Supplementary Appendix to "Delegation and Nonmonetary Incentives"

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Supplementary Appendix (NOT FOR PUBLICATION)

A1 Additional proofs

Proof of Claim 1: If $\theta_2 = \theta_1$, the statement is trivial, so assume $\theta_2 > \theta_1$. Denote $y_i = y(\theta_i)$, $m_i = m(\theta_i)$ for $i = 1, 2$. By (6), we have

$$l^a(\theta_1, y_1) + m_1 \leq l^a(\theta_1, y_2) + m_2;$$
$$l^a(\theta_2, y_2) + m_2 \leq l^a(\theta_2, y_1) + m_1.$$

These may be rewritten as

$$\begin{aligned} m_2 - m_1 &\geq l^a(\theta_1, y_1) - l^a(\theta_1, y_2); \\ m_2 - m_1 &\leq l^a(\theta_2, y_1) - l^a(\theta_2, y_2). \end{aligned} \tag{A1}$$

Consequently,

$$l^a(\theta_2, y_1) - l^a(\theta_2, y_2) \geq l^a(\theta_1, y_1) - l^a(\theta_1, y_2), \tag{A2}$$

Suppose that $\theta_2 > \theta_1$, but $y_2 < y_1$. Then

$$l^a(\theta_2, y_1) - l^a(\theta_2, y_2) = \int_{y_2}^{y_1} \frac{\partial l^a(\theta_2, y)}{\partial y} dy$$
$$< \int_{y_2}^{y_1} \frac{\partial l^a(\theta_1, y)}{\partial y} dy = l^a(\theta_1, y_1) - l^a(\theta_1, y_2),$$

where the inequality follows from the assumption that $y_2 < y_1$ and the single-crossing condition: $\frac{\partial l^a(\theta_2,y)}{\partial y} - \frac{\partial l^a(\theta_1,y)}{\partial y} = \int_{\theta_1}^{\theta_2} \frac{\partial l^a(\theta,y)}{\partial \theta \partial y} d\theta < 0$. But this contradicts (2*). ■

Proof of Claim 2: Suppose $\{y^*, m^*\}_{\theta\in\Theta}$ minimizes L^p (8) subject to (9)–(11), and $\inf_{\theta\in\Theta} m^*(\theta) = \varepsilon > 0$. Consider m' such that $\forall\theta \in \Theta : m'(\theta) = m^*(\theta) - \varepsilon$. Then $\{y^*, m'\}_{\theta\in\Theta}$ satisfies (9)–(11), has $L^p(y^*(\cdot), m'(\cdot)) = L^p(y^*(\cdot), m^*(\cdot))$, and has $\inf_{\theta\in\Theta} m'(\theta) = 0$. ■

Proof of Claim 3: By (10), we have

$$\begin{aligned} L^a(\theta_1) &\leq l^a(\theta_1, y_2) + m_2 \\ &= L^a(\theta_2) + (l^a(\theta_1, y_2) - l^a(\theta_2, y_2)) \\ &\leq L^a(\theta_2) + |\theta_2 - \theta_1| \Delta_\theta, \end{aligned}$$

so

$$L^a(\theta_1) - L^a(\theta_2) \leq |\theta_2 - \theta_1| \Delta_\theta.$$

Similarly,

$$L^a(\theta_2) - L^a(\theta_1) \leq |\theta_2 - \theta_1| \Delta_\theta,$$

which imply claim (i).

To establish claims (ii) and (iii), note that from (10) we have, for any ε,

$$\begin{aligned} L^a(\theta_0) &\leq l^a(\theta_0, y(\theta_0+\varepsilon)) + m(\theta_0+\varepsilon) \\ &= L^a(\theta_0+\varepsilon) + l^a(\theta_0, y(\theta_0+\varepsilon)) - l^a(\theta_0+\varepsilon, y(\theta_0+\varepsilon)), \end{aligned}$$

and similarly

$$\begin{gathered} L^a(\theta_0+\varepsilon) \leq l^a(\theta_0+\varepsilon, y(\theta_0)) + m\left(\theta_0+\varepsilon^2\right) \\ \leq L^a\left(\theta_0+\varepsilon^2\right) + l^a\left(\theta_0+\varepsilon, y\left(\theta_0+\varepsilon^2\right)\right) - l^a\left(\theta_0+\varepsilon^2, y\left(\theta_0+\varepsilon^2\right)\right) \\ \leq L^a(\theta_0) + \varepsilon^2\Delta_\theta + l^a\left(\theta_0+\varepsilon, y\left(\theta_0+\varepsilon^2\right)\right) - l^a\left(\theta_0+\varepsilon^2, y\left(\theta_0+\varepsilon^2\right)\right). \end{gathered}$$

Hence,

$$\begin{gathered} \frac{l^a(\theta_0+\varepsilon, y(\theta_0+\varepsilon)) - l^a(\theta_0, y(\theta_0+\varepsilon))}{\varepsilon} \leq \frac{L^a(\theta_0+\varepsilon) - L^a(\theta_0)}{\varepsilon} \\ \leq \frac{\varepsilon^2\Delta_\theta + l^a\left(\theta_0+\varepsilon, y\left(\theta_0+\varepsilon^2\right)\right) - l^a\left(\theta_0+\varepsilon^2, y\left(\theta_0+\varepsilon^2\right)\right)}{\varepsilon} \end{gathered}$$

Both the left-hand side and the right-hand side have the same limit $\frac{\partial l^a\left(\theta_0, \lim_{\theta\to\theta_0+} y(\theta)\right)}{\partial\theta}$ as $\varepsilon \to 0+$, hence

$$\frac{d^r L^a(\theta_0)}{d\theta} = \lim_{\varepsilon\to 0+} \frac{L^a(\theta_0+\varepsilon) - L^a(\theta_0)}{\varepsilon} = \frac{\partial l^a\left(\theta_0, \lim_{\theta\to\theta_0+} y(\theta)\right)}{\partial\theta}.$$

We can prove the formula for the left derivative similarly. Since $\frac{\partial l^a(\theta_0, y)}{\partial\theta}$ is strictly monotonic in y, we have $\frac{\partial l^a\left(\theta_0, \lim_{\theta\to\theta_0-} y(\theta)\right)}{\partial\theta} = \frac{\partial l^a\left(\theta_0, \lim_{\theta\to\theta_0+} y(\theta)\right)}{\partial\theta}$ if and only if $\lim_{\theta\to\theta_0-} y(\theta) = \lim_{\theta\to\theta_0+} y(\theta)$. Since $y(\theta)$ is monotonic, this is equivalent to continuity of $y(\theta)$ at θ_0, so the result on continuity follows. ■

A2 Example with overshooting in the optimal contract

Under the regularity conditions in Parts 2 and 3 of Theorem 6, the optimal contract has the intuitive feature that the implemented action is always between the ideal points of the principal and the agent. Moreover, both the action scheme and the amount of money burning are continuous and weakly increasing functions of the state. Below we show that if the regularity conditions do not hold, the optimal contract might not have any of the above features (besides the implemented action being weakly increasing in the state, which is a general property by Claim 1). Moreover, we show that the violations of these properties are interrelated.

The next theorem establishes that if the optimal contract involves no overshooting (that is, if the implemented policy is always between the players' ideal points) then both the implemented action and money burning are continuous and increasing in the state.

Theorem A1 *Assume that $(y^*(\cdot), m^*(\cdot))$ is an optimal contract, and $\theta \leq y^*(\theta) \leq \theta + b(\theta)$ for every $\theta \in (0,1)$. Then both $y^*(\cdot)$ and $m^*(\cdot)$ are continuous and weakly increasing on $(0,1)$.*

We prove the above result by showing that the type of deviation considered in the proof of Part 2 of Theorem 6, that is making the jump more gradual by offering an in-between option to types around the jump point, increases the expected utility of the principal for arbitrary convex loss functions, as long as the jump is in between the ideal points of the players.

