta > 0$. Then by (3.47), we get

$$d(\beta\omega_1 + \tilde{\omega}_2) = \beta(\beta\omega_1 + \tilde{\omega}_2) \wedge dy$$

and

$$d(-\beta\omega_1 + \tilde{\omega}_2) = -\beta(-\beta\omega_1 + \tilde{\omega}_2) \wedge dy\,.$$

So there exist functions u and v such that $\beta\omega_1 + \tilde{\omega}_2 = e^{-\beta y}du$ and $-\beta\omega_1 + \tilde{\omega}_2 = e^{\beta y}dv$. Thus

$$\tilde{\omega}_2 = \frac{e^{-\beta y}du + e^{\beta y}dv}{2}$$

and

$$\omega_1 = \frac{e^{-\beta y} du - e^{\beta y} dv}{2\beta}.$$

Now we take a new coframing. After scaling $\omega_4 \to \beta\omega_4$, $\omega_3 \to \frac{1}{\beta}\omega_3$ and $\omega_1 \to \frac{1}{\beta}\omega_1$, then the structure equation is transformed to

$$d\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix} = \begin{bmatrix} \omega_2 \wedge \omega_3 \\ \omega_3 \wedge \omega_4 \\ 0 \\ -\omega_2 \wedge \omega_3 \end{bmatrix}. \tag{3.53}$$

Define $\omega_0 = \omega_1 + \omega_4$, $\tilde{\omega}_2 = \omega_2 + \omega_4$ and $\tilde{\omega}_4 = \omega_2 - \omega_4$. Then $(\omega_0, \tilde{\omega}_2, \omega_3, \tilde{\omega}_4)$ is a new coframing. In this new coframing, after dropping tildes, the structure equation is

$$d\begin{bmatrix} \omega_0 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix} = \begin{bmatrix} 0 \\ \omega_3 \wedge \omega_2 \\ 0 \\ -\omega_3 \wedge \omega_4 \end{bmatrix}. \tag{3.54}$$

Let

$$\omega = \begin{bmatrix} \omega_0 & 0 & 0 & 0 \\ 0 & -\omega_3 & 0 & \omega_2 \\ 0 & 0 & \omega_3 & \omega_4 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.55}$$

be a matrix-valued 1-form. Then from (3.54)

$$d\omega = -\omega \wedge \omega. \tag{3.56}$$

Thus ω is a Maurer-Cartan form of a Lie group G. The connected and simply-connected Lie group corresponding to the Maurer-Cartan form in (3.55) is isomorphic to

$$G = \left\{ \left[\begin{matrix} c & 0 & 0 & 0 \\ 0 & t^{-1} & 0 & r \\ 0 & 0 & t & s \\ 0 & 0 & 0 & 1 \end{matrix} \right] \middle| \text{ where } r, s \in \mathbb{R} \text{ and } c > 0 \text{ and } t > 0 \right\}. \tag{3.57}$$

Theorem 3.3.4. *There exists a co-compact lattice of G.*

Proof. Let (X_1, X_2, X_3, X_4) be the left-invariant vectors dual to the left-invariant 1-forms $(\omega_1, \omega_3, -\omega_2, \omega_0)$, respectively. Then the nontrivial brackets are

$$[X_1, X_3] = X_1\,,$$

$$[X_2, X_3] = -X_2\,.$$

By the classification results of [Bock (2016)], there exists a co-compact lattice. □

We will give an explicit way to construct a lattice. Consider a subgroup $H \subset G$, where

$$H = \left\{ \begin{bmatrix} t^{-1} & 0 & r \\ 0 & t & s \\ 0 & 0 & 1 \end{bmatrix} \middle| \text{ where } r, s \in \mathbb{R} \text{ and } t > 0 \right\} \tag{3.58}$$

and the inclusion map of H to G is

$$\begin{bmatrix} t^{-1} & 0 & r \\ 0 & t & s \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & t^{-1} & 0 & r \\ 0 & 0 & t & s \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.59}$$

Then $G \cong \mathbb{R} \times H$ as a group. Let $N \subset H$ be the subgroup

$$N = \left\{ \begin{bmatrix} 1 & 0 & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{bmatrix} \middle| \text{ where } r, s \in \mathbb{R} \right\}. \tag{3.60}$$