Proof of Theorem A1: Note that the requirement that $\frac{\frac{\partial l^a(\theta,y_1)}{\partial\theta}-\frac{\partial l^a(\theta,y_0)}{\partial\theta}}{l^p(\theta,y_0)-l^p(\theta,y_1)} > \frac{\frac{\partial l^a(\theta,y_1)}{\partial\theta}-\frac{\partial l^a(\theta,y_2)}{\partial\theta}}{l^p(\theta,y_2)-l^p(\theta,y_1)}$ for every $\theta \in (0,1)$ and $\inf_{\theta\in(\theta_0,1]} y^*(\theta) \geq y_2 > y_0 > y_1 \geq \theta$ holds whenever:

$$\frac{\frac{\partial l^a(\theta_0,\hat{y}_1)}{\partial\theta}-\frac{\partial l^a(\theta_0,y)}{\partial\theta}}{l^p(\theta_0,y)-l^p(\theta_0,\hat{y}_1)} \tag{A3}$$

is decreasing in y for $\inf_{\theta\in(\theta_0,1]} y^*(\theta) \geq y > \hat{y}_1$, for every $\theta_0 \in (0,1)$ and $\hat{y}_1 \geq \theta_0$. If $y^*(\theta) \leq \theta + b(\theta)$ for every $\theta \in (0,1)$, then $\inf_{\theta\in(\theta_0,1]} y^*(\theta) \leq \theta + b(\theta)$ for every $\theta \in (0,1)$. Then for every $\inf_{\theta\in(\theta_0,1]} y^*(\theta) \geq y > \hat{y}_1$, the numerator of (A3) is decreasing in y, while the denominator of (A3) is increasing in y, implying that (A3) is decreasing in y. The same arguments used in the proof of Part 2 of Theorem 6 then imply the above result. ■

Next, we construct an example in which the optimal contract indeed involves overshooting and discontinuities, as well as non-monotonicity of money burning. We also provide an intuitive explanation why overshooting is optimal for the principal.

To start with, consider a specification of the model in which the loss functions are of the form: $l^p(\theta, y(\theta)) = 2(y(\theta) - \theta)^2$, and $l^a(\theta, y(\theta)) = (y(\theta) - \theta - 0.05)^2$. This is a special case of the class of loss functions considered in Section 5, with $A = 2$ and $b = 0.05$. Moreover, temporarily assume that $f(\theta) = \frac{3}{2}$ for $\theta \in [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ and $f(\theta) = 0$ for $\theta \in (\frac{1}{3}, \frac{2}{3})$ (below we change the example so that the density is strictly positive everywhere). It is easy to see that in this example the principal can solve its optimization separately for the regions $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. Using the results obtained in Section 4, the optimal contract specifies:

$$y^*(\theta) = \begin{cases} \theta + 0.05 & \text{if } \theta \leq \frac{7}{30}; \\ \frac{1}{6} + \frac{\theta}{2} & \text{if } \frac{7}{30} < \theta < \frac{1}{3} \end{cases}$$

and

$$y^*(\theta) = \begin{cases} \theta + 0.05 & \text{if } \frac{2}{3} \leq \theta \leq \frac{9}{10}; \\ \theta + \frac{1-\theta}{A} & \text{if } \frac{9}{10} < \theta. \end{cases}$$

Using (15), the amount of money burning implied by $y^*()$ at state $\theta = \frac{1}{3}$ is $\frac{1}{200}$ (twice the area between $y^*(\theta)$ and the agent's ideal line $\theta + 0.05$). Note that at this state the agent prefers action $y^*\left(\frac{1}{3}\right) = \frac{1}{3}$ and money burning $\frac{1}{200}$ to action $y^*\left(\frac{2}{3}\right) = \frac{43}{60}$ and money burning 0, and therefore the above $y^*(\theta)$ is incentive-compatible on $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ as long as $m^*\left(\frac{2}{3}\right) = 0$. On interval $(\frac{1}{3}, \frac{2}{3})$ the prior density is 0, therefore it does not matter how $y^*()$ and $m^*()$ are specified. For example the following specification achieves the optimum:

$$y^*(\theta) = \begin{cases} \frac{1}{20}\sqrt{2} + \frac{23}{60} & \text{if } \frac{1}{3} < \theta \leq 0.404\,04; \\ \theta + 0.05 & \text{if } 0.404\,04 < \theta < \frac{2}{3}. \end{cases}$$

This contract specifies overshooting at $\theta = 0.404\,04$, in order to bring the level of money burning back to 0. It is easy to verify that the utility that the above contract yields to the principal is bounded away from any contract that does not specify overshooting at any point of Θ (which would imply that the amount of money burning is monotonically increasing). Modify now the above example such that $f(\theta) = \frac{3}{2} - 2\varepsilon$ for $\theta \in [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ and $f(\theta) = \varepsilon$ for $\theta \in (\frac{1}{3}, \frac{2}{3})$. For small enough $\varepsilon > 0$ the above contract (which is still incentive-compatible, since the latter does not depend on the prior distribution) continues to yield a strictly higher payoff to the principal than any contract that does not specify overshooting. This establishes that the optimal contract requires overshooting (and hence a non-monotonic money burning scheme) in the modified example, too. It can also be shown that in this example the optimal contract requires a discontinuity at some state in $(\frac{1}{3}, \frac{2}{3})$.

The intuition behind the optimal contract involving overshooting is the following: if the implemented action is kept between the optimal points of the principal and the agent, the amount of prescribed money burning is increasing, and if the implemented action is kept strictly below the agent's ideal point the money burning is strictly increasing. The only way the principal can decrease money burning at some state in an incentive compatible way is if he prescribes an overshooting action. In the above example, this becomes optimal in the region $(\frac{1}{3}, \frac{2}{3})$, where the density of the prior is low. The optimal policy involves increasing money burning in low states, then overshooting in the region of unlikely states, and finally increasing money burning again in high states. Intuitively, the principal sacrifices utility in the unlikely states, in order to better align incentives in the more likely states and at the same time do not accumulate too high levels of money burning.

A3 Results for the Case with Contingent Transfers

We start by establishing properties of the optimal contract that are analogous to properties obtained in the case without transfers.

Claim A2 *There exists a solution to problem* (18) *subject to constraints* (19)–(21)*. Moreover, under the conditions stated in Theorem 6, for any optimal contract* $(y^*(\cdot), m^*(\cdot), t^*(\cdot))$ *the following hold:*

(i) $y^*(\cdot)$ *is weakly increasing on* $[0,1]$ *and continuous on* $(0,1)$*;*

(ii) $\theta \leq y^*(\theta) \leq \theta + b(\theta)$ *for all* $\theta \in [0,1]$*;*

(iii) for any $\theta \in (0,1)$*,*

$$\frac{dL^a(\theta)}{d\theta} = \frac{\partial l^a(\theta, y(\theta))}{\partial \theta},$$

and

$$L^a(\theta_2) - L^a(\theta_1) = \int_{\theta_1}^{\theta_2} \frac{\partial l^a(\theta, y(\theta))}{\partial \theta} d\theta;$$

(iv) for any $\theta_1, \theta_2 \in [0,1]$*,*

$$(m^*(\theta_2) - t^*(\theta_2)) - (m^*(\theta_1) - t^*(\theta_1)) =$$
$$l^a(\theta_1, y^*(\theta_1)) - l^a(\theta_2, y^*(\theta_2)) + \int_{\theta_1}^{\theta_2} \left(\frac{\partial l^a(\theta, y^*(\theta))}{\partial \theta} \right) d\theta.$$

Proof of Claim A2. The proofs of these results follow closely similar proofs for the case without conditional transfers, and are omitted.

The next result states that there is essentially no state at which there is both money burning and nonzero conditional transfer. An optimal contract specifies conditional transfers in low states (if there is a region with nonzero transfers). These transfers are decreasing in the state and at some point reach 0. At the right of this point an optimal contract might specify money burning, such that money burning is increasing in the state in this region.

Claim A3 *Suppose that the conditions stated in Theorem 6 hold. Then any two solutions to problem* (18) *subject to constraints* (19)–(21) *specify the same* $(y(\theta), m(\theta), t(\theta))$ *at almost every* $\theta \in \Theta$*. Moreover, there exists a solution which satisfies the following:*

(i) Either $m^*(\theta) = 0$ or $t^*(\theta) = 0$, for every $\theta \in \Theta$. Moreover, $m^*(\theta)$ is non-decreasing in θ and $t^*(\theta)$ is non-increasing in θ; in particular, $m^*(0) = 0$.

(ii) Either there exists $\theta_0 \in \Theta$ such that $m^*(\theta_0) = 0$ and $t^*(\theta_0) = 0$, or $t^*(\theta) > 0$ for all θ.

Proof of Claim A3. (i) Suppose that $m^*(\theta) > 0$ and $t^*(\theta) > 0$ for a positive measure of θ. Then there exists $\varepsilon > 0$ such that the measure of the set $\{\theta : m^*(\theta) > \varepsilon, t^*(\theta) > \varepsilon\}$ is positive. For all such θ's, let $m'(\theta) = m^*(\theta) - \varepsilon$ and $t'(\theta) = t^*(\theta) - \varepsilon$; in other cases, let $m'(\theta) = m^*(\theta)$ and $t'(\theta) = t^*(\theta)$. Then the contract $(y^*(\cdot), m'(\cdot), t'(\cdot))$ would satisfy all constraints and yield a higher payoff to the principal than $(y^*(\cdot), m^*(\cdot), t^*(\cdot))$, which is impossible. This contradiction proves that either $m^*(\theta) = 0$ or $t^*(\theta) = 0$ for almost all θ. Now, from Claim A2 we get that function $m^*(\theta) - t^*(\theta)$ is nondecreasing, and may without loss of generality assumed to be continuous. Therefore, we must have $t^*(\theta) = 0$ whenever $m^*(\theta) - t^*(\theta) > 0$ and $m^*(\theta) = 0$ whenever $m^*(\theta) - t^*(\theta) < 0$. Consequently, $m^*(\theta) = \max\{m^*(\theta) - t^*(\theta), 0\}$ and is therefore nondecreasing, while $t^*(\theta) = \max\{t^*(\theta) - m^*(\theta), 0\}$ is nonincreasing.

(ii) Without loss of generality, we may restrict attention to contracts with $m^*(0) = 0$. If this were not the case, we could take $m'(\theta) = m^*(\theta) - m(0) \geq 0$ since $m^*(\theta)$ is nondecreasing, and we would get a contract $(y^*(\cdot), m'(\cdot), t^*(\cdot))$ which would satisfy all constraints and have $m'(0) = 0$. Given that, consider function $t^*(\theta)$. If $t^*(\theta) = 0$ for some, then consider the supremum θ_0 of such points. By continuity of $t^*(\cdot)$ and $m^*(\cdot)$, we must have that $t^*(\theta_0) = m^*(\theta_0) = 0$. This completes the proof. ■

Proof of Theorem 11. Existence follows from Balder (1996), similarly to Theorem 6. Uniqueness follows from that the objective function is strictly convex in $y(\cdot)$, and the constraints are also convex (this may not immediately obvious for (25), but notice that local IC constraints in the are linear in $y(\cdot)$ and are sufficient for global IC constraints to hold; consequently, if (25) holds for some $y_1(\cdot)$ and $y_2(\cdot)$, then it also holds for any linear combination). Once the optimal $y^*(\cdot)$ is determined uniquely, $m(\theta) - t(\theta)$ is also determined uniquely up to an additive constant, and then Lemma A3 implies that $m(\theta) = \max(m(\theta) - t(\theta), 0)$ and $t(\theta) = \max(t(\theta) - m(\theta), 0)$, and thus the additive constant is also uniquely determined from the condition that either (24) or (26) must bind at the optimum. These considerations imply uniqueness.

Notice that the problem (23) s.t. (24)–(26) is equivalent to (27) s.t. (28)–(29) in the following sense. If some $(y^*(\cdot), m^*(\cdot), t^*(\cdot))$ solves (23) subject to the constraint, then there exist θ_0 and $t_0 = t(\theta_0)$ such that (28) and (29) are satisfied, and the value of the minimands in (23) and (27) are identical; this follows from Claim A3 and Claim A2 (it should be noted, however, that θ_0 need not be determined uniquely). Vice versa, if some $(y^*(\cdot), \theta_0^*, t_0^*)$ solve (27) s.t. (28)–(29), then we can define $m(\theta) = 0$ for $\theta \leq \theta_0^*$, $t(\theta) = t_0$ for $\theta \geq \theta_0^*$, and the values of $m(\theta)$ for $\theta > \theta_0^*$ and $t(\theta)$ for $\theta < \theta_0^*$ are defined so as to satisfy part (iv) of Claim A2; in that case, $(y^*(\cdot), m(\cdot), t(\cdot))$ will satisfy (24)–(26) and the values of the two minimands will again be the same. Consequently, it suffices to solve the problem (27) s.t. (28)–(29), which we do in what follows in each of the cases separately.

Part 1. Suppose that $\tilde{w} - \tilde{u} < -Ab\frac{1+A-Ab}{(A+1)^2}$. Here, similarly to the first case of Theorem 7, we solve the problem (27) s.t. (28) while ignoring (29), and then show that (29) is also satisfied. We use Luenberger's Theorem in the following way. We take the Lagrange multiplier $\lambda = 1$. We then take $y^*(\theta) = \frac{b}{A+1} + \theta$, $\theta_0^* = 1$, and $t_0^* = \tilde{u} - Ab\frac{1+A-Ab}{(A+1)^2}$. We need to establish that this combination of $y(\cdot)$, θ_0, and t_0 minimizes the following Lagrangian:

$$\min_{(y(\cdot),\theta_0,t_0)\in\mathcal{F}\times[0,1]\times\mathbb{R}} \left(\begin{array}{c} \int_0^1 A(y(\theta)-\theta)^2 d\theta + \int_0^{\theta_0} \left((y(\theta)-\theta-b)^2 - 2(y(\theta)-\theta-b)\theta\right) d\theta \\ -(y(\theta_0)-\theta_0-b)^2\theta_0 + t_0 + \lambda\left((y(\theta_0)-\theta_0-b)^2\right. \\ \left. -\int_0^1 2(y(\theta)-\theta-b)(1-\theta)d\theta + \int_0^{\theta_0} 2(y(\theta)-\theta-b)d\theta - t_0 + \tilde{u}\right) \end{array} \right).$$

Substituting $\lambda = 1$, simplifying and eliminating the constant, we need to prove that it minimizes

$$\min_{(y(\cdot),\theta_0,t_0)\in\mathcal{F}\times[0,1]\times\mathbb{R}} \left(\begin{array}{c} \int_0^1 A(y(\theta)-\theta)^2 d\theta + \int_0^{\theta_0} (y(\theta)-\theta-b)^2 + 2(y(\theta)-\theta-b)(1-\theta)d\theta \\ -\int_0^1 2(y(\theta)-\theta-b)(1-\theta)d\theta + (y(\theta_0)-\theta_0-b)^2(1-\theta_0) + \tilde{u}. \end{array} \right)$$

For a given $y(\cdot) \in \mathcal{F}$, this expression is nondecreasing in θ_0; indeed, if $y(\cdot)$ is differentiable at θ_0, then the derivative of this expression equals $2(y(\theta_0)-\theta_0-b)(1-\theta_0)y'(\theta_0) \geq 0$ (because $y(\theta_0) \leq \theta_0 + b$ and $y(\cdot)$ is nondecreasing), and if it is not differentiable, then a similar bound is true for the lower right-hand derivative, which is also nonnegative. Thus, if for some $y(\cdot)$, θ_0, and t_0 the Lagrangian is smaller than under $y^*(\cdot), \theta_0^*, t_0^*$, then it is also the case for $\theta_0^* = 1$. But if $\theta_0 = 1$, the expression becomes

$$\int_0^1 \left(A(y(\theta)-\theta)^2 + (y(\theta)-\theta-b)^2\right) d\theta + \tilde{u},$$

which is minimized if $y(\theta) = \frac{b}{A+1} + \theta$, because this $y(\theta)$ minimizes the function under the integral for every θ. This means that the Lagrangian is indeed minimized at $y^*(\cdot), \theta_0^*, t_0^*$ (as it does not explicitly depend on t_0).

It is straightforward to check that (28) is satisfied as equality. Luenberger's theorem now implies that $y^*(\cdot), \theta_0^*, t_0^*$ solve (27) s.t. (28). It remains to check that (29) is satisfied, but this is equivalent to $\tilde{w} - \tilde{u} + Ab\frac{1+A-Ab}{(A+1)^2} \leq 0$, which is true in this case. Thus, $y^*(\cdot), \theta_0^*, t_0^*$ are the unique solution in this case.