Lemma 3.3.5. *N is a normal subgroup of H.*

Proof. Let $h = \begin{bmatrix} t^{-1} & 0 & x \\ 0 & t & y \\ 0 & 0 & 1 \end{bmatrix}$ and $n = \begin{bmatrix} 1 & 0 & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{bmatrix}$ be any elements of H and N, respectively. Then

$$h^{-1}nh = \begin{bmatrix} 1 & 0 & t \cdot r \\ 0 & 1 & t^{-1} \cdot s \\ 0 & 0 & 1 \end{bmatrix} \in N. \tag{3.61}$$

Hence, N is a normal subgroup of H. □

Thus

$$H/N \cong \left\{ \begin{bmatrix} t^{-1} & 0 \\ 0 & t \end{bmatrix} \middle| \text{ where } t > 0 \right\}$$

is a quotient group. Let $L_1 = \langle \vec{v}_1, \vec{v}_2 \rangle \subset \mathbb{R}^2$ be a lattice of the normal subgroup N, to be determined later. We need to find a lattice L_2 of H/N such that the lattice of the group H is

$$L = \left\{ \begin{bmatrix} \gamma & \vec{v} \\ 0 & 1 \end{bmatrix} \middle| \text{ where } \gamma \in L_2, \vec{v} \in L_1 \right\}.$$

From the multiplication rule of the group H, this is equivalent to $\gamma\vec{v} \in L_1$ for any $\gamma \in L_2$ and any $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in L_1$. Hence we need to find $a_1, a_2, a_3, a_4 \in \mathbb{Z}$ and $c > 0$ and $c \neq 1$ such that

$$\begin{aligned} \gamma_c \vec{v}_1 &= a_1 \vec{v}_1 + a_2 \vec{v}_2 \,, \\ \gamma_c \vec{v}_2 &= a_3 \vec{v}_1 + a_4 \vec{v}_2 \,, \end{aligned} \tag{3.62}$$

where γ_c is the linear transform with transformation matrix $\begin{bmatrix} c^{-1} & 0 \\ 0 & c \end{bmatrix}$. Since $\langle \gamma_c \vec{v}_1, \gamma_c \vec{v}_2 \rangle$ form a new basis for the lattice L_1, then $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in SL_2(\mathbb{Z})$.

Thus (3.62) is equivalent to

$$\begin{bmatrix} c^{-1} & 0 \\ 0 & c \end{bmatrix} = (\vec{v}_1, \vec{v}_2) \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix} (\vec{v}_1, \vec{v}_2)^{-1} \tag{3.63}$$

So we can choose any matrix $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in SL_2(\mathbb{Z})$ such that $(A+D)^2 - 4 > 0$, then we have two real eigenvalues $\lambda_1 > \lambda_2 > 0$. We can set $c = \lambda_1$. If $(\vec{v}, \vec{w})$ are eigenvectors of S with eigenvalues (λ_1, λ_2), then we can set $(\vec{v}_1, \vec{v}_2) \propto (\vec{v}, \vec{w})^{-1}$.

Example 3.3.1. *Take* $S = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \in SL_2(\mathbb{Z})$*, then* $c = \frac{3+\sqrt{5}}{2}$*. We can take*

$\vec{v} = \begin{bmatrix} 1 \\ \frac{-1-\sqrt{5}}{2} \end{bmatrix}$ *and* $\vec{w} = \begin{bmatrix} 1 \\ \frac{-1+\sqrt{5}}{2} \end{bmatrix}$. *Thus*

$$(\vec{v}_1, \vec{v}_2) \propto (\vec{v}, \vec{w})^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{-1+\sqrt{5}}{2} & -1 \\ \frac{1+\sqrt{5}}{2} & 1 \end{bmatrix}.$$

Then the lattice of the Lie group H can be

$$L = \left\{ \left[\begin{array}{ccc} \left(\frac{3+\sqrt{5}}{2}\right)^{-m_0} & 0 & m_1\left(\frac{-1+\sqrt{5}}{2}\right) - m_2 \\ 0 & \left(\frac{3+\sqrt{5}}{2}\right)^{m_0} & m_1\left(\frac{1+\sqrt{5}}{2}\right) + m_2 \\ 0 & 0 & 1 \end{array} \right] \middle| \textit{ where } m_0, m_1, m_2 \in \mathbb{Z} \right\}. \tag{3.64}$$

Thus for $b < 0$, there exists a lattice Γ such that $G/\Gamma \cong S^1 \times H/L$ is compact.