Part 2. Suppose that $-Ab\frac{1+A-Ab}{(A+1)^2} \leq \tilde{w} - \tilde{u} \leq b + b^2$. Here, we again use Luenberger's theorem, this time applying it to the problem (27) s.t. (28) and (29) directly. We take Lagrange multipliers λ as defined in Subsection below and $\mu = 1-\lambda$, and set up the following minimization problem:

$$\min_{(y(\cdot),\theta_0,t_0)\in\mathcal{F}\times[0,1]\times\mathbb{R}} \left(\begin{array}{c} \int_0^1 A\left(y\left(\theta\right)-\theta\right)^2 d\theta + \int_0^{\theta_0}\left(\left(y\left(\theta\right)-\theta-b\right)^2 - 2\left(y\left(\theta\right)-\theta-b\right)\theta\right)d\theta \\ -\left(y\left(\theta_0\right)-\theta_0-b\right)^2\theta_0 + t_0 + \lambda\left(\left(y\left(\theta_0\right)-\theta_0-b\right)^2\right. \\ \left. -\int_0^1 2\left(y\left(\theta\right)-\theta-b\right)\left(1-\theta\right)d\theta + \int_0^{\theta_0} 2\left(y\left(\theta\right)-\theta-b\right)d\theta - t_0 + \tilde{u}\right) \\ +\mu\left(1-t_0\right) \end{array} \right). \tag{A4}$$

Then we take $y^*(\cdot)$ and θ_0^* as defined in the corresponding case of the respective characterization below, and let $t_0^* = \tilde{w}$. It is then straightforward to verify that both (28) and (29) hold as equality. It is less straightforward to verify, but nevertheless is true that this combination of $y^*(\cdot)$ and θ_0^* minimizes the Lagrangian (A4) (and t_0^* cancels out). To see this, the following auxiliary results (which are analogous to Lemma 1 and may be proved similarly) are of help: If $(y^*(\cdot), \theta_0^*, t_0^*)$ solves (A4), then for θ satisfying $y^*(0) < y^*(\theta) < y^*(\theta_0)$, it must be that $y^*(\theta) = \min\{z(\theta), \theta + b\}$, where $z(\theta) = 1 - \left(1 - \frac{\lambda}{A}\right)(1-\theta)$, and for θ satisfying $y^*(\theta_0) < y^*(\theta) < y^*(1)$, it must be that $y^*(\theta) = \min\{x(\theta), \theta + b\}$, where $x(\theta) = \frac{b}{1+A} + \left(1 + \frac{1-\lambda}{1+A}\right)\theta$. The problem then reduces to optimizing over θ_0 and possible values of $y(\theta_0)$; the corresponding calculations are extremely tedious and are available upon request.

Given the above, Luenberger's theorem implies that $y^*(\cdot)$, θ_0^*, and $t_0^* = \tilde{w}$ are indeed solutions to (27) s.t. (28) and (29).

Part 3. Suppose that $\tilde{w} - \tilde{u} > b + b^2$. This case is similar to the corresponding case of Theorem 7. Notice that the value of (27) cannot be smaller than $\tilde{w}$, as $t_0 \geq \tilde{w}$, and

$$\int_0^{\theta_0}\left(\left(y\left(\theta\right)-\theta-b\right)^2 - 2\left(y\left(\theta\right)-\theta-b\right)\theta\right)d\theta \geq \left(y\left(\theta_0\right)-\theta_0-b\right)^2\theta_0$$

as follows from integrating the second term of the integral by parts. This value is achieved for $\theta_0 = 0$, $t_0 = \tilde{w}$, and $y(\theta) = \theta$ for all θ. This automatically satisfies (29), and, moreover, the left-hand side of (28) is

$$(-b)^2 - \int_0^1 2(-b)(1-\theta)\,d\theta - \tilde{w} + \tilde{u} = b^2 + b - \tilde{w} + \tilde{u} \leq 0,$$

so (28) is also satisfied. Thus, this contract is indeed optimal. ■

Proof of Theorem 11. Analogously to Theorem 8, the proof follows from the explicit characterization given in Theorem 11 and is omitted. ■

Proof of Theorem 13. Analogously to Theorem 10, the proof follows from the explicit characterization given in Theorem 11 and is omitted. ■

A4 Explicit Characterization of Optimal Contract with Contingent Transfers

Here, we provide the explicit characterization of the optimal solution of problem (27) s.t. (28)–(29), and thus of problem (23) s.t. (24)–(26) (the expressions for $m^*(\cdot)$ and $t^*(\cdot)$ are straightforward to find, as argued in the proof of Theorem 11).

Define the following functions of A and b, wherever the functional form is a valid expression (informally, these correspond to similarly indexed lines on the Figures below):

$f_1(A,b) = -Ab\frac{1+A-Ab}{(A+1)^2}$;

$f_2(A,b) = \frac{1}{6}\left(\left(\frac{Ab}{A+1}\right)^2 - 4\frac{Ab}{A+1} + 1 - \left(\frac{Ab}{A+1}+1\right)\sqrt{1-6\frac{Ab}{A+1}+\left(\frac{Ab}{A+1}\right)^2}\right)$;

$f_3(A,b) = \frac{1}{3} - b + b^2 - \frac{1}{3}\frac{\left(A+1-2b-Ab+b\sqrt{A^2+2b-1}\right)^2}{(A+1-b)(A+1)}$;

$f_4(A,b) = \frac{1}{3}b^3\frac{1-A-A^2}{(1-A)(1-A^2)}$;

$f_5(A,b) = \frac{b}{3(1+A)}\left(\frac{1}{2} - Ab - \left(A+\frac{1}{2}\right)\sqrt{1-4Ab}\right)$;

$f_6(A,b) = \frac{b}{3(1+A)}\left(\frac{1}{2} - Ab + \left(A+\frac{1}{2}\right)\sqrt{1-4Ab}\right)$;

$f_7(A,b) = \frac{1}{3} + Ab\frac{1+A+Ab}{(A+1)^2} - \frac{1}{3}\left(\frac{A-b+1}{A+1}\right)^3\left(\frac{1}{1-\frac{b(A+1)}{A+2Ab+1}\left(1-\sqrt{1-\frac{A+2Ab+1}{(A+1)^3}}\right)}\right)^2$

$f_8(A,b) = b + b^2$;

$f_9(A,b) = \frac{1}{6}\left(\left(\frac{Ab}{A+1}\right)^2 - 4\frac{Ab}{A+1} + 1 + \left(\frac{Ab}{A+1}+1\right)\sqrt{1-6\frac{Ab}{A+1}+\left(\frac{Ab}{A+1}\right)^2}\right)$;

$f_{10}(A,b) = b^2 - b + \frac{1}{3}$;

$f_{11}(A,b) = Ab\frac{1+A+Ab}{(A+1)^2} + \frac{1}{3}$.

In addition, define the following functions of A and b (informally, these correspond to values of the Lagrange multiplier λ, wherever applicable).

$\lambda_1(A,b) = 1$;

$\lambda_2(A,b) = \frac{1+Ab-A+\sqrt{(A+1)^2-Ab(6+6A-Ab)}}{2}$;

$\lambda_3(A,b) = 1 - \frac{b}{A+\sqrt{A^2+2b-1}}$;

$\lambda_4(A,b) = A$;

$\lambda_5(A,b) = \frac{1+\sqrt{1-4Ab}}{2}$;

$\lambda_6(A,b) = \frac{1-\sqrt{1-4Ab}}{2}$;

$\lambda_7(A,b) = A\left(1-\frac{A-b+1}{\sqrt{(A+1)(A+2Ab+1)}}\right)$

$\lambda_8(A,b) = 0$;

$\lambda_9(A,b) = \frac{1+Ab-A-\sqrt{(A+1)^2-Ab(6+6A-Ab)}}{2}$;

$\lambda_{10}(A,b) = \frac{A}{2b-1}$;

$\lambda_{11}(A,b) = A\frac{2b-A-1}{A+2Ab+1}$.

Finally, define the following expressions:
$x(\theta, A, b, \lambda) = \frac{b}{1+A} + \left(1 + \frac{1-\lambda}{1+A}\right)\theta;$
$z(\theta, A, b, \lambda) = 1 - \left(1 - \frac{\lambda}{A}\right)(1 - \theta);$

Case 1: "Social optimum"

If $\tilde{w} - \tilde{u} < f_1(A, b)$, then:

$$y^*(\theta) = \theta + \frac{b}{A+1}. \tag{A5}$$

The values $\theta_0^* = 1$, $t_0^* = \tilde{u} - Ab\frac{A-Ab+1}{(A+1)^2}$.

In this case, there is no money burning $m(\theta) = 0$ for all θ; transfer $t(\theta) > \tilde{w}$ in all states, with smallest amount $t(1) = t_0^* = \tilde{u} - Ab\frac{A-Ab+1}{(A+1)^2} > \tilde{w}$.

Case 2: "Transfer for low actions, then a cap"

If $b < 2 - \sqrt{2}$, $A < 1 - b$ and $f_1(A, b) \leq \tilde{w} - \tilde{u} < f_2(A, b)$, then:

$$y^*(\theta) = \begin{cases} x(\theta) & \text{if } \theta < q \\ x(q) & \text{if } \theta \geq q \end{cases}, \tag{A6}$$

$$\text{where } q = \frac{(A+\lambda) - \sqrt{\frac{(1-\lambda)(2Ab(A+1)-(1-\lambda)(A+\lambda))}{A-\lambda+2}}}{A+1},$$

where λ is the unique solution to

$$(x(q(\lambda)) - b)^2 - \int_0^{q(\lambda)} 2(x(q(\lambda)) - \theta - b)(1-\theta)\,d\theta - \int_{q(\lambda)}^1 2(x(\theta) - \theta - b)(1-\theta)\,d\theta + \tilde{u} = \tilde{w}$$

such that $\lambda_2(A, b) < \lambda \leq \lambda_1(A, b)$.

If $b < 2 - \sqrt{2}$, $A \geq 1 - b$ and $f_1(A, b) \leq \tilde{w} - \tilde{u} < f_3(A, b)$, then $y^*(\theta)$ is given by (A6), where λ is such that $\lambda_3(A, b) < \lambda \leq \lambda_1(A, b)$.

If $b \geq 2 - \sqrt{2}$, $A < \frac{3-2\sqrt{2}}{b-3+2\sqrt{2}}$ and $f_1(A, b) \leq \tilde{w} - \tilde{u} < f_2(A, b)$, then $y^*(\theta)$ is given by (A6), where λ is such that $\lambda_2(A, b) < \lambda \leq \lambda_1(A, b)$.

If $2 - \sqrt{2} \leq b < 1$, $1 - b \leq A < \frac{3-2\sqrt{2}}{b-3+2\sqrt{2}}$ and $f_9(A, b) \leq \tilde{w} - \tilde{u} < f_3(A, b)$, then $y^*(\theta)$ is given by (A6), where λ is such that $\lambda_3(A, b) < \lambda \leq \lambda_9(A, b)$.

If $1 \leq b < 4 - 2\sqrt{2}$, $A < b - 1$ and $f_9(A, b) \leq \tilde{w} - \tilde{u} < f_{11}(A, b)$, then $y^*(\theta)$ is given by (A6), where λ is such that $\lambda_{11}(A, b) < \lambda \leq \lambda_9(A, b)$.

If $1 \leq b < 4 - 2\sqrt{2}$, $b - 1 \leq A < \frac{3-2\sqrt{2}}{b-3+2\sqrt{2}}$ and $f_9(A, b) \leq \tilde{w} - \tilde{u} < f_3(A, b)$, then $y^*(\theta)$ is given by (A6), where λ is such that $\lambda_3(A, b) < \lambda \leq \lambda_9(A, b)$.

If $2 - \sqrt{2} \leq b < 4 - 2\sqrt{2}$, $A \geq \frac{3-2\sqrt{2}}{b-3+2\sqrt{2}}$ and $f_1(A, b) \leq \tilde{w} - \tilde{u} < f_3(A, b)$, then $y^*(\theta)$ is given by (A6), where λ is such that $\lambda_3(A, b) < \lambda \leq \lambda_1(A, b)$.

If $b \geq 4 - 2\sqrt{2}$, $A < \frac{3-2\sqrt{2}}{b-3+2\sqrt{2}}$ and $f_9(A, b) \leq \tilde{w} - \tilde{u} < f_{11}(A, b)$, then $y^*(\theta)$ is given by (A6), where λ is such that $\lambda_{11}(A, b) < \lambda \leq \lambda_9(A, b)$.

If $b \geq 4 - 2\sqrt{2}$, $\frac{3-2\sqrt{2}}{b-3+2\sqrt{2}} \leq A < b - 1$ and $f_1(A,b) \leq \tilde{w} - \tilde{u} < f_{11}(A,b)$, then $y^*(\theta)$ is given by (A6), where λ is such that $\lambda_{11}(A,b) < \lambda \leq \lambda_1(A,b)$.

If $b \geq 4 - 2\sqrt{2}$, $A \geq b - 1$ and $f_1(A,b) \leq \tilde{w} - \tilde{u} < f_3(A,b)$, then $y^*(\theta)$ is given by (A6), where λ is such that $\lambda_3(A,b) < \lambda \leq \lambda_1(A,b)$.

The values $\theta_0^* = q$ (or any number on $[q,1]$), $t_0^* = 0$.

Furthermore, in all these cases there is no money burning: $m(\theta) = 0$; there is minimal transfer $t(\theta) = \tilde{w}$ for $\theta \geq q$, and there is higher than minimal transfer $t(\theta) > \tilde{w}$ for $\theta < q$.

Case 3: "Transfer for low actions, then some actions with no transfer, then a cap"

If $b < 1$, $A < 1 - b$ and $f_2(A,b) \leq \tilde{w} - \tilde{u} < f_4(A,b)$, then:

$$y^*(\theta) = \begin{cases} x(\theta) & \text{if } \theta < \frac{Ab}{1-\lambda} \\ \theta + b & \text{if } \frac{Ab}{1-\lambda} \leq \theta \leq 1 - \frac{2Ab}{A+\lambda} \\ 1 + \frac{\lambda - A}{A+\lambda} b & \text{if } \theta > 1 - \frac{2Ab}{A+\lambda} \end{cases}, \tag{A7}$$

where λ is the unique solution to

$$\int_0^{\frac{Ab}{1-\lambda}} 2(x(\theta) - \theta - b)\theta d\theta - \int_{1+\frac{\lambda-A}{A+\lambda}b}^1 2\left(1 + \frac{\lambda - A}{A+\lambda} b - \theta - b\right)(1-\theta)\,d\theta + \tilde{u} = \tilde{w}$$

such that $\lambda_4(A,b) < \lambda \leq \lambda_2(A,b)$.

If $b < 1$, $1 - b \leq A < \frac{3-2\sqrt{2}}{b-3+2\sqrt{2}}$ and $f_2(A,b) \leq \tilde{w} - \tilde{u} < f_9(A,b)$, then $y^*(\theta)$ is given by (A7), where λ is such that $\lambda_9(A,b) < \lambda \leq \lambda_2(A,b)$.

If $b \geq 1$, $A < \frac{3-2\sqrt{2}}{b-3+2\sqrt{2}}$ and $f_2(A,b) \leq \tilde{w} - \tilde{u} < f_9(A,b)$, then $y^*(\theta)$ is given by (A7), where λ is such that $\lambda_9(A,b) < \lambda \leq \lambda_2(A,b)$.

The values $\theta_0^* = \frac{Ab}{1-\lambda}$ (or any number on $\left[\frac{Ab}{1-\lambda}, 1 - \frac{2Ab}{A+\lambda}\right]$), $t_0^* = 0$.

Furthermore, in all these cases there is no money burning: $m(\theta) = 0$; there is minimal transfer $t(\theta) = \tilde{w}$ for $\theta \geq \frac{Ab}{1-\lambda}$, and there is higher than minimal transfer $t(\theta) > \tilde{w}$ for $\theta < \frac{Ab}{1-\lambda}$.

Case 4: "Constant action"

If $b \geq 1$, $A < b - 1$ and $f_{11}(A,b) \leq \tilde{w} - \tilde{u} < f_{10}(A,b)$, then:

$$y^*(\theta) = \frac{1}{2} + \frac{\lambda b}{A+\lambda}, \tag{A8}$$

where λ is the unique solution to

$$\left(\frac{1}{2} + \frac{\lambda b}{A+\lambda} - b\right)^2 - \int_0^1 2\left(\frac{1}{2} + \frac{\lambda b}{A+\lambda} - \theta - b\right)(1-\theta)\,d\theta + \tilde{u} = \tilde{w}$$

such that $\lambda_{10}(A,b) < \lambda \leq \lambda_{11}(A,b)$. In this case, the contract may be simplified to

$$y^*(\theta) = \frac{1}{2} + b - \sqrt{\tilde{w} - \tilde{u} - \frac{1}{12}}.$$

The values $\theta_0^* = 0$ (or any number on $[0,1]$), $t_0^* = 0$.

Furthermore, in this case there is no money burning: $m(\theta) = 0$; there is minimal transfer $t(\theta) = \tilde{w}$ for all θ.