Remark 3.3.1. *In our analysis of the existence of a lattice for H, we know that the different lattices correspond to*

1. *scaling or change of basis for eigenvectors of a matrix in $SL_2(\mathbb{Z})$*
2. *different matrices in $SL_2(\mathbb{Z})$ such that the absolute value of trace is greater than 2*

3.3.1.3 $b > 0$

Set $b = \beta^2$, where $\beta > 0$. Then by (3.47), there exist functions u and v such that

$$i\beta\omega_1 + \tilde{\omega}_2 = e^{-i\beta y} d(u + iv)$$

Take the real and imaginary part of the 1-form, we can get

$$\omega_1 = \frac{1}{\beta}(\cos\beta y\, dv - \sin\beta y\, du)$$

and

$$\tilde{\omega}_2 = \cos\beta y\, du + \sin\beta y\, dv\,.$$

Theorem 3.3.6. *There exists a co-compact lattice.*

Proof. Let (e_1, e_2, e_3, e_4) be the left-invariant vectors dual to the left-invariant 1-forms $(\beta\omega_1, \omega_2, \beta\omega_3, \frac{1}{\beta}\omega_4)$, respectively. Then the nontrivial brackets are

$$[e_2, e_3] = e_1 + e_4\,,$$

$$[e_3, e_4] = e_2\,.$$

Define $X_1 = e_1 + e_4, X_2 = e_2, X_3 = e_3, X_4 = e_1$, then the nontrivivial brackets are

$$[X_1, X_3] = -X_2\,,$$

$$[X_2, X_3] = X_1\,.$$

By the classification results of [Bock (2016)], there exists a co-compact lattice. □

So if $b > 0$, we get a compact quotient that supports a homogeneous Lagrangian Engel structure.

In summary, in case 1 we can get a compact quotient if and only if $a = 0$.

3.3.2 Analysis of Case 2

The structure equation is

$$d\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix} = \begin{bmatrix} \omega_2 \wedge \omega_3 + a\omega_1 \wedge \omega_3 + b\omega_1 \wedge \omega_4 \\ b\omega_1 \wedge \omega_3 + \omega_3 \wedge \omega_4 + b\omega_2 \wedge \omega_4 \\ 0 \\ 0 \end{bmatrix},$$

where a and b are constants. We can assume $b \neq 0$, otherwise, this is a special case of case 1.

From the structure equation,

$$d(\omega_1 \wedge \omega_2 \wedge \omega_3) = 2b\,\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4.$$

Since $b \neq 0$, there does not exist a compact quotient that supports a homogeneous Lagrangian Engel structure in case 2.

3.3.3 Analysis of Case 3

The structure equation is

$$d\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix} = \begin{bmatrix} \omega_2 \wedge \omega_3 + a\omega_1 \wedge \omega_3 - \frac{1}{4}a^2\omega_1 \wedge \omega_4 \\ -\frac{1}{4}a^2\omega_1 \wedge \omega_3 + \omega_3 \wedge \omega_4 - \frac{1}{4}a^2\omega_2 \wedge \omega_4 \\ \frac{1}{2}a^2b(\omega_1 \wedge \omega_3 - \omega_2 \wedge \omega_4) + ab\omega_2 \wedge \omega_3 - \frac{1}{4}a^3b\omega_1 \wedge \omega_4 \\ ab(\omega_1 \wedge \omega_3 - \omega_2 \wedge \omega_4) + 2b\omega_2 \wedge \omega_3 - \frac{1}{2}a^2b\omega_1 \wedge \omega_4 \end{bmatrix},$$

where a and b are constants. Since

$$d(\omega_1 \wedge \omega_2 \wedge \omega_3) = -\frac{a^2}{2}\,\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4,$$

there does not exist a compact quotient that supports a homogeneous Lagrangian Engel structure of case 3 if $a \neq 0$. In the following, we assume $a = 0$ and the structure equation is

$$d\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix} = \begin{bmatrix} \omega_2 \wedge \omega_3 \\ \omega_3 \wedge \omega_4 \\ 0 \\ 2b\omega_2 \wedge \omega_3 \end{bmatrix}.$$

This is a special case of case 1 with $a = 0$. There exists a compact quotient for any b.