Case 5: "Transfer for low actions, then some actions with no transfer, then money burning"

If $b < 1$, $A < 1 - b$ and $f_4(A,b) \le \tilde{w} - \tilde{u} < f_6(A,b)$, then:

$$y^*(\theta) = \begin{cases} x(\theta) & \text{if } \theta < \frac{Ab}{1-\lambda} \\ \theta + b & \text{if } \frac{Ab}{1-\lambda} \le \theta \le 1 - \frac{Ab}{\lambda} \\ z(\theta) & \text{if } \theta > 1 - \frac{Ab}{\lambda} \end{cases}, \tag{A9}$$

where λ is the unique solution to

$$\int_0^{\frac{Ab}{1-\lambda}} 2(x(\theta) - \theta - b)\theta d\theta - \int_{1-\frac{Ab}{\lambda}}^1 2(z(\theta) - \theta - b)(1-\theta)d\theta + \tilde{u} = \tilde{w}$$

such that $\lambda_6(A,b) < \lambda \le \lambda_4(A,b)$.

The values $\theta_0^* = \frac{Ab}{1-\lambda}$ (or any number on $\left[\frac{Ab}{1-\lambda}, \frac{Ab}{\lambda}\right]$), $t_0^* = 0$.

Furthermore, in this case there is no money burning $m(\theta) = 0$ for $\theta \le 1 - \frac{Ab}{\lambda}$ and positive money-burning $m(\theta) > 0$ for $\theta \ge 1 - \frac{Ab}{\lambda}$; there is minimal transfer $t(\theta) = \tilde{w}$ for $\theta \ge \frac{Ab}{1-\lambda}$, and there is higher than minimal transfer $t(\theta) > \tilde{w}$ for $\theta < \frac{Ab}{1-\lambda}$.

Case 6: "Transfer for low actions, then some actions with no incentive, then money burning"

If $b < \frac{1}{2}$, $1 - b \le A < \frac{1}{4b}$ and $f_3(A,b) \le \tilde{w} - \tilde{u} < f_5(A,b)$, then:

$$y^*(\theta) = \begin{cases} x(\theta) & \text{if } \theta < q_1 \\ x(q_1) & \text{if } q_1 \le \theta \le q_2 \\ z(\theta) & \text{if } \theta > q_2 \end{cases}, \tag{A10}$$

where $q_1 = \dfrac{A+1}{(A-\lambda+2)\left(1 + \frac{1-\lambda}{\lambda}\sqrt{\frac{A-\lambda}{A-\lambda+2}}\right)} - \dfrac{b}{(A-\lambda+2)\left(1 + \frac{A+1}{A}\sqrt{\frac{A-\lambda}{A-\lambda+2}}\right)}$

and $q_2 = \dfrac{\left(A^2 - 2A\lambda + 2A - 1\right)\lambda}{(A-\lambda)(A\lambda+1) + A(1-\lambda)(A-\lambda+2)\sqrt{\frac{A-\lambda}{A-\lambda+2}}} + \dfrac{b}{\frac{(A+1)(A-\lambda)}{A} + (A-\lambda+2)\sqrt{\frac{A-\lambda}{A-\lambda+2}}}$,

where λ is the unique solution to

$$(x(q_1) - q_1 - b)^2 + \int_0^{q_1} 2(x(\theta) - \theta - b)\theta d\theta - \int_{q_1}^{q_2} 2(x(q_1) - \theta - b)(1-\theta)d\theta$$
$$- \int_{q_2}^1 2(z(\theta) - \theta - b)(1-\theta)d\theta + \tilde{u} = \tilde{w}$$

such that $\lambda_5(A,b) < \lambda \le \lambda_3(A,b)$.

If $b < \frac{1}{2}$, $A < \frac{1}{4b}$ and $f_6(A,b) \le \tilde{w} - \tilde{u} < f_7(A,b)$, then $y^*(\theta)$ is given by (A10), where λ is such that $\lambda_7(A,b) < \lambda \le \lambda_6(A,b)$.

If $b < \frac{1}{2}$, $A \geq \frac{1}{4b}$ and $f_3(A,b) \leq \tilde{w} - \tilde{u} < f_7(A,b)$, then $y^*(\theta)$ is given by (A10), where λ is such that $\lambda_7(A,b) < \lambda \leq \lambda_3(A,b)$.

If $\frac{1}{2} \leq b < 1$, $A < 1-b$ and $f_6(A,b) \leq \tilde{w} - \tilde{u} < f_7(A,b)$, then $y^*(\theta)$ is given by (A10), where λ is such that $\lambda_7(A,b) < \lambda \leq \lambda_6(A,b)$.

If $\frac{1}{2} \leq b < 1$, $A \geq 1-b$ and $f_3(A,b) \leq \tilde{w} - \tilde{u} < f_7(A,b)$, then $y^*(\theta)$ is given by (A10), where λ is such that $\lambda_7(A,b) < \lambda \leq \lambda_3(A,b)$.

If $b \geq 1$, $A \geq b-1$ and $f_3(A,b) \leq \tilde{w} - \tilde{u} < f_7(A,b)$, then $y^*(\theta)$ is given by (A10), where λ is such that $\lambda_7(A,b) < \lambda \leq \lambda_3(A,b)$.

The values $\theta_0^* = q_1$ (or any number on $[q_1, q_2]$), $t_0^* = 0$.

Furthermore, in this case there is no money burning $m(\theta) = 0$ for $\theta \leq q_2$ and positive money-burning $m(\theta) > 0$ for $\theta > q_2$; there is minimal transfer $t(\theta) = \tilde{w}$ for $\theta \geq q_1$, and there is higher than minimal transfer $t(\theta) > \tilde{w}$ for $\theta < q_1$.

Case 7: "Free action, then money-burning"

If $b < 1$ and $f_7(A,b) \leq \tilde{w} - \tilde{u} < f_8(A,b)$, then:

$$y^*(\theta) = \begin{cases} z(q) & \text{if } \theta \leq q \\ z(\theta) & \text{if } \theta > q \end{cases}, \tag{A11}$$

$$\text{where } q = \frac{\sqrt{\frac{\lambda}{A-\lambda}\left(2A^2 b - \lambda(A+\lambda)\right)} - \lambda}{A},$$

where λ is the unique solution to

$$(z(q) - b)^2 - \int_0^q 2(z(q) - \theta - b)(1-\theta)\,d\theta - \int_q^1 2(z(\theta) - \theta - b)(1-\theta)\,d\theta + \tilde{u} = \tilde{w}$$

such that $\lambda_8(A,b) < \lambda \leq \lambda_7(A,b)$.

If $b \geq 1$, $A < b-1$ and $f_{10}(A,b) \leq \tilde{w} - \tilde{u} < f_8(A,b)$, then $y^*(\theta)$ is given by (A11), where λ is such that $\lambda_8(A,b) < \lambda \leq \lambda_{10}(A,b)$.

If $b \geq 1$, $A \geq b-1$ and $f_7(A,b) \leq \tilde{w} - \tilde{u} < f_8(A,b)$, then $y^*(\theta)$ is given by (A11), where λ is such that $\lambda_8(A,b) < \lambda \leq \lambda_7(A,b)$.

The values $\theta_0^* = q$ (or any number on $[0, q]$), $t_0^* = 0$.

Furthermore, in this case there is no money burning $m(\theta) = 0$ for $\theta \leq q$ and positive money-burning $m(\theta) > 0$ for $\theta > q$; there is minimal transfer $t(\theta) = \tilde{w}$ for all θ.

Case 8: "Principal's ideal action"

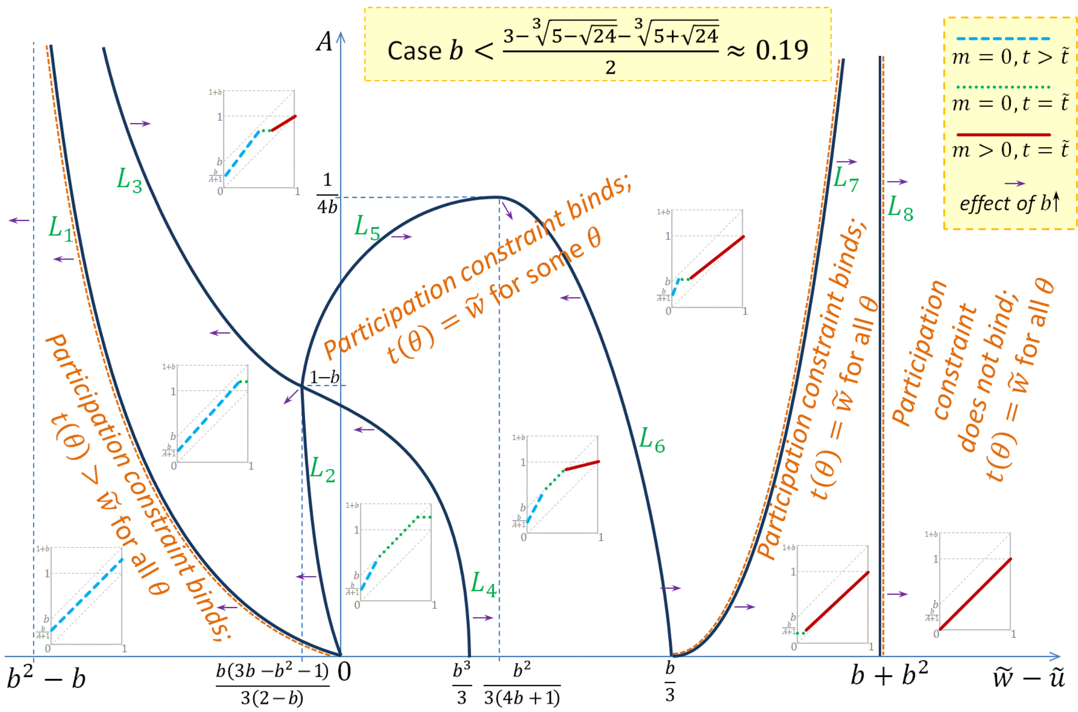
If $\tilde{w} - \tilde{u} \geq f_8(A,b)$, then

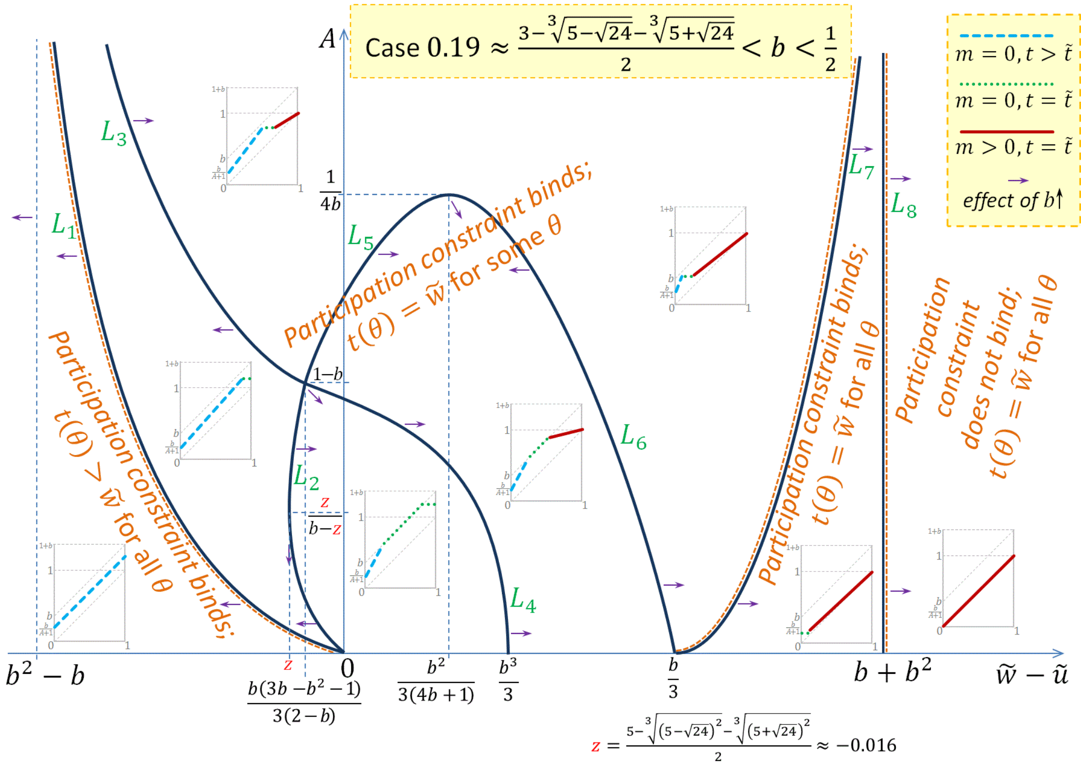
$$y^*(\theta) = \theta. \tag{A12}$$

The values $\theta_0^* = 0$, $t_0^* = 0$.

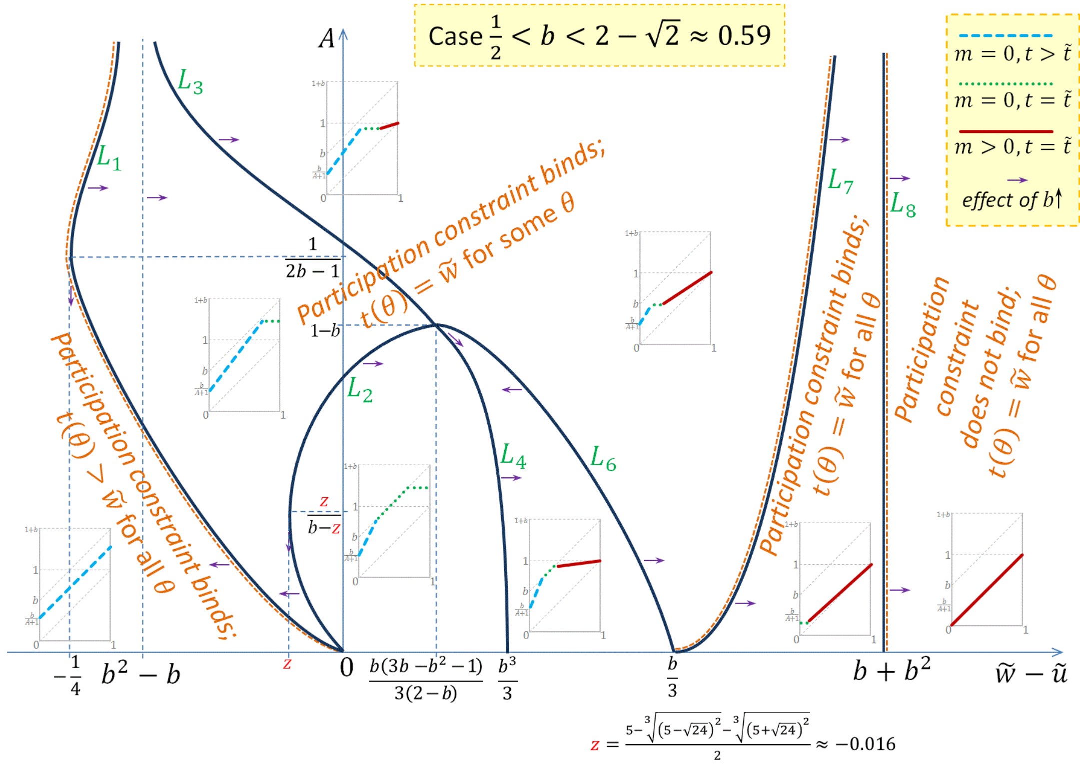
In this case, there is no money burning $m(\theta) = 0$ for $\theta = 0$ and positive money-burning $m(\theta) > 0$ for $\theta > 0$; there is minimal transfer $t(\theta) = \tilde{w}$ for all θ.