3.3.4 Analysis of Case 4

The structure equation is

$$d\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix} = \begin{bmatrix} \omega_2 \wedge \omega_3 + b\omega_1 \wedge \omega_3 + a\omega_1 \wedge \omega_4 \\ (a^2 + \frac{ab^2}{4})\omega_1 \wedge \omega_2 + a\omega_1 \wedge \omega_3 + \omega_3 \wedge \omega_4 + a\omega_2\omega_4 \\ (2a^3 + \frac{1}{2}a^2b^2)\omega_{12} + \frac{ab^2}{2}\omega_{13} + b\omega_{23} + a^2b\omega_{14} + 2a^2\omega_{24} \\ ab(-a - \frac{1}{4})\omega_{12} + ab\omega_{13} + (a - \frac{1}{4})\omega_{23} + (a^2 - \frac{ab^2}{4})\omega_{14} - ab\omega_{24} \end{bmatrix},$$

where a and b are constants.

By the structure equation

$$d(\omega_1 \wedge \omega_2 \wedge \omega_3) = 2a\,\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4\,,$$

$$d(\omega_1 \wedge \omega_2 \wedge \omega_4) = -b\,\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4$$

If $a \neq 0$ or $b \neq 0$, there does not exist a compact quotient that supports a homogeneous Lagrangian Engel structure in case 4. If $a = 0$ and $b = 0$, this is a special case of case 1 with compact quotients.

3.3.5 Analysis of Case 5

The structure equation is

$$d\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix} = \begin{bmatrix} \omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_4 \\ a\omega_1 \wedge \omega_2 + \omega_3 \wedge \omega_4 \\ a(\omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_4) \\ -a\omega_1 \wedge \omega_2 - \omega_3 \wedge \omega_4 \end{bmatrix},$$

where a is a constant.

By the structure equation

$$d(\omega_1 \wedge \omega_2 \wedge \omega_4) = -2\,\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4.$$

Thus there does not exist a compact quotient that supports a homogeneous Lagrangian Engel structure in case 5.

3.3.6 Analysis of Case 6

The structure equation is

$$d\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix} = \begin{bmatrix} \omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_4 \\ \omega_3 \wedge \omega_4 \\ 0 \\ a\omega_2 \wedge \omega_3 - \omega_3 \wedge \omega_4 \end{bmatrix},$$

where a is a constant.

By the structure equation

$$d(\omega_1 \wedge \omega_2 \wedge \omega_4) = -2\,\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4.$$

Thus there does not exist a compact quotient that supports a homogeneous Lagrangian Engel structure in case 6.

□

4

Conclusion

There are some open and interesting problems related to Engel structures and other compatible geometric structures. Let us list a few:

1. A *symplectic Engel structure* (M, Ω, D) is a 4-manifold M endowed with *a symplectic form* Ω *and an Engel 2-plane field* D *that is symplectic for* Ω.

   In the dissertation, we studied the geometry of complex Engel structures and Lagrangian Engel structures. The obvious problem is to classify homogeneous symplectic Engel structures and compact homogeneous symplectic Engel structures.

2. S. T. Yau proved the following theorem [Yau (1976)]:

   Theorem 4.0.1. *Let* M *be a compact two dimensional complex manifold whose tangent bundle is trivial in the topological sense. Then either*

   - *(i)* M *is a ruled surface of genus 1*
   - *(ii)* M *is covered by the complex torus or an elliptic fiber bundle over a compact curve of genus* > 1

- *(iii) M is the quotient space of $\mathbb{C}^2$ by some volume-preserving affine transformation group. The first Chern class of M is zero and the first Betti number is three.*
- *(iv) The first Betti number of M is one.*

Do there exist compatible complex Engel structures on these 4 classes of manifolds? If there exist complex Engel structures, can they be derived from certain variations of homogenous Engel structures?

3. Given a parallelizable 4-manifold M and a hypersurface $S \subset M$ with a contact structure on S, does there exist an Engel structure D on M, such that the contact structure can be realized as a transversal contact structure of the characteristic line field of D?
4. Is it possible to find a compact Engel manifold such that there exists only a finite family of closed integral curves of the characteristic line field?

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Biography

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