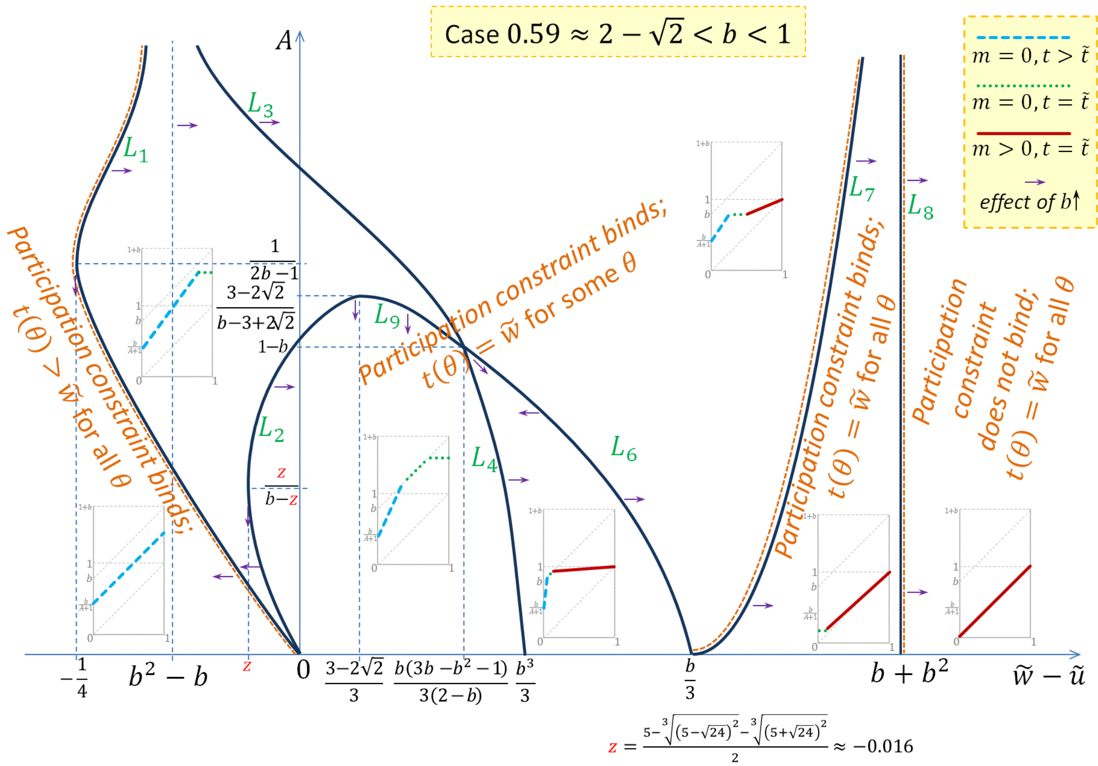
A5 Illustraions for the Case with Conditional Transfers



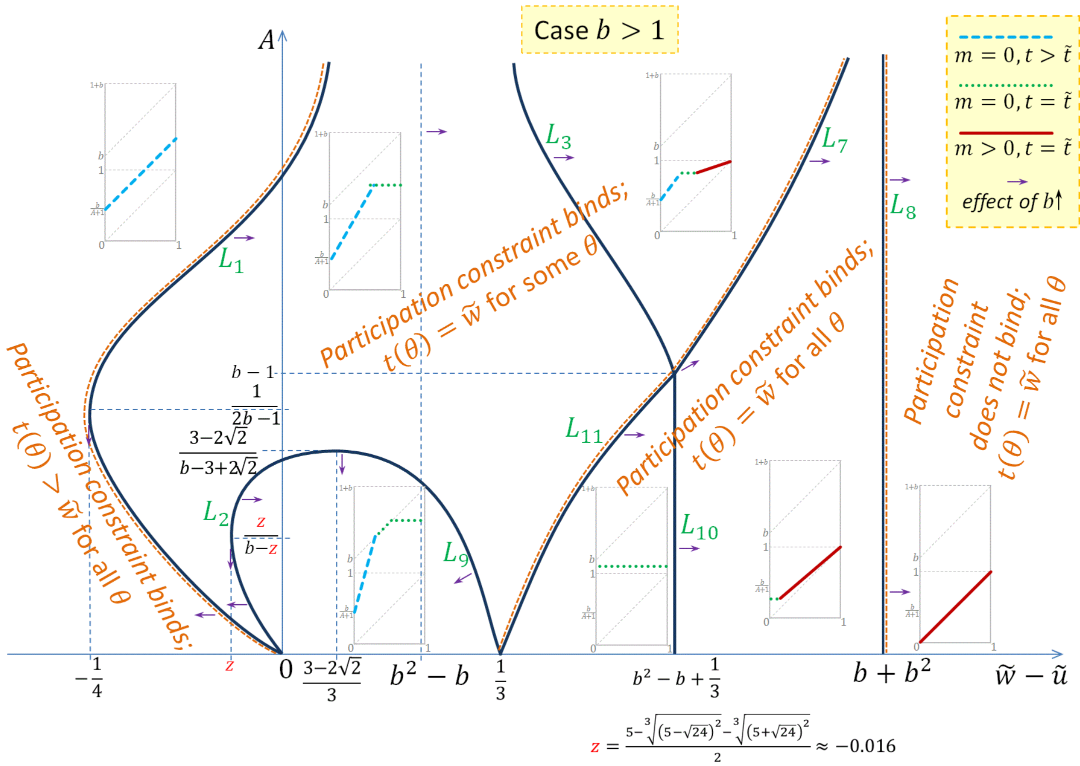


Case 0.19 ≈ (3−∛(5−√24)−∛(5+√24))/2 < b < 1/2
m = 0, t > t̃
m = 0, t = t̃
m > 0, t = t̃
effect of b↑
A
L1
L2
L3
L4
L5
L6
L7
L8
1/4b
1−b
z/(b−z)
Participation constraint binds; t(θ) > w̃ for all θ
Participation constraint binds; t(θ) = w̃ for some θ
Participation constraint binds; t(θ) = w̃ for all θ
Participation constraint does not bind; t(θ) = w̃ for all θ
b² − b
z
0
b(3b − b² − 1)/3(2 − b)
b²/3(4b + 1)
b³/3
b/3
b + b²
w̃ − ũ
z = (5 − ∛((5−√24)²) − ∛((5+√24)²))/2 ≈ −0.016


Case $\frac{1}{2} < b < 2-\sqrt{2} \approx 0.59$
$m=0, t>\tilde{t}$
$m=0, t=\tilde{t}$
$m>0, t=\tilde{t}$
effect of $b\uparrow$
A
L_1
L_2
L_3
L_4
L_6
L_7
L_8
$\frac{1}{2b-1}$
$1-b$
$\frac{z}{b-z}$
Participation constraint binds; $t(\theta) = \tilde{w}$ for some θ
Participation constraint binds; $t(\theta) > \tilde{w}$ for all θ
Participation constraint binds; $t(\theta) = \tilde{w}$ for all θ
Participation constraint does not bind; $t(\theta) = \tilde{w}$ for all θ
$-\frac{1}{4}$
b^2-b
z
0
$\frac{b(3b-b^2-1)}{3(2-b)}$
$\frac{b^3}{3}$
$\frac{b}{3}$
$b+b^2$
$\tilde{w}-\tilde{u}$
$z = \frac{5-\sqrt[3]{(5-\sqrt{24})^2}-\sqrt[3]{(5+\sqrt{24})^2}}{2} \approx -0.016$



Case 0.59 ≈ 2 − √2 < b < 1
m = 0, t > t̃
m = 0, t = t̃
m > 0, t = t̃
effect of b↑
A
L1
L2
L3
L4
L6
L7
L8
L9
Participation constraint binds; t(θ) > w̃ for all θ
Participation constraint binds; t(θ) = w̃ for some θ
Participation constraint binds; t(θ) = w̃ for all θ
Participation constraint does not bind; t(θ) = w̃ for all θ
1/(2b−1)
(3−2√2)/(b−3+2√2)
1−b
z/(b−z)
−1/4
b² − b
z
0
(3−2√2)/3
b(3b − b² − 1)/(3(2−b))
b³/3
b/3
b + b²
w̃ − ũ
z = (5 − ∛((5−√24)²) − ∛((5+√24)²))/2 ≈ −0.016


Case $b > 1$
$m = 0, t > \tilde{t}$
$m = 0, t = \tilde{t}$
$m > 0, t = \tilde{t}$
effect of b↑
A
L_1
L_2
L_3
L_7
L_8
L_9
L_{10}
L_{11}
Participation constraint binds; $t(\theta) = \tilde{w}$ for some θ
Participation constraint binds; $t(\theta) = \tilde{w}$ for all θ
Participation constraint binds; $t(\theta) > \tilde{w}$ for all θ
Participation constraint does not bind; $t(\theta) = \tilde{w}$ for all θ
$b-1$
$\frac{1}{2b-1}$
$\frac{3-2\sqrt{2}}{b-3+2\sqrt{2}}$
$\frac{z}{b-z}$
$-\frac{1}{4}$
z
0
$\frac{3-2\sqrt{2}}{3}$
$b^2 - b$
$\frac{1}{3}$
$b^2 - b + \frac{1}{3}$
$b + b^2$
$\tilde{w} - \tilde{u}$
$z = \frac{5-\sqrt[3]{(5-\sqrt{24})^2}-\sqrt[3]{(5+\sqrt{24})^2}}{2} \approx -0.016$